Useful stuff*

Jonas Nygaard Eriksen**

Asset Pricing, Fall 2021

*The note contains a brief and incomplete review of useful stuff in Asset Pricing and is prepared for use only in the Master's course "Asset Pricing". The note is work-in-progress (and an experiment from my side), so if you find errors and/or have suggestions for things to include, please do not hesitate to let me know. Please do not cite, circulate, or use for purposes other than this course.

**CREATES, Department of Economics and Business Economics, Aarhus University, Fuglesangs Allé 4, DK-8210 Aarhus V, Denmark. E-mail: jeriksen@econ.au.dk.

Contents

1 Means, variances, and covariances **1**

2 Jensen's inequality **1**

3 Taylor approximation **2**

4 Lognormal distribution **2**

5 Log utility as a special case of power utility **3**

6 Continuously compounded returns **4**

7 Optimization **5**

7.1 Unconstrained optimization 6

7.2 Constrained optimization 9

1. Means, variances, and covariances

This section presents a series of useful formulas for means, variances, and covariances. Assume that X and Y are scalar random variables and that a and b are constants. We then have the following relations.

$$\mathbb{E}\left[aX\right] = a\mathbb{E}\left[X\right] \tag{1}$$

$$\text{Var}\left[X\right] = \mathbb{E}\left[\left(X - \mathbb{E}\left[X\right]\right)^2\right] \tag{2}$$

$$\text{Var}\left[X\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}\left[X\right]^2 \tag{3}$$

$$\mathbb{E}\left[X^2\right] = \text{Var}\left[X\right] + \mathbb{E}\left[X\right]^2 \tag{4}$$

$$\text{Var}\left[aX\right] = a^2\text{Var}\left[X\right] \tag{5}$$

$$\text{Var}\left[X + Y\right] = \text{Var}\left[X\right] + \text{Var}\left[Y\right] + 2\text{Cov}\left[X,Y\right] \tag{6}$$

$$\text{Var}\left[X - Y\right] = \text{Var}\left[X\right] + \text{Var}\left[Y\right] - 2\text{Cov}\left[X,Y\right] \tag{7}$$

$$\text{Cov}\left[X,Y\right] = \mathbb{E}\left[XY\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right] \tag{8}$$

$$\mathbb{E}\left[XY\right] = \text{Cov}\left[X,Y\right] + \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right] \tag{9}$$

$$\text{Cov}\left[X,X\right] = \text{Var}\left[X\right] \tag{10}$$

$$\text{Cov}\left[aX,bY\right] = ab\text{Cov}\left[X,Y\right] \tag{11}$$

$$\text{Cov}\left[a + X,b + Y\right] = \text{Cov}\left[X,Y\right] \tag{12}$$

We can remark that $\text{Cov}\left[X,Y\right] = 0$ implies that X and Y are uncorrelated. If X and Y are independent, then by extension we have that $\text{Cov}\left[X,Y\right] = 0$. If $\text{Cov}\left[X,Y\right] = 0$, then X and Y are independent if they are both normally distributed. Importantly, this is not generally true for any distribution.

2. Jensen's inequality

Consider a concave function $f\left(\cdot\right)$ and numbers $x_1,x_2,\ldots,x_n$. Assume that we have weights $\omega_1,\omega_2,\ldots,\omega_n$ so that $\sum_{i=1}^n \omega_i = 1$. Then Jensen's inequality for a concave function states that (reverse for a convex function)

$$f\left(\sum_{i=1}^n \omega_i x_i\right) \geq \sum_{i=1}^n \omega_i f\left(x_i\right) \tag{13}$$

In the special case that $\omega_i = 1/n$, this becomes

$$f\left(\frac{1}{n}\sum_{i=1}^n x_i\right) \geq \frac{1}{n}\sum_{i=1}^n f\left(x_i\right) \tag{14}$$

Or using the expectations operator, we can write the inequality as

$$f\left(\mathbb{E}\left[X\right]\right) \geq \mathbb{E}\left[f\left(X\right)\right] \tag{15}$$

This is an important mathematical result that we will make use of frequently in the course. For example, we can use it to link risk aversion and concavity of a utility function.

3. Taylor approximation

The Taylor series of a function is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point. Consider a function $f\left(x\right)$, then a Taylor approximation around the point a is defined as

$$\begin{aligned} f\left(x\right) &= \sum_{n=0}^{\infty} \frac{f^{(n)}\left(a\right)}{n!}\left(x-a\right)^{n} & (16)\\ &= f\left(a\right) + \frac{f'\left(a\right)}{1!}\left(x-a\right) + \frac{f''\left(a\right)}{2!}\left(x-a\right)^{2} + \ldots, & (17) \end{aligned}$$

where ! denotes the factorial. We will again frequently make use of this device, but mostly limit outselves to first- and second-order approximations. A second-order approximation can be written more compactly as

$$f\left(x\right) = f\left(a\right) + f'\left(a\right)\left(x-a\right) + \tfrac{1}{2}f''\left(a\right)\left(x-a\right)^{2} \tag{18}$$

