CHAPTER 4

Measuring Risk and Risk Aversion

Chapter Outline

4.1 Introduction

We argued in Chapter 1 that the desire of investors to avoid risk, i.e., to smooth their consumption across states of nature and, for that reason, to avoid variations in the value of their portfolio holdings, is one of the primary motivations for financial contracting. But we have not thus far imposed restrictions on the VNM (von Neumann−Morgenstern) expected utility representation of investor preferences, which necessarily guarantee such behavior. For that to be the case, it must be further specialized. This is the subject of the present chapter to define, precisely, the notion of risk aversion and then to discuss its implications for the form of $U(\,)$.

4.2 Measuring Risk Aversion

What does the term *risk aversion* imply about an agent's utility function? Consider a financial contract where the potential investor either receives an amount of money h with probability $\frac{1}{2}$ or must pay an amount h with probability $\frac{1}{2}$. Our most basic sense of risk aversion must imply that for any level of personal wealth Y, a risk-averse investor would not wish to own such a security. In utility terms, this must mean

$$U(Y) > \left(\frac{1}{2}\right) U(Y+h) + \left(\frac{1}{2}\right) U(Y-h) = EU\left(\tilde{Y}\right)$$

where the expression on the right-hand side of the inequality sign is the VNM-expected utility associated with the random wealth levels:

$$Y+h, \ \text{probability} = \tfrac{1}{2}$$

$$Y-h, \ \text{probability} = \tfrac{1}{2}$$

This inequality can only be satisfied for all wealth levels Y if the agent's utility function has the form suggested in Figure 4.1. When this is the case, we say the utility function is strictly concave.

The important characteristic implied by this and similarly shaped utility functions is that the slope of the graph of the function decreases as the agent becomes wealthier (as Y increases). That is, the marginal utility (MU), represented by the derivative $\mathrm{d}(U(Y))/\mathrm{d}(Y) \equiv U'(Y)$, decreases with greater Y. Equivalently, for twice differentiable utility functions, $\mathrm{d}^2(U(Y))/\mathrm{d}(Y)^2 \equiv U''(Y) < 0$. For this class of functions, the latter is indeed a necessary and sufficient condition for risk aversion.

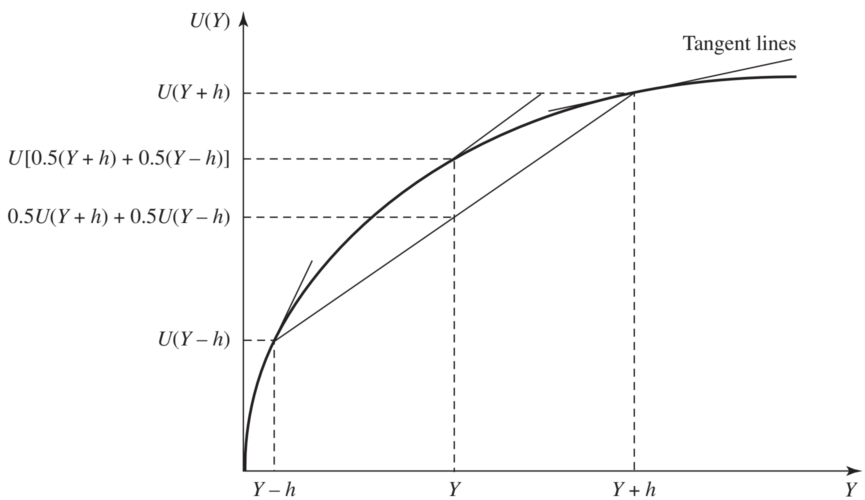

Figure 4.1
A strictly concave utility function.

As the discussion indicates, both consumption smoothing and risk aversion are directly related to the notion of decreasing *MU* of wealth. Whether it is envisaged across time or states, decreasing *MU* basically implies that income (or consumption) deviations from a fixed average level diminish rather than increase utility. Essentially, the positive deviations do not help as much as the negative ones hurt.

Risk aversion can also be represented in terms of indifference curves. Figure 4.2 illustrates the case of a simple situation with two states of nature. If consuming c, i.e., c_1 in state 1 and c_2 in state 2, represents a certain level of expected utility $EU = k_1$ and consuming c^*, i.e., c_1^* in state 1 and c_2^* in state 2, permits achieving the same level of expected utility, then the convex-to-the-origin indifference curve that is the appropriate translation of a strictly concave utility function indeed implies that the expected utility level generated by the average consumption is $(c + c^*)/2$ in both states (in this case a certain consumption level) is larger $(= k_2)$ than k_1.

We would like to be able to measure the degree of an investor's aversion to risk. This will allow us to compare whether one investor is more risk averse than another and to understand how an investor's risk aversion affects his investment behavior (e.g., the composition of his portfolio).

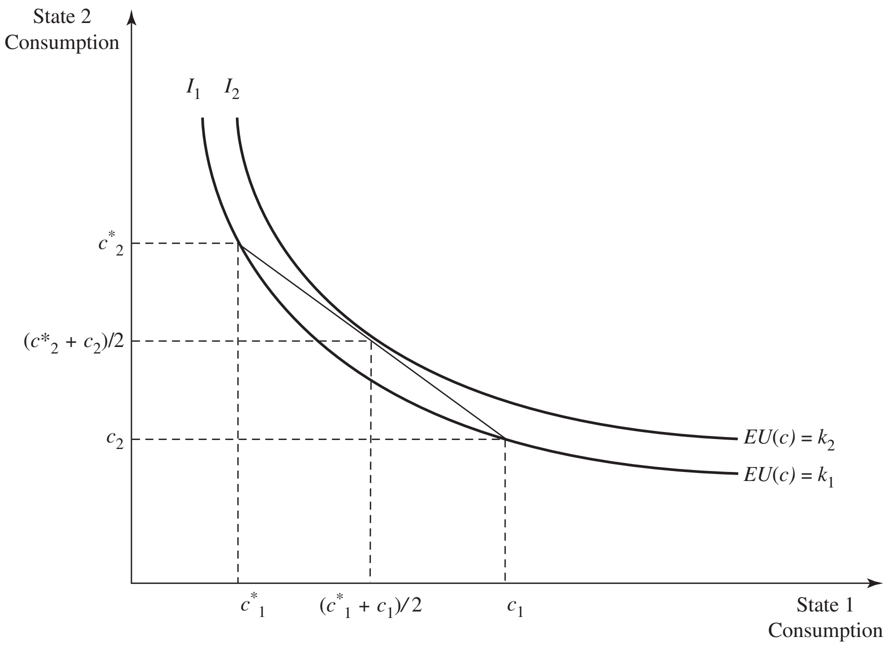

Figure 4.2
Indifference curves.

As a first attempt toward this goal, and since $U''(\ ) < 0$ implies risk aversion, why not simply say that investor A is more risk averse than investor B, if and only if $|U''_A(Y)| \geq |U''_B(Y)|$, for all income levels Y? Unfortunately, this approach leads to the following inconsistency. Recall that the preference ordering described by a utility function is invariant to linear transformations. In other words, suppose $U_A(\ )$ and $\overline{U}_A(\ )$ are such that $\overline{U}_A(\ ) = a + bU_A(\ )$ with $b > 0$. These utility functions describe the identical ordering and thus must display identical risk aversion. Yet, if we use the above measure, we have

$$|\overline{U}''_A(Y)| > |U''_A(Y)|, \text{ if } b > 1$$

This implies that investor A is more risk averse than he is himself, which must be a contradiction.

We therefore need a measure of risk aversion that is invariant to linear transformations. Two widely used measures of this sort have been proposed by, respectively, Pratt (1964) and Arrow (1971):

(i) absolute risk aversion = $U''(Y)/U'(Y) \equiv R_A(Y)$
(ii) relative risk aversion = $YU''(Y)/U'(Y) \equiv R_R(Y)$.

Both of these measures have simple behavioral interpretations. Note that instead of speaking of risk aversion, we could use the inverse of the measures proposed above and speak of risk tolerance. This latter terminology may be preferable on some occasions.

4.3 Interpreting the Measures of Risk Aversion

4.3.1 Absolute Risk Aversion and the Odds of a Bet

Consider an investor with wealth level Y who is offered—at no charge—an investment involving winning or losing an *amount* h, with probabilities π and $1 - \pi$, respectively. Note that any investor will accept such a bet if π is high enough (especially if $\pi = 1$) and reject it if π is small enough (surely if $\pi = 0$). Presumably, the willingness to accept this opportunity will also be related to his level of current wealth, Y. Let $\pi = \pi(Y,h)$ be that probability at which the agent is indifferent between accepting or rejecting the investment. It is shown below that

$$\pi(Y,h) \cong \tfrac{1}{2} + \left(\tfrac{1}{4}\right) hR_A(Y) \tag{4.1}$$

where the symbol $\cong$ represents "is approximately equal to."

