Contents

1 Perturbed flat FLRW metric

Add more info about decomposition theorem and gauge
Ref: Dodelson

The FLRW metric describes a flat, isotropic, homogeneous, and expanding universe. FLRW metric is of the form

$$\bar{g}_{00} = -t;\ \bar{g}_{0i} = \bar{g}_{io} = 0;\ \bar{g}_{ii} = -a(t)^2 \tag{1.1}$$

where, a(t) is the scale factor whose value is $= 1$ at present time and < 1 at earlier time.

The perturbations to the flat FLRW metric can be thought of as a sparse distribution of weak gravitational objects in the spacetime. The perturbed spacetime can be written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{1.2}$$

where $h_{\mu\nu}$ is the perturbation and $h_{\mu\nu} \ll g_{\mu\nu}$.

1.1 Perturbation metric and Gauge freedom

Suppose that we have a vector $\vec{v}$ at a point P in a 3D space. We can decompose this vector into parallel $(v_{\|})$ and perpendicular$(v_{\perp})$ components with respect to the position vector of point P. For these components we have $\bigtriangledown \times v_{\|} = 0$, and $\vec{\bigtriangledown} \cdot v_{\perp} = 0$. Therefore we can add a gradient term in $v_{\|}$ and a curl term in $v_{\perp}$ without changing the curl and divergence of $\vec{v}$.

Similarly, scalar-vector-tensor decomposition theorem states that the most generalized linear perturbations of the FLRW metric can be decomposed into four scalar, two vector and a tensor field. Similar to the three dimensional vector decomposition case, here also we have freedom of choosing the components in such a way that the physics remains unchanged. It is termed as Gauge freedom. In Newtonian gauge, we set two of the scalar components $= 0$, and set the remaining two scalar components equal to 2Φ and 2Ψ, where Ψ and Ψ are perturbations to the flat FLRW metric. Ψ corresponds to the Newtonian potential, and Φ corresponds to the spatial curvature. Both Ψ and Φ are small, therefore all the second and higher order terms are neglected.

If we consider only the scalar perturbations, the perturbed metric becomes

$$g_{00} = -(1 + 2\Psi(\vec{x}, t));\ g_{0i} = g_{io} = 0;\ g_{ii} = a(t)^2(1 + 2\Phi(\vec{x}, t)) \tag{1.3}$$

In overdense regions, $\Phi > 0$ and $\Psi < 0$ with this sign convention.

2 Motion of photon in the perturbed flat-FLRW spacetime

Let us consider a photon moving through the spacetime. Let's assume that the position four-vector is $x^\mu(t, \vec{x})$ and momementum four-vector is $P^\mu(P^0, \vec{P})$.

$$P^\mu = \frac{dx^\mu}{d\lambda} \tag{2.1}$$

where λ is a parameter which increases monotonically. While moving through the spacetime, the photon follows a null geodesic. To solve for the trajectory taken by the photon, we need to solve the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} \tag{2.2}$$

$\mu, \alpha,\ \beta$ run from 0 to 3.

To describe the spacetime we use the perturbed FLRW metric

$$g_{00} = -(1+2\Psi(\vec{x},t));\ g_{0i}\ =\ g_{io} = 0;\ g_{ii} = a(t)^2(1+2\Phi(\vec{x},t)) \tag{2.3}$$

For details on the metric and gauge go to Section 1.

Let's deal with the Chrishtoffel symbols first. The Christoffel symbols are defined as

$$\Gamma^\mu_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{dg_{\alpha\nu}}{dx^\beta} + \frac{dg_{\beta\nu}}{dx^\alpha} - \frac{dg_{\alpha\beta}}{dx^\nu}\right] \tag{2.4}$$

where μ, ν, α, and β run from 0 to 3.

For the spatial part of the geodesic equation, we have

$$\begin{aligned}\Gamma^i_{00} &= \frac{\Psi_{,i}}{a^2}\\ \Gamma^i_{\alpha 0} &= \delta^i_\alpha(H+\Phi_{,0}\,)\\ \Gamma^i_{\alpha\beta} &= \delta^i_\alpha\Phi_{,\beta} + \delta^i_\beta\Phi_{,\alpha} - \delta^{i\nu}\delta_{\alpha\beta}\Phi_{,\nu}\end{aligned} \tag{2.5}$$

where $_{,i}$ in the subscript denotes derivative w.r.t. x^i, $H = \frac{1}{a}\frac{da}{dt}$ is the Hubble parameter.

Four-momentum of the photon satifies

$$\begin{aligned}P^\mu P_\nu = 0 &= g_{\mu\nu}P^\mu P^\nu\\ &= -(1+2\Phi(\vec{x},t))(P^0)^2 + p^2\end{aligned}$$

where we have defined

$$p = g_{\mu\nu}P^\mu P^\nu \tag{2.6}$$

Therefore, by using the fact that Ψ and Φ are small, we get

$$P^0 = p(1-\Psi) \tag{2.7}$$

We substitute this expression of to replace P^0 in the following equations.