4. Lognormal distribution

A variable x is said to follow a lognormal distribution if $\ln x$ is normally distributed. Let $\ln x \sim \mathcal{N}\left(\mu_x,\sigma_x^2\right)$, then expected values are determined as

$$\begin{aligned} \mathbb{E}\left[x\right] &= \exp\left\{\mu_x + \frac{1}{2}\sigma_x^2\right\} & (19)\\ \mathbb{E}\left[x^a\right] &= \exp\left\{a\mu_x + \frac{1}{2}a^2\sigma_x^2\right\}. & (20) \end{aligned}$$

Moreover, we can compute the variance of a lognormally distributed variable as $\text{var}\left[x\right] = \mathbb{E}\left[x^2\right] - \mathbb{E}\left[x\right]^2 = \left(\mathbb{E}\left[x\right]\right)^2\left(\exp\left\{\sigma_x^2\right\} - 1\right)$, giving us the following relations for the variances

$$\begin{aligned} \text{var}\left[x\right] &= \exp\left\{2\mu_x + \sigma_x^2\right\}\left(\exp\left\{\sigma_x^2\right\} - 1\right) & (21)\\ \text{var}\left[x^a\right] &= \exp\left\{2a\mu_x + a^2\sigma_x^2\right\}\left(\exp\left\{a^2\sigma_x^2\right\} - 1\right). & (22) \end{aligned}$$

Suppose further that x and y are two iid lognormally distributed variables, then

$$\mathbb{E}\left[x^a y^b\right] = \exp\left\{a\mu_x + b\mu_y + \frac{1}{2}\left(a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\sigma_{xy}\right)\right\}. \tag{23}$$

5. Log utility as a special case of power utility

Consider the classic case of an investor with a constant relative risk aversion (CRRA) power utility function of the form

$$U(Y) = \frac{Y^{1-\gamma}}{1-\gamma} \tag{24}$$

where $\gamma = R_R(Y)$ denotes the Arrow-Pratt coefficient of relative risk aversion. We are interested in studying the behavior of the function when letting $\gamma \to 1$. To study such a problem, we make use of L'Hôpital's Rule.

L'Hôpital's Rule Suppose that $f(\gamma)$ and $g(\gamma)$ are differentiable functions. If $f(\gamma) \to 0$ and $g(\gamma) \to 0$ as $\gamma \to c$, then

$$\lim_{\gamma \to c} \frac{f(\gamma)}{g(\gamma)} = \lim_{\gamma \to c} \frac{f'(\gamma)}{g'(\gamma)} \tag{25}$$

Deriving the result Start by subtracting a constant from the utility function in (24), which is inconsequential as it has no effect on marginal utility and investor behavior

$$U(Y) = \frac{Y^{1-\gamma}}{1-\gamma} - \frac{1}{1-\gamma} = \frac{Y^{1-\gamma}-1}{1-\gamma} \tag{26}$$

Next, let $f(\gamma) = Y^{1-\gamma} - 1$ and $g(\gamma) = 1-\gamma$ to verify that, as $\gamma \to 1$, we obtain

$$\lim_{\gamma \to 1} f(\gamma) = Y^{1-\gamma} - 1 = 0 \tag{27}$$

$$\lim_{\gamma \to 1} f(\gamma) = 1 - \gamma = 0 \tag{28}$$

and that L'Hôpital's Rule is indeed applicable to the problem due to the indeterminate form $\frac{0}{0}$ at $\gamma = 1$. We can then further rewrite the expression in (25) as follows

$$U(Y) = \frac{Y^{1-\gamma}-1}{1-\gamma} = \frac{e^{\ln\left(Y^{1-\gamma}\right)}-1}{1-\gamma} = \frac{e^{(1-\gamma)\ln(Y)}-1}{1-\gamma} = \frac{f(\gamma)}{g(\gamma)} \tag{29}$$

The first derivatives with respect to γ are

$$f'(\gamma) = -\ln(Y) \cdot e^{(1-\gamma)\ln(Y)} \tag{30}$$

$$g'(\gamma) = -1 \tag{31}$$

Finally, applying L'Hôpital's Rule we find the desired results

$$\lim_{\gamma\to 1}\frac{-\ln(Y)\cdot e^{(1-\gamma)\ln(Y)}}{-1}=\frac{-\ln(Y)}{-1}=\ln(Y) \tag{32}$$

that CRRA power utility with $\gamma=1$ collapses to log utility in which $U(Y)=\ln(Y)$.

6. Continuously compounded returns

We wish to study the effect on interest rates when letting the compound frequency $m\to\infty$ for

$$\left(1+\frac{r}{m}\right)^{mt} \tag{33}$$

It turns out, that it is useful to start by taking the natural logarithm to the expression and re-arranging as follows

$$\ln\left(1+\frac{r}{m}\right)^{mt} \tag{34}$$

$$\Downarrow$$

$$mt\ln\left(1+\frac{r}{m}\right) \tag{35}$$

$$\Downarrow$$

$$\frac{t\ln\left(1+\frac{r}{m}\right)}{\frac{1}{m}} \tag{36}$$

The next step is to investigate what happens if we let $m\to\infty$. A useful trick in this respect is to use L'Hôpital's Rule.

