Contents

1 Extending Hu's quadratic estimator to three dimensions

Objective To understand eq. 22 to 31 in Zahn and Zaldarriaga 2006

1.1 Introduction

- Unlike CMB, we can use information from multiple planes in the case of 21 cm signal.
- But the different planes would be correlated. No simple way to take this correlation into account while constructing the estimator.
- Instead, we divide the 3D temperature fluctuations into radial and transverse parts.

1.2 Notation

We use the terminology and notation as described in Hogg 2000 arXiv:9905116.

Observed volume

- We define the observed volume as
  - located at redshift z.
  - having a width of solid angle $d\Omega$ along the transverse direction as seen by the observer
  - having the depth along the line of sight D_C = comoving distance between redshift z and z', where $z < z'$.
- The distance between the observer and the volume is D_M. It is called transverse comoving distance as per the definition in Hogg 2000.
- The transverse comoving size of the volume is S.

Intensity field Field $I(\mathbf{x})$ at position $\mathbf{x}$ in comoving space

$$\begin{aligned} I(\mathbf{r}) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \tilde{\mathrm{I}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \int \frac{d^2\mathbf{l}}{(2\pi)^2} \int \frac{dk_\parallel}{2\pi} \frac{\tilde{\mathrm{I}}(k_\perp, k_\parallel)}{D_M^2} e^{i(\mathbf{l}\cdot\boldsymbol{\theta} + k_\parallel x_\parallel)} \end{aligned} \tag{1.1}$$

where $\mathbf{k}$ is the wave vector.

1.3 Formalism

We divide $\mathbf{k}$ into radial (parallel) and transverse components: $k_\parallel$ and $k_\perp$. Corresponding to this, we also have $x_\parallel$ and $x_\perp$.

Discretize $k_\perp$ Using

$$\theta = \frac{S}{D_M} \tag{1.2}$$

and

$$\mathbf{l} = \frac{2\pi}{\vec{\theta}} \tag{1.3}$$

we get

$$\begin{aligned} \mathbf{k}_\perp &= \frac{2\pi}{\mathbf{S}} \\ &= \frac{2\pi}{\vec{\theta} D_M} \\ &= \frac{\mathbf{l}}{D_M} \end{aligned} \tag{1.4}$$

Discretize $k_\parallel$ Discretizing D_C into many slices along the line of sight. j is the discretizing factor.

$$k_\parallel = j\frac{2\pi}{D_C}; \delta(k_\parallel - k'_\parallel) = \frac{D_C}{2\pi}\delta_{j_1,j_2} \tag{1.5}$$

where j represents the j^{th} element of the volume.

Final expression of Intensity with discrete k

$$I(\mathbf{r}) = \int \frac{d^2\mathbf{l}}{(2\pi)^2} \sum_j \left(\frac{\tilde{\mathrm{I}}(k_\perp, k_\parallel)}{D_M^2 D_C} \right) e^{i\mathbf{k}\cdot\mathbf{r}} \tag{1.6}$$

Simplify Define

$$\hat{I}(\mathbf{l}, k_\parallel) \equiv \frac{\tilde{\mathrm{I}}(\mathbf{k}_\perp, k_\parallel)}{D_M^2 D_C} \tag{1.7}$$

Starting with

$$\tilde{\mathrm{I}}(\mathbf{k}_\perp, \kappa_\parallel; \tau) = \int dx_\parallel \int d^2\vec{x}_\perp e^{-i\mathbf{k}_\parallel \cdot x_\parallel} e^{-i\mathbf{k}_\perp \cdot x_\perp} I(\vec{x}_\perp, x_\parallel; \tau) \tag{1.8}$$

$$\begin{aligned} \langle I(\mathbf{k}_\perp, k_\parallel) I^*(\mathbf{k}'_\perp, k'_\parallel)\rangle &= \delta(\mathbf{k} - \mathbf{k}')(2\pi)^3 P(\mathbf{k}_\perp, k_\parallel) \\ &= (2\pi)^2 \delta(\mathbf{l} - \mathbf{l}') D_M^2 (2\pi) \delta(k_\parallel - k'_\parallel) P(\mathbf{k}_\perp, k_\parallel) \end{aligned} \tag{1.9}$$

