Lecture notes: The Capital Asset Pricing Model (CAPM)

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Learning outcomes: students should

- understand the difference between mean-variance analysis and the CAPM, in particular regarding the assumptions made in the latter;
- be able to solve for the required return of common stock using the capital asset pricing model (CAPM);
- have the ability to interpret the meaning of a stock's beta;
- be able to appraise critically the CAPM.

Reading:

Bodie, Kane and Marcus (2008) (Chapter 9)

Supplementary reading:

Grinblatt and Titman (2002) (Chapter 5), Levy and Post (2005) (Chapter 10)

Sharpe (1964), Black (1972)

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1 Risk-return paradox?

From the notes on Mean Variance Analysis we have learned that riskier assets should tend to have higher returns. Using historical data, we arrive at the following descriptive statistics:[1]

	annual return	standard deviation
100-stock portfolio	10.9 percent	4.45 percent
XYZ PLC	5.4 percent	7.26 percent.

The 100-stock portfolio offers a higher expected return than XYZ PLC even though it has a lower standard deviation.

Does this contradict market efficiency? Shouldn't prices adjust to move returns to adequately reflect the risk inherent in the asset?

An individual asset's contributions to portfolio risk

What is of interest to an investor is the variance of his or her portfolio in the end. As we have already seen, adding an individual asset to a portfolio can help in reducing the portfolio's variance. Therefore, to understand what is going on in the example, it is useful to first understand how an individual asset affects a portfolio's risk, as measured by portfolio variance. From the last lectures we know that

$$Var[\tilde{r}_P] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i\, w_j\, Cov[\tilde{r}_i, \tilde{r}_j]. \tag{1}$$

Thus the contribution of an asset i, say GM, to portfolio risk is

$$w_i^2\, Var[\tilde{r}_i] + w_i \left[\sum_{j\neq i} w_j\, Cov[\tilde{r}_i, \tilde{r}_j]\right].$$

What becomes apparent is that an asset's variance is not necessarily the most important contributor to portfolio variance since it appears only in one term, whereas there are n-1 covariance terms. The value of an asset to a portfolio lies in its "usefulness" for reducing portfolio variance, which is mainly driven through covariances with other assets, rather than in the individual asset's own variance. This leads us in the right direction. However, to properly address the "paradox", we need to have a model of how assets are priced in a market.

[1] This example is borrowed from Copeland, Weston and Shastri (2004), Chapters 5 and 6.

2 The Capital Asset Pricing Model (CAPM)

Jack Treynor, William Sharpe, John Lintner, and Jan Mossin contributed to developing an equilibrium theory of asset pricing. For his work on asset pricing Sharpe received the 1990 Nobel prize[2]

2.1 Assumptions of the basic version of the CAPM

Assumptions from Markowitz' Portfolio Selection theory (see lecture notes Mean Variance Analysis) plus

- homogeneous beliefs

  → all investors analyze securities in the same way and share the same views on the statistical properties of asset returns.

- all investors can borrow and lend at the risk-free rate.

2.2 Derivation of the CAPM

(This section draws heavily from Copeland et al. (2004), Chapter 6.C.)

Efficiency of the market portfolio

Suppose that asset prices are at some fixed level. Since investors take prices as given and have homogeneous beliefs they will all carry out the same mean variance analysis and thus all perceive the same opportunity set of risky assets. Thus, all investors will hold efficient portfolios. The ***market portfolio*** is just the sum of all individual holdings. Since these individual holdings are all efficient, the market portfolio is also an efficient portfolio.

Market clearing

For a market equilibrium to exist, asset prices must adjust until the market clears, i.e. the demand for each asset i in the market has to equal the supply of this asset i at the equilibrium price for that asset. Thus, in equilibrium the market portfolio will be made up of all marketable assets proportional to their market value (which equals the price of a share times the number of shares). That is, equilibrium asset weights in the market portfolio are:

$$w_i = \frac{\text{market value of asset } i}{\text{market value of all assets}}. \tag{2}$$

[2] The prize was shared with Harry Markowitz, whose contribution we already saw, and Merton Miller, whose theory we will see later in the course.

