CHAPTER 11

Arrow–Debreu Pricing, Part II

Chapter Outline

11.1 Introduction 325
11.2 Market Completeness and Complex Securities 326
11.3 Constructing State-Contingent Claims Prices in a Risk-Free World: Deriving the Term Structure 330
11.4 The Value Additivity Theorem 335
11.5 Using Options to Complete the Market: An Abstract Setting 337
11.6 Synthesizing State-Contingent Claims: A First Approximation 343
11.7 Recovering Arrow–Debreu Prices from Options Prices: A Generalization 345
11.8 Arrow–Debreu Pricing in a Multiperiod Setting 352
11.9 Conclusions 357
References 358
Appendix 11.1: Forward Prices and Forward Rates 358

11.1 Introduction

Chapter 9 presented the Arrow–Debreu asset pricing theory from the equilibrium perspective. With the help of a number of modeling hypotheses and building on the concept of market equilibrium, we showed that the price of a future contingent dollar can appropriately be viewed as the product of three main components: a pure time discount factor, the probability of the relevant state of nature, and an intertemporal marginal rate of substitution reflecting the collective (market) assessment of the scarcity of consumption in the future relative to today. This important message is one that we confirmed with the CCAPM of Chapter 10. Here, however, we adopt the alternative arbitrage perspective and revisit the same Arrow–Debreu pricing theory. Doing so is productive precisely because, as we have stressed before, the design of an Arrow–Debreu security is such that once its price is available, whatever its origin and makeup, it provides the answer to the key valuation question: what is a unit of the future state-contingent numeraire worth today? As a result, it constitutes the essential piece of information necessary to price arbitrary cash flows. Even if the equilibrium theory of Chapter 9 were all wrong, in the sense that the hypotheses made there turn out to be a very poor description of reality and that, as a consequence, the prices of Arrow–Debreu securities are not well described by Eq. (9.1), it remains true that if such securities are traded, their prices constitute the essential building blocks (in the sense of our Chapter 2 bicycle pricing analogy) for valuing any arbitrary risky cash flow.

Section 11.2 develops this message and goes further, arguing that the detour via Arrow–Debreu securities is useful even if no such security is actually traded. In making this argument, we extend the definition of the complete market concept. Section 11.3 illustrates the approach in the abstract context of a risk-free world where we argue that any *risk-free* cash flow can be easily and straightforwardly priced as an equivalent portfolio of *date-contingent* claims. These latter instruments are, in effect, discount bonds of various maturities.

Our main interest, of course, is to extend this approach to the evaluation of *risky* cash flows. To do so requires, by analogy, that for each future date-state the corresponding contingent cash flow be priced. This, in turn, requires that we know, for each future *date-state*, the price today of a security that pays off in that date-state and only in that date-state. This latter statement is equivalent to the assumption of market completeness.

In this chapter, we take on the issue of completeness in the context of securities known as options. Our goal is twofold. First, we want to give the reader an opportunity to review an important element of financial theory—the theory of options. A special appendix to this chapter, available on this text's website, describes the essentials for the reader in need of a refresher. Second, we want to provide a concrete illustration of the view that the recent expansion of derivative markets constitutes a major step in the quest for the "Holy Grail" of achieving a complete securities market structure. We will see, indeed, that options can, in principle, be used relatively straightforwardly to complete the markets. Furthermore, even in situations where this is not practicable, we can use option pricing theory to value risky cash flows in a manner as though the financial markets were complete. Our discussion will follow the outline suggested by the following two questions:

1. How can options be used to complete the financial markets? We will first answer this question in a simple, highly abstract setting. Our discussion closely follows Ross (1976).
2. What is the link between the prices of market-quoted options and the prices of Arrow–Debreu securities? We will see that it is indeed possible to infer Arrow–Debreu prices from option prices in a practical setting conducive to the valuation of an actual cash-flow stream. Here our discussion follows Banz and Miller (1978) and Breeden and Litzenberger (1978).

11.2 Market Completeness and Complex Securities

In this section we pursue, more systematically, the important issue of market completeness first addressed in Chapter 1 when we discussed the optimality property of a general competitive equilibrium. Let us start with two definitions.

1. **Completeness.** Financial markets are said to be *complete* if, for each state of nature θ, there exists a market for contingent claim or Arrow–Debreu security θ—in other words, for a claim promising delivery of one unit of the consumption good (or, more generally, the numeraire) if state θ is realized, and nothing otherwise. Note that this definition takes a form specifically appropriate to models where there is only one consumption good and several date-states. This is the usual context in which financial issues are addressed.
2. **Complex security.** A complex security is one that pays off in more than one state of nature.

Suppose the number of states of nature $N = 4$; an example of a complex security is $S = (5,2,0,6)$ with payoffs 5,2,0, and 6, respectively, in states of nature 1,2,3, and 4. If markets are complete, we can immediately price such a security since

$$(5, 2, 0, 6) = 5(1, 0, 0, 0) + 2(0, 1, 0, 0) + 0(0, 0, 1, 0) + 6(0, 0, 0, 1),$$

in other words, since the complex security can be replicated by a portfolio of Arrow–Debreu securities, the price of security S, q_S, must be

$$q_S = 5q_1 + 2q_2 + 6q_4.$$

We are appealing here to the law of one price or, equivalently, to a condition of no arbitrage.[1] This is the first instance of our using the second main approach to asset pricing, the arbitrage approach, which is our exclusive focus in Chapters 11–13. We are pricing the complex security on the basis of our knowledge of the prices of its components. The relevance of the Arrow–Debreu pricing theory resides in the fact that it provides the prices for what can be argued are the essential components of any asset or cash flow.

Effectively, the argument can be stated in the following proposition.

Proposition 11.1 If markets are complete, any complex security or any cash-flow stream can be replicated as a portfolio of Arrow–Debreu securities.

If markets are complete in the sense that prices exist for all the relevant Arrow–Debreu securities, then the "no arbitrage" condition implies that any complex security or cash flow can also be priced using Arrow–Debreu prices as fundamental elements. The portfolio, which is easily priced using the (Arrow–Debreu) prices of its individual components, is essentially the same good as the cash flow or the security it replicates: it pays the same amount of the consumption good in each and every state. Therefore, it should bear the same

[1] This is stating that the equilibrium prices of two separate units of what is essentially the same good should be identical. If this were not the case, a riskless and costless arbitrage opportunity would open up: buy extremely large amounts at the low price and sell them at the high price, forcing the two prices to converge. When applied across two different geographical locations (which is not the case here: our world is a point in space), the law of one price may not hold because of transport costs rendering the arbitrage costly.

price. This is a key result underlying much of what we do in the remainder of this chapter and our interest in Arrow–Debreu pricing.

If this equivalence is not observed, an arbitrage opportunity—the ability to make unlimited profits with no initial investment—will exist. By taking positions to benefit from the arbitrage opportunity, however, investors will expeditiously eliminate it, thereby forcing the price relationships implicitly asserted in Proposition 11.1. To illustrate how this would work, let us consider the prior example and postulate the following set of prices: $q_1 = \$0.86$, $q_2 = \$0.94$, $q_3 = \$0.93$, $q_4 = \$0.90$, and $q_{(5,2,0,6)} = \$9.80$. At these prices, the law of one price fails, since the price of the portfolio of state claims that exactly replicates the payoff to the complex security does not coincide with the complex's security's price:

$$q_{(5,2,0,6)} = \$9.80 < \$11.58 = 5q_1 + 2q_2 + 6q_4.$$

We see that the complex security is relatively undervalued vis-à-vis the state claim prices. This suggests acquiring a positive amount of the complex security while selling (short) the replicating portfolio of state claims. Table 11.1 illustrates a possible combination.

So the arbitrageur walks away with \$1.78 while (1) having made no investment of her own wealth and (2) without incurring any future obligation (perfectly hedged). She will thus replicate this portfolio as much as she can. But the added demand for the complex security will, *ceteris paribus*, tend to increase its price while the short sales of the state claims will depress their prices. This will continue (the arbitrage opportunity will persist) as long as the pricing relationships are not in perfect alignment.

Suppose now that only complex securities are traded and that there are M of them and N states. The following is true.

Proposition 11.2 If $M = N$, and all the M complex securities are linearly independent, then (i) it is possible to infer the prices of the Arrow–Debreu state-contingent claims from the complex securities' prices and (ii) markets are effectively complete.[2]

Table 11.1: An arbitrage portfolio

$t = 0$		$t = 1$ Payoffs			
Security	**Cost**	θ_1	θ_2	θ_3	θ_4
Buy 1 complex security	−\$9.80	5	2	0	6
Sell short 5 (1,0,0,0) securities	\$4.30	−5	0	0	0
Sell short 2 (0,1,0,0) securities	\$1.88	0	−2	0	0
Sell short 6 (0,0,0,1) securities	\$5.40	0	0	0	−6
Net	\$1.78	0	0	0	0

[2] When we use the language "linearly dependent," we are implicitly regarding securities as N-vectors of payoffs.

The hypothesis of linear independence can be interpreted as a requirement that there exist N truly different securities for completeness to be achieved. Thus, it is easy to understand that if among the N complex securities available, one security, A, pays (1, 2, 3) in the three relevant states of nature, and the other, B, pays (2, 4, 6), only $N-1$ truly distinct securities are available: B does not permit any different redistribution of purchasing power across states than A permits. More generally, the linear independence hypothesis requires that no one complex security can be replicated as a portfolio of some of the other complex securities. The reader will remember that we made the same hypothesis at the beginning of Section 8.4.

