Last Time:

- Local Stability
- Taylor Integration
- Runge-Kutta

Today:

- Higher-Order RK Method
- Stiffness + Stability of RK methods

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* Higher-Order RK Methods:

- We've seen how to build 2nd order RK methods. The general strategy is:

  1) Choose a number of stages
  2) Choose implicit/explicit
  3) Write down integrator step with undetermined weights
  4) Write down Taylor expansion of integrator step
  5) Choose weights to match Taylor expansion to desired order.

- Steps 4-5 become very messy past ~4th order.

- Weight choices are not unique - can optimize for accuracy, efficiency, etc.

- Most famous explicit RK methods are 4^{th} order w/ 4 stages:

$$f_1 = f(x_n)$$
$$f_2 = f(x_n + h a_{21} f_1)$$
$$f_3 = f(x_n + h a_{31} f_1 + h a_{32} f_2)$$
$$f_4 = f(x_n + h a_{41} f_1 + h a_{42} f_2 + h a_{43} f_3)$$

$$x_{n+1} = x_n + h[b_1 f_1 + b_2 f_2 + b_3 f_3 + b_4 f_4]$$

- Can organize coefficients into a matrix A and vector b called "Butcher Tableau"

$$\text{times when } \dot{x} = f(t,x) \text{ is evaluated} \left\{ \begin{array}{c|cccc} 0 & & & & \\ c_2 & a_{21} & & & \\ c_3 & a_{31} & a_{32} & & \\ c_4 & a_{41} & a_{42} & a_{43} & \\ \hline & b_1 & b_2 & b_3 & b_4 \end{array} \right.$$

- A is lower triangular for explicit methods
full for implicit.

- Two examples of 4^{th} order methods:

$$\underbrace{\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}}_{\text{"Classic RK4"}} \qquad \underbrace{\begin{array}{c|cccc} 0 & & & & \\ 1/3 & 1/3 & & & \\ 2/3 & -1/3 & 1 & & \\ 1 & 1 & -1 & 1 & \\ \hline & 1/8 & 3/8 & 3/8 & 1/8 \end{array}}_{\text{"3/8 rule"}}$$

- Also common to use a pair of RK methods of different orders (e.g. 2/3, 4/5, 7/8) together and adapt h to ensure solutions match within some tolerance. Provides automatic error control.

* Stiff ODEs and Stability

- Some systems are difficult to simulate. They require very small steps to not "blow up" even if the system smooth and stable.

- Often due to large mechanical stiffness / damping but not always.

- No precise mathematical definition.

- Let's look at a scalar test problem:

$$\dot{x} = ax, \quad a \in \mathbb{C}$$

- Remember: system is stable when $\mathrm{Re}[a] < 0$ and exact solution is:

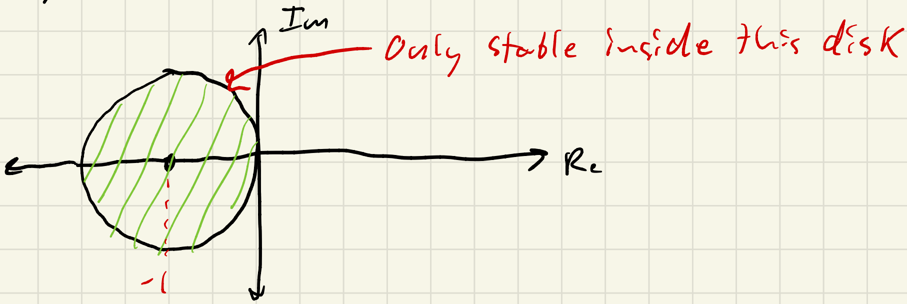
$$x(t) = e^{at} x_0$$

- Let's plug this into our RK methods and check stability of the discretized system.