The higher his measure of absolute risk aversion, the more favorable odds the investor will demand in order to be willing to accept the investment. If $R_A^1(Y) \geq R_A^2(Y)$, for agents 1 and

2, respectively, then investor 1 will always demand more favorable odds than investor 2, and in this sense investor 1 is more risk averse.

It is useful to examine the magnitude of this probability. Consider, for example, the family of utility-of-money functions of the form

$$U(Y) = -\frac{1}{v}\mathrm{e}^{-vY} \tag{4.2}$$

where v is a parameter.

For this case,

$$\pi(Y,h) \cong \tfrac{1}{2} + \left(\tfrac{1}{4}\right)hv$$

In other words, the odds requested are independent of the level of initial wealth (Y). On the other hand, the more wealth at risk (h), the greater the odds of a favorable outcome demanded. This expression advances the parameter v as the appropriate measure of the degree of *absolute* risk aversion for these preferences.

Let us now derive Eq. (4.1). By definition, $\pi(Y,h)$ must satisfy

$$\underbrace{U(Y)}_{\substack{\text{utility if he}\\ \text{foregoes the bet}}} = \underbrace{\pi(Y,h)U(Y+h) + [1-\pi(Y,h)]U(Y-h)}_{\text{expected utility if the investment is accepted}} \tag{4.3}$$

By an approximation (Taylor's Theorem), we know that:

$$U(Y+h) = U(Y) + hU'(Y) + \frac{h^2}{2}U''(Y) + H_1$$

$$U(Y-h) = U(Y) - hU'(Y) + \frac{h^2}{2}U''(Y) + H_2$$

where H_1, H_2 are remainder terms of order higher than h^2. Substituting these quantities into Eq. (4.3) gives

$$\begin{aligned} U(Y) = \pi(Y,h)&\left[U(Y) + hU'(Y) + \frac{h^2}{2}U''(Y) + H_1\right] \\ + \quad (1-\pi(Y,h))&\left[U(Y) - hU'(Y) + \frac{h^2}{2}U''(Y) + H_2\right] \end{aligned} \tag{4.4}$$

Collecting terms gives

$$U(Y) = U(Y) + (2\pi(Y,h) - 1)[hU'(Y)] + \frac{h^2}{2}U''(Y) + \underbrace{\pi(Y,h)H_1 + (1 - \pi(Y,h))H_2}_{=_{\mathrm{def}} H(\mathrm{small})}$$

Since the remainder term H is small—it is a weighted average of terms of order higher than h^2 and is, thus, itself of order higher than h^2—it can be ignored.

Solving for $\pi(Y,h)$ then yields

$$\pi(Y,h) = \frac{1}{2} + \frac{h}{4}\left[\frac{-U''(Y)}{U'(Y)}\right] \tag{4.5}$$

Utility functions for which $R_A(Y)$ is constant are referred to as displaying constant absolute risk aversion (CARA), with Eq. (4.2) being a principal case in point: $R_A(Y) = \nu$.

4.3.2 Relative Risk Aversion in Relation to the Odds of a Bet

Consider now an investment opportunity similar to the one just discussed except that the amount at risk is a *proportion* of the investor's wealth. In other words, $h = \theta Y$, where θ is the fraction of wealth at risk. By a derivation almost identical to the one presented above, it can be shown that

$$\pi(Y,\theta) \cong \frac{1}{2} + \frac{1}{4}\theta R_R(Y) \tag{4.6}$$

If $R_R^1(Y) \geq R_R^2(Y)$, for investors 1 and 2, then investor 1 will always demand more favorable odds, for any level of wealth, when the fraction θ of his wealth is at risk.

It is also useful to illustrate this measure by an example. Another popular family of VNM utility functions (for reasons to be detailed in the next chapters) has the form[1]

$$U(Y) = \frac{Y^{1-\gamma}}{1-\gamma}, \quad \text{for } 0 < \gamma \text{ and } \gamma \neq 1 \tag{4.7}$$

$$U(Y) = \ln Y, \quad \text{if } \gamma = 1$$

[1] Utility-of-money functions $U(Y)$ arise as the solution to problems of the form (3.3). Consider a utility of consumption function $u(c_1, c_2)$, defined over two goods c_1 and c_2 of the form $u(c_1, c_2) = -c_1^{-\gamma_1}c_2^{-\gamma_2}$, where $\gamma_1 > 0$ and $\gamma_2 > 0$. The resulting $U(Y)$ function has the form of (4.7): $U(Y) = \kappa(p_1,p_2,\gamma_1,\gamma_2)1/(1-\gamma)Y^{1-\gamma}$, where $\gamma = 1 + \gamma_1 + \gamma_2$ and $\kappa()$ is a function dependent on the consumption goods prices and the γ_1, γ_2 parameters. It is a positive constant from the investor's perspective (he takes prices as "given"). See the web notes to this chapter for details.

In the latter case, the probability expression becomes

$$\pi(Y,\theta) \cong \frac{1}{2} + \frac{1}{4}\theta$$

In this case, the requested odds of winning are not a function of initial wealth (Y) but depend upon θ, the fraction of wealth that is at risk: the lower the fraction θ, the more investors are willing to consider entering into a bet that is close to being fair (a risky opportunity where the probabilities of success or failure are both $\frac{1}{2}$). In the former, more general, case the analogous expression is

$$\pi(Y,\theta) \cong \frac{1}{2} + \frac{1}{4}\theta\gamma \tag{4.8}$$

Since $\gamma > 0$, these investors demand a higher probability of success. Furthermore, if $\gamma_2 > \gamma_1$, the investor characterized by $\gamma = \gamma_2$ will always demand a higher probability of success than will an agent with $\gamma = \gamma_1$, for the same fraction of wealth at risk. In this sense, a higher γ denotes a greater degree of *relative* risk aversion for this investor class.

Utility-of-money functions for which $R_R(Y)$ is constant are said to display constant relative risk aversion (CRRA). Utility form Eq. (4.7) is the work horse representative of this class ($R_R(Y) = \gamma$).

4.3.3 Risk Neutral Investors

One class of investors deserves special mention at this point. They are significant, as we shall later see, for the influence they have on the financial equilibria in which they participate. This is the class of investors who are risk neutral and who are identified with utility functions of a linear form

$$U(Y) = cY + d$$

where c and d are constants and $c > 0$.

Both of our measures of the degree of risk aversion, when applied to this utility function give the same result:

$$R_A(Y) \equiv 0 \text{ and } R_R(Y) \equiv 0$$

Whether measured as a proportion of wealth or as an absolute amount of money at risk, such investors do not demand better than even odds when considering risky investments of the type under discussion. They are indifferent to risk and are concerned only with an asset's expected payoff.

4.4 Risk Premium and Certainty Equivalence

The context of our discussion thus far has been somewhat artificial because we were seeking especially convenient probabilistic interpretations for our measures of risk aversion. More generally, a risk-averse agent ($U''(\ ) < 0$) will always value an investment at something less than the expected value of its payoffs. Consider an investor, with current wealth Y, evaluating an uncertain risky payoff $\tilde{Z}$. For any distribution function F_z,

$$U(Y + E\tilde{Z}) \geq E[U(Y + \tilde{Z})]$$

provided that $U''(\ ) < 0$. This is a direct consequence of a standard mathematical result known as Jensen's inequality.

Theorem 4.1 (Jensen's inequality) Let $g(\ )$ be a concave function on the interval (a,b), and $\tilde{x}$ be a random variable such that $\text{Prob}\{\tilde{x} \in (a,b)\} = 1$. Suppose the expectations $E(\tilde{x})$ and $Eg(\tilde{x})$ exist, then

$$E[g(\tilde{x})] \leq g[E(\tilde{x})]$$

Furthermore, if $g(\ )$ is strictly concave and $\text{Prob}\{\tilde{x} = E(\tilde{x})\} \neq 1$, then the inequality is strict.

This theorem applies irrespective of whether the interval (a,b) on which $g(\ )$ is defined is finite or infinite. If a and b are both finite, the interval can be open or closed at either endpoint. If $g(\ )$ is convex, the inequality is reversed. See De Groot (1970).

To put it differently, if an uncertain payoff is available for sale, a risk-averse agent will only be willing to buy it at a price less than its expected payoff. This statement leads to a pair of useful definitions. The (maximal) certain sum of money a person is willing to pay to acquire an uncertain opportunity defines his **certainty equivalent** (CE) for that risky prospect; the difference between the CE and the expected value of the prospect is a measure of the uncertain payoff's **risk premium**. It represents the maximum amount the agent would be willing to pay to avoid the investment or gamble.

