Asset Pricing

Term structure models

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Outline

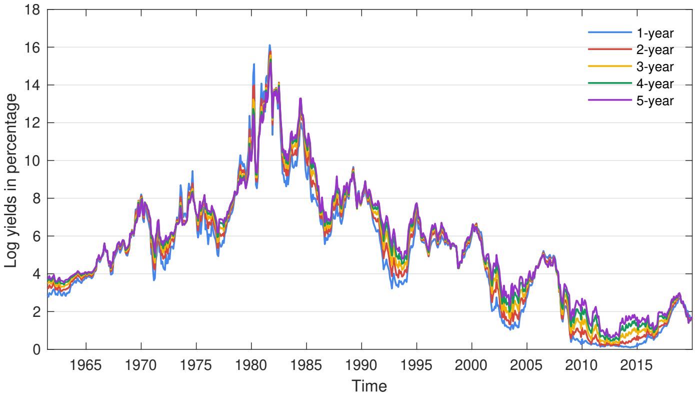
1. Affine term structure models
   - Homoskedastic single-factor model
   - Square-root single-factor model
   - Two-factor model

2. The nominal term structure

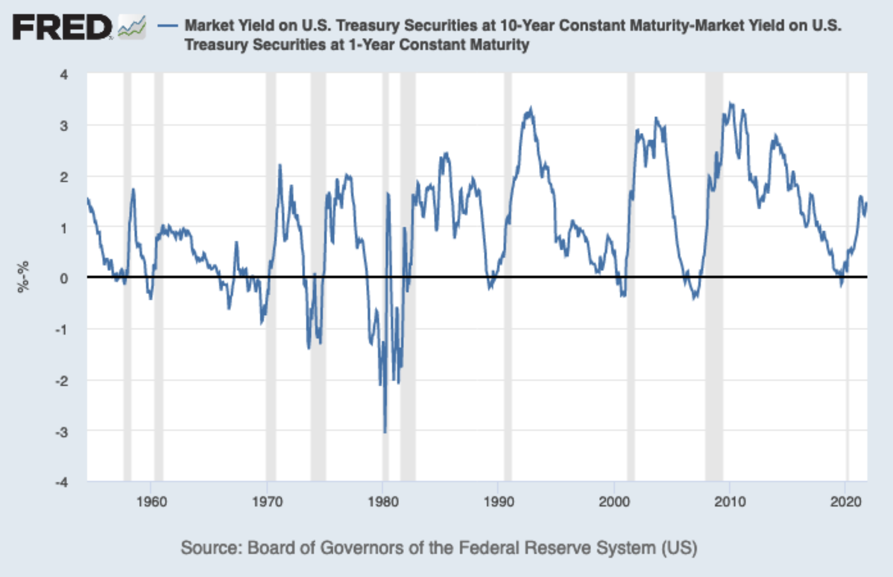
3. Chan et al. (1992): Evidence on the short rate process

The term structure of interest rates

* We are interested in developing fully specified equilibrium models of the term structure of interest rates using a discrete time approach
* The models should be able to match stylized facts about the yield curve
  - Upward sloping on average, but occasionally inverts and features humps
  - Yields move together and are relatively persistent across time



Yield curve inversion



A familiar asset pricing equation

Asset pricing equation

We will depart from the general asset pricing condition introduced in connection with the consumption-based model, i.e.

$$1 = \mathbb{E}_t \left[\left(1 + R_{i,t+1} \right) M_{t+1} \right] \tag{1}$$

where $1 + R_{i,t+1}$ is the gross real return on some asset i and M_{t+1} is a real stochastic discount factor

- This condition implies that the expected real return on any asset is negatively related to its covariance with M_{t+1}
- Fixed income instruments has deterministic cash flows, so covariance with M_{t+1} can only occur because of discount rate variation
- This variation in discount rates is driven by the time series behavior of M_{t+1}, implying that term structure models are time series models for M_{t+1}

Relation to the consumption-based framework

- In previous topics, we have spent some time looking at one-period yields. How is this connected to this framework?
- Recall that the gross real one-period yield (gross risk-free rate $R_{f,t+1}$ in CCAPM) is defined as

$$1 + Y_t^1 = \frac{1}{\mathbb{E}_t\left[M_{t+1}\right]} \tag{2}$$

- Assuming lognormality of the SDF, we obtain the log real risk free rate

$$y_t^1 = -\mathbb{E}_t\left[m_{t+1}\right] - \frac{1}{2}\mathsf{Var}_t\left[m_{t+1}\right] \tag{3}$$

where $m_{t+1} \equiv \ln M_{t+1}$ is the log real stochastic discount factor

An asset pricing formula for discount bonds

- The gross one-period holding-period return on an n-period real discount bond is given by the expression

$$1 + R^n_{t+1} = \frac{P^{n-1}_{t+1}}{P^n_t} \tag{4}$$

- Substituting (4) into (1) gives us a pricing equation that naturally lends itself to a recursive approach

$$P^n_t = \mathbb{E}_t \left[P^{n-1}_{t+1} M_{t+1} \right] \tag{5}$$

- We can solve this equation forward to obtain an equation for the n-period bond price as the expected product on n stochastic discount factors

$$P^n_t = \mathbb{E}_t \left[M_{t+1} M_{t+2} \ldots M_{t+n} \right] \tag{6}$$

Lognormal model for discount bond prices

Lognormal model for discount bond prices

Suppose that M_{t+1} and real bond prices P_t^n are jointly conditionally lognormal and heteroskedastic in (5), then we obtain the following model

$$p_t^n = \mathbb{E}_t\left[p_{t+1}^{n-1} + m_{t+1}\right] + \frac{1}{2}\mathrm{Var}_t\left[p_{t+1}^{n-1} + m_{t+1}\right] \tag{7}$$

