lent the radio set. Mr B. C. Browne, M.A., Demonstrator in that Department, laid aside personal work to assist me in mine, and overcame the difficulties encountered in rhythmic signal comparisons. My cousin, Miss Ethel Pierrepont, took up the duties of recorder; without her aid it would have been impossible for me to carry out the observations. I am very much indebted to Mr Frank Mercer, St Albans, who has placed one of his admirable break-circuit chronometers at my disposal for upwards of a year. The performance of this chronometer exceeded what I had believed possible with a portable timepiece. I also thank Brigadier M. N. Macleod, D.S.O., M.C., Director-General of the Ordnance Survey, and Major M. Hotine, R.E., for helpful co-operation and interest.

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The cascade theory of electronic showers

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1. Bhabha and Heitler (1937) and also Carlson and Oppenheimer (1937) have recently proposed a very ingenious theory of showers which explains nearly all the observed phenomena; both these treatments, however, contain approximations the validity of which cannot always be justified. The calculations of the first two authors make use of the method of successive approximations whose exactness is difficult to estimate, while the second authors replace one of the fundamental equations by a different one, and it is impossible to see how this replacement may distort the final result. Consequently, we have thought it useful to make the calculation in a more rigorous way; the fundamental physical ideas, however, are exactly the same as those of the authors mentioned above.

2. Consider a fast electron penetrating into a layer of matter; as long as its energy is below a certain order of magnitude, the loss of energy of the electron will be determined mainly by ionization; for higher energies, however, loss of energy by radiation (Bremsstrahlung) becomes more important. This critical energy lies in the region of about $\epsilon = 750/\bar{Z}$ million volts, where $\bar{Z}$ denotes a certain average nuclear charge (there may be several nuclei of different kinds present). Since the collision radiation, which is determined by the interaction between the electron and the nuclear field, increases as Z^2, while the ionization is approximately proportional to the number of electrons in the material, the most suitable definition of $\bar{Z}$ is

$$\bar{Z} = \Sigma_i n_i Z_i(Z_i + 1)/N, \tag{1}$$

where n_i is the number per c.c. of nuclei having the charge Z_i, and N is the total number of electrons per c.c. In the numerator we have put $Z_i(Z_i+1)$ instead of Z_1^2, since in the case of collision radiation we have to consider not only interactions with the nuclei but also with the electrons; it is important to remember this in the case of hydrogen. This same critical energy ϵ is also that for which the absorption of photons by pair production rather than by the Compton effect begins to become important.

3. Consider a particle (electron, positron) or photon moving normally to the boundary of a sufficiently thick layer of matter. If the energy of the particle inside this layer is high enough, both photons (produced by collision radiation) and new "material particles" (produced by pair production) will appear—in other words a shower will be generated. Let us consider how the number of particles $\Pi(E)\,dE$ and the number of photons $\Gamma(E)\,dE$, with energies in the range dE and at distance x from the surface of the layer, will change in consequence of pair production and collision radiation (we assume the energy E to be so high that the Compton effect and ionization can be neglected).

Let $\gamma(E, E')dE'$ be the probability per unit path length of the production of an electron of energy between E' and $E'+dE'$ by a photon of energy E, and let $\pi(E, E')dE'$ be the probability of the radiation of a photon of energy between E' and $E'+dE'$ by an electron of energy E. The rate of change of $\Pi(E)$ caused by pair production will consequently be

$$\left(\frac{d\Pi(E)}{dx}\right)_P = 2\int_E^\infty \Gamma(u)\,\gamma(u, E)\,du. \tag{2}$$

The factor 2 occurs because there are always two particles (electron and positron) created with equal probabilities. Similarly the rate of change of $\Pi(E)$ caused by radiation will be

$$\left(\frac{d\Pi(E)}{dx}\right)_R = \int_E^\infty \Pi(u)\,\pi(u,\,u-E)\,du - \int_0^E \Pi(E)\,\pi(E,\,E-u)\,du. \tag{3}$$

The first term gives the number of particles of energy greater than E, which have lost energy by radiating a photon, thus bringing them into the energy range between E and $E+dE$; the second term gives the number of particles which leaves this interval by radiating a photon. For the rate of change of the number of photons $\Gamma(E)$ we have in a similar way

$$\left(\frac{d\Gamma(E)}{dx}\right)_P = -\int_0^E \Gamma(E)\,\gamma(E,\,u)\,du, \tag{4}$$

$$\left(\frac{d\Gamma(E)}{dx}\right)_R = \int_E^\infty \Pi(u)\,\pi(u,\,E)\,du. \tag{5}$$

Thus for the total value of the change of the numbers of material particles and photons, we shall have:

$$\frac{d\Pi(E)}{dx} = 2\int_E^\infty \Gamma(u)\,\gamma(u,\,E)\,du + \int_E^\infty \Pi(u)\,\pi(u,\,u-E)\,du - \int_0^E \Pi(E)\,\pi(E,\,E-u)\,du, \tag{6}$$

$$\frac{d\Gamma(E)}{dx} = \int_E^\infty \Pi(u)\,\pi(u,\,E)\,du - \int_0^E \Gamma(E)\,\gamma(E,\,u)\,du. \tag{7}$$

This is the basis of the cascade theory of showers.

