CHAPTER 13

The Martingale Measure: Part II

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13.1 Introduction

We continue our study of risk-neutral valuation, by extending the settings to one with many time periods. This will be accomplished in two very different ways. First, we revisit the concept in the CCAPM setting. Recall that this is a discrete time, general equilibrium framework: preferences and endowment processes must be specified, and no-trade prices computed. We will demonstrate that, here as well, assets may be priced equal to the present value, discounted at the risk-free rate of interest, of their expected payoffs when expectations are computed using the set of risk-neutral probabilities. We would expect this to be possible. The CCAPM is an equilibrium model (hence there are no-arbitrage opportunities and thus a set of risk-neutral probabilities must exist) with complete markets (hence this set is unique).

Second, we extend the idea to the partial equilibrium setting of equity derivatives (e.g., equity options) valuation. The key to derivatives pricing is to have an accurate model of the underlying price process. We hypothesize such a process (it is not derived from underlying fundamentals—preferences, endowments, etc.; rather, it is a pure statistical model) and demonstrates that, in the presence of *local* market completeness and *local* no-arbitrage situations, there exists a transformation of measure by which all derivatives

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written on that asset may be priced equal to the present value, discounted at the risk-free rate, of their expected payoffs computed using this transformed measure.[1] The Black–Scholes formula, for example, may be derived in this way.

13.2 Discrete Time Infinite Horizon Economies: A CCAPM Setting

As in the previous chapter, time evolves according to $t = 0, 1, \ldots, T, T + 1, \ldots$. We retain the context of a single good endowment economy and presume the existence of a complete markets Arrow–Debreu financial structure. In period t, any one of N_t possible states, indexed by θ_t, may be realized.

We will assume that a period t event is characterized by two quantities:

i. The actually occurring period t event as characterized by θ_t.
ii. The unique history of events $(\theta_1, \theta_2, \ldots, \theta_{t-1})$ that precedes it.

Requirement (ii), in particular, suggests an evolution of uncertainty similar to that of a tree structure in which the branches never join (two events always have distinct prior histories). Although this is a stronger assumption than what underlies the CCAPM, it will allow us to avoid certain notational ambiguities; subsequently, assumption (ii) will be dropped. We are interested more in the idea than in any broad application, so generality is not an important consideration.

Let $\pi(\theta_t, \theta_{t+1})$ represent the probability of state θ_{t+1} being realized in period $t + 1$, given that θ_t is realized in period t. The financial market is assumed to be complete in the following sense: at every date t, and for every state θ_t, there exists a short-term contingent claim that pays one unit of consumption if state θ_{t+1} is realized in period $t + 1$ (and nothing otherwise). We denote the period t, state θ_t price of such a claim by $q(\theta_t, \theta_{t+1})$.

Arrow–Debreu long-term claims (relative to $t = 0$) are not formally traded in this economy. Nevertheless, they can be synthetically created by dynamically trading short-term claims. (In general, more frequent trading can substitute for fewer claims.) To illustrate, let $q(\theta_0, \theta_{t+1})$ represent the period $t = 0$ price of a claim to one unit of the numeraire, if and only if, event θ_{t+1} is realized in period $t + 1$. It must be the case that

$$q(\theta_0, \theta_{t+1}) = \prod_{s=0}^{t} q(\theta_s, \theta_{s+1}), \tag{13.1}$$

where $(\theta_0, \ldots, \theta_t)$ is the unique prior history of θ_{t+1}. By the uniqueness of the path to θ_{t+1}, $q(\theta_0, \theta_{t+1})$ is well defined. By no-arbitrage arguments, if the long-term Arrow–Debreu

[1] By *local* we mean that valuation is considered only in the context of the derivative, the underlying asset (a stock), and a risk-free bond.

security were also traded, its price would conform to Eq. (13.1). Arrow–Debreu securities can thus be effectively created via dynamic (recursive) trading, and the resulting financial market structure is said to be dynamically complete.[2] By analogy, the price in period t, state θ_t, of a security that pays one unit of consumption if state θ_{t+J} is observed in period $t + J$, $q(\theta_t, \theta_{t+J})$, is given by

$$q(\theta_t, \theta_{t+J}) = \prod_{s=t}^{t+J-1} q(\theta_s, \theta_{s+1}).$$

It is understood that θ_{t+J} is feasible from θ_t; that is, given that we are in state θ_t in period t, there is some positive probability for the economy to find itself in state θ_{t+J} in period $t + J$. Otherwise the claims price must be zero.

Since our current objective is to develop risk-neutral pricing representations, a natural next step is to define risk-free bond prices and associated risk-free rates. Given that the current date state is (θ_t,t), the price, $q^b(\theta_t,t)$, of a risk-free one-period (short-term) bond is given by (no arbitrage)

$$q^b(\theta_t, t) = \sum_{\theta_{t+1}=1}^{N_{t+1}} q(\theta_t, \theta_{t+1}); \tag{13.2}$$

Note here that the summation sign applies across all N_{t+1} future states of nature. The corresponding risk-free rate must satisfy

$$(1 + r_f(\theta_t)) = \{q^b(\theta_t, t)\}^{-1}$$

Pricing a k-period risk-free bond is similar:

$$q^b(\theta_t, t + k) = \sum_{\theta_{t+k}=1}^{N_{t+k}} q(\theta_t, \theta_{t+k}). \tag{13.3}$$

The final notion is that of an *accumulation factor*, denoted by $g(\theta_t,\theta_{t+k})$, and defined for a specific path $(\theta_t, \theta_{t+1},\ldots, \theta_{t+k})$ as follows:

$$g(\theta_t, \theta_{t+k}) = \prod_{s=t}^{t+k-1} q^b(\theta_s, s + 1). \tag{13.4}$$

[2] This fact suggests that financial markets may need to be "very incomplete" if incompleteness *per se* is to have a substantial effect on equilibrium asset prices and, for example, have a chance to resolve some of the puzzles uncovered in Chapter 10. See Telmer (1993).

The idea being captured by the accumulation factor is this: An investor who invests one unit of consumption in short-term risk-free bonds from date t to $t + k$, continually rolling over his investment, will accumulate $[g(\theta_t, \theta_{t+k})]^{-1}$ units of consumption by date $t + k$, if events $\theta_{t+1},\ldots, \theta_{t+k}$ are realized. Alternatively,

$$[g(\theta_t, \theta_{t+k})]^{-1} = \prod_{s=0}^{k-1}(1 + r_f(\theta_{t+s})). \tag{13.5}$$

Note that from the perspective of date t, state θ_t, the factor $[g(\theta_t, \theta_{t+k})]^{-1}$ is an uncertain quantity as the actual state realizations in the succeeding time periods are not known at period t. From the $t = 0$ perspective, $[g(\theta_t, \theta_{t+k})]^{-1}$ is in the spirit of a (conditional) forward rate.

Let us illustrate with the two-date forward accumulation factor. We take the perspective of the investor investing one unit of the numeraire in a short-term risk-free bond from date t to $t + 2$. His first investment is certain since the current state θ_t is known and it returns $(1 + r_f(\theta_t))$. At date $t + 1$, this sum will be invested again in a one-period risk-free bond with return $(1 + r_f(\theta_{t+1}))$ contracted at $t + 1$ and received at $t + 2$. From the perspective of date t, this is indeed an uncertain quantity. The compounded return on the investment is: $(1 + r_f(\theta_t))(1 + r_f(\theta_{t+1}))$. This is the inverse of the accumulation factor $g(\theta_t, \theta_{t+1})$ as spelled out in Eq. (13.5).

Let us next translate these ideas directly into the CCAPM settings.

13.3 Risk-Neutral Pricing in the CCAPM

We make two additional assumptions in order to restrict our current setting to the context of the CCAPM.

A13.1 There is one agent in the economy with time-separable VNM preferences represented by

$$\mathrm{U}(\tilde{c}) = E_0\left(\sum_{t=0}^{\infty} U(\tilde{c}_t, t)\right),$$

where $U(\tilde{c}_t, t)$ is a family of, strictly increasing, concave, differentiable period utility functions, with $U_1(c_t,t) > 0$ for all t, $\tilde{c}_t = c(\theta_t)$ is the uncertain period t consumption, and E_0 the expectations operator conditional on date $t = 0$ information.

This treatment of the agent's preferences is quite general. For example, $U(c_t,t)$ could be of the form $\delta^t U(c_t)$ as in earlier chapters. Alternatively, the period utility function could itself be changing through time in deterministic fashion, or some type of habit formation could be postulated. In all cases, it is understood that the set of feasible consumption sequences will be such that the sum exists (is finite).

A13.2 Output in this economy, $\tilde{Y}_t = Y_t(\theta_t)$ is exogenously given and, by construction, represents the consumer's income. In equilibrium it represents his consumption as well.