* There are two potential roads ahead from here
    1. Specify a time series process for m_{t+1}
    2. Specify and use an economic model for m_{t+1} (e.g., the power utility model)
* We will follow the first approach and specify a time series process for m_{t+1}
* We will consider discrete time versions of popular term structure models: the Vasicek (1977), the Cox et al. (1985), and the Longstaff and Schwartz (1992) models of the term structure

Homoskedastic single-factor model

The Vasicek model

The discrete time version of the Vasicek (1977) model assumes that the negative of the log stochastic discount factor $-m_{t+1}$ evolves as (note difference from CLM)

$$-m_{t+1} = x_t + \frac{1}{2}\beta^2\sigma^2 + \beta\varepsilon_{t+1} \tag{8}$$

where $\frac{1}{2}\beta^2\sigma^2$ is a Jensen's Inequality adjustment that eliminates effects of SDF volatility on the level of interest rates and where the latent state variable x_t follows a homoskedastic AR(1) process with mean μ and persistence ϕ

$$x_{t+1} = (1-\phi)\,\mu + \phi x_t + \varepsilon_{t+1} \tag{9}$$

where ε_{t+1} is a normally distributed innovation with constant variance σ^2

Identifying the state variable

- The latent state variable x_t measures the state of the economy in the model and is therefore of interest to identify
- We can then determine the price of a one-period real bond (Since $p^0_{t+1} = 0$) by substituting (8) into (7)

$$\begin{aligned} p^1_t &= \mathbb{E}_t\left[m_{t+1}\right] + \tfrac{1}{2}\text{Var}_t\left[m_{t+1}\right] \\ &= \mathbb{E}_t\left[-x_t - \tfrac{1}{2}\beta^2\sigma^2 - \beta\varepsilon_{t+1}\right] + \tfrac{1}{2}\text{Var}_t\left[-x_t - \tfrac{1}{2}\beta^2\sigma^2 - \beta\varepsilon_{t+1}\right] \\ &= -x_t - \tfrac{1}{2}\beta^2\sigma^2 + \tfrac{1}{2}\beta^2\sigma^2 \\ &= -x_t \end{aligned} \tag{10}$$

- Finally, using that $y^1_t = -p^1_t$, we find that the latent state variable is the one-period yield!

$$y^1_t = x_t \tag{11}$$

Relation to the continuous time model

- The original Vasicek (1977) model is written in continuous time where the short rate (r) follows an Ornstein-Uhlenbeck process. Specifically,

$$dr = \kappa\,(\theta - r)\,dt + \sigma d\mathcal{B} \tag{12}$$

  where κ, θ, and σ are constants and $\mathcal{B}$ is a Brownian motion

- The resemblance between continuous and discrete time is very clear when realizing that $x_t = y_t^1$ and (9) implies

$$y_{t+1}^1 - y_t^1 = (1-\phi)(\mu - y_t^1) + \varepsilon_{t+1} \tag{13}$$

The price function

We guess that the price function for an n-period bond is linear (or *affine*) in the state variable x_t

$$-p_t^n = ny_t^n = A_n + B_n x_t \tag{14}$$

- We already know that this guess is satisfied for $n = 0$ and $n = 1$ with coefficients $A_0 = B_0 = A_1 = 0$ and $B_1 = 1$
- Under this conjecture, we can write (7) as (using (8)–(9) and our guess)

$$\begin{aligned}\mathbb{E}_t\left[p_{t+1}^{n-1} + m_{t+1}\right] = &-A_{n-1} - B_{n-1}(1-\phi)\mu \\ &-(B_{n-1}\phi + 1)x_t - \tfrac{1}{2}\beta^2\sigma^2\end{aligned} \tag{15}$$

and

$$\mathrm{Var}_t\left[p_{t+1}^{n-1} + m_{t+1}\right] = (\beta + B_{n-1})^2\sigma^2 \tag{16}$$

The Vasicek coefficients

The Vasicek coefficients

Using the lognormal model in (7), we can verify our guess with the following coefficients

$$B_n = 1 + \phi B_{n-1} = (1 - \phi^n) / (1 - \phi) \tag{17}$$

$$A_n = A_{n-1} + B_{n-1} (1 - \phi) \mu - B_{n-1} \beta \sigma^2 - \tfrac{1}{2} B_{n-1}^2 \sigma^2 \tag{18}$$

