Asset Pricing

Mean-variance analysis

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The portfolio selection problem

* Consider the portfolio choice problem of an investor with initial wealth Y_0 that has to allocate her funds across two investment alternatives

  1. What money amount ϕ to invest in a risky asset with an uncertain rate of return $\widetilde{r}$
  2. What money amount $(Y_0 - \phi)$ to invest in a risk-free asset that pays r_f

* The investor's (uncertain) end-of-period wealth $\widetilde{Y}_1$ satisfies

$$\begin{aligned} \widetilde{Y}_1 &= (Y_0 - \phi)(1 + r_f) + \phi(1 + \widetilde{r}) && (1) \\ &= Y_0(1 + r_f) + \phi(\widetilde{r} - r_f) && (2) \end{aligned}$$

  where the first term in (2) is the investor's return on wealth when the entire portfolio is invested in the riskless asset and the second term is the difference in return gained by investing the money amount ϕ in the risky asset

The maximization problem

- The portfolio choice problem of a von Neumann-Morgenstern (VNM) utility maximizer that cares only about end-of-the-single-period consumption is

$$\max_{\phi} \mathbb{E}\left[U\left(\widetilde{Y}_1\right)\right] = \mathbb{E}\left[U\left(Y_0\left(1+r_f\right)+\phi\left(\widetilde{r}-r_f\right)\right)\right] \tag{3}$$

- Under risk aversion ($U'(Y) > 0$ and $U''(Y) < 0$), the necessary and sufficient first-order condition (FOC) for the problem in (3) is

$$\mathbb{E}\left[U'\left(\widetilde{Y}_1\right)\left(\widetilde{r}-r_f\right)\right] = 0 \tag{4}$$

where we can note that the second-order condition for a maximum

$$\mathbb{E}\left[U''\left(\widetilde{Y}_1\right)\left(\widetilde{r}-r_f\right)^2\right] < 0 \tag{5}$$

is automatically satisfied by the concavity of the utility function

The participation principle

- The FOC in (4) can provide a set of intuitive relations for portfolio allocations

Portfolio allocation and expected returns

A risk averse individual will invest in the risky asset if and only if the expected return on the risky asset exceeds that of the risk-free rate

$$\phi > 0 \quad \Leftrightarrow \quad \mathbb{E}\left[\tilde{r}\right] > r_f \tag{6}$$

$$\phi = 0 \quad \Leftrightarrow \quad \mathbb{E}\left[\tilde{r}\right] = r_f \tag{7}$$

$$\phi < 0 \quad \Leftrightarrow \quad \mathbb{E}\left[\tilde{r}\right] < r_f \tag{8}$$

- This demonstrates that even risk averse agents will always allocate some positive investment to the risky asset if its risk premium is positive
- The participation principle implies that risk aversion alone is insufficient to explain non-participation in risky asset markets with positive risk premia

Investment behavior over wealth and risk aversion

- Arrow (1971) demonstrates the intuitive result that less risk averse individuals always invest more in the risky asset

Portfolio allocation and risk aversion

Suppose that for all wealth levels $Y > 0$ we have that $R_A^1(Y) > R_A^2(Y)$ $\left(R_R^1(Y) > R_R^2(Y)\right)$, where $R_A^i(Y)$ $\left(R_R^i(Y)\right)$ denotes the coefficient of absolute (relative) risk aversion for investor $i = 1{,}2$, then $\phi_1 < \phi_2$ (Arrow, 1971)

- We can similarly show that the (approximate) risk premium required for full investment, i.e., $\phi = Y_0$, in the risky asset depends on the individual's aversion to risk

$$\mathbb{E}\left[\widetilde{r} - r_f\right] \approx R_A\left(Y_0\left(1 + r_f\right)\right) Y_0 \mathbb{E}\left[\left(\widetilde{r} - r_f\right)^2\right] \tag{9}$$

Risky amount and absolute risk aversion

- We can similarly relate the size of the amount ϕ invested in the risky asset to the initial wealth and degree of risk aversion exhibited by the individual
- Absolute risk aversion indicates how the investor's money amount ϕ invested in the risky asset changes with initial wealth

Absolute risk aversion

Let $R_A(Y) = -\frac{U''(Y)}{U'(Y)}$ be the Pratt-Arrow coefficient of absolute risk aversion, then the risky amount behaves as follows as a function of initial wealth (Arrow, 1971)

Risk aversion	Coefficient	Investment behavior
Decreasing absolute	$R'_A(Y) < 0$	$\phi'(Y_0) > 0$
Constant absolute	$R'_A(Y) = 0$	$\phi'(Y_0) = 0$
Increasing absolute	$R'_A(Y) > 0$	$\phi'(Y_0) < 0$

The arguments behind the investment behavior

* We can prove the above claims using a series of arguments. First, define the derivative of ϕ with respect to initial wealth Y_0 as

$$\phi'(Y_0) = \frac{d\phi}{dY_0} = \frac{(1+r_f)\,\mathbb{E}\left[U''\left(\widetilde{Y}_1\right)(\widetilde{r}-r_f)\right]}{-\mathbb{E}\left[U''\left(\widetilde{Y}_1\right)(\widetilde{r}-r_f)^2\right]} \tag{10}$$

* The denominator is always positive by assumption, so we can focus on the sign of the numerator for the analysis

* Suppose that we have two possible realizations for returns $r_H > r_f > r_L$ so that $Y_H \geq Y_0(1+r_f) \geq Y_L$ for $\phi \geq 0$ and DARA, then

$$\mathbb{E}\left[U''\left(\widetilde{Y}_1\right)(\widetilde{r}-r_f)\right] \geq -\underbrace{\mathbb{E}\left[U'\left(\widetilde{Y}_1\right)(\widetilde{r}-r_f)\right]}_{\text{FOC}} R_A\left(Y_0(1+r_f)\right) \geq 0 \tag{11}$$

Risky share and relative risk aversion

- Relative risk aversion indicates how the investor's fraction of wealth ϕ/Y invested in the risky asset changes with initial wealth
- Define the elasticity measuring the proportional increase in the risky asset for an increase in initial wealth as

$$\eta = \frac{d\phi/\phi}{dY_0/Y_0} = \frac{d\phi}{dY_0}\frac{Y_0}{\phi} \tag{12}$$

Relative risk aversion

Let $R_R(Y) = -Y\frac{U''(Y)}{U'(Y)}$ be the Pratt-Arrow coefficient of relative risk aversion, then the risky fraction behaves as follows (Arrow, 1971)

Risk aversion	Coefficient	Investment behavior
Decreasing relative	$R_R'(Y) < 0$	$\eta > 1$
Constant relative	$R_R'(Y) = 0$	$\eta = 1$
Increasing relative	$R_R'(Y) > 0$	$\eta < 1$

Augmenting the problem to many assets

- More generally, investors have the possibility of allocating funds to a wide array of different risky assets
- If we admit N risky asset with returns $\{\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_N\}$, then the investor's end-of-period wealth becomes

$$\widetilde{Y}_1 = Y_0 \left(1 + r_f\right) + \sum_{i=1}^{N} \phi_i \left(\widetilde{r}_i - r_f\right) \tag{13}$$

- The portfolio choice problem of the VNM utility maximizer becomes

$$\max_{\{\phi_1, \phi_2, \ldots, \phi_N\}} \mathbb{E}\left[U\left(Y_0 \left(1 + r_f\right) + \sum_{i=1}^{N} \phi_i \left(\widetilde{r}_i - r_f\right)\right)\right] \tag{14}$$

Introducing portfolio weights

- It is often convenient to define portfolio weights rather than money values. Let $\omega_i \equiv \phi_i / Y_0$ denote the proportion of wealth invested in the ith risky asset
- With ω_i being the key decision variable, we can write (14) as follows

$$\max_{\{\omega_1, \omega_2, \ldots, \omega_N\}} \mathbb{E}\left[U\left(Y_0\left(1+r_f\right)+\sum_{i=1}^{N} \omega_i Y_0\left(\widetilde{r}_i-r_f\right)\right)\right] \tag{15}$$

which can be equivalently written as

$$\max_{\{\omega_1, \omega_2, \ldots, \omega_N\}} \mathbb{E}\left[U\left(Y_0\left\{\left(1+r_f\right)+\sum_{i=1}^{N} \omega_i\left(\widetilde{r}_i-r_f\right)\right\}\right)\right] \tag{16}$$

