Last Time:

- Rigid Bodies
- Reference Frames
- Attitude Representations
- Rotation Matrices

Today:

- Linear Systems Review
- A little group theory
- Rotation Matrix Kinematics
- Quaternion Geometry

Linear Systems Review:

- Scalar Case:

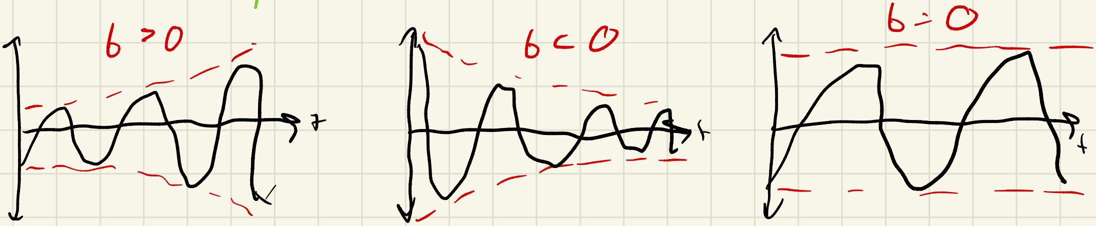
$$\dot{x} = ax \Rightarrow x(t) = e^{at} x_0 \quad , \quad a \in \mathbb{R}$$

$$a > 0 \Rightarrow e^{at} \to \infty \Rightarrow \text{unstable}$$
$$a < 0 \Rightarrow e^{at} \to 0 \Rightarrow \text{stable}$$

- Complex Scalar Case

$$\dot{x} = ax = (b + ic)\, x \Rightarrow x(t) = e^{(b+ci)t} x_0$$

$$= \underbrace{e^{bt}}_{\text{"envelope"}} \underbrace{[\cos(ct) + i\sin(ct)]}_{\text{"oscillation"}} x_0$$



$\Rightarrow$ real part determines stability
imaginary part determines frequency

(physically, simple harmonic oscillator)

- Some interesting 2×2 Matrices:

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow J^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

$\Rightarrow$ J and I act exactly like i and 1 (complex numbers)

$A = bI + cJ$ and $a = b + ci$ are "isomorphic"
"functionally identical"

Unsurprisingly, $\text{eig}\left(\begin{bmatrix} b & c \\ -c & b \end{bmatrix}\right) = b \pm ci$

$$\dot{x} = Ax \Rightarrow x(t) = e^{At}x_0$$

$$e^{At} = e^{bt}[\cos(ct)I + \sin(ct)J]$$

Matrix exponention

- General Case:

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n \Rightarrow x(t) = e^{At}x_0$$

formally define e^{At} by power series:

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 \ldots$$

- Assuming $A = VDV^{-1}$ (nice eigendecomposition)

$$e^{At} = I + VDV^{-1} + \frac{1}{2}VD[\cancel{V^{-1}V}]DV^{-1}t^2 \ldots$$

$$\overset{\shortparallel}{\frac{1}{2}VD^2V^{-1}t^2 \ldots}$$

$$= Ve^{Dt}V^{-1}$$

$\Rightarrow$ In eigenbasis system decouples into lots of simple harmonic oscillators

$\Rightarrow$ Eigenvalues tell us stability properties.

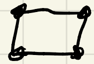
A Little Group Theory

- A group of elements with:

1) A multiplication operation
2) An identity element
3) An inverse

that is closed under multiplication (multiplying any 2 elements always gives another group element).

Examples:

- Positive reals under standard multiplication
- Integers under addition
- Complex numbers
- Invertible Matrices $GL(n)$
- Discrete symmetry groups e.g. C_4 
- Rotations in $\mathbb{R}^3$ $SO(3)$
- Rigid body transformations $SE(3)$ (translation + rotation)

- All useful groups have matrix representations.

- Formalizes what we saw with complex numbers and J/I matrices.

