Received: 17 April 2020 | Revised: 2 October 2020 | Accepted: 29 October 2020

DOI: 10.1002/rnc.5321

RESEARCH ARTICLE

WILEY

Event-triggered control of input-affine nonlinear interconnected systems using multiplayer game

Vignesh Narayanan[1] | Hamidreza Modares[2] | Sarangapani Jagannathan[3]

[1]Department of Electrical and Systems Engineering, Washington University, St. Louis, Missouri, USA

[2]Department of Mechanical Engineering, Michigan State University, East Lansing, Michigan, USA

[3]Department of Electrical and Computer Engineering, Missouri University of Science and Technology, Rolla, Missouri, USA

Correspondence
Vignesh Narayanan, Department of Electrical and Systems Engineering, Washington University, St. Louis, MO 63130, USA.
Email: vnxv4@umsystem.edu

Funding information
Division of Civil, Mechanical and Manufacturing Innovation, Grant/Award Number: 1547042; Division of Electrical, Communications and Cyber Systems, Grant/Award Number: 1406533

Abstract

In this article, we present a decentralized control scheme for regulating input-affine nonlinear interconnected systems. In particular, we propose a codesign strategy to synthesize a control policy and an event-triggering threshold at each subsystem of an interconnected system to simultaneously optimize the subsystem performance and reduce the computational burden on the controllers by enforcing aperiodic dynamic feedback. To this end, we formulate a differential game at every subsystem to design a decentralized control scheme in which we treat the control policy as the minimizing player and model the effect of interconnection inputs and the error introduced due to aperiodic feedback as a team of adversarial players. We then employ the solution to the proposed game for designing both the control policy and the event-triggering threshold at each subsystem. With the proposed approach, we also derive the conditions that guarantee the input-to-state stability of the overall system by leveraging the well-known small-gain theorem. Moreover, we show that these conditions, expressed in terms of the attenuation constants and penalty matrices introduced in the formulated game, are obtained as linear inequalities even when the dynamics of the subsystems are nonlinear. Finally, we illustrate the applicability of the proposed scheme to regulate interconnected systems using numerical examples.

KEYWORDS

decentralized control, event-triggered control, interconnected systems, optimal control

1 | INTRODUCTION

Control of large-scale interconnected systems is an active area of research due to its relevance in numerous emerging applications, including smart-grids, cyber-enabled manufacturing, traffic networks, and large-scale robotic systems. A centralized control scheme for such applications demands high communication cost, is prone to single-point-of-failure, and requires an accurate model of the structural interconnections.[1] On the other hand, in a decentralized control scheme, the interconnected system is decomposed into its subsystems, equipped with independent controllers, and are controlled using locally available feedback information.[1-3] In such decentralized control schemes, the control feedback loop at each subsystem is closed locally, and hence, a dedicated communication network to share information among the subsystems is not required. However, decentralized controllers may lead to unsatisfactory performance, and even instability, especially either if the interconnection strength is strong[3,4] or in the presence of unstable fixed modes.[5]

To systematically deal with the stability issues arising in decentralized controllers, robust control approaches based on small-gain theorem were introduced.[6-9] In these approaches,[6-9] an input-to-state stability (ISS) gain operator was derived, and conditions were imposed on this ISS gain operator to guarantee the existence of an ISS Lyapunov function for the overall system. However, they did not consider the task of optimizing the transient response performance of the subsystems. Optimality was later included as a design criterion in the decentralized control framework.[2,10-12] Besides optimality, it is also desirable to reduce the communication cost if the feedback loop consists of a local communication network at each subsystem, remove redundant control updates, and minimize the associated computational cost by using event-triggered controllers.[13-16] In many emerging applications, a reduction in computations and control updates can significantly reduce the implementation cost (e.g., in terms of battery life in wireless sensor networks). In this context, designing decentralized optimal controllers for a nonlinear interconnected system with event-triggered feedback is challenging since the small-gain theorem, one of the common theoretical framework that facilitates control design for an interconnected system, leads to nonlinear ISS gains.[9] The resulting (nonlinear) ISS gain conditions are much harder to evaluate when compared with its counterpart for linear systems.[9] Additionally, the error due to aperiodic feedback degrades the performance of the system. As a consequence, achieving a trade-off between performance optimization and computational cost for a nonlinear interconnected controlled system is crucial. Therefore, in this article, we propose a decentralized optimal control scheme for input-affine nonlinear (and linear) interconnected systems. The proposed control scheme undertakes the task of optimizing the transient response performance of each subsystem with a reduced computational cost and ensures the overall stability of the interconnected system.

In particular, to mitigate the aforementioned challenges, in the proposed approach, we consider each subsystem to be independent and model the effects of interconnections and the event-triggering errors as adversarial inputs. In this context, the control design problem can be viewed as a noncooperative multiplayer dynamic game,[17-20] where the control input tries to minimize the subsystem cost competing against a team of two adversarial players. The two adversarial players model the effect of interconnections and the event-triggering errors, both of which strive to maximize the performance cost. We propose to use the Nash solution to this dynamic game for synthesizing the control policy and the event-triggering threshold at each subsystem. Moreover, with the decentralized control policy at each subsystem, the conditions that warrant stability of the closed-loop subsystems using local feedback information are established. Specifically, we derive the sufficient conditions that need to be satisfied to ensure that each controlled subsystem is ISS with respect to the interconnection inputs, and to guarantee that the resulting overall system is also ISS.

The sufficient conditions that are derived within the proposed framework not only ensure ISS of each subsystem and the overall system but also yield linear inequality constraints that are obtained in terms of the attenuation parameters. These sufficient conditions are much easier to verify than the nonlinear small-gain condition[9] for an input-affine nonlinear dynamical system. Besides these sufficient conditions, we derive a rank condition using the attenuation parameters of the proposed controller, which is analogous to the rank condition derived for linear systems using the traditional small-gain theorem.[9] For completeness, we also show that the proposed control scheme ensures that the controlled subsystems satisfy the L_2 optimality conditions to ensure a desired performance. Finally, we provide two numerical examples to demonstrate the applicability of the proposed framework, where we employ a numerical method in an iterative manner[20,21] to solve the game.

The contributions of this article include the: (1) development of a game-theoretic framework for the codesign of the decentralized optimal control policy and the event-triggering mechanisms; (2) deriving easily verifiable conditions to ensure ISS of the overall system for input-affine nonlinear interconnected systems; (3) design and verification of the proposed control scheme for linear and nonlinear systems via numerical examples. In the analysis presented in this article, standard mathematical notations are used; Euclidean norm is used for vectors, and unless explicitly stated, Frobenius norm is used for matrices.

2 | BACKGROUND AND PROBLEM STATEMENT

In this section, we first introduce the dynamics of the interconnected system considered in this work followed by a brief review of some important definitions and results from the existing literature.

2.1 System Description

Consider an interconnected system composed of N subsystems each with the dynamics given by

$$\dot{x}_i(t) = f_i(x_i(t)) + g_i(x_i(t))u_i(t) + \sum_{\substack{j=1\\ j\neq i}}^{N} \Delta_{ij}(x_i(t))x_j(t), \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$, $x_i(0) = x_{i0} \in \mathbb{R}^{n_i}$ is the state vector of the i^{th} subsystem; $u_i \in \mathbb{R}^{m_i}$ is the control input, $f_i(x_i) \in \mathbb{R}^{n_i}, g_i(x_i) \in \mathbb{R}^{n_i \times m_i}$ are nonlinear maps representing the internal dynamics, satisfying $f_i(0) = 0$, and the input gain of the i^{th} subsystem, respectively. The effect of the interconnection between the i^{th} and the j^{th} subsystems on the i^{th} subsystem dynamics is captured by the function $\Delta_{ij}(x_i, x_j) = \Delta_{ij}(x_i)x_j \in \mathbb{R}^{n_i \times n_j}$. From here on, we denote the control input as $u_i(x_i(t))$, a feedback policy, since we will only consider such policies in the design. The time dependence in the notations corresponding to the functions $f_i, g_i, \Delta_{ij}, u_i$ is suppressed in the following for simplicity. The overall augmented system description is given by

$$\dot{\nu}(t) = F(\nu) + G(\nu)\mu(\nu), \quad \nu(0) = \nu_0, \tag{2}$$

where $\nu = (x_1', \dots, x_N')' \in \mathbb{R}^n$, $F = \left(\left(f_1 + \sum_{j=2}^{N} \Delta_{1j}x_j\right)', \dots, \left(f_N + \sum_{j=1}^{N-1} \Delta_{Nj}x_j\right)'\right)'$, $G = diag(g_1, \dots, g_N)$, and $\mu = (u_1'(x_1), \dots, u_N'(x_N))'$ with $n = \sum_{i=1}^{N} n_i$ and $m = \sum_{i=1}^{N} m_i$.

The subsystem dynamics described by (1) can be used to model a wide variety physical processes. For example, interconnected systems with diffusive coupling can be represented as in (1) (e.g., interconnected oscillator network[22] or interconnected phase models[23]). In this article, we present a case study using an example in Section 4 to demonstrate the applicability of the proposed scheme for a broader class of systems when the function modeling the effect of interconnections is a general nonlinear function, that is, $\Delta_{ij}(x_i)x_j$ is replaced with $\Delta_{ij}(x_i, x_j)$ for each $i, j = 1, \dots, N$.

2.2 Event-triggered control

In the event-based aperiodic feedback framework, the feedback information is not utilized continuously by the subsystem controllers to calculate and implement $u_i(x_i)$. This reduces the computational burden, and in addition, if a local communication network is used in each subsystem to close the feedback loop from its sensors to the actuators, this event-triggering mechanism can reduce the local communication burden as well. Thus, in this control framework, the feedback information is utilized only at certain discrete event-based sampling instants to update the control action. Therefore, to represent these discrete sampling instants at the i^{th} subsystem, we introduce a sequence of time instants $\{t_k^i\}_{k\in\{0,\mathbb{N}\}}$, for $i = 1, 2, \dots, N$ such that $0 = t_0^i < t_1^i < \dots$, and the control input is in the form of $u_i(x_i) = u_i(\breve{x}_i(t))$, where $\breve{x}_i(t) = x_i(t_k^i)$, $\forall t \in [t_k^i, t_{k+1}^i)$, $\forall k \in \{0, \mathbb{N}\}$.

The event-triggering error is then defined as the difference between the actual state and the states available at the controller, and is given by

$$e_i(t) = \breve{x}_i(t) - x_i(t), \quad \forall t \in [t_k^i, t_{k+1}^i). \tag{3}$$

The dynamics of each subsystem can be rewritten in an event-triggered framework as

$$\dot{x}_i(t) = f_i(x_i) + g_i(x_i)u_i(x_i) + \sum_{\substack{j=1\\ j\neq i}}^{N} \Delta_{ij}(x_i)x_j(t) + g_i(x_i)\eta_i(x_i), \tag{4}$$

where

$$\eta_i(x_i) = [u_i(\breve{x}_i(t)) - u_i(x_i(t))], \quad t \in [t_k^i, t_{k+1}^i), \quad k = 0, 1, \dots . \tag{5}$$

The difference between the continuously updated control policy $u_i(x_i(t))$ and the event-based control policy $u_i(\check{x}_i(t))$ in (5) is a function of the event-triggering error $e_i(t)$, and the dynamics in (4) can be viewed as the equation representing the slope of $x_i(t)$ with the internal inputs due to interconnections and external inputs due to the event-triggering error and the control input.

Assumption 1. The functions f_i, g_i, Δ_{ij} are Lipschitz continuous functions, and without loss of generality, we assume that $x_i = 0$ is a unique equilibrium point for $i = 1, \dots, N$. The overall state vector is considered to be available as measurements.