4. The expressions for $\gamma(E, E')$ and $\pi(E, E')$ have been given by Bethe and Heitler (1934). For the region of energies which concern us here, the screening can be considered as complete and we have

$$\gamma(E, E') = A\,\frac{E'^2 + (E-E')^2 + \frac{2}{3}E'(E-E')}{E^3}, \tag{8}$$

$$\pi(E, E') = A\,\frac{E^2 + (E-E')^2 - \frac{2}{3}E(E-E')}{E^2 E'}, \tag{9}$$

where $$A = \tfrac{4}{137}(e^2/mc^2)^2\,N\bar{Z}\log 183\,\bar{Z}^{-\frac{1}{3}}\ \text{cm.}^{-1}. \tag{10}$$

Putting in numerical values we find

$$A = 1{\cdot}19 \times 10^{-26} N\bar{Z}(1 - 0{\cdot}067 \log \bar{Z}). \tag{10'}$$

It is convenient to follow Carlson and Oppenheimer (1937) and measure lengths x in terms of the dimensionless unit, so that the length is defined by

$$x = t/A. \tag{11}$$

We obtain the lengths in cm. which correspond to unit value of t for various materials, shown in Table I, together with the values of $\bar{Z}$ and ϵ.

TABLE I

	$\bar{Z}$	ϵ (volts)	Equivalent thickness corresponding to $t=1$ cm.
Al	14	50×10^6	9·2
Pb	83	9×10^6	0·52
H_2O	7·6	100×10^6	38·0
Air	8·3	90×10^6	39·0 (height of water barometer)

The height of the atmosphere is $t = 26$. When this is done, all the material constants disappear from the fundamental equations, which then assume the form

$$\frac{d\Pi(E)}{dt} = \int_E^\infty \Pi(u)\frac{u^2+E^2-\frac{2}{3}uE}{u^2(u-E)}\,du - \int_0^E \Pi(E)\frac{E^2+u^2-\frac{2}{3}uE}{E^2(E-u)}\,du + \int_E^\infty \Gamma(u)\frac{2u^2-\frac{8}{3}uE+\frac{8}{3}E^2}{u^3}\,du, \tag{6*}$$

$$\frac{d\Gamma(E)}{dt} = \int_E^\infty \Pi(u)\frac{\frac{4}{3}u^2-\frac{4}{3}uE+E^2}{u^2E}\,du - \int_0^E \Gamma(E)\frac{\frac{4}{3}u^2-\frac{4}{3}uE+E^2}{E^3}\,du. \tag{7*}$$

For the discussion of these equations we introduce a new independent variable s in the place of E, in the following way: if $f(E)$ is any function of E, and f_s is defined by

$$f_s = \int_0^\infty f(E)\,E^s dE, \quad (0 \leqslant s < \infty), \tag{12}$$

then f_s is defined uniquely in terms of $f(E)$ and the converse may also be proved. Thus to any equation valid in a complete range of E there corresponds an equation in s, and vice versa. We now transform each term of equation (6*) and (7*) to the new variables s, and also write $\xi = E/u$

$$\int_0^\infty E^s dE \int_{\substack{0\\E}}^\infty \Pi(u) \frac{u^2+E^2-\frac{2}{3}uE}{u^2(u-E)}\,du - \int_0^\infty \Pi(E)\,E^s dE \int_0^E \frac{E^2-u^2-\frac{2}{3}uE}{E^2(E-u)}\,du$$
$$= -\int_0^\infty \Pi(u)\,u^s du \int_0^1 \frac{(1-\xi^s)}{1-\xi}(1-\tfrac{2}{3}\xi+\xi^2)\,d\xi,$$

$$\int_0^\infty E^s dE \int_E^\infty \Gamma(u)\,\frac{2u^2-\frac{8}{3}uE+\frac{8}{3}E^2}{u^3}\,du = \int_0^\infty \Gamma(u)\,u^s \int_0^1 \xi^s(\tfrac{8}{3}\xi^2-\tfrac{8}{3}\xi+2)\,d\xi,$$

$$\int_0^\infty E^s dE \int_E^\infty \Pi(u)\,\frac{\frac{4}{3}u^2-\frac{4}{3}uE+E^2}{u^2E}\,du = \int_0^\infty \Pi(u)\,u^s du \int_0^1 \xi^{s-1}(\xi^2+\tfrac{4}{3}\xi-\tfrac{4}{3})\,d\xi,$$

$$\int_0^\infty \Gamma(E)\,E^s dE \int_0^E \frac{\frac{4}{3}u^2-\frac{4}{3}uE+E^2}{E^3}\,du = \int_0^\infty \Gamma(E)\,E^s dE \int_0^1 (\tfrac{4}{3}/\xi^2-\tfrac{4}{3}/\xi+1)\,d(1/\xi).$$

