Asset Pricing

Capital Asset Pricing Model (CAPM)

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Outline

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The Capital Asset Pricing Model (CAPM)

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The Treynor (1961)–Sharpe (1964)–Lintner (1965)–Mossin (1966) Capital Asset Pricing Model (CAPM) is an equilibrium model defined by the linear relation

$$\mathbb{E}\left[\widetilde{r}_i\right] = r_f + \beta_{iM}\left(\mathbb{E}\left[\widetilde{r}_M\right] - r_f\right) \tag{1}$$

where $\widetilde{r}_i$ is the return on asset i, r_f is the risk-free rate, $\widetilde{r}_M$ is the return on the market portfolio, and $\beta_{iM} = \frac{\sigma_{iM}}{\sigma_M^2}$ measures the systematic risk of the ith asset

- The Capital Asset Pricing Model (CAPM) is an equilibrium theory built on the foundation of the mean-variance model (Markowitz, 1952)
- The CAPM therefore assumes that investors have mean-variance preferences. That is, either investors have quadratic utility, (log) returns are normally distributed, or the remainder term $\mathbb{E}\left[H_3\right]$ is small and negligible

CAPM as an equilibrium theory

* As an equilibrium theory, the CAPM assumes that supply equals demand in financial markets. Yet, the equilibrium imposed is somewhat peculiar

  1. The CAPM is a theory of financial equilibrium only, it does not provide a link between expected returns and the *real* economy
  2. It assumes that observed prices are equilibrium prices, i.e., that supply equals demand, but makes no attempt to model supply and demand functions explicitly
  3. The demand for assets is instead described by the mean-variance model
     - If individual i invests ω_{ij} of initial wealth Y_{0i} in asset j, then we can interpret the value of the holding $\omega_{ij}Y_{0i}$ as the individual's demand for asset j
     - If there are I individuals, then total holdings $\sum_{i=1}^{I}\omega_{ij}Y_{0i} = p_jQ_j$ can be interpreted as aggregate demand, where p_j is the equilibrium price and Q_j number of shares outstanding so that p_jQ_j corresponds to the market capitalization of asset j
  4. The CAPM in (1) therefore expresses equilibrium in terms of relationships between the return distributions of individual assets and that of the tangency portfolio

Assumptions and implications

The assumptions of the CAPM

1. Investors are rational, risk averse, and mean-variance optimizers
2. Investors have homogeneous expectations
3. All assets are tradable and infinitely divisible
4. There is perfect competition (i.e., investors are price takers)
5. There are no transaction costs and (personal) taxes
6. Unlimited short selling is allowed
7. Unlimited lending and borrowing at the risk-free rate

* These guarantee that the mean-variance efficient frontier is the same for every investor and, by the separation theorem, that all investors' optimal portfolios are of the same structure: a fraction of wealth is invested in the riskless asset, the rest in the tangency portfolio

Market equilibrium

- We can make the following set of observations from the above discussion
  - All investors choose mean-variance efficient portfolios by allocating wealth between the risk-free asset and the tangency portfolio
  - By the separation theorem, all investors demand risky assets in the same relative proportions given by the tangency portfolio. This must be aggregate demand
  - All existing risky assets must belong to the tangency portfolio T by the definition of equilibrium
    - If some asset k is not in T, then supply would exceed demand, which is inconsistent with the assumed financial equilibrium
    - Similarly, the share of any asset j in T must correspond to the ratio of the market value of that asset $p_j Q_j$ to the market value of all assets $\sum_{j=1}^{J} p_j Q_j$
    - Thus, the **tangency portfolio $\mathbf{T}$ must be none other than the market portfolio $\mathbf{M}$**

CAPM implication

A central and important prediction of the CAPM is therefore that the market portfolio is mean-variance efficient because it is located on the efficient part of the portfolio frontier (if $r_f < A/C$).

The Capital Market Line (CML)

The Capital Market Line (CML)

* In equilibrium, all efficient portfolios are located on the efficient frontier of all asset, which we now refer to as the Capital Market Line (CML)

* The slope of the CML is $(\mu_M - r_f)/\sigma_M$, which can be interpreted as the equilibrium price of risk per unit of standard deviation risk σ_p such that for efficient portfolios we have the simple linear relation

$$\mu_p = r_f + \frac{\mu_M - r_f}{\sigma_M}\sigma_p \tag{2}$$

* The CML applies to efficient portfolios, but is silent about the expected returns for arbitrary assets j that do not belong to the efficient frontier

* It therefore seems natural to study what we can say about inefficient assets within the framework as well

Sharpe's derivation of the CAPM

- Sharpe (1964) provides an early and classic reference for a traditional derivation of the Capital Asset Pricing Model
- Consider a portfolio with a fraction $1 - \alpha$ of wealth invested in asset j and a fraction α invested in the market portfolio

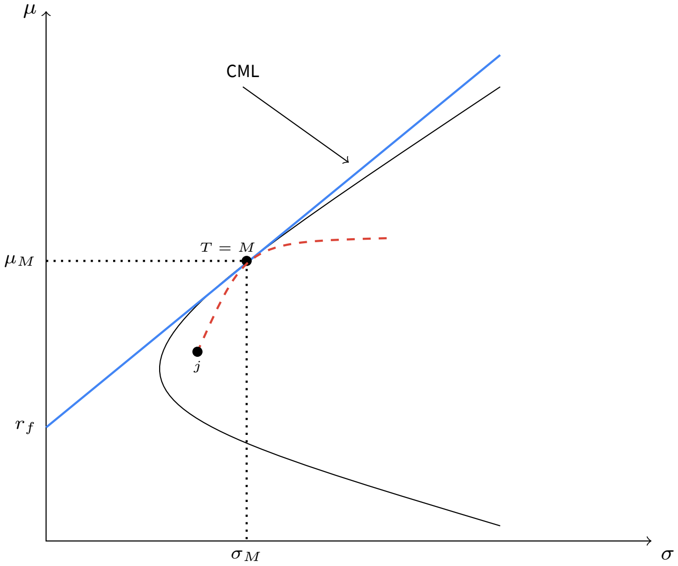
$$\mu_p = \alpha\mu_M + (1-\alpha)\,\mu_j \tag{3}$$

$$\sigma_p^2 = \alpha^2\sigma_M^2 + (1-\alpha)^2\,\sigma_j^2 + 2\alpha\,(1-\alpha)\,\sigma_{jM} \tag{4}$$