Note that the Lipschitz continuity assumption[9] is required to guarantee uniqueness of the solution for the initial value problem (1), that is, to have unique solution trajectories $x_i(t)$ for each subsystem for a given initial condition in $\mathbb{R}^{n_i}$.

2.3 | Input-to-state stability (ISS)

The overall objective of this article is to derive conditions to ensure stability of the equilibrium point at origin for the interconnected system (2) such that the subsystems in (1) satisfy certain performance criterion with event-sampled feedback. To guarantee stability, we use the notion of ISS and small-gain theorem. Some definitions and existing results on ISS and small-gain theorem are briefly reviewed in this subsection.

Definition 1. Consider the interconnected system (2) and assume that for each subsystem (4) there exists a function $J_i : \mathbb{R}^{n_i} \to \mathbb{R}^+$ which is proper, positive definite, and locally Lipschitz continuous on $\mathbb{R}^{n_i}$ such that $J_i(0) = 0$. The function J_i is called an input-to-state practically stable (ISpS) Lyapunov function for the corresponding i^{th} subsystem if there exists a monotone aggregate function μ_i, a positive constant $\bar{z}_i > 0$, $\gamma_{ij} \in \mathbb{K}_\infty \cup 0, \forall j \in \{1, \dots, N\}, j \neq i$, $\varphi_i \in \mathbb{K} \cup 0$, and a positive definite function α_i^k such that

$$J_i(x_i) \geq \mu_i(\gamma_{i1}(J_1(x_1)), \dots, \gamma_{iN}(J_N(x_N)), \varphi_i(\|e_i\|) + \bar{z}_i) \Rightarrow \dot{J}_i \leq \alpha_i^k(\|x_i\|),$$

where $\mu_i : \mathbb{R}^{N+1} \to \mathbb{R}$ is a monotone aggregate function, $\mathbb{K} = \{\hat{f} : \mathbb{R}_+ \to \mathbb{R}_+, \hat{f}$ is continuous, strictly increasing and $\hat{f}(0) = 0\}$, and $\mathbb{K}_\infty = \{\hat{f} \in \mathbb{K}, \hat{f}$ is unbounded$\}$. When $\bar{z} = 0$, the Lyapunov function is ISS.[7,9]

The functions γ_{ij} and φ_i are referred to as the ISS gains,[7,9] and the effect of interconnections from the j^{th} subsystem on the i^{th} subsystem is captured by γ_{ij}, while the effect of any external input is captured by φ_i. A common design practice[24] for finding a stabilizing event-triggering condition is to enforce the inequality

$$\varphi_i(\|e_i(t)\|) \leq \vartheta_i \alpha_i^k(\|x_i(t)\|) + \beta_i, \tag{6}$$

and trigger an event when this inequality is violated, where $0 \leq \vartheta_i < 1$, $\beta_i \geq 0$.

The following lemma presents sufficient conditions to guarantee ISS of a large-scale interconnected system using small-gain theorem.[9]

Lemma 1. *Consider the interconnected systems composed of N interconnected ISS subsystems as in (1) with the ISS gain function* $\Gamma := (\gamma_{ij})_{i,j=1,2,\dots,N}$. *If the map* Γ *is linear and satisfies*

$$\rho(\Gamma) < 1, \tag{7}$$

where $\rho(\cdot)$ *is the spectral radius operator, then the overall system is ISS for any essentially bounded external input applied to the system.*

Remark 1. If the rank condition in Lemma 1 is satisfied, then there exists $\hat{\alpha}_1, \dots, \hat{\alpha}_N > 0$ such that $\forall r \in \mathbb{R}_+^n \setminus \{0\} : \Gamma(I + diag(\hat{\alpha}_1, \dots, \hat{\alpha}_n))r < r$, and there exists a function $\phi \in \mathbb{K}_\infty$ such that the augmented gain matrix $\Gamma_\eta(\psi(r), \phi(r)) < \psi(r),\ \forall r > 0$ is satisfied. Furthermore, an ISS Lyapunov function for the overall system exists and it is given by

$$J(x) = \max_{i=1,2,\dots,N} \psi_i^{-1}(J_i(x_i)).$$

Specifically, if the Lyapunov function is differentiable,

$$J(x) \geq \max\{\psi_i^{-1}(\varphi_i(\|e_i\|)),\ i = 1, 2, \dots, N\} \Rightarrow \nabla J(x)\dot{x} \leq -\alpha(\|x\|),$$

where α is a positive definite function and ∇J is the gradient of the Lyapunov function (see for, e.g., theorem 5.3[9]).

It should be noted that (7) is a matrix equation for linear interconnected systems which is easy to verify. However, for nonlinear systems, Γ is a nonlinear map and the resulting inequality is, in general, not easy to check. Therefore, as part of the main results, in this article, we present a more generic sufficient condition to ensure ISS of the overall system. Specifically, instead of using the *max* operator (as presented in theorems 5.2 and 5.3 of the existing literature,[9] and summarized in Remark 1) to construct an ISS Lyapunov function for the overall system, we design the ISS Lyapunov function for the overall system as a weighted sum of ISS Lyapunov functions of each subsystem.

2.4 | Performance objective

For the considerations of performance optimization, we define the performance output to be controlled at each subsystem as

$$\|\zeta_i(t)\|^2 = Q_i(x_i) + u_i' R_i u_i, \tag{8}$$

where $Q_i(\cdot) > 0, R_i > 0$ with $Q_i(0) = 0$. Note that the dynamics of each subsystem can be represented as

$$\dot{x}_i(t) = \hat{F}(x_i, u_i) + \overline{\Delta}(x_i)\overline{d}_i(t), \tag{9}$$

where $\hat{F}(x_i, u_i) = f_i(x_i) + g_i(x_i)u_i(x_i)$, with f_i, g_i, u_i defined as in (1), $\overline{d}_i$ is an augmented vector of all the interconnection inputs with the corresponding nonlinear gain $\overline{\Delta}_i$, each with appropriate dimensions. With this compact representation of the subsystem dynamics (9), we can view the inputs $\overline{d}_i$ as adversarial to the subsystem i. To attenuate the effects of the adversarial inputs on the performance output of the subsystem (9), a controller should be designed such that the following L_2 gain condition[17] is satisfied.

Definition 2. The i^{th} subsystem (9) is said to have an L_2 gain from the adversarial inputs to the controlled output less than or equal to σ_i, if for any initial state $x_i(0)$ and control policy $u_i(x_i)$, its response corresponding to all $\overline{d}_i \in L_2[t, \infty)$ satisfies

$$\int_t^\infty \|\zeta_i(\tau)\|^2 d\tau \leq \sigma_i^2 \int_t^\infty \|\overline{d}_i(\tau)\|^2 d\tau + \overline{\kappa}(x_{i0}), \tag{10}$$

where σ_i represents the bound on the ratio between the L_2-norms of the defined performance output and the interconnection inputs, $\overline{\kappa}$ is a bounded function such that $\overline{\kappa}(0) = 0$.

In the next section, we will present the proposed event-triggered control framework, derive the ISS stability results, and analyze the performance of the controlled system in terms of the L_2 gain conditions.

3 | CONTROLLER DESIGN

In this section, we introduce the formulation of the game that is proposed in this article to codesign the decentralized control policy and the event-triggering threshold at each subsystems. In particular, we model the effect of the interconnections between subsystems and the error due to event-triggered feedback as internal and external adversarial inputs, respectively. We first consider just the design of control policy $u_i(x_i)$ at each subsystem to attenuate the effect of the internal adversarial inputs, and utilize Lyapunov conditions to derive the event-triggering thresholds. Later, we incorporate the event design problem into the game-theoretic framework to present our overall control scheme.

3.1 | Game-theoretic formulation

We begin by assuming continuous availability of feedback information at each controller and formulate a noncooperative game to develop a decentralized optimal control scheme. We will then introduce the artifacts of the event-triggering mechanism to complete the proposed framework.

The objective of the controller at the i^{th} subsystem is to attenuate the effect of adversarial inputs to the dynamics. In other words, the controller should guarantee the attenuation condition (10) with $\zeta_i(t)$ given by (8) and the adversarial inputs denoted by $\overline{d}_i(t)$ for $i=1,\ldots,N$, where $\overline{d}_i(t)$ models the effect of interconnection inputs on the subsystem dynamics. Before proceeding further, we define $\overline{d}_i$ as a vector composed of $N-1$ components of the form $d_j(t)\in\mathbb{R}^{n_j}, j=1,\ldots,N$, and $j\neq i$, that is, $\overline{d}_i=(d_1', d_2', \ldots, d_{\overline{i}}', \ldots, d_N')'$, where $d_{\overline{i}}$ indicates that the component d_i is omitted in $\overline{d}_i$. We treat these newly introduced d_j for each j to be a hypothetical adversarial signal modeling the effect of the interconnections. In particular, consider the dynamics for subsystem i with the adversarial inputs d_j such that

$$\dot{x}_i(t)=f_i(x_i)+g_i(x_i)u_i(t)+\sum_{\substack{j=1\\ j\neq i}}^{N}\Delta_{ij}(x_i)d_j(t). \tag{11}$$

In the following, we will use the dynamics given in (11), and formulate a game with respect to the performance output given by (8) for each $i=1,\ldots,N$. We will then use the solution to this game to find the control policies, which will be employed in (1). Furthermore, we will derive the conditions for stability with the proposed controls for the interconnected system given in (2) with the subsystems (1). Formally, using the adversarial inputs d_j, we can rewrite (10) to obtain constraint

$$\int_t^\infty [Q_i(x_i)+u_i'R_iu_i]d\tau\le\sigma_i^2\int_t^\infty\sum_{j=1,j\neq i}^{N}[d_j'(\tau)d_j(\tau)]d\tau+\overline{\kappa}(x_i(0)). \tag{12}$$

Since $\overline{\kappa}$ is fixed for an initial state, we will neglect this bias in the cost in our analysis.[25] Let the augmented cost function over the infinite horizon for the i^{th} subsystem be defined as

$$\mathcal{J}_i=\int_t^\infty\left[Q_i(x_i)+u_i'R_iu_i-\sigma_i^2\sum_{\substack{j=1\\ j\neq i}}^{N}d_j'd_j\right]d\tau. \tag{13}$$

To fix ideas, we model the effect of the adversarial inputs to the i^{th} subsystem due to the interconnections using $d_j\in L_2[0,\infty)$, where $j=1,\ldots,N$ and $j\neq i$. Let $\mathcal{U}_i$ and $\mathcal{V}_i$ for $i=1,\ldots,N$ denote the set of policies, that is, $u_i\in\mathcal{U}_i$ and $\overline{d}_i\in\mathcal{V}_i$ for each $j\neq i$. We consider the feasible sets $(\mathcal{U}_i,\mathcal{V}_i)$ of this optimization problem to be sets of $L_2[0,\infty)$ signals. Therefore, the theoretical adversarial policies and the control policies are square integrable, and hence, result in finite cost provided the control policy stabilizes the subsystem. In this context, we call the set of policies $(u_i,\overline{d}_i)$ for each i to be admissible if they result in a finite cost (13) and if they stabilize the system (9) at origin.[18,26]

Note that the task of minimizing the performance index (13) subject to the constraint of the subsystem dynamics is equivalent to finding optimal policies, $u_i^*\in\mathcal{U}_i$ and $\overline{d}_i^*\in\mathcal{V}_j$ that satisfies the bounded L_2 gain.[25] This noncooperative game has a unique solution under certain conditions (locally in the neighborhood of the origin) if the following Nash condition holds

$$V_i^*(x_i)=\min_{u_i\in\mathcal{U}_i}\max_{\overline{d}_i\in\mathcal{V}_j}\mathcal{J}(u_i,\overline{d}_i)=\max_{\overline{d}_i\in\mathcal{V}_i}\min_{u_i\in\mathcal{U}_i}\mathcal{J}(u_i,\overline{d}_i), \tag{14}$$

where $i=1,2,\ldots,N$, V_i^* is the optimal value function for the i^{th} subsystem. Specifically, for a linear system, the infinite horizon game admits a unique saddle point solution under certain conditions (see section 4.4[17]). However, for a nonlinear system, the infinite horizon game may not have a global solution. Moreover, even when a local solution exists, the solution V_i^* may be smooth only under the stabilizability and zero-state detectability conditions (lemmas 1–3[27,28]). To proceed further, we will consider a neighborhood χ_i for each $i=1,\ldots,N$ such that the initial states $x_{i0}\in\chi_i$, the equilibrium point or the zero vector in $\mathbb{R}^{n_i}$ for each i is also in χ_i. We will assume that a local solution to the game exists in χ_i.[18]

It is important to note here that the interaction inputs d_j model the effects of the states of the neighboring subsystems. Therefore, they cannot be explicitly updated based on the solution of the game. However, they provide a theoretical bound that can be used to quantify the subsystem performance. This, in principle, follows a similar spirit of designing controllers that attenuate the effect of external disturbance.[18,26] The adversarial inputs in our framework correspond to the interaction inputs while in the existing literature, the external disturbance is modeled as the adversarial inputs.[18,26] As a consequence of the proposed modeling viewpoint, the solution to the game correspond to a control input that attenuates the effect of the worst-case adversarial inputs. In the following, we derive these policies and demonstrate the advantage of the proposed game.

