CHAPTER 15

An Intuitive Overview of Continuous Time Finance

Chapter Outline

15.1 Introduction

If we think of stock prices as arising from the equilibration of traders' demands and supplies, then the binomial model is implicitly one in which security trading occurs at discrete time intervals, however short, and this is, in fact, factually what actually happens. It will be mathematically convenient, however, to abstract from this intuitive setting and hypothesize that trading takes place "continuously." This is consistent with the notion of continuous compounding. But it is not fully realistic: It implies that an uncountable number of individual transactions may transpire in any interval of time, however small, which is physically impossible.

Continuous time finance is principally concerned with techniques for the pricing of derivative securities under the fiction of continuous trading. These techniques frequently allow closed-form pricing solutions to be obtained—at the cost of working in a context that is less intuitive than discrete time. In the present chapter, we hope to convey some idea as to how this is done.

We will need first to develop a continuous time model of a stock's price evolution through time. Such a model must respect the statistical regularities that are known to characterize, empirically, equity returns as first formalized in Section 7.5.2:

1. Stock prices are lognormally distributed, which means that returns (continuously compounded) are normally distributed.
2. For short time horizons, stock returns are independently and identically distributed (iid) over nonoverlapping time intervals.

After we have faithfully represented these equity regularities in a continuous time setting, we will move on to a consideration of derivatives pricing. In doing so, we aim to give some idea how the principles of risk-neutral valuation carry over to this specialized setting. The discussion aims at intuition; no attempt is made to be mathematically complete.

In all cases, this intuition has its origins in the discrete time context. This leads us initially to review the notion of a random walk.

15.2 Random Walks and Brownian Motion

Consider a time horizon composed of N adjacent time intervals, each of duration Δt, and indexed successively by t_0, t_1, t_2, ... t_N, i.e.,

$$t_i - t_{i-1} = \Delta t, \ i = 1, 2, \ldots, N$$

We define a discrete time stochastic process, $\tilde{x}$, on this succession of time indices by

$$\begin{aligned} x(t_0) &= 0 \\ \tilde{x}(t_{j+1}) &= \tilde{x}(t_j) + \tilde{\varepsilon}(t_j)\sqrt{\Delta t}, \\ &\quad j = 0, 1, 2, \ldots, N-1 \end{aligned}$$

where, for all $j, \tilde{\varepsilon}(t_j) \sim N(0,1)$. It is further assumed that the random factors $\tilde{\varepsilon}(t_j)$ are independent of one another which implies

$$E(\tilde{\varepsilon}(t_j)\tilde{\varepsilon}(t_i)) = 0, \ i \neq j$$

This is a specific example of a random walk, particular in the sense that the uncertain disturbance term follows a specific distribution.[1]

We are interested in understanding the behavior of a random walk over extended time periods. More precisely, we want to characterize the statistical properties of the incremental difference

$$x(t_k) - x(t_j) \text{ for any } j < k.$$

Clearly,

$$\tilde{x}(t_k) - x(t_j) = \sum_{i=j}^{k-1} \tilde{\varepsilon}(t_i)\sqrt{\Delta t}$$

Since the random disturbances $\tilde{\varepsilon}(t_i)$ all have mean zero,

$$E(\tilde{x}(t_k) - x(t_j)) = 0$$

Furthermore,

$$\begin{aligned}
\operatorname{var}(\tilde{x}(t_k) - x(t_j)) &= E\left(\sum_{i=j}^{k-1} \tilde{\varepsilon}(t_i)\sqrt{\Delta t}\right)^2 \\
&= E\left(\sum_{i=j}^{k-1} [\tilde{\varepsilon}(t_i)]^2 \sqrt{\Delta t}\right) \\
&\text{(by independence)} \\
&= \sum_{i=j}^{k-1} (1)\Delta t = (k-j)\Delta t, \\
&\text{since } E[\tilde{\varepsilon}(t_i)]^2 = 1.
\end{aligned}$$

If we identify

$$x_{t_j} = \ln q^e_{t_j}$$

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[1] A very simple random walk is of the form $\tilde{x}(t_{j+1}) = x(t_j) + \tilde{n}(t_i)$, where for all $j = 0, 1, 2 \ldots$

$$\tilde{n}(t_j) = \begin{cases} +\,1, & \text{if a coin is flipped and heads appears} \\ -\,1, & \text{if a coin is flipped and tails appears.} \end{cases}$$

At each time interval $x(t_j)$, either increases or diminishes by one depending on the outcome of the coin toss. Suppose we think of $x(t_0) \equiv 0$ as representing the center of the sidewalk where an intoxicated person staggers one step to the right or to the left of the center in a manner that is consistent with independent coin flips (heads implies to the right). This example is the source of the term *random walk*.

where $q^e_{t_j}$ is the price of the stock at time t_j, then this simple random walk model becomes a candidate for our model of stock price evolution beginning from $t = 0$: At each node t_j, the logarithm of the stock's price is distributed normally, with mean $\ln q^e_{t_0}$ and variance $j\,\Delta t$.[2]

Since the discrete time random walk is so respectful of the empirical realities of stock returns, it is natural to seek its counterpart for "continuous time." This is referred to as a *Brownian motion* (or a Wiener process), and it represents the limit of the discrete time random walk as we pass to continuous time, i.e., as $\Delta t \mapsto 0$. It is represented symbolically by

$$\mathrm{d}z = \tilde{\varepsilon}(t)\sqrt{\mathrm{d}t}$$

where $\tilde{\varepsilon}(t) \sim N(0,1)$, and for any times t, t' where $t \neq t'$, and $\tilde{\varepsilon}(t), \tilde{\varepsilon}(t')$ are independent. We used the word *symbolically* not only because the term $\mathrm{d}z$ does not represent a differential in the terminology of ordinary calculus, but because we make no attempt here to describe how such a limit is taken. Following what is commonplace notation in the literature, we don't write a $\sim$ over z even though it represents a random quantity.

More formally, a stochastic process $z(t)$ defined on $[0, T]$ is a Brownian motion provided the following three properties are satisfied:

1. for any $t_1 < t_2, z(t_2) - z(t_1)$ is normally distributed with mean zero and variance $t_2 - t_1$;
2. for any $0 \leq t_1 < t_2 \leq t_3 < t_4$, $z(t_4) - z(t_3)$ is statistically independent of $z(t_2) - z(t_1)$; and
3. $z(t_0) \equiv 0$ with probability one.

A Brownian motion is a very unusual stochastic process, and we can only give a hint about what is actually transpiring as it evolves. Three of its properties are presented below:

1. First, a Brownian motion is a continuous process. If we were able to trace out a sample path $z(t)$ of a Brownian motion, we would not see any jumps.[3]
2. However, the sample path is not at all *smooth* and is, in fact, as "jagged as can be," which we formalize by saying that it is nowhere differentiable. A function must be essentially smooth if it is to be differentiable. That is, if we magnify a segment of its time path sufficiently, it will appear approximately linear. This *smoothness* is totally absent in a Brownian motion.
3. Lastly, a Brownian motion is of *unbounded variation*. This is perhaps the least intuitive of its properties. This conveys the idea that if we could take one of those mileage wheels that are drawn along a route on a map to assess the overall distance (each revolution of the wheel corresponding to a fixed number of kilometers) and apply it to

[2] To be absolutely clear in our use of terminology, the random walk is a model of a stock's rate of return evolution, and thus, indirectly, its price evolution. Differences in logs of stock prices represent a rate of return.