Suppose the following securities are traded:

$$(3,2,0) \quad (1,1,1) \quad (2,0,2)$$

at equilibrium prices \$1.00, \$0.60, and \$0.80, respectively. It is easy to verify that these three securities are linearly independent. We can then construct the Arrow–Debreu prices as follows. Consider, for example, the security (1,0,0):

$$\begin{aligned} &(1,0,0) = w_1(3,2,0) + w_2(1,1,1) + w_3(2,0,2) \\ &\text{Thus, } \ 1 = 3w_1 + w_2 + 2w_3 \\ &\qquad\quad\ \ 0 = 2w_1 + w_2 \\ &\qquad\quad\ \ 0 = w_2 + 2w_3 \end{aligned}$$

Solve: $w_1 = \frac{1}{3}, \ \ w_2 = -\frac{2}{3}, \ \ w_3 = \frac{1}{3}$, and $q_{(1,0,0)} = \frac{1}{3}(1.00) + \left(-\frac{2}{3}\right)(0.60) + \frac{1}{3}(0.80) = 0.1966$

Similarly, we could replicate (0,1,0) and (0,0,1) with portfolios ($w_1 = 0, \ \ w_2 = 1,$ $w_3 = -\frac{1}{2}$) and $\left(w_1 = \ -\frac{1}{3}, \ \ w_2 = \frac{2}{3}, \ \ w_3 = \frac{1}{6}\right)$, respectively, and price them accordingly.

Expressed in a more general way, the reasoning just completed amounts to searching for a solution of the following system of equations:

$$\begin{pmatrix} \mathbf{3} & \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} \end{pmatrix} \begin{pmatrix} w_1^1 & w_1^2 & w_1^3 \\ w_2^1 & w_2^2 & w_2^3 \\ w_3^1 & w_3^2 & w_3^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Of course, this system has solution $\begin{pmatrix} \mathbf{3} & \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ only if the matrix of security payoffs can be inverted, which requires that it be of full rank, or that its determinant be nonzero, or that all its lines or columns be linearly independent.

Now suppose the number of linearly independent securities is strictly less than the number of states. In this case the securities markets are fundamentally incomplete: there may be some assets that cannot be unambiguously priced (see Chapter 12 upcoming). Furthermore, risk-sharing opportunities are less than if the securities markets were complete and, in general, social welfare is lower than what it would be under complete markets: some gains from exchange cannot be exploited due to the lack of instruments permitting these exchanges to take place.

We conclude this section by revisiting the project valuation problem. In the light of the Arrow–Debreu pricing approach, how should we value an uncertain cash-flow stream such as:

$$
\begin{array}{ccccccc}
t= & 0 & 1 & 2 & 3 & \dots & T\,? \\
 & -I_0 & \tilde{C}F_1 & \tilde{C}F_2 & \tilde{C}F_3 & \dots & \tilde{C}F_T
\end{array}
$$

This cash-flow stream is akin to a complex security since it pays in multiple states of the world. Let us specifically assume that there are N states at each date t, $t = 1,\dots,T$ and let us denote $q_{t,\theta}$ the price of the Arrow–Debreu security promising delivery of one unit of the numeraire if state θ is realized at date t. Similarly, let us identify as $CF_{t,\theta}$ the cash flow associated with the project in the same occurrence. Then pricing the complex security in the manner of Arrow–Debreu pricing means valuing the project as in Eq. (11.1).

$$NPV = -I_0 + \sum_{t=1}^{T}\sum_{\theta=1}^{N} q_{t,\theta} CF_{t,\theta}. \tag{11.1}$$

Although this is a demanding procedure, it is a pricing approach that is fully general and involves no approximation. For this reason it constitutes an extremely useful benchmark.

In a risk-free setting, the concept of the state-contingent claim has a very familiar real-world counterpart. In fact, the notion of the term structure is simply a reflection of "date-contingent" claims prices. We pursue this idea in the next section.

11.3 Constructing State-Contingent Claims Prices in a Risk-Free World: Deriving the Term Structure

Suppose we are considering risk-free investments and risk-free securities exclusively. In this setting—where we ignore risk—the "states of nature" about which we have been speaking correspond to *future time periods.* This section shows that the process of computing the term structure from the prices of coupon bonds is akin to recovering Arrow–Debreu prices from the prices of complex securities.

Under this interpretation, the Arrow–Debreu state-contingent claims correspond to risk-free discount bonds of various maturities, as seen in Table 11.2.

These are Arrow–Debreu securities because they pay off in one state (the period of maturity) and zero in all other time periods (states).

Table 11.2: Risk-free discount bonds as Arrow—Debreu securities

Current Bond Price	Future Cash Flows					
$t=0$	1	2	3	4	...	T
$-q_1$	\$1000					
$-q_2$		\$1000				
...						
$-q_T$						\$1000
The cash flow of a "j-period discount bond" is given by:						
$t=0$	1	...	j	$j+1$	...	T
$-q_j$	0	0	\$1000	0	0	0

Table 11.3: Present and future cash flows for two coupon bonds

Bond Type	Cash Flow at Time t					
	$t=0$	1	2	3	4	5
$7\frac{7}{8}$% bond:	− 1097.8125	78.75	78.75	78.75	78.75	1078.75
$5\frac{5}{8}$% bond:	− 1002.8125	56.25	56.25	56.25	56.25	1056.25

In the United States at least, securities of this type are not issued for maturities longer than 1 year. Rather, only interest-bearing or coupon bonds are issued for longer maturities. These are complex securities by our definition: they pay off in many states of nature. But we know that if we have enough distinct complex securities we can compute the prices of the Arrow—Debreu securities even if they are not explicitly traded. So we can also compute the prices of these zero coupon or discount bonds from the prices of the coupon or interest-bearing bonds, assuming no arbitrage opportunities in the bond market.

For example, suppose we wanted to price a 5-year discount bond coming due in November 2019 (we view $t=0$ as November 2014), and we observe two coupon bonds being traded that mature at the same time:

i. $7\frac{7}{8}$ % bond priced at $109\frac{25}{32}$ or \$1097.8125/\$1000 of face value
ii. $5\frac{5}{8}$ % bond priced at $100\frac{9}{32}$ or \$1002.8125/\$1000 of face value

The coupons of these bonds are, respectively,

$$0.07875 \times \$1000 = \$78.75/\text{year}$$
$$0.05625 \times \$1000 = \$56.25/\text{year}^3$$

and their cash flows are shown in Table 11.3.

[3] In fact, interest is paid every 6 months on this sort of bond, a refinement that would double the number of periods without altering the argument in any way. Actual default free rates are of course much lower in 2014 than these numbers suggest.

Table 11.4: Eliminating intermediate payments

Bond	Cash Flow at Time t					
	$t=0$	1	2	3	4	5
$-1\times 7\frac{7}{8}\%$: $+1.4\times 5\frac{5}{8}\%$:	+1097.8125 −1403.9375	−78.75 78.75	−78.75 78.75	−78.75 78.75	−78.75 78.75	−1078.75 1478.75
Difference:	−306.125	0	0	0	0	400.00

Note that we want somehow to eliminate the interest payments (to create a discount bond). Notive that $78.75/56.25 = 1.4$, and consider the following strategy: sell one $7\frac{7}{8}\%$ bond while simultaneously buying 1.4 unit of $5\frac{5}{8}\%$ bonds. The corresponding cash flows are found in Table 11.4.

The net cash flow associated with this strategy thus indicates that the $t = 0$ price of a \$400 payment in 5 years is \$306.25. This price is implicit in the pricing of our two original coupon bonds. Consequently, the price of \$1000 in 5 years must be

$$\$306.125 \times \frac{1000}{400} = \$765.3125$$

Alternatively, the price today of \$1.00 in 5 years is \$0.7653125. In the notation of our earlier discussion we have the following securities:

$$\begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{array} \begin{bmatrix} 78.75 \\ 78.75 \\ 78.75 \\ 78.75 \\ 1078.75 \end{bmatrix} \text{ and } \begin{bmatrix} 56.25 \\ 56.25 \\ 56.25 \\ 56.25 \\ 1056.25 \end{bmatrix} \text{ and we consider}$$

$$-\frac{1}{400}\begin{bmatrix} 78.75 \\ 78.75 \\ 78.75 \\ 78.75 \\ 1078.75 \end{bmatrix} + \frac{1.4}{400}\begin{bmatrix} 56.25 \\ 56.25 \\ 56.25 \\ 56.25 \\ 1056.25 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This is an Arrow–Debreu security in the riskless context we are considering in this section.

If there are enough coupon bonds with different maturities with pairs coming due at the same time and with different coupons, we can thus construct a complete set of Arrow–Debreu securities and their implicit prices. Notice that the payoff patterns of the two bonds are fundamentally different: They are linearly independent of one another. This is a requirement, as per our earlier discussion, for being able to use them to construct a fundamentally new payoff pattern, in this case, the discount bond.

Implicit in every discount bond price is a well-defined rate of return notion. In the case of the prior illustration, for example, the implied 5-year compound risk-free rate is given by

$$\$765.3125(1+r_5)^5 = \$1000, \text{ or}$$
$$r_5 = 0.0549$$

This observation suggests an intimate relationship between discounting and Arrow–Debreu date pricing. Just as a full set of date claims prices should allow us to price any risk-free cash flow, the rates of return implicit in the Arrow–Debreu prices must allow us to obtain the same price by discounting at the equivalent family of rates. This family of rates is referred to as *term structure of interest rates.*

Definition 11.1 The term structure of interest rates r_1,r_2, . . . is the family of interest rates corresponding to risk-free discount bonds of successively greater maturity; that is, r_i is the rate of return on a risk-free discount bond maturing i periods from the present.

