IEEE TRANSACTIONS ON
NEURAL NETWORKS AND LEARNING SYSTEMS



# Event-Sampled Direct Adaptive Neural Network Output- and State-Feedback Control of Uncertain Strict-Feedback System

Nathan Szanto, N. Vignesh, and S. Jagannathan

This research is supported, in part, by Intelligent systems center, Rolla and NSF ECCS #1406533. Nathan Szanto, N. Vignesh and S. Jagannathan are with the Department of Electrical and Computer Engineering, Missouri University of Science and Technology, Rolla. (ns6cf@mst.edu, vnxv4@mst.edu, and sarangap@mst.edu)

***Abstract*—In this work, a novel event-triggered implementation of a tracking controller for a nonlinear strict-feedback system with uncertain dynamics is presented. Approximations of the unknown nonlinear system dynamics are used in the backstepping approach in order to generate the control input. Neural networks (NNs) with event-sampled inputs are used as the means for obtaining the approximations. The system state vector is assumed to be unknown and is estimated with a NN observer. Using the estimated state vector in the backstepping design approach, an event-sampled controller is introduced. With the proposed design, it is demonstrated that the closed-loop system exhibits input-to-state-like stability (ISS) when the feedback, which is continuously accessed by the controller, is injected with bounded measurement errors. Further, for the event triggered control execution, an event-trigger condition is derived using Lyapunov analysis and uniform ultimate bounded (UUB) regulation of the tracking errors, NN weights and the observer errors is demonstrated. Finally, results are also provided for the restricted case when all the states are assumed to be measurable. The effectiveness of the proposed controllers is demonstrated with simulation results.**

## I. Introduction

GIVEN their inherent ability to approximate unknown functions, neural networks (NNs) have gained popularity in online control applications that involve nonlinear systems with uncertain dynamics. Additionally, many practical systems can be represented in strict-feedback form wherein each state dynamics are written in terms of nonlinear functions. When there are unknown dynamics in strict-feedback systems, subsystems are considered individually and the unknown nonlinear function corresponding to each subsystem is approximated using a NN. A detailed control design procedure using this backstepping approach is presented in [1]; moreoever, the work in [3] is able to employ the backstepping design in a robotic system application. However, the efforts [1] and [3] rely on complete knowledge of the system state vector; unfortunately, this type of information is not always readily available.

In situations where the state vector is not available, an additional NN can be used as an observer to estimate the unknown states. However, unlike linear systems, the separation principle does not hold for nonlinear systems. Therefore, independent design of an observer and a controller do not guarantee stable operation of the system when they operate together. As an example, the work in [4] models a quadrotor UAV system in strict-feedback form and circumvents the need for velocity sensors with an observer; the derivation presented in [4] shows how Lyapunov theory can be used to demonstrate the stable design of a controller with an observer. In general, designing an output-feedback controller with backstepping is an approach that has been implemented in a number of different applications [4]-[8].

The works in [1]-[8] present design techniques for controllers wherein the control law is executed with continuous/periodic feedback. An alternative to the traditional periodic/continuous feedback paradigm is introduced in [9] wherein the control law is executed with event based feedback. When event based feedback is used, the controller is not updated at a fixed frequency, but only when there is an event. It can be quickly observed that an advantage to this approach is a reduction in computations; also communication overhead is reduced, if a communication network closes the feedback loop. Further, with aperiodic control updates, it can be demonstrated that computations can be reduced without having to compromise the fidelity of the controller's performance.

The efforts in [9] introduces the basic concepts of event based sampling, controller implementation and demonstrates how they can be applied in a general control system when the dynamics are known. Since the publication of [9], significant developments have been made in the area of event-triggered control implementation and they have been considered in numerous contexts: The efforts have addressed input-state stability [15], output-feedback systems [12], and trajectory tracking applications [14]. In the works [9],[12],[14],[15], the dynamics are taken to be known and the controllers did not incorporate any adaptive elements; consequently, the effects of event-based feedback in adaptive control are not explored. Finally, the work in [10],[11],[13],[16]-[18], considers system with unknown dynamics and the concepts of event-based sampling is extended to adaptive NN control. In any work pertaining to event based control implementation, an event-execution law to determine the feedback instants is required; a particularly useful conclusion given in [16] is the presentation of an execution law used in tandem with NN approximations.

The results in [16], however, are limited to input affine systems. Additionally, the execution law that was derived in [16] requires information on the estimates of the adaptive NN weights. This creates the need for an additional "mirror estimator" at the event-triggering mechanism. Similarly, the results in [17]-[18] develop control schemes for input affine systems in the context of optimal control design, [13] focuses on linear interconnected systems, and [10], [11] develops event triggered controllers in a discrete time framework. Unlike the input affine systems, the effects of event sampling for the nonlinear dynamics in strict feedback form are expected to affect the dynamics of each subsystem. Moreover, the inclusion of NNs, which allows for relaxing the requirement of complete knowledge of the system dynamics to design controllers will require event based approximation at each subsystem. Finally, the number of applications for which a controller could be employed can be further increased with the implementation of an observer, alleviating the need for full knowledge of the state vector.

The first main result of this paper is to demonstrate that with the backstepping output feedback based controller design using NNs, the closed loop system which includes a NN observer and controller admits a Lyapunov function which satisfies local input-to-state like stability (ISS) [21].This establishes the fact that, for any bounded, external input to the closed-loop system, the tracking error, the observer estimation error, and the NN weight estimation errors are locally uniformly ultimately bounded (UUB). Next, the event triggered control execution is considered. Here, it is demonstrated that the event-based execution of the controller results in an external input in the form of an event-triggering error. Finally, using the Lyapunov design, a novel event-triggering mechanism is proposed to ensure the boundedness of this event-triggering error. The trade-off between the NN approximation accuracy and frequency of events are discussed. Next, the theoretical results are extended for the case when all the states are measured. Finally, in order to demonstrate the effectiveness of the proporsed controllers, simulation results are provided.

Digital Object Identifier: 10.1109/TNNLS.2017.2678922

The contributions of this paper are: a) the design of strict-feedback controller incorporating adaptive NNs to learn the unknown nonlinear dynamics in the event-triggered feedback framework; b) develop adaptive observers to reconstruct the system internal states using the measured outputs; c) design of an event-execution rule for the strict-feedback system which is computationally efficient than the rule presented in [16]; d) finally, stability analysis for output feedback case using Lyapunov theory and a brief analysis of the state-feedback case [22] is presented as a corollary.

The remainder of this paper is organized as follows: System description and the problem statement are presented in Section II; The output feedback controller design and a brief summary of state feedback results are detailed in Section III; The theoretical design is verified using a numerical example via simulations. The performance of the proposed event based NN controllers and the continuously implemented controller are compared and these results are presented in Section IV; conclusions follow in Section V.

## II. Background and Problem Statement

In this section, the notations used in this paper are introduced first. Next, the approximation property of the artificial neural networks is recalled and finally, the nonlinear system dynamics in strict-feedback form and the control objectives are defined.

### A. Notations

The vector $\overline{x}_i = [x_1, \ldots, x_i]^T, \; i \leq n$, denotes the partial state vector. Since this work involves adaptive NN weights, a hat operator will be used to denote the estimated values; a tilde will be used to denote the difference between the estimate and actual value, specifically $\tilde{(\cdot)} = \hat{(\cdot)} - (\cdot)$, denoting that the error in the estimate is the difference between the estimate and the actual value. For brevity, any function $f_1(x_1)$ will simply be written as $f_1$. Finally, the norm operator, $\|\cdot\|$, when used with vectors will denote the Euclidean norm of the vector and for matrices, its Frobenius norm [19].

### B. Neural networks as function approximators

In this paper, NNs are used as function approximators and are designed to learn the unknown nonlinear functions in the system dynamics. Let $h(X)$ be a smooth, unknown function. Using the *universal approximation property* of NN, the approximation of $h(X)$ is given as

$$h(X) = W^T \varphi\left(V^T X\right) - \varepsilon, \tag{1}$$

where, $\varphi$ is the activation function, $V^T$ is the input layer weight matrix. In this paper, these weights are chosen to take random values to form a random vector functional link (RVFL) network [19]; $W^T$ denotes the target/ideal values of the NN weights and satisfy the bounds $\|W\| \leq W_M$; $\varepsilon$ - the bounded NN reconstruction/approximation error satisfying $\|\varepsilon\| < \varepsilon_M$. Finally, the activation functions satisfy $\varphi(0) = 0$ and $\|\varphi\| \leq \sqrt{N}$, where $N$ is the number of neurons in the hidden layer of the NN.

In the next subsection, the strict-feedback system is defined first and the control objectives are presented.

### C. Problem Statement

Consider the nonlinear system in strict feedback form given by

$$\begin{aligned} \dot{x}_i &= f_i(\overline{x}_i) + g_i(\overline{x}_i)\, x_{i+1}, && 1 \leq i \leq n-1 \\ \dot{x}_n &= f_n(\overline{x}_n) + g_n(\overline{x}_n)\, u, && n \geq 2 \\ y &= x_1. \end{aligned} \tag{2}$$

In this work, the nonlinear functions, $f_i(\overline{x}_i)$ and $g_i(\overline{x}_i)$, $i = 1, \ldots, n$, are considered to be smooth and unknown. The design objective is to develop NN based adaptive controllers for (2) using event-sampled feedback such that:

1) The system output, $y$, follows the trajectory, $y_d$. The desired trajectory is generated using a reference model of the form

$$\begin{aligned} \dot{x}_{di} &= f_{di}(x_d), && 1 \leq i \leq m \\ y_d &= x_{d1}, \end{aligned} \tag{3}$$

where the nonlinear functions $f_{di}(\cdot)$, $i = 1, 2, \ldots, m$, are smooth and known, $x_d = [x_{d1}, \ldots, x_{dm}]^T \in R^m$ are the states and $y_d$ is the output.
2) The closed-loop signals, which include the tracking errors and the NN weight estimates remain locally uniformly ultimately bounded (UUB).

In the design and analysis of the controller presented in this paper, the following assumptions are made:

*Assumption 1 [1]:* The reference model generates bounded state trajectories and $x_d(t) \in S_d, \forall t \geq 0$.

*Assumption 2 [1]:* The function $g_i$ is positive definite and there exist constants $0 < g_{mi} \leq g_{Mi}$ such that $g_{mi} \leq |g_i(\cdot)| \leq g_{Mi}, \forall \overline{x}_n \in S \subset \mathbb{R}^n$.

*Remark* 1. The assumption that $g_i$ is non-zero in its domain is required to ensure that the control effectiveness is not lost and that the desired stabilizing virtual controllers are well defined (Section III.B).

*Assumption 3 [12]:* The state vector, $\overline{x}_n$, is not available whereas the system (2) is observable.

*Remark* 2. For nonlinear systems, observability cannot be easily verified as in the case of a linear system. Unlike the linear case, where the observability matrix or the observability Grammian can be used to verify the possibility of reconstructing the internal states from the measured outputs, the observability properties of a nonlinear system are determined with Lie algebra on the state space; a detailed study of this was presented in [20]. In this paper, we assume that there exists observer gains such that the Lyapunov equation associated with the observer proposed in this paper has a positive definite solution (section III.A).

*Assumption 4 [9]:* Transmission delays and computational time delays are assumed to be absent.

*Remark* 3. It is shown in the event-triggered control literature [9] that the effects of the small computational delays can be accommodated by appropriately choosing the design parameter ($\sigma$, to be defined later) in the event-triggering condition.

The primary objective of the results presented in this paper is to develop a backstepping controller with event-sampled feedback as well as the associated event-triggering condition based on stability requirements. In the next section, the controller using output feedback is derived first, followed by the state-feedback controller. Since the procedure for the two controllers is nearly identical, details for state-feedback will be largely omitted and only major conclusions will be presented.