[3] At times such as the announcement of a takeover bid, stock prices exhibit jumps. We will not consider such *jump processes*, although considerable research effort has been devoted to studying them, and to the pricing of derivatives written on them.

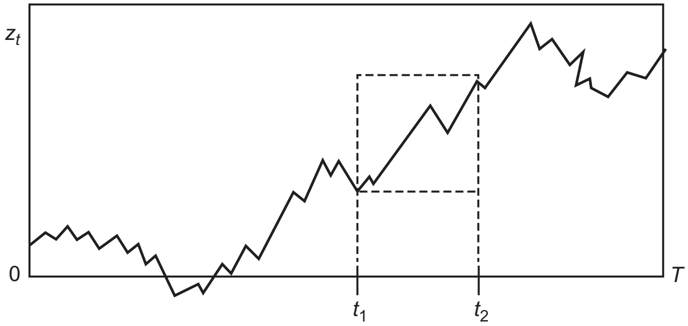

Figure 15.1
Approximate Brownian Motion Sample Path.

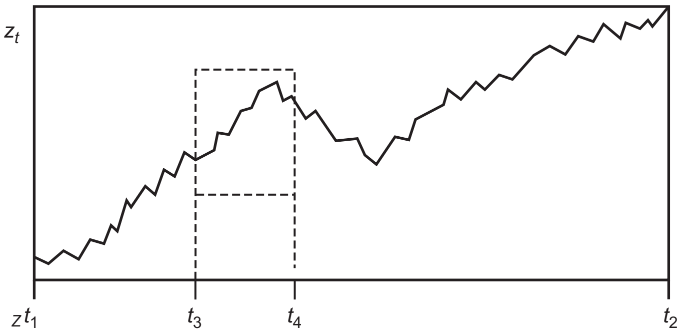

Figure 15.2
Approximate Brownian Motion Sample Path: Expanded Sub-Path of Figure 15.1.

the sample path of a Brownian motion, no matter how small the time interval, the mileage wheel would record an *infinite distance* (if it ever got to the end of the path!).[4]

One way of visualizing such a process is to imagine a rough sketch of a particular sample path where we connect its position at a sequence of discrete time intervals by straight lines. Figure 15.1 proposes one such path.

Suppose that we were next to enlarge the segment between time intervals t_1 and t_2. We would find something on the order of Figure 15.2.

Continue this process of taking a segment, enlarging it, taking another subsegment of that segment, enlarging it, etc. (in Figure 15.2 we could next enlarge the segment from t_3 to t_4). Under a typical differentiable function of bounded variation, we would eventually be enlarging such a small segment that it would appear as a straight line. With a Brownian motion, however, this will never happen. No matter how much we enlarge even a segment that corresponds to an arbitrarily short time interval, the same "sawtooth" pattern will appear, and there will be many, many "teeth."

[4] This reference to a 'mileage wheel' certainly dates your authors (now see Google Maps).

A Brownian motion represents a special case of a continuous process with independent increments. For such processes, the standard deviation per unit of time becomes unbounded as the time interval becomes small:

$$\lim_{\Delta\to 0}\frac{\sigma\sqrt{\Delta t}}{\Delta t}=\lim_{\Delta\to 0}\frac{\sigma}{\sqrt{\Delta t}}=\infty$$

No matter how small the time period, proportionately, a **lot** of variation remains. This constitutes our abstraction of a random walk to a context of continuous trading.[5]

15.3 More General Continuous Time Processes

A Brownian motion will be the principal building block of our description of the continuous time evolution of a stock's price—it will be the *engine* or *source* of the uncertainty. To it is often added a deterministic component intended to capture the "average" behavior through time of the process. Together we have something of the form

$$\begin{aligned} \mathrm{d}x(t) &= a\,\mathrm{d}t + b\tilde{\varepsilon}(t)\sqrt{\mathrm{d}t} \\ &= a\,\mathrm{d}t + b\,\mathrm{d}z \end{aligned} \tag{15.1}$$

where the first component is the deterministic one and a is referred to as the drift term.

This is an example of a generalized Brownian motion or, to use more common terminology, a generalized Wiener process. If there were no uncertainty, $x(t)$ would evolve deterministically: if we integrate

$$\mathrm{d}x(t)=a\,\mathrm{d}t,\ \text{ we obtain}$$
$$x(t)=x(0)+at$$

The solution to Eq. (15.1) is thus of the form

$$x(t)=x(0)+at+bz(t) \tag{15.2}$$

where the properties of $z(t)$ were articulated earlier (recall conditions 1, 2, and 3 of the definition). These imply that

$$\begin{aligned} E(x(t)) &= x(0)+at, \\ \mathrm{var}(x(t)) &= b^2t,\ \text{and} \\ \mathrm{SD}(x(t)) &= b\sqrt{t} \end{aligned}$$

[5] The name Brownian motion comes from a nineteenth century physicist named Brown, who studied the behavior of dust particles floating on the surface of water. Under a microscope, dust particles are seen to move randomly about in a manner similar to the sawtooth pattern shown except that the motion can be in any 360° direction. The interpretation of the phenomena is that the dust particles experience the effect of random collisions by water molecules.

Equation (15.2) may be further generalized to allow the coefficients to depend upon the time and the current level of the process:

$$\mathrm{d}x(t) = a(x(t), t)\mathrm{d}t + b(x(t), t)\mathrm{d}z \tag{15.3}$$

In this latter form, it is referred to as an Ito process after one of the earliest and most important developers of this field. An important issue in the literature—but one we will eschew—is to determine the conditions on $a(x(t), t)$ and $b(x(t), t)$ in order for Eq. (15.3) to have a solution. Equations (15.1) and (15.3) are generically referred to as *stochastic differential equations*.

Given this background, we now return to the original objective of modeling the behavior of a stock's price process.

15.4 A Continuous Time Model of Stock Price Behavior

Let us now restrict our attention only to those stocks that do not pay dividends, so that stock returns are exclusively identified with price changes (we will maintain this assumption throughout the chapter). Our basic discrete time model formulation is

$$\ln q^e(t + \Delta t) - \ln q^e(t) = \mu\, \Delta t + \sigma\tilde{\varepsilon}\sqrt{\Delta t} \tag{15.4}$$

Note that the stochastic process is imposed on differences in the logarithm of the stock's price. Equation (15.4) thus asserts that the continuously compounded return to the stock's ownership over the time period t to $t + \Delta t$ is distributed normally with mean $\mu\, \Delta t$ and variance $\sigma^2\, \Delta t$.