3 Weak gravitational lensing

Ref: Dodelson

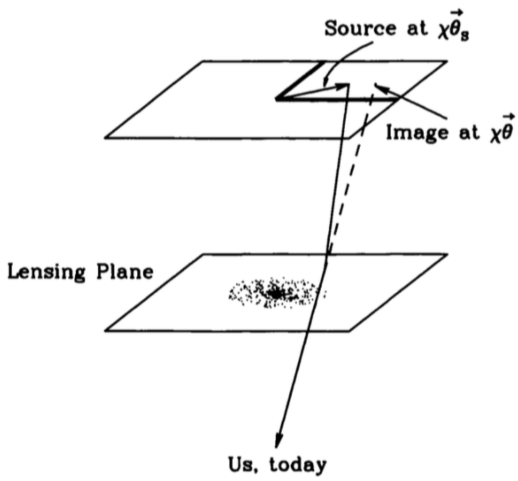

Figure 1: Schematic showing the deflection in the path of a photon due to gravitational lensing

Let's consider a situation where the path of photon gets deflected due to the presence of matter as shown in Figure 1.

In the source plane we have defined a set of mutually perpendicular axes denoted by superscript 1 and 2. Therefore, the position of a general point in the source plane will be denoted using two components: $(\chi^{\theta^1}, \chi^{\theta^2})$. Here χ and θ are labels. They denote that the point is located at χ comoving distance away from the observer and makes an angle θ with the line joining the observer and the center of source plane.

Here $\chi^{\vec{\theta}_S}$ is the actual position of the photon on the source plane and $\chi^{\vec{\theta}}$ is the apparent position of the source. The spatial position of the photon on the source plane is defined as $(\chi^{\theta^1}, \chi^{\theta^2}, \chi)$. Where χ is the comoving radial distance between the source and observer, and $|\chi^{\vec{\theta}}|$ is the comoving distance of the photon from the origin on the source plane.

The comoving distance *"from the observer to the source"* is given by the relation

$$\chi(a) = \int_{t(a)}^{t_0} \frac{dt'}{a(t')} = \int_a^1 \frac{da'}{a'H(a')} \tag{3.1}$$

In terms of axis, the χ vector extends from observer towards source. As photon moves towards the observer, t' increases, $a(t')$ increases and $|\chi|$ decreases. Therefore,

$$\frac{d\chi}{dt} = \frac{-1}{a} \tag{3.2}$$

We need to rewrite the geodesic equation in terms of the chosen coordinate system. Therefore simplifying the LHS using

$$\begin{aligned} \frac{d\chi^{\theta^i}}{d\lambda} &= \frac{dt}{d\lambda}\frac{d\chi}{dt}\frac{d\chi^{\theta^i}}{d\chi} \\ &= P^0 \frac{-1}{a}\frac{d\chi^{\theta^i}}{d\chi} \\ &= \frac{-1}{a} p(1-\Psi)\frac{d\chi^{\theta^i}}{d\chi} \\ &\simeq \frac{-p}{a}\frac{d\chi^{\theta^i}}{d\chi} \end{aligned} \tag{3.3}$$

Similarly,

$$\frac{d\chi}{d\lambda} = \frac{-p}{a}(1-\Psi) \tag{3.4}$$

Therefore, the LHS of the geodesic equation becomes

$$\frac{d^2\chi^{\theta^i}}{d\lambda^2} = \frac{p}{a}\frac{d}{d\chi}\left(\frac{p}{a}\frac{d\chi^{\theta^i}}{d\chi}\right) \tag{3.5}$$

On RHS we have,

$$\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\alpha}{d\lambda} = \Gamma^i_{\alpha\beta}\frac{p^2}{a^2}(1-\Psi)^2\frac{dx^\alpha}{d\chi}\frac{dx^\beta}{d\chi} \tag{3.6}$$

Therefore,

$$\begin{aligned}\Gamma^i_{0\beta}\frac{dt}{d\lambda}\frac{dx^\beta}{d\lambda} &= \delta^i_\beta(H+\Phi_{,0})\frac{p^2}{a^2}(1-\Psi)^2\frac{dt}{d\chi}\frac{dx^\beta}{d\chi} \\ &= \frac{p^2}{a^2}(-a)H\frac{d\chi^{\theta^i}}{d\chi}\end{aligned} \tag{3.7}$$

In the second equality, using the delta function, we replace x^β with the spatial transverse part. Upto first order, we neglect Ψ and substitute $\frac{dt}{d\chi} = -a$.

Now,

$$\begin{aligned}\Gamma^i_{00}(\frac{dt}{d\lambda})^2 &= \frac{\Psi_{,i}}{a^2}(\frac{dt}{d\chi})^2 \\ &= \Psi_{,i}\frac{p^2}{a^2} \qquad = -\Phi_{,i}\frac{p^2}{a^2}\end{aligned} \tag{3.8}$$

where the second equality holds since in the late universe there are no anisotropic stresses so $\Phi = -\Psi$. [dodelson]I don't understand this anisotropy part.

Similarly,

$$\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = (\delta^i_\alpha\Phi_{,\beta}+\delta^i_\beta\Phi_{,\alpha}-\delta_{\alpha\beta}\Phi_{,i})\frac{p^2}{a^2}(1-\Psi)^2\frac{dx^\alpha}{d\chi}\frac{dx^\beta}{d\chi} \tag{3.9}$$

Here, if x^α, x^β are transverse coordinates, then the resulting term after multiplication with Φ would be very small. Therefore, α and β are equal to 3. Therefore,

$$\begin{aligned}\Gamma^i_{33}\frac{dx^3}{d\lambda}\frac{dx^3}{d\lambda} &= (\delta^i_3\Phi_{,3}+\delta^i_3\Phi_{,3}-\delta_{33}\Phi_{,i})\frac{p^2}{a^2}(1-\Psi)^2\frac{dx^3}{d\chi}\frac{dx^3}{d\chi} \\ &= -\Phi_{,i}\frac{p^2}{a^2}\end{aligned} \tag{3.10}$$

The first two terms vanish because $i \neq 3$.