Let us make this notion more precise. The context of the discussion is as follows. Consider an agent with current wealth Y and utility function $U(\ )$ who has the opportunity to acquire an uncertain investment $\tilde{Z}$ with expected value $E\tilde{Z}$. The CE of the risky investment $\tilde{Z}$, $\text{CE}(Y, \tilde{Z})$, and the corresponding risk or insurance premium, $\Pi(Y, \tilde{Z})$, are the solutions to the following equations:

$$EU(Y + \tilde{Z}) = U(Y + \text{CE}(Y, \tilde{Z})) \tag{4.9a}$$

$$= U(Y + E\tilde{Z} - \Pi(Y, \tilde{Z})) \tag{4.9b}$$

which implies

$$\mathrm{CE}(Y,\tilde{Z}) = E\tilde{Z} - \Pi(Y,\tilde{Z}) \text{ or } \Pi(Y,\tilde{Z}) = E\tilde{Z} - \mathrm{CE}(Y,\tilde{Z})$$

These concepts are illustrated in Figure 4.3.

It is intuitively clear that there is a direct relationship between the size of the risk premium and the degree of risk aversion of a particular individual. This link can be made quite easily. For simplicity, the derivation that follows applies to the case of an actuarially fair prospect $\tilde{Z}$, one for which $E\tilde{Z} = 0$. Using Taylor series approximations, we can develop the left-hand side (LHS) and right-hand side (RHS) of the definitional Eqs (4.9a) and (4.9b).

LHS:

$$EU(Y+\tilde{Z}) = EU(Y) + E\left[\tilde{Z}U'(Y)\right] + E\left[\frac{1}{2}\tilde{Z}^2U''(Y)\right] + EH(\tilde{Z}^3)$$
$$= U(Y) + \frac{1}{2}\sigma_{\tilde{Z}}^2 U''(Y) + EH(\tilde{Z}^3)$$

RHS:

$$U(Y - \Pi(Y,\tilde{Z})) = U(Y) - \Pi(Y,\tilde{Z})U'(Y) + H(\Pi^2)$$

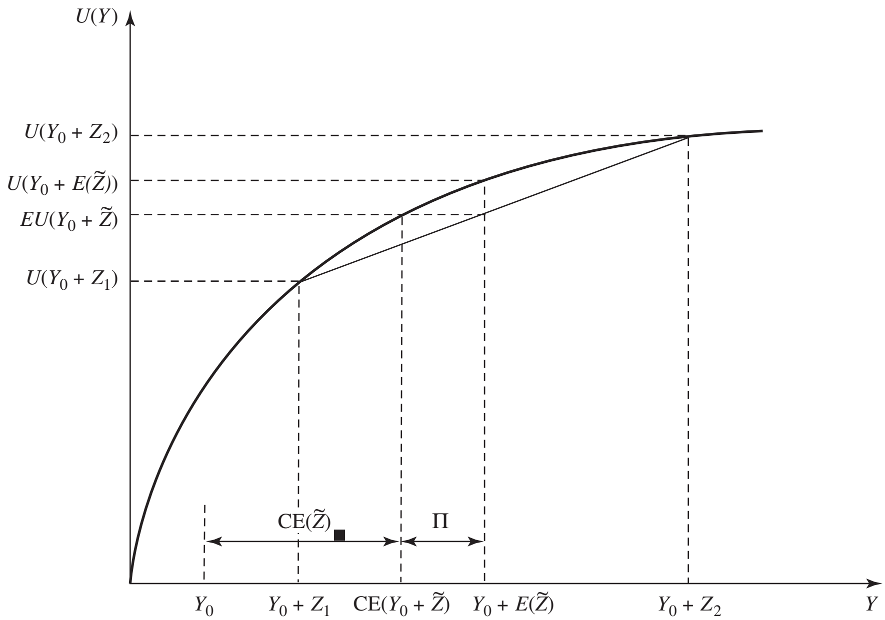

Figure 4.3
CE and the risk premium: an illustration.

or, ignoring the terms of order Z^3 or Π^2 or higher ($EH(\tilde{Z}^3)$ and $H(\Pi^2)$),

$$\Pi(Y,\tilde{Z}) \cong \frac{1}{2}\sigma_{\tilde{Z}}^2\left(\frac{-U''(Y)}{U'(Y)}\right) = \frac{1}{2}\sigma_{\tilde{Z}}^2 R_A(Y)$$

To illustrate, consider our earlier example in which $U(Y) = (Y^{1-\gamma})/(1-\gamma)$, and suppose $\gamma = 3$, $Y = \$500{,}000$, and

$$\tilde{Z} = \begin{cases} \$100,000 \text{ with probability } = \frac{1}{2} \\ -\$100,000 \text{ with probability } = \frac{1}{2} \end{cases}$$

For this case, the approximation specializes to

$$\Pi(Y,\tilde{Z}) = \frac{1}{2}\sigma_{\tilde{Z}}^2\frac{\gamma}{Y} = \frac{1}{2}(100,000)^2\left(\frac{3}{500,000}\right) = \$30,000$$

To confirm that this approximation is a good one, we must show that

$$U(Y - \Pi(Y,\tilde{Z})) = U(500,000 - 30,000) \cong \frac{1}{2}U(600,000) + \frac{1}{2}U(400,000) = EU(Y + \tilde{Z})$$

or

$$(4.7)^{-2} \cong \frac{1}{2}(6)^{-2} + \frac{1}{2}(4)^{-2}$$

or

$$0.0452694 \cong 0.04513; \text{confirmed}$$

Note also that for this preference class, the insurance premium is directly proportional to the parameter γ.

Can we convert these ideas into statements about rates of return? Let the equivalent risk-free return be defined by

$$U(Y(1 + r_{\mathrm{f}})) = U(Y + \mathrm{CE}(Y,\tilde{Z}))$$

The random payoff $\tilde{Z}$ can also be converted into a rate of return distribution via $\tilde{Z} = \tilde{r}Y$, or, $\tilde{r} = \tilde{Z}/Y$. Therefore, r_f is defined by the equation

$$U(Y(1 + r_{\mathrm{f}})) \equiv EU(Y(1 + \tilde{r}))$$

By risk aversion, $E\tilde{r} > r_f$. We thus define the utility-specific rate of return risk premium Π^r as $\Pi^r = E\tilde{r} - r_f$, or $E\tilde{r} = r_f + \Pi^r$, where Π^r depends on the degree of risk aversion of the agent in question. We conclude this section by computing the rate of return premium in a particular case. Suppose that $U(Y) = \ln Y$ and that the random payoff $\tilde{Z}$ satisfies

$$\tilde{Z} = \begin{cases} \$100,000 \;\; \text{with probability} = \dfrac{1}{2} \\ -\$50,000 \;\; \text{with probability} = \dfrac{1}{2} \end{cases}$$

from a base of $Y = \$500{,}000$. The risky rate of return implied by these numbers is clearly

$$\tilde{r} = \begin{cases} 20\% \;\; \text{with probability} = \dfrac{1}{2} \\ -10\% \;\; \text{with probability} = \dfrac{1}{2} \end{cases}$$

with an expected return of 5%. The $\mathrm{CE}(Y, \tilde{Z})$ must satisfy

$$\ln(500,000 + \mathrm{CE}(Y, \tilde{Z})) = \frac{1}{2}\ln(600,000) + \frac{1}{2}\ln(450,000), \;\; \text{or}$$

$$\mathrm{CE}(Y, \tilde{Z}) = e^{\frac{1}{2}\ln(600,000) + \frac{1}{2}\ln(450,000)} - 500,000$$

$$\mathrm{CE}(Y, \tilde{Z}) = 19,618, \;\; \text{so that}$$

$$(1 + r_f) = \frac{519,618}{500,000} = 1.0392$$

The utility-specific rate of return risk premium is thus 5−3.92% = 1.08%. Let us be clear: This rate of return risk premium does not represent a market or equilibrium premium.[2] Rather, it reflects personal preference characteristics and corresponds to the premium over the risk-free rate necessary to compensate, utility-wise, a specific individual, with the postulated preferences and initial wealth, for engaging in the risky investment.

4.5 Assessing the Degree of Relative Risk Aversion

Suppose that agents' utility functions are of the form $U(Y) = (Y^{1-\gamma})/(1-\gamma)$ class. As noted earlier, a quick calculation informs us that $R_R(Y) \equiv \gamma$, and we say that $U(\ )$ is of the CRRA

[2] Accordingly, we use a different symbol, Π^r, than the one used in Chapter 2 for the market risk premium (π).

class. To get a feeling as to what this measure means, consider the following uncertain payoff:

$$\begin{cases} \$50,000 \text{ with probability } \pi = 0.5 \\ \$100,000 \text{ with probability } \pi = 0.5 \end{cases}$$

Assuming your utility function is of the type just noted, what would you be willing to pay for such an opportunity (i.e., what is the CE for this uncertain prospect) if your current wealth were Y? The interest in asking such a question resides in the fact that, given the amount you are willing to pay, it is possible to infer your coefficient of relative risk aversion $R_R(Y) = \gamma$, provided your preferences are adequately represented by the postulated functional form. This is achieved with the following calculation.

