The lognormal power utility model[*]

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[*]This note provides an overview of the lognormal power utility version of the consumption-based asset pricing model. The content draws heavily on the textbook treatments of Campbell, Lo and MacKinlay (1997), Cochrane (2005), and Campbell (2017) as well as the surveys of Campbell (2013) and Mehra (2012). The note is prepared for use only in the Master's course "Asset Pricing". Please do not cite, circulate, or use for purposes other than this course.

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1. Introduction

The intertemporal Consumption-based Capital Asset Pricing Model (CCAPM) of Rubinstein (1976), Lucas (1978), and Breeden (1979) relates asset prices to the consumption and savings decision of investors. The Capital Asset Pricing Model (CAPM) of Treynor (1961), Sharpe (1964), Lintner (1965), Mossin (1966), and Black (1972) and the Arbitrage Pricing Theory (APT) of Ross (1976), which are static one-period models, ignore such considerations. Instead, they treat asset prices as being determined by the portfolio choices of investors whose preferences are defined over future one-period wealth. Implicitly, these models assume that wealth uniquely determines consumption so that preferences defined over consumption are equivalent to preferences defined over wealth. In other words, investors consume all their wealth after the single period. This simplification is ultimately unsatisfactory, although it has allowed us to take significant steps forward in our understanding of the determination of expected returns. Real-world investors are faced with making their portfolio choices and consumption decisions simultaneously over many periods, which requires an intertemporal framework. Intertemporal equilibrium models of asset pricing address this issue directly, and therefore offer the potential to answer two particular questions otherwise left unresolved by the one-period models. First, what determines the risk-free interest rate (or more generally the rate of return on a zero-beta asset)? Second, what drives the compensation that investors demand for bearing risk? In the CAPM, both the risk-free rate (or zero-beta return) and the risk premium associated with bearing market risk are exogenous parameters. The model is silent about their origins. In the APT, we replace the single market risk with a vector of factor risk prices, but the model is similarly silent about their origins. As we shall see in this note, the consumption-based framework is able to yield some insight into the determinants of these important and fundamental quantities.

Consumption-based asset pricing models frequently aggregate individual investors into a single representative, utility-maximizing agent whose utility derives from aggregate (per capita) consumption in the economy (Constantinides, 1982). In models of this kind, the stochastic discount factor (SDF) equals the intertemporal marginal rate of substitution for the representative investor. The investor wishes to maximize the present value of the expected discounted utility of an infinite consumption stream, where the first-order condition — the Euler equation — provides the conditions for optimal consumption and portfolio choices of the representative agent as well as a way to link asset returns and consumption. In this note, we will discuss a commonly used consumption-based model in which the representative agent has time-separable power utility. In this representation, a single parameter governs both risk aversion and the elasticity of intertemporal substitution, defined as the willingness of the representative agent to adjust planned consumption growth in response to investment opportunities, within the model. In fact, the elasticity of intertemporal substitution is the reciprocal of risk aversion, implying that risk-averse investors must also be unwilling to adjust their consumption growth rates to changes in interest rates. The model explains the risk premia on assets by their covariance with aggregate (per capita) consumption growth multiplied by the coefficient of relative risk aversion for the representative agent. While the consumption-based model with a representative agent equipped with power utility provides a set of simple and intuitive predictions, it fails to explain even the most basic aspects of financial markets. In particular, evaluating the model using U.S. consumption data reveals two important puzzles: (i) the equity premium puzzle of Mehra and Prescott (1985) and (ii) the risk-free rate puzzle of Weil (1989). The equity premium is the statement that the lognormal power utility model is unable to explain the observed equity premium unless investors are implausible risk averse. However, if we allow investors to be highly risk averse, i.e., disliking growth, then the model generates an excessively high real risk-free rate. The puzzle can equivalent be states in terms of the rate of preference, which becomes negative under reasonable levels of risk aversion. In a nutshell, the model cannot match the observed behavior that risk averse individuals, who dislike growth, save even when interest rates are low. Hansen and Jagannathan (1991) (HJ) show that the equity premium puzzle can be cast in terms of the volatility of the SDF and derive a simple lower bound on the SDF volatility that needs to be satisfied for the model to resolve the equity premium puzzle.

The rest of the note progresses as follows. Section 2 outlines the consumption-based framework. Section 3 develops the lognormal power utility model and its predictions for the risk-free rate and the equity premium. Section 4 prepares to model along the lines of Mehra and Prescott (1985). Section 5 discusses the puzzles of the model. Finally, Section 6 derives the HJ bound.

