Asset Pricing

Expected utility and risk aversion

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Choosing among risky prospects

Choosing among risky prospects

The central question of interest is: "How do individuals choose between risky alternatives"? That is, how do we model investors' demand for assets?

* Assets have uncertain future payoffs, so any theory of asset demand must specify investor preferences over future uncertain money payments
* Expected utility is the standard framework and workhorse for modeling investor choices over risky assets. To use it, we need to
  1. Define individual preferences and analyze conditions that lead to an expected utility function representation
  2. Analyze the link between the choice of utility function, risk aversion, and risk premia for different assets
  3. Understand the limitations and weaknesses of the expected utility framework

State-by-state dominance

* A first, but incomplete, way of ranking risky alternatives is dominance, where we can think of risk as the uncertainty (randomness) in future cash flows
* Consider a three asset, two state example, where the probability of either state occurring is $\pi_{\text{Bad}} = \pi_{\text{Good}} = \frac{1}{2}$

		Future payoff	
	Price today	Bad state	Good state
Asset 1	1000	1050	1200
Asset 2	1000	500	1600
Asset 3	1000	1050	1600

* Assuming that the typical individual is non-satiated in consumption, then asset 3 dominates assets 1 and 2 state-by-state (strongest form)
* Most of the cases are not so trivial (how to rank assets 1 and 2?)

Mean-variance dominance

- Another type of dominance is mean-variance dominance. In our example, the expected returns and standard deviations are

	Expected return	Standard deviation
Asset 1	12.5%	7.5%
Asset 2	5.0%	55.0%
Asset 3	32.5%	27.5%

- Mean-variance dominance is neither as strong nor as general as state-by-state dominance, and it does not provide a complete ranking either
- Asset 3, for instance, mean-variance dominates asset 2, but not 1, although it dominated both on a state-by-state basis
- A criterion for selecting investments of equal magnitude, which plays a prominent role in portfolio theory (next topic), is
  1. For investments of the same $\mathbb{E}[r]$, choose the one with the lowest σ
  2. For investments of the same σ, choose the one with the highest $\mathbb{E}[r]$

Sharpe ratio dominance

* Consider the below state-contingent rates of return with $\pi_{\text{Bad}} = \pi_{\text{Good}} = \frac{1}{2}$

	Bad state	Good state	$\mathbb{E}[r]$	σ	SR
Asset 4	3%	5%	4%	1%	4.00
Asset 5	2%	8%	5%	3%	1.67

* No dominance in either state-by-state or mean-variance terms

* Instead, one could specify the terms at which the investor is willing to substitute expected return for a given risk reduction

* For example, a given individual may rank asset by the index (which we with a slight abuse of terminology can think of as the Sharpe (1966) ratio)

$$\text{SR} = \frac{\mathbb{E}[r]}{\sigma} \tag{1}$$

* Other individuals may have different preferences, which would yield a different index. In the end, it depends on the investor's risk aversion

Ranking by expected payoffs

- A frequently applied criterion for measuring the attractiveness of an asset is the expected value of its future uncertain payoff

Expected value

Consider an asset that offers a single random payoff $\widetilde{x}$ at some future date, and suppose that this payoff has a discrete distribution with N possible outcomes $\{x_1, \ldots, x_N\}$ and corresponding probabilities $\{\pi_1, \ldots, \pi_N\}$, where $\sum_{i=1}^{N} \pi_i = 1$ and $\pi_i \geq 0$. The expected value of the payoff is then

$$\mathbb{E}\left[\widetilde{x}\right] = \sum_{i=1}^{N} \pi_i x_i \tag{2}$$

- Nicholas Bernoulli cleverly illustrated that expected value is insufficient to capture individuals' valuations of gambles in a letter to mathematician Pierre Raymond de Montmort in 1713

St. Petersburg Paradox (Nicholas Bernoulli, 1713)

- Consider Paul's expected value of the following coin flipping game:

  "*Peter tosses a coin and continues to do so until it lands "heads". He agrees to pay Paul two ducats if he gets heads on the very first throw, four ducats if he gets it on the second, eight if on the third, sixteen if on the fourth, and so on, so that on each additional throw the number of ducats he must pay is doubled*"

St. Petersburg Paradox (Nicholas Bernoulli, 1713)

If the number of coin flips taken to first arrive at heads is i, then $\pi_i = \left(\frac{1}{2}\right)^i = 2^{-i}$ and $x_i = 2^i$ so that the expected payoff equals

$$\mathbb{E}[\widetilde{x}] = \sum_{i=1}^{\infty} \pi_i x_i = \sum_{i=1}^{\infty} 2^{-i} 2^i \tag{3}$$

$$= \frac{1}{2}2 + \frac{1}{4}4 + \frac{1}{8}8 + \frac{1}{16}16 + \cdots \tag{4}$$

$$= \sum_{i=1}^{\infty} 1 = \infty \tag{5}$$

St. Petersburg and expected utility

* The "paradox" is that the expected value is infinite, but intuitively, few individuals would pay more than a moderate amount to play the game
* Daniel Bernoulli, a cousin to Nicholas Bernoulli, proposed a solution to the "paradox" in 1738 by introducing the concept of expected utility
* His main insight was that an individual's utility, or "happiness", from receiving a payoff could differ from the size of the payoff and that people cared more about the expected utility of an assets' payoffs

$$V \equiv \mathbb{E}\left[U\left(\widetilde{x}\right)\right] = \sum_{i=1}^{N} \pi_i U\left(x_i\right) \tag{6}$$

* He hypothesized that: "*the utility resulting from any small increase in wealth will be inversely proportionate to the quantity of goods previously possessed*"
* That is, Bernoulli effectively introduced the concept of diminishing marginal utility of wealth as a solution to the paradox (Machina, 1987)

