Asset Pricing

Fixed income securities

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Outline

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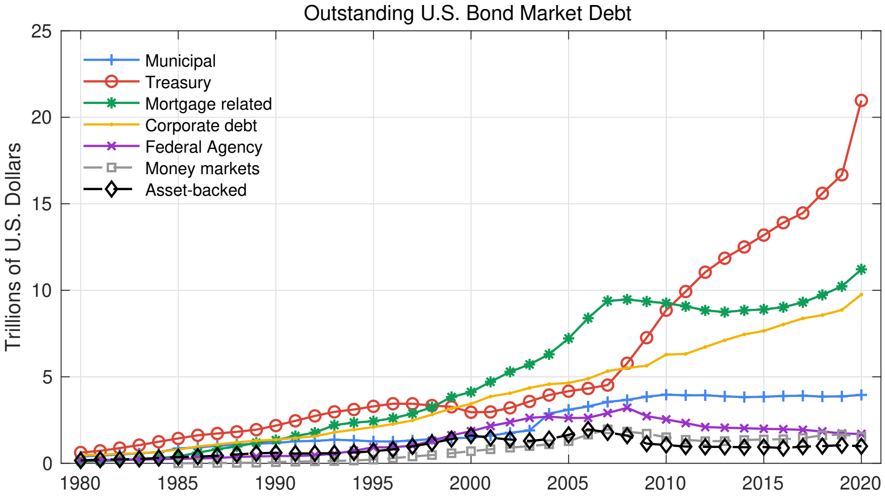
Introduction to fixed income markets

- We now turn our attention to the bond market and, in particular, the Treasury bond market. There are several reasons for studying this market
  - Fixed income markets have developed separately from equity markets, have their own institutional structure, terminology, and academic traditions
  - The market for Treasury securities is enormous in size and has recently experienced massive growth (see next slide)
  - Fixed income securities have no cash flow uncertainty → price variations can only originate from discount rate variations
  - Interest rates are one of the basic ingredients of applied work in finance
  - Conventional Treasury securities carry information about nominal discount rates and inflation-indexed securities (TIPS) about real discount rates
  - U.S. Treasury securities have a special place in financial markets due to their extremely low probability of default

Outstanding bond market debt

* Fixed income markets have seen impressive growth over the recent decades, and Treasuries have seen rapid growth ever since the housing collapse in 2007



Fundamental conceps and relations

* In principle, a fixed income security can be any security that make credible promises of deterministic payoffs at one or more dates in the future. Two cases are classics and central to asset pricing and finance in general

    1. Zero-coupon bonds (or discount bonds)

    2. Coupon-paying bonds

* Coupon bonds can be viewed as a portfolio of discount bonds and pricing follows a law of one price and no arbitrage principle

* Typically, zero-coupon bonds do not exist for longer maturities, but synthetic longer-term zero-coupon yields (prices) can be computed from long-term coupon bond (e.g., Treasury STRIPS)

Discount bond prices and yields

Price of a discount bond

Let Y_t^n denote the *per period* yield to maturity, then the time t price of a discount bond that makes a single nominal payment of $\$1$ at time $t+n$ is

$$P_t^n = \frac{1}{(1+Y_t^n)^n} \tag{1}$$

* Solving for the yield to maturity (or simply yield for short) Y_t^n gives us

$$1 + Y_t^n = (P_t^n)^{-\frac{1}{n}} \tag{2}$$

* Taking logs such that $y_t^n = \ln(1+Y_t^n)$ and $p_t^n = \ln P_t^n$ gives us a linear relation defined as

$$y_t^n = -\left(\frac{1}{n}\right) p_t^n \tag{3}$$

* The term structure of interest rates is then defined by a collection of yields on a given day (t) for bonds with different maturities (n)

Holding-period returns

Holding-period return

The one-period holding-period return R_{t+1}^n on an n–period bond purchased at time t and sold at time $t+1$ is

$$1 + R_{t+1}^n = \frac{P_{t+1}^{n-1}}{P_t^n} = \frac{\left(1 + Y_t^n\right)^n}{\left(1 + Y_{t+1}^{n-1}\right)^{n-1}} \tag{4}$$

* Taking logs so that $r_{t+1}^n = \ln\left(1 + R_{t+1}^n\right)$ we obtain the log holding-period return (using the relation in (3))

$$\begin{aligned} r_{t+1}^n &= p_{t+1}^{n-1} - p_t^n &(5)\\ &= ny_t^n - (n-1)\, y_{t+1}^{n-1} &(6)\\ &= y_t^n - (n-1)\left(y_{t+1}^{n-1} - y_t^n\right) &(7) \end{aligned}$$

revealing that the holding-period return is determined by the yield today and future changes in yields

Holding-period returns, bond prices, and yields

* Let $p_t^n = -r_{t+1}^n + p_{t+1}^{n-1}$ using the definition above and solve the difference equation forward to obtain (using that $p_t^0 = 0$)

$$p_t^n = -\sum_{i=0}^{n-1} r_{t+1+i}^{n-i} \tag{8}$$

* Using the relation in (3), we get the relation in terms of yields

$$y_t^n = \frac{1}{n}\sum_{i=0}^{n-1} r_{t+1+i}^{n-i} \tag{9}$$

* This tells us that the log yield to maturity on a discount bond equals the average log return per period of the bond if held to maturity

Forward rates

Forward rate

Bonds of different maturities can be combined to guarantee an interest rate on a fixed income investment to be made in the future. This forward rate is defined as

$$1 + F_t^n = \frac{1}{\left(P_t^n / P_t^{n-1}\right)} = \frac{P_t^{n-1}}{P_t^n} = \frac{\left(1 + Y_t^n\right)^n}{\left(1 + Y_t^{n-1}\right)^{n-1}} \tag{10}$$

where the superscript n refers to the number of period ahead that the one-period investment is realized (different from book)

* Moving to logs so that $f_t^n = \ln\left(1 + F_t^n\right)$ gives us the following relations

$$\begin{aligned} f_t^n &= p_t^{n-1} - p_t^n & (11) \\ &= n y_t^n - (n-1)\, y_t^{n-1} & (12) \\ &= y_t^{n-1} + n\left(y_t^n - y_t^{n-1}\right) & (13) \end{aligned}$$

  which informs us that the log forward rate curve lies above the yield curve when the yield curve is upward sloping

Forward rates, holding-period returns, and bond prices

- We can connect forward rates and holding-period returns by using the definition of a one-period holding-period return

$$
\begin{aligned}
r_{t+1}^{n} &= p_{t+1}^{n-1} - p_t^n && (14)\\
&= \left(p_{t+1}^{n-1} - p_{t+1}^{n}\right) + \left(p_{t+1}^{n} - p_t^n\right) && (15)\\
&= f_{t+1}^{n} - n\left(y_{t+1}^{n} - y_t^n\right) && (16)
\end{aligned}
$$