where $P(k_\perp, k_\parallel)$ is the 3D power spectrum of the intensity field, we get

$$\frac{\langle I(\mathbf{k}_\perp, k_\parallel) I^*(\mathbf{k}'_\perp, k'_\parallel)\rangle}{(D_C D_M^2)^2} = \frac{(2\pi)^2 \delta(l - l') D_M^2 (2\pi) \delta(k_\parallel - k'_\parallel) P(\mathbf{k}_\perp, k_\parallel)}{(D_C D_M^2)^2} \tag{1.10}$$

$$\langle \hat{I}(l, k_\parallel) \hat{I}^*(l', k'_\parallel)\rangle = \frac{(2\pi)^2 \delta(\mathbf{l} - \mathbf{l}') (2\pi) \delta(k_\parallel - k'_\parallel) P(\mathbf{k}_\perp, k_\parallel)}{(D_C^2 D_M^2)} \tag{1.11}$$

$$= \frac{(2\pi)^2 \delta(l - l') (2\pi) \delta(j - j') (D_C/2\pi) P(\mathbf{k}_\perp, k_\parallel)}{(D_C^2 D_M^2)} \tag{1.12}$$

$$\langle \hat{I}(\mathbf{l}, j\frac{2\pi}{D_C}) \hat{I}^*(\mathbf{l}', j'\frac{2\pi}{D_C})\rangle = (2\pi)^2 \delta(\mathbf{l} - \mathbf{l}') \delta(j - j') \frac{P(\mathbf{k}_\perp, j\frac{2\pi}{D_C})}{(D_C D_M^2)} \tag{1.13}$$

This is equation 27 in ZZ2006.

Definition of $C_{\mathbf{l},j}$ We now define the angular power spectrum for the seperate values of j,

$$C_{\mathbf{l},j} \equiv \frac{P\left(\sqrt{\frac{\mathbf{l}^2}{D_M^2} + (j\frac{2\pi}{D_C})^2}\right)}{D_C D_M^2} \tag{1.14}$$

where P now represents the spherically averaged power spectrum.

Now we are going to derive Equation 31 (Equation A21) of ZZ2006.

2 Lensing reconstruction noise

2.1 Mathematical form of a quadratic estimator

Let's start with a 3D intensity source field $I_s(\theta, D_M)$, which is the unlensed field of the matter present at a transverse comoving distance D_M away from the observer.

Due to the matter present between the source and observer, the signal experiences weak lensing and in this case the signal can be written as

$$I_o(\theta, D_M) = I_s(\theta + \delta\theta, D_M) \tag{2.1}$$

where I_o is the observed lensed field. This equation says that the field that we observe at coordinates (θ, D_M) is actually coming from $(\theta + \delta\theta, D_M)$, where $\delta\theta = \bigtriangledown\hat{\Psi}$ and $\hat{\Psi}$ is the *projected potential.*

Taylor expansion of RHS of previous equation gives

$$I_o(\theta, D_M) = I_s(\theta, D_M) + \delta\theta \cdot \vec{\bigtriangledown}_\theta I_s(\theta, D_M) + \ldots \tag{2.2}$$

The Fourier transform of this expression is

$$\begin{aligned}
\int d^2\theta\, e^{-i\mathbf{l}\cdot\theta}\, I_o(\theta, D_M) &= \int d^2\theta I_s(\theta, D_M) e^{-i\mathbf{l}\cdot\theta} + \int d^2\theta\, e^{-i\mathbf{l}\cdot\theta} \delta\theta \cdot \vec{\bigtriangledown}_\theta I_s(\theta, D_M) \\
\tilde{\mathrm{I}}_o(\mathbf{l}, D_M) &= \tilde{\mathrm{I}}_s(\mathbf{l}, D_M) + \underbrace{\int d^2\theta\, e^{-i\mathbf{l}\cdot\theta}\, \delta\theta \cdot \vec{\bigtriangledown}_\theta I_s(\theta, D_M)}_{A} \qquad (2.3)
\end{aligned}$$

