CHAPTER 12

The Martingale Measure: Part I

Chapter Outline

12.1 Introduction 361
12.2 The Setting and the Intuition 362
12.3 Notation, Definitions, and Basic Results 364
12.4 Uniqueness 369
12.5 Incompleteness 372
12.6 Equilibrium and No Arbitrage Opportunities 375
12.7 Application: Maximizing the Expected Utility of Terminal Wealth 377
12.7.1 Portfolio Investment and Risk-Neutral Probabilities 377
12.7.2 Solving the Portfolio Problem 380
12.7.3 A Numerical Example 381
12.8 Conclusions 383
References 384
Appendix 12.1 Finding the Stock and Bond Economy That Is Directly Analogous to the Arrow–Debreu Economy in Which Only State Claims Are Traded 384
Appendix 12.2 Proof of the Second Part of Proposition 12.6 386

12.1 Introduction

We have already introduced the concept of risk-neutral valuation within the specialized contexts of Arrow-Debreu pricing (Chapter 9) and the CCAPM (Chapter 10). In the present chapter we revisit it from the perspective of arbitrage pricing.

As we will shortly see, the theory of risk-neutral valuation - also referred to Martingale pricing - is actually a theory based on preference-free pure arbitrage principles.[1] That is, it is free of the structural assumptions on preferences, expectations and endowments that make the CAPM and the CCAPM so restrictive. In this respect the present chapter will illustrate how far one can go in pricing financial assets while abstracting from the usual structural assumptions.

[1] The theory of risk-neutral valuation was first developed by Harrison and Kreps (1979). Pliska (1997) provides an excellent review of the notion in discrete time. This chapter is based on his presentation. For historical reasons the risk-neutral probability measure is also referred to as the Martingale measure.

Recall that risk-neutral valuation offers a unique perspective on the asset valuation problem. Rather than modify the denominator—the discount factor—to take account of the risky nature of a cash flow to be valued, or the numerator, by transforming the expected cash flows into their certainty equivalent, risk-neutral valuation simply corrects the *probabilities* with respect to which the expectation of the future cash flows is taken. This is done in such a way that discounting at the risk-free rate is legitimate. It is thus a procedure by which an asset valuation problem is transformed into one in which the asset's expected cash flow, computed now with respect to a new set of risk-neutral probabilities, can be discounted at the risk-free rate. The risk-neutral valuation methodology thus places an arbitrary valuation problem into a context in which all fairly priced assets earn the risk-free rate.

Risk-neutral probability distributions naturally assume a variety of specialized forms when restricted to the specific settings of Chapters 9 and 10. Harking back to Chapter 9, we first illustrate them in the context of the well-understood, finite time Arrow-Debreu complete markets setting, arriving at the same results as Section 9.6 but from a different angle. This strategy serves to clarify the very tight relationship between Arrow-Debreu pricing and Martingale pricing despite the apparent differences in terminology and perspective. Chapter 13 focuses on a similar set of issues but within the dynamic context of the CCAPM.

12.2 The Setting and the Intuition

Our setting for preliminary discussion is the particularly simple one with which we are now long familiar. There are two dates, $t = 0$ and $t = 1$. At date $t = 1$, any one of $j = 1,2,\ldots,J$ possible states of nature can be realized; denote the jth state by θ_j and its objective probability by π_j. We assume $\pi_j > 0$ for all θ_j.

Securities are competitively traded in this economy. There is a risk-free security that pays a fixed return r_f; its period t price is denoted by $q^b(t)$. We sometimes assume $q^b(0) = 1$, with its price at date 1 given by $q^b(1) \equiv q^b(\theta_j, 1) = (1 + r_f)$, for all states θ_j. Since the date 1 price of the security is $(1 + r_f)$ in any state, we can as well drop the first argument in the pricing function indicating the state in which the security is valued.[2]

Also traded are N fundamental risky securities, indexed $i = 1,2,\ldots,N$, which we think of as stocks.[3] The period $t = 0$ price of the ith such security is represented as $q_i^e(0)$. In period $t = 1$ its contingent payoff, given that state θ_j is realized, is given by $q_i^e\ (\theta_j,1)$.[4] It is also

2 In this chapter, it will be useful for the clarity of exposition to alter some of our previous notational conventions. One of the reasons is that we will want, symmetrically for all assets, to distinguish between their price at date 0 and their price at date 1 under any given state θ_j.

3 Fundamental securities are linearly independent.

4 In the notation of Chapter 9, $q_i^e(\theta_j, 1)$ is the cash flow associated with security i if state θ_j is realized, $CF^i(\theta_j)$.

assumed that investors may hold any linear combination of the fundamental risk-free and risky securities. No assumption is made, however, regarding the number of linearly independent securities vis-à-vis the number of states of nature: The securities market may or may not be complete. Neither is there any mention of agents' preferences. Otherwise the setting is standard Arrow–Debreu. Let **S** denote the set of all fundamental securities, the stocks and the bond, and linear combinations thereof.

For this setting, the existence of a set of risk-neutral probabilities or, in more customary usage, a risk-neutral probability measure, effectively means the existence of a set of state probabilities, $\pi_j^{RN} > 0$, $j = 1, 2, \ldots, J$ such that for each and every fundamental security $i = 1,2,\ldots,N$

$$q_i^e(0) = \frac{1}{(1+r_f)} E\pi^{RN}(q_i^e(\theta, 1)) = \frac{1}{(1+r_f)} \sum_{j=1}^{j} \pi_j^{RN} q_i^e(\theta_j, 1) \tag{12.1}$$

where the analogous relationship automatically holds for the risk-free security.