## III. Output-Feedback Controller

In this section, the observer design and its stability results are derived first. Subsequently, the controller design procedure is presented wherein a Lyapunov function, $V_n$, will be constructed using sub-Lyapunov functions, $V_i$, corresponding to individual subsystems ($i = 1, 2, \ldots$). For the state feedback controller, $V_n$ will be used to derive the conditions for guaranteeing closed-loop stability while, for output feedback, $V_n$ together with the Lyapunov function for the observer, $V_o$, will be analysed.

In this work, the event-triggering mechanism and the observer are co-located. The block diagram of the control scheme is presented in Fig. 1. The states are estimated continously whereas the estimated states are accessible to the controller only when an event occurs. As a result, the event-sampling measurement errors are explicitly present only in the controller design and not in the observer.

*Remark* 4. Although the observer NN makes use of continuously estimated values, the NN weights are updated only at events to reduce the computations. Due to this aperiodic weight updates, the approximation errors are driven by the event based measurement errors. The relationship between the event sampling errors and the reconstruction errors are derived in [11].

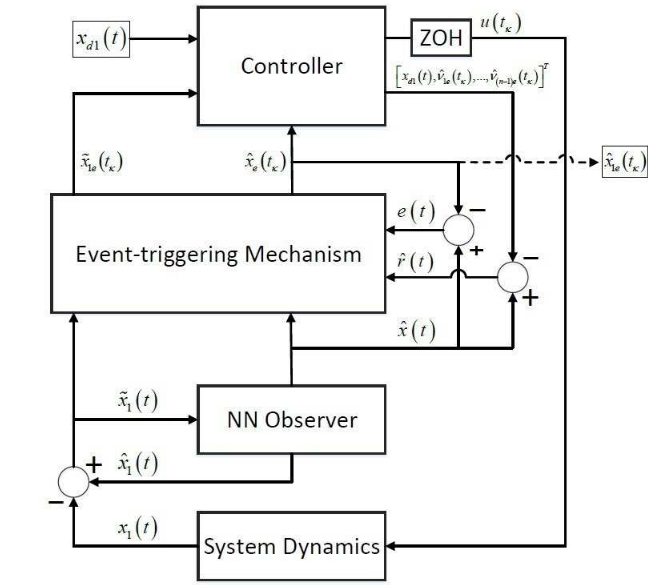



Fig. 1. NN Output-Feedback Control Structure

### A. NN Observer Design

In this section, a NN observer is designed to estimate the unknown state vector. In this work, the observer is located at the sensor. Begin by reformulating the system dynamics as

$$\begin{aligned} \dot{x}_i &= A_{oi}\overline{x}_i + f_{Ai}\left(\overline{x}_i\right) + g_i\left(\overline{x}_i\right)x_{i+1}, && 1 \le i \le n-1 \\ \dot{x}_n &= A_{on}\overline{x}_n + f_{An}\left(\overline{x}_n\right) + g_n\left(\overline{x}_n\right)u, && n \ge 2 \\ y &= cx. \end{aligned} \tag{4}$$

where $f_i$ in (2) is expressed as a sum of linear and nonlinear terms: $f_i\left(\overline{x}_i\right) = A_{oi}\overline{x}_i + f_{Ai}\left(\overline{x}_i\right)$ and $A_{oi}, \forall i = 1, 2, .., n$ are matrices of appropriate dimensions. Next, defining the following terms: $c = [1, 0, \ldots, 0] \in \mathbb{R}^{1\times n}$ and $F$ such that the system dynamics (4) can be rewritten as

$$\begin{aligned} \dot{x} &= F \\ y &= cx. \end{aligned} \tag{5}$$

Now, introduce an observer NN with ideal weights and denote

$$F = A_o x + W_o^T \varphi_o\left(X_o\right) - \varepsilon_o \tag{6}$$

with $X_o = \left[1, x^T, \tilde{x}_1\right]$, $A_o$ is the augmented matrix composed of $A_{oi}, \forall i = 1, 2, .., n$ and the observer estimation error is given by, $\tilde{x}_1 = \hat{x}_1 - x_1$. It is desired that the matrix $A_o$ renders the pair $(c, A_o)$ observable. Next, introduce the estimated weights and use the following observer

$$\begin{aligned} \dot{\hat{x}} &= \hat{F} \\ \hat{F} &= A_o\hat{x} + \hat{W}_o^T\varphi_o\left(\hat{X}_o\right) - Lc\tilde{x} \end{aligned} \tag{7}$$

with $L = [l_1, \ldots, l_n]^T \in \mathbb{R}^{n\times 1}$, is the observer gain vector chosen such that $A_o - Lc$ is Hurwitz, $\hat{X}_o = \left[1, \hat{x}^T, \tilde{x}_1\right]^T$. Observing that $\dot{\tilde{x}} = \hat{F} - F$, the observer estimation error dynamics are given by

$$\dot{\tilde{x}} = \hat{W}_o^T\varphi_o\left(\hat{X}_o\right) + A\tilde{x} - W_o^T\varphi_o\left(X_o\right) + \varepsilon_o. \tag{8}$$

For brevity, denote $A = A_o - Lc$, $\hat{\varphi} \triangleq \varphi\left(\hat{X}\right)$ and $\varphi \triangleq \varphi\left(X\right)$ and define $\tilde{\varphi} = \hat{\varphi} - \varphi$. Next, add and subtract $W_o^T\hat{\varphi}_o$ to (8) and rearrange terms in order to find an expression strictly in terms of observer estimation errors, NN weight estimation errors, and bounded terms. Denote the bounded term by $\xi \triangleq W_o^T\tilde{\varphi}_o + \varepsilon_o$ and note that $\|\xi\| \le \bar{\xi}_M$, where $\bar{\xi}_M = 2W_{Mo}\sqrt{N_o} + \varepsilon_{Mo}$, revealing

$$\dot{\tilde{x}} = A\tilde{x} + \tilde{W}_o^T\hat{\varphi}_o + \xi. \tag{9}$$

Next, stability results of the NN observer is presented in the following theorem. The results of Theorem 1 will be used in demonstrating the local ISS-like behavior of the closed-loop system.

*Remark* 5. In the following theorem, the control input, $u$, is assumed to be admissable; the assumption is relaxed in subsequent theorems.

**Theorem 1** *(NN Observer Boundedness):* Let the NN observer be defined by (7) with estimation error dynamics given by (9). Furthermore, let the NN observer weights be tuned by

$$\dot{\hat{W}}_o = F_o\left[-\hat{\varphi}_o I_n\tilde{x}_1 - \alpha_o\hat{W}_o\right] \tag{10}$$

where $F_o = F_o^T > 0, I_n = [1, 1, .., 1] \in \mathbb{R}^{1\times n}$ and $\alpha_o > 0$ are design parameters. Then, there exists a constant design parameter $A$ such that the observer estimation errors, $\tilde{x}$, and the NN observer estimation errors, $\tilde{W}_o$, are locally UUB, with the bounds being functions of the NN reconstruction error and other bounded terms.

*Proof:* Consider the following positive-definite Lyapunov candidate with $P_o > 0$:

$$V_o = \frac{1}{2}\tilde{x}^T P_o\tilde{x} + \frac{1}{2}\text{tr}\left\{\tilde{W}_o^T F_o^{-1}\tilde{W}_o\right\} \tag{11}$$

whose first derivative is given by $\dot{V}_o = \tilde{x}^T P_o\dot{\tilde{x}} + \text{tr}\left\{\tilde{W}_o^T F_o^{-1}\dot{\tilde{W}}_o\right\}$. Substitution of the error dynamics from (9) and using the NN update law given by (10) shows that

$$\begin{aligned} \dot{V}_o = &\ \tilde{x}^T P_o A\tilde{x} + \tilde{x}^T P_o\tilde{W}_o^T\hat{\varphi}_o \\ &+ \tilde{x}^T P_o\xi + \text{tr}\left\{-\tilde{W}_o^T\hat{\varphi}_o I_n c\tilde{x} - \alpha_o\tilde{W}_o^T\hat{W}_o\right\}. \end{aligned} \tag{12}$$

Since $A$ is Hurwitz, there exists a positive definite matrix $\tilde{A}$ such that the Lyapunov equation $\frac{1}{2}A^T P_o + \frac{1}{2}P_o A = -\frac{1}{2}\tilde{A}$ is satisfied. In order to proceed, recall the bounds on the ideal NN weights, the NN activation functions, and the NN reconstruction error. Using properties of the matrix trace operation, note that $\tilde{x}^T P_o\tilde{W}_o^T\hat{\varphi}_o + \text{tr}\left\{-\tilde{W}_o^T\hat{\varphi}_o I_n c\tilde{x} - \alpha_o\tilde{W}_o^T\hat{W}_o\right\} \le 2\left\|\tilde{W}\right\|\|\tilde{x}\|\sqrt{N_o}(\sqrt{n} + \|P_o\|) + \text{tr}\left\{-\alpha_o\tilde{W}_o^T\hat{W}_o\right\}$. Now, using the Young's inequality ($ab \le \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$, $\forall a, b, \epsilon > 0$), we have $2\left\|\tilde{W}\right\|\|\tilde{x}\|\sqrt{N_o}(\sqrt{n} + \|P_o\|) \le \frac{\|\tilde{W}\|^2}{2}4N_o(\sqrt{n} + \|P_o\|)^2 + \frac{\|\tilde{x}\|^2}{2}$ ($\epsilon = 1$). Using these results reveals

$$\begin{aligned} \dot{V}_o \le &- \left\|\bar{A}\right\|\|\tilde{x}\|^2 + \|\tilde{x}\|\xi_M \\ &- \bar{\alpha}_o\left\|\tilde{W}_o\right\|^2 + \alpha_o W_{Mo}\left\|\tilde{W}_o\right\|, \end{aligned} \tag{13}$$

where $\left\|\bar{A}\right\| = \left\|\frac{\tilde{A}}{2}\right\| - \frac{1}{2}$, $\|P_o\xi\| \le \xi_M$ and $\bar{\alpha}_o = \alpha_o - 2N_o(\sqrt{n} + \|P_o\|)^2$. In order to further simplify (13), apply Young's inequality to the terms $W_{Mo}\left\|\tilde{W}_o\right\|$ with $\epsilon = \bar{\alpha}_o$ and $\|\tilde{x}\|\xi_M$ with $\epsilon = 2/\bar{A}$, to obtain

$$\dot{V}_o \le -\left\|\bar{A}\right\|\|\tilde{x}\|^2 + \frac{\left\|\bar{A}\right\|}{4}\|\tilde{x}\|^2 + \frac{2\xi_M^2}{2\left\|\bar{A}\right\|} - \frac{\bar{\alpha}_o}{2}\left\|\tilde{W}_o\right\|^2 + \alpha_o^2\frac{W_{Mo}^2}{2\bar{\alpha}_o}.$$

Finally, group the bounded terms together and denote $\zeta_o \triangleq \frac{1}{2}\left[\frac{\alpha_o^2 W_{Mo}^2}{\bar{\alpha}_o} + \frac{2\xi_M^2}{\|\bar{A}\|}\right]$ to reveal the first derivative of the Lyapunov candidate as

$$\dot{V}_o \le -\frac{3}{4}\left\|\bar{A}\right\|\|\tilde{x}\|^2 - \frac{1}{2}\bar{\alpha}_o\left\|\tilde{W}_o\right\|^2 + \zeta_o. \tag{14}$$

Finally, (14) is less than zero when the design parameters $L, \alpha_o$ are selected such that $\left\|\bar{A}\right\|,\ \bar{\alpha}_o > 0$ and the following inequalities hold:

$$\|\tilde{x}\| > \sqrt{\frac{4\zeta_o}{3\left\|\bar{A}\right\|}} \quad \text{or} \quad \left\|\tilde{W}_o\right\| > \sqrt{\frac{2\zeta_o}{\bar{\alpha}_o}} \tag{15}$$

It can, therefore, be concluded [19] that $\dot{V}_o$ is less than zero outside a set defined by (15). This implies that, with properly selected gains, the observer estimation and the approximation errors are locally uniformly ultimately bounded. ■

*Remark* 6. By observation of (15) and the definition for $\zeta_o$, it can be seen that the bound on $\|\tilde{x}\|$ can be made arbitrarily small by choosing $\|A\|$ to be large and $\alpha_o$ to be small. Similar conclusions also apply for the bounds on $\left\|\tilde{W}_o\right\|$. Note that the weight update rule (10) uses $\tilde{x}_1$ as a forcing function to tune the weights.