- Similarly, let $p_t^n = -f_t^n + p_t^{n-1}$ using the forward rate definition and solve the difference equation forward to obtain the price of a discount bond

$$p_t^n = -\sum_{i=0}^{n-1} f_t^{n-i} \tag{17}$$

or an expression for yields using the relation in (3)

$$y_t^n = \frac{1}{n}\sum_{i=0}^{n-1} f_t^{n-i} \tag{18}$$

Coupon bond prices

Coupon bond price

Let $Y_{c,t}^{n}$ denote the *per period* yield to maturity and C the per period coupon rate, then the time t price of a coupon bond with a face value of $\$1$ and n periods to maturity is

$$P_{c,t}^{n} = \sum_{i=1}^{n} \frac{C}{\left(1 + Y_{c,t}^{n}\right)^{i}} + \frac{1}{\left(1 + Y_{c,t}^{n}\right)^{n}} \tag{19}$$

- The yield is again the discount rate that equates the present value of the bond's cash flows to its price
- There is no analytical solution for $Y_{c,t}^{n}$, but it can be found numerically
- It always has a unique solution because bonds make only positive payments
- Since the yield to maturity corresponding to a given bond is unique, it is often used as an alternative way of quoting bond prices

Yield to maturity as a measure of return

- Unlike the yield to maturity on a discount bond, the yield to maturity on a coupon bond does not necessarily equal the per period return if the bond is held to maturity

- Multiplying the pricing equation in (19) with $\left(1+Y_{c,t}^{n}\right)^{n}$ reveals why

$$P_{c,t}^{n}\left(1+Y_{c,t}^{n}\right)^{n}=\sum_{i=1}^{n}C\left(1+Y_{c,t}^{n}\right)^{n-i}+1 \tag{20}$$

- The yield to maturity equals the per period return on the coupon bond held to maturity only if coupons are reinvested at a rate equal to the yield to maturity

- Generally, however, there is no reason why one should be able to reinvest the coupon payments at the yield to maturity

- Note that the return is not defined until one specifies the reinvestment strategy for coupons received prior to maturity

Special cases

* The coupon bond price in (19) can be written more compactly as an annuity

$$P_{c,t}^{n} = \frac{C}{Y_{c,t}^{n}} \left[1 - \left(\frac{1}{1+Y_{c,t}^{n}} \right)^{n} \right] + \left(\frac{1}{1+Y_{c,t}^{n}} \right)^{n} \tag{21}$$

which provides us with a some intuitive relations and a special case (par bond)

- For $C = Y_{c,t}^{n}$ the price reduced to $P_{c,t}^{n} = 1$, i.e., par
- For $C > Y_{c,t}^{n}$ we have that $P_{c,t}^{n} > 1$
- For $C < Y_{c,t}^{n}$ we have that $P_{c,t}^{n} < 1$

* Finally, from (21), we see that letting the maturity $n \to \infty$ gives us the price of a perpetuity

$$P_{c,t}^{\infty} = \frac{C}{Y_{c,t}^{\infty}} \tag{22}$$

Per period holding-period return

Per period holding-period return

The per period holding-period return $R^n_{c,t+1}$ on a coupon bond with n periods to maturity is

$$1 + R^n_{c,t+1} = \frac{P^{n-1}_{c,t+1} + C}{P^n_{c,t}} \tag{23}$$

- If the bond is traded outside payment dates, then we need to account for accrued interest (AI)

  $$1 + R^n_{c,t+\tau} = \frac{P^{n-\tau}_{c,t+\tau} + C + AI_{c,t+\tau}}{P^n_{c,t} + AI_{c,t}} \tag{24}$$

  where AI, which refers to the accumulation of coupon between the most recent payment and the sale of the bond, is determined as

  $$AI_{c,t} = \frac{\tau - t_1}{t_2 - t_1} C \tag{25}$$

- $P^n_{c,t}$ is known as the flat/quoted/clean price and $P^n_{c,t} + AI_{c,t}$ is known as the full/invoice/dirty price, respectively

Risk measures and management

Interest rate risk

The dominant risk factor affecting the valuation of most fixed income securities is interest rate risk, which refers to the change in the fixed income prices (or yields) caused by an interest rate change

- Knowing how fixed income security prices react to interest rate changes is important for several reasons, including

  1. Traders wishing to hedge a bond with another bond, or portfolio, need to know how the bonds respond to interest rate changes
  2. If an investor has a particular view on future interest rates, it is of interest to know how bond prices will react whether you are right or wrong
  3. Risk managers need to know how volatile their portfolios are
  4. Important for asset liability management. To minimize risk, the sensitivity of the assets should equal the sensitivity of the liabilities to changes in interest rates

Duration

Duration

Duration can be viewed from two different perspectives for securities with fixed and known payments

1. As a measure of the sensitivity of prices of fixed income securities to small parallel shifts in yields
2. As a measure of the average time to payment from holding a fixed income security

* Suppose that we are interested in measuring the length of time a bondholder has invested money in fixed income securities
* For discount bonds this can simply be measured by the maturity of the bond
* For coupon bonds, maturity is an imperfect measure because much of a coupon bond's value comes from payments that are made prior to maturity

Macaulay duration

Macaulay duration

Macaulay's duration is a measure of the average length of time for which money is invested in a coupon bond, where the present values of each payment, evaluated using the coupon bond yield, are used to construct the average

$$D_{c,t}^{n} = \frac{\sum_{i=1}^{n} \frac{iC}{\left(1+Y_{c,t}^{n}\right)^{i}} + \frac{n}{\left(1+Y_{c,t}^{n}\right)^{n}}}{P_{c,t}^{n}} \tag{26}$$

- Macaulay's duration is thus a weighted average of the maturities of the underlying package of discount of bonds
- Macaulay's duration trivially equals maturity for discount bonds ($D_t^n = n$), but is always less than maturity for coupon bond
- Duration declines as the coupon rate C and/or the yield $Y_{c,t}^{n}$ increases because they reduce the weights on more distant payments in the average

Special duration cases

- A par bond has price $P^n_{c,t} = 1$ and yield $Y^n_{c,t} = C$, so the Macaulay duration becomes

$$D^n_{c,t} = \frac{1 - \left(1 + Y^n_{c,t}\right)^{-n}}{1 - \left(1 + Y^n_{c,t}\right)^{-1}} \tag{27}$$

- A perpetuity with infinite maturity has yield $Y^\infty_{c,t} = C/P^\infty_{c,t}$, so Macaulay duration becomes

$$D^\infty_{c,t} = \frac{1 + Y^\infty_{c,t}}{Y^\infty_{c,t}} = 1 + \frac{P^\infty_{c,t}}{C} \tag{28}$$