Using $\delta\theta(\vec{D_M}) = \bigtriangledown\hat{\Psi}(\vec{D_M})$

$$\begin{aligned}
A &= \int d^2\theta\, e^{-i\mathbf{l}\cdot\theta}\, \vec{\bigtriangledown}_\theta \Psi \cdot \vec{\bigtriangledown}_\theta I_s(\theta, D_M) && (2.4) \\
&= \int d^2\theta\, e^{-i\mathbf{l}\cdot\theta}\, \frac{1}{(2\pi)^4} \vec{\bigtriangledown}_\theta \int d^2 l' e^{i\mathbf{l}'\cdot\theta} \tilde{\Psi}(\mathbf{l}', D_M) \cdot \vec{\bigtriangledown}_\theta \int d^2 l'' e^{i\mathbf{l}''\cdot\theta} \tilde{\mathrm{I}}_s(\mathbf{l}'', D_M) && (2.5) \\
&= \int d^2\theta\, e^{-i\mathbf{l}\cdot\theta}\, \frac{1}{(2\pi)^4} \int d^2 l' \int d^2 l'' \tilde{\Psi}(\mathbf{l}', D_M) \tilde{\mathrm{I}}_s(\mathbf{l}'', D_M) \vec{\bigtriangledown}_\theta e^{i\mathbf{l}'\cdot\theta} \cdot \vec{\bigtriangledown}_\theta e^{i\mathbf{l}''\cdot\theta} && (2.6) \\
&= -\int d^2\theta\, e^{-i\mathbf{l}\cdot\theta}\, \frac{1}{(2\pi)^4} \int d^2 l' \int d^2 l'' \tilde{\Psi}(\mathbf{l}', D_M) \tilde{\mathrm{I}}_s(\mathbf{l}'', D_M) e^{i(\mathbf{l}'+\mathbf{l}'')\cdot\theta} \mathbf{l}' \cdot \mathbf{l}'' && (2.7) \\
&= -\int d^2\theta\, \frac{1}{(2\pi)^4} \int d^2 l' \int d^2 l'' e^{i(-\mathbf{l}+\mathbf{l}'+\mathbf{l}'')\cdot\theta} \mathbf{l}' \cdot \mathbf{l}''\ ](\mathbf{l}', D_M) \tilde{\mathrm{I}}_s(\mathbf{l}'', D_M) && (2.8)
\end{aligned}$$

Integrating over θ

$$A = -\int d^2l' \frac{1}{(2\pi)^2}\int d^2l''\delta(-\mathbf{l}+\mathbf{l}'+\mathbf{l}'')\mathbf{l}'\cdot\mathbf{l}''\,\tilde{\Psi}(\mathbf{l}',D_M)\tilde{\mathrm{I}}_s(\mathbf{l}'',D_M) \tag{2.9}$$

Integrating over $\mathbf{l}'$

$$A = -\frac{1}{(2\pi)^2}\int d^2l''\,\mathbf{l}''\cdot(\mathbf{l}-\mathbf{l}'')\,\tilde{\Psi}(\mathbf{l}-\mathbf{l}'',D_M)I_s(\mathbf{l}'',D_M) \tag{2.10}$$

Therefore, we get

$$\tilde{\mathrm{I}}_o(\mathbf{l},k) = \tilde{\mathrm{I}}_s(\mathbf{l},k) - \int d^2l'\frac{1}{(2\pi)^2}\tilde{\mathrm{I}}_s(\mathbf{l}',k)\tilde{\Psi}(\mathbf{l}-\mathbf{l}')(\mathbf{l}-\mathbf{l}')\cdot\mathbf{l}' \tag{2.11}$$

