Asset Pricing

Consumption-based Capital Asset Pricing Model (CCAPM)

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Consumption-based Capital Asset Pricing Model

Consumption-based Capital Asset Pricing Model (CCAPM)

The Rubinstein (1976), Lucas (1978), and Breeden (1979) Consumption-based Capital Asset Pricing Model (CCAPM) implies the following central pricing relation

$$\mathbb{E}_t\left[\widetilde{r}_{j,t+1}\right] - r_{f,t+1} = -\delta\left(1 + r_{f,t+1}\right)\mathsf{cov}_t\left[\frac{U'\left(\widetilde{c}_{t+1}\right)}{U'\left(c_t\right)}, \widetilde{r}_{j,t+1}\right] \tag{1}$$

where δ is a subjective time discount factor, $U'\left(c_t\right)$ denotes marginal utility of consumption, and the expression $\delta\frac{U'\left(\widetilde{c}_{t+1}\right)}{U'\left(c_t\right)}$ is the intertemporal marginal rate of substitution (IMRS) at which an investor is willing to forego consumption at time t in order to consume at time $t+1$

- An asset whose payoff covaries negatively with the IMRS tends to payoff poorly when the marginal utility of consumption is high and consumption itself is low
- This makes the asset undesirable (risky) and investors therefore demand compensation for taking on consumption risk
- The desirability of an asset therefore reflects its ability to smooth consumption

Throwback to consumption smoothing and risk aversion

* Recall that risk averse investors want to *smooth* consumption over time, where risk aversion means that investors prefer to avoid fair lotteries over certain outcomes

$$U(Y) > \mathbb{E}\left[U\left(Y + \widetilde{h}\right)\right] \tag{2}$$

where we can define the fair lottery $\widetilde{h}$ as follows

$$\widetilde{h} = \begin{cases} +h & \text{with probability } \frac{1}{2}, \\ -h & \text{with probability } \frac{1}{2} \end{cases} \tag{3}$$

* This is satisfied for increasing and (strictly) concave utility functions $U(\cdot)$, with $U'(\cdot) > 0$ and $U''(\cdot) < 0$, so that investors are risk averse and exhibit diminishing marginal utility of consumption

* Expected utility is of the von Neumann and Morgenstern (1944) (VNM) type, implying preferences that are consistent with their axiomatic description

Consumption smoothing and intertemporal aspects

* In order to smooth consumption over states and time, risk averse investors buy and sell financial assets
    - Individuals save today so that they can consume more in the future
    - Individuals save for future consumption by investing in financial assets
* The link between individuals' consumption and demand for financial assets implies that asset returns must be related to consumption decisions
* This intertemporal aspect cannot be captured by static, one-period models like the CAPM which
    1. have no dynamic or intertemporal aspects whatsoever
    2. assumes that investors only consume and maximize end-of-period wealth
    3. takes the risk-free rate and the market portfolio as exogenous
* The CAPM cannot explain the behavior of long-horizon investors, the risk-free rate, or the aggregate market return. Enter the consumption-based model

A single representative agent

- Consumption-based asset pricing models frequently aggregate individual investors into a single representative, utility-maximizing agent whose utility derives from aggregate (per capita) consumption

A single representative agent

If financial markets are competitive and complete, and individuals' preferences satisfy the VNM axioms for expected utility, then a representative agent economy exists with the same aggregate consumption series as the heterogeneous-agent economy and the same asset price functions (Constantinides, 1982)

- This assumption allows us to characterize outcomes in financial markets by studying a single representative agent, but limits our ability to study

  1. How investors use financial markets to diversify away idiosyncratic risks
  2. How trading volume affect asset prices (since trading volume will be zero)
  3. How differences in preferences and risk aversion help determine asset prices

- The assumption does, however, make it possible to obtain a clearer view of how equilibrium asset prices reflect aggregate risk

Intertemporal choice problem of an infinitely-lived agent

Intertemporal choice problem

The intertemporal choice problem of a representative investor, whose utility is described by a time-separable multi-period VNM expected utility function, that wishes to maximize the present value of expected discounted utility of an infinite consumption stream is

$$\max \mathbb{E}_t \left[\sum_{t=0}^{\infty} \delta^t U\left(\widetilde{c}_t\right) \right], \quad 0 < \delta < 1 \tag{4}$$

where $\delta = (1+\tau)^{-1}$ is a subjective discount factor and τ is the subjective rate of time preference, and $\widetilde{c}_t$ is aggregate consumption (per capita)

* The subjective discount factor $\delta = (1+\tau)^{-1}$ describes the impatience of the representative investor
* If δ is small (τ is high), investors are highly impatient and prefer consumption now versus consumption in the future

The bequest motive (Barro, 1974)

* The assumption of an infinite horizon is unrealistic if taken literally, but can be justified by a bequest motive as in Barro (1974)

* Suppose that each "generation t" cares not only about their own lifetime consumption $\widetilde{c}_t$, but also about the utility of their children, then

$$V_t = U\left(\widetilde{c}_t\right) + \delta V_{t+1} \tag{5}$$

where V_t is total utility of generation t and δ measures the strength of the bequest motive

* Continuing this argument eventually lands us in the total utility form assumed by Lucas (1978), i.e.,

$$\begin{aligned} V_t &= U\left(\widetilde{c}_t\right) + \delta U\left(\widetilde{c}_{t+1}\right) + \delta^2 U\left(\widetilde{c}_{t+2}\right) + \ldots & (6) \\ &= \sum_{t=0}^{\infty} \delta^t U\left(\widetilde{c}_t\right) & (7) \end{aligned}$$

The no-trade equilibrium and some remarks

No-trade equilibrium

In a representative agent economy, there can necessarily be no trade in financial assets (who is the single individual going to trade with?). Thus, the equilibrium is characterized by being a no-trade equilibrium

* Financial markets are thus in equilibrium, if and only if, at the prevailing prices, supply equals demand *and* both are simultaneously zero
* The important question is therefore: "What prices must assets assume so that the amount the representative agent *must* hold exactly equals what she *wants* to hold?" At these prices, no further trade is utility enhancing
* Note that this no-trade equilibrium concept necessarily implies that trading volume, by construction, is equal to zero

The Lucas (1978) fruit tree model

The Lucas (1978) fruit tree model

Lucas (1978) imagines an exchange (endowment) economy in which the only source of consumption is the fruit that grows on trees. The fruit is perishable (cannot be stored) and since the representative agent derives utility from consumption, it is all eaten (consumed)

* Let Y_t denote the number of fruits produced by the tree during period t
* Let z_t denote the share of the tree held by the representative investor at the beginning of period t
* Let q_t^e denote the time t real price of the tree in terms of consumption (consumption is the numeraire and the price of consumption is 1)
* The representative investor's share of the tree z_t entitles the agent to the fraction $z_t Y_t$ of the fruit

Remarks on the model setup

- Output (the quantity of fruit produced by the tree) arises exogenously and is seen as a stochastic variable (depends on the weather)
- In an exchange (endowment) economy without investment, the dividend (number of fruits) Y_t equals total available output in the economy
- This provides a link between financial markets and the real economy
- Finally, note that, in equilibrium, we must have that investors do not wish to increase nor decrease the holdings of trees
- In the Lucas tree model economy, we have the following interpretations