The expected utility model

The expected utility model

The first complete axiomatic treatment of the expected utility framework can be attributed to mathematicians von Neumann and Morgenstern (1944). They worked out the conditions under which an individual's preferences could be described by an expected utility function

* Expected utility is the workhorse of modern asset pricing and it will be central to many of the models developed in this course

* The expected utility theorem provides a set of hypothesis under which an investor's preference ranking over risky prospects may be represented by a utility index that combines two essential elements linearly, i.e.,

  1. The preference ordering on the ex post payoffs

  2. The respective probabilities of the payoffs

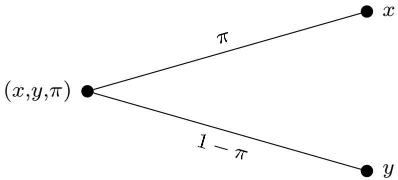
Lotteries

Lotteries

We can define a lottery as an asset that has a risky payoff. A generic lottery will be denoted $L = (x,y,\pi)$, which is to be understood as follows

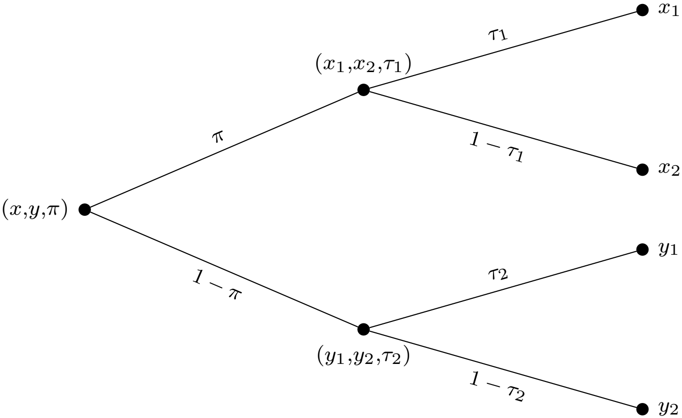
* The lottery offers a payoff equal to x with probability π
* The lottery offers a payoff equal to y with probability $1 - \pi$

This notion of a lottery is very general and encompasses a broad variety of different payoffs structures.



Compound lottery

* Our definition of a lottery asset includes compound lotteries defined as $L = (x,y,\pi) = ((x_1,x_2,\tau_1), (y_1,y_2,\tau_2), \pi)$, which implies that x and y are themselves lotteries



Preference orderings

Preference relations

There exists a preference relation, represented by the symbol $\succeq$, that describes investors' ability to compare and rank different lottery assets. We can consider three different scenarios

1. If an individual strictly prefers lottery L to lottery L^*, we write

$$L \succ L^* \quad \text{or} \quad L^* \prec L \tag{7}$$

2. If an individual is indifferent between the two lotteries we write

$$L \sim L^* \tag{8}$$

3. If an individual weakly prefers lottery L to lottery L^*, we write

$$L \succeq L^* \quad \text{or} \quad L^* \preceq L \tag{9}$$

Axioms of expected utility

Axioms of expected utility

Let $L = (x,y,\pi)$, $L^* = (x,y,\pi^*)$, and $L^{**} = (x,y,\pi^{**})$ denote three different lotteries. The von Neumann and Morgenstern (1944) axioms of rational behavior can then be stated as

1. **Completeness:** For any two lotteries L^* and L, either $L^* \succ L$, $L^* \prec L$, or $L^* \sim L$
2. **Transitivity:** If $L^{**} \succeq L^*$ and $L^* \succeq L$, then $L^{**} \succeq L$
3. **Continuity:** If $L^{**} \succeq L^* \succeq L$, there exists some $\lambda \in [0,1]$ such that $L^* \sim \lambda L^{**} + (1-\lambda)\, L$, where $\lambda L^{**} + (1-\lambda)\, L$ denotes a compound lottery
4. **Independence:** For any two lotteries L and L^*, $L^* \succ L$ if and only if for all $\lambda \in (0,1]$ and all L^{**} it holds that $\lambda L^* + (1-\lambda)\, L^{**} \succ \lambda L + (1-\lambda)\, L^{**}$
5. **Dominance:** Let $x \succ y$. Then $L^* \succ L$ if and only if $\pi^* > \pi$

Discussion and relation to choice under certainty

- The notion of economic rationality can be summarized by the first three of the above axioms (equivalent to consumer choice under certainty)
- In fact, axioms 1–3 are analogues to those used by Debreu (1954) to establish the existence of a continuous, time-invariant, real-valued utility function for choice under certainty (e.g., over consumer goods)
- The fourth axiom, independence, is new to the first three, but its linearity property is essential for preferences to be consistent with expected utility
  - Suppose that an individual chooses $L^* \succ L$. Axiom 4 implies that the choice of

    $$\lambda L^* + (1 - \lambda) L^{**} \quad \text{and} \quad \lambda L + (1 - \lambda) L^{**} \tag{10}$$

    is equivalent to a gamble in which with probability $(1 - \lambda)$ both lotteries pay L^{**}, and with probability λ the first compound lottery pays L^* and the second L
  - This is equivalent to being asked, prior to the gamble, if one would prefer L^* or L

Expected utility theorem

Expected utility theorem

If the von Neumann-Morgernstern (VNM) axioms are satisfied, then there exists an expected utility function $\mathbb{E}\left[U\left(\cdot\right)\right] \equiv \mathbb{U}\left(\cdot\right)$ defined on the space of lotteries so that

$$L \succeq L^{*} \Leftrightarrow \mathbb{U}\left(L\right) \geq \mathbb{U}\left(L^{*}\right) \tag{11}$$

where $\mathbb{U}\left(L\right) = \pi U\left(x\right) + \left(1-\pi\right) U\left(y\right)$ is the VNM expected utility function and $U\left(x\right) = \mathbb{U}\left(x,y,1\right)$ the Bernoulli utility (of money) function.

