The expectations hypothesis*

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*This note provides an overview of the expectations hypothesis, its implications for fixed income analysis, and how to test its empirical validity. The note draws on several sources, including the textbooks of Campbell, Lo, and MacKinlay (1997), Cochrane (2005), and Campbell (2017), but is mostly based on the academic articles cited throughout. The note is prepared for use only in the Master's course "Asset Pricing". Please do not cite, circulate, or use for purposes other than this course.

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"If the attractiveness of an economic hypothesis is measured by the number of papers which statistically reject it, the expectations theory of the term structure is a knockout."

Froot (1989)

1. Introduction

The expectations hypothesis (EH) of the term structure of interest rates is a simple proposition that relates long-term interest rates to current and future expected short-term interest rates. The basic idea and motivation of the hypothesis, which dates back at least to Fisher (1896), is that lenders (bond investors) and borrowers (bond issuers) decide between long-term and short-term bonds by comparing the yield on a long-term bond to the expected return on a roll-over strategy in short-term bonds. In a nutshell, they compare the n-period return on an n-period bond with the expected return from rolling over one-period bonds for n periods. In the absence of frictions, these investment should provide the same expected return (up to a constant risk premium) over the life of the long-term bond. This is the EH in words. A central prediction of the EH is therefore that there is no particularly good time to hold either long-term or short-term bonds because the risk premium for holding long-term bonds over short-term bonds, if any at all, is constant across time (Campbell, 2017).

The EH and its implications for the behavior of interest rates have been subject to extensive scrutiny in academia beginning with the seminal work by Macaulay (1938) and, despite its intuitive appeal, the EH has found remarkably little empirical support in the data. Selected empirical studies documenting sizable and predictable deviations from the EH include Fama and Bliss (1987), Campbell and Shiller (1991), Cochrane and Piazzesi (2005), Ludvigson and Ng (2009), Cieslak and Povala (2015), Eriksen (2017), Gargano, Pettenuzzo, and Timmermann (2019), Berardi, Markovich, Plazzi, and Tamoni (2020), and Borup, Eriksen, Kjær, and Thyrsgaard (2021).[1] The strong empirical rejections of the EH can originate from different sources, but one immediate and compelling explanation relates to the inherent assumption of time-invariant risk premia in the Treasury bond market implied by the EH. The EH can be formulated in several forms depending on the degree of restrictiveness placed on the assumption on risk premia, although all forms assume that risk premia are constant across time. In this note, we will limit ourselves to two formulations: (i) The pure expectations hypothesis (PEH) and (ii) the generalized expectations hypothesis (EH). The PEH rules out the existence of risk premia in the Treasury bond market, whereas the EH assumes the presence of a maturity specific, but time-invariant, risk premium. The latter version of the EH can be interpreted as a "generalized" EH because it nests other formulations such as the liquidity preference hypothesis. That risk

[1]This list is by no means exhaustive as the literature on the expectations hypothesis is vast. They are, however, "classics" in some sense and some of the authors whose contributions we will highlight in this note. Note, however, that papers such as Bauer and Hamilton (2018) and Ghysels, Horan, and Moench (2018) question the predictability.

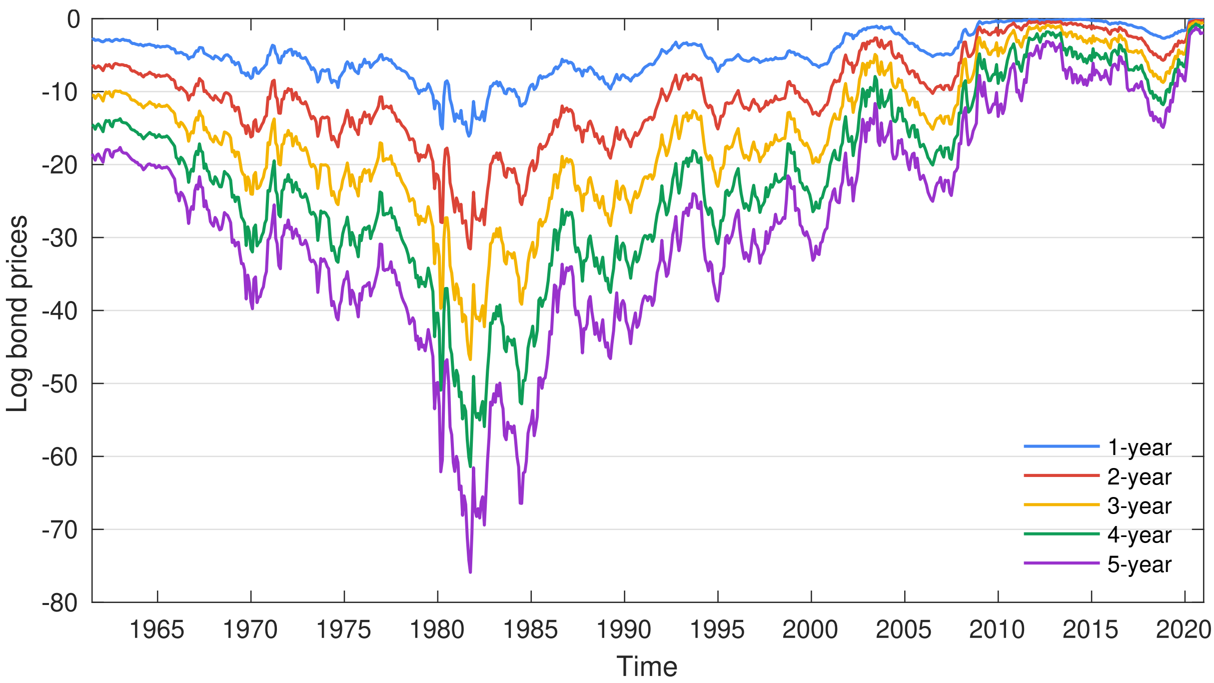
premia are time-invariant implies that the premium required by investors for bearing risk, such as interest rate risk, inflation risk, reinvestment risk, liquidity risk, and macroeconomic risk, is constant through time. As we shall illustrate in this note, such an assumption fares poorly in the data and the empirical evidence overwhelmingly favors time-varying bond risk premia.

Notation and preliminaries The (pure) expectations hypothesis is traditionally formulated in terms of Treasury discount bonds, which allows us to abstract from intermediate coupon payments. To set the stage, recall that the time t price of a discount bond with n periods left to maturity is

$$P_t^n = \frac{1}{(1+Y_t^n)^n}, \tag{1}$$

where Y_t^n denotes the time t zero-coupon bond yield for a discount bond that matures in n periods. We assume that the bonds make a unit nominal payment at maturity for simplicity throughout this note.

Figure 1: Time series dynamics of log discount bond prices



Although it is possible to formulate and evaluate the EH using simple yields, returns, and prices as in (1), there is an issue that the PEH cannot hold both in its one-period form and its n-period form due to Jensen's inequality (Campbell et al., 1997). Consequently, it has become standard practice to consider log (natural logarithm) bond prices and yields when testing and evaluating the empirical validity of the EH and its implications using Treasury bond data. We note that this is equivalent to assuming and working with continuous compounded yields.[2] The

[2]We will make use of the following logarithmic rules: (i) $\ln a/b = \ln a - \ln b$, (ii) $\ln a \cdot b = \ln a + \ln b$, and (iii) $\ln 1 + a \approx a$, if a is small.

log discount bond price in (1) can then be expressed as

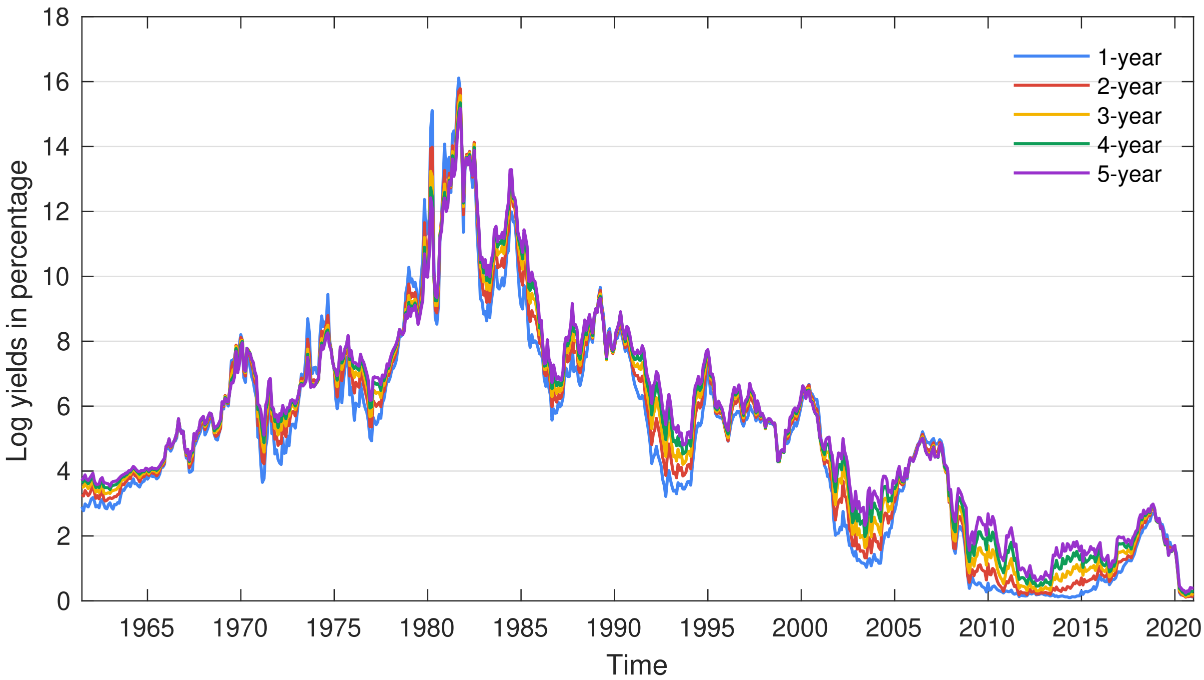
$$p_t^n = -ny_t^n, \tag{2}$$

where lowercase p_t^n denotes the time t log bond price of a discount bond with n periods to maturity. Figure 1 illustrates the time series dynamics of log discount bond prices. We note that discount bonds (or discount factors) were high during the 1980's, whereas they are currently low and close to zero for short maturities.[3] We also note that the log bond prices are decreasing with maturity n. We can go from log bond prices to log yields by re-arranging (2), i.e.,

$$y_t^n = -\left(\frac{1}{n}\right) p_t^n. \tag{3}$$

Figure 2 provides an illustration of the time series dynamics of continuously compounded monthly zero-coupon yields for bonds with 1, 2, 3, 4, and 5 years to maturity, respectively. We note that