- Explicit Euler:

$$x_{n+1} = x_n + h a x_n = (1 + ha) x_n$$

$$\text{stability} \Rightarrow |1 + ha| < 1$$

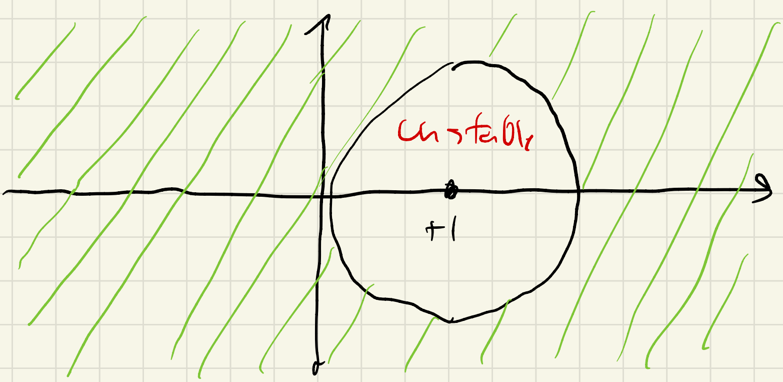


- Implicit Euler:

$$x_{n+1} = x_n + h a x_{n+1} \Rightarrow (1 - ha) x_{n+1} = x_n$$

$$\Rightarrow x_{n+1} = \left(\frac{1}{1 - ha}\right) x_n$$

$$\text{stability} \Rightarrow \left|\frac{1}{1 - ha}\right| < 1 \Rightarrow |1 - ha| > 1$$



artificially stabilizes unstable systems! (adds damping)

* Implicit Midpoint

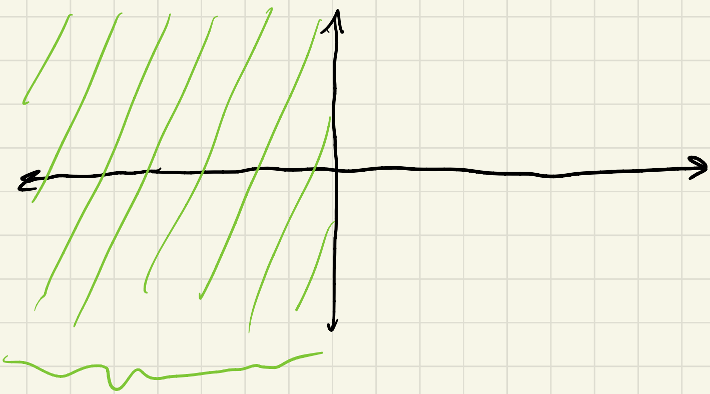
$$x_{n+1} = x_n + h\left(\tfrac{1}{2} a x_n + \tfrac{1}{2} a x_{n+1}\right)$$

$$= \left(1 + \tfrac{1}{2} ha\right) x_n + \tfrac{1}{2} a h x_{n+1}$$

$$\Rightarrow x_{n+1} = \left(\frac{1 + \tfrac{1}{2} ha}{1 - \tfrac{1}{2} ha}\right) x_n$$

stability $\Rightarrow \left| \dfrac{1 + \tfrac{1}{2} ha}{1 - \tfrac{1}{2} ha} \right| < 1$

$$\Rightarrow \underbrace{\left| 1 + \tfrac{1}{2} ha \right| < \left| 1 - \tfrac{1}{2} ha \right|}_{\text{true when } Re[ha] < 0}$$



Stability region is $Re[a] < 0$
Just like the real system!

- If an integrator's stability region contains the entire LHP it is called "A-stable" (A = "absolute")

- Can prove that explicit RK methods can <u>not</u> be A-stable

- Stability condition can be written in terms of Butcher Tableau:

$$\frac{\det(I - zA + zeb^T)}{\det(I - zA)} < 1 \qquad e = [1\ 1\ 1 \dots 1]^T \qquad z \in \mathbb{C} = h\alpha$$

* Definitive Reference:

Hairer, Norsett, Wanner, "Solving Ordinary Differential Equation."