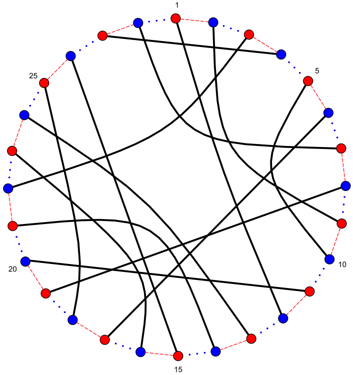
Book Embedding with Fixed Page Assignments



Daniel Hoske, 23rd October 2012

Department of Informatics, Institute of Theoretical Computer Science

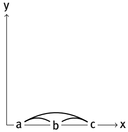
Advisors: Dipl.-Inform. Thomas Bläsius, Dr. Ignaz Rutter

Page embedding

Definition

Page embedding is planar embedding with

- vertices on a line and
- edges in half-plane above the line



Page embeddable = outerplanar

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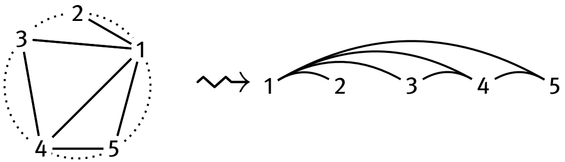
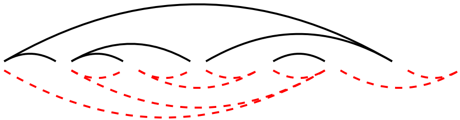


Book embedding

Definition

Book embedding of $G_i = (V, E_i), i \in \{1, \ldots, k\}$ consists of page embeddings for G_i with the same vertex positions.



Book embedding

Definition

Book embedding of $G_i = (V, E_i), i \in \{1, \ldots, k\}$ consists of page embeddings for G_i with the same vertex positions.

Problem: BOOK-EMBEDDING

Given: Vertex set V and edge sets $E_1, \ldots, E_k \subseteq \binom{V}{2}$.
Question: Is there a book embedding of (V, E_i)?

$k = 1$: embeddable = outerplanar

$k = 2$: decidable in $\mathcal{O}(n)$ [Hong and Nagamochi, 2009]

What happens for $k = 3$?

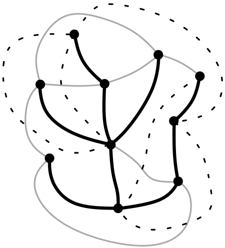
Motivation

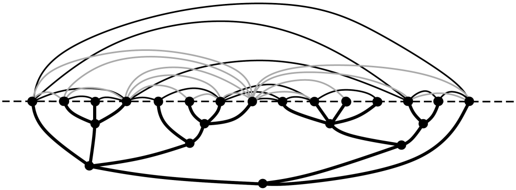
Problem: CONNECTED-SEFE

Given: Two graphs G_1 and G_2 on V where $G_1 \cap G_2$ is connected.

Question: Are there planar embeddings of G_1 and G_2 that coincide on $G_1 \cap G_2$?

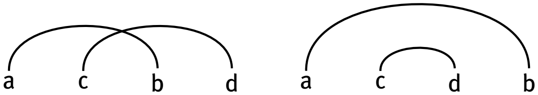
is equivalent to 2-page book embedding + a tree [Angelini et al., 2012]





Observations

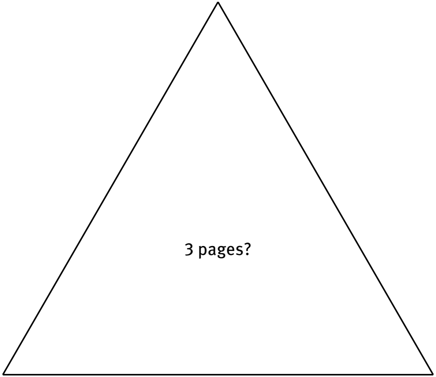
BOOK-EMBEDDING is an ordering problem:
Avoid suborder $a < c < b < d$ for $\{a, b\}, \{c, d\} \in E_i$



Mirror image and cyclic shifts of a valid order remain valid:

Results

2 pages: $\mathcal{O}(n)$



Contents

Gliederung

BOOK-EMBEDDING is NP-complete

Theorem

BOOK-EMBEDDING with matchings as pages is NP-complete.

Reduce from NP-complete problem BETWEENNESS [Opatrny, 1979].

Problem: BETWEENNESS

Given: Finite set $M := \{1, \dots, n\}$ and ordered triples $C \subseteq M^3$.
Question: Is there a total ordering $<$ of M such that $a < b < c$ or $a > b > c$ for all $(a, b, c) \in C$?

