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1 Extending Hu's quadratic estimator to three dimensions

Objective To understand eq. 22 to 31 in Zahn and Zaldarriaga 2006

1.1 Introduction

- Unlike CMB, we can use information from multiple planes in the case of 21 cm signal.
- But the different planes would be correlated. No simple way to take this correlation into account while constructing the estimator.
- Instead, we divide the 3D temperature fluctuations into radial and transverse parts.

1.2 Notation

We use the terminology and notation as described in [Hogg 2000 arXiv:9905116](#).

Observed volume

- We define the observed volume as
 - located at redshift z .
 - having a width of solid angle $d\Omega$ along the transverse direction as seen by the observer
 - having the depth along the line of sight $D_C =$ comoving distance between redshift z and z' , where $z < z'$.
- The distance between the observer and the volume is D_M . It is called transverse comoving distance as per the definition in Hogg 2000.
- The transverse comoving size of the volume is S .

Intensity field Field $I(\mathbf{x})$ at position \mathbf{x} in comoving space

$$\begin{aligned} I(\mathbf{r}) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \tilde{I}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \int \frac{d^2\mathbf{l}}{(2\pi)^2} \int \frac{dk_{\parallel}}{2\pi} \frac{\tilde{I}(k_{\perp}, k_{\parallel})}{D_M^2} e^{i(\mathbf{l}\cdot\theta + k_{\parallel}x_{\parallel})} \end{aligned} \tag{1.1}$$

where \mathbf{k} is the wave vector.

1.3 Formalism

We divide \mathbf{k} into radial (parallel) and transverse components: k_{\parallel} and k_{\perp} . Corresponding to this, we also have x_{\parallel} and x_{\perp} .

Discretize k_\perp Using

$$\theta = \frac{S}{D_M} \quad (1.2)$$

and

$$\mathbf{l} = \frac{2\pi}{\vec{\theta}} \quad (1.3)$$

we get

$$\begin{aligned} \mathbf{k}_\perp &= \frac{2\pi}{\mathbf{S}} \\ &= \frac{2\pi}{\vec{\theta} D_M} \\ &= \frac{\mathbf{l}}{D_M} \end{aligned} \quad (1.4)$$

Discretize k_\parallel Discretizing D_C into many slices along the line of sight. j is the discretizing factor.

$$k_\parallel = j \frac{2\pi}{D_C}; \delta(k_\parallel - k'_\parallel) = \frac{D_C}{2\pi} \delta_{j_1, j_2} \quad (1.5)$$

where j represents the j^{th} element of the volume.

Final expression of Intensity with discrete \mathbf{k}

$$I(\mathbf{r}) = \int \frac{d^2 \mathbf{l}}{(2\pi)^2} \sum_j \left(\frac{\tilde{I}(k_\perp, k_\parallel)}{D_M^2 D_C} \right) e^{i\mathbf{k} \cdot \mathbf{r}} \quad (1.6)$$

Simplify Define

$$\hat{I}(\mathbf{l}, k_\parallel) \equiv \frac{\tilde{I}(\mathbf{k}_\perp, k_\parallel)}{D_M^2 D_C} \quad (1.7)$$

Starting with

$$\tilde{I}(\mathbf{k}_\perp, \kappa_\parallel; \tau) = \int dx_\parallel \int d^2 \vec{x}_\perp e^{-i\mathbf{k}_\parallel \cdot x_\parallel} e^{-i\mathbf{k}_\perp \cdot x_\perp} I(\vec{x}_\perp, x_\parallel; \tau) \quad (1.8)$$

$$\begin{aligned} \langle I(\mathbf{k}_\perp, k_\parallel) I^*(\mathbf{k}'_\perp, k'_\parallel) \rangle &= \delta(\mathbf{k} - \mathbf{k}') (2\pi)^3 P(\mathbf{k}_\perp, k_\parallel) \\ &= (2\pi)^2 \delta(\mathbf{l} - \mathbf{l}') D_M^2 (2\pi) \delta(k_\parallel - k'_\parallel) P(\mathbf{k}_\perp, k_\parallel) \end{aligned} \quad (1.9)$$

where $P(k_\perp, k_\parallel)$ is the 3D power spectrum of the intensity field, we get

$$\frac{\langle I(\mathbf{k}_\perp, k_\parallel) I^*(\mathbf{k}'_\perp, k'_\parallel) \rangle}{(D_C D_M^2)^2} = \frac{(2\pi)^2 \delta(\mathbf{l} - \mathbf{l}') D_M^2 (2\pi) \delta(k_\parallel - k'_\parallel) P(\mathbf{k}_\perp, k_\parallel)}{(D_C D_M^2)^2} \quad (1.10)$$