Interpreting Macaulay's duration

* Macaulay's duration has the important property that it is the negative of the elasticity of a coupon bond's price with respect to its gross yield

$$D_{c,t}^{n} = -\frac{dP_{c,t}^{n}}{d\left(1+Y_{c,t}^{n}\right)} \frac{\left(1+Y_{c,t}^{n}\right)}{P_{c,t}^{n}} \tag{29}$$

* We can equivalently define the elasticity as the negative of the derivative of the log price with respect to the log yield

* This implies that the relations between holding-period returns, forward rates, and yields for zero-coupon bonds can be generalized to approximate relations for coupon bonds if we replace maturity with duration (Shiller et al., 1983)

$$r_{c,t+1}^{n} \approx D_{c,t}^{n} y_{c,t}^{n} - \left(D_{c,t}^{n} - 1\right) y_{c,t+1}^{n-1} \tag{30}$$

$$f_{t}^{n} \approx \frac{D_{c,t}^{n} y_{c,t}^{n} - D_{c,t}^{n-1} y_{c,t}^{n-1}}{D_{c,t}^{n} - D_{c,t}^{n-1}} \tag{31}$$

Modified duration

Modified duration

Modified duration is the negative of the derivative of the bond price with respect to the yield to maturity of the bond

$$MD^n_{c,t} = \frac{D^n_{c,t}}{\left(1 + Y^n_{c,t}\right)} = -\frac{dP^n_{c,t}}{dY^n_{c,t}} \frac{1}{P^n_{c,t}} \tag{32}$$

and measures the proportional sensitivity of a bond's price to a small parallel change in its yield

- Modified duration declines with yield for all bonds, even zero-coupon bonds whose Macaulay's duration is constant
- Modified duration works as a risk measure if we are willing to assume parallel yield curve shifts across the maturity spectrum
- Under parallel shifts, bond yields of all maturities move by the same amount so that a change in the zero-coupon yield is accompanied by an equal change in the coupon bond yield

Duration as a first-order approximation

* Consider a first-order Taylor approximation for the bond price around a small change in yields (i.e., $dY^n_{c_t} = 0$)

$$P^n_{c,t}\left(Y^n_{c,t} + dY^n_{c,t}\right) \approx P^n_{c,t} + \frac{dP^n_{c,t}}{dY^n_{c,t}} dY^n_{c,t} \tag{33}$$

* Subtracting and dividing with $P^n_{c,t}$ gives us

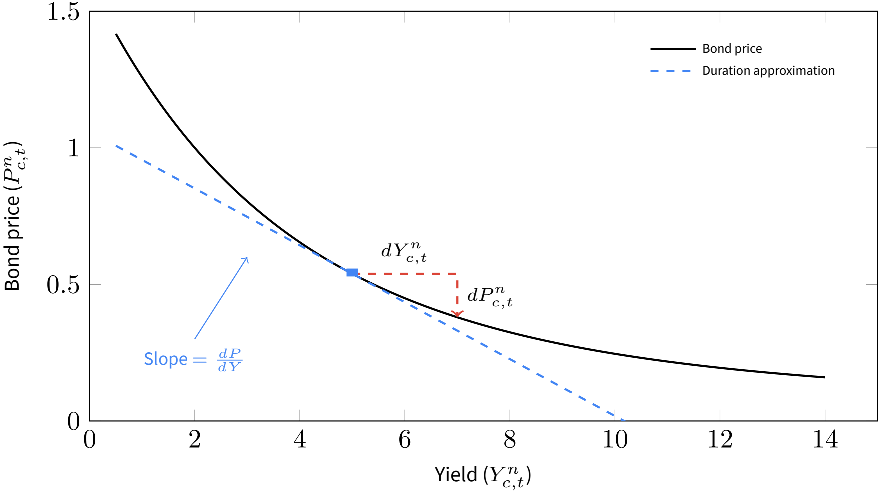
$$\frac{dP^n_{c,t}}{P^n_{c,t}} \approx \frac{1}{P^n_{c,t}} \frac{dP^n_{c,t}}{dY^n_{c,t}} dY^n_{c,t} \tag{34}$$

$$= -MD^n_{c,t} dY^n_{c,t} \tag{35}$$

* This implies that (modified) duration is a first-order approximation for price changes for infinitesimally small changes in yields

Approximating price changes using duration

- Consider the example of a 20-year coupon bond with $C = 2\%$, where we compute the duration for a current yield of $Y_{c,t}^{n} = 5\%$



A few remarks

* Macaulay's duration and modified duration both assume that cash flows are fixed and do not change when interest rates change
* This assumption is appropriate for Treasury securities but not for callable securities such as corporate bonds or mortgage-backed securities, or for securities with default risk if the probability of default varies with the level of interest rates
* By modelling the way in which cash flows vary with interest rates, it is possible to calculate the sensitivity of prices to interest rates for these more complicated securities: this sensitivity is known as *effective duration*

Convexity

The convexity of a security measures the percentage change in the price of the security due to the curvature of the price with respect to the yield

$$C_{c,t}^{n} = \frac{d^2 P_{c,t}^{n}}{d\left(Y_{c,t}^{n}\right)^2}\frac{1}{P_{c,t}^{n}} = \frac{\sum_{i=1}^{n}\frac{i(i+1)C}{\left(1+Y_{c,t}^{n}\right)^{i+2}} + \frac{n(n+1)}{\left(1+Y_{c,t}^{n}\right)^{n+2}}}{P_{c,t}^{n}} \tag{36}$$

* Convexity can be used in a second-order Taylor expansion

$$P_{c,t}^{n}\left(Y_{c,t}^{n} + dY_{c,t}^{n}\right) \approx P_{c,t}^{n} + \frac{dP_{c,t}^{n}}{dY_{c,t}^{n}}dY_{c,t}^{n} + \frac{1}{2}\frac{d^2 P_{c,t}^{n}}{d\left(Y_{c,t}^{n}\right)^2}\left(dY_{c,t}^{n}\right)^2 \tag{37}$$