Recall that equilibrium-contingent claims prices in the CCAPM economy are no-trade prices, supporting the consumption sequences $\{\tilde{c}_t\}$ in the sense that at these prices, the representative agent does not want to purchase any claims; that is, at the prevailing contingent-claims prices, his existing consumption sequence is optimal. The loss in period t utility experienced by purchasing a contingent claim $q(\theta_t, \theta_{t+1})$ is exactly equal to the resultant increase in expected utility in period $t+1$. There is no benefit to further trade. More formally,

$$U_1(c(\theta_t), t)q(\theta_t, \theta_{t+1}) = \pi(\theta_t, \theta_{t+1})U_1(c(\theta_{t+1}), t+1), \text{ or}$$
$$q(\theta_t, \theta_{t+1}) = \pi(\theta_t, \theta_{t+1})\left\{\frac{U_1(c(\theta_{t+1}), t+1)}{U_1(c(\theta_t), t)}\right\}. \tag{13.6}$$

Equation (13.7) corresponds to the state claim pricing in Section 10.4. State probabilities and intertemporal rates of substitution appear once again as the determinants of equilibrium Arrow–Debreu prices. Note that the more general utility specification adopted in this chapter does not permit bringing out explicitly the element of time discounting embedded in the intertemporal marginal rates of substitution. A short-term risk-free bond is thus priced according to

$$q^b(\theta_t, t+1) = \sum_{\theta_{t+1}=1}^{N_{t+1}} q(\theta_t, \theta_{t+1}) = \frac{1}{U_1(c(\theta_t), t)} E_t\{U_1(c(\theta_{t+1}), t+1)\}. \tag{13.7}$$

Risk-neutral valuation is in the spirit of discounting at the risk-free rate. Accordingly, we may ask: at what probabilities must we compute the expected payoff to a security in order to obtain its price by discounting that payoff at the risk-free rate? But which risk-free rates are we speaking about? In a multiperiod context, there are two possibilities, and the alternative we choose will govern the precise form of the probabilities themselves.

The spirit of the dilemma is portrayed in Figure 13.1, which illustrates the case of a $t = 3$ period cash flow.

Under the first alternative, the cash flow is discounted at a series of consecutive short (one-period) rates, while in the second we discount back at the term structure of

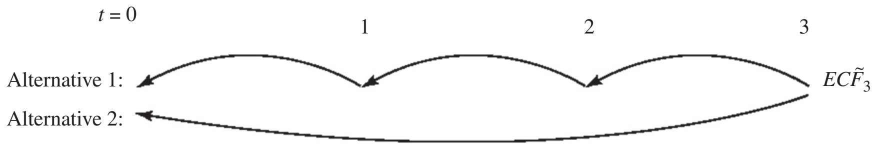

Figure 13.1
Two possibilities of discounting a $t=3$ period cash flow.

multiperiod discount bonds. These methods provide the same price, although the form of the risk-neutral probabilities will differ substantially. Here we offer a discussion of alternative 1; alternative 2 is considered in Appendix 13.1.

Since the one-period state claims are the simplest securities, we will first ask what the risk-neutral probabilities must be in order that they be priced equal to the present value of their expected payoff, discounted at the risk-free rate.[3] As before, let these numbers be denoted by $\pi^{RN}(\theta_t, \theta_{t+1})$. They are defined by:

$$q(\theta_t, \theta_{t+1}) = \pi(\theta_t, \theta_{t+1}) \left\{ \frac{U_1(c(\theta_{t+1}), t+1)}{U_1(c(\theta_t), t)} \right\} = q^b(\theta_t, t+1)\left[\pi^{RN}(\theta_t, \theta_{t+1})\right].$$

The second equality reiterates the tight relationship found in Chapter 12 between Arrow–Debreu prices and risk-neutral probabilities. Substituting Eq. (13.7) for $q^b(\theta_t, t+1)$ and rearranging terms, one obtains:

$$\begin{aligned} \pi^{RN}(\theta_t, \theta_{t+1}) &= \pi(\theta_t, \theta_{t+1}) \left\{ \frac{U_1(c(\theta_{t+1}), t+1)}{U_1(c(\theta_t), t)} \right\} \frac{U_1(c(\theta_t), t)}{E_t\{U_1(c(\theta_{t+1}), t+1)\}} \\ &= \pi(\theta_t, \theta_{t+1}) \left\{ \frac{U_1(c(\theta_{t+1}), t+1)}{E_t U_1(c(\theta_{t+1}), t+1)} \right\}. \end{aligned} \tag{13.8}$$

Since $U(c(\theta_t), t)$ is assumed to be strictly increasing, $U_1 > 0$ and $\pi^{RN}(\theta_t, \theta_{t+1}) > 0$ (without loss of generality we may assume $\pi(\theta_t, \theta_{t+1}) > 0$). Furthermore, by construction, $\sum_{\theta_{t+1}=1}^{N_{t+1}} \pi^{RN}(\theta_t, \theta_{t+1}) = 1$. The set $\{\pi^{RN}(\theta_t, \theta_{t+1})\}$ thus defines a set of conditional (on θ_t) risk-neutral transition probabilities. As in our earlier more general setting, if the representative agent is risk neutral, $U_1(c(\theta_t), t) \equiv$ constant for all t, and $\pi^{RN}(\theta_t, \theta_{t+1})$

[3] Recall that since all securities can be expressed as portfolios of state claims, we can use the state claims alone to construct the risk-neutral probabilities.

coincides with $\pi(\theta_t, \theta_{t+1})$, the true probability. Using these transition probabilities, makes it possible to discount expected future consumption flows at the intervening risk-free rates. Notice how the risk-neutral probabilities are related to the true probabilities: They represent the true probabilities scaled up or down by the relative consumption scarcities in the different states. For example, if, for some state θ_{t+1}, the representative agent's consumption is usually low, his marginal utility of consumption in that state will be much higher than average marginal utility and thus

$$\pi^{RN}(\theta_t, \theta_{t+1}) = \pi(\theta_t, \theta_{t+1}) \left\{ \frac{U_1(c(\theta_{t+1}), t+1)}{E_t U_1(c(\theta_{t+1}), t+1)} \right\} > \pi(\theta_t, \theta_{t+1}).$$

The opposite will be true if a state has a relative abundance of consumption. When we compute expected payoffs to assets using risk-neutral probabilities, we are thus implicitly taking into account both the (no-trade) relative equilibrium scarcities (prices) of their payoffs and their objective relative scarcities. This allows discounting at the risk-free rate: No further risk adjustment need be made to the discount rate as all such adjustments have been implicitly undertaken in the expected payoff calculation.

To gain a better understanding of this notion, let us go through a few examples.

Example 13.1 Denote a stock's associated dividend stream by $\{d(\theta_t)\}$. Under the basic state-claim valuation perspective of Section 11.2, its ex-dividend price at date t, given that θ_t has been realized, is:

$$q^e(\theta_t, t) = \sum_{s=t+1}^{\infty} \sum_{j=1}^{N_s} q(\theta_t, \theta_s(j)) d(\theta_s(j)), \tag{13.9}$$

or, with a recursive representation,

$$q^e(\theta_t, t) = \sum_{\theta_{t+1}} q(\theta_t, \theta_{t+1}) \{ q^e(\theta_{t+1}, t+1) + d(\theta_{t+1}) \} \tag{13.10}$$

Equation (13.10) may also be expressed as

$$q^e(\theta_t, t) = q^b(\theta_t, t+1) E_t^{RN} \{ q^e(\tilde{\theta}_{t+1}, t+1) + d(\tilde{\theta}_{t+1}) \}, \tag{13.11}$$

where E_t^{RN} denotes the expectation taken with respect to the relevant risk-neutral transition probabilities; equivalently,

$$q^e(\theta_t, t) = \frac{1}{1 + r_f(\theta_t)} E_t^{RN} \{ q^e(\tilde{\theta}_{t+1}, t+1) + d(\tilde{\theta}_{t+1}) \}.$$

Returning again to the present value expression, Eq. (13.9), we have

$$\begin{aligned} q^e(\theta_t, t) &= \sum_{s=t+1}^{\infty} E_t^{RN}\{g(\theta_t, \tilde{\theta}_s)d(\tilde{\theta}_s)\} \\ &= \sum_{s=t+1}^{\infty} E_t^{RN} \frac{d(\tilde{\theta}_s)}{\prod_{j=0}^{s-1}(1 + r_f(\tilde{\theta}_{t+j}))}. \end{aligned} \tag{13.12}$$

What does Eq. (13.12) mean? Any state $\hat{\theta}_s$ in period $s \geq t+1$ has a unique sequence of states preceding it. The product of the risk-neutral transition probabilities associated with the states along the path defines the (conditional) risk-neutral probability of $\hat{\theta}_s$ itself. The product of this probability and the payment as $d(\hat{\theta}_s)$ is then discounted at the associated accumulation factor—the present value factor corresponding to the risk-free rates identified with the succession of states preceding $\hat{\theta}_s$. For each $s \geq t+1$, the expectation represents the sum of all these terms, one for each θ_s feasible from θ_t.

Since the notational intensity tends to obscure what is basically a very straightforward idea, let us turn to a small numerical example.