Figure 2: Time series dynamics of log discount bond yields



log yields move opposite of log bond prices. Yields are high during the 1980's, which was a period characterized by high inflation, and low following the 2008 financial crisis and throughout the zero-lower bound period, whereas they rise with the lift-off in interest rates post 2015. Last, we note that they return to around zero somewhat quickly as a response to the outbreak of the covid-19 pandemic. These log zero-coupon yield series will be at the center of our attention throughout this note, and we will use them to compute quantities such as forward rates and holding-period returns using standard formulas.

[3] The log bond prices can be converted to discount factors using the following relation $Z_t^n = \exp\{p_t^n/100\}$.

We can further, from either log discount bond prices or yields, define the log forward rate at time t for loans between time $t+n-1$ and $t+n$ as

$$\begin{aligned} f_t^n &= p_t^{n-1} - p_t^n && (4) \\ &= ny_t^n - (n-1)\, y_t^{n-1} && (5) \\ &= y_t^{n-1} + n\left(y_t^n - y_t^{n-1}\right), && (6) \end{aligned}$$

which, since $p_t^0 = 0$, implies the relation $f_t^1 = y_t^1 = -p_t^1$ and where the second equality in (5) follows directly from the definition of log bond prices in (2). It turns out that the expressions in (4)–(6) allow us to make certain statements about the mechanics of the spot and forward interest rate market. First, and from (4), we see that the forward rate will be positive whenever log discount bond prices fall with maturity. Secondly, and from (6), we have that the time t forward rate will be above the $n-1$ period discount bond yield whenever the n-period yield is above the $n-1$ period yield, that is, when the term structure is upward sloping and vice versa when it is downward sloping.

As a final component, we can introduce the concept of a holding-period return in the discount bond market, which we define as follows

$$\begin{aligned} r_{t+1}^n &= p_{t+1}^{n-1} - p_t^n && (7) \\ &= ny_t^n - (n-1)\, y_{t+1}^{n-1} && (8) \\ &= y_t^n - (n-1)\left(y_{t+1}^{n-1} - y_t^n\right), && (9) \end{aligned}$$

where the holding-period return r_{t+1}^n in (7)–(9) can be viewed as a trading strategy in which we purchase today (at time t) an n-period bond and subsequently sell it one period later as an $n-1$ period bond.[4] The holding-period return is a risky return because we are buying and selling the discount bond prior to maturity and, accordingly, are subject to interest rate risk.

The remainder of this note continues as follows. Section 2 introduces the expectations hypothesis and its implications. Section 3 discusses yield-based regression tests of the EH. Section 4 tests the EH using predictive regressions for bond risk premia. Finally, Section 5 provides a brief introduction to yield curve factors.

2. The expectations hypothesis

The expectations hypothesis (EH) exists in different formulations, but our focus will be on two particular versions: (i) the log pure expectations hypothesis (PEH) and (ii) the generalized log expectations hypothesis (EH), which is somewhat weaker in its restrictiveness. As discussed in Section 1, investors, to decide on investment strategies, compare the n-period return on an

[4]Note that one needs to account for the coupon payments as well when dealing with coupon bonds. Here we simply omit it due to this note's exclusive focus on discount bonds.

n-period bond with the expected return from rolling over one-period bonds for n periods. The PEH states that such investment should provide identical expected returns

$$y_t^n = \frac{1}{n}\mathbb{E}_t \underbrace{\left(y_t^1 + y_{t+1}^1 + y_{t+2}^1 + \cdots + y_{t+n-1}^1\right)}_{\text{Roll-over strategy}} \tag{10}$$

$$= \frac{1}{n}\sum_{j=0}^{n-1}\mathbb{E}_t\left[y_{t+j}^1\right], \tag{11}$$

where the left-hand side is the n-period return on an n-period bond and the right-hand side is the expected return from rolling over one-period bonds for n periods.

The generalized EH, on other hand, postulates that long-term yields are given as the average of expected future short-term yields over the lifetime of the long-term bond plus a time-invariant term premium. Formally,

$$y_t^n = \frac{1}{n}\sum_{j=0}^{n-1}\mathbb{E}_t\left[y_{t+j}^1\right] + \Phi^n, \tag{12}$$

where Φ^n denotes the maturity-specific term premium component. We will refer to this as a constant risk premium for now, but provide more details as the note progresses. We note that the differences between the two versions lies in way they treat Φ^n. The PEH in (11) predicts that the risk premium is identically zero, whereas the EH in (12) allows it to be positive and depend on n, but requires that the term premium is time-invariant. Although the EH therefore appears most reasonable in terms of its restrictions on term premia, it is still strongly rejected in the data as bond risk premia are found, empirically, to vary substantially over time. Section 4 contains a discussion of possible explanations for this observed time variation.

Further implications of the expectations hypothesis In addition to the implication that (11) should hold under the PEH and (12) under the EH, the (pure) log expectations hypothesis further implies two important relations in fixed income contexts that again differ in their approach to the allowance for a risk premium. First, they imply that the forward rate should be an unbiased estimator of expected future spot rates

$$f_t^n = \mathbb{E}_t\left[y_{t+n-1}^1\right] + RP^n, \tag{13}$$

where the risk premium RP^n should be identically zero under the PEH, but may be positive and depend on n under the EH. Moreover, the (P)EH implies that the expected holding-period returns are equal on bonds of all maturities. Formally,

$$\mathbb{E}_t\left[r_{t+1}^n\right] = y_t^1 + RP^n, \tag{14}$$

where r_{t+1}^n is the holding-period return from (7). It is important to emphasize that the assumption of a constant risk premium distinguishes the model from the tautology that it would have been if we had also allowed for an arbitrary, time-varying risk premium. That is, it only has theoretical implications as long as we impose some kind of testable restrictions on the risk premium components. Finally, we note that the EH statements in (12)–(14) are all mathematically equivalent (see Appendix A.5 for proofs).

The rest of this note will deal with different ways of testing the EH and its implications in (12)–(14) empirically to see if they are supported or rejected by the data. To this end, Section 3 discusses yield-based regression tests as discussed by Campbell and Shiller (1991) and Section 4 considers predictive regressions for bond risk premia directly using the models of Fama and Bliss (1987), Cochrane and Piazzesi (2005), and Ludvigson and Ng (2009).

3. Yield-based regression tests

This section outlines a set of yield-based regression tests for evaluating the validity of the predictions of the (pure) expectations hypothesis. To motivate the general usefulness of these tests, consider a simple, but intuitively powerful, variable that contains information about the shape of the yield curve: the yield spread. The yield spread between an n period discount bond and the one-period (risk-free) bond is given by

