Contents

1 Extending Hu's quadratic estimator to three dimensions

Objective To understand eq. 22 to 31 in Zahn and Zaldarriaga 2006

1.1 Introduction

- Unlike CMB, we can use information from multiple planes in the case of 21 cm signal.
- But the different planes would be correlated. No simple way to take this correlation into account while constructing the estimator.
- Instead, we divide the 3D temperature fluctuations into radial and transverse parts.

1.2 Notation

We use the terminology and notation as described in Hogg 2000 arXiv:9905116.

Observed volume

- We define the observed volume as
  - located at redshift z.
  - having a width of solid angle $d\Omega$ along the transverse direction as seen by the observer
  - having the depth along the line of sight $\mathcal{L}$ = comoving distance between redshift z and z', where $z < z'$ (D_C in Hogg 2000).
- The distance between the observer and the volume is χ. It is called transverse comoving distance as per the definition in Hogg 2000 (D_M in this paper).
- The transverse comoving size of the volume is S.

Intensity field Field $I(\mathbf{x})$ at position $\mathbf{x}$ in comoving space

$$\begin{aligned} I(\mathbf{r}) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \tilde{\mathrm{I}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \int \frac{d^2\mathbf{l}}{(2\pi)^2} \int \frac{dk_\parallel}{2\pi} \frac{\tilde{\mathrm{I}}(k_\perp, k_\parallel)}{\chi^2} e^{i(\mathbf{l}\cdot\boldsymbol{\theta} + k_\parallel x_\parallel)} \end{aligned} \tag{1.1}$$

where $\mathbf{k}$ is the wave vector.

1.3 Formalism

We divide $\mathbf{k}$ into radial (parallel) and transverse components: $k_\parallel$ and $k_\perp$. Corresponding to this, we also have $x_\parallel$ and $x_\perp$.

Discretize $k_\perp$ Using

$$\theta = \frac{S}{\chi} \tag{1.2}$$

and

$$\mathbf{l} = \frac{2\pi}{\vec{\theta}} \tag{1.3}$$

we get

$$\begin{aligned} \mathbf{k}_\perp &= \frac{2\pi}{\mathbf{S}} \\ &= \frac{2\pi}{\vec{\theta}\chi} \\ &= \frac{\mathbf{l}}{\chi} \end{aligned} \tag{1.4}$$

Discretize $k_\parallel$ Discretizing $\mathcal{L}$ into many slices along the line of sight. j is the discretizing factor.

$$k_\parallel = j\frac{2\pi}{\mathcal{L}}; \delta(k_\parallel - k'_\parallel) = \frac{\mathcal{L}}{2\pi}\delta_{j_1,j_2} \tag{1.5}$$

where j represents the j^{th} element of the volume.

Final expression of Intensity with discrete k

$$I(\mathbf{r}) = \int \frac{d^2\mathbf{l}}{(2\pi)^2} \sum_j \left(\frac{\tilde{\mathrm{I}}(k_\perp, k_\parallel)}{\chi^2 \mathcal{L}} \right) e^{i\mathbf{k}\cdot\mathbf{r}} \tag{1.6}$$

Simplify Define

$$\hat{I}(\mathbf{l}, k_\parallel) \equiv \frac{\tilde{\mathrm{I}}(\mathbf{k}_\perp, k_\parallel)}{\chi^2 \mathcal{L}} \tag{1.7}$$

Starting with

$$\tilde{\mathrm{I}}(\mathbf{k}_\perp, \kappa_\parallel; \tau) = \int dx_\parallel \int d^2\vec{x}_\perp e^{-i\mathbf{k}_\parallel \cdot x_\parallel} e^{-i\mathbf{k}_\perp \cdot x_\perp} I(\vec{x}_\perp, x_\parallel; \tau) \tag{1.8}$$

$$\begin{aligned} \langle I(\mathbf{k}_\perp, k_\parallel) I^*(\mathbf{k}'_\perp, k'_\parallel) \rangle &= \delta(\mathbf{k} - \mathbf{k}')(2\pi)^3 P(\mathbf{k}_\perp, k_\parallel) \\ &= (2\pi)^2 \delta(\mathbf{l} - \mathbf{l}') \chi^2 (2\pi) \delta(k_\parallel - k'_\parallel) P(\mathbf{k}_\perp, k_\parallel) \end{aligned} \tag{1.9}$$

where $P(k_\perp, k_\parallel)$ is the 3D power spectrum of the intensity field, we get

$$\frac{\langle I(\mathbf{k}_\perp, k_\parallel) I^*(\mathbf{k}'_\perp, k'_\parallel) \rangle}{(\mathcal{L}\chi^2)^2} = \frac{(2\pi)^2 \delta(l - l') \chi^2 (2\pi) \delta(k_\parallel - k'_\parallel) P(\mathbf{k}_\perp, k_\parallel)}{(\mathcal{L}\chi^2)^2} \tag{1.10}$$

$$\langle \hat{I}(l, k_\parallel) \hat{I}^*(l', k'_\parallel) \rangle = \frac{(2\pi)^2 \delta(\mathbf{l} - \mathbf{l}') (2\pi) \delta(k_\parallel - k'_\parallel) P(\mathbf{k}_\perp, k_\parallel)}{(\mathcal{L}^2 \chi^2)} \tag{1.11}$$

$$= \frac{(2\pi)^2 \delta(l - l') (2\pi) \delta(j - j') (\mathcal{L}/2\pi) P(\mathbf{k}_\perp, k_\parallel)}{(\mathcal{L}^2 \chi^2)} \tag{1.12}$$

$$\langle \hat{I}(\mathbf{l}, j\frac{2\pi}{\mathcal{L}}) \hat{I}^*(\mathbf{l}', j'\frac{2\pi}{\mathcal{L}}) \rangle = (2\pi)^2 \delta(\mathbf{l} - \mathbf{l}') \delta(j - j') \frac{P(\mathbf{k}_\perp, j\frac{2\pi}{\mathcal{L}})}{(\mathcal{L}\chi^2)} \tag{1.13}$$

This is equation 27 in ZZ2006.

Definition of $C_{\mathbf{l},j}$ We now define the angular power spectrum for the seperate values of j,

$$C_{\mathbf{l},j} \equiv \frac{P\left(\sqrt{\frac{\mathbf{l}^2}{\chi^2} + (j\frac{2\pi}{\mathcal{L}})^2}\right)}{\mathcal{L}\chi^2} \tag{1.14}$$

where P now represents the spherically averaged power spectrum.

Now we are going to derive Equation 31 (Equation A21) of ZZ2006.

2 Lensing reconstruction noise

2.1 Mathematical form of a quadratic estimator

Let's start with a 3D intensity source field $I_s(\theta, \chi)$, which is the unlensed field of the matter present at a transverse comoving distance χ away from the observer.