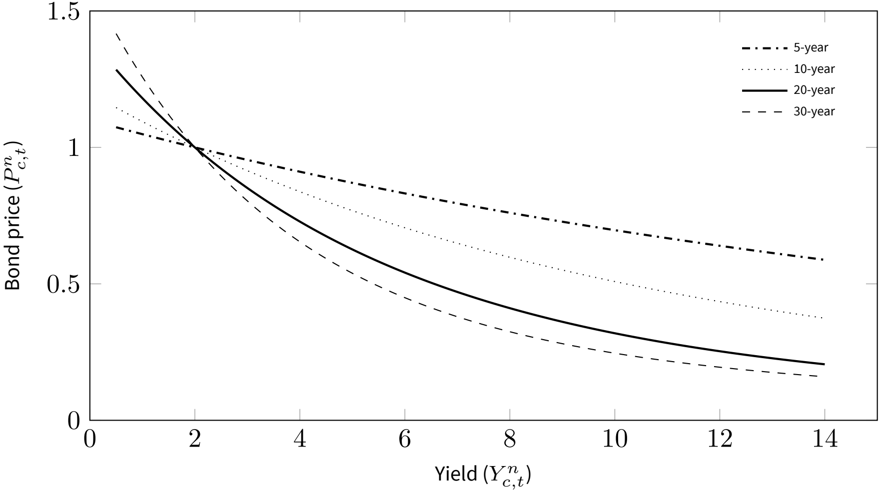
giving us the following percentage price change approximation

$$\frac{dP_{c,t}^{n}}{P_{c,t}^{n}} \approx \frac{1}{P_{c,t}^{n}}\frac{dP_{c,t}^{n}}{dY_{c,t}^{n}}dY_{c,t}^{n} + \frac{1}{2}\frac{1}{P_{c,t}^{n}}\frac{d^2 P_{c,t}^{n}}{d\left(Y_{c,t}^{n}\right)^2}\left(dY_{c,t}^{n}\right)^2 \tag{38}$$

$$= -MD_{c,t}^{n}dY_{c,t}^{n} + \frac{1}{2}C_{c,t}^{n}\left(dY_{c,t}^{n}\right)^2 \tag{39}$$

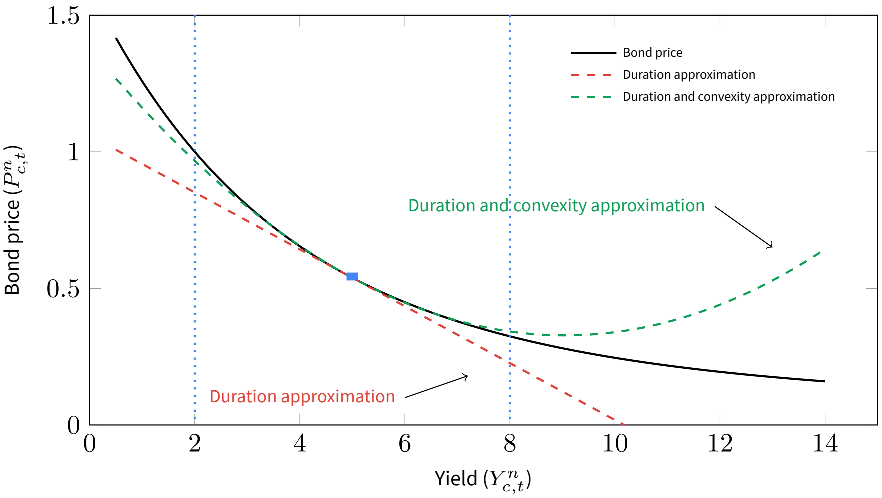
The non-linear relation between prices and rates

* Consider the below coupon bond prices for different maturities as a function of yields



Duration and convexity approximations

- Consider the example of a 20-year coupon bond with $C = 2\%$, where we compute the duration and convexity for a current yield of $Y^n_{c,t} = 5\%$



Immunization

Immunization

A cash flow or portfolio is said to be immunized (against interest rate risk) if the value of the cash flow or portfolio is not negatively affected by any possible change in the term structure of interest rates

* The classic immunization problem is therefore that of finding a cash flow or portfolio with the same sensitivity to interest rate movements
* Complete immunization can be achieved by investing in a cash flow or portfolio that perfectly replicates the cash flows
    - A single future payment can be replicated by an investment in a default free discount bond
    - A stream of future payments can be replicated by a portfolio a discount bonds that matches the payment

Dynamic immunization strategies

- It is not always possible to construct a static hedge portfolio in which case we match price, duration, and convexity using a dynamically rebalanced portfolio
- Let W_t^n denote the price of the cash flow or portfolio that we wish to immunize and let $D_{w,t}^n$ denote its duration and $C_{w,t}^n$ its convexity, then the problem is

$$W_t^n = \sum_{j=1}^{N} X_j P_{j,t}^n \tag{40}$$

$$D_{w,t}^n = \sum_{j=1}^{N} \frac{P_{j,t}^n}{W_t^n} X_j D_{j,t}^n \tag{41}$$

$$C_{w,t}^n = \sum_{j=1}^{N} \frac{P_{j,t}^n}{W_t^n} X_j C_{j,t}^n \tag{42}$$

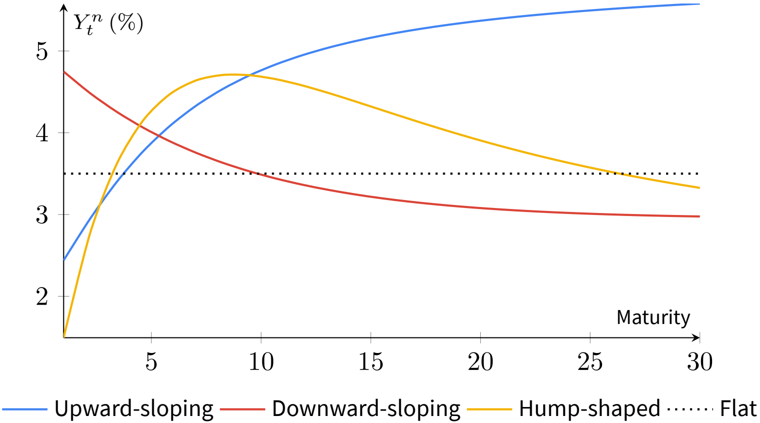
where X_j denote the face amount to buy of security j and N denotes the number of securities needed for the immunization

- for duration matching, we need $N = 2$
- for duration and convexity matching, we need $N = 3$

The zero-coupon term structure

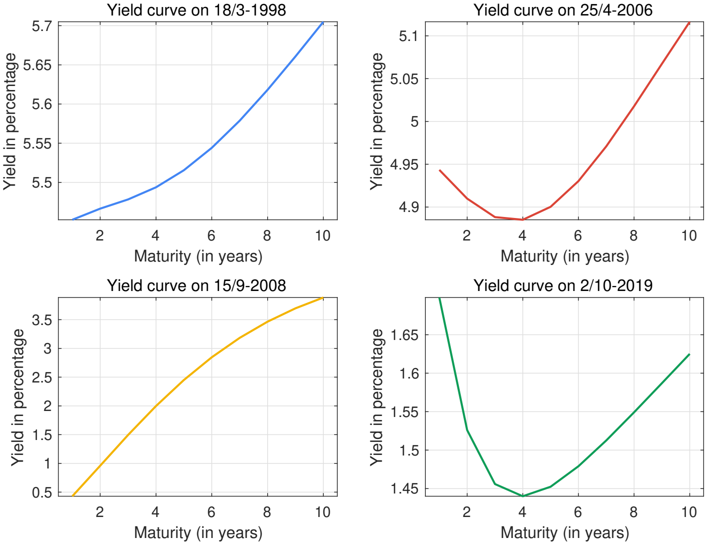
The zero-coupon term structure

The zero-coupon term structure at a given time (day) t defines the relation between the level of zero-coupon yields (interest rates) and their time to maturity n



