CHAPTER 9

Arrow–Debreu Pricing, Part I

Chapter Outline

9.1 Introduction

As interesting and popular as it is, the CAPM is a very limited theory of equilibrium pricing. We will devote the next chapters to reviewing alternative theories, each of which goes beyond the CAPM in one direction or another. The Arrow–Debreu pricing theory discussed in this chapter is a full general equilibrium theory as opposed to the partial equilibrium static view of the CAPM. Although also static in nature, it is applicable to a multiperiod setup and can be generalized to a broad set of situations. In particular, it is free of any preference restrictions and any distributional assumptions on returns. The Consumption CAPM considered subsequently (Chapter 10) is a fully dynamic construct. It is also an equilibrium theory, though of a somewhat specialized nature. With the Risk-Neutral Valuation Model and the Arbitrage Pricing Theory (APT), taken up in Chapters 12–14, we will move into the domain of arbitrage-based theories, after observing, however, that the Arrow–Debreu pricing theory itself may also be interpreted from the arbitrage perspective (Chapter 11).

The Arrow–Debreu model takes a more standard equilibrium view than the CAPM: it is explicit in stating that equilibrium means supply equals demand in every market. It is a very general theory accommodating production and, as already stated, very broad hypotheses on preferences. Moreover, no restriction on the distribution of returns is necessary. We will not, however, fully exploit the generality of the theory: In keeping with the objective of this text, we shall often limit ourselves to illustrating the theory with examples.

Intermediate Financial Theory.

Arrow–Debreu modeling will be especially useful for equilibrium security pricing, especially the pricing of complex securities that pay returns in many different time periods and states of nature, such as common stocks or 30-year government coupon bonds. The theory will also enrich our understanding of project valuation because of the formal equivalence, underlined in Chapter 2, between a project and an asset. In so doing we will move beyond a pure equilibrium analysis and start using the concept of arbitrage. It is in the light of a set of no-arbitrage relationships that the Arrow–Debreu pricing takes its full force. As noted earlier, the arbitrage perspective on the Arrow–Debreu theory will be developed in Chapter 11.

9.2 Setting: An Arrow–Debreu Economy

In the basic setting that we shall use, the following apply:

1. There are two dates: 0, 1. This setup, however, is fully generalizable to multiple periods; see discussion that follows.
2. There are N possible states of nature at date 1, which we index by $\theta = 1,2,\ldots, N$ with probabilities π_θ.[1]
3. There is one perishable (nonstorable) consumption good.
4. There are K agents, indexed by $k = 1,\ldots, K$, with preferences:

$$U_0^k(c_0^k) + \delta^k \sum_{\theta=1}^{N} \pi_\theta U^k(c_\theta^k);$$

5. Agent k's endowment is described by the vector $\{e_0^k, (e_\theta^k)_{\theta=1,2,\ldots,N}\}$.

In this description, c_θ^k denotes agent k's consumption of the sole consumption good in state θ, U is the real-valued utility representation of agent k's period preferences, and δ^k is the agent's time discount factor. In fact, the theory allows for more general preferences than the time-additive expected utility form. Specifically, we could adopt the following representation of preferences:

$$U^k(c_0^k, c_{\theta_1}^k, c_{\theta_2}^k, \ldots, c_{\theta_N}^k).$$

This formulation allows not only for a different way of discounting the future (implicit in the relative taste for present consumption relative to all future consumption), but it also permits heterogeneous, subjective views on the state probabilities (again implicit in the

[1] In the present chapter and in most of this text going forward, we assume that all agents hold the same (objective) probability beliefs. Such an assumption is most appropriate to a context where the economy's probabilistic structure over endowments and states does not change, allowing agents to learn the true structure, revising their own beliefs accordingly. In Chapter 18, the topic of heterogeneous beliefs is considered.

representation of relative preference for, say, $c^k_{\theta_2}$ vs $\cdot\, c^k_{\theta_3}$). In addition, it assumes neither time-additivity nor an expected utility representation. Since our main objective is not generality, we choose to work with the more restrictive but easier to manipulate time-additive expected utility form.

In this economy, the only traded securities are of the following type: One unit of security θ, with price q_θ, pays one unit of consumption if state θ occurs and nothing otherwise.
Its payout can thus be summarized by a vector with all entries equal to zero except for column θ where the entry is 1: (0,. . .,0, 1, 0,. . .0). These primitive securities are called **Arrow–Debreu securities**,[2] or *state-contingent claims* or simply *state claims*. Of course, the consumption of any individual k if state θ occurs equals the number of units of security θ that he holds. This follows from the fact that buying the relevant contingent claim is the only way for a consumer to secure purchasing power at a future date-state (recall that the good is perishable). An agent's decision problem can then be characterized by:

$$\text{(P)} \qquad \max_{(c^k_0, c^k_1, \ldots, c^k_N)} \quad U^k_0(c^k_0) + \delta^k \sum_{\theta=1}^{N} \pi_\theta U^k(c^k_\theta)$$

$$\text{s.t} \quad c^k_0 + \sum_{\theta=1}^{N} q_\theta c^k_\theta \le e^k_0 + \sum_{\theta=1}^{N} q_\theta e^k_\theta$$

$$c^k_0, c^k_1, \ldots, c^k_N \ge 0$$

The first inequality constraint will typically hold with equality in a world of nonsatiation. That is, the total value of goods and security purchases made by the agent (the left-hand side of the inequality) will exhaust the total value of his endowments (the right-hand side).

Equilibrium for this economy is a set of contingent claim prices (q_1, q_2,. . ., q_N) such that

1. at those prices $(c^k_0, \ldots, c^k_N)$ solve problem (P), for all k, and
2. $\sum_{k=1}^{K} c^k_0 = \sum_{k=1}^{K} e^k_0, \sum_{k=1}^{K} c^k_\theta = \sum_{k=1}^{K} e^k_\theta,$ for every θ.