This is clearly a lognormal model:

$$\ln(q^e(t + \Delta t)) \sim N(\ln q^e(t) + \mu\, \Delta t, \sigma\sqrt{\Delta t})$$

It is a more general formulation than a pure random walk as it admits the possibility that the mean increase in the logarithm of the price is positive. The continuous time analogue of Eq. (15.4) is

$$\mathrm{d} \ln q^e(t) = \mu\, \mathrm{d}t + \sigma\, \mathrm{d}z \tag{15.5}$$

Following Eq. (15.2), it has the solution

$$\ln q^e(t) = \ln q^e(0) + \mu t + \sigma z(t) \tag{15.6}$$

where

$$E \ln q^e(t) = \ln q^e(0) + \mu t, \text{ and}$$
$$\operatorname{var} q^e(t) = \sigma^2 t$$

Since the ln $q^e(t)$ on average grows linearly with t (so that, on average, $q^e(t)$ will grow exponentially), Eqs. (15.5) and (15.6) are, together, referred to as a geometric Brownian motion (GBM). It is clearly a lognormal process: ln $q^e(t) \sim N(\ln q^e(0) + \mu t, \sigma\sqrt{t})$, and the parameters μ and σ can be estimated exactly as was discussed in Boxes 3.1 and 7.2 with the maintained assumption that time is measured in years.

While Eq. (15.6) is a complete description of the evolution of the logarithm of a stock's price, we are rather interested in the evolution of the price itself. Passing from a continuous time process on ln $q^e(t)$ to one on $q^e(t)$ is not a trivial matter, however, and we need some additional background to make the conversion correctly. This is considered in the next few paragraphs.

The essence of lognormality is the idea that if a random variable $\tilde{y}$ is distributed normally, then the random variable $\tilde{w} = \mathrm{e}\tilde{y}$ is distributed lognormally. Suppose, in particular, that $\tilde{y} \sim N(\mu_y, \sigma_y)$. A natural first question is: How are μ_w and σ_w related to μ_y and σ_y when $\tilde{w} = \mathrm{e}^{\tilde{y}}$? We first note that

$$\mu_w \neq \mathrm{e}^{\mu_y}, \text{ and } \sigma_w \neq \mathrm{e}^{\sigma_y}$$

As noted back in Section 6.6, it is actually the case that:

$$\mu_w = \mathrm{e}^{\mu_y + 1/2\sigma_y^2} \tag{15.7}$$

and

$$\sigma_w = \mathrm{e}^{\mu_y + 1/2\sigma_y^2}(\mathrm{e}^{\sigma_y^2} - 1)^{1/2}. \tag{15.8}$$

These formulae are not obvious, but we can at least shed some light on Eq. (15.8): Why should the variance of y have an impact on the mean of $\tilde{w}$? To see why this is so, let us remind ourselves of the shape of the lognormal distribution, as found in Figure 15.3.

Suppose there is an increase in variance. Since this distribution is pinched off to the left at zero, a higher variance of y can only imply (within the same class of distributions) that probability is principally shifted to higher values of $\tilde{w}$. But this will have the simultaneous effect of increasing the mean of $\tilde{w}$. The variance of y and the mean of w cannot be specified independently. The mean and standard deviation of the lognormal variable $\tilde{w}$ are thus each related to both the mean and variance of $\tilde{y}$ as per the relationships in Eqs. (15.7) and (15.8).

These results allow us to express the mean and standard deviation of $q^e(t)$ (by analogy, $\tilde{w}$) in relation to ln $q^e(t) + \mu t$ and $\sigma^2 t$ (by analogy, the mean and variance of $\tilde{y}$) via Eqs. (15.5) and (15.6):

$$\begin{aligned} Eq^e(t) &= \mathrm{e}^{\ln q^e(0) + (\mu + 1/2\sigma^2)t} \\ &= q^e(0)\mathrm{e}^{(\mu + 1/2\sigma^2)t} \end{aligned} \tag{15.9}$$

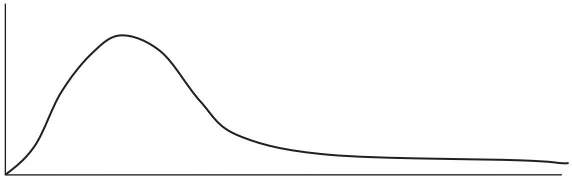

Figure 15.3
The lognormal density.

$$\begin{aligned} \mathrm{SD}q^e(t) &= e^{\ln q^e(0)+(\mu+1/2\sigma^2)t}(e^{\sigma^2 t}-1)^{1/2} \\ &= q^e(0)e^{(\mu+1/2\sigma^2)t}(e^{\sigma^2 t}-1)^{1/2} \end{aligned} \tag{15.10}$$

We are now in a position, at least at an intuitive level, to pass from a stochastic differential equation describing the behavior of $\ln q^e(t)$ to one that governs the behavior of $q^e(t)$. If $\ln q^e(t)$ is governed by Eq. (15.5), then

$$\frac{dq^e(t)}{q^e(t)} = (\mu + 1/2\sigma^2)dt + \sigma\, dz(t) \tag{15.11}$$

where $dq^e(t)/q^e(t)$ can be interpreted as the instantaneous (stochastic) rate of price change. Rewriting Eq. (15.11) slightly differently yields

$$dq^e(t) = (\mu + 1/2\sigma^2)q^e(t)dt + \sigma q^e(t)dz(t) \tag{15.12}$$

which informs us that the stochastic differential equation governing the stock's price represents an Ito process since the coefficients of dt and $dz(t)$ are both time dependent (and stochastic).

We would also expect that if $q^e(t)$ were governed by

$$dq^e(t) = \mu q^e(t)dt + \sigma q^e(t)dz(t), \ \textit{then} \tag{15.13}$$

$$d \ln q^e(t) = (\mu + 1/2\sigma^2)dt + \sigma\, dz(t) \tag{15.14}$$

Equations (15.13) and (15.14) are fundamental to what follows.

15.5 Simulation and European Call Pricing

15.5.1 Ito processes

Ito processes and their constituents, most especially the Brownian motion, are difficult to grasp at this abstract level and it will assist our intuition to describe how we might simulate a discrete time approximation to them.

Suppose we have estimated $\hat{\mu}$ and $\hat{\sigma}$ for a stock's return process as per the web notes to Chapter 12. Recall that these estimates are derived from daily price data properly scaled up to reflect the fact that in this literature it is customary to measure time in years. We have two potential stochastic differential equations to guide us—Eqs. (15.13) and (15.14)—and each has a discrete time approximate counterpart.

i. Discrete time counterpart to Eq. (15.13)

If we approximate the stochastic differential $dq^e(t)$ by the change in the stock's price over a short interval of time Δt, we have

$$\begin{aligned} q^e(t+\Delta t) - q^e(t) &= \hat{\mu} q^e(t)\Delta t + \hat{\sigma} q^e(t)\tilde{\varepsilon}(t)\sqrt{\Delta t}, \text{ or} \\ q^e(t+\Delta t) &= q^e(t)[1 + \hat{\mu}\Delta t + \hat{\sigma}\tilde{\varepsilon}(t)\sqrt{\Delta t}] \end{aligned} \tag{15.15}$$

There is a problem with this representation, however, because for any $q^e(t)$, the price in the next period, $q^e(t+\Delta t)$, is normally distributed (recall that $\tilde{\varepsilon}(t) \sim N(0,1)$) rather than lognormal as a correct match to the data requires. In particular, there is the unfortunate possibility that the price could go negative, although for small time intervals Δt, this is exceedingly unlikely.

ii. Discrete time counterpart to Eq. (15.14)