We can systematically recover the term structure from coupon bond prices provided we know the prices of coupon bonds of all different maturities. To illustrate, suppose we observe risk-free government bonds of 1,2,3,4-year maturities all selling at par with coupons, respectively, of 6%, 6.5%, 7.2%, and 9.5%. We can construct the term structure as follows:[4]

r_1: Since the 1-year bond sells at par, we have $r_1 = 6\%$;
r_2: By definition, we know that the 2-year bond is priced such that $1000 = 65/(1 + r_1) + 1065/(1+r_2)^2$ which, given that $r_1 = 6\%$, solves for $r_2 = 6.5113\%$.
r_3: This is derived accordingly as the solution to

$$1000 = \frac{75}{(1 + r_1)} + \frac{72}{(1+r_2)^2} + \frac{1072}{(1+r_3)^3}$$

With $r_1 = 6\%$ and $r_2 = 6.5113\%$, the solution is $r_3 = 7.2644\%$. Finally, given these values for r_1 to r_3, r_4 solves:

$$1000 = \frac{95}{(1 + r_1)} + \frac{95}{(1+r_2)^2} + \frac{95}{(1+r_3)^3} + \frac{1072}{(1+r_4)^4}, \text{ that is, } r_4 = 9.935\%.$$

[4] A bond selling at par is selling at its face value, typically $1000.

Note that these rates are the counterpart to the date-contingent claim prices (see Table 11.5).

Once we have the discount bond prices (the prices of the Arrow–Debreu claims) we can then price all other risk-free securities; for example, suppose we wished to price a 4-year 8% bond:

$$\begin{array}{ccccc} t=0 & 1 & 2 & 3 & 4 \\ -q_0^{8\%\ \text{bond}}(?) & 80 & 80 & 80 & 1080 \end{array}$$

and suppose also that we have available the discount bonds described in Tables 11.5 and 11.6.

Then the portfolio of discount bonds (Arrow–Debreu claims) which replicates the 8% bond cash flow is (Table 11.7):

$$\{0.08 \times \text{1-yr bond},\ 0.08 \times \text{2-yr bond},\ 0.08 \times \text{3-yr bond},\ 1.08 \times \text{4-yr bond}\}.$$

Table 11.5: Date claim prices versus discount bond prices

	Price of an N Year Claim	Corresponding Discount Bond Price ($1000 Denomination)
$N=1$	$q_1 = \$1/1.06 = \$0.94339$	$943.39
$N=2$	$q_2 = \$1/(1.065113)^2 = \$0.88147$	$881.47
$N=3$	$q_3 = \$1/(1.072644)^3 = \$0.81027$	$810.27
$N=4$	$q_4 = \$1/(1.09935)^4 = \$0.68463$	$684.63

Table 11.6: Discount bonds as Arrow–Debreu claims

Bond	Price ($t=0$)	CF Pattern			
		$t=1$	2	3	4
1-yr discount	−$943.39	$1000			
2-yr discount	−$881.47		$1000		
3-yr discount	−$810.27			$1000	
4-yr discount	−$684.63				$1000

Table 11.7: Replicating the discount bond cash flow

Bond	Price ($t=0$)	CF Pattern			
		$t=1$	2	3	4
08 1-yr discount	(0.08)(− 943.39) = −$75.47	$80 *(80 date 1 A–D claims)*			
08 2-yr discount	(0.08)(− 881.47) = −$70.52		$80 *(80 date 2 A–D claims)*		
08 3-yr discount	(0.08)(− 810.27) = −$64.82			$80	
1.08 4-yr discount	(1.08)(− 684.63) = −$739.40				$1080

Thus:

$$q_0^{8\%\ \text{bond}} = 0.08(\$943.39) + 0.08(\$881.47) + 0.08(\$810.27) + 1.08(\$684.63) = \$950.21.$$

Notice that we are emphasizing, in effect, the equivalence of the term structure of interest rates with the prices of date-contingent claims. Each defines the other. This is especially apparent in Table 11.5.

Let us now extend the above discussion to consider the evaluation of arbitrary risk-free cash flows: any such cash flow can be evaluated as a portfolio of Arrow–Debreu securities.

For example:

$t = 0$	1	2	3	4
	60	25	150	300

We want to price this cash flow today ($t = 0$) using the Arrow–Debreu prices we have calculated in Table 11.5.

$$\begin{aligned} q_0 &= (\$60 \text{ at } t = 1)\left(\frac{\$0.94339 \text{ at } t = 0}{\$1 \text{ at } t = 1}\right) + (\$25 \text{ at } t = 2)\left(\frac{\$0.88147 \text{ at } t = 0}{\$1 \text{ at } t = 2}\right) + \cdots \\ &= (\$60)\frac{1.00}{1 + \mathrm{r}_1} + (\$25)\frac{1.00}{(1 + \mathrm{r}_2)} + \cdots \\ &= (\$60)\frac{1.00}{1.06} + (\$25)\frac{1.00}{(1.065113)^2} + \cdots \end{aligned}$$

The second equality underlines the fact that *evaluating risk-free projects as portfolios of Arrow–Debreu state-contingent securities is equivalent to discounting at the term structure:*

$$= \frac{60}{(1 + r_1)} + \frac{25}{(1 + r_2)} + \frac{150}{(1 + r_3)} + \cdots, \text{ etc.}$$

In effect, we treat a risk-free project as a risk-free coupon bond with (potentially) differing coupons. There is an analogous notion of forward prices and its more familiar counterpart, the forward rate. We discuss this extension in Appendix 11.1.

11.4 The Value Additivity Theorem

In this section, we present an important result illustrating the power of the Arrow–Debreu pricing apparatus to generate one of the main lessons of the CAPM. Let there be two assets (complex securities) a and b with corresponding date 1 payoffs $\tilde{z}_a$ and $\tilde{z}_b$, respectively, and equilibrium (t = 0) prices q_a and q_b. Suppose a third asset, c, turns out to be a linear combination of a and b. By that we mean that the payoff to c can be replicated by a portfolio of a and b. One can thus write

$$\tilde{z}_c = A\tilde{z}_a + B\tilde{z}_b, \text{for some constant coefficients } A \text{ and } B \tag{11.2}$$

Then the proposition known as the **Value Additivity Theorem** asserts that the same linear relationship must hold for the date 0 prices of the three assets:

$$q_c = Aq_a + Bq_b$$

Let us first prove this result and then discuss its implications. The proof easily follows from our discussion in Section 11.2 on the pricing of complex securities in a complete market Arrow–Debreu world. Indeed, for our two securities a, b, one must have:

$$q_i = \sum_s q_s z_{si}, \quad i = a,\ b \tag{11.3}$$

where q_s is the price of an Arrow–Debreu security that pays one unit of consumption in state s (and zero otherwise) and z_{si} is the payoff of asset i in state s

But then, the pricing of c must respect the following relationships:

$$q_c = \sum_s q_s z_{sc} = \sum_s q_s(Az_{sa} + Bz_{sb}) = \sum_s (Aq_s z_{sa} + Bq_s z_{sb}) = Aq_a + Bq_b$$

The first equality follows from the fact that c is itself a complex security and can thus be priced using Arrow–Debreu prices (i.e., an equation such as Eq. (11.3) applies); the second directly follows from Eq. (11.2); the third is a pure algebraic expansion that is feasible because our pricing relationships are fundamentally linear; the fourth again follows from Eq. (11.3).

Now this is easy enough, but why is it interesting? Think of a and b as being two stocks with negatively correlated returns; we know that c, a portfolio of these two stocks, is much less risky than either one of them. But q_c is a linear combination of q_a and q_b. Thus, the fact that they can be combined in a less risky portfolio has implications for the pricing of the two independently riskier securities and their equilibrium returns. Specifically, it cannot be the case that q_c would be *high* because it corresponds to a desirable, riskless, claim while the q_a and q_b would be *low* because they are risky.

To see this more clearly, let us take an extreme example. Suppose that a and b are *perfectly* negatively correlated. For an appropriate choice of A and B, say A^* and B^*, the resulting portfolio, call it d, will have zero risk; that is, it will pay a constant amount in each and every state of nature. What should the price of this riskless portfolio be? Intuitively, its price must be such that purchasing d units at q_d will earn the riskless rate of return.

But how could the risk of a and b be remunerated while, simultaneously, d would earn the riskless rate and the Value Additivity Theorem would hold? The answer is that this is not possible. Therefore, there cannot be any remuneration for risk in the pricing of a and b. The prices q_a and q_b must be such that the expected return on a and b is the riskless rate. This is true despite the fact that a and b are two risky assets (they do not pay the same amount in each state of nature).

In formal terms, we have just asserted that the two terms of the Value Additivity Theorem $\tilde{z}_d = A^* \tilde{z}_a + B^* \tilde{z}_b$ and $q_d = A^* q_a + B^* q_b$, together with the fact that d is risk free,

$$\frac{E\tilde{z}_d}{q_d} = 1 + r_f, \quad \text{force}$$
$$\frac{E\tilde{z}_a}{q_a} = \frac{E\tilde{z}_b}{q_b} = 1 + r_f.$$

What we have obtained in this very general context is a confirmation of one of the main results of the CAPM: diversifiable risk is not priced. If risky assets a and b can be combined in a riskless portfolio, that is, if their risk can be diversified away, their return cannot exceed the risk-free return. Note that we have made no assumption here on utility functions or on the return expectations held by agents. On the other hand, we have explicitly assumed that markets are complete and that, consequently, each and every complex security can be priced (by arbitrage) as a portfolio of Arrow–Debreu securities.

It thus behooves us to describe how Arrow–Debreu state claim prices might actually be obtained in practice. This is the subject of the remaining sections of Chapter 11.