Computing the correlation function (and dropping k)

$$\begin{aligned}\langle\tilde{I}_o(\mathbf{l})\tilde{I}_o{}^*(\mathbf{m})\rangle_{\mathbf{m}\neq\mathbf{l}} &= -\int d^2l'\frac{1}{(2\pi)^2}\tilde{\Psi}(\mathbf{l}-\mathbf{l}')(\mathbf{l}-\mathbf{l}')\cdot\mathbf{l}'\langle\tilde{\mathrm{I}}_s(\mathbf{l}')\tilde{\mathrm{I}}_s^*(\mathbf{m})\rangle \\ &\quad -\int d^2l'\frac{1}{(2\pi)^2}\tilde{\Psi}^*(\mathbf{m}-\mathbf{l}')(\mathbf{m}-\mathbf{l}')\cdot\mathbf{l}'\langle\tilde{\mathrm{I}}_s^*(\mathbf{l}')\tilde{\mathrm{I}}_s(\mathbf{m})\rangle \\ &= -\tilde{\Psi}(\mathbf{l}-\mathbf{m})(\mathbf{l}-\mathbf{m})\cdot\mathbf{m}P_m \\ &\quad -\tilde{\Psi}^*(\mathbf{m}-\mathbf{l})(\mathbf{m}-\mathbf{l})\cdot\mathbf{l}P_l\end{aligned} \tag{2.12}$$

Now, assuming that $\mathbf{m}=\mathbf{l}-\mathbf{L}$ (and restoring k)

$$\langle\tilde{I}_o(\mathbf{l},k)\tilde{I}_o^*(\mathbf{l}-\mathbf{L},k)\rangle_{\mathbf{L}=\mathbf{l}-\mathbf{m}} = -\tilde{\Psi}(\mathbf{L},k)\mathbf{L}\cdot(\mathbf{l}-\mathbf{L})P_{l-L,k}+\tilde{\Psi}^*(-\mathbf{L},k)\mathbf{L}\cdot\mathbf{l}P_{l,k} \tag{2.13}$$

$$= -\tilde{\Psi}(\mathbf{L},k)\mathbf{L}\cdot[(\mathbf{l}-\mathbf{L})P_{l-L,k}+\cdot\mathbf{l}P_{l,k}] \tag{2.14}$$

Therefore we can define the quadratic estimator as

$$\hat{\tilde{\Psi}}(\mathbf{L}) \propto \tilde{\mathrm{I}}_o(\mathbf{l})\tilde{\mathrm{I}}_o(\mathbf{l}-\mathbf{L}) \tag{2.15}$$

2.2 Deriving the estimator

We therefore start with a quadratic estimator $\hat{\Psi}(\mathbf{L})$ for $\tilde{\psi}(\mathbf{L})$, i.e. of the form

$$\hat{\Psi}(\mathbf{L}) = \int\frac{d^2l}{(2\pi)^2}\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}F(\mathbf{l},k_1,k_2,\mathbf{L})\tilde{\mathrm{I}}_o(\mathbf{l},k_1)\tilde{\mathrm{I}}(\mathbf{L}-\mathbf{l},k_2) \tag{2.16}$$

(notice that $F(\mathbf{l},k_1,k_2,\mathbf{L}) = F(\mathbf{l},k_2,k_1,\mathbf{L})$). Because $\delta\hat{\Psi}(\mathbf{L}) = \delta\hat{\Psi}^*(-\mathbf{L})$ it can also be shown that

$$F(\mathbf{l},k_1,k_2,\mathbf{L}) = F^*(-\mathbf{l},-k_1,-k_2,-\mathbf{L}) \tag{2.17}$$