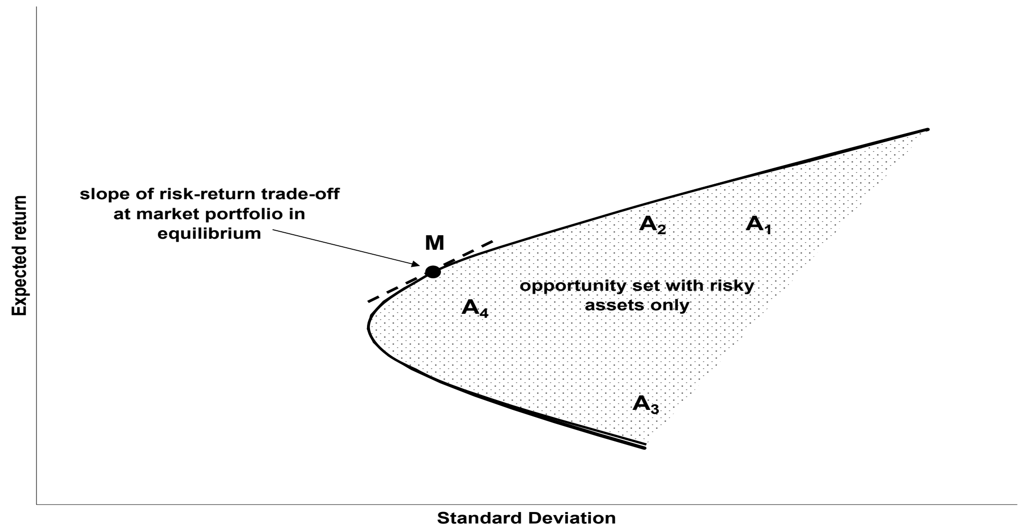

Figure 1: Slope of the equilibrium risk-return trade-off at point M

First equilibrium condition

Since investors have access to the risk-free asset, they will combine a long (or short position) in the risk-free asset with holdings in risky securities. They have homogeneous beliefs and thus all derive the same tangency portfolio, which therefore is the market portfolio (see previous paragraph). This again gives rise to a capital allocation line, which in the case of the CAPM is called the ***Capital Market Line (CML)***. We know that the slope of this line is given by:

$$\frac{E[\tilde{r}_M] - r_f}{\sigma_M}. \tag{3}$$

This provides us with our first equilibrium condition.

Second equilibrium condition

We know that investors carry out mean-variance analysis and in the end combine the risk-free asset with the tangency portfolio composed of risky assets. Since all investors carry out the same mean-variance analysis they arrive at the same tangency portfolio. As we have said, market clearing in equilibrium implies that this tangency portfolio is the market portfolio. This fact we can use to derive another condition about the risk-return trade-off that the market portfolio must offer in equilibrium – yielding our second equilibrium condition.

The thought experiment is as follows: suppose all investors use the market portfolio as the portfolio of risky assets (to be then combined with the risk-free asset) but one investor chooses the following risky portfolio (to be then combined with the risk-free asset): weight $(1-a)$ on the market portfolio and weight a on some risky asset i, where a can be either positive or negative. We then look how changing a affects the risky portfolio's mean and variance. This gives us the risk-return trade-off for any value a. Then we use the fact that in equilibrium $a = 0$ (i.e., the investor we are looking at will in equilibrium optimally choose to hold as portfolio of risky assets exactly the market portfolio given prices which lead to market clearing) to pin down the exact risk-return trade-off that must hold in equilibrium: otherwise, there would be excess demand or supply for asset i as we have already discussed.

Consider a portfolio of which proportion a is invested in risky asset i and $(1-a)$ in the market portfolio. Then we have:

$$E[\tilde{r}_P] = a\, E[\tilde{r}_i] + (1-a)\, E[\tilde{r}_M], \tag{4}$$

$$\sigma[\tilde{r}_P] = \left[a^2\, \sigma_i^2 + (1-a)^2\, \sigma_M^2 + 2\, a(1-a)\, \sigma_{iM}\right]^{1/2}. \tag{5}$$

To see how the expected return and standard deviation change if we alter the proportion a invested in asset i, we take the derivative with respect to a:

$$\frac{\partial\, E[\tilde{r}_P]}{\partial\, a} = E[\tilde{r}_i] - E[\tilde{r}_M], \tag{6}$$

$$\begin{aligned} \frac{\partial\, \sigma[\tilde{r}_P]}{\partial\, a} &= \frac{1}{2}\left[a^2\, \sigma_i^2 + (1-a)^2\, \sigma_M^2 + 2\, a(1-a)\, \sigma_{iM}\right]^{-1/2} \\ &\times \left[2\, a\, \sigma_i^2 - 2\, \sigma_M^2 + 2\, a\, \sigma_M^2 + 2\, \sigma_{iM} - 4\, a\, \sigma_{iM}\right]. \end{aligned} \tag{7}$$