  1. Shares in trees are like shares in stocks
  2. The fruit represents the dividend paid by stocks

The process for exogenous dividends

* In the Lucas model, dividends are exogenously given and stochastic, but stationary. In each period, they can take on one of N possible values

$$Y_t \in \left\{Y^1, Y^2, \ldots, Y^N\right\} \tag{8}$$

* The randomness in dividends is governed by a Markov chain in which the probabilities for dividends at time $t+1$ are allowed to depend on the outcome for dividends at time t, but not earlier periods

$$\pi_{ij} = \mathbb{P}\left[Y_{t+1} = Y^j \,\middle|\, Y_t = Y^i\right] \tag{9}$$

* This specifications allows for serial correlation in dividends so that high (low) dividends may be followed by high (low) dividends in the next period

The transition matrix

* Under the assumption of a finite number of possible dividend values and time-invariant probabilities, we can write the transition matrix $\mathbf{T}$ as

$$\mathbf{T} = \begin{array}{c} \\ Y_t^1 \\ Y_t^2 \\ \vdots \\ Y_t^N \end{array} \begin{array}{c} \begin{array}{cccc} Y_{t+1}^1 & Y_{t+1}^2 & \dots & Y_{t+1}^N \end{array} \\ \begin{pmatrix} \pi_{11} & \pi_{12} & \dots & \pi_{1N} \\ \pi_{21} & \pi_{22} & \dots & \pi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{N1} & \pi_{N2} & \dots & \pi_{NN} \end{pmatrix} \end{array} \tag{10}$$

* In a continuous-state version, one would have an output process described by a probability transition function

$$G\left(Y_{t+1}|Y_t\right) = \mathbb{P}\left[Y_{t+1} \leq Y^j | Y_t = Y^i\right] \tag{11}$$

Consumption and portfolio choice problem

- Faced with uncertainty about future dividends, the representative investor decides how much to consume ($\widetilde{c}_t$) and the share of the tree to buy (z_{t+1}) in each period $t = 0,1,2,\ldots$ to maximize the VNM expected utility function

$$\max_{\{z_{t+1}\}} \mathbb{E}_t \left[\sum_{t=0}^{\infty} \delta^t U\left(\widetilde{c}_t\right) \right], \quad 0 < \delta < 1 \tag{12}$$

subject to the budget constraints (note typo in book)

$$c_t + q_t^e z_{t+1} \leq z_t Y_t + q_t^e z_t \tag{13}$$

$$z_t \leq 1, \quad \forall t \tag{14}$$

- As a technical assumption, assume that the utility function is strictly concave with $\lim_{c_t \to 0} U'\left(c_t\right) = \infty$ so that a zero consumption level is never optimal

The Euler equation (Solution)

The Euler equation

The solution to the maximization problem in (12) provides us with the following Euler equation (also the representative investor's first-order condition)

$$U'(c_t)\, q_t^e = \delta \mathbb{E}_t \left[U'(\widetilde{c}_{t+1}) \left(\widetilde{q}_{t+1}^e + \widetilde{Y}_{t+1} \right) \right] \tag{15}$$

where $U'(c_t)$, as usual, denotes marginal utility of consumption

- The Euler equation in (15) describes an optimum for the representative investor's consumption and portfolio choice problem
- It equates marginal cost and benefit of current versus future consumption
  - **LHS:** Marginal utility loss from sacrificing q_t^e units of consumption today to purchase one additional unit of the asset
  - **RHS:** Expected discounted marginal utility gain associated with obtaining $\widetilde{q}_{t+1}^e + \widetilde{Y}_{t+1}$ for consumption in the next period

Remarks on the equilibrium

- Assuming that there is only one tree per investor in the economy, then for the representative agent economy to be in equilibrium, we must have that

  1. $z_t = z_{t+1} = z_{t+2} + \ldots \equiv 1$, i.e., the representative investor owns the entire tree
  2. $c_t = Y_t$, i.e., ownership of the entire three entitles the agent to the entire fruit

- That is, in equilibrium, prices must be such that the representative investor is willing to hold all shares and consume all fruits from the single tree

- This implies that we can write the Euler equation equivalently as (using 2)

$$U'(Y_t)\, q_t^e = \delta \mathbb{E}_t \left[U'\left(\widetilde{Y}_{t+1}\right)\left(\widetilde{q}_{t+1}^e + \widetilde{Y}_{t+1}\right)\right] \tag{16}$$

A stock pricing formula via recursive substitution

- The Euler equation in (15) is the fundamental equation of the CCAPM and we can use it to determine the today-price of any asset. Write (15) as

$$U'(c_t)\, q_t^e = \delta \mathbb{E}_t \left[U'(\widetilde{c}_{t+1})\, \widetilde{Y}_{t+1} \right] + \delta \mathbb{E}_t \left[U'(\widetilde{c}_{t+1})\, \widetilde{q}^e_{t+1} \right] \tag{17}$$

- This equation holds for all t, so we consider the same relation one period later

$$U'(\widetilde{c}_{t+1})\, \widetilde{q}^e_{t+1} = \delta \mathbb{E}_{t+1} \left[U'(\widetilde{c}_{t+2})\, \widetilde{Y}_{t+2} \right] + \delta \mathbb{E}_{t+1} \left[U'(\widetilde{c}_{t+2})\, \widetilde{q}^e_{t+2} \right] \tag{18}$$

Law of iterated expectations

For any random variable X_{t+2} that becomes known at time $t+2$ we have that

$$\mathbb{E}_t \left[\mathbb{E}_{t+1} \left[X_{t+2} \right] \right] = \mathbb{E}_t \left[X_{t+2} \right] \tag{19}$$

so that my today expectation of my future expectation of X_{t+2} is the same as my today expectation of X_{t+2}

A stock pricing formula via recursive substitution

- Substitute (18) into (17) and using the law of iterated expectations yields

$$\begin{aligned} U'(c_t)\, q_t^e = {} & \delta \mathbb{E}_t \left[U'(\widetilde{c}_{t+1})\, \widetilde{Y}_{t+1} \right] + \delta^2 \mathbb{E}_t \left[U'(\widetilde{c}_{t+2})\, \widetilde{Y}_{t+2} \right] \\ & + \delta^2 \mathbb{E}_t \left[U'(\widetilde{c}_{t+2})\, \widetilde{q}_{t+2}^e \right] \end{aligned} \tag{20}$$

- Continuing in this fashion recursively eventually yields

$$q_t^e = \mathbb{E}_t \left[\sum_{h=0}^{\infty} \delta^h \frac{U'(\widetilde{c}_{t+h})}{U'(c_t)} \widetilde{Y}_{t+h} \right] \tag{21}$$

which resembles the standard discounting formula except that discounting takes place using the intertemporal marginal rates of substitution

- To rule out speculative bubbles, we usually impose a so-called "transversality condition" of the form

$$\lim_{h \to \infty} \mathbb{E}_t \left[\delta^h \frac{U'(\widetilde{c}_{t+h})}{U'(c_t)} \widetilde{q}_{t+h}^e \right] = 0 \tag{22}$$

Generalizing the Lucas model

- While the Lucas model usually assumes a single tree (or asset), we can easily generalize the model by introducing more, say J, assets each producing an exogenous output $\widetilde{Y}_{j,t}$