$$s_t^n = y_t^n - y_t^1, \tag{15}$$

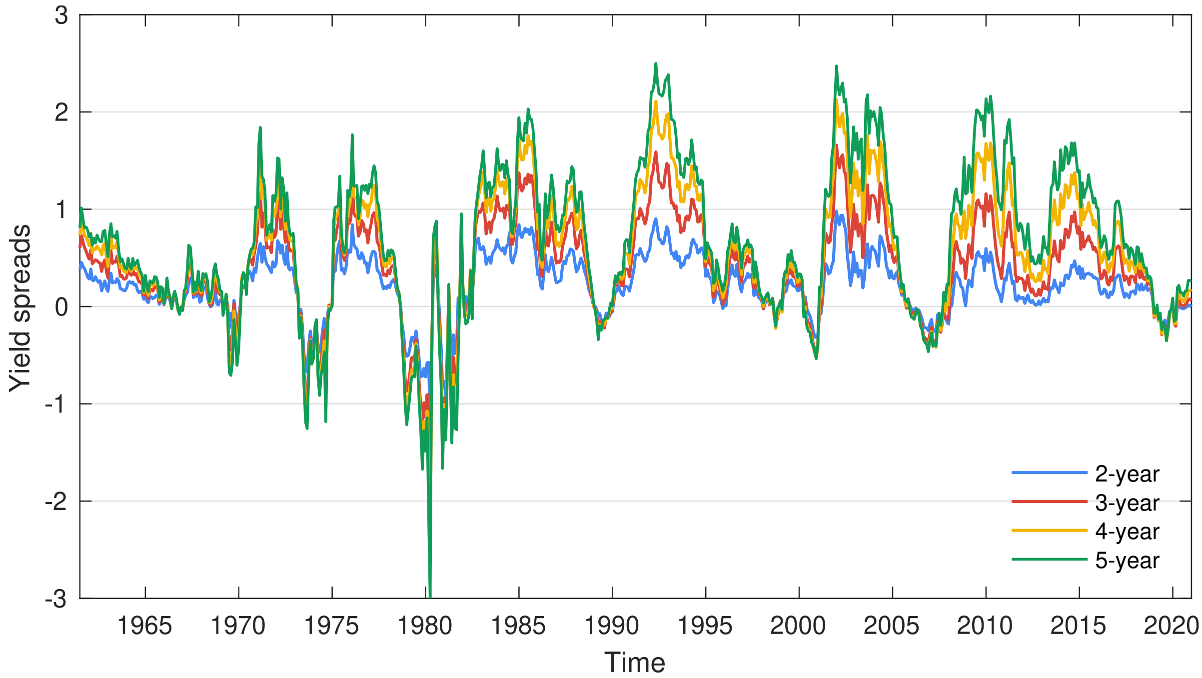
where y_t^n denotes the log zero-coupon yield from (3). The yield spread is proportional to the slope of the term structure of interest rates and informs us about market expectations. If the spread is positive, then we have an upward sloping term structure and vice versa if s_t^n is negative. Figure 3 illustrates the time series dynamics of yields spreads and their variation over time. Importantly for our purpose here, yield spreads vary significantly through time.

The expectations hypothesis implies that the yield spread s_t^n should consist of a constant risk premium plus an optimal forecast of changes in future yields. To see this, note that the definition of log holding-period returns in (8) implies that $y_t^n = \frac{1}{n}\sum_{i=0}^{n-1} r_{t+i+1}^{n-1}$ from which we can show that (See Appendix A.1 and A.2, respectively)

$$\begin{aligned} s_t^n &= \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}_t\left[y_{t+i}^1 - y_t^1\right] + \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}_t\left[r_{t+1+i}^{n-i} - y_{t+i}^1\right] \qquad (16)\\ &= \frac{1}{n}\sum_{i=1}^{n-1} \mathbb{E}_t\left[(n-i)\,\Delta y_{t+i}^1\right] + \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}_t\left[r_{t+1+i}^{n-i} - y_{t+i}^1\right], \qquad (17) \end{aligned}$$

where, under the expectations hypothesis, the second term on the right-hand side is a constant. The term is constant because it is a term premium component equal to Φ^n. In this case, the first

Figure 3: Time series dynamics of zero-coupon yields



equality tells us that time variations in the yield spread is due to changing expectations about future short-term yields over the life of the long-term bond. Said differently, the yield spread is an optimal predictor of future expected short-term yields over the life of the long-term bond. The second equality restates this insight in terms of the sum of one-period changes in one-period yields over the life of the long-term bond.

3.1. Testable yield implications

This sections develops two testable implications based on the above discussion by invoking the EH prediction of a constant risk premia. First, we can use the definition of a log holding-period return in (9) together with the EH implication in (14) to show that (see Appendix A.3)

$$s_t^n = (n-1)\,\mathbb{E}_t\left[y_{t+1}^{n-1} - y_t^n\right] + \mathbb{E}_t\left[r_{t+1}^n - y_t^1\right]. \tag{18}$$

The expression in (18) informs us that variation in the yield spread can originate from two different sources: (i) changing expectations about future changes in long-term rates over the life of the short-term bond or (ii) changing expectations to future risk premia. The latter is constant under the EH as implied by the restriction in (14), so that (18) reduces to

$$s_t^n = (n-1)\,\mathbb{E}_t\left[y_{t+1}^{n-1} - y_t^n\right] + RP^n. \tag{19}$$

This equation implies a testable implication stating that a high yield spread should predict increases in the long-term yield over the life of the short-term bond. That is, the yield spread is an optimal forecast of short-term changes in long-term yields.

Second, using the expression for the yield spread in (17) together with the assumption that term premia are constant, we can show that (see Appendix A.4)

$$s_t^n = \sum_{i=1}^{n-1} \mathbb{E}_t \left[\left(1 - \frac{i}{n}\right) \Delta y_{t+i}^1 \right] + \Phi^n, \tag{20}$$

which implies the testable restriction that a high yield spread should predict changes in short-term rates over the life of the long-term bond. In what follows, we will presume, or impose, that risk premia are time-invariant and test whether yield spreads contain information about expected future yield changes. This leads us to the famous Campbell and Shiller (1991) regressions.

3.2. Campbell-Shiller long rate regressions

The Campbell and Shiller (1991) long-rate regression departs from the implication in (19), which can be re-arranged to provide an expression for expected yield changes as a function of scaled yield spreads and risk premia

$$\mathbb{E}_t \left[y_{t+1}^{n-1} - y_t^n \right] = -\frac{RP^n}{n-1} + \frac{s_t^n}{n-1}, \tag{21}$$

whose implications can be conveniently framed and tested in a linear regression model

$$y_{t+1}^{n-1} - y_t^n = \alpha^n + \beta^n \left(\frac{s_t^n}{n-1} \right) + \varepsilon_{t+1}^n, \tag{22}$$

where $\alpha^n = -\frac{RP^n}{n-1}$. Under the EH, we should expect to find $\beta^n = 1$ for all n and α^n (zero) negative for all n for the (pure) expectations hypothesis. Note that modeling the risk premium as a constant in the regression is equivalent to imposing the restriction that risk premia are time-invariant. This implies that deviations from the EH implications can be interpreted as evidence in favor of time-varying risk premia. Intuitively, the regression implies that a high yield spread should forecast increases in long-term yields over the life of the short-term bond, where the increase should be proportional to the yield spread.

Table 1: Campbell-Shiller long rate regressions

	2-year	3-year	4-year	5-year
α^n	-0.155	-0.076	-0.016	0.031
$\text{se}^{\text{NW}}(\alpha^n)$	(0.238)	(0.221)	(0.207)	(0.196)
β^n	-0.497	-0.767	-1.029	-1.273
$\text{se}^{\text{NW}}(\beta^n)$	(0.564)	(0.629)	(0.670)	(0.698)
$\text{t}^{\text{NW}}(\beta^n = 1)$	-2.656	-2.808	-3.026	-3.259
R^2 (%)	0.94%	1.89%	3.00%	4.12%

Table 1 reports parameter estimates for testing the model using our yield data, and provides strong evidence against the predictions of the EH. In particular, all slope coefficients β^n are negative for all n and decreasing monotonically with maturity. In a nutshell, the coefficients are all far from one and even have the wrong sign. The implication being that yield spreads, i.e., $s_t^n = y_t^n - y_t^1$, tend to predict interest rate changes in the wrong direction. Moreover, for all maturities, we strongly reject the null of β^n being equal to one, where we use Newey and West (1987) HAC standard errors with 18 lags throughout. Moreover, we see that the intercept α^n is positive for the 5-year bond, which suggest a negative risk premium.

The regression in (22) contains the same information as a regression of the excess log holding-period return onto the yield spread s_t^n. In fact, the slope would equal $(1 - \beta^n)$ and the negative results for yield changes therefore imply a strong and positive relation between yield spreads and future bond excess returns.[5] We investigate the predictability of bond risk premia in more detail in Section 4.

3.3. Campbell-Shiller short rate regressions

The Campbell and Shiller (1991) short-rate regression departs from the implication in (20), which can be re-arranged to provide an expression for expected short-term yield changes over the life of the long-term bond as a function of the yield spread and term premium

$$\sum_{i=1}^{n-1} \mathbb{E}_t \left[\left(1 - \frac{i}{n}\right) \Delta y_{t+i}^1 \right] = s_t^n - \Phi^n. \tag{23}$$

This facilitates a test of the implication that a high yield spread should forecasts long-term increases in short-term rates. In order to test this implication, we define an ex post value of short-term rate changes, also known as a *perfect foresight spread*, as follows

$$s_t^{n,\star} = \sum_{i=1}^{n-1} \left(1 - \frac{i}{n}\right) \Delta y_{t+i}^1, \tag{24}$$

where $\Delta y_{t+i}^1 = y_{t+i}^1 - y_t^1$. We can then regress (24) onto the time t observable yield spread from (15)

$$s_t^{n,\star} = \delta^n + \theta^n s_t^n + \nu_{t+1}^n, \tag{25}$$

where, under the EH, we should expect to find $\theta^n = 1$ for all n. Table 2 provides empirical evidence that weakly supports the EH prediction in (24), but not unambiguously. Specifically, while the coefficients θ^n are positive for all n, we are able to reject the null hypothesis of a unit coefficient on the 2-year bond, a rejection of the expectations hypothesis. Intuitively, this implies

[5] Andreasen, Engsted, Møller, and Sander (2021) provide recent evidence that expected excess bond returns display a positive correlation with the slope of the yield curve (i.e., the yield spread) in expansions, but a negative correlation in recessions. That is, the relation depends on the state of the economy. See also Borup et al. (2021) for a recent discussion of time-varying predictability in bond markets.