Now that we have found out all the terms on RHS. Let's simplify LHS

$$\begin{aligned}\frac{d^2\chi^{\theta^i}}{d\lambda^2} &= \frac{p}{a}\frac{d}{d\chi}\left(\frac{p}{a}\frac{d\chi^{\theta^i}}{d\chi}\right) \\ &= \frac{p}{a^2}\frac{dp}{d\chi}\frac{d\chi^{\theta^i}}{d\chi} - \frac{p^2}{a^3}\frac{da}{d\chi}\frac{d\chi^{\theta^i}}{d\chi} + \frac{p^2}{a^2}\frac{d^2\chi^{\theta^i}}{\chi^2} \\ &= \frac{-2p^2}{a^3}\frac{da}{d\chi}\frac{\chi^{\theta^i}}{d\chi} + \frac{p^2}{a^2}\frac{d^2\chi^{\theta^i}}{d\chi^2}\end{aligned} \tag{3.11}$$

In the second equality, we just opened the brackets using the chain rule for differentiation. In the third equality, in the first term we used equation 3.4 to replace p with $-a\frac{d\chi}{d\lambda}$ upto zeroth order.

Equating equation 3.11, 3.7, 3.8, and 3.10

$$\frac{-2p^2}{a^3}\frac{da}{d\chi}\frac{\chi^{\theta^i}}{d\chi}+\frac{p^2}{a^2}\frac{d^2\chi^{\theta^i}}{d\chi^2}=\frac{p^2}{a^2}\left[2\Phi_{,i}+aH\frac{d\chi^{\theta^i}}{d\chi}\right] \tag{3.12}$$

First term on LHS ad second term on RHS cancel and we are left with (inserting c^2)

$$\frac{d^2\chi^{\theta^i}}{d\chi^2}=\frac{2\Phi_{,i}}{c^2} \tag{3.13}$$

3.1 Reduced deflection angle and lensing potential

Ref: Bartelmann, Maturi (arxiv:1612.06535)

After integration, we get

$$\frac{d\chi^{\theta^i}}{d\chi}=\int_0^{\chi}\frac{2\Phi_{,i}}{c^2}d\chi+constant \tag{3.14}$$

$$\chi^{\theta^i}\Big|_{limits}=\int_0^{\chi}d\chi''\int_0^{\chi''}d\chi'\frac{2\Phi_{,i}}{c^2} \tag{3.15}$$

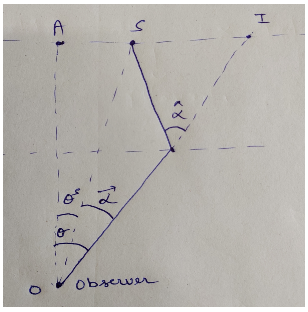

Figure 2: Schematic showing the angular coordinates of source and image

In the schematic, we can see that

$$\frac{\chi^{\theta^i}}{\chi}=\theta \tag{3.16}$$

$$\frac{\chi^{\theta^i_S}}{\chi}=\theta_S \tag{3.17}$$

This comes from the definition of angular diameter distance

$$\beta=\frac{l}{D_A} \tag{3.18}$$

where β is the angle subtended by an object of physical size l and at D_A is the angular diameter distance between the source and the observer.

We have $a\chi^{\theta^i} = l$ and $a\chi = D_A^{\chi}$. a's cancel and we get equation 3.16 and 3.17. Therefore, on dividing 3.15 by χ on both sides we get

$$\theta_S^i - \theta^i = \frac{1}{\chi}\int_0^{\chi} d\chi'' \int_0^{\chi''} d\chi' \frac{2\Phi_{,i}}{c^2} \tag{3.19}$$

writing $\int_0^{\chi''} d\chi' 2\Phi_{,i} \equiv f(\chi'')$. Therefore

$$\begin{aligned}
\theta_S^i - \theta^i &= \frac{1}{\chi}\int_0^{\chi} d\chi'' f(\chi'') \\
&= \frac{1}{\chi}\left([\chi'' f(\chi'')]_0^{\chi} - \int_0^{\chi} d\chi'' \frac{df(\chi'')}{d\chi''}\chi'' \right) \\
&= \frac{1}{\chi}\chi f(\chi) - \frac{1}{\chi}\int_0^{\chi} d\chi'' \frac{df(\chi'')}{d\chi''}\chi'' \\
&= \int_0^{\chi} d\chi' \frac{2\Phi_{,i}}{c^2} - \frac{1}{\chi}\int_0^{\chi} d\chi' \frac{2\Phi_{,i}}{c^2}(x(\chi'))\chi' \\
&= \int_0^{\chi} d\chi' \frac{2\Phi_{,i}}{c^2}(\vec{x}(\chi'))\left(1 - \frac{\chi'}{\chi}\right)
\end{aligned} \tag{3.20}$$

where the second equality has been obtained by doing integration by parts.
Therefore, we finally have the result

$$\theta_S^i - \theta^i = \frac{2}{c^2}\int_0^{\chi} d\chi' \Phi_{,i}\,(\vec{x}(\chi'))\left(1 - \frac{\chi'}{\chi}\right) \tag{3.21}$$

This result refers to the situation when the source is at a fixed distance from the observer. In the case of large scale structure lenses, we would also have to put a redshift distribution function of sources. refine.