Due to the matter present between the source and observer, the signal experiences weak lensing and in this case the signal can be written as

$$I_o(\theta, \chi) = I_s(\theta + \delta\theta, \chi) \tag{2.1}$$

where I_o is the observed lensed field. This equation says that the field that we observe at coordinates (θ, χ) is actually coming from $(\theta + \delta\theta, \chi)$, where $\delta\theta = \bigtriangledown\hat{\Psi}$ and $\hat{\Psi}$ is the *projected potential.*

Taylor expansion of RHS of previous equation gives

$$I_o(\theta, \chi) = I_s(\theta, \chi) + \delta\theta \cdot \vec{\bigtriangledown}_\theta I_s(\theta, \chi) + \dots \tag{2.2}$$

The Fourier transform of this expression is

$$\begin{aligned} \int d^2\theta\, e^{-i\mathbf{l}\cdot\theta}\, I_o(\theta, \chi) &= \int d^2\theta I_s(\theta, \chi) e^{-i\mathbf{l}\cdot\theta} + \int d^2\theta\, e^{-i\mathbf{l}\cdot\theta} \delta\theta \cdot \vec{\bigtriangledown}_\theta I_s(\theta, \chi) \\ \tilde{\mathrm{I}}_o(\mathbf{l}, \chi) &= \tilde{\mathrm{I}}_s(\mathbf{l}, \chi) + \underbrace{\int d^2\theta\, e^{-i\mathbf{l}\cdot\theta}\, \delta\theta \cdot \vec{\bigtriangledown}_\theta I_s(\theta, \chi)}_{A} \qquad (2.3) \end{aligned}$$

Using $\delta\theta(\vec{\chi}) = \bigtriangledown\hat{\Psi}(\vec{\chi})$

$$\begin{aligned} A &= \int d^2\theta\, e^{-i\mathbf{l}\cdot\theta}\, \vec{\bigtriangledown}_\theta \Psi \cdot \vec{\bigtriangledown}_\theta I_s(\theta, \chi) & (2.4) \\ &= \int d^2\theta\, e^{-i\mathbf{l}\cdot\theta}\, \frac{1}{(2\pi)^4} \vec{\bigtriangledown}_\theta \int d^2l' e^{i\mathbf{l}'\cdot\theta} \tilde{\Psi}(\mathbf{l}', \chi) \cdot \vec{\bigtriangledown}_\theta \int d^2l'' e^{i\mathbf{l}''\cdot\theta} \tilde{\mathrm{I}}_s(\mathbf{l}'', \chi) & (2.5) \\ &= \int d^2\theta\, e^{-i\mathbf{l}\cdot\theta}\, \frac{1}{(2\pi)^4} \int d^2l' \int d^2l'' \tilde{\Psi}(\mathbf{l}', \chi) \tilde{\mathrm{I}}_s(\mathbf{l}'', \chi) \vec{\bigtriangledown}_\theta e^{i\mathbf{l}'\cdot\theta} \cdot \vec{\bigtriangledown}_\theta e^{i\mathbf{l}''\cdot\theta} & (2.6) \\ &= -\int d^2\theta\, e^{-i\mathbf{l}\cdot\theta}\, \frac{1}{(2\pi)^4} \int d^2l' \int d^2l'' \tilde{\Psi}(\mathbf{l}', \chi) \tilde{\mathrm{I}}_s(\mathbf{l}'', \chi) e^{i(\mathbf{l}'+\mathbf{l}'')\cdot\theta} \mathbf{l}' \cdot \mathbf{l}'' & (2.7) \\ &= -\int d^2\theta\, \frac{1}{(2\pi)^4} \int d^2l' \int d^2l'' e^{i(-\mathbf{l}+\mathbf{l}'+\mathbf{l}'')\cdot\theta} \mathbf{l}' \cdot \mathbf{l}''\ ](\mathbf{l}', \chi) \tilde{\mathrm{I}}_s(\mathbf{l}'', \chi) & (2.8) \end{aligned}$$

Integrating over θ

$$A = -\int d^2l' \frac{1}{(2\pi)^2} \int d^2l'' \delta(-\mathbf{l} + \mathbf{l}' + \mathbf{l}'')\mathbf{l}' \cdot \mathbf{l}'' \, \tilde{\Psi}(\mathbf{l}', \chi)\tilde{\mathrm{I}}_s(\mathbf{l}'', \chi) \tag{2.9}$$

Integrating over $\mathbf{l}'$

$$A = -\frac{1}{(2\pi)^2} \int d^2l'' \, \mathbf{l}'' \cdot (\mathbf{l} - \mathbf{l}'') \, \tilde{\Psi}(\mathbf{l} - \mathbf{l}'', \chi) I_s(\mathbf{l}'', \chi) \tag{2.10}$$

Therefore, we get

$$\tilde{\mathrm{I}}_o(\mathbf{l}, k) = \tilde{\mathrm{I}}_s(\mathbf{l}, k) - \int d^2l' \frac{1}{(2\pi)^2} \tilde{\mathrm{I}}_s(\mathbf{l}', k)\tilde{\Psi}(\mathbf{l} - \mathbf{l}')(\mathbf{l} - \mathbf{l}') \cdot \mathbf{l}' \tag{2.11}$$

Computing the correlation function (and dropping k)

$$\begin{aligned} \langle \tilde{I}_o(\mathbf{l})\tilde{I}_o{}^*(\mathbf{m})\rangle_{\mathbf{m}\neq\mathbf{l}} = &-\int d^2l' \frac{1}{(2\pi)^2} \tilde{\Psi}(\mathbf{l} - \mathbf{l}')(\mathbf{l} - \mathbf{l}') \cdot \mathbf{l}' \langle \tilde{\mathrm{I}}_s(\mathbf{l}')\tilde{\mathrm{I}}_s^*(\mathbf{m})\rangle \\ &- \int d^2l' \frac{1}{(2\pi)^2} \tilde{\Psi}^*(\mathbf{m} - \mathbf{l}')(\mathbf{m} - \mathbf{l}') \cdot \mathbf{l}' \langle \tilde{\mathrm{I}}_s^*(\mathbf{l}')\tilde{\mathrm{I}}_s(\mathbf{m})\rangle \\ &= -\tilde{\Psi}(\mathbf{l} - \mathbf{m})(\mathbf{l} - \mathbf{m}) \cdot \mathbf{m} P_m \\ &- \tilde{\Psi}^*(\mathbf{m} - \mathbf{l})(\mathbf{m} - \mathbf{l}) \cdot \mathbf{l} P_l \end{aligned} \tag{2.12}$$

Now, assuming that $\mathbf{m} = \mathbf{l} - \mathbf{L}$ (and restoring k)

$$\langle \tilde{I}_o(\mathbf{l}, k)\tilde{I}_o^*(\mathbf{l} - \mathbf{L}, k)\rangle_{\mathbf{L}=\mathbf{l}-\mathbf{m}} = -\tilde{\Psi}(\mathbf{L}, k)\mathbf{L} \cdot (\mathbf{l} - \mathbf{L})P_{l-L,k} + \tilde{\Psi}^*(-\mathbf{L}, k)\mathbf{L} \cdot \mathbf{l} P_{l,k} \tag{2.13}$$

$$= -\tilde{\Psi}(\mathbf{L}, k)\mathbf{L} \cdot [(\mathbf{l} - \mathbf{L})P_{l-L,k} + \cdot \mathbf{l} P_{l,k}] \tag{2.14}$$

Therefore we can define the quadratic estimator as

$$\hat{\tilde{\Psi}}(\mathbf{L}) \propto \tilde{\mathrm{I}}_o(\mathbf{l})\tilde{\mathrm{I}}_o(\mathbf{l} - \mathbf{L}) \tag{2.15}$$