- As α varies we can trace a frontier that passes through j and M, but never crosses the CML (Why?) so must be tangent to the CML at M
- Defining $\beta_{jM} = \sigma_{jM}/\sigma_M^2$ and using that $\sigma_{jM} = \rho_{jM}\sigma_j\sigma_M$ one can show that

$$\mu_j = \underbrace{r_f + \beta_{jM}\left(\mu_M - r_f\right)}_{\text{Security Market Line}} = r_f + \left(\frac{\mu_M - r_f}{\sigma_M}\right)\rho_{jM}\sigma_j \tag{5}$$

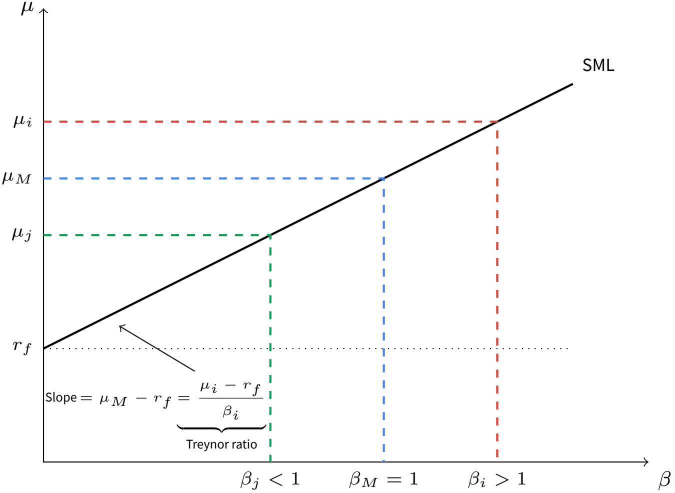
Capital Market Line (CML)



Remarks on systematic, idiosyncratic, and total risk

* The statement in (5) contains one of the most important lessons of the CAPM: investors are only compensated for a fraction ρ_{jM} of the risk of asset j
* The remaining fraction, $(1 - \rho_{jM})\,\sigma_j$, is known as idiosyncratic risk and has no "price" since investors can diversify it away by investing optimally
* Thus, the risk premium on any given asset is given as the market price of risk $(\mu_M - r_f)\,/\sigma_M$ multiplied by the quantity of risk $\rho_{jM}\sigma_j = \beta_{jM}\sigma_M \leq \sigma_j$
* $\rho_{jM}\sigma_j = \beta_{jM}\sigma_M$ measures the marginal contribution of asset j to the overall market portfolio, i.e. it is the systematic risk of asset j
* An efficient portfolio is one for which all diversifiable risk have been eliminated, i.e., where $\rho_{jM} = 1$
* For an efficient portfolio, total risk and systematic risk are one and the same

Security Market Line (SML)



Another look at systematic versus idiosyncratic risk

* We can write (5) as a linear regression for realized returns by allowing for a mean-zero error term $\widetilde{\varepsilon}_j$ such that

$$\widetilde{r}_j = r_f + \beta_{jM}\left(\widetilde{r}_M - r_f\right) + \widetilde{\varepsilon}_j \tag{6}$$

* Taking the variance operator to each side and using that Var $[r_f] = 0$ and that β_{jM} is a constant we obtain

$$\sigma_j^2 = \underbrace{\beta_{jM}^2 \sigma_M^2}_{\text{Systematic}} + \underbrace{\sigma_\varepsilon^2}_{\text{idiosyncratic}} \tag{7}$$

* Taking expectations to (6) yields the Security Market Line (SML) again

$$\mu_j = r_f + \beta_{jM}\left(\mu_M - r_f\right) \tag{8}$$

Review of the mean-variance model

* Recall the portfolio choice problem for N risky assets and one risk-free asset

$$\min_{\boldsymbol{\omega}} \ \frac{1}{2}\boldsymbol{\omega}^\top \mathbf{V}\boldsymbol{\omega} \tag{9}$$

$$\text{s.t. } \boldsymbol{\omega}^\top \boldsymbol{\mu} + \left(1 - \boldsymbol{\omega}^\top \mathbf{1}\right) r_f = \mu_p$$

where the optimal portfolio weights for the N risky assets are

$$\boldsymbol{\omega}_p^* = \frac{\mu_p - r_f}{H}\mathbf{V}^{-1}\left(\boldsymbol{\mu} - r_f \mathbf{1}\right) \tag{10}$$

with the constants A, B, C, and H defined by the expressions

$$A = \mathbf{1}^\top \mathbf{V}^{-1}\boldsymbol{\mu} \tag{11}$$

$$B = \boldsymbol{\mu}^\top \mathbf{V}^{-1}\boldsymbol{\mu} > 0 \tag{12}$$

$$C = \mathbf{1}^\top \mathbf{V}^{-1}\mathbf{1} > 0 \tag{13}$$

$$H = B - 2Ar_f + Cr_f^2 \tag{14}$$

Charecterizing efficient frontier portfolios

- The variance of any frontier portfolio can be written as

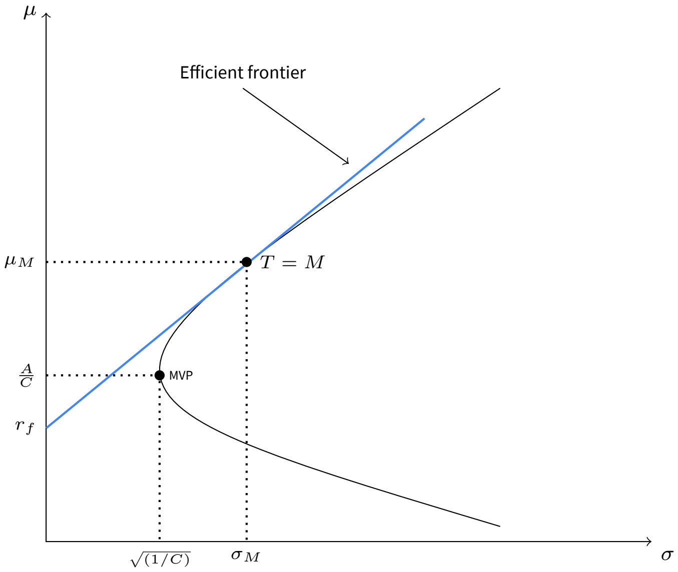
$$\sigma_p^2\left(\mu_p\right) = \boldsymbol{\omega}_p^{*\top}\mathbf{V}\boldsymbol{\omega}_p^* = \frac{\left(\mu_p - r_f\right)^2}{H} \tag{15}$$

