Journal of Discrete Algorithms 14 (2012) 150–172

Contents lists available at SciVerse ScienceDirect

Journal of Discrete Algorithms

www.elsevier.com/locate/jda

Testing the simultaneous embeddability of two graphs whose intersection is a biconnected or a connected graph ☆

Patrizio Angelini [a], Giuseppe Di Battista [a], Fabrizio Frati [a,b], Maurizio Patrignani [a], Ignaz Rutter [c,*]

[a] Dipartimento di Informatica e Automazione, Università Roma Tre, Italy
[b] School of Information Technologies, The University of Sydney, Australia
[c] Institute of Theoretical Informatics, Karlsruhe Institute of Technology (KIT), Germany

ARTICLE INFO

Article history:
Available online 8 December 2011

Keywords:
Simultaneous embedding
Planarity
Linear-time algorithm
SPQR-tree

ABSTRACT

In this paper we study the time complexity of the problem *Simultaneous Embedding with Fixed Edges* (SEFE), that takes two planar graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ as input and asks whether a planar drawing Γ_1 of G_1 and a planar drawing Γ_2 of G_2 exist such that: (i) each vertex $v \in V$ is mapped to the same point in Γ_1 and in Γ_2; (ii) every edge $e \in E_1 \cap E_2$ is mapped to the same Jordan curve in Γ_1 and Γ_2.
First, we give a linear-time algorithm for SEFE when the *intersection graph* of G_1 and G_2, that is the planar graph $G_{1\cap 2} = (V, E_1 \cap E_2)$, is *biconnected*. Second, we show that SEFE, when $G_{1\cap 2}$ is *connected*, is equivalent to a suitably-defined *book embedding* problem. Based on this equivalence and on recent results by Hong and Nagamochi, we show a linear-time algorithm for the SEFE problem when $G_{1\cap 2}$ is a *star*.
© 2011 Elsevier B.V. All rights reserved.

1. Introduction

Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two graphs on the same set of vertices. A *simultaneous embedding* of G_1 and G_2 consists of two planar drawings Γ_1 and Γ_2 of G_1 and G_2, respectively, such that any vertex $v \in V$ is mapped to the same point in each of the two drawings. Because of the applications to several visualization tasks and because of the interesting related theoretical problems, constructing simultaneous graph embeddings has recently grown to be a distinguished research topic in graph drawing.

The two main variants of the simultaneous embedding problem are the *geometric simultaneous embedding* and the *simultaneous embedding with fixed edges*. The former requires straight-line drawings of the input graphs, while the latter relaxes this constraint by just requiring the edges that are common to distinct graphs to be represented by the same Jordan curve in all the drawings. Geometric simultaneous embedding turns out to have limited usability, as testing whether two planar graphs admit a geometric simultaneous embedding is $\mathcal{NP}$-hard [9] and as geometric simultaneous embeddings do not always exist if the input graphs are three paths [4], if they are two outerplanar graphs [4], if they are two trees [15], and even if they are a tree and a path [2].

☆ Work partially supported by the MIUR, project AlgoDEEP 2008TFBWL4, and by the ESF project 10-EuroGIGA-OP-003 "Graph Drawings and Representations".
* Corresponding author.
E-mail addresses: angelini@dia.uniroma3.it (P. Angelini), gdb@dia.uniroma3.it (G. Di Battista), frati@dia.uniroma3.it (F. Frati), patrigna@dia.uniroma3.it (M. Patrignani), rutter@kit.edu (I. Rutter).

1570-8667/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.jda.2011.12.015

On the other hand, a Simultaneous Embedding with Fixed Edges (SEFE) always exists for much larger graph classes. Namely, a tree and a path always have a SEFE with few bends per edge [8]; an outerplanar graph and a path or a cycle always have a SEFE with few bends per edge [7]; a planar graph and a tree always have a SEFE [12].

The main open question about SEFE is whether testing the existence of a SEFE of two planar graphs is doable in polynomial time or not. A number of known results are related to this problem. Namely:

- Gassner et al. proved that testing whether three planar graphs admit a SEFE is $\mathcal{NP}$-hard and that SEFE is in $\mathcal{NP}$ for any number of input graphs [14];
- Fowler et al. characterized the planar graphs that always have a SEFE with any other planar graph and proved that testing whether two outerplanar graphs admit a SEFE is in $\mathcal{P}$ [11];
- Fowler et al. showed how to test in polynomial time whether two planar graphs admit a SEFE if one of them contains at most one cycle [10];
- Jünger and Schulz characterized the graphs $G_{1\cap 2}$ that allow for a SEFE of any two planar graphs G_1 and G_2 whose intersection graph is $G_{1\cap 2}$ [20];
- Angelini et al. showed how to test whether two planar graphs admit a SEFE if one of them has a fixed embedding [1].

In this paper, we show the following results:

In Section 3 we show a linear-time algorithm for the SEFE problem when the intersection graph $G_{1\cap 2}$ of G_1 and G_2 is biconnected. Our algorithm exploits the SPQR-tree decomposition of $G_{1\cap 2}$ in order to test whether a planar embedding of $G_{1\cap 2}$ exists that allows the edges of G_1 and G_2 not in $G_{1\cap 2}$ to be drawn in such a way that no two edges of the same graph intersect. Haeupler et al. [17] independently found a different linear-time algorithm for the same problem, based on PQ-trees.

In Section 4 we show that the SEFE problem, when $G_{1\cap 2}$ is connected, is equivalent to a suitably-defined book embedding problem. Namely, we show that, for every instance G_1, G_2 of SEFE such that $G_{1\cap 2}$ is connected, there exists a graph G', whose edges are partitioned into two sets E'_1 and E'_2, and a set of hierarchical constraints on the vertices of G', such that G_1 and G_2 have a SEFE if and only if G' admits a 2-page book embedding in which the edges of E'_1 are in one page, the edges of E'_2 are in another page, and the order of the vertices in V' along the spine respects the hierarchical constraints. Based on this characterization and on recent results by Hong and Nagamochi [19] concerning 2-page book embeddings with the edges assigned to the pages in the input, we prove that linear time suffices to solve the SEFE problem when $G_{1\cap 2}$ is a star.

2. Preliminaries

2.1. Drawings and embeddings

A *drawing* of a graph is a mapping of each vertex to a distinct point of the plane and of each edge to a simple Jordan curve connecting its endpoints. A drawing is *planar* if the curves representing its edges do not cross except, possibly, at common endpoints. A graph is *planar* if it admits a planar drawing. Two drawings of the same graph are *equivalent* if they determine the same circular ordering of edges around each vertex. A *planar embedding* (or just *embedding*) is an equivalence class of planar drawings. A planar drawing partitions the plane into topologically connected regions, called *faces*. The unbounded face is the *outer face*.

For a subgraph H of a graph G with planar embedding $\mathcal{E}$ we denote by $\mathcal{E}|_H$ the embedding of H induced by $\mathcal{E}$, and by ∂H the set of vertices of H that are adjacent to a vertex of $G - H$. The following lemma is a very basic tool for manipulating embeddings.

Lemma 1 *(Patching Lemma). Let $G = (V, E)$ be a biconnected planar graph with embedding $\mathcal{E}$ and let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two edge-disjoint biconnected subgraphs of G with $V_1 \cup V_2 = V$ and with the property that all the vertices of G_2 are in a single face f of $\mathcal{E}|_{G_1}$ (vertices in $V_1 \cap V_2$ are on the boundary of f). Further, let $\mathcal{E}'_2$ be an embedding of G_2 with the property that all the vertices of ∂G_2 are incident to the outer face of $\mathcal{E}'_2$ and appear in the same order as in $\mathcal{E}|_{G_2}$. Then there exists a planar embedding $\mathcal{F}$ of G with $\mathcal{F}|_{G_1} = \mathcal{E}|_{G_1}$ and $\mathcal{F}|_{G_2} = \mathcal{E}'_2$.*

Proof. We prove the statement by induction on the number of vertices in $V_1 \cap V_2$.

For the base case, suppose that $V_1 \cap V_2 = \emptyset$. Let E' be the set of edges having one end-vertex in V_1 and the other one in V_2. Remove from G all the edges of E' and change the embedding of G_2 to $\mathcal{E}'_2$. Since the two embeddings $\mathcal{E}|_{G_2}$ and $\mathcal{E}'_2$ look the same from the outside, that is, the order of the vertices in ∂G_2 along the outer face of $\mathcal{E}|_{G_2}$ and along the outer face of $\mathcal{E}'_2$ is the same, the edges in E' can be reinserted in a planar way, thus yielding the claimed embedding $\mathcal{F}$ of G.

For the inductive case, suppose that $V_1 \cap V_2 \neq \emptyset$ and let $u \in V_1 \cap V_2$. Since all the vertices of G_2 are in a single face of $\mathcal{E}|_{G_1}$ and since G_1 and G_2 are biconnected, edges of G_1 and of G_2 do not alternate around u. Hence, the edges of G_1 (resp. of G_2) incident to u form an interval in the cyclic ordering of edges around u. We can therefore split u into two vertices u_1 and u_2 connected by edge (u_1, u_2) such that u_i is connected to all the neighbors of u in G_i for $i = 1, 2$. Call G' the resulting graph and modify G_1 and G_2 by renaming vertex u to u_i in G_i for $i = 1, 2$. Graphs G_1 and G_2 share one vertex less than

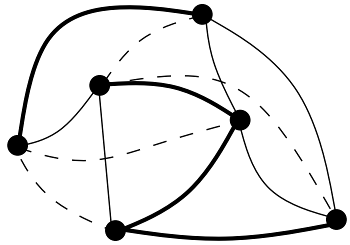

Fig. 1. A SEFE of two planar graphs. The edges that belong to both graphs are represented by solid fat segments, while the edges that belong to only one of the two graphs are represented by thin solid and dashed segments, respectively.

before and hence, by induction, there exists an embedding $\mathcal{F}'$ of G' with $\mathcal{F}'|_{G_1} = \mathcal{E}|_{G_1}$ and $\mathcal{F}'|_{G_2} = \mathcal{E}'_2$. We now undo the splitting operation by contracting edge (u_1, u_2). This results in the claimed embedding $\mathcal{F}$ of G. $\square$

2.2. Simultaneous embeddings

A *Simultaneous Embedding with Fixed Edges* (SEFE) of k planar graphs $G_1 = (V, E_1), G_2 = (V, E_2), \ldots, G_k = (V, E_k)$ consists of k drawings $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ such that: (i) Γ_i is a planar drawing of G_i, for $1 \leqslant i \leqslant k$; (ii) any vertex $v \in V$ is mapped to the same point in every drawing Γ_i, for $1 \leqslant i \leqslant k$; (iii) any edge $e \in E_i \cap E_j$ is mapped to the same Jordan curve in Γ_i and in Γ_j, for $1 \leqslant i, j \leqslant k$. The problem of testing whether k graphs admit a SEFE is called the SEFE problem. A SEFE of two planar graphs is depicted in Fig. 1.

Given two planar graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, the *intersection graph* of G_1 and G_2 is the planar graph $G_{1\cap 2} = (V, E_1 \cap E_2)$; further, the *exclusive subgraph* of G_1 (resp. of G_2) is the graph $G_{1\setminus 2} = (V, E_1 \setminus E_2)$ (resp. $G_{2\setminus 1} = (V, E_2 \setminus E_1)$). The *exclusive edges* of G_1 (of G_2) are the edges in $G_{1\setminus 2}$ (resp. in $G_{2\setminus 1}$). The *inclusive edges* of G_1 and G_2 are the edges in $G_{1\cap 2}$.

Jünger and Schulz [20] show that the SEFE problem can be equivalently stated in terms of embeddings. Namely, two graphs G_1 and G_2 whose intersection graph $G_{1\cap 2}$ is connected admit a SEFE if and only if there exist planar embeddings $\mathcal{E}_1$ and $\mathcal{E}_2$ of G_1 and G_2, respectively, such that $\mathcal{E}_1|_{G_{1\cap 2}} = \mathcal{E}_2|_{G_{1\cap 2}}$ holds, that is the two embeddings coincide when restricted to the intersection graph.

2.3. Book embeddings

A *book embedding* of a graph $G = (V, E)$ consists of a total ordering $\prec$ of the vertices in V and of an assignment of the edges in E to *pages* of a book, in such a way that no two edges (a, b) and (c, d) are assigned to the same page if $a \prec c \prec b \prec d$. A *k-page book embedding* is a book embedding using k pages. A *constrained k-page book embedding* is a k-page book embedding in which the assignment of edges to the pages is part of the input.

2.4. Connectivity and the SPQR-tree

A graph is *connected* if every pair of vertices is connected by a path. A graph G is *biconnected* (resp. *triconnected*) if removing any two vertices (resp. any three vertices) leaves G connected. In order to handle the decomposition of a biconnected graph into its triconnected components, we use the *SPQR-tree*, a data structure introduced by Di Battista and Tamassia (see, e.g., [5,6]). A biconnected planar graph and its SPQR-tree are depicted in Fig. 2. In the following we give a brief introduction to SPQR-trees and their use as a succinct representation of all embeddings of biconnected planar graphs.

A *separation pair* of G is a pair of vertices whose removal disconnects the graph. A *split pair* of G is either a separation pair or a pair of adjacent vertices. The SPQR-tree of a biconnected planar graph G is a tree $\mathcal{T}$, whose leaves correspond bijectively to the edges of G. The leaves of $\mathcal{T}$ are also called Q-nodes. We consider the rooted version of SPQR-trees, where the tree is rooted in an arbitrary Q-node, corresponding to a *reference edge* of G. The SPQR-tree is constructed by recursively decomposing G along split pairs, starting with the split pair defined by the reference edge. We refer to the vertices of $\mathcal{T}$ as nodes, in order to distinguish them from the vertices of G. Each node μ is associated with a multigraph $skel(\mu)$, the *skeleton* of μ. Each vertex of $skel(\mu)$ is also a vertex of G, and each edge uv in $skel(\mu)$ represents a corresponding split pair $\{u, v\}$ in G. The edges of the skeletons are either *virtual edges* representing a subgraph of G containing the end-vertices of the virtual edge or *real edges*, which are edges that also belong to G.

The skeleton of the Q-node corresponding to an edge (u, v) contains the two vertices u and v and two parallel edges between them, one real edge representing the edge (u, v) and one virtual edge representing the rest of the graph. Note that in our definition of the SPQR-tree only the skeletons of Q-nodes contain real edges, all other edges of the skeletons are virtual edges. An SPQR-tree has three types of internal nodes, namely S-nodes, P-nodes, and R-nodes. An S-node (or series node) is a node whose skeleton is a cycle of length $k \geqslant 3$. A P-node (or parallel node) is a node whose skeleton has two vertices and $k \geqslant 3$ parallel edges. An R-node (or rigid node) is a node whose skeleton is a simple 3-connected graph. It is assumed that no two S-nodes and no two P-nodes are adjacent in $\mathcal{T}$.

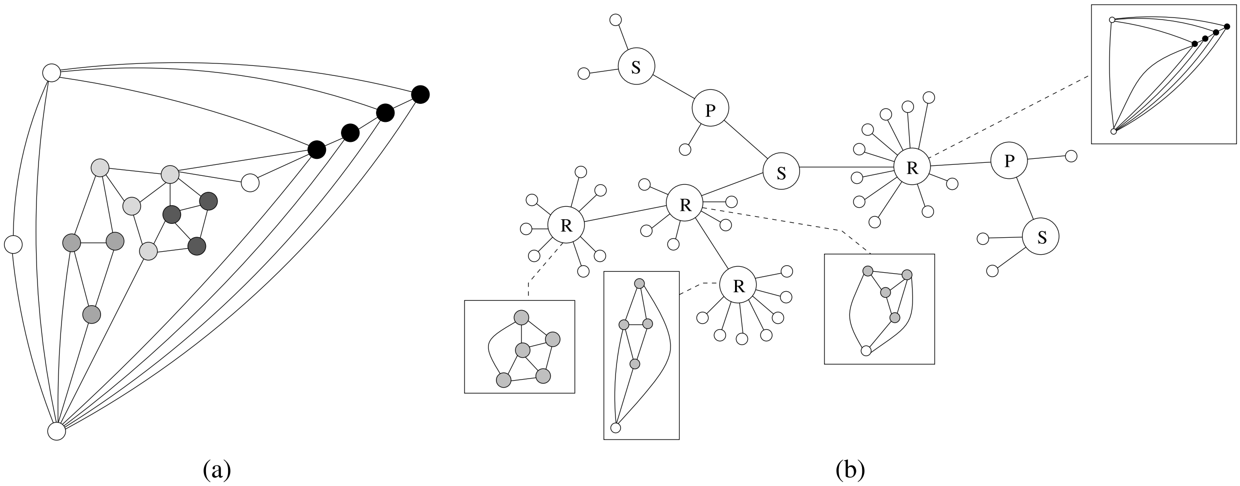

Fig. 2. (a) A biconnected planar graph and (b) its SPQR-tree, rooted at any Q-node adjacent to the R-node whose internal vertices are black. Q-nodes are represented by white circles. The skeletons of the R-nodes of the tree are represented inside the boxes. The virtual edge representing the parent of a node μ in the skeleton of μ is drawn as a dotted line.

Two distinct skeletons $skel(\mu)$ and $skel(\mu')$ share a virtual edge if and only if μ and μ' are adjacent in $\mathcal{T}$. Rooting the tree at a Q-node determines for each node μ different from the root one special virtual edge, namely the one that $skel(\mu)$ shares with its parent; its end-vertices are the *poles* of $skel(\mu)$. We denote by $u(\mu)$ and $v(\mu)$ the two poles of $skel(\mu)$.

The *pertinent graph* $G(\mu)$ of a node μ of $\mathcal{T}$ is the subgraph of G represented by the subtree of $\mathcal{T}$ rooted at μ. The pertinent graph of a Q-node different from the root is simply the edge represented by the Q-node. For any other node μ the pertinent graph $G(\mu)$ is obtained by merging the pertinent graphs of its children. The pertinent graph of the root is G itself. We say that a vertex v of G *belongs* to a node μ of $\mathcal{T}$ if v is a vertex of $G(\mu)$. In this case we also say that μ *contains* v.

Clearly, each skeleton $skel(\mu)$ can be obtained as a minor of G by contracting the pertinent graph of each child of μ and the rest of the graph to a single edge. Hence, for a planar graph G all the skeletons are planar. Moreover, a planar embedding $\mathcal{E}$ of G in which the reference edge lies on the outer face induces a unique planar embedding $\mathcal{E}(\mu)$ for each skeleton μ. Embedding $\mathcal{E}(\mu)$ is such that the virtual edge that μ shares with its parent is on the outer face. Conversely, the merging process described above shows that, given embeddings for all the skeletons, they can be merged into a unique planar embedding of G. More generally, specifying the embeddings of all the skeletons of the nodes belonging to the subtree of $\mathcal{T}$ rooted in a node μ, defines a unique planar embedding of $G(\mu)$. Since the skeletons of S- and Q-nodes are cycles, they have a unique embedding. For a P-node μ whose skeleton consists of k parallel edges, we may arbitrarily reorder its edges, so that the virtual edge that μ shares with its parent is on the outer face. For an R-node μ the skeleton is triconnected, and hence has a unique embedding up to flip.