2.2 Deriving the estimator

We therefore start with a quadratic estimator $\hat{\Psi}(\mathbf{L})$ for $\tilde{\psi}(\mathbf{L})$, i.e. of the form

$$\hat{\Psi}(\mathbf{L}) = \int \frac{d^2l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} F(\mathbf{l}, k_1, k_2, \mathbf{L})\tilde{\mathrm{I}}_o(\mathbf{l}, k_1)\tilde{\mathrm{I}}(\mathbf{L} - \mathbf{l}, k_2) \tag{2.16}$$

(notice that $F(\mathbf{l}, k_1, k_2, \mathbf{L}) = F(\mathbf{l}, k_2, k_1, \mathbf{L})$). Because $\delta\hat{\Psi}(\mathbf{L}) = \delta\hat{\Psi}^*(-\mathbf{L})$ it can also be shown that

$$F(\mathbf{l}, k_1, k_2, \mathbf{L}) = F^*(-\mathbf{l}, -k_1, -k_2, -\mathbf{L}) \tag{2.17}$$

We want to find F such that it minimizes the variance of $\tilde{\Psi}(\mathbf{L})$ under the condition that its ensemble average recovers the lensing field, $\langle\hat{\Psi}(\mathbf{L})\rangle_{\mathrm{I}} = \tilde{\psi}(\mathbf{L})$. This becomes (to first order in $\tilde{\psi}$)

$$\begin{aligned} \langle\hat{\Psi}(\mathbf{L})\rangle_{\mathrm{I}} = &\int \frac{d^2l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} F(\mathbf{l}, k_1, k_2, \mathbf{L})(2\pi)\delta(k_1 + k_2) \\ &\times \left[C_{l,k}\tilde{\psi}(\mathbf{L})\mathbf{L} \cdot \mathbf{l} + C_{L-l,k}\tilde{\psi}(\mathbf{L})\mathbf{L} \cdot (\mathbf{L} - \mathbf{l})\right], \end{aligned} \tag{2.18}$$

where e.g. $C_{l,k}$ is the power in a mode with angular component l and radial component k. With the requirement that $\langle \hat{\Psi}(\mathbf{L})\rangle_{\mathrm{I}} = \tilde{\psi}(\mathbf{L})$ this leads to the normalization condition

$$\int \frac{d^2l}{(2\pi)^2}\int \frac{dk_1}{2\pi}\int \frac{dk_2}{2\pi} F(\mathbf{l},k_1,k_2,\mathbf{L})(2\pi)\delta^D(k_1+k_2)\left[C_{l,k_1}\mathbf{L}\cdot\mathbf{l} + C_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})\right] = 1 \tag{2.19}$$

The condition of minimization of the variance gives

$$\begin{aligned}\langle ||\hat{\Psi}(\mathbf{L})||^2\rangle_{\tilde{\mathrm{I}}} = &\int \frac{d^2l}{(2\pi)^2}\int \frac{dk_1}{2\pi}\int \frac{dk_2}{2\pi}(2\pi)^2\delta(0)F(\mathbf{l},k_1,k_2,\mathbf{L})F^*(\mathbf{l}',k_1',k_2',\mathbf{L}')C^{tot}_{l,k_1}C^{tot}_{L-l,k_2}\\ &+\int \frac{d^2l}{(2\pi)^2}\int \frac{dk_1}{2\pi}\int \frac{dk_2}{2\pi}(2\pi)^2\delta(0)F(\mathbf{l},k_1,k_2,\mathbf{L})F^*(\mathbf{L}-\mathbf{l}',k_2',k_1',\mathbf{L}')C^{tot}_{l,k_1}C^{tot}_{L-l,k_2}\end{aligned} \tag{2.20}$$

but from 2.16 we see with the substitution $\mathbf{L}-\mathbf{l}\rightarrow\mathbf{l}$ that $F^*(\mathbf{L}-\mathbf{l},k_2,k_1,\mathbf{L}) = F^*(\mathbf{l},k_1,k_2,\mathbf{L})$ hence

$$\langle ||\hat{\Psi}(\mathbf{L})||^2\rangle = 2(2\pi)^2\delta(0)\int \frac{d^2l}{(2\pi)^2}\int \frac{dk_1}{2\pi}\int \frac{dk_2}{2\pi}F(\mathbf{l},k_1,k_2,\mathbf{L})F^*(\mathbf{l},k_1,k_2,\mathbf{L})C^{tot}_{l,k_1}C^{tot}_{l,k_2}\,. \tag{2.21}$$

Both real and imaginary part of $||F||^2 = F_R^2 + F_I^2$ contribute to this variance, however the condition for the minimization will only pick out the real part. (**Does this really matter?**) The solution is found by minimizing the function

$$\langle ||\hat{\Psi}(\mathbf{L})||^2\rangle - A_R\times(\,\text{Equation }2.19) \tag{2.22}$$

with respect to F, where A_R is a Lagrangian multiplier. In steps,

$$\frac{\partial(\,\text{Eq. }2.19)}{\partial F(\mathbf{l},k_1,k_2,\mathbf{L})} = A_R\frac{d^2l}{(2\pi)^2}\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}(2\pi)\delta^D(k_1+k_2)[C_{l,k_1}\mathbf{L}\cdot\mathbf{l}+C_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})] \tag{2.23}$$

and

$$\frac{\partial\langle ||\hat{\Psi}(\mathbf{L})||^2\rangle}{\partial F(\mathbf{l},k_1,k_2,\mathbf{L})} = 2(2\pi)^2\delta(0)\frac{d^2l}{(2\pi)^2}\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}2F_R(\mathbf{l},k_1,k_2,\mathbf{L})C^{tot}_{l,k_1}C^{tot}_{L-l,k_2} \tag{2.24}$$

so

$$4(2\pi)^2\delta(0)F_R(\mathbf{l},k_1,k_2,\mathbf{L}) = A_R(2\pi)\delta^D(k_1+k_2)\frac{[C_{l,k_1}\mathbf{L}\cdot\mathbf{l}+C_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]}{C^{tot}_{l,k_1}C^{tot}_{L-l,k_2}} \tag{2.25}$$

Therefore,

$$F_R(\mathbf{l},k_1,k_2,\mathbf{L}) = \frac{1}{4(2\pi)\delta(0)}A_R\delta^D(k_1+k_2)\frac{[C_{l,k_1}\mathbf{L}\cdot\mathbf{l}+C_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]}{C^{tot}_{l,k_1}C^{tot}_{L-l,k_2}} \tag{2.26}$$

and by inserting this into the normalization condition 2.19 we get that

$$\begin{aligned}1 &= \int \frac{d^2l}{(2\pi)^2}\int \frac{dk_1}{2\pi}\int \frac{dk_2}{2\pi}\frac{1}{4\delta(0)}A_R\delta^D(k_1+k_2)\frac{[C_{l,k_1}\mathbf{L}\cdot\mathbf{l}+C_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{C^{tot}_{l,k_1}C^{tot}_{L-l,k_2}}\delta^D(k_1+k_2)\\ &= \int \frac{d^2l}{(2\pi)^2}\int \frac{dk_1}{2\pi}\frac{\delta(0)}{4\delta(0)}A_R\frac{[C_{l,k_1}\mathbf{L}\cdot\mathbf{l}+C_{L-l,k_2=-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{C^{tot}_{l,k_1}C^{tot}_{L-l,k_2=-k_1}}\end{aligned} \tag{2.27}$$

Assuming that I can just cancel the factors of $\delta(0)$,

$$1 = \int \frac{d^2l}{(2\pi)^2} \int dk_1 \frac{1}{4 \times 2\pi} A_R \frac{[C_{l,k_1} \mathbf{L} \cdot \mathbf{l} + C_{L-l,-k_1} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]^2}{C^{tot}_{l,k_1} C^{tot}_{L-l,-k_1}} \tag{2.28}$$

Discretizing the integal over dk_1. Assumming that we have dicretized the space in the radial direction in the blocks of length $\mathcal{L}$ such that $k = \frac{2\pi}{\mathcal{L}}$.