For the objective function as given in (13), and for any admissible policies,[26] we can derive the following Hamiltonian for the i^{th} subsystem

$$H_i(x_i, u_i, d_j) = Q_i(x_i) + u_i' R_i u_i - \sigma_i^2 \sum_{j=1, j\neq i}^{N} d_j' d_j + V_{ix}' \left[f_i(x_i) + g_i(x_i)u_i(x_i) + \sum_{\substack{j=1 \\ j\neq i}}^{N} \Delta_{ij}(x_i) d_j \right], \tag{15}$$

where $V_{ix} = \partial V_i / \partial x_i$, and V_i is the value function corresponding to the policies u_i, d_j. Applying the stationarity condition[18,26] given by $\partial H_i/\partial u_i = 0$ and $\partial H_i/\partial d_j = 0$, the optimal control and the worst-case adversarial inputs are obtained as

$$u_i^*(x_i) = -\frac{1}{2} R_i^{-1} g_i' V_{ix}^*, \tag{16}$$

$$d_j^*(x_i) = \frac{1}{2\sigma_i^2} \Delta_{ij}' V_{ix}^*, \quad \forall i, j = 1, 2, \ldots, N, \ i \neq j \tag{17}$$

where d_j^* is the worst-case adversarial inputs that can be injected into the i^{th} subsystem without affecting the performance of the i^{th} subsystem. Since the states of the neighboring subsystems are affecting the performance of the i^{th} subsystem, d_j^* provides a theoretical worst-case bound on the states of the neighboring subsystems such the cost function (13) is optimized. Substituting the control input and the adversarial input, the following Hamiltonian Jacobi Isaacs (HJI) equation is obtained

$$H_i(u_i^*, d_j^*) \triangleq Q_i(x_i) - \frac{1}{4} V_{ix}^{*\prime} g_i R_i^{-1} g_i' V_{ix}^* + \frac{1}{4\sigma_i^2} V_{ix}^{*\prime} \left(\sum_{j=1, j\neq i}^{N} \Delta_{ij} \Delta_{ij}' \right) V_{ix}^* + V_{ix}^{*\prime} f_i(x_i(t)) = 0. \tag{18}$$

In the lemma that follows, it is demonstrated that the i^{th} input-affine nonlinear subsystem (1) is ISS when the optimal control policy $u_i^*(x_i)$ taking the form as in (16) is applied to it.

Lemma 2. *Consider the input-affine nonlinear interconnected system with N subsystems given by (4) with $x_{i0} \in \chi_i \subseteq \mathbb{R}^{n_i}$. Let $u_i^*(x_i)$ be applied to the subsystem. Let (16) and (17) be Lipschitz continuous in χ_i with the Lipschitz constant $C_{iu}, C_{id} > 0$, respectively, and let $L_{u_i} = max\{C_{iu}, C_{id}\}$. Then, there exists an ISS Lyapunov function defined in χ_i, a positive constant L_x, positive scalar c_i, such that if*

$$\|x_i\| \geq \frac{1}{c_i} \left(2\sigma_i^2 L_{u_i} \sum_{\substack{j=1 \\ j\neq i}}^{N} \|x_j\| + 2\|R_i\| L_{u_i}^2 \|e_i\| \right),$$

then

$$\dot{J}_i \leq -(L_x - c_i)\|x_i\|^2.$$

Proof. See Appendix. ■

Similar results can be derived for a linear interconnected system given by

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + \sum_{i=1, i\neq j}^{N} A_{ij} x_j(t), \tag{19}$$

where A_i, B_i, A_{ij} are linear maps.

Corollary 1. *Consider the interconnected system with N subsystems given by (19). Let $u_i^*(x_i)$ be an optimal control policy for the i^{th} subsystem. Then, there exists a positive definite Lyapunov function $J_i(x_i) = x_i'(t) P_i x_i(t)$ which is an ISS Lyapunov* *function (satisfying Definition 1), with symmetric positive definite matrix P_i and ISS gains*

$$\gamma_{ij}(x_j) = 2\frac{\|P_i\|^{3/2}}{c_i}\sum_{j=1,j\neq i}^{N}\left(\frac{\|A_{ij}\|}{[\lambda_{\min}(P_j)]^{1/2}}x_j^{1/2}\right), \quad i,j = 1, \dots, N$$

with $\gamma_{ii} = 0$.

Proof. See Appendix. ■

Remark 2. The functions γ_{ij} in the proof of Corollary 1 (and similarly in Lemma 2) are obtained as a function of the interconnection terms. If $\gamma_{ij} = 0$, the subsystems i and j are structurally decoupled, and when $\gamma_{ij} \neq 0$, the ISS Lyapunov gains indicate the influence of these interconnections on the subsystem states. To analyze the stability and robustness of the interconnected system, these gains are collected into a gain[9] matrix $\Gamma = (\gamma_{ij})_{i,j=1,2,\dots,N}$. Furthermore, note that the set χ_i denotes a neighborhood of the equilibrium point at origin and the results of the Lemma 2 suggests that when the subsystems are initialized in this neighborhood, they remain in the neighborhood provided the error $\|e_i\|$ is bounded, and as the error goes to zero, the states converge to the origin asymptotically.

Next, assuming that the ISS gain operator Γ is a linear map, easily verifiable conditions are proposed so that the small-gain condition (Lemma 1) is satisfied. Later, it is demonstrated that employing the proposed game-theoretic approach to develop controllers at each subsystem, the ISS gain operator becomes a matrix composed of σ_i^2 and, hence, a linear operator, even when the subsystem dynamics and interconnections are nonlinear.

Theorem 1. *Consider the interconnected with linear dynamics given by (19). Let the results of Corollary 1 hold, and let*

$$\Gamma = (\gamma_{ij})_{i,j=1,2,\dots,N},$$

for the interconnected systems (1) with linear dynamics. If $\|\Gamma\|_\infty < 1$ or $\|\Gamma\|_2 < 1$, then, the overall interconnected system (2) is ISS for any essentially bounded external input where $\|\cdot\|_\infty$, $\|\cdot\|_2$ are the standard matrix norms.[29]

Proof. See Appendix. ■

Note that the results of the Theorem 1 can be used together with the results of the small-gain theorem, as summarized in Lemma 1 and Remark 1, to construct an ISS Lyapunov function for the interconnected system using the ISS Lyapunov functions of the subsystems. The results from Reference 9 (briefly summarized in Remark 1), provide a method to construct ISS Lyapunov function for the overall system from the ISS subsystem Lyapunov function using the *max* operation. In the next theorem, an ISS Lyapunov function for the overall system is developed using the weighted sum of the ISS Lyapunov functions of subsystems.

Theorem 2. *Let Assumption 1 hold. Consider the interconnected system (1) with $x_{i0} \in \chi_i$. Let $V_i^* : \chi_i \to \mathbb{R}$ be a locally smooth solution to the HJI equation (18), and let the control policy $u_i^*(x_i)$ given by (16) be applied to each subsystem, and an event be triggered on violation of the condition*

$$\|e_i\|^2 \leq \frac{\bar{c}_i(L_x - L_{Jx})}{2\|R_i\|^2 L_{u_i}^2}\|x_i\|^2 \tag{20}$$

with $0 < \bar{c}_i < 1$. Let $\sigma_i > \sigma_i^*$ for all $i = 1, 2, \dots, N$. Then, the i^{th} subsystem and the overall system are ISS if $Q_i(\cdot) \geq 0, R_i > 0$ and σ_i are chosen such that

$$L_x > L_{Jx} + \sigma_i^2(N-1), \tag{21}$$

where $L_{Jx}, L_x > 0$.

Proof. See Appendix. ■

Note that for the linear subsystem (19), the HJI equation (18) reduces to the game algebraic Riccati equation, and the optimal policy will be a linear feedback policy with the Kalman gain matrix.[26]

Corollary 2. *Consider the interconnected system (1) with linear dynamics as in (19). Let (A_i, B_i) be controllable, $(A_i, \sqrt{Q_i})$ be observable and $\sigma_i > \sigma_i^*$ for all $i = 1, 2, \dots, N$. Furthermore, let the control policy $u_i^*(x_i)$ as in (16) be applied to each subsystem, and an event be triggered on violation of the condition*

$$\|e_i\|^2 \leq \frac{\overline{c}_i(L_x - L_{Jx})}{2\|R_i\|^2\overline{K}_i^2}\|x_i\|^2 \tag{22}$$

with $0 < \overline{c}_i < 1$ and $\overline{K}_i$ is the maximum singular value of the Kalman gain matrix. Then,

- *a solution to the game algebraic Riccati equation exists;*
- *the i^{th} subsystem and the overall system are ISS if $Q_i \geq 0, R_i > 0$ and σ_i are chosen such that*

$$L_x > L_{Jx} + \sigma_i^2(N-1), \tag{23}$$

where $L_{Jx} > 0$.

The first part of the proof of Corollary 2 provides the conditions for the existence of a Nash solution for a two-player zero-sum game[17,18] (σ_i^* is the H_∞ gain of the system). The second part of the proof is a direct extension of the results of Theorem 2 for the linear dynamical systems. Therefore, the proof of the Corollary 2 is not detailed here. It is important to note that the condition (20) (and (22)) can be referred to as the relative triggering condition[24] and the event-triggering mechanism is Zeno-free (see sec. 4.A[24]) under the conditions prescribed in Theorem 2. Moreover, the design parameter $\overline{c}_i$ is arbitrarily chosen to modify the frequency of events.

The control policy from Theorem 2 is optimally designed without accounting for the event-triggering error and later, the event-triggering conditions are designed to satisfy stability requirements. However, it is desirable to find the control policy which accounts for the error due to event-triggering with some performance guarantees. In the next section, we modify the game proposed in this section to explicitly account for the effect of event-triggered control implementation.