$$\langle \hat{I}(\mathbf{l}, k_\parallel) \hat{I}^*(\mathbf{l}', k'_\parallel) \rangle = \frac{(2\pi)^2 \delta(\mathbf{l} - \mathbf{l}') (2\pi) \delta(k_\parallel - k'_\parallel) P(\mathbf{k}_\perp, k_\parallel)}{(D_C^2 D_M^2)} \quad (1.11)$$

$$= \frac{(2\pi)^2 \delta(\mathbf{l} - \mathbf{l}') (2\pi) \delta(j - j') (D_C / 2\pi) P(\mathbf{k}_\perp, k_\parallel)}{(D_C^2 D_M^2)} \quad (1.12)$$

$$\langle \hat{I}(\mathbf{l}, j \frac{2\pi}{D_C}) \hat{I}^*(\mathbf{l}', j' \frac{2\pi}{D_C}) \rangle = (2\pi)^2 \delta(\mathbf{l} - \mathbf{l}') \delta(j - j') \frac{P(\mathbf{k}_\perp, j \frac{2\pi}{D_C})}{(D_C D_M^2)} \quad (1.13)$$

This is equation 27 in ZZ2006.

Definition of $C_{1,j}$ We now define the angular power spectrum for the separate values of j ,

$$C_{1,j} \equiv \frac{P \left(\sqrt{\frac{l^2}{D_M^2} + (j \frac{2\pi}{D_C})^2} \right)}{D_C D_M^2} \quad (1.14)$$

where P now represents the spherically averaged power spectrum.

Now we are going to derive Equation 31 (Equation A21) of ZZ2006.

2 Lensing reconstruction noise

2.1 Mathematical form of a quadratic estimator

Let's start with a 3D intensity source field $I_s(\theta, D_M)$, which is the unlensed field of the matter present at a transverse comoving distance D_M away from the observer.

Due to the matter present between the source and observer, the signal experiences weak lensing and in this case the signal can be written as

$$I_o(\theta, D_M) = I_s(\theta + \delta\theta, D_M) \quad (2.1)$$

where I_o is the observed lensed field. This equation says that the field that we observe at coordinates (θ, D_M) is actually coming from $(\theta + \delta\theta, D_M)$, where $\delta\theta = \nabla \hat{\Psi}$ and $\hat{\Psi}$ is the *projected potential*.

Taylor expansion of RHS of previous equation gives

$$I_o(\theta, D_M) = I_s(\theta, D_M) + \delta\theta \cdot \vec{\nabla}_\theta I_s(\theta, D_M) + \dots \quad (2.2)$$

The Fourier transform of this expression is

$$\begin{aligned} \int d^2\theta e^{-i\mathbf{l}\cdot\theta} I_o(\theta, D_M) &= \int d^2\theta I_s(\theta, D_M) e^{-i\mathbf{l}\cdot\theta} + \int d^2\theta e^{-i\mathbf{l}\cdot\theta} \delta\theta \cdot \vec{\nabla}_\theta I_s(\theta, D_M) \\ \tilde{I}_o(\mathbf{l}, D_M) &= \tilde{I}_s(\mathbf{l}, D_M) + \underbrace{\int d^2\theta e^{-i\mathbf{l}\cdot\theta} \delta\theta \cdot \vec{\nabla}_\theta I_s(\theta, D_M)}_A \end{aligned} \quad (2.3)$$

Using $\delta\theta(\vec{D}_M) = \nabla \hat{\Psi}(\vec{D}_M)$

$$A = \int d^2\theta e^{-i\mathbf{l}\cdot\theta} \vec{\nabla}_\theta \hat{\Psi} \cdot \vec{\nabla}_\theta I_s(\theta, D_M) \quad (2.4)$$

$$= \int d^2\theta e^{-i\mathbf{l}\cdot\theta} \frac{1}{(2\pi)^4} \vec{\nabla}_\theta \int d^2l' e^{i\mathbf{l}'\cdot\theta} \tilde{\Psi}(\mathbf{l}', D_M) \cdot \vec{\nabla}_\theta \int d^2l'' e^{i\mathbf{l}''\cdot\theta} \tilde{I}_s(\mathbf{l}'', D_M) \quad (2.5)$$