The CE, the maximum amount you are willing to pay for this prospect, is defined by the equation

$$\frac{(Y+\text{CE})^{1-\gamma}}{1-\gamma} = \frac{\frac{1}{2}(Y+50,000)^{1-\gamma}}{1-\gamma} + \frac{\frac{1}{2}(Y+100,000)^{1-\gamma}}{1-\gamma}$$

Assuming zero initial wealth ($Y = 0$), we obtain the following sample results (clearly, $\text{CE} > 50{,}000$):

$$\begin{array}{ll} \gamma = 0 & \text{CE} = 75,000 \text{ (risk neutrality)} \\ \gamma = 1 & \text{CE} = 70,711 \\ \gamma = 2 & \text{CE} = 66,667 \\ \gamma = 5 & \text{CE} = 58,566 \\ \gamma = 10 & \text{CE} = 53,991 \\ \gamma = 20 & \text{CE} = 51,858 \\ \gamma = 30 & \text{CE} = 51,209 \end{array}$$

Alternatively, if we suppose a current wealth of $Y = \$100{,}000$ and a degree of risk aversion of $\gamma = 5$, the equation results in a CE = \$66,532.

4.6 The Concept of Stochastic Dominance

In response to dissatisfaction with the standard ranking of risky prospects based on mean and variance, a theory of choice under uncertainty with general applicability has been developed. In this section, we show that the postulates of expected utility lead to a definition of two weaker alternative concepts of dominance with wider applicability than the concept of state-by-state dominance. These are of interest because they circumscribe the situations in which rankings among risky prospects are preference free, or can be defined independently of the specific trade-offs (among return, risk, and other characteristics of probability distributions) implicit in the form of an agent's utility function.

Table 4.1: Sample investment alternatives

Payoffs	**10**	**100**	**2000**
Probability Z_1 Probability Z_2	0.4 0.4	0.6 0.4	0 0.2
	$EZ_1 = 64, \quad \sigma_{z_1} = 44$ $EZ_2 = 444, \quad \sigma_{z_2} = 779$		

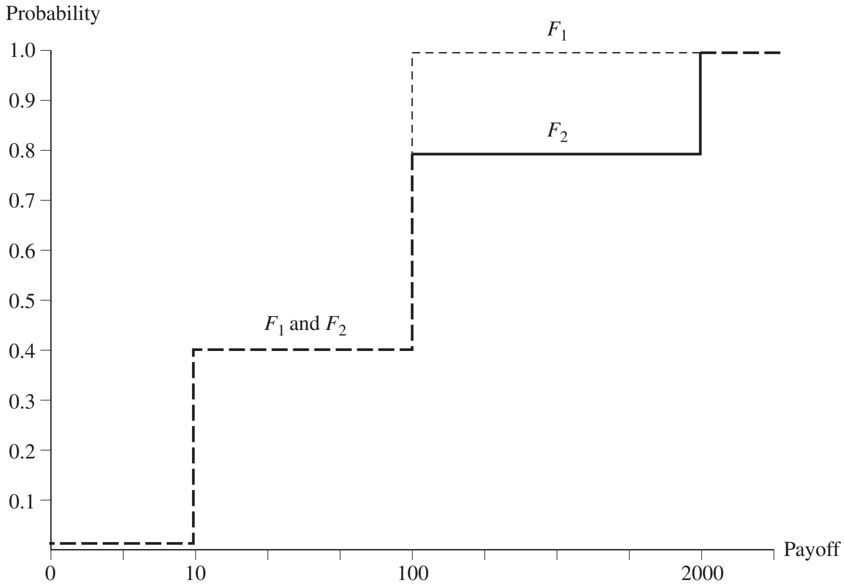

Figure 4.4
An example of FSD.

We start with an illustration. Consider two investment alternatives, $\tilde{Z}_1$ and $\tilde{Z}_2$, with the characteristics outlined in Table 4.1.

First, observe that under standard mean–variance analysis, these two investments cannot be ranked. Although investment $\tilde{Z}_2$ has the greater mean, it also has the greater variance. Yet, all of us would clearly prefer to own investment 2. It at least matches investment 1 and has a positive probability of exceeding it.

To formalize this intuition, let us examine the cumulative probability distributions associated with each investment, $F_1(\overline{Z})$ and $F_2(\overline{Z})$ where $F_i(\overline{Z}) = \text{Prob}(\tilde{Z}_i \leq \overline{Z})$.

In Figure 4.4, we see that $F_1(\cdot)$ always lies above $F_2(\cdot)$. This observation leads to Definition 4.1.

Definition 4.1 Let $F_A(\tilde{x})$ and $F_B(\tilde{x})$, respectively, represent the cumulative distribution functions of two random variables (cash payoffs) that, without loss of generality, assume

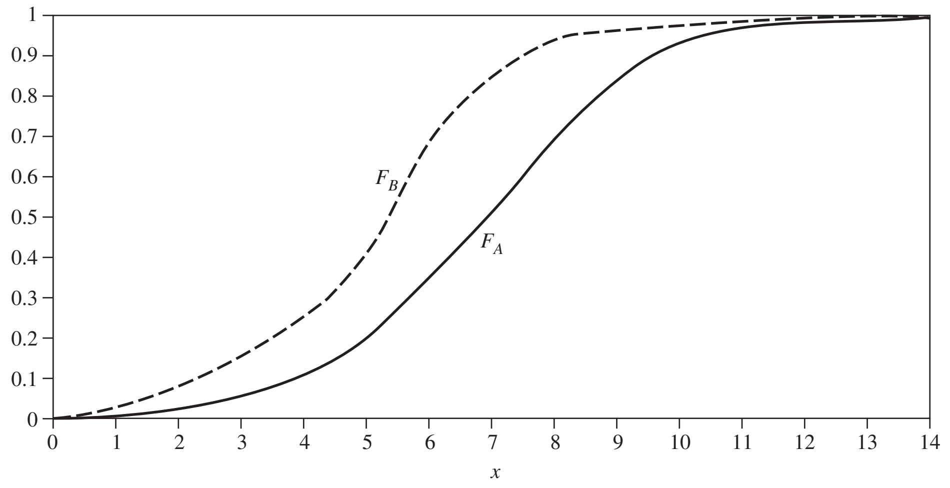

Figure 4.5
FSD: a more general representation.

values in the interval $[a,b]$. We say that $F_A(\tilde{x})$ first-order stochastically dominates (FSD) $F_B(\tilde{x})$ if and only if $F_A(x) \leq F_B(x)$ for all $x \in [a,b]$.

Distribution A in effect assigns more probability to higher values of x; in other words, higher payoffs are more likely. Accordingly, the distribution functions of A and B generally conform to the following pattern: if F_A FSD F_B, then F_A is everywhere below and to the right of F_B as represented in Figure 4.5. By this criterion, investment 2 in Figure 4.4 first order stochastically dominates investment 1. It should, intuitively, be preferred. Theorem 4.2 summarizes our intuition in this latter regard.

Theorem 4.2 Let $F_A(\tilde{x})$ and $F_B(\tilde{x})$, be two cumulative probability distributions for random payoffs $\tilde{x} \in [a, b]$. Then $F_A(\tilde{x})$ FSD $F_B(\tilde{x})$ if and only if $E_A U(\tilde{x}) \geq E_B U(\tilde{x})$ for all nondecreasing utility functions $U(\ )$.

Proof See the Appendix.

Although it is not equivalent to state-by-state dominance (see Exercise 4.8), FSD is an extremely strong condition. As is the case with the former, state-by-state dominance, it is so strong a concept that it induces only a very incomplete ranking among uncertain prospects. Can we find a broader measure of comparison, for instance, which would make use of the hypothesis of risk aversion as well? Consider the two independent investments in Table 4.2.[3]

[3] In this example, contrary to the previous one, the two investments considered are statistically independent.