- VNM utility functions are linear in probabilities and unique up to a linear monotonic transformation, i.e., $\mathbb{V}\left(\cdot\right) = a\mathbb{U}\left(\cdot\right) + b$ with $a > 0$ is also a VNM utility function (but not for non-linear transformations)
- A non-linear transformation does not always respect preference orderings, so VNM utility functions are cardinal (in contrast to the ordinal utility functions often found in standard microeconomic representations)
- Additional assumptions on $U\left(\cdot\right)$ will identity investors' risk preferences, and we will consider increasing and concave candidates

St. Petersburg paradox revisited

- To see how expected utility may solve the St. Petersburg paradox, suppose that an individual's utility is given by the natural logarithm of a monetary outcome (the increasing and concave function used by Bernoulli himself)

$$U(x) = \ln(x) \tag{12}$$

- In this case, the expected utility of the coin flipping game is

$$\begin{aligned} \mathbb{U} &= \sum_{i=1}^{\infty} \pi_i U(x_i) = \sum_{i=1}^{\infty} 2^{-i} \ln\left(2^i\right) = \sum_{i=1}^{\infty} 2^{-i} i \ln(2) \\ &= \ln(2) \underbrace{\sum_{i=1}^{\infty} 2^{-i} i}_{=2} = 2\ln(2) \cong 1.3863 \end{aligned} \tag{13}$$

implying that a certain payment of $\exp(2\ln(2)) = 4$ ducats has the same expected utility as playing the St. Petersburg game

Super St. Petersburg

- Yet, the VNM utility framework is not a complete resolution of the paradox as one can construct a Super St. Petersburg (Menger, 1934)
- Note that the St. Petersburg paradox has infinite expected value because the probability of winning and the payment declines/grows at the same rate 2^i
- To get unbounded (infinite) utility, we can define a Super St. Petersburg in which the winning payoffs increase at a rate $x_i = U^{-1}\left(2^i\right)$ such that expected utility increases at a rate 2^i
- For log utility, let $x_i = \exp\left(2^i\right)$ such that $U\left(x_1\right) = 2, U\left(x_2\right) = 4, U\left(x_3\right) = 8$ and so on. Then expected utility becomes

$$\mathbb{U} = \sum_{i=1}^{\infty} \pi_i U_i = \sum_{i=1}^{\infty} 2^{-i} \ln\left(\exp\left(2^i\right)\right) = \infty \tag{14}$$

How troublesome is a Super St. Petersburg?

* Should we be concerned that infinite expected utility (and valuations) can arise for any utility function if we simply let prizes grow fast enough?
* One could argue that St. Petersburg games are unrealistic, particularly ones where the payoffs are assumed to grow very rapidly
* The reason being that any individual offering this game would need to have infinite wealth, whereas any individual at most have finite wealth
* This effectively sets an upper bound on the amount of prizes that could feasibly be paid, making expected utility, and even the expected utility of the Super St. Petersburg, finite

The Allais paradox

- Allais (1953) provides experimental evidence on a simple choice problem that produces an inconsistency between observed choices and the predictions of expected utility theory
- This famous paradox is known today as the Allais paradox and its main point is to show that the independence axiom (Axiom 4) may not be valid

The Allais paradox

Consider the following set of lottery choices

1. $L^1 = (10000, 0, 1)$ versus $L^2 = (15000, 0, 0.9)$
2. $L^3 = (10000, 0, 0.1)$ versus $L^4 = (15000, 0, 0.09)$

When individuals are asked to rank these payoff, we typically observe the following preference orderings

$$L^1 \succ L^2 \quad \text{and} \quad L^4 \succ L^3 \tag{15}$$

Allais paradox explained

* To understand why this paradox violates the independence axiom, define a new lottery $L^0 = (0, 0, 1)$ and use the structure of compound lotteries to illustrate that

$$L^3 = \left(L^1, L^0, 0.1\right) \quad \text{and} \quad L^4 = \left(L^2, L^0, 0.1\right) \tag{16}$$

* By the independence axiom, the ranking between L^1 and L^2, on the one hand, and L^3 and L^4, on the other, should therefore be identical

* This is the Allais paradox to which there are a number of possible reactions

  1. Yes, my choices were inconsistent, let me think again and revise them
  2. No, I will stick to my choices. The following are missing from the theory
     - The pleasure of gambling
     - The notion of regret

Behavioral contradictions

* The notion of a "rational investor" underlies most of financial theory

Rational investor (homo economicus)

A rational investor is simply an investor with two essential attributes

1. Preferences over random money payoffs are described by expected utility
2. The probabilities assigned to these payoffs are objective in the sense that they incorporate all past and present information available to the investor in a manner that respects correct statistical procedure

* Yet, there are various behavioral and cognitive biases that seem to contradict this definition, including
    - Framing, Loss aversion, Overconfidence, Anchoring, Confirmation bias, herding, Overreactions, Underreactions, preference reversals, and so on …
    - If you are interested, the popular books by Kahneman (2011), Thaler (2015), and Lewis (2016) are great reads for the development of behavioral finance

Framing and loss aversion

* Consider the classic example from Kahneman and Tversky (1979) in which you are presented with the following lotteries

    1. In addition to whatever you own, you have been given $\$1{,}000$, and are asked to choose between

    $$L^A = (1000, 0, 0.5) \quad \text{and} \quad L^B = (500, 0, 1) \tag{17}$$

    2. In addition to whatever you own, you have been given $\$2{,}000$, and are asked to choose between

    $$L^C = (-1000, 0, 0.5) \quad \text{and} \quad L^D = (-500, 0, 1) \tag{18}$$

* A majority of subjects choose $B \succ A$ and $C \succ D$, despite A and C and B and D being equivalent when taking into account the differing initial payments

