Last Time:

- Calculus of Variations Pt. 2
- Dynamics from Energy
- Lagrangian Mechanics
- Least-Action Principle

Today:

- Interpretation of Least Action
- Manipulator Equation
- Non-Conservative
- Constraints + Coordinates

Interpreting Least Action:

- Last time we saw the EL equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

- Is the FON condition for:

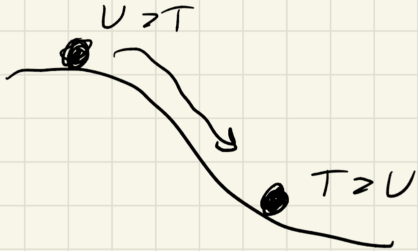
$$\min_{q(t)} \underbrace{S}_{\text{action}} = \int_{t_0}^{t_f} \underbrace{L(q,\dot{q})}_{\text{Lagrangian}} \, dt$$

- If we define $L = T - U$, this gives:

$$\min_{q(t)} \underbrace{\left[\int_{t_0}^{t_f} T(q,\dot{q})\, dt\right]}_{\approx \text{ Avg. kinetic energy}} - \underbrace{\left[\int_{t_0}^{t_f} U(q)\, dt\right]}_{\approx \text{ Avg. Potential energy}}$$

- System is conservative $\Rightarrow$ can only trade U for T and vice-versa.

- Think about a ball rolling down hill:



- U decreases, T increases

  $\Rightarrow$ Dynamics tries to bring average U and T closer together.

- At a minimum of the action, we have:

$$\underbrace{\frac{d}{dq}\left[\int_{t_0}^{t_f} T(q,\dot{q})\, dt\right]}_{\text{Change in kinetic energy}} = \underbrace{\frac{d}{dq}\left[\int_{t_0}^{t_f} V(q)\, dt\right]}_{\text{Change in Potential energy}}$$

- This is basically a re-statement of conservation of energy.

- This is only true at a local min of S

  $\Rightarrow$ True physical trajectories must minimize the action.

The Manipulator Equation:

- All mechanical systems have a Lagrangian of the form:

$$L = T - U = \frac{1}{2}\dot{q}^T \underbrace{M(q)}_{\text{Mass Matrix}} \dot{q} - U(q)$$

Mass Matrix
"Generalized Inertia Tensor"

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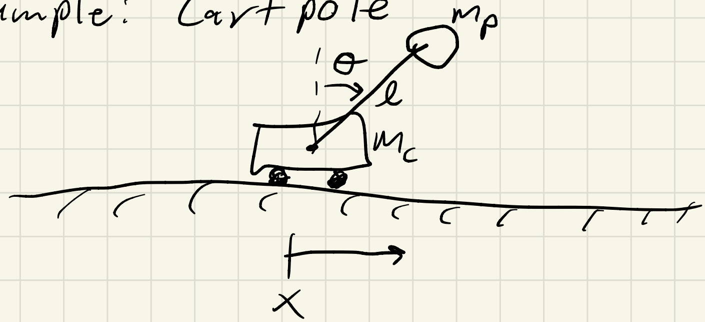
* Example: Cartpole



Pole mass position: $\begin{bmatrix} x + l\sin(\theta) \\ l\cos(\theta) \end{bmatrix}$

$$U = m_p g l \cos(\theta)$$

$$T = \frac{1}{2} m_c \dot{x}^2 + \frac{1}{2} m_p \begin{bmatrix} \dot{x} + l\cos(\theta)\dot{\theta} \\ -l\sin(\theta)\dot{\theta} \end{bmatrix}^T \begin{bmatrix} '' \\ '' \end{bmatrix}$$

$$= \frac{1}{2}(m_c + m_p)\dot{x}^2 + \frac{1}{2} m_p l^2 \dot{\theta}^2 + m_p \dot{x} l \cos(\theta)\dot{\theta}$$

$$T = \frac{1}{2} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}^T \underbrace{\begin{bmatrix} (m_c + m_p) & m_p l\cos(\theta) \\ m_p l\cos(\theta) & m_p l^2 \end{bmatrix}}_{M(q)} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}$$