Example 13.2 Let us value a two-period equity security, where $U(c_t, t) \equiv U(c_t) = \ln c_t$ for the representative agent (no discounting). The evolution of uncertainty is given by Figure 13.2 where

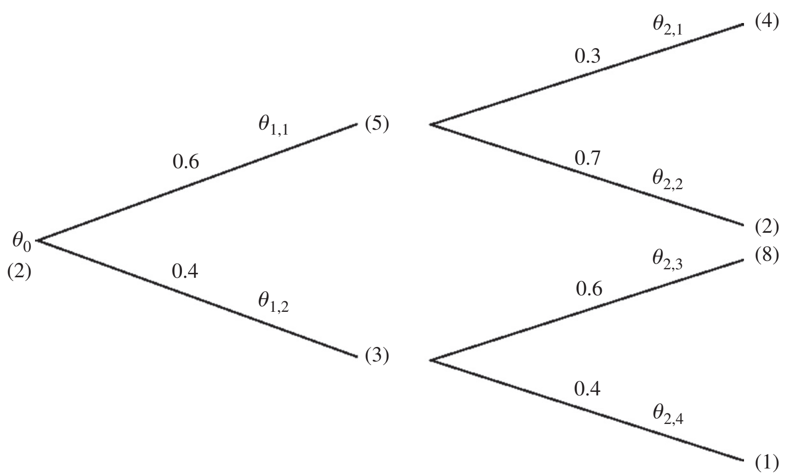

Figure 13.2
The structure of security payoffs—two periods—two states at each node.

$$\pi(\theta_0, \theta_{1,1}) = 0.6 \qquad \pi(\theta_{1,1}, \theta_{2,1}) = 0.3$$

$$\pi(\theta_0, \theta_{1,2}) = 0.4 \qquad \pi(\theta_{1,1}, \theta_{2,2}) = 0.7$$

$$\pi(\theta_{1,2}, \theta_{2,3}) = 0.6$$

$$\pi(\theta_{1,2}, \theta_{2,4}) = 0.4$$

The consumption at each node, which equals the dividend, is represented as the quantity in parentheses. To valuate this asset risk neutrally, we consider three stages.

1. Compute the (conditional) risk-neutral probabilities at each node.

$$\pi^{RN}(\theta_0, \theta_{1,1}) = \pi(\theta_0, \theta_{1,1})\left\{\frac{U_1(c(\theta_{1,1}))}{E_0\{U_1(c_1(\tilde{\theta}_1))\}}\right\} = \frac{\left(\frac{1}{5}\right)}{\left(0.6\left(\frac{1}{5}\right) + 0.4\left(\frac{1}{3}\right)\right)} = 0.4737$$

$$\pi^{RN}(\theta_0, \theta_{1,2}) = 1 - \pi^{RN}(\theta_0, \theta_{1,1}) = 0.5263$$

$$\pi^{RN}(\theta_{1,1}, \theta_{2,1}) = \pi(\theta_{1,1}, \theta_{2,1})\left\{\frac{\frac{1}{4}}{\left(0.3\left(\frac{1}{4}\right) + 0.7\left(\frac{1}{2}\right)\right)}\right\} = 0.1765$$

$$\pi^{RN}(\theta_{1,1}, \theta_{2,2}) = 1 - \pi^{RN}(\theta_{1,1}, \theta_{2,1}) = 0.8235$$

$$\pi^{RN}(\theta_{1,2}, \theta_{2,4}) = \pi(\theta_{1,2}, \theta_{2,4})\left\{\frac{1}{\left(0.6\left(\frac{1}{8}\right) + 0.4(1)\right)}\right\} = 0.8421$$

$$\pi^{RN}(\theta_{1,2}, \theta_{2,3}) = 0.1579$$

2. Compute the conditional bond prices.

$$q^b(\theta_0, 1) = \frac{1}{U_1(c_0)}E_0\{U_1(c_1(\tilde{\theta}))\} = \frac{1}{\left(\frac{1}{2}\right)}\left\{0.6\left(\frac{1}{5}\right) + 0.4\left(\frac{1}{3}\right)\right\} = 0.5066$$

$$q^b(\theta_{1,1}, 2) = \frac{1}{\left(\frac{1}{5}\right)}\left\{0.3\left(\frac{1}{4}\right) + 0.7\left(\frac{1}{2}\right)\right\} = 2.125$$

$$q^b(\theta_{1,1}, 2) = \frac{1}{\left(\frac{1}{3}\right)}\left\{0.6\left(\frac{1}{8}\right) + 0.4\left(\frac{1}{1}\right)\right\} = 1.425$$

3. Value the asset.

$$
\begin{aligned}
q^e(\theta_0, 0) = \sum_{s=1}^{2} E_0^{RN}\{g(\theta_0, \tilde{\theta}_s)d_s(\tilde{\theta}_s)\} \\
&= q^b(\theta_0, 1)\{\pi^{RN}(\theta_0, \theta_{1,1})(5) + \pi^{RN}(\theta_0, \theta_{1,2})(3)\} \\
&\quad + q^b(\theta_0, 1)q^b(\theta_{1,1}, 2)\{\pi^{RN}(\theta_0, \theta_{1,1})\pi^{RN}(\theta_{1,1}, \theta_{2,1})(4) \\
&\quad + \pi^{RN}(\theta_0, \theta_{1,1})\pi^{RN}(\theta_{1,1}, \theta_{2,2})(2)\} \\
&\quad + q^b(\theta_0, 1)q^b(\theta_{1,2}, 2)\{\pi^{RN}(\theta_0, \theta_{1,2})\{\pi^{RN}(\theta_{1,2}, \theta_{2,3})(8) \\
&\quad + \pi^{RN}(\theta_0, \theta_{1,2})\pi^{RN}(\theta_{1,2}, \theta_{2,4})(1)\} \\
&= 4.00
\end{aligned}
$$

At a practical level this appears to be a messy calculation at best, but it is not obvious how we might compute the no-trade equilibrium asset prices more easily. The Lucas tree methodologies, for example, do not apply here as the setting is not infinitely recursive. This leaves us to solve for the equilibrium prices by working back through the tree and solving for the no-trade prices at each node. It is not clear that this will be any less involved.

Sometimes, however, the risk-neutral valuation procedure does allow for a very succinct, convenient representation of specific asset prices or price interrelationship. A case in point is that of a long-term discount bond.

Example 13.3 To price at time t, state θ_t, a long-term discount bond maturing in date $t+k$, observe that the corresponding dividend $d_{t+k}(\theta_{t+k}) \equiv 1$ for every θ_{t+k} feasible from state θ_t. Applying Eq. (13.12) yields

$$
q^b(\theta_t, t+k) = E_t^{RN} g(\theta_t, \tilde{\theta}_{t+k}), \text{ or}
$$

$$
\frac{1}{(1+r_f(\theta_t, t+k))^k} = E_t^{RN}\left\{\frac{1}{\prod_{s=t}^{t+k-1}(1+r_f(\theta_s, s+1))}\right\} \tag{13.13}
$$

Equation (13.13), in either of its forms, informs us that the long-term rate is the expectation of the short rates taken with respect to the risk-neutral transition probabilities. This is generally not true if the expectation is taken with the ordinary or true probabilities.

At this point we draw this formal discussion to a close. We now have an idea what risk-neutral valuation might mean in a CCAPM context. Appendix 13.1 briefly discusses the second valuation procedure and illustrates it with the pricing of call and put options.

We thus see that the notion of risk-neutral valuation carries over easily to a CCAPM context. This is not surprising: The key to the existence of a set of risk-neutral probabilities is the presence of a complete set of securities markets, which is the case with the CCAPM. In fact, the somewhat weaker notion of dynamic completeness was sufficient.

We next turn our attention to equity derivatives pricing. The setting is much more specialized and not one of general equilibrium (though not inconsistent with it). One instance of this specialization is that the underlying stock's price is presumed to follow a specialized stochastic process. The term structure is also presumed to be flat. These assumptions, taken together, are sufficient to generate the existence of a unique risk-neutral probability measure, which can be used to value any derivative security written on the stock. That these probabilities are uniquely identified with the specific underlying stock has led us to dub them *local*.

13.4 The Binomial Model of Derivatives Valuation

Under the binomial abstraction we imagine a many-period world in which, at every date-state node only a stock and a bond are traded. With only two securities to trade, dynamic completeness requires that at each node there be only two possible succeeding states. For simplicity, we will also assume that the stock pays no dividend, in other words, that $d(\theta_t) \equiv 0$ for all $t \leq T$. Lastly, in order to avoid any ambiguity in the risk-free discount factors, it is customary to require that the risk-free rate be constant across all dates and states. We formalize these assumptions as follows:

A13.3 The risk-free rate is constant;

$$q^b(\theta_t, t+1) = \frac{1}{1+r_f} \quad \text{for all } t \leq T.$$

A13.4 The stock pays no dividends: $d(\theta_t) \equiv 0$ for all $t \leq T$.

A13.5 The rate of return to stock ownership follows an i.i.d. process of the form:

$$q^e(\theta_{t+1}, t+1) = \begin{cases} uq^e(\theta_t, t), & \text{with probability } \pi \\ dq^e(\theta_t, t), & \text{with probability } 1-\pi, \end{cases}$$

where u (up) and d (down) represent gross rates of return. In order to preclude the existence of an arbitrage opportunity, it must be the case that

$$u > R_f > d,$$

where, in this context, $R_f = 1 + r_f$.