3.2 | Designing events in the game formulation

In this section, we explicitly consider the effect of the event-triggered control implementation on the subsystem dynamics within the game-theoretic formulation introduced in the previous section. To develop such a control scheme, the effect of event-triggering errors on the subsystem dynamics is modeled as adversarial input, and it is paired with the adversarial interconnection inputs.

In particular, to include the event-triggering error as an adversarial input, in addition to the subsystem interconnections, the performance (13) is first modified. To do this, we introduce the threshold policy $\hat{\eta}_i \in L_2[t, \infty)$ and the sets $\mathcal{W}_i$ for $i = 1, \dots, N$ such that $\hat{\eta}_i \in \mathcal{W}_i$. Then the cost function (13) is rewritten as

$$\mathcal{J}_i = \int_t^\infty \left[Q_i(x_i) + u_i' R_i u_i - \sigma_i^2 \sum_{\substack{j=1, \\ j \neq i}}^N d_j' d_j - \sigma_{i0}^2 \hat{\eta}_i' \hat{\eta}_i \right] d\tau. \tag{24}$$

The new term $\hat{\eta}_i$ is used to model the effect of the event-triggering error η_i in (5) on the subsystem cost. The modified noncooperative game concerns with minimizing the cost (24) subject to the constraint

$$\dot{x}_i(t) = f_i(x_i) + g_i(x_i)u_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^N \Delta_{ij}(x_i)d_j(t) + g_i(x_i)\hat{\eta}_i, \tag{25}$$

where d_j for $j = 1, \dots, N$ are defined as in (11). This noncooperative game has a unique solution under certain conditions (locally in the neighborhood of the origin) if the following Nash condition holds

$$V_i^*(x_i(t)) = \min_{u_i \in \mathcal{U}_i} \max_{\overline{d}_i \in \mathcal{V}_j, \hat{\eta}_i \in \mathcal{W}_i} \mathcal{J}(u_i, \overline{d}_i) = \max_{\overline{d}_i \in \mathcal{V}_i, \hat{\eta}_i \in \mathcal{W}_i} \min_{u_i \in \mathcal{U}_i} \mathcal{J}(u_i, \overline{d}_i), \tag{26}$$

where $i, j = 1, 2, \dots, N$, $j \neq i$, and V_i^* is the optimal value function for the i^{th} subsystem. Similar to (15), (16), and (17), we can derive the HJI, and the optimal policies using the stationarity conditions on the HJI for the newly formulated game in this section. However, note that in addition to u_i^*, d_j^*, we will obtain an optimal policy corresponding to $\hat{\eta}_i$.

The next theorem presents the main results for the new game resulting from the modified cost functional (24) under the assumption that a solution to the associated game exists. It will later be demonstrated that for the case of linear systems, the saddle point solution to the proposed game exists under a set of conditions on the penalty matrices of control policy (R_i) and event-triggering error, and the standard controllability and observability requirements in Corollary 2. Furthermore, extension of similar conditions for input-affine nonlinear system will be discussed.

Theorem 3. *Let Assumption 1 hold. Consider the interconnected system (1) with* $x_{i0} \in \chi_i \subseteq \mathbb{R}^{n_i}$. *Let the solution to the game with the cost function (24) and the dynamic constraints (25) exists locally in* χ_i *and be defined as* $V_i^*(x_i)$ *for each subsystem. Let the optimal policy (16) be applied to each subsystem, where the* V_i^* *now corresponds to the solution of the new game as in (26) and the event-based feedback updates be triggered on violation of the condition*

$$\|u_i^*(\breve{x}_i) - u_i^*(x_i)\| \leq \|d_{i0}^*\|, \tag{27}$$

where $d_{i0}^* = \frac{1}{2\sigma_{i0}^2} g_i'(x_i) V_{ix}^*$ *is the optimal policy corresponding to* $\hat{\eta}_i$ *obtained from the set* $\mathcal{W}_i$. *Then, the* i^{th} *subsystem and the overall system are locally ISS when* $Q_i(\cdot) \geq 0, R_i > 0$, *and* σ_i *are chosen such that*

$$\overline{\delta}_i > \sigma_i^2 (N-1),$$

where $\overline{\delta}_i > 0$.

Proof. See Appendix. ∎

Note that the results of Theorems 2 and 3 do not impose any assumptions on the strength of the interconnections and are rather generic. The design parameters $\overline{q}_i, \delta_i, \sigma_i$ are chosen based on sufficient conditions obtained in Theorems 2 and 3. Leveraging the well-known small-gain theorem and the rank condition derived in Theorem 1, which ensures ISS of the overall system, we derive alternative conditions for stability of the interconnected system next.

Corollary 3. *Let the interconnected system (2) be composed of subsystem dynamics (1). Then, the* i^{th} *subsystem and the overall system are ISS if* $\sigma_i^2 < \frac{1}{N-1}$, *or* $\sum_{i=1}^{N-1} \sigma_i^2 < 1$, $\forall i = 1, \dots, N$ *and events are triggered based on the condition in Theorem 3 (or Theorem 2).*

Proof. See Appendix. ∎

Observe that the sufficient conditions that are obtained in Corollary 3 are much more conservative than the conditions obtained in Theorems 2 and 3 in choosing the design parameter. However, if the number of subsystems is small, then the condition obtained in Corollary 3 is easy to satisfy when compared with the nonlinear inequalities.[6,9] The results in Theorem 3 can be extended to a linear dynamical system, and they are summarized in the next corollary.

Corollary 4. *Consider an linear system with the dynamics given by*

$$\dot{x}(t) = Ax(t) + Bu(t) + B\eta(t) + \overline{\Delta} d(t), \quad x(0) = x_0,$$

where $A, B, \overline{\Delta}$ *are linear maps of appropriate dimensions and* $d(t), \eta(t)$ *are bounded adversarial external inputs. Now consider a cost function of the form (24) with* $Q \geq 0, R, \sigma, \sigma_o > 0$ *as*

$$\mathcal{J}(x) = \int_t^{\infty} [x'Qx + u'Ru - \sigma^2 d'd - \sigma_0^2 \eta'\eta] d\tau.$$

Let (A, B) *be controllable,* $(A, \sqrt{Q})$ *be observable, and* $\sigma_o^2 > \lambda_{\max}(R)$, *then,*

- *the solution to the multiplayer game with the players $u(t), d(t), \eta(t)$ with minimizer $u^*(t)$ and maximizers*

$$d^*(t) = \frac{1}{\sigma^2}\overline{\Delta}' P^* x(t), \quad \eta^*(t) = \frac{1}{\sigma_o^2} B' P^* x(t) \tag{28}$$

exists with $P^* > 0$ being the solution to the associated algebraic Riccati equation;

- *the control policy*

$$u^*(t) = -R^{-1} B' P^* x(t)$$

asymptotically stabilizes the system when an event is triggered such that $\|u_i^*(\breve{x}) - u_i^*(x)\|^2 \leq \eta^{*'} \eta^*$ holds for all $t \in \mathbb{R}^+$.

Remark 3. Note that by introducing the event-triggering error to the traditional multiplayer/two-player game, an additional condition on the design variable σ_o and R is obtained. This condition implies that to achieve fewer events, more control effort is required. This further implies that as the threshold for triggering an event is increased, the cost is increased and an optimal performance can be achieved by choosing $\sigma_o = \sigma_o^*$ (H_∞ gain of the system) and R can be chosen appropriately by satisfying the conditions in Corollary 4. Hence, by adding the third player to the game, through modeling the event-triggering threshold as an adversarial player, we can get a novel design approach to codesign controller and the event-triggering mechanism. Furthermore, the performance cost explicitly accounts for the adversarial effects due to interconnections and the event-triggering, resulting in a robust control performance.

Remark 4. For the case of input-affine nonlinear systems, existence of global saddle point solution cannot be easily guaranteed for a two-player game. However, local solution to the HJI equation exists for a two-player zero-sum game under the stabilizability, detectability conditions,[17,27] and when the attenuation constant is chosen appropriately (see ch. 10[26]). For the proposed multiplayer game, an additional condition given by $\sigma_{i0}^2 > \lambda_{\max}(R_i)$ as obtained in Theorem 3 (Corollary 4) is required to be satisfied.

Next, it is demonstrated that, if the solution for the HJI equation exists, the solution to the proposed game satisfies the L_2 gain condition.

Theorem 4. *Assume that the optimal value function for the i^{th} subsystem, that is, $V^*(x_i) \in C^1$, a positive semidefinite solution to the HJI equation, exists for each subsystem. Then, the minimizing control policy, u_i^*, obtained as a solution to the game makes the closed-loop subsystem to have L_2 gain less than or equal to σ_i.*

Proof. See Appendix. ∎

Remark 5. Note that if the optimal value function is chosen as the Lyapunov function for each subsystem, the ISS gain matrix will have σ_i^2 as the off-diagonal entries. Thus, the ISS gain operator is always linear and the rank condition can be imposed on this matrix to choose the attenuation constant σ_i (Corollary 3).

Remark 6. The existence of a unique solution for the HJI equation satisfying the condition derived in Corollary 3 cannot always be guaranteed, especially when the number of subsystems increases, as the parameter σ_i satisfying $\frac{1}{\sqrt{N}} > \sigma_i > \sigma_i^*$ may not exist. Therefore, conditions derived in the Theorems 2 and 3 can be chosen for designing controllers at each subsystem while ensuring stability of the overall system. In this case, the ISS Lyapunov function for the overall system will be a weighted sum of ISS Lyapunov functions of individual subsystems. The constants $L_{Jx}, L_x, \overline{\delta}_i$ are functions of the design parameters Q_i, R_i, σ_i. Since the Lyapunov function for the overall system is obtained as a weighted sum of Lyapunov functions of the subsystems (with weights uniformly chosen as 1), the additional term $\sigma_i^2(N-1)$ is seen in (23) and (21). This condition, instead of restricting the design choice of the attenuation parameter σ_i alone (Corollary 3), demands higher penalty for subsystem states as more subsystems are added (as N increases). A suitable choice of weights to obtain the overall Lyapunov function may relax this further. Furthermore, in order to obtain a solution to the Hamilton Jacobi equations at each subsystem, computation methods such as approximate dynamic programming can be employed.[11,18]

Remark 7. For a linear system with minimum phase zero, the value of $\sigma_i^* = 0$[30] and for a nonminimum phase system, σ_i^* depends on the solution of a pair of Lyapunov equations. It was demonstrated that by using suitable choice of the penalty matrices in the performance index, a saddle point solution can be derived.[31]

4 | SIMULATION RESULTS

The proposed control schemes are evaluated in this section using three examples.

Example 1. First, we consider an example of linear interconnected systems. Specifically, we consider 10 inverted pendulums connected by spring.[32] The dynamics of the i^{th} subsystem are

$$\dot{x}_i = \begin{pmatrix} 0 & 1 \\ \frac{q}{l} - \frac{a_i k}{ml^2} & 0 \end{pmatrix} x_i + \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix} u_i + \sum_{j \in N_i} \begin{pmatrix} 0 & 0 \\ \frac{h_{ij} k}{ml^2} & 0 \end{pmatrix} x_j,$$

where N_i is the set of subsystems in the neighborhood of the i^{th} system. The system parameters used in this simulation are given by $a_i = 2, \forall\, i \in \{2, \dots, 9\}, a_i = 1\ (i = 1, 10), l = 2,\ g = 10,\ m = 1,\ k = 5$ and each subsystem is connected with the adjacent subsystem only (for example subsystem i is connected with subsystems $i-1, i+1$, and subsystem corresponding to $i = 10, i = 1$ are connected with subsystems $i = 9, i = 2$, respectively). The parameter $h_{ij} = 1$ when the subsystems i, j are coupled. For this coupling configuration, we have subsystems 1,10 connected to one other subsystem, while the other subsystems are coupled with two other subsystems.