To gain some intuition as to what might be necessary, at a minimum, to guarantee the existence of such probabilities, first observe that in our setting the π_j^{RN} represent strictly positive numbers that must satisfy a large system of equations of the form

$$q_i^e(0) = \pi_1^{RN}\left(\frac{q_i^e(\theta_1, 1)}{1+r_f}\right) + \cdots + \pi_j^{RN}\left(\frac{q_i^e(\theta_J, 1)}{1+r_f}\right), \quad i = 1, 2, \ldots, N, \tag{12.2}$$

together with the requirement that $\pi_j^{RN} > 0$ for all j and $\sum_{j=1}^{J} \pi_j^{RN} = 1$.[5]

Such a system most certainly will not have a solution if there exist two fundamental securities, s and k, with the same $t = 0$ price, $q_s^e(0) = q_k^e(0)$, for which one of them, say k, pays as much as s in every state, and strictly more in at least one state; in other words,

$$q_k^e(\theta_j, 1) \geq q_s^e(\theta_j, 1) \quad \text{for all } j, \quad \text{and} \quad q_k^e(\theta_{\hat{j}}, 1) > q_s^e(\theta_{\hat{j}}, 1) \tag{12.3}$$

for at least one $j = \hat{J}$. Eq. (12.2) corresponding to securities s and k would, for any set $\{\pi_j^{RN}: j = 1, 2, \ldots, N\}$, have the same left-hand sides, yet different right-hand sides, implying no solution to the system. But two such securities cannot themselves be consistently priced because, together, they constitute an *arbitrage opportunity*: Short one unit of security s, long one unit of security k, and pocket the difference $q_k^e(\theta_{\hat{j}}, 1) - q_s^e(\theta_{\hat{j}}, 1) > 0$ if state $\hat{j}$ occurs; replicate the transaction many times over. These remarks suggest, therefore, that the existence of a risk-neutral measure is, in some intimate way, related to the absence of arbitrage opportunities in the financial markets. This is, in fact, the case, but first some notation, definitions, and examples are in order.

---

[5] Compare this system of equations with those considered in Section 11.2 when extracting Arrow–Debreu prices from a complete set of prices for complex securities.

12.3 Notation, Definitions, and Basic Results

Consider a portfolio, P, composed of n_p^b risk-free bonds and n_p^i units of risky security i, $i = 1,2,\ldots,N$. No restrictions will be placed on n_p^b, n_p^i: Short sales of these assets are permitted; they can, therefore, take negative values, and fractional share holdings are acceptable. The value of this portfolio at $t = 0$, $V_P(0)$, is given by

$$V_P(0) = n_p^b q^b(0) + \sum_{i=1}^{N} n_P^i q_i^e(0), \tag{12.4}$$

while its value at $t = 1$, given that state θ_j, is realized is

$$V_P(\theta_j, 1) = n_p^b q^b(1) + \sum_{i=1}^{N} n_P^i q_i^e(\theta_j, 1). \tag{12.5}$$

With this notation we are now in a position to define our basic concepts.

Definition 12.1 A portfolio P in $\mathbf{S}$ constitutes an arbitrage opportunity provided the following conditions are satisfied:

$$\begin{aligned} &\text{(i) } V_P(0) = 0, \\ &\text{(ii) } V_P(\theta_j, 1) \geq 0, \quad \text{for all } j \in \{1, 2, \ldots, J\}, \\ &\text{(iii) } V_P(\theta_{\hat{J}}, 1) > 0, \quad \text{for at least one } \hat{J} \in \{1, 2, \ldots, J\}. \end{aligned} \tag{12.6}$$

This is the standard sense of an arbitrage opportunity: With no initial investment and no possible losses (thus no risk), a strictly positive profit can be made in at least one state. Our second crucial definition is Definition 12.2.

Definition 12.2 A probability measure $\{\pi_j^{RN}\}_{j=1}^{J}$ defined on the set of states θ_j, $j = 1,2,\ldots,J$, is said to be a risk-neutral probability measure if

$$\begin{aligned} &(i)\ \pi_j^{RN} > 0, \quad \text{for all } j = 1, 2, \ldots, J, \text{ and} \\ &\text{(ii) } q_i^e(0) = E_{\pi^{RN}}\left\{\frac{\tilde{q}_i^e(\theta, 1)}{1 + r_f}\right\}, \end{aligned} \tag{12.7}$$

for all fundamental risky securities $i = 1,2,\ldots,N$ in $\mathbf{S}$.

Table 12.1: Fundamental securities for example 12.1

Period $t=0$ Prices	Period $t=1$ Payoffs		
		θ_1	θ_2
$q^b(0)$: 1	$q^b(1)$:	1.1	1.1
$q^e(0)$: 4	$q^e(\theta_j,1)$:	3	7

Table 12.2: Fundamental securities for example 12.2

Period $t=0$ Prices	Period $t=1$ Payoffs			
		θ_1	θ_2	θ_3
$q^b(0)$: 1	$q^b(1)$:	1.1	1.1	1.1
$q_1^e(0)$: 2	$q_1^e(\theta_j,1)$:	3	2	1
$q_2^e(0)$: 3	$q_2^e(\theta_j,1)$:	1	4	6

Both elements of this definition are crucial. Not only must each individual security be priced equal to the present value of its expected payoff, the latter computed using the risk-neutral probabilities (and thus it must also be true of portfolios of them), but these probabilities must also be strictly positive. To find them, if they exist, it is necessary only to solve the system of equations implied by part (ii) of Eq. (12.7) of the risk-neutral probability definition. We illustrate this idea in the Examples 12.1 through 12.4.

Example 12.1 There are two periods and two fundamental securities, a stock and a bond, with prices and payoffs presented in Table 12.1.

By the definition of a risk-neutral probability measure, it must be the case that simultaneously

$$4 = \pi_1^{RN}\left(\frac{3}{1.1}\right) + \pi_2^{RN}\left(\frac{7}{1.1}\right)$$
$$1 = \pi_1^{RN} + \pi_2^{RN}$$

Solving this system of equations, we obtain $\pi_1^{RN} = 0.65,\ \pi_2^{RN} = 0.35$.