Approximating $d \ln q^e(t)$ by successive log values of the price over small time intervals Δt yields

$$\begin{aligned} \ln q^e(t+\Delta t) - \ln q^e(t) &= (\hat{\mu} - 1/2\hat{\sigma}^2)\Delta t + \hat{\sigma}\tilde{\varepsilon}(t)\sqrt{\Delta t}, \text{ or} \\ \ln q^e(t+\Delta t) &= \ln q^e(t) + (\hat{\mu} - 1/2\hat{\sigma}^2)\Delta t + \hat{\sigma}\tilde{\varepsilon}\sqrt{\Delta t} \end{aligned} \tag{15.16}$$

Here it is the logarithm of the price in period $t+\Delta t$ that is normally distributed, as required, and for this reason we'll limit ourselves to Eq. (15.16) and its successors. For simulation purposes, it is convenient to express Eq. (15.16) as

$$q^e(t+\Delta t) = q^e(t)e^{(\hat{\mu}-1/2\hat{\sigma}^2)\Delta t + \hat{\sigma}\tilde{\varepsilon}(t)\sqrt{\Delta t}} \tag{15.17}$$

It is easy to generate a possible sample path of price realizations for Eq. (15.17). First select an interval of time Δt, and the number of successive time periods of interest (this will be the length of the sample path), say N. Using a random number generator, next generate N successive draws from the standard normal distribution. By construction, these draws are independent and thus successive rates of return $(q^e(t+\Delta t)/q^e(t) - 1)$ will be statistically independent of one another. Let this series of N draws be represented by $\{\varepsilon_j\}_{j=1}^N$. The corresponding sample path (or "time series") of prices is thus created as per Eq. (15.18).

$$q^e(t_{j+1}) = q^e(t_j)e^{(\hat{\mu}-1/2\hat{\sigma}^2)\Delta t + \hat{\sigma}\tilde{\varepsilon}_j\sqrt{\Delta t}} \tag{15.18}$$

where $t_{j+1} = t_j + \Delta t$. This is not the price path we would use for derivatives pricing, however.

15.5.2 Binomial Model

Under the binomial model, European call valuation is undertaken in a context where the probabilities have been changed in such a way that all assets, including the underlying stock, earn the risk-free rate. The simulation-based counterpart to this transformation is to replace $\widehat{\mu}$ by ln $(1 + r_f)$ in Eqs. (15.17) and (15.18):

$$q^e(t + \Delta t) = q^e(t)\mathrm{e}^{(\ln(1+r_f)-1/2\hat{\sigma}^2)\Delta t+\sigma\tilde{\varepsilon}(t)\sqrt{\Delta t}} \tag{15.19}$$

where r_f is the 1-year risk-free rate (not continuously compounded) and ln $(1 + r_f)$ is its continuously compounded counterpart.

How would we proceed to price a call in this simulation context? Since the value of the call at expiration is exclusively determined by the value of the underlying asset at that time, we first need a representative number of possible risk-neutral prices for the underlying asset at expiration. The entire risk-neutral sample path—as per Eq. (15.18)—is not required. By representative we mean enough prices so that their collective distribution is approximately lognormal. Suppose it was resolved to create J sample prices (to be even reasonably accurate, $J > 1000$) at expiration, T years from now. Given random draws $\{\varepsilon_k\}_{k=1}^{j}$ from N (0,1), the corresponding underlying stock price realizations are $\{q_k^e(T)\}_{k=1}^{j}$ as given by

$$q_k^e(T) = q^e(0)\mathrm{e}^{(\ln(1+r_f)-1/2\hat{\sigma}^2)T+\sigma\tilde{\varepsilon}_j\sqrt{T}} \tag{15.20}$$

For each of these prices, the corresponding call value at expiration is

$$C_j^T = \max\{0, q_j^e(T) - E\},\ j = 1, 2, \ldots, J$$

The average expected payoff across all these possibilities is

$$C_{\text{Avg}}^T = \frac{1}{J}\sum_{j=1}^{J} C_k^T$$

Since under risk-neutral valuation, the expected payoff of any derivative asset in the span of the underlying stock and a risk-free bond is discounted back at the risk-free rate, our estimate of the call's value today (when the stock's price is $q^e(0)$) is

$$C^0 = \mathrm{e}^{-\ln(1+r_f)T} C_{\text{Avg}}^T \tag{15.21}$$

In the case of the Asian option considered earlier (Chapter 11) or some other path-dependent option, a large number of sample paths would need to be generated since the exercise price of the option (and thus its value at expiration) is dependent upon the entire sample path of underlying asset prices leading to it.

Monte Carlo simulation, as the previous method is called, is not the only pricing technique where the underlying idea is related to the notion of risk-neutral valuation. There are ways that stochastic differential equations can be solved directly.

15.6 Solving Stochastic Differential Equations: A First Approach

Monte Carlo simulation employs the notion of risk-neutral valuation but it does not, of course, provide closed-form solutions for derivatives prices, such as the Black−Scholes formula for a European call.[6] How are such closed-form expressions obtained? In what follows we provide a nontechnical outline of the first of two available methods. The context is unchanged: European call valuation on a non-dividend paying stock.

The idea is to obtain a partial differential equation whose solution, given the appropriate boundary condition, is the price of the call. This approach is due to Black and Scholes (1973) and, in a more general context, Merton (1973). The latter author's arguments will guide our discussion here.

In the same spirit as the replicating portfolio approach introduced in Section 13.4, Merton (1973) noted that the payoff to a call can be represented in continuous time by a portfolio of the underlying stock and a risk-free bond whose quantities are continuously adjusted. Given the stochastic differential equation that governs the stock's price (Eq. (15.13)) and another nonstochastic differential equation governing the bond's price evolution, it becomes possible to construct the stochastic differential equation governing the value of the replicating portfolio. This latter transformation is accomplished via an important theorem referred to in the literature as Ito's lemma. Using results from the stochastic calculus, this expression can be shown to imply that the value of the replicating portfolio must satisfy a particular partial differential equation. Together with the appropriate boundary condition (e.g., that $C(T) = \max\{q^e(T) - E, 0\}$), this partial differential equation (PDE) has a known solution—the Black−Scholes formula.

In what follows we begin with a brief overview of Merton's approach. This is illustrated in three steps.

15.6.1 The Behavior of Stochastic Differentials

In order to motivate what follows, we need to get a better idea of what the object $dz(t)$ means. It is clearly a random variable of some sort. We first explore its moments. Formally, $dz(t)$ is

[6] The estimate obtained using a Monte Carlo simulation will very likely closely approximate the Black−Scholes value to a high degree of precision, however, if the number of simulated underlying stock prices is large (≥10,000) and the parameters r_f, E, σ, and T used in each method are identical.

$$\lim_{\Delta t \mapsto 0} z(t + \Delta t) - z(t) \tag{15.22}$$

where we will not attempt to be precise as to how the limit is taken. We are reminded, however, that

$$E[z(t + \Delta t) - z(t)] = 0, \ \textit{and}$$
$$var[z(t + \Delta t) - z(t)] = \left(\sqrt{\Delta t}\right)^2 = \Delta t, \ \text{for all } \Delta t$$

It is not entirely surprising, therefore, that

$$E(\mathrm{d}z(\mathrm{t})) \equiv \lim_{\Delta t \mapsto 0} E[z(t + \Delta t) - z(t)] = 0 \text{ and} \tag{15.23}$$

$$\mathrm{var}(dz(t)) \equiv \lim_{\Delta t \mapsto 0} E[(z(t+\Delta t)-z(t))^2] = \mathrm{d}t \tag{15.24}$$

The object $\mathrm{d}z(t)$ may thus be viewed as denoting an infinitesimal random variable with zero mean and variance $\mathrm{d}t$.