Therefore, we can write

$$\int \frac{dk}{2\pi} = \frac{1}{\mathcal{L}} \sum_k \tag{2.29}$$

Therefore, we get

$$1 = \int \frac{d^2l}{(2\pi)^2} \frac{1}{\mathcal{L}} \sum_{k_1} \frac{1}{4} A_R \frac{[C_{l,k_1} \mathbf{L} \cdot \mathbf{l} + C_{L-l,-k_1} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]^2}{C^{tot}_{l,k_1} C^{tot}_{L-l,-k_1}} \tag{2.30}$$

which gives,

$$A_R = \frac{4\mathcal{L}}{\int \frac{d^2l}{(2\pi)^2} \sum_{k_1} \frac{[C_{l,k_1} \mathbf{L} \cdot \mathbf{l} + C_{L-l,-k_1} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]^2}{C^{tot}_{l,k_1} C^{tot}_{L-l,-k_1}}} \tag{2.31}$$

Substituting this expression in Equation 2.26, we get

$$\begin{aligned}
F_R(\mathbf{l}, k_1, k_2, \mathbf{L}) &= \frac{A_R}{4(2\pi)\delta(0)} \delta^D(k_1 + k_2) \frac{[C_{l,k_1} \mathbf{L} \cdot \mathbf{l} + C_{L-l,k_2} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]}{C^{tot}_{l,k_1} C^{tot}_{L-l,k_2}} \\
&= \frac{\mathcal{L}/2\pi\delta(0)}{\int \frac{d^2l}{(2\pi)^2} \sum_{k_1} \frac{[C_{l,k_1} \mathbf{L} \cdot \mathbf{l} + C_{L-l,-k_1} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]^2}{C^{tot}_{l,k_1} C^{tot}_{L-l,-k_1}}} \delta^D(k_1 + k_2) \frac{[C_{l,k_1} \mathbf{L} \cdot \mathbf{l} + C_{L-l,k_2} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]}{C^{tot}_{l,k_1} C^{tot}_{L-l,k_2}} \\
&= \frac{1}{\int \frac{d^2l}{(2\pi)^2} \sum_{k_1} \frac{[C_{l,k_1} \mathbf{L} \cdot \mathbf{l} + C_{L-l,-k_1} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]^2}{C^{tot}_{l,k_1} C^{tot}_{L-l,-k_1}}} \delta^D(k_1 + k_2) \frac{[C_{l,k_1} \mathbf{L} \cdot \mathbf{l} + C_{L-l,k_2} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]}{C^{tot}_{l,k_1} C^{tot}_{L-l,k_2}}
\end{aligned} \tag{2.32}$$

Now, substituting this expression in Equation 2.21. Taking $FF^* \equiv F^2$

$$
\begin{aligned}
\langle ||\hat{\Psi}(\mathbf{L})||^2\rangle &= 2(2\pi)^2\delta(0)\int\frac{d^2l}{(2\pi)^2}\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}F(\mathbf{l},k_1,k_2,\mathbf{L})F^*(\mathbf{l},k_1,k_2,\mathbf{L})C^{tot}_{l,k_1}C^{tot}_{l,k_2}\\
&= 2(2\pi)^2\delta(0)\int\frac{d^2l}{(2\pi)^2}\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}F^2(\mathbf{l},k_1,k_2,\mathbf{L})C^{tot}_{l,k_1}C^{tot}_{l,k_2}\\
&= 2(2\pi)^2\delta(0)\int\frac{d^2l}{(2\pi)^2}\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}\left(\frac{1}{\int\frac{d^2l}{(2\pi)^2}\sum_{k_1}\frac{[C_{l,k_1}\mathbf{L}\cdot\mathbf{l}+C_{L-l,-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{C^{tot}_{l,k_1}C^{tot}_{L-l,-k_1}}}\right.\\
&\left.\delta^D(k_1+k_2)\frac{[C_{l,k_1}\mathbf{L}\cdot\mathbf{l}+C_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]}{C^{tot}_{l,k_1}C^{tot}_{L-l,k_2}}\right)^2C^{tot}_{l,k_1}C^{tot}_{l,k_2}\\
&= 2(2\pi)^2\delta(0)\int\frac{d^2l}{(2\pi)^2}\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}\left(\frac{1}{\int\frac{d^2l}{(2\pi)^2}\sum_{k_1}\frac{[C_{l,k_1}\mathbf{L}\cdot\mathbf{l}+C_{L-l,-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{C^{tot}_{l,k_1}C^{tot}_{L-l,-k_1}}}\right)^2\\
&\left(\delta^D(k_1+k_2)\frac{[C_{l,k_1}\mathbf{L}\cdot\mathbf{l}+C_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]}{C^{tot}_{l,k_1}C^{tot}_{L-l,k_2}}\right)^2C^{tot}_{l,k_1}C^{tot}_{l,k_2}\\
&= 2(2\pi)^2\delta(0)\int\frac{d^2l}{(2\pi)^2}\frac{\sum_{k_1}}{\mathcal{L}}\frac{\sum_{k_2}}{\mathcal{L}}\left(\frac{1}{\int\frac{d^2l}{(2\pi)^2}\sum_{k_1}\frac{[C_{l,k_1}\mathbf{L}\cdot\mathbf{l}+C_{L-l,-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{C^{tot}_{l,k_1}C^{tot}_{L-l,-k_1}}}\right)^2\\
&\left(\delta^D(k_1+k_2)\frac{[C_{l,k_1}\mathbf{L}\cdot\mathbf{l}+C_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]}{C^{tot}_{l,k_1}C^{tot}_{L-l,k_2}}\right)^2C^{tot}_{l,k_1}C^{tot}_{l,k_2}\\
&= 2(2\pi)^2\delta(0)\frac{1}{\int\frac{d^2l}{(2\pi)^2}\sum_{k_1}\frac{[C_{l,k_1}\mathbf{L}\cdot\mathbf{l}+C_{L-l,-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{C^{tot}_{l,k_1}C^{tot}_{L-l,-k_1}}}
\end{aligned} \tag{2.33}
$$

With the definition

$$\langle\hat{\Psi}(\mathbf{L})\hat{\Psi}^*(L')\rangle = (2\pi)^2\delta^D(\mathbf{L}-\mathbf{L}')N^{\hat{\Psi}} \tag{2.34}$$