The SPQR-tree $\mathcal{T}$ of a graph G with n vertices and m edges has m Q-nodes and $O(n)$ S-, P-, and R-nodes. Also, the total number of vertices of the skeletons stored at the nodes of $\mathcal{T}$ is $O(n)$. Finally, SPQR-trees can be constructed and handled efficiently. Namely, given a biconnected planar graph G, the SPQR-tree $\mathcal{T}$ of G can be computed in linear time [5,6,16].

In the following, we will only refer to the SPQR-tree of the intersection graph $G_{1\cap 2}$ of two graphs G_1 and G_2. However, with a slight abuse of notation, we will denote by $G_1(\mu)$ (by $G_2(\mu)$) the subgraph of G_1 (of G_2) induced by the vertices in $G_{1\cap 2}(\mu)$ and by $G(\mu)$ the graph $G_1(\mu) \cup G_2(\mu)$, where μ is a node of $\mathcal{T}$.

3. Computing a SEFE when the intersection graph is biconnected

In this section we show an algorithm for deciding the existence of a SEFE of two planar graphs G_1 and G_2 whose intersection graph $G_{1\cap 2}$ is biconnected. According to the characterization of Jünger and Schulz [20], this amounts to finding planar embeddings of G_1 and G_2 that coincide on $G_{1\cap 2}$. We take a slightly different view and search for an embedding of $G_{1\cap 2}$ that can be extended to planar embeddings of G_1 and G_2, respectively. Once such an embedding of $G_{1\cap 2}$ has been found, a SEFE of G_1 and G_2 can be easily computed by independently extending the embedding of $G_{1\cap 2}$ to embeddings of G_1 and G_2. This can for example be done with the algorithm by Angelini et al. [1], which solves exactly this problem in linear time. However, in our algorithm the necessary information for finding these extensions is actually constructed on the way of finding the embedding of $G_{1\cap 2}$, and thus the algorithm of Angelini et al. is not strictly necessary.

The description of the algorithm consists of two parts. Each exclusive edge of G_1 or G_2 puts certain restrictions on the embedding of $G_{1\cap 2}$. We study these restrictions and derive necessary conditions on the embedding of $G_{1\cap 2}$ in terms of its SPQR-tree in Section 3.1. We further show that these necessary condition are actually sufficient, which results in a simple polynomial-time algorithm for testing the existence of a SEFE of G_1 and G_2 by a bottom-up traversal of the SPQR-

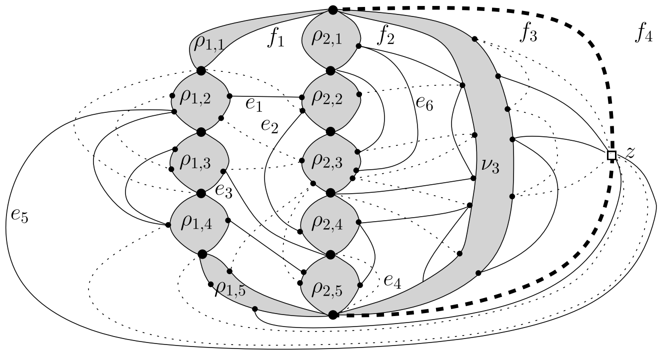

Fig. 3. A SEFE of graphs $G_1(\mu)$ and $G_2(\mu)$ when μ is a P-node with three children ν_1, ν_2, and ν_3. Also, ν_1 and ν_2 have children $\rho_{1,1},\dots,\rho_{1,5}$ and $\rho_{2,1},\dots,\rho_{2,5}$, respectively. For each visible node τ of μ, the interior of the cycle delimiting the outer face of $G_{1\cap 2}(\tau)$ is gray. Solid (dotted) edges are exclusive edges of G_1 (G_2). The dashed edge represents the rest of the graph.

tree of $G_{1\cap 2}$. In Section 3.2 we show how to improve the running time of the two main bottlenecks of the algorithm via dynamic programming; the resulting algorithm has linear running time.

3.1. A polynomial-time algorithm

Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two planar graphs whose intersection graph $G_{1\cap 2}$ is biconnected. Denote by $\mathcal{T}$ the SPQR-tree of $G_{1\cap 2}$.

To ease the description of the algorithm, we assume that $\mathcal{T}$ is rooted at any edge e of $G_{1\cap 2}$. This implies that e is adjacent to the outer face of any computed embedding of $G_{1\cap 2}$. Observe that this does not preclude the possibility of finding a SEFE of G_1 and G_2. Namely, consider any SEFE in the plane; “wrap” the SEFE around a sphere; project the SEFE back to the plane from a point in a face incident to e, thus obtaining a SEFE of G_1 and G_2 in which e is incident to the outer face of the embedding of $G_{1\cap 2}$. Further, if e is adjacent in $\mathcal{T}$ to an S-node, subdivide the edge of $\mathcal{T}$ connecting e to its only child by inserting a P-node. Observe that the described insertion of an artificial P-node ensures that the parent of any S-node is either an R-node or a P-node.

We classify the exclusive edges of G_1 or of G_2 into several types with respect to a node μ of $\mathcal{T}$, depending on whether its end-vertices belong to μ. An *internal edge* of a node $\mu \in \mathcal{T}$ is an exclusive edge e of G_1 or of G_2 such that both end-vertices of e belong to μ, at least one of them is not a pole of μ, and there exists no descendant of μ containing both the end-vertices of e. An *outer edge* of a node $\mu \in \mathcal{T}$ is an exclusive edge e of G_1 or of G_2 if exactly one end-vertex of e belongs to μ and this end-vertex is not a pole of μ. An *intra-pole edge* of a node $\mu \in \mathcal{T}$ is an exclusive edge e of G_1 or of G_2 if its end-vertices are the poles of μ. Observe that an exclusive edge e of G_1 or of G_2 can be an outer edge of a linear number of nodes of $\mathcal{T}$; also, e is an internal edge of at most one node of $\mathcal{T}$; moreover, e can be an intra-pole edge of a linear number of nodes of $\mathcal{T}$; however, e can be an intra-pole edge of at most one P-node of $\mathcal{T}$. In Fig. 3, edge e_1 is an internal edge of μ and an outer edge of $\rho_{1,2}$, of $\rho_{2,2}$, of ν_1, and of ν_2; edge e_2 is an internal edge of ν_2 and an outer edge of $\rho_{2,2}$ and $\rho_{2,4}$; edge e_3 is an internal edge of μ and an outer edge of $\rho_{1,3}$, ν_1, and ν_2; edge e_4 is an intra-pole edge of $\rho_{2,5}$; edge e_5 is an outer edge of $\rho_{1,2}$, of ν_1, and of μ.

The algorithm performs a bottom-up traversal of $\mathcal{T}$. When it visits a node μ of $\mathcal{T}$, either it concludes that a SEFE of G_1 and G_2 does not exist, or it determines a SEFE $\Gamma(\mu)$ of $G_1(\mu)$ and $G_2(\mu)$ such that, if a SEFE of G_1 and G_2 exists, there exists one in which the SEFE of $G_1(\mu)$ and $G_2(\mu)$ is $\Gamma(\mu)$. The rest of the graph, that is, the union of the graphs obtained from G_1 and G_2 by respectively removing the vertices of $G_1(\mu)$ and $G_2(\mu)$, except for $u(\mu)$ and $v(\mu)$, and their incident edges, will be placed in the same connected region of $\Gamma(\mu)$. This region is called the *outer face* of $\Gamma(\mu)$. The computed SEFE $\Gamma(\mu)$ of $G_1(\mu)$ and $G_2(\mu)$ has the property that all the outer edges of μ can be *drawn toward the outer face*, that is, a vertex z representing the contraction of the rest of the graph can be inserted into the outer face of $\Gamma(\mu)$ and all the outer edges of μ can be drawn with z replacing their end-vertex not in μ, still maintaining the planarity of the drawings of $G_1(\mu)$ and $G_2(\mu)$. An example of insertion of z in a SEFE of $G_1(\mu)$ and $G_2(\mu)$ is shown in Fig. 3.

The algorithm does not process any S-node directly, that is, the embedding choices for any S-node μ are deferred to the step in which the parent of μ is processed. Then, for every P-node and every R-node μ of $\mathcal{T}$, the *visible nodes* of μ are the children of μ that are not S-nodes plus the children of each child of μ that is an S-node. In Fig. 3 the visible nodes of the considered P-node are represented as gray regions. Intuitively, the visible nodes of a P-node or R-node μ are the descendants ν of μ such that, when μ is processed, an embedding of $G(\nu)$ has been already decided up to a flip of the whole embedding. In fact, when processing an R-node μ, a flip for the embedding of the pertinent graph $G(\nu)$ of each visible node ν is determined and, when processing a P-node μ, an ordering of the children of μ and a flip for the embedding of the pertinent graph $G(\nu)$ of each visible node ν are determined.

Some of the embedding choices that are taken when processing a P-node or an R-node μ are forced by the existence of exclusive edges connecting different visible nodes of μ, as will be stated in Lemmata 2 and 3. Some other embedding choices are arbitrary. However, such arbitrary choices do not alter the possibility of finding a SEFE of G_1 and G_2. Namely, we will prove that for any P-node or R-node μ any SEFE $\Gamma(\mu)$ of $G_1(\mu)$ and $G_2(\mu)$ can be extended to a SEFE of G_1 and G_2 if the latter SEFE exists, provided that $\Gamma(\mu)$ allows for drawing the outer edges of μ toward the outer face without creating crossings. Thus, when processing a P-node or an R-node μ there is no need for looking at the rest of the graph in order to decide an embedding of $G(\mu)$ such that the possibility of finding a SEFE of G_1 and G_2 is not precluded. We start with two necessary conditions on the embedding of the skeletons of the nodes of $\mathcal{T}$.

Lemma 2. *Let $\mathcal{E}_{1\cap 2}(\mu)$ be an embedding of $G_{1\cap 2}(\mu)$, with $\mu \in \mathcal{T}$, and let e be an internal edge of μ. Then, G_1 and G_2 have a SEFE in which the embedding of $G_{1\cap 2}(\mu)$ is $\mathcal{E}_{1\cap 2}(\mu)$ only if both end-vertices of e are incident to the same face of $\mathcal{E}_{1\cap 2}(\mu)$.*

Proof. Observe that e is an exclusive edge of either G_1 or G_2. The statement follows from the observation that, in any embedding $\mathcal{E}_{1\cap 2}(\mu)$ of $G_{1\cap 2}(\mu)$ in which the end-vertices of e are not both incident to the same face, edge e crosses at least one edge of $G_{1\cap 2}(\mu)$. As the edges of $G_{1\cap 2}(\mu)$ belong to both G_1 and G_2, either two edges of G_1 or two edges of G_2 cross (depending on whether $e \in G_1$ or $e \in G_2$). □

Lemma 3. *Let $\mathcal{E}_{1\cap 2}(\mu)$ be an embedding of $G_{1\cap 2}(\mu)$, with $\mu \in \mathcal{T}$, and let e be an outer edge incident to μ in a vertex $u(e)$. Then, G_1 and G_2 have a SEFE in which the embedding of $G_{1\cap 2}(\mu)$ is $\mathcal{E}_{1\cap 2}(\mu)$ only if $u(e)$ is on the outer face of $\mathcal{E}_{1\cap 2}(\mu)$.*

Proof. The statement follows from the observation that, in any embedding $\mathcal{E}_{1\cap 2}(\mu)$ of $G_{1\cap 2}(\mu)$ in which $u(e)$ is not incident to the outer face, edge e crosses at least one edge of $G_{1\cap 2}(\mu)$. As the edges of $G_{1\cap 2}(\mu)$ belong to both G_1 and G_2, either two edges of G_1 or two edges of G_2 cross (depending on whether $e \in G_1$ or $e \in G_2$). □

We now prove (in Lemma 4) that, in any SEFE Γ of G_1 and G_2 and for any node $\mu \in \mathcal{T}$ that is not an S-node, the outer face of $G(\mu)$ is (almost) the same. This will allow us to prove (in Lemma 5) that, if a SEFE of G_1 and G_2 exists, then (almost) *any* SEFE of $G_1(\mu)$ and $G_2(\mu)$ can be extended to a SEFE of G_1 and G_2.

Consider a SEFE $\Gamma = (\mathcal{E}_1, \mathcal{E}_2)$ of G_1 and G_2, and, for a node μ of $\mathcal{T}$, the outer face of $G_{1\cap 2}(\mu)$ in Γ. This face is delimited by a clockwise cycle C containing $u(\mu)$ and $v(\mu)$. Denote by $C_1(\Gamma, \mu)$ the circular list containing $u(\mu)$, $v(\mu)$, and all the vertices that are incident to exclusive edges of G_1 that are outer edges of μ, in the same order as they appear in C. Intuitively, $C_1(\Gamma, \mu)$ consists of the vertices of $\partial G_1(\mu)$ in their clockwise order of appearance along the outer face of $G_{1\cap 2}(\mu)$. List $C_2(\Gamma, \mu)$ is defined analogously, with G_1 replaced by G_2. We claim that, in each SEFE of G_1 and G_2 and for any node $\mu \in \mathcal{T}$ that is not an S-node, the lists $C_1(\Gamma, \mu)$ and $C_2(\Gamma, \mu)$ are essentially the same. Denote by C^{rev} the reverse of a circular list C.

Lemma 4. *For any two SEFE Γ and Γ' of G_1 and G_2 and for any node $\mu \in \mathcal{T}$ that is not an S-node, either $C_1(\Gamma, \mu) = C_1(\Gamma', \mu)$ and $C_2(\Gamma, \mu) = C_2(\Gamma', \mu)$ or $C_1(\Gamma, \mu) = C_1^{\text{rev}}(\Gamma', \mu)$ and $C_2(\Gamma, \mu) = C_2^{\text{rev}}(\Gamma', \mu)$ hold.*

Proof. Suppose, for a contradiction, that there exist two SEFES Γ and Γ' of (G_1, G_2) and a node μ of $\mathcal{T}$ that is not an S-node for which the statement does not hold. Then, consider any node μ of $\mathcal{T}$ such that: (i) the statement does not hold for μ; and (ii) the statement holds for all the descendants of μ in $\mathcal{T}$. We show that this implies that Γ or Γ' is actually not a SEFE of (G_1, G_2).

If μ is a Q-node then $C_1(\Gamma, \mu) = C_1(\Gamma', \mu) = C_2(\Gamma, \mu) = C_2(\Gamma', \mu) = [u(\mu), v(\mu)]$ and the statement holds, thus obtaining a contradiction.

Suppose that μ is an R-node. Since the statement holds for every visible node of μ and since *skel*(μ) has exactly one planar embedding, up to a reversal of the adjacency lists of all the vertices, there exists a visible node of μ that is flipped differently in Γ and Γ' and that has an outer edge e that is also an outer edge of μ; see Fig. 4a. Denote by $u(e)$ the end-vertex of e belonging to μ. Suppose that $u(e)$ is incident to the outer face of $G_{1\cap 2}(\mu)$ in Γ. Then, $u(e)$ is not incident to the outer face of $G_{1\cap 2}(\mu)$ in Γ', as shown in Fig. 4b. It follows that the edge e crosses $G_{1\cap 2}(\mu)$ in Γ', a contradiction.

Suppose that μ is a P-node. Then at most two children ν_x and ν_y of μ contain vertices of $\partial G_{1\cap 2}$ different from $u(\mu)$ and from $v(\mu)$, as otherwise a vertex of $\partial G_{1\cap 2}$ would not be incident to the outer face of $G_{1\cap 2}(\mu)$ in Γ and in Γ' and any outer edge of μ incident to such a vertex would cross $G_{1\cap 2}(\mu)$, thus contradicting the assumption that Γ and Γ' are SEFES of G_1 and G_2. The flips of ν_x and ν_y in Γ (if ν_x and ν_y are not S-nodes) or the flips of the children of ν_x and ν_y in Γ (if ν_x and ν_y are S-nodes) determine circular lists $C_1(\Gamma, \mu)$ and $C_2(\Gamma, \mu)$. An analogous statement holds with Γ' replacing Γ. Then, analogously to the R-node case, if a visible node of μ has an outer edge e that is also an outer edge of μ and such a node is flipped differently in Γ and Γ', then the end-vertex $u(e)$ of e in μ is not incident to the outer face of $G_{1\cap 2}(\mu)$ either in Γ or in Γ'. It follows that the edge e crosses $G_{1\cap 2}(\mu)$ in Γ or in Γ', a contradiction. □

Lemma 4 proves that the choice of an embedding $\mathcal{E}_{1\cap 2}(\mu)$ for $G_{1\cap 2}(\mu)$ does not restrict the possibility of finding a SEFE of G_1 and G_2 as long as the vertices in $\partial G_1(\mu)$ and $\partial G_2(\mu)$ are incident to the outer face of the computed SEFE of $G_{1\cap 2}(\mu)$. However, the condition that the vertices of $\partial G_1(\mu)$ and $\partial G_2(\mu)$ are incident to the outer face of $\mathcal{E}_{1\cap 2}(\mu)$ is not sufficient to guarantee that a SEFE of $G_1(\mu)$ and $G_2(\mu)$ can be extended to a SEFE of G_1 and G_2, if a SEFE of G_1 and G_2 exists. Namely,

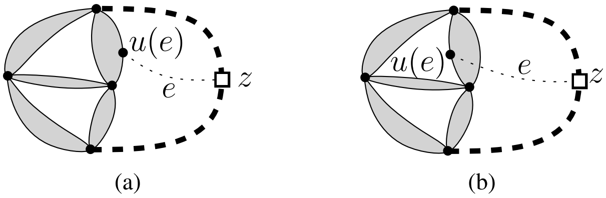

Fig. 4. (a) An embedding Γ of the skeleton *skel*(μ) of an R-node μ with an outer edge e whose end-vertex $u(e)$ is on the outer face of $G_{1\cap 2}(\mu)$. (b) An embedding Γ' of *skel*(μ) in which the visible node of μ containing $u(e)$ has a different flip than in Γ. Then the outer edge e must cross $G_{1\cap 2}(\mu)$.

it is also necessary that all the vertices of $\partial G_i(\mu)$ are incident to the outer face of $G_i(\mu)$, for $i = 1, 2$, as otherwise the outer edges of μ could not be drawn toward the outer face. A SEFE of $G_1(\mu)$ and $G_2(\mu)$ such that all the vertices of $\partial G_i(\mu)$ are incident to the outer face of $G_i(\mu)$, for $i = 1, 2$, is called *extendable*. We now show that any extendable SEFE of $G_1(\mu)$ and $G_2(\mu)$ can be extended to a full SEFE of G_1 and G_2, provided that a SEFE of G_1 and G_2 exists. In fact, we show a more general result, namely that we can replace the SEFE of $G_1(\mu)$ and $G_2(\mu)$ contained in an arbitrary SEFE $(\mathcal{E}_1, \mathcal{E}_2)$ of G_1 and G_2 with an arbitrary extendable SEFE $(\mathcal{E}_1^\mu, \mathcal{E}_2^\mu)$ of $G_1(\mu)$ and $G_2(\mu)$ (after possibly flipping SEFE $(\mathcal{E}_1^\mu, \mathcal{E}_2^\mu)$). Denote by $G_1 \setminus G_1(\mu)$ (by $G_2 \setminus G_2(\mu)$) the subgraph obtained from G_1 (resp. from G_2) by removing all the edges in $G_1(\mu)$ (resp. in $G_2(\mu)$) and all the vertices in $G_1(\mu)$ (resp. in $G_2(\mu)$), except for $u(\mu)$ and $v(\mu)$.