The portfolio return

* Finally, define the return on the portfolio with weights ω_i as

$$\widetilde{r}_p = \omega_f r_f + \sum_{i=1}^{N} \omega_i \widetilde{r}_i = r_f + \sum_{i=1}^{N} \omega_i \left(\widetilde{r}_i - r_f\right) \tag{17}$$

  where we use the fact that $\omega_f = 1 - \sum_{i=1}^{N} \omega_i$ to arrive at the second equality

* The portfolio choice problem in (16) can then be stated in terms of (17) as

$$\max_{\{\omega_1, \omega_2, \ldots, \omega_N\}} \mathbb{E}\left[U\left(Y_0\left(1 + \widetilde{r}_p\right)\right)\right] \tag{18}$$

  which, of course, corresponds directly to maximizing end-of-period wealth $\widetilde{Y}_1$

* The aim of mean-variance analysis is to solve the above portfolio choice problem under a set of simplifying assumptions, where the most important is that investors have mean-variance preferences

The mean-variance model

The mean-variance model

The mean-variance model of asset choice has been used extensively in finance since its development (Markowitz, 1952, 1959). We can make some observations

- A preference for expected returns and an aversion to variance is implied by the monotonicity and strict concavity of utility functions
- However, expected utility cannot be defined over only mean and variance for arbitrary return distributions and utility functions

- The key question is: "*What assumptions do we need to impose for investors to only care about mean and variance (and not higher-order moments)*"?
- To answer this, consider again a VNM utility maximizer with initial wealth Y_0 that invests in a portfolio to obtain end-of-period wealth $\widetilde{Y}_1 = Y_0 \left(1 + \widetilde{r}_p\right)$
- Let $\widetilde{R}_p = 1 + \widetilde{r}_p$ denote the gross return on the resulting portfolios, then

$$U\left(\widetilde{Y}_1\right) = U\left(Y_0\left(1 + \widetilde{r}_p\right)\right) = U\left(Y_0 \widetilde{R}_p\right) = U\left(\widetilde{R}_p\right) \tag{19}$$

Taylor series expansion for utility

- A Taylor series expansion of $U\left(\widetilde{R}_p\right)$ around the mean $\mathbb{E}\left[\widetilde{R}_p\right]$ yields

$$U\left(\widetilde{R}_p\right) = U\left(\mathbb{E}\left[\widetilde{R}_p\right]\right) + U'\left(\mathbb{E}\left[\widetilde{R}_p\right]\right)\left(\widetilde{R}_p - \mathbb{E}\left[\widetilde{R}_p\right]\right) + \frac{1}{2}U''\left(\mathbb{E}\left[\widetilde{R}_p\right]\right)\left(\widetilde{R}_p - \mathbb{E}\left[\widetilde{R}_p\right]\right)^2 + H_3 \tag{20}$$

  where the remainder term H_3 is given by

$$H_3 = \sum_{i=3}^{\infty} \frac{1}{i!} U'^{i}\left(\mathbb{E}\left[\widetilde{R}_p\right]\right)\left(\widetilde{R}_p - \mathbb{E}\left[\widetilde{R}_p\right]\right)^i \tag{21}$$

- The central question is how to deal with the remainder term H_3 so that it will be compatible with Markowitz's (1952) mean-variance framework

Mean-variance preferences

Mean-variance preferences

Take expectations to arrive at an expression for expected utility defined over gross portfolio returns

$$\mathbb{E}\left[U\left(\widetilde{R}_p\right)\right] = U\left(\mathbb{E}\left[\widetilde{R}_p\right]\right) + \frac{1}{2}U''\left(\mathbb{E}\left[\widetilde{R}_p\right]\right)\sigma^2_{\widetilde{R}_p} + \mathbb{E}\left[H_3\right] \tag{22}$$

which indicates a preference for expected returns and an aversion to variance for an individual with a strictly concave utility function ($U''(\cdot) < 0$), but not exclusively

* There are generally three ways to address the remainder term $\mathbb{E}\left[H_3\right]$
  1. Assume that is it small and negligible, so that it can be safely ignored
  2. Assume quadratic utility, which then allows for arbitrary return distributions
  3. Assume that returns are multivariate normally distributed, which then allows for arbitrary preferences
* See Markowitz (2010) for an excellent discussion of portfolio theory and its development over time

Quadratic utility

- For arbitrary return distributions, the mean-variance model can be motivated by assuming quadratic utility

$$U\left(\widetilde{R}_p\right) = \widetilde{R}_p - \frac{b}{2}\widetilde{R}_p^2, \quad b > 0 \tag{23}$$

for which $U'\left(\widetilde{R}_p\right) = 1 - b\widetilde{R}_p$, $U''\left(\widetilde{R}_p\right) = -b$, and $U'^{i}\left(\widetilde{R}_p\right) = 0, \forall i \geq 3$

- This ensures that $\mathbb{E}\left[H_3\right] = 0$ and we have that investors exhibit mean-variance preferences

$$\begin{aligned}
\mathbb{E}\left[U\left(\widetilde{R}_p\right)\right] &= U\left(\mathbb{E}\left[\widetilde{R}_p\right]\right) + \frac{1}{2}U''\left(\mathbb{E}\left[\widetilde{R}_p\right]\right)\sigma^2_{\widetilde{R}_p} \\
&= \mathbb{E}\left[\widetilde{R}_p\right] - \frac{b}{2}\mathbb{E}\left[\widetilde{R}_p\right]^2 - \frac{1}{2}b\sigma^2_{\widetilde{R}_p} \\
&= \mathbb{E}\left[\widetilde{R}_p\right] - \frac{b}{2}\left(\sigma^2_{\widetilde{R}_p} + \mathbb{E}\left[\widetilde{R}_p\right]^2\right)
\end{aligned} \tag{24}$$

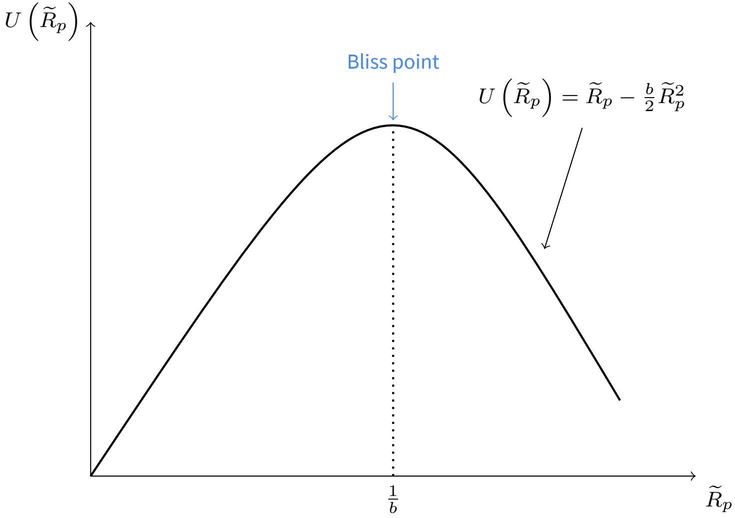
Drawbacks of quadratic utility

- Although quadratic utility leads to mean-variance preferences for any probability distribution of $\widetilde{R}_p$, it is not without problems
  1. Quadratic utility has negative marginal utility for levels of wealth larger than some "bliss point" (or satiation point) given by $\widetilde{R}_p > 1/b$
     - This seems unrealistic, unless "Mo' Money, Mo' Problems"
     - Anecdotal evidence suggests that investors never get tired of money
  2. Quadratic utility displays increasing absolute risk aversion (IARA)

$$R_A\left(\widetilde{R}_p\right) = -\frac{U''\left(\widetilde{R}_p\right)}{U'\left(\widetilde{R}_p\right)} = \frac{b}{1-b\widetilde{R}_p} \tag{25}$$

- Thus, quadratic utility seems restrictive and we may need something different
- Another choice is to make assumptions about the distribution of returns