2. The consumption-based framework

We begin our analysis of the consumption-based framework in the simplest possible way by considering the intertemporal choice problem of a representative investor who can trade freely in asset j and who maximizes the expectation of a time-separable utility function

$$\max_{\{z_{t+1}\}} \mathbb{E}_t \left[\sum_{t=0}^{\infty} \delta^t U(\widetilde{c}_t) \right], \tag{1}$$

where $0 \leq \delta = (1+\tau)^{-1} \leq 1$ is a deterministic time discount factor and τ denotes the rate of time preference, $\{z_{t+1}\}$ is the portfolio plan, $\widetilde{c}_t$ is the representative agent's consumption at time t, and $U(\widetilde{c}_t)$ is the utility of consumption.[1] The subjective discount factor δ also measures the investor's impatience. The smaller (bigger) δ (τ) is, the more the investor prefers consumption today relative to the future (the investor is more impatient). Let q_t^e denote the real price of a stock and Y_t the dividend, then the budget constraint for the problem is

$$c_t \leq z_t Y_t + q_t^e (z_t - z_{t+1}) \tag{2}$$

$$z_t \leq 1, \quad \forall t, \tag{3}$$

[1] The infinite horizon in the optimization problem can be motivated from a bequest motive as in Barro (1974).

which informs us that the representative agent cannot consume more than her income and proceeds from portfolio re-balancing. The Euler equation (the first-order condition for an optional consumption and savings plan) is

$$U'\left(c_t\right) q_{j,t}^e = \delta \mathbb{E}_t \left[U'\left(\widetilde{c}_{t+1}\right) \left(\widetilde{q}_{j,t+1}^e + \widetilde{Y}_{j,t+1} \right) \right], \tag{4}$$

where $U'\left(c_t\right)$ denotes the marginal utility of consumption. Consider the intertemporal choice problem of the representative agant at time t. The agent equates the marginal utility loss associated with buying one additional unit of the stock to the discounted expected utility gain from the additional consumption in the next period. In a nutshell, the investor must sacrifice consumption today in order to purchase the asset. This loss is $U'\left(c_t\right) q_{j,t}^e$. In exchange, the agent obtains $\widetilde{q}_{j,t+1}^e + \widetilde{Y}_{j,t+1}$ for consumption in the next period. The discounted expected gain is $\delta \mathbb{E}_t \left[U'\left(\widetilde{c}_{t+1}\right) \left(\widetilde{q}_{j,t+1}^e + \widetilde{Y}_{j,t+1} \right) \right]$. If the investor has reached an optimal consumption and portfolio plan, these quantities must be equal. Otherwise total utility can be improved. The Euler equation in (4) can be written more compactly as

$$1 = \mathbb{E}_t \left[\delta \frac{U'\left(\widetilde{c}_{t+1}\right)}{U'\left(c_t\right)} \frac{\left(\widetilde{q}_{j,t+1}^e + \widetilde{Y}_{j,t+1} \right)}{q_{j,t}^e} \right] = \mathbb{E}_t \left[M_{t+1} \widetilde{R}_{j,t+1} \right], \tag{5}$$

where we define $M_t = \delta \frac{U'\left(\widetilde{c}_{t+1}\right)}{U'\left(c_t\right)}$ as the stochastic discount factor (equivalently, the intertemporal marginal rate of substitution here) and $\widetilde{R}_{j,t+1} = \frac{\left(\widetilde{q}_{j,t+1}^e + \widetilde{Y}_{j,t+1} \right)}{q_{j,t}^e}$ as the gross return to asset j. Note that the intertemporal marginal rate of substitution, and hence the stochastic discount factor, are always positive in the model since marginal utilities are positive by assumption.

2.1. Power utility

Suppose that the representative agent has time separable power utility defined over aggregate per capita consumption

$$U\left(c_t\right) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}, \tag{6}$$

where γ is the Pratt-Arrow coefficient of relative risk aversion, and measures the curvature of the utility function. Note that when $\gamma = 1$, then (6) collapses to log utility, i.e., $U\left(c_t\right) = \ln\left(c_t\right)$. The power utility function has several important properties. First, it is scale-invariant: with constant return distributions, risk premia do not change over time as aggregate wealth and the scale of the economy increase. Second, if different agents in the economy have the same power utility function and can freely trade all the risks they face, then we can aggregate them into a representative investor with the same utility function even if they differ in their wealth levels. This provides some justification for the use of aggregate per capita consumption, rather than individual consumption, in the CCAPM.

A less desirable property of power utility is that it rigidly links two important concepts. Under power utility, the elasticity of intertemporal substitution (the derivative of planned log consumption growth with respect to the log interest rate), which we write as ψ, is the reciprocal of the coefficient of relative risk aversion γ. Hall (1988) argues that this linkage is inappropriate because the elasticity of intertemporal substitution concerns the willingness of an investor to move consumption between time periods — it is well-defined even if there is no uncertainty — whereas the coefficient of relative risk aversion concerns the willingness of an investor to move consumption between states of the world — it is well-defined even in a one-period model with no time dimension.

2.2. The power utility model

Taking the first derivative of (6) with respect to consumption shows that marginal utility is $U'(c_t) = c_t^{-\gamma}$ so that the Euler equation becomes

$$1 = \mathbb{E}_t \left[\delta \left(\frac{\widetilde{c}_{t+1}}{c_t} \right)^{-\gamma} \widetilde{R}_{j,t+1} \right], \tag{7}$$

which was first derived by Grossman and Shiller (1981), following the closely related continuous-time model of Breeden (1979). A typical objective of empirical research is to estimate the coefficient of relative risk aversion γ (or its reciprocal ψ) and to test the restrictions imposed by (7). It is easiest to do this if one assumes that asset returns and aggregate consumption are jointly lognormal. Although this implies constant expected excess log returns, and thus cannot fit the data, it is useful for building intuition and understanding methods that can be applied to more realistic models.