* Note that (11) then implies that we can model the full term structure as a function of the one-period yield

$$y_t^n = n^{-1} \left(A_n + B_n y_t^1 \right) \tag{19}$$

* Finally, $B_n = (1 - \phi^n) / (1 - \phi)$ measures the fall in the log bond price of an n-period bond when there is an increase in y_t^1
* This sensitivity increases in n until B_n approaches its limit $B_\infty = 1/(1 - \phi)$

Log excess holding-period return

Log excess holding-period return

The expected log excess holding-period return on an n-period bond is defined as $\mathbb{E}_t\left[r_{t+1}^n\right] - y_t^1 = \mathbb{E}_t\left[p_{t+1}^{n-1}\right] - p_t^n + p_t^1$, which in the Vasicek (1977) model becomes

$$\mathbb{E}_t\left[r_{t+1}^n\right] - y_t^1 = -\mathrm{Cov}_t\left[r_{t+1}^n, m_{t+1}\right] - \frac{1}{2}\mathrm{Var}_t\left[r_{t+1}^n\right] \tag{20}$$

$$= B_{n-1}\mathrm{Cov}_t\left[x_{t+1}, m_{t+1}\right] - \frac{1}{2}B_{n-1}^2\mathrm{Var}_t\left[x_{t+1}\right] \tag{21}$$

$$= -B_{n-1}\beta\sigma^2 - \frac{1}{2}B_{n-1}^2\sigma^2 \tag{22}$$

where the second term is an inequality adjustment term and the first term gives the risk premium as the amount of risk ($-B_{n-1}$) times the price of risk ($\beta\sigma^2$)

* The sign of β determines the sign of the overall bond risk premia: $\beta > 0$ ($\beta < 0$) delivers a negative (positive) risk premium on the n-period bond

* Suppose that $\beta > 0$: If $\varepsilon_{t+1}\uparrow$, then $x_{t+1}\uparrow$, $p_{t+1}^n\downarrow$, and $m_{t+1}\downarrow \rightarrow$ a positive covariance between m_{t+1} and $p_{t+1}^n \rightarrow$ negative risk premium (hedge value)

Sharpe ratio

Sharpe ratio

An alternative way to define the price of risk is to consider the Sharpe ratio of the n-period bond, which here is

$$SR = \frac{\mathbb{E}_t\left[r_{t+1}^n\right] - y_t^1 + \frac{1}{2}\mathsf{Var}_t\left[r_{t+1}^n\right]}{\sqrt{\mathsf{Var}_t\left[r_{t+1}^n\right]}} = \frac{-B_{n-1}\beta\sigma^2}{\sqrt{B_{n-1}^2\sigma^2}} = -\beta\sigma \tag{23}$$

- The Sharpe ratio for an n-period bond is constant and so is its expected log excess holding period returns
- This implies that the log expectations hypothesis holds in the model (but not in its pure form)
- Again, we see that the sign of β determines the sign of the Sharpe ratio

A consumption-based interpretation

A consumption-based SDF

Suppose that the investor has power utility $M_{t+1} = \delta \left(\frac{\widetilde{c}_{t+1}}{c_t} \right)^{-\gamma}$, where δ is the subjective discount factor and γ the coefficient of relative risk aversion. Taking logs

$$m_{t+1} = \ln \delta - \gamma \widetilde{g}_{t+1} \tag{24}$$

- It follows that the state variable x_t is a linear function of consumption growth

$$x_t \equiv \mathbb{E}_t \left[-m_{t+1} \right] = -\ln \delta + \gamma \mathbb{E}_t \left[\widetilde{g}_{t+1} \right] \tag{25}$$

- When $\beta > 0$, bond returns and consumption growth are negatively correlated, i.e. bonds help investors smooth consumption over time (vice versa if $\beta < 0$)

Square-root single-factor model

The Cox-Ingersoll-Ross (CIR) model

The discrete time version of the Cox et al. (1985) model assumes that the negative of the log stochastic discount factor $-m_{t+1}$ evolves as (note difference from CLM)

$$-m_{t+1} = x_t \left(1 + \frac{1}{2}\beta^2\sigma^2\right) + \beta\sqrt{x_t}\varepsilon_{t+1} \tag{26}$$

where the latent state variable x_t follows a square-root heteroskedastic process

$$x_{t+1} = (1-\phi)\,\mu + \phi x_t + \sqrt{x_t}\varepsilon_{t+1} \tag{27}$$

where ε_{t+1} is a normally distributed innovation with constant variance σ^2. The model allows for time-varying interest rate volatility and risk premia and delivers strictly positive short rates (for short enough time intervals)

* The continuous time version of the Cox et al. (1985) model is given by

$$dr = \kappa\,(\theta - r)\,dt + \sigma\sqrt{r_t}d\mathcal{B} \tag{28}$$

where κ, θ, and σ are constants and $\mathcal{B}$ is a Brownian motion

What about the state variable?