11.5 Using Options to Complete the Market: An Abstract Setting

Let us assume a finite number of possible future date-states indexed $i = 1,2,\ldots, N$. Suppose, for a start, that three states of the world are possible in date $T = 1$, yet only one security (a stock) is traded. The single security's payoffs are as follows:

State	Payoff
θ_1	1
θ_2	2
θ_3	3

Clearly, this unique asset is not equivalent to a complete set of state-contingent claims. Note that we can identify the payoffs with the ex-post price of the security in each of the three states: the security pays two units of the numeraire commodity in state 2, and we decide that its price then is \$2. This amounts to normalizing the ex-post, date 1, price of the commodity to \$1, much as we have done at date 0. On that basis, we can consider call

options written on this asset with exercise prices \$1 and \$2, respectively. These securities are contracts giving the right (but not the obligation) to purchase the underlying security tomorrow at prices \$1 and \$2, respectively. They are contingent securities in the sense that the right they entail is valuable only if the price of the underlying security exceeds the exercise price at expiration, and they are valueless otherwise. We think of the option expiring at $T = 1$, when the state of nature is revealed.[5] The *states of nature* structure enables us to be specific regarding what these contracts effectively promise to pay. Take the call option with exercise price \$1. If state 1 is realized, that option is a right to buy at \$1 the underlying security whose value is exactly \$1. The option is said to be *at the money*, and, in this case, the right in question is valueless. If state 2 is realized, however, the stock is worth \$2. The right to buy, at a price of \$1, something one can immediately resell for \$2 naturally has a market value of \$1. In this case, the option is said to be *in the money*.
In other words, at $T = 1$, when the state of nature is revealed, an option is worth the difference between the value of the underlying asset and its exercise price, if this difference is positive, and zero otherwise. The complete payoff vectors of these options at expiration are as follows:

$$C_T([1,2,3];1) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \begin{matrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{matrix} \quad \left\{ \begin{matrix} \text{at the money} \\ \text{in the money} \\ \text{in the money} \end{matrix} \right\}.$$

Similarly,

$$C_T([1,2,3];2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{matrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{matrix}$$

In our notation, $C_T\ (S;\ K)$ is the payoff to a call option written on security S with exercise price K at expiration date T. We use $C_t\ (S;\ K)$ to denote the option's market price at time $t \le T$. We frequently drop the time subscript to simplify notation when there is no ambiguity.

It remains now to convince ourselves that the three traded assets (the underlying stock and the two call options, each denoted by its payoff vector at $T=1$)

$$\begin{matrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{matrix} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

[5] In our simple two-date world there is no difference between an American option, which can be exercised at any date before the expiration date, and a European option, which can be exercised only at expiration.

constitute a complete set of securities markets for states (θ_1, θ_2, θ_3). This is so because we can use them to create all the state claims. Clearly $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is present.

To create $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, observe that

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = w_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + w_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + w_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where $w_1 = 0$, $w_2 = 1$, and $w_3 = -2$.

The vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ can be similarly created.

We have thus illustrated one of the main ideas of this chapter, and we need to discuss how general and applicable it is in more realistic settings. A preliminary issue is why trading call option securities C([1,2,3]; 1) and C([1,2,3]; 2) might be the preferred approach to completing the market, relative to the alternative possibility of directly issuing the Arrow–Debreu securities [1,0,0] and [0,1,0]? In the simplified world of our example, in the absence of transactions costs, there is, of course, no advantage to creating the options markets. In the real world, however, if a new security is to be issued, its issuance must be accompanied by costly disclosure as to its characteristics; in our parlance, the issuer must disclose as much as possible about the security's payoff in the various states. As there may be no agreement as to what the relevant future states are—let alone what the payoffs will be—this disclosure is difficult. And if there is no consensus as to its payoff pattern (i.e., its basic structure of payoffs), investors will not want to hold it, and it will not trade. But the payoff pattern of an option on an already traded asset is obvious and verifiable to everyone. For this reason, it is, in principle, a much less expensive new security to issue. Another way to describe the advantage of options is to observe that it is useful conceptually, but difficult in practice, to define and identify a single state of nature. It is more practical to define contracts contingent on a well-defined *range* of states. The fact that these states are themselves defined in terms of, or revealed through, market prices is another advantage of this type of contract.

Note that options are by definition in zero net supply; that is, in this context

$$\sum_j C_t^j([1,2,3];K) = 0$$

where $C_t^j([1,2,3]);K)$ is the value of call options with exercise price K, held by agent j at time $t \leq T$. This means that there must exist a group of agents with negative positions serving as the counterparty to the subset of agents with positive holdings. We naturally interpret those agents as agents who have *written* the call options.

We have illustrated the property that markets can be completed using call options. Now let us explore the generality of this result. Can call options always be used to complete the market in this way? The answer is not necessarily. It depends on the payoff to the underlying fundamental assets. Consider the asset:

$$\begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_3 \end{array} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

For any exercise price K, all options written on this security must have payoffs of the form:

$$C([2,2,3];K) = \begin{cases} \begin{bmatrix} 2-K \\ 2-K \\ 3-K \end{bmatrix} & \text{if } K \le 2 \\ \begin{bmatrix} 0 \\ 0 \\ 3-K \end{bmatrix} & \text{if } 2 < K \le 3 \end{cases}$$

Clearly, for any K,

$$\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 2-K \\ 2-K \\ 3-K \end{bmatrix}$$

have identical payoffs in state θ_1 and θ_2, and, therefore, they cannot be used to generate Arrow–Debreu securities

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

There is no way to complete the markets with options in the case of this underlying asset. This illustrates the following truth: it is not possible to write options that distinguish between two states if the underlying assets pay identical returns in those states.

The problem just illustrated can sometimes be solved *if we permit options to be written on **portfolios** of the basic underlying assets.* Consider the case of four possible states at $T = 1$, and suppose that the only assets currently traded are

$$\begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{array} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

It can be shown that it is not possible, using call options, to generate a complete set of securities markets using only these underlying securities. Consider, however, the portfolio composed of two units of the first asset and one unit of the second:

$$2\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

The portfolio pays a different return in each state of nature. Options written on the portfolio alone can thus be used to construct a complete set of traded Arrow–Debreu securities. The example illustrates a second general truth, which we will enumerate as Proposition 11.3.

Proposition 11.3 A necessary as well as sufficient condition for the creation of a complete set of Arrow–Debreu securities is that there exists a single portfolio with the property that options can be written on it and such that its payoff pattern distinguishes among all states of nature.

Returning to the example immediately above, we easily see that the created portfolio and the three calls to be written on it,

$$\begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \text{ plus } \underset{(K=3)}{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}} \text{ and } \underset{(K=4)}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}} \text{ and } \underset{(K=5)}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

are sufficient (i.e., constitute a complete set of markets in our four-state world). Combinations of the $(K = 5)$ and $(K = 4)$ vectors can create:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Combinations of this vector, and the $(K = 5)$ and $(K = 3)$ vectors can then create:

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{etc.}$$

Probing further we may inquire whether *the writing of* calls *on the underlying assets is always sufficient*, or whether there are circumstances under which other types of options may be necessary. Again, suppose there are four states of nature, and consider the following set of *primitive* securities:

$$\begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{array} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Because these assets pay either one or zero in each state, calls written on them will either replicate the asset itself, or give the zero payoff vector. The writing of call options will not help because they cannot further discriminate among states. But suppose we write a put option on the first asset with exercise price 1. A put is a contract giving the right, but not the obligation, to *sell* an underlying security at a prespecified exercise price on a given expiration date. The put option with exercise price 1 has positive value at $T = 1$ in those states where the underlying security has value less than 1. The put on the first asset with exercise price = \$1 thus has the following payoff:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = P_T([0, 0, 0, 1]; 1).$$

You can confirm that the securities plus the put are sufficient to allow us to construct (as portfolios of them) a complete set of Arrow–Debreu securities for the indicated four states. In general, one can prove Proposition 11.4.

Proposition 11.4 If it is possible to create, using options, a complete set of traded securities, simple put and call options written on the underlying assets are sufficient to accomplish this goal.

That is, portfolios of options are not required.

11.6 Synthesizing State-Contingent Claims: A First Approximation

The abstract setting of the preceding discussion aimed at conveying the message that options are natural instruments for completing the markets. In this section, we show how we can directly create a set of state-contingent claims, *as well as their equilibrium prices*, using option prices or option pricing formulas in a more realistic setting. The interest in doing so is, of course, to exploit the possibility, inherent in Arrow–Debreu prices, of pricing any complex security. In this section, we first approach the problem under the hypothesis that the price of the underlying security or portfolio can take only discrete values.

Assume that a risky asset is traded with current price S and future price S_T. It is assumed that S_T discriminates across all states of nature so that Proposition 11.3 applies. Without loss of generality, we may assume that S_T takes the following set of values:

$$S_1 < S_2 < \cdots < S_\theta < \cdots < S_N,$$

where S_θ denotes the price of this complex security if state θ is realized at date T. Assume also that call options are written on this asset with all possible exercise prices, and that these options are traded. Let us also assume that $S_\theta = S_{\theta-1} + \delta$ for every state θ. (This is not so unreasonable as stocks, say, are traded at prices that can differ only in multiples of a minimum price change).[6] Throughout the discussion we will fix the time to expiration and will not denote it notationally.

Consider, for any state $\hat{\theta}$, the following portfolio P:

Buy one call with $K = S_{\hat{\theta}-1}$
Sell two calls with $K = S_{\hat{\theta}}$
Buy one call with $K = S_{\hat{\theta}+1}$

At any point in time, the value of this portfolio, V_P, is

$$V_P = C(S, K = S_{\hat{\theta}-1}) - 2C(S, K = S_{\hat{\theta}}) + C(S, K = S_{\hat{\theta}+1}).$$

To see what this portfolio represents, let us examine its payoff *at expiration* (refer to Figure 11.1).

[6] Until recently, the minimum price change was equal to $\$\frac{1}{16}$ on the NYSE. At the end of 2000, *decimal pricing* was introduced whereby the prices are quoted to the nearest $\$\frac{1}{100}$ (1 cent).

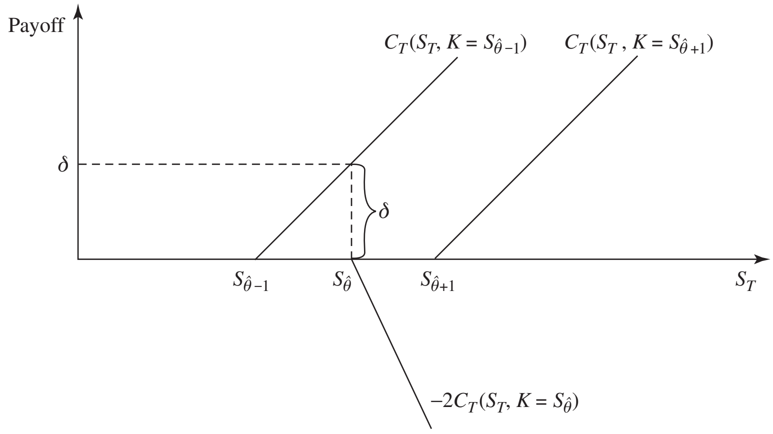

Figure 11.1
Payoff diagram for all options in the portfolio P.

