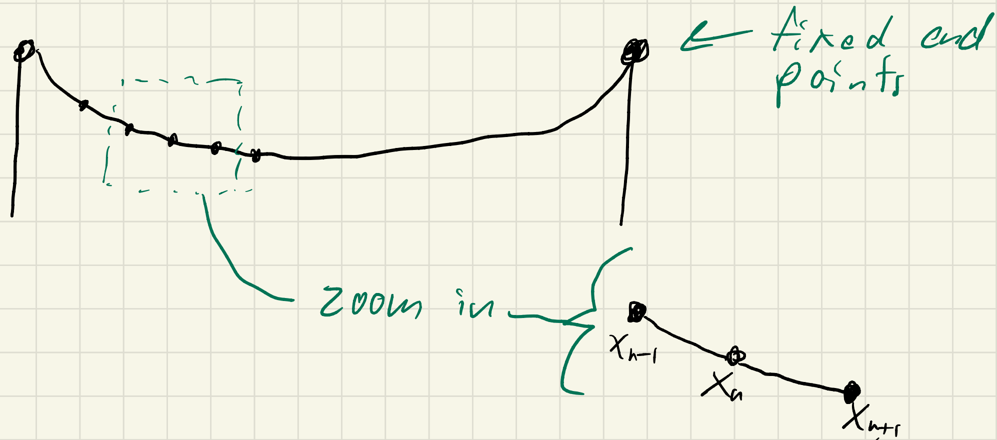
Last Time:

- Calculus of Variations Pt. 2
- Hanging Cable

Today:

- Calculus of Variations Pt. 2
- Dynamics from Energy
- Lagrangian Mechanics
- Least-Action Principle

Another look at the Hanging Cable:

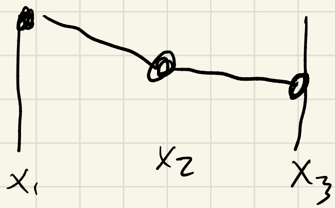


- If we zoom in a small subsection also has to be optimal (minimum potential energy).

- We can take any subsection and fix its endpoints to the true solution.

- Looks exactly like the original problem, just smaller

- Let's look at a sub-section with only 3 discrete points:



$$s_n^2 = \Delta x_n^2 + \Delta y_n^2$$
$$= \Delta x^2 + (y_{n+1} - y_n)^2$$
$$\Rightarrow s_n = \sqrt{1 + \left(\frac{y_{n+1} - y_n}{\Delta x}\right)^2} \Delta x$$

$$U_n = \tfrac{1}{2}(y_{n+1} + y_n)\rho g s_n = \tfrac{1}{2}(y_{n+1} + y_n)\rho g \sqrt{1 + \left(\frac{y_{n+1} - y_n}{\Delta x}\right)^2} \Delta x$$

$$U = \sum u_n = u(y_1, y_2) + u(y_2, y_3)$$

- Length Constraint:

$$S = \sum_n s_n = s(y_1, y_2) + s(y_2, y_3) = \ell_{1:3}$$

- Now we only have to minimize w.r.t. y_2

- KKT Conditions:

$$\frac{\partial}{\partial y_2}\left[U(y_1, y_2, y_3) + \lambda S(y_1, y_2, y_3)\right] = 0$$

$$\Rightarrow \frac{\partial}{\partial y_2}\left[u(y_1, y_2) + u(y_2, y_3) + \lambda s(y_1, y_2) + \lambda s(y_2, y_3)\right] = 0$$

$$\Rightarrow D_2 u(y_1, y_2) + D_1 u(y_2, y_3) + \lambda D_2 s(y_1, y_2) + \lambda D_1 s(y_2, y_3) = 0$$

"Discrete Euler-Lagrange Equation"

- This matches what we got last time by minimizing the whole sum.

- We got this by minimizing w.r.t. y_2, but I can use DEL to solve for any of y_1, y_2, y_3 given the other two.

- If we take continuum limit $\Delta x \to 0$

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$ } Continuous Euler-Lagrange Equation

$$\underbrace{L = u(y, \dot{y}) + \lambda s(y, \dot{y})}_{\text{Lagrangian}}$$

- The fact that every sub-segment has to be optimal lets us derive local optimality conditions (Euler-Lagrange Equation).

- This is called the "Principle of Optimality" or "Bellman's Principle" and underlies optimal control, dynamic programming, and RL.