Zero-coupon term structure examples

* Below are real-life examples of estimated zero-coupon yield curves



Zero-coupon discount factors

Zero-coupon discount factor

The zero-coupon discount factor between two dates, t and $t+n$, provides the term of exchange between a given amount of money at t versus a (certain) amount of money at a later date $t+n$. We denote the discount factor by Z_t^n

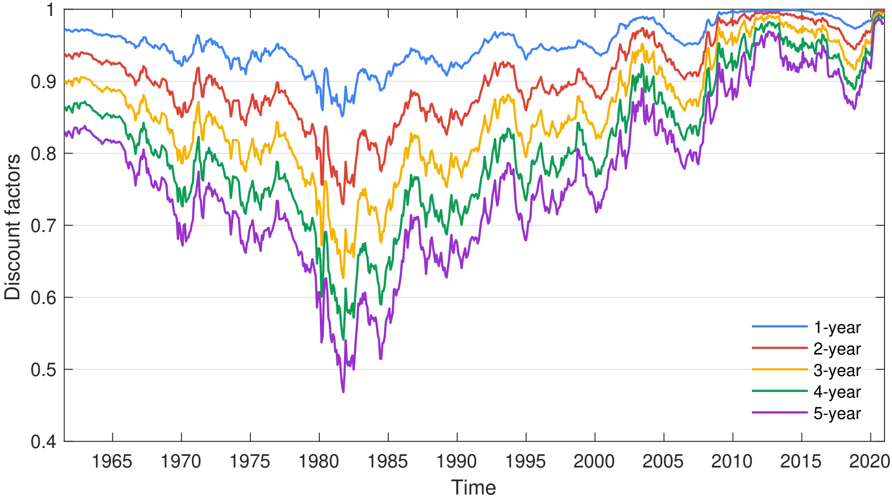
- At any given time t, the discount factor is lower, the longer the maturity n. That is, given two dates n_1 and n_2 with $n_1 < n_2$, it is always the case that

$$1 \geq Z_t^{n_1} \geq Z_t^{n_2} \geq 0 \tag{43}$$

- Discount factors can be viewed as a measure of the time value of money identified as the collective willingness of investors to receive a certain amount of money in the future in exchange for a lower amount today

- They are equivalent to discount bonds when available (i.e., $P_t^n \equiv Z_t^n$)

Discount factors over time



Extracing zero-coupon dicsount factors

- We will take our point of departure for pricing any fixed income asset in the no-arbitrage relation

$$P^n_{c,t} = Z^1_t C + Z^2_t C + \cdots + Z^n_t (1 + C) \tag{44}$$

$$= C \sum_{i=1}^{n} Z^i_t + Z^n_t \tag{45}$$

  where C denotes the per period coupon rate and Z^n_t is an n period discount factor that represents the present value of a future nominal dollar

- While it would be natural to extract discount factors directly from traded discount bonds P^n_t (Why?), this is typically not feasible for longer maturities

- As a results, we have to extract information about market zero-coupon bond prices for longer maturities from the prices of traded fixed income securities

  - Treasury bonds
  - Interest rate swaps (IRS)
  - Overnight index swap (OIS)

Bootstrapping

Iterative bootstrapping

Let t be a given date and suppose that there are N coupon bonds available with per period coupon rates C^n, maturities n, and prices $P^n_{c,t}$. If the coupon term structure is complete and regularly spaced, then the bootstrap methodology to estimating discount factors Z^n_t for every $i = 1, \ldots, n$ is as follows

1. The first discount factor Z^1_t is given by

$$Z^1_t = \frac{P^1_{c,t}}{1 + C^1}. \tag{46}$$

2. Any other discount factor Z^n_t for $i = 2, \ldots, n$ is given by

$$Z^n_t = \frac{P^n_{c,t} - C^n \sum_{i=1}^{n-1} Z^i_t}{1 + C^n} \tag{47}$$

Bootstrapping example data

* Consider the annualized coupon rates, maturities, and today prices of the following set of coupon-bonds maturing in regularly spaced intervals (semi-annually), all paying a face value $1

Coupon (Annualized)	Yield (Annualized)	Maturity (years)	Maturity (periods)	Price per face value
1.250%	0.149%	0.5	1	1.0055
4.875%	0.351%	1.0	2	1.0451
4.500%	0.574%	1.5	3	1.0586
4.750%	0.731%	2.0	4	1.0797
3.375%	0.992%	2.5	5	1.0587
3.500%	1.199%	3.0	6	1.0676
2.000%	1.543%	3.5	7	1.0155
2.250%	1.746%	4.0	8	1.0194
2.125%	1.931%	4.5	9	1.0083

Iterative procedure for obtaining discount factors

* The bootstrap methodology is a recursive approach in which we begin from the bond with the shortest maturity whose price is

$$P^1_{c,t} = Z^1_t \left(1 + C^1\right), \tag{48}$$

* Take the first bond, which matures in one period (six month), and plug its characteristics into the discount formula in (46), i.e.

$$Z^1_t = \frac{P^1_{c,t}}{1 + C^1} \tag{49}$$

$$= \frac{1.0055}{1 + 0.00625} = 0.99925 \tag{50}$$

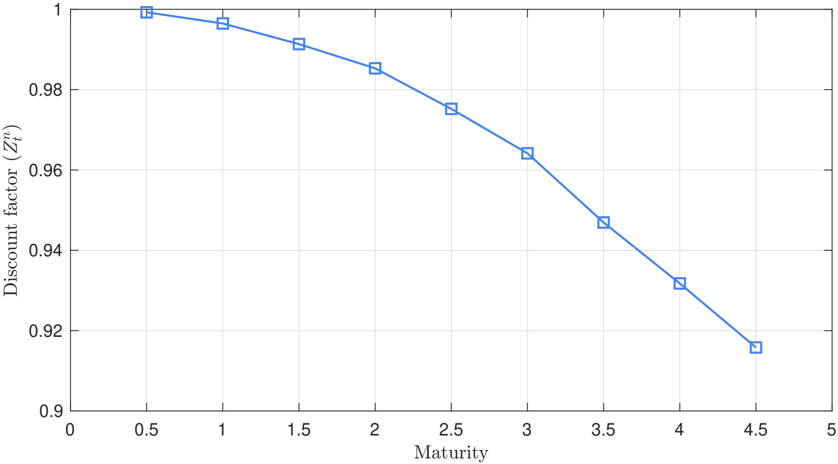
where we note that we use per period coupon rates, which with semi-annual payments are half the annualized coupon rates