Therefore, we finally have

$$
\begin{aligned}
\langle ||\hat{\Psi}(\mathbf{L})||^2\rangle &= \frac{2(2\pi)\mathcal{L}}{\int\frac{d^2l}{(2\pi)^2}\sum_{k_1}\frac{[C_{l,k_1}\mathbf{L}\cdot\mathbf{l}+C_{L-l,-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{C^{tot}_{l,k_1}C^{tot}_{L-l,-k_1}}}\\
&= (2\pi)^2\delta(0)N^{\hat{\Psi}}
\end{aligned} \tag{2.35}
$$

Which gives

$$N^{\hat{\Psi}}_L = \frac{1}{\int\frac{d^2l}{(2\pi)^2}\sum_{k_1}\frac{[C_{l,k_1}\mathbf{L}\cdot\mathbf{l}+C_{L-l,-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{2C^{tot}_{l,k_1}C^{tot}_{L-l,-k_1}}} \tag{2.36}$$

This is the lensing reconstruction noise that we will experience while measuring the lensing potential, where

$$C_{l,j} = (1+\mu_k^2)^2\frac{P(\sqrt{(l/\chi)^2+(2\pi j/\mathcal{L})^2})}{\chi^2\mathcal{L}} \tag{2.37}$$

and P is the 21cm power spectrum corresponding to the source redshift.

3 Pourtsidou et al. 2014

3.1 Cosmology and parameters

3.1.1 Cosmology

Parameters taken from Table 2 (68% limits) of Planck 2013.

$$
\begin{aligned}
h &= 0.673 \\
\Omega_{m_0} &= 0.315 \\
\Omega_b &= 0.02205/h^2 \\
\Omega_c &= 0.1199/h^2 \\
\Omega_l &= 0.685 \\
As &= 2.196 \times 10^{-9} \\
n_s &= 0.9603 \\
Y_{He} &= 0.24770 \\
\sigma_8 &= 0.829
\end{aligned}
$$

3.1.2 Experimental parameters

Parameters of SKA-like experiment used in P2014.

$$
\begin{aligned}
T_{sys} &= 50\ K \\
t_{obs} &= 2\ years \\
Bandwidth &= 40\ MHz \\
l_{max} &= 19900 \\
f_{cover} &= 0.06 \\
\Delta_L &= 36 \\
f_{sky} &= 0.2
\end{aligned}
$$

Other parameters To calculate $\bar{T}(z)$. Refer to Eq. 1 and Eq. 17-19 of P2014.

$$
\begin{aligned}
\alpha &= -1.3 \\
\phi^\star &= 0.0204 h^3\ Mpc^{-3} \\
M^\star &= \frac{3.47 \times 10^9}{h^2} M_\odot \\
\rho_c &= 2.7755 \times 10^{11} h^2 M_\odot Mpc^{-3} \\
\rho_{HI} &= \phi^\star M^\star \Gamma(\alpha + 2) \\
\Omega_{HI}(z) &= \rho_{HI}/\rho_c = 0.0004919
\end{aligned}
$$

The parameter Ω_{HI} is a function of z but I can't see how. Are $M^\star$ and $\phi^\star$ dependent on z? ρ_c is the present day critical density so this is not bringing in the z dependence. If we

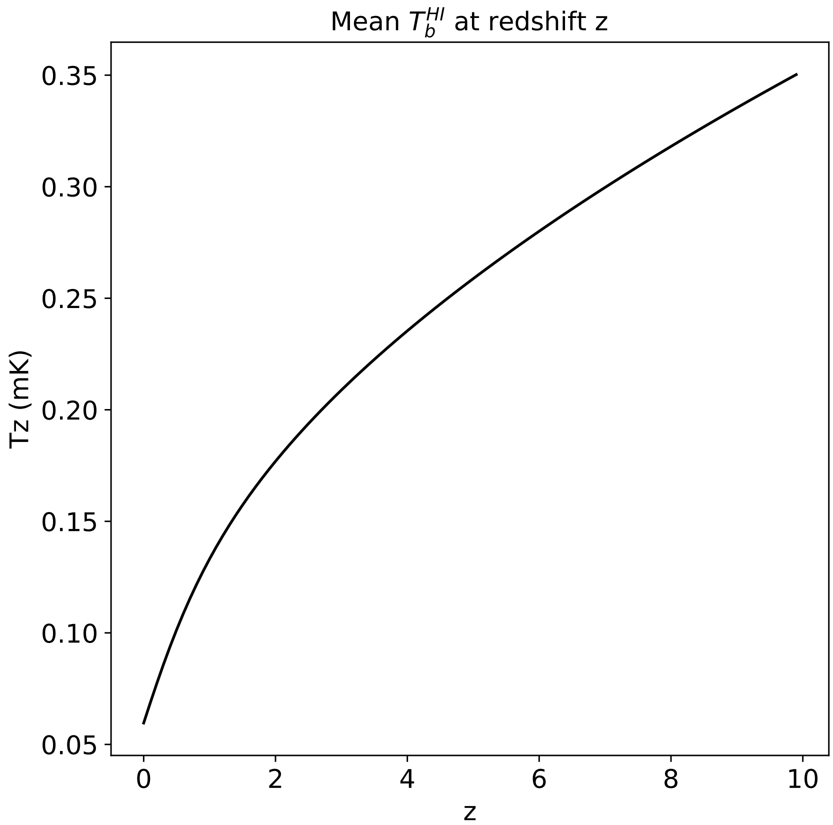


Figure 1: Variation of HI
brightness temperature with redshift.

look at the units of Ω_{HI}, we get.

$$\Omega_{HI}(z) = \frac{\rho_{HI}}{\rho_c} \tag{3.1}$$

$$\propto \frac{\phi^\star M^\star \Gamma(\alpha+2)}{\rho_c} \tag{3.2}$$

$$\propto \frac{h^3 Mpc^{-3} \times h^{-2} M_\odot}{h^2 M_\odot Mpc^{-3}} \tag{3.3}$$

$$\propto \frac{1}{h}, \tag{3.4}$$

Do we expect this h dependence of Ω_{HI}?

3.2 T_{HI}

The mean HI intensity temperature at redshift z (Eq. 1 of P2014) is written as

$$\bar{T}(z) = 180\Omega_{HI}(z)h\frac{(1+z)^2}{E(z)}\delta \text{ mK} \tag{3.5}$$

where $h = H_0/(100kms^{-1}Mpc^{-1})$, Ω_{HI} is the average HI density at redshift z realtive to the present day critical density. Fig. 1 shows the variation of $\bar{T}(z)$ with z. Here and in

Eq. 20 of P2014 I have taken

$$
\begin{aligned}
E(z) &= \frac{H(z)}{H_0} && (3.6)\\
&= \sqrt{\Omega_{m_0}(1+z)^3 + \Omega_l}. && (3.7)
\end{aligned}
$$

3.3 Matter power spectrum

I have used the following code to obtain the matter power spectrum using Python CAMB.

```
pars = model.CAMBparams(NonLinear = 0, WantTransfer = True, H0 = 100 * h, omch2 = omegach2, ombh2 = omegabh2, YHe = YHe)
pars.DarkEnergy.set_params(w = -1)
pars.set_for_lmax(lmax = l_ul)
pars.InitPower.set_params(ns = ns, As = As)
#results = camb.get_background(pars)
results = camb.get_results(pars)
k = np.linspace(10**(-5), k_max(l_ul, z_s) ,1000)
PK = get_matter_power_interpolator(pars, nonlinear=False, kmax = np.max(k), k_hunit = False, hubble_units = False)
```

Some of the important parameters are:

- `NonLinear = 0`: Out of the four options available in CAMB for choosing the type of power spectrum, `0` gives linear power spectrum with $\sigma_8 \sim 0.8$ (so the normalization is correct), `1` gives “Non-linear Matter Power (HALOFIT)” and the other two have CMB in their name so we don’t care. The `nonlinear = False` also specifies that we wanr to generate a linear power spectrum. Before the thesis submission we were using `NonLinear = 1` and `nonlinear = True`, and that’s why our convergence angular power spectrum was not matching with that of P2014.