- We can also view this as an optimal control problem:

$$\min_{y(x)} \quad U = \int_0^{\ell} u(y, \dot{y})\, dx$$ } Potential Energy

$$\text{s.t.} \quad \left.\begin{aligned} y(0) &= y_0 \\ y(\ell) &= y_n \end{aligned}\right\} \text{end point conditions}$$

$$s = \ell$$

$$\min_{\substack{x(t)\\u(t)}} \quad J = \int_{t_0}^{t_f} \ell(x,u)\, dt \;\} \text{ cost}$$

$$\text{s.t.} \quad \begin{aligned} \dot{x} &= f(x,u) && \} \text{ dynamics} \\ x(0) &= x_0 && \} \text{ initial state} \\ x(t_f) &= x_g && \} \text{ goal state} \end{aligned}$$

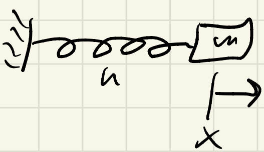
Correspondance:

$$\begin{aligned} U &\longrightarrow J \\ x &\longrightarrow t \\ y(x) &\longrightarrow x(t) \\ \dot{y}(x) &\longrightarrow u(t) \end{aligned}$$

$$\dot{y} = f(y,u) = u \quad \} \text{ "dynamics constraint"}$$

Dynamics from Energy:

- Let's look at a spring-mass system:



$$m\ddot{x} = -kx$$

$$\underbrace{T = \tfrac{1}{2} m \dot{x}^2}_{\text{Kinetic Energy}} \qquad \underbrace{U = \tfrac{1}{2} k x^2}_{\text{Potential Energy}}$$

- No damping $\Rightarrow$ Total energy is conserved

$$T + U = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} k x^2 = \underbrace{H}_{\text{constant}}$$

$$\Rightarrow \frac{d}{dt} H = m\dot{x}\ddot{x} + Kx\dot{x} = 0$$

- Divide by $\dot{x}$

$$\Rightarrow m\ddot{x} = -Kx$$

- This works for any system of the form:

$$H = \frac{1}{2}\dot{q}^T \underbrace{M}_{\text{constant mass matrix}} \dot{q} + U(q)$$

$$\Rightarrow \dot{H} = \dot{q}^T M \ddot{q} + \frac{\partial q}{\partial q}\dot{q} = 0 = \dot{q}^T(M\ddot{q} + \nabla U(q))$$

$$\Rightarrow M\ddot{q} + \nabla U(q) = 0$$

- Doesn't work if $M(q)$ is not constant
- This is not coordinat invariant
- A slightly different setup that does work:
  - Define $L = T - U$
  - Knowing $M\ddot{q} + \nabla U = 0$

$$\Rightarrow \underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)}_{M\ddot{q}} - \underbrace{\frac{\partial L}{\partial q}}_{-\nabla U} = 0$$

- To show that this is coordinate invariant.

$$q' = f(q), \quad \dot{q}' = \frac{\partial f}{\partial q}\dot{q} = J(q)\dot{q}$$

$$\Rightarrow \quad L(q', \dot{q}') = L(f(q), J(q)\dot{q})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} J(q)\right) - \frac{\partial L}{\partial q} J(q) \quad \frac{\partial L}{\partial \dot{q}}\left(\frac{\partial}{\partial q} J(q)\, \dot{q}\right) = 0$$

$$\Rightarrow \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right) J(q) + \cancel{\frac{\partial L}{\partial \dot{q}}\left(\frac{\partial J}{\partial q}\dot{q}\right)} - \frac{\partial L}{\partial q} J(q) - \cancel{\frac{\partial L}{\partial \dot{q}}\left(\frac{\partial J}{\partial q}\dot{q}\right)}$$

cancel

$$= 0$$

$$\Rightarrow \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right] J(q) = 0$$

$$\Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- Euler-Lagrange equation is exactly the same in both q and q' coordinates.

- We also know that is the first-order necessary conditions for an optimization problem!

$$\min_{q(t)} \quad \underbrace{S}_{\text{"Action"}} = \int_{t_0}^{t_f} \overbrace{L(q, \dot{q})}^{\text{"Lagrangian"}}\, dt$$

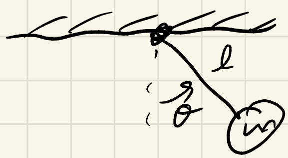
$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

- Minimization problem is called "Least-Action principle" or "Hamilton's Principle".

- Least-action is most general formulation of dynamics. Can derive others by making particular choices of L, q, etc.

- Major advantage over $F=ma$: I can pick whatever coordinates I want for q.

★ Example: Pendulum



$$T = \frac{1}{2} m v^2 = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$V = mgy = mgl(1-\cos(\theta))$$

$$L = T - V = \frac{1}{2} m l^2 \dot{\theta} - mgl(1-\cos(\theta))$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = m l^2 \ddot{\theta} + mgl \sin(\theta) = 0$$