There are several other useful relationships:

$$E(\mathrm{d}z(t)\mathrm{d}z(t)) \equiv var(\mathrm{d}z(t)) = \mathrm{d}t \tag{15.25}$$

$$\mathrm{var}(\mathrm{d}z(t)\mathrm{d}z(t)) \equiv \lim_{\Delta t \mapsto 0} E[(z(t+\Delta t)-z(t))^4 - (\Delta t)^2] \approx 0 \tag{15.26}$$

$$E(\mathrm{d}z(t)\mathrm{d}t) = \lim_{\Delta t \mapsto 0} E[(z(t + \Delta t) - z(t))\Delta t)] = 0 \tag{15.27}$$

$$\mathrm{var}(\mathrm{d}z(t)\mathrm{d}t) \equiv \lim_{\Delta t \to \infty} E[(z(t+\Delta t)-z(t))^2(\Delta t)^2] \approx 0 \tag{15.28}$$

Equations (15.28) and (15.26) imply, respectively, that Eqs. (15.25) and (15.27) are not only satisfied in expectation but with equality. Equation (15.25) is, in particular, quite surprising, as it argues that the square of a Brownian motion random process is effectively deterministic.

These results are frequently summarized by Table 15.1.

The expression $(\mathrm{d}t)^2$ is negligible in the table in the sense that it is very much smaller than $\mathrm{d}t$ and we may treat it as zero.

Table 15.1: Products of Stochastic Differentials

	$\mathrm{d}z$	$\mathrm{d}t$
$\mathrm{d}z$	$\mathrm{d}t$	0
$\mathrm{d}t$	0	0

The power of these results is apparent if we explore their implications for the computation of a quantity such as $(\mathrm{d}q^e(t))^2$:

$$\begin{aligned}(\mathrm{d}q^e(t))^2 &= (\mu \mathrm{d}t + \sigma\,\mathrm{d}z(t))^2\\ &= \mu^2(\mathrm{d}t)^2 + 2\mu\sigma\,\mathrm{d}t\,\mathrm{d}z(t) + \sigma^2(\mathrm{d}z(t))^2\\ &= \sigma^2\,\mathrm{d}t\end{aligned}$$

since, by the results in Table 15.1, $(\mathrm{d}t)(\mathrm{d}t) = 0$ and $\mathrm{d}t\,\mathrm{d}z(t) = 0$.

The object $\mathrm{d}q^e(t)$ thus behaves in the manner of a random walk in that its variance is proportional to the length of the time interval.

We will use these results in the context of Ito's lemma.

15.6.2 Ito's Lemma

A statement of this fundamental result is presented in Theorem 15.1.

Theorem All.2.1 (Ito's Lemma) Consider an Ito process $\mathrm{d}x(t)$ of form $\mathrm{d}x(t) = a(x(t), t)\mathrm{d}t + b(x(t), t)\mathrm{d}z(t)$, where $\mathrm{d}z(t)$ is a Brownian motion and consider a process $y(t) = F(x(t), t)$. Under quite general conditions, $y(t)$ satisfies the stochastic differential equation

$$\mathrm{d}y(t) = \frac{\partial F}{\partial x}\mathrm{d}x(t) + \frac{\partial F}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\mathrm{d}x(t))^2 \tag{15.29}$$

The presence of the rightmost term (which would be absent in a standard differential equation) is due to the unique properties of a stochastic differential equation. Taking advantage of the results in Table 15.1, let us specialize Eq. (15.29) to the standard Ito process where, for notational simplicity, we suppress the dependence of coefficients $a(\,)$ and $b(\,)$ on $x(t)$ and t:

$$\begin{aligned}\mathrm{d}y(t) = &\frac{\partial F}{\partial x}(a\,\mathrm{d}t + b\,\mathrm{d}z(t)) + \frac{\partial F}{\partial t}\mathrm{d}t\\ &+ \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(a\,\mathrm{d}t + b\,\mathrm{d}z(t))^2\\ = &\frac{\partial F}{\partial x}a\,\mathrm{d}t + \frac{\partial F}{\partial x}b\,\mathrm{d}z(t) + \frac{\partial F}{\partial x}\mathrm{d}t\\ &+ \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(a^2(\mathrm{d}t)^2 + ab\,\mathrm{d}t\,\mathrm{d}z(t) + b^2(\mathrm{d}z(t))^2)\end{aligned}$$

Note that $(\mathrm{d}t)^2 = 0, \mathrm{d}t\,\mathrm{d}z(t) = 0,$ and $(\mathrm{d}z(t))^2 = \mathrm{d}t$

Making these substitutions and collecting terms gives

$$dy(t) = \left(\frac{\partial F}{\partial x}a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}b^2\right)dt + \frac{\partial F}{\partial x}b\,dz(t) \tag{15.30}$$

As a simple application, let us take as given

$$dq^e(t) = \mu q^e(t)dt + \sigma q^e(t)dz(t)$$

and try to derive the relationship for $d\ln q^e(t)$.

Here we have

$$a(q^e(t), t) \equiv \mu q^e(t),$$

$$b(q^e(t), t) \equiv \sigma q^e(t) \text{ and}$$

$$\frac{\partial F}{\partial q^e(t)} = \frac{1}{q^e(t)} \text{ and } \frac{\partial^2 F}{\partial q^e(t)^2} = -\frac{1}{q^e(t)^2}$$

Lastly $(\partial F(\,))/(\partial t) = 0$.

Substituting these results into Eq. (15.30) yields

$$\begin{aligned} d\ln q^e(t) &= \left[\frac{1}{q^e(t)}\mu q^e(t) + 0 + \frac{1}{2}(-1)\left(\frac{1}{q^e(t)}\right)^2(\sigma q^e(t))^2\right]dt \\ &\quad + \frac{1}{q^e(t)}\sigma q^e(t)dz(t) \\ &= (\mu - 1/2\sigma^2)dt + \sigma\, dz(t) \end{aligned}$$

as was observed earlier (Eqn. 15.14).

This is the background.

15.6.3 The Black—Scholes Formula

In his derivation of the Black-Scholes formula, Merton (1973) requires four assumptions:

1. There are no market imperfections (perfect competition), transactions costs, taxes, short sales constraints, or any other impediment to the continuous trading of securities.
2. There is unlimited riskless borrowing and lending at the constant risk-free rate. If q^b is the period t price of a discount bond, then q^b is governed by the differential equation

$$dq^b(t) = r_f q^b(t)dt \quad \text{or}$$
$$q^b(t) = q^b(0)e^{r_f t}$$

3. The underlying stock's price dynamics is given by a GBM of the form

$$\begin{aligned}&dq^e(t) = \mu q^e(t)dt + \sigma q^e(t)dz(t),\\&q^b(0) > 0\end{aligned}$$

4. There are no arbitrage opportunities across the financial markets in which the call, the underlying stock, or the discount bond are traded.