- `k_unit = False` and `hubble_units = False`: Read `k` in Mpc units and output the power spectrum in Mpc units. `True` will enable Mpc/h units.

- `WantTransfer = True`: I need to find out what this option does, and if I can drop it. I am using this flag since the beginning.

- `results = camb.get_background(pars)` and `results = camb.get_results(pars)`: I am not sure what’s the difference between the two; both work. I was using `get_background` since the beginning because it is used in this CAMB demo. I switched to `get_results` because it can be used to find σ_8 by doing `results.get_sigma8()` after we have defined what `results` is as shown in the code.

- `l_ul`: It is the upper limit in $\int d^2l$ integrals in the lensing noise reconsruction expression (Eq. 14 of P2014). It is different from l_{max} defined in P2014 as the highest multipole visible to the instrument ($l_{max} = 19900$ in P2014). The integrals, I think, should ideally go upto $l = \infty$, so I usually take `l_ul = 70000` or higher. We need to take such large number because we have 21-cm power spectrum terms in the noise expression. I think that if we take a smaller value for `set_for_lmax` then the power spectrum would not be correct for large `l` values of the integral.

Figure 2 shows the plot of matter power spectrum $P_\delta(k)$ as a function of k.

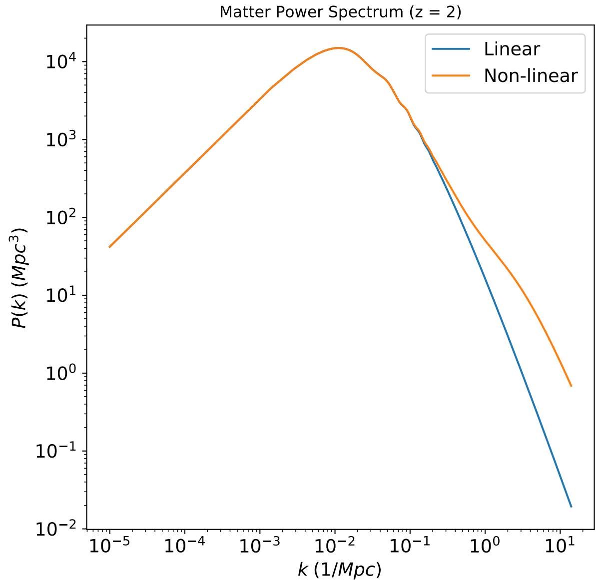


Figure 2: A comparison of the linear and non-linear power spectrum. The linear power spectrum has been used in P2014.

3.4 Power spectrum of 21-cm line

Defining the 21-cm power spectrum using Eq. 2 of P2014 as

$$P_{\Delta T_b}(k) = [\bar{T}(z)]^2(1 + f\mu_k^2)P_\delta(k), \tag{3.8}$$

where $P_\delta(k)$ is the underlying (dark) matter power spectrum, $f = \frac{d\, ln\, D}{ln\ a}$, D is the linear growth rate and μ_k is the cosine of the angle between wavevector $\mathbf{k}$ and the line of sight $\hat{z}$, and a is the scale factor $(1+z)^{-1}$.

I have ignored $(1 + f\mu_k^2)$ in my calculations. The resulting 21-cm power spectrum is shown in Figure 3a

3.5 Signam-to-noise ratio

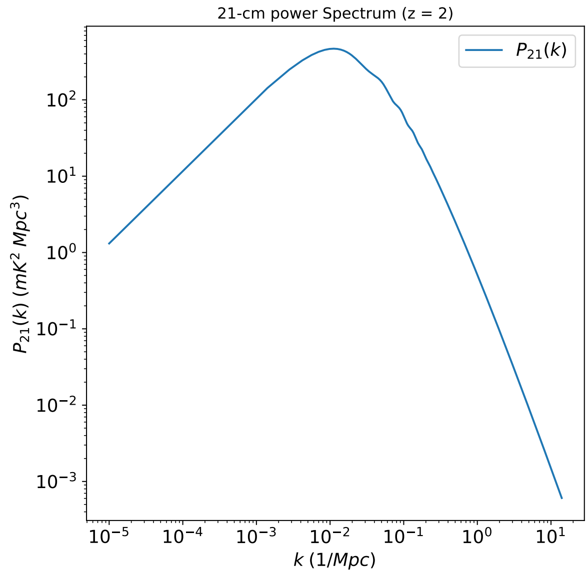
3.5.1 Signal

Prior to the thesis submission I hadn't really figured out how we are choosing our signal. I think I have now understood the concept of signal and noise in the context of lensing reconstruction. It's the following.

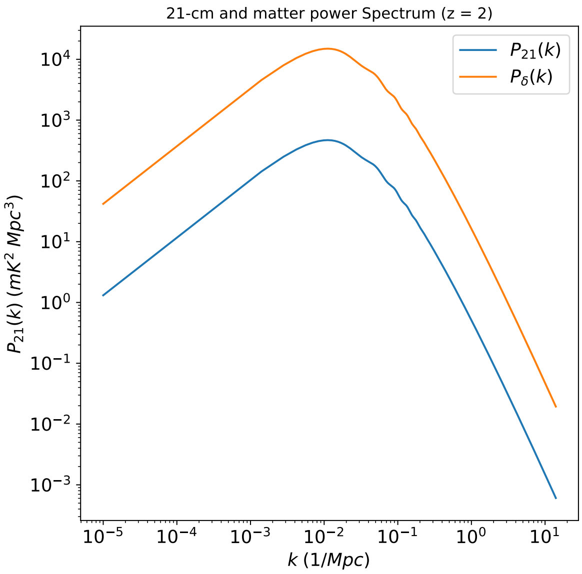
Figure 4 shows a typical lensing setup. The source is located at a transverse comoving distance [1] χ away from the observer. The vector sign in $\chi\vec{\theta}$ signifies that there are two components of the transverse distance $\chi\vec{\theta}$ in the source plane – one in the plane of the paper and the other perpendicular to it. Here, $\hat{\alpha}$ is called deflection angle and $\vec{\alpha}$ is called reduced deflection angle. We'll just need $\vec{\alpha}$ to understand our signal.

Knowledge of $\vec{\alpha}$ gives the observer the ability to map the images to the source. In other words, the observer reverses the lensing to obtain an unlensed distribution of sources. If we solve the geodesic equation for the above set up, assuming perturbed FLRW metric, we get

[1] As per the definition given in Hogg 2006.