With the derivation for the observer complete, the backstepping controller design is now considered.

### B. Backstepping Controller Design With Estimated States

In the previous subsection, the derivation of an observer was presented. In this subsection, the estimated state vector from the observer will be used to design the controller. In the design, each subsystem will be considered separately starting with subsystem 1; in order to avoid redundancies, detailed explanation for step 1 will be given and for the intermediate $i^{th}$ step as well as the final $n^{th}$ step, only the main results will be provided.

In the analysis that follows, the derivations will make use of the fact $\hat{x}_i = x_i + \tilde{x}_i, i = 1, \ldots, n$. The need for this subtle change arises in finding the tracking error dynamics: Evaluating $\dot{\hat{x}}_i$ would introduce the need to address the problem of evaluating the unknown functions, $f_i$ and $g_i$, at estimated states; the problem is circumvented by, instead, considering $\dot{x}_i$, from the system dynamics, and $\dot{\tilde{x}}_i$, from the observer dynamics.

Step 1: Define the tracking error, $\hat{r}_1 = \hat{x}_1 - x_{d1}$, whose derivative is

$$\dot{\hat{r}}_1 = f_1 + g_1 x_2 - \dot{x}_{d1} + \dot{\tilde{x}}_1. \tag{16}$$

The differential equation (16) represents the first subsystem dynamics and $x_2$ is considered to act as a virtual control input to this subsystem; let $\nu_1^* \triangleq x_2^*$ denote the ideal virtual control input. Now, define the Lyapunov function candidate, $V_{\hat{r}1} = \frac{1}{2}\hat{r}_1^2$, with the time derivative is given by

$$\dot{V}_{\hat{r}1} = \hat{r}_1 \dot{\hat{r}}_1 = \hat{r}_1 \left[f_1 + g_1 \nu_1 - \dot{x}_{d1} + \dot{\tilde{x}}_1\right]. \tag{17}$$

Define $\nu_1^* = -k_1 \hat{r}_1 - \frac{1}{g_1}\left[f_1 - \dot{x}_{d1}\right]$, with $k_1 > 0$, so that the Lyapunov derivative is simplified as follows:

$$\dot{V}_{\hat{r}1} = -k_1 g_1 \hat{r}_1^2 + \hat{r}_1 \dot{\tilde{x}}_1. \tag{18}$$

If the estimation error dynamics are bounded, note that (18) can be used to choose $k_1$ to satisfy the stability requirements.

However, a part of the virtual controller given by $\frac{1}{g_1}\left[f_1 - \dot{x}_{d1}\right]$ is unknown. Denote

$$h_1\left(\hat{X}_1\right) \triangleq \frac{1}{g_1}\left[f_1 - \dot{x}_{d1}\right], \tag{19}$$

where $\hat{X}_1 = \left[1, \hat{x}_1, \dot{x}_{d1}, \tilde{x}_1\right]^T$. Using a NN to approximate the nonlinear function $h_1$, the desired trajectory for the virtual control can be expressed as

$$\nu_1^* = -k_1 \hat{r}_1 - W_1^T \hat{\varphi}_1 + \varepsilon_1 \tag{20}$$

where it is understood that $\hat{\varphi}_1$ is evaluated at $\hat{X}_1$. Even though the function being approximated is in terms of actual states, $\hat{X}_1$ is taken to be the input, because, in practice, only estimated states are available. Since the ideal NN weights, $W_1$, are being estimated, $\hat{W}_1$, the estimated quantity, is introduced and the estimate of (20) is obtained as

$$\hat{\nu}_1 = -k_1 \hat{r}_1 - \hat{W}_1^T \hat{\varphi}_1. \tag{21}$$

Since the states of the observer are available to the controller only at event-based sampling instants $t_\kappa, t_{\kappa+1}, \ldots$, in the inter-event time period, $t_\kappa \le t < t_{\kappa+1}$, the difference between the current estimated state and the estimate at the last event will increase. This difference can be viewed as a measurement error. In general, the event-sampling measurement error corresponding to the $i^{th}$ subsystem is defined by

$$e_i\left(t\right) = \hat{x}_i\left(t\right) - \hat{x}_{ie}\left(t_\kappa\right), \forall t \in \left[t_\kappa, t_{\kappa+1}\right). \tag{22}$$

*Remark* 7. By definition (22), $e_i\left(t_\kappa\right) = 0$ at each event instant. This fact will be used later in the analysis. Furthermore, the event-triggering law, which determines these sampling instant, is designed so that measurement errors satisfy

$$e_i^2 \le \sigma_i \mu_i \hat{r}_i^2,\ i = 1, \ldots, n \tag{23}$$

where $0 < \mu_i < 1$ and $0 < \sigma_i < 1$.

*Remark* 8. Using a zero order hold (ZOH) at the actuator, a piece wise continuous control input is applied to the system. In the analysis, the last updated estimated states in the control input, $\hat{x}_{ie}\left(t_\kappa\right)$, can be replaced with event-sampling error and continuous states, using the relation in (22).

When the feedback is available only at event triggering instants, $\hat{x}_i$, is replaced with, $\hat{x}_{ie}$; by (22), $\hat{x}_{ie} = \hat{x}_i - e_i$. Furthermore, by using the definition for the estimated tracking error in tandem with the definition for the measurement error it can be concluded that, in general, $\hat{r}_{ie} = \hat{r}_i - e_i$.

Next, observe that that $x_2$ is acting as a virtual control input to the $\hat{r}_1$-subsystem and the desired virtual control derived from the NN output is updated only at event triggering instants; therefore, define the tracking error $r_2 = x_2 - \hat{\nu}_{1e}$ and use $x_2 = r_2 + \hat{\nu}_{1e}$ in the analysis. Moreover, for this same reason, it is only through the desired virtual control inputs which are injected into the system by the controller that the measurement errors are introduced. With this in consideration, observe that $r_2$ is introduced in the analysis, however, since the controller relies on estimated states, it is necessary to consider $\hat{r}_2$. This problem is easily addressed by noting that, in general, $r_i = \hat{r}_i - \tilde{x}_i$.

Now, consider the desired event-sampled virtual control

$$\hat{\nu}_{1e} = -k_1\left[\hat{r}_1 - e_1\right] - \hat{W}_1^T \hat{\varphi}_{1e}, \tag{24}$$

and use these results in the dynamics of the tracking error, giving

$$\begin{aligned} \dot{\hat{r}}_1 &= f_1 + g_1 x_2 - \dot{x}_{d1} + \dot{\tilde{x}}_1 \\ &= g_1\left[\hat{r}_2 - \tilde{x}_2 + \hat{\nu}_{1e} + \frac{f_1 - \dot{x}_{d1}}{g_1}\right] + \dot{\tilde{x}}_1 \\ &= g_1\left[\hat{r}_2 - k_1\hat{r}_1 - \hat{W}_1^T\hat{\varphi}_{1e} + W_1^T\varphi_1 - \varepsilon_1\right] \\ &\quad + g_1\left[k_1 e_1\right] - g_1\tilde{x}_2 + \dot{\tilde{x}}_1. \end{aligned} \tag{25}$$

Now, define the Lyapunov function candidate

$$V_1 = \frac{\hat{r}_1^2}{2\left\|A\right\|^2} + \frac{\operatorname{tr}\left\{\tilde{W}_1^T F_1^{-1}\tilde{W}_1\right\}}{2\left\|A\right\|^2} \tag{26}$$

and the weight update law

$$\dot{\hat{W}}_1 = F_1\left[\hat{r}_{1e}\hat{\varphi}_{1e} - \alpha_1\left|\hat{r}_{1e}\right|\hat{W}_1\right], \tag{27}$$

where $F_1 = F_1^T > 0$ and $\alpha_1 > 0$ are design parameters. Using (27), the time derivative of $V_1$ is found to be

$$\dot{V}_1 = \frac{1}{\|A\|^2}\hat{r}_1\dot{\hat{r}}_1 + \frac{1}{\|A\|^2}\mathrm{tr}\left\{\tilde{W}_1^T F_1^{-1}\dot{\hat{W}}_1\right\}$$
$$= \frac{1}{\|A\|^2}\Big[g_1\hat{r}_1\hat{r}_2 - k_1 g_1\hat{r}_1^2 - g_1\hat{r}_1\hat{W}_1^T\hat{\varphi}_{1e} + g_1\hat{r}_1 W_1^T\varphi_1$$
$$- g_1\hat{r}_1\varepsilon_1 + g_1\hat{r}_1\left[k_1 e_1\right] - g_1\hat{r}_1\tilde{x}_2 + \hat{r}_1\dot{\bar{x}}_1\Big]$$
$$+ \frac{1}{\|A\|^2}\mathrm{tr}\left\{\hat{r}_{1e}\tilde{W}_1^T\hat{\varphi}_{1e} - \alpha_1\left|\hat{r}_{1e}\right|\tilde{W}_1^T\hat{W}_1\right\} \tag{28}$$

In order to further simplify the expression for $\dot{V}_1$, introduce the temporary variables

$$A_1 = -k_1 g_1\hat{r}_1^2 - g_1\hat{r}_1\varepsilon_1$$
$$B_1 = -g_1\hat{r}_1\hat{W}_1^T\hat{\varphi}_{1e} + g_1\hat{r}_1 W_1^T\varphi_1 + \mathrm{tr}\left\{\hat{r}_{1e}\tilde{W}_1^T\hat{\varphi}_{1e} - \alpha_1\left|\hat{r}_{1e}\right|\tilde{W}_1^T\hat{W}_1\right\}$$

and consider them separately. Using completion of squares with respect to $\hat{r}_1$ reveals

$$A_1 \leq -\frac{1}{2}k_1 g_1\hat{r}_1^2 - \frac{1}{2}k_1 g_1\left[\hat{r}_1^2 - 2\hat{r}_1\frac{\varepsilon_1}{k_1} + \frac{\varepsilon_1^2}{k_1^2}\right] + \frac{g_1\varepsilon_1^2}{2k_1}$$
$$= -\frac{1}{2}k_1 g_1\hat{r}_1^2 - \frac{1}{2}k_1 g_1\left[\hat{r}_1 - \frac{\varepsilon_1}{k_1}\right]^2 + \frac{g_1\varepsilon_1^2}{2k_1}$$
$$\leq -\frac{1}{2}k_1 g_{1m}\hat{r}_1^2 + \frac{g_{1M}\varepsilon_{1M}^2}{2k_1}$$