For future reference note that the fundamental securities in this example define a complete set of financial markets for this economy, and that there are clearly no arbitrage opportunities among them.

Example 12.2 Consider next an analogous economy with three possible states of nature and three securities, as found in Table 12.2.

The relevant system of equations is now

$$2 = \pi_1^{RN}\left(\frac{3}{1.1}\right) + \pi_2^{RN}\left(\frac{2}{1.1}\right) + \pi_3^{RN}\left(\frac{1}{1.1}\right)$$
$$3 = \pi_1^{RN}\left(\frac{1}{1.1}\right) + \pi_2^{RN}\left(\frac{4}{1.1}\right) + \pi_3^{RN}\left(\frac{6}{1.1}\right)$$
$$1 = \pi_1^{RN} + \pi_2^{RN} + \pi_3^{RN}.$$

The solution to this set of equations,

$$\pi_1^{RN} = 0.3, \quad \pi_2^{RN} = 0.6, \quad \pi_3^{RN} = 0.1,$$

satisfies the requirements of a risk-neutral measure. By inspection, we again observe that this financial market is complete and that there are no arbitrage opportunities among the three securities.

Example 12.3 To see what happens when the financial markets are incomplete, consider the securities in Table 12.3.

For this example the relevant system is

$$2 = \pi_1^{RN}\left(\frac{1}{1.1}\right) + \pi_2^{RN}\left(\frac{2}{1.1}\right) + \pi_3^{RN}\left(\frac{3}{1.1}\right)$$
$$1 = \pi_1^{RN} + \pi_2^{RN} + \pi_3^{RN}$$

Because this system is underdetermined, there will be many solutions. Without loss of generality, first solve for π_2^{RN} and π_3^{RN} in terms of π_1^{RN}:

$$2.2 - \pi_1^{RN} = 2\pi_2^{RN} + 3\pi_3^{RN}$$
$$1 = \pi_1^{RN} = \pi_2^{RN} + \pi_3^{RN},$$

which yields the solution $\pi_3^{RN} = 0.2 + \pi_1^{RN}$ and $\pi_2^{RN} = 0.8 - 2\pi_1^{RN}$.

Table 12.3: Fundamental securities for example 12.3

Period $t = 0$ Prices	Period $t = 1$ Payoffs			
		θ_1	θ_2	θ_3
$q^b(0)$: 1	$q^b(1)$:	1.1	1.1	1.1
$q_1^e(0)$: 2	$q_1^e(\theta_j, 1)$:	1	2	3

In order for a triple $(\pi_1^{RN}, \pi_2^{RN}, \pi_3^{RN})$ to simultaneously solve this system of equations, while also satisfying the strict positivity requirement of risk-neutral probabilities, the following inequalities must hold:

$$\pi_1^{RN} > 0$$
$$\pi_2^{RN} = 0.8 - 2\pi_1^{RN} > 0$$
$$\pi_3^{RN} = 0.2 + \pi_1^{RN} > 0$$

By the second inequality $\pi_1^{RN} < 0.4$, and by the third $\pi_1^{RN} > -0.2$. In order that all probabilities be strictly positive, it must, therefore, be the case that

$$0 < \pi_1^{RN} < 0.4,$$

with π_2^{RN} and π_3^{RN} given by the indicated equalities.

In an incomplete market, therefore, there appear to be many risk-neutral probability sets: any triple $(\pi_1^{RN}, \pi_2^{RN}, \pi_3^{RN})$ where

$$(\pi_1^{RN}, \pi_2^{RN}, \pi_3^{RN}) \in \{(\lambda, 8 - 2\lambda, 0.2 + \lambda) : 0 < \lambda < 0.4\}$$

serves as a risk-neutral probability measure for this economy.

Example 12.4 Lastly, we may as well see what happens if the set of fundamental securities contains an arbitrage opportunity (see Table 12.4).

Any attempt to solve the system of equations defining the risk-neutral probabilities fails in this case. There is no solution. Notice also the implicit arbitrage opportunity: risky security 2 dominates a portfolio of one unit of the risk-free security and one unit of risky security 1, yet it costs less.

It is also possible to have a solution in the presence of arbitrage. In this case, however, at least one of the solution probabilities will be zero, disqualifying the set for the risk-neutral designation.

Table 12.4: Fundamental securities for example 12.4

Period $t = 0$ Prices	Period $t = 1$ Payoffs			
		θ_1	θ_2	θ_3
$q^b(0)$: 1	$q^b(1)$:	1.1	1.1	1.1
$q_1^e(0)$: 2	$q_1^e(\theta_j, 1)$:	2	3	1
$q_2^e(0)$: 2.5	$q_2^e(\theta_j, 1)$:	4	5	3

Together with our original intuition, these examples suggest that arbitrage opportunities are incompatible with the existence of a risk-neutral probability measure. This is the substance of the first main result.

Proposition 12.1 Consider the two-period setting described earlier in this chapter. Then there exists a risk-neutral probability measure on **S**, if and only if there are no arbitrage opportunities among the fundamental securities.

Proposition 12.1 tells us that, provided financial markets are characterized by the absence of arbitrage opportunities, our ambition to use distorted, risk-neutral probabilities to compute expected cash flows and discount at the risk-free rate has some legitimacy! Note, however, that the proposition admits the possibility that there may be many such measures, as in Example 12.3.

Proposition 12.1 also provides us, in principle, with a method for testing whether a set of fundamental securities contains an arbitrage opportunity. If the system of Eq. (12.7.ii) has no solution probability vector where all the terms are strictly positive, an arbitrage opportunity is present. Unless we are highly confident of the actual states of nature and the payoffs to the various fundamental securities in those states, however, this observation is of limited use. But even for a very large number of securities it is easy to check the solution vector computationally.