In Figure 2, S is the source, I is the apparent position of the source on the source plane. $\hat{\alpha}$ is the deflection angle. $\vec{\alpha}$ is the reduced deflection angle.

$$\vec{\theta_S} - \vec{\theta} \ = \vec{\alpha} \tag{3.22}$$

Therefore,

$$\begin{aligned}
\vec{\alpha} &= \vec{\nabla}_{\perp}\int_0^{\chi} d\chi' \frac{2}{c^2}\Phi(\vec{x}(\chi'))\left(1 - \frac{\chi'}{\chi}\right) \\
&= \vec{\nabla}_{\theta}\int_0^{\chi} d\chi' \frac{2}{c^2}\Phi(\vec{x}(\chi'))\left(1 - \frac{\chi'}{\chi}\right)\frac{1}{\chi'} \\
&= \vec{\nabla}_{\theta}\psi \\
\therefore \vec{\theta}_S &= \vec{\theta} + \vec{\nabla}_{\theta}\psi
\end{aligned} \tag{3.23}$$

where, ψ is defined as ***lensing potential***. We converted $\vec{\nabla}_{\perp}$ to $\vec{\nabla}_{\theta}$ because when we measure the position of sources, we measure the angle subtended by them. Therefore, we want all the derivatives with respect to the angular coordinates.

4 Convergence and shear

Ref: Bartelmann, Maturi (arxiv:1612.06535)

4.1 Convergence

Using the poisson equation we relate the gravitational potential with the over-density of the region. Similarly, here we want to relate the laplacian of lensing potential with a local quantity. Therefore, we start with

$$\vec{\nabla}_\theta \cdot \vec{\alpha} = \vec{\nabla}^2_\theta \psi = \frac{2}{c^2} \int_0^\chi \vec{\nabla}^2_\perp \left(1 - \frac{\chi'}{\chi}\right) \chi' \Phi(\vec{x}(\chi')) d\chi' \tag{4.1}$$

The complete Laplacian is

$$\vec{\nabla}^2 = \vec{\nabla}_\perp - \frac{\partial^2}{\partial(\chi')^2} \tag{4.2}$$

If the extent of lensing mass distribution is small compared to the cosmological distances (D_{LS}, D_S, and D_L), then

$$\int \frac{\partial^2 \Phi}{\partial(\chi')^2} d\chi' = \left.\frac{\partial \Phi}{\partial \chi'}\right|_{\text{end points}} = 0 \tag{4.3}$$

Therefore, we can replace $\vec{\nabla}^2_\perp$ with $\vec{\nabla}^2$ and we get

$$\vec{\nabla}_\theta \cdot \vec{\alpha} = \frac{2}{c^2} \vec{\nabla}^2 \int_0^\chi \left(1 - \frac{\chi'}{\chi}\right) \chi' \Phi(\vec{x}(\chi')) d\chi' \tag{4.4}$$

Using the poisson's equation

$$\vec{\nabla}^2 \Phi = 4\pi G \rho \tag{4.5}$$

we get

$$\begin{aligned} \vec{\nabla}^2_\theta \psi &= \frac{8\pi G}{c^2} \int_0^\chi \left(1 - \frac{\chi'}{\chi}\right) \chi' \rho(\vec{x}(\chi')) a^2 d\chi' \\ &= 2\kappa \end{aligned} \tag{4.6}$$

where κ is the convergence, defined as

$$\begin{aligned} \kappa &= \frac{1}{2} \vec{\nabla}^2_\theta \psi \\ &= \frac{4\pi G}{c^2} \int_0^\chi \left(1 - \frac{\chi'}{\chi}\right) \chi' \rho(\vec{x}(\chi')) a^2 d\chi' \end{aligned} \tag{4.7}$$

and a is the scale factor.

This expression shows that convergence is a geometrically weighted line-of-sight integral over mass density ρ. ρ is the fluctuation of mass density about its cosmological mean value $\bar{\rho}$ and not the entire mass density.refine The convergence κ thus describes the lensing effects of matter inhomogeneities in an otherwise homogeneous mean universe. Writing ρ as $\rho = \bar{\rho}\delta$, where δ is dimensionless. In terms of conventional cosmological parameters, the mean matter density is

$$\bar{\rho} = \frac{3H_0^2}{8\pi G} \Omega_{m0} a^{-3} \tag{4.8}$$

where H_0 is the Hubble constant quantifying the present expansion rate of the universe, and Ω_{m0} is the dimensionless matter-density parameter. Therefore, we get

$$\kappa(\chi^{\vec{\theta}}) = \frac{3}{2}\frac{H_0^2}{c^2}\Omega_{m0}\int_0^{\chi} d\chi'\chi'\frac{(\chi-\chi')}{\chi}\frac{\delta(\vec{\chi}')}{a} \tag{4.9}$$

for the convergence κ of an extended lens. It is called *effective convergence* because it corresponds to the convergence of a thin lens whose effects are equivalent to those caused by the actual extended matter distribution. Again, this expression is true for a fixed source. For extended large scale structure, we would have to put the source redshift distribution function in the expression.

$$\kappa(\chi^{\vec{\theta}}) = \int_0^{\chi} d\chi' W(\chi', \chi)\delta(\vec{\chi}')) \tag{4.10}$$

4.2 Statitstics of convergence field

$$\langle\tilde{\kappa}(\vec{l})\tilde{\kappa}(\vec{l'})\rangle = (2\pi)^2\delta_D^2(\vec{l}+\vec{l'})C_l^{\kappa\kappa} \tag{4.11}$$

Power spectrum depends only on the magnitude of the wave vector $\vec{l}$, not its direction. Analogous to the 3D case, each angular scale l has fluctuation with variance $l^2C_l^{\kappa\kappa}/2\pi$,. The variance of the fluctuations of the convergence field will be the logarithmic integral over all l of $l^2C_l^{\kappa\kappa}/2\pi$.