The simplification process for $B_1$ utilizes the bounds on the NN activation functions. With that, as well as properties of the matrix trace and norm operators as well as the conditions imposed by (23), the following conclusion can be made:

$$B_1 \leq \frac{1}{2}\hat{r}_1^2 - \alpha_1\left[2-\chi_1\right]\left|\hat{r}_1\right|\left\|\tilde{W}_1\right\|^2 + 2g_{1M}^2 W_{1M}^2 N_1$$
$$+ \left|\hat{r}_1\right|\left[\sqrt{N_1}\left[g_{1M}+\chi_1\right] + \chi_1\alpha_1 W_{1M}\right]\left\|\tilde{W}_1\right\|$$

where, in general, $\chi_i = \sqrt{\sigma_i\mu_i} + 1$.The expressions for $A_1$ and $B_1$ can be used to rewrite (28). Denoting the bounded term, $\zeta_1 \triangleq \frac{g_{1M}\varepsilon_{1M}^2}{2\|A\|^2 k_1} + \frac{2g_{1M}^2 W_{1M}^2 N_1}{\|A\|^2}$, and rearranging terms gives

$$\dot{V}_1 \leq \frac{g_1\hat{r}_1\hat{r}_2}{\|A\|^2} - \frac{\left|\hat{r}_1\right|}{\|A\|^2}\Bigg[\frac{1}{2}\left[k_1 g_{1m} - 1\right]\left|\hat{r}_1\right| + \alpha_1\left[2-\chi_1\right]\left\|\tilde{W}_1\right\|^2$$
$$- \left[\sqrt{N_1}\left[g_{1M}+\chi_1\right] + \chi_1\alpha_1 W_{1M}\right]\left\|\tilde{W}_1\right\|\Bigg]$$
$$+ \zeta_1 + \frac{1}{\|A\|^2}\left[g_1\hat{r}_1\left[k_1 e_1\right] - g_1\hat{r}_1\tilde{x}_2 + \hat{r}_1\dot{\bar{x}}_1\right] \tag{29}$$

Step 2: Let the tracking error be defined as $\hat{r}_2 = \hat{x}_2 - \hat{\nu}_{1e}$, whose derivative is given by $\dot{\hat{r}}_2 = f_2 + g_2 x_3 - \dot{\hat{\nu}}_{1e} + \dot{\tilde{x}}_2$. Here, $x_3$ is considered to be the virtual control input to the $\hat{r}_2$-subsystem and the ideal feedback control is given by

$$\nu_2^* = -r_1 - k_2 r_2 - \frac{1}{g_2}\left[f_2 - \dot{\hat{\nu}}_{1e}\right] \tag{30}$$

where $k_2 > 0$ is a design constant. Denote the unknown part with

$$h_2\left(\hat{X}_{2e}\right) \triangleq \frac{1}{g_2}\left[f_2 - \dot{\hat{\nu}}_{1e}\right]. \tag{31}$$

Note that the unknown part of $\nu_2^*$ is a function of $\bar{\hat{x}}_{2e}$ and $\dot{\hat{\nu}}_{1e}$; in turn, $\dot{\hat{\nu}}_{1e}$ is a function of $\hat{x}_{1e}$, $\dot{x}_{d1}$, and $\hat{W}_1$. To design the NN inputs, define

$$\hat{\psi}_{1e} = \frac{\partial\hat{\nu}_{1e}}{\partial x_d}\dot{x}_d + \frac{\partial\hat{\nu}_{1e}}{\partial\hat{W}_1}\left[F_1\left[\hat{r}_{1e}\hat{\varphi}_{1e} - \alpha_1\left|\hat{r}_{1e}\right|\hat{W}_1\right]\right] \tag{32}$$

and use it in the input, $\hat{X}_{2e} = \left[1, \bar{\hat{x}}_{2e}, \partial\hat{\nu}_{1e}/\partial x_{1e}, \hat{\psi}_{1e}, \tilde{x}_{1e}\right]^T$, for the NN approximating $h_2$, allowing (30) to be rewritten as

$$\nu_2^* = -\hat{r}_1 - k_2\hat{r}_2 - W_2^T\hat{\varphi}_2 + \varepsilon_2. \tag{33}$$

With event-sampling and estimated weights, the desired virtual control becomes

$$\hat{\nu}_{2e} = -\left[\hat{r}_1 - e_1\right] - k_2\left[\hat{r}_2 - e_2\right] - \hat{W}_2^T\hat{\varphi}_{2e}. \tag{34}$$

Using these results, dynamics of the tracking error are given by

$$\dot{\hat{r}}_2 = g_2\left[\hat{r}_3 - k_2\hat{r}_2 - \hat{r}_1 - \hat{W}_2^T\hat{\varphi}_{2e} + W_2^T\varphi_2 - \varepsilon_2\right]$$
$$+ g_2\left[k_2 e_2 + e_1\right] - g_2\tilde{x}_3 + \dot{\tilde{x}}_2. \tag{35}$$

Define the Lyapunov function candidate

$$V_2 = V_1 + \frac{\hat{r}_2^2}{2\|A\|^2} + \frac{\mathrm{tr}\left\{\tilde{W}_2^T F_2^{-1}\tilde{W}_2\right\}}{2\|A\|^2} \tag{36}$$

and the weight tuning law

$$\dot{\hat{W}}_2 = F_2\left[\hat{r}_{2e}\hat{\varphi}_{2e} - \alpha_2\left|\hat{r}_{2e}\right|\hat{W}_2\right], \tag{37}$$

where $F_2 = F_2^T > 0$ and $\alpha_2 > 0$ are design parameters.

*Remark* 9. Expression for the desired $i^{th}$ virtual control is given by

$$\hat{\nu}_{ie} = -\left[\hat{r}_{i-1} - e_{i-1}\right] - k_i\left[\hat{r}_i - e_i\right] - \hat{W}_i^T\hat{\varphi}_{ie}, \tag{38}$$

the dynamics of the tracking error become

$$\dot{\hat{r}}_i = g_i\left[\hat{r}_{i+1} - k_i\hat{r}_i - \hat{r}_{i-1} - \hat{W}_i^T\hat{\varphi}_{ie} + W_i^T\varphi_i - \varepsilon_i\right]$$
$$+ g_i\left[k_i e_i + e_{i-1}\right] - g_i\tilde{x}_{i+1} + \dot{\tilde{x}}_i, \tag{39}$$

the Lyapunov candidate is provided by

$$V_i = V_{i-1} + \frac{\hat{r}_i^2}{2\|A\|^2} + \frac{\mathrm{tr}\left\{\tilde{W}_i^T F_i^{-1}\tilde{W}_i\right\}}{2\|A\|^2}, \tag{40}$$

and the weight update law is given by

$$\dot{\hat{W}}_i = F_i\left[\hat{r}_{ie}\hat{\varphi}_{ie} - \alpha_i\left|\hat{r}_{ie}\right|\hat{W}_i\right], \tag{41}$$

where $F_i = F_i^T > 0$ and $\alpha_i > 0$ are design parameters.

Then, by introducing temporary variables and use a simplification procedure similar to Step 1, to get the expression

$$\dot{V}_2 \leq \frac{1}{\|A\|^2}\left[\hat{r}_1\hat{r}_2\left[g_1 - g_2\right] + g_2\hat{r}_2\hat{r}_3\right]$$
$$- \frac{1}{\|A\|^2}\sum_{j=1}^{2}\left|\hat{r}_j\right|\Bigg[\frac{1}{2}\left[k_j g_{jm} - 1\right]\left|\hat{r}_j\right| + \alpha_j\left[2-\chi_j\right]\left\|\tilde{W}_j\right\|^2$$
$$- \left[\sqrt{N_j}\left[g_{jM}+\chi_j\right] + \chi_j\alpha_j W_{jM}\right]\left\|\tilde{W}_j\right\|\Bigg]$$
$$+ \sum_{j=1}^{2}\zeta_j + \frac{1}{\|A\|^2}\sum_{j=1}^{2}\hat{r}_j\dot{\tilde{x}}_j - \frac{1}{\|A\|^2}\sum_{j=1}^{2}g_j\hat{r}_j\tilde{x}_j$$
$$+ \frac{1}{\|A\|^2}\left[g_1\hat{r}_1\left[k_1 e_1\right] + g_2\hat{r}_2\left[k_2 e_2 + e_1\right]\right] \tag{42}$$

where $\zeta_j = \frac{g_{jM}\varepsilon_{jM}^2}{2\|A\|^2 k_j} + \frac{2g_{jM}^2 W_{jM}^2 N_j}{\|A\|^2}$ is bounded.

Step $n$: Similar to the previous steps, start by defining the tracking error $\hat{r}_n = \hat{x}_n - \hat{\nu}_{(n-1)e}$, with its time-derivative $\dot{\hat{r}}_n = f_n + g_n u - \dot{\hat{\nu}}_{(n-1)e} + \dot{\tilde{x}}_n$. Here, the forcing function is the actual control input $u$ for the system and the ideal feedback control is given by

$$u^* = -r_{n-1} - k_n r_n - \frac{1}{g_n}\left[f_n - \dot{\hat{\nu}}_{(n-1)e}\right] \tag{43}$$

where $k_n > 0$ is a design constant. Denote the unknown part with

$$h_n\left(\hat{X}_{ne}\right) \triangleq \frac{1}{g_n}\left[f_n - \dot{\hat{\nu}}_{(n-1)e}\right]. \tag{44}$$

To obtain the NN inputs, define

$$\begin{aligned}\hat{\psi}_{(n-1)e} =& \sum_{j=1}^{n-1} \frac{\partial \hat{\nu}_{(n-1)e}}{\partial x_d}\dot{x}_d \\ &+ \sum_{j=1}^{n-1} \frac{\partial \hat{\nu}_{(n-1)e}}{\partial \hat{W}_j}\left[F_j\left[\hat{r}_{je}\hat{\varphi}_{je} - \alpha_j\left|\hat{r}_{je}\right|\hat{W}_j\right]\right]. \end{aligned}\tag{45}$$

Use a NN which takes as input $\hat{X}_{ne} = \left[1, \bar{\hat{x}}_{ne}, {}^{\partial\hat{\nu}_{(n-1)e}}\!/\!_{\partial x_{1e}}, \ldots, {}^{\partial\hat{\nu}_{(n-1)e}}\!/\!_{\partial x_{(n-1)e}}, \hat{\psi}_{(n-1)e}, \tilde{x}_{1e}\right]^T$ to approximate $h_n$, allowing (43) to be rewritten as

$$u^* = -\hat{r}_{n-1} - k_n\hat{r}_n - W_n^T\hat{\varphi}_n + \varepsilon_n. \tag{46}$$

Replacing the actual NN weights with its estimate and taking into account the errors due to event based feedback, the desired control input is given by

$$u_e = -\left[\hat{r}_{n-1} - e_{n-1}\right] - k_n\left[\hat{r}_n - e_n\right] - \hat{W}_n^T\hat{\varphi}_{ne} \tag{47}$$

and the dynamics of the tracking error become

$$\begin{aligned}\dot{\hat{r}}_n = g_n&\left[-k_n\hat{r}_n - \hat{r}_{n-1} - \hat{W}_n^T\hat{\varphi}_{ne} + W_n^T\varphi_n - \varepsilon_n\right] \\ &+ g_n\left[k_n e_n + e_{n-1}\right] + \dot{\tilde{x}}_n. \end{aligned}\tag{48}$$

Note that, since $u_e$ is the actual control applied to the system, there is no need to introduce an additional error term and, therefore, the $\tilde{x}_i$ term that exists in previous subystems is absent here.

