Last Time:

- History
- Newtonian Review
- Coordinate Invariance
- Pendulum

Today:

- State Space
- Euler Integration
- Energy
- Lyapunov Stability

★ State Space

- Pendulum from last time:

$$\ddot{\theta} = -\frac{g}{\ell} \sin(\theta)$$

- This is a 2^{nd}-order ODE:

$$\ddot{\theta} = f(\theta, \dot{\theta})$$

- To predict the system's motion we need θ and $\dot{\theta}$. Let's combine these into a "state" vector:

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} , \quad \dot{x} = f(x)$$

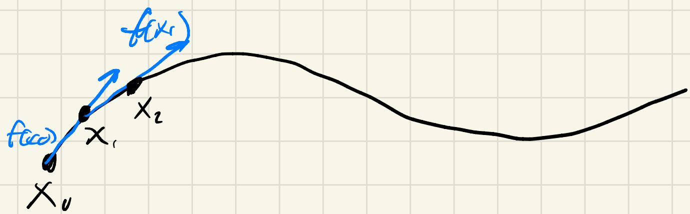
- We can always write ODEs in this 1^{st}-order form.

- This is the form that standard ODE solvers use.

* Our first "Simulator"

  - We usually can't solve these ODEs analytically, so we need numerical methods
  - The simplest one is Explicit Euler

$$x_{n+1} = x(t_n + \underbrace{h}_{\text{time step (small)}}) \approx x_n + h\dot{x} = x_n + hf(x_n)$$



  - Intuition: take small steps along tangent vector to solution curve $x(t)$.

* Take-Aways from Simulation

  - Simulations can produce unphysical behavior
  - Structural things can change due to discretization errors
  - You should probably never use explicit Euler

* Energy

- A scalar quantity that comes in 2 flavors

  * Kinetic (how much I'm moving)

    $$T = \frac{1}{2} m \vec{v} \cdot \vec{v} \quad \text{for a particle}$$

  * Potential (how much could I move based on where I am)

    $$U_{grav} = m g z \quad \text{for constant gravity}$$

    z ← height

- For a particle $m = 1$ that falls $\Delta z = 1$ from rest $\Rightarrow \Delta U = -g$, gain $T = g$ kinetic

  $$\Rightarrow T = g = \frac{1}{2} m v^2 \Rightarrow v = \sqrt{2g}$$

- Note that $U(x)$ is only ever a function of position (never velocity)

- Total Energy

  $$E = T + U$$

- Forces can be calculated from a potential

  $$F = -\nabla U(x)$$

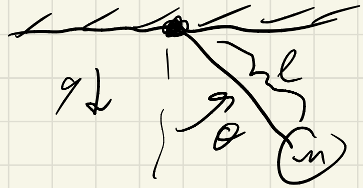
  $$\Rightarrow F_{spring} = -kx = -\frac{\partial}{\partial x} U_{spring}(x)$$

  $$\Rightarrow U_{spring} = \frac{1}{2} k x^2$$

- Forces that can be derived from a potential are called "conservative" since they don't change total energy (just trade for kinetic).

- Friction, drag, damping are all non-conservative. They depend on v and change E.

* Lyapunov stability



- Pendulum Energy:

$$T = \frac{1}{2} m \vec{v} \cdot \vec{v} = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$U = mgz = mgl(1 - \cos(\theta))$$

$$\Rightarrow E = T + U = \frac{1}{2} m l^2 \dot{\theta} + mgl(1 - \cos(\theta))$$

- Look at $\dot{E}$

$$\dot{T} = \frac{\partial T}{\partial \dot{\theta}} \ddot{\theta} = m l^2 \dot{\theta} \underbrace{\left[\frac{-g}{l} \sin(\theta) \right]}_{\text{plug in dynamics}} = -mgl \sin(\theta) \dot{\theta}$$

$$\dot{U} = \frac{\partial U}{\partial \theta} \dot{\theta} = mgl \sin(\theta) \dot{\theta}$$

$$\Rightarrow \dot{E} = \dot{T} + \dot{U} = 0 \Rightarrow \text{energy is conserved}$$