Sharpe and Treynor's insight: in equilibrium, the market portfolio is already invested in asset i with a weight of w_i as defined above. Note that a situation where one investor decides to invest a proportion $a \neq 0$ cannot be an equilibrium since there would be excess demand for asset i or excess supply of asset i. Prices would adjust until there is zero excess demand and all assets are held by some investors. Thus, evaluating the derivatives in equations (6) and (7) at $a = 0$ allows us to determine the equilibrium price relationship in the market portfolio M in the mean standard deviation plane.

$$\left.\frac{\partial\, E[\tilde{r}_P]}{\partial\, a}\right|_{a=0} = E[\tilde{r}_i] - E[\tilde{r}_M], \tag{8}$$

$$\left.\frac{\partial\, \sigma[\tilde{r}_P]}{\partial\, a}\right|_{a=0} = \frac{1}{2}\left[\sigma_M^2\right]^{-1/2}\left[-2\, \sigma_M^2 + 2\, \sigma_{iM}\right] = \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}. \tag{9}$$

In the mean standard deviation diagram, the slope (or the risk-return trade-off) in point M, representing the market portfolio, becomes (see figure 1):

$$\left.\frac{d\, E[\tilde{r}]}{d\, \sigma[\tilde{r}]}\right|_{\text{point M}} = \left.\frac{\partial\, E[\tilde{r}]/\partial\, a}{\partial\, \sigma[\tilde{r}]/\partial\, a}\right|_{a=0} = \frac{E[\tilde{r}_i] - E[\tilde{r}_M]}{(\sigma_{iM} - \sigma_M^2)/\sigma_M}. \tag{10}$$

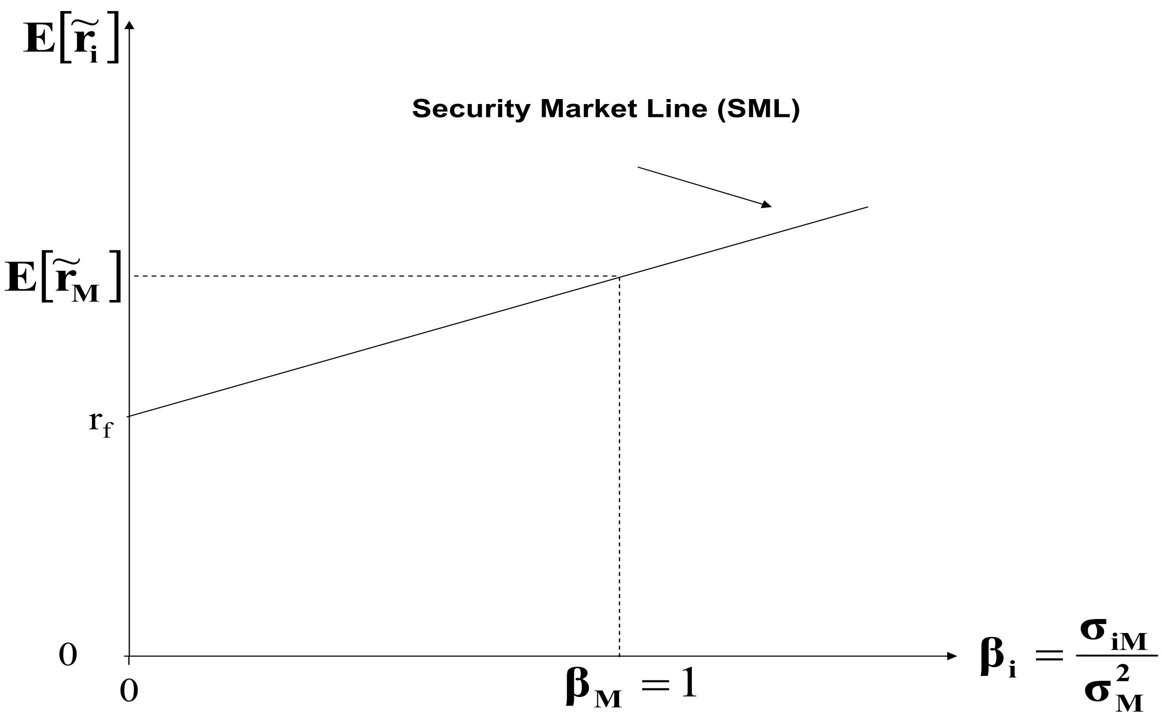

Figure 2: Security market line

But since the market portfolio must also be the investor's tangency portfolio, equation (10) must also give the slope of the CML, which provides us with a second equilibrium condition.