(a)



(b)

Figure 3: (a) 21-cm power spectrum; (b) Comparison of 21-cm power spectrum with matter power spectrum

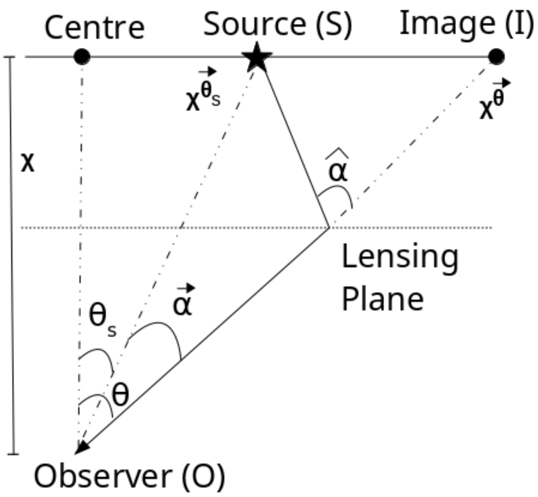


Figure 4: Schematic of a typical lensing setup. Only one lensing plane is shown for simplicity. Practically, the volume between the observer and the source can be thought of as a stack of many such planes of a certain thickness depending on the bin size we choose.

$$\vec{\alpha} = \vec{\nabla}_\theta \int_0^{\chi_s} d\chi' \frac{2}{c^2} \Phi(\theta, \chi') \left(\frac{\chi_s - \chi'}{\chi' \chi_s} \right) = \vec{\nabla}_\theta \psi, \tag{3.9}$$

where χ' represents the position of a particular lensing plane relative to the observer, Φ is the actual gravitational potential due to the mass distribution, and ψ is called lensing potential.

But the problem is that we don't know $\vec{\alpha}$ for any distribution. CONCERTO is going to record the mass distribution and shape features in its data and we can't find $\vec{\alpha}$ from that data directly. So we need to relate $\vec{\alpha}$ to another quantity that can be computed from the data.

Figure 4 gives

$$\theta - \theta_s = \vec{\alpha} \tag{3.10}$$

The following Jacobi matrix then gives the map to go from the source to the image. It basically tells us about the transformations by which the image differs from the shape of source.

$$\begin{aligned} \mathcal{A}_{ij}(\chi_S, \theta) &\equiv \frac{\partial \theta_S^i}{\partial \theta^j} && (3.11) \\ &= \delta_{ij} - \psi_{,ij} && (3.12) \\ \implies \mathcal{A} &= \underbrace{(1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{isotropic} + \underbrace{\begin{pmatrix} -\gamma_1 & -\gamma_2 \\ -\gamma_2 & \gamma_1 \end{pmatrix}}_{anisotropic} && (3.13) \end{aligned}$$

Here γ_i are the components of shear and κ is convergence. Since we are dealing with cosmological weak lensing, we ignore the contribution from shear and take the isotropic term only. I am sure there's a better explanation for this, like convergence and shear are almost the same for weak lensing, but I am not going into that for now.

Question When we say that we using taking convergence power spectrum in our calculations, does it mean that we are ignoring shear in the sense that it's not even there? Or does it mean that we are not considering the effect of shear on the map but it's there, and we are just looking at the effect of convergence?

Figure 5 shows the effect of convergence for positive and negative values of κ in Eq. (3.13). From this figure we see that convergence is a quantity that can be measured from the intensity mapping data (more on this later). Therefore, if we can relate α with κ then the statistics of κ will also be applicable to α.

κ is defined as

$$\kappa(\theta, \chi) = \frac{1}{2} \vec{\nabla}_\theta \cdot \vec{\alpha}(\theta, \chi). \tag{3.14}$$

Using this realation, we have connected a quantity $\vec{\alpha}$, which couldn't be computed from the data, with a quantity that can be computed via different statistical methods.

The quantity that we have chosen to compute is the power spectrum $P^\kappa(k)$ of convergence, which is just the two-point correlation function of κ in Fourier space. Assuming

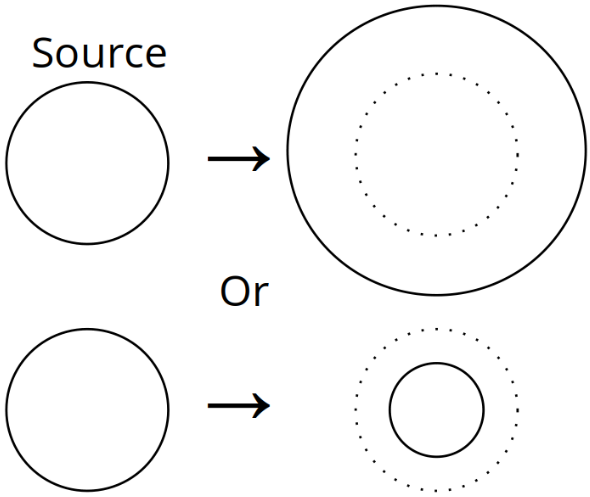


Figure 5: Effect of convergence – magnification for $\kappa < 1$ and demagnification for $\kappa > 1$.

that $\vec{l}$ is the Fourier conjugate to position $\vec{\theta}$, we can write the angular power spectrum of the two quantities as

$$C^{\kappa\kappa}(L) = \frac{1}{4}L(L+1)C^{\alpha\alpha}(L), \tag{3.15}$$

$$\implies C^{\alpha\alpha}(L) = \frac{4}{L(L+1)}C^{\kappa\kappa}(L) \tag{3.16}$$

where L is the observed Fourier mode.

Error in thesis Figure (3.16) is the reason for why we have $4/L(L+1)$ in equation 20 of P2014.

Eq. (3.16) gives us the statistic we wanted. We can now compute the displacement angle power spectrum from CONCERTO's data indirectly using convergence κ. So the final expression of our signa (in Pourtsidou's notation) is

$$C^{\alpha\alpha}(L) \equiv C^{\delta\theta\delta\theta}(L) = \frac{9H_0^4\Omega_{m0}^2}{L(L+1)c^4}\int_0^{\chi_s} d\chi' \left[\frac{(\chi_s - \chi')}{\chi_s}\frac{1}{a}\right]^2 P_\delta\left(\frac{L}{\chi'}\right). \tag{3.17}$$

Figure 6 shows the plot of this expression. Our plot (signal only) now matches with that of P2014. Earlier we were using non-linear power spectrum and there was a lot of confusion about the units. Everywhere in P2014 they have used Mpc/h units so I was expecting the same units in the plot. We were also under the impression that they have used non-linear power spectrum. Overall there were these four free variables (k units, P(k) units, linear, and non-linear) that I was trying to mix and match to see what combination worked. In the end it turned out that they have used the linear power spectrum with Mpc units.