Attention is restricted to call pricing formulae, which are functions only of the stock's price currently and the time (so, for example, the possibility of past stock price dependence is ignored), i.e.,

$$C = C(q^e(t), t)$$

By a straightforward application of Ito's lemma, the call's price dynamics must be given by

$$\begin{aligned}dC = &\left[\mu q^e(t)\frac{\partial C}{\partial q^e(t)} + \frac{\partial C}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 C}{\partial q^e(t)^2}\right]dt\\&+ \sigma q^e(t)\frac{\partial C}{\partial q^e(t)}dz(t)\end{aligned}$$

which is of limited help since the form of $C(q^e(t), t)$ is precisely what is not known. The partials with respect to $q^e(t)$ and t of $C(q^e(t), t)$ must be somehow circumvented.

Following the replicating portfolio approach, Merton (1973) defines the value of the call in terms of the self-financing, continuously adjustable portfolio P composed of $\Delta(q^e(t),t)$ shares and $N(q^e(t), t)$ risk-free discount bonds:

$$V(q^e(t), t) = \Delta(q^e(t), t)q^e(t) + N(q^e(t), t)q^b(t) \tag{15.31}$$

By a straightforward application of Ito's lemma once again, the value of the portfolio must evolve according to (suppressing functional dependence in order to reduce the burdensome notation):

$$dV = \Delta dq^e + N dq^b + d\Delta q^e + dN q^b + (d\Delta)dq^e \tag{15.32}$$

Since $V(\,)$ is assumed to be self-financing, any change in its value can only be due to changes in the values of the constituent assets and not in the numbers of them. Thus it must be that

$$dV = \Delta dq^e + N dq^b \tag{15.33}$$

which implies that the remaining terms in Eq. (15.32) are identically zero:

$$d\,\Delta q^e + dN q^b + (d\Delta)dq^e \equiv 0 \tag{15.34}$$

But both $\Delta(\ )$ and $N(\ )$ are functions of $q^e(t)$ and t and thus Ito's lemma can be applied to represent their evolution in terms of $\mathrm{d}z(t)$ and $\mathrm{d}t$. Using the relationships of Table 15.1 and collecting terms, both those preceding $\mathrm{d}z(t)$ and those preceding $\mathrm{d}t$ must individually be zero.

Together these relationships imply that the value of the portfolio must satisfy partial differential:

$$1/2\sigma^2(q^e)^2 Vq^e q^e + r_f q^e Vq^e + V_t = r_f V \tag{15.35}$$

which can be shown to have as its solution the Black–Scholes formula when coupled with the terminal condition $V(q^e(T), T) = \max[0, q^e(T) - E]$.

15.7 A Second Approach: Martingale Methods

This method originated in the work of Harrison and Kreps (1979). It is popular as a methodology because it frequently allows for simpler computations than in the PDE approach. The underlying mathematics, however, are very complex and beyond the scope of this book. In order to convey a sense of what is going on, we present a brief heuristic argument that relies on the binomial abstraction.

Recall that in the binomial model, we undertook our pricing in a tree context where the underlying asset's price process had been modified. In particular, the true probabilities of the up and down states were replaced by the corresponding risk-neutral probabilities. All assets (including the underlying stock) displayed an expected return equal to the risk-free rate in the transformed setting.

Under GBM, the underlying price process is represented by an Ito stochastic differential equation of the form

$$\mathrm{d}q^e(t) = \mu q^e(t)\mathrm{d}t + \sigma q^e(t)\mathrm{d}z(t) \tag{15.36}$$

In order to transform this price process into a risk-neutral setting, two changes must be made.

1. The expression μ defines the mean return and it must be replaced by r_f. Only with this substitution will the mean return on the underlying stock become r_f. Note that r_f denotes the corresponding continuously compounded risk-free rate.
2. The standard Brownian motion process must be modified. In particular, we replace $\mathrm{d}z$ by $\mathrm{d}z^*$, where the two processes are related via the transformation:

$$\mathrm{d}z^* = \mathrm{d}z + (\mu - r_f)/\sigma$$

The transformed price process is thus

$$dq^e(t) = r_f q^e(t)dt + \sigma q^e(t)dz^*(t) \tag{15.37}$$

By Eq. (15.14), the corresponding process on $\ln q^e(t)$ is

$$d \ln q^e(t) = (r_f - 1/2\sigma^2)dt + \sigma\, dz^*(t) \tag{15.38}$$

Let T denote the expiration date of a simple European call option. In the same spirit as the binomial model, the price of a call must be the present value of its expected payoff at expiration under the transformed process.

Equation (15.38) informs us that in the transformed economy,

$$\ln\left(\frac{q^e(T)}{q^e(0)}\right) \sim N((r_f - (1/2)\sigma^2)T, \sigma^2 T) \tag{15.39}$$

Since

$$\text{Prob}_{\substack{\text{transformed}\\ \text{economy}}}(q^e(t) \geq E) = \text{Prob}_{\substack{\text{transformed}\\ \text{economy}}}(\ln q^e(t) \geq \ln E)$$

we can compute the call's value using the probability density implied by Eq. (15.39):

$$C = e^{-r_f T}\int_{\ln E}^{\infty}(e^s - E)f(s)ds$$

where $f(s)$ is the probability density of the log of the stock's price. This becomes

$$C = e^{-r_f T}\left(\frac{1}{\sqrt{2\pi\sigma^2 T}}\right)\int_{\ln E}^{\infty}(e^s - E) \times e^{-[s - \ln q^e(0) - r_f T + (\sigma^2 T/2)]^2/2\sigma^2 T}ds \tag{15.40}$$

which, when the integration is performed, yields the Black–Scholes formula.

15.8 Applications

We make reference to a number of applications that have been considered earlier in the text.

15.8.1 The Consumption–Savings Problem

This is a classic economic problem and we considered it fairly thoroughly in Chapter 4. Without the requisite math background, there is not a lot we can say about the continuous time analogue other than to set up the problem, but even that first step will be illuminating.

Suppose the risky portfolio (M) is governed by the following price process:

$$\mathrm{d}q^M(t) = q^M(t)[\mu_M \,\mathrm{d}t + \sigma_M \,\mathrm{d}z(t)]$$

$q^M(0)$ given, and the risk-free asset by

$$\mathrm{d}q^B(t) = r_f q^B(t)\mathrm{d}t,\ q^B(0) \text{ given}$$

If an investor has initial wealth $Y(0)$ and chooses to invest the proportion $w(t)$ (possibly continuously varying) in the risky portfolio, then his wealth $Y(t)$ will evolve according to

$$\begin{aligned}\mathrm{d}Y(t) = Y(t)[w(t)(\mu_M - r_f) + r_f]\mathrm{d}t \\ + Y(t)[w(t)\sigma \,\mathrm{d}z(t)] - c(t)\mathrm{d}t\end{aligned} \tag{15.41}$$

where $c(t)$ is his consumption path. With objective function

$$\max_{c(t),w(t)} E \int_0^T \mathrm{e}^{-\gamma t} U(c(t))\mathrm{d}t \tag{15.42a}$$

the investor's problem is one of maximizing Eq. (15.42a) subject to Eq. (15.41) and initial conditions on wealth and the constraint that $Y(t) \geq 0$ for all t.