Iterative procedure for obtaining discount factors

- We can then move along the maturities using the next maturing bond whose price is
$$P_{c,t}^{2} = Z_t^1 C^2 + Z_t^2 \left(1 + C^2\right) \tag{51}$$
- Using the iterative formula in (47) gives us the expression
$$Z_t^2 = \frac{P_{c,t}^{2} - Z_t^1 C^2}{1 + C^2} \tag{52}$$
which, when inserting the numbers, yields $Z_t^2 = 0.99648$ for the one-year discount factor
- And so on …
- This iterative procedure is known as bootstrapping or yield curve stripping. While simple, this approach can be quite cumbersome in practice

The bootstrapped discount curve

- Continuing in this iterative fashion for all available bonds results in the following zero-coupon discount curve



Putting bootstrap in matrix notation

- We can denote the $N \times 1$ vector of coupon bond prices and the $n \times 1$ vector of discount factors available at time t as

$$\mathbf{P} = \begin{bmatrix} P^1_{c,t} \\ P^2_{c,t} \\ \vdots \\ P^n_{c,t} \end{bmatrix}, \qquad \mathbf{Z} = \begin{bmatrix} Z^1_t \\ Z^2_t \\ \vdots \\ Z^n_t \end{bmatrix} \tag{53}$$

- As above, we can then write the price of all coupon bonds simultaneously as

$$\mathbf{P} = \mathbf{C} \times \mathbf{Z} \tag{54}$$

where the cash flow matrix is defined as

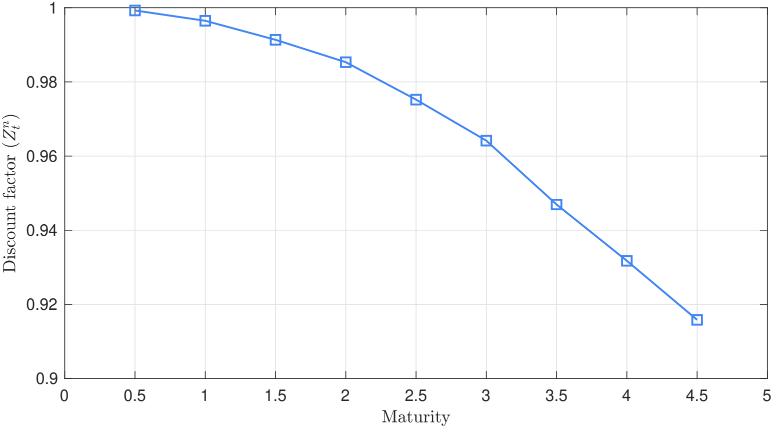
$$\mathbf{C} = \begin{bmatrix} CF^1_1 & CF^1_2 & \cdots & CF^1_n \\ CF^2_1 & CF^2_2 & \cdots & CF^2_n \\ \vdots & \vdots & \ddots & \vdots \\ CF^N_1 & CF^N_2 & \cdots & CF^N_n \end{bmatrix} \tag{55}$$

Obtaining the bootstrapped discount curve

- In the event that $\mathbf{C}$ is a square and invertible matrix ($n = N$), then we can find the unique solution for $\mathbf{Z}$ as follows

$$\mathbf{Z} = \mathbf{C}^{-1} \times \mathbf{P} \tag{56}$$

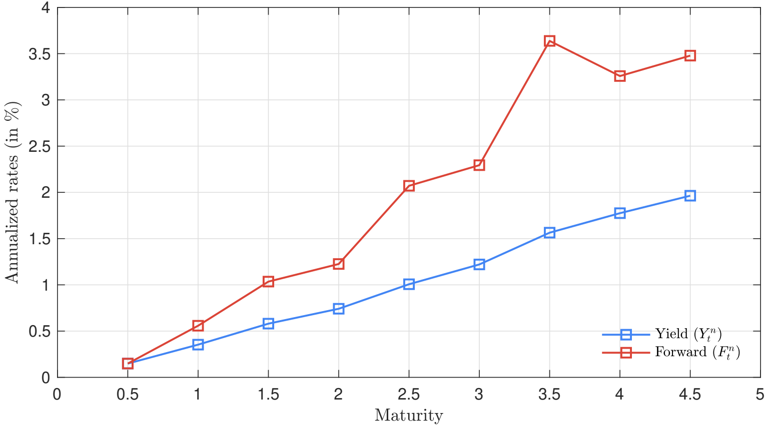
which is completely equivalent to the recursive procedure outlined earlier



Bootstrapped yield and forward rate curves

* Using the relations between interest rates and discount factors, we can obtain the bootstrapped per period yield and forward rate curves

$$Y_t^n = (Z_t^n)^{-\frac{1}{n}} - 1 \quad \text{and} \quad F_t^n = \frac{Z_t^{n-1}}{Z_t^n} - 1$$



Cross-sectional regressions

* While the bootstrap methodology is intuitively simple and straightforward to implement, it has two particular drawbacks for practical uses

  1. It requires that the coupon term structure is complete, i.e., that the number of bonds equal maturities $n = N$
  2. It requires bonds to be observable in regularly spaced maturities (e.g., six months)

* In practice, we are often faced with either a more-than-complete ($n < N$) or incomplete coupon term structure ($n > N$)
* The regression methodology is well-suited to deal with the case of a more-than-complete coupon term structure (say, for short-term curves)
* While one could revert to bootstrapping by throwing away bonds to reach the $n = N$ case, this also means throwing away information (not recommended)

Estimating the discount curve using a linear regression

Cross-sectional regression for the zero-coupon discout curve

The cross-sectional regression approach begins from the standard pricing equation in (54) and allows for a $N \times 1$ vector of small pricing errors ε (Why?) such that

$$\mathbf{P} = \mathbf{C} \times \mathbf{Z} + \varepsilon \tag{57}$$

- The above representation reminds us of the standard linear regression model in matrix form, which implies that we can obtain consistent estimates of $\mathbf{Z}$ using Ordinary Least Squares (OLS) as follows

$$\mathbf{Z} = \left(\mathbf{C}^\top \mathbf{C}\right)^{-1} \mathbf{C}^\top \times \mathbf{P} \tag{58}$$

- Importantly, for this procedure to work, we need a more-than-complete coupon term structure, otherwise we cannot solve the system
- If we are interested in discount factors for longer maturities, this might not be feasible and different approaches are warranted