Now, define the final Lyapunov function

$$V_n = V_{n-1} + \frac{\hat{r}_n^2}{2\|A\|^2} + \frac{\text{tr}\left\{\tilde{W}_n^T F_n^{-1}\tilde{W}_n\right\}}{2\|A\|^2} \tag{49}$$

and the weight update law

$$\dot{\hat{W}}_n = F_n\left[\hat{r}_{ne}\hat{\varphi}_{ne} - \alpha_n\left|\hat{r}_{ne}\right|\hat{W}_n\right], \tag{50}$$

where $F_n = F_n^T > 0$ and $\alpha_n > 0$ are design parameters. Then, introducing temporary variables and following the simplification procedure similar to the previous step reveals

$$\begin{aligned}\dot{V}_n \leq& \\ &\frac{1}{\|A\|^2}\sum_{j=2}^{n}\hat{r}_{j-1}\hat{r}_j\left[g_{j-1} - g_j\right] \\ &- \frac{1}{\|A\|^2}\sum_{j=1}^{n}\left|\hat{r}_j\right|\left[\frac{1}{2}\left[k_j g_{jm} - 1\right]\left|\hat{r}_j\right| + \alpha_j\left[2 - \chi_j\right]\left\|\tilde{W}_j\right\|^2\right. \\ &\left. - \left[\sqrt{N_j}\left[g_{jM} + \chi_j\right] + \chi_j\alpha_j W_{jM}\right]\left\|\tilde{W}_j\right\|\right] \\ &+ \sum_{j=1}^{n}\zeta_j + \frac{1}{\|A\|^2}\sum_{j=1}^{n}\hat{r}_j\dot{\tilde{x}}_j - \frac{1}{\|A\|^2}\sum_{j=1}^{n-1}g_j\hat{r}_j\tilde{x}_{j+1} \\ &+ \frac{1}{\|A\|^2}\left[g_1\hat{r}_1\left[k_1 e_1\right] + \sum_{j=2}^{n}g_j\hat{r}_j\left[k_j e_j + e_{j-1}\right]\right] \end{aligned}\tag{51}$$

Now, in order to simplify (51), introduce the following temporary variables:

$$\begin{aligned} T_1 &= \frac{1}{\|A\|^2}\sum_{j=2}^{n}\hat{r}_{j-1}\hat{r}_j\left[g_{j-1} - g_j\right] \\ T_2 &= \frac{1}{\|A\|^2}\left(\sum_{j=2}^{n}g_j\hat{r}_j\left[k_j e_j + e_{j-1}\right] + g_1\hat{r}_1\left[k_1 e_1\right]\right) \\ T_3 &= \frac{1}{\|A\|^2}\sum_{j=1}^{n}\hat{r}_j\dot{\tilde{x}}_j \\ T_4 &= -\frac{1}{\|A\|^2}\sum_{j=1}^{n-1}g_j\hat{r}_j\tilde{x}_{j+1} \end{aligned}$$

First, consider $T_1$ and $T_2$. In order to simplify these terms, begin by defining $g_M = \max\{g_{1M}, \ldots, g_{nM}\}$ and $g_m = \min\{g_{1m}, \ldots, g_{nm}\}$. Expanding and rearranging terms and invoking Young's inequality reveals $T_1 \leq \frac{1}{2\|A\|^2}\sum_{j=1}^{n}\left[1 + g_M^2\right]\hat{r}_j^2$ and $T_2 \leq \frac{1}{\|A\|^2}\sum_{j=1}^{n}\left[g_M^2\right]\hat{r}_j^2 + \frac{1}{2\|A\|^2}\sum_{j=1}^{n}\left[k_j^2 + 1\right]e_j^2$. Using these results in (51) reveals

$$\begin{aligned}\dot{V}_n \leq& -\frac{1}{\|A\|^2}\sum_{j=1}^{n}\left|\hat{r}_j\right|\left[\frac{1}{2}\left[k_j g_m - 3g_M^2 - 2\right]\left|\hat{r}_j\right|\right. \\ &+ \alpha_j\left[2 - \chi_j\right]\left\|\tilde{W}_j\right\|^2 - \left[\sqrt{N_j}\left[g_M + \chi_j\right]\right. \\ &\left.\left. + \chi_j\alpha_j W_{jM}\right]\left\|\tilde{W}_j\right\|\right] + \sum_{j=1}^{n}\zeta_j + \frac{1}{2\|A\|^2}\sum_{j=1}^{n}\left[k_j^2 + 1\right]e_j^2 \\ &+ \frac{1}{\|A\|^2}\sum_{j=1}^{n}\hat{r}_j\dot{\tilde{x}}_j - \frac{1}{\|A\|^2}\sum_{j=1}^{n-1}g_j\hat{r}_j\tilde{x}_{j+1} \end{aligned}\tag{52}$$

Now, considering $T_3$, recall the observer estimation error dynamics from (9). Using Young's inequality with these dynamics gives

$$\begin{aligned} T_3 \leq& \frac{1}{2\|A\|^2}\sum_{j=1}^{n}\left[\frac{N_o}{\|A\|^2} + 2\right]\hat{r}_j^2 \\ &+ \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\left\|\tilde{W}_o\right\|^2 + \frac{1}{2\|A\|^2}\xi_M^2 \end{aligned}$$

Finally, for $T_4$, use a similar procedure to discover

$$T_4 \leq \frac{g_M^2}{2\|A\|^4}\|\hat{r}\|^2 + \frac{1}{2}\|\tilde{x}\|^2.$$

The expressions for $T_3$ and $T_4$ are only useful in the presence of the closed-loop. Therefore, proceed by focusing on the closed-loop dynamics.

### C. Closed-Loop Output-Feedback Dynamics

Consider the closed-loop Lyapunov candidate, $V = V_o + V_n$ and define the following bounded terms:

$$\zeta = \zeta_o + \sum_{j=1}^{n} \zeta_j + \frac{1}{2\|A\|^2}\xi_M^2$$
$$\eta_j = \frac{[g_M + \chi_j]\sqrt{N_j} + \chi_j \alpha_j W_{jM}}{\alpha_j [2 - \chi_j]}$$
$$\delta_j = k_j g_m - 3g_M^2 - \frac{g_M^2}{\|A\|^2} - \frac{N_o}{\|A\|^2} - 4$$
$$\beta_j = k_j^2 + 1$$

Furthermore, recall the bound, $|e_j| \le B_{ej}$. Finally, with completion of squares with respect to $\left\|\tilde{W}_j\right\|$, the final Lyapunov candidate for the closed-loop system is given by

$$\begin{aligned}\dot{V} \le &-\frac{1}{4}\left[3\left\|\bar{A}\right\| - 4\right]\|\tilde{x}\|^2 - \frac{1}{2}\left[\bar{\alpha}_o - 1\right]\left\|\tilde{W}_o\right\|^2 \\ &- \sum_{j=1}^{n} |\hat{r}_j| \left[\frac{1}{2}\delta_j |\hat{r}_j| + \alpha_j [2-\chi_j]\left[\left\|\tilde{W}_j\right\| - \frac{\eta_j}{2}\right]^2\right. \\ &\left. - \frac{\eta_j^2 \alpha_j [2-\chi_j]}{4}\right] + \zeta + \frac{1}{2\|A\|^2}\sum_{j=1}^{n}\beta_j B_{ej}^2. \end{aligned} \quad (53)$$

The ISS like behavior of the closed loop system with the continuous time implementation of the backstepping controller is demonstrated in the next theorem.

**Theorem 2** *(Output-Feedback Input-to-State Stability):* Let the NN observer be defined by (7) with estimation error dynamics given by (9). Consider the NN observer weight tuning given by (10). Let the *Assumptions 1-4* hold. Consider the dynamics of the tracking errors (25), (35), (39), and (48). Let the actual and the desired virtual control inputs be given by (47), (24), (34), and (38), respectively. Define the weight adaptation rule for the NN as (27), (37), (41), and (50). Finally, let $|e_i| \le B_{ei}$, $i = 1, 2, \ldots, n$. Then, there exists, $\|A\|$ and $k_i$ such that the observer estimation error, $\tilde{x}$, the tracking errors, $\hat{r}_i$, and the NN weight estimation errors, $\tilde{W}_o$ and $\tilde{W}_i$, are locally uniformly ultimately bounded, with bounds defined as functions of NN reconstruction errors and measurement errors.

*Proof:* Consider the following positive-definite Lyapunov candidate describing the closed loop:

$$V = V_o + V_n \quad (54)$$

where $V_o$ was defined in (11) and $V_n$ was defined in (49). The first derivative of $V$ is given by $\dot{V} = \dot{V}_o + \dot{V}_n$. In Theorem 1, $\dot{V}_o$ was found to be bounded above by (14); in further derivations, $\dot{V}_n$ was found to be bounded above by (52). Making use of the observer estimation error dynamics given by (9), the results from (14) and (52) are able to be connected. Based on the results given by (53), choose the design constants satisfying the following conditions:

$$\left\|\bar{A}\right\| > \frac{4}{3} \quad \text{and} \quad k_i > \Gamma_i \quad (55)$$

where $\Gamma_i = \frac{1}{g_m}\left[3g_M^2 + \frac{g_M^2}{\|A\|^2} + \frac{N_o}{\|A\|^2} + 4\right]$ for all $i = 1, \ldots, n$. It can be concluded that $\dot{V} < 0$ as long as the controller gains satisfy (55) and if the following conditions are satisfied:

$$\|\tilde{x}\| > \sqrt{\frac{4\zeta + 2B_E}{3\left\|\bar{A}\right\| - 4}} \quad \text{or} \quad \left\|\tilde{W}_o\right\| > \sqrt{\frac{2\zeta + B_E}{[\bar{\alpha}_o - 1]}}$$

or

$$|\hat{r}_j| > \frac{\eta_j^2 \alpha_j [2-\chi_j]}{2\delta_j} \quad \text{or} \quad \left\|\tilde{W}_j\right\| > \eta_j \quad (56)$$

where $B_E = \frac{1}{\|A\|^2}\sum_{j=1}^{n}\beta_j B_{ej}^2$. It can, therefore, be concluded [19] that $\dot{V}$ is less than zero whenever the tracking errors, observer errors and NN weight estimation errors are outside their respective bounds. This implies that the observer estimation error, the tracking errors, and the NN estimation errors are locally UUB. ■

*Remark* 10. The result in Theorem 2 reveals that the system with the backstepping controller exhibits local input-to-state stable behavior. This can be noticed by considering an augmented vector, $z = \left[\hat{r}_1, \ldots, \hat{r}_n, \left\|\tilde{W}_1\right\|, \ldots, \left\|\tilde{W}_n\right\|\right]^T$, and observing that (53) can be written in the form $\dot{V}(z) \le -\Lambda(\|z\|) + \gamma(\|E\|)$, where $\gamma$ is viewed as a positive definite function of inputs to the system and the inputs are in the form of measurement errors and the NN approximation errors. Therefore, in summary, the system with the backstepping controller satisfies local ISS property [9]. However, for the event sampled implementation of the controller presented in this paper, the measurement error is due to the difference between the continuously estimated states at the observer and the estimated states available at the controller.

Next, the derivation of the event-execution rule and the boundedness of the measurement error due to event based feedback is presented.

**Theorem 3** *(Overall Stability and Boundedness of Measurement Error):* Let the NN observer be defined by (7) with estimation error dynamics given by (9). Select the NN observer weight tuning given by (10). Let the *Assumptions 1-4* hold. Consider the dynamics of the tracking errors (25), (35), (39), and (48). Choose the actual control input and the desired virtual control input be given by (47), (24), (34), and (38), respectively. Let the weights of the NN approximator be updated at the event-triggering instants based on the rules given by (27), (37), (41), and (50). Finally, let the error due to event-sampling satisfy (23). Then, there exists design parameters, $\|A\|$ and $k_i$, such that the observer estimation error, $\tilde{x}$, the tracking errors, $\hat{r}_i$, and the NN weight estimation errors, $\tilde{W}_o$ and $\tilde{W}_i$, are locally uniformly ultimately bounded during inter-event periods.

*Proof:* In this proof, two cases are considered. In the first case the analysis is carried out when the measurement errors are set to zero and in the second case stability analysis during the inter-event period is carried out.