L'Hôpital's Rule Suppose that $f(m)$ and $g(m)$ are differentiable functions. If $f(m)\to 0$ and $g(m)\to 0$ as $m\to\infty$, then

$$\lim_{m\to\infty}\frac{f(m)}{g(m)}=\lim_{m\to\infty}\frac{f'(m)}{g'(m)} \tag{37}$$

Applying the rule Let us see how we can apply this result to our problem at hand. We start by taking the limit to (36), i.e.

$$\lim_{m\to\infty}\frac{t\ln\left(1+\frac{r}{m}\right)}{\frac{1}{m}} \tag{38}$$

We notice that when $m\to\infty$ both the numerator and the denominator goes to zero, that is

$$\lim_{m\to\infty}t\ln\left(1+\frac{r}{m}\right)\to 0 \tag{39}$$

$$\lim_{m\to\infty}\frac{1}{m}\to 0 \tag{40}$$

implying that we can make use of L'Hôpital's Rule. To do so, we begin by determining the first derivative of the functions $f(m)$ and $g(m)$ with respect to m, i.e.

$$\frac{\partial f(m)}{\partial m} = \frac{\partial t \ln\left(1+\frac{r}{m}\right)}{\partial m} = \frac{t}{\left(1+\frac{r}{m}\right)}\left(-\frac{r}{m^2}\right) \tag{41}$$

$$\frac{\partial g(m)}{\partial m} = \frac{\partial \frac{1}{m}}{\partial m} = -\frac{1}{m^2} \tag{42}$$

Thus, by L'Hôpital's Rule, (38) can be written equivalently as

$$\lim_{m\to\infty} \frac{\frac{t}{\left(1+\frac{r}{m}\right)}\left(-\frac{r}{m^2}\right)}{-\frac{1}{m^2}} = \lim_{m\to\infty} r \cdot \frac{t}{\left(1+\frac{r}{m}\right)} \tag{43}$$

Then, letting $m \to \infty$, we get the following result

$$\lim_{m\to\infty} r \cdot \frac{1t}{\left(1+\frac{r}{m}\right)} = rt \tag{44}$$

The final step is then to realize that since we started out by taking the natural logarithm, we need to take the exponential to cancel out the log, i.e. we get the final results (where we abstract away from the m subscript on the rates)

$$e^{rt} \tag{45}$$

7. Optimization

One frequently encounters a myriad of different optimization (i.e. maximization and/or minimization) problems in economics and finance, where the goal is to determine the largest (or smallest) value a given function can take. These problems can either be unconstrained or constrained by a series of conditions that the solution is required to satisfy (e.g. return or volatility targets or budget constraints). As a classic example of an optimization problem, consider an investor that wishes to form a portfolio of risky assets. Investors are generally fond of returns, so an unconstrained problem could be to construct a portfolio that delivers the highest possible expected return. Investors, however, also care about the risk of their portfolios and, as such, tend to dislike volatility (a measure of risk). To take this aspect into account, the investor may instead consider a constrained optimization problem in which the objective is to maximize the return of the portfolio of risky assets subject to a target volatility level. A similar, and related, solution is to minimize volatility for a given target rate of return. In any case, we impose a set of constraints that the optimal solution must satisfy.

7.1. Unconstrained optimization

In an unconstrained optimization problem, agents face the problem of maximizing (or minimizing) some function $F(x)$ over all possible values of x. Formally,

$$\max_x \; F(x) \tag{46}$$

We, broadly speaking, have two alternative approaches to solve such a problem. First, we can search over all possible values of x to locate the maximum, but this can become cumbersome. A second, and preferred, option is to rely on calculus and solve for the optimal value of x, which we will refer to as $x^\star$, that satisfies the optimization problem.

Solution If $x^\star$ solves the problem in (46), then $x^\star$ is a critical point of $F(\cdot)$ so that

$$F'(x) = 0 \tag{47}$$

where $F'(x)$ denotes the first derivative of the objective function. This is known as the first-order condition (FOC) and is a necessary condition for an optimal solution. Note that the same FOC also characterizes the value $x^\star$ that minimizes a function, should that be the objective instead.

Graphical illustration Figures 1-3 provide a visual illustration of this principle. Figure 1 illustrates the case in which $x < x^\star$. In this case, we have that $F'(x) > 0$ so that $F(x)$ can be increased by increasing x. Figure 2 illustrates the case in which $x > x^\star$. In this case, we have that $F'(x) < 0$ so that $F(x)$ can be increased by decreasing x. Finally, Figure 3 illustrates the case in which $x = x^\star$. In this case, we have that $F'(x) = 0$ so that $F(x)$ is maximized (or minimized).

On the need for a second-order condition To verify that a maximum or minimum has been achieved, one can investigate the second-order conditions. In particular, for an unconstrained maximization problem we need $F'(x) = 0$ and $F''(x) < 0$ and for an unconstrained minimization problem we need $F'(x) = 0$ and $F''(x) > 0$. These first- and second-order conditions are necessary and sufficient conditions for achieving a (local) maximum or minimum.