There are effectively only two possible future states in this model ($\theta_t \in \{\theta_1, \theta_2\}$ where θ_1 is identified with u and θ_2 identified with d). Thus, the evolution of the stock's price can be represented by a simple tree structure as seen in Figure 13.3.

Why such a simple setting should be of use is not presently clear, but it will become so shortly.

In this context, the risk-neutral probabilities can be easily computed from Eq. (13.11), specialized to accommodate $d(\theta_t) \equiv 0$:

$$\begin{aligned} q^e(\theta_t, t) &= q^b(\theta_t, t+1)E_t^{RN}\{q^e(\tilde{\theta}_{t+1}, t+1)\} \\ &= q^b(\theta_t, t+1)\{\pi^{RN} uq^e(\theta_t, t) + (1-\pi^{RN})dq^e(\theta_t, t)\} \end{aligned} \tag{13.14}$$

This implies

$$\begin{aligned} &R_f = \pi^{RN}u + (1-\pi^{RN})d, \text{ or} \\ &\pi^{RN} = \frac{R_f - d}{u - d}. \end{aligned} \tag{13.15}$$

The power of this simple context is made clear when comparing Eq. (13.15) with Eq. (13.8). Here risk-neutral probabilities can be expressed without reference to marginal rates of substitution, that is, to agents' preferences.[4] This provides an immense

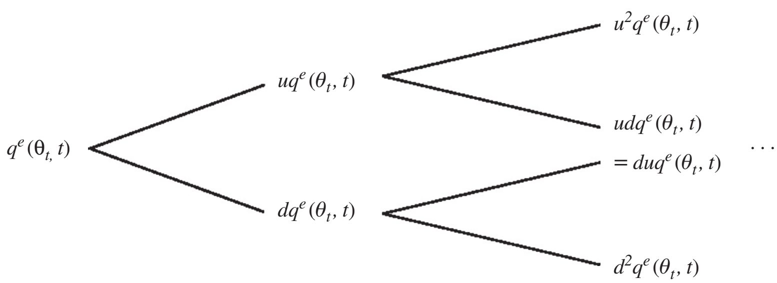

Figure 13.3
A binomial tree structure.

[4] Notice that the risk-neutral probability distribution is i.i.d. as well.

simplification, which all derivative pricing will exploit in one way or another. Of course, the same is true for one-period Arrow–Debreu securities since they are priced equal to the present value of their respective risk-neutral probabilities:

$$q(\theta_t, \theta_{t+1} = u) = \left(\frac{1}{R_f}\right)\left(\frac{R_f - d}{u - d}\right), \text{ and}$$

$$q(\theta_t, \theta_{t+1} = d) = \left(\frac{1}{R_f}\right)\left(1 - \frac{R_f - d}{u - d}\right) = \left(\frac{1}{R_f}\right)\left(\frac{u - R_f}{u - d}\right).$$

Furthermore, since the risk-free rate is assumed constant in every period, the price of a claim to one unit of the numeraire to be received $T - t > 1$ periods from now if state θ_T is realized is given by

$$q(\theta_t, \theta_T) = \frac{1}{(1 + r_f(\theta_t, T))^{T-t}} \sum_{\{\theta_t, \ldots, \theta_{T-1}\} \in \Omega} \prod_{s=t}^{T-1} \pi^{RN}(\theta_s, \theta_{s+1}),$$

where Ω represents the set of all time paths $\{\theta_t, \theta_{t+1}, \ldots, \theta_{T-1}\}$ leading to θ_T. In the binomial setting this becomes

$$q(\theta_t, \theta_T) = \frac{1}{(R_f)^{T-t}} \binom{T-t}{s} (\pi^{RN})^s (1 - \pi^{RN})^{T-t-s}, \tag{13.16}$$

where s is the number of intervening periods in which the u state is observed on any path from θ_t to θ_T. The expression $\binom{T-t}{s}$ represents the number of ways s successes (u moves) can occur in $T - t$ trials. A standard result states

$$\binom{T-t}{s} = \frac{(T-t)!}{s!(T-t-s)!}.$$

The explanation is as follows. Any possible period T price of the underlying stock will be identified with a unique number of u and d realizations. Suppose, for example, that s_1 u-realizations are required. There are then $\binom{T-t}{s_1}$ possible paths, each of which has exactly s_1 u and $(T - t - s_1)$ d states, leading to the prespecified period T price. Each path has the common risk-neutral probability $(\pi^{RN})^{s_1}(1 - \pi^{RN})^{T-t-s_1}$. As an example, suppose $T - t = 3$, and the particular final price is the result of two up moves and one down move. Then, there are $3 = \frac{3!}{2!1!} = \frac{3 \cdot 2 \cdot 1}{(2 \cdot 1)(1)}$ possible paths leading to that final state: *uud*, *udu*, and *duu*.

To illustrate the simplicity of this setting we again consider several examples.

Example 13.4 A European call option revisited: let the option expire at $T > t$; the price of a European equity call with exercise price K, given the current date-state (θ_t, t), $C_e(\theta t, t)$ is

$$C_e(\theta_t, t) = \left(\frac{1}{R_f}\right)^{T-t} E_t^{RN}\left(\max\left\{q^e(\theta_T, T) - K, 0\right\}\right)$$
$$= \left(\frac{1}{R_f}\right)^{T-t} \sum_{s=0}^{T-t} \binom{T-t}{s} (\pi^{RN})^s (1-\pi^{RN})^{T-t-s} \left(\max\left\{q^e(\theta_t, t) u^s d^{T-t-s} - K, 0\right\}\right)$$

When taking the expectation, we sum over all possible values of $s \leq T - t$, thus weighting each possible option payoff by the risk-neutral probability of attaining it.

Define the quantity $\hat{s}$ as the minimum number of intervening up states necessary for the underlying asset, the stock, to achieve a price in excess of K. The prior expression can then be simplified to:

$$C_e(\theta_t, t) = \frac{1}{(R_f)^{T-t}} \sum_{s=\hat{s}}^{T-t} \binom{T-t}{s} (\pi^{RN})^s (1-\pi^{RN})^{T-t-s} \left[q^e(\theta_t, t) u^s d^{T-t-s} - K\right], \tag{13.17}$$

or

$$C_e(\theta_t, t) = \frac{1}{(R_f)^{T-t}} \sum_{s=\hat{s}}^{T-t} \binom{T-t}{s} (\pi^{RN})^s (1-\pi^{RN})^{T-t-s} q^e(\theta_t, t) u^s d^{T-t-s}$$
$$- \sum_{s=\hat{s}}^{T-t} \binom{T-t}{s} (\pi^{RN})^s (1-\pi^{RN})^{T-t-s} K \tag{13.18}$$

The first term within the braces of Eq. (13.18) is the risk-neutral expected value at expiration of the acquired asset if the option is exercised, whereas the second term is the risk-neutral expected cost of acquiring it. The difference is the risk-neutral expected value of the call's payoff (value) at expiration.[5] To value the call today, this quantity is then put on a present value basis by discounting at the risk-free rate R_f.

This same valuation can also be obtained by working backward, recursively, through the tree. Since markets are complete, in the absence of arbitrage opportunities any asset—the call included —is priced equal to its expected value in the succeeding time period discounted at R_f. This implies

$$C_e(\theta_t, t) = q^b(\theta_t, t) E^{RN} C_e(\tilde{\theta}_{t+1}, t+1). \tag{13.19}$$

Let us next illustrate how this fact may be used to compute the call's value in a simple three-period example.

[5] Recall that there is no actual transfer of the security. Rather, this difference $q^e(\theta_T, T) - K$ represents the amount of money the writer (seller) of the call must transfer to the buyer at the expiration date if the option is exercised.

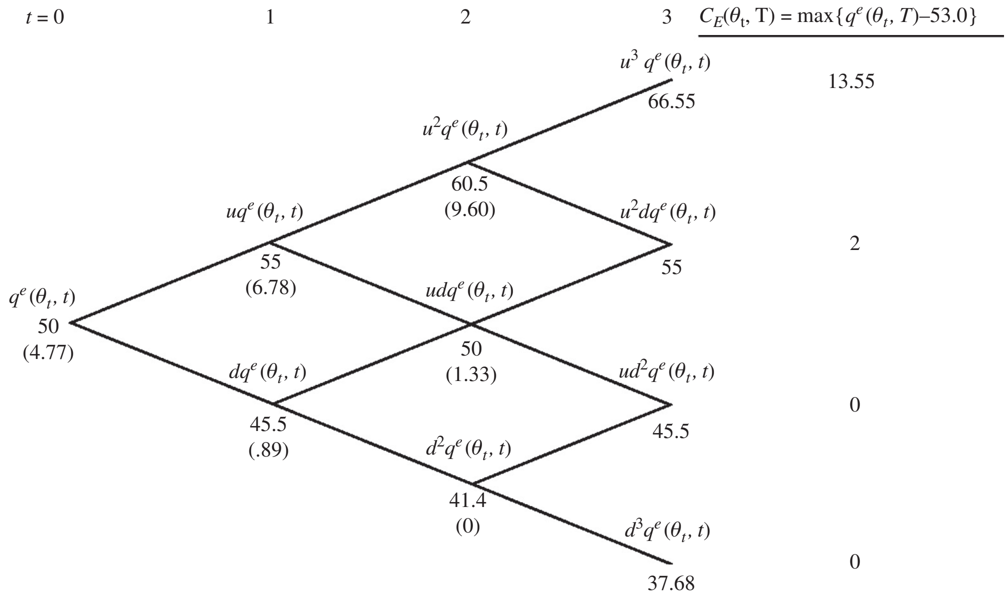

Figure 13.4
The binomial tree of Example 13.5.