3. Power utility in a lognormal model

This section develops the consumption-based model for a representative investor with time separable power utility defined over per capita consumption and lognormal gross consumption growth. This model belongs to the constant relative risk aversion (CRRA) model class. Define $\widetilde{G}_{t+1} = 1 + \widetilde{g}_{t+1} = \frac{\widetilde{c}_{t+1}}{c_t}$ as the gross growth rate of consumption, then we can write the Euler equation in (7) as

$$1 = \mathbb{E}_t \left[\delta G_{t+1}^{-\gamma} \widetilde{R}_{j,t+1} \right]. \tag{8}$$

and the SDF as $M_{t+1} = \delta G_{t+1}^{-\gamma}$. The SDF will be conditionally lognormal if consumption is conditionally lognormal so that the expected value of the SDF becomes

$$\mathbb{E}_t \left[M_{t+1} \right] = \delta \exp \left\{ -\gamma \mathbb{E}_t \left[\widetilde{g}_{t+1} \right] + \frac{1}{2} \gamma^2 \sigma_g^2 \right\} \tag{9}$$

and the log SDF is $m_{t+1} = \ln \delta - \gamma \widetilde{g}_{t+1}$, where we notice the Jensen's Inequality difference. We will make the assumption that gross consumption growth and asset returns are jointly conditionally lognormal and homoskedastic.[2] In this case, and using the properties of the lognormal distribution (see Appendix A), we can obtain an expression for the Euler equation

$$1 = \delta \exp \left\{ -\gamma \mathbb{E}_t \left[\widetilde{g}_{t+1} \right] + \mathbb{E}_t \left[\widetilde{r}_{j,t+1} \right] + \frac{1}{2} \left(\gamma^2 \sigma_g^2 + \sigma_j^2 - 2\gamma \sigma_{jg} \right) \right\}, \tag{10}$$

where $\widetilde{g}_{t+1}$ denotes the log consumption growth rate, $\widetilde{r}_{j,t+1}$ the log return as asset j, and σ_g^2, σ_j^2, and σ_{jg} denote the unconditional variances of consumption growth and asset returns, respectively, and their covariance. Finally, taking logs to (10) yields the expression

$$0 = \ln \delta - \gamma \mathbb{E}_t \left[\widetilde{g}_{t+1} \right] + \mathbb{E}_t \left[\widetilde{r}_{j,t+1} \right] + \frac{1}{2} \left(\gamma^2 \sigma_g^2 + \sigma_j^2 - 2\gamma \sigma_{jg} \right), \tag{11}$$

which was first derived by Hansen and Singleton (1983) and has both time-series and cross-sectional implications. We explore these in the following sections.

3.1. The risk-free rate

This section develops an expression for the log real risk-free rate that allows us to make statements about it determination and drivers. Consider the Euler equation for a riskless asset whose return satisfies

$$1 = \mathbb{E}_t \left[\delta \widetilde{G}_{t+1}^{-\gamma} \right] (1 + r_{f,t+1}), \tag{12}$$

where we can move the risk-free rate outside of the expectations operator because it is known at time t. Maintaining the assumptions from above, and using the properties of the lognormal distribution, we can use (9) to obtain

$$1 = \delta \exp \left\{ -\gamma \mathbb{E}_t \left[\widetilde{g}_{t+1} \right] + \frac{1}{2} \gamma^2 \sigma_g^2 \right\} (1 + r_{f,t+1}). \tag{13}$$

Next, we use that $\frac{1}{\exp\{x\}} = \exp\{-x\}$, which enables us to isolate for the gross risk-free rate

$$1 + r_{f,t+1} = \frac{1}{\delta} \exp \left\{ \gamma \mathbb{E}_t \left[\widetilde{g}_{t+1} \right] - \frac{1}{2} \gamma^2 \sigma_g^2 \right\}, \tag{14}$$

where we note that, in a world with no uncertainty ($\sigma_g^2 = 0$) and no consumption growth ($\mathbb{E}_t \left[\widetilde{g}_{t+1} \right] = 0$), the gross risk-free rate is simply $1/\delta$. Last, taking logs to (14) immediately gives us an expression for the log risk-free rate in the lognormal power utility model

$$r_{f,t+1} = -\ln \delta + \gamma \mathbb{E}_t \left[\widetilde{g}_{t+1} \right] - \frac{1}{2} \gamma^2 \sigma_g^2. \tag{15}$$

[2] Specifically, we assume that the gross growth rate of consumption is lognormally distributed, but homoskedastic, $\ln \widetilde{G}_{t+1} \sim \mathcal{N} \left(\mathbb{E}_t \left[\widetilde{g}_{t+1} \right], \sigma_g^2 \right)$ and that the gross asset return is as well $\ln \widetilde{R}_{j,t+1} \sim \mathcal{N} \left(\mathbb{E}_t \left[\widetilde{r}_{j,t+1} \right], \sigma_j^2 \right)$.

The risk-free real rate is linear in expected consumption growth with slope coefficient equal to the coefficient of relative risk aversion. The equation can be reversed to express expected consumption growth as a linear function of the risk-free real rate, with slope coefficient $\psi = \frac{1}{\gamma}$. Importantly, this relation between expected consumption growth and the interest rate is what defines the elasticity of intertemporal substitution. We note that the real risk-free interest rate, in the time series domain, is

1. increasing in the rate of time preference: $-\ln\delta$ (impatience term)
2. increasing in consumption growth: $\gamma \mathbb{E}_t\left[\widetilde{g}_{t+1}\right]$ (intertemporal substitution)
3. decreasing in consumption risk: $\frac{1}{2}\gamma^2\sigma_g^2$ (precautionary savings).