Illustration: Quadratic utility



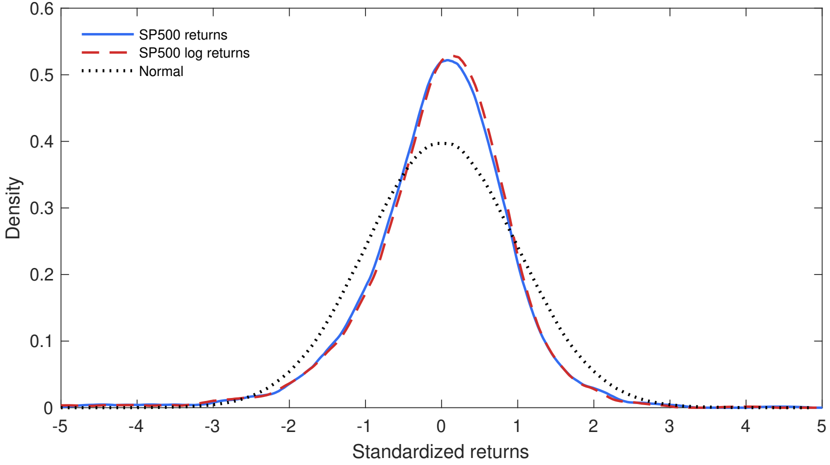
Normally distributed returns

- For arbitrary preferences, the mean-variance model can be motivated by assuming that returns are multivariate normally distributed
- The normal distribution has several attractive features
  1. The normal distribution is fully characterized by mean and variance, so $\mathbb{E}\left[H_3\right]$ is a function of mean and variance only
  2. A weighted average of normal random variables, $\widetilde{r}_p$, is also normally distributed
- Yet, there are also some drawbacks, including
  1. Is a normal distribution a realistic description of asset returns?
  2. Normally distributed returns are inconsistent with the limited liability feature, i.e. $\widetilde{r}_i \geq -1$, for most financial instruments and stocks in particular
  3. Compounding problems: The product of normally distributed random variables is not itself normally distributed
- One can instead consider continuously compounded returns ($Y_0 e^r \geq 0$)

$$\widetilde{r}^c_{i,t} = \ln\left(\frac{\widetilde{q}^e_{i,t+1} + \widetilde{\mathsf{div}}_{i,t+1}}{q^e_{i,t}}\right) \sim \mathcal{N}\left(\mu_i, \sigma_i^2\right) \tag{26}$$

Empirical return distribution

* Consider the empirical distribution of S&P500 (log) returns compared to a normal distribution



Assumptions going forward

* To summarize the above discussions, we will work under the following assumptions going forward

  1. For all risky assets under consideration for portfolio inclusion, $\widetilde{r}_{i,t} \sim \mathcal{N}\left(\mu_i, \sigma_i^2\right)$
  2. Individual asset returns and portfolio returns are continuously compounded
  3. The investor's utility function $U\left(\cdot\right)$ is "homogeneous" so that the same optimal portfolio proportions, once determined, apply for all investable wealth levels
  4. The investor's VNM utility function $\mathbb{U}\left(\cdot\right)$, defined over portfolio return distributions $\widetilde{r}_p$, can be expressed in the form $\mathbb{U}\left(\mu_p, \sigma_p\right)$

* Moreover, investors have one-period investment horizons and take the random returns $\widetilde{r}_{i,t}$ as exogenously given and know the values μ_i and σ_i

Portfolio choice with two assets

Portfolio choice with two assets

Consider a risk averse investor with mean-variance preferences that wishes to maximize expected utility of end-of-period wealth by investing in an optimal portfolio. The investor

- has a preference for expected returns (μ_p), but an aversion to volatility (σ_p)
- wishes to holds portfolios that mean-variance dominate other portfolios, i,e., he seeks portfolios with
  - $\mu_1 \geq \mu_2$ and $\sigma_1 < \sigma_2$, or
  - $\mu_1 > \mu_2$ while $\sigma_1 \leq \sigma_2$

- This allows us to define the efficient portfolio frontier as the locus of all non-dominated portfolios in mean-standard deviation (μ–σ) space
- By definition, no (rational) mean-variance investor would choose to hold a portfolio not located on the efficient frontier

Formulas for mean and variance

* Suppose that there are two assets, asset 1 and asset 2, with expected returns μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively, and return correlation $\rho_{1,2}$

* The expected return on this two-asset portfolio is given by

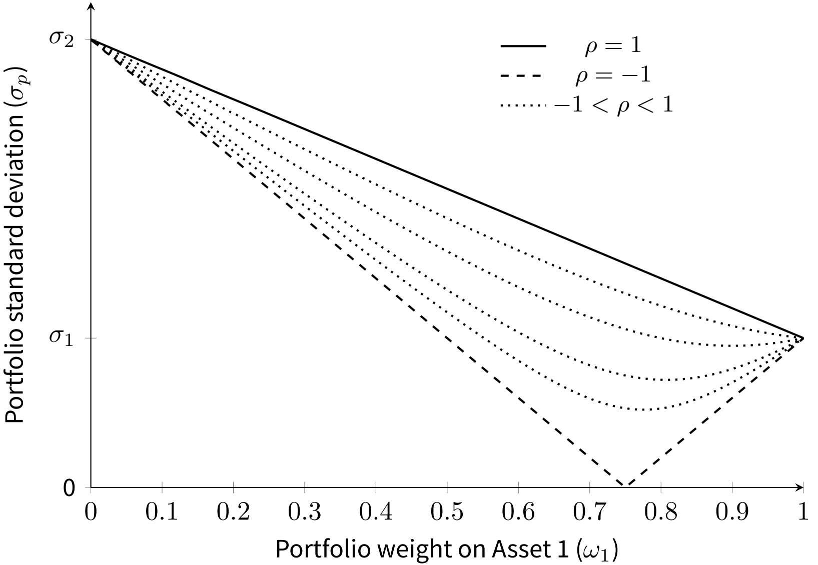
$$\mu_p = \omega_1 \mu_1 + (1 - \omega_1) \mu_2 \tag{27}$$

* The standard deviation of the portfolio is given by

$$\sigma_p = \left[\omega_1^2 \sigma_1^2 + (1 - \omega_1)^2 \sigma_2^2 + 2\omega_1 (1 - \omega_1) \sigma_1 \sigma_2 \rho_{1,2} \right]^{\frac{1}{2}} \tag{28}$$

  from which we immediately notice that the efficient frontier will depend upon the value of $\rho_{1,2}$. That is, there are potentials for diversification benefits

Diversification for different levels of return correlation



Case 1: Perfect positive correlation ($\rho_{1,2} = 1$)

* In the case where the two risky assets are perfectly positively correlated ($\rho_{1,2} = 1$), the efficient frontier becomes linear

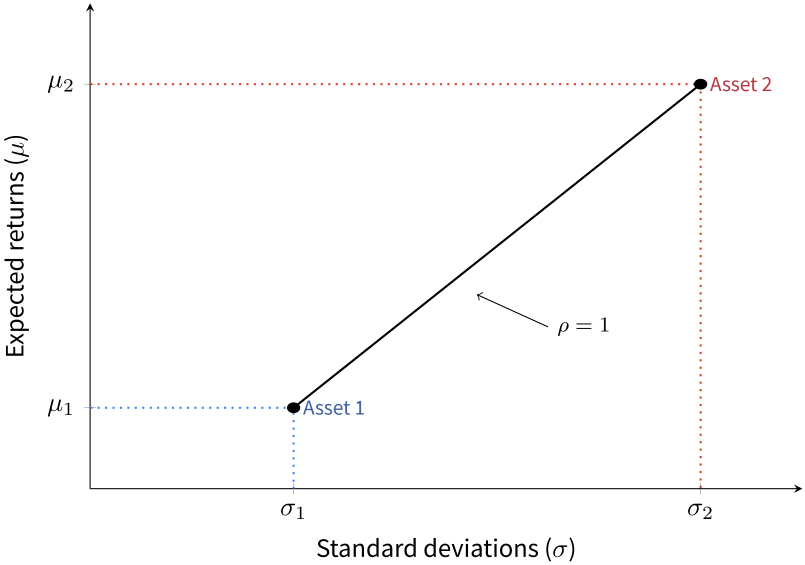
* Since the two assets are essentially identical, there is no diversification gain. To see this, note that the standard deviation in (28) becomes a perfect square such that

$$\begin{aligned}\sigma_p &= \left[\omega_1^2\sigma_1^2 + (1-\omega_1)^2\,\sigma_2^2 + 2\omega_1\,(1-\omega_1)\,\sigma_1\sigma_2\right]^{\frac{1}{2}} \\ &= \omega_1\sigma_1 + (1-\omega_1)\,\sigma_2 \qquad (29)\end{aligned}$$