Lemma 5 *(Simultaneous Patching Lemma). Let G_1 and G_2 be two planar graphs such that $G_{1\cap 2}$ is biconnected, let $\mathcal{T}$ be the SPQR-tree of $G_{1\cap 2}$, and let μ be a node of $\mathcal{T}$ that is not an S-node. Let $(\mathcal{E}_1^\mu, \mathcal{E}_2^\mu)$ be an extendable* SEFE *of $G_1(\mu)$ and $G_2(\mu)$ and let $(\mathcal{E}_1, \mathcal{E}_2)$ be a* SEFE *of G_1 and G_2. Then there exists a* SEFE *$(\mathcal{E}'_1, \mathcal{E}'_2)$ of G_1 and G_2 such that*:

1. *$\mathcal{E}'_i$ and $\mathcal{E}_i$ coincide on $G_i \setminus G_i(\mu)$ for $i = 1, 2$, and*
2. *$\mathcal{E}'_i$ coincides with $\mathcal{E}_i^\mu$ on $G_i(\mu)$, for $i = 1, 2$, or $\mathcal{E}'_i$ coincides with the flip of $\mathcal{E}_i^\mu$ on $G_i(\mu)$, for $i = 1, 2$.*

Proof. We wish to show that we can replace the SEFE of $G_1(\mu)$ and $G_2(\mu)$ in $(\mathcal{E}_1, \mathcal{E}_2)$ by (a flip of) the SEFE $(E_1^\mu, \mathcal{E}_2^\mu)$ of $G_1(\mu)$ and $G_2(\mu)$. To this end, we apply the Patching Lemma, i.e. Lemma 1, to both G_1 and G_2, replacing in $\mathcal{E}_i$ the embedding of $G_i(\mu)$ with the embedding $\mathcal{E}_i^\mu$, for $i = 1, 2$ (after possibly flipping $\mathcal{E}_1^\mu$ and $\mathcal{E}_2^\mu$). The resulting embeddings $\mathcal{E}'_1$ and $\mathcal{E}'_2$ coincide on $G_{1\cap 2}(\mu)$ and on $G_{1\cap 2} \setminus G_{1\cap 2}(\mu)$, and hence on $G_{1\cap 2}$, i.e., $(\mathcal{E}'_1, \mathcal{E}'_2)$ is a SEFE of G_1 and G_2. Moreover, by construction, $\mathcal{E}'_i$ and $\mathcal{E}_i$ coincide on $G_i \setminus G_i(\mu)$, for $i = 1, 2$, and either $\mathcal{E}'_i$ coincides with $\mathcal{E}_i^\mu$ on $G_i(\mu)$, for $i = 1, 2$, or $\mathcal{E}'_i$ coincides with the flip of $\mathcal{E}_i^\mu$ on $G_i(\mu)$, for $i = 1, 2$. It remains to show that the conditions for applying the Patching Lemma are satisfied.

Clearly, $G_i(\mu)$ and $G_i \setminus G_i(\mu)$ are edge-disjoint subgraphs of G_i by construction, and the union of the vertex sets of $G_i(\mu)$ and $G_i \setminus G_i(\mu)$ is the vertex set of G_i. Additionally, all the vertices of $G_i(\mu)$ are in a single face of $\mathcal{E}_i|_{G_1 \setminus G_1(\mu)}$, with the common vertices (that is, $u(\mu)$ and $v(\mu)$) on the boundary of this face. Moreover, all the vertices of $\partial G_i(\mu)$ are on the outer face of $\mathcal{E}_i^\mu$, since $(\mathcal{E}_1^\mu, \mathcal{E}_2^\mu)$ is an extendable SEFE of $G_1(\mu)$ and $G_2(\mu)$. To satisfy the conditions of Lemma 1, we need to ensure that the vertices in $\partial G_i(\mu)$ occur in the same order along the outer faces of $\mathcal{E}_i^\mu$ and of $\mathcal{E}_i$ restricted to $G_i(\mu)$, for $i = 1, 2$. Due to Lemma 4, this order is fixed up to simultaneous reversal, and thus after possibly flipping both $\mathcal{E}_1^\mu$ and $\mathcal{E}_2^\mu$ the condition is satisfied. Then the Patching Lemma can be applied, yielding the claimed SEFE. $\square$

Lemma 5 shows that, for a non-S-node $\mu \in \mathcal{T}$ with visible nodes $\mu_1, \dots, \mu_k$, we can choose arbitrary extendable SEFES $(\mathcal{E}(G_1(\mu_i)), \mathcal{E}(G_2(\mu_i)))$ of $G_1(\mu_i)$ and $G_2(\mu_i)$ up to a flip, for $i = 1, \dots, k$, without altering the possibility of finding a SEFE of $(G_1(\mu), G_2(\mu))$. Therefore, when processing μ we assume that the visible nodes $\mu_1, \dots, \mu_k$ have fixed extendable SEFES $(\mathcal{E}(G_1(\mu_i)), \mathcal{E}(G_2(\mu_i)))$, for $i = 1, \dots, k$, and we want to test whether an extendable SEFE of $(G_1(\mu), G_2(\mu))$ exists. Observe that the computation of an extendable SEFE of μ implies choosing an embedding of *skel*(μ) and a flip for the SEFES $(\mathcal{E}(G_1(\mu_i)), \mathcal{E}(G_2(\mu_i)))$, for $j = 1, \dots, k$. Lemmata 2 and 3 give necessary conditions that the embedding of *skel*(μ) has to satisfy to lead to an extendable SEFE of $(G_1(\mu), G_2(\mu))$. An embedding of *skel*(μ) satisfying these conditions is a *compatible* embedding. We now show that given any compatible embedding $\mathcal{E}(skel(\mu))$ of *skel*(μ), if $G_1(\mu)$ and $G_2(\mu)$ admit an extendable SEFE, then they admit an extendable SEFE in which the embedding of *skel*(μ) is $\mathcal{E}(skel(\mu))$.

Theorem 1. *Let G_1 and G_2 be two planar graphs whose intersection graph $G_{1\cap 2}$ is biconnected and let $\mathcal{T}$ be the SPQR-tree of $G_{1\cap 2}$. Let μ be any node of $\mathcal{T}$ that is not an S-node and let $\mu_1, \dots, \mu_k$ be the visible nodes of μ. Assume that an extendable* SEFE *$(\mathcal{E}(G_1(\mu_i)), \mathcal{E}(G_2(\mu_i)))$ of $(G_1(\mu_i), G_2(\mu_i))$ exists, for each $i = 1, \dots, k$, and assume that a compatible embedding $\mathcal{E}(skel(\mu))$ of skel(μ) exists. Then, if $G_1(\mu)$ and $G_2(\mu)$ admit an extendable* SEFE*, they admit an extendable* SEFE *in which the embedding of skel(μ) is $\mathcal{E}(skel(\mu))$ and the* SEFE *of $(G_1(\mu_i), G_2(\mu_i))$ is either $(\mathcal{E}(G_1(\mu_i)), \mathcal{E}(G_2(\mu_i)))$ or its flip, for each $i = 1, \dots, k$.*

Proof. If $(G_1(\mu), G_2(\mu))$ do not admit an extendable SEFE then there is nothing to prove. Hence, assume that $(G_1(\mu), G_2(\mu))$ admit an extendable SEFE $(\mathcal{E}'(G_1(\mu)), \mathcal{E}'(G_2(\mu)))$. Let $\mathcal{E}'(skel(\mu))$ be the embedding of *skel*(μ) in $(\mathcal{E}'(G_1(\mu)), \mathcal{E}'(G_2(\mu)))$.

We show how to transform $(\mathcal{E}'(G_1(\mu)), \mathcal{E}'(G_2(\mu)))$ into an extendable Sefe $(\mathcal{E}(G_1(\mu)), \mathcal{E}(G_2(\mu)))$ of $(G_1(\mu), G_2(\mu))$ such that the embedding of $skel(\mu)$ in $(\mathcal{E}(G_1(\mu)), \mathcal{E}(G_2(\mu)))$ is $\mathcal{E}(skel(\mu))$ and the Sefe of $(G_1(\mu_i), G_2(\mu_i))$ is either $(\mathcal{E}(G_1(\mu_i)), \mathcal{E}(G_2(\mu_i)))$ or its flip, for each $i = 1, \ldots, k$.

If μ is an R-node, then the embedding of $skel(\mu)$ is unique up to a flip, hence $\mathcal{E}'(skel(\mu))$ and $\mathcal{E}(skel(\mu))$ coincide up to a flip of $(\mathcal{E}'(G_1(\mu)), \mathcal{E}'(G_2(\mu)))$. Moreover, by Lemma 5, the Sefe of $(G_1(\mu_i), G_2(\mu_i))$ can be set to be either $(\mathcal{E}(G_1(\mu_i)), \mathcal{E}(G_2(\mu_i)))$ or its flip, without changing the rest of the graph. Thus, after the Sefe of $(G_1(\mu_i), G_2(\mu_i))$ is set to be either $(\mathcal{E}(G_1(\mu_i)), \mathcal{E}(G_2(\mu_i)))$ or its flip, for $i = 1, \ldots, k$, the claimed Sefe is obtained.

If μ is a P-node, then an embedding of $skel(\mu)$ is a clockwise ordering of the virtual edges $e_1, \ldots, e_\ell$ of $skel(\mu)$. Consider the graph O whose vertices are $e_1, \ldots, e_\ell$ and that contains an edge (e_i, e_j) if the children of μ corresponding to e_i and e_j share an outer edge. Observe that, by Lemmata 2 and 3, e_i and e_j are adjacent in any compatible embedding of $skel(\mu)$. Since a Sefe of $(G_1(\mu), G_2(\mu))$ exists, graph O is either a cycle or a disjoint union of paths and isolated vertices. If O is a cycle, then the clockwise ordering of $e_1, \ldots, e_\ell$ in $\mathcal{E}'(skel(\mu))$ and in $\mathcal{E}(skel(\mu))$ is the same up to a flip of $(\mathcal{E}'(G_1(\mu)), \mathcal{E}'(G_2(\mu)))$. Otherwise, denote by $O_1, \ldots, O_r$ the connected components of O and, for $i = 1, \ldots, r$, let $G_1(O_i)$, $G_2(O_i)$, and $G_{1\cap 2}(O_i)$ be the corresponding subgraphs of G_1, G_2, and $G_{1\cap 2}$, respectively. The virtual edges of $skel(\mu)$ belonging to the same connected component O_i of O form an interval both in the clockwise ordering of $e_1, \ldots, e_\ell$ defining $\mathcal{E}(skel(\mu))$ and in the clockwise ordering of $e_1, \ldots, e_\ell$ defining $\mathcal{E}'(skel(\mu))$. Hence, $\mathcal{E}'(skel(\mu))$ and $\mathcal{E}(skel(\mu))$ may differ only for the clockwise order in which the different components of O occur and for the flip of the Sefe of $(G_1(O_i), G_2(O_i))$, for each connected component O_i of O. However, for $j \neq i$, $G_1(O_i)$ and $G_2(O_i)$ share with $G_1(O_j)$ and $G_2(O_j)$ only vertices $u(\mu)$ and $v(\mu)$. Therefore, the Sefes of $(G_1(O_1), G_2(O_1)), \ldots, (G_1(O_r), G_2(O_r))$ in $(\mathcal{E}'(G_1(\mu)), \mathcal{E}'(G_2(\mu)))$ can be ordered and independently flipped as in $\mathcal{E}(skel(\mu))$, therefore obtaining a Sefe of $(G_1(\mu), G_2(\mu))$ in which the embedding of $skel(\mu)$ is $\mathcal{E}(skel(\mu))$. Finally, by Lemma 5, the Sefe of $(G_1(\mu_i), G_2(\mu_i))$ can be set to be either $(\mathcal{E}(G_1(\mu_i)), \mathcal{E}(G_2(\mu_i)))$ or its flip, without changing the rest of the graph. Thus, after the Sefe of $(G_1(\mu_i), G_2(\mu_i))$ is set to be either $(\mathcal{E}(G_1(\mu_i)), \mathcal{E}(G_2(\mu_i)))$ or its flip, for $i = 1, \ldots, k$, the claimed Sefe is obtained. □

Theorem 1 suggests a very simple polynomial-time algorithm to test the existence of a Sefe of two planar graphs G_1 and G_2 whose intersection graph is biconnected; an outline of this algorithm is given as Algorithm 1. Namely, perform a bottom-up traversal of the SPQR-tree $\mathcal{T}$ of $G_{1\cap 2}$ and compute an extendable Sefe $(\mathcal{E}(G_1(\mu)), \mathcal{E}(G_2(\mu)))$ of $(G_1(\mu), G_2(\mu))$ for each node $\mu \in \mathcal{T}$ that is not an S-node. When processing a node $\mu \in \mathcal{T}$ that is not an S-node, an extendable Sefe $(\mathcal{E}(G_1(\mu_i)), \mathcal{E}(G_2(\mu_i)))$ of $G_1(\mu_i)$ and $G_2(\mu_i)$ is already fixed up to a flip for each visible node μ_i of μ.

Algorithm 1: SefeBico.

```
Input: Graphs G1 and G2
Output: Sefe of G1 and G2
Compute the SPQR-tree T of G1∩2 with root ρ;
Compute, for each node μ of T, its internal, outer, and intra-pole edges;
for each node μ in T in bottom-up order do
    if μ is a Q-node or an S-node then continue;
    else compute a compatible embedding E(skel(μ)) of skel(μ), stop if none exists;
    flip μ's visible nodes to find an extendable Sefe of G1(μ) and G2(μ), stop if not possible;
return the computed Sefe of G1(ρ) and G2(ρ);
```

A compatible embedding $\mathcal{E}(skel(\mu))$ of $skel(\mu)$ is then found as follows: If μ is an R-node, then $\mathcal{E}(skel(\mu))$ is the only (up to a flip) planar embedding of $skel(\mu)$; if μ is a P-node, then $\mathcal{E}(skel(\mu))$ is defined by a circular ordering $\mathcal{O}$ of the virtual edges of $skel(\mu)$ such that two virtual edges whose corresponding children of μ share an outer edge are consecutive in $\mathcal{O}$; observe that if such an ordering $\mathcal{O}$ does not exist, then, by Lemmata 2 and 3, G_1 and G_2 have no Sefe.

Next, we determine flips for the Sefes $(\mathcal{E}(G_1(\mu_i)), \mathcal{E}(G_2(\mu_i)))$, for each visible node μ_i of μ, and we determine a face of $\mathcal{E}(skel(\mu))$ in which each outer edge of μ, each internal edge of μ, the intra-pole edge of μ, and each internal edge of any S-node child of μ is embedded. Observe that such choices completely specify a Sefe $(\mathcal{E}(G_1(\mu)), \mathcal{E}(G_2(\mu)))$ of $(G_1(\mu), G_2(\mu))$. It is not hard to see that any internal edge of μ, any internal edge of an S-node child of μ, and any outer edge of μ can be embedded in at most two different faces. On the other hand, an analogous statement does not hold for intra-pole edges. In particular, an intra-pole edge of a P-node could possibly be embedded in a linear number of faces. The following lemma shows how to efficiently handle intra-pole edges.

Lemma 6. *Let (G_1, G_2) be an instance of* Sefe *with $G_{1\cap 2}$ biconnected and let (G'_1, G'_2) be the instance obtained from (G_1, G_2) by removing all the exclusive edges that are intra-pole edges. Let $\Gamma' = (\mathcal{E}'_1, \mathcal{E}'_2)$ be a* Sefe *of (G'_1, G'_2). Then, there exists a* Sefe *of (G_1, G_2) if and only if the intra-pole edges can be reinserted into $(\mathcal{E}'_1, \mathcal{E}'_2)$ without creating crossings. Moreover, such a reinsertion can be performed in linear time.*

Proof. Let $e^\star$ be an intra-pole edge belonging to G_1. If the end-vertices of $e^\star$ share a face in $\mathcal{E}'_1$ we simply embed $e^\star$ into this face. Note that this procedure never causes a crossing between two intra-pole edges of G_1. We proceed analogously for

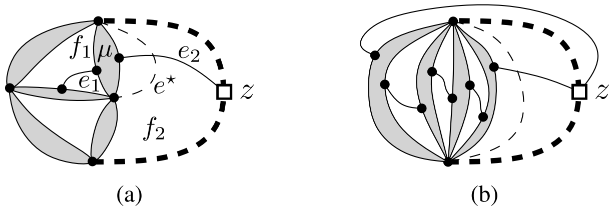

Fig. 5. Illustration of the cases in which reinserting the intra-pole edge $e^\star$ of some node into an embedding fails. (a) shows an S-node μ inside its parent, which is an R-node. (b) shows a P-node where the intra-pole edge $e^\star$ cannot be added in a planar way.

$\mathcal{E}_2'$ and the intra-pole edges belonging to G_2. This either results in the claimed SEFE of G_1 and G_2 or we find an intra-pole edge $e^\star$ belonging to G_1 (to G_2), whose end-vertices do not share a common face in $\mathcal{E}_1'$ (in $\mathcal{E}_2'$).

We prove that in the latter case G_1 and G_2 do not admit a SEFE. Assume that $e^\star$ is such an intra-pole edge belonging, without loss of generality, to G_1. Let μ the top-most node of the SPQR-tree $\mathcal{T}$ of $G_{1\cap 2}$ for which $e^\star$ is an intra-pole edge.

Observe that μ is not a Q-node, as otherwise $e^\star$ would not be an exclusive edge of G_1.

If μ is an S-node, then its parent μ' must be an R-node, as otherwise we would have two adjacent S-nodes in $\mathcal{T}$ (if the parent of μ is an S-node) or $e^\star$ would also be an intra-pole edge of the parent of μ (if the parent of μ is a P-node). Therefore, edge $e^\star$ can be embedded in at most two of the faces of the unique (up to a flip) embedding of *skel*(μ'). It follows that there exist outer edges e_1 and e_2 of μ belonging to G_1 that are embedded in f_1 and f_2, respectively. Edges e_1 and e_2 are therefore either internal edges or outer edges of μ'. This situation is depicted in Fig. 5a. Hence, the embeddings of e_1 into f_1 and of e_2 into f_2 are forced, and the end-vertices of $e^\star$ do not share a face in any SEFE of G_1' and G_2', contradicting the assumption that G_1 and G_2 admit a SEFE.