- Based on this we can bound the region the system can move in in the state space

Given E_0 at $t=0$

$$T = \frac{1}{2} m l^2 \dot{\theta}^2 \Rightarrow \|\dot{\theta}(t)\| \leq \sqrt{\frac{2}{ml^2} E_0} \quad \underbrace{\forall}_{\text{"for all"}} t$$

$$U = mgl(1-\cos(\theta)) \Rightarrow \|\theta(t)\| \leq \cos^{-1}\left(\frac{E_0}{mgl} - 1\right) \quad \forall t$$

- When a system is guaranteed to stay within some region we call it "Lyapunov Stable"

- We can formalize/generalize this:

  * If we can find a function $V(x) : \mathbb{R}^n \to \mathbb{R}$

    1) $V(x) \geq 0 \quad \forall x, \quad V(0) = 0$

    2) $V(x) \to \infty$ as $\|x\| \to \infty$

    3) $\dot{V}(x) = \frac{\partial V}{\partial x} \underbrace{f(x)}_{\dot{x}} \leq 0 \quad \forall x$

  $\Rightarrow$ The system is stable about the origin

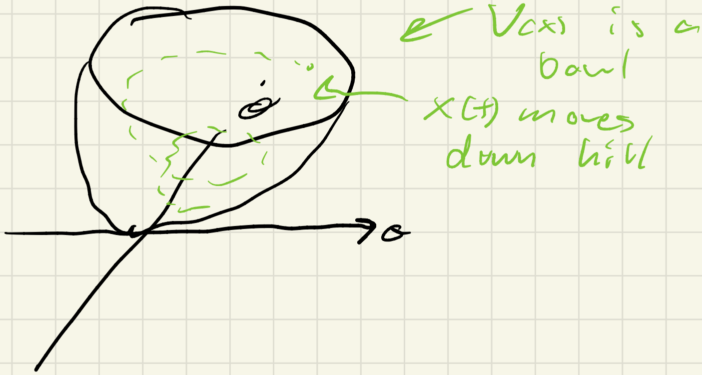
  - We can always change coordinates to make some other point x^* the origin

$$\tilde{x} = x - x^*$$

- $\dot{V}(x) = 0 \Rightarrow$ "Lyapunov stable"

- $\dot{V}(x) < 0 \Rightarrow$ "Asymptotically stable"

~ Intuition:



* Pendulum with Euler Integration

- We can do the same analysis in discrete time by looking at ΔE (or ΔV) instead of $\dot{E}$ or $\dot{V}$

$$\Delta T_n = T_{n+1} - T_n = \frac{1}{2} m l^2 (\dot{\theta}_{n+1}^2 - \dot{\theta}_n^2)$$

$$\Delta V_n = V_{n+1} - V_n = mgl(\cos(\theta_n) - \cos(\theta_{n+1}))$$

- Plug in Euler integrated dynamics:

$$\begin{bmatrix} \theta_{n+1} \\ \dot{\theta}_{n+1} \end{bmatrix} = \begin{bmatrix} \theta_n + h\dot{\theta}_n \\ \dot{\theta}_{n+1} - h\frac{g}{l}\sin(\theta_n) \end{bmatrix}$$

$$\Rightarrow \Delta T_n = \frac{1}{2} m l^2 \left(\cancel{\dot{\theta}_n^2} - 2h\frac{g}{l}\sin(\theta_n)\dot{\theta}_n + h^2\frac{g^2}{l^2}\sin^2(\theta_n) - \cancel{\dot{\theta}_n^2}\right)$$

$$= -hmgl\sin(\theta_n)\dot{\theta}_n + \frac{1}{2}h^2 mg^2\sin^2(\theta_n)$$

$$\Delta V_n = mgl[\cos(\theta_n) - \underbrace{\cos(\theta_n + h\dot{\theta}_n)}]$$

Taylor expand assuming $h \ll 1$

$$\simeq \cos(\theta_n) - \sin(\theta_n) h\dot{\theta}_n$$

$$\simeq hmgl\sin(\theta_n)\dot{\theta}_n$$

$$\Delta E_n = \Delta T_n + \Delta U_n = \underbrace{\tfrac{1}{2} h^2 m g^2 \sin^2(\theta_n)}_{\text{Always Positive!!}} \geq 0$$

$\Rightarrow$ The discretized system is <u>always</u> unstable!