- In this case, the Euler equation must be satisfied for each asset price $q^e_{j,t}$

$$U'(c_t)\, q^e_{j,t} = \delta \mathbb{E}_t \left[U'(\widetilde{c}_{t+1}) \left(\widetilde{q}^e_{j,t+1} + \widetilde{Y}_{j,t+1} \right) \right] \tag{23}$$

  where, in equilibrium, aggregate consumption is then the sum of outputs, i.e., $c_t = \sum_{j=1}^{J} Y_{j,t}$ and the price of asset j is given by the pricing formula

$$q^e_{j,t} = \mathbb{E}_t \left[\sum_{h=0}^{\infty} \delta^h \frac{U'(\widetilde{c}_{t+h})}{U'(c_t)} \widetilde{Y}_{j,t+h} \right] \tag{24}$$

The Euler equation in terms of returns

- In the spirit of traditional asset pricing formulas, we want to understand what determines the risk premium on a risky asset (as in the CAPM and APT)
- Let the gross return $\widetilde{R}_{j,t+1} = 1 + \widetilde{r}_{j,t+1}$, with $\widetilde{r}_{j,t+1}$ being the simple return on asset j, be defined as follows

$$\widetilde{R}_{j,t+1} = 1 + \widetilde{r}_{j,t+1} = \frac{\widetilde{q}^{e}_{j,t+1} + \widetilde{Y}_{j,t+1}}{q^{e}_{j,t}} \tag{25}$$

- Re-arranging the Euler equation in (15) then reveals that the return on any asset j must satisfy

$$1 = \mathbb{E}_t \left[\delta \frac{U'\left(\widetilde{c}_{t+1}\right)}{U'\left(c_t\right)} \widetilde{R}_{j,t+1} \right] = \mathbb{E}_t \left[\delta \frac{U'\left(\widetilde{c}_{t+1}\right)}{U'\left(c_t\right)} \left(1 + \widetilde{r}_{j,t+1}\right) \right] \tag{26}$$

The stochastic discount factor (SDF)

The stochastic discount factor (SDF)

We can introduce a more compact notation by writing (26) as

$$1 = \mathbb{E}_t \left[M_{t+1} \left(1 + \widetilde{r}_{j,t+1} \right) \right] \tag{27}$$

where $M_{t+1} = \delta \frac{U'(\widetilde{c}_{t+1})}{U'(c_t)}$ is referred to as the stochastic discount factor (SDF)

- In the standard version of the consumption-based model, the SDF equals the intertemporal marginal rate of substitution of consumption for the representative agent
- The holy grail of consumption-based asset pricing is to determine the appropriate from of the SDF M_{t+1} in (27)
- Finally, we note that if (27) holds for all t, it must also hold unconditionally

$$1 = \mathbb{E} \left[M_{t+1} \left(1 + \widetilde{r}_{j,t+1} \right) \right] \tag{28}$$

An expression for the risk-free rate

- Consider first a riskless asset, e.g., a Treasury (discount) bond in zero net supply, that pays a risk-free rate $r_{f,t+1}$ known in advance at time t. The Euler equation is

$$1 = \mathbb{E}_t \left[M_{t+1} \left(1 + r_{f,t+1} \right) \right] \tag{29}$$

- Re-arranging the Euler equation in (29) yields an expression for the risk-free rate of return

$$1 + r_{f,t+1} = R_{f,t+1} = \frac{1}{\mathbb{E}_t \left[M_{t+1} \right]} \tag{30}$$

- We can similarly derive an expression for the price of the riskless asset

$$q_t^b = \frac{1}{1 + r_{f,t+1}} = \mathbb{E}_t \left[M_{t+1} \right] \tag{31}$$

An expression for the expected risky return

- Consider again the Euler equation for a risky asset j given by

$$1 = \mathbb{E}_t \left[M_{t+1} \left(1 + \widetilde{r}_{j,t+1} \right) \right] \tag{32}$$

- Using that $\mathbb{E}_t [XY] = \mathbb{E}_t [X] \, \mathbb{E}_t [Y] + \mathrm{cov}_t [X, Y]$ we can write (32) as

$$1 = \mathbb{E}_t [M_{t+1}] \, \mathbb{E}_t \left[\left(1 + \widetilde{r}_{j,t+1} \right) \right] + \mathrm{cov}_t \left[M_{t+1}, \widetilde{r}_{j,t+1} \right] \tag{33}$$

- Making use of the expression for the risk-free rate in (31), we can write

$$1 = \frac{1 + \mathbb{E}_t \left[\widetilde{r}_{j,t+1} \right]}{1 + r_{f,t+1}} + \mathrm{cov}_t \left[M_{t+1}, \widetilde{r}_{j,t+1} \right] \tag{34}$$

- Isolating for the gross return on the risky asset yields

$$1 + \mathbb{E}_t \left[\widetilde{r}_{j,t+1} \right] = \left(1 + r_{f,t+1} \right) \left(1 - \mathrm{cov}_t \left[M_{t+1}, \widetilde{r}_{j,t+1} \right] \right) \tag{35}$$

The central pricing relation of the CCAPM

The central pricing relation

Re-arranging the expression in (35) and inserting the definition of M_{t+1} yields the central pricing relation of the CCAPM

$$\mathbb{E}_t\left[\widetilde{r}_{j,t+1}\right] - r_{f,t+1} = -\left(1 + r_{f,t+1}\right)\mathsf{cov}_t\left[M_{t+1}, \widetilde{r}_{j,t+1}\right] \tag{36}$$

$$= -\delta\left(1 + r_{f,t+1}\right)\mathsf{cov}_t\left[\frac{U'\left(\widetilde{c}_{t+1}\right)}{U'\left(c_t\right)}, \widetilde{r}_{j,t+1}\right] \tag{37}$$

* If $\mathsf{cov}_t\left[\frac{U'(\widetilde{c}_{t+1})}{U'(c_t)}, \widetilde{r}_{j,t+1}\right] < 0$, then asset j pays off poorly in bad states and well in good states, making it undesirable for consumption smoothing purposes
* If $\mathsf{cov}_t\left[\frac{U'(\widetilde{c}_{t+1})}{U'(c_t)}, \widetilde{r}_{j,t+1}\right] > 0$, then the asset provide consumption insurance by paying off in bad states when the investor values additional consumption
* Risk averse investors desire a smooth consumption stream and require compensation for holding assets that covary negatively with their IMRS

Interpreting the pricing relation

- Since $U(\cdot)$ is a strictly concave utility function (investors are risk averse), we have that the investor's intertemporal marginal rate of substitution (IMRS)

$$\text{IMRS}_{t+1} = M_{t+1} = \delta \frac{U'(\widetilde{c}_{t+1})}{U'(c_t)} \tag{38}$$

  is high whenever consumption $\widetilde{c}_{t+1}$ is low relative to c_t and vice versa

- That is, the investor's IMRS is inversely related to the business cycle: The IMRS is high during recessions (when consumption is low) and low during expansions (when consumption is high)

- This provides a direct link to the real economy through the representative investor's consumption stream

Replacing marginal utility with consumption itself

* To connect the CCAPM and the CAPM, suppose that the representative investor has quadratic utility $U(c_t) = ac_t - \frac{b}{2}c_t^2$ with $a,b > 0$. Then

$$\mathbb{E}_t\left[\widetilde{r}_{j,t+1}\right] - r_{f,t+1} = -\delta\left(1 + r_{f,t+1}\right)\mathsf{cov}_t\left[\frac{a - b\widetilde{c}_{t+1}}{a - bc_t}, \widetilde{r}_{j,t+1}\right] \quad (39)$$

$$= \left(1 + r_{f,t+1}\right)\frac{b\delta}{a - bc_t}\mathsf{cov}_t\left[\widetilde{c}_{t+1}, \widetilde{r}_{j,t+1}\right] \quad (40)$$

