Last Time:

- Discrete Legendre Transform
- Variational Integrators w/ Constraints
- " " w/ External Forces

Today:

- Contact Dynamics
- Discrete Mechanics with Impacts

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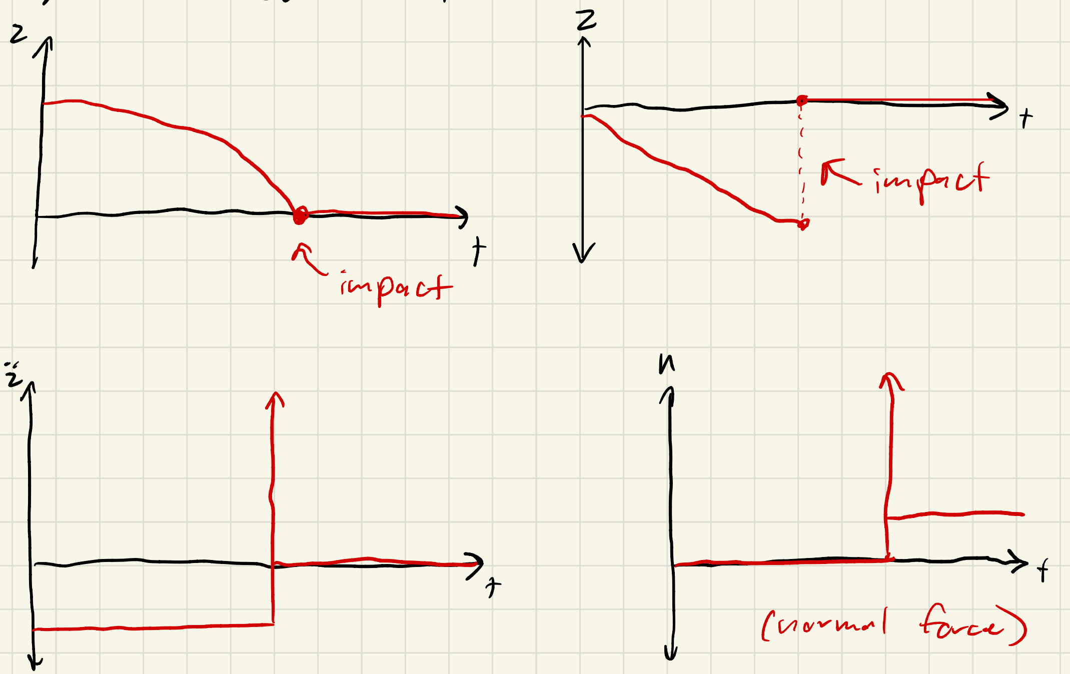
Contact Dynamics:

- Why is this hard?
- Imagine a falling particle that hits the ground (no friction):



- Impacts (and stiction) cause velocity discontinuities. For ideal rigid bodies this leads to infinite forces + accelerations.

- Strictly speaking, $F = ma$ is not well defined in thes scenarios

- In reality, things are not perfectly rigid, but practically this is still difficult.

- Makes modeling, simulation, control, learning hard.

- Three Broad Categories of simulation techiques:

1) Smooth Contact Models

    - Smooth all contact forces with e.g. nonlinear spring-damper model

    - Pros: Easy to implement, differentiable

    - Cons: Not very accurate: interpenetration, creep, stiff ODEs

    - MuJoCo does this

## 2) Hybrid / Event-Driven Methods

- Explicitly detect impact events while integrating ODEs. Perform discontinuous "jump" update, then proceed with smooth integration.

- Pros: Can use standard ODE tools, not stiff, captures qualitatively correct physics.

- Cons: Scales poorly with number of contacts, issues with simultaneous impacts, not differentiable.

- Common in control

## 3) Time-Stepping Methods

- Solve an optimization problem at each time step to compute contact forces that satisfy interpenetration constraints.

- Pros: Scales well in number of contact points, No issues with simultaneous impacts, qualitatively correct physics.

- Cons: Computationally expensive. Not differentiable

- Used by Bullet, Dart, Gazebo

☆ We're going to focus on 3)

# Discrete Mechanics with Impacts:

- Review KKT conditions for inequality constrained optimization problem:

$$\left.\begin{array}{ll} \min\limits_{x} & f(x) \\ \text{s.t.} & c(x) \geq 0 \end{array}\right\} \Rightarrow \underbrace{L(x,\lambda) = f(x) - \lambda^T c(x)}_{\text{"Lagrangian" from optimization}}$$

- KKT Conditions:

$$\frac{\partial L}{\partial x} = \frac{\partial f}{\partial x} - \lambda^T \frac{\partial c}{\partial x} = 0 \quad \text{(stationarity)}$$

"Complementarity condition"

$\lambda \perp c(x)$

$$\left\{\begin{array}{ll} c(x) \geq 0 & \text{(primal feasibility)} \\ \lambda \geq 0 & \text{(dual feasibility)} \\ \lambda \odot c(x) = 0 & \text{(compl. slackness)} \end{array}\right.$$

Hadamard product (element-wise)

- Complementarity conditions define switching behavior for active vs. inactive constraints.

- Now let's apply this to the falling particle

- Lets define a signed-distance function that return distances between closest points.:


$$\phi(q) = [0 \;\; 1]\, q$$

- Note: $\phi(q) = 0$ indicates contact
  $\phi(q) < 0$ indicates penetration (bad)

- Recall Lagrangian for a particle:

$$L(q, \dot{q}) = \frac{1}{2} m \dot{q}^T \dot{q} - mg[0 \;\; 1]\, q$$

$$L_d(q_n, q_{n+1}) = h L\left(\frac{q_{n+1} + q_n}{2}, \frac{q_{n+1} - q_n}{h}\right)$$

- Now we write down Least-Action s.t. interpenetration constraint:

$$\min_{q_{1:N}} \sum_{n=1}^{N-1} L_d(q_n, q_{n+1})$$

$$\text{s.t.} \quad \phi(q_n) \geq 0 \quad \forall K$$

- Looks almost like joint constraints. KKT conditions:

$$D_2 L_d(q_{n-1}, q_n) + D_1 L_d(q_n, q_{n+1}) + h n_n^T D_1 \phi(q_n) = 0$$

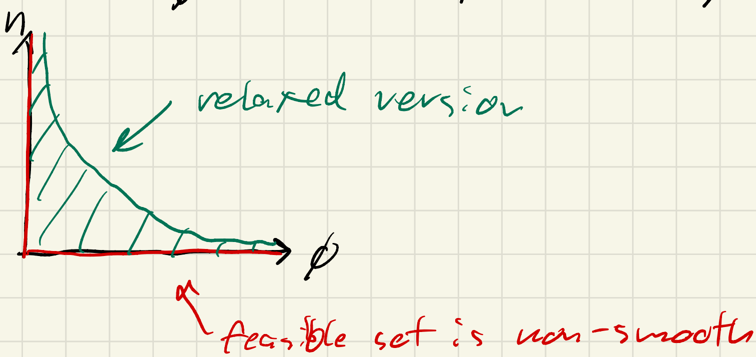
stay above the ground $\qquad \phi(q_{n+1}) \geq 0$

normal force can only push $\qquad n_n \geq 0$

no force off the ground $\qquad n_n \odot \phi(q_n) = 0$

★ Quick notes on solving this:

- Feasible region for complementarity:




- Relax with a slack variable:

$$n\,\phi(q) = 0 \implies n\,\phi(q) \leq s, \quad s \geq 0$$