If μ is an R-node, consider the embedding of *skel*(μ) induced by $\mathcal{E}_1'$. Analogously to the S-node case, $e^\star$ must be embedded in one of the two faces f_1 and f_2 of *skel*(μ) that are incident to the edge that μ shares with its parent, as these are the only two faces incident to both the end-vertices of $e^\star$ in the only embedding of *skel*(μ). Since neither of these faces is available for the insertion of $e^\star$, there exist outer edges e_1 and e_2 of μ, both belonging to G_1, that are embedded in f_1 and f_2, respectively. However, edges e_1 and e_2 have to be embedded in f_1 and f_2, respectively, in any SEFE of G_1' and G_2', thus contradicting the assumption that G_1 and G_2 admit a SEFE.

Finally, if μ is a P-node, then $e^\star$ can be potentially embedded in any face of *skel*(μ), where *skel*(μ) has the embedding induced by $\mathcal{E}_1'$. Let $e_1, \ldots, e_k$ be the order of the virtual edges around $u(\mu)$ in the embedding of *skel*(μ) induced by $\mathcal{E}_1'$. Let e_1 be the virtual edge representing the rest of the graph in *skel*(μ). Assume that an index $1 < i < k$ exists such that edges e_i and e_{i+1} are not connected by any internal edge of μ. Then, $e^\star$ can be embedded in the face of *skel*(μ) delimited by e_i and e_{i+1}. Analogously, the face between e_k and e_1 and the face between e_1 and e_2 have to contain outer edges of μ belonging to G_1 in order for $e^\star$ not to be embeddable in any of such faces. The existence of all such internal and outer edges of μ completely determines the embedding of *skel*(μ) in any SEFE of G_1' and G_2', hence a SEFE of G_1 and G_2 does not exist, a contradiction. Fig. 5b shows an example of such a situation.

Finally, since the insertion process only requires to identify a common face of two vertices for each intra-pole edge, it can be implemented to run in linear time. □

In the following we therefore assume that (G_1, G_2) has no intra-pole edges. Once a SEFE for this instance has been found, the intra-pole edges can easily be reinserted in linear time. We now show how to find flips of the visible nodes of μ and embeddings of the internal edges of μ, of the outer edges of μ, and of the outer edges of the visible nodes of μ into faces of *skel*(μ) that result in an extendable SEFE of $G_1(\mu)$ and $G_2(\mu)$. Let *skel*$'(\mu)$ be the graph obtained from *skel*(μ) by replacing each virtual edge corresponding to an S-node ν with a path whose edges correspond to the children of ν. Let $\mathcal{E}(skel'(\mu))$ be the embedding of *skel*$'(\mu)$ obtained from $\mathcal{E}(skel(\mu))$ by replacing each virtual edge corresponding to an S-node with its associated path.

Note that all the internal edges of μ and all the outer edges of μ have to be embedded in a unique face of *skel*$'(\mu)$. Embedding such edges possibly determines the flips of the visible nodes containing their end-vertices. Observe that constraints stemming from different edges might enforce different flips on the same visible node; in this case we conclude that a SEFE does not exist. Fixing the flip of a node may in turn determine the face in which an edge has to be embedded. If an edge has no faces left to be embedded in (because of the flips of the visible nodes containing its end-vertices), we conclude that an extendable SEFE of $G_1(\mu)$ and $G_2(\mu)$ does not exist. If an edge has only one face left to be embedded in, we embed it there, again possibly fixing the flips of the visible nodes containing its end-vertices. This process stops in one of the following conditions: Either (1) all the internal edges of μ, all the outer edges of μ, and all the outer edges of the visible nodes of μ are embedded, or (2) some of the outer edges of the visible nodes of μ are not yet embedded (such edges are in fact internal edges for S-nodes children of μ) and the components containing the end-vertices of all such edges have not yet been flipped. In the former case we can arbitrarily choose the flips of the visible nodes that have not yet been fixed and obtain an extendable SEFE of $G_1(\mu)$ and $G_2(\mu)$. For the latter case, we show how to construct a 2SAT formula whose satisfying assignments are in one-to-one correspondence with the flips of the non-yet-flipped visible nodes and with the embeddings of the non-yet-embedded edges that yield an extendable SEFE of $G_1(\mu)$ and $G_2(\mu)$. We remark that each of the non-yet-flipped visible nodes has two possible flips and each non-yet-embedded edge can possibly be embedded in two faces.

The 2SAT formula contains one variable x_ν for each visible node ν of μ whose flip is not yet fixed and, for each exclusive edge e that is not yet embedded and may be embedded in two faces f_1 and f_2 of $skel'(\mu)$, it contains variables $x_e^{f_1}$ and $x_e^{f_2}$. The meaning of the variables is that $x_\nu = \texttt{true}$ if and only if the SEFE of $(G_1(\nu), G_2(\nu))$ is $(\mathcal{E}(G_1(\nu)), \mathcal{E}(G_2(\nu)))$, and $x_\nu = \texttt{false}$ if and only if their SEFE is the flip of $(\mathcal{E}(G_1(\nu)), \mathcal{E}(G_2(\nu)))$. The variable x_e^f is true if and only if e is embedded in the face f, with $f = f_1, f_2$. The set of clauses comprising the 2SAT formula consists of two subsets of clauses, the *consistency part* and the *planarity part*.

The consistency part expresses the constraints arising from the fact that embedding an exclusive edge into a face may require a certain flip of the visible nodes containing its end-vertices and vice versa. The consistency part is as follows. First, for each edge e that can be embedded in faces f_1 and f_2, we introduce the constraints $x_{e^*}^{f_1} \vee x_{e^*}^{f_2}$ and $\neg x_{e^*}^{f_1} \vee \neg x_{e^*}^{f_2}$ to ensure that e is embedded in exactly one of these faces. Moreover, if embedding an edge e into a face f implies a certain flip of a visible node of μ that contains an end-vertex of e, then we express this as an implication, which is a single 2SAT clause. Analogously, we can express as 2SAT clauses the implications that certain flips of visible nodes may have on the embeddings of the edges. It is not hard to see that the consistency part of the formula has size linear in the size of $skel'(\mu)$ and in the number of exclusive edges that need to be embedded.

The planarity part of the formula expresses the constraint that the resulting embedding should be planar. For each pair of non-embedded edges e_1 and e_2 of G_1 (of G_2) that would cross if they were embedded in the same face f, we add the constraint $\neg x_{e_1}^f \vee \neg x_{e_2}^f$ to express that at least one of them must not be embedded in f. Clearly, the planarity part has at most quadratic size in the number of exclusive edges that need to be embedded.

By construction, the formula is satisfiable if and only if $G_1(\mu)$ and $G_2(\mu)$ admit an extendable SEFE and such a SEFE can be constructed from a satisfying truth assignment. Since 2SAT can be solved efficiently [3], in fact in linear time, this yields a polynomial-time algorithm. We have the following theorem.

Theorem 2. *Given two graphs G_1 and G_2 on n vertices, where $G_{1\cap 2}$ is biconnected, it can be decided in $O(n^3)$ time whether G_1 and G_2 admit a* SEFE*. Moreover, if a* SEFE *exists, it can be computed in the same running time.*

Proof. The correctness of Algorithm 1 descends from Theorem 1. It remains to analyze the running time. The computation of the SPQR-tree takes linear time.

It is not hard to see that, given an exclusive edge e, all the nodes μ for which e is an internal, an outer, or an intra-pole edge can be computed in linear time by a traversal of the SPQR-tree. Hence, computing all the internal, outer, and intra-pole edges for all the nodes μ takes $O(n^2)$ time. In particular, each skeleton may have only linearly many such edges, and therefore finding a compatible embedding takes $O(n)$ time for each node. Although even the total size of $skel'(\mu)$ of all non-S-nodes μ is only $O(n)$, $O(n)$ exclusive edges may be internal, outer, or intra-pole for each of them. Hence, processing a non-S-node may require the consideration of $O(n)$ exclusive edges. Since the constructed 2SAT formula for finding the flips of the visible nodes and the embeddings of the edges is quadratic, also the existence of corresponding flips can be checked in $O(n^2)$ time for each non-S-node μ. Since the number of such nodes is in $O(n)$ the algorithm takes $O(n^3)$ time. □

Obviously, the main bottlenecks concerning the running time of the algorithm are 1) the distribution of the edges over the skeletons of the nodes, 2) the fact that the number of exclusive edges relevant for deciding an embedding of $skel'(\mu)$ and flips of its children may be $O(n)$, for each non-S-node μ, and 3) the fact that the constructed formula may have quadratic size in the number of exclusive edges that are relevant for deciding the embedding of $skel'(\mu)$. We will improve on all these bottlenecks in the next section.

3.2. A linear-time algorithm

We now show how to improve the running time of the algorithm described in Section 3.1 to linear.

We first sketch the main ideas. At a first glance, it might actually seem that quadratic running time is unavoidable, since each of the possibly linearly-many exclusive edges may appear as an internal, outer, or intra-pole edge of linearly many nodes of the SPQR-tree $\mathcal{T}$. However, many of these exclusive edges have their end-vertices in the same components of $skel'(\mu)$, where μ is a non-S-node of $\mathcal{T}$. The key idea here is to treat all the exclusive edges connecting two visible nodes ν_1 and ν_2 of μ as one edge. Hence, the exclusive edges are processed simultaneously (as opposed to distributing them to the skeletons one by one) and, for each node μ of $\mathcal{T}$ and each pair (ν_1, ν_2) of visible nodes of μ, only one representative edge (for each of G_1 and G_2) with end-vertices in ν_1 and in ν_2 is maintained when computing an extendable SEFE of $(G_1(\mu), G_2(\mu))$. Since $G_1(\mu)$ and $G_2(\mu)$ are planar, this already implies that the size of $skel'(\mu)$ together with all the representative edges is $O(|skel'(\mu)|)$, and thus sums up to $O(n)$ for all nodes of $\mathcal{T}$; this resolves the second bottleneck. Another observation concerns the processing of a node μ. Here the crucial step is the construction of a 2SAT formula, consisting of the consistency part and the planarity part, as described in the previous section. First, note that for the construction of both parts, it is sufficient to know for each node how embedding an edge in a face determines a flip of the children containing its end-vertices, and vice versa. On the other hand, the knowledge of a specific order of these

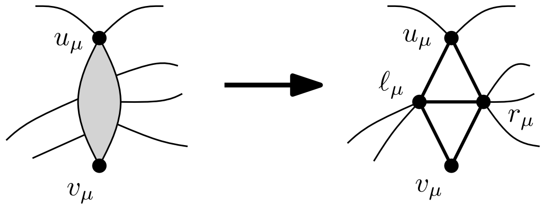

Fig. 6. The gadget of a node of μ that is not an S-node, represented as a thick gray edge. The thin edges attaching to μ are the exclusive edges of μ. After the transformation they are attached to either ℓ_μ or r_μ. Note that this transformation neglects exactly the order of the attachment vertices, but keeps the information about implications between flipping the node and embedding its adjacent edges in a face.

end-vertices along the outer face of the embedding of the pertinent graph of the same child (although it is unique by Lemma 4 for non-S-nodes) is not relevant for the construction of the formula, and can hence be neglected. We exploit this by introducing the *model* of nodes of the SPQR-tree, which contains this condensed information. We then show that deciding embeddings of all models is equivalent to solving the original SEFE problem. We further show that the models of all nodes can be computed in linear time; this removes the first bottleneck of the algorithm. The third issue with the construction of the 2SAT formula is the potential quadratic size of the planarity part. First, it should be noted that, for any non-S-node μ, the internal and outer edges have a unique face in which they can be embedded. If two such edges cross, then this crossing cannot be avoided and hence, if they belong to the same graph G_1 or G_2, a SEFE does not exist. Thus, the only edges for which we need the planarity constraints are the internal edges of the S-nodes that are children of μ. We show how to construct a planarity formula of linear size for each of such S-nodes. This improves the total size of the 2SAT formulas to linear and thus removes the third bottleneck of the algorithm.

We now describe the necessary modifications to Algorithm 1 in detail. First, we remark that, when a node μ is processed during the bottom-up traversal of the SPQR-tree $\mathcal{T}$ of $G_{1\cap2}$, an embedding $\mathcal{E}(G_{1\cap2}(\mu))$ of the pertinent graph $G_{1\cap2}$ of μ is fixed, up to a flip of the whole graph. This embedding determines a partition of the outer edges of μ into two sets of edges, that we call *left* and *right* edges, according to the position of their end-vertex on the outer face. In fact, Lemma 4 shows that the partition into left and right edges of any node that is not an S-node is unique, although flipping the embedding swaps left and right edges. Let μ be a node that is not an S-node with an embedding $\mathcal{E}(G_{1\cap2}(\mu))$ that allows for an extendable SEFE of $(G_1(\mu), G_2(\mu))$. Let the *gadget* of μ be a graph with four vertices, namely its poles u_μ and v_μ, and two vertices ℓ_μ and r_μ, called *attachment vertices*, and with five edges, namely (u_μ, ℓ_μ), (u_μ, r_μ), (ℓ_μ, v_μ), (r_μ, v_μ), and (ℓ_μ, r_μ). See Fig. 6. The gadget of μ succinctly describes the behavior of the pertinent graph $G_{1\cap2}(\mu)$ of μ with respect to the rest of the graph when its embedding $\mathcal{E}(G_{1\cap2}(\mu))$ has been fixed up to a flip. Namely, the only embedding choice for the gadget concerns its flip and, regardless of this choice, the two attachment vertices ℓ_μ and r_μ lie on opposite sides of the outer face. This behavior corresponds to the fact that all the left edges have to be attached to one of the two paths connecting u_μ and v_μ and comprising the outer face of $\mathcal{E}(G_{1\cap2}(\mu))$, while all the right edges have to be attached to the other of the two paths connecting u_μ and v_μ and comprising the outer face of $\mathcal{E}(G_{1\cap2}(\mu))$.

In order to find an embedding of the skeleton of each non-S-node μ and to decide the flips of the embeddings $(\mathcal{E}(G_1(\mu_i)), \mathcal{E}(G_2(\mu_i)))$ of each visible node μ_i of μ, we introduce, for each node μ of $\mathcal{T}$, the *model* of μ, denoted by $M(\mu)$. The model $M(\mu)$ consists of (1) a *frame*, which is composed of the gadgets of the visible nodes of μ arranged as in the skeleton of μ, and of (2) the exclusive edges that *occur in* μ, i.e., the internal edges of μ, the intra-pole edge of μ, the outer edges of μ, and the internal edges of the S-node children of μ.

We construct the frame by replacing in *skel*(μ) each virtual edge that corresponds to an S-node ν with a path of length equal to the number of children of ν. Moreover, we subdivide the virtual edge of *skel*(μ) representing the rest of the graph with a vertex z representing the outer face. Finally, we replace each edge corresponding to a visible node ν of μ with its gadget. Note that the model $M(\mu)$ is defined also when μ is an S-node; in such a case the gadgets of the nodes that are children of μ appear both in $M(\mu)$ and in the model of the parent of μ. However, we will use the models of S-nodes and of non-S-nodes in different steps of the algorithm, thus guaranteeing that the embedding choices are coherent.

In order to explain how to handle the exclusive edges that occur in μ we need some more definitions. Let a be a vertex of $G_{1\cap2}$ that is incident to an exclusive edge e occurring in μ. We define the *representative* of a in $M(\mu)$ as follows. If a does not belong to μ, then its representative is z. If a is a vertex of *skel*(μ) or a cutvertex of an S-node child of μ, then its representative is a itself. If none of the previous cases applies, then a belongs to the pertinent graph of a unique visible node ν of μ. In this case e is an outer edge of ν and therefore a lies on the outer face of any embedding of $G_{1\cap2}(\nu)$ that allows for an extendable SEFE of $G_1(\nu)$ and $G_2(\nu)$. If a lies on the clockwise path from $u(\nu)$ to $v(\nu)$ along the outer face of an embedding of $G_{1\cap2}(\mu)$, then its representative is ℓ_ν, otherwise its representative is r_ν. Note that the partition of the outer edges of ν into those having ℓ_ν as representative and those having r_ν as representative is unique, by Lemma 4, and it does not depend on the actual embedding of $G_{1\cap2}(\nu)$. Flipping the actual embedding of $G_{1\cap2}(\nu)$ maintains the same partition but swaps ℓ_ν with r_ν.

We now add the exclusive edges occurring in μ to the model $M(\mu)$. For any exclusive edge (u, v) of G_1 or of G_2 occurring in μ we add to $M(\mu)$ the edge between the representatives of its end-vertices. Fig. 7 shows the model of the node μ presented in Fig. 3.

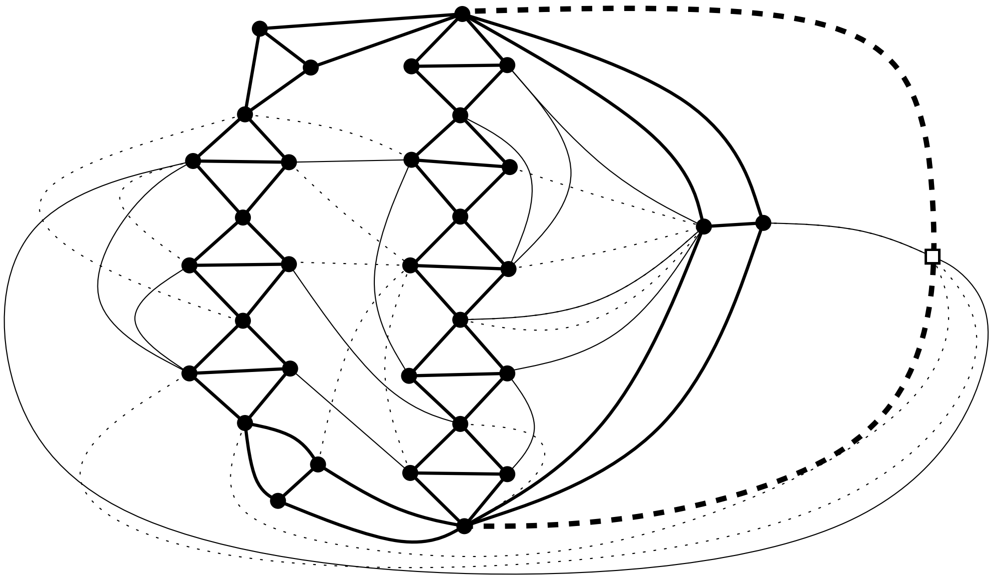

Fig. 7. The model of the node μ shown in Fig. 3.