We want to find F such that it minimizes the variance of $\tilde{\Psi}(\mathbf{L})$ under the condition that its ensemble average recovers the lensing field, $\langle\hat{\Psi}(\mathbf{L})\rangle_{\mathrm{I}} = \tilde{\psi}(\mathbf{L})$. This becomes (to first order in $\tilde{\psi}$)

$$\begin{aligned}\langle\hat{\Psi}(\mathbf{L})\rangle_{\mathrm{I}} = &\int\frac{d^2l}{(2\pi)^2}\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}F(\mathbf{l},k_1,k_2,\mathbf{L})(2\pi)\delta(k_1+k_2) \\ &\times\Big[\mathrm{P}_{l,k}\tilde{\psi}(\mathbf{L})\mathbf{L}\cdot\mathbf{l}+\mathrm{P}_{L-l,k}\tilde{\psi}(\mathbf{L})\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})\Big]\,,\end{aligned} \tag{2.18}$$

where e.g. $\mathrm{P}_{l,k}$ is the power in a mode with angular component l and radial component k. With the requirement that $\langle\hat{\Psi}(\mathbf{L})\rangle_{\mathrm{I}} = \tilde{\psi}(\mathbf{L})$ this leads to the normalization condition

$$\int \frac{d^2l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} F(\mathbf{l}, k_1, k_2, \mathbf{L})(2\pi)\delta^D(k_1+k_2)\left[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l} + \mathrm{P}_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})\right] = 1 \tag{2.19}$$

The condition of minimization of the variance gives

$$\begin{aligned}\langle||\hat{\Psi}(\mathbf{L})||^2\rangle_{\tilde{\mathrm{I}}} &= \int \frac{d^2l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi}(2\pi)^2\delta(0)F(\mathbf{l}, k_1, k_2, \mathbf{L})F^*(\mathbf{l}', k_1', k_2', \mathbf{L}')\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,k_2} \\ &+ \int \frac{d^2l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi}(2\pi)^2\delta(0)F(\mathbf{l}, k_1, k_2, \mathbf{L})F^*(\mathbf{L}-\mathbf{l}', k_2', k_1', \mathbf{L}')\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,k_2}\end{aligned} \tag{2.20}$$

but from 2.16 we see with the substitution $\mathbf{L}-\mathbf{l}\to\mathbf{l}$ that $F^*(\mathbf{L}-\mathbf{l}, k_2, k_1, \mathbf{L}) = F^*(\mathbf{l}, k_1, k_2, \mathbf{L})$ hence

$$\langle||\hat{\Psi}(\mathbf{L})||^2\rangle = 2(2\pi)^2\delta(0)\int \frac{d^2l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} F(\mathbf{l}, k_1, k_2, \mathbf{L})F^*(\mathbf{l}, k_1, k_2, \mathbf{L})\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_2}\,. \tag{2.21}$$

Both real and imaginary part of $||F||^2 = F_R^2 + F_I^2$ contribute to this variance, however the condition for the minimization will only pick out the real part. (**Does this really matter?**) The solution is found by minimizing the function

$$\langle||\hat{\Psi}(\mathbf{L})||^2\rangle - A_R \times (\,\text{Equation } 2.19) \tag{2.22}$$

with respect to F, where A_R is a Lagrangian multiplier. In steps,

$$\frac{\partial(\,\text{Eq. } 2.19)}{\partial F(\mathbf{l}, k_1, k_2, \mathbf{L})} = A_R \frac{d^2l}{(2\pi)^2}\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}(2\pi)\delta^D(k_1+k_2)[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l} + \mathrm{P}_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})] \tag{2.23}$$

and

$$\frac{\partial\langle||\hat{\Psi}(\mathbf{L})||^2\rangle}{\partial F(\mathbf{l}, k_1, k_2, \mathbf{L})} = 2(2\pi)^2\delta(0)\frac{d^2l}{(2\pi)^2}\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}2F_R(\mathbf{l}, k_1, k_2, \mathbf{L})\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,k_2} \tag{2.24}$$

so

$$4(2\pi)^2\delta(0)F_R(\mathbf{l}, k_1, k_2, \mathbf{L}) = A_R(2\pi)\delta^D(k_1+k_2)\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l} + \mathrm{P}_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,k_2}} \tag{2.25}$$

