Last Time:

- Linear Systems
- Group Theory
- Rotation Matrices
- Quaternion Geometry

Today:

- More Quaternions
- Kinetic Energy
- Inertia
- Euler's Equation

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* Quaternions:

- Geometric Picture:

- In terms of axis-angle vector:

$$q = \begin{bmatrix} \cos(\theta/2) \\ \hline r\sin(\theta/2) \end{bmatrix} \begin{matrix} \leftarrow \text{"scalar part"} \\ \leftarrow \text{"vector part"} \end{matrix}$$

$$\theta = \|\phi\| = \text{angle}$$

$$r = \frac{\phi}{\|\phi\|} = \text{axis of rotation}$$

- Note:

$$q^T q = 1 \Rightarrow$$ rotations are unit quaternions (easy to normalize)

q and $-q$ correspond to the same rotation "double cover" of $SO(3)$.

* Opperations on quaternions are analogous to rotation matrices

- Quaternion Multiplication:

$$q_1 * q_2 = \begin{bmatrix} s_1 \\ \hline V_1 \end{bmatrix} * \begin{bmatrix} s_2 \\ \hline V_2 \end{bmatrix} = \begin{bmatrix} s_1 s_2 - V_1^T V_2 \\ \hline s_1 V_2 + s_2 V_1 + V_1 \times V_2 \end{bmatrix}$$

$$= \underbrace{\left[\begin{array}{c|c} s_1 & -V_1^T \\ \hline V_1 & s_1 I + \hat{V}_1 \end{array}\right]}_{L(q_1)} \begin{bmatrix} s_2 \\ \hline V_2 \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} s_2 & -V_2^T \\ \hline V_2 & s_2 I - \hat{V}_2 \end{array}\right]}_{R(q_2)} \begin{bmatrix} s_1 \\ \hline V_1 \end{bmatrix}$$

- Quaternion Identity

$$\theta = 0 \Rightarrow \begin{bmatrix} \cos(0) \\ r\sin(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Quaternion Conjugate:

$$q^+ = \begin{bmatrix} s \\ \hline -V \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} 1 & 0 \\ \hline 0^T & -I \end{array}\right]}_{T} \begin{bmatrix} s \\ \hline V \end{bmatrix}$$ (opposite rotation, like Q^T)

- Rotating a Vector:

$$\underbrace{\begin{bmatrix} 0 \\ {}^{N}x \end{bmatrix}}_{\text{"hat" map for quaternions}} = q * \begin{bmatrix} 0 \\ {}^{B}x \end{bmatrix} * q^{+} = L(q) R^{T}(q) \begin{bmatrix} 0 \\ {}^{B}x \end{bmatrix}$$

$$= \underbrace{R^{T}(q) L(q)}_{\text{gives } Q} \begin{bmatrix} 0 \\ {}^{B}x \end{bmatrix}$$

$$\hat{x} = H\,{}^{N}x, \quad H = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

note: $L(q^{+}) = L^{T}(q)$, $R(q^{+}) = R^{T}(q)$

- Quaternion Kinematics:

- Look at small incremental rotation:

$$q' = q * \Delta q = q * \begin{bmatrix} \cos(\theta/2) \\ r \sin(\theta/2) \end{bmatrix} \approx q * \underbrace{\begin{bmatrix} 1 \\ \frac{1}{2}\phi \end{bmatrix}}_{\text{small axis-angle}}$$

small-angle approximation

$$= q * \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2}\phi \end{bmatrix} \right) = q + q * \begin{bmatrix} 0 \\ \frac{1}{2}\phi \end{bmatrix}$$

- Plug in $\phi = \omega \Delta t$

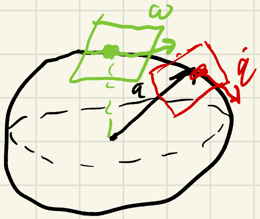
$$q' - q = q * \begin{bmatrix} 0 \\ \frac{1}{2}\omega \Delta t \end{bmatrix} \Rightarrow \frac{q' - q}{\Delta t} = q * \begin{bmatrix} 0 \\ \frac{1}{2}\omega \end{bmatrix}$$

- Take limit $\Delta t \to 0$

$$\dot{q} = \frac{1}{2} q * \begin{bmatrix} 0 \\ \omega \end{bmatrix} = \frac{1}{2} L(q) H \omega$$

(compare to $\dot{Q} = Q\hat{\omega}$ for rotation matrices)

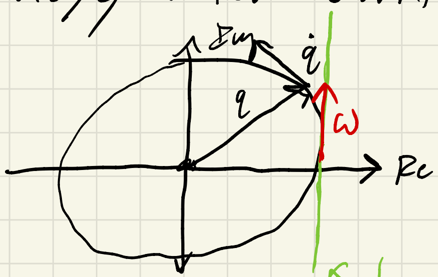
- Geometry:



$\dot{q} \in \mathbb{R}^n$ lies in tangent plane to sphere

ω is always written in tangent plane at identity ("north pole"/ body frame). Kinematics rotates it to tangent plane at current $q(t)$

- Analogy with Unit Complex numbers in 2D



$$q = \cos(\theta) + i\sin(\theta)$$

$$\Rightarrow \quad \underbrace{q = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}}_{q^T q = 1}$$

$$\dot{q} = \frac{\partial q}{\partial \theta}\dot{\theta} = \frac{\partial q}{\partial \theta}\omega = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}\omega = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} 0 \\ \omega \end{bmatrix}$$