* As a next step, we want to introduce a portfolio that is mostly highly correlated with consumption ~~growth~~ and whose return we denote by $\widetilde{r}_{c,t+1}$
* That is, suppose that we can construct a factor-mimicking portfolio for real per capita consumption growth
* In the terminology of Breeden et al. (1989), this is also referred to as the maximum correlation portfolio (MCP)

A linear Consumption-CAPM (CCAPM)

* Using that the MCP is a portfolio like any other, we can write

$$\mathbb{E}_t\left[\widetilde{r}_{c,t+1}\right] - r_{f,t+1} = \left(1 + r_{f,t+1}\right)\frac{b\delta}{a - bc_t}\text{cov}_t\left[\widetilde{c}_{t+1}, \widetilde{r}_{c,t+1}\right] \tag{41}$$

* Dividing (40) with (41) and re-arranging yields the consumption-CAPM

$$\mathbb{E}_t\left[\widetilde{r}_{j,t+1}\right] - r_{f,t+1} = \frac{\beta_{j,c_t}}{\beta_{c,c_t}}\left(\mathbb{E}_t\left[\widetilde{r}_{c,t+1}\right] - r_{f,t+1}\right) \tag{42}$$

where $\beta_{j,c_t} = \frac{\text{cov}_t\left[\widetilde{r}_{j,t+1}, \widetilde{c}_{t+1}\right]}{\text{var}_t\left[\widetilde{c}_{t+1}\right]}$ is the consumption-β for asset j and $\beta_{c,c_t} = \frac{\text{cov}_t\left[\widetilde{r}_{c,t+1}, \widetilde{c}_{t+1}\right]}{\text{var}_t\left[\widetilde{c}_{t+1}\right]}$ is the consumption-β of portfolio c, the MCP

* Breeden et al. (1989) detail the construction of the MCP and show that the MCP can be built such that $\beta_{c,c_t} = 1$ and a direct analogue of the CAPM obtains

Power utility in the consumption-based model

Time-seperable power utility

Suppose that the representative agent has time-separable power utility defined over aggregate per capita consumption

$$U\left(c_t\right) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} \tag{43}$$

where $\gamma = -c_t \frac{U''(c_t)}{U'(c_t)} \geq 0$ is the Pratt-Arrow measure of relative risk aversion

* Since γ is constant, this class of utility functions is known as the CRRA class
* The power utility function has two important properties for our purposes
  1. It is scale-invariant: with constant return distributions, risk premia do not change over time as aggregate wealth and the scale of the economy increase
  2. If agents in the economy have the same power utility function and can freely trade all risks, then we can aggregate them into a representative investor with the same utility function even if they differ in their wealth levels

The Euler equation under power utility

The Euler equation under power utility

Under power utility, we can write the stochastic discount factor (SDF) as follows

$$M_{t+1} = \delta \left(\frac{\widetilde{c}_{t+1}}{c_t} \right)^{-\gamma} \tag{44}$$

such that the Euler equation becomes

$$1 = \mathbb{E}_t \left[\delta \left(\frac{\widetilde{c}_{t+1}}{c_t} \right)^{-\gamma} (1 + \widetilde{r}_{j,t+1}) \right] \tag{45}$$

- The prevailing consensus is that γ should be somewhere between 2 and 5 (although this is not a settled debate by any means)
- This is perhaps the most well-known form of the consumption-based model, and the most frequently tested specification in empirical applications

Modeling consumption growth as a lognormal variable

* To compare the prediction of the CCAPM with real data, Mehra and Prescott (1985) modify the Lucas model to allow for variations in consumption growth rather than consumption itself

* Let $\widetilde{G}_{t+1} = 1 + \widetilde{g}_{t+1} = \frac{\widetilde{c}_{t+1}}{c_t}$ denote the gross growth rate in consumption and assume that it is conditionally lognormally distributed and homoskedastic

$$\ln \widetilde{G}_{t+1} \sim \mathcal{N}\left(\mathbb{E}_t\left[\widetilde{g}_{t+1}\right], \sigma_g^2\right) \tag{46}$$

* This implies that the approximation $\ln \widetilde{G}_{t+1} = \ln\left(1 + \widetilde{g}_{t+1}\right) \approx \widetilde{g}_{t+1}$ is approximately normally distributed

* Since $\widetilde{G}_{t+1} = \exp\left\{\ln \widetilde{G}_{t+1}\right\}$ by definition, Jensen's Inequality implies that the mean and variance is not simply $\exp\left\{\mu_g\right\}$ and $\exp\left\{\sigma_g^2\right\}$ since the exponential function is convex

Properties of the lognormal distribution

Properties of the lognormal distribution

A variable x is said to follow a lognormal distribution if $\ln x$ is normally distributed. Let $\ln x \sim \mathcal{N}\left(\mu_x, \sigma_x^2\right)$, then

$$\mathbb{E}\left[x\right] = \exp\left\{\mu_x + \frac{1}{2}\sigma_x^2\right\} \tag{47}$$

$$\mathbb{E}\left[x^a\right] = \exp\left\{a\mu_x + \frac{1}{2}a^2\sigma_x^2\right\} \tag{48}$$

$$\text{var}\left[x\right] = \exp\left\{2\mu_x + \sigma_x^2\right\}\left(\exp\left\{\sigma_x^2\right\} - 1\right) \tag{49}$$

$$\text{var}\left[x^a\right] = \exp\left\{2a\mu_x + a^2\sigma_x^2\right\}\left(\exp\left\{a^2\sigma_x^2\right\} - 1\right) \tag{50}$$

Suppose further that x and y are two lognormally distributed variables, then

$$\mathbb{E}\left[x^a y^b\right] = \exp\left\{a\mu_x + b\mu_y + \frac{1}{2}\left(a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\sigma_{xy}\right)\right\} \tag{51}$$

The SDF is conditional lognormal

* We can write the stochastic discount factor (SDF) using the definition of $\widetilde{G}_{t+1}$ more compactly as

$$M_{t+1} = \delta \widetilde{G}_{t+1}^{-\gamma} \tag{52}$$

* The SDF will be conditionally lognormal if consumption is conditionally lognormal so that the expected value of the SDF becomes

$$\mathbb{E}_t \left[M_{t+1} \right] = \delta \exp \left\{ -\gamma \mathbb{E}_t \left[\widetilde{g}_{t+1} \right] + \frac{1}{2} \gamma^2 \sigma_g^2 \right\} \tag{53}$$

* The log SDF is $m_{t+1} = \ln \delta - \gamma \widetilde{g}_{t+1}$, where the Jensen's inequality difference becomes immediately clear

* The variance of the SDF is then given by the expression (using the properties of the lognormal distribution)

$$\begin{aligned} \mathrm{Var}_t \left[M_{t+1} \right] &= \exp \left\{ -2\gamma \mathbb{E}_t \left[\widetilde{g}_{t+1} \right] + \gamma^2 \sigma_g^2 \right\} \left(\exp \left\{ \gamma^2 \sigma_g^2 \right\} - 1 \right) && (54) \\ &= \mathbb{E}_t \left[M_{t+1} \right]^2 \left(\exp \left\{ \gamma^2 \sigma_g^2 \right\} - 1 \right). && (55) \end{aligned}$$