Therefore,

$$F_R(\mathbf{l}, k_1, k_2, \mathbf{L}) = \frac{1}{4(2\pi)\delta(0)}A_R\delta^D(k_1+k_2)\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l} + \mathrm{P}_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,k_2}} \tag{2.26}$$

and by inserting this into the normalization condition 2.19 we get that

$$\begin{aligned}1 &= \int \frac{d^2l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi}\frac{1}{4\delta(0)}A_R\delta^D(k_1+k_2)\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l} + \mathrm{P}_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,k_2}}\delta^D(k_1+k_2) \\ &= \int \frac{d^2l}{(2\pi)^2} \int \frac{dk_1}{2\pi}\frac{\delta(0)}{4\delta(0)}A_R\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l} + \mathrm{P}_{L-l,k_2=-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,k_2=-k_1}}\end{aligned} \tag{2.27}$$

Assuming that I can just cancel the factors of $\delta(0)$,

$$1=\int\frac{d^2l}{(2\pi)^2}\int dk_1\frac{1}{4\times 2\pi}A_R\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l}+\mathrm{P}_{L-l,-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,-k_1}} \tag{2.28}$$

Discretizing the integal over dk_1. Assumming that we have dicretized the space in the radial direction in the blocks of length D_C such that $k=\frac{2\pi}{D_C}$.
Therefore, we can write

$$\int\frac{dk}{2\pi}=\frac{1}{D_C}\sum_k \tag{2.29}$$

Therefore, we get

$$1=\int\frac{d^2l}{(2\pi)^2}\frac{1}{D_C}\sum_{k_1}\frac{1}{4}A_R\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l}+\mathrm{P}_{L-l,-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,-k_1}} \tag{2.30}$$

which gives,

$$A_R=\frac{4D_C}{\int\frac{d^2l}{(2\pi)^2}\sum_{k_1}\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l}+\mathrm{P}_{L-l,-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,-k_1}}} \tag{2.31}$$

Substituting this expression in Equation 2.26, we get

$$\begin{aligned}F_R(\mathbf{l},k_1,k_2,\mathbf{L})&=\frac{A_R}{4(2\pi)\delta(0)}\delta^D(k_1+k_2)\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l}+\mathrm{P}_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,k_2}}\\&=\frac{D_C/2\pi\delta(0)}{\int\frac{d^2l}{(2\pi)^2}\sum_{k_1}\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l}+\mathrm{P}_{L-l,-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,-k_1}}}\delta^D(k_1+k_2)\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l}+\mathrm{P}_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,k_2}}\\&=\frac{1}{\int\frac{d^2l}{(2\pi)^2}\sum_{k_1}\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l}+\mathrm{P}_{L-l,-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,-k_1}}}\delta^D(k_1+k_2)\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l}+\mathrm{P}_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,k_2}}\end{aligned} \tag{2.32}$$

Now, substituting this expression in Equation 2.21. Taking $FF^*\equiv F^2$

$$
\begin{aligned}
\langle||\hat{\Psi}(\mathbf{L})||^2\rangle &= 2(2\pi)^2\delta(0)\int\frac{d^2l}{(2\pi)^2}\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}F(\mathbf{l},k_1,k_2,\mathbf{L})F^*(\mathbf{l},k_1,k_2,\mathbf{L})\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_2} \\
&= 2(2\pi)^2\delta(0)\int\frac{d^2l}{(2\pi)^2}\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}F^2(\mathbf{l},k_1,k_2,\mathbf{L})\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_2} \\
&= 2(2\pi)^2\delta(0)\int\frac{d^2l}{(2\pi)^2}\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}\left(\frac{1}{\int\frac{d^2l}{(2\pi)^2}\sum_{k_1}\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l}+\mathrm{P}_{L-l,-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,-k_1}}}\right. \\
&\left.\delta^D(k_1+k_2)\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l}+\mathrm{P}_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,k_2}}\right)^2\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_2} \\
&= 2(2\pi)^2\delta(0)\int\frac{d^2l}{(2\pi)^2}\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}\left(\frac{1}{\int\frac{d^2l}{(2\pi)^2}\sum_{k_1}\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l}+\mathrm{P}_{L-l,-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,-k_1}}}\right)^2 \\
&\left(\delta^D(k_1+k_2)\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l}+\mathrm{P}_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,k_2}}\right)^2\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_2} \\
&= 2(2\pi)^2\delta(0)\int\frac{d^2l}{(2\pi)^2}\frac{\sum_{k_1}}{D_C}\frac{\sum_{k_2}}{D_C}\left(\frac{1}{\int\frac{d^2l}{(2\pi)^2}\sum_{k_1}\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l}+\mathrm{P}_{L-l,-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,-k_1}}}\right)^2 \\
&\left(\delta^D(k_1+k_2)\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l}+\mathrm{P}_{L-l,k_2}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,k_2}}\right)^2\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_2} \\
&= 2(2\pi)^2\delta(0)\frac{1}{\int\frac{d^2l}{(2\pi)^2}\sum_{k_1}\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l}+\mathrm{P}_{L-l,-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,-k_1}}}
\end{aligned}
\tag{2.33}
$$