$$= \int d^2\theta e^{-i\mathbf{l}\cdot\theta} \frac{1}{(2\pi)^4} \int d^2l' \int d^2l'' \tilde{\Psi}(\mathbf{l}', D_M) \tilde{I}_s(\mathbf{l}'', D_M) \vec{\nabla}_\theta e^{i\mathbf{l}'\cdot\theta} \cdot \vec{\nabla}_\theta e^{i\mathbf{l}''\cdot\theta} \quad (2.6)$$

$$= - \int d^2\theta e^{-i\mathbf{l}\cdot\theta} \frac{1}{(2\pi)^4} \int d^2l' \int d^2l'' \tilde{\Psi}(\mathbf{l}', D_M) \tilde{I}_s(\mathbf{l}'', D_M) e^{i(\mathbf{l}' + \mathbf{l}'')\cdot\theta} \mathbf{l}' \cdot \mathbf{l}'' \quad (2.7)$$

$$= - \int d^2\theta \frac{1}{(2\pi)^4} \int d^2l' \int d^2l'' e^{i(-\mathbf{l} + \mathbf{l}' + \mathbf{l}'')\cdot\theta} \mathbf{l}' \cdot \mathbf{l}''](\mathbf{l}', D_M) \tilde{I}_s(\mathbf{l}'', D_M) \quad (2.8)$$

Integrating over θ

$$A = - \int d^2 l' \frac{1}{(2\pi)^2} \int d^2 l'' \delta(-\mathbf{l} + \mathbf{l}' + \mathbf{l}'') \mathbf{l}' \cdot \mathbf{l}'' \tilde{\Psi}(\mathbf{l}', D_M) \tilde{\mathbf{I}}_s(\mathbf{l}'', D_M) \quad (2.9)$$

Integrating over \mathbf{l}'

$$A = - \frac{1}{(2\pi)^2} \int d^2 l'' \mathbf{l}'' \cdot (\mathbf{l} - \mathbf{l}'') \tilde{\Psi}(\mathbf{l} - \mathbf{l}'', D_M) I_s(\mathbf{l}'', D_M) \quad (2.10)$$

Therefore, we get

$$\tilde{\mathbf{I}}_o(\mathbf{l}, k) = \tilde{\mathbf{I}}_s(\mathbf{l}, k) - \int d^2 l' \frac{1}{(2\pi)^2} \tilde{\mathbf{I}}_s(\mathbf{l}', k) \tilde{\Psi}(\mathbf{l} - \mathbf{l}') (\mathbf{l} - \mathbf{l}') \cdot \mathbf{l}' \quad (2.11)$$

Computing the correlation function (and dropping k)

$$\begin{aligned} \langle \tilde{I}_o(\mathbf{l}) \tilde{I}_o^*(\mathbf{m}) \rangle_{\mathbf{m} \neq \mathbf{l}} &= - \int d^2 l' \frac{1}{(2\pi)^2} \tilde{\Psi}(\mathbf{l} - \mathbf{l}') (\mathbf{l} - \mathbf{l}') \cdot \mathbf{l}' \langle \tilde{\mathbf{I}}_s(\mathbf{l}') \tilde{\mathbf{I}}_s^*(\mathbf{m}) \rangle \\ &\quad - \int d^2 l' \frac{1}{(2\pi)^2} \tilde{\Psi}^*(\mathbf{m} - \mathbf{l}') (\mathbf{m} - \mathbf{l}') \cdot \mathbf{l}' \langle \tilde{\mathbf{I}}_s^*(\mathbf{l}') \tilde{\mathbf{I}}_s(\mathbf{m}) \rangle \\ &= - \tilde{\Psi}(\mathbf{l} - \mathbf{m}) (\mathbf{l} - \mathbf{m}) \cdot \mathbf{m} P_m \\ &\quad - \tilde{\Psi}^*(\mathbf{m} - \mathbf{l}) (\mathbf{m} - \mathbf{l}) \cdot \mathbf{l} P_l \end{aligned} \quad (2.12)$$

Now, assuming that $\mathbf{m} = \mathbf{l} - \mathbf{L}$ (and restoring k)

$$\langle \tilde{I}_o(\mathbf{l}, k) \tilde{I}_o^*(\mathbf{l} - \mathbf{L}, k) \rangle_{\mathbf{L} = \mathbf{l} - \mathbf{m}} = - \tilde{\Psi}(\mathbf{L}, k) \mathbf{L} \cdot (\mathbf{l} - \mathbf{L}) P_{l-L, k} + \tilde{\Psi}^*(-\mathbf{L}, k) \mathbf{L} \cdot \mathbf{l} P_{l, k} \quad (2.13)$$

$$= - \tilde{\Psi}(\mathbf{L}, k) \mathbf{L} \cdot [(\mathbf{l} - \mathbf{L}) P_{l-L, k} + \mathbf{l} P_{l, k}] \quad (2.14)$$