For $S_T \leq S_{\hat\theta-1}$, the value of our options portfolio, P, is zero. A similar situation exists for $S_T \geq S_{\hat\theta+1}$ since the loss on the two written calls with $K = S_{\hat\theta}$ exactly offsets the gains on the other two calls. In state $\hat\theta$, the value of the portfolio is δ corresponding to the value of $C_T(S_{\hat\theta}, K = S_{\hat\theta-1})$, the other two options being worthless when the underlying security takes value $S_{\hat\theta}$. The payoff from such a portfolio thus equals:

$$\text{Payoff to } P = \begin{cases} 0 & \text{if} \quad S_T < S_{\hat\theta} \\ \delta & \text{if} \quad S_T = S_{\hat\theta} \\ 0 & \text{if} \quad S_T > S_{\hat\theta}\,; \end{cases}$$

in other words, it pays a positive amount δ in state $\hat\theta$, and nothing otherwise. That is, it replicates the payoff of the Arrow–Debreu security associated with state $\hat\theta$ up to a factor (in the sense that it pays δ instead of 1). Consequently, the current price of the state $\hat\theta$ contingent claim (i.e., one that pays \$1 if state $\hat\theta$ is realized and nothing otherwise) must be

$$q_{\hat\theta} = \frac{1}{\delta}\left[C(S, K = S_{\hat\theta\,-1}) + C(S, K = S_{\hat\theta\,+1}) - 2C(S, K = S_{\hat\theta}\,)\right].$$

Even if these calls are not traded, if we identify our relevant states with the prices of some security—say the market portfolio—then we can use readily available option pricing formulas (such as the famous Black and Scholes formula) to obtain the necessary call prices and, from them, compute the price of the state-contingent claim. We explore this idea further in the next section.

11.7 Recovering Arrow–Debreu Prices from Options Prices: A Generalization

By the CAPM, the only relevant risk is systematic risk. We may interpret this to mean that the only states of nature that are economically or financially relevant are those that can be identified with different values of the market portfolio.[7] The market portfolio thus may be selected to be the complex security on which we write options, portfolios of which will be used to replicate state-contingent payoffs. The conditions of Proposition 11.1 are satisfied, guaranteeing the possibility of completing the market structure.

In Section 11.6, we considered the case where the underlying asset assumed a discrete set of values. If the underlying asset is the market index quoted in \$.01 units, the number of potential discrete values may be quite large, and the discrete calculations accordingly involved. Can economies of procedure be achieved by assuming the underlying asset assumes a continuum of values? How to accommodate this generalization is discussed below.

1. Suppose that S_T, the price of the underlying portfolio (we may think of it as a proxy for M), assumes a *continuum* of possible values. We want to price an Arrow–Debreu security that pays \$1 if $\tilde{S}_T \in [-\delta/2 + \hat{S}_T, \hat{S}_T + \delta/2]$, in other words, if S_T assumes any value in a range of width δ, centered on $\hat{S}_T$. We are thus identifying our states of nature with **ranges of possible values** for the market portfolio. Here the subscript T refers to the future date at which the Arrow–Debreu security is to pay \$1 if the relevant state is realized.
2. Let us construct the following portfolio for some small positive number $\varepsilon > 0$,

$$\text{Buy one call with } K = \hat{S}_T - \frac{\delta}{2} - \varepsilon$$

$$\text{Sell one call with } K = \hat{S}_T - \frac{\delta}{2}$$

$$\text{Sell one call with } K = \hat{S}_T + \frac{\delta}{2}$$

$$\text{Buy one call with } K = \hat{S}_T + \frac{\delta}{2} + \varepsilon.$$

Figure 11.2 depicts what this portfolio pays *at expiration*.[8]

---

[7] That is, diversifiable risks have zero market value (see Chapter 8 and Section 11.4). At an individual level, personal risks are, of course, also relevant. They can, however, be insured or diversified away. Insurance contracts are often the most appropriate to cover these risks. Recall our discussion of this issue in Chapter 1.

[8] The option position corresponding to this portfolio is known as a *butterfly spread* in the jargon of options traders.

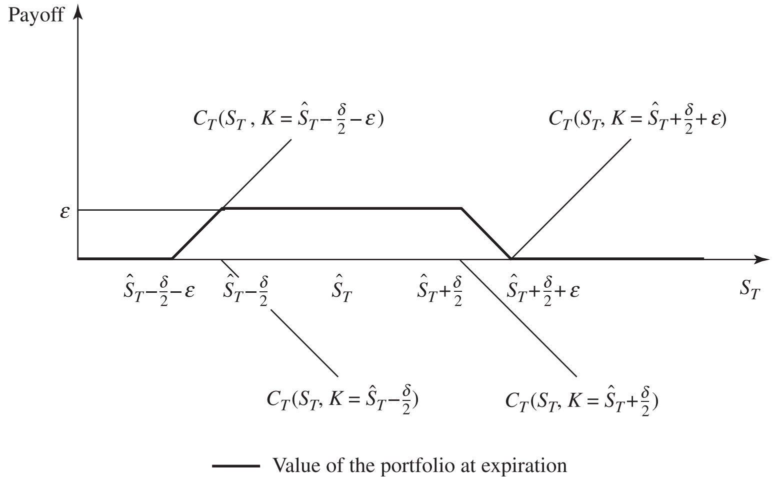

Figure 11.2
Payoff diagram: portfolio of options.

Observe that our portfolio pays ε on a range of states and 0 almost everywhere else. By purchasing $1/\varepsilon$ units of the portfolio, we will mimic the payoff of an Arrow–Debreu security, except for the two small diagonal sections of the payoff line where the portfolio pays something between 0 and ε. This undesirable feature (since our objective is to replicate an Arrow–Debreu security) will be dealt with by using a standard mathematical trick involving taking limits.

3. Let us thus consider buying $1/\varepsilon$ units of the portfolio. The total payment, when $\hat{S}_T - \delta/2 \leq S_T \leq \hat{S}_T + \delta/2$, is $\varepsilon \cdot 1/\varepsilon \equiv 1$, for any choice of ε. We want to let $\varepsilon \mapsto 0$, so as to eliminate payments in the ranges $S_T \in [\hat{S}_T - \delta/2 - \varepsilon,\ \hat{S}_T - \delta/2)$ and $S_T \in (\hat{S}_T + \delta/2,\ \hat{S}_T + \delta/2 + \varepsilon]$. The value of $1/\varepsilon$ units of this portfolio is:

$$\frac{1}{\varepsilon}\left\{C\left(S, K = \hat{S}_T - \frac{\delta}{2} - \varepsilon\right) - C\left(S, K = \hat{S}_T - \frac{\delta}{2}\right)\right.$$
$$\left. - \left[C\left(S, K = \hat{S}_T + \frac{\delta}{2}\right) - C\left(S, K = \hat{S}_T + \frac{\delta}{2} + \varepsilon\right)\right]\right\},$$

where a minus sign indicates that the call was sold (thereby reducing the cost of the portfolio by its sale price). On balance, the portfolio will have a positive price as it represents a claim on a positive cash flow in certain states of nature. Let us assume that the pricing function for a call with respect to changes in the exercise price can be differentiated. (This property is true, in particular, in the case of the Black and Scholes option pricing formula.) We then have:

$$
\begin{aligned}
&\lim_{\varepsilon \mapsto 0} \frac{1}{\varepsilon}\left\{ C\left(S, K=\hat{S}_T-\frac{\delta}{2}-\varepsilon\right)-C\left(S, K=\hat{S}_T-\frac{\delta}{2}\right)\right.\\
&\quad\left.-\left[C\left(S, K=\hat{S}_T+\frac{\delta}{2}\right)-C\left(S, K=\hat{S}_T+\frac{\delta}{2}+\varepsilon\right)\right]\right\}\\
&= -\lim_{\varepsilon \mapsto 0}\left\{\underbrace{\frac{C\left(S, K=\hat{S}_T-\frac{\delta}{2}-\varepsilon\right)-C\left(S, K=\hat{S}_T-\frac{\delta}{2}\right)}{-\varepsilon}}_{\leq 0}\right\}\\
&\quad+\lim_{\varepsilon \mapsto 0}\left\{\underbrace{\frac{C\left(S, K=\hat{S}_T+\frac{\delta}{2}+\varepsilon\right)-C\left(S, K=\hat{S}_T+\frac{\delta}{2}\right)}{\varepsilon}}_{\leq 0}\right\}\\
&= C_2\left(S, K=\hat{S}_T+\frac{\delta}{2}\right)-C_2\left(S, K=\hat{S}_T-\frac{\delta}{2}\right).
\end{aligned}
$$

Here the subscript 2 indicates the partial derivative with respect to the second argument (K), evaluated at the indicated exercise prices. In summary, the limiting portfolio has a payoff at expiration as represented in Figure 11.3 and a (current) price $C_2(S, K=\hat{S}_T+\delta/2)-C_2(S, K=\hat{S}_T-\delta/2)$ that is positive since the payoff is positive. We have thus priced an Arrow–Debreu state-contingent claim one period ahead, given

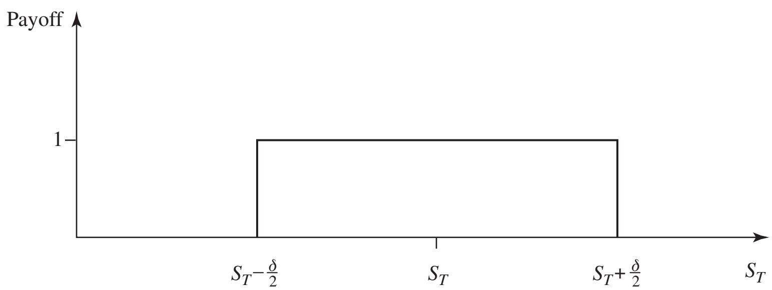

Figure 11.3
Payoff diagram for the limiting portfolio.

that we define states of the world as coincident with ranges of a proxy for the market portfolio.