A classic result allows us to transform this problem into one that can be solved much more easily:

$$\begin{aligned}&\max_{c(t),w(t)} E \textstyle\int_0^T \mathrm{e}^{-\gamma t} U(c(t))\mathrm{d}t \\ &\text{s.t. } PV_0(c(t)) = E^* \textstyle\int_0^T \mathrm{e}^{-r_f t} c(t)\mathrm{d}t \leq Y(0)\end{aligned} \tag{15.42b}$$

where E^* is the transformed risk-neutral measure under which the growth rate of the risky portfolio is r_f.

In what we have presented so far, all the notation is directly analogous to that of Chapter 4: $U(\ )$ is the investor's utility of (instantaneous) consumption, γ his (instantaneous) discount rate, and T his time horizon.

15.8.2 An Application to Portfolio Analysis

Here we hope to give a hint of how to extend the portfolio analysis of Chapters 5 and 6 to a setting where trading is (hypothetically) continuous and individual security returns follow GBMs.

Let there be $i = 1,2, \ldots, N$ equity securities, each of whose return is governed by the process in Eq. (15.43).

$$\frac{\mathrm{d}q_i^e(t)}{q_i^e(t)} = \mu_i \,\mathrm{d}t + \sigma_i \,\mathrm{d}z_i(t) \tag{15.43}$$

where $\sigma > 0$. These processes may also be correlated with one another in a manner that we can represent precisely. Conducting a portfolio analysis in this setting has been found to have two principal advantages. First, it provides new insights concerning the implications of diversification for long-run portfolio returns and, second, it allows for an easier solution to certain classes of problems. We will note these advantages with the implicit understanding that the derived portfolio rules must be viewed as guides for practical applications. Literally interpreted they will imply, for example, continuous portfolio rebalancing—at an unbounded total expense, if the cost of doing each rebalancing is positive—which is absurd. In practice, one would rather employ them weekly or perhaps daily.

The stated objective is to maximize the expected rate of appreciation of a portfolio's value, or equivalently, to maximize its expected terminal value, which is the terminal wealth of the investor who owns it. Most portfolio managers would be familiar with this goal.

To get an idea of what this simplest criterion implies, and to make it more plausible in our setting, we first consider the discrete time equivalent (and, by implication) the discrete time approximation to GBM.

15.8.2.1 Digression to Discrete Time

Suppose a CRRA investor has initial wealth $Y(0)$ at time $t = 0$ and is considering investing in any or all of a set of stocks whose returns are iid. Since the rate of expected appreciation of the portfolio is its expected rate of return, and since the return distributions of the available assets are iid, the investor's optional portfolio proportions will be invariant to the level of his wealth, and the distribution of his portfolio's returns will itself be iid. At the conclusion of his planning horizon, T periods from the present, the investor's wealth will be

$$Y_T = Y_0 \prod_{s=1}^{T} \tilde{R}_s^P \tag{15.44}$$

where $\tilde{R}_s^P$ denotes the (iid) gross portfolio return in period s. It follows that

$$\begin{aligned} \ln\left(\frac{Y_T}{Y_0}\right) &= \sum_{s=1}^{T} \ln \tilde{R}_s^P \text{ and} \\ \ln\left(\frac{Y_T}{Y_0}\right)^{1/T} &= \left(\frac{1}{T}\right)\sum_{s=1}^{T} \ln \tilde{R}_s^P \end{aligned} \tag{15.45}$$

Note that whenever we introduce the log we effectively convert to continuous compounding within the time period. As the number of periods in the time horizon grows without bound, $T \mapsto \infty$, by the law of large numbers,

$$\left(\frac{Y_T}{Y_0}\right)^{1/T} \mapsto e^{E\ln \tilde{R}^P} \quad \text{or} \tag{15.46}$$

$$Y_T \mapsto Y_0\, e^{TE\ln \tilde{R}^P} \tag{15.47}$$

Consider an investor with a many-period time horizon who wishes to maximize her expected terminal wealth under continuous compounding. The relationship in Eq. (15.47) informs her that:

1. it is sufficient, under the aforementioned assumptions, for her to choose portfolio proportions that maximize $E \ln \tilde{R}^P$, the expected logarithm of the one-period return, and
2. by doing so the average growth rate of her wealth will approach a deterministic limit.

Before returning to the continuous time setting, let us present a brief classic example, one in which an investor must decide what fractions of his wealth to assign to a highly risky stock and to a risk-free asset (actually, the risk-free asset is equivalent to keeping money in a shoebox under the bed). For an amount Y_0 invested in either asset, the respective returns are found in Figure 15.4.

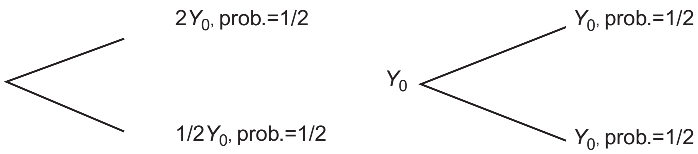

Figure 15.4
Two alternative asset returns.

Let w represent the proportion in the stock and note that the expected gross return to either asset under continuous compounding is *zero*:

Stock: $E \ln R^e = 1/2 \ln(2) + 1/2 \ln(1/2) = 0$
Shoebox: $E \ln R^{sb} = 1/2 \ln(1) + 1/2 \ln(1) = 0.$

With each asset paying the same expected return, and the stock being wildly risky, at first appearance the shoebox would seem the way to go. But according to Eq. (15.47), the investor ought to allocate his wealth between the two assets so as to maximize the expected log of the portfolio's one-period gross return:

$$\max E \ln \tilde{R}^P = \max_w \{1/2 \ln(2w + (1-w)) + 1/2 \ln(1/2w + (1-w))\}$$

A straightforward application of the calculus yields $w = 8/4$, with consequent portfolio returns in each state as shown in Figure 15.5.

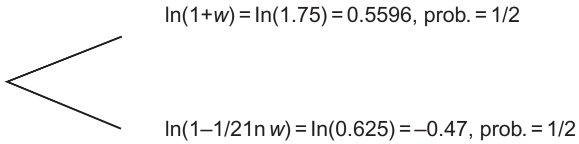

Figure 15.5
Optimal portfolio returns in each state.

As a result, $E \ln \tilde{R}^P = 0.0448$ with an effective risk-free period return (for a very long time horizon) of 4.5% ($e^{0.448} = 1.045$).

This result is surprising and the intuition is not obvious. Briefly, the optimal proportions of $w = 3/4$ and $1-w = 1/4$ reflect the fact that by always keeping a fixed fraction of wealth in the risk-free asset, the worst wealth trajectories can be avoided. By frequent trading, although each asset has an expected return of zero, the indicated combination will yield an expected return that is strictly positive, and over a long time horizon, effectively riskless. As first noted in Chapter 13, frequent trading expands market opportunities.

15.8.2.2 Return to Continuous Time

The previous setup applies directly to a continuous time setting as all of the fundamental assumptions are satisfied. In particular, there are a very large number of periods (an uncountable number, in fact) and the returns to the various securities are iid through time. Let us make the added generalization that the individual asset returns are correlated through their Brownian motion components. By an application of Ito's lemma, we may write

$$\mathrm{cov}(\mathrm{d}z_i, \mathrm{d}z_j) = E(\mathrm{d}z_i(t)\mathrm{d}z_j(t)) = \sigma_{ij}\mathrm{d}t$$

where σ_{ij} denotes the (i,j) entry of the (instantaneous) variance−covariance matrix.