*Case 1.* Begin by observing that the expression for $\dot{V}_o$ remains identical to (14) and the definition for the bounded term, $\zeta_o$, also remains unchanged. Next, the dynamics of tracking errors with the measurement errors set to zero are given as

$$\begin{aligned}\dot{\hat{r}}_1 &= g_1\left[\hat{r}_2 - k_1\hat{r}_1 - \tilde{W}_1^T\hat{\varphi}_1 + W_1^T\tilde{\varphi}_1 - \varepsilon_1\right] - g_1\tilde{x}_2 + \dot{\tilde{x}}_1 \\ \dot{\hat{r}}_i &= g_i\left[\hat{r}_{i+1} - k_i\hat{r}_i - \hat{r}_{i-1} - \tilde{W}_i^T\hat{\varphi}_i + W_i^T\tilde{\varphi}_i - \varepsilon_i\right] \\ &\qquad - g_i\tilde{x}_{i+1} + \dot{\tilde{x}}_i \\ \dot{\hat{r}}_n &= g_n\left[-k_n\hat{r}_n - \hat{r}_{n-1} - \tilde{W}_n^T\hat{\varphi}_n + W_n^T\tilde{\varphi}_n - \varepsilon_n\right] + \dot{\tilde{x}}_n \end{aligned} \quad (57)$$

with $i = 2, \ldots, n-1$. In the Lyapunov functions given in (26), (36), (40), and (49), use the error dynamics and following a simplification process similar to what was presented in Section III-B gives

$$\begin{aligned}\dot{V}_n \le &\frac{1}{\|A\|^2}\sum_{j=2}^{n}\hat{r}_{j-1}\hat{r}_j[g_{j-1} - g_j] - \frac{1}{\|A\|^2}\sum_{j=1}^{n}\left[k_j g_{jm}|\hat{r}_j|\right. \\ &\left. + \alpha_j\left\|\tilde{W}_j\right\|^2 - \left[\sqrt{N_j}[g_{jM} + 1] + \alpha_j W_{jM}\right]\left\|\tilde{W}_j\right\| - \zeta_{j1}\right]|\hat{r}_j| \\ &+ \frac{1}{\|A\|^2}\sum_{j=1}^{n}\hat{r}_j\dot{\tilde{x}}_j - \frac{1}{\|A\|^2}\sum_{j=1}^{n-1} g_j\hat{r}_j\tilde{x}_{j+1} \end{aligned} \quad (58)$$

where $\zeta_{j1} = g_{jM}\varepsilon_{jM} + 2g_{jM}W_{jM}\sqrt{N_j}$ is a constant.

For further simplification, the first term and the last two terms of (58) are considered separately. Simplifying the first terms allows the expression for $\dot{V}_n$ to be simplified, but the results for the simplification of the last two terms only make sense in the presence of the closed loop. Therefore, consider the Lyapunov candidate (54). First, define the following bounded terms:

$$\zeta_{B1} = \sum_{j=1}^{n} \zeta_{j1} + \xi_M$$
$$\eta_{j1} = \frac{\sqrt{N_j}}{\alpha_j}\left[g_M + 1\right] + W_{jM}$$
$$\delta_{j1} = 2k_j g_m - g_M^2 - \frac{g_M^2}{\|A\|^2} - \frac{N_o}{\|A\|^2} - 2$$

Simplifying and rearranging terms in (58) and $\dot{V}_o$ from Theorem 1 gives

$$\begin{aligned}\dot{V} \leq &-\frac{1}{4}\left[3\left\|\bar{A}\right\| - 4\right]\|\tilde{x}\|^2 - \frac{1}{2}\left[\bar{\alpha}_o - 1\right]\left\|\tilde{W}_o\right\|^2 \\ &- \frac{1}{\|A\|^2}\sum_{j=1}^{n}\left[\frac{1}{2}\delta_{j1}\left|\hat{r}_j\right| + \alpha_j\left[\left\|\tilde{W}_j\right\| - \frac{\eta_{j1}}{2}\right]^2\right. \\ &\left. - \left[\zeta_{B1} + \frac{\eta_{j1}^2\alpha_j}{4}\right]\right] + \zeta_o. \end{aligned} \tag{59}$$

Choose the controller gain such that the following conditions are satisfied:

$$\left\|\bar{A}\right\| > \frac{4}{3} \quad \text{and} \quad k_i > \Gamma_{i1} \tag{60}$$

where $\Gamma_{i1} = \frac{1}{2g_m}\left[g_M^2 + \frac{g_M^2}{\|A\|^2} + \frac{N_o}{\|A\|^2} + 2\right],\ \forall i = 1, \ldots, n$. Then, it can be observed from (59) that $\dot{V}$ is less than zero when the following inequalities hold:

$$\|\tilde{x}\| > \sqrt{\frac{4\zeta_o}{3\left\|\bar{A}\right\| - 4}} \quad \text{or} \quad \left\|\tilde{W}_o\right\| > \sqrt{\frac{2\zeta_o}{\left[\bar{\alpha}_o - 1\right]}}$$

or

$$\left|\hat{r}_j\right| > \frac{4\zeta_{B1} + \eta_{j1}^2\alpha_j}{2\delta_{j1}}$$

or

$$\left\|\tilde{W}_j\right\| > \frac{\eta_{j1}}{2} + \sqrt{\frac{4\zeta_{B1} + \alpha_j\eta_{j1}^2}{4\alpha_j}}. \tag{61}$$

*Case 2.* Now, consider the time between events, $t \in [t_\kappa, t_{\kappa+1})$. In this period, $e_i(t) \neq 0$ and $\dot{\hat{W}} = 0$ for the weights of all the NNs.

As before, begin by considering the observer estimation error dynamics and notice that they remain identical to (9). The primary difference in this analysis arises in the Lyapunov candidate, (11), noting that its first derivative simplifies to

$$\begin{aligned}\dot{V}_o &= \tilde{x}^T P_o \dot{\tilde{x}} + \cancelto{0}{\operatorname{tr}\left\{\tilde{W}_o^T F_o^{-1}\dot{\tilde{W}}_o\right\}} \\ &= \tilde{x}^T P_o \dot{\tilde{x}}. \end{aligned} \tag{62}$$

Use (9) in (62) and on simplification we arrive at

$$\dot{V}_o \leq -\frac{1}{4}\left[\left\|\tilde{A}\right\| - 2N_o\right]\|\tilde{x}\|^2 + \frac{1}{2}\left\|P_o\tilde{W}_o\right\|^2 + \zeta_{o2} \tag{63}$$

where $\zeta_{o2} = \frac{\xi_M^2}{\|A\|}$.

Similar to the observer estimation error dynamics, the dynamics of the tracking error from (25), (35), (39), and (48) remain unchanged and the Lyapunov first derivative will have the terms, $\hat{r}_i\dot{\hat{r}}_i$ for all $i = 1, \ldots, n$. Following a similar simplification process reveals

$$\begin{aligned}\dot{V}_n \leq &-\frac{1}{2\|A\|^2}\sum_{j=1}^{n}\left[k_j g_{jm} - 4g_{jM}^2 - 1\right]\hat{r}_j^2 \\ &+ \frac{1}{2\|A\|^2}\sum_{j=1}^{n}\left[k_j^2 + 1\right]e_j^2 + \frac{1}{2\|A\|^2}\sum_{j=1}^{n}N_j\left[W_{jM} + \left\|\hat{W}_j\right\|\right]^2 \\ &+ \sum_{j=1}^{n}\zeta_{j2} + \frac{1}{\|A\|^2}\sum_{j=1}^{n}\hat{r}_j\dot{\tilde{x}}_j - \frac{1}{\|A\|^2}\sum_{j=1}^{n-1}g_j\hat{r}_j\tilde{x}_{j+1} \end{aligned} \tag{64}$$

where $\zeta_{j2} = \frac{g_M\varepsilon_{jM}^2}{2\|A\|^2 k_j}$ is a constant. Since the weights of the NN are not being tuned between events, the term $\frac{1}{2}\sum_{j=1}^{n}N_j\left[W_{jM} + \left\|\hat{W}_j\right\|\right]^2$ will remain constant between events, depending on the last updated instant, $\hat{W}_j = \hat{W}_j(t_\kappa)$.

As before, the final terms in (64) can be simplified and incorporated in the closed-loop dynamics, revealing

$$\begin{aligned}\dot{V} \leq &-\frac{1}{4}\left[\left\|\tilde{A}\right\| - 2N_o - 4\right]\|\tilde{x}\|^2 \\ &- \frac{1}{2\|A\|^2}\sum_{j=1}^{n}\left[k_j g_m - 4g_M^2 - \frac{g_M^2}{\|A\|^2} - \frac{N_o}{\|A\|^2} - 3\right]\hat{r}_j^2 \\ &+ \frac{1}{2\|A\|^2}\sum_{j=1}^{n}\left[k_j^2 + 1\right]e_j^2 + \zeta_{B2} \end{aligned} \tag{65}$$

where $\zeta_{B2} = \left\|P_o\tilde{W}_o\right\|^2 + \frac{1}{2}\sum_{j=1}^{n}N_j\left[W_{jM} + \left\|\hat{W}_j\right\|\right]^2 + \sum_{j=1}^{n}\zeta_{j2} + \zeta_{o2}$ is bounded (recall that the observer NN weight estimation errors also remain bounded as a result of the weights remaining constant during the inter-event period).

To handle the measurement error due to event based feedback, consider

$$\mu_j = \frac{1}{k_j^2 + 1}, \tag{66}$$

for the event trigger condition (23). Using (66) and (23) in (65), to get the expression for $\dot{V}$:

$$\dot{V} \leq -\frac{1}{4}\left[\left\|\tilde{A}\right\| - 2N_o - 4\right]\|\tilde{x}\|^2 - \frac{1}{2\|A\|^2}\sum_{j=1}^{n}\delta_{j2}\hat{r}_j^2 + \zeta_{B2} \tag{67}$$

where $\delta_{j2} = k_j g_m - 4g_M^2 - \frac{g_M^2}{\|A\|^2} - \frac{N_o}{\|A\|^2} - \sigma_j - 3$. When controller gains are selected such that

$$\left\|\tilde{A}\right\| > 2N_o + 4 \quad \text{and} \quad k_i > \Gamma_{i2} \tag{68}$$

where $\Gamma_{i2} = \frac{1}{g_m}\left[4g_M^2 + \frac{g_M^2}{\|A\|^2} + \frac{N_o}{\|A\|^2} + \sigma_i + 3\right]$ for all $i = 1, \ldots, n$, it can be concluded that $\dot{V}$ is less than zero given the following inequalities hold:

$$\|\tilde{x}\| > \sqrt{\frac{4\zeta_{B2}}{\left\|\tilde{A}\right\| - 2N_o - 4}} \quad \text{or} \quad \left|\hat{r}_j\right| > \|A\|\sqrt{\frac{2\zeta_{B2}}{\delta_{j2}}}. \tag{69}$$

In this case, the selection of an event-triggering rule eliminated the explicit presence of measurement errors. Due to the terms $\sigma_j$, in the $\delta_{j2}$ term of (69) bounds obtained in Theorem 2 is different from (69).

In Case 1, the stability of the system was demonstrated when the measurement errors are zero and when the NN's are updated continuously and it was shown that all closed-loop signals remain bounded. Then, in Case 2, it was shown how all signals in the system remain bounded during periods of time when there are nonzero measurement errors and when the NN weight estimates are held. In connecting these two cases, one may consider the dynamics that exist at the moments of transition. This can be accomplished by considering the jump dynamics [17]. It is not difficult to show that the Lyapunov stability conditions are satisfied by considering the first difference of the Lyapnuov function at event sampling instants. See [16], [17] for details. Hence, the bounds that exist for the observer estimation errors, the tracking errors, and the NN weight estimation errors are decreasing during the inter-event periods as well as in the jump dynamics.

Cases 1 and 2 reveals that the weight estimation errors of the NN and the tracking errors are UUB. Define the bounding terms

$$r_{jB} = \max\left\{\frac{4\zeta_{j1} + \alpha_j \eta_{j1}^2}{2\delta_{j1}}, \sqrt{\frac{2\zeta_{B2}}{\delta_{j2}}}\right\}$$

$$W_{jB} = \frac{1}{2}\left[\sqrt{\frac{4\zeta_{j1} + \alpha_j \eta_{j1}^2}{\alpha_j}} + \eta_{j1}\right].$$

Next, using (68) to choose the design constants, $\dot{V} < 0$ when

$$|r_j| > r_{jB} \quad \text{or} \quad \left\|\tilde{W}_j\right\| > W_{jB}. \tag{70}$$

■

*Remark* 11. By Theorems 2 and 3, the tracking errors in terms of estimated states, $\hat{r}_i,\ i = 1, \dots, n$, are shown to be bounded. From Theorem 1, the observer estimation errors, $\|\tilde{x}\|$, are also bounded. Together, these results imply that the actual tracking errors, $r_i$, are also bounded.