4.3 Errors on the convergence power spectrum

4.4 Shear

Using the relation

$$\vec{\theta}_S = \vec{\theta} - \vec{\nabla}_\theta\psi \tag{4.12}$$

we get the Jacobian matrix as 4

$$\begin{aligned} A_{ij}(\chi_S, \theta) &\equiv \frac{\partial\theta_S^i}{\partial\theta^j} \\ &= \delta_{ij} - \psi_{,ij} \end{aligned} \tag{4.13}$$

where subscripts in ψ_{ij} denote derivatives w.r.t. components of $\vec{\theta}$.
We split the jacobi matrix into isotropic and anisotropic (trace-free) part

$$tr(A) = 2 - \vec{\nabla}^2\psi = 2(1-\kappa) \tag{4.14}$$

Where, $tr(\psi_{,ij})$ gives the diagonal elements which is equal to $\vec{\nabla}^2\psi$
Using this we obtain the shear matrix by removing the isotropic part from A

$$\begin{aligned} \Gamma &= \left(A - \frac{1}{2}[tr(A)]I\right) \\ &= ([\delta_{ij} - \psi_{,ij}]_{\text{matrix}} - (1-k)I) \\ &= -[\psi_{ij}]_{\text{matrix}} - (1-k)I \\ &= -\left([\psi_{ij}]_{\text{matrix}} + (1-k)I\right) \end{aligned} \tag{4.15}$$

We have $k = \frac{1}{2}(\psi_{11} + \psi_{22})$, using this we finally get
It's components are

$$\begin{aligned}
&\Gamma_{11} = -\gamma_1 = -\frac{1}{2}(\psi_{11} - \psi_{22}) \\
&\Gamma_{22} = \gamma_1 \\
&\Gamma_{12} = \Gamma_{21} = -\gamma_2 = -\psi_{12}
\end{aligned} \tag{4.16}$$

Therefore,

$$\begin{aligned}
A &= \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_{12} \\ -\gamma_{21} & 1-\kappa+\gamma_1 \end{pmatrix} \\
&= (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\gamma_1 & -\gamma_{12} \\ -\gamma_{21} & \gamma_1 \end{pmatrix}
\end{aligned} \tag{4.17}$$

Convergence causes an isotropic magnification of angular size in the neighborhood of $\vec{\theta}$ while shear produces anisotropy.

For a small circular source, its lensed image is an ellipse, with major and minor axes

$$a = (1-\kappa-\gamma_1)^{-1} \text{and } b = (1-\kappa+\gamma_1)^{-1} \tag{4.18}$$

5 Cosmological weak gravitational lensing

Ref: Bartelmann, Maturi (arxiv:1612.06535)

5.1 Limber's approximation

If the quantity $x(\vec{\theta})$ defined in two dimensions is a projection

$$x(\vec{\theta}) = \int_0^{\chi} d\chi' w(\chi') y(\vec{x}(\chi')) \tag{5.1}$$

of a quantity $y(\vec{r})$ defined in three dimensions with a weight function $w(\chi')$, then the angular power spectrum of x is given by

$$C_x(l) = \int_0^{\chi} d\chi' \frac{w^2(\chi')}{(\chi')^2} P_y\left(\frac{l}{\chi'}\right) \tag{5.2}$$

where $P_y(k)$ is the power spectrum of y, taken at the three dimensional wave number $k = l/\chi'$. The condition of applicability of the approximation is that y must vary on length scales much smaller than the typical length scale of the weight.

5.2 Convergence power spectrum

Using Equation 4.9, we write the two point function of the convergence field

$$\langle \kappa(\chi^{\vec{\theta}})\kappa^*(\chi^{\vec{\theta'}})\rangle = \frac{9H_0^2\Omega_{m0}^2}{4c^4}\int_0^{\chi_S} d\chi \int_0^{\chi_S} d\chi' \left(\frac{\chi_S - \chi}{\chi_S}\right)\left(\frac{\chi_S - \chi'}{\chi_S}\right)\chi\chi' \frac{1}{a^2}\langle \delta(\vec{\chi})\delta^*(\vec{\chi'})\rangle \tag{5.3}$$

Here,

$$\langle\delta(\vec{\chi})\delta^*(\vec{\chi}')\rangle = \langle\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}e^{ik\chi}e^{-ik'\chi'}\delta(\vec{k})\delta^*(\vec{k}')\rangle \quad (5.4)$$

$$= \int\frac{d^3k}{(2\pi)^3}\int_0^k\frac{d^3k'}{(2\pi)^3}e^{ik\chi}e^{-ik'\chi'}\langle\delta(\vec{k})\delta^*(\vec{k}')\rangle \quad (5.5)$$

$$= \int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}e^{ik\chi}e^{-i(k'+k-k)\chi'}\langle\delta(\vec{k})\delta^*(\vec{k}')\rangle \quad (5.6)$$

$$= \int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}e^{ik(\chi-\chi')}e^{i(k-k')\chi'}\langle\delta(\vec{k})\delta^*(\vec{k}')\rangle \quad (5.7)$$

$$= \int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}e^{ik(\chi-\chi')}e^{-i(k'-k)\chi'}\langle\delta(\vec{k})\delta^*(\vec{k}')\rangle \quad (5.8)$$