Note that here the agents are solving for desired future and present consumption holdings rather than holdings of Arrow–Debreu securities. This is justified because, as just noted, there is a one-to-one relationship between the amount consumed by an individual in a given state θ and his holdings of the Arrow–Debreu security corresponding to that particular state θ, the latter being a promise to deliver one unit of the consumption good if that state occurs.

Note also that nothing in this formulation inherently restricts matters to two periods, if we define our notion of a state somewhat more richly, as a *date-state* pair. Consider three

[2] So named after the originators of modern equilibrium theory: see Arrow (1951) and Debreu (1959).

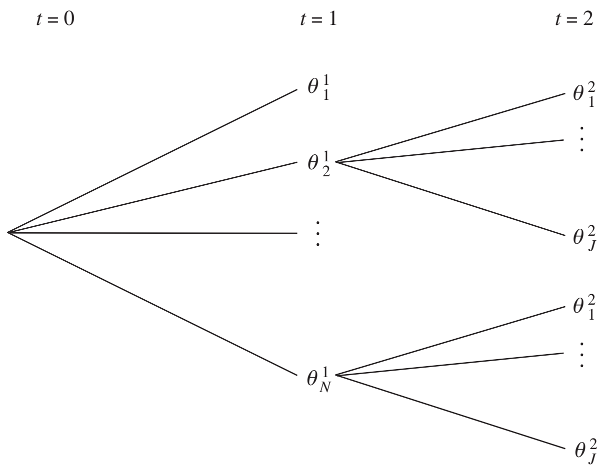

Figure 9.1
The structure of an economy with two dates and $(1 + NJ)$ states.

periods, for example. There are N possible states in date 1 and J possible states in date 2, regardless of the state achieved in date 1. Define $\hat{\theta}$ new states to be of the form $\hat{\theta}_s = (k, \theta_k^j)$, where k denotes the state in date 1 and θ_k^j denotes the state j in date 2, conditional that state k was observed in date 1 (refer to Figure 9.1). So $(1, \theta_1^5)$ would be a state and $(2, \theta_2^3)$ another state. Under this interpretation, the number of *states* expands to $1 + NJ$, with:

1: the date 0 state
N: the number of date 1 states
J: the number of date 2 states

With minor modifications, we can thus accommodate many periods and states. In this sense, our model is fully general and can represent as complex an environment as we might desire. In this model, the real productive side of the economy is in the background. We are, in effect, viewing that part of the economy as invariant to securities trading. The unusual and unrealistic aspect of this economy is that all trades occur at $t = 0$.[3] We will relax this assumption in Chapter 10.

9.3 Competitive Equilibrium and Pareto Optimality Illustrated

Let us now develop an example. The essentials are found in Table 9.1.

There are two dates and, at the future date, two possible states of nature with probabilities $\frac{1}{3}$ and $\frac{2}{3}$. It is an exchange economy, and the issue is to share the existing endowments

[3] Interestingly, this is less of a restriction for project valuation than for asset pricing.

Table 9.1: Endowments and preferences in our reference example

Agents	Endowments			Preferences
	$t=0$	$t=1$		
		θ_1	θ_2	
Agent 1	10	1	2	$\frac{1}{2}c_0^1 + 0.9\left(\frac{1}{3}\ln\left(c_1^1\right) + \frac{2}{3}\ln\left(c_2^1\right)\right)$
Agent 2	5	4	6	$\frac{1}{2}c_0^2 + 0.9\left(\frac{1}{3}\ln\left(c_1^2\right) + \frac{2}{3}\ln\left(c_2^2\right)\right)$

between two individuals. Their (identical) preferences are linear in date 0 consumption, with constant marginal utility equal to $\frac{1}{2}$. This choice is made for ease of computation, but great care must be exercised in interpreting the results obtained in such a simplified framework. Date 1 preferences are concave and identical. The discount factor is 0.9. Let q_1 be the price of a unit of consumption in date 1 state 1, and q_2 the price of one unit of the consumption good in date 1 state 2. We will solve for optimal consumption directly, knowing that this will define the equilibrium holdings of the securities. The prices of these consumption goods coincide with the prices of the corresponding state-contingent claims; period 0 consumption is taken as the numeraire, and its price is normalized to 1. This means that all prices are expressed in units of period 0 consumption: q_1, q_2 are prices for the consumption good at date 1, in states 1 and 2, respectively, measured in units of date 0 consumption. They can thus be used to add up or compare units of consumption at different dates and in different states, making it possible to add different date cash flows, with the q_i being the appropriate weights. This, in turn, permits computing an individual's wealth. Thus, in the previous problem, agent 1's wealth, which equals the present value of his current and future endowments, is $10 + 1q_1 + 2q_2$, while agent 2's wealth is $5 + 4q_1 + 6q_2$.

The respective agent problems are:

Agent 1.

$$\max \frac{1}{2}(10 + 1q_1 + 2q_2 - c_1^1 q_1 - c_2^1 q_2) + 0.9\left(\frac{1}{3}\ln(c_1^1) + \frac{2}{3}\ln(c_2^1)\right)$$
$$\text{s.t. } c_1^1 q_1 + c_2^1 q_2 \leq 10 + q_1 + 2q_2, \text{ and } c_1^1, c_2^1 \geq 0$$

Agent 2.

$$\max \frac{1}{2}(5 + 4q_1 + 6q_2 - c_1^2 q_1 - c_2^2 q_2) + 0.9\left(\frac{1}{3}\ln(c_1^2) + \frac{2}{3}\ln(c_2^2)\right)$$
$$\text{s.t. } c_1^2 q_1 + c_2^2 q_2 \leq 5 + 4q_1 + 6q_2 \text{ and } c_1^2, c_2^2 \geq 0$$

Note that in this formation, we have substituted out for the date 0 consumption; in other words, the first term in the max expression stands for $\frac{1}{2}(c_0)$, where we have substituted for c_0 its value obtained from the constraint: $c_0 + c_1^1 q_1 + c_2^1 q_2 = 10 + 1q_1 + 2q_2$. With this trick, the only constraints remaining are the nonnegativity constraints requiring consumption to be nonnegative in all date-states.