* The latent state variable x_t still measures the state of the economy in the model and, as we will see, the risk premia

* We can determine the price of a one-period real bond as before by substituting (26) into (7)

$$\begin{aligned} p_t^1 &= \mathbb{E}_t\left[m_{t+1}\right] + \tfrac{1}{2}\mathsf{Var}_t\left[m_{t+1}\right] \\ &= \mathbb{E}_t\left[-x_t - \tfrac{1}{2}\beta^2 x_t \sigma^2 - \beta\sqrt{x_t}\varepsilon_{t+1}\right] + \tfrac{1}{2}\mathsf{Var}_t\left[-x_t - \tfrac{1}{2}\beta^2 x_t \sigma^2 - \beta\sqrt{x_t}\varepsilon_{t+1}\right] \\ &= -x_t - \tfrac{1}{2}\beta^2 x_t \sigma^2 + \tfrac{1}{2}\beta^2 x_t \sigma^2 \\ &= -x_t \end{aligned} \tag{29}$$

* Finally, using that $y_t^1 = -p_t^1$, we again find that the latent state variable equals the one-period yield

$$y_t^1 = x_t \tag{30}$$

The CIR coefficients

The price function

We again guess that the price function for an n-period bond is linear (or *affine*) in the state variable x_t

$$-p_t^n = n y_t^n = A_n + B_n x_t \tag{31}$$

* We already know that this guess is satisfied for $n = 0$ and $n = 1$ with coefficients $A_0 = B_0 = A_1 = 0$ and $B_1 = 1$

* The remaining coefficients obey the following structure

$$B_n = 1 + B_{n-1}\phi - \tfrac{1}{2}B_{n-1}^2\sigma^2 - B_{n-1}\beta\sigma^2 \tag{32}$$
$$A_n = A_{n-1} + B_{n-1}(1-\phi)\,\mu \tag{33}$$

where we note that σ^2 has moved from A_n to B_n because the variance is proportional to the state variable

Log excess holding-period return

Log excess holding-period return

The expected log excess holding-period return on an n-period bond is defined as $\mathbb{E}_t\left[r_{t+1}^n\right] - y_t^1 = \mathbb{E}_t\left[p_{t+1}^{n-1}\right] - p_t^n + p_t^1$, which in the Cox et al. (1985) model becomes

$$\mathbb{E}_t\left[r_{t+1}^n\right] - y_t^1 = -\text{Cov}_t\left[r_{t+1}^n, m_{t+1}\right] - \tfrac{1}{2}\text{Var}_t\left[r_{t+1}^n\right] \tag{34}$$

$$= B_{n-1}\text{Cov}_t\left[x_{t+1}, m_{t+1}\right] - \tfrac{1}{2}B_{n-1}^2\text{Var}_t\left[x_{t+1}\right] \tag{35}$$

$$= \left(-B_{n-1}\beta\sigma^2 - \tfrac{1}{2}B_{n-1}^2\sigma^2\right) x_t \tag{36}$$

where we find the expected log excess returns is proportional to the state variable x_t (or equivalently y_t^1)

- The sign of β still determines the sign of the bond risk premia: $\beta > 0$ ($\beta < 0$) delivers a negative (positive) risk premium on the n-period bond
- Note that this implies that bond risk premia varies over time in CIR term structure model

Sharpe ratio

Sharpe ratio

The price of interest rate risk, or the Sharpe ratio of the n-period bond, is here equal to

$$SR = \frac{\mathbb{E}_t\left[r^n_{t+1}\right] - y^1_t + \frac{1}{2}\mathsf{Var}_t\left[r^n_{t+1}\right]}{\sqrt{\mathsf{Var}_t\left[r^n_{t+1}\right]}} = \frac{-B_{n-1}\beta x_t\sigma^2}{\sqrt{B^2_{n-1}x_t\sigma^2}} = -\beta\sigma\sqrt{x_t} \qquad (37)$$

- The Sharpe ratio for an n-period bond is proportional to the square root of the state variable x_t and equivalently short-term yields y^1_t
- Clearly, this implies that risk premia varies over time with the state variable
- Accordingly, the log expectations hypothesis is violated within the Cox et al. (1985) model

Two-factor model

The Longstaff-Schwartz model

Longstaff and Schwartz (1992) presents a simple two-factor model in which the negative of the stochastic discount factor evolves as

$$-m_{t+1} = x_{1,t}\left(1 + \tfrac{1}{2}\beta^2\sigma_1^2\right) + x_{2,t} + \beta\sqrt{x_{1,t}}\varepsilon_{t+1} \tag{38}$$

where the latent state variables follow square-root heteroskedastic processes

$$x_{1,t+1} = (1-\phi_1)\,\mu_1 + \phi_1 x_{1,t} + \sqrt{x_{1,t}}\varepsilon_{1,t+1} \tag{39}$$

$$x_{2,t+1} = (1-\phi_2)\,\mu_2 + \phi_2 x_{2,t} + \sqrt{x_{2,t}}\varepsilon_{2,t+1} \tag{40}$$

- Proceeding in the usual way, we find the one-period bond prices are driven by the two latent state variables

$$p_t^1 = \mathbb{E}_t\left[m_{t+1}\right] + \tfrac{1}{2}\mathsf{Var}_t\left[m_{t+1}\right] = -x_{1,t} - x_{2,t} \tag{41}$$

- Thus, the one-period yield is no longer sufficient to measure the state of the economy within the model