The *impatience* term shows that $\delta < 1$ indicates a preference for consumption today relative to the future (i.e., the investor is impatient). The more impatient the representative agent is, the higher is the real risk-free rate. The intuition is as follows: if agents prefer consumption today, then it will take a high risk-free rate to incentivize them to save for the future. The *intertemporal substitution* term highlights that the log real risk-free rate is increasing in consumption growth. The intuition is straightforward: if $\mathbb{E}_t\left[\widetilde{g}_{t+1}\right] > 0$, then future consumption is likely to be high. A risk averse agent will prefer to smooth lifetime consumption by borrowing against future consumption. This desire to smooth consumption is increasing in risk aversion γ. Consequently, we need a high risk-free rate to discourage borrowing against future consumption. Last, the *precautionary savings* term illustrates that the log risk-free rate is decreasing in uncertainty. In an uncertain world ($\sigma_g^2 > 0$), risk averse agents prefer to hedge against future unfavorable consumption outcomes by saving for bad times. This effect is increasing in both risk aversion and consumption volatility. The intuition is, in order to fend off precautionary savings, that the risk-free rate must fall to encourage consumption today.

3.2. The risky (excess) return

This section turns to a discussion of the risky (excess) return within the lognormal power utility model. The expected (log) return on a risky asset j follows directly from (11) and is given as

$$\mathbb{E}_t\left[\widetilde{r}_{j,t+1}\right] = -\ln\delta + \gamma\mathbb{E}_t\left[\widetilde{g}_{t+1}\right] - \frac{1}{2}\left(\gamma^2\sigma_g^2 + \sigma_j^2 - 2\gamma\sigma_{jg}\right). \tag{16}$$

Subtracting the risk-free rate in (15) from (16) yields the log risk premium

$$\mathbb{E}_t\left[\widetilde{r}_{j,t+1}\right] - r_{f,t+1} + \frac{1}{2}\sigma_j^2 = \gamma\sigma_{jg}. \tag{17}$$

The variance term on the left-hand side of (17) is a Jensen's Inequality adjustment arising from the fact that we are describing expectations of log returns. We can eliminate the need for this adjustment by rewriting the equation in terms of the log of the expected ratio of gross returns to the gross risk-free rate, i.e.,

$$\ln \frac{1+\mathbb{E}_t\left[\widetilde{r}_{j,t+1}\right]}{1+r_{f,t+1}} = \gamma\sigma_{jg}. \tag{18}$$

Equation (18) demonstrates that risk premia are determined by the coefficient of relative risk aversion (price of risk) times the covariance of asset j with consumption growth (quantity of risk). The risk premium is therefore increasing linearly with risk aversion and the exposure to consumption growth. In line with the standard logic in the consumption-based framework, we see that assets with a positive covariance with consumption growth command a positive risk premium, whereas assets with a negative covariance obtains a negative risk premium as it hedges against unfavorable consumption outcomes.

As a final remark, we note that the lognormal model presented here implies time-invariant risk premia on risky assets. This is a highly implausible implication, but nonetheless provides us the opportunity to explore the model and develop intuition about the basic mechanisms.

4. Market clearing in the Lucas model

This section develops a version of the lognormal power utility model that is consistent with the Lucas (1978) single fruit tree model. This is the version considered in the seminal paper by Mehra and Prescott (1985) on the equity premium puzzle. The model considers a single tree whose output then determines consumption. Accordingly, suppose that there is a single tree that produces a stochastic number of fruits over time. Under power utility, the Euler equation in (4) can then be re-arranged into a price formula for the single fruit tree as

$$q_t^e = \delta\mathbb{E}_t\left[\left(\frac{\widetilde{c}_{t+1}}{c_t}\right)^{-\gamma}\left(\widetilde{q}_{t+1}^e + \widetilde{Y}_{t+1}\right)\right] = \delta\mathbb{E}_t\left[\widetilde{G}_t^{-\gamma}\left(\widetilde{q}_{t+1}^e + \widetilde{Y}_{t+1}\right)\right]. \tag{19}$$

Suppose that the real stock price q_t^e is proportional (homogeneous of degree one) to dividends Y_{t+1} so that we can represent equity prices as

$$q_t^e = \nu Y_t. \tag{20}$$

This is also the line of attack considered in Danthine and Donaldson (2015), but we provide some additional details on the argument here for completeness following Mehra (2012). For example, we can show that (19) can be written as (using (20))

$$\nu Y_t = \delta\mathbb{E}_t\left[\widetilde{G}_t^{-\gamma}\left(\nu+1\right)\widetilde{Y}_{t+1}\right], \tag{21}$$

which immediately tells us that (dividing through with Y_t)

$$\nu = \delta \mathbb{E}_t \left[\widetilde{G}_t^{-\gamma} (\nu + 1) \frac{\widetilde{Y}_{t+1}}{Y_t} \right]. \tag{22}$$