Although we have calculated the risk-neutral probabilities with respect to the prices and payoff of the fundamental securities only, the analogous relationship must hold for arbitrary portfolios in **S**—all linear combinations of the fundamental securities—in the absence of arbitrage opportunities. This result is formalized in Proposition 12.2.

Proposition 12.2 Suppose the set of securities **S** is free of arbitrage opportunities. Then for any portfolio $\hat{P}$ in **S**

$$V_{\hat{P}}(0) = \frac{1}{(1+r_f)} E_{\pi^{RN}} \tilde{V}_{\hat{P}}(\theta, 1) \tag{12.8}$$

for any risk-neutral probability measure π^{RN} on **S**.

Proof Let $\hat{P}$ be an arbitrary portfolio in **S**, and let it be composed of $n^b_{\hat{P}}$ bonds and $n^i_{\hat{P}}$ shares of fundamental risky asset i. In the absence of arbitrage, $\hat{P}$ must be priced equal to the value of its constituent securities. In other words,

$$V_{\hat{P}}(0) = n^b_{\hat{P}} q^b(0) + \sum_{i=1}^{N} n^i_{\hat{P}} q^e_i(0) = n^b_{\hat{P}} E_{\pi^{RN}} \left(\frac{q^b(1)}{1+r_f} \right) + \sum_{i=1}^{N} n^i_{\hat{P}} E_{\pi^{RN}} \left(\frac{\tilde{q}^e_i(\theta, 1)}{1+r_f} \right),$$

for any risk-neutral probability measure π^{RN},

$$= E_{\pi^{RN}}\left\{\frac{n_{\hat{P}}^b q^b(1) + \sum_{i=1}^{N} n_{\hat{P}}^i \tilde{q}_i^e(\theta, 1)}{1 + r_f}\right\} = \frac{1}{(1 + r_f)} E_{\pi^{RN}}(\tilde{V}_{\hat{P}}(\theta, 1)).$$

Proposition 12.2 is merely a formalization of the obvious fact that if every security in the portfolio is priced equal to the present value, discounted at r_f, of its expected payoffs computed with respect to the risk-neutral probabilities, the same must be true of the portfolio itself. This follows from the linearity of the expectations operator and the fact that the portfolio is valued as the sum total of its constituent securities, which must be the case in the absence of arbitrage opportunities.

A multiplicity of risk-neutral measures on **S** does not compromise this conclusion in any way, because each of them assigns the same value to the fundamental securities and thus to the portfolio itself via Eq. (12.8). For completeness, we note that a form of a converse to Proposition 12.2 is also valid.

Proposition 12.3 Consider an arbitrary period $t = 1$ payoff $\tilde{x}(\theta, 1)$ and let M represent the set of all risk-neutral probability measures on the set **S**. Assume **S** contains no arbitrage opportunities. If

$$\frac{1}{(1 + r_f)} E_{\pi^{RN}} \tilde{x}(\theta, 1) = \frac{1}{(1 + r_f)} E_{\hat{\pi}^{RN}} \tilde{x}(\theta, 1) \quad \text{for any} \quad \pi^{RN}, \hat{\pi}^{RN} \in M,$$

then there exists a portfolio in **S** with the same $t = 1$ payoff as $\tilde{x}(\theta, 1)$.

It would be good to be able to dispense with the complications attendant to multiple risk-neutral probability measures on **S**. When this is possible it is the subject of Section 12.4.

12.4 Uniqueness

Examples 12.1 and 12.2 both possessed unique risk-neutral probability measures. They were also complete markets models. This illustrates an important general proposition.

Proposition 12.4 Consider a set of securities **S** without arbitrage opportunities. Then **S** is complete if and only if there exists exactly one risk-neutral probability measure.

Proof Let us prove one side of the proposition, as it is particularly revealing. Suppose **S** is complete and there were two risk-neutral probability measures, $\{\pi_j^{RN}: j = 1, 2, \ldots, J\}$ and $\{\vec{\pi}_J^{RN}: j = 1, 2, \ldots, J\}$. Then there must be at least one state $\hat{J}$ for which $\pi_{\hat{j}}^{RN} \neq \vec{\pi}_{\hat{j}}^{RN}$. Since the market is complete, one must be able to construct a portfolio P in **S** such that

$$V_P(0) > 0, \quad \text{and} \quad \begin{cases} V_P(\theta_j, 1) = 0 & j \neq \hat{j} \\ V_P(\theta_j, 1) = 1 & j \neq \hat{j} \end{cases}.$$

This is simply the statement of the existence of an Arrow–Debreu security associated with $\theta_{\hat{j}}$.

But then $\{\pi_J^{RN}: j = 1, 2, \ldots, J\}$ and $\{\vec{\pi}_j^{RN}: j = 1, 2, \ldots, J\}$ cannot both be risk-neutral measures since, by Proposition 12.2,

$$\begin{aligned} V_P(0) &= \frac{1}{(1 + r_f)} E_{\pi^{RN}} \tilde{V}_P(\theta, 1) = \frac{x_{\hat{j}}^{RN}}{1 + r_f} \\ &\neq \frac{\vec{\pi}_{\hat{j}}^{RN}}{(1 + r_f)} = \frac{1}{(1 + r_f)} E_{\vec{\pi}^{RN}} \tilde{V}_P(\theta, 1) \\ &= V_P(0), \ \text{ a contradiction.} \end{aligned}$$

Thus, there cannot be more than one risk-neutral probability measure in a complete market economy.

We omit a formal proof of the other side of the proposition. Informally, the idea is as follows: if the market is not complete, then the fundamental securities do not span the space. Hence, the system of Eq. (12.6) contains more unknowns than equations, yet they are all linearly independent (no arbitrage). There must be a multiplicity of solutions and hence a multiplicity of risk-neutral probability measures.