A Sefe of a model $M(\mu)$ is an embedding of the model such that crossings only occur between pairs of exclusive edges where one edge stems from G_1 and the other one from G_2. Notice that there is a one-to-one correspondence between the Sefes of $M(\mu)$ and the extendable Sefes of $G_1(\mu)$ and $G_2(\mu)$. Namely, an embedding of the frame of μ corresponds to an embedding of *skel*(μ) plus a possible flip of the Sefe $(\mathcal{E}(G_1(\mu_i)), \mathcal{E}(G_2(\mu_i)))$, for each visible node μ_i for which a Sefe $(\mathcal{E}(G_1(\mu_i)), \mathcal{E}(G_2(\mu_i)))$ has already been decided; an embedding in $M(\mu)$ of the edges that occur in μ corresponds to an embedding in an extendable Sefe of $(G_1(\mu), G_2(\mu))$ of the edges that occur in μ. Moreover, if $(G_1(\mu), G_2(\mu))$ has an extendable Sefe $(\mathcal{E}(G_1(\mu)), \mathcal{E}(G_2(\mu)))$, then the same embedding and flipping choices lead to a Sefe of $M(\mu)$. The converse is, in general, not true. In fact, $M(\mu)$ may allow for a Sefe, while the same embedding and flip choices do not lead to an extendable Sefe of $(G_1(\mu), G_2(\mu))$. However, the next lemma shows that in this case $(G_1(\mu), G_2(\mu))$ do not allow for an extendable Sefe at all. Hence, once a Sefe of $M(\mu)$ has been determined, the algorithm has to check whether the resulting embedding for $G_{1\cap 2}$ allows for an extendable Sefe.

Lemma 7. *Let G_1 and G_2 be two planar graphs whose intersection graph $G_{1\cap 2}$ is biconnected. Let μ be a non-S-node of the SPQR-tree $\mathcal{T}$ of $G_{1\cap 2}$, let $M(\mu)$ be the model of μ, and let $\mathcal{E}_{M(\mu)}$ be a Sefe of $M(\mu)$. Suppose that (G_1, G_2) has a Sefe. Then the embedding and the flipping choices induced by $\mathcal{E}_{M(\mu)}$ lead to an extendable Sefe of $(G_1(\mu), G_2(\mu))$.*

Proof. Assume, for a contradiction, that the embedding of $(G_1(\mu), G_2(\mu))$ (together with the edges toward the outer face) induced by $\mathcal{E}_{M(\mu)}$ has a crossing. Observe that such a crossing can only involve two edges e_1 and e_2 that share an attachment vertex in $M(\mu)$, but do not share any vertex in $(G_1(\mu), G_2(\mu))$, as otherwise such a crossing would appear in $M(\mu)$ as well. Then, such edges are both incident to the same path that connects $u(\mu_i)$ and $v(\mu_i)$ along the outer face of the embedding of $(G_1(\mu_i), G_2(\mu_i))$, for some visible node μ_i of μ. First, this rules out the possibility that one of e_1 and e_2 is an intra-pole edge of μ. Second, if one of the crossing edges is an internal or an outer edge of μ, then the face of the embedding of *skel*(μ) in which it has to be embedded is fixed. By Lemma 4, the order of the attachment vertices of all these edges around this face is fixed (up to a flip) and therefore the crossing occurs in every embedding of $(G_1(\mu), G_2(\mu))$, thus contradicting the assumption that (G_1, G_2) admits a Sefe. Third, if both the end-vertices of e_1 and both the end-vertices of e_2 belong to the same S-node child of μ, then there are two faces in which e_1 and e_2 can be embedded. However, again by Lemma 4, in any extendable Sefe of $(G_1(\mu), G_2(\mu))$ the clockwise order of the vertices of the outer face of $G_{1\cap 2}(\mu)$ is the same, thus e_1 and e_2 cross in every embedding of $(G_1(\mu), G_2(\mu))$, thus contradicting the assumption that (G_1, G_2) admits a Sefe. □

Lemma 7 shows that, in order to find an extendable Sefe of $G_1(\mu)$ and $G_2(\mu)$, if one exists, it is enough to deal with the models of the non-S-nodes of the SPQR-tree of $G_{1\cap 2}$. Unfortunately it still seems difficult to compute all the models in linear time. We thus omit from the model some of the exclusive edges, the *superfluous edges*, which do not create any relevant embedding constraints. We call *partial model* a graph that can be obtained from the model of a node μ by removing some or all its superfluous edges. Consider an exclusive edge e of $M(\mu)$ in G_1 (in G_2) whose one end-vertex is a pole of μ and whose other end-vertex w is either the left attachment vertex ℓ_ν or the right attachment vertex r_ν of some visible

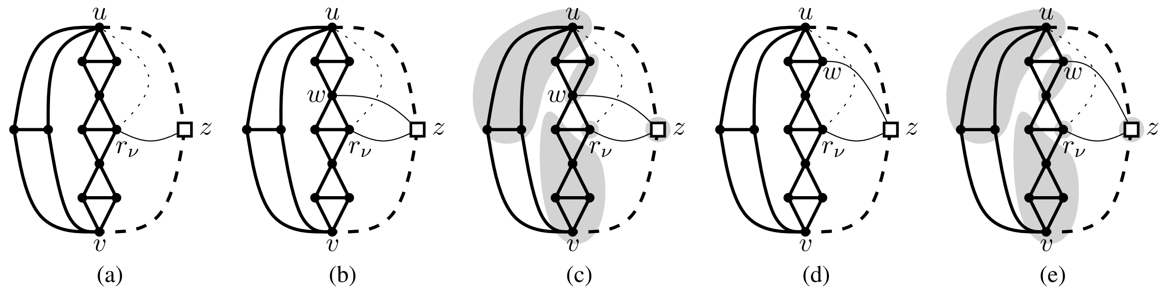

Fig. 8. Illustration of the reinsertion of a superfluous edge into a partial model of a non-S-node μ. Edges of the frame are drawn as thick black lines and the superfluous edge (r_ν, u) in G_1 is dashed. Other exclusive edges of G_1 are drawn as thin black lines. (a) Edge (r_ν, z) must exist since (r_ν, u) is superfluous. (b)–(e) If (r_ν, z) cannot be inserted in a planar way, then there exists an exclusive edge (w, z) of G_1 crossing (r_ν, u). Vertex w is either a pole of a visible node of μ (b) or a left/right vertex of some visible node (d). In both cases, a suitable contraction yields K_5 (c), (e), thus contradicting the planarity of G_1.

node ν of μ. Edge e is *superfluous* if $M(\mu)$ also contains edge (w, z), where z is the vertex of $M(\mu)$ representing the outer face. We now show that model $M(\mu)$ admits a SEFE if and only if any partial model of μ does.

Lemma 8. *Let μ be a node of the SPQR-tree that is not an S-node and let $M(\mu)$ be its model. Let further $M'(\mu)$ be a partial model of μ. Then $M(\mu)$ admits a* SEFE *if and only $M'(\mu)$ does.*

Proof. One direction is trivial: If $M(\mu)$ admits a SEFE, then a SEFE of $M'(\mu)$ can be obtained from the one of $M(\mu)$ by removing the superfluous edges of $M(\mu)$ that are not present in $M'(\mu)$.

Next, consider a SEFE $(\mathcal{E}'_1, \mathcal{E}'_2)$ of $M'(\mu)$, where u and v are the vertices of $M'(\mu)$ and of $M(\mu)$ corresponding to the poles of *skel*(μ). We show that all the superfluous edges can be added to $(\mathcal{E}'_1, \mathcal{E}'_2)$ without creating crossings. We show how to add the superfluous edges in G_1; the superfluous edges in G_2 can be added analogously. Let e be a superfluous edge of G_1, i.e., one of the end-vertices of e is either u or v, say u, and the other end-vertex is either the left or the right attachment vertex of some visible node ν of μ, say r_ν. The other cases are symmetric; see Fig. 8a for an example.

Since e is superfluous, the partial model contains an edge (r_ν, z) of G_1. Because of edge (r_ν, z), the embedding of the frame of μ in $(\mathcal{E}'_1, \mathcal{E}'_2)$ has a face f shared by r_ν and z. Since μ is not an S-node, the frame contains exactly one such a face. Now suppose that e causes a crossing when embedded in f. This implies that there exists an exclusive edge e' of G_1 that is embedded in f and that separates u from r_ν. Since $(\mathcal{E}'_1, \mathcal{E}'_2)$ is a SEFE, edge e' does not cross edge (r_ν, z). Hence, vertex z must be one of the end-vertices of e'. Let w be the other end-vertex of e'. Vertex w is either a pole of a visible node of μ (Fig. 8b) or a left/right attachment vertex of some gadget of a visible node of μ (Fig. 8d). In both cases, suitable contractions on the edges of G_1 lead to a K_5, as shown in Figs. 8c and 8e, thus contradicting the fact that G_1 is planar. □

This proves that it is in fact enough to work with partial models of the nodes in order to determine compatible embeddings of the skeletons and flips of their visible nodes. Thus models and partial models can be used interchangeably, and we will also denote partial models of μ by $M(\mu)$ in the following. We modify the algorithm from the previous section as described in Algorithm 2. The correctness already follows from Lemmata 7 and 8.

Algorithm 2: SefeBicoLinear.

Input: Graphs G_1 and G_2
Output: SEFE of G_1 and G_2

Compute the SPQR-tree $\mathcal{T}$ of $G_{1\cap 2}$;
Compute for each non-S-node μ of $\mathcal{T}$ a partial model $M(\mu)$;
for *each node μ in $\mathcal{T}$ in bottom-up order* **do**
  if *μ is a Q- or S-node* **then** continue;
  else find a compatible embedding $\mathcal{E}(\mu)$ of *skel*(μ), stop if none exists;
  compute a corresponding SEFE of $M(\mu)$ of the partial model of μ, stop if not possible;
$\mathcal{E} \leftarrow$ embedding of $G_{1\cap 2}$ determined by the embeddings of the skeletons;
$\mathcal{E}_1 \leftarrow$ resulting embedding of G_1 extending $\mathcal{E}$; $\mathcal{E}_2 \leftarrow$ resulting embedding of G_2 extending $\mathcal{E}$;
if *extension $\mathcal{E}_1$ or $\mathcal{E}_2$ does not exist* **then** stop;
return $(\mathcal{E}_1, \mathcal{E}_2)$;

It remains to show that the algorithm can be implemented to run in linear time. As already mentioned in the outline of this section, it is not necessary to keep track of more than one edge of G_1 and of more than one edge of G_2 connecting the same two vertices of $M(\mu)$. Hence, even if a linear number of nodes of $\mathcal{T}$ might exist having a linear number of outer edges each, $M(\mu)$ only contains $O(|M(\mu)|)$ such edges, and hence the total size of all the models is linear. The two main issues are the computation of the (partial) models for all the nodes of $\mathcal{T}$ and testing the existence of a SEFE of a model. We first show how to compute all the models of all the nodes of $\mathcal{T}$ in linear time.

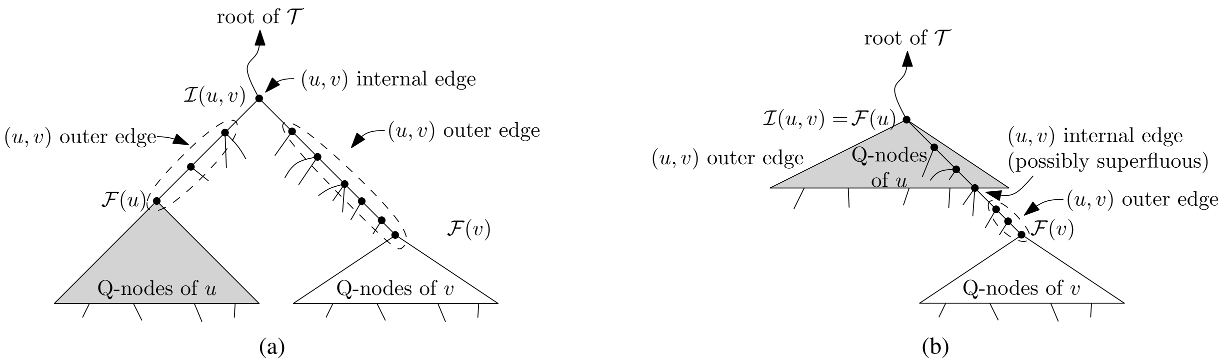

Fig. 9. Illustration of how an exclusive edge (u, v) occurs in different skeletons of the SPQR-tree. If $\mathcal{F}(u)$ and $\mathcal{F}(v)$ are not ancestor/descendant of one another, as in (a), then (u, v) appears as an internal edge at the lowest common ancestor of $\mathcal{F}(u)$ and $\mathcal{F}(v)$. Moreover, it appears as an outer edge on the paths from $\mathcal{F}(u)$ and $\mathcal{F}(v)$ to $\mathcal{I}(u, v)$, respectively. If $\mathcal{F}(v)$ is a descendant of $\mathcal{F}(u)$, as in (b), then (u, v) appears as an outer edge on the path from $\mathcal{F}(v)$ to the first node ρ containing u. Edge (u, v) appears as an internal edge in ρ. In this case, if there is a corresponding outer edge present in the model of ρ, (u, v) is superfluous and may be omitted from such a model.

For the computation of the models we require a data structure that, given a node μ and a vertex v, allows us to find the vertex or the virtual edge representing v in *skel*(μ).

Lemma 9. *Let G be an n-vertex biconnected planar graph and let $\mathcal{T}$ be its SPQR-tree. When traversing $\mathcal{T}$ in bottom-up order, it is possible to maintain in $O(n)$ time a data structure that, for every node μ of $\mathcal{T}$, allows to query, for a given vertex v of $G_{1\cap 2}$, the gadget containing the representative of v in $M(\mu)$ in amortized constant time.*

Proof. This can be done with a simple application of a union-find data structure. The main observation that is required to achieve $O(n)$ time is that the sequence of union operations only depends on $\mathcal{T}$ and is therefore known in advance. Hence, the $O(n)$ time version of union find by Gabow and Tarjan [13] applies. □

The following lemma shows how to compute partial models of all the non-S-nodes of $\mathcal{T}$ in $O(n)$ time.

Lemma 10. *Let G_1 and G_2 be two n-vertex planar graphs whose intersection graph $G_{1\cap 2}$ is biconnected and let $\mathcal{T}$ be the SPQR-tree of $G_{1\cap 2}$. It is possible to compute in $O(n)$ time a partial model of all the non-S-nodes of $\mathcal{T}$ or to conclude that (G_1, G_2) does not admit any* SEFE.

Proof. Let $\mathcal{T}$ be rooted at an arbitrary Q-node. We assume that all the exclusive edges that are intra-pole edges have been removed from G_1 and G_2. First, observe that the frame of each non-S-node $\mu \in \mathcal{T}$ can be easily computed by replacing each virtual edge representing a visible node ν of μ in *skel*(μ) with the gadget of ν and by replacing the virtual edge representing the rest of the graph with a path composed of two edges. Since the total number of virtual edges in $\mathcal{T}$ is $O(n)$ and since each gadget has constant size, the frames of all the non-S-nodes of $\mathcal{T}$ can be computed in $O(n)$ time.

Next, we determine in which nodes of $\mathcal{T}$ an exclusive edge occurs as an internal or an outer edge. We define the *first node* of a vertex u, denoted by $\mathcal{F}(u)$, as the lowest node of $\mathcal{T}$ such that $G_{1\cap 2}$ contains u and u is not a pole of $\mathcal{F}(u)$. Observe that $\mathcal{F}(u)$ is the lowest common ancestor of all the Q-nodes of $\mathcal{T}$ that represent edges incident to u, possibly excluding the root of $\mathcal{T}$ if it is among these Q-nodes. We now study how an exclusive edge (u, v) traverses the skeletons of $\mathcal{T}$. We distinguish two cases, based on the relative positions of $\mathcal{F}(u)$ and $\mathcal{F}(v)$ in $\mathcal{T}$. If $\mathcal{F}(u)$ and $\mathcal{F}(v)$ are not proper ancestor/descendant of one another, then their lowest common ancestor $\mathcal{I}(u, v)$ is either distinct from both $\mathcal{F}(u)$ and $\mathcal{F}(v)$, or we have $\mathcal{F}(u) = \mathcal{F}(v) = \mathcal{I}(u, v)$. In both cases, edge (u, v) is an internal edge of node $\mathcal{I}(u, v)$ and it is an outer edge of all the nodes on the paths from $\mathcal{F}(u)$ and from $\mathcal{F}(v)$ to $\mathcal{I}(u, v)$, respectively; see Fig. 9a. If $\mathcal{F}(u)$ is an ancestor of $\mathcal{F}(v)$, we define $\mathcal{I}(u, v) = \mathcal{F}(u)$. Let further ρ denote the first node on the path from $\mathcal{F}(v)$ to $\mathcal{F}(u)$ for which $G_{1\cap 2}(\rho)$ contains u. Edge (u, v) is an outer edge of all the nodes on the path from $\mathcal{F}(v)$ to ρ, excluding ρ, and an internal edge of ρ. Hence, u is a pole of ρ and, if ρ has an outer edge whose end-vertex in ρ belongs to the same visible node of ρ as v, the edge (u, v) is superfluous, and may be omitted from the partial model of ρ, see Fig. 9b.

Note that, using Harel and Tarjan's lowest common ancestor data structure [18], we can compute the first vertex $\mathcal{F}(u)$ of each vertex u in $O(\deg(u))$ time, thus in total $O(n)$ time over all the vertices of $G_{1\cap 2}$. Analogously, $\mathcal{I}(u, v)$ can be computed for each exclusive edge (u, v) in constant time, thus in total $O(n)$ time over all the exclusive edges of G_1 and G_2. We now distribute the information conveyed in values $\mathcal{F}(u)$ and in values $\mathcal{I}(u, v)$ over $\mathcal{T}$. Namely, for each node μ we build a list F_μ of all the edges that have an end-vertex whose first node is μ and a list I_μ of all the edges (u, v) for which $\mathcal{I}(u, v)$ is μ. Observe that, given that the first node of all vertices can be computed in $O(n)$ time and given that $\mathcal{I}(u, v)$ can be computed for all the exclusive edges in $O(n)$ time, lists F_μ and I_μ can be constructed in $O(n)$ time for all the nodes μ of $\mathcal{T}$.

In order to compute the edges that have to be part of the (partial) model of a node μ, we assume that each visible node μ has been equipped with two lists L_ν and R_ν, containing all the outer edges of ν and, possibly, some superfluous internal edges of ν or its descendants. We now describe how we can exploit the previous observations to quickly compute a partial model $M(\mu)$ for μ. Recall that the frame of μ has already been computed, hence the goal is (i) to identify the exclusive edges of the partial model $M(\mu)$ and (ii) to compute the partition of the outer edges of μ into left and right edges, represented by the lists L_μ and R_μ. We remark that such lists have to contain all the outer edges of μ plus, possibly, some superfluous internal edges of μ or its descendants.