The FOCs state that the intertemporal rate of substitution between future (in either state) and present consumption (i.e., the ratio of the relevant marginal utilities) should equal the price ratio. The latter is effectively measured by the price of the Arrow–Debreu security, the date 0 price of consumption being the numeraire. These first order conditions (FOCs) are (assuming interior solutions)

$$\text{Agent 1:}\begin{cases} c_1^1\colon \dfrac{q_1}{2} = 0.9\left(\dfrac{1}{3}\right)\dfrac{1}{c_1^1} \\[2ex] c_2^1\colon \dfrac{q_2}{2} = 0.9\left(\dfrac{2}{3}\right)\dfrac{1}{c_2^1} \end{cases} \qquad \text{Agent 2:}\begin{cases} c_1^2\colon \dfrac{q_1}{2} = 0.9\left(\dfrac{1}{3}\right)\dfrac{1}{c_1^2} \\[2ex] c_2^2\colon \dfrac{q_2}{2} = 0.9\left(\dfrac{2}{3}\right)\dfrac{1}{c_2^2} \end{cases}$$

while the market-clearing conditions read: $c_1^1 + c_1^2 = 5$ and $c_2^1 + c_2^2 = 8$. Each of the FOCs is of the form $(q_\theta/1) = ((0.9)\pi_\theta(1/c_\theta^k))/(1/2),\ k,\theta = 1, 2$, or

$$q_\theta = \frac{\delta\pi_\theta \dfrac{\partial U^k}{\partial c_\theta^k}}{\dfrac{\partial U_0^k}{\partial c_0^k}},\ k,\theta = 1,2. \tag{9.1}$$

Together with the market-clearing conditions, Eq. (9.1) reveals the determinants of the equilibrium Arrow–Debreu security prices. It is of the form:

$$\frac{\text{Price of the good if state } \theta \text{ is realized}}{\text{Price of the good today}} = \frac{MU_\theta^k}{MU_0^k}.$$

In other words, the ratio of the price of the Arrow–Debreu security to the price of the date 0 consumption good must equal (at an interior solution; see Box 9.1) the ratio of the marginal utility of consumption tomorrow if state θ is realized to the marginal utility of today's consumption (the latter being constant at $\frac{1}{2}$). This is the *marginal rate of substitution* (MRS) between the contingent consumption in state θ and today's consumption. From this system of equations, one clearly obtains $c_1^1 = c_1^2 = 2.5$ and $c_2^1 = c_2^2 = 4$ from which one, in turn, derives:

$$q_1 = \frac{1}{\frac{1}{2}}(0.9)\left(\frac{1}{3}\right)\left(\frac{1}{c_1^1}\right) = 2(0.9)\left(\frac{1}{3}\right)\left(\frac{1}{2.5}\right) = (0.9)\left(\frac{1}{3}\right)\left(\frac{4}{5}\right) = 0.24$$

$$q_2 = \frac{1}{\frac{1}{2}}(0.9)\left(\frac{2}{3}\right)\left(\frac{1}{c_2^1}\right) = 2(0.9)\left(\frac{2}{3}\right)\left(\frac{1}{4}\right) = (0.9)\left(\frac{2}{3}\right)\left(\frac{4}{8}\right) = 0.3$$

BOX 9.1 Interior Versus Corner Solutions

We have described the *interior* solution to the maximization problem. By that restriction we generally mean the following: the problem under maximization is constrained by the condition that consumption at all dates should be nonnegative. There is no interpretation given to a negative level of consumption, and, generally, even a zero consumption level is precluded. Indeed, when we make the assumption of a log utility function, the marginal utility at zero is infinity, meaning that by construction the agent will do all that is in his power to avoid that situation. Effectively, an equation such as Eq. (9.1) will never be satisfied for finite and nonzero prices with log utility and period one consumption level equal to zero; that is, it will never be optimal to select a zero consumption level. Such is not the case with the linear utility function assumed to prevail at date 0. Here it is conceivable that, no matter what, the marginal utility in either state at date 1 (the numerator in the RHS of Eq. (9.1)) will be larger than $\frac{1}{2}$ times the Arrow–Debreu price (the denominator of the RHS in Eq. (9.1) multiplied by the state price). Intuitively, this would be a situation where the agent derives more utility from the good tomorrow than from consuming today, even when his consumption level today is zero. Fundamentally, the interior optimum is one where he would like to consume less than zero today to increase even further consumption tomorrow, something that is impossible. Thus the only solution is at a corner, that is, at the boundary of the feasible set, with $c_0^k = 0$ and the condition in Eq. (9.1) taking the form of an inequality.

In the present case, we can argue that corner solutions cannot occur with regard to future consumption (because of the log utility assumption). The full and complete description of the FOCs for problem (P) spelled out in Section 9.2 is then

$$q_\theta \frac{\partial U_0^k}{\partial c_0^k} \leq \delta \pi_\theta \frac{\partial U^k}{\partial c_\theta^k}, \text{ and if } c_0^k > 0, \text{ and } k, \theta = 1, 2. \tag{9.2}$$

In line with our goal of being as transparent as possible, we will often, in the sequel, satisfy ourselves with a description of interior solutions to optimizing problems, taking care to ascertain, ex-post, that the solutions do indeed occur at the interior of the choice set. This can be done in the present case by verifying that the optimal c_0^k is strictly positive for both agents at the interior solutions, so that Eq. (9.1) must indeed apply.

Notice how the Arrow–Debreu state-contingent prices reflect probabilities, on the one hand, and marginal rates of substitution (taking the time discount factor into account and computed at consumption levels compatible with market clearing) and thus relative scarcities, on the other. The prices computed above differ in that they take account both of the different state probabilities $\left(\frac{1}{3}\text{ for state 1, }\frac{2}{3}\text{ for state 2}\right)$ and the differing marginal utilities as a result of the differing total quantities of the consumption good available in state 1 (5 units) and in state 2 (8 units). In our particular formulation, the total amount of goods available at date 0 is made irrelevant by the fact that date 0 marginal utility is constant. Note that if the date 1 marginal utilities were constant, as would be the case with

Table 9.2: Post-trade equilibrium consumptions

		$t=1$	
	$t=0$	θ_1	θ_2
Agent 1	9.04	2.5	4
Agent 2	5.96	2.5	4
Total	15.00	5.0	8

linear (risk-neutral) utility functions, the goods endowments would not influence the Arrow–Debreu prices, which would then be exactly proportional to the state probabilities.