Concealed in the proof of Proposition 12.4 is an important observation: the price of an Arrow–Debreu security that pays one unit of payoff if event $\theta_{\hat{j}}$ is realized and nothing otherwise must be $\frac{\pi_{\hat{j}}^{RN}}{1 + r_f}$, the present value of the corresponding risk-neutral probability. In general,

$$q_j(0) = \frac{\pi_j^{RN}}{(1 + r_f)}$$

where $q_j(0)$ is the $t = 0$ price of a state claim paying 1 if and only if state θ_j was realized. Provided the financial market is complete, risk-neutral valuation is nothing more than valuing an uncertain payoff in terms of the value of a replicating portfolio of Arrow–Debreu claims. Notice, however, that we thus identify the all-important Arrow–Debreu prices without having to impose any of the economic structure of Chapter 9; in particular, knowledge of the agents' preferences is not required. This approach can be likened to describing the Arrow–Debreu pricing theory from the perspective of Proposition 11.2. It is possible, and less restrictive, to limit our inquiry to extracting Arrow–Debreu prices from the prices of a (complete) set of complex securities and proceed from there to price arbitrary cash flows. In the absence of further structure, nothing can be said, however, about the determinants of Arrow–Debreu prices (or risk-neutral probabilities).

Let us illustrate with the data of our second example. There we identified the unique risk-neutral measure to be:

$$\pi_1^{RN} = 0.3, \quad \pi_2^{RN} = 0.6, \quad \pi_3^{RN} = 0.1,$$

Together with $r_f = 0.1$, these values imply that the Arrow–Debreu security prices must be

$$q_1(0) = \frac{0.3}{1.1} = 0.27;\ q_2(0) = \frac{0.6}{1.1} = 0.55;\ q_3(0) = \frac{0.1}{1.1} = 0.09.$$

Conversely, given a set of Arrow–Debreu claims with strictly positive prices, we can generate the corresponding risk-neutral probabilities and the risk-free rate. As noted in earlier chapters, the period zero price of a risk-free security (one that pays one unit of the numeraire in every date $t = 1$ state) in this setting is given by

$$q_{r_f} = \sum_{j=1}^{J} q_j(0),$$

and thus

$$(1 + r_f) = \frac{1}{q_{r_f}} = \frac{1}{\sum_{j=1}^{J} q_j(0)}$$

We define the risk-neutral probabilities $\{\pi^{RN}(\theta)\}$ according to

$$\pi_j^{RN} = \frac{q_j(0)}{\sum_{j=1}^{J} q_j(0)} \tag{12.9}$$

Clearly, $\pi_j^{RN} > 0$ for each state j (since $q_j(0) > 0$ for every state) and, by construction $\sum_{j=1}^{J} \pi_j^{RN} = 1$. As a result, the set $\{\pi_j^{RN}\}$ qualifies as a risk-neutral probability measure.[6]

Referring now to the example developed in Section 9.3, let us recall that we had found a complete set of Arrow–Debreu prices to be $q_1(0) = 0.24$; $q_2(0) = 0.3$; this means, in turn, that the unique risk-neutral measure for the economy there described is

$$\pi_1^{RN} = \frac{0.24}{0.54} = 0.444, \quad \pi_2^{RN} = \frac{0.3}{0.54} = 0.556.$$

For complete markets we see that the relationship between strictly positively priced state claims and the risk-neutral probability measure is indeed an intimate one: each implies the other. Since, in the absence of arbitrage possibilities, there can exist only one set of state claims prices, and thus only one risk-neutral probability measure, Proposition 12.4 is reconfirmed.

12.5 Incompleteness

What about the case in which **S** is an incomplete set of securities? By Proposition 12.4 there will be a multiplicity of risk-neutral probabilities, but these will all give the same valuation to elements of **S** (Proposition 12.2). Consider, however, a time $t = 1$ bounded state-contingent payoff vector $\tilde{x}(\theta,1)$ that does *not* coincide with the payoff to any portfolio in **S**. By Proposition 12.4, different risk-neutral probability measures will assign different values to this payoff: essentially, its price is not well defined. It is possible, however, to establish *arbitrage bounds* on the value of this claim. For any risk-neutral probability π^{RN}, defined on **S**, consider the following quantities:

$$H_x = \inf\left\{ E_{\pi^{RN}}\left[\frac{\tilde{V}_P(\theta,1)}{1+r_f}\right] : V_P(\theta_j,1) \geq x(\theta_j,1), \forall j = 1,2,\ldots J \text{ and } P \in \mathbf{S} \right\}$$
$$L_x = \sup\left\{ E_{\pi^{RN}}\left[\frac{\tilde{V}_P(\theta,1)}{1+r_f}\right] : V_P(\theta_j,1) \leq x(\theta_j,1), \forall j = 1,2,\ldots J \text{ and } P \in \mathbf{S} \right\} \tag{12.10}$$

In these evaluations we don't care what risk-neutral measure is used because any one of them gives identical valuations for all portfolios in **S**. Since, for some γ, $\gamma q^b(1) > x(\theta_j,1)$,

[6] Note that we also assured at the same expression as (12.9) back in Chapter 9.

for all j, H_x is bounded above by $\gamma q^b(0)$, and hence is well defined (an analogous comment applies to L_x). The claim is that the no arbitrage price of x, $q^x(0)$ lies in the range

$$L_x \le q^x(0) \le H_x$$

To see why this must be so, suppose that $q^x(0) > H_x$ and let P^* be any portfolio in $\mathbf{S}$ for which

$$\begin{aligned} &q^x(0) > V_{P^*}(0) > H_x, \text{ and} \\ &V_{P^*}(\theta_j, 1) \ge x(\theta_j, 1), \quad \text{for all } \theta_j,\ j = 1, 2, \ldots N. \end{aligned} \tag{12.11}$$