- Separate representation from function/structure

- Continuous groups (as opposed to discrete groups) are called Lie groups

- 3D rotations are called $SO(3)$

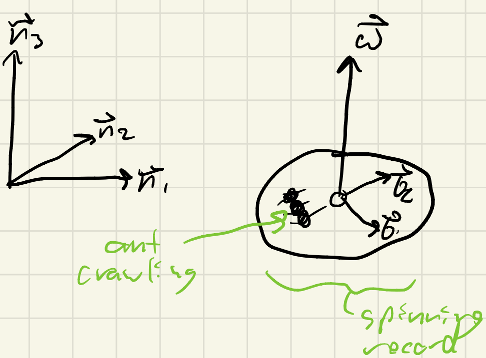
$$\underbrace{\text{Special}}_{\det(Q)=1}\ \underbrace{\text{Orthogonal}}_{Q^TQ=I}\ \text{group in}\ \underline{\text{3D}}$$

★ Rotation Matrix Kinematics

- How do I integrate $\omega(t)$ from a gyro?

$$\omega(t) \xrightarrow[?]{} \dot{Q}(t) \longrightarrow Q(t)$$

- Velocities in a rotating reference frame:



$${}^N\dot{x} = Q({}^B\dot{x} + {}^B\omega \times {}^Bx) \quad (1)$$

$$\underbrace{{}^B\dot{x} = Q^T\,{}^N\dot{x} - {}^B\omega \times {}^Bx}_{\text{"Kinematic transport theorem"}}$$

- Think about a vector fixed in body frame

$${}^Nx = Q\,{}^Bx \Rightarrow {}^N\dot{x} = \dot{Q}\,{}^Bx + \cancelto{0}{Q\,{}^B\dot{x}}$$

$$(1)\quad {}^N\dot{x} = Q(\omega \times {}^Bx) \Rightarrow \dot{Q}\,{}^Bx = Q(\omega \times {}^Bx)$$

$$\underbrace{\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}}_{\text{skew-symmetric}} \Rightarrow \underbrace{\hat{\omega} x = \omega \times x}_{\text{"hat map"}}$$

$$\hat{\omega}^T = -\hat{\omega}$$

$$\Rightarrow \dot{Q}\,{}^Bx = Q\,\hat{\omega}\,{}^Bx \Rightarrow \boxed{\dot{Q} = Q\hat{\omega}}$$

- Linear 1st order ODE

$$\dot{Q} = Q\hat{\omega} \Leftrightarrow \dot{x} = Ax$$

- For constant ω:

$$Q(t) = Q_0 \underbrace{e^{\hat{\omega}t}}_{\text{Matrix exponential}}$$

- For small Δt

$$e^{\hat{\omega}\Delta t} \approx I + \hat{\omega}\Delta t$$

$$\Rightarrow e^{\hat{\omega}\Delta t}x \approx x + \Delta t(\omega \times x)$$

  - $\omega\Delta t$ is an axis-angle vector

$$\phi \approx \omega\Delta t$$

- Exponential maps from axis-angle vectors to rotation matrices

- Useful for easily sampling random rotations

- You can also go the other way:

$$Q = e^{\hat{\phi}} \Leftrightarrow \hat{\phi} = \log(Q)$$

- A little more group theory:

  - Axis-angle vectors / skew-symmetric matrices are Lie <u>algebra</u> $\underbrace{so(3)}_{\text{lower case}}$ corresponding to $SO(3)$

- From Taylor series

$$e^{\hat{\phi}} \simeq I + \underbrace{\hat{\phi}}_{} + \ldots$$

Lie algebra is the linearization of the group at the identity

- While group is <u>not</u> a vector space, the Lie algebra <u>is</u>.

- The Lie algebra $\longleftrightarrow$ group connection lets us use standard vector math ideas and translate to the group.

* Quaternions:

  - Standard rotation representation used in simulation

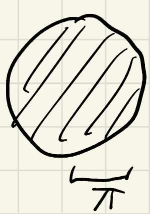
  - No singularities + more efficient than rotation matrices.

- Geometry:

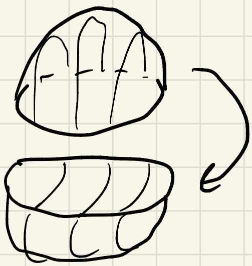
  - Set of all possible axis-angle vectors

$$\|\phi\| \leq \pi \quad (\text{ball in } \mathbb{R}^3)$$

- Visualize as disc in 2D:



- There is a discontinuous jump at $\pm\pi$
- We want to get rid of the jump
- Stretch disc up out of plane into hemisphere:



- Make a copy, flip it, and glue it on underneath to form a sphere
- Now, instead of jumping at $+/-\pi$, we can continue smoothly onto "southern hemisphere."