Example 13.5 Let $u = 1.1$, $d = \frac{1}{u} = 0.91$, $q^e(\theta_t,t) = \$50$, $K = \$53$, $R_f = 1.05$, $T - t = 3$.

$$\pi^{RN} = \frac{R_f - d}{u - d} = \frac{1.05 - 0.91}{1.1 - 0.91} = 0.70$$

The numbers in parentheses in Figure 13.4 are the recursive values of the call, working backward in the manner of Eq. (13.19). These are obtained as follows:

$$C_e(u^2, t+2) = \frac{1}{1.05}\{0.70(13.55) + 0.30(2)\} = 9.60$$

$$C_e(ud, t+2) = \frac{1}{1.05}\{0.70(2) + 0.30(0)\} = 1.33$$

$$C_e(u, t+1) = \frac{1}{1.05}\{0.70(9.60) + 0.30(1.33)\} = 6.78$$

$$C_e(d, t+1) = \frac{1}{1.05}\{0.70(1.33) + 0.30(0)\} = 0.89$$

$$C_e(\theta_t, t) = \frac{1}{1.05}\{0.70(6.78) + 0.30(0.89)\} = 4.77$$

Table 13.1: Payoff pattern—Asian option

t	$t+1$	$t+2$	...	$T-1$	T
0	0	0		0	$\max\{q^e_{AVG}(\theta_T, T) - K, 0\}$

For a simple call, its payoff at expiration is dependent only upon the value of the underlying asset (relative to K) at that time, regardless of its price history. For example, the value of the call when $q^e(\theta_T,T) = 55$ is the same if the price history is (50,55,50,55) or (50,45.5,50,55).

For other derivatives, however, this is not the case; they are *path dependent.* An Asian (path-dependent) option is a case in point. Nevertheless, the same valuation methods apply: its expected payoff is computed using the risk-neutral probabilities, and then discounted at the risk-free rate.

Example 13.6 A path-dependent option: we consider an Asian option for which the payoff pattern assumes the form outlined in Table 13.1.

Where $q^e_{AVG}\ (\theta_T, T)$ is the average price of the stock along the path from $q^e(\theta_t, t)$ to, and including, $q^e(\theta_T, T)$. We may express the period t value of such an option as

$$C_A(\theta_t, t) = \frac{1}{(R_f)^{T-t}} E_t^{RN} \max\{q^e_{AVG}(\theta_T, T) - K, 0\}$$

A simple numerical example with $T - t = 2$ follows. Let $q^e(\theta_t, t) = 100$, $K = 100$, $u = 1.05$, $d = \frac{1}{u} = 0.95$, and $R_f = 1.005$. The corresponding risk-neutral probabilities are

$$\pi^{RN} = \frac{R_f - d}{u - d} = \frac{1.005 - 0.95}{1.05 - 0.95} = 0.55; 1 - \pi^{RN} = 0.45$$

With two periods remaining, the possible evolutions of the stock's price and corresponding option payoffs are those found in Figure 13.5.

Thus,

$$C_A(\theta_t, t) = \frac{1}{(1.005)^2}\{(0.55)^2(5.083) + (0.55)(0.45)(1.67)\} = \$1.932$$

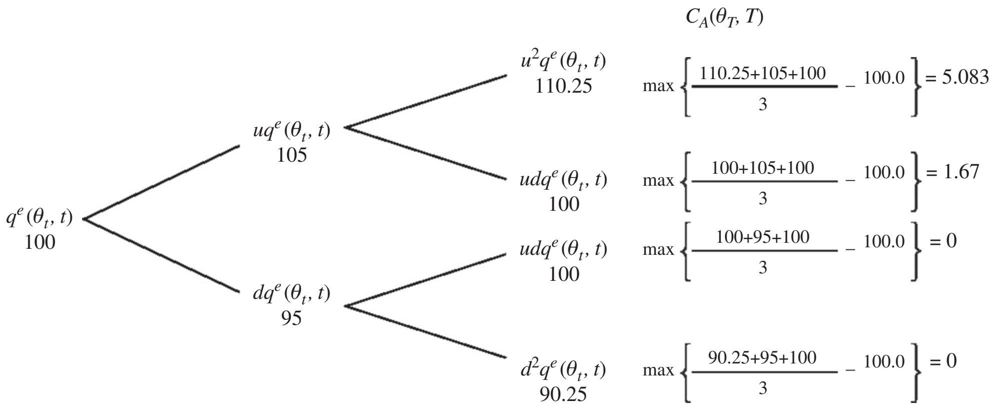

Figure 13.5
Evolution of the stock's price and the Asian option payoffs.

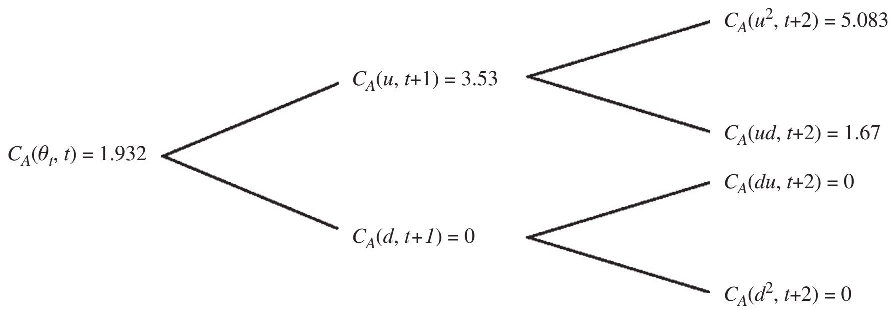

Figure 13.6
Computing recursively the value of the Asian option.

Note that we may as well work backward, recursively, in the price/payoff tree as shown in Figure 13.6,

$$\text{where } C_A(\theta_{t+1} = u, t+1) = 3.53 = \frac{1}{(1.005)}\{0.55(5.083) + 0.45(1.67)\}, \text{ and}$$

$$C_A(\theta_t, t) = \frac{1}{(1.005)}\{0.55(3.53) + 0.45(0)\} = \$1.932.$$

A number of fairly detailed comments are presently in order. Note that with a path-dependent option it is not possible to apply, naively, a variation on Eq. (13.18). Unlike with straightforward calls, the value of this type of option is not the same for all paths leading to the same final-period asset price.

Who might be interested in purchasing such an option? For one thing, they have payoff patterns similar in spirit to an ordinary call, but are generally less expensive (there is less upward potential in the average than in the price itself). This feature has contributed to the usefulness of path-dependent options in foreign exchange trading. Consider a firm that needs to provide a stream of payments (say, perhaps, for factory construction) in a foreign currency. It would want protection against a rise in the value of the foreign currency relative to its own because such a rise would increase the cost of the payment stream in terms of the firm's own currency. Since many payments are to be made, what is of concern is the average price of the foreign currency rather than its price at any specific date. By purchasing the correct number of Asian calls on the foreign currency, the firm can create a payment for itself if, on average, the foreign currency's value exceeds the strike price—the level above which the firm would like to be insured. By analogous reasoning, if the firm wished to protect the average value of a stream of payments, it was receiving in a foreign currency, the purchase of Asian puts would be one alternative.

We do not want to lose sight of the fact that risk-neutral valuation is a direct consequence of the dynamic completeness (at each node there are two possible future states and two securities available for trade) and the no-arbitrage assumption, a connection that is especially apparent in the binomial setting. Consider a call option with expiration one period from the present. Over this period the stock's price behavior and the corresponding payoffs to the call option are as found in Figure 13.7.

By the assumed dynamic completeness we know that the payoff to the option can be replicated on a state-by-state basis by a position in the stock and the bond. Let this position be characterized by a portfolio of Δ shares and a bond investment of value B

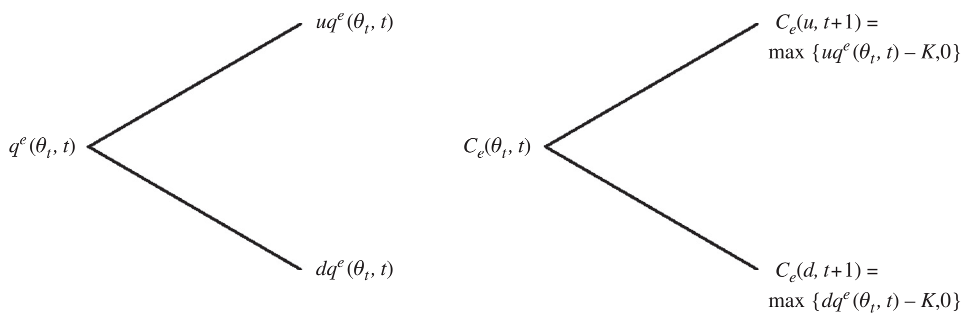

Figure 13.7
A call option one period to expiration and the underlying stock's price.