In order to compute this information, we consider the left and right edges of the visible nodes of μ, which have already been computed, and combine them depending on the skeleton of μ. Note that the partition of the outer edges of the visible nodes of μ is only used to compute the partial model of μ and will not be used in subsequent steps. Based on this strategy, our algorithm computes the partition of the outer edges of μ into lists L_μ and R_μ efficiently. We implement these lists so that each edge appears in at most two lists, one for each end-vertex, and any given edge can be removed in $O(1)$ time from a list, without the need of finding its location inside the list. This is achieved by storing the previous- and next-pointers that are used to maintain the list with the edges. We initialize the list of each Q-node to an empty list. In the following we show how compute the exclusive edges of a partial model $M(\mu)$ and lists L_μ and R_μ in $O(n)$ time for all non-S-nodes μ of $\mathcal{T}$.

The algorithm for processing a non-S-node μ works in three steps. In the first step, we perform a preprocessing that ensures the following two properties. 1) All non-superfluous exclusive edges of $M(\mu)$ that are not outer edges of μ are contained in I_μ, or in F_μ, or in I_σ, or in F_σ, for some S-node σ child of μ, or in an auxiliary list A to be defined later. 2) For each such edge, we can compute its representatives in $M(\mu)$ in amortized $O(1)$ time. In the second step, we traverse all these lists to compute all the non-superfluous exclusive edges of μ that are not outer edges of μ. We add to $M(\mu)$ edges representing the computed non-superfluous exclusive edges of μ. In the third step, we identify the outer edges of μ, add to $M(\mu)$ edges representing them, and concatenate them according to $skel(\mu)$ to form the lists L_μ and R_μ.

We start by describing the first step. We first note that, given an exclusive edge (u, v), we can compute the gadgets of $M(\mu)$ containing the representatives of its end-vertices in amortized $O(1)$ time, by Lemma 9. This information is sufficient to find out whether (u, v) is an internal edge of μ, an internal edge of an S-node child of μ, or an outer edge of μ. The last case applies if exactly one end-vertex is represented by z. The former two cases apply if the representatives of u and v lie in distinct gadgets of $M(\mu)$. If neither of the cases applies, then (u, v) is *not relevant* for $M(\mu)$, that is, no edge will be added to $M(\mu)$ because of the existence of (u, v). On the other hand, given a *relevant* edge, that is, an edge that has to be represented in $M(\mu)$, we have to compute the representatives of its end-vertices. Observe that such an information cannot be derived from Lemma 9, that only enables us to find the gadgets containing such representatives. To actually find such representatives, observe that if the edge belongs to list L_ν (to list R_ν) of some visible node ν of μ, then the representative of the end-vertex u belonging to ν is ℓ_ν (is r_ν, respectively). We would thus like to traverse lists L_ν and R_ν of the visible nodes ν of μ to find the correct attachments. The problem is that the lists L_ν and R_ν of the visible nodes ν of μ cannot be traversed entirely. In fact, such lists contain internal edges of μ and internal edges of the S-nodes children of μ (such edges are in fact the ones we search for), contain superfluous edges of descendant nodes of μ (which we can remove for cleanup when found), and contain potentially linearly-many outer edges of μ (which we cannot afford to consider individually, as there is a linear number of such edges in a linear number of nodes). The key observation is that in any instance that admits a SEFE, for any visible node ν of μ only one of the two lists L_ν and R_ν may contain outer edges, otherwise both ℓ_ν and r_ν would have to share a face with the vertex z representing the outer face, which is not possible for P- and R-nodes.

We exploit this idea by performing a preprocessing step in which we traverse (part of) lists L_ν and R_ν of all the visible nodes ν of μ. For each internal edge of μ and each internal edge of the S-nodes children of μ that we find while traversing a list L_ν or R_ν, we annotate the correct vertex representing it (i.e., either ℓ_ν or r_ν, depending on whether we found it in L_ν or R_ν). We additionally add such an edge to the aforementioned auxiliary list A to be processed later. If we find a superfluous edge of a descendant of μ (which can be easily detected from Lemma 9), we remove it from the list. Since each edge may only be treated twice as an internal edge or as a superfluous edge, the running time for this computation over all the nodes of $\mathcal{T}$ is linear. If an edge is internal to μ, or if it is internal to an S-node child of μ, or if it is superfluous for a descendant of μ, we remove it from the list where it was found. Finally, whenever we are traversing a list L_ν or R_ν and we encounter an outer edge of μ, we immediately stop the traversal. As discussed above, for each visible node ν of μ we completely traverse at least one of the lists L_ν and R_ν, and hence at least one of these lists will be empty after the preprocessing. We have thus identified the representative vertices of all the edges in the list that is completely traversed, and of some of the edges in the list that was not fully traversed.

We claim that at this point all the non-superfluous internal edges of μ and of the S-nodes children of μ are contained either in the list I_μ, or in a list I_σ of an S-node σ child of μ, or in the list F_μ, or in a list F_σ for an S-node σ child of μ, or in the auxiliary list A. We prove the claim. Consider a non-superfluous edge (u, v), that is internal to μ or to one of the S-nodes children of μ. If the first nodes $\mathcal{F}(u)$ and $\mathcal{F}(v)$ of its end-vertices are not ancestor/descendant of one another, then (u, v) is an edge of $\mathcal{I}(u, v)$, as illustrated in Fig. 9a and, by the definition of the I-lists, it is contained in I_μ or in I_σ, for some S-node σ child of μ. Otherwise, assume without loss of generality that $\mathcal{F}(u)$ is an ancestor of $\mathcal{F}(v)$, as illustrated in Fig. 9b. In this case u is a pole of $skel(\mu)$, since u is shared by the skeleton of μ and the skeleton of its parent. If v is a vertex of $skel(\mu)$, then it is contained either in F_ν or in a list F_σ, for some S-node σ child of μ. Otherwise, the representative vertex of v in $M(\mu)$ is either ℓ_ν or r_ν, say ℓ_ν, for some visible node ν. In such a case (u, v) is an outer edge of ν. If the list L_ν has been completely traversed in the preprocessing step, then (u, v) is contained in the auxiliary list A, by construction of A. Otherwise, the traversal of L_ν has been stopped because an outer edge e of μ was found. Observe that one of the representatives of e in $M(\mu)$ is ℓ_ν, and the other one is z, given that such an edge is an outer edge of μ. Hence, in this case (u, v) is superfluous. This concludes the proof of the claim. Moreover, this finishes the description of the first step of the algorithm for processing μ. In fact, as we just proved, lists I_μ, F_μ, I_σ, F_σ, and A satisfy property (1). Moreover, the representatives of the end-vertices of all the edges in I_μ, F_μ, I_σ, F_σ, and A can be computed in amortized $O(1)$ time. Namely, if an end-vertex of any such an edge belongs to *skel*(μ), then its representative is the end-vertex itself. Otherwise, the representative of such an end-vertex has been computed when traversing a list of outer edges of a visible node of μ.

We now describe the second step. We traverse lists I_μ, $\mathcal{F}_\mu$, A, I_σ (for each S-node σ child of μ), and F_ν (for each visible node ν of μ), in order to add all the non-superfluous internal edges of μ and internal edges of the S-nodes children of μ to $M(\mu)$. If we encounter an edge that is marked as processed, we simply remove it from the currently traversed list. For each unprocessed edge (u, v) in any of these lists, we consider the representatives of u and v in $M(\mu)$ and find out whether (u, v) is actually an internal edge of μ, an internal edge of one of the S-nodes children of μ, or a superfluous edge. This step can be performed in $O(1)$ time due to property (ii) of the considered lists, as guaranteed in the first step. If (u, v) is superfluous, we remove it from the currently traversed list and from all the lists L_ν and R_ν of outer edges of the visible nodes of μ that may still contain it (recall that the removal of an edge from these lists can be performed in $O(1)$ time). If (u, v) is an internal edge of μ or an internal edge of one of the S-nodes children of μ, we add a corresponding edge to $M(\mu)$, if such an edge is not already present in $M(\mu)$, we mark (u, v) as processed, and remove it from the currently traversed list and from all the lists L_ν and R_ν of outer edges of the visible nodes of μ that may still contain it. If (u, v) is an outer edge of μ, which can only happen if the currently traversed list is F_μ or F_ν, for some visible node ν of μ, we leave it in the list. After such a processing, all the non-superfluous edges internal to μ or internal to an S-node child of μ have been added to the model of μ. This concludes the second step of the algorithm.

In the third step of the algorithm, we find the outer edges of μ and partition them into lists L_μ and R_μ. We also add edges to $M(\mu)$ corresponding to the outer edges of μ. By the computation performed in the second step of the algorithm, all the edges still in F_μ and in F_σ, for each S-node σ child of μ, are outer edges, and each non-empty list L_ν or R_ν of a visible node ν of μ contains at least one outer edge of μ. Conversely, any outer edge of μ is either incident to a vertex of *skel*$'(\mu)$, in which case it is contained in F_μ or in F_σ, for some S-node σ child of μ, or it is also an outer edge of some visible node ν of μ, in which case it is contained either in L_ν or in R_ν. We consider each list F_μ and F_σ, for each S-node σ child of μ, and, for each edge in such lists, we add a corresponding edge to the model, if not yet present. Observe that each edge in such lists connects a vertex of *skel*(μ), whose representative is the vertex itself, with a vertex not in μ whose representative is z. Hence, each such an edge can be added in $O(1)$ to the model. We also consider each list L_ν (each list R_ν) and if such a list is not empty, independently of the number of outer edges contained in it, we add to $M(\mu)$ an edge (ℓ_ν, z) (respectively (r_ν, z)). Notice that we do not traverse lists L_ν and R_ν, instead we just check whether they are non-empty, which can be done in $O(1)$ time per list. We have thus computed a partial model $M(\mu)$ for μ. It remains to compute the partition of the outer edges of μ into lists L_μ and R_μ.

We first check whether all the outer edges of μ can be embedded on the outer face. If μ is an R-node, this amounts to checking whether all the visible nodes ν of μ that have a non-empty list L_ν or R_ν can be flipped so that the corresponding representative vertex is incident to the outer face, and whether all the vertices v incident to edges in F_μ are incident to the outer face. If this is not the case, then the instance does not admit any SEFE by Lemma 3. If μ is a P-node, we reorder the virtual edges such that all the visible nodes ν of μ with non-empty lists L_ν or R_ν are adjacent to the outer face. If this is not possible, then the instance does not admit any SEFE, by Lemma 3. Then, to compute L_μ we traverse the boundary of the outer face from u_μ to v_μ in counterclockwise order and concatenate all the non-empty lists of visible nodes encountered on the way. We also traverse the list F_μ and the lists F_σ for each S-node σ child of μ, and add the edges to this list if their representative vertex in the model lies on the path from u_μ to v_μ along the outer face. We compute R_μ analogously by a traversal along the counterclockwise boundary of the outer face from v_μ to u_μ. This concludes the third step of the algorithm.

Since each step of the algorithm can be performed in linear time, as described above, the total running time is linear. □

Next, we deal with the problem of finding a SEFE of a given model $M(\mu)$ of a non-S-node μ in time $O(|M(\mu)|)$.

Lemma 11. *Given a model $M(\mu)$ of a non-S-node μ of the SPQR tree $\mathcal{T}$ of $G_{1\cap 2}$, it is possible to find a* SEFE *of $M(\mu)$ or to decide that no* SEFE *of $M(\mu)$ exists in $O(|M(\mu)|)$ time.*

Proof. We remove all the intra-pole edges from the model and reinsert them afterward. The correctness of this step follows from Lemma 6. After the removal of the intra-pole edges, each exclusive edge can be embedded in at most two different faces of the model. First, we find a compatible embedding of *skel*(μ).

If μ is an R-node, we simply check, for each exclusive edge (u, v) in $M(\mu)$, whether u and v share a face in the unique embedding of *skel*(μ). If μ is a P-node, we build an auxiliary graphs O containing one vertex for each child of μ. Two vertices ν_1 and ν_2 of O are connected by an edge if there is an exclusive edge in $M(\mu)$ connecting two vertices belonging ν_1 and ν_2, respectively. Since the number of exclusive edges in the model of μ is $O(|M(\mu)|)$, graph O can be constructed in $O(|M(\mu)|)$ time. A compatible embedding exists if and only if O is either a collection of disjoint paths or a single cycle.

Once O has been constructed, this can easily be checked (and a corresponding embedding can be constructed) in $O(|M(\mu)|)$ time.

Second, we describe how to flip the gadgets of the visible nodes of μ and how to embed the exclusive edges of $M(\mu)$, in order to obtain a SEFE of $M(\mu)$. To this end, we fix a *default flip* for each gadget of a visible node of μ and we construct a 2SAT formula φ_μ, analogously to the previous section. As before, the formula φ_μ contains one variable x_ν for each visible node ν of μ and, for each exclusive edge e of $M(\mu)$ that may be embedded in two faces f_1 and f_2 of $skel'(\mu)$, it contains variables $x_e^{f_1}$ and $x_e^{f_2}$. The meaning of the variables is as follows: $x_\nu = \texttt{true}$ means that ν must be flipped, while $x_\nu = \texttt{false}$ means that ν keeps its default flip; moreover, x_e^f is true if and only if e is embedded in the face f, with $f = f_1, f_2$.

Formula φ_μ consists of a consistency part and of a planarity part. For the consistency part, φ_μ contains clauses ensuring consistency between the choices done for the flips of the gadgets and the choices done for the embeddings of the edges. Namely, if an edge e can be embedded in a face f of $skel'(\mu)$ if and only if the gadget of a visible node ν is flipped, we add to φ_μ clauses $x_\nu \vee \neg x_e^f$ and $\neg x_\nu \vee x_e^f$. For the planarity part, formula φ_μ contains clauses such that a SEFE of $M(\mu)$ exists only if φ_μ is satisfiable. Namely, we will construct φ_μ so that it is satisfiable if and only if there exists a flip of all the gadgets of the visible nodes of μ such that: (1) for every exclusive edge e in $M(\mu)$, the end-vertices of e share a common face in $skel'(\mu)$; and (2) no two exclusive edges e_1 and e_2 in $M(\mu)$ cross if they belong to the same graph G_1 or G_2 and if at least one of them is an internal edge of an S-node child of μ. Hence, a satisfying assignment to the variables of φ_μ does not rule out the possibility that the embedding of $M(\mu)$ corresponding to such an assignment has a crossing between two internal edges of μ. However, such a kind of crossings cannot be avoided, as any internal edge can be embedded in exactly one face of $skel'(\mu)$. We now describe in detail how to construct the planarity part of formula φ_μ.

Consider each exclusive edge (u, v) of $M(\mu)$ such that u and v belong to gadgets of visible nodes of μ associated with different children of μ. The embedding of (u, v) is restricted to a unique face f. We then add the clause x_e^f to φ_μ in order to encode this.

Next, we take care of each pair of edges that belong to the same graph G_1 or G_2 and such that at least one of them is an internal edge of an S-node child ν of μ. First, in order to express that each such edge e has to be embedded in exactly one of the two faces f_1 and f_2 incident to ν in $skel'(\mu)$, we add to φ_μ clauses $x_e^{f_1} \vee x_e^{f_2}$ and $\neg x_e^{f_1} \vee \neg x_{e_2}^{f}$. In order to add planarity constraints for such edges, we could pick all the pairs of such edges and, for each pair of edges e_1 and e_2 that would cross when embedded in the same face f, add clauses $x_{e_1}^f \vee x_{e_2}^f$ and $\neg x_{e_1}^f \vee \neg x_{e_2}^f$. However, since there may be a quadratic number of such edge pairs, this would result in a formula of quadratic size. In the following we show that the same set of constraints can in fact be represented by a number of clauses that is linear in $|M(\mu)|$. More specifically, we will handle separately the constraints concerning edges internal to distinct S-nodes children of μ. In order to handle the constraints concerning edges internal to an S-node ν, we will consider the model $M(\nu)$. The models $M(\nu)$ of all the S-nodes ν children of μ can be easily computed in $O(|M(\mu)|)$ time from $M(\mu)$.

We show how to add to φ_μ clauses expressing planarity constraints stemming from pairs of edges of G_1 in $M(\nu)$, for an S-node ν child of μ. Planarity constraints stemming from pairs of edges of G_2 in $M(\nu)$ can be handled analogously. Let f_1 and f_2 be the two faces of $skel'(\mu)$ incident to ν. Let $u_1 = u(\nu), u_2, \ldots, u_m = v(\nu)$ be the vertices of $skel(\nu)$, ordered as they occur from pole $u(\nu)$ to pole $v(\nu)$. Let H_j be the gadget containing vertices u_j and u_{j+1}, for each $1 \leqslant j \leqslant m-1$, and let ℓ_i and r_i denote the left and right vertices of gadget H_i, respectively. We remark that the vertices of $M(\nu)$ are the vertices of gadgets $H_1, \ldots, H_m$ plus a vertex z that represents the outer face and that is connected to u_1 and u_m. Remove from $M(\nu)$ all the exclusive edges in G_2. Consider the SPQR-tree $\mathcal{T}_1(\nu)$ of the resulting graph $G_1(\nu)$.

First, we observe the following. Since each gadget H_i of a child ν_i of ν is triconnected, we have that, in any node ξ of $\mathcal{T}_1(\nu)$, either the five edges of H_i are represented by five different edges of $skel(\xi)$, that is, ξ is an R-node whose skeleton contains five virtual edges corresponding to the five Q-nodes composing H_i, or the five edges of H_i belong to the pertinent graph of one virtual edge of $skel(\xi)$. Moreover, let C denote the directed cycle $u_1, \ell_1, r_1, u_2, \ldots, u_{m-1}, \ell_{m-1}, r_{m-1}, u_m, z$. Since C is Hamiltonian, it follows that C defines a corresponding directed Hamiltonian cycle C_ξ in $skel(\xi)$ of each R-node ξ of $\mathcal{T}_1$.

Second, let e_1 and e_2 be two exclusive edges of G_1 that belong to $M(\nu)$ and that are such that, if they are both embedded in f_1 or both embedded in f_2, then they cross. We claim that there exists an R-node ξ of $\mathcal{T}_1(\nu)$ whose skeleton has two virtual edges g_1 and g_2 corresponding to Q-nodes e_1 and e_2. That is, if e_1 and e_2 cannot be embedded in the same face of $skel'(\mu)$, then e_1 and e_2 are edges of the skeleton $skel(\xi)$ of the same R-node ξ. To prove the claim, it suffices to observe that $M(\nu)$ contains K_4 as a minor on the end-vertices of e_1 and e_2, i.e., it contains six vertex-disjoint (except at the end-vertices) paths, each connecting a distinct pair among the end-vertices of e_1 and e_2. To see this, consider the cycle C and the edges e_1 and e_2. Since e_1 and e_2 cross if they are embedded in the same face, their end-vertices alternate along C, and contracting the subpaths of C between consecutive end-vertices of e_1 and e_2 to single edges yields the claimed K_4.

The previous two observations are what we need in order to be able to express all the planarity constraints concerning internal edges of ν with a number of clauses that is linear in $M(\nu)$. We first give an informal description of why this is true, and we later formally explain which clauses have to be introduced in φ_μ.