3.5.2 Noise

Just like we have found a quantity κ to indirectly measure the deflection angle power spectru, we also need to find the noise in the measurement of $\vec{\alpha}$. In Eq, (2.15) we saw that $\Psi \equiv$ lensing potential is a natural choice for an estimator. So in Eq. (2.36) we found the lensing reconstruction noise corresponding to the lensing potential measurement. But

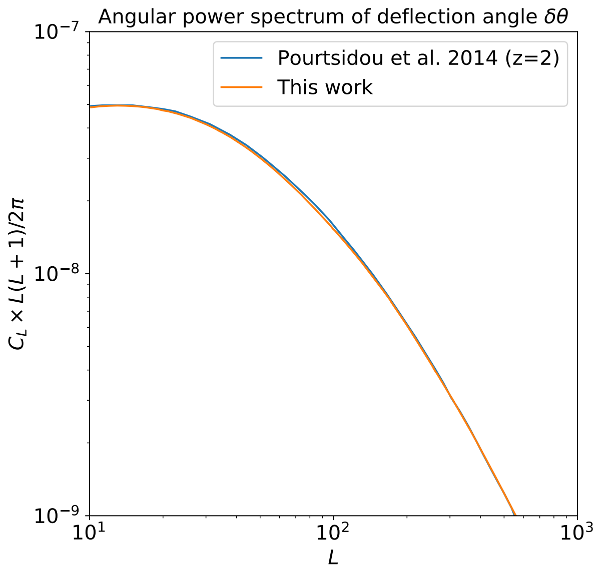


Figure 6: Our signal now matches with that of P2014.

as we just discussed, we are measuring the deflection angle power spectrum, so we need to convert that noise accordingly.

The lensing potential ψ and the deflection angle $\vec{\alpha}$ are related as

$$\vec{\alpha} = \vec{\nabla}_\theta \psi, \tag{3.18}$$

and the definition of lensing reconstruction noise $(N[L])$ is

$$\langle \hat{\tilde{\Psi}}(L)\hat{\tilde{\Psi}}(L') \rangle = 2\pi\delta(L - L')N^{\psi}(L), \tag{3.19}$$

therefore, using the relation between $\vec{\alpha}$ and ψ we write the lensing deflection angle reconstruction noise as

$$N^{\delta\theta}(L) = L^2 N^{\psi}(L). \tag{3.20}$$

In this equation we have indirectly found the noise in the measurement of displacement field power spectrum through the noise in the lensing potential measurement. This is a general result.

3.5.3 Error bars

The error in measurement of the deflection angle power spectrum is given by

$$\Delta C^{\delta\theta\delta\theta} = \sqrt{\frac{2}{(2L+1)\Delta L f_{sky}}}(C^{\delta\theta\delta\theta}(L) + N^{\delta\theta}(L)). \tag{3.21}$$

Error in thesis As I mentioned before Eq. (3.17), I took $C^{\kappa\kappa}(L)$ in place of $C^{\delta\theta\delta\theta}(L)$.

3.5.4 Gaussian noise

We have derived the expression of $N^{\psi}(L)$ for Gaussian matter distribution in equation (3.20). Using that, we find the reconstruction noise for corresponding to the displacement angle power spectrum to be

$$N^{\delta\theta} = \frac{L^2}{\int \frac{d^2l}{(2\pi)^2} \sum_j \frac{[C_{l,j}\mathbf{L}\cdot\mathbf{l} + C_{L-l,j}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{2C^{tot}_{l,j}C^{tot}_{L-l,j}}} \tag{3.22}$$

where $C_{l,j}$ is the discretized 21-cm power spectrum defined as

$$C_{l,j} = \frac{P(\sqrt{(l/\chi^2) + (2\pi j/\mathcal{L})})}{\chi^2\mathcal{L}} \tag{3.23}$$

3.5.5 Instrumental Noise

$$C_l^N = \frac{(2\pi)^3 T_{sys}^2}{B t_{obs} f_{cover}^2 l_{max}(\nu)^2} \tag{3.24}$$

3.5.6 Gaussian + Poisson noise

Mass moments and number density of galaxies The number density of galaxies in the mass range dM is given by Schechter function

$$\frac{dn}{dM}dM = \phi^\star \left(\frac{M}{M^\star}\right)^\alpha exp\left[\frac{-M}{M}\right]\frac{dM}{M^\star} \tag{3.25}$$

where α is the slope, $M^\star$ is characteristic mass and $\phi^\star$ is normalization.

Using this we calculate the mass density of HI sources as

$$\rho_{HI} = \phi^\star M^\star \int \left(\frac{M}{M^\star}\right)^{\alpha+1} exp\left[-\frac{M}{M^\star}\right]\frac{dM}{M^\star} \tag{3.26}$$

$$= \phi^\star M^\star \Gamma(\alpha+2) \tag{3.27}$$

where Γ denotes the Gamma function.

Average number density of galaxies The given expression of Schechter function has a normalization constant. So is

$$\int \frac{dn}{dM}dM = 1? \tag{3.28}$$

If not, then is

$$\bar{\eta} = \int \frac{dn}{dM}dM \tag{3.29}$$

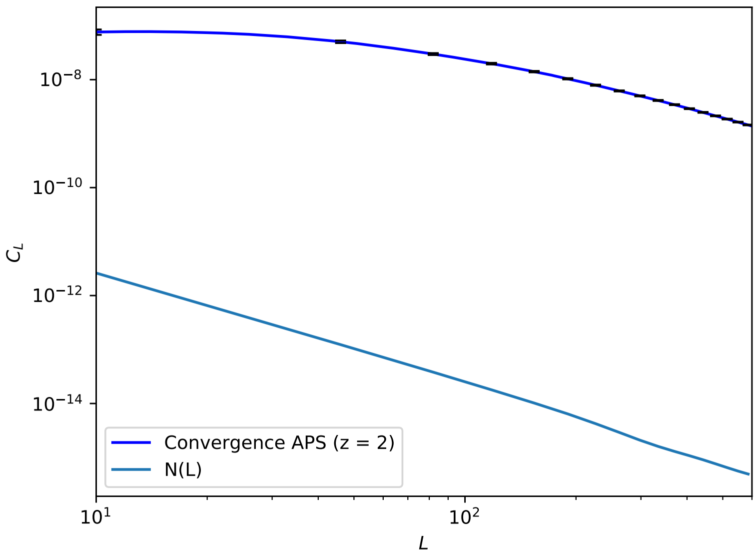
$$= \int \phi^\star x^\alpha e^{-\alpha}dx \tag{3.30}$$

$$= \int \phi^\star x^{(\alpha+1)-1}e^{-\alpha}dx \tag{3.31}$$

$$= \phi^\star\Gamma(\alpha+1) \tag{3.32}$$

Here the integral blows up for $\alpha < -1$ but we still went ahead and wrote the Gamma function. Here for $\alpha = -1.3$ in our case, the argument of Γ is negative. For negative non-integer values, the Gamma function is defined as

$$\Gamma(1+x) = \frac{1}{1+x}\Gamma(2+x), \tag{3.33}$$



where $x < 1$. Now we can calculate $\bar{\eta}$ as

$$\bar{\eta} = \phi^\star \frac{\Gamma(2+\alpha)}{1+\alpha} \tag{3.34}$$
$$= -4.326 \tag{3.35}$$

for $\alpha = -1.3$.

Mass moments Defining the mass moment as

$$\langle M \rangle = \frac{\Phi^\star M^\star \int x^{\alpha+1} e^{-x} dx}{\bar{\eta}} \tag{3.36}$$
$$= \frac{\phi^\star M^\star \Gamma[\alpha+2]}{\bar{\eta}} \tag{3.37}$$

Therefore the n^{th} moment is

$$\langle M^n \rangle = \frac{\phi^\star M^{\star n} \Gamma(\alpha+n+1)}{\bar{\eta}} \tag{3.38}$$

Here if $\bar{\eta}$ is negative then all the mass moments will be negative. But these are physical quantities. They can not have negative values.