Let's write $\langle\delta(\vec{k})\delta^*(\vec{k}')\rangle \equiv \xi(\vec{k}_d)$, where, $\vec{k}_d \equiv \vec{k}-\vec{k}'$ Therefore,

$$\langle\delta(\vec{\chi})\delta^*(\vec{\chi}')\rangle = \int\frac{d^3\vec{k}}{(2\pi)^3}e^{i\vec{k}(\chi-\chi')}\int\frac{d^3\vec{k}_d}{(2\pi)^3}e^{i(\vec{k}_d)\chi'}\xi(\vec{k}_d) \quad (5.9)$$

Writing $\int\frac{d\vec{k}_d}{(2\pi)^3}e^{i(\vec{k}_d)\chi'}\xi(\vec{k}_d) \equiv P(\vec{\chi}')$, where we $P(\vec{\chi}')$ is the power spectrum. Thus,

$$\langle\delta(\vec{\chi})\delta^*(\vec{\chi}')\rangle = \delta(\vec{\chi}-\vec{\chi}')P(\vec{\chi}') \quad (5.10)$$

Using this result, we further get

$$\langle\kappa(\vec{\chi}^{\theta})\kappa^*(\vec{\chi}^{\theta'})\rangle = \frac{9H_0^2\Omega_{m0}^2}{4c^4}\int_0^{\chi_S}d\chi\int_0^{\chi_S}d\chi'\left(\frac{\chi_S-\chi}{\chi_S}\right)\left(\frac{\chi_S-\chi'}{\chi_S}\right)\chi\chi'$$
$$\frac{1}{a^2}\delta(\vec{\chi}-\vec{\chi}')\int\frac{d\vec{k}_d}{(2\pi)^3}e^{i(\vec{k}_d)\chi'}\xi(\vec{k}_d)) \quad (5.11)$$

$$= \frac{9H_0^2\Omega_{m0}^2}{4c^4}\int_0^{\chi_S}d\chi\int_0^{\chi_S}d\chi'\left(\frac{\chi_S-\chi}{\chi_S}\right)\left(\frac{\chi_S-\chi'}{\chi_S}\right)\chi\chi'$$
$$\frac{1}{a^2}\delta(\vec{\chi}-\vec{\chi}')P(\vec{\chi}') \quad (5.12)$$

$$= \frac{9H_0^2\Omega_{m0}^2}{4c^4}\int_0^{\chi_S}d\chi\left(\frac{\chi_S-\chi}{\chi_S}\right)^2\chi^2\frac{1}{a^2}P(\vec{\chi}) \quad (5.13)$$

Instead of finding out the convergence in real space, let's find write down the convergence and its correlation function in Fourier space. Let's begin with convergence,

$$\kappa(\chi^{\vec{\theta}}) = \frac{3}{2}\frac{H_0^2}{c^2}\Omega_{m0}\int_0^{\chi_S}d\chi\chi\frac{(\chi_S-\chi)}{\chi_S}\frac{\delta(\vec{\chi})}{a} \quad (5.14)$$

We rewrite the equation as

$$\kappa(\vec{\theta}) = \int_0^{\chi_S}d\chi W(\chi_S,\chi)\delta(\vec{\theta}\chi,\chi) \quad (5.15)$$

In Fourier space,

$$\begin{aligned}
\kappa(\vec{l}) &= \int d^2\theta e^{-i\vec{l}\cdot\vec{\theta}} \int_0^{\chi_S} d\chi W(\chi,\chi_S) \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \delta(\vec{k}) && (5.16)\\
&= \int d^2\theta e^{-i\vec{l}\cdot\vec{\theta}} \int_0^{\chi_S} d\chi W(\chi,\chi_S) \int \frac{d^2\vec{k}_\perp}{(2\pi)^2} \int \frac{d\vec{k}_\parallel}{2\pi} e^{i\vec{k}\cdot\vec{x}} \delta(\vec{k}_\perp, k_\parallel) && (5.17)\\
&= \int_0^{\chi_S} d\chi W(\chi,\chi_S) \int \frac{d^2\vec{k}_\perp}{(2\pi)^2} \int \frac{d\vec{k}_\parallel}{2\pi} \int d^2\theta e^{-i(\vec{l}-\vec{k}_\perp\chi)\cdot\vec{\theta}} e^{i\vec{k}_\parallel\cdot\vec{x}} \delta(\vec{k}_\perp, k_\parallel) && (5.18)
\end{aligned}$$

The θ integral gives a delta function

$$\begin{aligned}
\kappa(\vec{l}) &= \int_0^{\chi_S} d\chi W(\chi,\chi_S) \int \frac{d^2\vec{k}_\perp}{(2\pi)^2} \int \frac{d\vec{k}_\parallel}{2\pi} (2\pi)^2 \delta(\vec{l}-\vec{k}_\perp\chi) e^{i\vec{k}_\parallel\cdot\vec{x}} \delta(\vec{k}_\perp, k_\parallel) && (5.19)\\
&= \int_0^{\chi_S} d\chi W(\chi,\chi_S) \int \frac{d^2\vec{l}}{(\chi)^2} \int \frac{d\vec{k}_\parallel}{2\pi} \delta(\vec{l}-\vec{k}_\perp\chi) e^{i\vec{k}_\parallel\cdot\vec{x}} \delta(\vec{k}_\perp, k_\parallel) && (5.20)\\
&= \int_0^{\chi_S} d\chi \frac{W(\chi,\chi_S)}{\chi^2} \int d^2\vec{l} \int \frac{d\vec{k}_\parallel}{2\pi} \delta(\vec{l}-\vec{k}_\perp\chi) e^{i\vec{k}_\parallel\cdot\vec{x}} \delta(\vec{k}_\perp, k_\parallel) && (5.21)\\
&= \int_0^{\chi_S} d\chi \frac{W(\chi,\chi_S)}{\chi^2} \int \frac{d\vec{k}_\parallel}{2\pi} e^{i\vec{k}_\parallel\cdot\vec{x}} \delta(\frac{\vec{l}}{\chi}, k_\parallel) && (5.22)
\end{aligned}$$