Power utility in a lognormal model

* Suppose that $\widetilde{G}_{t+1}$ and $\widetilde{R}_{j,t+1}$ are jointly conditionally lognormal and homoskedastic, then we can take logs to the Euler equation for the CRRA, power utility case

$$1 = \mathbb{E}_t \left[\delta \widetilde{G}_{t+1}^{-\gamma} \left(1 + \widetilde{r}_{j,t+1}\right) \right] \tag{56}$$

and obtain a lognormal model with power utility using the properties of the lognormal distribution

$$0 = \ln \delta - \gamma \mathbb{E}_t \left[\widetilde{g}_{t+1} \right] + \mathbb{E}_t \left[\widetilde{r}_{j,t+1} \right] + \frac{1}{2} \left(\gamma^2 \sigma_g^2 + \sigma_j^2 - 2\gamma \sigma_{jg} \right) \tag{57}$$

where σ_{jg} denotes the constant covariance between the return on asset j and consumption growth

* The expression in (57) was first derived by Hansen and Singleton (1983) and has both time series and cross-sectional implications

The risk-free rate in the lognormal model

* Consider first the Euler equation for a riskless asset whose return satisfies

$$1 = \mathbb{E}_t \left[\delta \widetilde{G}_{t+1}^{-\gamma} \right] (1 + r_{f,t+1}) \tag{58}$$

* Using the relation $\frac{1}{\exp\{x\}} = \exp\{-x\}$ and the properties of the lognormal distribution we get

$$1 + r_{f,t+1} = \frac{1}{\delta} \exp \left\{ \gamma \mathbb{E}_t \left[\widetilde{g}_{t+1} \right] - \frac{1}{2} \gamma^2 \sigma_g^2 \right\} \tag{59}$$

* Finally, take logs to obtain the prediction of the lognormal power utility model for the equilibrium real risk-free rate

$$r_{f,t+1} = -\ln \delta + \gamma \mathbb{E}_t \left[\widetilde{g}_{t+1} \right] - \frac{1}{2} \gamma^2 \sigma_g^2 \tag{60}$$

Interpreting the risk-free rate expression

- We note that the risk-free rate is determined by three factors, where the risk-free rate is
  - increasing in the rate of time preference $-\ln\delta$ (i.e., when investors are impatient)
  - increasing in consumption growth $\gamma\mathbb{E}_t\left[\widetilde{g}_{t+1}\right]$ (intertemporal substitution)
  - declining in consumption risk $\gamma^2\sigma_g^2$ (precautionary savings)
- Interest rates are high when people are impatient, when δ is low and τ is high. If everyone wants to consume now, it takes a high interest rate to convince them to save
- Interest rates are high when consumption growth is high. In times of high interest rates, it pays investors to consume less now, invest more, and consume more in the future
- The representative investor is more likely to save if consumption volatility is high to avoid low consumption states by smoothing consumption. This drives down interest rates

The risky return in the lognormal model

- The log risky return in the lognormal model follows directly from (57)

$$\mathbb{E}_t\left[\widetilde{r}_{j,t+1}\right] = -\ln\delta + \gamma\mathbb{E}_t\left[\widetilde{g}_{t+1}\right] - \frac{1}{2}\left(\gamma^2\sigma_g^2 + \sigma_j^2 - 2\gamma\sigma_{jg}\right) \tag{61}$$

- Subtracting the log risk-free rate from each side gives us an expression for the risk premium on any risky asset j

$$\mathbb{E}_t\left[\widetilde{r}_{j,t+1}\right] - r_{f,t+1} + \frac{1}{2}\sigma_j^2 = \gamma\sigma_{jg} \tag{62}$$

- We can eliminate the Jensen's Inequality adjustment by rewriting (62) as

$$\ln\frac{1+\mathbb{E}_t\left[\widetilde{r}_{j,t+1}\right]}{1+r_{f,t+1}} = \gamma\sigma_{jg} \tag{63}$$

  which again highlights that the risk premium is determined by the investor's relative risk aversion and the covariance with consumption growth

- Note that one can interpret γ as the price of risk and σ_{jg} as the amount of risk

Market clearing condition in the Lucas model

- Recall that the Lucas fruit tree model assumes $\widetilde{c}_{t+1} = \widetilde{Y}_{t+1}$ such that consumption growth and dividend growth is identical
- Using that prices are proportional to dividends, i.e., $q_t^e = \nu Y_t$, we can write the return on the market portfolio (the single tree in the Lucas model) as follows

$$\widetilde{R}_{t+1} \equiv 1 + \widetilde{r}_{t+1} = \frac{\widetilde{q}_{t+1}^e + \widetilde{Y}_{t+1}}{q_t^e} = \frac{\nu + 1}{\nu} \frac{\widetilde{Y}_{t+1}}{Y_t} = \frac{\nu + 1}{\nu} \widetilde{G}_{t+1} \tag{64}$$

- Since returns and consumption growth is now perfectly correlated, we have the following expression for the log risk premium in the Lucas world

$$\ln \frac{1 + \mathbb{E}_t \left[\widetilde{r}_{t+1} \right]}{1 + r_{f,t+1}} = \gamma \sigma_g^2 \tag{65}$$

which is the product of the coefficient of relative risk aversion and the variance of the growth rate of consumption (see logNormalModel.pdf note for details)

Testing the CCAPM and its implications empirically

* As for the CAPM and the APT, we now turn to a discussion of the empirical validity of the Consumption-based asset pricing model
* We will pay exclusive attention to the representative agent model in which the investor exhibit CRRA power utility
* We will start with a famous example by Mehra and Prescott (1985) whose results indicate that the CCAPM is unable to match even basic aspects of data
* This is known as the equity premium puzzle, where the puzzle is that the model is unable to match the historical equity risk premium for reasonable parameter values (including γ)
* Next, we will consider the result of Weil (1989) that simply raising the value of γ is not a viable solution as it gives rise to the risk-free rate puzzle
* We end with a discussion of the Hansen and Jagannathan (1991) lower bounds on the volatility of the SDF

The equity premium puzzle

* Consider the empirical data from Mehra and Prescott (1985) reproduced in the Table below (note that the book is a bit loose with the numbers)

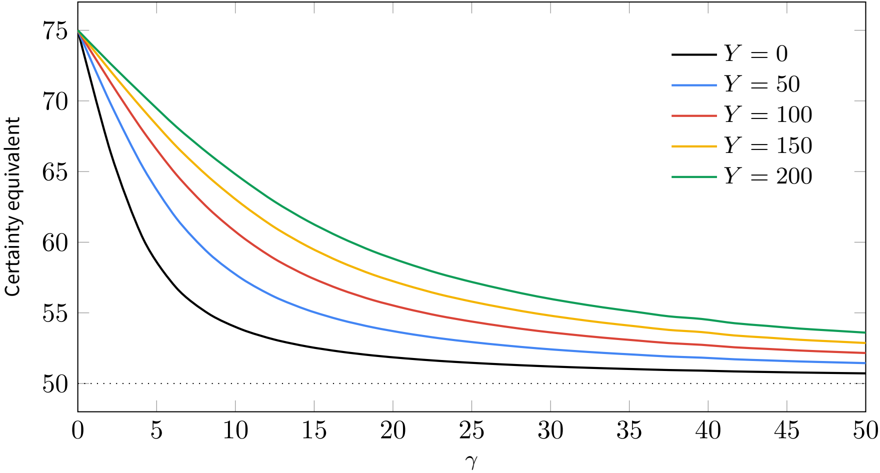
	Mean (%)	Standard deviation (%)
Consumption growth	1.83%	3.57%
Risk-free return	0.80%	5.67%
Risky return	6.98%	16.54%
Equity premium	6.18%	16.67%

