1 Distance and Redshift Integrals

Here $E(z) = H(z)/H_0$, $\mathcal{D}$ is angular diameter distance. χ is comoving radial distance

$$
\begin{aligned}
\int_0^{z_s} f\left(z, \frac{\ell}{\mathcal{D}}\right) \frac{1}{E(z)} dz &= \int_0^{z_s} f\left(z, \frac{\ell}{\mathcal{D}}\right) \frac{1}{E(z)} dz & (1) \\
&= \int f\left(z, \frac{\ell}{\mathcal{D}}\right) \frac{1}{E(z)} \frac{1}{a^2} da & (2) \\
&= \int f\left(z, \frac{\ell}{\mathcal{D}}\right) \frac{1}{E(z)} \frac{1}{a^2} \frac{da}{dt} dt & (3) \\
&= \int f\left(z, \frac{\ell}{\mathcal{D}}\right) \frac{1}{E(z)} \frac{1}{a} \frac{\dot{a}}{a} dt & (4) \\
&= \int f\left(z, \frac{\ell}{\mathcal{D}}\right) \frac{1}{E(z)} \frac{1}{a} H(z) dt & (5) \\
&= \int f\left(z, \frac{\ell}{\mathcal{D}}\right) \frac{1}{E(z)} \frac{1}{a} \frac{cH(z)}{c} dt & (6) \\
&= \int f\left(z, \frac{\ell}{\mathcal{D}}\right) \frac{1}{a} \frac{cH_0}{c} dt & (7) \\
&= \frac{H_0}{c} \int f\left(z, \frac{\ell}{\mathcal{D}}\right) \frac{cdt}{a} & (8) \\
&= \frac{H_0}{c} \int f\left(z, \frac{\ell a}{\mathcal{D}a}\right) \frac{cdt}{a} & (9) \\
&= \frac{H_0}{c} \int_0^{\chi_s} f\left(z, \frac{\ell}{\chi a}\right) d\chi & (10) \\
&= \frac{H_0}{c} \int_0^{\chi_s} f\left(z, \frac{\ell(1+z)}{\chi}\right) d\chi & (11) \\
& & (12)
\end{aligned}
$$