Table 2: Campbell-Shiller short rate regressions

	2-year	3-year	4-year	5-year
δ^n	-0.062	-0.219	-0.421	-0.589
$\mathrm{se}^{\mathrm{NW}}\ (\delta^n)$	(0.124)	(0.240)	(0.300)	(0.332)
θ^n	0.207	0.476	0.685	0.769
$\mathrm{se}^{\mathrm{NW}}\ (\theta^n)$	(0.283)	(0.332)	(0.294)	(0.251)
$\mathrm{t}^{\mathrm{NW}}\ (\theta^n = 1)$	-2.796	-1.576	-1.072	-0.919
R^2 (%)	0.66%	4.15%	9.83%	13.92%

that the spread between a long-term n-period bond and the one-period bond does not translate one-to-one into expected future short-term yield changes. However, although the remaining coefficients are not one, we are unable to reject the null hypothesis that the coefficients could be one. Put differently, the yield spread does forecast future short rate changes, but the subsequent changes in short rates are too small to enforce the predictions of the expectations hypothesis. In the end, our results indicate that the term structure almost always gives a forecast in the wrong direction for the short-term change in the yield on long-term bonds, but provides a forecast in the right direction for the longer-term change in short-term rates.

As in the previous case, the regression in (25) contains the same information as a regression of $1/n$ times the excess n-period return on an n-period bond into the yield spread, where the coefficient would equal $(1 - \theta^n)$, indicating that yield spreads do predict excess returns. It is in this sense that we can say that the yield-based tests points to time-varying risk premia in the Treasury bond market.

Closing remaks on yield-based tests The empirical finding that the EH is rejected in the data has important implications for our understanding of the dynamics in fixed income markets and the behavior of interest rates in general. The rejection implies that forward rates are not unbiased estimators of expected future spot rates as suggested in (13) and that long-term yields are not equivalent to the sum of expected future short-term yields plus a constant term premium as predicted in (12). The yield-based tests all implicitly assume that risk premia are constant over time, and the next section therefore turns to an investigation of the sources of time variations in bond risk premia (or log holding-period excess returns). In this framework, one directly constructs bond risk premia ex post and then tests, in a predictive regression framework, whether they vary with a set of predetermined explanatory variables.

4. Predicting bond risk premia

This section addresses the implications of the (pure) expectations hypothesis from a different angle. According to (17), variations in the yield spread much originate from either (i) changing expectations to short term rates, (ii) changing expectations to future excess holding-period returns, or (iii) a combination of the two. In sum, the yield spread must predict either yield changes or excess returns. The conclusions from the previous section suggest that a large part of the variation in s_t^n arises from risk premia whose predictable variations are often gauged using predictive regressions for bond risk premia.

To examine the expectations hypothesis implication of constant risk premia in (14), we start by defining an *ex post* measure of bond risk premia — or log excess holding-period returns. We can define bond risk premia, denoted rx_{t+1}^n, directly from the implication in (14) as

$$rx_{t+1}^n = r_{t+1}^n - y_t^1, \tag{26}$$

for $n > 2$, and where r_{t+1}^n denotes the log holding-period return and y_t^1 is the one-year zero-coupon yield, which we will treat as a risk-free rate of return. As discussed above, this excess return can be interpreted as originating from a trading strategy in which we purchase today (at time t) an n-period bond and sell it one period later (time $t+1$) as an $n-1$ period bond, where the initial purchase is financed by selling a one-period riskless bond.

Figure 4 illustrates the time series dynamics of bond risk premia for our yield series and delivers preliminary evidence consistent with time-varying risk premia in the Treasury bond market. Bond risk premia appears to vary substantially over time throughout the entire sample period. However, if the EH were to hold true, then we should expect rx_{t+1}^n to equal a constant in expectation that may differ across maturities n. Empirically speaking, there is amble evidence that risk premia are time-varying and contain a predictable component, and I provide a brief overview of this research area below.

You may wonder how the risk premium in (26) is connected to the EH formulation in (12). If so, good on you. It turns out that the term premium component Φ^n from (12) can be written (see also (17)) as a particular function of the bond risk premia from (26), i.e.,

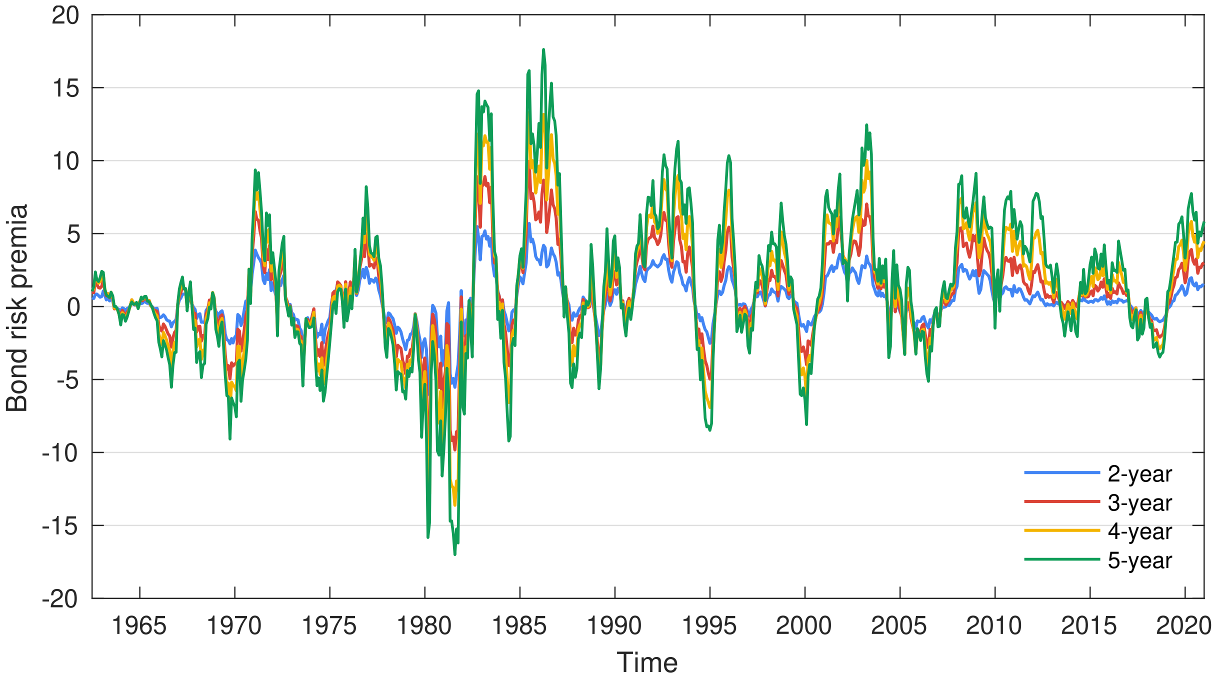
$$\Phi^n = \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{E}_t\left[rx_{t+j}^{n-j}\right], \tag{27}$$

which implies that we can write the expectations hypothesis equivalently as

$$y_t^n = \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{E}_t\left[y_{t+j}^1\right] + \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{E}_t\left[rx_{t+j}^{n-j}\right]. \tag{28}$$

Intuitively, (28) tells us that we can think of Φ^n as being equivalent to the sum of expected future

Figure 4: Time series dynamics of bond risk premia



holding-period excess return on discount bonds with declining maturities. That is, the yield on the long-term n-period bonds equals the returns and risk premia from the roll-over strategy. We can note that this is a so-called accounting identity that holds true by construction. Thus, if rx_{t+1}^n varies over time, then Φ^n will vary over time as well.

4.1. Predictive regressions for bond risk premia

We follow a long tradition in the finance literature and consider a standard linear predictive regression framework to gauge the predictability of bond risk premia

$$rx_{t+1}^n = \alpha^n + \boldsymbol{x}_t' \boldsymbol{\beta}^n + \varepsilon_{t+1}^n, \tag{29}$$

where $\boldsymbol{x}_t$ denotes a vector of time t observable variables that we believe can predict and/or explain the observed time variations in future bond risk premia and α^n and $\boldsymbol{\beta}^n$ are constant parameters to be estimated using Ordinary Least Squares (OLS). Under the EH, we should expect to find $\boldsymbol{\beta}^n = \mathbf{0}$ for all n. In this case, the model collapses to a simple no-predictability, constant mean model, i.e., $rx_{t+1}^n = \alpha^n + \varepsilon_{t+1}^n$, which is consistent with the constant expected excess bond return implied by the EH.

The empirical literature has uncovered a broad collection of candidates for $\boldsymbol{x}_t$ that explains part, but not all, of the time-variation in bond risk premia. One strand of research relates such variations to forward spreads (Fama and Bliss, 1987), yield spreads (Campbell and Shiller, 1991), and forward rates (Cochrane and Piazzesi, 2005). A second strand of research moves beyond the information embedded in the yield curve itself by arguing that bond risk premia should vary with macroeconomic variables and innovations (Ilmanen, 1995, Cooper and Priestley, 2009, Ludvigson and Ng, 2009, Joslin, Priebsch, and Singleton, 2014, Cieslak and Povala, 2015, Eriksen, 2017, Gargano et al., 2019, Berardi et al., 2020, Bianchi, Büchner, and Tamoni, 2021, Zhao, Zhou, and Zhu, 2021). As such, this strand of research is representative of a risk-based explanation of time-varying risk premia in that they are related to the macroeconomic environment. We will consider examples of both views in this note.