* Using that aggregate log consumption is growing at $\mu_g = 1.752\%$ annually with a standard deviation of $\sigma_g = 3.505\%$, we can determine the coefficient of relative risk aversion needed to match the equity risk premium

$$\begin{aligned} \gamma &= \frac{\ln\left\{1 + \mathbb{E}_t\left[\widetilde{r}_{t+1}\right]\right\} - \ln\left\{1 + \mathbb{E}_t\left[r_{f,t+1}\right]\right\}}{\sigma_g^2} \\ &= \frac{\ln\{1.0698\} - \ln\{1.008\}}{0.03505^2} = 48.44 \end{aligned} \tag{66}$$

A high γ implies extreme risk aversion

- A value of $\gamma = 48.44$ implies an implausible level of risk aversion among investors that will pay only a modest amount for a gamble $L = (100,50,0.5)$



The risk-free rate puzzle

- One response to the equity premium puzzle is to consider larger values for the coefficient of relative risk aversion γ. Maybe investors are simply more risk averse than we thought
- However, Weil (1989) demonstrates that the high value of γ required to match the equity premium results in yet another puzzle
- First consider the following "reasonable" values $\delta = 0.99$ and $\gamma = 5$

$$r_{f,t+1} = -\ln \delta + \gamma \mathbb{E}_t \left[\widetilde{g}_{t+1}\right] - \frac{1}{2}\gamma^2 \sigma_g^2 \tag{67}$$

$$= -\ln\{0.99\} + 5 \cdot 0.01752 - \frac{1}{2} 5^2 \cdot 0.03505^2 = 8.23\% \tag{68}$$

where we see that the risk-free rate is much higher than the one observed empirically of 0.8%

- Since consumption growth is positive, agents have a strong desire to borrow to smooth consumption, so it takes a high interest to keep them from doing so

The risk-free rate puzzle

* Suppose that we want to solve for δ using the empirically observed risk-free rate of $r_f = 0.8\%$ and $\gamma = 5$, then we find

$$\ln \delta = -r_{f,t+1} + \gamma \mathbb{E}_t \left[\widetilde{g}_{t+1} \right] - \frac{1}{2} \gamma^2 \sigma_g^2 \tag{69}$$

$$= -\ln \{1.008\} + 5 \cdot 0.01752 - \frac{1}{2} 5^2 \cdot 0.03505^2 = 0.064 \tag{70}$$

* This implies a discount factor of $\delta = 1.07$, which is greater than one and corresponding to a negative rate of time preference (i.e., negative τ)

* Thus, the model "solves" the problem by having agents prefer consumption tomorrow over consumption today, which reduces their desire to borrow so that the risk-free rate can remain low

A knife-edge case with a high γ

* Consider again the previous case with $\gamma = 48.44$, in this case we obtain

$$\ln \delta = -r_{f,t+1} + \gamma \mathbb{E}_t \left[\widetilde{g}_{t+1}\right] - \frac{1}{2}\gamma^2 \sigma_g^2 \tag{71}$$

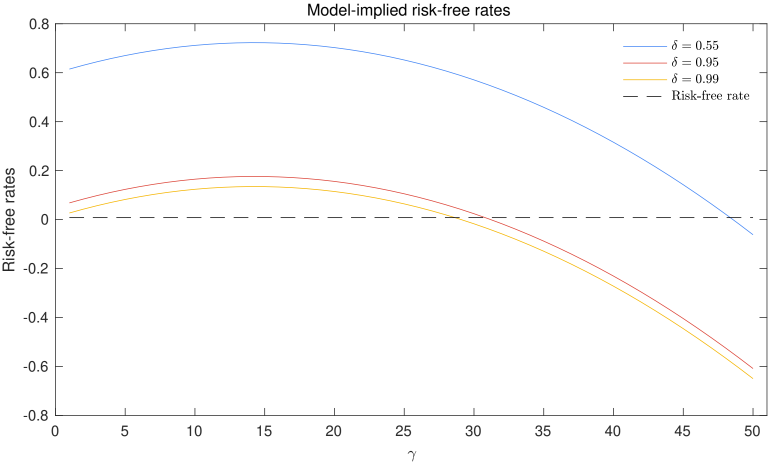
$$= -\ln\{1.008\} + 48.44 \cdot 0.01752 - \frac{1}{2} 48.44^2 \cdot 0.03505^2 = -0.60 \tag{72}$$

* This yields $\delta = 0.55$ and we are back to a positive rate of time preference
* As such, the CCAPM can match both the average risk-free rate and the equity risk premium with $\delta = 0.55$ and $\gamma = 48.44$
* The problem, however, is that a $\delta = 0.55$ implies that, in a world of certainty, investors would discount the future by 45% per period $\Rightarrow$ Investors are extremely impatient

Illustrating the risk-free rate puzzle

* We can illustrate the risk-free rate puzzle as follows using that the risk-free rate in the lognormal power utility model is defined as

$$r_{f,t+1} = -\ln\delta + \gamma\mathbb{E}_t\left[\widetilde{g}_{t+1}\right] - \frac{1}{2}\gamma^2\sigma_g^2 \tag{73}$$



Kocherlakota (1996)

- Kocherlakota (1996) assumes the existence of two assets: Stocks and short-term bonds with gross returns $\widetilde{R}^S_{t+1}$ and R^b_{t+1}, respectively
- Assuming that investors have CRRA power utility, we have the following Euler equations

$$E\left[\delta\left(\frac{\widetilde{c}_{t+1}}{c_t}\right)^{-\gamma}\left(\widetilde{R}^S_{t+1} - R^b_{t+1}\right)\right] = 0 \tag{74}$$

$$E\left[\delta\left(\frac{\widetilde{c}_{t+1}}{c_t}\right)^{-\gamma} R^b_{t+1}\right] = 1 \tag{75}$$

- If we collect historical data on $\widetilde{R}^S_{t+1}$, R^b_{t+1}, and $\widetilde{c}_{t+1}$, we can form the sample counterparts

$$e^S_{t+1} = \delta\left(\frac{\widetilde{c}_{t+1}}{c_t}\right)^{-\gamma}\left(\widetilde{R}^S_{t+1} - R^b_{t+1}\right) \tag{76}$$

$$e^b_{t+1} = \delta\left(\frac{\widetilde{c}_{t+1}}{c_t}\right)^{-\gamma} R^b_{t+1} - 1 \tag{77}$$

A simple t-test for the Euler equation

- If the CCAPM with CRRA utility holds, we must have $\mathbb{E}\left[e^S_{t+1}\right] = \mathbb{E}\left[e^b_{t+1}\right] = 0$, which we can test using a simple t-test

$$t = \frac{\overline{e}}{se\left(\overline{e}\right)} \tag{78}$$

  where both $\overline{e}$ and $se\left(\overline{e}\right)$ can be obtained from regressing the time series of e^S_{t+1} or e^b_{t+1} on a constant

