Last Time:

- Higher-Order RK Methods
- Stiff ODEs
- Stability of RK Methods

Today:

- Rigid Bodies
- Reference Frames
- Attitude Representations
- Rotation Matrices

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* Rigid Bodies

- So far we've talked about particles
- Next step: "rigid bodies" that have finite volume
- Definition: A collection of N particles whose relative distances are fixed:

$$\|\underbrace{r_i}_{\text{position } \in \mathbb{R}^3} - r_j\|_2 = \underbrace{c_{ij}}_{\text{distance } \in \mathbb{R}}, \quad i,j \in [1 \dots N]$$

- Never true in reality. Approximation is valid when $\underbrace{\omega_{flex}}_{\text{Natural frequencies of flexible dynamics}} \gg \underbrace{\omega_{RB}}_{\text{Natural frequencies of rigid body dynamics/controller}}$

Examples:

1) Rocket has bending modes $1\sim10$ Hz so controller must be filtered / band-limited to avoid exciting flexible dynamics (not rigid)

2) Car on highway $\Rightarrow$ rigid body model OK vs. race car where roll dynamics, suspension, tire deformation matter.

- If flexible dynamics are much faster than control inputs, can ignore $\Rightarrow$ rigid

- Rigid body models are extremely useful across robotics, control, aerospace etc.

★ Transformations → Configuration / Pose

- Particle only has position but rigid body has more configuration / pose information

- We could just keep track of N particle positions, but that's inefficient

- What are all the transformations I can do to the N particles that will respect the relative distance constraint.

Translation: $r_i' = r_i + v \ , \ \forall i \ , \ v \in R^3$

$$\|r_i' - r_j'\| = \|r_i + v - r_j - v\| = \|r_i - r_j\| = C_{ij}$$

Rotation: $r_i' = Q r_i \;\forall i, \; Q \in SO(3)$

$$\|r_i' - r_j'\| = \|Q r_i - Q r_j\| = \|\underbrace{Q(r_i - r_j)}_{\text{preserves lengths}}\|$$

$$= \|r_i - r_j\| = c_{ij}$$

- Now we can just keep track of a single translation + rotation instead of N positions

* Reference Frames + Attitude:

  - Set of mutually-orthogonal basis vectors that form a right-handed coordinate system.

  - Two kinds:

    - "Inertial/Newtonian" ⇒ Newton's 2nd law holds

    - "Body-fixed"/"Body" ⇒ attached to a (possibly moving) rigid body

  - For our purposes, rigid body always comes with a reference frame.

What is Attitude?

- Rotation between body-fixed frame and inertial frame

- A Lie group $SO(3)$ (more on this later)

☆ How do we write down Attitude?

- Rotation Matrix:

  9 Numbers, 6 constraints, no singularities
  linear/bilinear kinematics, redundant

- Quaternions:

  4 numbers, 1 constraint, no singularities,
  linear/bilinear kinematics, double cover

- Euler Angles:

  3 numbers, 0 constraints, singularities,
  nonlinear kinematics and composition

- Axis-Angle Vector:

  3 numbers, singularities, nonlinear kinematics
  + composition

- Gibbs/Rodrigues Vector:

  3 number, singularities, quadratic
  kinematics/composition

- Modified Rodrigues Parameters:

  3 numbers, singularity at 360°, quartic
  kinematics/composition

- We will primarily use rotation matrices + quaternions

- Don't use Euler Angles!

* Rotation Matrices

  - Linear Transformation between body and inertial frame:

Body frame

$$\underset{\text{Newtonian Frame}}{{}^{N}V} = Q\, {}^{B}V$$

  - Write $\vec{v}$ explicitly in components

$$\vec{v} = \begin{bmatrix} \vec{n}_1 \\ \vec{n}_2 \\ \vec{n}_3 \end{bmatrix}^T \begin{bmatrix} {}^{N}V_1 \\ {}^{N}V_2 \\ {}^{N}V_3 \end{bmatrix} = \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix}^T \begin{bmatrix} {}^{B}V_1 \\ {}^{B}V_2 \\ {}^{B}V_3 \end{bmatrix}$$

$${}^{N}V = \begin{bmatrix} \vec{n}_1 \cdot \vec{v} \\ \vec{n}_2 \cdot \vec{v} \\ \vec{n}_3 \cdot \vec{v} \end{bmatrix} = \vec{n} \cdot \vec{v} = \vec{n} \cdot (\vec{b}^T\, {}^{B}V)$$

$$= \underbrace{(\vec{n} \cdot \vec{b}^T)}_{Q}\, {}^{B}V$$

$$\Rightarrow Q = \begin{bmatrix} \vec{n}_1 \\ \vec{n}_2 \\ \vec{n}_3 \end{bmatrix} \cdot \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} = \begin{bmatrix} \vec{n}_1 \cdot \vec{b}_1 & \vec{n}_1 \cdot \vec{b}_2 & \vec{n}_1 \cdot \vec{b}_3 \\ \vec{n}_2 \cdot \vec{b}_1 & \vec{n}_2 \cdot \vec{b}_2 & \vec{n}_2 \cdot \vec{b}_3 \\ \vec{n}_3 \cdot \vec{b}_1 & \vec{n}_3 \cdot \vec{b}_2 & \vec{n}_3 \cdot \vec{b}_3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} {}^{N}b_1 & {}^{N}b_2 & {}^{N}b_3 \end{bmatrix}}_{B \text{ basis in } N \text{ components}} = \underbrace{\begin{bmatrix} {}^{B}n_1^T \\ {}^{B}n_2^T \\ {}^{B}n_3^T \end{bmatrix}}_{N \text{ basis in } B \text{ components}}$$

* What is the inverse of Q?

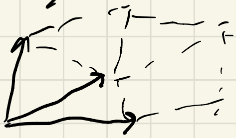
$$Q^TQ = \begin{bmatrix} {}^{N}b_1^T \\ {}^{N}b_2^T \\ {}^{N}b_3^T \end{bmatrix} \begin{bmatrix} {}^{N}b_1 & {}^{N}b_2 & {}^{N}b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow Q^TQ = I \Rightarrow \boxed{Q^T = Q^{-1}}$$

"Orthogonal Matrix"

* What is the determinant of Q?

- measures volume scaling ("stretching")
- Gives signed volume spanned by columns



$Vol = 1$

- Negative determinant implies reflection

$$\det(Q) = 1$$

"special" $\Rightarrow \det(Q) = 1$

$SO(3)$ $\quad \mathbb{R}^{3\times 3}$

"Orthogonal"