(for simplicity of notation we suppress the dependence of these latter quantities on the current state and date). Replication requires

$$uq^e(\theta_t, t)\Delta + R_f B = Ce(u, t+1), \text{ and}$$
$$dq^e(\theta_t, t)\Delta + R_f B = Ce(d, t+1),$$

from which follows

$$\Delta = \frac{C_e(u, t+1) - C_e(d, t+1)}{(u-d)q^e(\theta_t, t)}, \text{ and}$$

$$B = \frac{uC_e(d, t+1) - dC_e(u, t+1)}{(u-d)R_f},$$

By the no-arbitrage assumption:

$$\begin{aligned}
C_e(\theta_t, t) &= \Delta q^e(\theta_t, t) + B \\
&= \left\{ \frac{C_e(u, t+1) - C_e(d, t+1)}{(u-d)q^e(\theta_t, t)} q^e(\theta_t, t) \frac{uC_e(d, t+1) - dC_e(u, t+1)}{(u-d)R_f} \right\} \\
&= \left(\frac{1}{R_f}\right) \left\{ \left(\frac{R_f - d}{u-d}\right) C_e(u, t+1) + \left(\frac{u - R_f}{u-d}\right) C_e(d, t+1) \right\} \\
&= \left(\frac{1}{R_f}\right) \left\{ \pi^{RN} C_e(u, t+1) + (1 - \pi^{RN}) C_e(d, t+1) \right\},
\end{aligned}$$

which is just a specialized case of Eq. (13.18).

Valuing an option (or other derivative) using risk-neutral valuation is thus equivalent to pricing its replicating portfolio of stock and debt. Working backward in the tree corresponds to recomputing the portfolio of stock and debt that replicates the derivative's payoffs at each of the succeeding nodes. In the earlier example of the Asian option, the value 3.53 at the intermediate u node represents the value of the portfolio of stocks and bonds necessary to replicate the option's values in the second-period nodes leading from it (5.083 in the u state, 1.67 in the d state).

Let us see how the replicated portfolio evolves in the case of the Asian option written on a stock.

$$\Delta_u = \frac{C_A(u^2, t+2) - C_A(ud, t+2)}{(u-d)q^e(\theta_t, t)} = \frac{5.083 - 1.67}{(1.05 - 0.95)(105)} = 0.325$$

$$B_u = \frac{uC_A(ud, t+2) - dC_A(u^2, t+2)}{(u-d)R_f} = \frac{(1.05)(1.67) - (0.95)(5.083)}{(1.05 - 0.95)(1.005)} = -30.60$$

$$\left.\begin{array}{l}\Delta_d = 0 \\ B_d = 0\end{array}\right. \text{(all branches leading from the "}d\text{" node result in zero option value)}$$

$$\Delta = \frac{C_A(u, t+1) - C_A(d, t+1)}{(u-d)q^e(\theta_t, t)} = \frac{3.53 - 0}{(1.05 - 0.95)(100)} = 0.353$$

$$B = \frac{uC_A(d, t+2) - dC_A(u, t+1)}{(u-d)R_f} = \frac{(1.05)(0) - (0.95)(3.53)}{(1.05 - 0.95)(1.005)} = -33.33$$

We interpret these numbers as follows. In order to replicate the value of the Asian option, regardless of whether the underlying stock's price rises to \$105 or falls to \$95.20, it is necessary to construct a portfolio composed of a loan of \$33.33 at R_f in conjunction with a long position of 0.353 share. The net cost is

$$0.353(100) - 33.33 = \$1.97,$$

the cost of the call, except for rounding errors. To express this idea slightly differently, if you want to replicate, at each node, the value of the Asian option, borrow \$33.33 (at R_f) and, together with your own capital contribution of \$1.97, take this money and purchase 0.353 share of the underlying stock.

As the underlying stock's value evolves through time, this portfolio's value will evolve so that at any node it matches exactly the call's value. At the first u node, for example, the portfolio will be worth \$3.53. Together with a loan of \$30.60, this latter sum will allow the purchase of 0.325 share, with no additional capital contribution required. Once assembled, the portfolio is entirely self-financing, no additional capital need be added, and none may be withdrawn (until expiration).

This discussion suggests that Asian options represent a levered position in the underlying stock. To see this, note that at the initial node the replicating portfolio consists of a \$1.97 equity contribution by the purchaser in conjunction with a loan of \$30.60. This implies a debt/equity ratio of $\frac{\$30.60}{1.97} \cong 15.5$! For the analogous straight call, with the same exercise price as the Asian and the same underlying price process, the analogous quantities are, respectively, \$3.07 and \$54.47, giving a debt/equity ratio of approximately 18. Call-related securities are thus attractive instruments for speculation! For a relatively small cash outlay, a stock's entire upward potential (within a limited span of time) can be purchased.

Under this pricing perspective, there are no-arbitrage opportunities within the universe of the underlying asset, the bond, or any derivative asset written on the underlying asset.

We were reminded of this fact in the prior discussion! The price of the call at all times equals the value of the replicating portfolio. It does not, however, preclude the existence of such opportunities among different stocks or among derivatives written on different stocks.

These discussions make apparent the fact that binomial risk-neutral valuation views derivative securities, and call options in particular, as redundant assets, redundant in the sense that their payoffs can be replicated with a portfolio of preexisting securities.

The presence or absence of these derivatives is deemed not to affect the price of the underlying asset (the stock) on which they are written. This is in direct contrast to our earlier motivation for the existence of options: their desirable property in assisting in the completion of the market. In principle, the introduction of an option has the potential of changing all asset values if it makes the market more complete.

This issue has been examined fairly extensively in the literature. From a theoretical perspective, Detemple and Selden (1991) construct a mean variance example where there is one risky asset, one risk-free asset, and an incomplete market. There the introduction of a call option is shown to increase the equilibrium price of the risky asset. In light of our earlier discussions, this is not entirely surprising: The introduction of the option enhances opportunities for risk sharing, thereby increasing demand and consequently the price of the risky asset. This result can be shown not to be fully applicable to all contexts, however. On the empirical side, Detemple and Jorion (1990) examine a large sample of options introductions over the period 1973 to 1986 and find that, on average, the underlying stock's price rises 3% as a result and its volatility diminishes.

13.5 Continuous Time: An Introduction to the Black–Scholes Formula

Although the binomial model presents a transparent application of risk-neutral valuation, it is not clear that it represents the accurate description of the price evolution of any known security. We deal with this issue presently.

Fat tails aside, there is ample evidence to suggest that stock prices may be modeled as being lognormally distributed; more formally,

$$\ln q^e(\theta_T, T) \sim N(\ln q^e(\theta_t, t) + \mu(T-t), \sigma\sqrt{T-t}),$$

where μ and σ denote, respectively, the mean and standard deviation of the continuously compounded rate of return over the reference period, typically 1 year. Regarding t as the present time, this expression describes the distribution of stock prices at some time T in the future given the current price $q^e(\theta_t, t)$. The length of the time horizon $T - t$ is measured in years.

The key result is this: properly parameterized, the distribution of final prices generated by the binomial distribution can arbitrarily well approximate the prior lognormal distribution when the number of branches becomes very large. More precisely, we may imagine a binomial model in which we divide the period $T-t$ into n subintervals of equal length $\Delta T(n) = (T-t)/n$. If we adjust u, d, p (the true probability of a u price move), and R_f appropriately, then as $n \to \infty$, the distribution of period T prices generated by the binomial model will converge in probability to the hypothesized lognormal distribution. The adjustment requires that

$$u(n) = e^{\sigma\sqrt{\Delta t(n)}}, \quad d(n) = \frac{1}{u(n)}, \quad p = \frac{e^{\mu\Delta t(n)} - d(n)}{u(n) - d(n)}, \quad \text{and} \qquad (13.20)$$
$$R_f(n) = (R_f)^{\frac{1}{n}}$$

For this identification, the binomial valuation formula for a call option, Eq. (13.18), converges to the Black–Scholes formula for a European call option written on a nondividend paying stock:

$$C_e(\theta_t, t) = q^e(\theta_t, T)N(d_1) - Ke^{-\hat{r}_f(t-t)}N(d_2) \qquad (13.21)$$

where N ( ) is the cumulative normal probability distribution function,

$$\hat{r}_f = \ell n(R_f)$$

$$d_1 = \frac{\ell n\left(\frac{q_e(\theta_t,t)}{K}\right) + (T-t)\left(\hat{r}_f + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-1}$$

Cox and Rubinstein (1979) provide a detailed development and proof of this equivalence, but we can see the rudiments of its origin in Eq. (13.18), which we now present, modified to make apparent its dependence on the number of subintervals n:

$$C_e(\theta_t, t; n) = \frac{1}{(R_f(n))^n}\left\{\sum_{s=a(n)}^{n}\binom{n}{s}(\pi(n)^{RN})^s(1-\pi(n)^{RN})^{n-s}q^e(\theta_t, t)\right. \qquad (13.22)$$
$$\left. - K\sum_{s=a(n)}^{n}\binom{n}{s}(\pi(n)^{RN})^s(1-\pi(n)^{RN})^{n-s}\right\}$$

where $\pi(n)^{RN} = (R_f(n) - d(n))/(u(n) - d(n))$ g and a(n) is the minimum number of up moves for the option to have positive payoff.