- Kocherlakota (1996) find that the coefficient of relative risk aversion γ needs to exceed 8.5 for $\overline{e}^S$ to be indistinguishable from zero (assuming $\delta = 1$)

- Similarly, and even for values of γ as low as 1, the risk-free rate puzzle is present in the data when setting $\delta = 0.99$

Hansen-Jagannathan (HJ) bounds

* Hansen and Jagannathan (1991) provide another perspective on the equity premium puzzle by deriving lower bounds on the volatility of the stochastic discount factor (SDF)

* Recall that we can write the general Euler equation as

$$\begin{aligned} 1 &= \mathbb{E}_t\left[M_{t+1}\left(1+\widetilde{r}_{j,t+1}\right)\right] \\ &= \mathbb{E}_t\left[M_{t+1}\right]\mathbb{E}_t\left[\left(1+\widetilde{r}_{j,t+1}\right)\right] + \mathrm{cov}_t\left[M_{t+1}, \widetilde{r}_{j,t+1}\right] \\ &= \mathbb{E}_t\left[M_{t+1}\right]\mathbb{E}_t\left[\left(1+\widetilde{r}_{j,t+1}\right)\right] + \rho\left[M_{t+1}, \widetilde{r}_{j,t+1}\right]\sigma_{M_{t+1}}\sigma_{\widetilde{r}_{j,t+1}} \end{aligned} \tag{79}$$

* Re-arranging and using that $1 + r_{f,t+1} = \mathbb{E}_t\left[M_{t+1}\right]^{-1}$ we obtain

$$\frac{\mathbb{E}_t\left[\widetilde{r}_{j,t+1}\right] - r_{f,t+1}}{\sigma_{\widetilde{r}_{j,t+1}}} = -\rho\left[M_{t+1}, \widetilde{r}_{j,t+1}\right]\frac{\sigma_{M_{t+1}}}{\mathbb{E}_t\left[M_{t+1}\right]} \tag{80}$$

which relates the Sharpe ratio of asset j to the behavior of the stochastic discount factor M_{t+1}

Obtaining the HJ bound

Hansen-Jagannathan (HJ) bound

Since –$1 \leq \rho\left[M_{t+1}, \widetilde{r}_{j,t+1}\right] \leq 1$, the expression in (80) provides a lower bound on the behavior of M_{t+1}, giving rise to the inequality

$$\left|\frac{\mathbb{E}_t\left[\widetilde{r}_{j,t+1}\right] - r_{f,t+1}}{\sigma_{\widetilde{r}_{j,t+1}}}\right| \leq \frac{\sigma_{M_{t+1}}}{\mathbb{E}_t\left[M_{t+1}\right]} \tag{81}$$

- Using the data from Mehra and Prescott (1985) we can determine the following bound on the stochastic discount factor

$$\frac{\sigma_{M_{t+1}}}{\mathbb{E}_t\left[M_{t+1}\right]} \geq \frac{6.98\% - 0.80\%}{16.54\%} = 0.374 \tag{82}$$

- Next, let us investigate if this bound is satisfied by the CRRA utility model

Checking if the bound is satisfied for CRRA utility

- Recall that the stochastic discount factor under CRRA utility is given by

$$M_{t+1} = \delta \left(\frac{\widetilde{c}_{t+1}}{c_t} \right)^{-\gamma} = \delta \widetilde{G}_{t+1}^{-\gamma} \tag{83}$$

- Under the assumption of lognormal consumption growth, we have that (for $\delta = 0.99$, $\gamma = 2$, $\mu_g = 1.752\%$, and $\sigma_g^2 = 0.123\%$)

$$\mathbb{E}_t \left[M_{t+1} \right] = \delta \exp \left\{ -\gamma \mu_g + \frac{1}{2} \gamma^2 \sigma_g^2 \right\} = 0.96 \tag{84}$$

- Suppose further that $\sigma_M = 0.00234$ (see the appendix), then

$$\frac{\sigma_{M_{t+1}}}{\mathbb{E}_t \left[M_{t+1} \right]} = \frac{0.00234}{0.96} = 0.0024 \tag{85}$$

  which is orders of magnitude lower than 0.374 and the bound is violated. That is, the SDF is simply not volatile enough to explain the equity premium puzzle

HJ bound in the lognormal power utility model

HJ bound in the lognormal power utility model

The HJ bound in the lognormal power utility model becomes

$$\left| \frac{\mathbb{E}_t\left[\widetilde{r}_{j,t+1}\right] - r_{f,t+1}}{\sigma_{\widetilde{r}_{j,t+1}}} \right| \leq \frac{\sigma_{M_{t+1}}}{\mathbb{E}_t\left[M_{t+1}\right]} = \sqrt{\exp\left\{\gamma^2 \sigma_g^2\right\} - 1} \approx \gamma \sigma_g, \tag{86}$$

- The bound implies that $\sigma_{M_{t+1}} = \gamma \sigma_g$ for $\mathbb{E}_t\left[M_{t+1}\right] \approx 1$, which is the equity risk premium all over again
- If σ_g is too low, then we need implausibly high levels of risk aversion to satisfy the HJ lower bound on the volatility of a valid SDF
- The condition makes use of the expected value and variance of the SDF under conditional lognormality

Diagnostics and extensions

- To diagnose the model, consider the following version of the Euler equation for the market portfolio

$$\mathbb{E}_t\left(\widetilde{r}_{M,t+1}\right) - r_{f,t+1} = -\left(1 + r_{f,t+1}\right)\rho\left[M_{t+1}, \widetilde{r}_{M,t+1}\right]\sigma_{M_{t+1}}\sigma_{\widetilde{r}_{M,t+1}} \qquad (87)$$

with $M_{t+1} = \delta\frac{U'(\widetilde{c}_{t+1})}{U'(c_t)}$ being the stochastic discount factor

- It is immediately clear the the equity premium depends on the following three quantities

  1. The standard deviations of the IMRS (SDF), which depends on consumption
  2. The standard deviation of the return on the market portfolio
  3. The correlation between the two quantities

Problems with the consumption data

* One often cited cause of the poor performance of the CCAPM with CRRA utility is the data on aggregate consumption

1. Aggregate consumption simply varies to little (i.e., it is too smooth). Alternative ideas are explored in Savov (2011) and Kroencke (2017) using garbage data and unfiltered consumption data, respectively
2. A related issue is timing. Should we measure consumption when we buy the goods or when we actually consume the goods?
3. Similarly, as pointed out in Engsted and Møller (2015), there is a timing question related to whether we should use end-of-period or beginning-of-period consumption?
4. Limited participation in the stock market implies that aggregate per capita consumption is not necessarily representative of investors that are active in financial markets (see, e.g., Vissing-Jørgensen (2002) and Malloy et al. (2009))
5. Revisions to the consumption data may smooth out short-term fluctuations in consumption growth relevant for asset pricing, suggesting that first-release data may be more appropriate (Borup and Schütte, 2021)

Savov (2011) and his garbage measure

- Savov (2011) asks a key question: It is the model or the data?
- He uses municipal solid waste (MSE), or garbage, from the EPA as a new measure for consumption to address the issue