Therefore we can define the quadratic estimator as

$$\hat{\tilde{\Psi}}(\mathbf{L}) \propto \tilde{\mathbf{I}}_o(\mathbf{l}) \tilde{\mathbf{I}}_o(\mathbf{l} - \mathbf{L}) \quad (2.15)$$

2.2 Deriving the estimator

We therefore start with a quadratic estimator $\hat{\tilde{\Psi}}(\mathbf{L})$ for $\tilde{\psi}(\mathbf{L})$, i.e. of the form

$$\hat{\tilde{\Psi}}(\mathbf{L}) = \int \frac{d^2 l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} F(\mathbf{l}, k_1, k_2, \mathbf{L}) \tilde{\mathbf{I}}_o(\mathbf{l}, k_1) \tilde{\mathbf{I}}(\mathbf{L} - \mathbf{l}, k_2) \quad (2.16)$$

(notice that $F(\mathbf{l}, k_1, k_2, \mathbf{L}) = F(\mathbf{l}, k_2, k_1, \mathbf{L})$). Because $\delta\hat{\tilde{\Psi}}(\mathbf{L}) = \delta\hat{\tilde{\Psi}}^*(-\mathbf{L})$ it can also be shown that

$$F(\mathbf{l}, k_1, k_2, \mathbf{L}) = F^*(-\mathbf{l}, -k_1, -k_2, -\mathbf{L}) \quad (2.17)$$

We want to find F such that it minimizes the variance of $\hat{\tilde{\Psi}}(\mathbf{L})$ under the condition that its ensemble average recovers the lensing field, $\langle \hat{\tilde{\Psi}}(\mathbf{L}) \rangle_{\mathbf{I}} = \tilde{\psi}(\mathbf{L})$. This becomes (to first order in $\tilde{\psi}$)

$$\begin{aligned} \langle \hat{\tilde{\Psi}}(\mathbf{L}) \rangle_{\mathbf{I}} &= \int \frac{d^2 l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} F(\mathbf{l}, k_1, k_2, \mathbf{L}) (2\pi) \delta(k_1 + k_2) \\ &\quad \times \left[P_{l, k} \tilde{\psi}(\mathbf{L}) \mathbf{L} \cdot \mathbf{l} + P_{L-l, k} \tilde{\psi}(\mathbf{L}) \mathbf{L} \cdot (\mathbf{L} - \mathbf{l}) \right], \end{aligned} \quad (2.18)$$

where e.g. $P_{l,k}$ is the power in a mode with angular component l and radial component k . With the requirement that $\langle \hat{\Psi}(\mathbf{L}) \rangle_I = \tilde{\psi}(\mathbf{L})$ this leads to the normalization condition

$$\int \frac{d^2 l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} F(\mathbf{l}, k_1, k_2, \mathbf{L}) (2\pi) \delta^D(k_1 + k_2) [P_{l,k_1} \mathbf{L} \cdot \mathbf{l} + P_{L-l,k_2} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})] = 1 \quad (2.19)$$

The condition of minimization of the variance gives

$$\begin{aligned} \langle ||\hat{\Psi}(\mathbf{L})||^2 \rangle_I &= \int \frac{d^2 l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} (2\pi)^2 \delta(0) F(\mathbf{l}, k_1, k_2, \mathbf{L}) F^*(\mathbf{l}', k'_1, k'_2, \mathbf{L}') \tilde{P}_{l,k_1}^{\text{tot}} \tilde{P}_{L-l,k_2}^{\text{tot}} \\ &\quad + \int \frac{d^2 l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} (2\pi)^2 \delta(0) F(\mathbf{l}, k_1, k_2, \mathbf{L}) F^*(\mathbf{L} - \mathbf{l}', k'_2, k'_1, \mathbf{L}') \tilde{P}_{l,k_1}^{\text{tot}} \tilde{P}_{L-l,k_2}^{\text{tot}} \end{aligned} \quad (2.20)$$