The result for the output-feedback controller (Theorems 2 and 3) can be easily realized for a state-feedback controller. As a matter of fact, the derivations for the state-feedback case become simpler due to the fact that an observer becomes unnecessary and the dynamics that result from observer estimation vanishes from the analysis. A detailed presentation for the derivation of the state-feedback controller would be highly redundant and, therefore, only major conclusions will be provided. See [22] for more details.

**Corollary 1.** Consider the tracking error dynamics given by

$$\begin{aligned}
\dot{r}_1 &= g_1\left[r_2 - k_1 r_1 - \hat{W}_1^T \varphi_{1e} + W_1^T \varphi_1 - \varepsilon_1\right] + g_1\left[k_1 e_1\right]\\
\dot{r}_i &= g_i\left[r_{i+1} - k_i r_i - r_{i-1} - \hat{W}_i^T \varphi_{ie} + W_i^T \varphi_i - \varepsilon_i\right]\\
&\qquad + g_i\left[k_i e_i + e_{i-1}\right]\\
\dot{r}_n &= g_n\left[-k_n r_n - r_{n-1} - \hat{W}_n^T \varphi_{ne} + W_n^T \varphi_n - \varepsilon_n\right]\\
&\qquad + g_n\left[k_n e_n + e_{n-1}\right].
\end{aligned} \tag{71}$$

Define the actual control and the desired virtual control inputs as

$$\begin{aligned}
\hat{\nu}_{1e} &= -k_1\left[r_1 - e_1\right] - \hat{W}_1^T \varphi_{1e}\\
\hat{\nu}_{ie} &= -\left[r_{i-1} - e_{i-1}\right] - k_i\left[r_i - e_i\right] - \hat{W}_i^T \varphi_{ie}\\
u_e &= -\left[r_{n-1} - e_{n-1}\right] - k_n\left[r_n - e_n\right] - \hat{W}_n^T \varphi_{ne}.
\end{aligned} \tag{72}$$

Select the NN weight tuning given by

$$\dot{\hat{W}}_i = F_i\left[r_{ie}\varphi_{ie} - \alpha_i\left|r_{ie}\right|\hat{W}_i\right] \; i = 1, \dots, n \tag{73}$$

where $F_i = F_i^T > 0$ and $\alpha_i > 0$ are design parameters; Let *Assumptions 1,2,4* hold. If $|e_i| \le B_{ei},\ i = 1, 2, \dots, n$, then, $\exists k_i :$ $r_i,\ \tilde{W}_i$ are locally uniformly ultimately bounded. The bounds are obtained as explicit function of the NN reconstruction errors and the measurement errors. Further, if the event-sampling condition is chosen as

$$e_i^2 \le \sigma_i \mu_i r_i^2,\ i = 1, \dots, n \tag{74}$$

with $\mu_j = \frac{1}{k_j^2 + 1}$,. Then, $r_i$ and $\tilde{W}_i$ are locally uniformly ultimately bounded.

*Proof:* Choose the Lyapunov candidates that have identical forms as (26), (40), and (49); however, with the absence of an observer, it is not necessary to include the scaling term, $\frac{1}{\|A\|^2}$, that was incorporated in the output-feedback derivation in order to avoid controller gain bounds that are directly proportional to $\|A\|$. Using a procedure similar to what was presented in the output-feedback controller reveals

$$\begin{aligned}
\dot{V}_n \le &-\sum_{j=1}^{n}|r_j|\left[\frac{1}{2}\delta_{j3}|r_j| + \alpha_j\left[2 - \chi_j\right]\left[\left\|\tilde{W}_j\right\| - \frac{\eta_{j3}}{2}\right]^2\right.\\
&\left.- \frac{\eta_{j3}^2\alpha_j\left[2 - \chi_j\right]}{4}\right] + \frac{1}{2}\sum_{j=1}^{n}\beta_j B_{ej}^2 + \zeta_{FS1}
\end{aligned} \tag{75}$$

where

$$\zeta_{FS1} = \sum_{j=1}^{n}\zeta_{j3}, \quad \eta_{j3} = \frac{\left[g_M + \chi_j\right]\sqrt{N_j} + \chi_j\alpha_j W_{jM}}{\alpha_j\left[2 - \chi_j\right]}$$
$$\beta_j = k_j^2 + 1, \quad \delta_{j3} = k_j g_m - 3g_M^2 - 2.$$

with $\zeta_{j3} = \frac{g_{jM}\varepsilon_{jM}^2}{2k_j} + 2g_{jM}^2 W_{jM}^2 N_j$.

With (75) and an approach similar to what was presented in Theorem 2, it is easy to demonstrate that the system is ISS-like with respect to the measurement error, provided that controller gains are selected satisfying the following condition:

$$k_i > \Gamma_{i3} \tag{76}$$

where $\Gamma_{i3} = \frac{1}{g_m}\left[3g_M^2 + 2\right]$ for all $i = 1, \dots, n$. Bounds on the tracking errors and the NN weight estimation errors can be found in terms of bounded terms and measurement errors introduced by event-sampling.

Next, letting the event-sampling error satisfy the condition $e_i^2 \le \sigma_i \mu_i r_i^2,\ i = 1, \dots, n$ and choosing $\mu_j = \frac{1}{k_j^2 + 1}$, reveals

$$\dot{V}_n \le -\frac{1}{2}\sum_{j=1}^{n}\delta_{j4} r_j^2 + \zeta_{FS2} \tag{77}$$

where $\delta_{j4} = k_j g_m - 4g_M^2 - \sigma_j - 1$ and $\zeta_{FS2} = \frac{1}{2}\sum_{j=1}^{n} N_j\left[W_{jM} + \left\|\hat{W}_j\right\|\right]^2 + \sum_{j=1}^{n}\frac{g_M\varepsilon_{jM}^2}{2k_j}$ is bounded. The bound on the tracking error can, hence, be expressed as functions of bounded terms.

*Remark* 12. The event trigger condition derived in Theorem 3, Case 2, is in terms of the values that are continuously available at the event triggering mechanism. The event execution rule which uses NN weights can be derived by using Lipschitz continuity of the activation function [11]. Such a condition will cause frequent events to facilitate NN learning.

*Remark* 13. With the ISS stability conditions satisfied, it is not difficult to show that the event triggering mechanism will not exhibit zeno behavior. See [9][15] for details.

*Remark* 14. In the expressions for the bounds on the tracking errors and the NN weight estimation errors, note the frequent appearance of terms stemming from the unknown functions, $g_i(\cdot)$; in other words, many of the bounds are in terms of $g_M$ and/or $g_m$. The presence of these terms can be largely avoided by invoking an assumption on the boundedness of $|\dot{g}_i(\cdot)|$, as was done in [1]. With an assumption on $|\dot{g}_i(\cdot)|$ being bounded, and using Barbalat's lemma, it would not be difficult to show that the tracking errors converge to zero in the ideal case when all states are available and when errors from NN approximations are zero and all sigma-modification terms, $\alpha_i,\ i = 1, \dots n$, are zero.

*Remark* 15. A dead-zone operator can be used to avoid redundant events once the tracking error reaches its bounds [22].

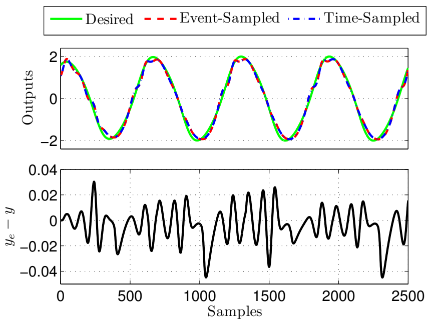



Fig. 2. Controller Outputs - Output Feedback

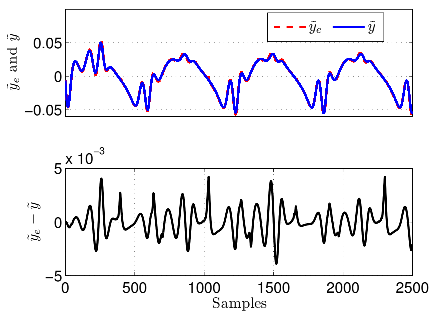



Fig. 3. Observer Estimation Errors

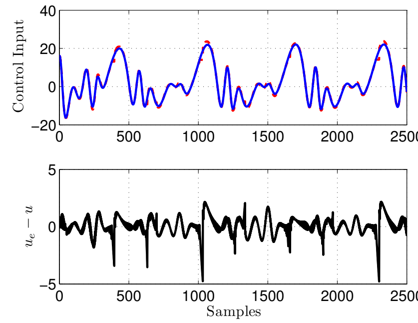



Fig. 4. Continuous (Solid), Event-Sampled (Dashed) - Output Feedback

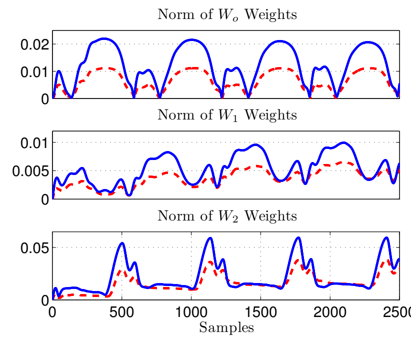



Fig. 5. Continuous (Solid), Event-Sampled (Dashed) - Output Feedback

## IV. Simulation Results

In this section, the analytical design will be verified using a numerical example via simulation. The simulation results will be presented as a comparative study of continuous/time-sampled control implementation with the proposed event-sampled control implementation for both output and state feedback.

Due to the strict-feedback form it is observed from the Lyapunov study that decentralized event generating mechanism can be designed at each subsystem. Alternatively, if the sensors are in proximity, an event-generating mechanism for the entire system can be used. Regardless of the approach taken for implementing the event triggering mechanism, the stability results remain valid. Here, a the control implementation with a single event triggering mechanism is presented.

Prior to giving results, details common to both simulations will be specified. The following system will be used to test the proposed controllers:

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2\\ \dot{x}_2 &= x_1 + x_2 + u\\ y &= x_1.\end{aligned} \tag{78}$$

The desired trajectory, $y_d$, is generated from the following van der Pol oscillator system:

$$\begin{aligned}\dot{x}_{1d} &= x_{d2}\\ \dot{x}_{2d} &= -x_{d1} + 0.4\left[1 - x_{d1}^2\right]x_{d2}\\ y_d &= x_{d1}.\end{aligned} \tag{79}$$

In both simulations, the controller NNs each contain 25 nodes and the event-execution parameters $\sigma_1 = \sigma_2 = 0.0008$ were selected. Initial conditions of $[x_1(0), x_2(0)]^T = [1.1, 0.9]^T$ and $[x_{d1}(0), x_{d2}(0)]^T = [1.6, 0.8]^T$ were used.

### A. Output Feedback Simulation Results

An observer NN with 10 nodes was implemented. Control gains of $k_1 = k_2 = 6.5$ and $l_1 = l_2 = 60$ and NN parameters of $F_1 = F_2 = 0.01$, $\alpha_1 = 140$, $\alpha_2 = 40$ $F_o = 0.2$, and $\alpha_o = 0.1$ were selected such that the conditions from Theorems 1,2, and 3 are satisfied. The trajectory of the system output is recorded in Fig. 2 for continuous feedback and event sampled feedback. The observer estimation errors are recorded in Fig. 3. The difference in the output trajectory for continuous and event sampled feedback is very less. When considering the control input (Fig. 4), it is found that the control effort is nearly identical. It can be seen from Fig. 5 that the NN weights with event-sampled controller are not updated as frequently. The benefits of event based control implementation can be viewed in Fig. 9. This plot is generated by executing the simulation with the sensor sampling set at 10 ms. It is observed that, the event-sampled controller provides satisfactory tracking performance using fewer than 1200 samples out of the available 2500 samples. Moreover, observe the linear nature of Fig. 9, indicating that the occurence of events is fairly evenly distributed.