Variance of one-period yield

* Longstaff and Schwartz (1992) show, however, that the conditional variance of the one-period yield is linear in the state variables

$$\text{Var}_t \left[y^1_{t+1} \right] = \sigma_1^2 x_{1,t} + \sigma_2^2 x_{2,t} \tag{42}$$

* Importantly, this implies that the one-period yield and the conditional variance of the one-period yield jointly summarizes the state of the economy

* That is, we can always state the model in terms of these two quantities

The Longstaff-Schwartz price function

The Longstaff-Schwartz price function

We guess that the price function for an n-period bond is linear (or *affine*) in the state variables $x_{1,t}$ and $x_{2,t}$

$$-p_t^n = ny_t^n = A_n + B_{1,n}x_{1,t} + B_{2,n}x_{2,t} \tag{43}$$

* As before, we know that this guess is satisfied for $n = 0$ and $n = 1$ for the coefficients $A_0 = B_{1,0} = B_{2,0} = 0$, $A_1 = 0$, and $B_{1,1} = B_{2,1} = 1$ and the remaining coefficients obey

$$B_{1,n} = 1 + B_{1,n-1}\phi_1 - \tfrac{1}{2}B_{1,n-1}^2\sigma_1^2 - B_{1,n-1}\beta\sigma_1^2 \tag{44}$$

$$B_{2,n} = 1 + B_{2,n-1}\phi_2 - \tfrac{1}{2}B_{2,n-1}^2\sigma_2^2 \tag{45}$$

$$A_n = A_{n-1} + B_{1,n-1}\left(1-\phi_1\right)\mu_1 + B_{2,n-1}\left(1-\phi_2\right)\mu_2 \tag{46}$$

Log excess holding-period return

Log excess holding-period return

The expected log excess holding-period return on an n-period bond in the Longstaff and Schwartz (1992) model is

$$\mathbb{E}_t\left[r^n_{t+1}\right] - y^1_t = -\mathsf{Cov}_t\left[r^n_{t+1}, m_{t+1}\right] - \frac{1}{2}\mathsf{Var}_t\left[r^n_{t+1}\right] \tag{47}$$

$$\begin{aligned} &= B_{1,n-1}\mathsf{Cov}_t\left[x_{1,t+1}, m_{t+1}\right] - \frac{1}{2}B^2_{1,n-1}\mathsf{Var}_t\left[x_{1,t+1}\right] \\ &\quad - \frac{1}{2}B^2_{2,n-1}\mathsf{Var}_t\left[x_{2,t+1}\right] \end{aligned} \tag{48}$$

$$= \left(-B_{1,n-1}\beta\sigma^2 - \frac{1}{2}B^2_{1,n-1}\sigma^2_1\right)x_{1,t} - \frac{1}{2}B^2_{2,n-1}\sigma^2 x_{2,t} \tag{49}$$

where we find the expected log excess returns is driven by both latent state variables

* The model therefore features time-varying risk premia, just like the Cox et al. (1985) model

* Note again that β determines the sign of the risk premium

The nominal term structure

- Almost all bonds are given in nominal terms, so it is relevant to discuss how the above models can be adapted to deal with this fact
- To do so, we begin by introducing some new notation
  - Q_t denotes the nominal price index at time t
  - $\Pi_{t+1} \equiv Q_{t+1}/Q_t$ denotes the gross rate of inflation from t to $t+1$
  - $P^n_{\$,t} = P^n_t Q_t$ is the nominal price of an n-period bond that pays a nominal dollar at maturity and P^n_t is now the real price of this nominal bond
- We begin from the general (real) asset pricing condition (stated below) and transform it into a nominal counterpart on the next slide

$$P^n_t = \mathbb{E}_t \left[P^{n-1}_{t+1} M_{t+1} \right] \tag{50}$$

The nominal stochastic discount factor

The nominal stochastic discount factor

We can write (50) using our new notation as

$$\frac{P^n_{\$,t}}{Q_t} = \mathbb{E}_t\left[\frac{P^{n-1}_{\$,t+1}}{Q_{t+1}} M_{t+1}\right] \tag{51}$$

which when multiplying through with Q_t becomes

$$\begin{aligned} P^n_{\$,t} &= \mathbb{E}_t\left[P^{n-1}_{\$,t+1} M_{t+1} \frac{Q_t}{Q_{t+1}}\right] \\ &= \mathbb{E}_t\left[P^{n-1}_{\$,t+1} \frac{M_{t+1}}{\Pi_{t+1}}\right] \\ &= \mathbb{E}_t\left[P^{n-1}_{\$,t+1} M_{\$,t+1}\right] \end{aligned} \tag{52}$$

* $M_{\$,t+1} = \frac{M_{t+1}}{\Pi_{t+1}}$ can be thought of as a *nominal* stochastic discount factor that prices nominal returns

Making use of the result

* The empirical literature on nominal bonds uses the above results in one of two ways when implementing the models

  1. Apply the primitive assumptions about M_{t+1} that we made earlier to $M_{\$,t+1}$ instead and reinterpret the real term structure models detailed above as nominal term structure models