We can then impose the market clearing condition from the Lucas model that $\widetilde{c}_{t+1} = \widetilde{Y}_{t+1}$, i.e., that dividends determine consumption, so that $\frac{\widetilde{Y}_{t+1}}{Y_t} = \widetilde{G}_{t+1}$ to obtain a solution for ν

$$\nu = \delta \mathbb{E}_t \left[\widetilde{G}_t^{-\gamma} (\nu + 1) \widetilde{G}_{t+1} \right] \tag{23}$$

$$= \delta \mathbb{E}_t \left[\widetilde{G}_t^{1-\gamma} (\nu + 1) \right] \tag{24}$$

$$= \frac{\delta \mathbb{E}_t \left[\widetilde{G}_t^{1-\gamma} \right]}{1 - \delta \mathbb{E}_t \left[\widetilde{G}_t^{1-\gamma} \right]}. \tag{25}$$

Using the price relation in (20), we can also write the gross return of the tree as a function of the gross growth rate of consumption

$$\widetilde{R}_{t+1} = 1 + \widetilde{r}_{t+1} = \frac{q_{t+1}^e + \widetilde{Y}_{t+1}}{q_t^e} = \frac{\nu + 1}{\nu} \frac{\widetilde{Y}_{t+1}}{Y_t} = \frac{\nu + 1}{\nu} \widetilde{G}_{t+1} \tag{26}$$

which, when taking expectations and using (25), reveals that the expected return on the tree is

$$\mathbb{E}_t \left[\widetilde{R}_{t+1} \right] = \frac{\nu + 1}{\nu} \mathbb{E}_t \left[\widetilde{G}_{t+1} \right] \tag{27}$$

$$= \left(1 + \frac{1}{\nu} \right) \mathbb{E}_t \left[\widetilde{G}_{t+1} \right] \tag{28}$$

$$= \left(1 + \frac{1 - \delta \mathbb{E}_t \left[\widetilde{G}_t^{1-\gamma} \right]}{\delta \mathbb{E}_t \left[\widetilde{G}_t^{1-\gamma} \right]} \right) \mathbb{E}_t \left[\widetilde{G}_{t+1} \right] \tag{29}$$

$$= \left(1 + \frac{1}{\delta \mathbb{E}_t \left[\widetilde{G}_t^{1-\gamma} \right]} - 1 \right) \mathbb{E}_t \left[\widetilde{G}_{t+1} \right] \tag{30}$$

$$= \frac{\mathbb{E}_t \left[\widetilde{G}_{t+1} \right]}{\delta \mathbb{E}_t \left[\widetilde{G}_t^{1-\gamma} \right]}. \tag{31}$$

Analogously, we can determine an expression for the risk-free rate as follows

$$R_{f,t+1} = \frac{1}{q_t^b} = \frac{1}{\delta \mathbb{E}_t \left[\widetilde{G}_t^{-\gamma} \right]}. \tag{32}$$

Suppose that we are interested in determining the risk premium, then we can take the ratio of the two and use the properties of the lognormal distribution to obtain

$$\frac{\mathbb{E}_t\left[\widetilde{R}_{t+1}\right]}{R_{f,t+1}} = \frac{\mathbb{E}_t\left[\widetilde{G}_{t+1}\right]\delta\mathbb{E}_t\left[\widetilde{G}_t^{-\gamma}\right]}{\delta\mathbb{E}_t\left[\widetilde{G}_t^{1-\gamma}\right]} \tag{33}$$

$$= \frac{\exp\left\{\mathbb{E}_t\left[\widetilde{g}_{t+1}\right]+\frac{1}{2}\sigma_g^2\right\}\delta\exp\left\{-\gamma\mathbb{E}_t\left[\widetilde{g}_{t+1}\right]+\frac{1}{2}\gamma^2\sigma_g^2\right\}}{\delta\exp\left\{(1-\gamma)\,\mathbb{E}_t\left[\widetilde{g}_{t+1}\right]+\frac{1}{2}\left(1-\gamma\right)^2\sigma_g^2\right\}} \tag{34}$$

$$= \frac{\delta\exp\left\{(1-\gamma)+\frac{1}{2}\sigma_g^2+\frac{1}{2}\gamma^2\sigma_g^2\right\}}{\delta\exp\left\{(1-\gamma)+\frac{1}{2}\sigma_g^2-\gamma\sigma_g^2+\frac{1}{2}\gamma^2\sigma_g^2\right\}} \tag{35}$$

$$= \exp\left\{\gamma\sigma_g^2\right\}, \tag{36}$$

which then implies that the log equity risk premium, which is constant in this model, is determined as

$$\ln\frac{\mathbb{E}_t\left[\widetilde{R}_{t+1}\right]}{R_{f,t+1}} = \gamma\sigma_g^2. \tag{37}$$

Equation (37) shows that risk premium on the single tree (the market) is determined by the coefficient of relative risk aversion times the variance of consumption growth. This is a testable implication that can be evaluated using data on market returns, the risk-free return, and aggregate consumption data. The empirical evidence is a clear rejection of the model, where two puzzles are especially troublesome: the equity risk premium puzzle (Mehra and Prescott, 1985) and the risk-free rate puzzle (Weil, 1989).