4. Suppose, for example, we have an uncertain payment with the following payoff at time T:

$$CF_T = \left\{ \begin{array}{ll} 0 & \text{if } S_T \notin \left[\hat{S}_T - \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2}\right] \\ 50,000 & \text{if } S_T \in \left[\hat{S}_T - \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2}\right] \end{array} \right\}$$

The value today of this cash flow is:

$$50,000 \cdot \left[C_2\left(S, K = \hat{S}_T + \frac{\delta}{2}\right) - C_2\left(S, K = \hat{S}_T - \frac{\delta}{2}\right)\right].$$

The formula we have developed is very general. In particular, for any arbitrary values S_T^1 and S_T^2, the price of an Arrow–Debreu contingent claim that pays \$1 if the underlying market portfolio assumes a value $S_T \in [S_T^1, S_T^2]$, is given by

$$q(S_T^1, S_T^2) = C_2(S, K = S_T^2) - C_2(S, K = S_T^1). \tag{11.4}$$

We value this quantity in Box 11.1 for a particular set of parameters making explicit use of the Black–Scholes option pricing formula.

BOX 11.1 Pricing A–D Securities with Black–Scholes

For calls priced according to the Black–Scholes option pricing formula, Breeden and Litzenberger (1978) prove that

$$\begin{aligned} q(S_T^1, S_T^2) &= C_2(S, K = S_T^2) - C_2(S, K = S_T^1) \\ &= e^{-rT}\{N(d_2(S_T^1)) - N(d_2(S_T^2))\} \end{aligned}$$

where

$$d_2(S_T^i) = \frac{\left[\ln\left(\frac{S_0}{S_T^i}\right) + \left(r_f - \gamma - \frac{\sigma^2}{2}\right)T\right]}{\sigma\sqrt{T}}$$

In this expression, T is the time to expiration, r_f the annualized continuously compounded riskless rate over that period, γ the continuous annualized portfolio dividend yield, σ the standard deviation of the continuously compounded rate of return on the underlying index portfolio, $N(\ )$ the standard normal distribution, and S_0 the current value of the index.

(Continued)

BOX 11.1 Pricing A–D Securities with Black–Scholes (Continued)

Suppose the not-continuously compounded risk-free rate is 0.06, the not-continuously compounded dividend yield is $\delta = 0.02$, $T = 0.5$ year, $S_0 = 1500$, $S_T^2 = 1700$, $S_T^1 = 1600$, $\sigma = 0.20$; then

$$\begin{aligned} d_2(S_T^1) &= \frac{\left\{ \ln\left(\frac{1500}{1600}\right) + \left[\ln(1.06) - \ln(1.02) - \frac{(0.20)^2}{2}\right](0.5)\right\}}{0.20\sqrt{0.5}} \\ &= \frac{\{-0.0645 + (0.0583 - 0.0198 - 0.02)(0.5)\}}{0.1414} \\ &= -0.391 \end{aligned}$$

$$\begin{aligned} d_2(S_T^2) &= \frac{\left\{ \ln\left(\frac{1500}{1700}\right) + (0.0583 - 0.0198 - 0.02)(0.5)\right\}}{0.1414} \\ &= \frac{\{(-0.1252 + 0.00925)\}}{0.1414} \\ &= -0.820 \end{aligned}$$

$$\begin{aligned} q(S_T^1, S_T^2) &= e^{-\ln(1.06)(0.5)}\{N(-0.391) - N(-0.820)\} \\ &= 0.9713\{0.3479 - 0.2061\} \\ &= 0.1381, \end{aligned}$$

or about \$0.14.

Suppose we wished to price an uncertain cash flow to be received one period from the present, where a period corresponds to a duration of time T. What do we do? Choose several ranges of the value of the market portfolio corresponding to the various states of nature that may occur—say three states: "recession," "slow growth," and "boom"—and estimate the cash flow in each of these states (see Figure 11.4). It would be unusual to have a large number of states, as the requirement of having to estimate the cash flows in each of those states is likely to exceed our forecasting abilities.

Suppose the cash-flow estimates are, respectively, CF_B, CF_{SG}, and CF_R, where the subscripts denote, respectively, "boom," "slow growth," and "recession." Then,

$$\text{Value of the } CF = V_{CF} = q(S_T^3, S_T^4)CF_B + q(S_T^2, S_T^3)CF_{SG} + q(S_T^1, S_T^2)CF_R,$$

where $S_T^1 < S_T^2 < S_T^3 < S_T^4$, and the Arrow–Debreu prices are estimated from option prices or option pricing formulas according to Eq. (11.4).

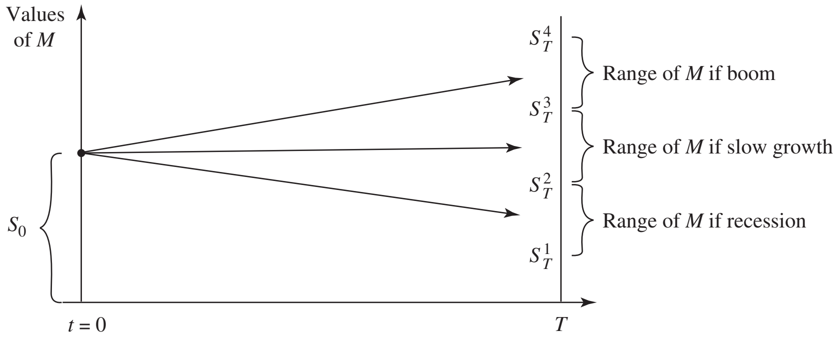

Figure 11.4
Constructing "states" as ranges of the future value of *M*.

We can go one (final) step further if we assume for a moment that the cash flow we wish to value can be described by a continuous function of the value of the market portfolio.

In principle, for a very fine partition of the range of possible values of the market portfolio, say $\{S_1, \ldots, S_N\}$, where $S_i < S_{i+1}$, $S_N = \max S_T$, and $S_1 = \min S_T$, we can price the Arrow–Debreu securities that pay off in each of these $N - 1$ states defined by the partition:

$$q(S_1, S_2) = C_2(S, S_2) - C_2(S, S_1)$$
$$q(S_2, S_3) = C_2(S, S_3) - C_2(S, S_2), \ldots, \text{ etc.}$$

Simultaneously, we could approximate a cash-flow function $CF(S_T)$ by a function that is constant in each of these ranges of S_T (a so-called step function); in other words, $\hat{C}F(S_T) = CF_i$, for $S_{i-1} \le S_T \le S_i$. For example,

$$\hat{C}F(S_T) = CF_i = \frac{CF(S, S_T = S_i) + CF(S, S_T = S_{i-1})}{2} \text{ for } S_{i-1} \le S_T \le S_i$$

This particular approximation is represented in Figure 11.5. The value of the approximate cash flow would then be

$$\begin{aligned} V_{CF} &= \sum_{i=1}^{N} \hat{C}F_i \cdot q(S_{i-1}, S_i) \\ &= \sum_{i=1}^{N} \hat{C}F_i [C_2(S, S_T = S_i) - C_2(S, S_T = S_{i-1})] \end{aligned} \tag{11.5}$$

Our approach is now clear. The precise value of the uncertain cash flow will be the sum of the approximate cash flows evaluated at the Arrow–Debreu prices as the norm of the

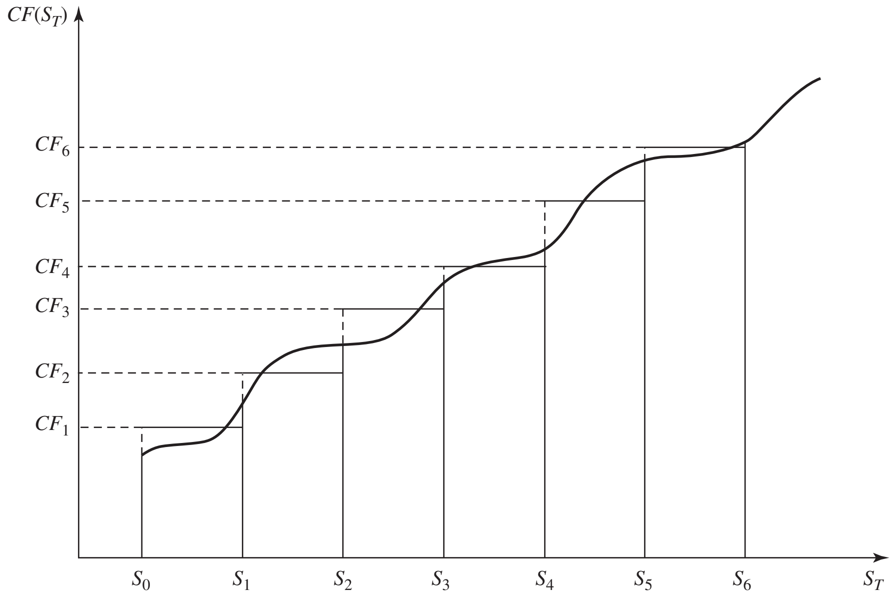

Figure 11.5
A discrete approximation to a continuous cash-flow function.

partition (the size of the interval $S_i - S_{i-1}$) tends to zero. It can be shown (and it is intuitively plausible) that the limit of Eq. (11.5) as $\max_i |S_{i+1} - S_i| \to 0$ is the integral of the cash-flow function multiplied by the second derivative of the call's price with respect to the exercise price. The latter is the continuum counterpart to the difference in the first derivatives of the call prices entering in Eq. (11.4).

$$\lim_{\substack{\max\limits_i |S_{i+1} - S_i| \to 0}} \sum_{i=1}^{N} \hat{C}F_i[C_2(S, S_T = S_{i+1}) - C_2(S, S_T = S_i)] = \int CF(S_T)C_{22}(S, S_T)dS_T. \tag{11.6}$$

As a particular case of a constant cash-flow stream, a risk-free bond paying \$1 in every state is then priced as per

$$q^{r_f} = \frac{1}{(1 + r_f)} = \int_0^\infty C_{22}(S, S_T)\mathrm{d}S_T. \tag{11.7}$$

See Box 11.2 for a numerical illustration of these ideas (Table 11.8).