4.2. Fama-Bliss forward spread regressions

In an early and seminal study, Fama and Bliss (1987) ask if forward rates (or forward spreads more precisely) contain information about expected future holding-period excess returns on discount bonds. To understand why the n-period forward spread, theoretically at least, is a prime candidate predictor, begin from the definition of the log forward rate in (4), subtract the one-period yield, add and subtract p_{t+1}^{n-1}, and re-arrange to obtain

$$\begin{aligned} f_t^n - y_t^1 &= p_t^{n-1} - p_t^n - y_t^1 && (30)\\ &= p_t^{n-1} - p_t^n - y_t^1 + p_{t+1}^{n-1} - p_{t+1}^{n-1} && (31)\\ &= \left(p_{t+1}^{n-1} - p_t^n - y_t^1\right) + \left(-p_{t+1}^{n-1} + p_t^{n-1}\right) && (32)\\ &= \left(r_{t+1}^n - y_t^1\right) + (n-1)\left(y_{t+1}^{n-1} - y_t^{n-1}\right), && (33) \end{aligned}$$

where the final expression makes use of the relations in (2) and (7). Finally, take conditional expectations at time t to obtain the expression

$$f_t^n - y_t^1 = \left(\mathbb{E}_t\left[r_{t+1}^n\right] - y_t^1\right) + (n-1)\left(\mathbb{E}_t\left[y_{t+1}^{n-1}\right] - y_t^{n-1}\right). \tag{34}$$

The forward spread in (34) consists of two distinct components: (i) the expected holding-period return on an n-period bond in excess of the one-period spot rate and (ii) the expected change in the yield on an $n-1$ period bond from time t to $t+1$. The expression in (34) is an accounting identity and provides us with some powerful insight into the forecasting mechanisms in fixed income markets. In particular, it tells us that, if the forward spread varies over time, then it must forecast at least one of the two components. That is, it must either forecast expected future excess holding-period returns, spot rate changes, or a combination of the two.

Fama and Bliss (1987) test the EH implication in (14) using the forward spread as a predictor of excess bond returns. In this setup, one runs maturity-specific regressions

$$rx_{t+1}^n = \alpha^n + \beta^n\left(f_t^n - y_t^1\right) + \varepsilon_{t+1}^n, \tag{35}$$

where rejections of the null hypothesis that $\beta^n = 0$ is seen as evidence of time-varying risk premia in the bond market. The results from running the regressions in (35) using our yield data are presented in Panel A of Table 3. We note that β^n is positive for all n, implying that

Table 3: Fama-Bliss regressions

	2-year	3-year	4-year	5-year
	Panel A: Risk premia regressions			
α^n	0.155	0.169	0.085	-0.064
$\text{se}^{\text{NW}}(\alpha^n)$	(0.238)	(0.456)	(0.648)	(0.820)
β^n	0.748	0.911	1.078	1.239
$\text{se}^{\text{NW}}(\beta^n)$	(0.282)	(0.347)	(0.383)	(0.406)
$\text{t}^{\text{NW}}(\beta^n = 0)$	2.656	2.624	2.810	3.053
R^2 (%)	7.91%	8.29%	9.25%	10.21%
	Panel B: Yield change regressions			
δ^n	-0.155	-0.169	-0.085	0.064
$\text{se}^{\text{NW}}(\delta^n)$	(0.238)	(0.456)	(0.648)	(0.820)
θ^n	0.252	0.089	-0.078	-0.239
$\text{se}^{\text{NW}}(\theta^n)$	(0.282)	(0.347)	(0.383)	(0.406)
$\text{t}^{\text{NW}}(\theta^n = 0)$	0.893	0.256	-0.202	-0.589
R^2 (%)	0.96%	0.09%	0.05%	0.42%

an increase in the forward spreads predicts higher bond risk premia going forward. Moreover, we see that the coefficients increase with bond maturity and that the coefficients are statically significant for all maturities as well. As in the Campbell and Shiller (1991) regressions, this is strong evidence against the EH.

It is worth noting that the accounting identity in (34) implies that a regression of the ex post change in the yield on an $n-1$ period bond from time t to $t+1$ onto the forward spread

$$(n-1)\left(y_{t+1}^{n-1} - y_t^{n-1}\right) = \delta^n + \theta^n\left(f_t^n - y_t^1\right) + \varepsilon_{t+1}^n \tag{36}$$

would result in slope coefficients exactly equal to $\theta^n = 1 - \beta^n$ and constant values equal to $\delta^n = -\alpha^n$. This is equivalent to the statement made for the Campbell and Shiller (1991) regressions and the informational content in the yield spread for yield changes and excess bond returns. Panel B of Table 3 illustrates this point. The important lesson here is that while the forward spreads, under the EH, is expected to forecast yield changes, as risk premia should be constant over time, Panel B reveals that they do not. Instead, forward spreads forecast bond risk premia, which implies that forward spread variations are driven by time-varying risk premia. This is surely inconsistent with the predictions of the expectations hypothesis.

4.3. The Cochrane-Piazzesi forward rate factor

Cochrane and Piazzesi (2005) extend the work of Fama and Bliss (1987) by considering the entire term structure of available forward rates. In our example, this corresponds to using the set $\{f_t^1, f_t^2, f_t^3, f_t^4, f_t^5\}$ of forward rates. A main, and seminal, innovation in Cochrane and Piazzesi

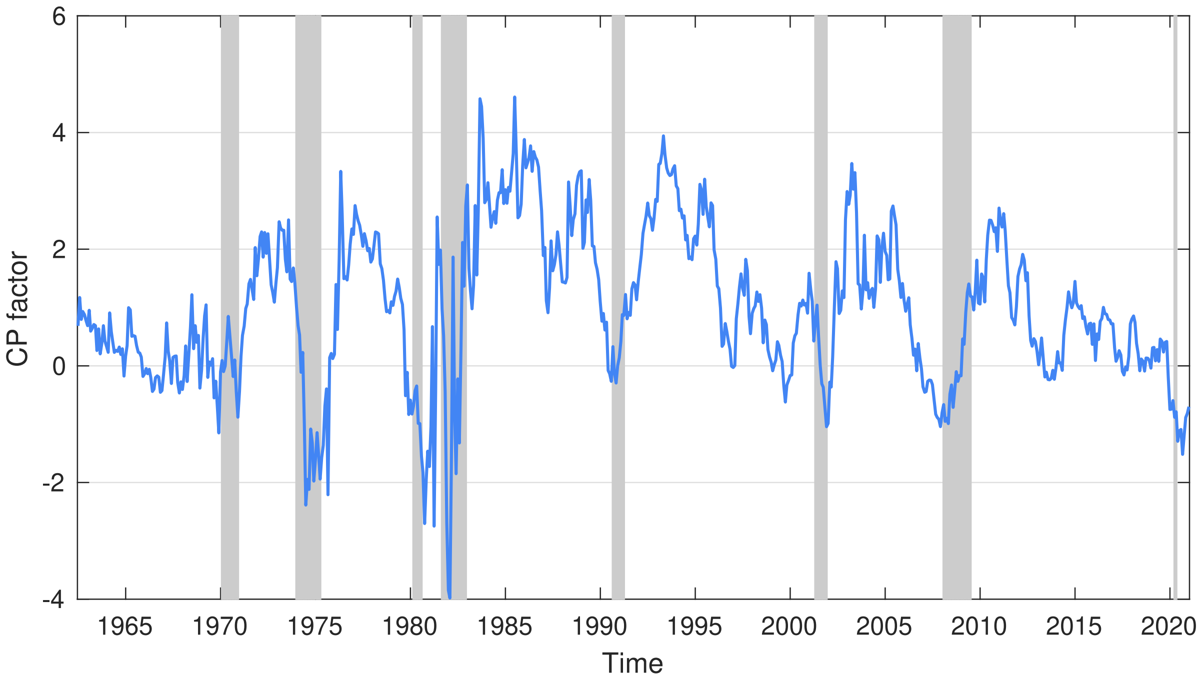
(2005) is to construct a single forecasting factor, labeled CP, that predicts bond risk premia for all maturities. The factor reveals that the same function of forward rates forecast bond risk premia for any n. To construct the single factor, Cochrane and Piazzesi (2005) regress average bond risk premia on the term structure of forward rates, including a constant, and define CP as the fitted values from the regression. Formally,

$$\frac{1}{4}\sum_{n=2}^{5} rx_{t+1}^{n} = \delta + \gamma_1 f_t^1 + \gamma_2 f_t^2 + \gamma_3 f_t^3 + \gamma_4 f_t^4 + \gamma_5 f_t^5 + \varepsilon_{t+1} \tag{37}$$

$$CP_t = \widehat{\delta} + \widehat{\gamma}_1 f_t^1 + \widehat{\gamma}_2 f_t^2 + \widehat{\gamma}_3 f_t^3 + \widehat{\gamma}_4 f_t^4 + \widehat{\gamma}_5 f_t^5, \tag{38}$$

where we use $\widehat{\gamma}$ to denote estimated parameters. Figure 5 plots the CP factor over time and illustrates that the factor varies considerably over time at a business cycle frequency. The variable

Figure 5: Time series dynamics of the Cochrane-Piazzesi (CP) factor



is weakly countercyclical and decreases in anticipation of recessions (shaded areas are NBER recessions). Evidence of β^n being statistically different from zero in the regression

$$rx_{t+1}^{n} = \alpha^n + \beta^n CP_t + \varepsilon_{t+1}^{n} \tag{39}$$

is taken as evidence against the expectations hypothesis and evidence in favor of bond risk premia varying over time. To emphasize, CP is the same for all $n = 2, \ldots, 5$ regressions, whereas Fama and Bliss (1987) use maturity-specific predictors, which implies that there exists a common component across discount bonds with different maturities.