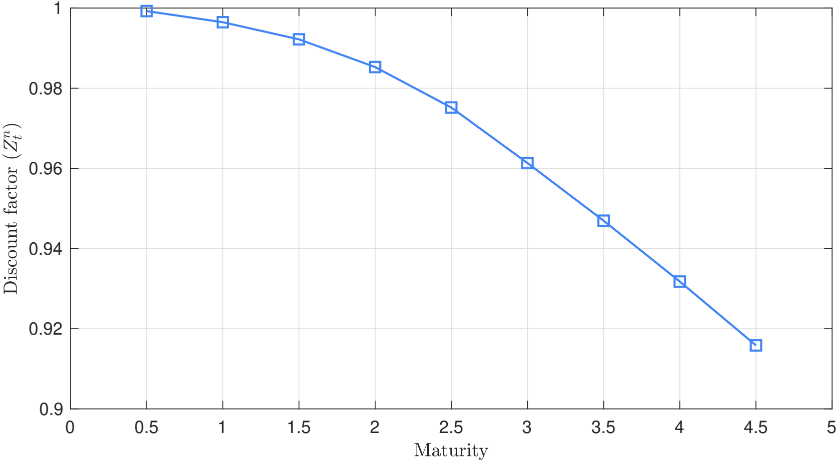
Data for cross-sectional regressions

* Consider the following bond data, where we now have access to two additional coupon bonds

Coupon (Annualized)	Yield (Annualized)	Maturity (years)	Maturity (periods)	Price per face value
1.250%	0.149%	0.5	1	1.0055
4.875%	0.351%	1.0	2	1.0451
4.500%	0.574%	1.5	3	1.0586
4.250%	0.461%	1.5	3	1.0566
4.750%	0.731%	2.0	4	1.0797
3.375%	0.992%	2.5	5	1.0587
3.500%	1.199%	3.0	6	1.0676
3.850%	1.383%	3.0	6	1.0723
2.000%	1.543%	3.5	7	1.0155
2.250%	1.746%	4.0	8	1.0194
2.125%	1.931%	4.5	9	1.0083

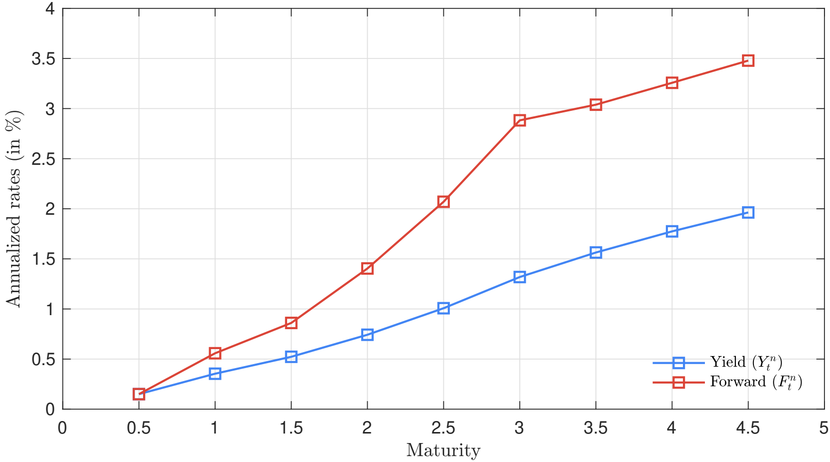
Regression-based discount curve

* The regression-based zero-coupon discount curve looks fine and slopes downwards as expected



Yield and forward rate curves based on regression

* Notice how the resulting yield and, in particular, the forward rate curve is now smoother, which is, in part, caused by allowing for small pricing errors



What is a parametric yield curve model?

* Nelson and Siegel (1987) present a parametric model for the zero-coupon term structure for the case of an incomplete coupon term structure

What do we mean by a parametric model?

In a parametric curve fitting model, one typically assumes a particular functional form for the discount factor

$$Z_t^n(\boldsymbol{\beta}) \tag{59}$$

as a function of periods to maturity n, where the $\boldsymbol{\beta}$ argument emphasizes the dependence on some parameter vector

* Rather than obtaining a full discount curve by interpolating between available data points, it may be better to consider a parametric model for the term structure of interest rates

* In this course, we will consider two particular functional forms for $Z_t^n(\boldsymbol{\beta})$, namely the Nelson and Siegel (1987) and Svensson (1994) models

The instantaneous forward rate

The instantaneous forward rate

Nelson and Siegel (1987) consider the following specification for the instantaneous log forward rate

$$f_t^n(0) = \theta_0 + \theta_1 e^{-\frac{n}{\lambda}} + \theta_2 (n/\lambda) e^{-\frac{n}{\lambda}} \tag{60}$$

which can be viewed as a constant plus a Laguerre function (a polynomial times an exponential decay term)

* This specification addresses a series of previous concerns in modeling approaches such as splines (McCulloch, 1971, 1975) and exponential splines (Vasicek and Fong, 1982) as highlighted by Shea (1984, 1985)
* Most notably, it ensures a positive forward rate that does not diverge, but converges to a fixed value when $n \to \infty$

The Nelson-Siegel (NS) model

Nelson-Siegel (NS) model

In the Nelson-Siegel framework, the discount factor takes the form

$$Z_t^n = e^{-ny_t^n}, \tag{61}$$

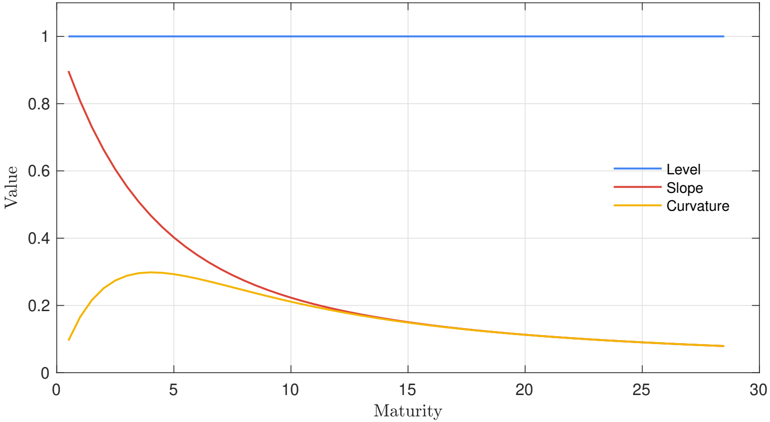
where the continuously compounded (log) zero-coupon yield is given by

$$y_t^n = \theta_0 + \theta_1 \frac{1 - e^{-\frac{n}{\lambda}}}{\frac{n}{\lambda}} + \theta_2 \left[\frac{1 - e^{-\frac{n}{\lambda}}}{\frac{n}{\lambda}} - e^{-\frac{n}{\lambda}} \right], \tag{62}$$

- The parameters of the Nelson-Siegel model can be interpreted as follows
  - θ_0 is interpreted as the long-run level of interest rates as $n \to \infty$
  - θ_1 governs the short-term component
  - θ_2 governs the medium-term component
  - λ is scale parameter that governs the decay rates and when the curvature component reaches its maximum

The Nelson-Siegel components

- As shown below, we can give the components of the model an economic interpretation as well
- This is important as it allows the model-user to make informed assessments of the particular shape and parameters of the fitted curve