The subsystems are open-loop unstable and the pair A_i, B_i is controllable. We verify the results obtained in Theorem 1, Corollary 2, Theorem 3 (for linear systems) using this example. From Theorem 1, the attenuation parameters should be chosen such that $\|\Gamma\|_P < 1\ (P = 2, \infty)$ where $\Gamma = \begin{pmatrix} 0 & \sigma_1^2 & 0 & \dots \\ \sigma_2^2 & 0 & \sigma_2^2 & 0 \\ & \vdots & \ddots & \\ 0 & \dots & \dots & \sigma_{10}^2 \end{pmatrix}$. This implies that $\sigma_i^2 < 1$ for $i = 1, 10$ and $\sigma_i^2 < 0.5$ for $i = \{2, \dots, 9\}$ (based on Theorem 1and Corollary 3). With $\sigma_i = 0.45$ for $i = \{2, \dots, 9\}$ and $\sigma_i = 0.85$ for $i = 1, 10$, the continuously updated control stabilized the system (not shown in the figures) as predicted by Theorem 1.

Alternatively, from Corollary 2, for event-triggered control implementation, the controller design parameters should satisfy (22) and (23). As noted at the end of the proof of Lemma 2, for linear systems, $L_x = 1 + Q_i + \|K_i\|^2(\|R_i\| + 2N\sigma_i^2)$, $L_x > L_{Jx} + \sigma_i^2(N-1)$ becomes $1 + Q_i + \|K_i\|^2 \left(\|R_i\| + 2N\sigma_i^2 - \sigma_i^2 - \frac{1}{2}\right) - \sigma_i^2(N-1) > 0$. The parameters used in the simulations are $R_i = 0.0005$, $Q_i = 75$ and $\sigma_i = 0.9$ for $i = 2, \dots, 9$ and $R_i = 0.001$, $Q_i = 50$ and $\sigma_i = 0.85$ for $i = 1, 2$. The Kalman gains for the selected Q_i, R_i, σ_i are $K_i = [22.2174\ 22.0048]$ for $i = 1, 10$ and $K_i = [26.6944\ 26.6378]$ for $i = \{2, \dots, 9\}$ and $c_i = 10^{-5}$ is chosen. Using these design parameters, the event-triggering condition was obtained as $\|e_i\|^2 \le 0.9\|x_i\|^2$.

The resulting state and control trajectories are recorded in Figure 1. For the multiplayer game-theoretic formulation, $\sigma_i = \sigma_{i0} = 0.1$ was fixed (satisfying $R^{-1} > I_m/\sigma_{i0}^2$, derived in Theorem 3), and the rest of the parameters were unchanged. The states and control trajectories corresponding to this case are plotted in Figure 2.

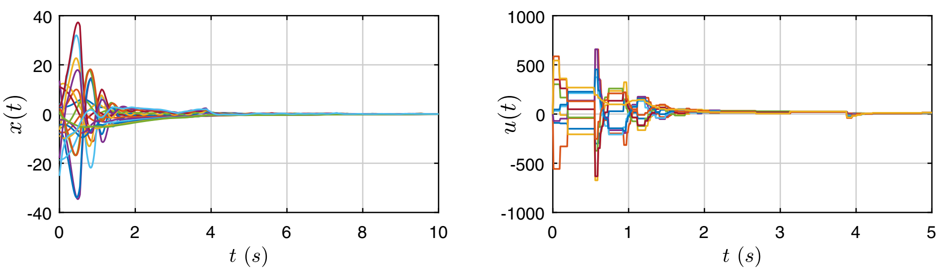

FIGURE 1 Example 1: State (left) and control (right) trajectories and event-triggering condition satisfying (22) [Colour figure can be viewed at wileyonlinelibrary.com]

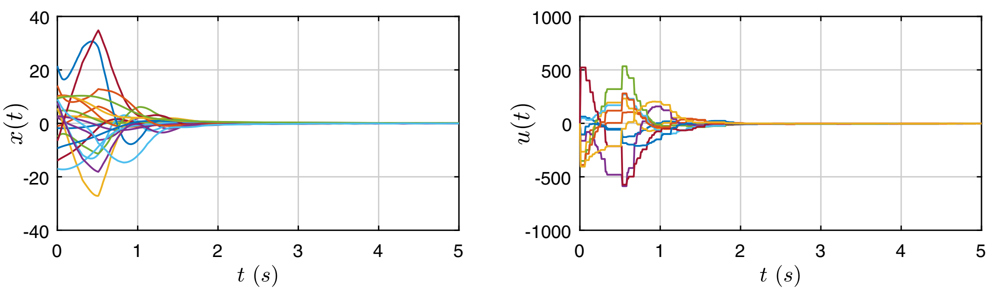

FIGURE 2 Example 1: State (left) and control (right) trajectories and event-triggering condition satisfying (27) [Colour figure can be viewed at wileyonlinelibrary.com]

It is observed that in both these cases, the controller stabilized the system when the control parameters were designed based on the sufficient conditions derived in Theorem 1- Corollary 3, Corollary 2, and Theorem 3. For the multiplayer game, the event-triggering mechanism is codesigned with the controller using the solution of the algebraic game Riccati equation. The detailed comparative analysis of the cost, and inter-event time are presented in Example 3.

Example 2. We consider here a network of interconnected Van der pol oscillators with a ring configuration and present the simulation results for two coupling functions.[22] Specifically, we consider the dynamics

$$\ddot{x}_i(t) + \varepsilon(x_i^2(t) - 1) + x_i(t) + au_i(t) = b(x_{i-1} - 2x_i + x_{i+1}), \quad i = 1, \dots, N, \tag{29}$$

where $\varepsilon > 0$. We consider 10 interconnected oscillators, $a, b = 1$ (case 1) and $a = \sin(x_i), b(.) = \sin(.)$ (case 2). Note that the input gain is constant and the interconnection is a linear function of x_i for case 1 $(a, b = 1)$, while for case 2, it is a bounded nonlinear function. Furthermore, the interconnections are of the form $\Delta_{ij}(x_i, x_j)$ satisfying Assumption 1 (note that for case 2, $\Delta_{ij}(.)$ is bounded). First, we use the approximate dynamic programming algorithm given in References 18,21 to solve the HJI equations corresponding to each subsystem.

The design parameters were chosen as $R_i = 10$, $Q_i = \begin{bmatrix} 1 & 0.6 \\ 0.4 & 1 \end{bmatrix}$, and $\sigma_i = 0.75$ for each subsystem. The approximate optimal cost function corresponding to each subsystem is estimated to be of the form $V_i^*(x_i) = W_1x_{i1}^2 + W_2x_{i2}^2 + W_3x_{i1}x_{i2} + W_4x_{i1}x_{i2}^2 + W_5x_{i1}^2x_{i2}$ with $W_k = \{2.5093, 2.6061, -0.9625, 0.4725, -0.1746\}$ for $k = 1, \dots, 5$. As noted at the end of the proof of Lemma 2, $L_x = 1 + q_i + \|L_{u_i}\|^2(\|R_i\| + 2N\sigma_i^2)$, $L_x > L_{Jx} + \sigma_i^2(N-1)$ becomes $1 + q_i + \|L_{u_i}\|^2\left(\|R_i\| + 2N\sigma_i^2 - \sigma_i^2 - \frac{1}{2}\right) - \sigma_i^2(N-1) > 0$. To compute L_{u_i}, the norm of W_k for $k = 1, \dots, 5$, were used, and the resulting control parameters satisfied the conditions in Lemma 2. The state and control trajectories corresponding to this controller are plotted in Figure 3.

Furthermore, using these design parameters, the event-triggering condition was obtained as $\|e_i\|^2 \le 0.9\|x_i\|^2$. The resulting state and control trajectories are recorded in Figure 4 for the case 1 where $a, b = 1$.

Without changing the design parameters, the control scheme was implemented for case 2, which corresponds to the case when $a = \sin(.)$, $b(x_i, x_j) = \sin(-2x_i + x_{i-1} + x_{i+1})$. Similar to case 1, the proposed controller stabilized the system with a similar control effort (Figure 5).

As pointed out in Section 3.1, the optimal adversarial input d_j^* is a theoretical worst-case input for the subsystem. This can be used to analyze the performance of the controlled system in comparison with an ideal system trajectory. In particular, to analyze the performance for case 2, using the optimal policies, define as ideal subsystem dynamics as $\dot{z}_i(t) = f_i + g_iu_i^* + \sum_{i=1, i\neq j}^{N} \Delta_{ij}d_j^*$ with $z_i(0) = x_{i0}$, and with continuous feedback, the actual system trajectory is given

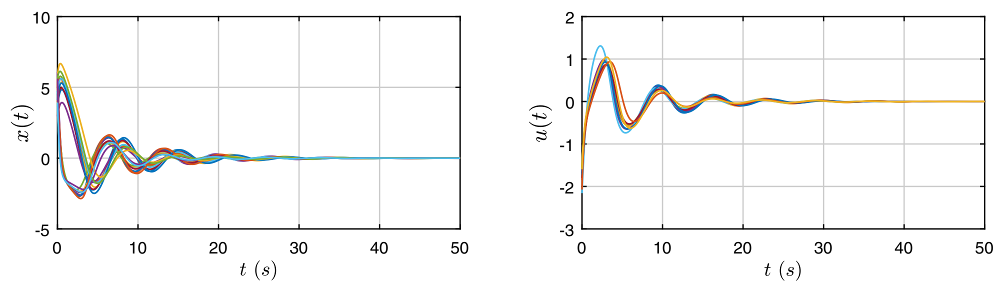

FIGURE 3 Example 2: State (left) and control (right) trajectories and design parameters satisfying conditions in Lemma 2, case 1 $(b = 1)$ [Colour figure can be viewed at wileyonlinelibrary.com]

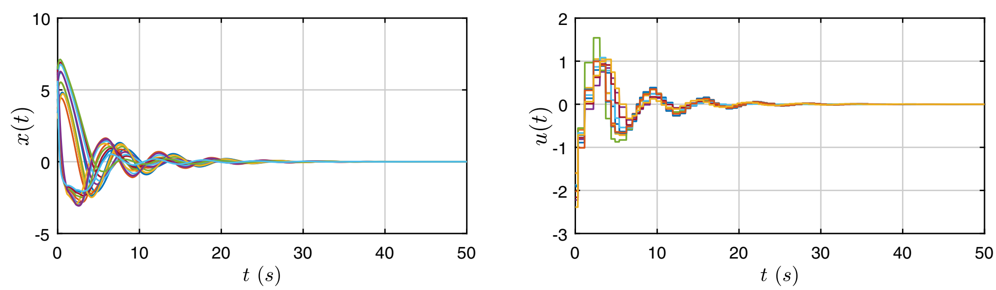

FIGURE 4 Example 2: State (left) and control (right) trajectories and event-triggering condition satisfying (20), case 1 $(b = 1)$ [Colour figure can be viewed at wileyonlinelibrary.com]

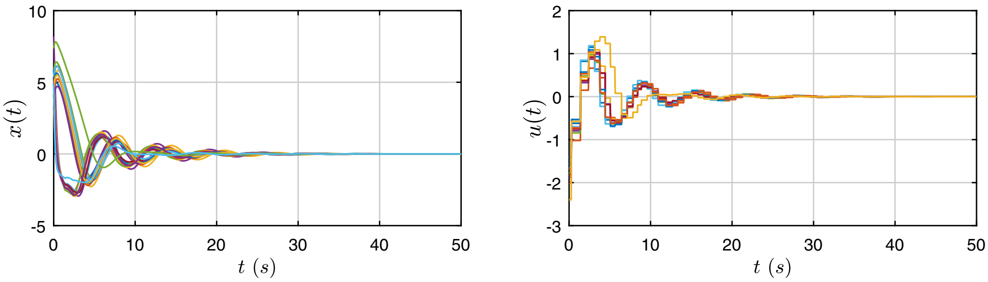

FIGURE 5 Example 2: State (left) and control (right) trajectories and event-triggering condition satisfying (20), case 2 [Colour figure can be viewed at wileyonlinelibrary.com]