As has been our custom, denote the portfolio's proportions for the N assets by $w_{\mathrm{i}}, \ldots, w_N$ and let the superscript P denotes the portfolio itself. As in earlier chapters, the process on the portfolio's instantaneous rate of return, $(\mathrm{d}Y^P(t))/(Y^P(t))$ will be the weighted average of the instantaneous constituent asset returns (as given in Eq. (15.43)):

$$\begin{aligned}\frac{\mathrm{d}Y^P(t)}{Y^P(t)} &= \sum_{i=1}^{N} w_i \frac{\mathrm{d}q_i^e(t)}{q_i^e(t)} = \sum_{i=1}^{N} w_i(\mu_i \mathrm{d}t + \mathrm{d}z_i(t)) \\ &= \left(\sum_{i=1}^{N} w_i \mu_i\right)\mathrm{d}t + \sum_{i=1}^{N} w_i \mathrm{d}z_i(t)\end{aligned} \tag{15.48}$$

where the variance of the stochastic term is given by

$$E\left(\sum_{i=1}^{N} w_i \mathrm{d}z_i(t)\right)^2 = E\left\{\left(\sum_{i=1}^{N} w_i \mathrm{d}z_i(t)\right)\left(\sum_{j=1}^{N} w_j \mathrm{d}z_j(t)\right)\right\}$$
$$= \left(\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}\right)\mathrm{d}t$$

Equation (15.48) describes the process on the portfolio's rate of return and we see that it implies that the portfolio's value, at any future time horizon T, will be lognormally distributed; furthermore, an uncountable infinity of periods will have passed. By analogy (and formally), our discrete time reflections suggest that an investor should, in this context, also choose portfolio proportion so as to maximize the mean growth rate, v_P, of the portfolio as given by

$$E\left\{\ln \frac{Y^P(T)}{Y(0)}\right\} = T v_P$$

Since the portfolio's value itself follows a Brownian motion (with drift $\sum_{i=1}^{N} w_i \mu_i$ and disturbance $\sum_{i=1}^{N} w_i \mathrm{d}z_i$),

$$E\left[\ln \frac{Y^P(t)}{Y(0)}\right] = \left(\sum_{i=1}^{N} w_i \mu_i\right)T - \frac{1}{2}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}\right)T, \text{and thus} \tag{15.49}$$

$$v_P = \frac{1}{T} E\left[\ln \frac{Y^P(t)}{Y(0)}\right] = \sum_{i=1}^{N} w_i \mu_i - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij} \tag{15.50}$$

The investor should choose portfolio proportions to maximize this latter quantity.

Without belaboring this development much further, it behooves us to recognize the message implicit in Eq. (15.50). This can be accomplished most straightforwardly in the context of an equally weighted portfolio, where each of the N assets is distributed independently of one another ($\sigma_{ij} = 0$ for $i \neq j$), and all have the same mean and variance $((\mu_i, \sigma_i) = (\mu, \sigma) i = 1, 2, \ldots, N$.

In this case Eq. (15.50) reduces to

$$v_P = \mu - \left(\frac{1}{2N}\right)\sigma^2 \tag{15.51}$$

with the direct implication that the more identical stocks the investor adds to the portfolio the higher the mean instantaneous return. In this sense, it is useful to search for many similarly volatile stocks whose returns are independent of one another: by combining them in a portfolio where we continually (frequently) rebalance to maintain equal proportions, not only will portfolio variance decline ($(1/2N)\ \sigma^2$), as in the discrete time case, but the mean return will also rise (which is *not* the case in discrete time!).

15.8.3 The Consumption CAPM in Continuous Time

Our final application concerns the consumption CAPM of Chapter 10, and the question we address is this: What is the equilibrium asset price behavior in a Mehra–Prescott asset pricing context when the growth rate in consumption follows a GBM? Specializing preferences to be of the customary form $U(c) = (c^{1-\gamma}/1-\gamma)$, pricing relationship (10.4) reduces to

$$\begin{aligned} P_t &= E_t\left\{ Y_t \sum_{j=1}^{\infty} \delta^j x_{i+j}^{1-\gamma} \right\} \\ &= Y_t \sum_{j=1}^{\infty} \delta^j E_t\{x_{i+j}^{1-\gamma}\} \end{aligned}$$

where x_{i+j} is the growth rate in output (equivalently, consumption in the Mehra–Prescott economy) from period j to period $j+1$.

We hypothesize that the growth rate x follows a GBM of the form

$$\mathrm{d}x = \mu x\ \mathrm{d}t + \sigma x\ \mathrm{d}z$$

where we interpret x_{i+j} as the discrete time realization of $x(t)$ at time $i+j$.

One result from statistics is needed. Suppose $\tilde{w}$ is lognormally distributed which we write $\tilde{w} \sim L(\xi, n)$ where $\xi = E\ \ln\ \tilde{w}$ and $\eta^2 = \mathrm{var}\ \ln\ \tilde{w}$. Then for any real number q,

$$E\{\tilde{w}^q\} = \mathrm{e}^{q\xi + 1/2q^2\eta^2}$$

By the process on the growth rate just assumed, $x(t) \sim L((\mu - 1/2\sigma^2)t, \sigma\sqrt{t})$ so that at time $t+j, x_{t+j} \sim L((\mu - 1/2\sigma^2)j, \sigma\sqrt{j})$. By this result,

$$\begin{aligned} E_i\{x_{i+j}^{1-\gamma}\} &= e^{(1-\gamma)(\mu - 1/2\sigma^2)j + 1/2(1-\gamma)^2\sigma^2 j} \\ &= e^{(1-\gamma)(\mu - 1/2\gamma\sigma^2)j} \end{aligned}$$

and thus,

$$\begin{aligned} q_t &= Y_t \sum_{j=1}^{\infty} \delta^j e^{(1-\gamma)(\mu - 1/2\gamma\sigma^2)j}, \\ &= Y_t \sum_{j=1}^{\infty} (\delta^j e^{(1-\gamma)(\mu - 1/2\gamma\sigma^2)j} \end{aligned}$$

which is well defined (the sum has a finite value) if $\beta\, e^{(1-\gamma)(\mu-1/2\gamma\sigma^2)} < 1$, which we will assume to be the case. Then

$$q_t = Y_t \frac{\beta\, e^{(1-\gamma)(\mu-1/2\gamma\sigma^2)}}{1+\beta\, e^{(1-\gamma)(\mu-1/2\gamma\sigma^2)}} \tag{15.52}$$

This is an illustration of the fact that working in continuous time often allows convenient closed solutions.

These remarks are taken from Mehra and Sah (2002). There is much that may be said. In particular, there are many more extensions of CCAPM style models to a continuous time setting. Another issue we have not addressed is the sense in which a continuous time price process (e.g., Eq. (15.13)) can be viewed as an equilibrium price process in the sense of that concept as presented in this book.

15.9 Final Comments

Continuous time is clearly different from discrete time, but does its use (as a derivatives pricing tool) enrich our economic understanding of the larger financial and macroeconomic reality? That is not clear. It does, however, make available valuation formulae that are completely unaccessible using discrete time methodologies. Expression (15.52) is a case in point.

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