* What is going on? Apparently, it matters whether you "frame" the outcomes as gains or losses relative to the reference wealth level (reference-dependence)

* Obviously, framing is irrelevant under the expected utility framework

Prospect theory

- At present, Kahneman and Tversky's (1979, 1992) prospect theory is the most highly developed behavioral theory of choice under uncertainty (see Barberis (2013) for a recent summary)

- Under cumulative prospect theory, investors evaluate uncertain gambles using decision weights δ rather than objective probabilities, i.e.,

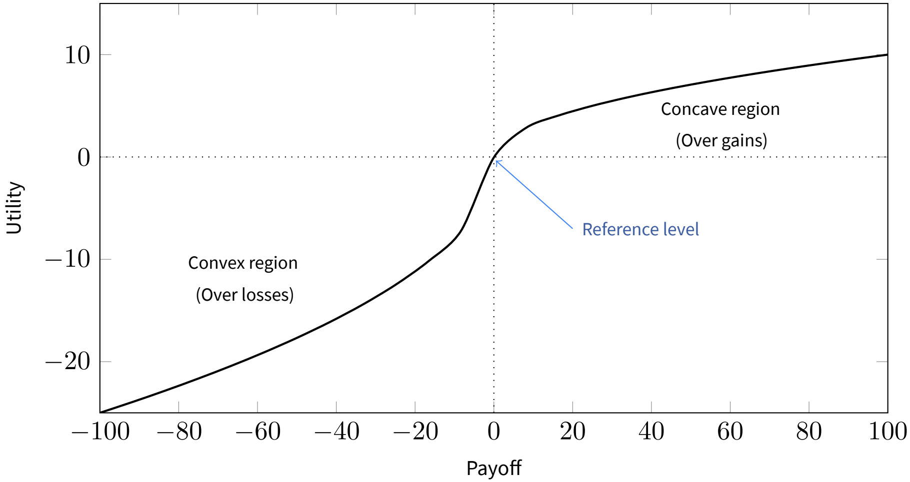
$$\sum_{i=1}^{N} \delta_i V(x_i), \tag{19}$$

where $V(0) = 0$ is referred to as a value function. There are four critical elements to prospect theory that separates it from expected utility

1. Reference dependence
2. Loss aversion
3. changing behavior relative to the reference point
4. probability weighting through decision weights

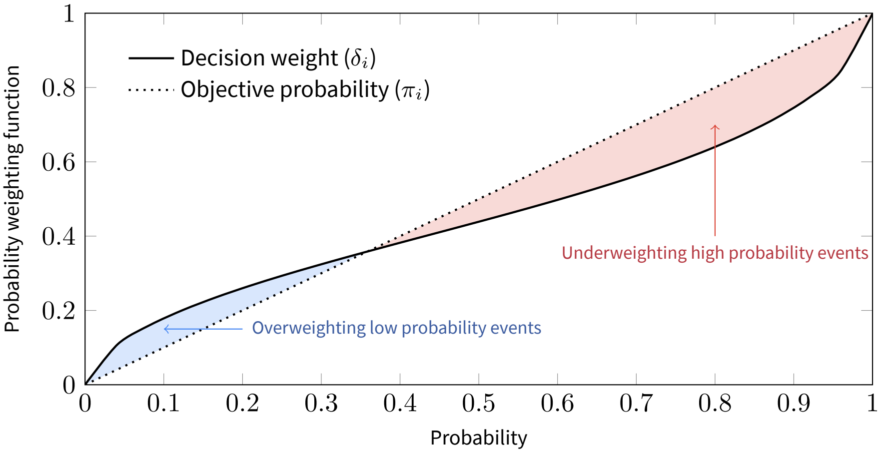
The value function

* At its core, prospect theory is about the shape of the value function and its implication for behavior around the reference point



The decision weights

- In prospect theory, individuals weight outcomes by transformed probabilities referred to as decision weights δ_i rather than objective probabilities π_i
- They should not be confused with erroneous beliefs as individuals in prospect theory are perfectly capable of understanding objective probabilities



Risk aversion

Risk aversion

An individual is said to be risk averse if the person is unwilling to accept a "fair" lottery $\widetilde{h}$, where a fair lottery is defined as one that has an expected value of zero

$$U\left(Y\right) > \mathbb{E}\left[U\left(Y+\widetilde{h}\right)\right] \tag{20}$$

- Investors want to avoid risk, i.e., they have a preference for smoothing their consumption stream across states of nature
- Since probabilities are assumed objective and independent of investor preferences, this idea must be implemented in the utility function
- Recall that Daniel Bernoulli solved the St. Petersburg paradox by proposing that utility functions should display diminishing marginal utility, i.e., that $U\left(\cdot\right)$ should be an increasing, but concave, function of wealth/consumption
- He recognized that this concavity implies that an individual is risk averse

Defining a fair lottery asset

A fair lottery

A fair, or pure risk, lottery asset is defined as a gamble with random payoff $\widetilde{h}$, where

$$\widetilde{h} = \begin{cases} +h & \text{with probability } \frac{1}{2} \\ -h & \text{with probability } \frac{1}{2}, \end{cases} \tag{21}$$

which we can write equivalently as $L = (h, \text{-}h, 0.5)$ using previous notation

- A fair lottery has an expected value of zero, which is satisfied here

$$\mathbb{E}\left[\widetilde{h}\right] = \frac{1}{2}h - \frac{1}{2}h = 0 \tag{22}$$

- We will mostly stick to this formulation of a fair lottery, but note that any lottery with expectation zero is fair
- The question is then whether an individual with preferences consistent with VNM expected utility would accept such a lottery?

