Last Time:

- Simulation with Constraints
- DAEs
- Baumgarte Stabilization

Today:

- Variational Integrators
- Momentum
- Legendre Transform
- Hamiltonian Mechanics

Variational Integrators:

- Rather than dealing with generic $\dot{x} = f(x)$, we specialize to Lagrangian systems
- Key idea: discretize/approximate Lagrangian/least-action priciple rather than ODE.
- By construction, automatically obey constraints and conserve momentum, energy, etc.

* Least-Action:

$$\min_{q(t)} S(q(t)) = \int_{t_0}^{t_f} L(q, \dot{q})\, dt$$

- Let's break this into a bunch of tiny pieces (exact so far):

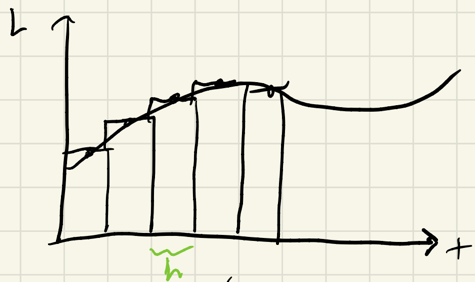
$$\min_{q(t)} S(q(t)) = \sum_{n=1}^{N-1} \int_{t_n}^{t_{n+1}} L(q, \dot{q})\, dt$$

$$t_n = t_0 + kh$$

- Now let's approximate the tiny integrals:

midpoint approximation

$$\int_{t_n}^{t_{n+1}} L(q,\dot{q})\,dt \approx \underbrace{h L\left(\frac{q_n+q_{n+1}}{2}, \frac{q_{n+1}-q_n}{h}\right)}_{\substack{\text{"Discrete Lagrangian"} \\ L_d(q_n,\, q_{n+1})}}$$



- Approximating an integral like this is called "quadrature."

- Well-developed theory very similar to RK methods (i.e. order conditions etc.)

- Now we get a discrete "Action Sum":

$$\min_{q_{1:N}} S_d(q_{1:N}) = \sum_{n=1}^{N-1} L_d(q_n, q_{n+1})$$

- This is exactly the same scenario we saw with the hanging cable. We can get local FON conditions by looking at a very short segment with only 3 knot points:

$$\frac{\partial}{\partial q_2}\left[L_d(q_1,q_2) + L_d(q_2,q_3)\right] = 0$$

$$\Rightarrow \boxed{D_2 L_d(q_1,q_2) + D_1 L_d(q_2,q_3) = 0}$$

"Discrete Euler-Lagrange Equation"

- We can use the DEL equation as an implicit integrator
- Given $q_1, q_2 \Rightarrow$ can solve for q_3 using Newton's method

* Pendulum Example:
  - Energy oscillates about true value $\Rightarrow$ Lyapunov stable
  - Energy behavior is independent of step size
  - Very good for long-term simulations
  - Turns out accuracy is 2^{nd} order. In general accuracy of the quadrature rule used in L_d = accuracy of variational integrator.
  - Easy to construct higher-order methods.

Momentum + Hamiltonian:

- We needed q_1 and q_2 to initialize our variational integrator.
- How do we do this with q_1 and v_1 instead?
- How do we get v_n in general?

- Remember the finite-diff velocities in L_d are defined at the midpoints, not knot points.

- We can trivially re-write the least-action principle as:

$$\min_{\substack{q(t)\\v(t)}} \int_{t_0}^{t_f} L(q,v)\,dt$$

$$\text{s.t. } \dot{q} = v$$ ← kinematic constraints

- Enforcing the kinematic constraint with a Lagrange multiplier gives KKT conditions:

$$\frac{\partial}{\partial q(t)}\left[\int_{t_0}^{t_f} L(q,v) + p^T(\dot{q}-v)\,dt\right] = 0$$

$$\frac{\partial}{\partial v(t)}\left[\qquad \text{''} \qquad\right] = 0$$

$$\dot{q} - v = 0$$

- Doing the standard integration by parts trick:

$$\frac{\partial L}{\partial q} - \dot{p} = 0$$

$$\frac{\partial L}{\partial v} - p = 0$$

$$\dot{q} = v$$

- Let's plug in $L = \frac{1}{2}v^T M v - U(q)$:

$$\Rightarrow \begin{cases} \dfrac{\partial L}{\partial q} = \underbrace{-\nabla U(q)}_{F} = \underbrace{\dot{p}}_{M\dot{v}} \\[2ex] \dfrac{\partial L}{\partial v} = \underbrace{Mv}_{\text{momentum}} = p \Rightarrow \text{Lagrange multiplier} = \text{Momentum} \end{cases}$$

- This version of the least-action principle is called the Hamilton-Pontryagin principle because of its connection to optimal control.

- We can replace v with p in these equations if we want:

$$\frac{\partial L}{\partial v} = M(q)v = p \Rightarrow v = M^{-1}p$$

$$\Rightarrow \int_{t_0}^{t_f} \frac{1}{2}v^T M(q) v - U(q) + p^T(\dot{q} - v)\, dt$$

$$= \int_{t_0}^{t_f} \frac{1}{2}p^T M^{-1}(q) p - U(q) + p^T(\dot{q} - M^{-1}(q)p)\, dt$$

$$= \int_{t_0}^{t_f} -\frac{1}{2}p^T M^{-1}(q) p - U(q) + p^T\dot{q}\, dt$$

- We can define the Hamiltonian:

$$H(q,p) = \frac{1}{2}p^T M^{-1}(q) p + U(q)$$

Such that:

$$-S = \int_{t_0}^{t_f} H(q,p) - p^T \dot{q} \, dt$$

- In these variables, standard FON conditions are:

$$\frac{\partial}{\partial q(t)} \left[\int_{t_0}^{t_f} H(q,p) - p^T \dot{q} \, dt \right] = 0$$

$$\frac{\partial}{\partial p(t)} \left[\quad " \quad \right] = 0$$

$$\boxed{\begin{aligned} \frac{\partial H}{\partial q} + \dot{p} &= 0 \\ \frac{\partial H}{\partial p} - \dot{q} &= 0 \end{aligned}}$$

$\leftarrow$ "Hamilton's Equations"

- In optimization terms, Hamiltonian is "dual form" of least-action problem (Lagrangian is "primal form").

- The change of variables $v \to p$ is called "Legendre Transform" in classical mechanics.