It turns out, however, that the first-order condition $F'(x) = 0$ is both necessary and sufficient for (most of) our applications. If $F''(x) < 0$ for all possible values of x, that is $x \in \mathbb{R}$, then the function $F(\cdot)$ is said to be concave. As wee will see during the course, concave functions arise naturally in economics and finance and is the form taken by nearly all applicable utility functions. We will therefore nearly always be able to work with the first-order condition $F'(x) = 0$ only.

Figure 1: Suboptimal solution for $x < x^{\star}$

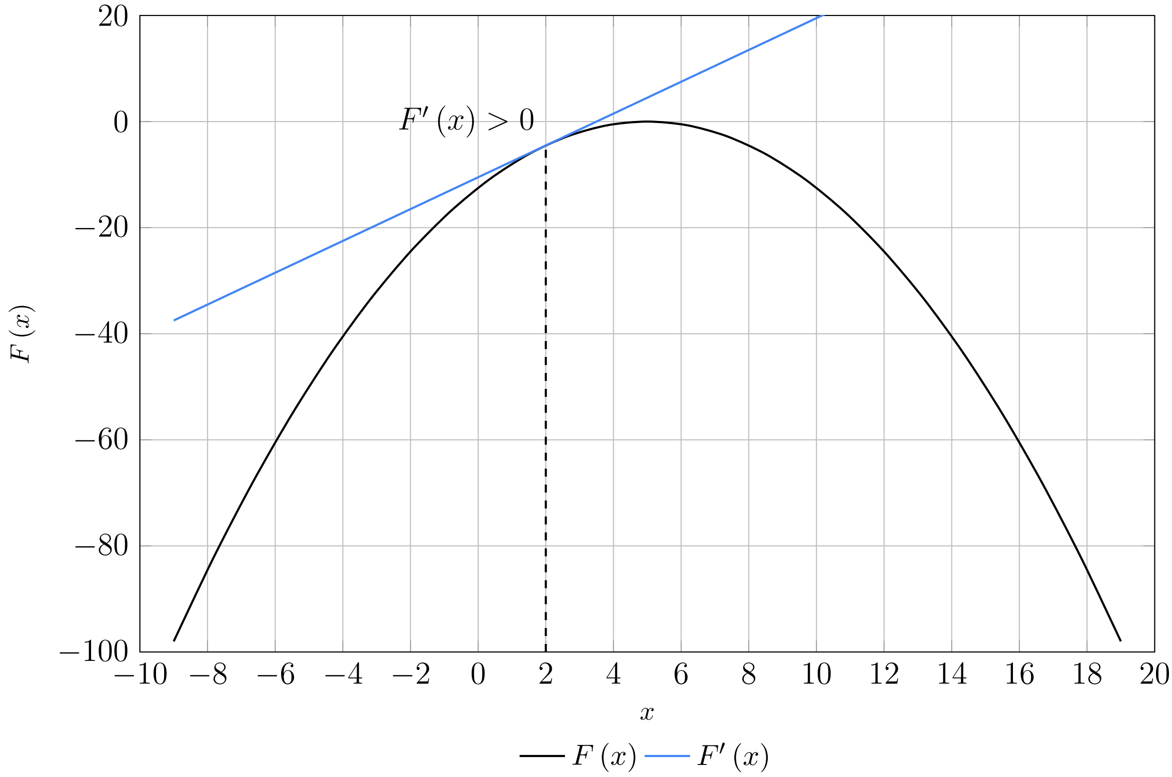


Figure 2: Suboptimal solution for $x > x^{\star}$

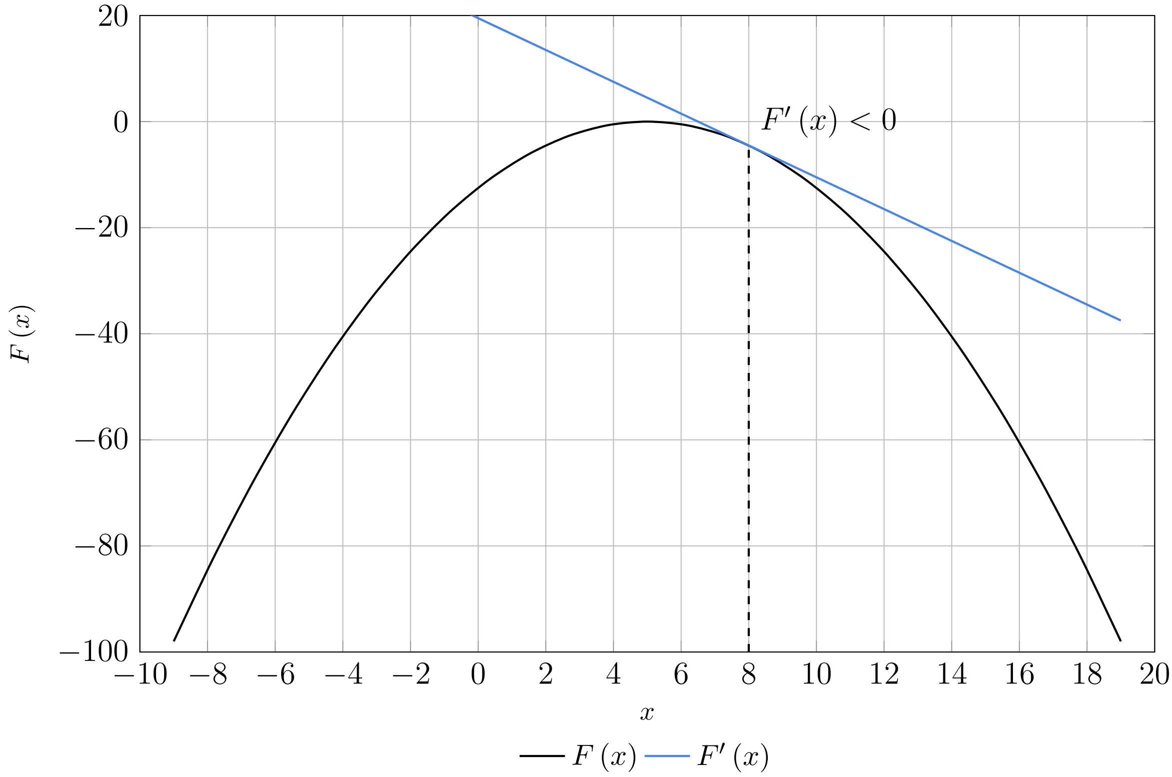
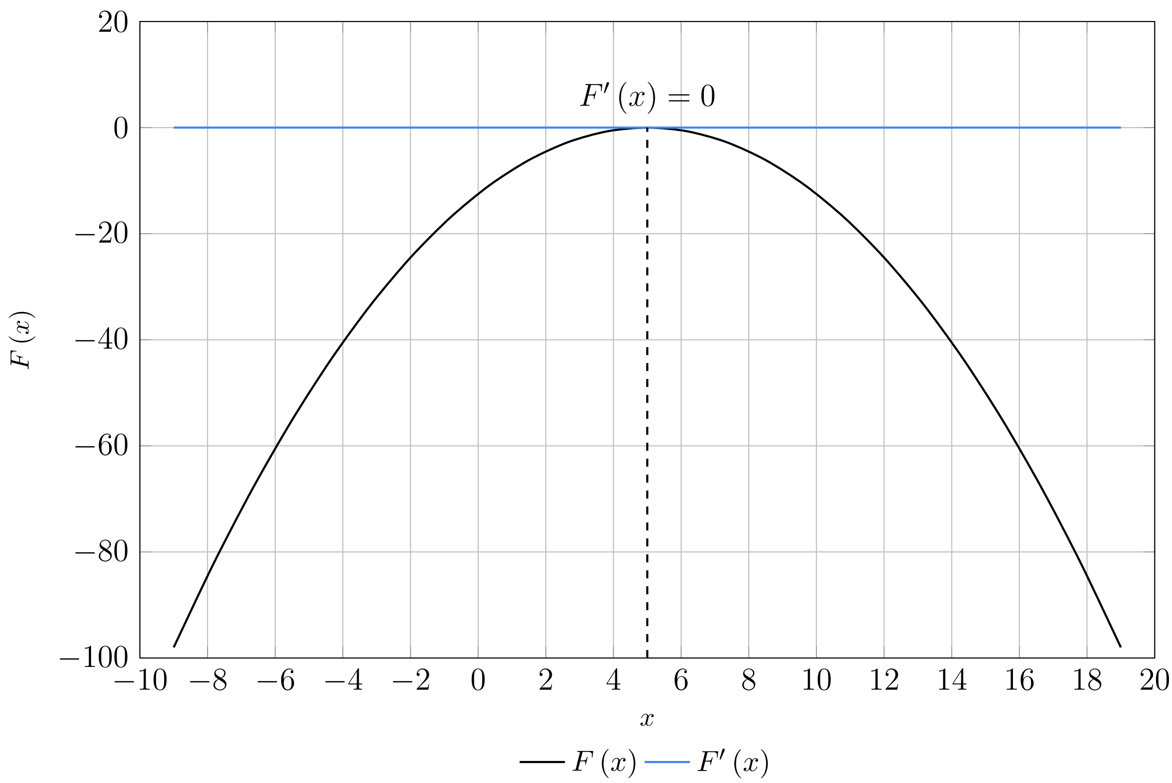


Figure 3: Optimal solution for $x = x^{\star}$



Example Consider the following maximization problem

$$\max_{x} \; F(x) = \left(-\frac{1}{2}\right)(x-\gamma)^2 \tag{48}$$

where γ is a real number ($\gamma \in \mathbb{R}$) that we can think of as some "target" that we want to achieve. This function is quadratic (and concave), so we only need to consider the first-order conditions, which is