Table 4 presents the results from estimating the predictive regression in (39) using our bond data. β^n is positive for all n, implying that an increase in CP predicts higher bond risk premia

Table 4: Cochrane-Piazzesi regressions

	2-year	3-year	4-year	5-year
α^n	0.080	0.061	-0.014	-0.128
$\text{se}^{\text{NW}}\ (\alpha^n)$	(0.216)	(0.390)	(0.542)	(0.678)
β^n	0.413	0.812	1.199	1.576
$\text{se}^{\text{NW}}\ (\beta^n)$	(0.125)	(0.235)	(0.334)	(0.423)
$\text{t}^{\text{NW}}\ (\beta^n = 0)$	3.307	3.455	3.595	3.725
R^2 (%)	11.01%	12.86%	14.45%	15.79%

going forward. Moreover, we see that the coefficient increase with bond maturity and that the coefficients are statically significant for all maturities as well. Importantly, this suggest that the same function of forward rates forecasts excess holding-period returns at all maturities. Longer maturities just have greater loadings on the same function.

It is important to emphasize here that the there is usually little to no efficiency loss associated with using CP over the full set of forward rates. Moreover, ene can show that the CP factor subsumes all the information in individual forward spreads for all n, implying that it is the better predictor. This is not surprising, however, as the maturity specific forward spread can be obtained by imposing particular restrictions on the parameter values when regressing bond risk premia onto the full set of forward rates. That is, the Fama and Bliss (1987) predictor is nested within the Cochrane and Piazzesi (2005) framework.

4.4. The Ludvigson-Ng macro factor

The empirical findings of Cochrane and Piazzesi (2005) significantly enhance our understanding of the Treasury bond market by introducing the concept of single factors in bond risk premia, but they do not inform us about the macroeconomic drivers of bond risk premia. Ludvigson and Ng (2009) make significant progress on this question and by addressing two empirical questions: (i) do movements in bond risk premia bear any direct relation to cyclical macroeconomic activity and (ii), if so, do macroeconomic fundamentals contain information about risk premia that is not already embedded in the zero-coupon term structure? The answer to both is a clear yes.[6] In fact, macroeconomic variables appear to capture and restore a countercyclical component in risk premia otherwise missed by forward rates (see also Berardi et al. (2020)).

A central innovation in Ludvigson and Ng (2009) is to use a large panel of macroeconomic variables that spans many different categories such as employment, prices, and production. There are two reasons for using a panel of data rather than a few selected variables. First, some macroeconomic driving variables may be latent and impossible to summarize with a few observable series. Second, macro variables are more likely than financial series to be imperfectly measured and

[6] Cooper and Priestley (2009) arrive at a similar conclusion using the output gap, which measures the difference between actual and potential output in an economy.

less likely to correspond to the precise economic concepts provided by theoretical models. To address these issues, they collect more than 130 macroeconomic variables and estimate macro factors using a dynamic factor model (Stock and Watson, 2002a,b, Bai and Ng, 2006). They argue that eight factors are enough to represent the information in the full panel and identify the optimal model for bond risk premia using the BIC criterion. Their preferred specification is presented below (referred to as $F5$ in their paper)

$$\frac{1}{4}\sum_{n=2}^{5} rx_{t+1}^{n} = \lambda + \delta_1 \widehat{F}_{1,t} + \delta_2 \widehat{F}_{1,t}^{3} + \delta_3 \widehat{F}_{3,t} + \delta_4 \widehat{F}_{4,t} + \delta_5 \widehat{F}_{8,t} + \nu_{t+1}, \tag{40}$$

where $\widehat{F}_{i,t}$ denotes the ith factor estimated using PCA and the LN factor is constructed as the fitted values from the regression analogous to the CP factor.

Figure 6: Time series dynamics of the Ludvigson-Ng (LN) factor

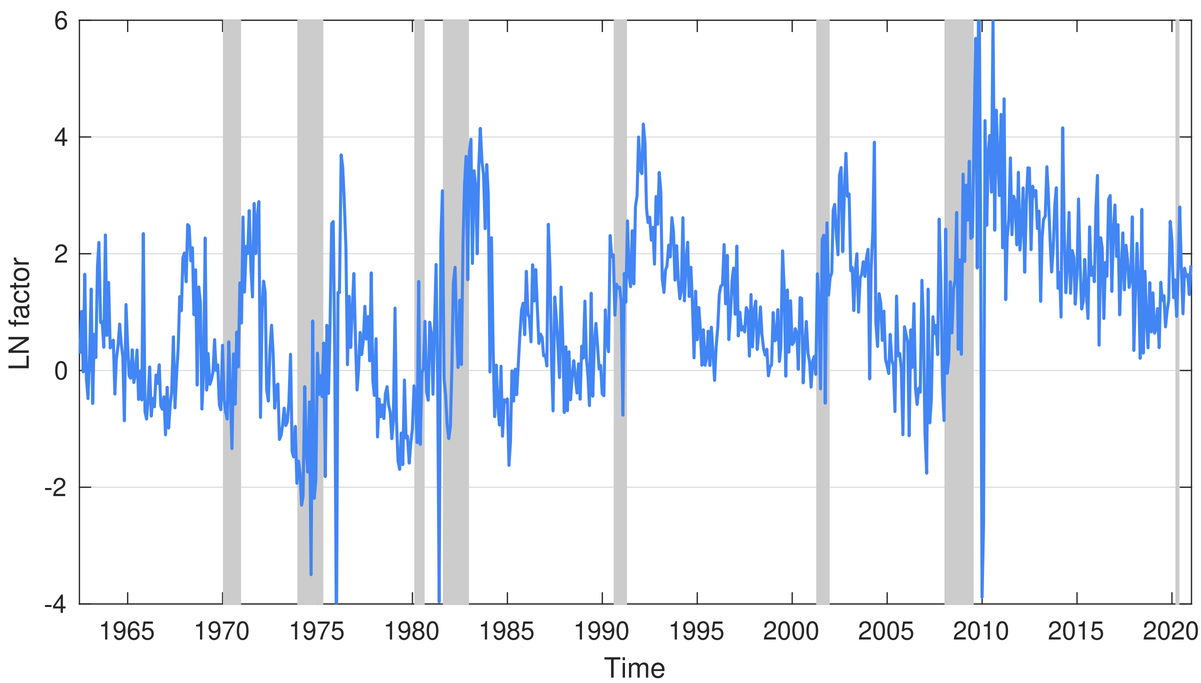


Figure 6 plots the LN factor over time. The factor is markedly cyclical and rises in bad times. Importantly, if LN predicts bond risk premia with a positive coefficient, then we have evidence of a cyclical, macroeconomic component in bond risk premia. To test this hypothesis, we run linear predictive regressions of the kind

$$rx_{t+1}^{n} = \alpha^{n} + \beta^{n} LN_t + \varepsilon_{t+1}^{n}, \tag{41}$$

where $\beta^n \neq 0$ is evidence against the EH and $\beta^n > 0$ is consistent with expected bond risk premia being countercyclical. That is, that investors demand a higher excess return in bad times.

Table 5 presents the empirical results from the predictive regressions. β^n is positive for all n, which confirms that bond risk premia are countercyclical, i.e., that they are higher in bad times.

Table 5: Ludvigson-Ng regressions

	2-year	3-year	4-year	5-year
α^n	0.030	0.016	-0.007	-0.038
$\text{se}^{\text{NW}}(\alpha^n)$	(0.206)	(0.361)	(0.494)	(0.614)
β^n	0.464	0.859	1.193	1.485
$\text{se}^{\text{NW}}(\beta^n)$	(0.099)	(0.159)	(0.203)	(0.243)
$\text{t}^{\text{NW}}(\beta^n = 0)$	4.676	5.407	5.876	6.105
R^2 (%)	16.45%	16.98%	16.88%	16.54%

This is consistent with canonical finance theory and the intuition from most asset pricing models (e.g., the consumption-based framework) that investors demand compensation for bearing recession risk. Moreover, similar to the CP factor, we see that the coefficient increase monotonically with maturity and that the coefficients are strongly statically significant for all maturities as well. This is more evidence that a single function of the same variables forecasts excess holding-period returns at all maturities, but with loadings that are increasing with maturity. That is, the same function of macro factors predict bond excess returns for all maturities.