Reduction from BETWEENNESS

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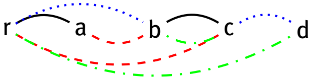
Map triple $(a, b, c) \in C$ to two new pages (r fixed new vertex):

For example: (a, b, c), (b, c, d)



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Map triple $(a, b, c) \in C$ to two new pages (r fixed new vertex):

BETWEENNESS $\Rightarrow$ BOOK-EMBEDDING:

Take r as first vertex

$\rightsquigarrow$ Orders $r < a < b < c$ or $r < c < b < a$ are valid

Reduction from BETWEENNESS

Problem: BETWEENNESS

Given: Finite set $M := \{1, \ldots, n\}$ and ordered triples $C \subseteq M^3$.
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Map triple $(a, b, c) \in C$ to two new pages (r fixed new vertex):

BOOK-EMBEDDING $\Rightarrow$ BETWEENNESS:

Rotate r to front

$\rightsquigarrow r < a < b < c$ or $r < c < b < a$ are the only valid orders

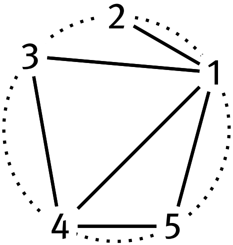
Connected pages

Theorem

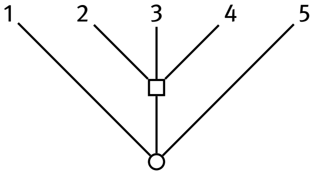
BOOK-EMBEDDING with connected pages can be solved in $\mathcal{O}(kn)$ time.

Idea:

Compute valid orders $\pi_i \subseteq \mathrm{Sym}(n)$ for single pages and intersect them.



Interludium: PQ-trees

$T =$ 

$$\pi(T) = \{12345, 14325, 52341, 54321, 15234, 15432,$$
$$51234, 51432, 23415, 43215, 23451, 43251\}$$

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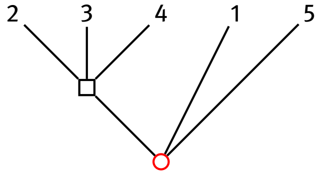
$T =$ 

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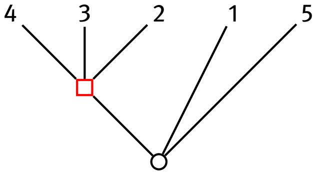
$T =$ 

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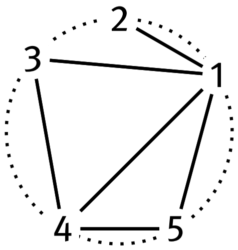

Connected pages

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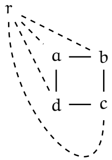
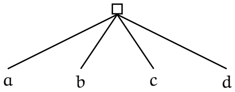
BOOK-EMBEDDING with connected pages can be solved in $\mathcal{O}(kn)$ time.

- Construct PQ-trees T_i on V representing valid orders of individual pages (V, E_i)





[Booth and Lueker et. al. 1976, Shih and Hsu 1993, Boyer and Myrvold 1999]

Connected pages

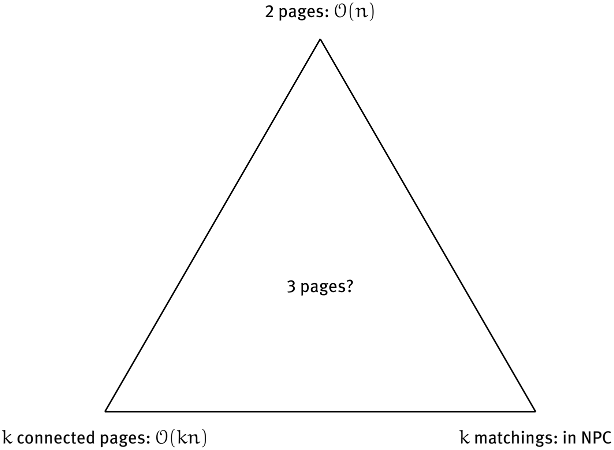
Theorem

BOOK-EMBEDDING with connected pages can be solved in $\mathcal{O}(kn)$ time.

- Construct PQ-trees T_i on V representing valid orders of individual pages (V, E_i)
- Intersect the T_i [Booth, 1975]

$\rightarrow$ Resulting PQ-tree T represents valid book orders

Results



2 pages: $\mathcal{O}(n)$
3 pages?
k connected pages: $\mathcal{O}(kn)$
k matchings: in NPC

Gliederung

Disjoint perfect matchings as pages

Problem: PERFECT-MATCHINGS-BOOK-EMBEDDING

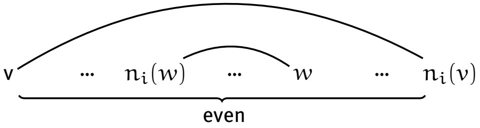
Given: Disjoint perfect matchings $E_1, \ldots, E_k$ on a vertex set V.