Table 4.2: Two independent investments

Investment 3		Investment 4	
Payoff	**Probability**	**Payoff**	**Probability**
4	0.25	1	0.33
5	0.50	6	0.33
9	0.25	8	0.33

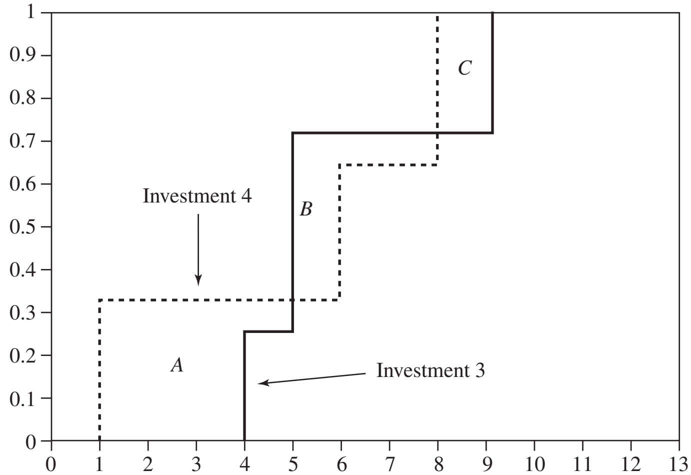

Figure 4.6
SSD illustrated.

Which of these investments is better? Clearly, neither investment (first order) stochastically dominates the other, as Figure 4.6 confirms. The probability distribution function corresponding to investment 3 is not everywhere below the distribution function of investment 4. Yet, we would probably prefer investment 3. Can we formalize this intuition (without resorting to the mean/variance criterion, which in this case accords with intuition: $ER_4 = 5$, $ER_3 = 5.75$; $\sigma_4 = 2.9$, and $\sigma_3 = 1.9$)? This question leads to a weaker notion of stochastic dominance that explicitly compares distribution functions.

Definition 4.2 Second-order stochastic dominance (SSD) Let $F_A(\tilde{x})$ and $F_B(\tilde{x})$ be two cumulative probability distributions for random payoffs in $[a,b]$. We say that $F_A(\tilde{x})$ SSD $F_B(\tilde{x})$ if and only if for any x:

$$\int_{-\infty}^{x} [F_B(t) - F_A(t)]\mathrm{d}t \geq 0$$

(with strict inequality for some meaningful interval of values of t).

Table 4.3: Investment 3 second-order stochastically dominates investment 4

Values of x	$\int_0^x f_x(t)dt$	$\int_0^x F_3(t)dt$	$\int_0^x f_4(t)dt$	$\int_0^x F_4(t)dt$	$\int_0^x [F_4(t) - F_3(t)]dt$
0	0	0	0	0	0
1	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
2	0	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
3	0	0	$\frac{1}{3}$	1	1
4	0.25	0.25	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{13}{12}$
5	0.75	1	$\frac{1}{3}$	$\frac{5}{3}$	$\frac{2}{3}$
6	0.75	1.75	$\frac{2}{3}$	$\frac{7}{3}$	$\frac{7}{12}$
7	0.75	2.5	$\frac{2}{3}$	3	$\frac{1}{2}$
8	0.75	3.25	1	4	$\frac{3}{4}$
9	1	4.25	1	5	0.75
10	1	5.25	1	6	0.75
11	1	6.25	1	7	0.75
12	1	7.25	1	8	0.75
13	1	8.25	1	9	0.75

The calculations in Table 4.3 reveal that, in fact, investment 3 SSD investment 4 (let $f_i(x)$, $i = 3,4$, denote the density functions corresponding to the cumulative distribution function $F_i(x)$). In geometric terms (Figure 4.6), this would be the case as long as area B is smaller than area A.

As Theorem 4.3 shows, this notion makes sense, especially for risk-averse agents:

Theorem 4.3 Let $F_A(\tilde{x})$ and $F_B(\tilde{x})$ be two cumulative probability distributions for random payoffs $\tilde{x}$ defined on $[a,b]$. Then, $F_A(\tilde{x})$ SSD $F_B(\tilde{x})$ if and only if $E_A U(\tilde{x}) \geq E_B U(\tilde{x})$ for all nondecreasing and concave U.

Proof See Laffont (1989), Section 2.5.

That is, all risk-averse agents will prefer the SSD asset. Of course, FSD implies SSD: if for two investments Z_1 and Z_2, Z_1 FSD Z_2, then it is also true that Z_1 SSD Z_2. But the converse is not true.[4]

4.7 Mean Preserving Spreads

Theorems 4.2 and 4.3 attempt to characterize a notion of "better/worse" that is relevant for probability distributions or random variables representing investment payoffs. But there are two aspects to such a comparison: the notion of "more or less risky" and the trade-off between risk and return. Let us now attempt to isolate the former effect by comparing only

[4] It turns out that SSD is a concept fundamental to understanding the benefits to diversification (see Chapter 6).

those probability distributions with identical means. We will then review Theorem 4.3 in the context of this latter requirement.

The concept of more or less risky is captured by the notion of a mean preserving spread. In our context, this notion can be informally stated as follows: Let $f_A(x)$ and $f_B(x)$ describe, respectively, the probability density functions on payoffs to assets A and B. If $f_B(x)$ can be obtained from $f_A(x)$ by removing some of the probability weight from the center of $f_A(x)$ and distributing it to the tails in such a way as to leave the mean unchanged, we say that $f_B(x)$ is related to $f_A(x)$ via a **mean preserving spread**. Figure 4.7 suggests what this notion would mean in the case of normal-type distributions with identical mean, yet different variances.

How can this notion be made both more intuitive and more precise? Consider a set of possible payoffs $\tilde{x}_A$ that are distributed according to $F_A()$. We further randomize these payoffs to obtain a new random variable $\tilde{x}_B$ according to

$$\tilde{x}_B = \tilde{x}_A + \tilde{z} \tag{4.10}$$

where, for any x_A value, $E(\tilde{z}) = \int z \, \mathrm{d}H_{x_A}(\tilde{z}) = 0$; in other words, we add some pure randomness to $\tilde{x}_A$. Let $F_B(\ )$ be the distribution function associated with $\tilde{x}_B$. We say that $F_B(\ )$ is a mean preserving spread of $F_A(\ )$. A simple example of this is as follows. Let

$$\tilde{x}_A = \begin{cases} 5 \text{ with probability } \frac{1}{2} \\ 2 \text{ with probability } \frac{1}{2} \end{cases}$$

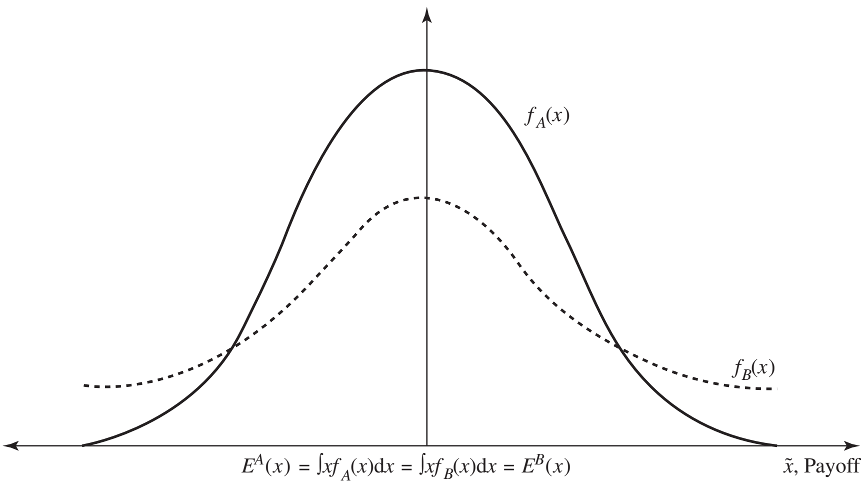

Figure 4.7
Mean preserving spread.

and suppose

$$\tilde{z} = \begin{cases} +1 \text{with probability } \dfrac{1}{2} \\ -1 \text{with probability } \dfrac{1}{2} \end{cases}$$

Then,

$$\tilde{x}_B = \begin{cases} 6 \text{ with probability } \dfrac{1}{4} \\ 4 \text{ with probability } \dfrac{1}{4} \\ 3 \text{ with probability } \dfrac{1}{4} \\ 1 \text{ with probability } \dfrac{1}{4} \end{cases}$$

Clearly, $E\tilde{x}_A = E\tilde{x}_B = 3.5$; we would also all agree that $F_B(\ )$ is intuitively riskier.

Our final theorem (Theorem 4.4) relates the sense of a mean preserving spread, as captured by Eq. (4.10), to our earlier results.

Theorem 4.4 Let $F_A(\ )$ and $F_B(\ )$ be two distribution functions defined on the same state space with identical means. The following statements are equivalent:

(i) $F_A(\tilde{x})$ SSD $F_B(\tilde{x})$
(ii) $F_B(\tilde{x})$ is a mean preserving spread of $F_A(\tilde{x})$ in the sense of Eq. (4.10).

Proof See Rothschild and Stiglitz (1970).

But what about distributions that are not stochastically dominant under either definition and for which the mean−variance criterion does not give a relative ranking? For example, consider (independent) investments 5 and 6 in Table 4.4.