The definite integrals with respect to ξ occurring in these expressions, we shall denote by $A(s)$, $B(s)$, $C(s)$ and D, i.e.

$$A(s) = \int_0^1 (1-\xi^s)(1-\xi)^{-1}(1-\tfrac{2}{3}\xi+\xi^2)\,d\xi = \tfrac{4}{3}\{\psi(s)+\gamma\} - \frac{s(5s+7)}{6(s+1)(s+2)}, \tag{13}$$

$$B(s) = \int_0^1 \xi^s(2-\tfrac{8}{3}\xi+\tfrac{8}{3}\xi^2)\,d\xi = 2/(1+s)-\tfrac{8}{3}/(2+s)+\tfrac{8}{3}/(3+s), \tag{14}$$

$$C(s) = \int_0^1 \xi^{s-1}(\xi^2+\tfrac{4}{3}\xi-\tfrac{4}{3})\,d\xi = \tfrac{4}{3}/s-\tfrac{4}{3}/(s+1)+1/(s+2), \tag{15}$$

$$D = \int_0^1 (\tfrac{4}{3}\xi^2-\tfrac{4}{3}\xi+1)\,d(1/\xi) = 7/9, \tag{16}$$

where $\psi(s)$ is defined by $\psi(s) = \dfrac{d}{ds}\log s!$, and γ is Euler's constant.

Thus in terms of the new variable s, the system of fundamental equations assumes the simple form

$$\left.\begin{aligned} \frac{d\Pi_s}{dt} &= -A(s)\,\Pi_s + B(s)\,\Gamma_s, \\ \frac{d\Gamma_s}{dt} &= C(s)\,\Pi_s - D\Gamma_s. \end{aligned}\right\} \tag{17}$$

5. As is well known, the solutions of (17) are given by

$$\left.\begin{aligned} \Pi_s(t) &= a_s \exp(-\lambda_s t) + b_s \exp(-\mu_s t), \\ \Gamma_s(t) &= c_s \exp(-\lambda_s t) + d_s \exp(-\mu_s t), \end{aligned}\right\} \tag{18}$$

where λ_s and μ_s are the roots of the secular equations

$$\begin{vmatrix} -A(s)+x & B(s) \\ C(s) & -D+x \end{vmatrix} = 0, \tag{19}$$

so that $\lambda(s) = \lambda_s = \frac{1}{2}\{A(s)+D\} - \frac{1}{2}\sqrt{[\{A(s)-D\}^2 + 4B(s)\,C(s)]}$

$$\mu(s) = \mu_s = \tfrac{1}{2}\{A(s)+D\} + \tfrac{1}{2}\sqrt{[\{A(s)-D\}^2 + 4B(s)\,C(s)]},$$

and a_s, b_s, c_s, d_s are constants determined by the initial conditions ($t=0$). The case which interests us is when the rate of change of the number of particles is large, i.e. when the exponents are large, and consequently we consider all quantities which do not vary exponentially as being of order of magnitude unity.† With this approximation we can neglect the solutions corresponding to μ_s, and approximately satisfy the initial conditions by using only the λ_s-solutions.

An examination of the secular equation (19) shows that for change of s in the interval $0 \leqslant s \leqslant \infty$, the function $\lambda(s)$ increases from $-\infty$ to the limiting value $\frac{7}{9}$.

6. If only a single photon passes through the layer with energy E_0, the initial conditions are

$$\Pi(E) = 0, \quad \Gamma(E) = \delta(E - E_0);$$

thus to our approximation we have

$$\int_0^\infty \{\Pi(E) + \Gamma(E)\}\, E^s dE = E_0^s \exp(-\lambda_s t). \tag{20}$$

Following Bhabha and Heitler (1937) we measure the energy by means of the dimensionless quantity y, so that

$$y = \log(E_0/E) \quad \text{and} \quad \eta = \log(E_0/\epsilon), \tag{21}$$

and η will then be the quantity corresponding to the critical energy. Let $P(y,t) = \exp\phi(y,t)$ be the distribution function with respect to y so that‡

$$\{\Gamma(E) + \Pi(E)\}\, dE = -P dy. \tag{22}$$

† The assumption that the exponentials are respectively large or small implies that the total number of secondary particles is large. This assumption, absolutely necessary, if radiation and creation of pairs are treated separately from ionization and Compton effect, is of course made (together with other assumptions which we have avoided) by Bhabha and Heitler and by Carlson and Oppenheimer.