but from 2.16 we see with the substitution $\mathbf{L} - \mathbf{l} \rightarrow \mathbf{l}$ that $F^*(\mathbf{L} - \mathbf{l}, k_2, k_1, \mathbf{L}) = F^*(\mathbf{l}, k_1, k_2, \mathbf{L})$ hence

$$\langle ||\hat{\Psi}(\mathbf{L})||^2 \rangle = 2(2\pi)^2 \delta(0) \int \frac{d^2 l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} F(\mathbf{l}, k_1, k_2, \mathbf{L}) F^*(\mathbf{l}, k_1, k_2, \mathbf{L}) \tilde{P}_{l,k_1}^{\text{tot}} \tilde{P}_{l,k_2}^{\text{tot}}. \quad (2.21)$$

Both real and imaginary part of $||F||^2 = F_R^2 + F_I^2$ contribute to this variance, however the condition for the minimization will only pick out the real part. (**Does this really matter?**) The solution is found by minimizing the function

$$\langle ||\hat{\Psi}(\mathbf{L})||^2 \rangle - A_R \times (\text{Equation 2.19}) \quad (2.22)$$

with respect to F , where A_R is a Lagrangian multiplier. In steps,

$$\frac{\partial (\text{Eq. 2.19})}{\partial F(\mathbf{l}, k_1, k_2, \mathbf{L})} = A_R \frac{d^2 l}{(2\pi)^2} \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} (2\pi) \delta^D(k_1 + k_2) [P_{l,k_1} \mathbf{L} \cdot \mathbf{l} + P_{L-l,k_2} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})] \quad (2.23)$$

and

$$\frac{\partial \langle ||\hat{\Psi}(\mathbf{L})||^2 \rangle}{\partial F(\mathbf{l}, k_1, k_2, \mathbf{L})} = 2(2\pi)^2 \delta(0) \frac{d^2 l}{(2\pi)^2} \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} 2F_R(\mathbf{l}, k_1, k_2, \mathbf{L}) \tilde{P}_{l,k_1}^{\text{tot}} \tilde{P}_{L-l,k_2}^{\text{tot}} \quad (2.24)$$

so

$$4(2\pi)^2 \delta(0) F_R(\mathbf{l}, k_1, k_2, \mathbf{L}) = A_R (2\pi) \delta^D(k_1 + k_2) \frac{[P_{l,k_1} \mathbf{L} \cdot \mathbf{l} + P_{L-l,k_2} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]}{\tilde{P}_{l,k_1}^{\text{tot}} \tilde{P}_{L-l,k_2}^{\text{tot}}} \quad (2.25)$$

Therefore,

$$F_R(\mathbf{l}, k_1, k_2, \mathbf{L}) = \frac{1}{4(2\pi) \delta(0)} A_R \delta^D(k_1 + k_2) \frac{[P_{l,k_1} \mathbf{L} \cdot \mathbf{l} + P_{L-l,k_2} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]}{\tilde{P}_{l,k_1}^{\text{tot}} \tilde{P}_{L-l,k_2}^{\text{tot}}} \quad (2.26)$$

and by inserting this into the normalization condition 2.19 we get that

$$\begin{aligned} 1 &= \int \frac{d^2 l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \frac{1}{4\delta(0)} A_R \delta^D(k_1 + k_2) \frac{[P_{l,k_1} \mathbf{L} \cdot \mathbf{l} + P_{L-l,k_2} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]^2}{\tilde{P}_{l,k_1}^{\text{tot}} \tilde{P}_{L-l,k_2}^{\text{tot}}} \delta^D(k_1 + k_2) \\ &= \int \frac{d^2 l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \frac{\delta(0)}{4\delta(0)} A_R \frac{[P_{l,k_1} \mathbf{L} \cdot \mathbf{l} + P_{L-l,k_2=-k_1} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]^2}{\tilde{P}_{l,k_1}^{\text{tot}} \tilde{P}_{L-l,k_2=-k_1}^{\text{tot}}} \end{aligned} \quad (2.27)$$

Assuming that I can just cancel the factors of $\delta(0)$,

$$1 = \int \frac{d^2 l}{(2\pi)^2} \int dk_1 \frac{1}{4 \times 2\pi} A_R \frac{[P_{l,k_1} \mathbf{L} \cdot \mathbf{l} + P_{L-l,-k_1} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]^2}{\tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{L-l,-k_1}^{\text{tot}}} \quad (2.28)$$

Discretizing the integral over dk_1 . Assuming that we have discretized the space in the radial direction in the blocks of length D_C such that $k = \frac{2\pi}{D_C}$.