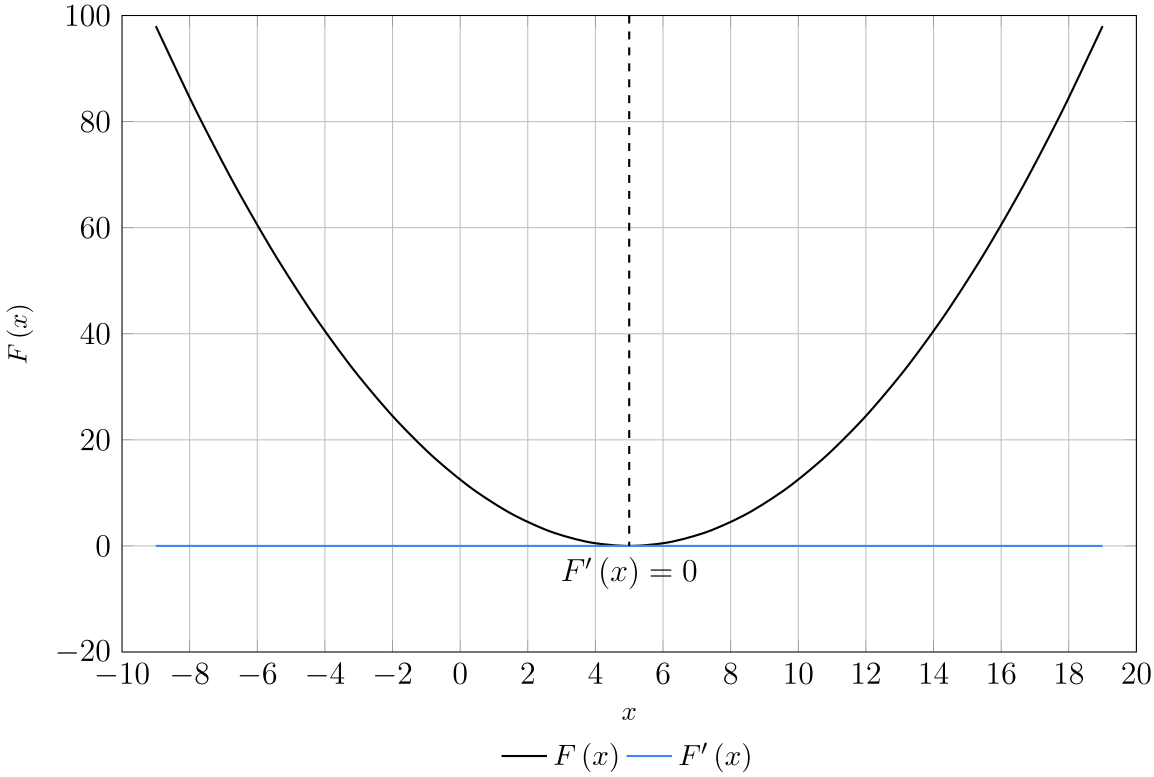
$$\frac{\partial F(x)}{\partial x} = -(x-\gamma) = 0 \tag{49}$$

which immediately leads us to the solution: $x^{\star} = \gamma$. This problem underlies Figures 1-3 with $\gamma = 5$. The problem in (48) can also be stated as a minimization problem by minimizing the negative of the function, i.e.

$$\min_{x} \; F(x) = \left(\frac{1}{2}\right)(x-\gamma)^2 \tag{50}$$

where we should find the same solution, i.e. $x^{\star} = \gamma$. This is illustrated in Figure 4.

Figure 4: Optimal solution for $x = x^{\star}$



7.2. Constrained optimization

In a constrained optimization exercise, on the other hand, we consider problems of the following form

$$\max_{x} \; F(x) \qquad \text{s.t. } c \geq G(x) \tag{51}$$

where c is some constant and $G(x)$ is some function of x. This is essentially the type of problem that one considers for optimal portolio selection. For instance, one can think about c as the target volatility level, $G(x)$ as the formula for the volatility of a portfolio, and $F(x)$ as the portfolio return. Alternatiely, c can denote the target portfolio return, $G(x)$ the portfolio return, and $F(x)$ the negative of volatility. As before, one can solve the problem either by searching over all possible values of x or use calculus. We will again prefer the latter. Although solving a constrained optimization problem may seem tricky, we can easily rely on the method developed by Joseph-Louis Lagrange for solving these kind of problems known as the Lagrange multiplier method.

Solution To find the $x^\star$ that solves the constrained optimization problem, define the Lagrangian

$$\mathcal{L} = F(x) + \lambda\left[c - G(x)\right] \tag{52}$$

where λ is the Lagrange multiplier (sometimes referred to as a shadow price in economics). If $x^\star$ maximizes $F(x)$ subject to $c \geq G(x)$, then there exist a value $\lambda^\star$ so that, together, $x^\star$ and $\lambda^\star$ satisfy the first-order condition

$$\frac{\partial \mathcal{L}}{\partial x} = F'(x^\star) - \lambda^\star G'(x^\star) = 0 \tag{53}$$

and the complementary slackness condition

$$\lambda^\star\left[c - G(x^\star)\right] = 0 \tag{54}$$

Non-binding constraints In the case where the constraint $c \geq G(x)$ is non-binding, then the complementary slackness condition

$$\lambda^\star\left[c - G(x^\star)\right] = 0 \tag{55}$$

requires that $\lambda^\star = 0$. And the first-order condition

$$F'(x^\star) - \lambda^\star G'(x^\star) = 0 \tag{56}$$

by extension requires that $F'(x^\star) = 0$. That is, we are squarely back in the setting of an unconstrained problem.