With the definition

$$
\langle\hat{\Psi}(\mathbf{L})\hat{\Psi}^*(L')\rangle = (2\pi)^2\delta^D(\mathbf{L}-\mathbf{L}')N^{\hat{\Psi}} \tag{2.34}
$$

Therefore, we finally have

$$
\begin{aligned}
\langle||\hat{\Psi}(\mathbf{L})||^2\rangle &= \frac{2(2\pi)D_C}{\int\frac{d^2l}{(2\pi)^2}\sum_{k_1}\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l}+\mathrm{P}_{L-l,-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,-k_1}}} \\
&= (2\pi)^2\delta(0)N^{\hat{\Psi}}
\end{aligned}
\tag{2.35}
$$

Which gives

$$
N^{\hat{\Psi}}_L = \frac{1}{\int\frac{d^2l}{(2\pi)^2}\sum_{k_1}\frac{[\mathrm{P}_{l,k_1}\mathbf{L}\cdot\mathbf{l}+\mathrm{P}_{L-l,-k_1}\mathbf{L}\cdot(\mathbf{L}-\mathbf{l})]^2}{2\tilde{\mathrm{P}}^{\mathrm{tot}}_{l,k_1}\tilde{\mathrm{P}}^{\mathrm{tot}}_{L-l,-k_1}}} \tag{2.36}
$$

This is equation A20 in ZZ2006.

3 Notes on lensing a Poisson distributed surface brightness by Metcalf and Alkistis

3.1 Equation 1

3.1.1 Discrete and Continuous FT

For a 3D signal, we have

$$\tilde{\mathrm{I}}_s(\mathbf{k}) = \int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} I_s(\mathbf{x}) \tag{3.1}$$

We are probing a volume of dimensions $(\theta D_M \times \theta D_M \times D_C)$, where D_C is the length of the volume along the line of sight between $z = z_1$ and $z = z_2$. We divide the it into $N_\perp$ parts along the transverse direction and $N_\parallel$ parts along the radial direction.

Discretizing $\tilde{\mathrm{I}}_s(k)$

$$\tilde{\mathrm{I}}_s(\mathbf{k}) = \sum_{t_1} \frac{\Theta_s D_M}{N_\perp} \sum_{t_2} \frac{\Theta_s D_M}{N_\perp} \sum_{p} \frac{D_C}{N_\parallel} e^{-i\mathbf{k}\cdot(\frac{D_C}{N_\parallel}p + \frac{\Theta_s}{N_\perp}t_1 + \frac{\Theta_s}{N_\perp}t_2)} I_s(\mathbf{x}) \tag{3.2}$$

Using

$$\begin{aligned} \Theta_s \times \Theta_s &\equiv \Omega_s && (3.3)\\ N_\perp^2 &\equiv N_\perp, && (3.4) \end{aligned}$$

we get,

$$\tilde{\mathrm{I}}_s(\mathbf{k}) = \sum_{\mathbf{x}} \frac{\Omega_s D_M^2}{N_\perp^2} \frac{D_C}{N_\parallel} e^{-i\mathbf{k}\cdot\mathbf{x}} I_s(\mathbf{x}) \tag{3.5}$$