- Taking the square root of each side of (15) and rearranging yields

$$\mu_p = r_f \pm \sigma_p H^{\frac{1}{2}} \tag{16}$$

which shows that the efficient frontier is linear and defined by a line that goes through r_f and the tangency portfolio

Illustrating the efficient frontier



Tangency portfolio

Tangency portfolio

The tangency portfolio is the only frontier portfolio composed only of risk assets, i.e., $\mathbf{1}^\top \boldsymbol{\omega}_T^* = 1$, whose weights are determined as

$$\boldsymbol{\omega}_T^* = \frac{1}{A - Cr_f}\mathbf{V}^{-1}\left(\boldsymbol{\mu} - r_f\mathbf{1}\right) \tag{17}$$

- To see this, note that $\mathbf{1}^\top \boldsymbol{\omega}_T^* = 1$ implies that

$$\mathbf{1}^\top \boldsymbol{\omega}_T^* = \frac{\mu_T - r_f}{H}\mathbf{1}^\top\mathbf{V}^{-1}\left(\boldsymbol{\mu} - r_f\mathbf{1}\right) = 1 \tag{18}$$

- Solving for μ_T yields the following expression

$$\mu_T = r_f + \frac{H}{A - Cr_f} \tag{19}$$

which when substituted back into (10) yields the final solution

Asset covariances with the tangency portfolio

* Let $\text{cov}\left[\widetilde{r}_q, \widetilde{r}_p\right] = \sigma_{q,p} = \boldsymbol{\omega}_q^\top \mathbf{V} \boldsymbol{\omega}_p$ denote the covariance between any portfolio q and any frontier portfolio p such that (using (10))

$$\sigma_{q,p} = \boldsymbol{\omega}_q^\top \mathbf{V} \frac{\mu_p - r_f}{H} \mathbf{V}^{-1} \left(\boldsymbol{\mu} - r_f \mathbf{1}\right) \tag{20}$$

$$= \frac{\mu_p - r_f}{H} \boldsymbol{\omega}_q^\top \left(\boldsymbol{\mu} - r_f \mathbf{1}\right) \tag{21}$$

$$= \frac{\left(\mu_p - r_f\right)\left(\mu_q - r_f\right)}{H} \tag{22}$$

* Solving for $\mu_q - r_f$, using that $\sigma_p^2 = \frac{\left(\mu_p - r_f\right)^2}{H}$, yields

$$\mu_q - r_f = \frac{H \sigma_{q,p}}{\mu_p - r_f} = \frac{\sigma_{q,p}}{\sigma_p^2} \left(\mu_p - r_f\right) \tag{23}$$

Obtaining the standard CAPM

- Recall that p is a frontier portfolio, so we can easily choose $p = T$. Moreover, in equilibrium, $T = M$, so we can also identify it as the market portfolio such that

$$\mu_q = r_f + \frac{\sigma_{qM}}{\sigma_M^2} \left(\mu_M - r_f \right) \tag{24}$$

- The final step is to use that $\beta_{qM} = \frac{\sigma_{qM}}{\sigma_M^2}$ such that we obtain

$$\mu_q = r_f + \beta_{qM} \left(\mu_M - r_f \right) \tag{25}$$

- Finally, noting that any risky asset j is itself a feasible portfolio, we obtain the Capital Asset Pricing Model (CAPM)

$$\mu_j = r_f + \beta_{jM} \left(\mu_M - r_f \right) \tag{26}$$

Economic interpretation of β_{jM}

The economic question

Suppose that Asset 1 is risker than Asset 2, that is, $\beta_1 > \beta_2$. Why does this imply that $\mu_1 > \mu_2$?

- Recall that all investors hold the market portfolio, so they like when markets are up, but dislike when market are down
- Due to having decreasing marginal utility (concave utility function), investors prefer additional payoffs in bad times to those in good times
- Thus, agents like asset with a low covariance with the market, i.e., those with a low (or even negative) beta because they provide consumption insurance
- Asset 2 offers a higher payoff in bad times, so investors are willing to pay more for such an asset (lower expected return)

Black's zero-beta CAPM

- What if no riskless asset exists so that all assets are risky, does a similar asset pricing relation then exists?
- Recall that one assumption underlying the standard CAPM was that investors could borrow and lend at the risk-free rate
- Fortunately, Fisher Black (1972), showed that there does indeed exists an asset pricing relation: the **zero-beta CAPM**

Proposition

For any frontier portfolio p, except for the minimum variance portfolio, there exists a unique frontier portfolio with which p has zero covariance

- We will refer to this portfolio as the zero-covariance portfolio relative to p and denote its vector of portfolio weights by $ZC(p)$
- Note that the MVP has a constant covariance of $\frac{1}{C}$ with all assets, resulting in the above exclusion

Proving the proposition

* To prove the above proposition, begin by setting the covariance between two frontier portfolios p and $ZC\left(p\right)$ equal to zero

$$\text{cov}\left[\widetilde{r}_p,\widetilde{r}_{ZC(p)}\right] = \sigma_{p,ZC(p)} = \frac{C}{D}\left[\mu_p - \frac{A}{C}\right]\left[\mu_{ZC(p)} - \frac{A}{C}\right] + \frac{1}{C} = 0 \qquad (27)$$

* Since A, C, and D are all scalars (numbers), we can solve for $\mu_{ZC(p)}$

$$\mu_{ZC(p)} = \frac{A}{C} - \frac{\frac{D}{C^2}}{\mu_p - \frac{A}{C}} \qquad (28)$$