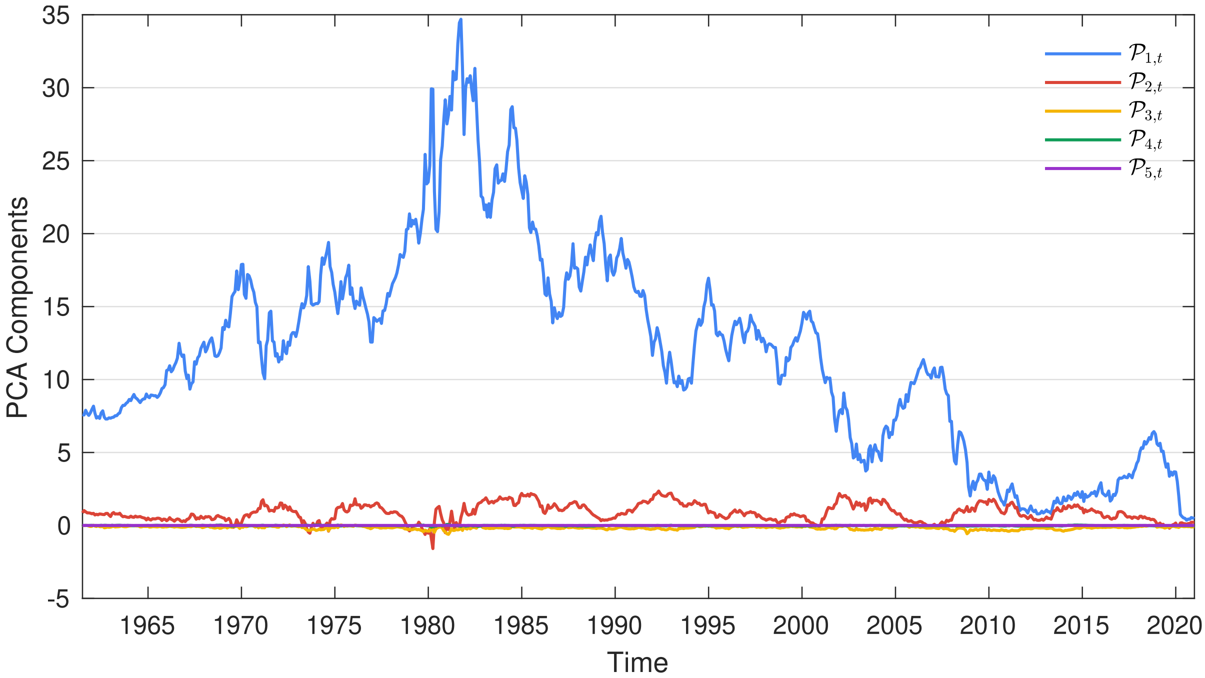
As a final note, the results here imply that risk premia in the Treasury bond market are driven by macroeconomic developments, or the business cycle. However, the findings in Ludvigson and Ng (2009) are not without controversy. Ghysels et al. (2018) argue that most of the predictive power originates from data revisions, which are not available to an investor in real time. That is, since LN is based on today-available revised macroeconomic variables, its out-of-sample predictive ability is overstated.[7] This is always a valid concern when using macroeconomic data in asset pricing studies, and particularly so for forecasting studies.

5. Yield curve factors

Researchers and practitioners alike are often interested in ways to efficiently summarize the entire yield curve for a multitude of purposes, including forecasting. Clearly, forecasting future yields for all possible maturities individually seems a tremendously tedious task. Instead, following the spirit of Litterman and Scheinkman (1991), it has become common practice to summarize the term structure by a small set of linear combinations of yields, e.g., principal components. This works as bond yields have a strong factor structure across maturities, something highlighted by the Cochrane and Piazzesi (2005) findings as well. An uncontroversial conclusion of the term structure literature is that the first few principal components of the covariance matrix of yields capture almost all of the variation in the term structure. These first three principal components are commonly referred to as *level*, *slope*, and *curvature*, respectively.

[7] Eriksen (2017), however, shows that the LN factor retains a high degree of predictive power when constructed using the vintage of historical data available to the investor at the time of forecasts construction.

Figure 7: Time series dynamics of yield curve factors



Estimating yield curve factors A common practice is to use principal components to estimate yield curve factors. One reason for this is that the method of principal components is both simple and powerful. First, collect all possible yields for different maturities at time t in the vector $\mathcal{Y}_t$. Principal components can be computed from both levels and changes in yields. While this exposition will consider levels of yields, obtaining principal components for changes requires only that one substitutes $\mathcal{Y}_t$ with $\Delta\mathcal{Y}_t$. Suppose that you have data on K different yields that are contained in the vector $\mathcal{Y}_t$, we can then write the covariance matrix of this set of yields as

$$\text{Cov}\left[\mathcal{Y}_t\right] = \Omega\Lambda\Omega^{\intercal}, \tag{42}$$

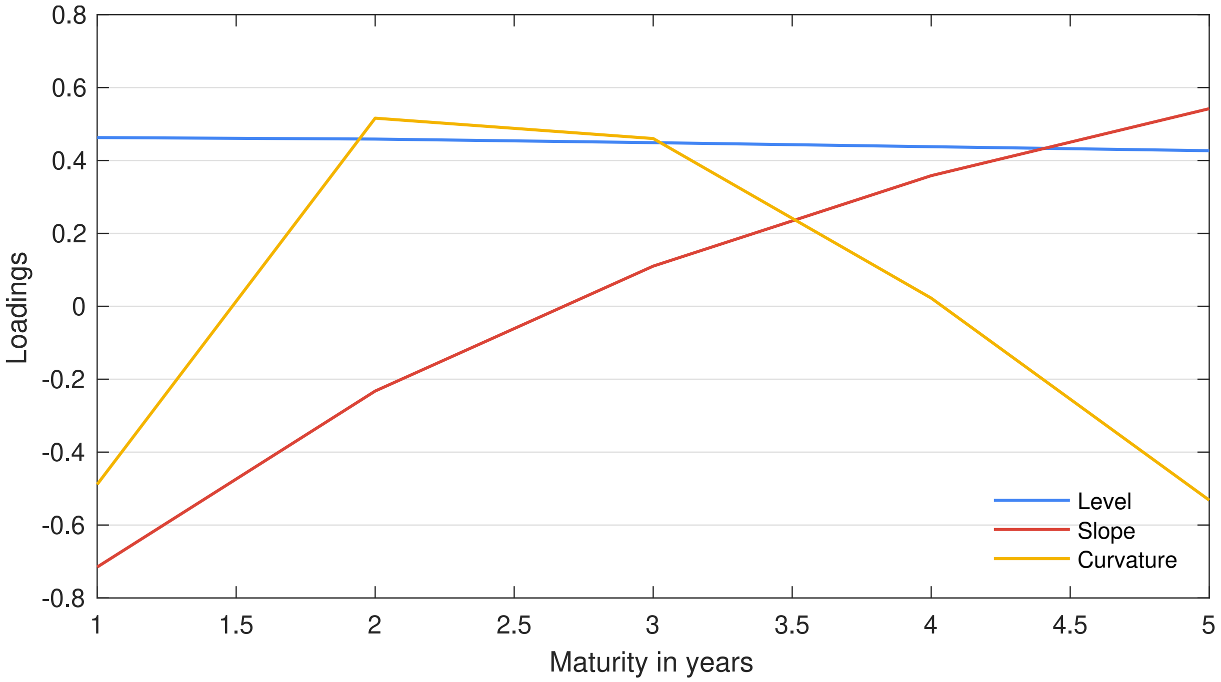
where Λ is a diagonal matrix of eigenvalues of the matrix $\text{Var}\left[\mathcal{Y}_t\right]$ and Ω is an orthogonal matrix (meaning that it satisfies $\Omega^{\intercal} = \Omega^{-1}$) whose columns are standardized eigenvectors.[8] Suppose that the eigenvalues are ordered from largest to smallest ($\Lambda_{1,1}$ is the largest), then $\Omega^{\intercal}_{(:,n)}\mathcal{Y}_t$ is the nth factor and the fraction of variance explained is $\Lambda_n / \left(\iota^{\intercal}\Lambda\iota\right)$. More generally, the principal components, call them $\mathcal{P}_t$, or yield curve factors, are then defined by

$$\mathcal{P}_t = \Omega^{\intercal}\mathcal{Y}_t. \tag{43}$$

Figure 7 illustrates the time series dynamics of the five yield curve factors $\mathcal{P}_t$ over time. As mentioned above, the first three components are often referred to as level, slope, and curvature. To see why, Figure 8 plots the loadings, i.e., $\Omega^{\intercal}$, for the first three factors. The loadings of the first

[8]This particular way of factorizing a matrix is known as an eigendecomposition and is implemented in most statistical software packages.

Figure 8: Yield curve factor loadings



principal component are near horizontal. This pattern means that changes in the first principal component correspond to parallel shifts in the yield curve. This principal component is therefore referred to as a level factor. The loadings of the second principal component is upward sloping and changes sign depending on the maturity. A change in the second component, as a result, moves short-maturity yields in the opposite direction of long-maturity yields. We therefore refer to this factor as a slope factor. The loading of the third principal component is hump shaped. The hump occurs at intermediate maturities. The third principal component therefore affects the curvature of the yield curve, which is why it is called the curvature factor. Comparing the shapes with the loadings from the Nelson and Siegel (1987) model reveals why we can think of these loadings in terms of yield curve factors.