Now, we proceed to find the correlation function

$$\begin{aligned}
\langle\kappa(l)\kappa(l')\rangle &= \langle \int_0^{\chi_S} d\chi \frac{W(\chi,\chi_S)}{\chi^2} \int \frac{d\vec{k}_\parallel}{2\pi} e^{i\vec{k}_\parallel\cdot\vec{x}} \delta(\frac{\vec{l}}{\chi}, k_\parallel) \times \\
&\quad \int_0^{\chi_S} d\chi' \frac{W(\chi',\chi_S)}{\chi'^2} \int \frac{d\vec{k}_\parallel}{2\pi} e^{i\vec{k}'_\parallel\cdot\vec{x}'} \delta(\frac{\vec{l'}}{\chi'}, k_\parallel)\rangle && (5.23)\\
&= \int_0^{\chi_S} d\chi \frac{W(\chi,\chi_S)}{\chi^2} \int_0^{\chi_S} d\chi' \frac{W(\chi',\chi_S)}{\chi'^2} \int \frac{d\vec{k}_\parallel}{2\pi} e^{i\vec{k}_\parallel\cdot\vec{x}} && (5.24)\\
&\quad \int \frac{d\vec{k}_\parallel}{2\pi} e^{i\vec{k}'_\parallel\cdot\vec{x}'} \langle \delta(\frac{\vec{l}}{\chi}, k_\parallel) \delta(\frac{\vec{l'}}{\chi'}, k'_\parallel)\rangle && (5.25)
\end{aligned}$$

We rewrite the term inside the angular brackets using the Equation 6.6.

$$\begin{aligned}
\langle\kappa(l)\kappa(l')\rangle &= \int_0^{\chi_S} d\chi \frac{W(\chi,\chi_S)}{\chi^2} \int_0^{\chi_S} d\chi' \frac{W(\chi',\chi_S)}{\chi'^2} \int \frac{d\vec{k}_\parallel}{2\pi} e^{i\vec{k}_\parallel\cdot\vec{x}} && (5.26)\\
&\quad \int \frac{d\vec{k}_\parallel}{2\pi} e^{i\vec{k}'_\parallel\cdot\vec{x}'} (2\pi)^2 \delta(\frac{\vec{l}}{\chi} + \frac{\vec{l'}}{\chi'}) P(\frac{\vec{l'}}{\chi'}, k'_z)(2\pi)\delta(k_\parallel + k'_\parallel) && (5.27)\\
&= \int_0^{\chi_S} d\chi \frac{W(\chi,\chi_S)}{\chi^2} \int_0^{\chi_S} d\chi' \frac{W(\chi',\chi_S)}{\chi'^2} && (5.28)\\
&\quad \int \frac{d\vec{k}_\parallel}{2\pi} e^{i\vec{k}_\parallel\cdot(\vec{x}-\vec{x}')} (2\pi)^2 \delta(\frac{\vec{l}}{\chi} + \frac{\vec{l'}}{\chi'}) P(\frac{\vec{l'}}{\chi'}, k_z) && (5.29)
\end{aligned}$$

Let's take a break and connect this expression with the angular power spectrum. The relation between the two is

$$\langle\kappa(\vec{l})\kappa(\vec{l'})\rangle = (2\pi)^2\delta(\vec{l}+\vec{l'})C_l^{kk} \quad (5.30)$$

Therefore,

$$C_l^{kk} = \int \frac{d^2l'}{(2\pi)^2} \langle \kappa(\vec{l})\kappa(\vec{l'}) \rangle \tag{5.31}$$

Substituting the expression of the term in angular brackets, we get

$$\begin{aligned}
C_l^{kk} &= \int \frac{d^2l'}{(2\pi)^2} \int_0^{\chi_S} d\chi \frac{W(\chi,\chi_S)}{\chi^2} \int_0^{\chi_S} d\chi' \frac{W(\chi',\chi_S)}{\chi'^2} && (5.32)\\
&\quad \int \frac{d\vec{k}_\parallel}{2\pi} e^{i\vec{k}_\parallel\cdot(\vec{\chi}-\vec{\chi}')} (2\pi)^2 \delta(\frac{\vec{l}}{\chi} + \frac{\vec{l'}}{\chi'}) P(\frac{\vec{l'}}{\chi'}, k_z) && (5.33)\\
&= \int_0^{\chi_S} d\chi \frac{W(\chi,\chi_S)}{\chi^2} \int_0^{\chi_S} d\chi' \frac{W(\chi',\chi_S)}{\chi'^2} && (5.34)\\
&\quad \int \frac{d\vec{k}_\parallel}{2\pi} e^{i\vec{k}_\parallel\cdot(\vec{\chi}-\vec{\chi}')} \int d^2l' \delta(\frac{\vec{l}}{\chi} + \frac{\vec{l'}}{\chi'}) P(\frac{\vec{l'}}{\chi'}, k_z) && (5.35)\\
&= \int_0^{\chi_S} d\chi \frac{W(\chi,\chi_S)}{\chi^2} \int_0^{\chi_S} d\chi' W(\chi',\chi_S) && (5.36)\\
&\quad \int \frac{d\vec{k}_\parallel}{2\pi} e^{i\vec{k}_\parallel\cdot(\vec{\chi}-\vec{\chi}')} P(\frac{\vec{l}}{\chi}, k_\parallel) && (5.37)
\end{aligned}$$