### B. State Feedback Simulation Results

Control gains of $k_1 = k_2 = 6.5$ and NN parameters of $F_1 = F_2 = 0.01$, $\alpha_1 = 140$, and $\alpha_2 = 40$ were selected such that the stability conditions from Corollary 1 are satisfied. The output trajectory plot is presented in Fig. 6 for event triggered feedback. The comparison with continuous feedback shows the benefit of event based control implementation. This becomes apparent with the comparison of the control efforts (Fig. 7). This reveals that with a reductions in the computations and feedback instants, satisfactory tracking performance can be achieved. For the learning/adaptation part, the NN weights (Fig. 8) with the event-sampled controller are not updated frequently resulting in a slower learning. Finally, for the primary benefit of event-triggered control implementation the cumulative events can be seen in Fig. 9: Using only 1300 samples out of the sampled 2500 measurements, a satisfactory tracking performance is achieved.

### C. Discussions on Event-Execution Parameter

The event-execution parameters, $\sigma_1$ and $\sigma_2$, were set at different values and the sensor sampling period was set at 10 ms. The

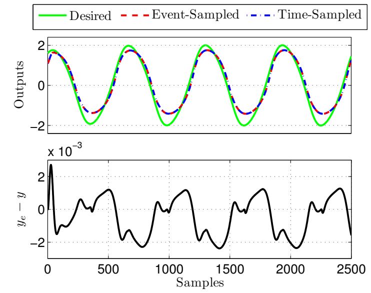



Fig. 6. Controller Outputs - State Feedback

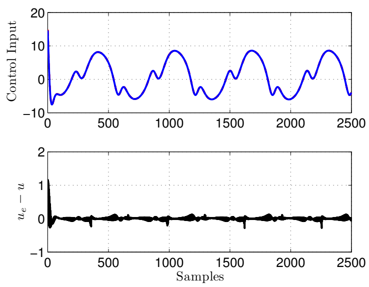



Fig. 7. Continuous (Solid), Event-Sampled (Dashed) - State Feedback

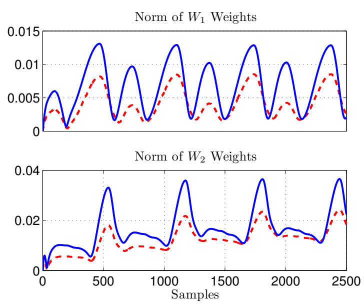



Fig. 8. Continuous (Solid), Event-Sampled (Dashed) - State Feedback

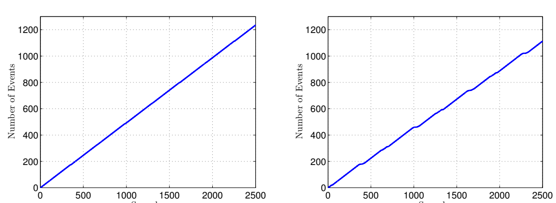



Fig. 9. Number of events: State feedback (L) and Output feedback (R)

number of feedback instants for these values are recorded in Table I; it is observed that as $\sigma_1$ and $\sigma_2$ are increased the number of computations are reduced as the event error is allowed to grow longer. However, as a consequence, the virtual and actual controllers are not frequently updated resulting in either an increase in control effort or degradation in the tracking performance.

## V. Conclusions

In this paper, the derivations for backstepping controllers using NNs that assume an event-sampling paradigm are presented: The first was an output-feedback controller that required an observer to estimate an unknown state vector and the second was a state-feedback controller which, in practice, would require the use of additional sensors in order to obtain full knowledge of the state vector. In both cases, system output tracked the desired trajectory with bounded errors and the errors in the estimation of NN weights are uniformly ultimately bounded with event-triggered control implementation; an event-triggering law is derived using Lyapunov theory. Due to the strict-feedback form, this event execution law can be implemented in a decentralizd manner if the sensors are spatially separated from each other. Simulation results demonstrated the advantage of the event based control execution scheme as the comparison plots with the continuous feedback control showed that even with the aperiodic and reduced feedback instants the tracking performance is unaffected.

TABLE I
Cumulative Events

| $\sigma_{1,2}$ | Total Samples | Number of Events | |
|---|---|---|---|
| | | Output-Feedback | State-Feedback |
| 0.0008 | 2500 | 1110 | 1235 |
| 0.008 | 2500 | 919 | 1188 |
| 0.08 | 2500 | 370 | 899 |

## References


[1] S. Ge and Cong Wang, "Direct adaptive NN control of a class of nonlinear systems," *IEEE Trans. Neural Netw.*, vol. 13, no. 1, pp. 214-221, 2002.

[2] D. Wang and J. Huang, "Neural network-based adaptive dynamic surface control for a class of uncertain nonlinear systems in strict-feedback form," *IEEE Trans. Neural Netw.*, vol. 16, no. 1, pp. 195-202, 2005.

[3] W. Chen, S. S. Ge, J. Wu and M. Gong, "Globally Stable Adaptive Backstepping Neural Network Control for Uncertain Strict-Feedback Systems With Tracking Accuracy Known a Priori," *IEEE Trans. Neural Netw. Learning Syst*, vol. 26, no. 9, pp. 1842-1854, Sept. 2015.

[4] T. Dierks and S. Jagannathan, "Output feedback control of a quadrotor UAV using neural networks," *IEEE Trans. Neural Netw.*, vol. 21, no. 1, pp. 50-66, 2010.

[5] Zhi Liu, Fang Wang, Yun Zhang, Xin Chen and C. Chen, "Adaptive fuzzy output-feedback controller design for nonlinear systems via backstepping and small-gain approach," *IEEE Trans. Cybern.*, vol. 44, no. 10, pp. 1714-1725, 2014.

[6] S. Tong and Y. Li, "Adaptive fuzzy output feedback tracking backstepping control of strict-feedback nonlinear systems with unknown dead zones," *IEEE Trans. Fuzzy Syst.*, vol. 20, no. 1, pp. 168-180, 2012.

[7] S. Tong, Y. Li and P. Shi, "Observer-based adaptive fuzzy backstepping output feedback control of uncertain MIMO pure-feedback nonlinear systems," *IEEE Trans. Fuzzy Syst.*, vol. 20, no. 4, pp. 771-785, 2012.

[8] S. Shanwei, "Output-feedback tracking control for a class of nonlinear non-minimum phase systems via backstepping method," *in 34th Chinese Control Conference*, 2015, pp. 475-480.

[9] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1680-1685, 2007.

[10] L. Dong; X. Zhong; C. Sun; H. He, "Adaptive event-triggered control based on heuristic dynamic programming for nonlinear discrete-time systems,"*IEEE Transactions on Neural Networks and Learning Systems*, to be published.

[11] A. Sahoo, H. Xu and S. Jagannathan, "Near optimal event-triggered control of nonlinear discrete-time systems using neurodynamic programming," *IEEE Trans. Neural Netw. Learning Syst*. vol. 27, no. 9, pp. 1801-1815, 2015.

[12] H. Zhang, G. Feng, H. Yan and Q. Chen, "Observer-Based Output Feedback Event-Triggered Control for Consensus of Multi-Agent Systems,"*IEEE Trans. on Industrial Electronics*, vol. 61, no. 9, pp. 4885-4894, Sept. 2014.

[13] V. Narayanan and S. Jagannathan, "Distributed adaptive optimal regulation of uncertain large-scale interconnected systems using hybrid Q-learning approach," *IET Control Theory Applications*,vol. 10, no. 12, pp. 1448-1457, 2015.

[14] P. Tallapragada and N. Chopra, "On event triggered tracking for nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 9, pp. 2343-2348, 2013.

[15] M. Mazo Jr. and P. Tabuada, "An ISS self-triggered implementation of linear controllers," in*Automatica*, Vol. 46-8, pp. 1310-1314, August 2010.

[16] A. Sahoo, H. Xu and S. Jagannathan, "Neural network-based adaptive event-triggered control of nonlinear continuous-time systems," *IEEE International Symposium on Intelligent Control (ISIC)*, Hyderabad, 2013, pp. 35-40.

[17] X. Zhong and H. He, “An event-triggered ADP control approach for continuous-time system with unknown internal states”, *IEEE Trans. Cybern.*, pp. 1-12, 2016.

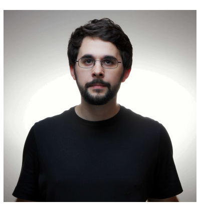
[18] V. Narayanan and S. Jagannathan, "Distributed event-sampled approximate optimal control of interconnected affine nonlinear continuous-time systems,"*American Control Conference (ACC)*, Boston, USA, 2016, pp. 3044-3049.

[19] F. Lewis, S. Jagannathan and A. Yeşildirek, *Neural Network Control of Robot Manipulators and Nonlinear Systems*. London: Taylor & Francis, 1999.

[20] R. Hermann and A. Krener, “Nonlinear controllability and observability”, *IEEE Trans. Automatic Control*, pp. 728-740, 1977.

[21] H.K. Khalil and J. W. Grizzle, *Nonlinear systems*. Vol. 3. New Jersey: Prentice hall, 1996.

[22] N. Szanto, N. Vignesh and S. Jagannathan, "Event-sampled direct adaptive NN state-feedback control of uncertain strict-feedback system," *Conference on Decision and Control (CDC)*, Las Vegas, NV, 2016, pp. 3395-3400.

**Nathan Szanto** received B.S. and M.S. degrees in Electrical Engineering at the Missouri University of Science and Technology (formerly the University of Missouri-Rolla), Rolla, in 20014 and 2016, respectively.

His interests include nonlinear control, neural network control, and mathematics. He also enjoys PCB design and working with electronics. Nathan currently works in consulting at Erdos Miller in Houston, TX.

**Vignesh Narayanan** (S’14) received the B.Tech. and M.Tech. degrees in electrical engineering from SASTRA University, Thanjavur and National Institute of Technology, Kurukshetra, India in 2012 and 2014, respectively. He is currently working toward the Ph.D. degree in the Department of Electrical and Computer Engineering in Missouri University of Science and Technology (formerly the University of Missouri), Rolla.

His research interests include nonlinear control, neural network control, and the control and coordination of autonomous ground and aerial vehicles.

**Dr. Sarangapani Jagannathan** (F’16) is at the Missouri University of Science and Technology (former University of Missouri-Rolla) where he is a Rutledge-Emerson Distinguished Professor of Electrical and Computer Engineering and Site Director for the NSF Industry/University Cooperative Research Center on Intelligent Maintenance Systems. His research interests include neural network control, adaptive event-triggered control/cyber-physical systems, prognostics, and autonomous systems/robotics. He has coauthored with his students 142 peer reviewed journal articles, 255 refereed IEEE conference articles, several book chapters, authored/co-edited 6 books, received 19 US patents, one patent defense publication and several pending. He graduated 25 doctoral and 30 M.S thesis students, and his total funding is in excess of $16 million with over 9.8$ million towards his shared credit from federal and industrial entities. He was a co-editor for the IET book series on control from 2010 until 2013 and now serving on many editorial boards including IEEE Systems, Man and Cybernetics. He received many awards including the 2000 NSF Career Award, 2001 Caterpillar Research Excellence Award, 2007 Boeing Pride Achievement Award, and has been on organizing committees of several IEEE Conferences. He is a Fellow of the IEEE, Institute of Measurement and Control, UK and Institution of Engineering and Technology (IET), UK.