Risk aversion and concave utility

* Intuition tells us that a risk averse investor would prefer to avoid such a lottery for any level of wealth Y

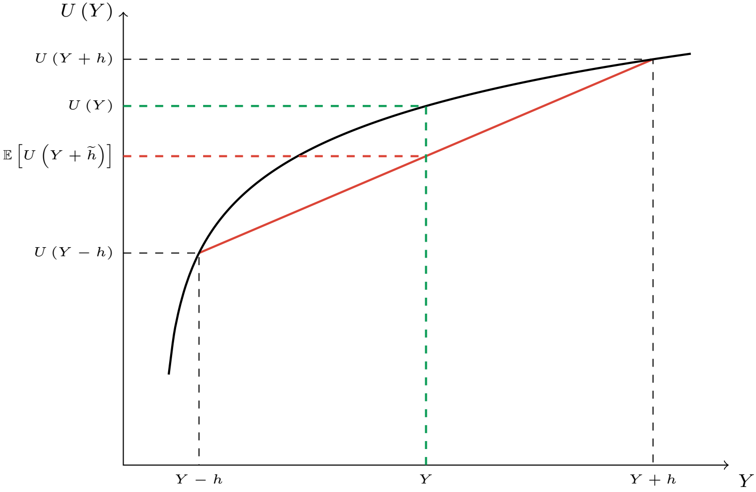
* The expected utility including the lottery is $\mathbb{E}\left[U\left(Y+\widetilde{h}\right)\right]$ and without it $\mathbb{E}\left[U\left(Y\right)\right]=U\left(Y\right)$. The lottery rejection must therefore imply

$$U\left(Y\right)>\mathbb{E}\left[U\left(Y+\widetilde{h}\right)\right]=\frac{1}{2}U\left(Y+h\right)+\frac{1}{2}U\left(Y-h\right) \tag{23}$$

which can only be satisfied for all levels of wealth Y if the utility function is (strictly) concave

  - For a differentiable function $U\left(\cdot\right)$, concavity is equivalent to non-increasing marginal utility: $U'\left(Y_1\right)\leq U'\left(Y_0\right)$ if $Y_1>Y_0$
  - Strict concavity is equivalent to decreasing marginal utility (strict inequality)
  - For a twice differentiable function, concavity is equivalent to $U''\left(Y\right)<0$ for all Y

Representing preference for smoothness



Risk aversion, concavity, and Jensen's Inequality

- We can show that (23) is equivalent to having a concave utility function. Using that $\widetilde{h}$ is a fair lottery, we can write (23) as

$$U\left(Y + \frac{1}{2}h - \frac{1}{2}h\right) > \frac{1}{2}U\left(Y + h\right) + \frac{1}{2}U\left(Y - h\right) \tag{24}$$

- It turns out that this coincides with the very definition of a concave function

Concave function

A function f is concave if, for any x and y and any $\lambda \in (0,1)$,

$$f\left(\lambda x + \left(1 - \lambda\right) y\right) \geq \lambda f\left(x\right) + \left(1 - \lambda\right) f\left(y\right) \tag{25}$$

If the inequality is strict, then the function is strictly concave

Risk aversion, concavity, and Jensen's Inequality

- We can also show that concavity implies unwillingness to accept a fair lottery using (Johan) Jensen's Inequality.

Jensen's Inequality for concave functions

Consider a concave function f and numbers $x_1, \ldots, x_n \in \mathbb{R}$, then

$$f\left(\mathbb{E}\left[X\right]\right) > \mathbb{E}\left[f\left(X\right)\right] \tag{26}$$

- Letting $f = U$ and $X = Y + \widetilde{h}$, where $\mathbb{E}\left[\widetilde{h}\right] = 0$, we have the desired result

$$U\left(Y\right) = U\left(\mathbb{E}\left[Y + \widetilde{h}\right]\right) > \mathbb{E}\left[U\left(Y + \widetilde{h}\right)\right] \tag{27}$$

How to measure the degree of risk aversion?

- We know from earlier that $U''(\cdot) < 0$ implies risk aversion and that $U''(\cdot)$ measures the curvature, so why not simply say that investor A is more risk averse than investor B for all wealth levels Y if and only if

$$|U''_A(Y)| \geq |U''_B(Y)| \tag{28}$$

- However, this approach is inconsistent. Recall the VNM expected utility is invariant to linear transformation such that we can define

$$\overline{U}_A(Y) = a + bU_A(Y), \quad \text{with } b > 0 \tag{29}$$

- $\overline{U}_A(Y)$ and $U_A(Y)$ must describe the same preference ordering and must display identical risk aversion, yet we have that

$$\left|\overline{U}''_A(Y)\right| \geq |U''_A(Y)| \quad \text{if } b > 1 \tag{30}$$

  implying that investor A is more risk averse than himself — nonsense

- This contradiction implies that we need risk aversion measures that are invariant to linear transformations

Risk aversion coefficients

- Two such widely used measures, attributable to Pratt (1964) and Arrow (1971), are absolute and relative risk aversion

Absolute risk aversion

The Arrow-Pratt measure of absolute risk-aversion (ARA), also known as the coefficient of absolute risk aversion, is defined as

$$R_A(Y) \equiv -\frac{U''(Y)}{U'(Y)} \tag{31}$$

Relative risk aversion

The Arrow-Pratt measure of relative risk-aversion (RRA), also known as the coefficient of relative risk aversion, is defined as

$$R_R(Y) \equiv -Y \cdot \frac{U''(Y)}{U'(Y)} \tag{32}$$

Absolute risk aversion and the odds of a bet

* Consider an investor with wealth Y who is offered the lottery $L = (h, \text{-}h, \pi)$
* Any investor will accept such a bet if π is high enough (especially if $\pi = 1$) and reject it if π is small enough (and surely if $\pi = 0$)
* Moreover, the willingness to accept the lottery may also be related to the current level of wealth

Proposition

Let $\pi = \pi\left(Y,h\right)$ be the probability at which the investor is indifferent between accepting and rejecting the lottery. Then

$$\pi\left(Y,h\right) \cong \frac{1}{2} + \frac{1}{4} h R_A\left(Y\right) \tag{33}$$