- Let's plug this into EL equation:

$$L = \frac{1}{2}\dot{q}^T M(q)\,\dot{q} - U(q)$$

$$\Rightarrow \begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{d}{dt}\left(M(q)\,\dot{q}\right) = M(q)\,\ddot{q} + \dot{M}(q)\,\dot{q} \\ \frac{\partial L}{\partial q} = \frac{-\partial U}{\partial q} + \frac{\partial}{\partial q}\left(\frac{1}{2}\dot{q}^T M(q)\,\dot{q}\right) \end{cases}$$

$$\Rightarrow M(q)\,\ddot{q} + \underbrace{\left[\dot{M}(q)\,\dot{q} - \frac{\partial}{\partial q}\left(\frac{1}{2}\dot{q}^T M(q)\,\dot{q}\right)\right]}_{C(q,\dot{q})} + \underbrace{\frac{\partial U}{\partial q}}_{G(q)} = 0$$

$C(q,\dot{q})$ "Coriolis term" (from q-dependence of M)

$G(q)$ "potential term"

$$\Rightarrow \underbrace{M(q)\,\ddot{q} + C(q,\dot{q}) + G(q) = 0}_{\text{"Manipulator Equation"}}$$

- Super common in robotics. Used in most dynamics software.
- Just a re-statement of EL eqn.
- Sometimes you will see:

$\underbrace{C(q,\dot{q})}_{\text{matrix}}\,\dot{q}$ ← this is not unique

- Sometimes you'll see $C(q,\dot{q})$ include both coriolis and potential. Called "dynamic bias" term.

- Manipulator form nicely shows worst-case computational complexity of computing dynamics:

$$\ddot{q} = \underbrace{-M(q)^{-1}\left(C(q,\dot{q}) + G(q)\right)}_{\substack{\text{solving an } n\times n \text{ linear system} \\ O(n^3)}}$$

- There's more to say about this ...

★ Example: Cartpole

$$L = \frac{1}{2}(m_c+m_p)\dot{x}^2 + m_p \dot{x}\, l\cos(\theta)\dot{\theta} + \frac{1}{2}m_p l^2 \dot{\theta}^2 - m_p l\cos(\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \overbrace{\begin{bmatrix} (m_c+m_p) & m_p l\cos(\theta) \\ m_p l\cos(\theta) & m_p l^2 \end{bmatrix}}^{M(q)} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} -m_p l \dot{\theta}^2 \sin(\theta) \\ 0 \end{bmatrix}}_{C(q,\dot{q})} + \underbrace{\begin{bmatrix} 0 \\ -m_p g l \sin(\theta) \end{bmatrix}}_{G(q)} = 0$$

Non-Conservative Forces:

- Conservative forces coming from $F_c = -\nabla U$ are already included in the EL eqn.

- From $F = ma$, we can guess what should happen with other forces:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F$$

"Generalized Forces" (defined in q coordinates)

$$M(q)\ddot{q} + C(q,\dot{q}) + G(q) = F$$

- From $F_c = -\nabla U$, we know we need a term inside S that "integrates F" whose variational derivative w.r.t. $q(t)$ is F:

$$\Rightarrow \frac{d}{dq(t)}\left[\int_{t_0}^{t_f} L(q,\dot{q}) + \underbrace{F(t)^T}_{} q(t)\, dt\right] = 0$$

note that this is not a function of $q(t)$, just t.

$$\Rightarrow \int_{t_0}^{t_f} D_1 L(q,\dot{q})\,\delta q + D_2 L(q,\dot{q})\,\delta\dot{q} + F^T \delta q \, dt$$

$$\Rightarrow \int_{t_0}^{t_f} D_1 L(q,\dot{q})\delta q - \frac{d}{dt}\left(D_2 L(q,\dot{q})\right)\delta q + F^T \delta q \, dt$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F$$

- This is called "Lagrange - D'Alembert Principle" or "Principle of virtual work".