Risk-return trade-off for the market portfolio

Using the above results, we can pin down the risk-return trade-off for the market portfolio using the two equilibrium conditions. Combining (3) and (10) one obtains an expression for the equilibrium pricing of risky assets:

$$\begin{aligned} \frac{E[\tilde{r}_M] - r_f}{\sigma_M} &= \frac{E[\tilde{r}_i] - E[\tilde{r}_M]}{(\sigma_{iM} - \sigma_M^2)/\sigma_M} \\ \Leftrightarrow \quad E[\tilde{r}_i] &= r_f + [E[\tilde{r}_M] - r_f] \frac{\sigma_{iM}}{\sigma_M^2}. \end{aligned} \tag{11}$$

Equation (11) is known as the ***Security Market Line (SML)*** and is the key result of the Capital Asset Pricing Model (CAPM)(see figure 2).

2.3 The Security Market Line (SML)

The SML gives the required rate of return, $E[\tilde{r}_i]$, on an asset, which consists of the risk-free rate and a risk premium. This premium is computed by multiplying the expected excess return of the market portfolio, $E[\tilde{r}_M] - r_f$, with the "quantity of risk", which is measured by the asset's beta: $\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$. Thus, the risk that is priced in the market is an individual asset's beta and **not** its standard deviation!

Note that the market portfolio has $\beta_M = 1$ since $\sigma_{MM} = \sigma_M^2$.

2.4 The "risk-return paradox" revisited

In our first discussion of the "risk-return" paradox in the beginning of this lecture we argued that an investor cares about an individual asset's contribution to portfolio variance. Now we have learned that the relevant measure of risk for asset pricing in the CAPM is an asset's beta. How do these two observations square?

We had seen that asset i's contribution to portfolio variance is given by (derive this on your own)

$$\begin{aligned} & w_i^2\, Var[\tilde{r}_i] + w_i \left[\sum_{j \neq i} w_j\, Cov[\tilde{r}_i, \tilde{r}_j] \right] \\ = \quad & w_i \left[\sum_{j=1}^{n} w_j\, Cov[\tilde{r}_i, \tilde{r}_j] \right] \\ = \quad & w_i\, Cov \left[\tilde{r}_i, \sum_{j=1}^{n} w_j\, \tilde{r}_j \right]. \end{aligned} \tag{12}$$

In a market equilibrium, this last expression in the covariance term is just the return on the market portfolio, $\tilde{r}_M = \sum_{j=1}^{n} w_j\, \tilde{r}_j$. Thus, asset i's contribution to the variance of the market portfolio is $w_i\, Cov[\tilde{r}_i, \tilde{r}_M]$. If we relate this to the market portfolio's variance, then we get $w_i\, \frac{Cov[\tilde{r}_i, \tilde{r}_M]}{\sigma_M^2} = w_i\, \beta_i$. Expanding the formula for the market portfolio's variance, it can easily be seen that the market portfolio's beta (which equals one!) can be written as $\beta_M = \sum_{i=1}^{n} w_i\, \beta_i$ (check this on your own). Thus, our expression above for the contribution of asset i to the market portfolio's variance, normalized by the overall variance of the market portfolio, is equal to the contribution of asset i to the market portfolio's beta.

Finally, to resolve the puzzle let's look at the betas for the two investments:

	annual return	standard deviation	beta
100-stock portfolio	10.9 percent	4.45 percent	1.11
XYZ PLC	5.4 percent	7.26 percent	0.71

Judging by the correct measure of risk, the asset's beta, XYZ is *less* risky than the 100-stock portfolio. This resolves the "paradox".

Two conclusions arise:

- The beta of an asset is the appropriate measure of its risk because its beta is proportional to the asset's contribution to the market portfolio's variance.

- Betas are additive (as we saw above): a portfolio's beta is simply the weighted sum of all assets' betas. Thus, all that is needed to measure the systematic risk of portfolios is knowledge of the component betas. It is not necessary to derive the efficient set, which would be required if we only accepted the assumptions of mean variance analysis.

2.5 Comparison of CML and SML

The Capital Market Line (CML) gives the expected rates of return for *efficient portfolios* as a function of the standard deviation of returns.