Proving the proposition

* By definition, the indifference probability $\pi\left(Y,h\right)$ must satisfy

$$U\left(Y\right)=\pi\left(Y,h\right)U\left(Y+h\right)+\left[1-\pi\left(Y,h\right)\right]U\left(Y-h\right) \tag{34}$$

* Taking a second-order Taylor approximation around $h=0$ yields

$$U\left(Y+h\right)\cong U\left(Y\right)+hU'\left(Y\right)+\frac{h^2}{2}U''\left(Y\right) \tag{35}$$

$$U\left(Y-h\right)\cong U\left(Y\right)-hU'\left(Y\right)+\frac{h^2}{2}U''\left(Y\right) \tag{36}$$

* Substituting the approximations back into (34) gives us

$$\begin{aligned}U\left(Y\right)\cong{}&\pi\left(Y,h\right)\left\{U\left(Y\right)+hU'\left(Y\right)+\frac{h^2}{2}U''\left(Y\right)\right\}\\&+\left[1-\pi\left(Y,h\right)\right]\left\{U\left(Y\right)-hU'\left(Y\right)+\frac{h^2}{2}U''\left(Y\right)\right\}\end{aligned} \tag{37}$$

Proving the proposition

* Collecting terms gives us

$$U(Y) \cong U(Y) + (2\pi(Y,h) - 1)\left[hU'(Y)\right] + \frac{h^2}{2}U''(Y) \tag{38}$$

* Solving for $\pi(Y,h)$ gives us the final result

$$\pi(Y,h) \cong \frac{1}{2} + \frac{1}{4}h\left[-\frac{U''(Y)}{U'(Y)}\right] \cong \frac{1}{2} + \frac{1}{4}hR_A(Y) \tag{39}$$

* Utility functions for which $R_A(Y)$ is constant are referred to as displaying constant absolute risk aversion (CARA). Let us consider an example

Example: Exponential utility

* Consider an investor with wealth Y and an exponential utility function

$$U\left(Y\right) = -\frac{1}{\nu}e^{-\nu Y} \tag{40}$$

* We can show that the coefficient of absolute risk aversion becomes

$$R_A\left(Y\right) = -\frac{U''\left(Y\right)}{U'\left(Y\right)} = -\frac{-\nu e^{-\nu Y}}{e^{-\nu Y}} = \nu \tag{41}$$

such that

$$\pi\left(Y{,}h\right) \cong \frac{1}{2} + \frac{1}{4}h\nu \tag{42}$$

* The odds demanded are thus independent of the level of initial wealth (Y), but do depend on the amount of wealth at risk (h)

Relative risk aversion and the odds of a bet

- Consider a lottery $L = (h, \text{-}h, \pi)$ in which the amount at risk is now a proportion θ of the investor's wealth, i.e., $h = \theta Y$. Following steps similar to above, we can show that

$$\pi\left(Y,\theta\right) \cong \frac{1}{2} + \frac{1}{4}\theta R_R\left(Y\right) \tag{43}$$

- If investor 1 and 2 have relative risk aversion coefficients $R_R^1\left(Y\right) \geq R_R^2\left(Y\right)$, respectively, then investor 1 will always demand more favorable odds, for any level of wealth, when the fraction θ of his wealth is at risk

- Utility functions for which $R_R\left(Y\right)$ is constant are referred to as displaying constant relative risk aversion (CRRA). Let us consider an example

Example: Power utility

* Consider an investor with wealth Y and a power utility function

$$U\left(Y\right)=\begin{cases}\dfrac{Y^{1-\gamma}}{1-\gamma}, & \text{for } 0<\gamma \text{ and } \gamma\neq 1\\ \ln Y, & \text{if } \gamma=1\end{cases} \tag{44}$$

* We can show that the coefficient of relative risk aversion becomes

$$R_R\left(Y\right)=-\frac{Y\cdot U''\left(Y\right)}{U'\left(Y\right)}=-Y\frac{-\gamma Y^{-\gamma-1}}{Y^{-\gamma}}=\gamma \tag{45}$$

such that

$$\pi\left(Y,\theta\right)\cong\frac{1}{2}+\frac{1}{4}\theta\gamma \tag{46}$$

* The odds demanded remain independent of the level of initial wealth, but depend on the fraction θ of wealth that is at risk

Risk-neutral investors

Risk-neutral investors

Consider a risk-neutral investor whose utility function is identified by a linear specification of the form

$$U(Y) = cY + d, \quad \text{with } c > 0 \tag{47}$$

from which is follows immediately that

$$R_A(Y) \equiv 0 \quad \text{and} \quad R_R(Y) \equiv 0 \tag{48}$$

* Risk-neutral investors do not demand better than even odds for a fair lottery, i.e., $\pi(Y,h) = \pi(Y,\theta) = \frac{1}{2}$. That is, they are indifferent between the sure thing and the fair lottery
* They are indifferent (neutral) to risk and are concerned only with an asset's expected payoff

Introducing the notion of a risk premium

Risk premium

A key question that comes to mind is: "How can we quantify risk aversion"?