Estimating the Nelson-Siegel parameters

1. Choose a set of sensible starting values for $\boldsymbol{\beta} = (\theta_0, \theta_1, \theta_2, \lambda)$
2. Compute model-implied yields and discount factors using (61) and (62)
3. Compute model-implied bond prices using (54), i.e.,

$$\mathbf{P}^{\mathsf{NS}} = \mathbf{C} \times \mathbf{Z}^{\mathsf{NS}} \tag{63}$$

   where we add the superscript NS to emphasize the dependence on the model

4. Estimate model parameters using NLS by minimizing the sum of squared errors between the model-implied prices and market prices

$$\boldsymbol{\beta}^{\star} = \arg\min_{\boldsymbol{\beta}} \sum_{i=1}^{N} \left[\omega_i \left(P_c^{i,\mathsf{Market}} - P_c^{i,\mathsf{NS}}\right)\right]^2 \tag{64}$$

   using some optimizer (e.g., *fminsearch*) by varying the parameter vector $\boldsymbol{\beta}$ while, potentially, imposing sensible restrictions such as $\theta_0, \lambda \geq 0$ (Why?)

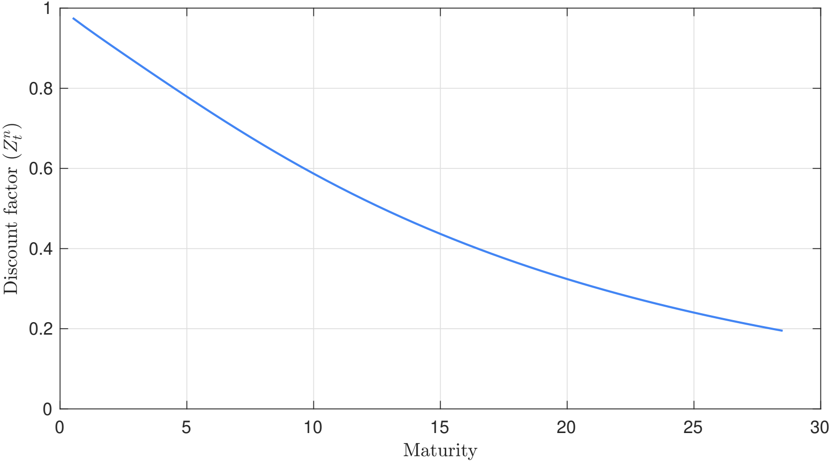
Data for NS model estimation

* Consider the following incomplete coupon term structure data for estimating the parameters of the Nelson-Siegel model

Coupon (Annualized)	Yield (Annualized)	Maturity (years)	Maturity (periods)	Price per face value
5.500%	5.00%	0.5	1	1.0024
5.875%	4.95%	1.0	2	1.0089
5.625%	4.88%	1.5	3	1.0107
5.250%	4.89%	2.5	5	1.0084
5.875%	4.98%	3.5	7	1.0284
6.500%	5.10%	5.5	11	1.0664
5.500%	5.15%	7.0	14	1.0204
6.500%	5.25%	9.0	18	1.0887
8.250%	5.48%	14.0	28	1.2683
6.125%	5.65%	28.5	57	1.0669

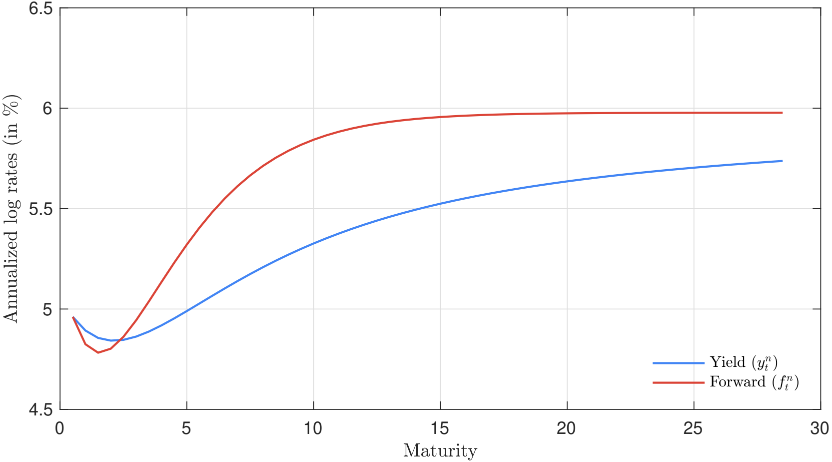
Nelson-Siegel discount curve

* The Nelson-Siegel zero-coupon discount curve looks fine and slopes downwards as expected



Nelson-Siegel yield and forward rate curves

* Using the above data, we obtain the below yield and forward rate curves



The Nelson-Siegel-Svensson (NSS) model

* Although the Nelson and Siegel (1987) model usually works quite well, it lacks the flexibility to match term structures that are *highly* nonlinear
* The reason being that the Nelson and Siegel (1987) model can only accommodate one "hump" in the forward rate curve
* At times, the curve is so nonlinear that a single "hump" is not enough, which may happen because the curve needs two "humps" (Gürkaynak et al., 2007)
  1. One at short maturities associated with monetary policy expectations
  2. another at long maturities to capture convexity effects

The NSS instantaneous forward rate

Svensson (1994) extends the Nelson and Siegel (1987) model by adding an additional term so that the instantaneous log forward rate becomes

$$f_t^n(0) = \theta_0 + \theta_1 e^{-\frac{n}{\lambda_1}} + \theta_2 (n/\lambda_1) e^{-\frac{n}{\lambda_1}} + \theta_3 (n/\lambda_2) e^{-\frac{n}{\lambda_2}} \tag{65}$$

The Nelson-Siegel-Svensson (NSS) model

The Nelson-Siegel-Svensson (NSS) model

In the Nelson-Siegel-Svensson framework, the discount factor takes the form

$$Z_t^n = e^{-ny_t^n}, \tag{66}$$

where the continuously compounded (log) zero-coupon yield is given by

$$y_t^n = \theta_0 + \theta_1 \frac{1 - e^{-\frac{n}{\lambda_1}}}{\frac{n}{\lambda_1}} + \theta_2 \left[\frac{1 - e^{-\frac{n}{\lambda_1}}}{\frac{n}{\lambda_1}} - e^{-\frac{n}{\lambda_1}} \right] + \theta_3 \left[\frac{1 - e^{-\frac{n}{\lambda_2}}}{\frac{n}{\lambda_2}} - e^{-\frac{n}{\lambda_2}} \right] \tag{67}$$

- The new parameters θ_3 and λ_2 are interpreted similarly to the parameters θ_2 and λ above
- The estimation of the model parameters is identical to the Nelson-Siegel case, except that the parameter vector now consists of 6 variables, i.e., the vector is now $\boldsymbol{\beta} = (\theta_0, \theta_1, \theta_2, \theta_3, \lambda_1, \lambda_2)$, instead of four

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