The date 0 consumptions, at the equilibrium prices, are given by

$$c_0^1 = 10 + 1(0.24) + 2(0.3) - 2.5(0.24) - 4(0.3) = 9.04$$
$$c_0^2 = 5 + 4(0.24) + 6(0.3) - 2.5(0.24) - 4(0.3) = 5.96$$

The post-trade equilibrium consumptions are found in Table 9.2.

This allocation is the best each agent can achieve at the equilibrium prices $q_1 = 0.24$ and $q_2 = 0.3$. Furthermore, at those prices, supply equals demand in each market, in every state and time period. These are the characteristics of a (general) competitive equilibrium.

In light of this example, it is interesting to return to some of the concepts discussed in our introductory chapter. In particular, let us confirm the (Pareto) optimality of the allocation emerging from the competitive equilibrium. Indeed, we have assumed as many markets as there are states of nature, so assumption H1 is satisfied. We have *de facto* assumed competitive behavior on the part of our two consumers (they have taken prices as given when solving their optimization problems), so H2 is satisfied. (Of course, in reality such behavior would not be privately optimal if indeed there were only two agents. Our example would not have changed materially had we assumed a large number of agents, but the notation would have become much more cumbersome.)

In order to guarantee the existence of an equilibrium, we need hypotheses H3 and H4 as well. H3 is satisfied in a weak form (no curvature in date 0 utility). Finally, ours is an exchange economy where H4 does not apply (or, if one prefers, it is trivially satisfied). Once the equilibrium is known to exist, as is the case here, H1 and H2 are sufficient to guarantee the optimality of the resulting allocation of resources. Thus, we expect to find that the above competitive allocation is Pareto optimal (PO); that is, it is impossible to rearrange the allocation of consumptions so that the utility of one agent is higher without diminishing the utility of the other agent.

One way to verify the optimality of the competitive allocation is to establish the precise conditions that must be satisfied for an allocation to be PO in the exchange economy context of our example. It is intuitively clear that the above Pareto superior real allocations will be impossible if the initial allocation maximizes the weighted sum of the two agents' utilities. That is, an allocation is optimal in our example if, for some weight λ it solves the following maximization problem.[4]

$$\max_{\{c_0^1, c_1^1, c_2^1\}} U^1\left(c_0^1, c_1^1, c_2^1\right) + \lambda U^2\left(c_0^2, c_1^2, c_2^2\right)$$

$$\text{s.t.}$$

$$c_0^1 + c_0^2 = 15;\;\; c_1^1 + c_1^2 = 5;\;\; c_2^1 + c_2^2 = 8,$$

$$c_0^1, c_1^1, c_2^1, c_0^2, c_1^2, c_2^2 \geq 0$$

This problem can be interpreted as the problem of a benevolent central planner constrained by an economy's total endowment (15, 5, 8) and weighting the two agents utilities according to a parameter λ, possibly equal to 1. The decision variables at his disposal are the consumption levels of the two agents in the two dates and the two states. With U_i^k denoting the derivative of agent k's utility function with respect to $c_i^k (i = 1, 2, 3)$, the FOCs for an interior solution to the above problem are found in Eq. (9.3).

$$\frac{U_0^1}{U_0^2} = \frac{U_1^1}{U_1^2} = \frac{U_2^1}{U_2^2} = \lambda \tag{9.3}$$

This condition states that, in a PO allocation, the ratio of the two agents' marginal utilities with respect to the three goods (i.e., the consumption good at date 0, the consumption good at date 1 if state 1, and the consumption good at date 1 if state 2) should be identical.[5] In an exchange economy this condition, properly extended to take account of the possibility of a corner solution, together with the condition that the agents' consumption adds up to the endowment in each date-state, is necessary and sufficient.

[4] In this discussion it is just as easy to work with the most general utility representation.

[5] Check that Eq. (9.3) implies that the MRS between any two pair of goods is the same for the two agents and refer to the definition of the contract curve (the set of PO allocations) in the Appendix to Chapter 1.

It remains to check that Eq. (9.3) is satisfied at the equilibrium allocation. We can rewrite Eq. (9.3) for the parameters of our example:

$$\frac{\frac{1}{2}}{\frac{1}{2}} = \frac{(0.9)\frac{1}{3}\frac{1}{c_1^1}}{(0.9)\frac{1}{3}\frac{1}{c_1^2}} = \frac{(0.9)\frac{2}{3}\frac{1}{c_2^1}}{(0.9)\frac{2}{3}\frac{1}{c_2^2}}$$

It is clear that the condition in Eq. (9.3) is satisfied since $c_1^1 = c_1^2; c_2^1 = c_2^2$ at the competitive equilibrium, which thus corresponds to the Pareto optimum with equal weighting of the two agents' utilities: $\lambda = 1$, and all three ratios of marginal utilities are equal to 1. Note that other Pareto optima are feasible, for example, one where $\lambda = 2$. In that case, however, only the latter two equalities can be satisfied: the date 0 marginal utilities are constant, which implies that no matter how agent consumptions are redistributed by the market or by the central planner, the first ratio of marginal utilities in Eq. (9.3) cannot be made equal to 2. This is an example of a corner solution to the maximization problem leading to Eq. (9.3).

In our example, agents are able to purchase consumption in any date-state of nature. This is the case because there are enough Arrow–Debreu securities; specifically, there is an Arrow–Debreu security corresponding to each state of nature. If this were not the case, the attainable utility levels would decrease: at least one agent, possibly both of them, would be worse off. If we assume that only the state 1 Arrow–Debreu security is available, then there is no way to make the state 2 consumption of the agents differ from their endowments. It is easy to check that this constraint does not modify their demand for the state 1 contingent claim, nor its price. The post-trade allocation, in that situation, is found in Table 9.3.