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* Kinetic Energy + Moment of Inertia

- Just add up all the particles in the body:

$$T = \sum_i \frac{1}{2} m_i \, {}^N\dot{r}_i^T \, {}^N\dot{r}_i$$

- Plug in rigid-body transformation:

$${}^N r_i(t) = Q(t)\underbrace{{}^B r_i}_{\text{constant}} + \underbrace{{}^N d(t)}_{\text{translation}} \Rightarrow {}^N\dot{r}_i = \dot{Q}\,{}^B r_i + {}^N\dot{d} = Q\hat{\omega}\,{}^B r_i + {}^N v$$

$$T = \sum_i m_i ({}^{\mathcal{}}Q\,{}^B\hat{\omega}\,{}^B r_i + {}^{\sim}v)^T (\ldots)$$

$$= \sum_i \tfrac{1}{2} m_i {}^B r_i^T \hat{\omega}^T \cancel{Q^T}^{=I} Q \hat{\omega} {}^B r_i + \tfrac{1}{2}\Big(\sum_i m_i\Big) {}^{\sim}v^{T\sim}v$$
$$+ {}^{\sim}v^T Q \hat{\omega} \underbrace{\Big(\sum_i m_i {}^B r_i\Big)}_{\text{Center of mass}}$$

- If I put origin of B frame at CoM, last term is zero

$$\Rightarrow T = \underbrace{\tfrac{1}{2} m {}^{\sim}v^{T\sim}v}_{\text{point mass term}} + \underbrace{\tfrac{1}{2} \sum_i m_i {}^B r_i {}^B\hat{\omega}^{T\,B}\hat{\omega}\, r_i}_{\text{new rotation term}}$$

Let's rearrange the rotation term:

$${}^B\hat{\omega}\,{}^B r_i = {}^B\omega \times {}^B r_i = -{}^B r_i \times {}^B\omega = -{}^B\hat{r}_i\,{}^B\omega$$

$$\Rightarrow T = \tfrac{1}{2} m {}^{\sim}v^{T\sim}v + \tfrac{1}{2} {}^B\omega^T \underbrace{\Big[\sum_i -m_i {}^B\hat{r}_i {}^B\hat{r}_i\Big)}_{J = \text{“moment of inertia matrix”}} {}^B\omega$$

$$T = \tfrac{1}{2} m {}^{\sim}v^{T\sim}v + \tfrac{1}{2} {}^B\omega^{T\,B} J\, {}^B\omega$$

- Note J is closely related to covariance of mass distribution

* Euler's Equation

- Translation + rotation de-couple in dynamics just like in kinetic energy (if we put origin of B at CoM)

- Analagous to $F = ma$:

$$\frac{d\,\overbrace{{}^N p}^{\text{momentum}}}{dt} = \underbrace{{}^N F}_{\text{force}} \qquad \frac{d\,\overbrace{{}^N l}^{\text{angular momentum}}}{dt} = \underbrace{{}^N \tau}_{\text{torque}}$$

- Angular momentum is most easily defined in B frame:

$$ {}^B l = \underbrace{{}^B J}_{\substack{\text{constant} \\ \text{in } B \text{ frame}}} {}^B\omega \Rightarrow {}^N l = Q J\, {}^B\omega$$

- Plug back in:

$$\frac{d\,{}^N l}{dt} = \frac{d}{dt}\left(Q J\, {}^B\omega\right) = Q J\, {}^B\dot{\omega} + \dot{Q} J\, {}^B\omega$$

$$= Q\, {}^B J\, {}^B\dot{\omega} + Q\, {}^B\hat{\omega} J\, {}^B\omega = {}^N\tau$$

$$\Rightarrow J\, {}^B\dot{\omega} + {}^B\hat{\omega} J\, {}^B\omega = Q^T\, {}^N\tau$$

$$\Rightarrow \boxed{J\, {}^B\dot{\omega} + {}^B\omega \times J\, {}^B\omega = {}^B\tau}$$ ← "Euler's Equation"

- Now we can simulate a rigid body by combining this with $F = ma$ for translation.

* Some Geometry:

- Assume un-forced case:

$$J\,{}^B\dot{\omega} + {}^B\omega \times J\,{}^B\omega = 0$$

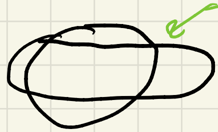
- Can re-write this in terms of ${}^B l = J\,{}^B\omega$

$$ {}^B\dot{l} + (J^{-1}\,{}^B l) \times {}^B l \Rightarrow {}^B\dot{l} = {}^B l \times J^{-1}\,{}^B l$$

- In the un-forced case $\|{}^B l\| =$ constant
  $\Rightarrow$ lies on "momentum sphere"

- Conservation of energy $\frac{1}{2}\,{}^B l^T J^{-1}\,{}^B l =$ constant
  $\Rightarrow$ lies on "Energy ellipsoid"

  $\Rightarrow {}^B l(t)$ lies on intersection of sphere and ellipsoid:



trajectories follow intersection

* Equilibria:

$$\left.\begin{aligned} {}^B\dot{l} &= 0 \quad \text{when } l \times J^{-1} l = 0 \\ \dot{\omega} &= 0 \quad \text{when } \omega \times J\omega = 0 \end{aligned}\right\} \Rightarrow \omega \text{ parallel to } l$$

$$\omega = \underbrace{\lambda}_{\text{scalar}} J\omega$$

$\Rightarrow$ equilibria correspond to $\pm$ eigenvectors of J
(6 total)