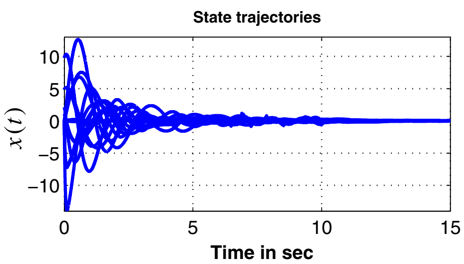

FIGURE 6 Convergence of state trajectories—Example 3 [Colour figure can be viewed at wileyonlinelibrary.com]

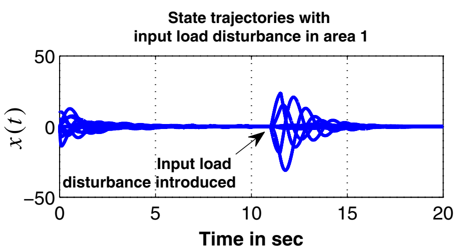

FIGURE 7 State trajectories with input load disturbance in area 1 [Colour figure can be viewed at wileyonlinelibrary.com]

by $\dot{x}_i(t) = f_i + g_i u_i^* + \sum_{i=1, i\neq j}^{N} \Delta_{ij}(x_i)x_j$ with $x_i(0) = x_{i0}$. The ideal subsystem dynamics generates the target trajectory which is optimal, and the performance bound for the subsystem with the proposed control scheme is in the neighborhood of the optimal trajectory with the bounds uniformly defined as $\|z_i(t) - x_i(t)\|_2$. Some of the application examples, for example, as shown in Reference 33 satisfy the assumption that $\Delta_{ij}(v) \leq h_{i1}\|v\| + h_{i2}$ for some positive constants h_{i1}, h_{i2}, and using this assumption, the system trajectories can be directly compared with the ideal trajectory to provide performance guarantee for subsystems with general nonlinear coupling function.

Example 3. (Load frequency control of three area power system) The states of the power system model at each subsystem under consideration are frequency change, incremental change in output power of the generator, change in governor valve position, incremental change in integral control, and tie-line power deviation. For the detailed dynamics see Reference 34.

The controller design parameters are chosen as $R_i = 0.1$, $Q_i = 0.4$, $\sigma_i = 0.9$. All the system states are regulated using the proposed controller. The convergence of the state trajectories can be seen in Figure 6. A load disturbance at area 1 is introduced as $[-0.06\ 0\ 0\ 0\ 0]$ at the 11th second. The evolution of the state trajectories under the input load disturbance can be seen in Figure 7. It is evident that the designed controller is robust as the states are regulated subsequently.

In the following, we present some comparative analysis of the proposed control scheme. The parameter σ_{io} is varied and the resulting performance of the system in terms of cumulative cost, number of events, and the interevent time (IET) are recorded in Table 1.

The parameter σ_{io} is varied to demonstrate the advantages of the proposed method (PM) with a traditional event-triggering approach (event-triggered LQR) and the results are summarized in Table 1 for the case when the system dynamics are linear. Due the space consideration, all the simulation figures are not included and the important results are summarized in Table 1. The cumulative cost corresponds to the value of the integral (24). Since the goal of the

TABLE 1 Analysis with event-triggering design parameter σ

	Avg. IET in s		Cumulative cost		Events	
σ_{io}	E.LQR	P.M	E.LQR	P.M	E.LQR	P.M
0.55	0.0585	0.0857	9.96E+03	1.38E+04	127	87
0.7	0.0602	0.0906	1.12E+04	7.38E+03	123	82
0.75	0.0615	0.0934	1.06E+04	6.70E+03	121	79
0.85	0.0619	0.1	1.33E+04	6.09E+03	119	74
0.95	0.0589	0.1015	1.72E+04	5.99E+03	127	73

TABLE 2 Analysis of decentralized optimal control scheme

	Avg. IET in s		Cumulative cost		Events	
σ_{io}	2-Pl	3-Pl	2-Pl	3-Pl	2-Pl	3-Pl
0.35	0.0576	0.0981	217.5966	151.4405	130	76
0.4	0.0512	0.0729	230.5441	144.3559	146	96
0.46	0.0567	0.0776	275.7723	126.436	128	95
0.6	0.0604	0.0699	394.1401	109.8525	124	104
0.65	0.0669	0.0647	861.1497	106.1883	110	115

proposed method is to develop easily verifiable conditions for guaranteeing ISS of the interconnected system, we proposed a game theoretic approach and demonstrated the resulting linear conditions in Theorems 2 and 3. The terms 2-Pl and 3-Pl in Table 2 denotes 2-player and 3-player game, respectively; σ_{io} for 2-Pl corresponds to the parameter $\bar{c}_i$ and Avg. should be read as average. The additional term in the cost function, the third player, which maximizes the event-triggering threshold provides an explicit relationship between event-triggering error and the system performance and hence, a tractable trade-off is achieved between both. Table 2 therefore, presents a comparison between controllers obtained using the conditions in Theorems 2 (2-Pl) and 3 (3-Pl). From the analysis summarized in Tables 1 and 2, our proposed method can yield competitive performance by saving cost of control implementation, in addition to the easily verifiable sufficient conditions derived in the main results in Section 3.

5 | CONCLUSIONS

In this work, we proposed a novel noncooperative game-theoretic framework for event-triggered decentralized control of nonlinear interconnected systems. The proposed framework strives to optimize the control policy and the frequency of feedback in the presence of structural interactions. The sufficient conditions derived using the game-theoretic formulation resulted in ISS. These conditions are derived for both linear and input-affine nonlinear interconnected systems, and can be easily satisfied through an appropriate choice of the attenuation constant.

The existence of a Nash solution for the proposed game depends on the relationship between the control penalty matrices, and the attenuation constants, revealing an explicit relationship between the event-triggering threshold and the cost function. It is observed that the existing methods to construct the ISS Lyapunov function for the large-scale system based on *max* operation imposes a stringent condition on the attenuation constants while the proposed ISS Lyapunov function for the overall system using the weighted sum of Lyapunov functions of each subsystem relaxes this condition on the attenuation constant while demanding a higher penalty for the state-trajectories in the cost function. The net result is a unified design approach for event-triggered decentralized optimal control of nonlinear interconnected system.

ACKNOWLEDGMENT

Research supported in part by NSF ECCS#1406533, NSF CMMI #1547042 and Intelligent Systems Center, Rolla.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

ORCID

Vignesh Narayanan https://orcid.org/0000-0002-9505-7143
Hamidreza Modares https://orcid.org/0000-0001-5695-4709

REFERENCES

1. Šiljak DD. *Large-Scale Dynamic Systems: Stability and Structure*. Amserdam, Netherlands: North Holland Publishing Company; 1978.
2. Bakule L. Decentralized control: an overview. *Annu Rev Control*. 2008;32(1):87-98.
3. Antonelli G. Interconnected dynamic systems: an overview on distributed control. *IEEE Control Syst*. 2013;33(1):76-88.
4. Gusrialdi A, Hirche S. Communication topology design for large-scale interconnected systems with time delay. Paper presented at: Proceedings of the 2011 American Control Conference, San Francisco, CA: 2011:4508-4513; IEEE.
5. Wang S, Davison EJ. On the stabilization of decentralized control systems. *IEEE Trans Autom Control*. 1973;18(5):473-478.
6. Jiang ZP, Teel AR, Praly L. Small-gain theorem for ISS systems and applications. *Math Control Signals Syst*. 1994;7(2):95-120.
7. Jiang ZP, Mareels IMY, Wang Y. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica*. 1996;32(8):1211-1215.
8. Liu T, Jiang ZP. A small-gain approach to robust event-triggered control of nonlinear systems. *IEEE Trans Autom Control*. 2015;60(8):2072-2085.
9. Dashkovskiy SN, Rüffer B, Wirth FR. Small gain theorems for large scale systems and construction of ISS Lyapunov functions. *SIAM J Control Optim*. 2010;48(6):4089-4118.
10. Liu D, Wang D, Li H. Decentralized stabilization for a class of continuous-time nonlinear interconnected systems using online learning optimal control approach. *IEEE Trans Neural Netw Learn Syst*. 2014;25(2):418-428.
11. Jiang Y, Jiang ZP. Robust adaptive dynamic programming for large-scale systems with an application to multimachine power systems. *IEEE Trans Circ Syst II Exp Briefs*. 2012;59(10):693-697.
12. Gao W, Jiang Y, Jiang ZP, Chai T. Output-feedback adaptive optimal control of interconnected systems based on robust adaptive dynamic programming. *Automatica*. 2016;72:37-45.
13. Wang X, Lemmon M. On event design in event-triggered feedback systems. *Automatica*. 2011;47(10):2319-2322.
14. De Persis C, Sailer R, Wirth F. Parsimonious event-triggered distributed control: A Zeno free approach. *Automatica*. 2013;49(7):2116-2124.
15. Yang Y, Vamvoudakis KG, Modares H, Yin Y, Wunsch DC. Safe intermittent reinforcement learning with static and dynamic event generators. *IEEE Trans Neural Netw Learn Syst*. 2020;PP(99):1-15 (in press). https://doi.org/10.1109/TNNLS.2020.2967871.
16. Narayanan V, Jagannathan S. Distributed event-sampled approximate optimal control of interconnected affine nonlinear continuous-time systems. Paper presented at: Proceedings of the 2016 American Control Conference; 2016:3044-3049; Boston, MA: IEEE.
17. Başar T, Bernhard P. *H∞ Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*. Berlin, Germany: Springer Science & Business Media; 2008.
18. Vamvoudakis KG, Modares H, Kiumarsi B, Lewis FL. Game theory-based control system algorithms with real-time reinforcement learning: How to solve multiplayer games online. *IEEE Control Syst*. 2017;37(1):33-52.
19. Narayanan V, Sahoo A, Jagannathan S, George K. Approximate optimal distributed control of nonlinear interconnected systems using event-triggered nonzero-sum games. *IEEE Trans Neural Netw Learn Syst*. 2018;30(5):1512-1522.
20. Yang Y, Vamvoudakis KG, Modares H. Safe reinforcement learning for dynamical games. *Int J Robust Nonlinear Control*. 2020;30(9):3706-3726.
21. Narayanan V, Modares H, Jagannathan S, Lewis FL. Event-driven off-policy reinforcement learning for control of interconnected systems. *IEEE Trans Cybern*. 2020 (In press).
22. Barrón MA, Sen M. Synchronization of four coupled van der Pol oscillators. *Nonlinear Dyn*. 2009;56(4):357.
23. Kawato M, Suzuki R. Biological oscillators can be stopped—Topological study of a phase response curve. *Biol Cybern*. 1978;30(4):241-248. https://doi.org/10.1007/BF00361045.
24. Borgers DPN, Heemels WMH. Event-separation properties of event-triggered control systems. *IEEE Trans Autom Control*. 2014;59(10):2644-2656.
25. Modares H, Lewis FL, Jiang ZP. H_∞ tracking control of completely unknown continuous-time systems via off-policy reinforcement learning. *IEEE Trans Neural Netw Learn Syst*. 2015;26(10):2550-2562.
26. Lewis FL, Vrabie D, Syrmos VL. *Optimal Control*. Boca Raton, FL: John Wiley & Sons; 2012.
27. Van Der Schaft AJ. L2 gain analysis of nonlinear systems and nonlinear state-feedback H_∞ control. *IEEE Trans Automat Control*. 1992;37(6):770-784.
28. Vamvoudakis KG, Lewis FL. Online solution of nonlinear two-player zero-sum games using synchronous policy iteration. 49th IEEE Conference on Decision and Control (CDC), Atlanta, GA; 2010:3040-3047. https://doi.org/10.1109/CDC.2010.5717607
29. Ortega JM. *Matrix Theory: A Second Course*. University Series in Mathematics. New York, NY: Springer; 2013.
30. Chen BM. *Robust and H_∞ Control*. Berlin, Germany: Springer Science & Business Media; 2013.
31. Petersen I. Disturbance attenuation and H_∞ optimization: a design method based on the algebraic Riccati equation. *IEEE Trans Automat Control*. 1987;32(5):427-429.
32. Narayanan V, Jagannathan S. Event-triggered distributed approximate optimal state and output control of affine nonlinear interconnected systems. *IEEE Trans Neural Netw Learn Syst*. 2018;29(7):2846–2856. https://doi.org/10.1109/TNNLS.2017.2693205.
33. Guo Y, Hill DJ, Wang Y. Nonlinear decentralized control of large-scale power systems. *Automatica*. 2000;36(9):1275-1289.