5. The puzzles

We are now ready to examine the main puzzles of the standard model in which the representative agent has time separable power utility defined over aggregate per capita consumption We will treat each puzzle below and illustrate the necessary calculations needed to fully evaluate the model. We will use the data moments from Mehra and Prescott (1985) to illustrate the puzzles. Table 1 presents the descriptive statistics, where we note that this is the raw data.

Table 1: Data moments from Mehra and Prescott (1985)

	Mean (%)	Standard deviation (%)
Consumption growth	1.83%	3.57%
Risk-free return	0.80%	5.67%
Risky return	6.98%	16.54%
Equity premium	6.18%	16.67%

5.1. Equity premium puzzle

The equity premium puzzle of Mehra and Prescott (1985) is a quantitative puzzle in which the risk aversion needs to be implausible high to match the data. We can quickly see this in the model by solving for the coefficient of relative risk aversion in (37)

$$\gamma = \frac{\ln\left\{1+\mathbb{E}_t\left[\widetilde{r}_{t+1}\right]\right\} - \ln\left\{1+\widetilde{r}_{f,t+1}\right\}}{\sigma_g^2} \tag{38}$$

$$= \frac{\ln\{1.0698\} - \ln\{1.0080\}}{0.00123} \tag{39}$$

$$= 48.44 \tag{40}$$

where we find that we need $\gamma = 48.44$ to match the equity premium in the model. That is, we need an implausible high risk aversion before the model can match basic aspects of the data: the equity premium. Again, this is a quantitative, not qualitative, puzzle. The number $\sigma_g^2 = 0.00123$ can be obtained by solving the system

$$1.0183 = \exp\left\{\mu_g + \frac{1}{2}\sigma_g^2\right\} \tag{41}$$

$$0.0357^2 = \exp\left\{2\mu_g + \sigma_g^2\right\}\left(\exp\left\{\sigma_g^2\right\} - 1\right). \tag{42}$$

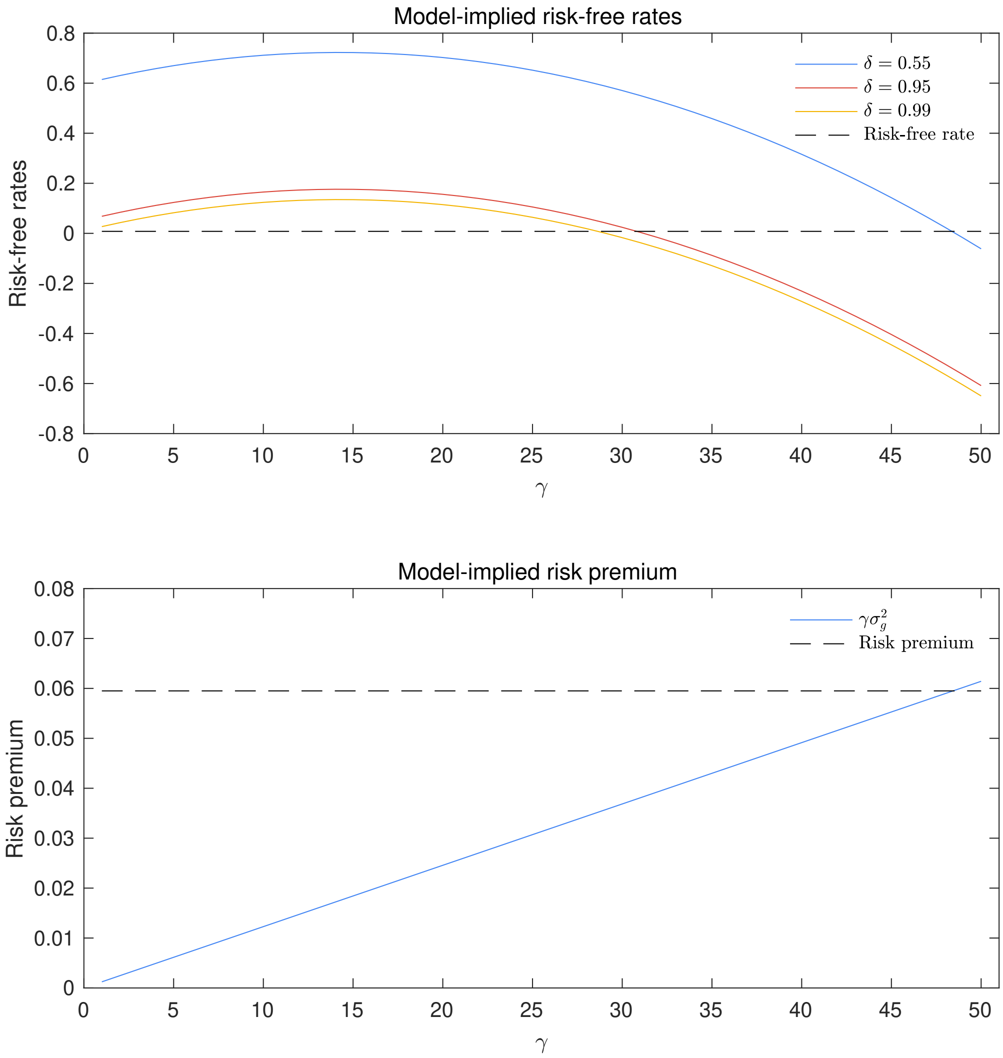
Square the first equation to obtain $1.0183^2 = \exp\left\{\mu_g + \frac{1}{2}\sigma_g^2\right\}^2 = \exp\left\{2\mu_g + \sigma_g^2\right\}$ and insert into the second to obtain $0.0357^2 = 1.0183^2\left(\exp\left\{\sigma_g^2\right\} - 1\right)$ and finally solve to find $\sigma_g^2 = 0.03505^2 = 0.00123$. Last, use this value and the first equation to find $\mu_g = 0.01752$.