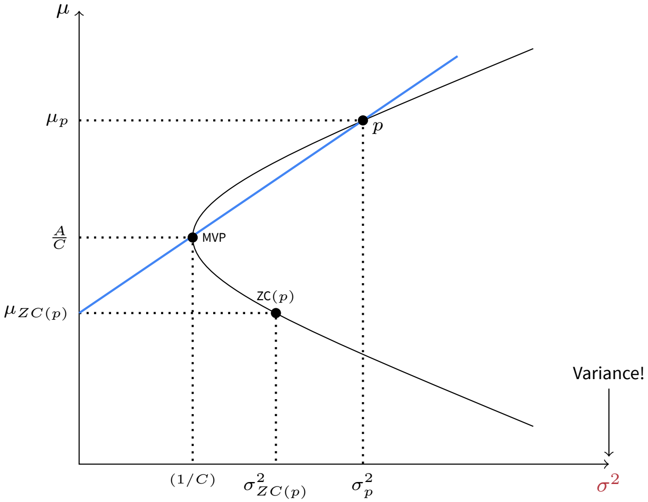
A statement about efficiency

Since A,C,$D > 0$, we can use (28) to make statements about the location of $\mu_{ZC(p)}$ for p other than the minimum variance portfolio. If p is an efficient portfolio, then $ZC\left(p\right)$ is an inefficient portfolio (and vice versa if p is an inefficient portfolio)

$$\mu_p > \frac{A}{C} \quad \Leftrightarrow \quad \mu_{ZC(p)} < \frac{A}{C} \qquad (29)$$

Illustrating the location of $ZC\,(p)$

* We can determine the location of $ZC\,(p)$ geometrically in mean-variance $\left(\mu - \sigma^2\right)$ space (and not standard deviation!)



Getting a line from $ZC(p)$ to p

- The line joining a frontier portfolio p and the minimum variance portfolio (MVP) can be shown to be

$$\mathbb{E}\left[\widetilde{r}\right] = \frac{A}{C} - \frac{\frac{D}{C^2}}{\mu_p - \frac{A}{C}} + \frac{\mu_p - \frac{A}{C}}{\sigma_p^2 - \frac{1}{C}}\sigma^2\left(\widetilde{r}\right) \tag{30}$$

- If $\sigma^2\left(\widetilde{r}\right) = 0$, then

$$\mathbb{E}\left[\widetilde{r}\right] = \frac{A}{C} - \frac{\frac{D}{C^2}}{\mu_p - \frac{A}{C}} = \mu_{ZC(p)} \tag{31}$$

  implying that the line joining a frontier portfolio p and the MVP has an intercept equal to the expected return on the zero-covariance portfolio

Covariance: Frontier portfolio p and any portfolio q

- Recall from our discussion of the efficient frontier from N risky asset and no riskless asset that frontier weights are given by

$$\boldsymbol{\omega}_p^* = \underbrace{\frac{C\mu_p - A}{D}}_{\lambda} \mathbf{V}^{-1}\boldsymbol{\mu} + \underbrace{\frac{B - A\mu_p}{D}}_{\gamma} \mathbf{V}^{-1}\mathbf{1} \tag{32}$$

- Let p be a frontier portfolio other than the MVP and let q be any portfolio (which might not be on the portfolio frontier). Their covariance is

$$\begin{aligned}
\sigma_{p,q} &= \boldsymbol{\omega}_p^\top \mathbf{V}\boldsymbol{\omega}_q \\
&= \left[\lambda \mathbf{V}^{-1}\boldsymbol{\mu} + \gamma \mathbf{V}^{-1}\mathbf{1}\right]^\top \mathbf{V}\boldsymbol{\omega}_q \\
&= \lambda \boldsymbol{\mu}^\top \mathbf{V}^{-1}\mathbf{V}\boldsymbol{\omega}_q + \gamma \mathbf{1}^\top \mathbf{V}^{-1}\mathbf{V}\boldsymbol{\omega}_q \\
&= \lambda \mu_q + \gamma
\end{aligned} \tag{33}$$

Defining the return on the portfolio q

- Substituting in the expressions for λ and γ in (33) yields

$$\sigma_{p,q} = \lambda\mu_q + \gamma = \frac{C\mu_p - A}{D}\mu_q + \frac{B - A\mu_p}{D} \tag{34}$$

- Solving for μ_q yields

$$\mu_q = \frac{A\mu_p - B}{C\mu_p - A} + \sigma_{p,q}\frac{D}{C\mu_p - A} \tag{35}$$

$$= \frac{A}{C} - \frac{\frac{D}{C^2}}{\mu_p - \frac{A}{C}} + \frac{\sigma_{p,q}}{\sigma_p^2}\left[\frac{1}{C} + \frac{\left(\mu_p - \frac{A}{C}\right)^2}{\frac{D}{C}}\right]\frac{D}{C\mu_p - A} \tag{36}$$

$$= \mu_{ZC(p)} + \beta_{qp}\left(\mu_p - \frac{A}{C} + \frac{\frac{D}{C^2}}{\mu_p - \frac{A}{C}}\right) \tag{37}$$

$$= \mu_{ZC(p)} + \beta_{qp}\left(\mu_p - \mu_{ZC(p)}\right) \tag{38}$$

Obtaining the zero-beta CAPM

- The previous slide tells us that the expected return on any portfolio q is a linear combination of the expected return on a frontier portfolio p and its zero-covariance portfolio $ZC\left(p\right)$
- As before, since p is a frontier portfolio, we can simply choose M (provided that the market portfolio is not the MVP) such that

$$\mu_q = \mu_{ZC(M)} + \beta_{qM}\left(\mu_M - \mu_{ZC(M)}\right) \tag{39}$$

Black's zero-beta CAPM

Finally, since any single risky asset j is itself a feasible portfolio, we can rewrite (39) to obtain Black's (1972) zero-beta CAPM

$$\mu_j = \mu_{ZC(M)} + \beta_{jM}\left(\mu_M - \mu_{ZC(M)}\right) \tag{40}$$

which holds if the market portfolio is an efficient portfolio (recall (29)).