Therefore, we can write

$$\int \frac{dk}{2\pi} = \frac{1}{D_C} \sum_k \quad (2.29)$$

Therefore, we get

$$1 = \int \frac{d^2 l}{(2\pi)^2} \frac{1}{D_C} \sum_{k_1} \frac{1}{4} A_R \frac{[P_{l,k_1} \mathbf{L} \cdot \mathbf{l} + P_{L-l,-k_1} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]^2}{\tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{L-l,-k_1}^{\text{tot}}} \quad (2.30)$$

which gives,

$$A_R = \frac{4D_C}{\int \frac{d^2 l}{(2\pi)^2} \sum_{k_1} \frac{[P_{l,k_1} \mathbf{L} \cdot \mathbf{l} + P_{L-l,-k_1} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]^2}{\tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{L-l,-k_1}^{\text{tot}}}} \quad (2.31)$$

Substituting this expression in Equation 2.26, we get

$$\begin{aligned} F_R(\mathbf{l}, k_1, k_2, \mathbf{L}) &= \frac{A_R}{4(2\pi)\delta(0)} \delta^D(k_1 + k_2) \frac{[P_{l,k_1} \mathbf{L} \cdot \mathbf{l} + P_{L-l,k_2} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]}{\tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{L-l,k_2}^{\text{tot}}} \\ &= \frac{D_C/2\pi\delta(0)}{\int \frac{d^2 l}{(2\pi)^2} \sum_{k_1} \frac{[P_{l,k_1} \mathbf{L} \cdot \mathbf{l} + P_{L-l,-k_1} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]^2}{\tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{L-l,-k_1}^{\text{tot}}}} \delta^D(k_1 + k_2) \frac{[P_{l,k_1} \mathbf{L} \cdot \mathbf{l} + P_{L-l,k_2} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]}{\tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{L-l,k_2}^{\text{tot}}} \\ &= \frac{1}{\int \frac{d^2 l}{(2\pi)^2} \sum_{k_1} \frac{[P_{l,k_1} \mathbf{L} \cdot \mathbf{l} + P_{L-l,-k_1} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]^2}{\tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{L-l,-k_1}^{\text{tot}}}} \delta^D(k_1 + k_2) \frac{[P_{l,k_1} \mathbf{L} \cdot \mathbf{l} + P_{L-l,k_2} \mathbf{L} \cdot (\mathbf{L} - \mathbf{l})]}{\tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{L-l,k_2}^{\text{tot}}} \end{aligned} \quad (2.32)$$