We know that such a P^* exists because the set

$$\mathbf{S}_x = \{P{:}P \in \mathbf{S}, V_P(\theta_j, 1) \ge x(\theta_j, 1), \quad \text{for all } j = 1, 2, \ldots, J\}$$

is closed. Hence there is a $\hat{P}$ in $\mathbf{S}_x$ such that $E_{\pi^{RN}} \frac{\tilde{V}_{\hat{P}}(\theta,1)}{(1+r_f)} = H_x$. By the continuity of the expectations operator, we can find a $\lambda > 1$ such that $\lambda\hat{P}$ is in $\mathbf{S}_x$ and[7]

$$q^x(0) > \frac{1}{1+r_f} E_{\pi^{RN}} \tilde{V}_{\lambda\hat{P}}(\theta, 1) = \lambda \frac{1}{1+r_f} E_{\pi^{RN}} \tilde{V}_{\hat{P}}(\theta, 1) = \lambda H_x > H_x.$$

Since $\lambda > 1$, for all j, $V_{\lambda\hat{P}}(\theta_j, 1) > V_{\hat{P}}(\theta_j, 1) \ge x(\theta_j, 1)$; let $P^* = \lambda\hat{P}$. Now the arbitrage argument: sell the security with title to the cash flow $x(\theta_j,1)$, and buy the portfolio P^*. At time $t = 0$, you receive, $q^x(0) - V_{P^*}(0) > 0$, while at time $t = 1$ the cash flow from the portfolio, by Eq. (12.11), fully covers the obligation under the short sale in every state. In other words, there is an arbitrage opportunity. An analogous argument demonstrates that $L_x \le q^x(0)$.

In some cases it is readily possible to solve for these bounds.

Example 12.5 Revisit, for example, our earlier Example 12.3, and consider the payoff

	θ_1	θ_2	θ_3
$x(\hat{\theta}_j, 1)$:	0	0	1

This security is most surely not in the span of the securities (1.1, 1.1, 1.1) and (1, 2, 3), a fact that can be confirmed by observing that the system of equations implied by equating

$$(0, 0, 1) = a(1.1, 1.1, 1.1) + b(1, 2, 3),$$

[7] By $\lambda\hat{P}$ we mean a portfolio with constituent bonds and stocks in the proportions $(\lambda n^b_{\hat{P}}, \lambda n^i_{\hat{P}})$.

in other words, the system:

$$\begin{aligned}0 &= 1.1a + b\\ 0 &= 1.1a + 2b\\ 1 &= 1.1a + 3b\end{aligned}$$

has no solution. But any portfolio in **S** can be expressed as a linear combination of (1.1, 1.1, 1.1) and (1, 2, 3) and thus must be of the form

$$a(1.1, 1.1, 1.1) + b(1, 2, 3) = (a(1.1) + b, a(1.1) + 2b, a(1.1) + 3b)$$

for some *a, b* real numbers.

We also know that in computing H_x, L_x, any risk-neutral measure can be employed. Recall that we had identified the solution of Example 12.3 to be

$$(\pi_1^{RN}, \pi_2^{RN}, \pi_3^{RN}) \in \{(\lambda, 0.8 - 2\lambda, 0.2 + \lambda): 0 < \lambda < 0.4\}$$

Without loss of generality, choose $\lambda = 0.2$; thus

$$(\pi_1^{RN}, \pi_2^{RN}, \pi_3^{RN}) = (0.2, 0.4, 0.4).$$

For any choice of *a, b* (thereby defining a $\tilde{V}_P(\theta; 1)$)

$$\begin{aligned}E_{\pi^{RN}}\left[\frac{\tilde{V}_P(\theta; 1)}{(1 + r_f)}\right] &= \frac{0.2\{(1.1)a + b\} + 0.4\{(1.1)a + 2b\} + 0.4\{(1.1)a + 3b\}}{1.1}\\ &= \frac{(1.1)a + (2.2)b}{1.1} = a + 2b.\end{aligned}$$

Thus,

$$H_x = \inf_{a,b,\in R}\{(a + 2b): a(1.1) + b \geq 0, a(1.1) + 2b \geq 0, \text{ and } a(1.1) + 3b \geq 1\}$$

Similarly,

$$L_x = \sup_{a,b,\in R}\{(a + 2b): a(1.1) + b \leq 0, a(1.1) + 2b \leq 0, \; a(1.1) + 3b \leq 1\}$$

Because the respective sets of admissible pairs are closed in R^2, we can replace inf and sup by, respectively, min and max.

Solving for H_x, L_x thus amounts to solving small linear programs. The solutions, obtained via MATLAB, are detailed in Table 12.5.

Table 12.5: Solutions for H_x and L_x

	H_x	L_x
a*	−0.4545	−1.8182
b*	0.5	1
H_x	0.5455	
L_x		0.1818

Table 12.6: The exchange economy of section 9.3—endowments and preferences

	Endowments			Preferences
	$t=0$	$t=1$		
Agent 1	10	1	2	$U^1(c_0, c_1) = \frac{1}{2}c_0^1 + 0.9(\frac{1}{3}\ln(c_1^1) + \frac{2}{3}\ln(c_2^1))$
Agent 2	5	4	6	$U^2(c_0, c_1) = \frac{1}{2}c_0^2 + 0.9(\frac{1}{3}\ln(c_1^2) + \frac{2}{3}\ln(c_2^2))$

The value of the security (state claim), we may conclude, lies in the interval (0.1818, 0.5455).

Before turning to the applications, there is one additional point of clarification.

12.6 Equilibrium and No Arbitrage Opportunities

Thus far we have made no reference to financial equilibrium, in the sense discussed in earlier chapters. Clearly, equilibrium implies no arbitrage opportunities: The presence of an arbitrage opportunity will induce investors to assume arbitrarily large short and long positions, which is inconsistent with the existence of equilibrium. The converse is also clearly not true. It could well be, in some specific market, that supply exceeds demand or, conversely, without this situation opening up an arbitrage opportunity in the strict sense understood in this chapter. In what follows the attempt is made to convey the sense of risk-neutral valuation as an equilibrium phenomenon.