Question: Is there a book embedding of (V, E_i)?

Theorem

Necessary: $G := (V, E_1 \cup \cdots \cup E_k)$ *is bipartite.*

- Even number of vertices between adjacent vertices in valid order



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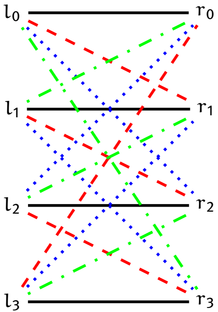
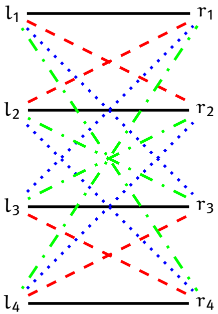
$\Rightarrow$ Vertices with even and odd indexes form bipartition

Bipartite examples



k pages: Take the partition $E_i := \left\{\{l_j, r_{(j+i) \bmod k}\} : j \in \{1, \ldots, k-1\}\right\}$ of $K_{k,k}$

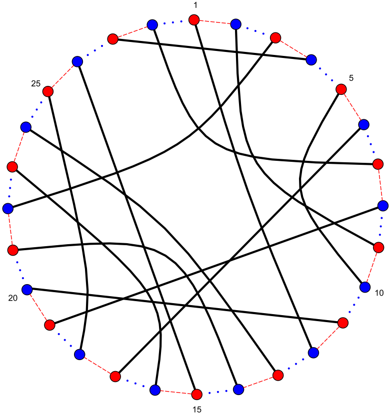
Bipartite counterexamples



$k \geqslant 4$ pages: Partition $K_{k,k}$ into disjoint perfect matchings that contain this counterexample

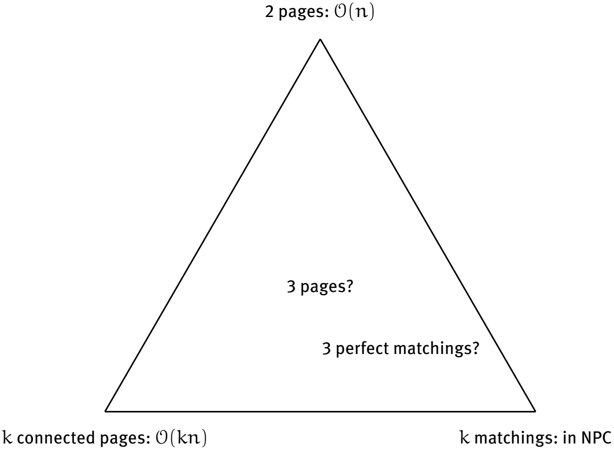
Bipartite counterexample for three pages

Smallest counterexample where two of the matchings form a cycle



Smallest unrestricted counterexample: $20 \leqslant n \leqslant 28$

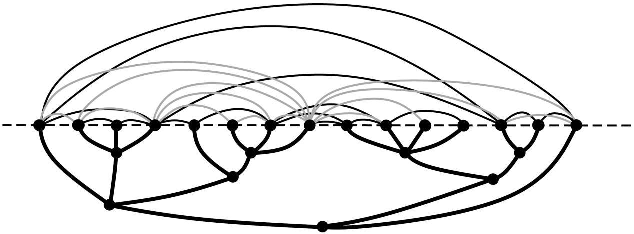
Results


2 pages: $\mathcal{O}(n)$
3 pages?
3 perfect matchings?
k connected pages: $\mathcal{O}(kn)$
k matchings: in NPC

Gliederung

Tree on the vertices

BOOK-EMBEDDING + tree:



Drawing the tree = Restricting permutations by a P-tree

Tree on the vertices

BOOK-EMBEDDING + tree:

Problem: P-TREE-BOOK-EMBEDDING

Given: BOOK-EMBEDDING instance I and a P-tree T with leaves V.

Question: Is there a total order $< \in \pi(T)$ solving I?

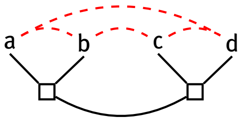
More structure (same for binary trees):

Problem: Q-TREE-BOOK-EMBEDDING

Given: BOOK-EMBEDDING instance I and a Q-tree T with leaves V.

Question: Is there a total order $< \in \pi(T)$ solving I?