Rearranging terms yields

$$C_e(\theta_t,t;n) = q^e(\theta_t,t)\sum_{s=a(n)}^{n}\binom{n}{s}\left(\frac{\pi(n)^{RN}}{R_f(n)}\right)^s\left(\frac{1-\pi(n)^{RN}}{R_f(n)}\right)^{n-s} - K\left(\frac{1}{R_f(n)}\right)^n\sum_{s=a}^{n}\binom{n}{s}(\pi(n)^{RN})^s(1-\pi(n)^{RN})^{n-s}, \tag{13.23}$$

which is of the general form

$$C_{e;}(\theta_t,t;n) = q_e(\theta_t,t)\times\text{Probability} - (\text{present value factor})\times K\times\text{Probability},$$

as per the Black–Scholes formula. Since, at each step of the limiting process (i.e., for each n, as $n \mapsto \infty$), the call valuation formula is fundamentally an expression of risk-neutral valuation, the same must be true of its limit. As such, the Black–Scholes formula represents the first hint at the translation of risk-neutral methods to the case of continuous time.

Let us conclude this section with a few more observations. The first concerns the relationship of the Black–Scholes formula to the replicating portfolio idea. Since at each step of the limiting process the call's value is identical to that of the replicating portfolio, this notion must carry over to the continuous time setting. This is indeed the case: in a context when investors may continuously and costlessly adjust the composition of the replicating portfolio, the initial position to assume (at time t) is one of $N(d_1)$ shares, financed in part by a risk-free loan of $Ke^{-R_fT}N(d_2)$. The net cost of assembling the portfolio is the Black–Scholes value of the call.

Notice also that neither the mean return on the underlying asset nor the true probabilities explicitly enter anywhere in the discussion.[6] None of this is surprising. The short explanation is simply that risk-neutral valuation abandons the true probabilities in favor of the risk-neutral ones, and, in doing so, all assets are determined to earn the risk-free rate. The underlying assets' mean return still matters, but it is now R_f. More intuitively, risk-neutral valuation is essentially no-arbitrage pricing. In a world with full information and without transaction costs, investors will eliminate all arbitrage opportunities regardless of their objective likelihood or of the mean returns of the assets involved.

It is sometimes remarked that to purchase a call option is to buy volatility, and we need to understand what this expression is intended to convey. Returning to the binomial approximation (in conjunction with Eq. (13.18)), we observe first that a larger σ implies the possibility of a higher underlying asset price at expiration, with the attendant higher call payoff. More formally, σ is the only statistical characteristic of the underlying stock's price

[6] They are implicitly present in the equilibrium price of the underlying asset.

process to appear in the Black–Scholes formula. Given r_f, K, and $q^e(\theta_t,t)$, there is a unique identification between the call's value and σ. For this reason, estimates of an asset's volatility are frequently obtained from its corresponding call price by inverting the Black–Scholes formula. This is referred to as an *implied volatility estimate*.

The use of risk-neutral methods for the valuation of options is probably the area in which asset pricing theory has made the most progress. Indeed, Merton and Scholes were awarded the Nobel Prize for their work (Fischer Black had died). So much progress has, in fact, been made that the finance profession has largely turned away from conceptual issues in derivatives valuation to focus on the development of fast computer valuation algorithms that mimic the risk-neutral methods. This, in turn, has allowed the use of derivatives, especially for hedging purposes, to increase so enormously over the past 30 years.

13.6 Dybvig's Evaluation of Dynamic Trading Strategies

Let us next turn to a final application of these methods: the evaluation of dynamic trading strategies. To do so, we retain the partial equilibrium setting of the binomial model, but invite agents to have preferences over the various outcomes. Note that under the pure pricing perspective of Section 13.4, preferences were irrelevant. All investors would agree on the prices of call and put options (and all other derivatives) regardless of their degrees of risk aversion, or their subjective beliefs as to the true probability of an up or down state. This simply reflects the fact that any rational investor, whether highly risk averse or risk neutral, will seek to profit by an arbitrage opportunity, whatever the likelihood, and that in equilibrium, assets should thus be priced so that such opportunities are absent. In this section our goal is different, and preferences will have a role to play. We return to Assumption A13.1.

Consider the optimal consumption problem of an agent who takes security prices as given and who seeks to maximize the present value of time-separable utility (A13.1). His optimal consumption plan solves

$$\begin{aligned}&\max E_0\left(\sum_{t=0}^{\infty} U(\tilde{c}_t, t)\right)\\&\text{s.t.} \sum_{t=0}^{\infty}\sum_{s\in N_t} q(\theta_0, \theta_t(s))c(\theta_t(s)) \le Y_0,\end{aligned} \tag{13.24}$$

where Y_0 is his initial period 0 wealth and $q(\theta_0, \theta_t(s))$ is the period $t = 0$ price of an Arrow–Debreu security paying one unit of the numeraire if state s is observed at time $t > 0$. Assuming a finite number of states and expanding the expectations operator to make explicit the state probabilities, we find that the Lagrangian for this problem is

$$L(\ ) = \sum_{t=0}^{\infty}\sum_{s=1}^{N_t}\pi(\theta_0, \theta_t(s))U(c(\theta_t(s), t))$$
$$+ \lambda\left(Y_0 - \sum_{t=0}^{\infty}\sum_{s=1}^{N_t}q(\theta_0, \theta_t(s))c(\theta_t(s))\right),$$

where $\pi(\theta_0, \theta_t(s))$ is the conditional probability of state s occurring, at time t and λ the Lagrange multiplier.

The first-order condition is

$$U_1(c(\theta_t(s)), t)\pi(\theta_0, \theta_t(s)) = \lambda q(\theta_0, \theta_t(s)).$$

By the concavity of $U(\ )$, if $\theta_t(1)$ and $\theta_t(2)$ are two states, then

$$\frac{q(\theta_0, \theta_t(1))}{\pi(\theta_0, \theta_t(1))} > \frac{q(\theta_0, \theta_t(2))}{\pi(\theta_0, \theta_t(2))}, \text{ if and only if } c(\theta_t(1), t) < c(\theta_t(2), t). \tag{13.25}$$

It follows that if

$$\frac{q(\theta_0, \theta_t(1))}{\pi(\theta_0, \theta_t(1))} > \frac{q(\theta_0, \theta_t(2))}{\pi(\theta_0, \theta_t(2))}, \text{ then } c(\theta_t(1)) = c(\theta_t(2)).$$

The $q(\theta_0, \theta_t(s))/\pi(\theta_0, \theta_t(s))$ ratio measures the relative scarcity of consumption in state $\theta_t(s)$: A high ratio in some state suggests that the price of consumption is very high relative to the likelihood of that state being observed. This suggests that consumption is scarce in the high $q(\theta_0, \theta_t(s))/\pi(\theta_0, \theta_t(s))$ states. A rational agent will consume less in these states and more in the relatively cheaper ones, as Eq. (13.25) suggests.

This observation is, in fact, quite general as Proposition 13.1 demonstrates.

Proposition 13.1 (Dybvig, 1988) Consider the consumption allocation problem described by Eq. (13.24). For any rational investor for which $U_{11}(c_t,t) < 0$, his optimal consumption plan is a decreasing function of $q(\theta_0,\theta_t(s))\pi(\theta_0, \theta_t(s))$. Furthermore, for any consumption plan with this monotonicity property, there exists a rational investor with concave period utility function $U(c_t,t)$ for which the consumption plan is optimal in the sense of solving Problem (13.24).

Dybvig (1988) illustrates the power of this result most effectively in the binomial context where the price-to-probability ratio assumes an especially simple form. Recall that in the binomial model the state at time t is completely characterized by the number of up states, u, preceding it. Consider a state $\theta_t(s)$ where s denotes the number of preceding up states. The true conditional probability of $\theta_t(s)$ is

$$\pi(\theta_0, \theta_t(s)) = \pi^s(1-\pi)^{t-s},$$

while the corresponding state claim has price

$$q(\theta_0, \theta_t(s)) = (R_f)^{-t}(\pi^{RN})^s(1-\pi^{RN})^{t-s}.$$

The price/probability ratio thus assumes the form

$$\frac{q(\theta_0, \theta_t(s))}{\pi(\theta_0, \theta_t(s))} = (R_f)^{-t}\left(\frac{\pi^{RN}}{\pi}\right)^s\left(\frac{1-\pi^{RN}}{1-\pi}\right)^{t-s} = (R_f)^{-t}\left(\frac{\pi^{RN}(1-\pi)}{(1-\pi^{RN})\pi}\right)^s\left(\frac{1-\pi^{RN}}{1-\pi}\right)^t.$$

We now specialize the binomial process by further requiring the condition in Assumption A13.6.