The resulting post-trade utilities are

Agent 1: $\frac{1}{2}(9.64) + 0.9\left(\frac{1}{3}\ \ln(2.5) + \frac{2}{3}\ln(2)\right) = 5.51$

Agent 2: $\frac{1}{2}(5.36) + 0.9\left(\frac{1}{3}\ \ln(2.5) + \frac{2}{3}\ln(6)\right) = 4.03$

In the case with two state-contingent claim markets, the post-trade utilities are both higher (illustrating a reallocation of resources that is said to be *Pareto superior* to the no-trade allocation):

Agent 1: $\frac{1}{2}(9.04) + 0.9\left(\frac{1}{3}\ \ln(2.5) + \frac{2}{3}\ln(4)\right) = 5.62$

Table 9.3: The post-trade allocation

		$t=1$	
	$t=0$	θ_1	θ_2
Agent 1	9.64	2.5	2
Agent 2	5.36	2.5	6
Total	15.00	5.0	8

Agent 2: $\frac{1}{2}(5.96) + 0.9(\frac{1}{3}\ln(2.5) + \frac{2}{3}\ln(4)) = 4.09$

When there is an Arrow—Debreu security available to trade corresponding to each state of nature, one says that the securities markets are complete.

9.4 Pareto Optimality and Risk Sharing

In this section and the next, we further explore the nexus between a competitive equilibrium in an Arrow—Debreu economy and Pareto optimality. We first discuss the risk-sharing properties of a PO allocation. We remain in the general framework of the example of the previous two sections but start with a different set of parameters. In particular, let the endowment matrix for the two agents be as shown in Table 9.4.

Assume further that each state is now equally likely with probability $\frac{1}{2}$. As before, consumption in period 0 cannot be stored and carried over into period 1. In the absence of trade, agents clearly experience widely differing consumption and utility levels in period 1, depending on what state occurs (see Table 9.5).

How could agents' utilities be improved? By concavity (risk aversion), this must be accomplished by reducing the spread of the date 1 income possibilities, in other words, lowering the risk associated with date 1 income. Because of symmetry, all date 1 income fluctuations can, in fact, be eliminated if agent 2 agrees to transfer two units of the good in state 1 against the promise to receive two units from agent 1 if state 2 is realized (see Table 9.6).

Table 9.4: The new endowment matrix

		$t=1$	
	$t=0$	θ_1	θ_2
Agent 1	4	1	5
Agent 2	4	5	1

Table 9.5: Agents' utility in the absence of trade

	State-Contingent Utility		
	θ_1	θ_2	Expected Utility in Period 1
Agent 1	$\ln(1) = 0$	$\ln(5) = 1.609$	$\frac{1}{2}\ln(1) + \frac{1}{2}\ln(5) = 0.8047$
Agent 2	$\ln(5) = 1.609$	$\ln(1) = 0$	$\frac{1}{2}\ln(1) + \frac{1}{2}\ln(5) = 0.8047$

Table 9.6: The desirable trades and post-trade consumptions

Date 1	Endowments Pre-trade		Consumption Post-trade	
	θ_1	θ_2	θ_1	θ_2
Agent 1	1	5 [$\Downarrow$2]	3	3
Agent 2	5 [$\Uparrow$2]	1	3	3

Now we can compare expected second-period utility levels before and after trade for both agents:

Before	After
0.8047	$\frac{1}{2}\ln(3) + \frac{1}{2}\ln(3) = 1.099 \cong 1.1$

In other words, expected utility has increased quite significantly, as anticipated.[6]

This feasible allocation is, in fact, *Pareto optimal.* In conformity with Eq. (9.3), the ratios of the two agents' marginal utilities are indeed equalized across states. More is accomplished in this perfectly symmetrical and equitable allocation: consumption levels and MU are equated across agents and states, but this is a coincidence resulting from the symmetry of the initial endowments.

Suppose the initial allocation is the one identified in Table 9.7.

Once again there is no aggregate risk: The total date 1 endowment is the same in the two states, but one agent is now richer than the other. Now consider the plausible trade outlined in Table 9.8.

Check that the new post-trade allocation is also PO. Although consumption levels and marginal utilities are not identical, the ratio of marginal utilities is the same across states (except at date 0 where, as before, we have a corner solution since the marginal utilities are given constants). Note that this PO allocation features perfect risk sharing as well. By that we mean that the two agents have constant date 1 consumption (two units for agent 1, four units for agent 2) independent of the realized state. This is a general characteristic of PO allocations in the absence of aggregate risk (and with risk-averse agents). If there is no aggregate risk, all PO allocations necessarily feature full mutual insurance.

This statement can be demonstrated, using the data of our problem. Equation (9.3) states that the ratio of the two agents' marginal utilities should be equated across states. This also

[6] With the specified utility function, expected utility has increased by 37%. Such quantification is not, however, compatible with the observation that expected utility functions are defined only up to a linear transformation. Instead of using $\ln c$ for the period utility function, we could equally well have used $(b + \ln c)$ to represent the same preference ordering. The quantification of the increase in utility pre- and post-trade would be affected.

Table 9.7: Another set of initial allocations

	$t=0$	$t=1$	
		θ_1	θ_2
Agent 1	4	1	3
Agent 2	4	5	3

Table 9.8: Plausible trades and post-trade consumptions

	Endowments Pre-trade		Consumption Post-trade	
Date 1	θ_1	θ_2	θ_1	θ_2
Agent 1	1	3 [$\Downarrow$1]	2	2
Agent 2	5 [$\Uparrow$1]	3	4	4

implies, however, that the MRS between state 1 and state 2 consumption must be the same for the two agents. In the case of log period utility:

$$\frac{1/c_1^1}{1/c_1^2} = \frac{1/c_2^1}{1/c_2^2} \Leftrightarrow \frac{1/c_1^1}{1/c_2^1} = \frac{1/c_1^2}{1/c_2^2}$$

The latter equality has the following implications:

1. If one of the two agents is fully insured—no variation in his date 1 consumption (i.e., MRS = 1)—the other must be as well.
2. More generally, if the MRS are to differ from 1, given that they must be equal between them, the low consumption–high MU state must be the same for both agents and similarly for the high consumption–low MU state. But this is impossible if there is no aggregate risk and total endowment is constant. Thus, as asserted, in the absence of aggregate risk, a PO allocation features perfectly insured individuals and MRS identically equal to 1.
3. If there is aggregate risk, however, the above reasoning also implies that, at a Pareto optimum, it is shared "proportionately." This is literally true if agents' preferences are homogeneous. Refer to the competitive equilibrium of Section 9.3 for an illustration.
4. Finally, if agents are differentially risk averse, in a PO allocation the less risk averse will typically provide some insurance services to the more risk averse. This is most easily illustrated by assuming that one of the two agents, say agent 1, is risk neutral. By risk neutrality, agent one's marginal utility is constant. But then the marginal utility of agent 2 should also be constant across states. For this to be the case, however, agent 2's income uncertainty must be fully absorbed by agent 1, the risk-neutral agent.
5. More generally, optimal risk sharing dictates that the agent most tolerant of risk bears a disproportionate share of it.

9.5 Implementing PO Allocations: On the Possibility of Market Failure

Although to achieve the desired allocations, the agents of our previous section could just effect a handshake trade, real economic agents typically interact only through impersonal security markets or through deals involving financial intermediaries. One reason for this practice is that, in an organized security market, the contracts implied by the purchase or sale of a security are enforceable. This is important: without an enforceable contract, if state 1 occurs, agent 2 might retreat from his ex-ante commitment and refuse to give up the promised consumption to agent 1, and vice versa if state 2 occurs. Accordingly, we now address the following question: What securities could empower these agents to achieve the optimal allocation for themselves?

Consider the Arrow–Debreu security with payoff in state 1 and call it security Q to clarify the notation below. Denote its price by q_Q, and let us compute the demand by each agent for this security denoted $z_Q^i, i = 1, 2$. The price is expressed in terms of period 0 consumption. We otherwise maintain the setup of the preceding section. Thus,

$$\text{Agent 1 solves: } \max(4 - q_Q z_Q^1) + \left[\tfrac{1}{2}\ln(1 + z_Q^1) + \tfrac{1}{2}\ln(5)\right]$$

$$\text{s.t. } q_Q z_Q^1 \leq 4$$

$$\text{Agent 2 solves: } \max(4 - q_Q z_Q^2) + \left[\tfrac{1}{2}\ln(5 + z_Q^2) + \tfrac{1}{2}\ln(1)\right]$$

$$\text{s.t. } q_Q z_Q^2 \leq 4$$

Assuming an interior solution, the FOCs are, respectively,

$$-q_Q + \frac{1}{2}\left(\frac{1}{1 + z_Q^1}\right) = 0;\ -q_Q + \frac{1}{2}\left(\frac{1}{5 + z_Q^2}\right) = 0 \Rightarrow \frac{1}{1 + z_Q^1} = \frac{1}{5 + z_Q^2};$$

also $z_Q^1 + z_Q^2 = 0$ in equilibrium, hence, $z_Q^1 = 2; z_Q^2 = -2$; these represent the holdings of each agent and $q_Q = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{6}$. In effect, agent 1 gives up $q_Q z_Q^1 = \left(\frac{1}{6}\right)(2) = \frac{1}{3}$ unit of consumption at date 0 to agent 2 in exchange for 2 units of consumption at date 1 if state 1 occurs. Both agents are better off as revealed by the computation of their expected utilities post-trade:

$$\text{Agent 1 expected utility: } 4 - \tfrac{1}{3} + \tfrac{1}{2}\ln 3 + \tfrac{1}{2}\ln 5 = 5.013$$

$$\text{Agent 2 expected utility: } 4 + \tfrac{1}{3} + \tfrac{1}{2}\ln 3 + \tfrac{1}{2}\ln 1 = 4.879,$$

Table 9.9: Market allocation when both securities are traded

		$t=1$	
	$t=0$	θ_1	θ_2
Agent 1	4	3	3
Agent 2	4	3	3

though agent 2 only slightly so. Clearly agent 1 is made proportionately better off because security Q pays off in the state where his MU is highest. We may view agent 2 as the issuer of this security as it entails, for him, a future obligation.[7]

Let us denote R the other conceivable Arrow–Debreu security, one paying in state 2. By symmetry, it would also have a price of $\frac{1}{6}$, and the demand at this price would be $z_R^1 = -2,\ z_R^2 = +2$, respectively. Agent 2 would give up $\frac{1}{3}$ unit of period 1 consumption to agent 1 in exchange for 2 units of consumption in state 2.

Thus, if both security Q and R are traded, the market allocation will replicate the optimal allocation of risks, as seen in Table 9.9.

In general, it is possible to achieve the optimal allocation of risks provided the number of linearly independent securities equals the number of states of nature. By linearly independent we mean, again, that there is no security whose payoff pattern across states and time periods can be duplicated by a portfolio of other securities. This important topic will be discussed at length in Chapter 11. Here let us simply take stock of the fact that our securities Q, R are the simplest pair of securities with this property.

Although a complete set of Arrow–Debreu securities is sufficient for optimal risk sharing, it is not necessary in the sense that it is possible, by coincidence, for the desirable trades to be effected with a simplified asset structure. For our simple example, one security would allow the agents to achieve that goal because of the essential symmetry of the problem. Consider security Z with payoffs:

	θ_1	θ_2
Z	2	−2

Clearly, if agent 1 purchases one unit of this security ($z_Z^1 = 1$) and agent 2 sells one unit of this security ($z_Z^2 = -1$), optimal risk sharing is achieved. (At what price would this security sell?)

[7] In a noncompetitive situation, it is likely that agent 2 could extract a larger portion of the rent. Remember, however, that we maintain, throughout, the assumption of price-taking behavior for our two agents who are representatives of larger classes of similar individuals.