‡ When in the following we refer to the number of particles we mean both charged particles and light quanta.

The expression (20) then becomes

$$\int_{-\infty}^{\infty} \exp\{\phi(y,t)-sy\}\,dy = \exp(-\lambda_s t). \tag{23}$$

The integrand has a sharp maximum for a certain value of y, say $\bar{y}$; then $\bar{y}$ will be the root of the equation

$$\frac{\partial\phi(y,t)}{\partial y} = s. \tag{24}$$

Neglecting the non-exponential terms, we obtain from (23)

$$\phi(\bar{y},t)-s\bar{y} = -\lambda(s)\,t. \tag{25}$$

From equations (24) and (25), knowing the function $\phi(y,t)$, we could determine the function $\lambda(s)$; since, however, it is $\lambda(s)$ which is given, and we wish to find the function $\phi(y,t) = \log P(y,t)$, we differentiate equation (25) with respect to s, which gives

$$\frac{\partial\phi}{\partial\bar{y}}\cdot\frac{d\bar{y}}{ds}-\bar{y}-s\frac{d\bar{y}}{ds} = -\lambda'(s)\,t.$$

On account of (24) this simplifies to

$$\bar{y} = \lambda'(s)\,t. \tag{26}$$

We substitute this value $\bar{y}$ in equation (25), and write simply y instead of $\bar{y}$ in what follows.

The equations

$$\left.\begin{aligned} y &= t\lambda'(s), \\ \phi &= t\{s\lambda'(s)-\lambda(s)\}, \end{aligned}\right\} \tag{27}$$

give a parametric representation of ϕ as a function of y and t, and in fig. 1 we show the dependence of ϕ on y for the particular values $t = 10$ and $E_0/\epsilon = 100$ (for other values the scale of fig. 1 must be correspondingly altered). Since it follows from (24) that

$$\frac{\partial\phi}{\partial y} = s > 0,$$

we see then that the function increases monotonically, or in other words, the number of created particles observed in a given thickness t of the layer increases with y, i.e. with diminishing energy of these created particles.

The expression for ϕ ceases to be valid as soon as the energy of the created particles falls to the order of magnitude of the critical energy, since the ionization and Compton effect which we could neglect for higher energies, then become increasingly important. As can be seen from equations (8) and (9) photons of small energy are formed in much larger quantities, by collision radiation, than are material particles of small energy by pair production. The number of created photons with energies in the range $E', E'+dE'$ is proportional to $1/E'$. If $E' \ll \epsilon$ then the absorption of these photons is determined mainly by the Compton effect and not by pair production: consequently in this energy region the absorption coefficient is nearly proportional to $1/E'$ (as follows from the theory of the Compton effect), i.e. the numbers of photons $\Gamma(E)$ for $E < \epsilon$ is independent of the energy.

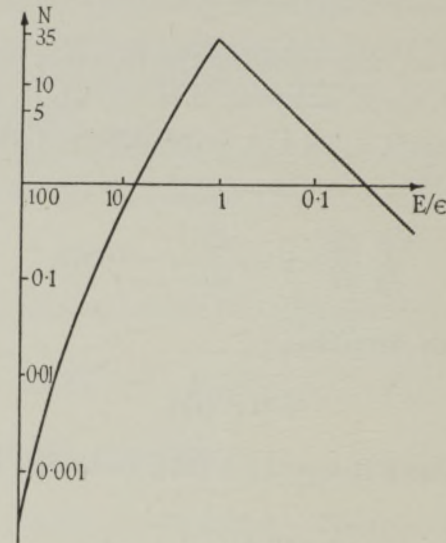

FIG. 1

As can be shown by a more detailed analysis, the same is true of the number of particles $\Pi(E)$ for $E < \epsilon'$. If we go over from E to the logarithmic measure y we find in consequence of (21) that the distribution function $P(y, t)$ for $y > \eta$ has the form

$$P(y, t) = \exp\{\text{constant} - y\}.$$

We see then that the number of particles for given penetration depth t increases with diminution of energy until the critical region is reached, when further diminution of the energy diminishes the number of particles. This means that for given penetration depth the majority of particles in a shower will have the energy ϵ.