- To investigate this, consider a risk averse investor with current wealth Y that face a lottery with random payoff $\widetilde{Z}$

$$\widetilde{Z} = \begin{cases} Z_1 & \text{with probability } \pi, \\ Z_2 & \text{with probability } 1-\pi \end{cases} \tag{49}$$

- We know from earlier that a risk averse investor is willing to pay less than the expected value of the lottery, i.e.,

$$U\left(\mathbb{E}\left[Y+\widetilde{Z}\right]\right) > \mathbb{E}\left[U\left(Y+\widetilde{Z}\right)\right] \tag{50}$$

- The endgame is therefore simply to introduce a "risk premium" for the lottery asset such that the investor obtains equal utility from either alternative

Risk premium and certainty equivalence

Pratt (1964) risk premium for a lottery asset

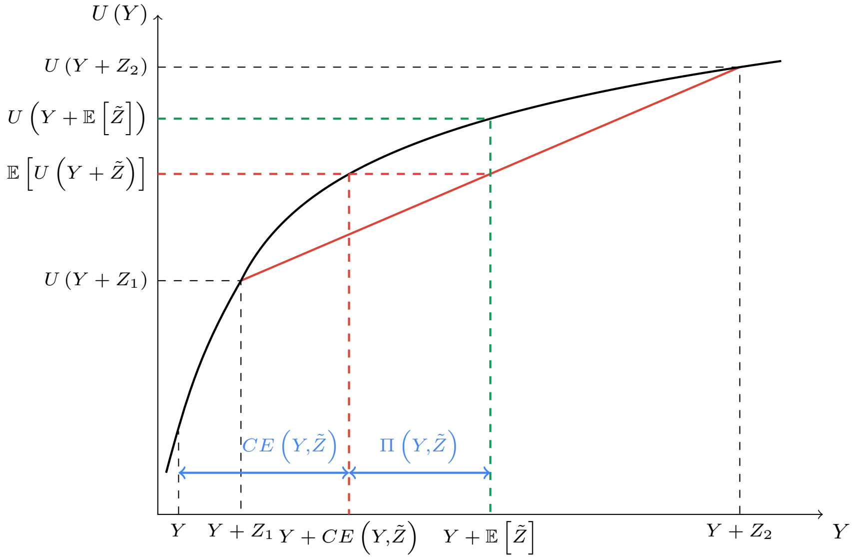
Let $\Pi\left(Y,\widetilde{Z}\right)$ denote the investor's risk premium for a lottery asset with uncertainty payoff $\widetilde{Z}$. $\Pi\left(Y,\widetilde{Z}\right)$ is the maximum insurance payment that an investor would pay to avoid the lottery risk, i.e.,

$$U\left(Y+\mathbb{E}\left[\widetilde{Z}\right]-\Pi\left(Y,\widetilde{Z}\right)\right)=\mathbb{E}\left[U\left(Y+\widetilde{Z}\right)\right] \tag{51}$$

* $\mathbb{E}\left[\widetilde{Z}\right]-\Pi\left(Y,\widetilde{Z}\right)=CE\left(Y,\widetilde{Z}\right)$ can be defined as the certain equivalent (CE), which denotes the maximal amount of money an investor is willing to pay to acquire the lottery
* Since $U\left(\cdot\right)$ is an increasing, concave function ($U'\left(\cdot\right)>0$), we must have that $CE\left(Y,\widetilde{Z}\right)<\mathbb{E}\left[\widetilde{Z}\right]$, which further implies that

$$U\left(Y+CE\left(Y,\widetilde{Z}\right)\right)<U\left(Y+\mathbb{E}\left[\widetilde{Z}\right]\right) \tag{52}$$

CE and risk premia with positive expected value



Example: Certainty equivalent

- Consider an investor with log utility $U(Y) = \ln Y$ and current wealth $Y = 1{,}000$. This investor is offered the lottery $L = (Z_1 = 200, Z_2 = 0, 0.5)$. How much is the investor willing to pay for this lottery?

- To answer this question, we need to compute the certainty equivalent that makes the following equality hold

$$U\left(Y + CE\left(Y, \widetilde{Z}\right)\right) = \mathbb{E}\left[U\left(Y + \widetilde{Z}\right)\right] \tag{53}$$

  Substituting in values yields

$$\ln(1{,}000 + CE) = \frac{1}{2}\ln(1{,}200) + \frac{1}{2}\ln(1{,}000) \tag{54}$$

  which implies that $CE = 95.45$, which indeed is lower than the expected value of the lottery

Pratt's approximation for the risk premium

Pratt's approximation for the risk premium

Pratt (1964) derives a simple approximation for the risk premium assuming that the lottery is fair and that $\widetilde{Z}$ is "small"

$$\Pi\left(Y,\widetilde{Z}\right) \cong \frac{1}{2}\sigma_{\widetilde{Z}}^{2}\left[-\frac{U''(Y)}{U'(Y)}\right] = \frac{1}{2}\sigma_{\widetilde{Z}}^{2}R_{A}(Y) \tag{55}$$

where $R_A(Y)$ is the Arrow-Pratt measure of absolute risk aversion

- We note that the risk premium $\Pi\left(Y,\widetilde{Z}\right)$ is increasing in the uncertainty of the lottery $\sigma_{\widetilde{Z}}^{2}$ and in the investor's coefficient of absolute risk aversion
- Pratt's definition of a risk premium is commonly interpreted as the payment that an individual is willing to make to forego the lottery

Deriving Pratt's risk premium approximation

- Consider the Pratt (1964) definition of a risk premium for a fair lottery

$$U\left(Y - \Pi\left(Y, \widetilde{Z}\right)\right) = \mathbb{E}\left[U\left(Y + \widetilde{Z}\right)\right] \tag{56}$$

- Take a first-order Taylor expansion of the LHS of (56) around $\Pi\left(Y, \widetilde{Z}\right) = 0$

$$U\left(Y - \Pi\left(Y, \widetilde{Z}\right)\right) \cong U\left(Y\right) - \Pi\left(Y, \widetilde{Z}\right) U'\left(Y\right) \tag{57}$$