To illustrate, let us return to the first example in Chapter 9. The basic data of that Arrow−Debreu equilibrium is provided in Table 12.6 and the $t=0$ corresponding equilibrium state prices are $q_1(0) = 0.24$ and $q_2(0) = 0.30$. In this case the risk-neutral probabilities are

$$\pi_1^{RN} = \frac{0.24}{0.54}, \quad \text{and} \quad \pi_2^{RN} = \frac{0.30}{0.54}.$$

Suppose a stock were traded where $q^e(\theta_1,1) = 1$, and $q^e(\theta_2,1) = 3$. By risk-neutral valuation (or equivalently, using Arrow−Debreu prices), its period $t=0$ price must be

$$q^e(0) = 0.54\left[\frac{0.24}{0.54}(1) + \frac{0.30}{0.54}(3)\right] = 1.14;$$

the price of the risk-free security is $q^b(0) = 0.54$.

Verifying this calculation is a bit tricky because, in the original equilibrium, this stock was not traded. Introducing such assets requires us to decide what the original endowments must be, that is, who owns what in period 0. We cannot just add the stock arbitrarily, as the wealth levels of the agents would change as a result, and, in general, this would alter the state prices, risk-neutral probabilities, and all subsequent valuations. The solution of this problem is to compute the equilibrium for a similar economy in which the two agents have the same preferences and in which the only traded assets are this stock and a bond. Furthermore, the initial endowments of these instruments must be such as to guarantee the same period $t = 0$ and $t = 1$ net endowment allocations as in the first equilibrium.

Let $\hat{n}_e^i, \hat{n}_b^i$ denote, respectively, the initial endowments of the equity and debt securities of agent i, $i = 1,2$. The equivalence noted previously is accomplished as outlined in Table 12.7 (see Appendix 12.1).

A straightforward computation of the equilibrium prices yields the same $q^e(0) = 1.14$, and $q^b(0) = 0.54$ as predicted by risk-neutral valuation.

We conclude this section with one additional remark. Suppose one of the two agents were risk neutral; without loss of generality let this be agent 1. Under the original endowment scheme, his problem becomes:

$$\max(10 + 1q_1(0) + 2q_2(0) - c_1^1 q_1(0) - c_2^1 q_2(0)) + 0.9\left(\tfrac{1}{3}c_1^1 + \tfrac{2}{3}c_2^1\right)$$

$$\text{s.t. } c_1^1 q_1(0) + c_2^1 q_2(0) \le 10 + q_1(0) + 2q_2(0)$$

Table 12.7: Initial holdings of equity and debt achieving equivalence with Arrow–Debreu equilibrium endowments

	$t = 0$		
	Consumption	$\hat{n}_e^i$	$\hat{n}_b^i$
Agent 1	10	$\frac{1}{2}$	$\frac{1}{2}$
Agent 2	5	1	3

The first-order conditions are

$$c_1^1: \quad q_1(0) = \frac{1}{3}.0.9$$

$$c_2^1: \quad q_2(0) = \frac{2}{3}.0.9$$

from which it follows that $\pi_1^{RN} = \frac{\frac{1}{3}0.9}{0.9} = \frac{1}{3}$ while $\pi_2^{RN} = \frac{\frac{2}{3}0.9}{0.9} = \frac{2}{3}$; that is, in equilibrium, the risk-neutral probabilities coincide with the true probabilities. This is the source of the term *risk-neutral probabilities*: if at least one agent is risk neutral, the risk-neutral probabilities and the true probabilities coincide.

We conclude from this example that risk-neutral valuation holds in equilibrium, as it must because equilibrium implies no arbitrage. The risk-neutral probabilities thus obtained, however, are to be uniquely identified with that equilibrium, and it is meaningful to use them only for valuing securities that are elements of the participants' original endowments.

12.7 Application: Maximizing the Expected Utility of Terminal Wealth

12.7.1 Portfolio Investment and Risk-Neutral Probabilities

Risk-neutral probabilities are intimately related to the *basis* or the set of fundamental securities in an economy. Under no arbitrage, given the prices of fundamental securities, we obtain a risk-neutral probability measure, and vice versa. This raises the possibility that it may be possible to formulate any problem in wealth allocation, for example, the classic consumption-savings problem, in the setting of risk-neutral valuation. In this section we consider a number of these connections.

The simplest portfolio allocation problem with which we have dealt involves an investor choosing a portfolio so as to maximize the expected utility of his period $t = 1$ (terminal) wealth (we retain, without loss of generality, the two-period framework). In our current notation, this problem takes the form: choose portfolio P, among all feasible portfolios (i.e., P must be composed of securities in $\mathbf{S}$ and the date-0 value of this portfolio (its acquisition price) cannot exceed initial wealth) so as to maximize expected utility of terminal wealth, which corresponds to the date-1 value of P:

$$\begin{aligned} &\max_{\{n_P^b, n_P^i, i=1,2,\ldots,N\}} EU(\tilde{V}_P(\theta, 1)) \\ &\text{s.t. } V_P(0) = V_0, P \in \mathbf{S}, \end{aligned} \tag{12.12}$$

where V_0 is the investor's initial wealth, $U(\ )$ is her period utility function, assumed to have the standard properties, and n_p^b, n_p^i, are the positions (not proportions, but units of indicated assets) in the risk-free asset and the risky asset $i = 1,2,\ldots,N$, respectively, defining portfolio P. It is not obvious that there should be a relationship between the solvability of this problem and the existence of a risk-neutral measure, but this is the case.

Proposition 12.5 If Eq. (12.12) has a solution, then there are no arbitrage opportunities in **S**. Hence there exists a risk-neutral measure on **S**.