The distribution of particles along the path of penetration into the layer of matter is given by the function which is shown graphically in fig. 2 for

$\eta = \log 100$. For other values of η the scale has to be correspondingly changed. We give in Table II the corresponding values of s, t/y, ϕ/y and λ. The curve has a maximum at the point $t = t_m$, for which the derivative $\partial\phi/\partial t$ vanishes. From (25) we have

$$d\phi - \eta\, ds - s\, d\eta = -\lambda\, dt - t\, d\lambda.$$

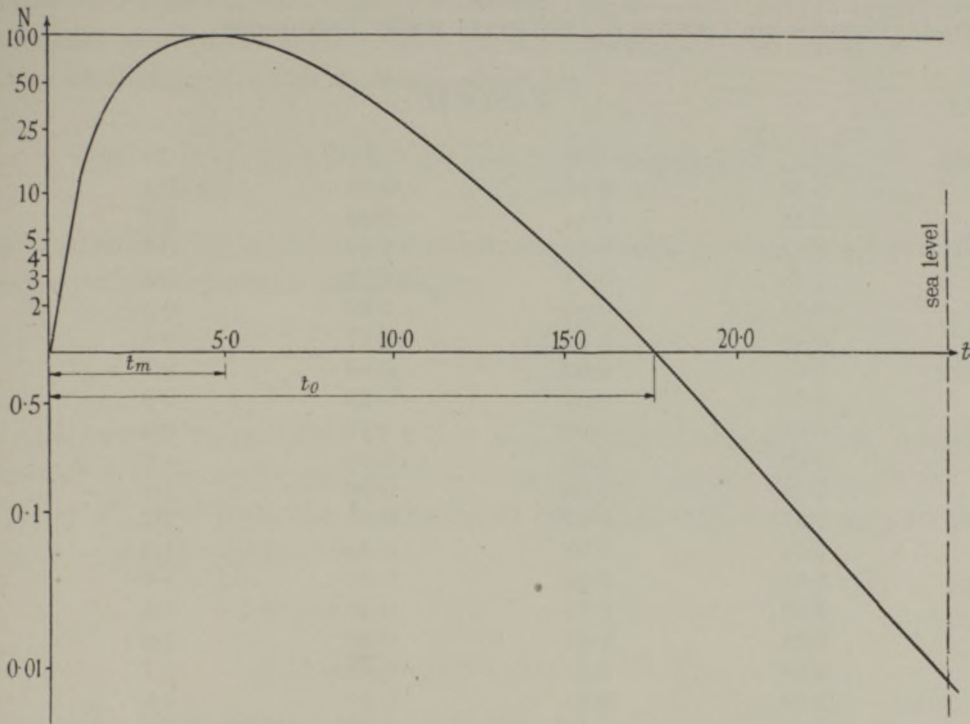

FIG. 2

Since $d\eta = 0$ and, on account of (26), $\eta\, ds = t\, d\lambda$, we obtain

$$\frac{\partial\phi}{\partial t} = -\lambda(s). \tag{28}$$

From the law of conservation of energy it follows that

$$\frac{d}{dt}\{\Gamma_1 + \Pi_1\} = 0;$$

hence $\lambda(1) = 0$; this can of course also be proved by a direct calculation.

7. Equation (26) gives for the position of the maximum

$$t_m = \eta/\lambda'(1), \quad \phi_{\max} = \eta,$$

i.e. the maximum number of particles will be observed at a depth

$$t_m = 1{\cdot}01 \log(E/\epsilon),$$

and will be equal to E_0/ϵ particles. The length of the showers track will be, expressed in t-units,

$$t_0 = 3{\cdot}8 \log(E/\epsilon),$$

which gives for air $1{\cdot}48 \log E_0/100$ m. of water barometer.

Table II

t/y	ϕ/y	λ	s
0·04	0·25	−3·76	0·1
0·11	0·44	−2·26	0·2
0·19	0·59	−1·56	0·3
0·28	0·71	−1·12	0·4
0·37	0·80	−1·80	0·5
0·48	0·88	−0·57	0·6
0·60	0·93	−0·38	0·7
0·72	0·97	−0·23	0·8
0·86	0·99	−0·11	0·9
1·01	1·00	0·00	1·0
1·18	0·99	0·09	1·1
1·37	0·97	0·17	1·2
1·58	0·92	0·24	1·3
1·81	0·86	0·30	1·4
2·07	0·78	0·35	1·5
2·36	0·67	0·39	1·6
2·69	0·53	0·44	1·7
3·07	0·36	0·47	1·8
3·49	0·16	0·50	1·9
3·96	−0·08	0·53	2·0
4·49	−0·37	0·55	2·1
5·08	−0·70	0·57	2·2
5·73	−1·08	0·59	2·3
6·46	−1·52	0·61	2·4
7·29	−2·03	0·62	2·5
8·17	−2·67	0·63	2·6
...	...	...	...
∞	$-\infty$	0·78	∞

Let us calculate the fraction of the number of particles which, having penetrated the atmosphere with an initial energy of 10^{10} V, can be observed at sea-level; we have $\eta = \log 100 = 4{\cdot}6$ and $t/\eta = 26/4{\cdot}6 = 5{\cdot}65$, and from Table II we find the corresponding value of $\phi/\eta = -1{\cdot}03$ so that $\phi = -4{\cdot}74$. Consequently the required fraction is $\exp(-4{\cdot}74)$, or slightly less than 1 %. As regards the particles penetrating the atmosphere with energies of 3×10^9 V, only 1 in 1000 can reach sea-level.