So far we have implicitly assumed that the creation of these securities is costless. In reality, the creation of a new security is an expensive proposition: disclosure documents, promotional materials, and so on, must be created, and the agents most likely to be interested in the security contacted. In this example, issuance will occur only if the cost of issuing Q and R does not exceed the (expected) utility gained from purchasing them. In this margin lies the investment banker's fee.

In the previous discussion we imagined each agent as issuing securities to the other simultaneously. More realistically, perhaps, we could think of the securities Q and R as being issued in sequence, one after the other (but both before period 1 uncertainty is resolved). Is there an advantage or disadvantage of going first, that is, of issuing the *first* security? Alternatively, we might be concerned with the fact that, although both agents benefit from the issuance of new securities, only the individual issuer pays the cost of establishing a new market. From this perspective, it is interesting to measure the net gains to trade for each agent. These quantities are summarized in Table 9.10.

In our example, this computation tells us that the issuer of the security gains less than the other party in the future trade. If agent 2 goes first and issues security Q, his net expected utility gain is 0.0783, which also represents the most he would be willing to pay his investment bank in terms of period 0 consumption to manage the sale for him. By analogy, the marginal benefit to agent 1 of *then* issuing security R is 0.0780. The reverse assignments would have occurred if agent 1 had gone first, due to symmetry in the agent endowments. That these quantities represent the upper bounds on possible fees comes from the fact that period 0 utility of consumption is the level of consumption itself.

The impact of all this is that each investment bank will, out of desire to maximize its fee potential, advise its client to issue his security second. No one will want to go first. Alternatively, if the effective cost of setting up the market for security Q is anywhere between 0.0783 and 0.2936, there is a possibility of *market failure*, unless agent 2 finds a

Table 9.10: The net gains from trade: expected utility levels and net trading gains (Gain to issuer in bold)

	No Trade	Trade Only Q		Trade Both Q and R	
	EU	EU	ΔEU[a]	EU	ΔEU[b]
Agent 1	4.8047	5.0206	0.2159	5.0986	**0.0780**
Agent 2	4.8047	4.883	**0.0783**	5.0986	0.2156
Total			0.2942		0.2936

[a]Difference in EU when trading Q only, relative to no trade.
[b]Difference in EU when trading both Q and R, relative to trading Q only.

way to have agent 1 share in the cost of establishing the market. We speak of market failure because the social benefit of setting up the market would be positive 0.2936 minus the cost itself—while the market might not go ahead if the private cost to agent 2 exceeds his own private benefit, measured at 0.0783 units of date 0 consumption. Of course, it might also be the case that the cost exceeds the total benefit. This is another reason for the market not to exist and, in general, for markets to be incomplete. But in this case, one would not talk of market failure. Whether the privately motivated decisions of individual agents lead to the socially optimal outcome—in this case the socially optimal set of securities—is a fundamental question in financial economics.

There is no guarantee that private incentives will suffice to create the social optimal set of markets. We have identified a problem of sequencing—the issuer of a security may not be the greatest beneficiary of the creation of the market—and as a result there may be a waiting game with suboptimal consequences. There is also a problem linked with sharing the cost of setting up a market. The benefits of a new market often are widely spread across a large number of potential participants, and it may be difficult to find an appropriate mechanism to have them share the initial setup cost, for example because of free rider or coordination problems. Note that in both cases, as well as in the situation where the cost of establishing a market exceeds the total benefit for individual agents, we anticipate that technical innovations leading to decreases in the cost of establishing markets will help alleviate the problem and foster a convergence toward a more complete set of markets.[8]

9.6 Risk-Neutral Valuations

The Arrow–Debreu pricing perspective allows us to introduce the "distorted probabilities" approach to asset valuation first mentioned in Chapter 2. In general, this asset pricing perspective goes under the title of "risk-neutral valuation." While our introduction to the idea in the paragraphs below may appear at first to be nothing more than a clever algebraic manipulation—a "trick"—it is a concept that will turn out to be of paramount usefulness, facilitating such diverse activities as options pricing and the empirical assessment of our Arrow–Debreu cum VNM-expected utility asset pricing theory.

[8] The larger issue here is whether making markets more complete uniformly improves agent welfare. By this we mean the following: consider our two period multi date-state setting and assume there is one state for which no state claim is traded: in that state all agents consume their endowments. Now open up claims trading to include one of the previously excluded states. Will the welfare of all the agents be improved? There is no guarantee this will happen. Transfers among agents could be affected, however, that would increase the welfare of everyone. But some agents may not agree to participate! See the Web Notes to this chapter for illustrative examples.

Table 9.11: Security payoffs and equilibrium date-state security prices

	$t=1$		
$t=0$	θ_1	θ_2	θ_3
$q_1 = 0.5$	1	0	0
$q_2 = 0.1$	0	1	0
$q_3 = 0.3$	0	0	1
$q^b = ?$	1	1	1
$q^e = ?$	2	1	4

While risk-neutral valuation does not require a VNM-expected utility preference specification (agents may be assumed to have ordinal representations $u^k(c_1^k, c_2^k, c_{\theta_1}^k, \ldots, c_{\theta_N}^k)$, as noted in Section 9.2), it does demand that financial markets are complete: for every future date-state there must exist a traded security that pays exclusively in that date-state. In our discussion below we take these prices as given, side-stepping their equilibrium determination, in order to focus on the pricing of more complex, multistate cash flows. For clarity and simplicity we retain the two period setting of the present chapter; multiperiod generalizations are treated in later chapters.

Consider the simple date-state payoff uncertainty and pricing structure of Table 9.11.