The market portfolio is efficient

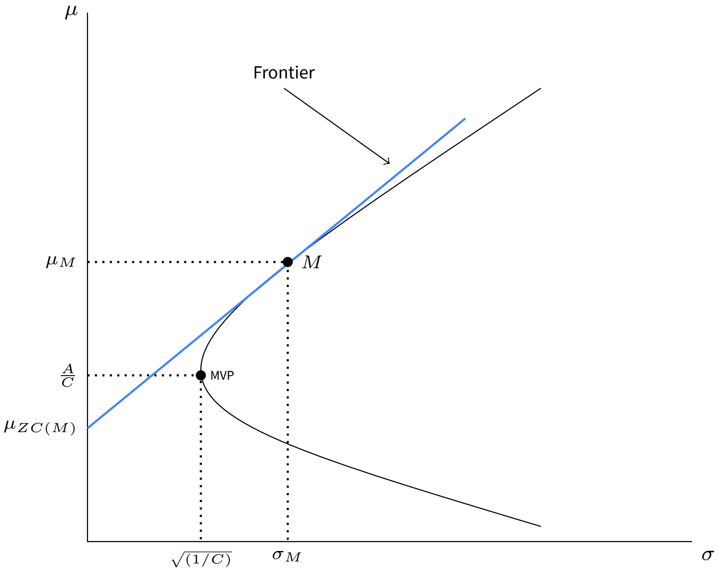
- Suppose that there are $i = 1, 2, \ldots, I$ investors in the economy owning a proportion W_i of total wealth

- Each investor i chooses an efficient frontier portfolio $\boldsymbol{\omega}^*_{i,p} = \mathbf{g} + \mathbf{h}\mu_{i,p}$, so that

$$
\begin{aligned}
\boldsymbol{\omega}_M &= \sum_{i=1}^{I} W_i \boldsymbol{\omega}^*_{i,p} && (41)\\
&= \sum_{i=1}^{I} W_i \left(\mathbf{g} + \mathbf{h}\mu_{i,p}\right) && (42)\\
&= \mathbf{g}\sum_{i=1}^{I} W_i + \mathbf{h}\sum_{i=1}^{I} W_i \mu_{i,p} && (43)\\
&= \mathbf{g} + \mathbf{h}\mu_M, && (44)
\end{aligned}
$$

implying that the market portfolio, being a linear combination of efficient frontier portfolios, must itself be an efficient frontier portfolio!!

Illustrating the zero-beta CAPM



Remarks on the zero-Beta CAPM

- Like the standard CAPM, the zero-beta CAPM is an equilibrium theory
- Both depend on the market portfolio M being efficient and observable
- The name zero-beta CAPM originates from the observation that

$$\beta_{M,ZC(M)} = \frac{\text{cov}\left[\widetilde{r}_m, \widetilde{r}_{ZC(M)}\right]}{\sigma^2_{ZC(M)}} = 0 \tag{45}$$

  which holds by construction of $ZC\left(M\right)$ as a zero-covariance portfolio with respect to the market portfolio M

Roll's (1977) critique

Roll's (1977) critique

Roll's (1977) critique is an important analysis of the validity of empirical tests of the CAPM that centers on two observations

1. **Mean-variance tautology:** Any mean-variance efficient portfolio p satisfies the CAPM relation exactly
$$\mu_j = r_f + \beta_{jp}\left(\mu_p - r_f\right) \tag{46}$$
implying that any test of the CAPM is really just a test for the mean-variance efficiency of the chosen proxy for the market portfolio

2. **Unobservability of the market portfolio:** The market should, in theory, contain every single risky assets, including stocks, bonds, currencies, housing, consumer durables, precious metals, human capital, and so forth

* Thus, if we reject the CAPM, we may simply be rejecting the mean-variance efficiency of the proxy, not the theory itself (joint hypothesis problem). That is, we learn nothing about the CAPM (since we cannot test it)

Testing the CAPM empirically

- Under the CAPM, *all* assets should lie on the Security Market Line (SML)

$$\mathbb{E}\left[\widetilde{r}_i\right] = r_f + \beta_{iM}\left(\mathbb{E}\left[\widetilde{r}_M\right] - r_f\right) \tag{47}$$

  which gives rise to the following set of implications

  1. Investors are well diversified (they hold units of the market portfolio), implying that idiosyncratic risks are irrelevant
  2. The relation between the asset's expected return and β_{iM} is linear
  3. Only β_{iM} matters for the determination of an asset's expected return
  4. An asset with $\beta_{iM} = 0$ has an expected return of r_f
  5. An asset with $\beta_{iM} = 1$ has an expected return of μ_M

- Empirical evidence suggest a strong rejection of the CAPM, making it an empirical failure (Is this surprising?)

Fama-MacBeth two-pass cross-sectional regressions

- Fama and MacBeth (1973) develop a simple, but powerful, approach to testing the CAPM in the cross-sectional dimension
- We can re-write the model in a way that facilitates cross-sectional testing by defining $\gamma_M = \mathbb{E}\left[\widetilde{r}_M\right] - r_f$ so that

$$\mathbb{E}\left[\widetilde{r}_i\right] = r_f + \gamma_M \beta_{iM} \tag{48}$$

- In this formulation, we can view β_{iM} as the amount of risk and γ_M as the price of risk per unit of β_{iM} risk
- Note that this implies a testable implication that the estimated price of risk should equal the mean excess return of the factor (if traded)

Working with excess returns and implications

Beta-pricing representation

It is often convenient to work with excess returns instead, which gives rise to the following specification

$$\mathbb{E}\left[\widetilde{r}_i\right] - r_f = \gamma_0 + \gamma_M \beta_{iM} \tag{49}$$

* In this case, the CAPM implications are as follows
  1. β_{iM} is the only thing that matters ($\gamma_0 = 0$)
  2. There is a positive and linear relationship between risk and return ($\gamma_M > 0$)
* We can also test if β_{iM} is the only source of systematic risk by adding more factors to the model

$$\mathbb{E}\left[\widetilde{r}_i\right] - r_f = \gamma_0 + \gamma_M \beta_{iM} + \gamma_X X_i \tag{50}$$

where the additional null hypothesis is $\gamma_X = 0$ for the CAPM to hold true

Estimation of the time series β

First-pass regression

In the first-pass, time series regression, we estimate the β for each asset or portfolio $i = 1, \ldots, N$ using the regression

$$\widetilde{r}_{t,i} - \widetilde{r}_{t,f} = \alpha_i + \beta_{iM} \left(\widetilde{r}_{t,M} - \widetilde{r}_{t,f} \right) + \widetilde{\varepsilon}_{t,i} \tag{51}$$