Consider any two edges e_1 and e_2 that cross when both embedded in f_1 or in f_2. Let e_1^a and e_1^b be the end-vertices of e_1, and let e_2^a and e_2^b be the end-vertices of e_2. Then, consider an R-node ξ in $\mathcal{T}_1(\nu)$ whose skeleton contains e_1 and e_2, which exists by the previous observation. Now consider the cycle C_ξ in $skel(\xi)$, see Fig. 10. The cycle C_ξ separates f_1

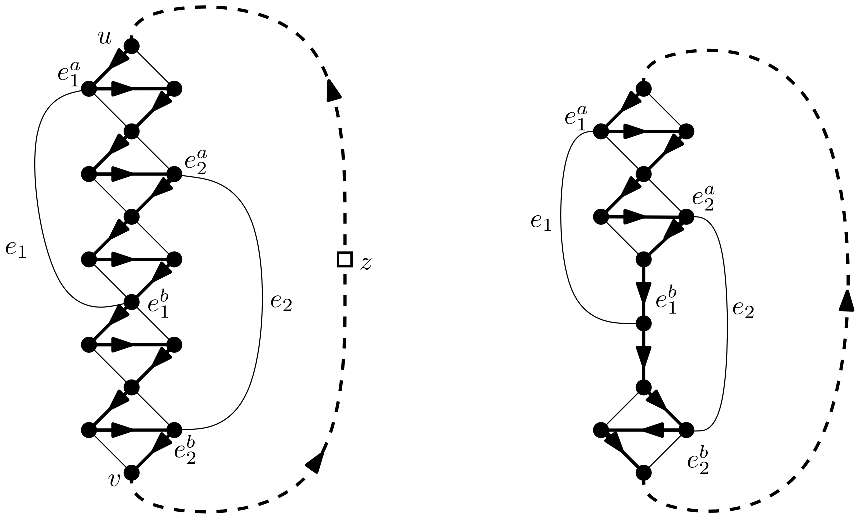

Fig. 10. Part of the model $M(\nu)$, on the left, where C is shown by thick lines, the edges incident to z are dashed. Part of the skeleton $skel(\xi)$ of an R-node ξ, on the right, where C_ξ is shown by thick lines, the virtual edge representing the rest of the graph is dashed.

from f_2. Since $skel(\xi)$ has exactly two embeddings, which are one the flip of the other, e_1 and e_2 are embedded in different faces in any planar embedding of $skel(\xi)$. More generally, any two internal edges of ν that belong to the same skeleton $skel(\xi)$ of a node ξ of $\mathcal{T}_1(\nu)$ either are always embedded in the same face or are always embedded in different faces in any planar embedding of $skel(\xi)$. Thus, in order to capture all the planarity constraints among internal edges of a node ν that occur in $skel(\xi)$, it suffices to choose any internal edge of ν in $skel(\xi)$, say e_1, and express the constraints of all the other internal edges of ν in $skel(\xi)$ with respect to e_1.

We now formally describe which clauses to introduce in φ_μ. For each R-node ξ of $\mathcal{T}_1(\nu)$, consider the set $\{e_1, e_2, \ldots, e_p\}$ of internal edges of ν that are edges of $skel(\xi)$. In any embedding of $skel(\xi)$, cycle C_ξ splits these edges into two disjoint subsets S_1 and S_2, such that all exclusive edges in S_1 are embedded on one side of C_ξ and the edges of S_2 are embedded on the other side of C_ξ. Without loss of generality we assume that $e_1 \in S_1$. Now, we express that all edges of S_1 must be embedded in the same face as e_1 by adding the clauses $x_{e_1}^{f_1} \vee \neg x_e^{f_1}$ and $\neg x_{e_1}^{f_1} \vee x_e^{f_1}$, for each edge $e \in S_1 \setminus \{e_1\}$. Analogously, we specify that all edges of S_2 must be embedded in the face different from the one where e_1 is embedded by adding the clauses $x_{e_1}^{f_1} \vee x_e^{f_1}$ and $\neg x_{e_1}^{f_1} \vee \neg x_e^{f_1}$, for each edge $e \in S_2$.

The resulting formula φ_μ is linear in the size of $M(\mu)$ and can be constructed in $O(|M(\mu)|)$ time. Satisfiability of φ_μ and, in the positive case, a corresponding assignment and the resulting flips can be tested in $O(|\varphi_\mu|)$ time [3]. Clearly, if φ_μ is not satisfiable, the constraints contradict each other and a SEFE does not exist. Otherwise, we get a SEFE of $M(\mu)$, which concludes the proof of the lemma. □

To solve the SEFE problem for graphs G_1 and G_2 with biconnected intersection graph we now perform three steps. First, we compute the SPQR-tree $\mathcal{T}$ of $G_{1\cap 2}$ and, for each non-S-node μ of $\mathcal{T}$, we compute in linear time its model $M(\mu)$. This is possible by Lemma 10. Second, we construct SEFEs for all models, each in time proportional to the model size. As the total size of all the models is linear, by Lemma 11 this can be done in linear time, too. Third, we construct from the SEFEs of the models an embedding of $G_{1\cap 2}$. This can be easily done in linear time. It remains to check whether the exclusive edges can be inserted in linear time, thus yielding a SEFE of G_1 and G_2. This can either be done by using the algorithm of Angelini et al. [1], or as follows. Note that the SEFEs of the models tell us, for each exclusive edge, the face of $G_{1\cap 2}$ in which it is embedded. We then traverse all the faces of $G_{1\cap 2}$ and check whether all the exclusive edges of G_1 (of G_2) assigned to it can embedded in a planar way inside that face. Since each face of $G_{1\cap 2}$ is bounded by a simple cycle C, this can be done by a simple scan along C, which checks that no two edges have alternating end-vertices. We obtain the following.

Theorem 3. *Given two graphs G_1 and G_2 on n vertices, where $G_{1\cap 2}$ is biconnected, it can be decided in linear time whether G_1 and G_2 admit a* SEFE*. Moreover, if a* SEFE *exists, it can be computed in the same running time.*

4. The intersection graph is connected

In this section we show that the SEFE problem, when the intersection graph is connected, is equivalent to a 2-page book embedding problem defined as follows. Let G be a graph, let (E_1, E_2) be a partition of its edge set, and let T be a rooted tree whose leaves are exactly the vertices of G. Note that this implies that the internal nodes of T are disjoint from the vertices of G. The problem PARTITIONED T-COHERENT 2-PAGE BOOK EMBEDDING with input (G, E_1, E_2, T) asks: Does a 2-page

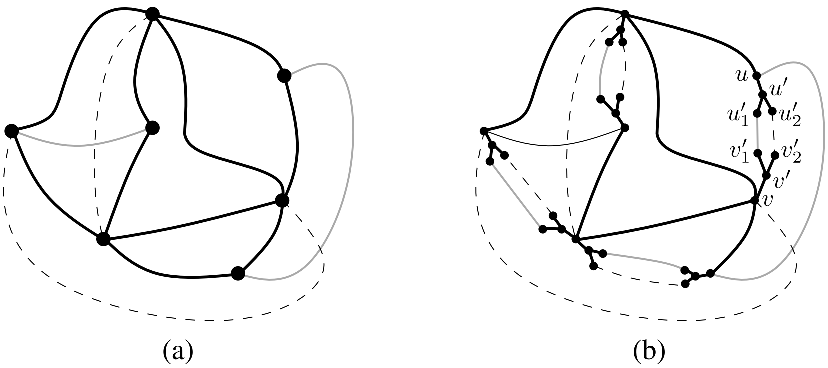

Fig. 11. (a) A SEFE Γ of G_1, G_2. (b) A SEFE Γ' of G'_1, G'_2 constructed from Γ.

book embedding of G exist in which the edges of E_1 lie in one page, the edges of E_2 lie in the other page, and, for every internal vertex $t \in T$, the vertices of G in the subtree of T rooted at t appear consecutively in the vertex ordering of G defined in the book embedding? See Fig. 14 for an example.

We now show how to transform an instance $G_1 = (V, E_1), G_2 = (V, E_2)$ of SEFE in which $G_{1\cap 2}$ is connected into an instance of PARTITIONED T-COHERENT 2-PAGE BOOK EMBEDDING. This transformation consists of three steps.

In the first step, we transform an instance G_1, G_2 of SEFE such that $G_{1\cap 2}$ is connected into an equivalent instance G'_1, G'_2 of SEFE such that $G'_{1\cap 2}$ is a tree. To this end, start with graphs G'_1, G'_2 having the same vertex set of G_1, G_2 and having no edges. Partition the edges of $G_{1\cap 2}$ into two sets $E'_{1\cap 2}$ and $E''_{1\cap 2}$, in such a way that the edges of $E'_{1\cap 2}$ induce a spanning tree of $G_{1\cap 2}$ and the edges of $E''_{1\cap 2}$ are the edges in $G_{1\cap 2}$ not in $E'_{1\cap 2}$. The edges of $E'_{1\cap 2}$ belong to $G'_{1\cap 2}$, that is, they are added both to G'_1 and to G'_2. Also, add to G'_1 each exclusive edge of G_1 and add to G'_2 each exclusive edge of G_2. Next, for each edge (u, v) in $E''_{1\cap 2}$, we introduce the *gadget of* (u, v), which consists of vertex u, of vertex v, of the new vertices u', u'_1, u'_2, v', v'_1, and v'_2, of edges (u, u'), (u', u'_1), (u', u'_2), (v, v'), (v', v'_1), (v', v'_2) in $G'_{1\cap 2}$ (that is, such edges are added to both to G'_1 and to G'_2), of the exclusive edge (u'_1, v'_1) in G'_1, and of the exclusive edge (u'_2, v'_2) in G'_2. We have the following:

Lemma 12. *G_1, G_2 is a positive instance of* SEFE *if and only if G'_1, G'_2 is a positive instance of* SEFE*. Moreover, G'_1, G'_2 is such that the intersection graph $G'_{1\cap 2}$ is a tree with $O(n)$ vertices.*

Proof. We prove that $G'_{1\cap 2}$ is a tree with $O(n)$ vertices. The edges of $E'_{1\cap 2}$ induce a spanning tree of $G_{1\cap 2}$. Moreover, the edges that are introduced in $G'_{1\cap 2}$ for each edge in $E''_{1\cap 2}$ induce two trees spanning the newly introduced vertices and each one attached to exactly one vertex of $E'_{1\cap 2}$. Since the number of vertices and edges introduced for each edge in $E''_{1\cap 2}$ is constant, it follows that the intersection graph $G'_{1\cap 2}$ has $O(n)$ vertices. Next, we prove that G_1, G_2 is a positive instance of SEFE if and only if G'_1, G'_2 is a positive instance of SEFE.

First, suppose that G_1, G_2 is a positive instance of SEFE. Consider any SEFE Γ of G_1, G_2. Construct a SEFE Γ' of G'_1, G'_2 as follows (see Fig. 11). The edges that are common to G'_1, G'_2 and to G_1, G_2 (that is, the edges in $E'_{1\cap 2}$ plus the exclusive edges of G_1 and G_2) have the same drawing in Γ' as they have in Γ. The edges of the gadget of (u, v) (such edges replace the edge (u, v) of $E''_{1\cap 2}$) have a drawing that is arbitrarily close to the one of (u, v). Hence, as edge (u, v) does not intersect any other edge in Γ, its corresponding edges do not intersect any other edge in Γ', which is hence a SEFE of G'_1, G'_2.

Second, suppose that G'_1, G'_2 is a positive instance of SEFE. Consider any SEFE Γ' of G'_1, G'_2. For each edge (u, v) in $E''_{1\cap 2}$ consider the drawing of the gadget of (u, v) in Γ'. The edges of the gadget that are inclusive edges of G'_1, G'_2 do not cross any other edge in Γ'. On the other hand, edge (u'_1, v'_1) (resp. (u'_2, v'_2)) could cross exclusive edges of G'_2 (resp. of G'_1).

We show that Γ' can be modified into a SEFE of G'_1, G'_2 such that (u'_1, v'_1) does not cross any exclusive edge of G'_2, except possibly for (u'_2, v'_2), and such that (u'_2, v'_2) does not cross any exclusive edge of G'_1, except possibly for (u'_1, v'_1). Consider the curve $\gamma_1(u, v)$ composed of the inclusive edges (u', u'_1) and (v', v'_1) and of the exclusive edge (u'_1, v'_1). Also, consider the curve $\gamma_2(u, v)$ composed of the inclusive edges (u', u'_2) and (v', v'_2) and of the exclusive edge (u'_2, v'_2). See Fig. 12a. Such curves subdivide the plane into closed regions $R_1, R_2, \ldots, R_k$. Observe that both u and v belong to the same region, say R_1, as otherwise the path connecting u and v composed of edges in $E'_{1\cap 2}$ would intersect $\gamma_1(u, v)$ or $\gamma_2(u, v)$, by the Jordan curve theorem, thus contradicting the assumption that Γ' is a SEFE of G'_1, G'_2. Also, observe that both u' and v' lie on the border of R_1, as otherwise the path composed of edge (u, u'), of edge (v, v'), and of the path connecting u and v composed of edges in $E'_{1\cap 2}$ would intersect $\gamma_1(u, v)$ or $\gamma_2(u, v)$, again by the Jordan curve theorem, thus contradicting the assumption that Γ' is a SEFE of G'_1, G'_2. Then, denote by $R_{\sum}$ the closed region $\bigcup_{i=2}^{k} R_i$. See Fig. 12b. The border of $R_{\sum}$ consists of a curve B_1 clockwise connecting v' to u' and of curve B_2 clockwise connecting u' to v'. Observe that $R_{\sum}$ contains no vertex w different from u', u'_1, u'_2, v', v'_1, and v'_2, as otherwise the path from w to u that is composed of edges in $E'_{1\cap 2}$ would cross $\gamma_1(u, v)$ or $\gamma_2(u, v)$, again by the Jordan curve theorem, thus contradicting the assumption that Γ' is a SEFE of G'_1, G'_2. Since the vertices of the gadget different from u and v have no incident exclusive edge other than

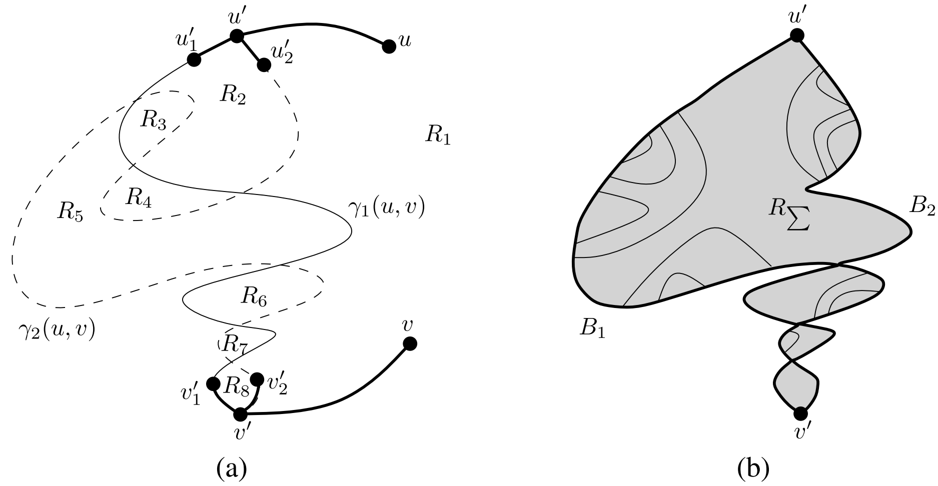

Fig. 12. (a) Curves $\gamma_1(u,v)$ and $\gamma_2(u,v)$ and regions $R_1, R_2, \ldots, R_k$. (b) Region $R_\sum$, curves B_1 and B_2, and the curves in S.

(u'_1, v'_1) and (u'_2, v'_2), it follows that the intersection of $R_\sum$ with the exclusive edges of G'_1 different from (u'_1, v'_1) is a set S of curves whose end-vertices are on the border of $R_\sum$. If a curve s in S has one end-point in B_1 and the other end-point in B_2 then s crosses $\gamma_1(u,v)$, thus contradicting the assumption that Γ' is a Sefe of G'_1, G'_2. Hence, all the curves in S have both end-points on B_1 or both end-points on B_2. If a curve having both end-points on B_1 exists, then there exists a curve s having both end-points on B_1 and such that the closed region delimited by s and B_1 contains no curve in S. Then, s can be replaced by a curve lying in the interior of R_1 arbitrarily close to B_1. The resulting drawing is clearly still a Sefe of G'_1, G'_2 with one less curve into S. Thus, iterating such a modification we eventually get a Sefe of G'_1, G'_2 in which no exclusive edge of G'_1 different from (u'_1, v'_1) intersects B_1. Modifying analogously the curves in S intersecting B_2 and modifying analogously the drawings of the exclusive edges of G'_2 different from (u'_2, v'_2), we eventually get a Sefe of G'_1, G'_2 in which no exclusive edge of G'_1 and G'_2 different from (u'_1, v'_1) and from (u'_2, v'_2) crosses $R_\sum$; hence in such a Sefe no exclusive edge of G'_1 and G'_2 different from (u'_1, v'_1) and from (u'_2, v'_2) crosses (u'_1, v'_1) or (u'_2, v'_2).

After the described modification has been done for all the edges corresponding to an edge (u,v) in $E''_{1\cap 2}$, we obtain a Sefe Γ'' of G'_1, G'_2 and we draw each edge (u,v) as the concatenation of the drawings of edges (u,u'), (u',u'_1), (u'_1,v'_1), (v'_1,v'), (v',v) in Γ''. Observe that no edge of G'_1 nor of G'_2 crosses (u,u'), (u',u'_1), (v'_1,v'), (v',v), since Γ'' is a Sefe. Moreover, no exclusive edge of G'_1 nor of G'_2 different from (u'_2, v'_2) crosses (u'_1, v'_1) by the construction of Γ''. Hence, removing from Γ'' all the vertices of the gadget of (u,v), except for u and v, and removing all the edges of the gadget of (u,v), for each edge (u,v) in $E''_{1\cap 2}$, we get a Sefe Γ of G_1, G_2. □

In the second step, we transform instance G_1, G_2 of Sefe into an equivalent instance G'_1, G'_2 of Sefe such that $G'_{1\cap 2}$ is a tree and all the exclusive edges of G'_1 and of G'_2 are incident only to leaves of $G'_{1\cap 2}$. To this end, we modify every edge $(u,v) \in G_{1\setminus 2}$ such that u is not a leaf of $G_{1\cap 2}$ as follows. We subdivide edge (u,v) with a new vertex u'; we add edge (u,u') to E_2, so that u' is a leaf in the intersection graph of the two modified graphs. Symmetrically, we subdivide every edge $(u,v) \in G_{2\setminus 1}$ such that u is not a leaf of $G_{1\cap 2}$ with a new vertex u' and we add edge (u,u') to E_1, so that u' is a leaf in the intersection graph of the two modified graphs. Note that the exclusive edges of G_1 and G_2 that are incident to two non-leaf vertices are subdivided twice. Denote by G'_1 and by G'_2 the resulting graphs. We have the following:

Lemma 13. *G_1, G_2 is a positive instance of Sefe if and only if G'_1, G'_2 is a positive instance of Sefe. Further, $G'_{1\cap 2}$ is a tree and all the exclusive edges of G'_1 and of G'_2 are incident only to leaves of $G'_{1\cap 2}$. Moreover, $G'_{1\cap 2}$ has $O(n)$ vertices.*

Proof. $G_{1\cap 2}$ is a tree, by assumption. When an exclusive edge (u,v) in G_1 (resp. in G_2) such that u is not a leaf of $G_{1\cap 2}$ is subdivided with a vertex u' and edge (u,u') is added to E_2 (resp. to E_1), an edge is inserted into $G_{1\cap 2}$ connecting an internal vertex of $G_{1\cap 2}$ with a new leaf of $G_{1\cap 2}$, namely u'. Hence, $G_{1\cap 2}$ remains a tree after such a modification and thus $G'_{1\cap 2}$ is a tree. Moreover, each of these modifications decreases the number of incidences between internal nodes of $G_{1\cap 2}$ and exclusive edges by 1. Hence, after all such modifications have been performed, all the exclusive edges are incident only to leaves of $G'_{1\cap 2}$. Each exclusive edge is subdivided at most twice. Since the number of edges of $G_{1\setminus 2}$ and $G_{2\setminus 1}$ is $O(n)$, then $G'_{1\cap 2}$ has $O(n)$ vertices. We now prove that G_1, G_2 is a positive instance of Sefe if and only if G'_1, G'_2 is a positive instance of Sefe.