In this case we are left to compare distributions by computing their respective expected utilities. That is to say, the ranking between these two investments is preference dependent. Some risk-averse individuals will prefer investment 5, while other risk-averse individuals will prefer investment 6. This is not bad. There remains a systematic basis of comparison. The task of the investment advisor is made more complex, however, as he will have to elicit more information on the preferences of his client if he wants to be in position to provide adequate advice.

Table 4.4: Two investments; No dominance

Investment 5		Investment 6	
Payoff	Probability	Payoff	Probability
1	0.25	3	0.33
7	0.5	5	0.33
12	0.25	8	0.34

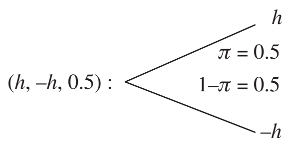

Figure 4.8
A fair lottery.

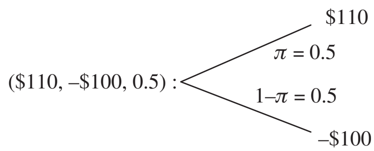

Figure 4.9
An often-refused lottery.

4.8 An Unsettling Observation About Expected Utility

Our definition of a risk-averse investor is one who is unwilling to accept the lottery depicted in Figure 4.8 at any wealth level. As noted in the discussion of loss aversion, however, there is also substantial experimental evidence (see, for example, Kahneman and Tversky, 1979) that risk-averse investors also refuse lotteries of the form depicted in Figure 4.9 at all wealth levels.

Building on this experimental observation, Rabin (2000) goes on to show analytically that the rejection of the lottery depicted in Figure 4.9 at all wealth levels implies unrealistically extreme risk aversion over large gambles. To illustrate, he shows that an investor with concave utility who rejects ($110, −$100, 0.5) for all wealth levels less than or equal to $300,000 will *necessarily* reject ($850,000,000,000, −$2000, 0.5) at a wealth level of $290,000 (Rabin, 2000, Table 2). On the face of it, such a rejection seems a bit absurd: even at a relatively modest wealth level of $290,000, most of us would be willing to hazard a loss of \$2000 against a 50% chance at an \$850 billion jackpot. Rabin's (2000) argument rests on the observation that if the investor rejects (\$110, −\$100,0.5) at all wealth levels, his marginal utility of wealth must be decreasing geometrically as his wealth level increases. A very large offsetting payment is thus required to induce the acceptance of a lottery with even modest losses as wealth increases.[5,6]

Rabin's observations suggest that if the VNM-expected utility framework is to be appropriate for evaluating large "gambles," investors must be essentially risk neutral over small ones. Yet, they are not, at least in experimental contexts. Apparently, investors' preference for small gambles are guided by some other preference structure such as loss aversion. It is at present unclear what this inconsistency means for important economic behavior such as the consumption-savings decision.

4.9 Applications: Leverage and Risk

Here we make more precise the notion of leverage first introduced in Section 2.5.3.

The word leverage is both a noun and a verb. As a noun, it simply means "borrowed money," obtained in exchange for issuing a debt security. As a verb (as in "to leverage" an asset), it means to acquire an asset using both the investor's own money (his equity contribution E) and borrowed funds (the debt contribution D). The degree of leverage applied to an asset (the relative extent of debt financing) is variously measured by the (D/E) ratio or the debt/value ratio (D/V), where V is the market value of all claims to the underlying asset's cash flows.

One of the hallmarks of the financial crisis was the enormous levels of indebtedness across many different institutions that it revealed. Whether on the part of financial institutions or individual households (and, in some cases, sovereign states), the years leading up to the financial crisis were often marked by dramatic increases in leverage. We want to understand why this was the case, and a natural way to initiate the exploration is first to consider the effects of leverage on an asset's risk and return.

Consider an investor with equity capital E, who is interested to acquire asset A, with intrinsic rate of return distribution $r_A(\tilde{\theta})$. An unlevered (all-equity) investment in A would naturally imply that the return on the investor's equity, $r_{\mathrm{u}}^{\mathrm{e}}(\tilde{\theta})$ would satisfy

$$r_{\mathrm{u}}^{\mathrm{e}}(\theta) = r_A(\theta) \text{ for every state } \theta$$

[5] Note that Rabin's (2000) results require only concavity and are not specific to any functional form. Accordingly, we invite the reader to confirm the rejection of (\$850 billion, −\$2000, 0.5) at a wealth level of \$290,000 for any of the utility functions considered in this chapter.

[6] See this chapter's web notes for the details of Rabin's (2000) argument.

If, however, the investor borrows an amount D at a constant rate r_D to help acquire the asset, then the return on his now "levered equity," $r^e_L(\tilde{\theta})$, is given by[7]

$$r^e_L(\theta) = r^e_u(\theta) + (D/E)(r^e_u(\theta) - r_d) \tag{4.11}$$

It follows from Eq. (4.11) that

$$E\tilde{r}^e_L = E\tilde{r}^e_u + (D/E)(E\tilde{r}^e_u - r_d), \text{ and} \tag{4.12}$$

$$\sigma_{\tilde{r}^e_L} = (1 + D/E)\sigma_{\tilde{r}^e_u} \tag{4.13}$$

where we assume the debt is risk free. Equations (4.12) and (4.13) make clear that an increase in leverage, as measured by an increase in (D/E) will increase both the expected return and risk of leveraged equity.

Intuitively, what mechanism lies behind Eqs. (4.12) and (4.13)? Leverage clearly only makes sense if $E\tilde{r}^e_u > r_d$: on average, the underlying asset's return must exceed the cost of borrowing. It follows that there must be a preponderance of states (or such states have a preponderance of the probability) for which $r^e_u(\theta) > r_d$. By Eq. (4.11), in those states $\hat{\theta}$ where $r^e_u(\hat{\theta}) > r_d$, the equity investor not only receives what the underlying asset itself pays, $r^e_u(\hat{\theta})$, on *his* invested capital, E, but also the return surplus the asset earns above the financing cost, $(r^e_u(\hat{\theta}) - r_d)$, on the debt financed portion of the asset's cost, D, in those states. This surplus is the source of the increase in expected returns for levered versus unlevered equity. In the event, the asset pays a return below the borrowing cost; however, the investor's return is lowered because the asset return shortfall relative to r_d must be made up from his own residual payment. Therein lies the source of the increase in risk that leverage promotes. Note that when we speak of the investor's return, we mean only the return on his own capital invested in the project.

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[7] The logic behind Eq. (4.12) is as follows: Consider two distinct financings for the purchase of asset A, one all equity, the other a mixture of debt and (leveraged) equity. If an investor owns all the unlevered equity (the first case financing) or all the debt and all the equity (the second case financing), in either case the return to his portfolio securities must be the same and coincide with the return on the underlying asset itself, $r_A(\tilde{\theta})$. Why? In either case, the investor would, in the aggregate, receive all the asset's free cash flows. Algebraically, this discussion can be summarized as:

$$r_A(\theta) = \underbrace{r^e_u(\theta)}_{\textit{all equity financing's return}} = \underbrace{\left(\frac{E}{D+E}\right) r^e_L(\theta) + \left(\frac{D}{D+E}\right) r_d}_{\textit{average return to levered equity and debt under partial debt financing}}$$

Rearranging the above equation to solve for $r^e_L(\theta)$ gives Eq. (4.11). (We have anticipated future chapters by relying on the fact that the return on a portfolio is the value-weighted average of the returns to its constituent assets.) Note that we first introduced relationship (4.11) in Chapter 2.

4.9.1 An Example

Let us imagine an investor who purchases a \$300,000 home in Phoenix, Arizona, with a \$20,000 down payment. The investor foresees $E\tilde{r}_A = 10\%$ (house price appreciation) with $\sigma_{\tilde{r}_A} = 5\%$. Net of assorted tax benefits, the mortgage cost, r_d, is 5%. In this case, $D/E =$ \$280,000/\$20,000 = 14, and $E\tilde{r}^{\mathrm{e}}_{\mathrm{L}} = E\tilde{r}^{\mathrm{e}}_{\mathrm{u}} + D/E(E\tilde{r}^{\mathrm{e}}_{\mathrm{u}} - r_d) = 0.10 + 14(.10 - 0.05) = 0.80$ or 80% relative to $E\tilde{r}_A$, a huge increase. Risk, however, also rises dramatically:

$$\sigma_{\tilde{r}^{\mathrm{e}}_{\mathrm{L}}} = \left(1 + \frac{D}{E}\right)\sigma_{\tilde{r}^{\mathrm{e}}_{\mathrm{u}}} = (1 + 14)(0.05) = 0.75 \text{ or } 75\%$$

Note that this investor has only a \$20,000 equity investment in the house with $\tilde{r}^{\mathrm{e}}_{\mathrm{L}}$ representing the return on that amount alone. Relative to an all-equity purchase of the house, note also that expected returns rose by a factor of 8 while risk increased by a factor of 15.

