Last Time:

- Variational Integrator
- Momentum
- Legendre Transform
- Hamiltonian Mechanics

Today:
- Discrete Legendre Transform
- Variational Integrators with Constraints
- " with non-conservative forces

Discrete Legendre Transform:

- Last time we saw that momentum is:

$$\frac{\partial L}{\partial \dot{q}} = p$$

- We can also re-write the EL Equation in dual form as:

$$H(q,p) = \frac{1}{2} p^T M^{-1}(q) p + U(q) \quad \} \text{ Hamiltonian}$$

$$\underbrace{\frac{\partial H}{\partial q} + \dot{p} = 0 \qquad \frac{\partial H}{\partial p} - \dot{q} = 0}_{\text{Hamilton's Equations}}$$

- Now we'll look at this in discrete time

- We can define 2 different versions of the discrete Legendre transform:

$$\frac{\partial L}{\partial \dot q} = p \Rightarrow \begin{cases} \dfrac{\partial L_d(q_n, q_{n+1})}{\partial q_{n+1}} = p^+_{k+1} & \text{"left"} \\[2ex] \dfrac{\partial L_d(q_n, q_{n+1})}{\partial q_n} = -p^-_n & \text{"right"} \end{cases}$$

- Plug in $L_d = \frac{h}{2}\left(\frac{q_{n+1}-q_n}{h}\right)^T M \left(\frac{q_{n+1}-q_n}{h}\right) - hU\left(\frac{q_{n+1}+q_n}{2}\right)$

$$D_2 L_d(q_n, q_{n+1}) = M\left(\frac{q_{n+1}-q_n}{h}\right) - \frac{h}{2}\nabla U\left(\frac{q_{n+1}+q_n}{2}\right) = p^+_{n+1}$$

$$D_1 L_d(q_n, q_{n+1}) = \underbrace{-M\left(\frac{q_{n+1}-q_n}{h}\right)}_{\text{midpoint momentum}} - \underbrace{\frac{h}{2}\nabla U\left(\frac{q_{n+1}+q_n}{2}\right)}_{\text{"half time step correction"}} = -p^-_n$$

- While v_n are defined at midpoints, p_n^+ and p_n^- are defined at the knot points

- Using these definitions, we can re-interpret the DEL equation:

$$\underbrace{D_2 L_d(q_{n-1}, q_n)}_{p_n^+} + \underbrace{D_1 L_d(q_n, q_{n+1})}_{-p_n^-} = 0$$

$$\Rightarrow p_n^+ = p_n^-$$

- DEL is just a momentum balance

- Now we can initialize our variational integrator given $q(0)$ and $v(0)$:

$$p_0^+ = M(q(0))\,v(0)$$

$$\Rightarrow \quad p_0^+ + D_1 L_d(q_0, q_1) = 0$$

- We can also get correct velocities on the knot points:

$$v_n = M(q_n)^{-1} p_n$$

* Pendulum example
  - Exactly matches implicit midpoint

DEL with Constraints:

- Least Action with Constraints:

$$\min_{q(t)} \int_{t_0}^{t_f} L(q, \dot{q})\, dt$$

$$\text{s.t.} \quad C(q) = 0$$

- Discretize:

$$\min_{q_{1:N}} \sum_{n=1}^{N-1} L_d(q_n, q_{n+1})$$

$$\text{s.t.} \quad C(q_n) = 0 \quad \forall k$$

- KKT Conditions:

Force applied over $t_n \to t_{n+1}$

to satisfy constraint at t_n

$$\frac{\partial}{\partial q_{1:N}} \left[\sum_{n=1}^{N-1} L_d(q_n, q_{n+1}) + h\lambda_n^T C(q_{n+1}) \right] = 0$$

$$C(q_n) = 0 \quad \forall\, k$$

- We can play the same indexing tricks as before:

$$\begin{aligned}
\delta S_d = {} & D_1 L_d(\cancel{q_1}, q_2)\Delta q_1 + D_2 L_d(q_1, q_2)\Delta q_2 + h\lambda_1^T D_1 C(q_2)\Delta q_2 \\
& + \sum_{n=2}^{N-2} D_1 L_d(q_n, q_{n+1})\Delta q_n + D_2 L_d(q_n, q_{n+1})\Delta q_{n+1} + \\
& \qquad h\lambda_n^T D_1 C(q_{n+1})\Delta q_{n+1} \\
& + D_1 L_d(q_{N-1}, q_N)\Delta q_{N-1} + D_2 L_d(\cancel{q_{N-1}}, q_N)\Delta q_N + h\lambda_{N-1}^T D_1 \cancel{C(q_N)}\Delta q_N \\
= {} & \sum_{n=2}^{N-1} D_2 L_d(q_{n-1}, q_n)\Delta q_n + D_1 L_d(q_n, q_{n+1})\Delta q_n + h\lambda_{n-1}^T D_1 C(q_n)\Delta q_n \\
& = 0
\end{aligned}$$

$$\Rightarrow \begin{cases} D_2 L_d(q_{n-1}, q_n) + D_1 L_d(q_n, q_{n+1}) + h\lambda_{n-1}^T D_1 C(q_n) = 0 \\ C(q_{n+1}) = 0 \end{cases}$$

* Pendulum example
  - Exact constraint satisfaction

DEL with Non-Conservative Forces:

- We've already seen Lagrange-D'Alembert:

$$\min_{q(t)} \int_{t_0}^{t_f} L(q,\dot{q}) + F(t)^T q(t) \, dt$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F$$

- We can discretize this in the usual way:

$$\min_{q_{1:N}} \sum_{n=1}^{N-1} L_d(q_n, q_{n+1}) + \frac{h}{2} F(t_n + h/2)^T (q_n + q_{n+1})$$

- Applying the same indexing trick again, we get the forced DEL equation:

$$D_2 L_d(q_{n-1}, q_n) + D_1 L_d(q_n, q_{n+1}) + \frac{h}{2} F(t_n - h/2) + \frac{h}{2} F(t_n + h/2) = 0$$

- Note that we can evaluate $F(t)$ as a function of $q(t)$ and/or $\dot{q}(t)$ if necessary

* Pendulum Example