The Security Market Line (SML) is valid both for individual assets and efficient portfolios. It gives the required rate of return on an asset as a function of its beta. In practice, this relation can be used as a benchmark to determine the "fair" expected return of a risky asset. The difference between this fair and the actually expected returns is captured by a stock's alpha, α_i. We will return to this a bit later when we discuss how to translate the CAPM into an empirical specification.

2.6 The equilibrium excess return

Now that we have learned that in equilibrium only systematic risk, i.e. the market portfolio's variance, is priced we can turn to the question of what the price for risk will actually be.

We start off by considering an individual investor's asset allocation decision. We know that his or her portfolio will be a combination of the risk-free asset and the market portfolio only. Denote by y_i the proportion of the market portfolio in the investor's overall portfolio and let the investor have risk-aversion coefficient A_i. Investor i determines the optimal weight y_i by solving:

$$\begin{aligned}\max_{y_i} U(\tilde{r}_P(y_i)) &= E[\tilde{r}_P(y_i)] - \frac{A_i}{2}\,\sigma_P(y_i)^2 \\ &= r_f + y_i\ (E[\tilde{r}_M] - r_f) - \frac{A_i}{2}\,{y_i}^2\,\sigma_M^2.\end{aligned}$$

The first-order condition for this problem is (check that the second-order condition also holds!):

$$E[\tilde{r}_M] - r_f - A_i\,y_i\,\sigma_M^2 \overset{!}{=} 0$$

$$\Leftrightarrow \quad y_i = \frac{E[\tilde{r}_M] - r_f}{A_i\,\sigma_M^2}.$$

A useful way of rewriting this is in terms of an investor's ***risk tolerance***, τ_i, defined as the ***inverse*** of an investor's risk aversion coefficient:[3]

$$\textbf{(risk tolerance)} \qquad \tau_i = \frac{1}{A_i}.$$

That is,

$$y_i = \frac{E[\tilde{r}_M] - r_f}{\sigma_M^2}\,\tau_i. \tag{13}$$

Thus, an investor's optimal position in the (risky) market portfolio is inversely proportional to the market (or systematic) risk (as measured by the variance of the market portfolio), and proportional to the the investor's risk tolerance and the market portfolio's expected excess return.

In equilibrium the asset market must clear, i.e. the average position in the market portfolio must be 100 percent. Thus,

$$\text{(equilibrium condition)} \qquad \bar{y} \equiv \frac{1}{n}\left(\sum_{i=1}^{n} y_i\right) = 1.$$

Combining this with (13) we get[4]

$$\begin{aligned} \frac{1}{n}\left(\sum_{i=1}^{n} \frac{E[\tilde{r}_M] - r_f}{\sigma_M^2}\,\tau_i\right) &= 1 \\ \Leftrightarrow \quad E[\tilde{r}_M] - r_f &= \frac{\sigma_M^2}{\bar{\tau}}. \end{aligned} \tag{14}$$

What equation (14) tells us is that the equilibrium excess return of the market portfolio – the "price of risk" – is related to the market portfolio's variance by the average investor risk tolerance. The higher the average risk tolerance, the lower the price of risk.

Question:

If all investors are risk-neutral, what will the price of risk be?

Equation (14) then no longer is defined since $\bar{\tau}$ is not defined when $\bar{A} = 0$. However, we can answer the question by looking at the limit of (14) when $\bar{A} \to 0 \Leftrightarrow \bar{\tau} \to \infty$. Then the price of risk goes to zero, in line with what intuition suggests: risk-neutral investors only care about the expected return on an investment.

2.7 An empirical specification of the basic CAPM

If one translates the relation in equation (11) into an empirical model, the excess return of an asset i in period $t = 1, ..., T$, written as $\tilde{r}_{i,t} - r_f$, is given by a regression equation with

[3]Note that a very risk-averse investor, with high coefficient of risk aversion A_i, has a low risk tolerance. Lower coefficients of risk aversion translate into higher risk-tolerance.

[4]In Bodie et al. (2008), p. 298 (6th edition: p.285) the formula (9.2) is wrong if we do not have $A_1 = A_2 = ... = A_n = \bar{A}$ since then $\frac{1}{n}\left(\sum_{i=1}^{n} 1/A_i\right) \neq 1/\bar{A}$.