How does this enormous risk manifest itself? Consider a decline in house prices of 10% (not a large decline; in the recent housing crisis, Phoenix area homes declined in value by nearly 50% on average), and let $\hat{\theta}$ denotes the specific state in which the 10% decline is experienced; then

$$\begin{aligned} r^{\mathrm{e}}_{\mathrm{L}}(\hat{\theta}) &= r^{\mathrm{e}}_{\mathrm{u}}(\hat{\theta}) + \left(\frac{D}{E}\right)(r^{\mathrm{e}}_{\mathrm{u}}(\hat{\theta}) - r_d) \\ &= -0.10 + (14)(-0.10 - 0.05) \\ &= -2.20 \end{aligned}$$

which means that the investor "lost" 220% of his initial wealth. While an all-equity investor can lose at most his entire investment ($\tilde{r}^{\mathrm{e}}_{\mathrm{u}}(\theta) \geq -1.00$), a leveraged investor can lose even more because of the debt he must personally repay.[8] This occurs as follows:[9]

[8] This statement is country specific as regards the responsibility of the mortgagee, and applies more to Europe than to the United States. In the USA most home mortgages are 'non-recourse', meaning that the mortgagee can walk away from his mortgage and surrender his house to the lender without the lender being able to seize his other assets to cover any discrepancy between the home's market value and the loan's outstanding balance. In this sense the purchase of a home is closely similar to an equity investment in much of the USA (particulars are governed by state law).

[9] The situation may be analyzed alternatively by observing that the investor effectively owns the following portfolio P:

$$P = \{1 \text{ Phoenix house, } 1 \text{ loan}\} \text{ where } w_{\text{house}} = \frac{\$300,000}{\$20,000} = 15 \text{ and } w_{\text{loan}} = \frac{-\$280,000}{\$20,000} = -14; \text{ thus}$$

$$\begin{aligned} r_P(\hat{\theta}) &= w_{house} r_{house}(\hat{\theta}) + w_{loan} r_d \\ &= 15(-.10) - 14(.05) \\ &= -2.20 = r^e_L(\hat{\theta}) \end{aligned}$$

If the event $\hat{\theta}$ is realized, this mortgage is said to go "underwater."

$$\begin{aligned}\text{Loss on the value of the house: } -0.10\ (\$300,000) &= -\$30,000\\ \text{Mortgage interest payment: } -0.05\ (\$280,000) &= \underline{-\$14,000}\\ \text{Total losses/outflows} &= -\$44,000\end{aligned}$$

$$1 + r_{\mathrm{L}}^{\mathrm{e}}(\hat{\theta}) = \frac{-\$44,000}{\$20,000} = -2.2, \quad \text{or} \quad -220\%$$

4.9.2 Is Leverage a Good Thing?

From our discussion at the start of Chapter 3, we know it is generally impossible to rank two assets directly, one of which has both a higher mean and a higher standard deviation of returns. The leveraged asset return has this feature relative to its unleveraged counterpart. But what about the Sharpe ratio? We observe

$$\frac{E\tilde{r}_{\mathrm{L}}^{\mathrm{e}} - r_d}{\sigma_{\tilde{r}_{\mathrm{L}}^{\mathrm{e}}}} = \frac{E\tilde{r}_{\mathrm{u}}^{\mathrm{e}} + \frac{D}{E}(E\tilde{r}_{\mathrm{u}}^{\mathrm{e}} - r_d) - r_d}{\left(1 + \frac{D}{E}\right)\sigma_{\tilde{r}_{\mathrm{u}}^{\mathrm{e}}}} \tag{4.14}$$

$$= \frac{\left(1 + \frac{D}{E}\right)E\tilde{r}_{\mathrm{u}}^{\mathrm{e}} - \left(1 + \frac{D}{E}\right)r_d}{\left(1 + \frac{D}{E}\right)\sigma_{\tilde{r}_{\mathrm{u}}^{\mathrm{e}}}} = \frac{E\tilde{r}_{\mathrm{u}}^{\mathrm{e}} - r_d}{\sigma_{\tilde{r}_{\mathrm{u}}^{\mathrm{e}}}} \tag{4.15}$$

i.e., leverage has no effect on the Sharpe ratio.[10]

Since a higher Sharpe ratio is viewed as indicative of superior investment performance, the fact that an increase in debt leaves this fundamental measure unchanged must be viewed as a discouragement to its use. Yet, in the years immediately preceding the 2008 financial crisis, household, firm, and government debt levels all rose to record levels in both the United States and certain European countries (e.g., Ireland, Spain). At the time of its collapse (September 15, 2008) Lehman Brothers Corporation had a (D/E) of roughly 50. What was going on?

There are, of course, various tax incentives for debt; in particular, the home mortgage interest deduction in the US. Homeowners may also be willing to shoulder the risk of debt financing because they have a strong preference for home ownership, and debt financing makes this preference more immediately accessible. Firms also benefit from the tax deductability of interest payments. Prior to the financial crisis, home mortgages in the United States, however, usually began with a (D/E) = 4 (a 20% down payment). Yet in the last years of the housing boom, mortgages were being written with no down payment at all ($D/E = \infty$). Since tax incentives to debt financing have been in place for a long time, other incentives must have been at work.

[10] The ratio $E\tilde{r}_L^e/\sigma_{\tilde{r}_L^e}$ actually declines as (D/E) increases.

To propose one such possible incentive, it is necessary to be more precise regarding the effects of leverage on equity return distributions. This is the focus of Theorem 4.4. It basically asserts that in the favorable states of nature, $\{\theta\colon\ \tilde{r}_{\mathrm{u}}^{\mathrm{e}}(\theta) > r_d\}$, the leveraged return distribution FSD its unlevered counterpart (and thus also more highly leveraged return distributions will FSD less highly leveraged ones on the set of favorable states).

Theorem 4.5 Let Θ denote the underlying space of events, and consider the following partition of Θ:

$$\mathrm{A} = \{\theta \in \Theta : r_{\mathrm{u}}^{\mathrm{e}}(\theta) > r_d\}, \text{ and } \Theta - A = \{\theta \in \Theta : r_{\mathrm{u}}^{\mathrm{e}}(\theta) \le r_d\}$$

Furthermore, let $F_{L/B}$ and $F_{u/B}$ denote the conditional distribution functions restricted to some set $B \subseteq \Theta$ for $\tilde{r}_{\mathrm{L}}^{\mathrm{e}}(\theta)$ and $\tilde{r}_{\mathrm{u}}^{\mathrm{e}}(\theta)$ respectively. Then,

(i) $F_{L/A}(\tilde{r}_{\mathrm{L}}^{\mathrm{e}}(\theta))$ FSD $F_{u/A}(\tilde{r}_{\mathrm{u}}^{\mathrm{e}}(\theta))$
(ii) $F_{u/\Theta-A}(\tilde{r}_{\mathrm{u}}^{\mathrm{e}}(\theta))$ FSD $F_{L/\Theta-A}(\tilde{r}_{\mathrm{L}}^{\mathrm{e}}(\theta))$

Proof We demonstrate only (i), as (ii) follows similarly.

$$\begin{aligned}
F_{L/A}(\hat{r}) &= \frac{\mathrm{Prob}(\tilde{r}_{\mathrm{L}}^{\mathrm{e}}(\theta) \le \hat{r} : \theta \in A)}{\mathrm{Prob}(\theta \in A)} \\
&= \frac{\mathrm{Prob}\left(\tilde{r}_{\mathrm{u}}^{\mathrm{e}}(\theta) + \dfrac{D}{E}(\tilde{r}_{\mathrm{u}}^{\mathrm{e}}(\theta) - r_d) \le \hat{r} : \theta \in A\right)}{\mathrm{Prob}(\theta \in A)} \\
&= \frac{\mathrm{Prob}\left(\tilde{r}_{\mathrm{u}}^{\mathrm{e}}(\theta) \le \dfrac{\hat{r} + \dfrac{D}{E} r_d}{\left(1 + \dfrac{D}{E}\right)} : \theta \in A\right)}{\mathrm{Prob}(\theta \in A)} \\
&< \frac{\mathrm{Prob}\left(\tilde{r}_{\mathrm{u}}^{\mathrm{e}}(\theta) \le \dfrac{\hat{r} + \dfrac{D}{E} \hat{r}}{\left(1 + \dfrac{D}{E}\right)} : \theta \in A\right)}{\mathrm{Prob}(\theta \in A)} \\
&< \frac{\mathrm{Prob}(\tilde{r}_{\mathrm{u}}^{\mathrm{e}}(\theta) \le \hat{r} : \theta \in A)}{\mathrm{Prob}(\theta \in A)} \\
&= F_{u/A}(\hat{r})
\end{aligned}$$

A simple paraphrase of Theorem 4.5 is as follows: in high return states, $\{\theta : \tilde{r}_{\mathrm{u}}^{\mathrm{e}}(\theta) > r_d\}$, leverage makes them even higher; in low return states $\{\theta : \tilde{r}_{\mathrm{u}}^{\mathrm{e}}(\theta) < r_d\}$ it makes them even lower. Accordingly, agents will desire high leverage if they can somehow ignore the low return states ($\{\theta : \tilde{r}_{\mathrm{u}}^{\mathrm{e}}(\theta) < r_d\}$), or if a belief bias leads them to underestimate the low return state probabilities. Both investors and their managers may be guilty of overoptimistic beliefs, but limited liability managers, whose compensation is linked principally to payoffs only in the favorable states, are uniquely positioned to ignore the low return states altogether.