Upon normalizing this result with a factor of $D_M^2 D_C$, we get

$$\tilde{I}_s(\mathbf{k}) = \sum_{\mathbf{x}} \frac{\Omega_s}{N_\perp N_\parallel} e^{-i\mathbf{k}\cdot\mathbf{x}} I_s(\mathbf{x}) \tag{3.6}$$

3.2 Equation 2

Writing the inverse FT, we get

$$I_s(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{\mathrm{I}}_s(\mathbf{k}) \tag{3.7}$$

Upon discretization, we get

$$I_s(\mathbf{x}) = \frac{1}{(2\pi)^3} \sum_{\mathbf{k}} \frac{2\pi}{D_C} \frac{(2\pi)^2}{\Omega_s D_M^2} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{\mathrm{I}}_s(\mathbf{k}) \tag{3.8}$$

Absorbing the factor of $D_M^2 D_C$ into $\tilde{\mathrm{I}}(\mathbf{k})$.

$$I_s(\mathbf{x}) = \sum_{\mathbf{k}} \frac{1}{\Omega_s} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{\mathrm{I}}_s(\mathbf{k}) \tag{3.9}$$

3.3 Equation 5

starting with

$$\tilde{\mathrm{I}}_s(\mathbf{k}) = \int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} I_s(\mathbf{x}) \tag{3.10}$$

we get

$$\begin{aligned}
\langle \tilde{\mathrm{I}}_s(k)\tilde{\mathrm{I}}^*(k')\rangle &= \int d^3x \int d^3x' \langle I_s(\mathbf{x}) I_s'(\mathbf{x}')\rangle e^{-i\mathbf{k}\mathbf{x}} e^{i\mathbf{k}'\mathbf{x}'} && (3.11)\\
&= \int d^3x \int d^3x' \int \frac{d^3k''}{(2\pi)^3} P(\mathbf{k}'') e^{i\mathbf{k}''(\mathbf{x}-\mathbf{x}')} e^{-i\mathbf{k}\mathbf{x}} e^{i\mathbf{k}'\mathbf{x}'} && (3.12)\\
\langle \tilde{I}_s(k)\tilde{I}_s^*(k')\rangle &= \sum_{\mathbf{x}} \frac{\Omega_s}{N_\parallel N_\perp} \sum_{\mathbf{x}'} \frac{\Omega_s}{N_\parallel N_\perp} \sum_{\mathbf{k}''} \frac{P(\mathbf{k}'')}{\Omega_s D^2 \mathcal{L}} e^{i\mathbf{k}''(\mathbf{x}-\mathbf{x}')} e^{-i\mathbf{k}\mathbf{x}} e^{i\mathbf{k}'\mathbf{x}'} && (3.13)\\
&= \sum_{\mathbf{x}} \frac{\Omega_s}{N_\parallel N_\perp} \sum_{\mathbf{x}'} \frac{\Omega_s}{N_\parallel N_\perp} \sum_{\mathbf{k}''} \frac{P(\mathbf{k}'')}{\Omega_s D^2 \mathcal{L}} e^{-i(\mathbf{k}-\mathbf{k}'')\mathbf{x}} e^{-i(\mathbf{k}''-\mathbf{k}')\mathbf{x}'} && (3.14)\\
&= \Omega_s \frac{P(\mathbf{k}'')}{D^2\mathcal{L}} \delta^k(\mathbf{l},\mathbf{l}')\delta^k(j,j') && (3.15)\\
&= \Omega_s C_{\mathbf{l},j} \delta^k(\mathbf{l},\mathbf{l}')\delta^k(j,j') && (3.16)
\end{aligned}$$

where $C_{lj} \equiv \frac{P(\mathbf{k}'')}{D^2\mathcal{L}}$ is the discrete angular power spectrum.