BOX 11.2 Extracting Arrow–Debreu Prices from Option Prices: A Numerical Illustration

Let us now illustrate the power of the approach adopted in this and the previous section. For that purpose, Table 11.8 (adapted from Pirkner et al. (1999)) starts by recording call prices, obtained from the Black–Scholes formula for a call option, on an underlying index portfolio, currently valued at $S = 10$, for a range of strike prices going from $K = 7$ to $K = 13$ (columns 1 and 2). Column 3 computes the value of portfolio P of Section 11.6. Given that the difference between the exercise prices is always 1 (i.e., $\delta = 1$), holding exactly one unit of this portfolio replicates the \$1 payoff of the Arrow–Debreu security associated with $K = 10$. This is shown on the bottom line of column 7, which corresponds to $S = 10$. From column 3, we learn that the price of this Arrow–Debreu security, which must be equal to the value of the replicating portfolio, is \$0.184. Finally, the last two columns approximate the first and second derivatives of the call price with respect to the exercise price. In the current context, this is naturally done by computing the first and second differences (the price increments and the increments of the increments as the exercise price varies) from the price data given in column 2. This is a literal application of Eq. (11.4). One thus obtains the full series of Arrow–Debreu prices for states of nature identified with values of the underlying market portfolios ranging from 8 to 12, confirming that the \$0.184 price occurs when the state of nature is identified as $S = 10$ (or $9.5 < S < 10.5$).

Table 11.8: Pricing an Arrow–Debreu state claim

			Payoff if S_T =								
K	*C(S,K)*	Cost of Position	7	8	9	10	11	12	13	ΔC	$\Delta(\Delta C) = q_\theta$
7	3.354										
										−0.895	
8	2.459										0.106
										−0.789	
9	1.670	+1.670	0	0	0	1	2	3	4		0.164
										−0.625	
10	1.045	−2.090	0	0	0	0	−2	−4	−6		**0.184**
										−0.441	
11	0.604	+0.604	0	0	0	0	0	1	2		0.162
										−0.279	
12	0.325										0.118
										−0.161	
13	0.164										
		0.184	0	0	0	1	0	0	0		

11.8 Arrow–Debreu Pricing in a Multiperiod Setting

The fact that the Arrow–Debreu pricing approach is static makes it ideal for the pricing of one-period cash flows, and it is, quite naturally, in this context that most of our discussion has been framed. But as we have emphasized previously, it is formally equally appropriate for pricing multiperiod cash flows. The estimation (for instance, via option pricing formulas and the methodology introduced in the last two sections) of Arrow–Debreu prices for several periods ahead is inherently more difficult, however, and relies on more perilous assumptions than in the case of one period ahead prices. (This parallels the fact that the assumptions necessary to develop closed-form option pricing formulas are more questionable when they are used in the context of pricing long-term options.) Pricing long-term assets, whatever the approach adopted, requires making hypotheses to the effect that the recent past tells us something about the future, which, in ways to be defined and which vary from one model to the next, translates into hypotheses that some form of stationarity prevails. Completing the Arrow–Debreu pricing approach with an additional stationarity hypothesis provides an interesting perspective on the pricing of multiperiod cash flows. This is the purpose of the present section.

For notational simplicity, let us first assume that the same two states of nature (ranges of value of M) can be realized in each period and that all future state-contingent cash flows have been estimated. The structure of the cash flow is found in Figure 11.6.

Suppose also that we have estimated, using our formulas derived earlier, the values of the one-period state-contingent claims as follows:

$$\begin{array}{ccc} & & \text{Tomorrow} \\ & & \begin{array}{cc} 1 & \quad 2 \end{array} \\ \text{Today} & \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0.54 & 0.42 \\ 0.46 & 0.53 \end{bmatrix} = \mathbf{q} \end{array}$$

where q_{11} (=0.54) is the price today of an Arrow–Debreu claim paying \$1 if state 1 (a boom) occurs tomorrow, given that we are in state 1 (boom) today. Similarly, q_{12} is the

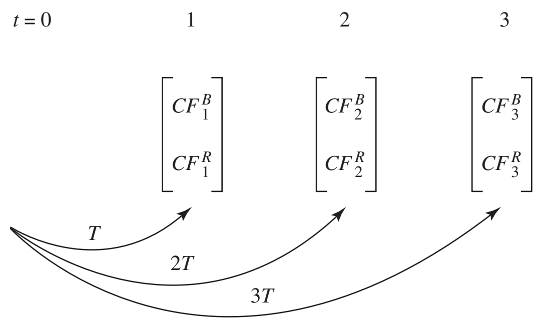

Figure 11.6
A multiperiod cash flow: two states of nature every period.

price today of an Arrow–Debreu claim paying $1 if state 2 (recession) occurs tomorrow given that we are in state 1 today. Note that these prices differ because the distribution of the value of M tomorrow differs depending on the state today.

Now let us introduce our stationarity hypothesis: Suppose that $\mathbf{q}$, the matrix of values, is invariant through time.[9] That is, the same two states of nature describe the possible futures at all future dates, and the contingent one-period prices remain the same. This allows us to interpret powers of the $\mathbf{q}$ matrix, $\mathbf{q}^2$, $\mathbf{q}^3$,... in a particularly useful way. Consider $\mathbf{q}^2$ (see also Figure 11.7):

$$\mathbf{q}^2 = \begin{bmatrix} 0.54 & 0.42 \\ 0.46 & 0.53 \end{bmatrix} \cdot \begin{bmatrix} 0.54 & 0.42 \\ 0.46 & 0.53 \end{bmatrix}$$

$$= \begin{bmatrix} (0.54)(0.54) + (0.42)(0.46) & (0.54)(0.42) + (0.42)(0.53) \\ (0.46)(0.54) + (0.53)(0.46) & (0.46)(0.42) + (0.53)(0.53) \end{bmatrix}$$

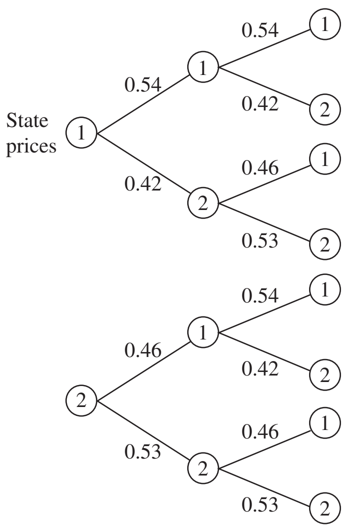

Figure 11.7
The evolution of state prices through time.

[9] If this were not the case, the approach in Figure 11.7 would carry on provided we would be able to compute forward Arrow–Debreu prices. In other words, the Arrow–Debreu matrix would change from date to date, and it would have to be time-indexed. Mathematically, the procedure described would carry over, but the information requirement would, of course, be substantially larger.

Table 11.9: State-contingent cash flows

$t=0$	1	2	3
state 1 *state 2*	$\begin{bmatrix} 42 \\ 65 \end{bmatrix}$	$\begin{bmatrix} 48 \\ 73 \end{bmatrix}$	$\begin{bmatrix} 60 \\ 58 \end{bmatrix}$

Note that there are two ways to be in state 1 two periods from now, given that we are in state 1 today. Therefore, the price today of \$1, if state 1 occurs in two periods, given we are in state 1 today is:

$$\underbrace{(0.54)(0.54)}_{\substack{\text{value of \$1 in 2 periods if state 1 occurs} \\ \text{and the intermediate state is 1}}} + \underbrace{(0.42)(0.46)}_{\substack{\text{value of \$1 in 2 periods if state 1 occurs} \\ \text{and the intermediate state is 2}}}$$

Similarly, $\mathbf{q}^2_{22} = (0.46)(0.42) + (0.53)(0.53)$ is the price today, if today's state is 2, of \$1 contingent on state 2 occurring in two periods. In general, for powers N of the matrix $\mathbf{q}$, we have the following interpretation for $\mathbf{q}^N_{ij}$: given that we are in state i today, it gives the price today of \$1, contingent on state j occurring in N periods. Of course, if we hypothesized three states, then the Arrow–Debreu matrices would be 3×3 and so forth.

How can this information be used in a "capital budgeting" problem? First we must estimate the cash flows. Suppose they are as outlined in Table 11.9.