Example of non-binding constraint Consider the same problem as above, but let us introduce a simple constraint

$$\begin{aligned} &\max_x \; F(x) = \left(-\frac{1}{2}\right)(x-5)^2 \\ &\text{s.t. } 7 \geq x \end{aligned} \tag{57}$$

The Lagrangian for this problem is

$$\mathcal{L} = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(7-x) \tag{58}$$

The first-order condition is

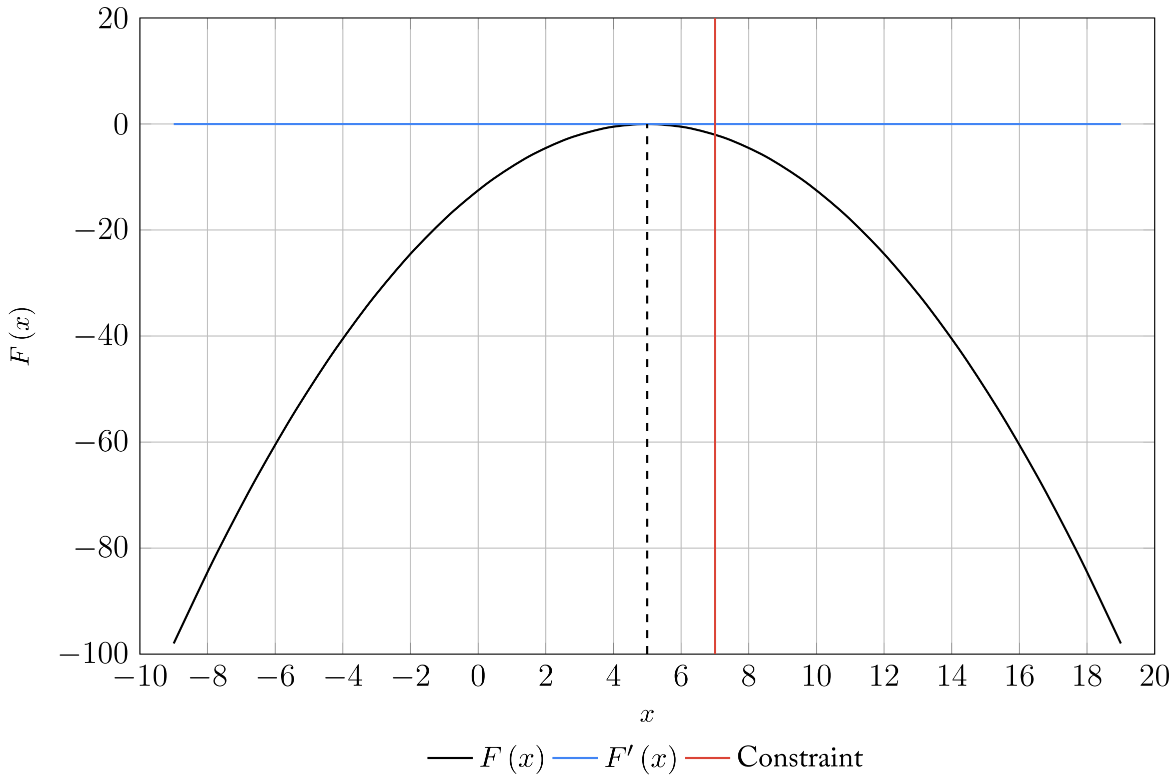
$$\frac{\partial \mathcal{L}}{\partial x} = -(x-5) - \lambda = 0 \tag{59}$$

and the complementary slackness condition is

$$\lambda (7 - x) = 0 \tag{60}$$

We can therefore see that the problem is solved by $x^\star = 5$ and $\lambda^\star = 0$ and that the constraint is non-binding since $x^\star = 5 < 7$ (the constraint, the red line, lies to the right of the optimal value of x). In this case, $F'(x) = 0$ as illustrated in Figure 5 by the blue line and optimization works essentially as in the unconstrained case.

Figure 5: Non-binding constraint solution for $x = x^\star$



Binding constraints In the case where the the constraint $c = G(x)$ is binding, then the complementary slackness condition

$$\lambda^\star [c - G(x^\star)] = 0 \tag{61}$$

puts no further restrictions on $\lambda^\star$. This implies that the first-order condition

$$F'(x^\star) - \lambda^\star G'(x^\star) = 0 \tag{62}$$

by extension requires that $F'(x^\star) = \lambda^\star G'(x^\star)$ for an optimal solution.