As earlier, q^b denotes the period $t = 0$ price of a risk-free bond paying one unit of consumption in each period 1 state, q^e is the price of some "equity security" with the indicated payoffs, and the q_θ, $\theta = 1, 2, 3$ represent the Arrow–Debreu state prices. Our objective is to study the equilibrium pricing of q^b and q^e. Since equilibrium tolerates no arbitrage opportunities, it must be that

$$q^b = q_1 + q_2 + q_3 = \sum_{i=1}^{3} q_\theta = 0.9,$$

which in turn implies a risk-free rate as per

$$q^b = 0.9 = \frac{1}{(1+r_f)}; \; r_f = 0.11 \textit{ or } 11\%.$$

In a like fashion, let us next price the "equity security":

$$\begin{aligned} q^e &= 2q_1 + q_2 + 4q_3 \; (= 2(0.5) + 1(0.1) + 4(0.3) = 2.3) \\ &= q^b\left[2\left(\frac{q_1}{q^b}\right) + 1\left(\frac{q_2}{q^b}\right) + 4\left(\frac{q_3}{q^b}\right)\right] \\ &= \frac{1}{(1+r_f)}\left[2\left(\frac{q_1}{q^b}\right) + 1\left(\frac{q_2}{q^b}\right) + 4\left(\frac{q_3}{q^b}\right)\right]. \end{aligned} \tag{9.4}$$

We pause here for a moment to make two observations. First, note that

$$\frac{q_\theta}{\sum_{\theta=1}^{3} q_\theta} > 0, \forall\theta. \text{ Second note also that}$$

$$\sum_{\theta=1}^{3}\left(\frac{q_\theta}{\sum\limits_{\theta=1}^{3} q_\theta}\right) = \frac{1}{q^b}\sum_{\theta=1}^{3} q_\theta = \frac{1}{q^b}\cdot q^b = 1.$$

Taken together, these relationships imply that the quantities $\left\{(q_\theta)/\left(\sum_{\theta=1}^{3} q_\theta\right)\right\}, i = 1, 2, 3$ have the structure of a probability density. Henceforth, we will refer to them as "risk-neutral" probabilities and employ the notation $\{\pi_\theta^{RN}\}_{\theta=1,2,3}$ where $\pi_\theta^{RN} = (q_\theta/\sum_{\theta=1}^{3} q_\theta)$.

Equation (9.4) can then be rewritten as

$$\begin{aligned} q^e &= \frac{1}{(1+r_f)}\left[2\pi_1^{RN} + 1\pi_2^{RN} + 4\pi_3^{RN}\right] \\ &= \frac{1}{(1+r_f)} E_{\pi^{RN}} CF_\theta^e \end{aligned}$$

with $E_{\pi^{RN}}$ denoting the expectations operators taken with respect to the risk-neutral probability distribution. This accomplishes our goal: The asset is priced as its "distorted probability" expected "cash flow," discounted at the risk-free rate. At this juncture we offer a number of clarifying comments.

1. The price of any asset whose payoffs lie in the span of the three Arrow–Debreu securities described in Table 9.11 (that is, any security whose payoffs can be replicated by some portfolio of these securities) can be represented as its expected cash flow, computed using the indicated risk-neutral probabilities, discounted at the risk-free rate.
2. No mention has been made of the objective probability distribution governing the future states because our development of the notion of risk-neutral valuation did not require that agents' preferences take the VNM-expected utility form. As will become apparent in the next chapter these probabilities are embedded in the relative Arrow–Debreu state claims prices. Present and future state-dependent endowments and any other quantities relevant for agent security demands (e.g., agent risk aversions) are embedded in the Arrow–Debreu prices as well.
3. When expectations of future cash flows are taken with respect to the risk-neutral probability distribution, all assets in the span of the Arrow–Debreu securities are deemed to earn the risk-free rate of return, an assertion that seems strikingly

counterintuitive: risk appears to be "ignored." In fact, this is not the case. When we impose greater economic structure (e.g., explicit objective probabilities and VNM-expected utility preferences) on the risk-neutral asset pricing theory, it will be shown that risk-neutral probabilities are "pessimistic" in the sense that they are larger than the objective probabilities for states where payoffs are small—the "bad" states—and smaller than the objective probabilities for states where payoffs are high, with the consequence that risk-neutral expected payoffs are numerically smaller than expected payoffs computed using the common objective probabilities. This adjustment compensates for discounting at the lower, not-risk-adjusted, risk-free rate in a risk-neutral pricing environment.

4. Our final comment concerns the origin of the title "risk-neutral" valuation given to the pricing perspective articulated above. Although there are a number of associations to which we could appeal to explore this designation, a principal one is as follows: as noted in the commentary following Eq. (9.1), if agents are risk neutral or if one agent is both risk neutral and has sufficient endowment to insure all other agents perfectly (no corner solution), then the Arrow−Debreu prices are directly proportional (by the common factor β) to to their respective objective probabilities. In this case, the risk-neutral and objective probabilities coincide. Under either pricing perspective, expected asset cash flows are discounted at the risk-free rate. Note that these comments continue to presume the expected utility representation of agent preferences.

From what has been presented in Section 9.6, the usefulness of the risk-neutral valuation concept is not immediately apparent: it appears to be simply a reformulation of Arrow−Debreu pricing theory and we know Arrow−Debreu securities are generally not presently traded in organized financial markets. When more structure is imposed on the economy (e.g., VNM-expected utility preferences), however, the concept leads to the notion of a stochastic discount factor which is simply a more structured form of the risk-neutral probability distribution. The stochastic discount factor represents the ultimate discounting perspective in modern finance theory, and big financial firms spend millions attempting to estimate it. Its properties can also be related to certain macro data series thereby allowing empirical tests of the theory.

9.7 Conclusions

The Arrow−Debreu asset pricing theory presented in this chapter is in some sense the father of all asset pricing relationships. It is fully general and constitutes an extremely valuable reference formulation. Conceptually, its usefulness is unmatched, which justifies our investing more in its associated apparatus. At the same time, it is one of the most abstract theories, and its usefulness in practice is impaired by the difficulty in identifying individual states of nature and by the fact that, even when a set of states can be identified, their actual realization cannot always be verified. As a result, it is difficult to write the appropriate conditional contracts. These problems go a long way in explaining why we do not see Arrow–Debreu securities being traded, a fact that does not strengthen the immediate applicability of the theory. In addition, as already mentioned, the static setting of the Arrow–Debreu theory is unrealistic for most applications. For all these reasons we cannot stop here, and we will explore a set of alternative, sometimes closely related, avenues for pricing assets in the following chapters.

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