  > MSW – otherwise known as trash or garbage – consists of everyday items such as product packaging, grass clippings, furniture, clothing, bottles, food scraps, newspapers, appliances, and batteries. Not included are materials that also may be disposed in landfills but are not generally considered MSW, such as construction and demolition materials, municipal wastewater treatment sludges, and nonhazardous industrial wastes

- The idea is the rates of garbage generation should be a better reflection of actual consumption relative to traditional expenditure measures from NIPA

	Paper	Glass	Metals	Plastics	Food	Yard	Other	Total	Less Yard
Proportion	36	7	9	7	10	16	13	100	84
St. dev.	4.85	4.21	3.27	9.58	8.42	2.61	3.11	2.48	2.88
Corr. R^M	56	9	38	8	14	−6	16	54	58
Cov. R^M	44	6	20	12	19	−3	8	22	27

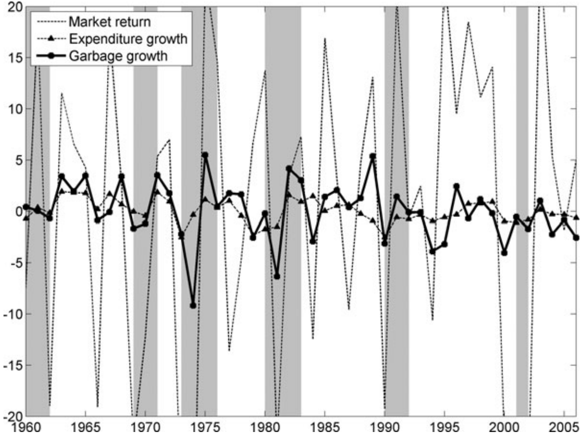
Properties of garbage data

* The standout feature of the data is that garbage growth is significantly more volatile and more highly correlated with stock returns compared to NIPA expenditure growth (or other competing consumption measures)

Panel A: Sample Moments							
	Garbage	Durables	Nondurables	Services	Nondur. & Serv.	P–J	Q4–Q4
Mean	1.47	4.62	1.67	2.55	2.21	4.96	2.21
	(0.36)	(0.91)	(0.23)	(0.24)	(0.21)	(0.60)	(0.22)
St. dev.	2.88	5.56	1.45	1.18	1.14	2.99	1.29
	(0.39)	(0.60)	(0.19)	(0.09)	(0.11)	(0.33)	(0.14)
Autocorr.	−14.51	28.90	22.09	51.58	40.01	67.72	32.78
	(11.54)	(11.49)	(12.23)	(11.09)	(10.84)	(6.01)	(11.16)
Corr. R^M	57.94	46.33	47.35	21.89	37.83	13.79	26.42
	(11.25)	(12.00)	(11.58)	(12.11)	(11.64)	(10.53)	(11.47)
Cov. R^M	26.86	41.47	11.04	4.15	6.92	6.64	5.49
	(10.32)	(14.32)	(4.27)	(2.48)	(2.70)	(4.84)	(2.71)
Garbage		42	51	45	53	13	36
Durables			78	57	74	61	65
Nondurables				57	85	54	72
Services					92	42	82
Nondur. & Serv.						53	87
P–J							61

Dynamics of garbage data

* Garbage growth is closely connected to the business cycle and falls sharply at the onset of recessions (NBER recession periods in gray shading)
* NIPA expenditure growth is much flatter and varies far less over time



Coefficients of relative risk aversion for garbage

* Savov (2011) fixes the subjective discount factor at $\beta = 0.95$ and estimates the coefficient of relative risk aversion γ using the Euler equation

$$E\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} R^e_{t+1}\right] = 0 \tag{88}$$

* These are essentially tests of the equity premium puzzle as we examine at which level of risk aversion that we can match the observed equity premium

	Garbage	Expenditure	P–J	Q4–Q4
RRA (γ)	17	81	66	85
(s.e.)	(9)	(49)		(71)
Adjusted γ	9	40	33	
Implied R^f	17	303	417	300
Pricing error	(0.00)	(0.00)	(4.30)	(0.00)

Fama-MacBeth with garbage

* Last, we consider FMB estimates of linear factor models using various consumption growth measures to examine risk prices for consumption risk

$$R^e_{i,t+1} = \alpha_i + \beta_{i,\Delta c}\left(\frac{c_t}{c_{t-1}}\right) + \epsilon_{i,t+1} \tag{89}$$

Garbage	Garbage × *cay*	Expenditure	Expenditure × *cay*	P–J	Q4–Q4	MRF	SMB	HML	r.m.s. (p)
2.44									3.42
(3.59)									(0.00)
2.06	0.34								3.21
(2.32)	(0.74)								(0.00)
		1.25							3.74
		(3.56)							(0.00)
		0.56	0.84						3.24
		(1.50)	(1.85)						(0.00)
2.38		0.57							3.42
(3.53)		(1.69)							(0.00)
				5.61					3.08
				(3.91)					(0.00)
					2.08				2.33
					(3.85)				(0.00)
						8.18			3.46
						(3.57)			(0.00)
1.92						8.01			3.41
(2.15)						(3.50)			(0.00)
		0.48				7.89			3.42
		(1.43)				(3.54)			(0.00)
						6.48	2.73	4.90	1.96
						(2.98)	(1.22)	(2.65)	(0.00)
−0.03						6.52	2.69	5.03	1.93
(0.04)						(2.99)	(1.21)	(2.74)	(0.00)
		0.10				6.51	2.72	4.82	1.95
		(0.34)				(2.99)	(1.22)	(2.59)	(0.00)

Overview of recent extensions

- Recent extensions of the canonical Consumption-CAPM reviewed thus far include (see Cochrane (2017) for an excellent and recent review)
  - Adding a disaster state as in Rietz (1988) and Barro (2006). The main idea centers on the notion of a very low probability state of the world (e.g., the Great Depression) in which consumption decreases extraordinarily. This makes investors require compensation for such events, providing one explanation for the equity premium puzzle
  - Allowing for habit formation in which marginal utility depends not only on aggregate consumption, but consumption relative to the past, i.e.

$$U(c_t, c_{t-1}) = \frac{(c_t - \chi c_{t-1})^{1-\gamma}}{1-\gamma} \tag{90}$$

    where $\chi \leq 1$ is a parameter. Models of this kind is explored in, among others, Campbell and Cochrane (1999)

Oveview of recent extensions

- Using a generalized utility function with so-called Epstein and Zin (1989) recursive preferences, i.e.,

$$U\left(c_t, \mathbb{E}_t\left[\widetilde{U}_{t+1}\right]\right) = \left[(1-\delta)\, c_t^{\frac{1-\gamma}{\theta}} + \delta\left(\mathbb{E}_t\left[\widetilde{U}_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}} \tag{91}$$

  where ψ is now the intertemporal elasticity of substitution and γ is the coefficient of relative risk aversion, where the connected between the two is given by

$$\theta = \frac{1-\gamma}{1-\frac{1}{\psi}} \tag{92}$$

  where we note that for $\gamma = \frac{1}{\psi}$ the model reduces to CRRA utility

- Alter the consumption process to include long-run risk as in Bansal and Yaron (2004). The main idea is that agent not only want compensation for short-run risk (as in the standard CCAPM), but also require compensation for shocks to long-run growth and volatility of consumption

- In sum, there is a whole new world of asset pricing models that builds on and extends the standard CRRA model in many different directions

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