A13.6 $\pi u + (1-\pi)d > R_f$, in other words, the expected return on the stock exceeds the risk-free rate.

Assumption A13.6 implies that

$$\pi > \frac{R_f - d}{u-d} = \pi^{RN},$$

so that

$$\frac{\pi^{RN}(1-\pi)}{(1-\pi^{RN})} < 1,$$

for any time t, and the price probability ratio $q(\theta_0,\theta_t(s))/\pi(\theta_0,\theta_t(s))$ is a decreasing function of the number of preceding up moves, s. By Proposition 13.1 the period t level of optimal, planned consumption across states $\theta_t(s)$ is thus an *increasing function of the number of up moves*, s, *preceding it.*

Let us now specialize our agent's preferences to assume that he is only concerned with his consumption at some terminal date T, at which time he consumes his wealth. Problem (13.24) easily specializes to this case:

$$\begin{aligned} &\max \sum_{s \in N_T} \pi(\theta_0, \theta_T(s))U(c(\theta_T(s))) \\ &\text{s.t.} \sum_{s \in N_T} q(\theta_0, \theta_T(s))c(\theta_T(s)) \le Y_0 \end{aligned} \tag{13.26}$$

$$q_e(\theta_0, 0) = 16, \qquad u = 2, \qquad d = \tfrac{1}{2}, \qquad R_f = 1$$

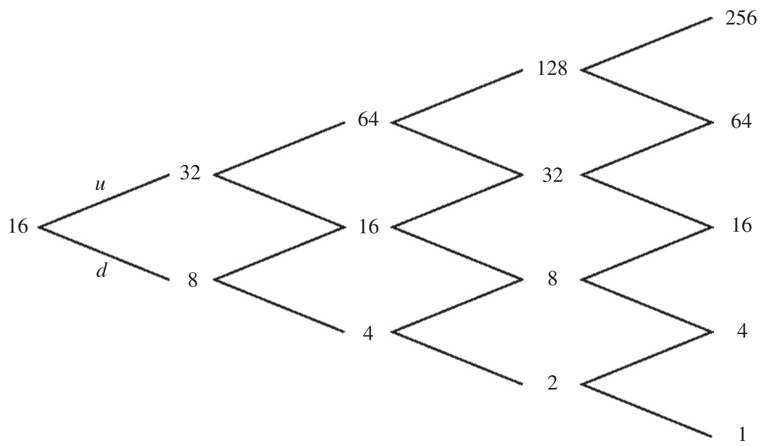

Figure 13.8
Binomial evolution of a stock's price over four periods.

In effect, we set $U(c_t,t) \equiv 0$ for $t \leq T$.

Remember also that a stock, from the perspective of an agent who is concerned only with terminal wealth, can be viewed as a portfolio of period t state claims. The results of Proposition 13.1 thus apply to this security as well.

Dybvig (1988) shows how these latter observations can be used to assess the optimality of many commonly used trading strategies. The context of his discussion is illustrated with the example in Figure 13.8 where the investor is presumed to consume his wealth at the end of the trading period.

For this particular setup, $\pi^{RN} = \frac{1}{3}$. He considers the following frequently cited equity trading strategies:

1. Technical analysis: buy the stock and sell it after an up move; buy it back after a down move; invest at R_f (zero in this example) when out of the market. But under this strategy

$$c_4(\theta_t(s)|uuuu) = \$32, \text{ yet}$$
$$c_4(\theta_t(s)|udud) = \$48; \text{ in other words,}$$

the investor consumes more in the state with the fewer preceding up moves, which violates the optimality condition. This cannot be an optimal strategy.

2. Stop-loss strategy: buy and hold the stock, sell only if the price drops to \$8, and stay out of the market. Consider, again, two possible evolutions of the stock's price:

$$c_4(\theta_t(s)|duuu) = \$8$$
$$c_4(\theta_t(s)|udud) = \$16.$$

Once again, consumption is not an increasing function of the number of up states under this trading strategy, which must, therefore, be suboptimal.

13.7 Conclusions

We have extended the notion of risk-neutral valuation to two important contexts: the dynamic setting of the general equilibrium consumption CAPM and the partial equilibrium binomial model. The return on our investment is particularly apparent in the latter framework. The reasons are clear: in the binomial context, which provides the conceptual foundations for an important part of continuous time finance, the risk-neutral probabilities can be identified independently from agents' preferences. Knowledge of the relevant intertemporal marginal rates of substitution, in particular, is superfluous. This is the huge dividend of the twin modeling choices of binomial framework and arbitrage pricing. It has paved the way for routine pricing of complex derivative-based financial products and for their attendant use in a wide range of modern financial contracts.

References

Cox, J., Rubinstein, M., 1979. Option Markets. Prentice-Hall, Upper Saddle River, NJ.
Detemple, J., Jorion, P., 1990. Option listing and stock returns: an empirical analysis. J. Bank. Financ. 14, 781–801.
Detemple, J., Selden, L., 1991. A general equilibrium analysis of option and stock market interactions. Int. Econ. Rev. 32, 279–303.
Dybvig, P.H., 1988. Inefficient dynamic portfolio strategies or how to throw away a million dollars in the stock market. Rev. Financ. Stud. 1, 67–88.
Telmer, C., 1993. Asset pricing puzzles and incomplete markets. J. Financ. 48, 1803–1832.

Appendix 13.1: Risk-Neutral Valuation When Discounting at the Term Structure of Multiperiod Discount Bond

Here we seek a valuation formula where we discount not at the succession of one-period rates but at the term structure. This necessitates a different set of risk-neutral probabilities with respect to which the expectation is taken.

Define the k-period, time-adjusted risk-neutral transition probabilities as:

$$\hat{\pi}^{RN}(\theta_t, \theta_{t+k}) = \left\{ \frac{\pi^{RN}(\theta_t, \theta_{t+k}) g(\theta_t, \theta_{t+k})}{q^b(\theta_t, \theta_{t+k})} \right\}$$

where $\pi^{RN}(\theta_t, \theta_{t+k}) = \prod_{s=t}^{t+k-1} \pi^{RN}(\theta_t, \theta_{s+1})$, and $\{\theta_t, \ldots, \theta_{t+k-1}\}$ is the path of states preceding θ_{t+k}. Clearly, the $\hat{\pi}^{RN}(\,)$ are positive since $\pi^{RN}(\,) \geq 0$, $g(\theta_t, \theta_{t+k}) > 0$ and $q^b(\theta_t, \theta_{t+k}) > 0$. Furthermore, by Eq. (13.13),

$$\begin{aligned} \sum_{\theta_{t+k}} \hat{\pi}^{RN}(\theta_t, \theta_{t+k}) &= \left(\frac{1}{q^b(\theta_t, \theta_{t+k})} \right) \sum_{\theta_{t+k}} \pi^{RN}(\theta_t, \theta_{t+k}) g(\theta_t, \theta_{t+k}) \\ &= \frac{q^b(\theta_t, \theta_{t+k})}{q^b(\theta_t, \theta_{t+k})} = 1. \end{aligned}$$

Let us now use this approach to price European call and put options. A European call option contract represents the right (but not the obligation) to buy some underlying asset at some prespecified price (referred to as the exercise or strike price) at some prespecified future date (date of contract expiration). Since such a contract represents a right, its payoff is as shown in Table A13.1, where T represents the time of expiration and K the exercise price.

Table A13.1: Payoff pattern—European call option

t	$t+1$	$t+2$	...	$T-1$	T
0	0	0		0	$\max\{q^e(\theta_T, T) - K, 0\}$

Let $C_e(\theta_t, t)$ denote the period t, state θ_t price of the call option. Clearly,

$$\begin{aligned} C_e(\theta_t, t) &= E_t^{RN}\{g(\theta_t, \tilde{\theta}_T)(\max\{q^e(\tilde{\theta}_t, T) - K, 0\})\} \\ &= q^b(\theta_t, T)\hat{E}_t^{RN}\{\max\{q^e(\tilde{\theta}_t, T) - K, 0\}\}, \end{aligned}$$

where $\hat{E}_t^{RN}$ denotes the expectations operator corresponding to the $\{\hat{\pi}^{RN}\}$. A European put option is the right to sell some underlying asset at some prespecified price (exercise price K) at some prespecified future date T. It is similarly priced according to

$$\begin{aligned} P_e(\theta_t, t) &= E_t^{RN}\{g(\theta_t, \tilde{\theta}_T)(\max\{K - q^e(\tilde{\theta}_t, T), 0\})\} \\ &= q^b(\theta_t, T)\hat{E}_t^{RN}\{\max\{K - q^e(\tilde{\theta}_T, T), 0\}\} \end{aligned}$$