5.2. Risk-free rate puzzle

One response to the equity premium puzzle is simply to accept that investors are (much) more risk averse than previously thought. However, accepting a high risk aversion in the lognormal model delivers implausible implications for the equilibrium riskless real interest rate. This observations is referred to as the risk-free rate puzzle (Weil, 1989). In a nutshell, the puzzle is that a high risk aversion also implies that investors dislike consumption growth. Yet, despite the low risk-free rates in the economy, individuals still save at a sufficiently fast rate to generate average per capita consumption growth of around two percent per year.

We can be more specific by considering the expression for the log risk-free rate in (15). In particular, we note that the risk-free rate is quadratic in risk aversion γ: risk aversion is multiplied with expected consumption growth and the negative of risk aversion squared is multiplied with one-half the variance of consumption growth. Since the variance of consumption growth is small (see Table 1), the linear term dominates for small to moderate levels of risk aversion and so higher risk aversion (in this region) increases the log real risk-free rate for a fixed rate of time preference. Intuitively, risk averse agents wish to smooth consumption over time by borrowing

Figure 1: The puzzles



against future consumption growth, which drives up the equilibrium risk-free rate. To fully appreciate the risk-free rate puzzle, consider what happens when risk aversion becomes large enough and the negative quadratic term dominates. Extremely risk averse consumers react to even modest uncertainty about future consumption with aggressive precautionary saving, driving down the equilibrium real interest rate for a fixed rate of time preference. Figure 1 illustrates these mechanics for the numbers used in this note for a broad set of $\gamma = \{1{,}2{,}3, \ldots, 50\}$ values and three values of $\delta = \{0.55, 0.95, 0.99\}$. The figure also immediately reveals another issue: with risk aversion within the appropriate range (1–10) and $\delta = 0.99$, the model predicts a risk-free rates much higher than the one empirically observed. The only way the model can solve this issue for an acceptable level of risk aversion is to set $\delta > 1$, suggesting that investors are patient and prefer consumption in the future rather than today. This is not something that seems supported in the real world.

A knife-edge case The parameter value $\delta = 0.55$ is of interest because it provides a knife-edge case achieved by offsetting the strong effects of intertemporal substitution and precautionary saving on the equilibrium real risk-free rate with a high risk aversion of $\gamma = 48.44$. This is clearly visible in Figure 1, where we see that these parameter values matches the equity premium and the risk-free rate simultaneously. Note, however, that this implies an extreme impatience where investors discount the future by 45% each period. Also, note that no other set of parameters will work – hence the wording of a knife-edge case.

6. The Hansen-Jagannathan bound in the lognormal model

Hansen and Jagannathan (1991) offer an alternative perspective on the equity premium puzzle by deriving a lower bound on the volatility of any valid SDF given a set of observed asset returns. We start from the generic Euler equation

$$
\begin{aligned}
1 &= \mathbb{E}_t \left[M_{t+1} \left(1 + \widetilde{r}_{j,t+1}\right) \right] \\
&= \mathbb{E}_t \left[M_{t+1} \right] \mathbb{E}_t \left[\left(1 + \widetilde{r}_{j,t+1}\right) \right] + \mathrm{cov}_t \left[M_{t+1}, \widetilde{r}_{j,t+1} \right] \\
&= \mathbb{E}_t \left[M_{t+1} \right] \mathbb{E}_t \left[\left(1 + \widetilde{r}_{j,t+1}\right) \right] + \rho \left[M_{t+1}, \widetilde{r}_{j,t+1} \right] \sigma_{M_{t+1}} \sigma_{\widetilde{r}_{j,t+1}},
\end{aligned} \tag{43}
$$

where we use the familiar relation that $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y] + \mathrm{Cov}[X, Y]$ in the second equality and the definition of a covariance in the third. Next, use the relation $1 + r_{f,t+1} = \mathbb{E}_t [M_{t+1}]^{-1}$ to obtain

$$
\frac{\mathbb{E}_t \left[\widetilde{r}_{j,t+1} \right] - r_{f,t+1}}{\sigma_{\widetilde{r}_{j,t+1}}} = -\rho \left[M_{t+1}, \widetilde{r}_{j,t+1} \right] \frac{\sigma_{M_{t+1}}}{\mathbb{E}_t \left[M_{t+1} \right]} \tag{44}
$$

Since $-1 \leq \rho\left[M_{t+1}, \widetilde{r}_{j,t+1}\right] \leq 1$, the expression in (44) provides a lower bound on the behavior of M_{t+1}, giving rise to the inequality

$$\underbrace{\left|\frac{\mathbb{E}_t\left[\widetilde{r}_{j,t+1}\right] - r_{f,t+1}}{\sigma_{\widetilde{r}_{j,t+1}}}\right|}_{\text{Sharpe Ratio}} \leq \underbrace{\frac{\sigma_{M_{t+1}}}{\mathbb{E}_t\left[M_{t+1}\right]}}_{\text{SDF Volatility}} \tag{45}$$

This is the Hansen and Jagannathan (1991) (HJ) lower bounds on the volatility of any valid SDF given a set of observable asset returns. Since $\mathbb{E}_t\left[M_{t+1}\right] = q_t^b \approx 1$, we see that the volatility of a valid SDF must equal or exceed the Sharpe ratio of the asset. Any SDF that fails to satisfy this bound will fail to fit the properties of the test assets.