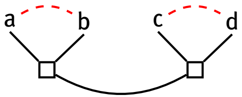
Example



- What happens to the forbidden suborder constraint?

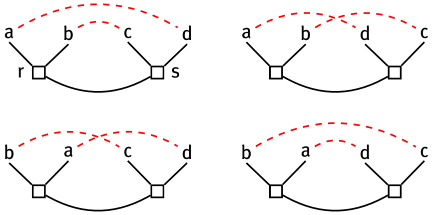
$\Rightarrow$ 2-CNF formula (Boolean equations) on orientation of Q-nodes

Example



- $\{a, b\}, \{c, d\}$: true

Example



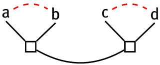
- $\{a, d\}, \{b, c\}$: $a < b \Leftrightarrow c < d$
- Fix reference orientation of inner nodes r
- Boolean variable o_r for being in reference orientation

$\Rightarrow$ $o_r \Leftrightarrow o_s$

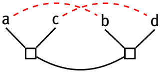
Book constraints to 2-CNF formula

Theorem

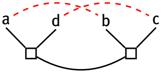
Q-TREE-BOOK-EMBEDDING is solvable in $\mathcal{O}(kn^2)$ *time.*



true



$o_{r(a,c)} \Leftrightarrow o_{r(b,d)}$ or $o_{r(a,c)} \Leftrightarrow \neg o_{r(b,d)}$



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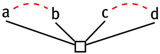
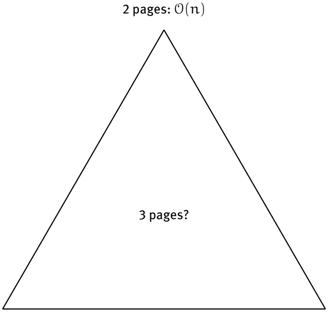


true or false

Conclusion and outlook



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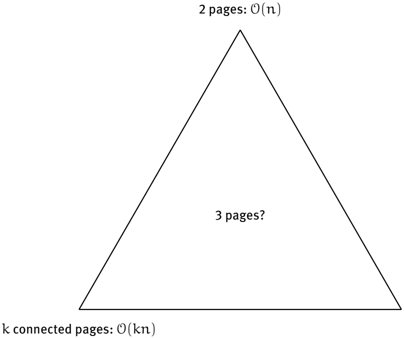

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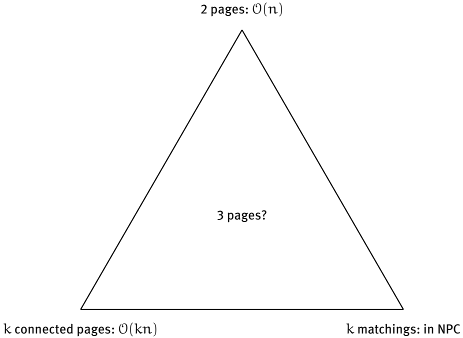

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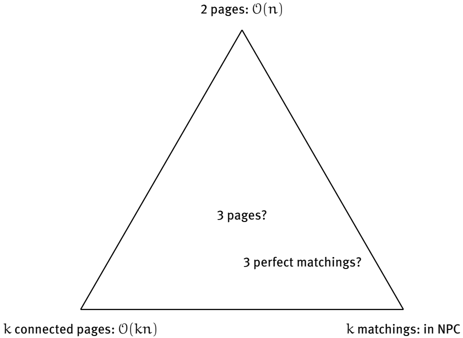

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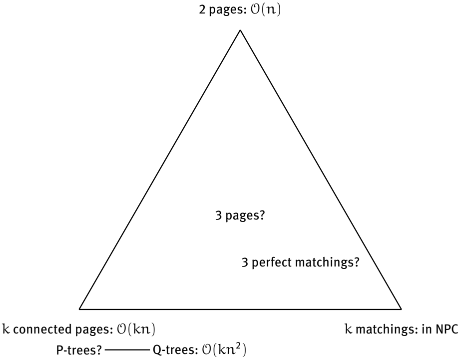

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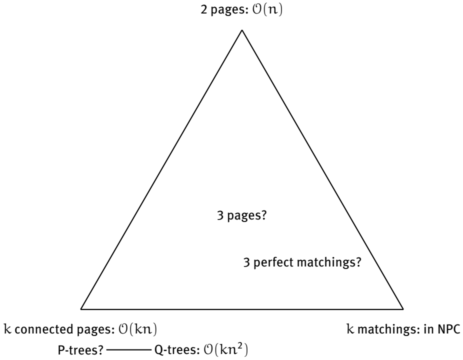

- Complexity for constant number of pages?
- Complexity when constrained by a P-tree?

Addendum: Multiple spines