Assuming that the $k_\perp \gg k_\parallel$, we can ignore the dependence of $P(\vec{k})$ on $k_\parallel$. Therefore, we get

$$\begin{aligned}
C_l^{kk} &= \int_0^{\chi_S} d\chi \frac{W(\chi,\chi_S)}{\chi^2} \int_0^{\chi_S} d\chi' W(\chi',\chi_S) && (5.38)\\
&\quad \int \frac{d\vec{k}_\parallel}{2\pi} e^{i\vec{k}_\parallel\cdot(\vec{\chi}-\vec{\chi}')} P(\frac{\vec{l}}{\chi}) && (5.39)\\
&= \int_0^{\chi_S} d\chi \frac{W(\chi,\chi_S)}{\chi^2} \int_0^{\chi_S} d\chi' W(\chi',\chi_S) && (5.40)\\
&\quad \delta(\vec{\chi} - \vec{\chi}') P(\frac{\vec{l}}{\chi}) && (5.41)\\
&= \int_0^{\chi_S} d\chi \frac{W(\chi,\chi_S)^2}{\chi^2} P(\frac{\vec{l}}{\chi}) && (5.42)
\end{aligned}$$

From equation 4.9, we have

$$w(\chi') = \frac{3H_0^2}{2c^2} \Omega_{m0} \frac{\chi(\chi_S - \chi)}{a\chi_S} \tag{5.43}$$

Therefore

$$C_\kappa(l) = \left(\frac{3H_0^2}{2c^2} \Omega_{m0} \right)^2 \int_0^{\chi_S} d\chi \left[\frac{(\chi_S - \chi)}{\chi_S} \frac{1}{a} \right]^2 P_\delta \left(\frac{\vec{l}}{\chi} \right) \tag{5.44}$$

6 Overdensity, power spectrum, and angular brackets

We define the over-density as

$$\delta(\mathbf{x}, t) \equiv \frac{\rho(\mathbf{x}, t) - \bar{\rho}(t)}{\bar{\rho}(t)} \tag{6.1}$$

where, $\bar{\rho}$ is the mean density at time t.

Therefore,

$$\langle \delta(\mathbf{x}) \rangle = 0 \tag{6.2}$$

We write the two point correlation function as

$$\xi(\mathbf{x}, \mathbf{y}) \equiv \langle \delta(\mathbf{x})\delta(\mathbf{y}) \rangle \tag{6.3}$$

Homogeneity requires that the correlation should depend only upon the magnitude of distance between the two positions

$$\xi(\mathbf{x}, \mathbf{y}) = \xi(\mathbf{x} - \mathbf{y}) \tag{6.4}$$

Isotropy requires

$$\xi(\mathbf{x}, \mathbf{y}) = \xi(|\mathbf{x} - \mathbf{y}|) \tag{6.5}$$

6.1 A note about the angular brackets

We will shortly calculate these integrals and show that they yield the power spectrum, but first let us reconsider the meaning of the angular brackets. So far, we have been thinking of this as an average over all space. We could still think that way, as long as switch from real space to Fourier space, but there is a more profound way to understand this averaging. Our current model of structure in the universe is that, early on, every Fourier mode $\tilde{\delta}(\vec{k})$ was a random variable drawn from a Gaussian distribution. Therefore, the angular brackets are perhaps best understood as the mean value of the quantity drawn from this distribution. For example, if the distribution were still Gaussian, then it would be proportional to $\exp\{-|\tilde{\delta}(\vec{k})|^2/2P(k)\}$, where $P(k)$ is the power spectrum. Then, the expected value of $|\tilde{\delta}(\vec{k})|^2$ would be proportional to $P(k)$. Today, due to the long-term effects of gravity, the distribution is no longer Gaussian, but there is still some underlying distribution from which the $\tilde{\delta}(\vec{k})$ are drawn. The angular brackets in Eq. (8.17) are best thought of then as referring to the expected value of $\tilde{\delta}(\vec{k})\,\tilde{\delta}(\vec{k}')$ when drawn from the distribution.

Figure 3

6.2 power spectrum

Now, we define the correlation function as

$$\begin{aligned}
\langle \delta(\kappa)\delta(\kappa) \rangle &= \int dx \int dy e^{-\kappa\cdot x} e^{-\kappa'\cdot y} \langle \delta(x)\delta(y) \rangle \\
&= \int dx \int dy e^{-\kappa\cdot x} e^{-\kappa'\cdot y} \xi(x-y) \\
&= \int dx \int dy e^{-(\kappa+\kappa')\cdot x} e^{\kappa'\cdot(x-y)} \xi(x-y) \\
&= \int dx e^{(\kappa+\kappa')\cdot x} \int dx_d e^{\kappa'\cdot(x_d)} \xi(x_d) \\
&= 2\pi\delta(\kappa+\kappa')P(k')
\end{aligned} \tag{6.6}$$

where, in the fourth equality $(x-y)$ was replaced by x_d. Minus sign coming from $-dx_d$ was cancelled by the change of variable $x \rightarrow -x$. $P(k)$ is called as the power spectrum. In this case it's a 3d power spectrum.