We can link this to the lognormal model and derive the HJ bound within the model. The SDF in the lognormal model takes the form $M_{t+1} = \delta \widetilde{G}_{t+1}^{-\gamma}$, where $\widetilde{G}_{t+1}$ follows a lognormal distribution. The expected value of the SDF is

$$\mathbb{E}_t\left[M_{t+1}\right] = \delta \exp\left\{-\gamma \mathbb{E}_t\left[\widetilde{g}_{t+1}\right] + \frac{1}{2}\gamma^2\sigma_g^2\right\} \tag{46}$$

and the variance of the SDF is given by the expression (see Appendix A)

$$\begin{aligned} \mathrm{Var}_t\left[M_{t+1}\right] &= \exp\left\{-2\gamma \mathbb{E}_t\left[\widetilde{g}_{t+1}\right] + \gamma^2\sigma_g^2\right\}\left(\exp\left\{\gamma^2\sigma_g^2\right\} - 1\right) && (47) \\ &= \mathbb{E}_t\left[M_{t+1}\right]^2\left(\exp\left\{\gamma^2\sigma_g^2\right\} - 1\right). && (48) \end{aligned}$$

This, of course, implies that the volatility of the SDF is

$$\sigma_{M_{t+1}} = \mathbb{E}_t\left[M_{t+1}\right]\sqrt{\exp\left\{\gamma^2\sigma_g^2\right\} - 1}. \tag{49}$$

This allows us to write the HJ bound in the lognormal model as follows

$$\left|\frac{\mathbb{E}_t\left[\widetilde{r}_{j,t+1}\right] - r_{f,t+1}}{\sigma_{\widetilde{r}_{j,t+1}}}\right| \leq \frac{\sigma_{M_{t+1}}}{\mathbb{E}_t\left[M_{t+1}\right]} = \sqrt{\exp\left\{\gamma^2\sigma_g^2\right\} - 1} \approx \gamma\sigma_g, \tag{50}$$

where the last approximate equality in (50) derives from the following logic. The log SDF for the power utility model is $m_{t+1} = \ln\delta - \gamma\mathbb{E}_t\left[\widetilde{g}_{t+1}\right]$. Assuming that $m_{t+1} \approx M_{t+1} - 1$, which is accurate for $\mathbb{E}_t\left[M_{t+1}\right]$ close to one and not too variable, then we have $\mathrm{Var}_t\left[M_{t+1}\right] \approx \mathrm{Var}_t\left[m_{t+1}\right] = \gamma^2\sigma_g^2$. This implies that the volatility of the SDF is $\sigma_{M_{t+1}} = \gamma\sigma_g$, giving us the approximation when $\mathbb{E}_t\left[M_{t+1}\right]$ is close to one. This is nothing more than the equity premium puzzle all over again. If σ_g is too low, then we need implausibly high levels of risk aversion to satisfy the HJ lower bound on the volatility of a valid SDF. That is, the model fails to match the observed equity premium for reasonable values of γ.

A. Properties of the lognormal distribution

A variable x is said to follow a lognormal distribution if $\ln x$ is normally distributed. Let $\ln x \sim \mathcal{N}\left(\mu_x, \sigma_x^2\right)$, then expected values are determined as

$$\mathbb{E}\left[x\right] = \exp\left\{\mu_x + \frac{1}{2}\sigma_x^2\right\} \tag{A.1}$$

$$\mathbb{E}\left[x^a\right] = \exp\left\{a\mu_x + \frac{1}{2}a^2\sigma_x^2\right\}. \tag{A.2}$$

Moreover, we can compute the variance of a lognormally distributed variable as $\mathrm{var}\left[x\right] = \mathbb{E}\left[x^2\right] - \mathbb{E}\left[x\right]^2 = \left(\mathbb{E}\left[x\right]\right)^2\left(\exp\left\{\sigma_x^2\right\} - 1\right)$, giving us the following relations for the variances

$$\mathrm{var}\left[x\right] = \exp\left\{2\mu_x + \sigma_x^2\right\}\left(\exp\left\{\sigma_x^2\right\} - 1\right) \tag{A.3}$$

$$\mathrm{var}\left[x^a\right] = \exp\left\{2a\mu_x + a^2\sigma_x^2\right\}\left(\exp\left\{a^2\sigma_x^2\right\} - 1\right). \tag{A.4}$$

Suppose further that x and y are two iid lognormally distributed variables, then

$$\mathbb{E}\left[x^a y^b\right] = \exp\left\{a\mu_x + b\mu_y + \frac{1}{2}\left(a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\sigma_{xy}\right)\right\}. \tag{A.5}$$

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