* The efficient frontier for $\rho_{1,2} = 1$ can then (assuming only positive weights) be expressed as follows

$$\mu_p = \mu_1 + \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}\,(\sigma_p - \sigma_1) \qquad (30)$$

Case 1: Illustration



Case 2: Perfect negative correlation ($\rho_{1,2} = -1$)

* In the case where the two risky assets are perfectly negatively correlated ($\rho_{1,2} = -1$), one can show that the minimum variance portfolio is risk-free and the efficient frontier is linear

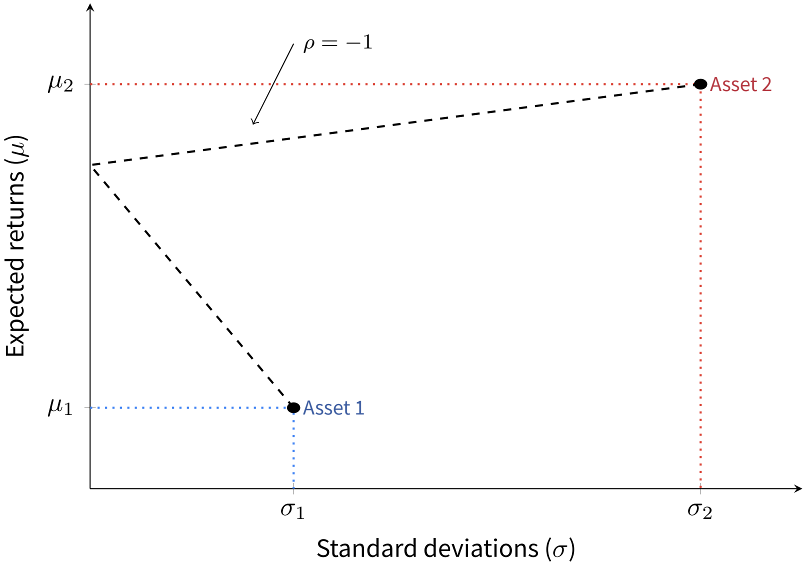
* To see this, note that the standard deviation in (28) again becomes a perfect square such that

$$\begin{aligned}\sigma_p &= \left[\omega_1^2\sigma_1^2 + (1-\omega_1)^2\,\sigma_2^2 - 2\omega_1\,(1-\omega_1)\,\sigma_1\sigma_2\right]^{\frac{1}{2}} \\ &= \omega_1\sigma_1 - (1-\omega_1)\,\sigma_2\end{aligned} \tag{31}$$

* The efficient frontier for $\rho_{1,2} = -1$ can then be expressed as follows

$$\mu_p = \frac{\sigma_1\mu_2 + \sigma_2\mu_1}{\sigma_1 + \sigma_2} \pm \frac{\mu_1 - \mu_2}{\sigma_1 + \sigma_2}\sigma_p \tag{32}$$

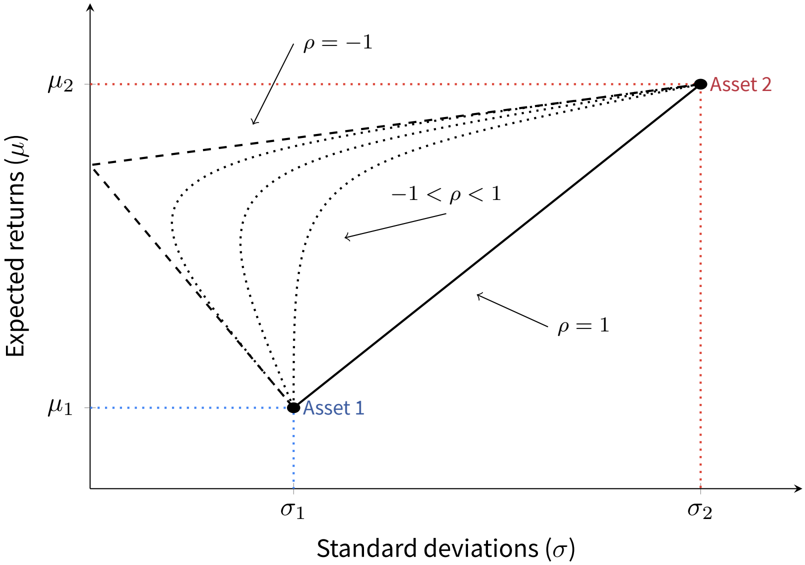
Case 2: Illustration



Case 3: Imperfect correlation ($-1 < \rho_{1,2} < 1$)

* When two risky assets are imperfectly correlated ($-1 < \rho < 1$), we have diversification benefits, i.e. $\sigma_p < \omega_1 \sigma_1 + (1 - \omega_1)\, \sigma_2$
* The efficient frontier must therefore stand left of the straight line obtained in Case 1 and gets farther away as the correlation moves towards –1
* Some portfolios will be dominated by other portfolios, implying that not all portfolios are efficient
* In this way, it makes sense to distinguish between the minimum variance frontier from the efficient frontier

Case 3: Illustration



Case 4: One risk-free and one risky asset

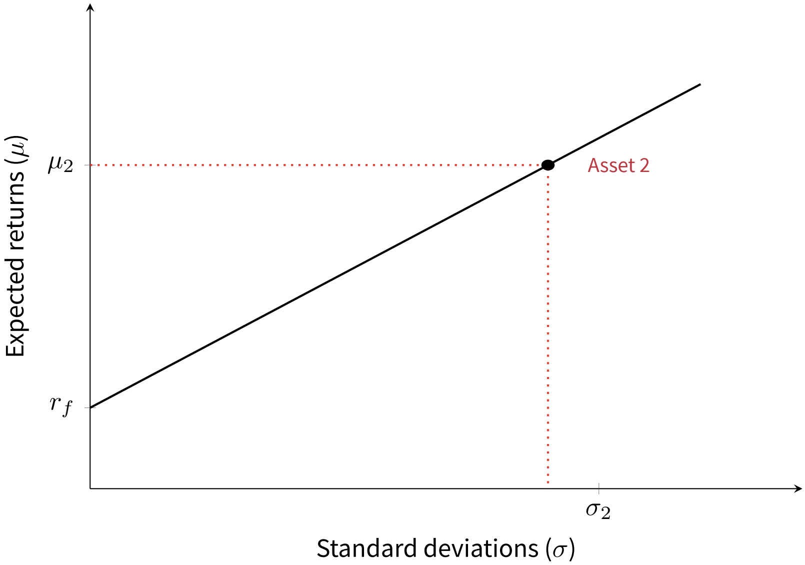
- If one of the two assets is riskless ($\sigma_1 = 0$, say), then the efficient frontier is a straight line originating from the vertical axis at the risk-free return
- In the absence of short selling restrictions, investors are able to borrow at the risk-free rate to leverage their holdings of the risky asset such that the portfolio can be made riskier than the risky asset
- To see this, suppose that Asset 1 is riskless, i.e. $\sigma_1 = 0$ and $\mu_1 = r_f$, such that the standard deviation of the portfolio becomes

$$\sigma_p = (1 - \omega_1)\,\sigma_2 \tag{33}$$

- The efficient frontier can then be expressed as follows

$$\mu_p = r_f + \frac{\mu_2 - r_f}{\sigma_2}\sigma_p \tag{34}$$

Case 4: Illustration



Case 5: N risky assets

* It is important to realize that a portfolio is an asset itself fully defined by its expected return, standard deviation, and correlation with other assets
* The immediate implication being that the two asset examples can be generalized to a setting in which the investor can choose among many assets
* If there are N risky, imperfectly correlated assets, then the efficient frontier of risky assets will take a "bullet" shape
* The key implication here is that adding an extra, imperfectly correlated asset to the set of existing assets improves diversification possibilities, which shifts the efficient frontier to the left