4.9.3 An Application to Executive Compensation

Certain private equity firms employ high leverage ratios in their managed portfolios, yet the portfolio managers of these firms are surely aware of high risk and unchanged Sharpe ratio that such leverage entails. Accordingly, the popular press has speculated that high leverage choices may simply represent a response of the managers of these firms to the way they are paid. For example, the "2 and 20" rule is pervasive: managers receive 2% of the amount invested as a one-time payment, plus 20% of net positive returns above some benchmark, computed annually, for the contracting period. If the investment returns fall short of the benchmark, the managers receive nothing. But, neither are they assessed any penalty payment: it is a limited liability contract.

In the simplified setting of the present discussion, the limited liability, incentive portion of the contract may be expressed as

$$\text{Incentive Comp}(\theta) = \begin{cases} \gamma_1 Y(\tilde{r}_A(\theta) - r_d); & r_A(\theta) > r_d \\ 0; & \text{otherwise} \end{cases} \tag{4.16}$$

where we choose, for simplicity, r_d as the benchmark. The quantity Y is the amount invested by the client, and $\tilde{r}_A(\theta)$ the uncertain return on some portfolio A of assets selected by the manager on the investor's behalf with, say, $\gamma_1 = 0.20$.

Let $r_{\mathrm{L}}^{\mathrm{e}}(\theta)$ represent the return on a leveraged portfolio of these assets with the degree of leverage measured by the (D/E) ratio. In a like fashion we naturally identify $r_A(\theta) = r_{\mathrm{u}}^{\mathrm{e}}(\theta)$. Define the set A' by

$$\begin{aligned} A' &= \{\theta : \theta \in \Theta : r_A(\theta) > r_d\} \\ &= \{\theta : \theta \in \Theta : \text{Incentive Comp}(\theta) > 0\} \end{aligned}$$

The following corollaries to Theorem 4.4 point in the direction of the likely behavior of a self-interested manager with compensation contract Eq. (4.16).

Corollary 4.5.1 Let $F^C_{L/A'}$ and $F^C_{u/A'}$ denote the distribution functions of the manager's incentive compensation restricted to the set A' when, respectively, he leverages the underlying portfolio of assets ($D/E > 0$) and when he does not ($D/E = 0$). Then,

(i) $F^C_{L/A'}$ FSD $F^C_{u/A'}$ for any ($D/E > 0$).

Since the manager's incentive compensation is zero on $\Theta - A'$, we can also claim

(ii) $F^C_{L/\Theta}$ FSD $F^C_{u/\Theta}$ for any ($D/E > 0$).[11]

Proof Adaptation of the proof of Theorem 4.5.

Corollary 4.5.2 Every manager with an increasing and concave VNM-expected utility representation and a limited liability incentive contract of the form (4.16) will employ leverage ($D/E > 0$).

Proof By Theorem 4.3 and Corollary 4.5.1 since FSD implies SSD.

The results in Corollaries 4.5.1–4.5.2 are really nothing more than the colloquial observation that asset managers with no "skin in the game" have little regard for (downside) risk.[12] These observations likely explain many instances where leverage ratios grew rapidly in the precrisis years when credit was readily available to "grow" the leverage.

4.10 Conclusions

The main topic of this chapter was the VNM-expected utility representation specialized to admit risk aversion. Two measures of the degree of risk aversion were presented. Both are functions of an investor's current level of wealth, and, as such, we would expect them to change as wealth changes. Is there any systematic relationship between $R_A(Y)$, $R_R(Y)$, and Y which it is reasonable to assume?

In order to answer that question, we must move away from the somewhat artificial setting of this chapter. As we will see in Chapter 5, systematic relationships between wealth and the measures of absolute and relative risk aversion are closely related to investors' portfolio behavior.

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11 This argument is largely unchanged if the manager also receives an additional fixed salary payment.

12 The theory of optimal contracting between investors (principals) and financial advisors (their agents) is not yet well understood and is not a general topic we consider. See Stracca (2006) for an excellent survey. It is known, however, that simple linear contracts such as Eq. (4.16) are not optimal for delegated portfolio managers. See Admati and Pfleiderer (1997).

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Appendix: Proof of Theorem 4.2

$\Rightarrow$ There is no loss in generality in assuming $U(\ )$ is differentiable, with $U'(\ ) > 0$.

Suppose $F_A(x)$ FSD $F_B(x)$, and let $U(\ )$ be a utility function defined on $[a,b]$ for which $U'(\ ) > 0$. We need to show that

$$E_A U(\tilde{x}) = \int_a^b U(\tilde{x}) \mathrm{d}F_A(\tilde{x}) > \int_a^b U(\tilde{x}) \mathrm{d}F_B(\tilde{x}) = E_B U(\tilde{x})$$

This result follows from integration by parts (recall the relationship $\int_a^b u\,\mathrm{d}v = uv\Big|_a^b - \int_a^b v\,\mathrm{d}u$).

$$\begin{aligned}
&\int_a^b U(\tilde{x})\mathrm{d}F_A(\tilde{x}) - \int_a^b U(\tilde{x})\mathrm{d}F_B(\tilde{x}) \\
&= U(b)F_A(b) - U(a)F_A(a) - \int_a^b F_A(\tilde{x})U'(\tilde{x})\mathrm{d}\tilde{x} \\
&- \{U(b)F_B(b) - U(a)F_B(a) - \int_a^b F_B(\tilde{x})U'(\tilde{x})\mathrm{d}\tilde{x}\} \\
&= -\int_a^b F_A(\tilde{x})U'(\tilde{x})\mathrm{d}\tilde{x} + \int_a^b F_B(\tilde{x})U'(\tilde{x})\mathrm{d}\tilde{x}, \\
&(\text{since } \ F_A(b) = F_B(b) = 1, \ \text{ and } \ F_A(a) = F_B(a) = 0) \\
&= \int_a^b [F_B(\tilde{x}) - F_A(\tilde{x})]U'(\tilde{x})\mathrm{d}\tilde{x} \geq 0
\end{aligned}$$

The desired inequality follows since, by the definition of FSD and the assumption that the marginal utility is always positive, both terms within the integral are positive. If there is some subset $(c,a) \subset [a,b]$ on which $F_A(x) > F_B(x)$, the final inequality is strict.

$\Leftarrow$ Proof by contradiction. If $F_A(\tilde{x}) \leq F_B(\tilde{x})$ is false, then there must exist an $\overline{x} \in [a, b]$ for which $F_A(\overline{x}) > F_B(\overline{x})$. Define the following nondecreasing function $\hat{U}(x)$ by

$$\hat{U}(x) = \begin{cases} 1 & \text{for } b \geq x > \tilde{x} \\ 0 & \text{for } a \ \ \leq x < \tilde{x} \end{cases}$$

We will use integration by parts again to obtain the required contradiction.

$$\begin{aligned}
&\int_a^b \hat{U}(\tilde{x})\mathrm{d}F_A(\tilde{x}) - \int_a^b \hat{U}(\tilde{x})\mathrm{d}F_B(\tilde{x}) \\
&= \int_a^b \hat{U}(\tilde{x})[\mathrm{d}F_A(\tilde{x}) - \mathrm{d}F_B(\tilde{x})] \\
&= \int_{\overline{x}}^b 1 \ \ [\mathrm{d}F_A(\tilde{x}) - \mathrm{d}F_B(\tilde{x})] \\
&= F_A(b) - F_B(b) - [F_A(\overline{x}) - F_B(\overline{x})] - \int_{\overline{x}}^b [F_A(\tilde{x}) - F_B(\tilde{x})](0)\mathrm{d}\tilde{x} \\
&= F_B(\overline{x}) - F_A(\overline{x}) < 0
\end{aligned}$$

Thus we have exhibited an increasing function $\hat{U}(x)$ for which $\int_a^b \hat{U}(\tilde{x})\mathrm{d}F_A(\tilde{x}) < \int_a^b U(\tilde{x})\mathrm{d}F_B(\tilde{x})$, a contradiction.