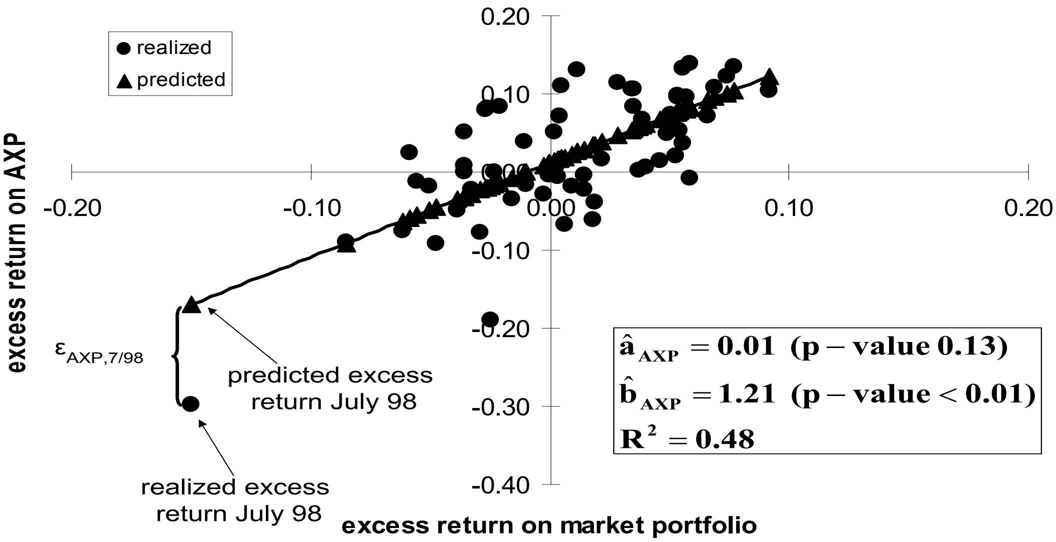

Figure 3: Estimated CAPM - American Express (AXP)

intercept a_i, the market portfolio's excess return, $\tilde{r}_{M,t} - r_f$ and a noise term $\tilde{\epsilon}_{i,t}$:[5]

$$\tilde{r}_{i,t} - r_f = a_i + b_i \left[\tilde{r}_{M,t} - r_f\right] + \tilde{\epsilon}_{i,t}. \tag{15}$$

If one estimates this model, the coefficient $\hat{b}_i$ provides an estimate of β_i in the CAPM. Figure 3 illustrates this for the excess returns on American Express (AXP).
The estimated beta for AXP is 1.21 (significant). If the CAPM holds, the estimated coefficient for the intercept should be zero, i.e. $\hat{a}_i = 0$. This hypothesis cannot be rejected for AXP stock at the 10-percent significance level. We will discuss such empirical tests in more detail later in the course.

2.8 Systematic and unsystematic risk

The CAPM tells us that in a market equilibrium the expected rate of return of an asset is given exactly by the Security Market Line. Even if an investment does not lie on the efficient set, this relation will hold. An investor is only concerned about the covariance of the asset with the market portfolio, which is that part of the asset's variance that cannot be shed through

[5] In the regression it is assumed that $E[\tilde{\epsilon}_{i,t}] = 0$, $Var[\tilde{\epsilon}_{i,t}] = \sigma^2_\epsilon$, and that $\tilde{\epsilon}_{i,t}$ is independent of the market return, $\tilde{r}_{M,t}$, i.e. $Cov[\tilde{r}_{M,t}, \tilde{\epsilon}_{i,t}] = 0$, and of other asset's noise terms.

diversification (***undiversifiable risk***). Thus, an asset's total risk can be decomposed into two parts:

total risk of asset i=systematic risk of asset i + unsystematic risk of asset i.

Using the empirical specification of the CAPM, the variance of returns on asset i can be written as (a_i is a constant!)

$$\underbrace{\sigma_i^2}_{\text{total risk}} = \underbrace{b_i^2\,\sigma_M^2}_{\text{systematic risk}} + \underbrace{\sigma_\epsilon^2}_{\text{unsystematic risk}}. \tag{16}$$

3 Extensions of the CAPM

The basic version of the CAPM that we derived relied on highly unrealistic assumptions. The benefit of making these assumptions was that we obtained a very tractable model to work with. The usefulness of such a model of course depends on the ***robustness*** of its predictions to changes in the assumptions made. It turns out that the implications of the CAPM are fairly resilient to many such alterations. For example, Black (1972) lifts the assumption that investors can lend/borrow at the risk-free rate. We will not have time to consider these extensions in class. For a treatment of some of these extensions see, e.g., Bodie et al. (2008), chapter 9.5 (6th edition 9.2) and Copeland et al. (2004), Chapter 6.G.

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