In the third column of Table II we have written down the values of λ. Now equation (28) shows that λ is the coefficient of absorption and we see that the limiting value of λ (for large t) is 7/9, which gives 2 per metre of water barometer in the case of air. If we estimate the energy of the particles entering the atmosphere as of order of magnitude 3×10^9–10^{10} V, we obtain 0·6 as the order of magnitude of λ, and consequently an absorption coefficient 1·5 per metre of water barometer. This then is the size of absorption coefficient we should expect to find for light particles at sea-level.

The total energy of the shower is given by

$$E(t) = \int_0^\infty \{\Gamma(E) + \Pi(E)\} \, E \, dE = E_0 \int_{-\infty}^\infty \exp\{\phi(y,t) - y\} \, dy. \tag{29}$$

The integrand has a sharp maximum at the point $y = \bar{y}$, where the derivative of the exponent vanishes, i.e.

$$\frac{\partial \phi}{\partial y} = 1 = s\left(\frac{t}{y}\right). \tag{30}$$

As can be seen from Table II $\bar{y} \fallingdotseq t$; if $\bar{y} < \eta$, $\phi(\bar{y}, t) = \bar{y}$ and the integral becomes of order unity, and we find that the total energy of the shower is of order E_0. For $\bar{y} > \eta$ the maximum of the integrand corresponds to the point $y = \eta$ and we have

$$E(t) = E_0, \quad t < \eta$$

$$E(t) = E_0 \exp\{\phi(\eta, t) - \eta\}, \quad t > \eta.$$

The graph of this function is shown in fig. 3.

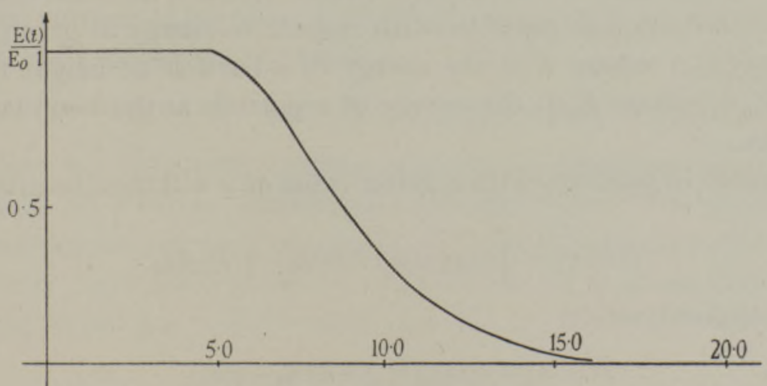

FIG. 3

8. In principle it is possible to deduce the spectral distribution of the particles entering the atmosphere and producing showers along their path

from the experimentally observable number of particles $N(t)$ per unit length of path. Let $\exp \xi(\eta)$ be the number of particles entering the atmosphere with energy E_0; each of these will produce a shower, so that the total number of particles $N(t)$ per unit length observed at depth t is

$$N(t) = \int \exp\{\xi(\eta) + \phi(\eta, t)\}\, d\eta. \tag{31}$$

The integrand has a maximum at the point $\eta = \bar{\eta}$, where the derivative $\xi'(\eta) + \dfrac{\partial \phi(\eta, t)}{\partial \eta}$ vanishes, and $\bar{\eta}$ of course depends on t. The majority of particles observed at different heights are formed by particles of different energies. In our approximation we can write

$$N(t) = \exp\{\xi(\bar{\eta}) + \phi(\bar{\eta}, t)\},$$

and we have
$$\frac{1}{N} \cdot \frac{dN}{dt} = \frac{\partial \phi}{\partial t} + \left\{\xi'(\bar{\eta}) + \frac{\partial \phi(\bar{\eta}, t)}{\partial \bar{\eta}}\right\} \frac{d\bar{\eta}}{dt} = -\lambda\left(\frac{t}{\bar{\eta}}\right). \tag{32}$$

The left-hand side of this equation can be obtained from experiment, and thus knowing $\lambda(t/\bar{\eta})$ we can determine for given depth t the corresponding value of the energy of the shower-producing particles. In the same column of Table II we find the corresponding value of $\phi(t/\bar{\eta})$, and finally obtain the required function as

$$\exp \xi(\eta) = N(t) \exp(-\phi(\bar{\eta}, t)). \tag{33}$$

$$t = t(\bar{\eta}).$$

9. We shall now consider the behaviour of cosmic particles in passing from air to a medium with a different atomic number, and first of all determine the distribution of particles with respect to energy at given height t. Let $\eta = \log(E/\epsilon)$, where E is the energy of a particle at height t, and let $\eta_0 = \log(E_0/\epsilon)$, where E_0 is the energy of a particle at the boundary of the atmosphere.