  2. Assume that the two components of $M_{\$,t+1}$, i.e. M_{t+1} and $1/\Pi_{t+1}$, are independent of each other so that we obtain a multifactor model

  $$m_{\$,t+1} = m_{t+1} - \pi_{t+1} \tag{53}$$

  where the log inflation rate π_{t+1} becomes the second state variable in the model

* Let us examine the implications of the second choice in more detail

Implications of the independence assumption

- Assuming independence of M_{t+1} and $1/\Pi_{t+1}$ implies that prices of nominal bonds ($P^n_{\$,t}$) just are equal to nominal prices of real bonds ($Q_t P^n_{\mathsf{R},t}$) multiplied by the expected real value of the $\$1$ payment at maturity

- To see this, start by considering the nominal price of a one-period bond

$$P^1_{\$,t} = \mathbb{E}_t\left[M_{\$,t+1}\right] = \mathbb{E}_t\left[M_{t+1}\right]\mathbb{E}_t\left[\frac{1}{\Pi_{t+1}}\right] = Q_t P^1_{\mathsf{R},t}\mathbb{E}_t\left[\frac{1}{Q_{t+1}}\right] \tag{54}$$

- We then guess that a similar relationship holds for all n and use that if $P^n_{\$,t} = Q_t P^n_{\mathsf{R},t}\mathbb{E}_t\left[1/Q_{t+n}\right]$ then

$$\begin{aligned}
P^n_{\$,t} &= \mathbb{E}_t\left[P^{n-1}_{\$,t+1} M_{\$,t+1}\right] = \mathbb{E}_t\left[Q_{t+1} P^{n-1}_{\mathsf{R},t+1}\mathbb{E}_t\left[\frac{1}{Q_{t+1+n-1}}\right] M_{t+1}\frac{Q_t}{Q_{t+1}}\right] \\
&= Q_t\mathbb{E}_t\left[P^{n-1}_{\mathsf{R},t+1} M_{t+1}\mathbb{E}_t\left[\frac{1}{Q_{t+n}}\right]\right] = Q_t P^n_{\mathsf{R},t}\mathbb{E}_t\left[\frac{1}{Q_{t+n}}\right]
\end{aligned} \tag{55}$$

where the last equality uses both the independence of real variables from price levels and that $P^n_{\mathsf{R},t} = \mathbb{E}_t\left[P^{n-1}_{\mathsf{R},t+1} M_{t+1}\right]$

Final remarks on independence

- The expression in (55) further implies that the expected real return on a nominal bond equals the expected real return on a real bond

$$\mathbb{E}_t\left[\frac{P^{n-1}_{\$,t+1}}{P^{n}_{\$,t}}\frac{Q_t}{Q_{t+1}}\right] = \mathbb{E}_t\left[\frac{Q_{t+1}P^{n-1}_{\mathsf{R},t+1}\mathbb{E}_{t+1}\left[1/Q_{t+1+n-1}\right]}{Q_t P^{n}_{\mathsf{R},t}\mathbb{E}_t\left[1/Q_{t+n}\right]}\frac{Q_t}{Q_{t+1}}\right]$$
$$= \mathbb{E}_t\left[\frac{P^{n-1}_{\mathsf{R},t+1}}{P^{n}_{\mathsf{R},t}}\right] \tag{56}$$

- We end by noting that while the independence assumption is extremely convenient, it is also unlikely to be realistic. Why??

Chan et al. (1992): Evidence on the short rate process

* Chan et al. (1992) undertake a comprehensive study of term structure models and their ability to match the dynamics of the short-term interest rate

1. Merton	$dr = \alpha dt + \sigma dZ$
2. Vasicek	$dr = (\alpha + \beta r)dt + \sigma dZ$
3. CIR SR	$dr = (\alpha + \beta r)dt + \sigma r^{1/2} dZ$
4. Dothan	$dr = \sigma r dZ$
5. GBM	$dr = \beta r dt + \sigma r dZ$
6. Brennan-Schwartz	$dr = (\alpha + \beta r)dt + \sigma r dZ$
7. CIR VR	$dr = \sigma r^{3/2} dZ$
8. CEV	$dr = \beta r dt + \sigma r^{\gamma} dZ$

The general framework

- The list of considered term structure models can all be represented by the following stochastic differential equation

$$dr = (\alpha + \beta r)\, dt + \sigma r^{\gamma} d\mathcal{B} \tag{57}$$

- The different term structure models then differ in the restrictions placed on the parameters

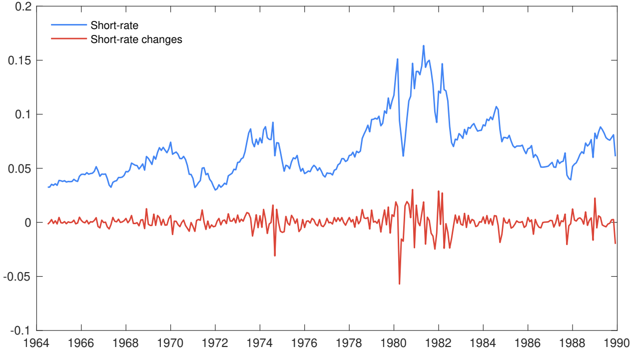
$$dr = (\alpha + \beta r)dt + \sigma r^{\gamma} dZ$$