First, suppose that a Sefe Γ of G_1, G_2 exists. Modify Γ to obtain a Sefe Γ' of G'_1, G'_2 as follows (see Figs. 13a and 13b). When an exclusive edge (u,v) in G_1 (resp. in G_2) such that u is not a leaf of $G_{1\cap 2}$ is subdivided with a vertex u' and edge (u,u') is added to E_2 (resp. to E_1), insert u' in Γ along edge (u,v) arbitrarily close to u. Since the drawing of G_1 in Γ is

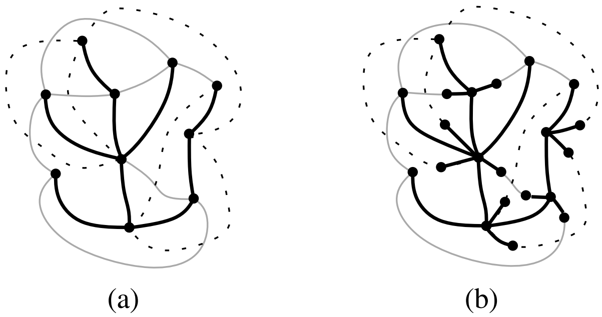

Fig. 13. (a) A SEFE Γ of G_1, G_2. (b) A SEFE Γ' of G'_1, G'_2.

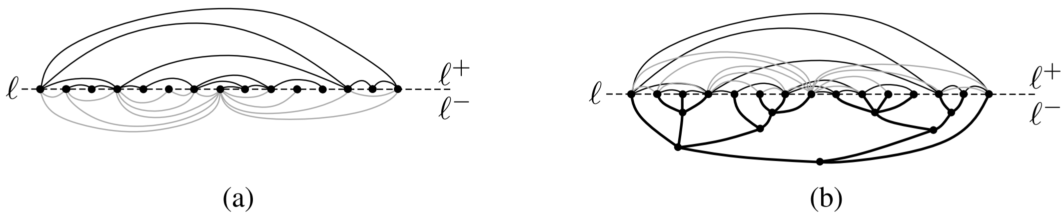

Fig. 14. (a) A PARTITIONED T-COHERENT 2-PAGE BOOK EMBEDDING of $(G, E_{1\setminus 2}, E_{2\setminus 1}, T)$. (b) The SEFE of G_1, G_2 obtained from the book embedding of $(G, E_{1\setminus 2}, E_{2\setminus 1}, T)$.

not modified and since the drawing of G_2 in Γ is modified by inserting an arbitrarily small edge incident to a vertex, the resulting drawing is a SEFE of the current graphs and hence Γ' is a SEFE of G'_1, G'_2.

Second, suppose that a SEFE Γ' of G'_1, G'_2 exists. A SEFE Γ of G_1, G_2 can be obtained by drawing each edge (u, v) of G_1 (resp. of G_2) exactly as in Γ'. Observe that (u, v) is subdivided never, once, or twice in G'_1 (resp. in G'_2); then, its drawing in Γ is composed of the concatenation of the one, two, or three curves representing the parts of (u, v) in Γ', respectively. That no two edges of G_1 (resp. of G_2) intersect in the resulting drawing Γ directly descends from the fact that no two edges of G'_1 (resp. of G'_2) intersect in Γ'. □

In the third step, we transform an instance G_1, G_2 of SEFE such that $G_{1\cap 2}$ is a tree and all the exclusive edges of G_1 and of G_2 are incident only to leaves of $G_{1\cap 2}$ into an equivalent instance of PARTITIONED T-COHERENT 2-PAGE BOOK EMBEDDING.

The input of PARTITIONED T-COHERENT 2-PAGE BOOK EMBEDDING consists of the graph G composed of all the vertices that are leaves of $G_{1\cap 2}$, of all the exclusive edges $E_{1\setminus 2}$ of G_1, and of all the exclusive edges $E_{2\setminus 1}$ of G_2. The partition of the edges of G is $(E_{1\setminus 2}, E_{2\setminus 1})$. Finally, tree T is $G_{1\cap 2}$, rooted at an arbitrary internal node. We have the following:

Lemma 14. *G_1, G_2 is a positive instance of* SEFE *if and only if $(G, E_{1\setminus 2}, E_{2\setminus 1}, T)$ is a positive instance of* PARTITIONED T-COHERENT 2-PAGE BOOK EMBEDDING.

Proof. First, suppose that $(G, E_{1\setminus 2}, E_{2\setminus 1}, T)$ is a positive instance of PARTITIONED T-COHERENT 2-PAGE BOOK EMBEDDING. See Fig. 14. An ordering of the vertices of G along a line ℓ exists such that the edges in $E_{1\setminus 2}$ are drawn on one side ℓ^+ of ℓ, the edges in $E_{2\setminus 1}$ are drawn on the other side ℓ^- of ℓ, no two edges in $E_{1\setminus 2}$ cross, and no two edges in $E_{2\setminus 1}$ cross. Move all the edges in $E_{2\setminus 1}$ to ℓ^+. Since such edges do not cross in ℓ^- and since the ordering of the vertices of G is not modified, the edges in $E_{2\setminus 1}$ still do not cross. Finally, construct a planar drawing of $G_{1\cap 2}$ in ℓ^-. This can always be done since, for each internal vertex t of $G_{1\cap 2}$, the vertices in the subtree of $G_{1\cap 2}$ rooted at t appear consecutively on ℓ. The resulting drawing is hence a SEFE of G_1, G_2.

Second, suppose that G_1, G_2 is a positive instance of SEFE. Consider any SEFE Γ of G_1, G_2 and consider an Euler Tour $\mathcal{E}$ of $G_{1\cap 2}$. See Figs. 15a and 15b. Construct a planar drawing of $\mathcal{E}$ in Γ as follows. Each edge of $\mathcal{E}$ is drawn arbitrarily close to the corresponding edge in $G_{1\cap 2}$. Each end-vertex t of an edge of $\mathcal{E}$ that is a leaf in $G_{1\cap 2}$ is drawn at the same point where it is drawn in Γ. Each end-vertex t of an edge of $\mathcal{E}$ that is not a leaf in $G_{1\cap 2}$ and that has two adjacent edges (t, t_1) and (t, t_2) in $\mathcal{E}$ (observe that $t_1 \neq t_2$ as t is an internal vertex of $G_{1\cap 2}$) is drawn arbitrarily close to the point where t is drawn in Γ, in the region "between" edges (t, t_1) and (t, t_2). Clearly, the resulting drawing of $\mathcal{E}$ is planar. Further, all the leaf vertices of $G_{1\cap 2}$ are drawn at the same point in Γ and in the drawing of $\mathcal{E}$. Moreover, all the exclusive edges of $G_{1\setminus 2}$ and all the exclusive edges of $G_{2\setminus 1}$ lie entirely outside $\mathcal{E}$, except for their end-vertices. Remove all the internal vertices and all the edges of $G_{1\cap 2}$ from the drawing. Move all the edges of $G_{2\setminus 1}$ inside $\mathcal{E}$. Since T is rooted at an arbitrary internal node, there exist two leaves ℓ_1 and ℓ_2 of T that belong to distinct subtrees of the root and are adjacent on $\mathcal{E}$. We cut open $\mathcal{E}$ by removing the edge (ℓ_1, ℓ_2) to obtain a linear ordering of the leaves. The resulting drawing proves that $(G, E_{1\setminus 2}, E_{2\setminus 1}, T)$ is

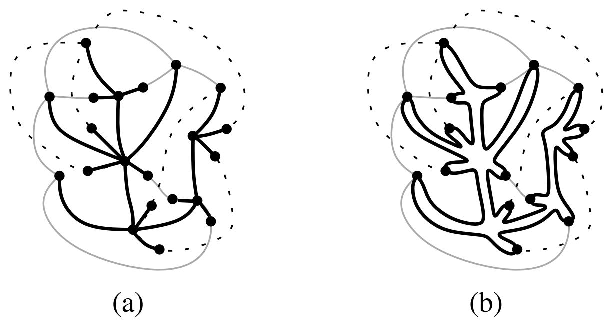

Fig. 15. (a) A SEFE Γ of G_1, G_2. (b) Euler Tour $\mathcal{E}$ of $G_{1\cap 2}$ and exclusive edges.

a positive instance of PARTITIONED T-COHERENT 2-PAGE BOOK EMBEDDING. Namely, all the edges in $E_{1\setminus 2}$ are on one side of $\mathcal{E}$ and all the edges in $E_{2\setminus 1}$ are on the other side of $\mathcal{E}$. No two edges in $E_{1\setminus 2}$ cross as they do not cross in Γ. No two edges in $E_{2\setminus 1}$ cross as they do not cross in Γ. Finally, we observe that all the leaf vertices in a subtree of $G_{1\cap 2}$ rooted at an internal vertex t of $G_{1\cap 2}$ appear consecutively in $\mathcal{E} - (\ell_1, \ell_2)$. In the circular ordering of the leaves induced by $\mathcal{E}$, this follows from the fact that the drawing of $G_{1\cap 2}$ in Γ is planar. The only two vertices that are adjacent in $\mathcal{E}$ but not in $\mathcal{E} - (\ell_1, \ell_2)$ are ℓ_1 and ℓ_2, however, they belong to distinct subtrees with respect to the root and thus there exists no proper subset of the vertices that includes ℓ_1 and ℓ_2 and that corresponds to an internal node of T. □

Given an instance (G, E_1, E_2, T) of PARTITIONED T-COHERENT 2-PAGE BOOK EMBEDDING, it is possible to construct an equivalent instance of SEFE as follows. Let G_1 be the graph whose vertex set is composed of the vertices of G and of the internal vertices of T, and whose edge set is composed of the edges of E_1 and of the edges of T. Analogously, let G_2 be the graph whose vertex set is composed of the vertices of G and of the internal vertices of T, and whose edge set is composed of the edges of E_2 and of the edges of T. Then the following lemma holds; the proof is almost verbatim the proof of Lemma 14.

Lemma 15. *(G, E_1, E_2, T) is a positive instance of* PARTITIONED T-COHERENT 2-PAGE BOOK EMBEDDING *if and only if G_1, G_2 is a positive instance of* SEFE.

Since both reductions can easily be performed in linear time we obtain the following.

Theorem 4. *The following two problems have the same time complexity*:

(1) PARTITIONED T-COHERENT 2-PAGE BOOK EMBEDDING;
(2) SEFE *for two graphs with connected intersection graph.*

The problem PARTITIONED T-COHERENT 2-PAGE BOOK EMBEDDING has been recently studied by Hong and Nagamochi [19] when T is a star. That is, the graph has the edges partitioned into two pages as part of the input, but there is no constraint on the order of the vertices in the required book embedding. In such a case, Hong and Nagamochi proved that the problem is $O(n)$-time solvable [19]. While their motivation was a connection to the c-planarity problem, Lemmata 12–14 together with Hong and Nagamochi's result imply that deciding whether a SEFE exists for two n-vertex graphs whose intersection graph is a star is a linear-time solvable problem.

Theorem 5. *The* SEFE *problem for two n-vertex graphs G_1 and G_2 is solvable in $O(n)$ time if the intersection graph $G_{1\cap 2}$ of G_1 and G_2 is a star.*

5. Conclusions

In this paper we have shown new results on the time complexity of the problem of deciding whether two planar graphs admit a SEFE.

First, we have shown that the SEFE problem can be solved in polynomial time if the intersection graph $G_{1\cap 2}$ of the input graphs G_1 and G_2 is biconnected. Using dynamic programming, we further refined our algorithm and gave a linear-time implementation.

Haeupler et al. [17] consider a slightly more general case of SEFE with biconnected intersection, where they allow two input graphs G_1 and G_2 with different vertex sets, i.e., they also allow exclusive vertices. Our approach also generalizes to the yet more general case where the intersection graph consists of a set of biconnected vertices plus a set of isolated vertices. The difference between these two cases is that the isolated vertices may be adjacent to both exclusive edges of G_1 and of G_2. It is clear that the combinatorial characterizations, and thus also the polynomial-time algorithm from Section 3.1

carry over to this setting. Is it also possible to adapt the linear-time algorithm to this case? Also, the following generalization of the SEFE problem with $G_{1\cap 2}$ biconnected seems worth tackling: What is the time complexity of computing a SEFE when $G_{1\cap 2}$ is *edge-biconnected*?

Second, we have shown that when $G_{1\cap 2}$ is connected the SEFE problem can be equivalently stated as a 2-page book embedding problem with edges assigned to the pages and with hierarchical constraints. Hence, pursuing an $\mathcal{NP}$-hardness proof for such a book embedding problem is a possible direction for trying to prove the $\mathcal{NP}$-hardness for the SEFE problem.

Acknowledgements

We thank Seok-Hee Hong for presenting us the results in [19]. Moreover, we thank the anonymous referees of the journal version of this paper for their helpful suggestions to improve the presentation of this paper.

References

[1] P. Angelini, G. Di Battista, F. Frati, V. Jelínek, J. Kratochvíl, M. Patrignani, I. Rutter, Testing planarity of partially embedded graphs, in: Symposium on Discrete Algorithms (SODA'10), 2010, pp. 202–221.

[2] P. Angelini, M. Geyer, M. Kaufmann, D. Neuwirth, On a tree and a path with no geometric simultaneous embedding, in: U. Brandes, S. Cornelsen (Eds.), Graph Drawing (GD'10), in: LNCS, vol. 6502, 2011, pp. 38–49.

[3] B. Aspvall, M.F. Plass, R.E. Tarjan, A linear-time algorithm for testing the truth of certain quantified boolean formulas, Information Processing Letters 8 (3) (1979) 121–123.

[4] P. Braß, E. Cenek, C.A. Duncan, A. Efrat, C. Erten, D. Ismailescu, S.G. Kobourov, A. Lubiw, J.S.B. Mitchell, On simultaneous planar graph embeddings, Computational Geometry 36 (2) (2007) 117–130.

[5] G. Di Battista, R. Tamassia, On-line maintenance of triconnected components with SPQR-trees, Algorithmica 15 (4) (1996) 302–318.

[6] G. Di Battista, R. Tamassia, On-line planarity testing, SIAM Journal on Computing 25 (5) (1996) 956–997.

[7] E. Di Giacomo, G. Liotta, Simultaneous embedding of outerplanar graphs, paths, and cycles, International Journal of Computational Geometry and Applications 17 (2) (2007) 139–160.

[8] C. Erten, S.G. Kobourov, Simultaneous embedding of planar graphs with few bends, Journal of Graph Algorithms and Applications 9 (3) (2005) 347–364.

[9] A. Estrella-Balderrama, E. Gassner, M. Jünger, M. Percan, M. Schaefer, M. Schulz, Simultaneous geometric graph embeddings, in: S.H. Hong, T. Nishizeki, W. Quan (Eds.), Graph Drawing (GD'07), in: LNCS, vol. 4875, 2007, pp. 280–290.

[10] J.J. Fowler, C. Gutwenger, M. Jünger, P. Mutzel, M. Schulz, An SPQR-tree approach to decide special cases of simultaneous embedding with fixed edges, in: I.G. Tollis, M. Patrignani (Eds.), Graph Drawing (GD'08), in: LNCS, vol. 5417, 2008, pp. 157–168.

[11] J.J. Fowler, M. Jünger, S.G. Kobourov, M. Schulz, Characterizations of restricted pairs of planar graphs allowing simultaneous embedding with fixed edges, in: H. Broersma, T. Erlebach, T. Friedetzky, D. Paulusma (Eds.), Graph-Theoretic Concepts in Computer Science (WG'08), in: LNCS, vol. 5344, 2008, pp. 146–158.

[12] F. Frati, Embedding graphs simultaneously with fixed edges, in: M. Kaufmann, D. Wagner (Eds.), Graph Drawing (GD'06), in: LNCS, vol. 4372, 2006, pp. 108–113.

[13] H.N. Gabow, R.E. Tarjan, A linear-time algorithm for a special case of disjoint set union, Journal of Computer and System Sciences 30 (1985) 209–221.

[14] E. Gassner, M. Jünger, M. Percan, M. Schaefer, M. Schulz, Simultaneous graph embeddings with fixed edges, in: F.V. Fomin (Ed.), Graph-Theoretic Concepts in Computer Science (WG'06), in: LNCS, vol. 4271, 2006, pp. 325–335.

[15] M. Geyer, M. Kaufmann, I. Vrt'o, Two trees which are self-intersecting when drawn simultaneously, Discrete Mathematics 307 (4) (2009) 1909–1916.

[16] C. Gutwenger, P. Mutzel, A linear time implementation of SPQR-trees, in: J. Marks (Ed.), Graph Drawing (GD'00), in: LNCS, vol. 1984, 2001, pp. 77–90.

[17] B. Haeupler, K.R. Jampani, A. Lubiw, Testing simultaneous planarity when the common graph is 2-connected, in: O. Cheong, K.-Y. Chwa, K. Park (Eds.), Proceedings of the 21st Symposium on Algorithms and Computation (ISAAC'10), in: LNCS, vol. 6507, Springer, Heidelberg/Berlin, 2010, pp. 410–421.

[18] D. Harel, R.E. Tarjan, Fast algorithms for finding nearest common ancestors, SIAM Journal on Computing 13 (2) (1984) 338–355.

[19] S.H. Hong, H. Nagamochi, Two-page book embedding and clustered graph planarity, Tech. Report 2009-004, Department of Applied Mathematics & Physics, Kyoto University, 2009.

[20] M. Jünger, M. Schulz, Intersection graphs in simultaneous embedding with fixed edges, Journal of Graph Algorithms and Applications 13 (2) (2009) 205–218.