Case 6: One risk-free and N risky assets

- If there a N risky assets and a risk-free one, then the efficient frontier once again becomes a straight line given by

$$\mu_p = r_f + \frac{\mu_T - r_f}{\sigma_T}\sigma_p \tag{35}$$

- Any investor will choose the tangency portfolio (T) on the efficient frontier to combine with the risk-free asset (similar to Case 4)
- The tangency portfolio (T) can be identified as the portfolio along the efficient frontier of risky assets that has the highest Sharpe ratio
- If we allow for short selling of the riskless asset, the efficient frontier can again extend beyond the tangency portfolio

Two-fund separation with risky assets

- Tobin's (1958) two-fund separation theorem is a central result in the mean-variance model as pioneered by Markowitz (1952)
- We know that an optimal portfolio is one that maximizes the investor's (mean-variance) utility of end-of-period wealth
- The two-fund separation theorem for the case of N risky asset states that

  "*Every portfolio on the mean-variance portfolio frontier can be replicated by a combination of any two frontier portfolios; and an individual will be indifferent between choosing among the N financial assets, or choosing a combination of just two frontier portfolios*"

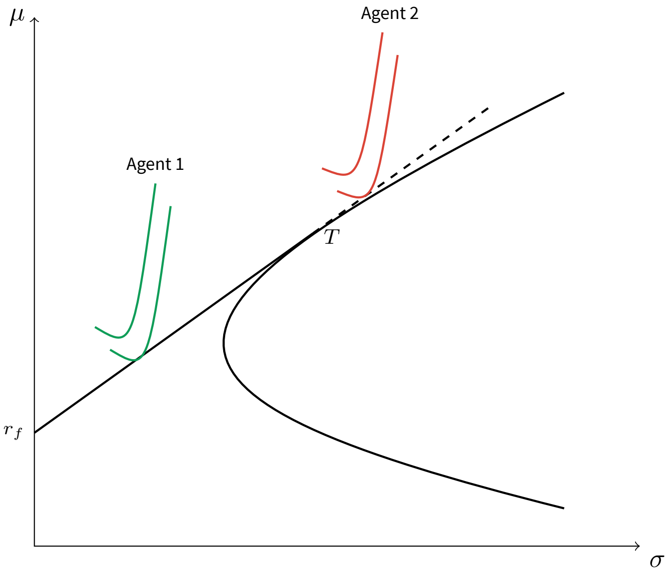
- This powerful finding implies that investors can form their individually preferred frontier portfolios by trading in as little as two frontier portfolios

Two-fund separation with a risk-free asset

- With a risk-free asset, the efficient frontier becomes linear and all tangency points must lie on this line, regardless of the investor's risk aversion
- The two-fund separation theorem obtained is even stronger here and implies that investors will now invest in the same two "funds":

  "*Any portfolio on the efficient frontier can be replicated by a portfolio that holds only the riskless asset and a portfolio on the risky-asset-only frontier. This result implies, if all investor agree on the distribution of asset returns (homogeneous beliefs) and, therefore, the linear efficient frontier, that they will all choose to hold risky asset in the same relative proportion given by the tangency portfolio*"
- This is a hugely important results and implies that *all* investors hold the tangency portfolio and the allocation depends on their risk aversion

Two-fund separation and indifference curves



Portfolio selection with risky assets

- Suppose that there are $N \geq 2$ risky assets traded in a frictionless economy and that their returns are linearly independent

- Define the $N \times 1$ vector of expected asset returns as

$$\mathbb{E}\left(\widetilde{\mathbf{r}}\right) \equiv \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix} \tag{36}$$

- Similarly, define the $N \times N$ variance-covariance matrix of returns as

$$\mathbf{V} \equiv \mathsf{Cov}\left(\widetilde{\mathbf{r}}\right) = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1,N} \\ \vdots & \ddots & \vdots \\ \sigma_{N,1} & \cdots & \sigma_N^2 \end{bmatrix} \tag{37}$$

Portfolio selection with risky assets

* We can also define the $N \times 1$ vector of portfolio weights as

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_N \end{bmatrix} \tag{38}$$

* Finally, let $\mathbf{1}$ denote a $N \times 1$ vector of ones, let

$$\mu_p = \boldsymbol{\omega}^\top \boldsymbol{\mu} \tag{39}$$

be a scalar denoting the required rate of return on the portfolio, and let

$$\sigma_p^2 = \boldsymbol{\omega}^\top \mathbf{V} \boldsymbol{\omega} \tag{40}$$

be a scalar denoting the variance of the return on the resulting portfolio

Portfolio choice problem for N risky assets

Portfolio choice problem for N risky assets

The problem of finding the efficient frontier can now be stated as an optimization problem where the objective is to choose a portfolio with minimum variance for a given expected rate of return

$$\begin{aligned} \min_{\boldsymbol{\omega}} \ & \frac{1}{2}\boldsymbol{\omega}^\top \mathbf{V} \boldsymbol{\omega} \\ \text{s.t. } & \boldsymbol{\omega}^\top \boldsymbol{\mu} = \mu_p \\ & \boldsymbol{\omega}^\top \mathbf{1} = 1 \end{aligned} \tag{41}$$

* Any portfolio, characterized by $\boldsymbol{\omega}_p^*$, that satisfies the above constraints is a frontier portfolio

* Note that the absence of non-negativity constraints implies the allowance of short selling

The Lagrangian and first-order conditions

* We can restate the problem as $\min\limits_{\{\boldsymbol{\omega},\lambda,\gamma\}} \mathcal{L}$, where $\mathcal{L}$ is the Lagrangian

$$\mathcal{L} = \frac{1}{2}\boldsymbol{\omega}^\top \mathbf{V}\boldsymbol{\omega} + \lambda\left(\mu_p - \boldsymbol{\omega}^\top \boldsymbol{\mu}\right) + \gamma\left(1 - \boldsymbol{\omega}^\top \mathbf{1}\right) \tag{42}$$

where the necessary and sufficient first-order conditions (FOCs) are

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\omega}} = \mathbf{V}\boldsymbol{\omega} - \lambda\boldsymbol{\mu} - \gamma\mathbf{1} = \mathbf{0} \tag{43}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \mu_p - \boldsymbol{\omega}^\top \boldsymbol{\mu} = 0 \tag{44}$$

$$\frac{\partial \mathcal{L}}{\partial \gamma} = 1 - \boldsymbol{\omega}^\top \mathbf{1} = 0 \tag{45}$$

Solution

* The solution to the optimization problem in (42) can be shown to be

$$\boldsymbol{\omega}_p^* = \underbrace{\frac{C\mu_p - A}{D}}_{\lambda} \mathbf{V}^{-1}\boldsymbol{\mu} + \underbrace{\frac{B - A\mu_p}{D}}_{\gamma} \mathbf{V}^{-1}\mathbf{1} \tag{46}$$

where

$$A = \mathbf{1}^\top \mathbf{V}^{-1}\boldsymbol{\mu} \tag{47}$$

$$B = \boldsymbol{\mu}^\top \mathbf{V}^{-1}\boldsymbol{\mu} > 0 \tag{48}$$

$$C = \mathbf{1}^\top \mathbf{V}^{-1}\mathbf{1} > 0 \tag{49}$$

$$D = BC - A^2 > 0 \tag{50}$$

An affine function for optimal portfolio weights

* We can equivalently write (46) as an affine function as follows

$$\boldsymbol{\omega}_p^* = \mathbf{g} + \mathbf{h}\mu_p \tag{51}$$

where

$$\mathbf{g} = \frac{1}{D}\left[B\mathbf{V}^{-1}\mathbf{1} - A\mathbf{V}^{-1}\boldsymbol{\mu}\right] \tag{52}$$

$$\mathbf{h} = \frac{1}{D}\left[C\mathbf{V}^{-1}\boldsymbol{\mu} - A\mathbf{V}^{-1}\mathbf{1}\right] \tag{53}$$

* We note that any frontier portfolio can be generated by (51) and that any portfolio that can be represented by (51) is a frontier portfolio

Interpreting the vectors $\mathbf{g}$ and $\mathbf{g}+\mathbf{h}$

- It turns out that the vectors $\mathbf{g}$ and $\mathbf{g}+\mathbf{h}$ can be interpreted as portfolio weights corresponding to two easily recognizable frontier portfolios

- First, we claim that $\mathbf{g}$ is a vector of portfolio weights corresponding to a frontier portfolio with a zero expected returns. Too see this, set $\mu_p = 0$ in (51)

$$\boldsymbol{\omega}_p^* = \mathbf{g} + \mathbf{h} \cdot 0 = \mathbf{g} \tag{54}$$