Now, substituting this expression in Equation 2.21. Taking $FF^* \equiv F^2$

$$\begin{aligned}
\langle ||\hat{\Psi}(\mathbf{L})||^2 \rangle &= 2(2\pi)^2 \delta(0) \int \frac{d^2 l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} F(\mathbf{l}, k_1, k_2, \mathbf{L}) F^*(\mathbf{l}, k_1, k_2, \mathbf{L}) \tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{l,k_2}^{\text{tot}} \\
&= 2(2\pi)^2 \delta(0) \int \frac{d^2 l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} F^2(\mathbf{l}, k_1, k_2, \mathbf{L}) \tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{l,k_2}^{\text{tot}} \\
&= 2(2\pi)^2 \delta(0) \int \frac{d^2 l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \left(\frac{1}{\int \frac{d^2 l}{(2\pi)^2} \sum_{k_1} \frac{[\mathbf{P}_{l,k_1} \mathbf{L} \cdot \mathbf{l} + \mathbf{P}_{L-l,-k_1} \mathbf{L} \cdot (\mathbf{L}-\mathbf{l})]^2}{\tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{L-l,-k_1}^{\text{tot}}}} \right. \\
&\quad \left. \delta^D(k_1 + k_2) \frac{[\mathbf{P}_{l,k_1} \mathbf{L} \cdot \mathbf{l} + \mathbf{P}_{L-l,k_2} \mathbf{L} \cdot (\mathbf{L}-\mathbf{l})]}{\tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{L-l,k_2}^{\text{tot}}} \right)^2 \tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{l,k_2}^{\text{tot}} \\
&= 2(2\pi)^2 \delta(0) \int \frac{d^2 l}{(2\pi)^2} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \left(\frac{1}{\int \frac{d^2 l}{(2\pi)^2} \sum_{k_1} \frac{[\mathbf{P}_{l,k_1} \mathbf{L} \cdot \mathbf{l} + \mathbf{P}_{L-l,-k_1} \mathbf{L} \cdot (\mathbf{L}-\mathbf{l})]^2}{\tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{L-l,-k_1}^{\text{tot}}}} \right)^2 \\
&\quad \left(\delta^D(k_1 + k_2) \frac{[\mathbf{P}_{l,k_1} \mathbf{L} \cdot \mathbf{l} + \mathbf{P}_{L-l,k_2} \mathbf{L} \cdot (\mathbf{L}-\mathbf{l})]}{\tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{L-l,k_2}^{\text{tot}}} \right)^2 \tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{l,k_2}^{\text{tot}} \\
&= 2(2\pi)^2 \delta(0) \int \frac{d^2 l}{(2\pi)^2} \frac{\sum_{k_1}}{D_C} \frac{\sum_{k_2}}{D_C} \left(\frac{1}{\int \frac{d^2 l}{(2\pi)^2} \sum_{k_1} \frac{[\mathbf{P}_{l,k_1} \mathbf{L} \cdot \mathbf{l} + \mathbf{P}_{L-l,-k_1} \mathbf{L} \cdot (\mathbf{L}-\mathbf{l})]^2}{\tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{L-l,-k_1}^{\text{tot}}}} \right)^2 \\
&\quad \left(\delta^D(k_1 + k_2) \frac{[\mathbf{P}_{l,k_1} \mathbf{L} \cdot \mathbf{l} + \mathbf{P}_{L-l,k_2} \mathbf{L} \cdot (\mathbf{L}-\mathbf{l})]}{\tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{L-l,k_2}^{\text{tot}}} \right)^2 \tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{l,k_2}^{\text{tot}} \\
&= 2(2\pi)^2 \delta(0) \frac{1}{\int \frac{d^2 l}{(2\pi)^2} \sum_{k_1} \frac{[\mathbf{P}_{l,k_1} \mathbf{L} \cdot \mathbf{l} + \mathbf{P}_{L-l,-k_1} \mathbf{L} \cdot (\mathbf{L}-\mathbf{l})]^2}{\tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{L-l,-k_1}^{\text{tot}}}}
\end{aligned} \tag{2.33}$$

With the definition

$$\langle \hat{\Psi}(\mathbf{L}) \hat{\Psi}^*(L') \rangle = (2\pi)^2 \delta^D(\mathbf{L} - \mathbf{L}') N^{\hat{\Psi}} \tag{2.34}$$

Therefore, we finally have

$$\begin{aligned}
\langle ||\hat{\Psi}(\mathbf{L})||^2 \rangle &= \frac{2(2\pi) D_C}{\int \frac{d^2 l}{(2\pi)^2} \sum_{k_1} \frac{[\mathbf{P}_{l,k_1} \mathbf{L} \cdot \mathbf{l} + \mathbf{P}_{L-l,-k_1} \mathbf{L} \cdot (\mathbf{L}-\mathbf{l})]^2}{\tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{L-l,-k_1}^{\text{tot}}}} \\
&= (2\pi)^2 \delta(0) N^{\hat{\Psi}}
\end{aligned} \tag{2.35}$$

Which gives

$$N_L^{\hat{\Psi}} = \frac{1}{\int \frac{d^2 l}{(2\pi)^2} \sum_{k_1} \frac{[\mathbf{P}_{l,k_1} \mathbf{L} \cdot \mathbf{l} + \mathbf{P}_{L-l,-k_1} \mathbf{L} \cdot (\mathbf{L}-\mathbf{l})]^2}{2\tilde{\mathbf{P}}_{l,k_1}^{\text{tot}} \tilde{\mathbf{P}}_{L-l,-k_1}^{\text{tot}}}} \tag{2.36}$$

This is equation A20 in ZZ2006.

3 Notes by Metcalf and Alkistis

3.1 Equation 1

3.1.1 Discrete and Continuous FT

For a 3D signal, we have

$$\tilde{I}_s(\mathbf{k}) = \int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} I_s(\mathbf{x}) \quad (3.1)$$

We are probing a volume of dimensions $(\theta D_M \times \theta D_M \times D_C)$, where D_C is the length of the volume along the line of sight between $z = z_1$ and $z = z_2$. We divide the it into N_\perp parts along the transverse direction and N_\parallel parts along the radial direction.