- We often use portfolios as test assets to reduce noise from imprecisely estimated betas for individual assets
- While Fama and MacBeth (1973) originally used rolling five-year regressions, we will use full sample betas instead for convenience (Fama and French, 1992, Cochrane, 2005)
- The implicit assumption of constant betas is unlikely to hold for individual assets, but more reasonable for portfolios with constant characteristics

Estimation of the cross-sectional risk premium γ

Second-pass regression

In the second-pass, we run cross-sectional regressions for each period $t = 1, \dots, T$ of all assets against estimated betas

$$\widetilde{r}_{t,i} - \widetilde{r}_{t,f} = \gamma_{t,0} + \gamma_{t,M}\widehat{\beta}_{iM} + \eta_{t,i} \tag{52}$$

where the estimated value $\widehat{\beta}_{iM}$ from the first-pass regression as the regressor

- This provides us with a time series of estimates $\{\widehat{\gamma}_{t,0}, \widehat{\gamma}_{t,M}\}$, which can be used to form final estimates of γ_0 and γ_M as follows

$$\widehat{\gamma}_j = \frac{1}{T}\sum_{t=1}^{T}\widehat{\gamma}_{t,j} \tag{53}$$

- A similar estimator can be used to obtain average pricing errors using $\widehat{\eta}_i$

Estimation of standard errors for hypothesis testing

- Fama and MacBeth (1973) propose to use the time series of cross-sectional estimates to estimate standard errors that account for cross-sectional correlation

$$\sigma^2\left(\gamma_j\right) = \frac{1}{T^2}\sum_{t=1}^{T}\left(\widehat{\gamma}_{t,j} - \widehat{\gamma}_j\right)^2 \tag{54}$$

- We can then conduct testing and inference for the null hypothesis $\mathcal{H}_0 : \gamma_j = 0$ in the standard way, i.e.,

$$t\left(\gamma_j\right) = \frac{\widehat{\gamma}_j}{\sigma\left(\gamma_j\right)} \xrightarrow{d} \mathcal{N}\left(0,1\right) \quad \text{as} \quad T \to \infty \tag{55}$$

where the test statistic is student t distributed with $T - K - 1$ degrees of freedom in small samples and asymptotically standard normal

Drawback of the Fama-MacBeth approach

* The Fama-MacBeth approach, while simple and highly useful, does have several problems

1. The inputs, $\mathbb{E}[\widetilde{r}_i]$, $\mathbb{E}[\widetilde{r}_M]$ and β_{iM}, are unobservable and have to be estimated
2. This gives rise to a so-called errors-in-variables problem as we are using estimated β_{iM}s in the second-stage cross-sectional regression
   - The errors-in-variables problem biases standard errors and biases γ_M downwards
   - To reduce the problem, one can either group stocks into portfolio to get better β_{iM} estimates (not a complete resolution)
   - or we can explicitly adjust standard errors to account for the bias introduced by the errors-in-variables problem using, e.g., Shanken (1992) (outside the scope of this course, see Kroencke and Thimme (2021) for a recent overview and review)
3. The approach does not account for autocorrelation and heteroskedasticity
   - One can solve this by using the Generalized Method of Moments approach of Hansen (1982) instead (outside the scope of this course)
4. There may be issues with general measurement errors and omitted variable biases (Giglio and Xiu, 2021)

Getting γ in a single regression

* If we are only interested in the estimates of $\widehat{\gamma}_0$ and $\widehat{\gamma}_M$, then we can simply run one single cross-sectional regression on time series averages, i.e.

$$\mathbb{E}\left[\widetilde{r}_i\right] - \widetilde{r}_f = \gamma_0 + \gamma_M \widehat{\beta}_{iM} + \eta_i \tag{56}$$

  where we, under rational expectations, can replace $\mathbb{E}\left[\widetilde{r}_i\right] - \widetilde{r}_f$ with the sample average

$$\mathbb{E}\left[\widetilde{r}_i\right] - \widetilde{r}_f = \frac{1}{T}\sum_{t=1}^{T} \widetilde{r}_{t,i} - \widetilde{r}_{t,f} \tag{57}$$

* This will provide us with numerically equivalent estimates of the factor risk premiums

* Note, however, that standard OLS standard errors will be very wrong!

Banz (1981) and the size effect

The size premium

The size premium of Banz (1981) refers to the empirical observation that smaller firms earns higher risk adjusted returns, on average, than larger firms

- Banz began a grand tradition by sorting stocks according to characteristics unrelated to a stock's market beta
- Consider his seminal approach below, in which we form 25 portfolios by double sorting on size and market beta
- As is standard, we annualize monthly excess returns by multiplying with 12

	Size (ME)				
β	Small	ME-2	ME-3	ME-4	Large
Low-β	8.79	8.47	8.22	8.12	6.25
β-2	10.46	10.38	10.47	9.44	6.78
β-3	10.62	11.35	9.93	8.99	6.36
β-4	11.63	10.56	9.54	7.39	6.41
High-β	8.74	8.32	8.69	9.07	5.38

Testing if the size effect matters

* Suppose that we replicate the results of Banz (1981) by running his second-pass regression of the form

$$\widetilde{r}_{t,i} - \widetilde{r}_{t,f} = \gamma_{t,0} + \gamma_{t,M}\widehat{\beta}_{iM} + \gamma_{t,ME}\left[\frac{ME_{i,t-1} - ME_{M,t-1}}{ME_{M,t-1}}\right] + \eta_{t,i} \quad (58)$$

where $ME_{i,t-1}$ is the average market value of the stocks in portfolio i and $ME_{M,t-1}$ the average market value of all stocks in the market portfolio, and the test assets are the 25 size-β sorted portfolios