34. Alrifai MT, Hassan MF, Zribi M. Decentralized load frequency controller for a multi-area interconnected power system. *Int J Electr Power Energy Syst*. 2011;33(2):198-209.

How to cite this article: Narayanan V, Modares H, Jagannathan S. Event-triggered control of input-affine nonlinear interconnected systems using multiplayer game. *Int J Robust Nonlinear Control*. 2020;1–21. https://doi.org/10.1002/rnc.5321

APPENDIX A. PROOF OF LEMMA 2

We will choose the optimal cost function as the Lyapunov candidate function for each subsystem, that is, let $J_i(x_i(t)) = V_i^*(x_i(t))$. Taking the time derivative of this candidate function and substituting the dynamics as in (4) with the control u_i^*, yields

$$\dot{J}_i = J_{ix}' \left[f_i(x_i) + g_i(x_i)u_i^* + \sum_{j=1, j\neq i}^{N} \Delta_{ij}(x_i)x_j + g_i(x_i)(\breve{u}_i^* - u_i^*) \right]. \tag{A1}$$

Adding and subtracting $\sum_{j=1,j\neq i}^{N} \Delta_{ij}(x_i)d_j^*$ to (A1), we get

$$\dot{J}_i = J_{ix}' \left[f_i(x_i) + g_i(x_i)u_i^* + \sum_{j=1,j\neq i}^{N} \Delta_{ij}(x_i)d_j^* + \sum_{j=1,j\neq i}^{N} \Delta_{ij}(x_i)(x_j - d_j^*) + g_i(x_i)(\breve{u}_i^* - u_i^*) \right].$$

From the HJI equation (18), we have

$$J_{ix}' \left[f_i(x_i) + g_i(x_i)u_i^* + \sum_{j=1,j\neq i}^{N} \Delta_{ij}(x_i)d_j^* \right] = -Q_i(x_i) - u_i^{*\prime} R_i u_i^* + \sigma_i^2 \sum_{j=1,j\neq i}^{N} d_j^{*\prime} d_j^*. \tag{A2}$$

Using (A2), the time derivative of the Lyapunov function becomes

$$\dot{J}_i = -Q_i(x_i) - u_i^{*\prime} R_i u_i^* + \sigma_i^2 \sum_{j=1,j\neq i}^{N} d_j^{*\prime} d_j^* + J'_{ix} \left[\sum_{j=1,j\neq i}^{N} \Delta_{ij}(x_i)(x_j - d_j^*) + g_i(x_i)(\breve{u}_i^* - u_i^*) \right]. \tag{A3}$$

Recalling the definitions from (16) and (17), we have $2R_i u_i^*(x_i) = -g_i' V_{ix}^*$ and $2\sigma_i^2 d_j^*(x_i) = \Delta_{ij}' V_{ix}^*$. Using these equalities in (A3), we get

$$\dot{J}_i = -Q_i(x_i) - u_i^{*\prime} R_i u_i^* - \sigma_i^2 \sum_{j=1,j\neq i}^{N} d_j^{*\prime} d_j^* + 2\sigma_i^2 \sum_{j=1,j\neq i}^{N} d_j^{*\prime} x_j - 2u_i^{*\prime} R_i(\breve{u}_i^* - u_i^*). \tag{A4}$$

Using the Euclidean norm, we can simplify (A4) to obtain

$$\dot{J}_i \leq -\overline{Q}_i(x_i) + 2\sigma_i^2 \sum_{\substack{j=1 \\ j\neq i}}^{N} \|d_j^*\| \|x_j\| + 2\|R_i\| \|u_i^*\| \|\breve{u}_i^* - u_i^*\|, \tag{A5}$$

where $\overline{Q}_i(x_i) = Q_i(x_i) + u_i^{*\prime} R_i u_i^* + \sigma_i^2 \sum_{j=1,j\neq i}^{N} d_j^{*\prime} d_j^*$. Using the Lipschitz property of u_i^*, d_j^*, we have

$$\dot{J}_i \leq -\overline{Q}_i(x_i) + 2\sigma_i^2 L_{u_i} \|x_i\| \sum_{\substack{j=1 \\ j\neq i}}^{N} \|x_j\| + 2\|R_i\| L_{u_i}^2 \|x_i\| \|e_i\|. \tag{A6}$$

For some $L_x > 0$, it holds that $\overline{Q}_i(x_i) \geq L_x\|x_i\|^2$ (for $x_i \neq 0$) and hence, we get

$$\dot{J}_i \leq -L_x\|x_i\|^2 + \|x_i\| \left(2\sigma_i^2 L_{u_i} \sum_{\substack{j=1 \\ j\neq i}}^{N} \|x_j\| + 2\|R_i\| L_{u_i}^2 \|e_i\| \right). \tag{A7}$$

Furthermore, for any $0 < c_i < L_x$, it can be concluded that

$$\|x_i\| \geq \frac{1}{c_i} \left(2\sigma_i^2 L_{u_i} \sum_{\substack{j=1 \\ j\neq i}}^{N} \|x_j\| + 2\|R_i\| {L_{u_i}}^2 \|e_i\| \right)$$

implies that the first derivative of the Lyapunov function

$$\dot{J}_i \leq -(L_x - c_i)\|x_i\|^2.$$

Note that $\overline{Q}_i(x_i) > Q_i(x_i)$. Since $Q_i(x_i) > 0$ (for $x_i \neq 0$), $R_i > 0, \sigma_i > 0$ are design parameters, they can be chosen such that $\overline{Q}_i(x_i) \geq L_x\|x_i\|^2$ (for example $Q_i(x) = x_i' q_i x_i$ for some $q_i > 0$ implies, L_x can be selected as minimum singular value of q_i). This concludes the proof.

APPENDIX B. PROOF OF COROLLARY 1

Noting that for the case of linear system, the optimal cost is quadratic, we have $J_i = x_i' P_i x_i$, where P_i is a symmetric positive definite solution for the algebraic game Riccati equation with $Q_i(x_i) = x_i' Q_i x_i$ for $Q_i > 0$. Using this definition of the value function, we have $u_i^* = -\frac{1}{2} R_i^{-1} B_i' P_i x_i$, $d_j^* = \frac{1}{2\sigma_i^2} A_{ij}' P_i x_i$. Following a similar procedure as in the proof of Lemma 2, for the case of linear system dynamics, we can arrive at

$$\dot{J}_i = -x_i' \overline{Q}_i x_i + 2 \sum_{\substack{j=1 \\ j\neq i}}^{N} (x_i' P_i A_{ij} x_j) + 2x_i' P_i B_i K_i e_i,$$

where $\overline{Q}_i = Q_i + \frac{1}{4} P_i B_i R_i^{-1} B_i' P_i + \frac{1}{4\sigma_i^2} \sum_{j=1, j\neq i}^{N} P_i A_{ij} A_{ij}' P_i$. For some $0 < c_i < \overline{Q}_i$, we can get the desired gains as $\gamma_{ij}(s) = 2\frac{\|P_i\|^{3/2}}{c_i} \sum_{\substack{j=1 \\ j\neq i}}^{N} \left(\frac{\|A_{ij}\|}{[\lambda_{\min}(P_j)]^{1/2}} s^{1/2} \right)$, $\gamma_{ii} = 0$ with $\overline{\alpha}_i^{\kappa}(s) = (\lambda_{\min}(\overline{Q}_i) - c_i)s^2$, $\eta_i(s) = 2\frac{\|P_i\|^{3/2}}{c_i} \|B_i K_i\| s$, and $\mu_i(s) = \sum_{j=1}^{N+1} s_j^2$. This completes the proof.

APPENDIX C. PROOF OF THEOREM 1

Let that the operator norm of Γ be denoted as $\|\Gamma\|$. The spectral radius of Γ, $\rho(\Gamma) = \max\{|\sigma_i|\}, i = 1, \ldots, N$. Using the relationship of operator norm and the singular values of Γ, it is a fact that $\rho(\Gamma) \leq \|\Gamma\|$ for any operator norm.[29] Therefore, $\|\Gamma\|_\infty < 1$ or $\|\Gamma\|_2 < 1$ implies that the spectral radius of Γ will be less than one. Then, the existence of an ISS Lyapunov function for the overall system follows from the results in Reference 9 (theorem 5.3).

APPENDIX D. PROOF OF THEOREM 2

Consider the Lyapunov function for the overall system as $J(\nu) = \sum_{i=1}^{N} J_i(x_i)$, where $J_i(x_i)$ is chosen as the optimal cost function of the i^{th} subsystem as in the Proof of Lemma 2. From the proof of Lemma 2, we have

$$\dot{J}_i \leq -L_x\|x_i\|^2 + \|x_i\| \left(2\sigma_i^2 L_{u_i} \sum_{\substack{j=1 \\ j\neq i}}^{N} \|x_j\| + 2\|R_i\| L_{u_i}^2 \|e_i\| \right). \tag{D1}$$

Using the Young's inequality, we get

$$\dot{J}_i \leq -\left(L_x - \left(\sigma_i^2 + \frac{1}{2}\right) L_{u_i}^2\right) \|x_i\|^2 + \sigma_i^2 \sum_{j=1, j\neq i}^{N} \|x_j\|^2 + 2\|R_i\|^2 L_{u_i}^2 \|e_i\|^2. \tag{D2}$$

Furthermore, using the event-triggering condition (20) with $L_{Jx} = \left(\sigma_i^2 + \frac{1}{2}\right) L_{u_i}^2$, $0 < \bar{c}_i < 1$, we get

$$\dot{J}_i \leq -(1-\bar{c}_i)(L_x - L_{Jx})\|x_i\|^2 + \sigma_i^2 \sum_{\substack{j=1 \\ j\neq i}}^{N} \|x_j\|^2.$$

Now, using the Lyapunov function for the overall system $\dot{J} = \sum_{i=1}^{N} \dot{J}_i$, we have

$$\dot{J} \leq -\sum_{i=1}^{N} (1-\bar{c}_i)(L_x - L_{Jx} - \sigma_i^2(N-1))\|x_i\|^2.$$

Thus, the Lyapunov function is negative definite as long as $L_x > L_{Jx} + \sigma_i^2(N-1)$. This completes the proof.