The number of particles with a given value of η will then be given by

$$N(t, \eta) = \int \exp\{\xi(\eta_0) + \phi(\eta_0 - \eta, t)\}\, d\eta_0, \tag{34}$$

or to our approximation

$$N(t, \eta) = \exp\{\xi(\bar{\eta}_0) + \phi(\bar{\eta}_0 - \eta, t)\}, \tag{34*}$$

where η_0 is the root of the equation

$$\xi'(\bar{\eta}_0) + \frac{\partial \phi(\bar{\eta}_0 - \eta, t)}{\partial \eta_0} = 0. \tag{35}$$

If we are not concerned with particles with very high energies, we can assume that η is small compared with η_0, and retain only the linear term in the expansion of the expression in the exponent in powers of η; we then have

$$\left(\frac{\partial}{\partial\eta}\log N(t,\eta)\right)_{\eta=0} = \{\xi'(\bar{\eta}_0)+\phi(\bar{\eta}_0-\eta,t)\}\frac{\partial\bar{\eta}_0}{\partial\eta}-\frac{\partial\phi(\bar{\eta}_0,t)}{\partial\bar{\eta}_0}$$

or,
$$\log N(t,\eta) = \text{const.} - s(t/\bar{\eta}_0)\,\eta, \tag{36}$$

where the constant of course depends on t but not on η. The quantity s in this formula is connected directly with the absorption coefficient λ at the given height, and can be calculated from it by means of the table. We obtain finally

$$N(t,\eta) = N_0\exp\{-s\eta\}, \tag{37}$$

and going over to the ordinary scale of energy, we have evidently

$$dN = \text{const.}\, dE/E^{s+1} \tag{38}$$

(this formula applies of course only for $E \gg \epsilon$). For example, for $t = 10$, $E_0 = 10^{10}$ e-volts, we obtain from the table $s \sim 1{\cdot}5$, for $t = 20$, $E_0 = 10^{10}$ e-volts we find $s \sim 2{\cdot}0$.

We now determine the number of showers produced by a particle passing from air through a not too thin layer of some material; let

$$\log(\epsilon/\epsilon') = \log(\bar{Z}'/\bar{Z}) = \zeta,$$

where ϵ and $\bar{Z}$ refer to air and ϵ' and $\bar{Z}'$ to the layer, then we can write $\eta' = \eta+\zeta$. As was shown in § 7, the showers will emerge from the layer only if $\eta' > t/3{\cdot}8$ or $\eta > t/3{\cdot}8-\zeta$.

Since the number of particles diminishes with energy, for negative values of ζ (i.e. $\bar{Z}' < \bar{Z}$) we can simply write

$$\eta = t/3{\cdot}8+|\zeta| \quad\text{and}\quad N = N_0\exp\{-s(t/3{\cdot}8+|\zeta|)\}. \tag{39}$$

We see from this that with increase of the layer thickness the number of showers falls off with an absorption coefficient $s/3{\cdot}8$. For positive ζ we have to distinguish between two cases: (1) $t < 3{\cdot}8\,\zeta$ and (2) $t > 3{\cdot}8\,\zeta$; in the first case the lower limit given by the formula for η is negative, and we have simply to put $\eta = 0$, from which $N = N_0$; in the second case we obtain as before

$$N = N_0\exp\{-s(t/3{\cdot}8-\zeta)\}. \tag{40}$$

To calculate the number of particles emerging from the given layer, we have

$$N = N_0\int\exp\{-s\eta+\phi(\eta+\zeta,t)\}\,d\eta. \tag{41}$$

For negative ζ the integrand always has a maximum for a value of η different from zero, and in our approximation we can write

$$N_0 = N_0 \exp\{-s\bar{\eta}+\phi(\bar{\eta}+\zeta,t)\}, \tag{42}$$

where $\bar{\eta}$ is the root of the equation

$$s = \frac{\partial\phi(\eta+\zeta,t)}{\partial\eta} = s\left(\frac{t}{\eta+\zeta}\right). \tag{43}$$

