Last Time:

- Quaternions
- Kinetic Energy
- Inertia
- Euler's Equation

Today:

- Stability of spinning bodies
- Numerical simulation of 3D Rotations
- Newton Euler Dynamics and SE(3)

Stability of Spinning Bodies:

* Equilibria

- Assume $\|\ell\| = $ const, rigid body has 6 "equilibrium spins":
- Equilibrium spin $\Rightarrow$ ${}^B\dot{\ell} = {}^B\dot{\omega} = 0$

$$\underbrace{J\,{}^B\dot{\omega}}_{{}^B\dot{\ell}} = -\omega \times J\omega = 0 \Rightarrow \omega \times J\omega = 0$$

- Cross product $= 0$ if ${}^B\omega$ is parallel to $J\,{}^B\omega$

$$\Rightarrow J\,{}^B\omega = \lambda\,{}^B\omega \quad , \lambda \in \mathbb{R}$$

$\Rightarrow$ equilibria are $\pm$ eigenvectors of J

* Principle Axes

- Since kinetic energy ≥ 0, J must be Positive definite:

$$\tfrac{1}{2}\omega^T J \omega > 0 \quad \forall \omega \neq 0 \Rightarrow J \succ 0$$

- Positive-definite matrices have positive real eigenvalues

$\Rightarrow$ we can always diagonalize J

- Eigenbasis of J called "Principle axes" of rigid body. Typically correspond to symmetry axes of body.

- In principle axes, Euler's equation becomes:

$$^P J_{11}\, {^P\dot{\omega}_1} + ({^P J_{33}} - {^P J_{22}})\, {^P\omega_2}\, {^P\omega_3} = {^P\tau_1}$$

$$J_{22}\, \dot{\omega}_2 + (J_{11} - J_{33})\, \omega_1 \omega_3 = \tau_2$$

$$J_{33}\, \dot{\omega}_3 + (J_{22} - J_{11})\, \omega_1 \omega_2 = \tau_2$$

- Now equilibria lie along $\vec{p}_i$ basis vectors

* Stability

- Linearize about each axis and check local stability.

- Assume $J_{11} < J_{22} < J_{33}$

* $\omega_1 = \omega_0 \gg \omega_2, \omega_3$

$$\dot{\omega}_2 = \underbrace{\omega_0 \left(\frac{J_{33} - J_{11}}{J_{22}}\right)}_{\alpha_1 > 0} \omega_3$$

$$\dot{\omega}_3 = \underbrace{\omega_0 \left(\frac{J_{11} - J_{22}}{J_{33}}\right)}_{\alpha_2 < 0} \omega_2$$

$$\Rightarrow \underbrace{\frac{d}{dt}\begin{bmatrix} \omega_2 \\ \omega_3 \end{bmatrix}}_{\dot{x}} \underset{=}{=} \underbrace{\begin{bmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \omega_2 \\ \omega_3 \end{bmatrix}}_{x}$$

} Looks like SHO

$$\text{eig}(A) = \underbrace{\pm\sqrt{\alpha_1 \alpha_2}}_{\text{pure imaginary}} \Rightarrow \text{marginally stable}$$

* $\omega_2 = \omega_0 \gg \omega_1, \omega_3$:

$$\dot{\omega}_1 = \underbrace{\omega_0 \left(\frac{J_{22} - J_{33}}{J_{11}}\right)}_{\alpha_1 < 0} \omega_3 \, , \quad \dot{\omega}_3 = \underbrace{\omega_0 \left(\frac{J_{11} - J_{22}}{J_{33}}\right)}_{\alpha_2 < 0} \omega_1$$

$$\Rightarrow \text{eig}(A) = \underbrace{\pm\sqrt{\alpha_1 \alpha_2}}_{\text{pure real}} \Rightarrow \text{unstable!}$$