- Next, we claim that that $\mathbf{g}+\mathbf{h}$ is the vector of portfolio weights of a frontier portfolio with an expected return of 1, i.e.

$$\boldsymbol{\omega}_p^* = \mathbf{g} + \mathbf{h} \cdot 1 = \mathbf{g} + \mathbf{h} \tag{55}$$

The frontier portfolios $\mathbf{g}$ and $\mathbf{g} + \mathbf{h}$ generates the frontier

Proposition

The entire set of frontier portfolios can be generated by $\mathbf{g}$ and $\mathbf{g} + \mathbf{h}$

* Let q denote a frontier portfolio with expected return μ_q, then we know from (51) that the following must hold

$$\boldsymbol{\omega}_q^* = \mathbf{g} + \mathbf{h}\mu_q \tag{56}$$

* Consider the portfolio weights on $\mathbf{g}$ and $\mathbf{g} + \mathbf{h}$: $\{(1 - \mu_q), \mu_q\}$, then we can show that

$$(1 - \mu_q)\,\mathbf{g} + \mu_q\,(\mathbf{g} + \mathbf{h}) = \mathbf{g} + \mathbf{h}\mu_q = \boldsymbol{\omega}_q^* \tag{57}$$

  so that the above chosen portfolio indeed generates the frontier portfolio q

* Since the portfolio q is arbitrarily chosen, we have shown that the entire portfolio frontier can be generated by the two frontier portfolios $\mathbf{g}$ and $\mathbf{g} + \mathbf{h}$

Any two frontier portfolios can generate the frontier

Proposition (Portfolio separation)

The portfolio frontier can be generated by *any* two distinct frontier portfolios, not just the frontier portfolios $\mathbf{g}$ and $\mathbf{g}+\mathbf{h}$

- Let p_1 and p_2 be two distinct frontier portfolios $(\mu_1 \neq \mu_2)$ and let q be *any* frontier portfolio, then there exists a unique real number α such that

$$\mu_q = \alpha\mu_1 + (1-\alpha)\,\mu_2 \tag{58}$$

- Now, consider the portfolio weights on p_1 and p_2: $\{\alpha, (1-\alpha)\}$, where we make use of the result in (51)

$$\begin{aligned}
\alpha\boldsymbol{\omega}^*_{p_1} + (1-\alpha)\,\boldsymbol{\omega}^*_{p_2} &= \alpha\left(\mathbf{g}+\mathbf{h}\mu_1\right) + (1-\alpha)\left(\mathbf{g}+\mathbf{h}\mu_2\right) && (59)\\
&= \mathbf{g} + \mathbf{h}\left[\alpha\mu_1 + (1-\alpha)\,\mu_2\right] && (60)\\
&= \mathbf{g} + \mathbf{h}\mu_q && (61)\\
&= \boldsymbol{\omega}^*_q && (62)
\end{aligned}$$

- This shows that the portfolio frontier can be generated by any two distinct frontier portfolios as stated in the portfolio separation result

Variance of the frontier portfolio

* The variance of any frontier portfolio can be written as

$$\sigma_p^2(\mu_p) = \boldsymbol{\omega}_p^{*\top} \mathbf{V} \boldsymbol{\omega}_p^* \tag{63}$$

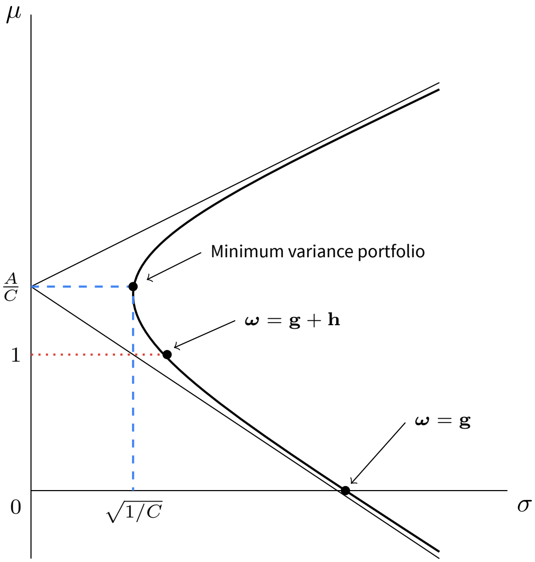
$$= \lambda \mu_p + \gamma \tag{64}$$

$$= \frac{C}{D}\left(\mu_p - \frac{A}{C}\right)^2 + \frac{1}{C} \tag{65}$$

* We know that $C, D > 0$, so the variance possess the following properties

1. The expected return on the minimum variance portfolio is $\frac{A}{C}$
2. The variance of the minimum variance portfolio is given by $\frac{1}{C}$
3. Equation (65) defines a parabola with vertex $\left(\frac{1}{C}, \frac{A}{C}\right)$ in expected return-variance space and a hyperbola in expected return-standard deviation space

Frontier portfolios: $\mu - \sigma$ space



Characterizing efficient portfolios

Efficient portfolios

Efficient portfolios are those frontier portfolios for which the expected return μ_p exceeds $^A/_C$, the expected return on the minimum variance portfolio

* Let $\boldsymbol{\omega}_i$, $i = 1,2,\ldots,m$ be m frontier portfolios and α_i, $i = 1,2,\ldots,m$ be real numbers so that $\sum_{i=1}^{m} \alpha_i = 1$, then

$$\sum_{i=1}^{m} \alpha_i \boldsymbol{\omega}_i = \sum_{i=1}^{m} \alpha_i \left(\mathbf{g} + \mathbf{h}\mu_i \right) = \mathbf{g} + \mathbf{h} \sum_{i=1}^{m} \alpha_i \mu_i \tag{66}$$

  any linear combination of frontier portfolios is also a frontier portfolio

* If the portfolios $\boldsymbol{\omega}_i$, $i = 1,2,\ldots,m$ are efficient, and if α_i, $i = 1,2,\ldots,m$ are non-negative, then

$$\sum_{i=1}^{m} \alpha_i \mu_i \geq \sum_{i=1}^{m} \alpha_i \frac{A}{C} = \frac{A}{C} \tag{67}$$

  any linear combination of efficient portfolios will also be an efficient portfolio and the set of efficient portfolios is a convex set

Portfolio selection with a risk-free asset

Portfolio choice problem for N risky assets and one risk-free

Consider a setup with N risky assets with expected return vector $\boldsymbol{\mu}$ and one risk-free asset with expected return r_f. The portfolio optimization problem can then be written as

$$\begin{aligned} &\min_{\boldsymbol{\omega}} \ \frac{1}{2}\boldsymbol{\omega}^\top \mathbf{V}\boldsymbol{\omega} \\ &\text{s.t. } \boldsymbol{\omega}^\top \boldsymbol{\mu} + \left(1 - \boldsymbol{\omega}^\top \mathbf{1}\right) r_f = \mu_p \end{aligned} \tag{68}$$

- As before, we can once again form the Lagrangian instead as

$$\mathcal{L} = \frac{1}{2}\boldsymbol{\omega}^\top \mathbf{V}\boldsymbol{\omega} + \delta\left(\mu_p - \boldsymbol{\omega}^\top \boldsymbol{\mu} - \left(1 - \boldsymbol{\omega}^\top \mathbf{1}\right) r_f\right) \tag{69}$$

with necessary and sufficient FOCs

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\omega}} = \mathbf{V}\boldsymbol{\omega} - \delta\left(\boldsymbol{\mu} - r_f \mathbf{1}\right) = 0 \tag{70}$$

$$\frac{\partial \mathcal{L}}{\partial \delta} = \mu_p - \boldsymbol{\omega}^\top \boldsymbol{\mu} - \left(1 - \boldsymbol{\omega}^\top \mathbf{1}\right) r_f = 0 \tag{71}$$

Solution

* The solution to the optimization problem in (69) can be shown to be

$$\boldsymbol{\omega}_p^* = \frac{\mu_p - r_f}{H}\mathbf{V}^{-1}\left(\boldsymbol{\mu} - r_f\mathbf{1}\right) \tag{72}$$

where

$$\begin{aligned} H &= \left(\boldsymbol{\mu} - r_f\mathbf{1}\right)^\top \mathbf{V}^{-1}\left(\boldsymbol{\mu} - r_f\mathbf{1}\right) && (73)\\ &= B - 2Ar_f + Cr_f^2 && (74)\end{aligned}$$

and A, B, and C are defined as before, i.e.