Now we have (see equation (25))

$$\phi(\bar{\eta}+\zeta,t)-(\bar{\eta}+\zeta)s = -t\lambda(s),$$

hence
$$N = N_0 \exp\{s\zeta - t\lambda(s)\}. \tag{44}$$

In other words the number of particles continues to diminish in the material with the same absorption coefficient λ as in air. For $t = 0$,

$$N = N_0 \exp\{s\zeta\}, \tag{45}$$

i.e. since ζ is negative for a transition from air to a medium of smaller atomic number, the number of particles falls discontinuously (actually in a thin layer $t \ll 1$) by a factor $\exp s\,|\,\zeta\,|$. For a transition from air to lithium we have $\bar{Z}' = 4$, $\zeta \sim -0{\cdot}76$, which for $s \sim 1{\cdot}5$ gives a diminution of the number of particles by a factor 3.

Passing to a medium with larger Z, we have two cases according as to whether equation (43) has or has not a positive solution for $\bar{\eta}$. In the former case we evidently obtain the same result as for negative ζ, but in the latter which occurs when $s(t/\zeta) < s$ we must write

$$N = N_0 \exp\phi(\zeta,t). \tag{46}$$

This expression has a maximum for $t \cong \zeta$, as was shown, this maximum being

$$N = N_0 \exp\zeta = N_0(\bar{Z}'/Z). \tag{47}$$

We see therefore that for passage through a medium with Z larger than in air the number of particles at first increases, 10 times for lead, for instance, and then falls, reaching after a certain point the absorption coefficient for air.

10. The showers observed at great heights should evidently be ascribed to the so-called soft component of the penetrating radiation, i.e. particles produced by penetration of photons or electrons. Correspondingly the number of showers leaving a lead plate has, as we know, a maximum for a certain thickness of lead, falling off with further increase of the thickness

corresponding to the finite length of path of the showers. The number of showers at and below the earth's surface tends to a certain limit with increase of the thickness of the lead plate. As has already been pointed out, such showers can be related to secondary particles created during the passage of the hard component through matter and such particles may evidently be either electrons or photons. From the relativistic scattering formula, we can obtain for the number of fast electrons produced per unit path of the "semi-heavy" particle the following expression (we assume the velocity of the particle to be close to that of light)

$$dI = \frac{2\pi}{m} N \frac{e^4}{c^2} \frac{dE}{E^2}. \tag{48}$$

Since the number of particles increases rapidly with decrease of their energies, the number of observed showers should be very sensitive to the minimum energy of the shower which the apparatus can detect. In order to obtain a very rough estimate of the order of magnitude, we assume arbitrarily that this lower limit is $E \fallingdotseq \epsilon$ and the length of path of the shower is $t = 1$. It is seen from the calculation that the atomic number of the medium drops out of the final expression, so that the number of showers is independent of the nature of the medium, and we obtain 3 or 4 as the order of magnitude of the number of showers per 100 semi-heavy particles. Thus if the particle producing the shower is an electron we should expect that the number of showers should be independent of the mass and energy of the penetrating particle, and independent of the properties of the medium in which the shower is created, but that it should be strongly dependent on the particular experimental arrangement used for detecting the showers.

On the other hand, if it is a proton which produces the shower, calculation shows that the number of showers should depend chiefly on the mass of the penetrating particle, and also to some extent on its energy and the atomic number of the medium, but much less on the particular experimental arrangement.

The relative importance of these two possibilities depends very much on what we assume about the mass of the initial particle. Since experimentally it is found that the number of showers is independent of the medium, we may conclude that it is the first possibility which is correct.

Summary

1. Starting from the physical idea proposed by Bhabha and Heitler, and Carlson and Oppenheimer, on the cascade production of showers, a more rigorous treatment of the problem is given (§§ 1–5).

2. A numerical table permits the calculation of the number of shower particles as a function of depth of penetration for any given initial energy of the creating particle, and also the energy distribution for shower particles at a given depth (§ 6).

3. The showers possess a definite range (§ 7).

4. The behaviour of a shower passing from air to a medium with a different atomic number is studied (§ 9).

5. The consequences arising from the hypothesis of the new "semi-heavy" particles are discussed.

References

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Polish on metals

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[Plates 12, 13]

Introduction

In the first year of this century, Beilby (1901) made microscopic and visual observations on the polished surfaces of metals. He stated that such surfaces were covered with a glass-like layer and finally reached the conclusion (Beilby 1921) that this polished layer was similar to a super-cooled liquid, the metallic atoms being in the positions which they would occupy, at a given instant, in the liquid state. His ideas did not gain general acceptance, however, amongst metallographers and other workers—thus, while Desch (1922) supports the Beilby theory, yet Tammann (1925) is definitely opposed to it. This question as to the structure of the polish layer on metals can only be decided definitely by X-rays or electron diffraction, although an experiment described by Finch, Quarrell and