Model	α	β	σ^2	γ
Merton		0		0
Vasicek				0
CIR SR				$1/2$
Dothan	0	0		1
GBM	0			1
Brennan-Schwartz				1
CIR VR	0	0		$3/2$
CEV	0			

Data

- The Treasury bill yield data are monthly and cover the period from June 1964 to December 1989. All yields are expressed in annualized form and are obtained from the Center for Research in Security Prices (CRSP)

Variables	N	Mean	Standard Deviation	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
r_t	307	0.06715	0.02675	0.95	0.91	0.86	0.82	0.80	0.78
$r_{t+1} - r_t$	306	0.00009	0.00821	-0.08	0.07	-0.12	-0.14	-0.03	-0.04



The econometric approach

- To decide on a term structure model, one can perform and examination of their explanatory power for the dynamics of short-term interest rates

- In the single-factor models the one-period yield inherits the AR(1) dynamics of the state variable, which implies that we in general terms can write up the process for the one-period yield as (Chan et al., 1992)

$$y_{t+1}^{1} - y_{t}^{1} = \alpha + \beta y_{t}^{1} + \varepsilon_{t+1} \tag{58}$$

  where

$$\mathbb{E}_t\left[\varepsilon_{t+1}\right] = 0 \qquad \text{and} \qquad \mathbb{E}_t\left[\varepsilon_{t+1}^{2}\right] = \sigma^2\left(y_t^1\right)^{2\gamma} \tag{59}$$

- This model nests the single factor models outlined above: the Vasicek (1977) model has $\gamma = 0$ and the Cox et al. (1985) model has $\gamma = 0.5$

- This econometric specification is a discrete-time approximation of the continues-time specification in (57)

Estimation by Generalized Method of Moment (GMM)

- The parameter vector $\boldsymbol{\theta} = \{\alpha, \beta, \sigma^2, \gamma\}$ can be estimated using GMM (Hansen, 1982) with moment conditions implied by (59) when using a constant and y_t^1 as instruments

$$f_t(\boldsymbol{\theta}) = \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1} y_t^1 \\ \varepsilon_{t+1}^2 - \sigma^2 \left(y_t^1\right)^{2\gamma} \\ \left(\varepsilon_{t+1}^2 - \sigma^2 \left(y_t^1\right)^{2\gamma}\right) y_t^1 \end{bmatrix} \tag{60}$$

- Under the null hypothesis that the restrictions in (58)–(59) are true, then we should find

$$\mathbb{E}\left[f_t(\boldsymbol{\theta})\right] = \mathbf{0} \tag{61}$$

- We detail the GMM estimation procedure for their setup on the next slide

The GMM procedure

- The GMM procedure replaces $\mathbb{E}\left[f_t\left(\boldsymbol{\theta}\right)\right]$ with its sample counterpart

$$g_T\left(\boldsymbol{\theta}\right) = \frac{1}{T}\sum_{t=1}^{T} f_t\left(\boldsymbol{\theta}\right) \tag{62}$$

- Let $W_T\left(\boldsymbol{\theta}\right)$ be a weighting matrix, then the parameters can be estimated by minimizing the quadratic form

$$J_T\left(\boldsymbol{\theta}\right) = g_T\left(\boldsymbol{\theta}\right)' W_T\left(\boldsymbol{\theta}\right) g_T\left(\boldsymbol{\theta}\right) \tag{63}$$

- The weighting matrix differ between the case of the unrestricted specification and all other models
  - For the unrestricted model, $J_T\left(\boldsymbol{\theta}\right)$ attains zero for all choices of $W_T\left(\boldsymbol{\theta}\right)$ so we can simply use $W_T\left(\boldsymbol{\theta}\right) = I$, which is the identity matrix
  - The choice of $W_T\left(\boldsymbol{\theta}\right)$ matters for all nested model in the overidentified parameter vector and Hansen (1982) suggests to use $W_T\left(\boldsymbol{\theta}\right) = S^{-1}\left(\boldsymbol{\theta}\right)$, where $S\left(\boldsymbol{\theta}\right)$ is an estimate of the covariance matrix of the moment condition matrix obtained using the Newey and West (1987) estimator

Estimation results

* Chan et al. (1992) finds that α and β are small and often statistically insignificant, implying that short-rates are random walks
* They also find that γ is large and precisely estimated so that they can rule out models with $\gamma < 1$ — which includes our single-factor models

$$r_{t+1} - r_t = \alpha + \beta r_t + \varepsilon_{t+1}$$

$$E[\varepsilon_{t+1}] = 0, \qquad E[\varepsilon_{t+1}^2] = \sigma^2 r_t^{2\gamma}$$