- The non-conservative term has units of work (force × distance):

$$W = \int_{t_0}^{t_f} F(t)^T q(t)\, dt$$

- Variational derivative of this term:

$$\underbrace{\delta W}_{\text{"Virtual work"}} = \int_{t_0}^{t_f} F \cdot \underbrace{\delta q}_{\text{"Virtual displacement"}}\, dt$$

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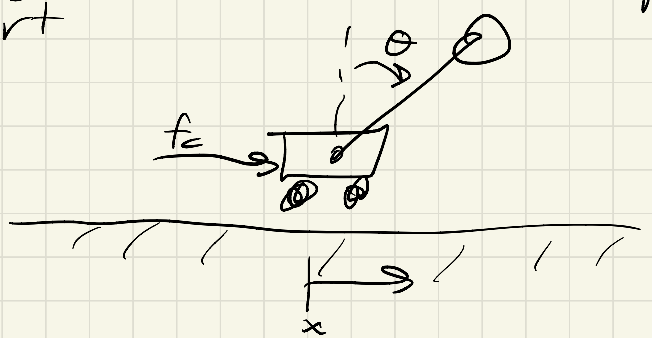
* Cartpole:

  - Assume we have a motor that applies f_c to cart



$$M(x,\theta) \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + C(q,\dot{q}) + G(q) = \begin{bmatrix} f_c \\ 0 \end{bmatrix}$$

Constraints + Coordinates:

- So far we've been dealing with minimal coordinates.

- Often dynamics can be simpler with more coordinates.

- Now we have to deal with constraints to remove "fake" degrees of freedom.

- We've already seen how to handle constraints in the EL equation:

$$\min_{q(t)} \int_{t_0}^{t_f} L(q,\dot{q})\, dt$$

$$\text{s.t.} \quad C(q) = 0$$

$$\Rightarrow \frac{\partial}{\partial q(t)}\left[\int_{t_0}^{t_f} L(q,\dot{q}) + \lambda^T_{(t)} C(q)\, dt\right] = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} - \lambda^T \underbrace{\frac{\partial C}{\partial q}}_{\text{"constraint Jacobian" } J(q)} = 0 \quad (\text{and } C(q)=0)$$

"constraint force"

$$\Rightarrow M(q)\ddot{q} + C(q,\dot{q}) + G(q) = F + J^T_{(q)}\lambda$$

- We have to solve for both $\ddot{q}$ and λ now

- To do this, differentiate $C(q)$ twice w.r.t. time:

$$\frac{d^2 c}{dt^2} = \frac{d}{dt}\left(\frac{\partial c}{\partial q}\dot{q}\right) = J(q)\,\ddot{q} + \frac{\partial}{\partial q}\left(J(q)\,\dot{q}\right)\dot{q} = 0$$

- Now we can combine the acceleration constraint with the dynamics:

$$\begin{bmatrix} M(q) & -J(q)^T \\ J(q) & 0 \end{bmatrix}\begin{bmatrix} \ddot{q} \\ \lambda \end{bmatrix} = \begin{bmatrix} -C(q,\dot{q}) - G(q) + F \\ -\frac{\partial}{\partial q}\left(J(q)\,\dot{q}\right)\dot{q} \end{bmatrix}$$

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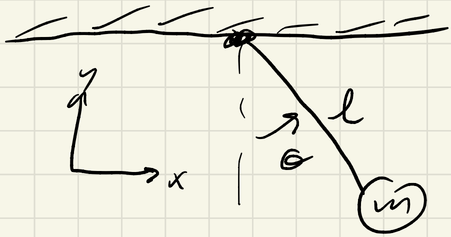
* Example: Pendulum



$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy$$

$$C(x,y) = x^2+y^2-\ell^2 = 0$$

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}}_{M}\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ mg \end{bmatrix} = \underbrace{\begin{bmatrix} 2x \\ 2y \end{bmatrix}}_{J(q)^T}\lambda$$

$$\frac{d^2 C}{dt^2} = \frac{d}{dt}\left(J(q)\,\dot{q}\right) = \frac{d}{dt}\left(\begin{bmatrix} 2x & 2y \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}\right) = \begin{bmatrix} 2x & 2y \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + 2(\dot{x}^2+\dot{y}^2) = 0$$

$$\Rightarrow \begin{bmatrix} M & -J(q)^T \\ J(q) & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \\ -2(\dot{x}^2 + \dot{y}^2) \end{bmatrix}$$