- Take a second-order Taylor expansion of the RHS of (56) around $\widetilde{Z} = 0$ and use that the lottery is fair so that $\sigma^2_{\widetilde{Z}} = \mathbb{E}\left[\widetilde{Z}^2\right]$ is the lottery's variance

$$\begin{aligned}\mathbb{E}\left[U\left(Y + \widetilde{Z}\right)\right] &\cong \mathbb{E}\left[U\left(Y\right) + \widetilde{Z}U'\left(Y\right) + \frac{1}{2}\widetilde{Z}^2 U''\left(Y\right)\right] \\ &= U\left(Y\right) + \frac{1}{2}\sigma^2_{\widetilde{Z}} U''\left(Y\right)\end{aligned} \tag{58}$$

- Equating (57) and (58) then yields the Pratt (1964) risk premium approximation

Arrow's approximation of the risk premium

* Arrow (1971) derives an equivalent approximation by defining the risk premium θ through the probabilities

$$U(Y) = \frac{1}{2}(1+\theta)U(Y+Z) + \frac{1}{2}(1-\theta)U(Y-Z) \tag{59}$$

* Take a second-order Taylor approximation around $Z = 0$

$$U(Y) \cong \frac{1}{2}(1+\theta)\left[U(Y) + ZU'(Y) + \frac{1}{2}Z^2U''(Y)\right] \tag{60}$$

$$+ \frac{1}{2}(1-\theta)\left[U(Y) - ZU'(Y) + \frac{1}{2}Z^2U''(Y)\right] \tag{61}$$

* Collecting terms yields

$$U(Y) \cong U(Y) + \theta ZU'(Y) + \frac{1}{2}Z^2U''(Y) \tag{62}$$

* Last, solve for the risk premium θ, which is the fraction of Z one is willing to forego to avoid the lottery

$$\theta \cong \frac{1}{2}ZR_A(Y) \tag{63}$$

Example: Power utility

- Consider an investor with power utility given by $U(Y) = \frac{Y^{1-\gamma}}{1-\gamma}$, with $\gamma = 3$ and initial wealth of $Y = \$500$, who faces the fair lottery

$$\widetilde{Z} = \begin{cases} +\$100 & \text{with probability } \frac{1}{2}, \\ -\$100 & \text{with probability } \frac{1}{2} \end{cases} \tag{64}$$

- Using (55), we can compute the (approximate) Pratt risk premium as

$$\Pi\left(Y, \widetilde{Z}\right) = \frac{1}{2}\sigma^2_{\widetilde{Z}}\frac{\gamma}{Y} = \frac{1}{2}100^2\frac{3}{500} = \$30 \tag{65}$$

- Using (63), we can compute the (approximate) Arrow risk premium as

$$\theta = \frac{1}{2}Z\frac{\gamma}{Y} = \frac{1}{2}100\frac{3}{500} = 0.3 \tag{66}$$

which when converted into a monetary amount yields $\Pi\left(Y, \widetilde{Z}\right) = Z\theta = \$30$

Assessing the degree of relative risk aversion

* Suppose that investors' utility functions are of the power form

$$U(Y) = \begin{cases} \dfrac{Y^{1-\gamma}}{1-\gamma}, & \text{for } 0 < \gamma \text{ and } \gamma \neq 1 \\ \ln Y, & \text{if } \gamma = 1 \end{cases} \tag{67}$$

which implies that $R_R(Y) = \gamma$ such that $U(\cdot)$ belong to the CRRA class

* To better understand the implication of γ, consider the following uncertain payoff

$$\widetilde{Z} = \begin{cases} \$50 & \text{with probability } \frac{1}{2}, \\ \$100 & \text{with probability } \frac{1}{2} \end{cases} \tag{68}$$

* What would you be willing to pay for such a lottery if your current wealth is Y?

Certainty equivalents and CRRA

- Depending on the amount you are willing to pay for the lottery (your CE), it is possible to infer your coefficient of relative risk aversion $R_R(Y) = \gamma$ under power utility from the expression

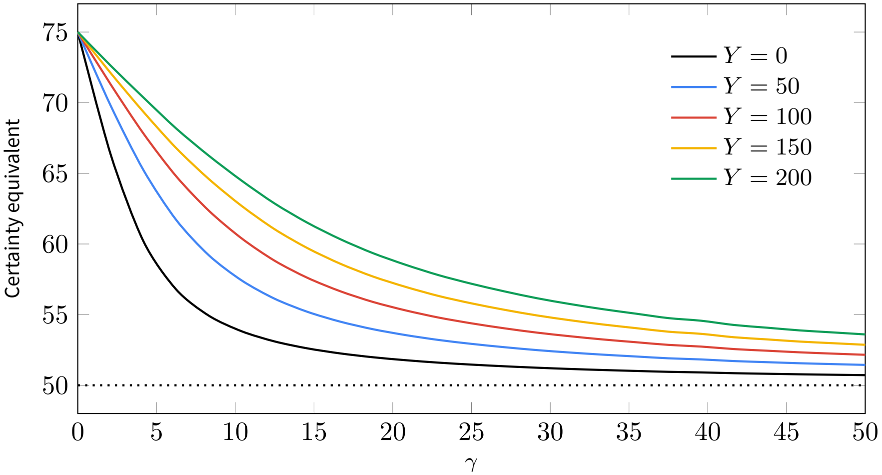
$$\frac{(Y + CE)^{1-\gamma}}{1-\gamma} = \frac{1}{2}\frac{(Y + 50)^{1-\gamma}}{1-\gamma} + \frac{1}{2}\frac{(Y + 100)^{1-\gamma}}{1-\gamma} \tag{69}$$

- If we assume zero current wealth ($Y = 0$), then we obtain the following sample values

$\gamma = 0$	CE = 75 (risk-neutrality)
$\gamma = 1$	CE = 70.711
$\gamma = 2$	CE = 66.667
$\gamma = 5$	CE = 58.566
$\gamma = 10$	CE = 53.991
$\gamma = 20$	CE = 51.858
$\gamma = 30$	CE = 51.209

Certainty equivalent under various risk coefficients

* Below we plot certainty equivalents for various levels of current wealth



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