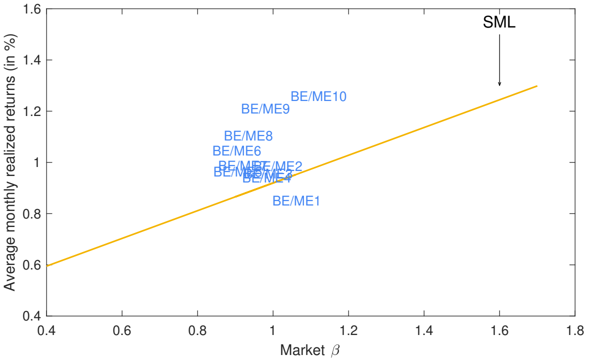
	γ_0	γ_M	γ_{ME}	R^2 (%)
Estimate	0.81	-0.07	-0.07	61.29
s.e.	(0.17)	(0.23)	(0.03)	
tstat	[4.92]	[-0.30]	[-2.68]	

The value premium

Value premium

The value premium (Stattmann, 1980, Rosenberg et al., 1985) refers to the empirical observation that stocks with high book-to-market equity (BE/ME) ratios (value stocks), on average, earn higher rates of returns than do low BE/ME (growth) stocks

- We can illustrate the value premium by comparing the realized mean returns to those predicted by the CAPM (updated Figure 3 in Fama and French (2004))



Fama and French (1992) and the book-to-market effect

* In another fatal blow to the CAPM, Fama and French (1992) document that

  1. The market β does a poor job at explained the cross-section of average asset returns
  2. Size and book-to-market equity capture the cross-sectional variation in average asset returns quite well
  3. The combination of size and book-to-market equity absorbs the explanatory power of leverage (Bhandari, 1988) and the earnings-price (E/P) ratio (Basu, 1983)
  4. The book-to-market equity ratio seems to be the most important variable

* This article lays the foundation for the infamous Fama-French 3-factor model, which we will review in more detail later in this course

Fama and French (1992) Fama-MachBeth results

β	ln(ME)	ln(BE/ME)	ln(A/ME)	ln(A/BE)	E/P Dummy	E(+)/P
0.15 (0.46)						
	−0.15 (−2.58)					
−0.37 (−1.21)	−0.17 (−3.41)					
		0.50 (5.71)				
			0.50 (5.69)	−0.57 (−5.34)		
					0.57 (2.28)	4.72 (4.57)
	−0.11 (−1.99)	0.35 (4.44)				
	−0.11 (−2.06)		0.35 (4.32)	−0.50 (−4.56)		
	−0.16 (−3.06)				0.06 (0.38)	2.99 (3.04)
	−0.13 (−2.47)	0.33 (4.46)			−0.14 (−0.90)	0.87 (1.23)
	−0.13 (−2.47)		0.32 (4.28)	−0.46 (−4.45)	−0.08 (−0.56)	1.15 (1.57)

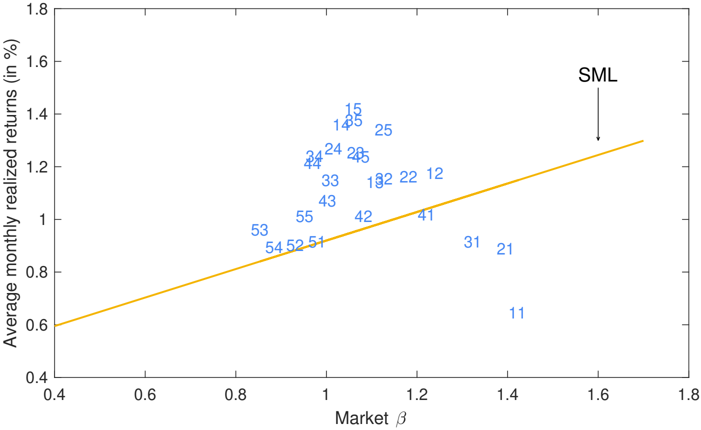
Size and value sorted portfolios

- Fama and French (1992) proceed to construct a set of portfolios sorted on size and book-to-market equity (commonly referred to as value), which we show below for a 5×5 intersection

	Book-to-market equity (BE/ME)				
Size (ME)	Low	2	3	4	High
	Panel A: Mean annualized returns				
Small	3.19	9.56	9.16	11.76	12.50
2	6.12	9.40	10.49	10.66	11.53
3	6.44	9.31	9.24	10.33	11.97
4	7.67	7.60	8.31	10.02	10.30
Large	6.43	6.28	6.98	6.18	7.60
	Panel B: Time series betas				
Small	1.42	1.24	1.11	1.03	1.06
2	1.40	1.18	1.07	1.02	1.13
3	1.32	1.13	1.01	0.98	1.06
4	1.22	1.08	1.00	0.97	1.08
Large	0.98	0.93	0.85	0.89	0.95

Size and value are detrimental to the CAPM

* As before, let us gauge how CAPM fares in capturing size and value effects simultaneously (first number is size quantile, second is value quantile)



Testing the CAPM on size-value portfolios

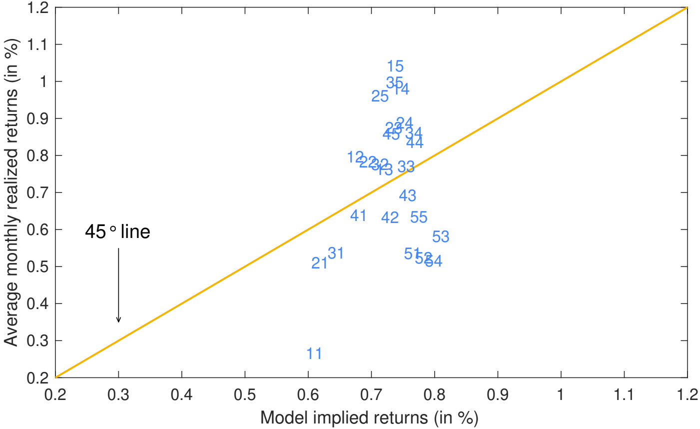
- Suppose that we want to test the CAPM using the 25 size-value portfolios as test assets over the period 1963:07 to 2019:12 (to avoid Covid-19 influence)

- If we estimate the CAPM using the Fama and MacBeth (1973) method, we obtain the following results

	γ_0	γ_M	R^2 (%)
Estimate	1.11	-0.35	7.38
s.e.	(0.37)	(0.40)	
t-stat	[2.99]	[-0.87]	