Discretizing $\tilde{I}_s(k)$

$$\tilde{I}_s(\mathbf{k}) = \sum_{t_1} \frac{\Theta_s D_M}{N_\perp} \sum_{t_2} \frac{\Theta_s D_M}{N_\perp} \sum_p \frac{D_C}{N_\parallel} e^{-i\mathbf{k}\cdot(\frac{D_C}{N_\parallel} p + \frac{\Theta_s}{N_\perp} t_1 + \frac{\Theta_s}{N_\perp} t_2)} I_s(\mathbf{x}) \quad (3.2)$$

Using

$$\Theta_s \times \Theta_s \equiv \Omega_s \quad (3.3)$$

$$N_\perp^2 \equiv N_\perp, \quad (3.4)$$

we get,

$$\tilde{I}_s(\mathbf{k}) = \sum_{\mathbf{x}} \frac{\Omega_s D_M^2}{N_\perp^2} \frac{D_C}{N_\parallel} e^{-i\mathbf{k}\cdot\mathbf{x}} I_s(\mathbf{x}) \quad (3.5)$$

Upon normalizing this result with a factor of $D_M^2 D_C$, we get

$$\tilde{I}_s(\mathbf{k}) = \sum_{\mathbf{x}} \frac{\Omega_s}{N_\perp N_\parallel} e^{-i\mathbf{k}\cdot\mathbf{x}} I_s(\mathbf{x}) \quad (3.6)$$

3.2 Equation 2

Writing the inverse FT, we get

$$I_s(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{I}_s(\mathbf{k}) \quad (3.7)$$

Upon discretization, we get

$$I_s(\mathbf{x}) = \frac{1}{(2\pi)^3} \sum_{\mathbf{k}} \frac{2\pi}{D_C} \frac{(2\pi)^2}{\Omega_s D_M^2} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{I}_s(\mathbf{k}) \quad (3.8)$$

Absorbing the factor of $D_M^2 D_C$ into $\tilde{I}(\mathbf{k})$.

$$I_s(\mathbf{x}) = \sum_{\mathbf{k}} \frac{1}{\Omega_s} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{I}_s(\mathbf{k}) \quad (3.9)$$

3.3 Equation 5

starting with

$$\tilde{I}_s(\mathbf{k}) = \int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} I_s(\mathbf{x}) \quad (3.10)$$

we get

$$\langle \tilde{I}_s(k) \tilde{I}_s^*(k') \rangle = \int d^3x \int d^3x' \langle I_s(\mathbf{x}) I_s'(\mathbf{x}') \rangle e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{k}'\cdot\mathbf{x}'} \quad (3.11)$$

$$= \int d^3x \int d^3x' \int \frac{d^3k''}{(2\pi)^3} P(\mathbf{k}'') e^{i\mathbf{k}''(\mathbf{x}-\mathbf{x}')} e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{k}'\cdot\mathbf{x}'} \quad (3.12)$$

$$\langle \tilde{I}_s(k) \tilde{I}_s^*(k') \rangle = \sum_{\mathbf{x}} \frac{\Omega_s}{N_{\parallel} N_{\perp}} \sum_{\mathbf{x}'} \frac{\Omega_s}{N_{\parallel} N_{\perp}} \sum_{\mathbf{k}''} \frac{P(\mathbf{k}'')}{\Omega_s D^2 \mathcal{L}} e^{i\mathbf{k}''(\mathbf{x}-\mathbf{x}')} e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{k}'\cdot\mathbf{x}'} \quad (3.13)$$

$$= \sum_{\mathbf{x}} \frac{\Omega_s}{N_{\parallel} N_{\perp}} \sum_{\mathbf{x}'} \frac{\Omega_s}{N_{\parallel} N_{\perp}} \sum_{\mathbf{k}''} \frac{P(\mathbf{k}'')}{\Omega_s D^2 \mathcal{L}} e^{-i(\mathbf{k}-\mathbf{k}'')\cdot\mathbf{x}} e^{-i(\mathbf{k}''-\mathbf{k}')\cdot\mathbf{x}'} \quad (3.14)$$

$$= \Omega_s \frac{P(\mathbf{k}'')}{D^2 \mathcal{L}} \delta^k(\mathbf{l}, \mathbf{l}') \delta^k(j, j') \quad (3.15)$$

$$= \Omega_s C_{1,j} \delta^k(\mathbf{l}, \mathbf{l}') \delta^k(j, j') \quad (3.16)$$

where $C_{lj} \equiv \frac{P(\mathbf{k}'')}{D^2 \mathcal{L}}$ is the discrete angular power spectrum.