Model	α	β	σ^2	γ	χ^2 Test (p-value)	d.f.	R_1^2	R_2^2
Unrestricted	0.0408	−0.5921	1.6704	1.4999			0.0259	0.2046
	(1.85)	(−1.55)	(0.77)	(5.95)				
Merton	0.0055	0.0	0.0004	0.0	6.7579	2	0.0000	0.0000
	(1.44)		(7.27)		(0.0341)			
Vasicek	0.0154	−0.1779	0.0004	0.0	8.8467	1	0.0132	0.0000
	(0.76)	(−0.50)	(6.53)		(0.0029)			
CIR SR	0.0189	−0.2339	0.0073	0.5	6.1512	1	0.0164	0.0546
	(0.94)	(−0.66)	(7.55)		(0.0131)			
Dothan	0.0	0.0	0.1172	1.0	5.6018	3	0.0000	0.1340
			(7.99)		(0.1327)			
GBM	0.0	0.1101	0.1185	1.0	3.1541	2	−0.0096	0.1329
		(1.50)	(8.04)		(0.2066)			
Brennan-Schwartz	0.0242	−0.3142	0.1185	1.0	2.2172	1	0.0202	0.1395
	(1.24)	(−0.92)	(8.09)		(0.1364)			
CIR VR	0.0	0.0	1.5778	1.5	6.2067	3	0.0000	0.2049
			(8.00)		(0.1019)			
CEV	0.0	0.1026	0.5207	1.2795	3.0801	1	−0.0098	0.1801
		(1.52)	(0.62)	(4.15)	(0.0793)			

Monetary policy and structural breaks

* Shifts in monetary policy (e.g. from monetary to interest rate targeting in October 1979) may induce structural breaks in the interest rate process

* The framework of Chan et al. (1992) allows us to empirically evaluate the presence of breaks by introducing a dummy variable D_t that takes the value 1 in one regime and 0 in the other

$$y_{t+1}^1 - y_t^1 = (\alpha + D_t\delta_1) + (\beta + D_t\delta_2)\, y_t^1 + \varepsilon_{t+1} \tag{64}$$

  where

$$\mathbb{E}_t\left[\varepsilon_{t+1}\right] = 0 \qquad \text{and} \quad \mathbb{E}_t\left[\varepsilon_{t+1}^2\right] = \left(\sigma^2 + D_t\delta_3\right)\left(y_t^1\right)^{2(\gamma + D_t\delta_4)} \tag{65}$$

  and where $\delta_i, i = 1, \ldots, 4$ are parameters associated with the dummy shift variables. Thus, testing whether they differ from zero is the object of interest

* We can estimate this version of the model and its eight parameters using GMM and instruments $\left\{1, y_t^1, D_t, D_t y_t^1\right\}$

Results for structural breaks

$$r_{t+1} - r_t = (\alpha + D_t \delta_1) + (\beta + D_t \delta_2) r_t + \varepsilon_{t+1}$$

$$E[\varepsilon_{t+1}] = 0, \qquad E[\varepsilon_{t+1}^2] = (\sigma^2 + D_t \delta_3) r_t^{2(\gamma + D_t \delta_4)}$$

	System Parameters				Dummy Parameters				χ^2 Test	
Model	α	β	σ^2	γ	δ_1	δ_2	δ_3	δ_4	(p-value)	d.f.
Unrestricted	0.0174	−0.2213	1.3846	1.4808	0.0608	−0.0751	−0.2082	−0.0641	2.2939	4
	(0.87)	(−0.53)	(0.45)	(3.83)	(1.20)	(−1.01)	(−0.06)	(−0.12)	(0.6818)	
Merton	0.0069	0.0	0.0002	1.0	−0.0021	0.0	0.0006	0.0	13.813	2
	(1.93)		(6.83)		(−0.21)		(3.66)		(0.0010)	
Vasicek	−0.0009	0.1612	0.0002	0.0	0.0374	−0.6019	0.0006	0.0	14.380	2
	(−0.05)	(0.42)	(6.83)		(0.79)	(−0.87)	(3.53)		(0.0024)	
CIR SR	0.0020	0.1025	0.0041	0.5	0.0421	−0.6336	0.0074	0.0	12.642	3
	(0.11)	(0.27)	(7.63)		(0.90)	(−0.92)	(3.34)		(0.0054)	
Dothan	0.0	0.0	0.0778	1.0	0.0	0.0	0.0666	0.0	6.0575	1
			(7.81)				(2.46)		(0.0138)	
GBM	0.0	0.1391	0.0819	1.0	0.0	−0.0862	0.0644	0.0	6.6672	2
		(1.89)	(8.10)			(−0.60)	(2.33)		(0.0356)	
Brennan-Schwartz	0.0078	−0.0174	0.0825	1.0	0.0522	−0.7138	0.0690	0.0	7.1759	3
	(0.43)	(−0.05)	(8.13)		(1.12)	(−1.04)	(2.47)		(0.0665)	
CIR VR	0.0	0.0	1.4390	1.5	0.0	0.0	0.1653	0.0	1.8559	1
			(7.97)				(0.49)		(0.1731)	
CEV	0.0	0.1311	0.4921	1.3059	0.0	−0.0796	−0.1352	−0.1244	1.4090	3
		(1.77)	(0.43)	(3.26)		(−0.56)	(−0.10)	(−0.21)	(0.7034)	

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