Pricing error plot for the CAPM

- Another way to illustrate the same failure as above is to plot realized returns against model implied returns, i.e., to visualize cross-sectional pricing errors
- Fitted values are computed as $\text{fit}_i = \widehat{\gamma}_0 + \widehat{\gamma}_M \widehat{\beta}_i$

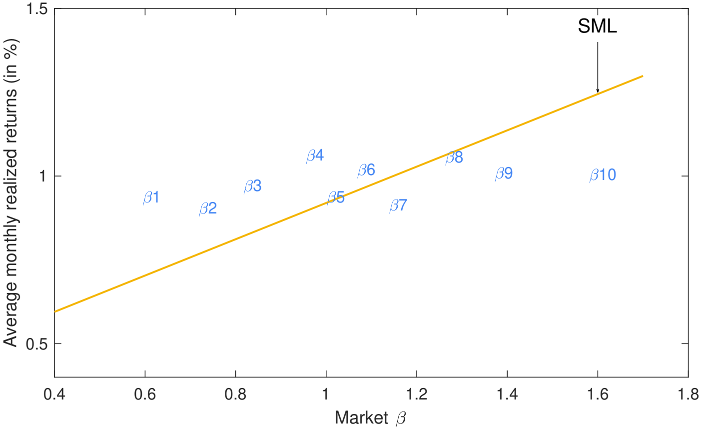


Low risk anomalies

- Recall that a central prediction of the CAPM is that assets with more undiversifiable risk (higher β) should earn higher expected returns
- However, there is a large body of research documenting that the empirical Security Market Line (SML) is *too* flat relative to the CAPM or even negative
- Generally speaking, low risk anomalies refer to puzzling relations between expected returns and
  1. Market betas
  2. Total risk (or volatility)
  3. Idiosyncratic risk (or volatility)

Low risk anomalies: Beta

* Black et al. (1972) and Frazzini and Pedersen (2014), among others, document that portfolios sorted on ex ante betas do not match up with the CAPM
* Fama and French (2004) also make this point in their Figure 2, and we will consider an updated version for ten beta-sorted portfolios below



Low risk anomalies: Volatility

- Portfolios of assets with high volatility should, according to the CAPM, earn higher expected rates of return
- However, Ang et al. (2006, 2009) show that (idiosyncratic) volatility negatively predicts equity returns and that stocks with high sensitivities to aggregate volatility risk earn low returns

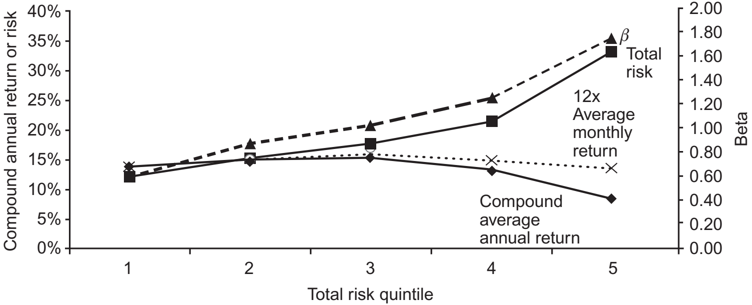


Figure 8.7

Low risk anomalies: Idiosyncratic volatility

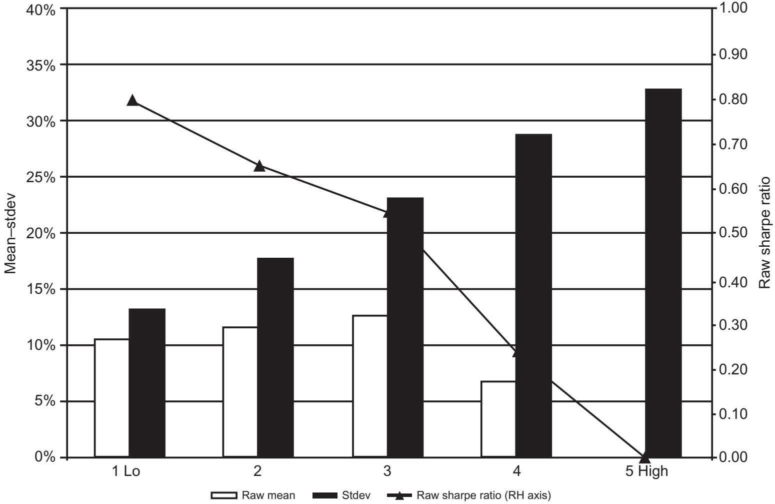


Figure 8.8

Average returns and idiosyncratic volatility: idiosyncratic volatility and subsequent returns.

Possible explanations for low risk anomalies

* Frazzini and Pedersen (2014) build on the idea of Black (1972) that restrictions to borrowing affect the shape of the security market line (SML)
  - I particular, leverage constrained investors (subject to borrowing constraints) bid up high-β asset which in turn generate low risk-adjusted returns
  - They further show that a simply strategy, known as betting against beta (BAB), which is long leveraged low-β assets and short high-β assets, produces significant positive risk-adjusted returns
* The flatness of the empirical SML may also be caused by high demand for "lottery stocks"
  - These are stocks of near-to-bankruptcy firms that have characteristics (low prices, tiny probabilities of gigantic upward price increases) similar to a lottery ticket
  - Kumar (2009) finds, at the aggregate level, that individual investors prefer stocks with lottery features, and like lottery demand, the demand for lottery-type stocks increases during economic downturns (see also Bali et al. (2017))
  - This demand will drive up prices of high-β assets, again attributing to the flatness of the empirical SML
* Schneider et al. (2020) argue that these low risk anomalies arise from a demand for compensation for skewness and co-skewness risk

Ressurections of the CAPM?

* Albeit the CAPM is usually strongly rejected in the data, there are instances when the model's predictions are resurrected

  1. The CAPM holds in January, but not during other months in the year (Tinic and West, 1984)
  2. The CAPM holds on days with important scheduled macroeconomic announcement such as monetary policy (Lucca and Moench, 2015, Savor and Wilson, 2014)
  3. The CAPM holds on days with Federal Open Market Committee (FOMC) press conferences, but not other days (Bodilsen et al., 2021)
  4. THe CAPM holds on days when influential firms disclose quarterly earnings news early in the earnings season (Chan and Marsh, 2021)
  5. The CAPM holds overnight, but not during the day (Hendershott et al., 2020)

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