Example of binding constraint Consider the problem

$$\max_x \ F(x) = \left(-\frac{1}{2}\right)(x-5)^2 \tag{63}$$
$$\text{s.t. } 4 \geq x$$

The Lagrangian for this problem is

$$\mathcal{L} = \left(-\frac{1}{2}\right)(x-5)^2 + \lambda(4-x) \tag{64}$$

The first-order condition is

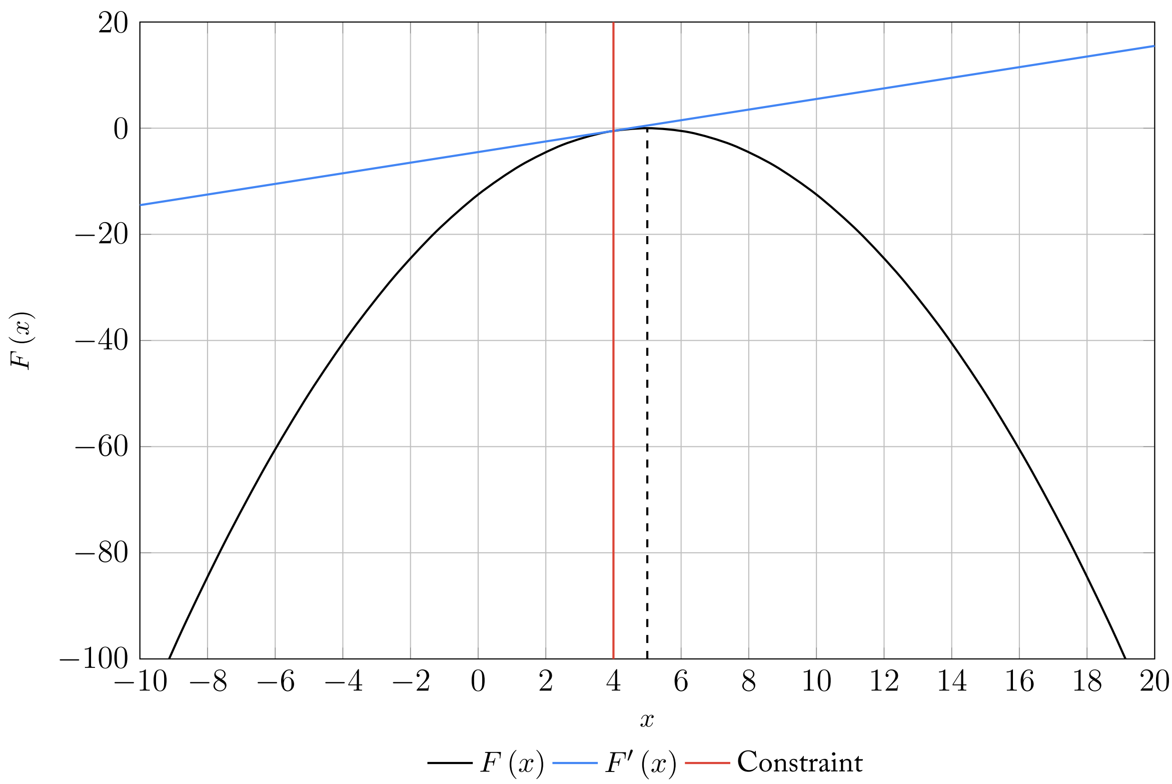
$$\frac{\partial \mathcal{L}}{\partial x} = -(x-5) - \lambda = 0 \tag{65}$$

and the complementary slackness condition is

$$\lambda(4-x) = 0 \tag{66}$$

We can therefore see that the problem is solved by $x^\star = 4$ and $\lambda^\star = 1$ and that the constraint is binding. In this case, $F'(x^\star) = 1$ and $\lambda^\star G'(x^\star) = 1$ as illustrated in Figure 6, where the red line is the constraint that is now binding.

Figure 6: Constrained solution for $x < x^\star$



Here, the solution has $F'(x^\star) = \lambda^\star G'(x^\star) > 0$ since the constraint is binding. $F'(x^\star) > 0$ indicates that we would like to increase x, but we are unable to do so due to the constraint. With a binding constraint, we therefore have that $F'(x^\star) \neq 0$, but we instead have that

$$F'(x^\star) - \lambda^\star G'(x^\star) = 0 \tag{67}$$

implying that the value $x^\star$ that solves the optimization problem is not a critical point in the objective function $F(x)$, but on the Lagrangian function $F(x) + \lambda [c - G(x)]$ instead.

Example of minimization problem Consider the following minimization problem

$$\begin{aligned} &\min_x \ F(x) = \left(\frac{1}{2}\right)(x-5)^2 \\ &\text{s.t. } 4 \geq x \end{aligned} \tag{68}$$

This problem corresponds to a situation in which we want to minimize, say, portfolio variance subject to a required rate of return x. The Lagrangian for this problem is

$$\mathcal{L} = \left(\frac{1}{2}\right)(x-5)^2 + \lambda(4-x) \tag{69}$$

The first-order condition is

$$\frac{\partial \mathcal{L}}{\partial x} = (x-5) - \lambda = 0 \tag{70}$$

and the complementary slackness condition is

$$\lambda(4-x) = 0 \tag{71}$$

We can therefore see that the problem is solved by $x^\star = 4$ and $\lambda^\star = -1$ and that the constraint is binding. In this case, $F'(x) = -1$ as illustrated in Figure 6, where the red line is the constraint.

Figure 7: Constrained solution for $x < x^{\star}$ in minimization