Book Embedding with Fixed Page Assignments

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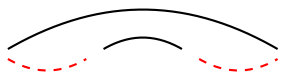
Introduction

Definition. A *page embedding* of a graph $G = (V, E)$ is a planar embedding of G such that the vertices of G lie on the *real line* $\mathbb{R} \times \{0\}$ and every edge lies in the *upper half-plane* $\{(x, y) \in \mathbb{R}^2 : y > 0\}$, apart from the edge's endpoints.



A book embedding with two pages.

Definition. A *book embedding* of graphs $G_1 = (V, E_1), \ldots, G_k = (V, E_k)$ on the same set of vertices consists of page embeddings for each of the graphs that coincide in their vertex positions.

Problem: BOOK-EMBEDDING
Given: A vertex set V and edge sets $E_1, \ldots, E_k \subseteq \binom{V}{2}$
Question: Is there a book embedding of $(V, E_1), \ldots, (V, E_k)$?

Definition. Let (V, E) be a graph and $<$ a total order on V. We call the condition that the suborder $a < c < b < d$ does not occur for any $\{a, b\}, \{c, d\} \in E$ the *book constraint for* E. If $<$ fulfils the book constraint for E, we say that it is a *valid book order for* E.

Lemma. *There is a page embedding for* $G = (V, E)$ *if and only if there is a valid book order* $<$ *for* E.

Lemma. *Let* $(V, E_1), \ldots, (V, E_k)$ *be a book embedding instance with valid book order* $< \in \mathrm{Sym}(V)$. *Then the cyclic shifts and the mirror image of* $<$ *are also valid.*

NP-completeness and connected pages

Problem: BETWEENNESS
Given: A finite set $M := \{1, \ldots, n\}$ and a set of ordered triples $C \subseteq M^3$.
Question: Is there a total ordering $<$ of M such that either $a < b < c$ or $a > b > c$ occurs for all $(a, b, c) \in C$?

Theorem. *There is a polynomial time reduction from* BETWEENNESS *to* BOOK-EMBEDDING. *Thus,* BOOK-EMBEDDING *is NP-complete.*

Problem: **CONNECTED-BOOK-EMBEDDING**
Given: A vertex set V and edge sets $E_1, \dots, E_k \subseteq \binom{V}{2}$ such that the graphs (V, E_i) are connected for $i \in \{1, \dots, k\}$.
Question: Is there a book embedding of $(V, E_1), \dots, (V, E_k)$?

Theorem. CONNECTED-BOOK-EMBEDDING *can be solved in linear time.*

Definition. A *PQ-tree* T on $M := \{1, \dots, n\}$ is a rooted, ordered tree with leaves M and inner nodes either of type P (◯) or type Q (□).

The tree represents a set of permutations $\pi(T) \subseteq \mathrm{Sym}(M)$ on its leaves as follows: The order of the children of a P-node can be permuted in any way, while the order of the children of a Q-node can only be reversed. The set $\pi(T)$ then consists exactly of the permutations of the leaves M that we can get by flipping the inner nodes in any of the specified valid ways.

Let a *P-tree* be a PQ-tree containing only P-nodes as inner nodes and a *Q-tree* be a PQ-tree only containing Q-nodes as inner nodes.

Disjoint perfect matchings

Problem: **PERFECT-MATCHINGS-BOOK-EMBEDDING**
Given: Disjoint perfect matchings $E_1, \dots, E_k$ on a vertex set V.
Question: Is there a book embedding of $(V, E_1), \dots, (V, E_k)$?

Theorem. *Let* $I := (V, E_1, \dots, E_k)$ *be an instance of* PERFECT-MATCHINGS-BOOK-EMBEDDING. *If the graph* $G := (V, E_1 \cup \dots \cup E_k)$ *is not bipartite, it has no book embedding.*

Tree on the vertices

Problem: **Q-TREE-BOOK-EMBEDDING**
Given: A BOOK-EMBEDDING instance I with vertices V and a Q-tree T with leaves V.
Question: Is there a total order $< \in \pi(T)$ solving I?

Theorem. Q-TREE-BOOK-EMBEDDING *can be solved in quadratic time.*