$$\begin{aligned} A &= \mathbf{1}^\top\mathbf{V}^{-1}\boldsymbol{\mu} && (75)\\ B &= \boldsymbol{\mu}^\top\mathbf{V}^{-1}\boldsymbol{\mu} > 0 && (76)\\ C &= \mathbf{1}^\top\mathbf{V}^{-1}\mathbf{1} > 0 && (77)\end{aligned}$$

Variance of the frontier portfolio with risk-free asset

- The variance of any frontier portfolio can be written as

$$\sigma_p^2\left(\mu_p\right) = \boldsymbol{\omega}_p^{*\top}\mathbf{V}\boldsymbol{\omega}_p^* = \frac{\left(\mu_p - r_f\right)^2}{H} \tag{78}$$

- Taking the square root of each side of (78) and rearranging yields

$$\mu_p = r_f \pm \sigma_p H^{\frac{1}{2}} \tag{79}$$

  which shows that the efficient frontier is linear and defined by a line that goes through r_f and the tangency portfolio with slope $H^{\frac{1}{2}}$

Tangency portfolio

Tangency portfolio

The tangency portfolio is the only frontier portfolio composed only of risky assets, i.e., $\mathbf{1}^\top \boldsymbol{\omega}_T^* = 1$, whose weights are determined as

$$\boldsymbol{\omega}_T^* = \frac{1}{A - Cr_f} \mathbf{V}^{-1} \left(\boldsymbol{\mu} - r_f \mathbf{1} \right) \tag{80}$$

- To see this, note that $\mathbf{1}^\top \boldsymbol{\omega}_T^* = 1$ implies that

$$\mathbf{1}^\top \boldsymbol{\omega}_T^* = \frac{\mu_T - r_f}{H} \mathbf{1}^\top \mathbf{V}^{-1} \left(\boldsymbol{\mu} - r_f \mathbf{1} \right) = 1 \tag{81}$$

- Solving for μ_T yields the following expression

$$\mu_T = r_f + \frac{H}{A - Cr_f} \tag{82}$$

which when substituted back into (72) yields the final solution

Portfolio separation with a risk-free asset

- Recall that the expected return on the minimum variance portfolio is $\frac{A}{C}$
- As long as $r_f \neq \frac{A}{C}$, then the separation principle applies:

  "*Any frontier portfolio can be replicated with one portfolio that is located on the risky-asset-only frontier and another portfolio that holds only the riskless asset*"

- We note that this statement implies the existence of three special cases, namely

  1. $r_f < \frac{A}{C}$
  2. $r_f > \frac{A}{C}$
  3. $r_f = \frac{A}{C}$

Special case: $r_f < A/C$

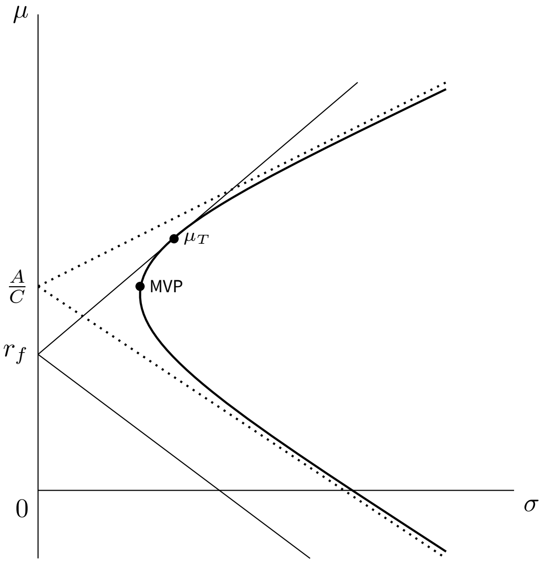
* Consider the first special case where $r_f < A/C$. Let r_f be the return on the risk-free asset and let μ_T denote the tangent point of the half line $r_f + \sigma_p H^{\frac{1}{2}}$ and the portfolio frontier of all risky assets

    1. Any portfolio on the line segment from r_f to μ_T is a convex combination of portfolio μ_T and the riskless asset
    2. Any portfolio on the half line $r_f + \sigma_p H^{\frac{1}{2}}$ beyond the tangent point μ_T involves short-selling the riskless asset and investing the proceeds in μ_T
    3. Any portfolio on the half line $r_f - \sigma_p H^{\frac{1}{2}}$ involves short-selling μ_T and investing the proceeds in the riskless asset

Two-fund separation theorem: $r_f < A/C$

Any portfolio on the efficient frontier can be replicated by a portfolio consisting of only the riskless asset and a portfolio of the risky-asset-only frontier. This result implies, if all investor agree on the distribution of asset returns (homogeneous beliefs) and, therefore, the linear efficient frontier, that they will all choose to hold risky asset in the same relative proportion given by the tangency portfolio

Illustration: $r_f < A/C$

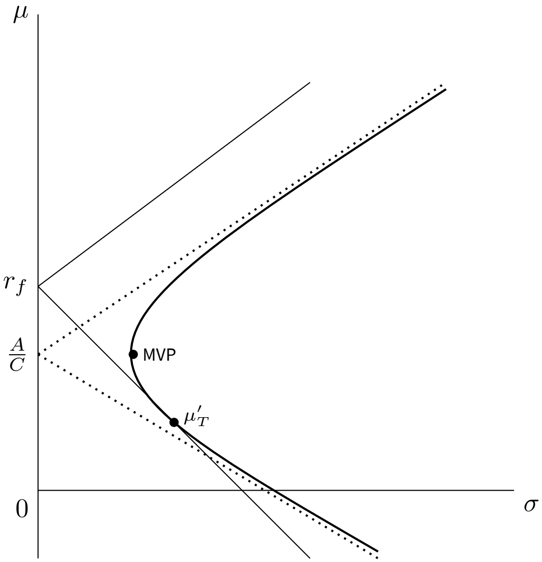


Special case: $r_f > A/C$

- The story for the second special case, $r_f > A/C$, is slightly different

  1. Any portfolio on the half line $r_f + \sigma_p H^{\frac{1}{2}}$ involves short-selling μ'_T and investing the proceeds in the riskless asset
  2. Any portfolio on the half line $r_f - \sigma_p H^{\frac{1}{2}}$, on the other hand, involves a long position in the portfolio μ'_T

Illustration: $r_f > A/C$



Special case: $r_f = A/C$

* Last, consider the special case where $r_f = A/C$, which implies that

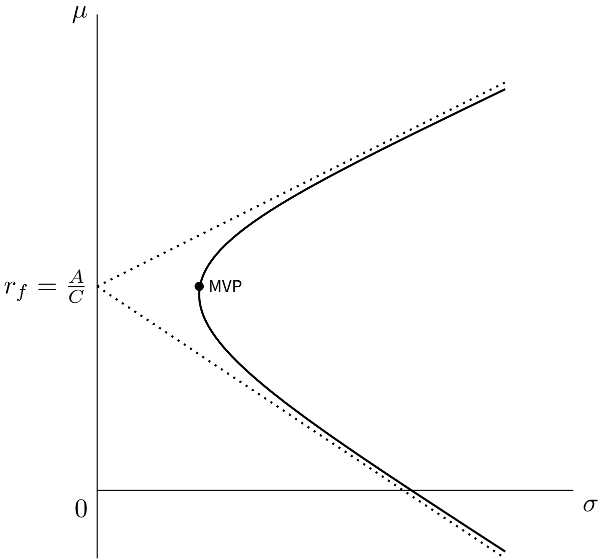
$$H = B - 2Ar_f + Cr_f^2 \tag{83}$$
$$= B - 2A\left(\frac{A}{C}\right) + C\left(\frac{A}{C}\right)^2 \tag{84}$$
$$= \frac{BC - A^2}{C} = \frac{D}{C} > 0 \tag{85}$$

* This in turn implies that we can write the risky asset portfolio frontier as

$$\mu_p = r_f \pm \sigma_p \left(\frac{D}{C}\right)^{\frac{1}{2}} \tag{86}$$

* In the other cases, the efficient frontier for all asset were generated by the riskless asset and the tangency portfolios μ_T and μ_T', but in this case there is no tangency portfolio

Illustration: $r_f = A/C$



Special case: $r_f = A/C$

* The question then becomes: "How is the efficient frontier of all assets generated?"