* Local Stability

- Often we can't find a Lyapunov function but we can still get weaker local stability results.

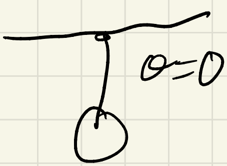
- We can do this around points, trajectories, orbits

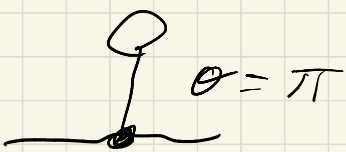
- Most commonly look at equilibrium points

  * Equilibrium $\Rightarrow$ $\dot{x} = f(x) = 0$ (system won't move)

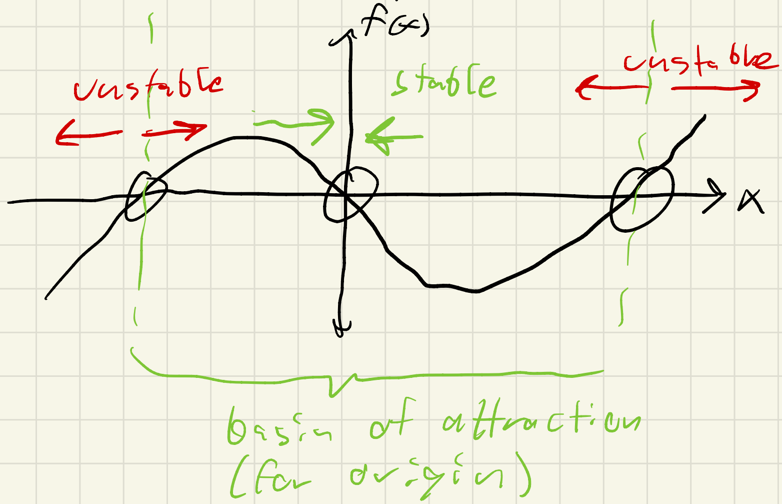
- For pendulum we have 2:

$$x = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{l}\sin(\theta) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ if } x^* = \begin{bmatrix} \pi n \\ 0 \end{bmatrix},\ \underbrace{n \in \mathbb{Z}}_{\text{integer}}$$





- Let's look at 1D system ($x \in \mathbb{R}$)



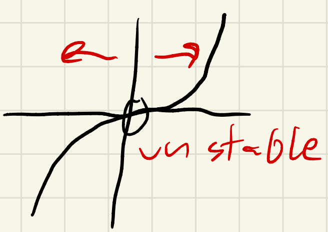
$$\left.\frac{\partial f}{\partial x}\right|_{x^*} < 0 \Rightarrow x^* \text{ is locally stable}$$

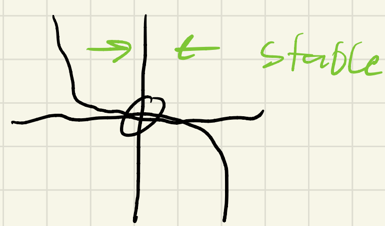
$$\left.\frac{\partial f}{\partial x}\right|_{x^*} > 0 \Rightarrow x^* \text{ is unstable}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x^*} = 0 \Rightarrow \text{inconclusive}$$

- Examples of inconclusive cases:

$f(x) = x^3$ $\qquad\qquad$ $f(x) = -x^3$





$$x^* = 0 \qquad \left.\frac{\partial f}{\partial x}\right|_{x^*} = 0$$