APPENDIX E. PROOF OF THEOREM 3

Choose the same Lyapunov function as in Theorem 2. The derivative of the Lyapunov function of the i^{th} subsystem with respect to time along the dynamics (4) is given by

$$\dot{J}_i = J_{ix}' \left[f_i(x_i) + g_i(x_i)u_i^* + g_i(x_i)(\breve{u}_i^* - u_i^*) + \sum_{j=1, j\neq i}^{N} \Delta_{ij}(x_i)x_j \right]. \tag{E1}$$

Since the optimal value function satisfies the HJI, we have

$$J_{ix}' \left[f_i(x_i) + g_i(x_i)u_i^* + \sum_{j=1, j\neq i}^{N} \Delta_{ij} d_j^* + g_i(x_i) d_{i0}^* \right] = -Q_i(x_i) - u_i^{*'} R_i u_i^* + \sigma_i^2 \sum_{j=1, j\neq i}^{N} d_j^{*'} d_j^* + \sigma_{i0}^2 d_{i0}^{*'} d_{i0}. \tag{E2}$$

Using (E2) in (E1), we get

$$\begin{aligned} \dot{J}_i = {} & -Q_i(x_i) - u_i^{*\prime} R_i u_i^* + \sigma_i^2 \sum_{j=1, j\neq i}^{N} d_j^{*\prime} d_j^* + \sigma_{i0}^2 d_{i0}^{*\prime} d_{i0}^* - J_{ix}' g_i(x_i) d_{i0}^* - J_{ix}' \sum_{\substack{j=1 \\ j\neq i}}^{N} \Delta_{ij}(x_i) d_j^* \\ & + J_{ix}' \sum_{\substack{j=1 \\ j\neq i}}^{N} \Delta_{ij} x_j + J_{ix}' g_i [u_i^*(\breve{x}_i) - u_i^*(x_i)]. \end{aligned} \tag{E3}$$

Using the definitions of d_j^*, d_{i0}^*, we get

$$\dot{J}_i = -Q_i(.) - u_i^{*\prime} R_i u_i^* - \sigma_i^2 \sum_{j=1, j\neq i}^{N} d_j^{*\prime} d_j^* - \sigma_{i0}^2 d_{i0}^{*\prime} d_{i0}^* + 2\sigma_i^2 \sum_{\substack{j=1 \\ j\neq i}}^{N} d_j^{*\prime} x_j + 2\sigma_{i0}^2 d_{i0}^{*\prime} [u_i^*(\breve{x}_i) - u_i^*(x_i)].$$

Applying the norm operator reveals

$$\dot{J}_i \leq -Q_i(.) - u_i^{*\prime} R_i u_i^* - \sigma_i^2 \sum_{j=1, j\neq i}^{N} d_j^{*\prime} d_j^* - \sigma_{i0}^2 d_{i0}^{*\prime} d_{i0}^* + 2\sigma_i^2 \sum_{\substack{j=1 \\ j\neq i}}^{N} \|d_j^*\| \|x_j\| + 2\sigma_{i0}^2 \|d_{i0}^{*\prime}\| \|u_i^*(\breve{x}_i) - u_i^*(x_i)\|.$$

Using the Young's inequality to simplify further, we get

$$\dot{J}_i \leq -Q_i(x_i) - u_i^{*\prime} R_i u_i^* - \sigma_i^2 \sum_{j=1, j\neq i}^{N} d_j^{*\prime} d_j^* - \sigma_{i0}^2 d_{i0}^{*\prime} d_{i0}^* + \sigma_i^2 \sum_{\substack{j=1 \\ j\neq i}}^{N} d_j^{*\prime} d_j^* + \sigma_{i0}^2 d_{i0}^{*\prime} d_{i0}^*$$
$$+ \sigma_i^2 \sum_{\substack{j=1 \\ j\neq i}}^{N} \|x_j\|^2 + \sigma_{i0}^2 \|u_i^*(\breve{x}_i) - u_i^*(x_i)\|^2.$$

The event-triggering condition (27) can be used to obtain

$$\dot{J}_i \leq -Q_i(x_i) - u_i^{*\prime} R_i u_i^* + \sigma_{i0}^2 d_{i0}^{*\prime} d_{i0}^* + \sigma_i^2 \sum_{\substack{j=1 \\ j\neq i}}^{N} \|x_j\|^2.$$

Define $\overline{\delta}_i > 0$ such that $-Q_i(x_i) - u_i^{*\prime} R_i u_i^* + \sigma_{i0}^2 d_{i0}^{*\prime} d_{i0}^* \leq -\overline{\delta}_i \|x_i\|^2$ to get

$$\dot{J}_i \leq -\overline{\delta}_i \|x_i\|^2 + \sigma_i^2 \sum_{\substack{j=1 \\ j\neq i}}^{N} \|x_j\|^2.$$

Now, using the Lyapunov function for the overall system $\dot{J} = \sum_{i=1}^{N} \dot{J}_i$, we have $\dot{J} \leq -\sum_{i=1}^{N} (\overline{\delta}_i - \sigma_i^2 (N-1)) \|x_i\|^2$. Thus, the Lyapunov function is negative definite as long as $\overline{\delta}_i > \sigma_i^2 (N-1)$. Observe that $u_i^{*\prime} R_i u_i^* - \sigma_{i0}^2 d_{i0}^{*\prime} d_{i0}^*$ can be simplified as $\frac{1}{4} V_{ix}^{*\prime} g_i(x_i)(R_i^{-1} - \frac{I_m}{\sigma_{i0}^2}) g_i'(x_i) V_{ix}^*$, where I_m is an identity matrix of appropriate dimensions. Choosing $R_i^{-1} > \frac{I_m}{\sigma_{i0}^2}$ and Q_i as in Lemma 2, $\overline{\delta}_i$ can be obtained. This concludes the proof.

APPENDIX F. PROOF OF COROLLARY 3

From Theorems 2 and 3, we have $\dot{J}_i \leq -\delta_i \|x_i\|^2 + \sigma_i^2 \sum_{j=1, j\neq i}^{N} x_j' x_j$, with $\delta_i = (1 - \overline{c}_i)(L_x - L_{Jx})$ (Theorem 2) or $\delta_i = \overline{\delta_i}$ (Theorem 3). The ISS gain matrix can be constructed such that $\Gamma = \begin{bmatrix} 0 & \sigma_1^2 & \sigma_1^2 & \dots \\ \sigma_2^2 & 0 & \sigma_2^2 & \sigma_2^2 \\ \vdots & \sigma_i^2 & \ddots & \sigma_i^2 \\ \sigma_N^2 & \dots & \sigma_N^2 & 0 \end{bmatrix}$ and choosing the L_2 gains such that either $\sum_{i=1}^{N-1} \sigma_i^2 < 1$ or $\sigma_i^2 < \frac{1}{N-1}$, $\forall i = 1, \dots, N$ will satisfy the equivalent of Theorem 1 for nonlinear systems (see Reference 9, theorem 5.3), and hence, ISS of the overall system follows.

APPENDIX G. PROOF OF THEOREM 4

This proof is an extension of the proof in an existing work[25] for the proposed game theoretic formulation. Consider the HJI equation with some admissible control u_i and interconnections

$$H_i(V_i^*, u_i, d_j) = Q_i(x_i) + u_i' R_i u_i - \sigma_i^2 \sum_{j=0, j\neq i}^{N} d_j' d_j + V_{ix}^{*'} \left[f_i + g_i u_i(t) + \sum_{\substack{j=1 \\ j\neq i}}^{N} \Delta_{ij} d_j(t) \right], \tag{G1}$$

where $\Delta_{i0} = g_i$ and $d_0 = d_{i0}$ for $i = 1, \dots, N$. Computing the difference in the Hamiltonians $H_i(V_i^*, u_i, d_j) - H_i(V_i^*, u_i^*, d_j^*)$, yields

$$H_i(V_i^*, u_i, d_j) - H_i(V_i^*, u_i^*, d_j^*) = u_i' R_i u_i - u_i^{*'} R_i u_i^* + \sigma_i^2 \left(\sum_{j=0, j\neq i}^{N} (d_j^{*'} d_j^* - d_j' d_j) \right) + V_{ix}^{*'} \left[(g_i u_i - g_i u_i^*) + \sum_{\substack{j=1 \\ j\neq i}}^{N} \Delta_{ij} d_j - \sum_{\substack{j=1 \\ j\neq i}}^{N} \Delta_{ij} d_j^* \right].$$

Using the definition of optimal policies gives

$$H_i(V_i^*, u_i, d_j) - H_i(V_i^*, u_i^*, d_j^*) = (u_i - u_i^*)' R_i (u_i - u_i^*) - \sigma_i^2 \sum_{j=0, j\neq i}^{N} (d_j^* - d_j)'(d_j^* - d_j). \tag{G2}$$

Using the fact that $H_i(V_i^*, u_i^*, d_j^*) = 0$, we get the relation

$$\begin{aligned} &Q_i(x_i) + u_i' R_i u_i - \sigma_i^2 \sum_{j=0, j\neq i}^{N} d_j' d_j + V_{ix}^{*'} \left[f_i + g_i u_i(t) + \sum_{\substack{j=1 \\ j\neq i}}^{N} \Delta_{ij} d_j(t) \right] \\ &= (u_i - u_i^*)' R_i (u_i - u_i^*) - \sigma_i^2 \sum_{\substack{j=1 \\ j\neq i}}^{N} (d_j^* - d_j)'(d_j^* - d_j). \end{aligned} \tag{G3}$$

Substituting $u_i = u_i^*$ in (G3) reveals

$$\begin{aligned} &Q_i(x_i) + u_i^{*\prime} R_i u_i^* - \sigma_i^2 \sum_{j=0, j\neq i}^{N} d_j' d_j + V_{ix}^{*'} \left[f_i x_i + g_i u_i^* + \sum_{\substack{j=1 \\ j\neq i}}^{N} \Delta_{ij} d_j(t) \right] \\ &= -\sigma_i^2 \sum_{\substack{j=1 \\ j\neq i}}^{N} (d_j^* - d_j)'(d_j^* - d_j) \leq 0. \end{aligned} \tag{G4}$$

Obtaining using the chain rule $\dot{V}_i^* = V_{ix}^{*'} \left[f_i x_i + g_i u_i^* + \sum_{\substack{j=1 \\ j\neq i}}^{N} \Delta_{ij} d_j(t) \right]$ and substituting it in G.4 reveals

$$\dot{V}_i^* \leq -Q_i(x_i) - u_i^{*'} R_i u_i^* + \sigma_i^2 \sum_{j=0, j\neq i}^{N} d_j' d_j. \tag{G5}$$

Integrating from both sides from 0 to $T > 0$ yields

$$V_i^*(x_i(T)) - V_i^*(x_i(0)) \leq \int_0^T \left(-Q_i(x_i) - u_i^{*'} R_i u_i^* + \sigma_i^2 \sum_{j=0, j\neq i}^{N} d_j' d_j \right) d\tau. \tag{G6}$$

Using the fact that $V_i^*(\cdot) \geq 0$, we have

$$0 \leq \int_0^T \left(-Q_i(x_i) - u_i^{*'} R_i u_i^* + \sigma_i^2 \sum_{j=0, j\neq i}^{N} d_j' d_j \right) d\tau + V_i^*(x_i(0)). \tag{G7}$$

Rearranging the last equation results in

$$\int_0^T (Q_i(x_i) + u_i^{*'} R_i u_i^*) d\tau \leq \int_0^T \left(\sigma_i^2 \sum_{j=0, j\neq i}^{N} d_j' d_j \right) d\tau + V_i^*(x_i(0)). \tag{G8}$$

This completes the proof.