* We can answer this by substituting $r_f = A/C$ into the expression

$$\boldsymbol{\omega}_p^* = \frac{\mu_p - r_f}{H} \mathbf{V}^{-1} \left(\boldsymbol{\mu} - r_f \mathbf{1} \right) \tag{87}$$

and multiplying with $\mathbf{1}^\top$ to obtain

$$\begin{aligned} \mathbf{1}^\top \boldsymbol{\omega}_p^* &= \frac{\mu_p - r_f}{H} \mathbf{1}^\top \mathbf{V}^{-1} \left(\boldsymbol{\mu} - \frac{A}{C} \mathbf{1} \right) && (88) \\ &= \frac{\mu_p - r_f}{H} \left(A - \frac{A}{C} C \right) && (89) \\ &= 0 && (90) \end{aligned}$$

* Any portfolio on the efficient frontier of all assets therefore involves investing everything in the riskless asset and holding an *arbitrage portfolio* of risky assets, that is, a portfolio whose weights sum to zero

Remarks on long-run portfolio management

Investment advice from the mean-variance model

The canonical portfolio problem and the mean-variance model are both one-period utility-of-terminal-wealth maximization problems. The investment advice implicit in these theories are astonishingly simple, but generally very powerful

1. **Be well diversified:** This implies that investors' risky portion of a portfolio should resemble the tangency portfolio. That is, we should eliminate idiosyncratic risk
2. **Be on the capital market line:** Investors should, according to their risk preferences, allocate wealth between the risk-free asset and the tangency portfolio

- The big question, however, is: what should investors do in the next period once this period's risky portfolio return realization has been observed?
- If investors behave myopically (i.e., short-sighted), they will simply repeat the maximization period-by-period with (potentially) updated inputs

Revisiting the one-period setting

* In the canonical portfolio problem, an investor with initial wealth Y_0 is deciding what money amount ϕ to invest in a risky asset and what amount $(Y_0 - \phi)$ to invest in a risk-free asset paying r_f

$$\begin{aligned} \widetilde{Y}_1 &= (Y_0 - \phi)(1 + r_f) + \phi(1 + \widetilde{r}) \\ &= Y_0 (1 + r_f) + \phi(\widetilde{r} - r_f) \end{aligned} \tag{91}$$

* In a one-period setting for an investor with a VNM utility function, the objective is to maximize expected utility of the end-of-the-single-period wealth (or consumption), i.e.

$$\max_{\phi} \mathbb{E}\left[U\left(\widetilde{Y}_1\right)\right] = \max_{\phi} \mathbb{E}\left[U\left(Y_0 (1 + r_f) + \phi(\widetilde{r} - r_f)\right)\right] \tag{92}$$

* However, most investors do not face one-period wealth problems, but rather multiperiod consumption problems in which we also have to take savings and labor income into account

Introducing the joint savings-consumption problem

- Consider a two-period economy in which the investor faces the following consumption-savings problems

$$\max_{S} U\left(Y_0 - S\right) + \delta \mathbb{E}\left[U\left(S\left(1+\widetilde{r}_p\right)\right)\right] \tag{93}$$
$$\text{s.t.} \quad Y_0 \geq S \geq 0$$

  where S denotes the amount saved and $\widetilde{r}_p$ denotes the return on a risky portfolio consisting of a risk-free asset and the market portfolio (for simplicity)

- In this model, the investor chooses how much to consume today of initial wealth $Y_0 - S$ and much to save S for future consumption

- As such, we can equivalent write the above expression in terms of consumption, i.e. $U\left(Y_0 - S\right) = U\left(C_0\right)$ and $U\left(S\left(1+\widetilde{r}_p\right)\right) = U\left(\widetilde{C}_1\right)$

The joint savings-consumption-portfolio problem

* Suppose that $\widetilde{r}$ is the return on a broad market portfolio, then we can combine the canonical portfolio problem and the savings-consumption problem to obtain

$$\max_{\{\phi,S\}} U\left(Y_0 - S\right) + \delta \mathbb{E}\left[U\left(S\left(1 + r_f\right) + \phi\left(\widetilde{r} - r_f\right)\right)\right] \tag{94}$$

* As in the canonical portfolio problem, we can use weights rather than money values by defining $\omega = \frac{\phi}{S}$ such that the problem becomes

$$\max_{\{\omega,S\}} U\left(Y_0 - S\right) + \delta \mathbb{E}\left[U\left(S\left\{\left(1 + r_f\right) + \omega\left(\widetilde{r} - r_f\right)\right\}\right)\right] \tag{95}$$

Introducing outside labor income

* Investors have only been endowed with some initial wealth Y_0 so far. Yet, investors (and most individuals in general) receive outside labor income

* Suppose that the investor receive labor income of L_0 in the initial period and $\widetilde{L}$ at the end of the two-period economy, then

$$\max_{\{\omega,S\}} U\left(Y_0 + L_0 - S\right) + \delta\mathbb{E}\left[U\left(S\left\{(1+r_f) + \omega\left(\widetilde{r} - r_f\right)\right\} + \widetilde{L}\right)\right] \tag{96}$$

* We can once again write the maximization problem in terms of consumption

$$\begin{aligned} &\max_{\{\omega,S\}} U\left(C_0\right) + \delta\mathbb{E}\left[U\left(\widetilde{C}_1\right)\right] \\ \text{s.t. } &C_0 \leq Y_0 + L_0 - S \\ &C_1 = S\left\{(1+r_f) + \omega\left(\widetilde{r} - r_f\right)\right\} + \widetilde{L} \end{aligned} \tag{97}$$

Extending the problem to multiple periods

- We can generalize (97) to a T-period VNM utility maximizing investor-saver with outside labor income as

$$\max_{\{\omega_t, S_t\}} \mathbb{E}\left[\sum_{t=0}^{T} \delta^t U\left(\widetilde{C}_t\right)\right] \tag{98}$$

$$\text{s.t. } C_0 \leq Y_0 + L_0 - S$$

$$C_t \leq S_{t-1}\left\{(1 + r_{f,t}) + \omega_{t-1}\left(\widetilde{r}_t - r_{f,t}\right)\right\} + \widetilde{L}_t - S_t$$

$$C_T = S_{T-1}\left\{(1 + r_{f,T}) + \omega_{T-1}\left(\widetilde{r}_T - r_{f,T}\right)\right\} + \widetilde{L}_T$$

- The problem in (98) is not easy to solve by any means, but imposing a set of assumptions and exploring special cases will provide some insights

The myopic solution

- Consider the special case in which the horizon has no effect on the investor's portfolio allocations so that the investor behaves myopically

- This special case needs special assumption that ensures that the investment opportunities are unchanged over time, e.g., that

  1. Investors exhibit power utility with a constant relative risk aversion (γ)
  2. The risk-free rate is constant over time, i.e., $r_{f,t} = r_f$
  3. Returns on the risky asset are independently and identically distributed (iid), which coupled with the lognormality assumption implies $\ln \widetilde{R} = \widetilde{r} \sim \mathcal{N}\left(\mu_r, \sigma_r^2\right)$
  4. Finally, we need outside labor income to be absent from the economy, i.e., $L_t \equiv 0$

Merton (1969,1971) Theorem

Consider the canonical multiperiod consumption-savings-portfolio allocation problem in (98). If the above assumptions are satisfied, then the proportion $\omega_t = \omega$ invested in the risky asset is time invariant (Merton, 1969, 1971)

The Merton ratio

The Merton ratio

One can show that the constant fraction $\omega^\star$, known as the Merton ratio, will equal

$$\omega^\star = \frac{\mu_T - r_f}{\gamma \sigma_T^2} \tag{99}$$

where we immediately observe that the fraction $\omega^\star$ invested in the tangency is time-invariant for a stationary environment and independent of the age and the prosperity of the investor

- This is a hugely important result as it delineates the conditions under which the static, one-period allocation continues to characterize the multiperiod allocation
- The Merton ratio is also the solution to the question: what fraction should an investor with relative risk aversion γ allocate to the tangency portfolio, i.e., where should she place herself of the capital allocation line

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