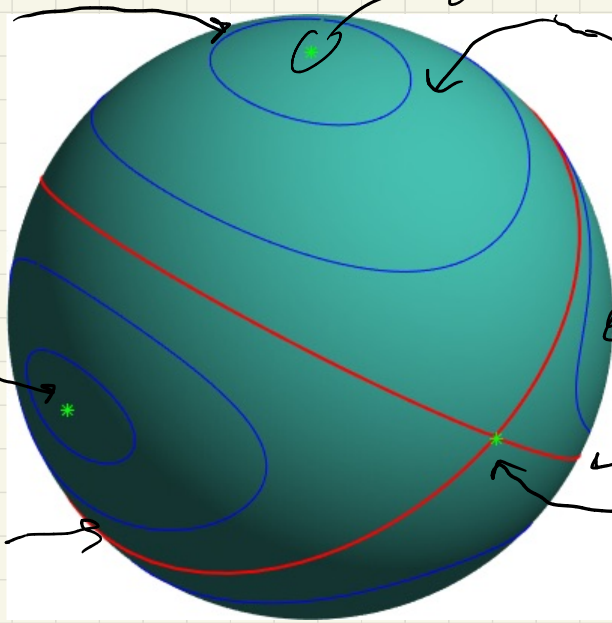
* $\omega_3 = \omega_0 \gg \omega_1, \omega_2$

$$\dot{\omega}_1 = \underbrace{\omega_0 \left(\frac{J_{22} - J_{33}}{J_{11}}\right)}_{\alpha_1 < 0} \omega_3, \quad \dot{\omega}_2 = \underbrace{\omega_0 \left(\frac{J_{22} - J_{11}}{J_{33}}\right)}_{\alpha_2 > 0} \omega_1$$

$$\Rightarrow \text{eig}(A) = \underbrace{\pm\sqrt{\alpha_1 \alpha_2}}_{\text{pure imaginary}} \Rightarrow \text{marginally stable}$$

- Spin along maximum/minimum axes (a.k.a major/minor axes) of inertia is Lyapunov stable.
- Spin about intermediate axis is always unstable.
- Geometry on Momentum Sphere:

$${}^B\dot{l} = {}^B l \times J^{-1}\, {}^B l$$



- Oscillatory motion about stable spin axis is called "nutation"

Numerical Simulation With 3D Rotations:

- Naive use of RK method on Q or q leads to numerical drift off of $SO(3)$ manifold (Q no longer orthogonal, q no longer unit norm)
- Standard Hacks:
  * Re-Normalize
    - Easy for q, expensive for Q (svd)
    - Only possible if you implement your own RK method
  * Baumgarte Stabilization
    - Apply proportional or PD control to constraint in the dynamics
    - Easy for q, harder for Q
    - Makes norm Lyapunov stable, but still oscillates about true value
    - Higher gain makes error small but makes ODE stiff.
    - Can apply with standard toolboxes like DifferentialEquations.jl

- Better Options:

  * Implicit Method with Explicit Constraint

    - Can combine implicit RK condition with constraint equation and simultaneously solve both with Newton.

    - Can be expensive

  * Runge-Kutta Munthe-Kaas Methods

    - Compute RK step on Lie algebra (axis-angle vectors) and use exp and multiplication to compute Q_{n+1} / q_{n+1}

    - Since group is closed under multiplication $\Rightarrow$ guaranteed to stay on group

    - Straight-forward to construct RK-MK method from any standard RK method

    - Can be explicit or implicit

+ Explicit Lie-Euler Method:

  - Standard Euler Method

$$\omega_{n+1} = \omega_n + h\dot{\omega}_n = h J^{-1}(\tau_n - \omega_n^{\times} J \omega_n)$$

$$Q_{n+1} = Q_n + h\dot{Q}_n = Q_n + hQ_n\hat{\omega}_n = Q_n\underbrace{(I + h\hat{\omega}_n)}_{\text{1st-order Taylor expansion of } \exp(h\hat{\omega}_n)}$$

- Key idea: replace 1st order Taylor expansion with exact exp function:

$$Q_{n+1} = Q_n \exp(h\hat{\omega}_n)$$

or

$$q_{n+1} = q_n * \exp\left(h\begin{bmatrix}0\\ \frac{1}{2}\omega_n\end{bmatrix}\right) = L(q_n)\,\mathrm{Exp}\left(\frac{h}{2}\begin{bmatrix}0\\ \omega_n\end{bmatrix}\right)$$

- Guaranteed to stay on Lie group

- Derivation gets messier for order ≥ 2, but same idea.