A. Derivations

A.1. Yields as a function of returns

Start from the definition of a holding-period return

$$r_{t+1}^{n} = ny_{t}^{n} - (n-1)\, y_{t+1}^{n-1} \tag{44}$$

and solve for the implied yield definition

$$ny_{t}^{n} = r_{t+1}^{n} + (n-1)\, y_{t+1}^{n-1}. \tag{45}$$

We can then solve this equation forward to obtain

$$\begin{aligned}
ny_{t}^{n} &= r_{t+1}^{n} + (n-1)\, y_{t+1}^{n-1} && (46)\\
&= r_{t+1}^{n} + \left[r_{t+2}^{n-1} + (n-2)\, y_{t+2}^{n-2}\right] && (47)\\
&= r_{t+1}^{n} + r_{t+2}^{n-1} + \left[r_{t+3}^{n-2} + (n-3)\, y_{t+3}^{n-3}\right] && (48)\\
&\vdots && (49)\\
&= r_{t+1}^{n} + r_{t+2}^{n-1} + r_{t+3}^{n-2} + \cdots + \left[r_{t+n-1}^{n-(n-2)} + (n-(n-1))\, y_{t+n}^{1}\right] && (50)\\
&= \sum_{i=0}^{n-1} r_{t+i+1}^{n-i} && (51)
\end{aligned}$$

Finally, re-arrange to obtain the result that log yields on a discount bond equals the average log holding-period return per period if the bond is held to maturity

$$y_{t}^{n} = \frac{1}{n} \sum_{i=0}^{n-1} r_{t+i+1}^{n-i} \tag{52}$$

A.2. Yield spreads

Recall that we can write the log yield as a function of one-period holding-period returns as

$$y_{t}^{n} = \frac{1}{n} \sum_{i=0}^{n-1} r_{t+1+i}^{n-i}. \tag{53}$$

Subtracting the yield on a one-period bond y_t^1 from both sides yields

$$y_{t}^{n} - y_{t}^{1} = \frac{1}{n} \sum_{i=0}^{n-1} \left(r_{t+1+i}^{n-i} - y_{t}^{1}\right) \tag{54}$$

where $y_t^n - y_t^1 = s_t^n$ is the yield spread. Adding and subtracting y_{t+i}^1 from the right-hand side and re-arranging yields

$$y_t^n - y_t^1 = \frac{1}{n}\sum_{i=0}^{n-1}\left[\left(y_{t+i}^1 - y_t^1\right) + \left(r_{t+1+i}^{n-i} - y_{t+i}^1\right)\right] \tag{55}$$

$$= \frac{1}{n}\sum_{i=0}^{n-1}\left(y_{t+i}^1 - y_t^1\right) + \frac{1}{n}\sum_{i=0}^{n-1}\left(r_{t+1+i}^{n-i} - y_{t+i}^1\right) \tag{56}$$

where $r_{t+1+i}^{n-i} - y_{t+i}^1$ is the excess holding-period returns on an $n-i$ year bond. We can write the first summation in (56) as (by expanding the terms using telescoping sums)

$$\frac{1}{n}\sum_{i=0}^{n-1}\left(y_{t+i}^1 - y_t^1\right) = \left(y_{t+1}^1 - y_t^1\right) + \left(y_{t+2}^1 - y_t^1\right) + \left(y_{t+3}^1 - y_t^1\right) + \cdots + \left(y_{t+n-1}^1 - y_t^1\right) \tag{57}$$

$$\begin{aligned} &= \left(y_{t+1}^1 - y_t^1\right) \\ &+ \underbrace{\left(y_{t+2}^1 - y_{t+1}^1\right) + \left(y_{t+1}^1 - y_t^1\right)}_{\left(y_{t+2}^1 - y_t^1\right)} \\ &+ \underbrace{\left(y_{t+3}^1 - y_{t+2}^1\right) + \left(y_{t+2}^1 - y_{t+1}^1\right) + \left(y_{t+1}^1 - y_t^1\right)}_{\left(y_{t+3}^1 - y_t^1\right)} \\ &+ \cdots + \underbrace{\left(y_{t+n-1}^1 - y_{t+n-2}^1\right) + \cdots + \left(y_{t+1}^1 - y_t^1\right)}_{\left(y_{t+n-1}^1 - y_t^1\right)} \end{aligned} \tag{58}$$

$$= (n-1)\,\Delta y_{t+1}^1 + (n-2)\,\Delta y_{t+2}^1 + \cdots + (n-(n-1))\,\Delta y_{t+n-1}^1 \tag{59}$$

$$= \sum_{i=1}^{n-1}(n-i)\,\Delta y_{t+i}^1 \tag{60}$$

Inserting back into (56) and taking conditional expectations gives us

$$s_t^n = \frac{1}{n}\sum_{i=1}^{n-1}\mathbb{E}_t\left[(n-i)\,\Delta y_{t+i}^1\right] + \frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}_t\left[r_{t+1+i}^{n-i} - y_{t+i}^1\right], \tag{61}$$

which recovers that the yield spread equals a weighted average of expected future interest rate changes over the life of the long-term bond plus a a risk premium component.

A.3. Long-rate regressions

As stated above, under the EH, the yield spread should consist of two components: (i) a constant risk premium and (ii) an optimal forecast of future changes in long-term yields over the life of the one-period bond. Too see this, consider the definition of the log holding-period returns

$$r_{t+1}^n = y_t^n - (n-1)\left(y_{t+1}^{n-1} - y_t^n\right) \tag{62}$$

and subtract the one-period yield y_t^1 from both sides (or insert the expression into (14)) to obtain

$$r_{t+1}^n - y_t^1 = \left(y_t^n - y_t^1\right) - (n-1)\left(y_{t+1}^{n-1} - y_t^n\right) \tag{63}$$

Re-arranging and letting $s_t^n = y_t^n - y_t^1$ gives us

$$s_t^n = (n-1)\left(y_{t+1}^{n-1} - y_t^n\right) + \left(r_{t+1}^n - y_t^1\right) \tag{64}$$

Under the EH, the second term on the right-hand side is a constant risk premium RP^n and, last, taking conditional expectations yields

$$s_t^n = (n-1)\,\mathbb{E}_t\left[y_{t+1}^{n-1} - y_t^n\right] + RP^n \tag{65}$$

A.4. Short-rate regressions

Under the EH, the second term on the right-hand side of (61) is a constant risk premium RP^n. This has the important implication that the yield spread (up to a constant) is the optimal forecaster of the change in the long-bond yield over the life of the short bond as well as the optimal forecaster of changes in short rates over the life of the long-term bond, i.e.,

$$s_t^n = \sum_{i=1}^{n-1} \mathbb{E}_t\left[\left(1 - \frac{i}{n}\right)\Delta y_{t+i}^1\right] + \Phi^n \tag{66}$$

A.5. Equivalence of statements

Last, we note that (56) implies the key relation (add y_t^1 to each side)

$$y_t^n = \frac{1}{n}\sum_{i=0}^{n-1} y_{t+i}^1 + \frac{1}{n}\sum_{i=0}^{n-1}\left(r_{t+1+i}^{n-i} - y_{t+i}^1\right) \tag{67}$$

$$= \frac{1}{n}\sum_{i=0}^{n-1} y_{t+i}^1 + \frac{1}{n}\sum_{i=0}^{n-1}\left(rx_{t+1}^{n-i}\right) \tag{68}$$

where the second term on the right-hand side is constant under the EH. Taking conditional expectations gives us

$$y_t^n = \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}_t\left[y_{t+i}^1\right] + \Phi^n \tag{69}$$

We can show that this statement is implied by the other implications as well. In fact, the statements in (12)– (14) are all mathematically equivalent. That is, for the log pure expectations hypothesis, the statements

$$y_t^1 = \mathbb{E}_t \left[r_{t+1}^n \right] \quad \Leftrightarrow \quad f_t^n = \mathbb{E}_t \left[y_{t+n-1}^1 \right] \quad \Leftrightarrow \quad y_t^n = \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}_t \left[y_{t+i}^1 \right] \tag{70}$$

are all mathematically equivalent. For example, consider first the statement that forward rates are unbiased estimators of expected future yields

$$f_t^n = \mathbb{E}_t \left[y_{t+n-1}^1 \right]. \tag{71}$$

Adding these up over n periods yields

$$\begin{aligned} & \mathbb{E}_t \left[y_t^1 + y_{t+1}^1 + y_{t+2}^1 + \cdots + y_{t+n-1}^1 \right] && (72) \\ & = f_t^1 + f_t^2 + f_t^3 + \cdots + f_t^n && (73) \\ & = \left(p_t^0 - p_t^1 \right) + \left(p_t^1 - p_t^2 \right) + \left(p_t^2 - p_t^3 \right) + \cdots + \left(p_t^{n-1} - p_t^n \right) && (74) \\ & = -p_t^n = n y_t^n, && (75) \end{aligned}$$

which recovers that long yields equal average expected future short rates. To show this, we have used the definition of a forward rate i.e. $f_t^n = p_t^{n-1} - p_t^n$, and that $p_t^0 = 0$.[9]

Similar, we can start from the statement that expected holding-period returns should be constant and equal across maturities

$$y_t^1 = \mathbb{E}_t \left[r_{t+1}^n \right] \tag{76}$$

and add them up over n periods to obtain

$$\begin{aligned} & \mathbb{E}_t \left[y_t^1 + y_{t+1}^1 + y_{t+2}^1 + \cdots + y_{t+n-1}^1 \right] && (77) \\ & = \mathbb{E}_t \left[r_{t+1}^n + r_{t+2}^{n-1} + r_{t+3}^{n-2} + \cdots + r_{t+n}^1 \right] && (78) \\ & = \sum_{i=0}^{n-1} \mathbb{E}_t \left[r_{t+1+i}^{n-1} \right] && (79) \\ & = n y_t^n && (80) \end{aligned}$$

which again recovers that long yields equal average expected future short rates.

[9]Recall that the nominal bond pays out \$1 at maturity and that $\ln P_t^0 = \ln 1 = 0$.

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