Then the present value (PV) of the cash flows, contingent on state 1 or state 2, are given by:

$$\begin{aligned} PV &= \begin{bmatrix} PV_1 \\ PV_2 \end{bmatrix} \\ &= \begin{bmatrix} 0.54 & 0.42 \\ 0.46 & 0.53 \end{bmatrix} \begin{bmatrix} 42 \\ 65 \end{bmatrix} + \begin{bmatrix} 0.54 & 0.42 \\ 0.46 & 0.53 \end{bmatrix}^2 \begin{bmatrix} 48 \\ 73 \end{bmatrix} + \begin{bmatrix} 0.54 & 0.42 \\ 0.46 & 0.53 \end{bmatrix}^3 \begin{bmatrix} 60 \\ 58 \end{bmatrix} \\ &= \begin{bmatrix} 0.54 & 0.42 \\ 0.46 & 0.53 \end{bmatrix} \begin{bmatrix} 42 \\ 65 \end{bmatrix} + \begin{bmatrix} 0.4848 & 0.4494 \\ 0.4922 & 0.4741 \end{bmatrix} \begin{bmatrix} 48 \\ 73 \end{bmatrix} + \begin{bmatrix} 0.4685 & 0.4418 \\ 0.4839 & 0.4580 \end{bmatrix} \begin{bmatrix} 60 \\ 58 \end{bmatrix} \\ &= \begin{bmatrix} 49.98 \\ 53.77 \end{bmatrix} + \begin{bmatrix} 56.07 \\ 58.23 \end{bmatrix} + \begin{bmatrix} 53.74 \\ 55.59 \end{bmatrix} = \begin{bmatrix} 159.79 \\ 167.59 \end{bmatrix}. \end{aligned}$$

This procedure can be expanded to include as many states of nature as one may wish to define. This amounts to choosing as fine a partition of the range of possible values of M as one wishes to choose. It makes no sense to construct a finer partition, however, if we have no real basis for estimating different cash flows in those states. For most practical problems, three or four states are probably sufficient. But an advantage of this method is that it forces one to think carefully about what a project cash flow will be in each state, and what the relevant states, in fact, are.

One may wonder whether this methodology implicitly assumes that the states are equally probable. That is not the case. Although the probabilities, which would reflect the likelihood of the value of the market portfolio M lying in the various intervals, are not explicit, they are built into the prices of the state-contingent claims.

We close this chapter by suggesting a way to tie the approach proposed here with our previous work in this chapter. Risk-free cash flows are special (degenerate) examples of risky cash flows. It is thus easy to use the method of this section to price risk-free flows. The comparison with the results obtained with the method of Section 11.3 then provides a useful check of the appropriateness of the assumptions made in the present context.

Consider our earlier example with Arrow—Debreu prices given by:

$$\begin{array}{ll} & \quad\;\; 1 \qquad\quad 2 \\ \text{State 1} & \multirow{2}{*}{$\begin{bmatrix} 0.54 & 0.42 \\ 0.46 & 0.53 \end{bmatrix}$} \\ \text{State 2} & \end{array}$$

If we are in state 1 today, the price of \$1 in each state tomorrow (i.e., a risk-free cash flow tomorrow of \$1) is $0.54 + 0.42 = 0.96$. This implies a risk-free rate of:

$$(1 + r_f^1) = \frac{1.00}{0.96} = 1.0416 \text{ or } 4.16\%.$$

To put it differently, $0.54 + 0.42 = 0.96$ is the price of a one-period discount bond paying \$1 in one period, given that we are in state 1 today. More generally, we would evaluate the following risk-free cash flow as:

$$\begin{array}{cccc} t = 0 & 1 & 2 & 3 \\ & 100 & 100 & 100 \end{array}$$

$$\begin{aligned} PV &= \begin{bmatrix} PV_1 \\ PV_2 \end{bmatrix} \\ &= \begin{bmatrix} 0.54 & 0.42 \\ 0.46 & 0.53 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} + \begin{bmatrix} 0.54 & 0.42 \\ 0.46 & 0.53 \end{bmatrix}^2 \begin{bmatrix} 100 \\ 100 \end{bmatrix} + \begin{bmatrix} 0.54 & 0.42 \\ 0.46 & 0.53 \end{bmatrix}^3 \begin{bmatrix} 100 \\ 100 \end{bmatrix} \\ &= \begin{bmatrix} 0.54 & 0.42 \\ 0.46 & 0.53 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} + \begin{bmatrix} 0.4848 & 0.4494 \\ 0.4922 & 0.4741 \end{bmatrix}^2 \begin{bmatrix} 100 \\ 100 \end{bmatrix} + \begin{bmatrix} 0.4685 & 0.4418 \\ 0.4839 & 0.4580 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} \end{aligned}$$

So

$$\begin{aligned} PV_1 &= [0.54 + 0.42]100 + [0.4848 + 0.4494]100 + [0.4685 + 0.4418]100 \\ &= [0.96]100 + [0.9342]100 + [0.9103]100 \\ &= 280.45 \end{aligned}$$

where [0.96] = price of a one-period discount bond given state 1 today, [0.9342] = price of a two-period discount bond given state 1 today, [0.9103] = price of a three-period discount bond given state 1 today. The *PV* given state 2 is computed analogously. Now this provides us with a **verification test**: if the price of a discount bond using this method does not coincide with the prices using the approach developed in Section 11.3 (which relies on quoted coupon bond prices), then this must mean that our states are not well defined or numerous enough or that the assumptions of the option pricing formulas used to compute Arrow–Debreu prices are inadequate.

11.9 Conclusions

This chapter has served two main purposes. First, it has provided us with a platform to think more in depth about the all-important notion of market completeness. Our demonstration that, in principle, a portfolio of simple calls and puts written on the market portfolio might suffice to reach a complete market structure suggests that the "Holy Grail" may not be totally out of reach. Caution must be exercised, however, in interpreting the necessary assumptions. Can we indeed assume that the market portfolio—and what do we mean by the latter—is an adequate reflection of all the economically relevant states of nature? And the time dimension of market completeness should not be forgotten. The most relevant state of nature for a Swiss resident of 40 years of age may be the possibility of a period of prolonged depression with high unemployment in Switzerland 25 years from now (when he nears retirement). Now extreme aggregate economic conditions would certainly be reflected in the Swiss Market Index (SMI), but options with 20-year maturities are not customarily traded. Is it because of a lack of demand (possibly meaning that our assumption as to the most relevant state is not borne out), or because the structure of the financial industry is such that the supply of securities for long horizons is deficient?[10]

The second part of the chapter discussed how Arrow–Debreu prices can be extracted from option prices (in the case where the relevant option is actively traded) or option pricing

[10] A forceful statement in support of a similar claim is found in Shiller (1993) (see also the conclusions to Chapter 1). For the particular example discussed here, it may be argued that shorting the SMI (Swiss Market Index) would provide the appropriate hedge. Is it conceivable to take a short SMI position with a 20-year horizon?

formulas (in the case where they are not). This discussion helps make Arrow–Debreu securities a less abstract concept. In fact, in specific cases the detailed procedure is fully operational and may indeed be the wiser route to evaluating risky cash flows. The key hypotheses are similar to those we have just discussed: the relevant states of nature are adequately distinguished by the market portfolio, a hypothesis that may be deemed appropriate if the context is limited to the valuation of risky cash flows. Moreover, in the case where options are not traded, the quality of the extracted Arrow–Debreu prices depends on the appropriateness of the various hypotheses embedded in the option pricing formulas to which one has recourse. This issue has been abundantly discussed in the relevant literature.

References

Banz, R., Miller, M., 1978. Prices for state-contingent claims: some estimates and applications. J. Bus. 51, 653–672.

Breeden, D., Litzenberger, R.H., 1978. Prices of state-contingent claims implicit in option prices. J. Bus. 51, 621–651.

Pirkner, C.D., Weigend, A.S., Zimmermann, H., 1999. Extracting Risk-Neutral Densities from Option Prices Using Mixture Binomial Trees. University of St-Gallen (Mimeographed).

Ross, S., 1976. Options and efficiency. Q. J. Econ. 90, 75–89.

Shiller, R.J., 1993. Macro Markets-Creating Institutions for Managing Society's Largest Economic Risks. Clarendon Press, Oxford.

Appendix 11.1: Forward Prices and Forward Rates

Forward prices and forward rates correspond to the prices of (rates of return earned by) securities to be issued in the future.

Let ${}_kf_\tau$ denote the (compounded) rate of return on a risk-free discount bond to be issued at a future date k and maturing at date $k+\tau$. These forward rates are defined by the equations:

$$
\begin{aligned}
(1+r_1)(1+{}_1f_1) &= (1+r_2)^2 \\
(1+r_1)(1+{}_1f_2)^2 &= (1+r_3)^3 \\
(1+r_2)^2(1+{}_2f_1) &= (1+r_3)^3, \text{ etc.}
\end{aligned}
$$

We emphasize that the forward rates are *implied* forward rates, in the sense that the corresponding contracts are typically not traded. However, it is feasible to *lock in* these forward rates—that is, to guarantee their availability in the future. Suppose we wished to lock in the 1-year forward rate 1 year from now. This amounts to creating a new security "synthetically" as a portfolio of existing securities, and it is accomplished by simply

Table 11.10: Locking in a forward rate

$t =$	0	1	2
Buy a 2-yr bond	-1000	65	1065
Sell short a 1-yr bond	$+1000$	-1060	
	0	-995	1065

undertaking a series of *long* and *short* transactions today. For example, take as given the implied discount bond prices of Table 11.5 and consider the transactions in Table 11.10.

The portfolio we have constructed has a zero cash flow at date 0, requires an investment of \$995 at date 1, and pays \$1065 at date 2. The gross return on the date 1 investment is

$$\frac{1065}{995} = 1.07035.$$

That this is exactly equal to the corresponding forward rate can be seen from the forward rate definition:

$$1 + {}_1f_1 = \frac{(1+r_2)^2}{(1+r_1)} = \frac{(1.065163)^2}{1.06} = 1.07035.$$

Let us scale back the previous transactions to create a \$1000 payoff for the forward security. This amounts to multiplying all of the indicated transactions by $1000/1065 = 0.939$.

This price (\$934.34) is the no arbitrage price of this forward bond—no arbitrage in the sense that if there were any other contract calling for the delivery of such a bond at a price different from \$934.34, an arbitrage opportunity would exist (see Table 11.11).[11]

Table 11.11: Creating a \$1000 payoff

$t =$	0	1	2
Buy 0.939 × 2-yr bonds	-939	61.0	1000
Sell short 0.939 × 1-yr bonds	$+939$	-995.34	
	0	-934.34	1000

[11] The approach of this section can, of course, be generalized to more distant forward rates.