Proof The idea is that an arbitrage opportunity is a costless way to endlessly improve upon the (presumed) optimum. So no optimum can exist. More formally, we prove the proposition by contradiction. Let $\hat{P} \in \mathbf{S}$ be a solution to Eq. (12.12), and let $\hat{P}$ have the structure $\{n_{\hat{P}}^b, n_{\hat{P}}^i : i = 1, 2, \ldots, N\}$. Assume also that there exists an arbitrage opportunity, in other words, a portfolio $\overleftrightarrow{P}$, with structure $\{n_{\overleftrightarrow{P}}^b, n_{\overleftrightarrow{P}}^i : i = 1, 2, \ldots, N\}$, such that $V_{\overleftrightarrow{P}}(0) = 0$ and $E\tilde{V}_{\overleftrightarrow{P}}(\theta, 1) > 0$. Consider the portfolio P^* with structure

$$\{n_{P^*}^b, n_{P^*}^i : i = 1, 2, \ldots, N\}$$

$$n_{P^*}^b, n_{\hat{P}}^b + n_{\overleftrightarrow{P}}^b \text{ and } n_{P^*}^i = n_{\hat{P}}^i + n_{\overleftrightarrow{P}}^i, \quad i = 1, 2, \ldots, N.$$

P^* is still feasible for the agent, and it provides strictly more wealth in at least one state. Since $U(\ )$ is strictly increasing,

$$EU(\tilde{V}_{P^*}(\theta, 1)) > EU(\tilde{V}_{\hat{P}}(\theta, 1)).$$

This contradicts $\hat{P}$ as a solution to Eq. (12.12). We conclude that there cannot exist any arbitrage opportunities and thus, by Proposition 12.1, a risk-neutral probability measure on **S** must exist.

Proposition 12.5 informs us that arbitrage opportunities are incompatible with an optimal allocation—the allocation can always be improved upon by incorporating units of the arbitrage portfolio. More can be said. The solution to the agents' problem can, in fact, be used to identify the risk-neutral probabilities. To see this, let us first rewrite the objective function in Eq. (12.12) as follows:

$$\begin{aligned}
&\max_{\{n_P^i : i = 1,2,\ldots,N\}} EU\left((1 + r_f)\left\{V_0 - \sum_{i=1}^{N} n_P^i q_i^e(0)\right\} + \sum_{i=1}^{N} n_P^i q_i^e(\theta, 1)\right) \\
&= \max_{\{n_P^i : i = 1,2,\ldots,N\}} \sum_{j=1}^{J} \pi_j U\left((1 + r_f)\left\{V_0 + \sum_{i=1}^{N} n_P^i \frac{q_i^e(\theta_j, 1)}{1 + r_f} - \sum_{i=1}^{N} n_P^i q_i^e(0)\right\}\right) \qquad (12.13) \\
&= \max_{\{n_P^i : i = 1,2,\ldots,N\}} \sum_{j=1}^{J} \pi_j U\left((1 + r_f)\left\{V_0 + \sum_{i=1}^{N} n_P^i \left(\frac{q_i^e(\theta_j, 1)}{1 + r_f} - q_i^e(0)\right)\right\}\right)
\end{aligned}$$

The necessary and sufficient first-order conditions for this problem are of the form:

$$0=\sum_{j=1}^{J}\pi_j U_1\left((1+r_f)\left\{V_0+\sum_{i=1}^{N}n_P^i\left(\frac{q_i^e(\theta_j,1)}{1+r_f}-q_i^e(0)\right)\right\}\right)(1+r_f)\left[\frac{q_i^e(\theta_j,1)}{1+r_f}-q_i^e(0)\right] \tag{12.14}$$

Note that the quantity $\pi_j U_1(V_P^1(\theta_j,1))(1+r_f)$ is strictly positive because $\pi_j>0$ and $U(\ )$ is strictly increasing. If we normalize these quantities, we can convert them into probabilities. Let us define

$$\breve{\pi}_j=\frac{\pi_j U_1(V_P(\theta_j,1))(1+r_f)}{\sum_{j=1}^{J}\pi_j U_1(V_P(\theta_j,1))(1+r_f)}=\frac{\pi_j U_1(V_P(\theta_j,1))}{\sum_{j=1}^{J}\pi_j U_1(V_P(\theta_j,1))},\quad j=1,2,\ldots,J.$$

Since $\breve{\pi}_j>0, j=1,2,\ldots,J,\quad \sum_{j=1}^{J}\breve{\pi}_j=1$ and by (12.14)

$$q_i^e(0)=\sum_{j=1}^{J}\breve{\pi}_j\frac{q_i^e(\theta_j,1)}{1+r_f};$$

these three properties establish the set $\{\breve{\pi}_j{:}j=1,2,\ldots,N\}$ as a set of risk-neutral probabilities.

We have just proved one-half of the following proposition:

Proposition 12.6 Let $\{n_{p^*}^b, n_{p^*}^i{:}i=1,2,\ldots,N\}$ be the solution to the optimal portfolio problem (12.12). Then the set $\{n_j^*{:}j=1,2,\ldots,J\}$, defined by

$$\pi_j^*=\frac{\pi_j U_1(V_{P^*}(\theta_j,1))}{\sum_{j=1}^{J}\pi_j U_1(V_{P^*}(\theta_j,1))}, \tag{12.15}$$

constitutes a risk-neutral probability measure on **S**. Conversely, if there exists a risk-neutral probability measure $\{\pi_j^{RN}{:}j=1,2,\ldots,J\}$ on **S**, there must exist a concave, strictly increasing, differentiable utility function $U(\ )$ and an initial wealth V_0 for which Eq. (12.12) has a solution.

Proof We have proved the first part. The proof of the less important converse proposition is relegated to Appendix 12.2.

12.7.2 Solving the Portfolio Problem

Now we can turn to solving Eq. (12.12). Since there is as much information in the risk-neutral probabilities as in the security prices, it should be possible to fashion a solution to Eq. (12.12) using that latter construct. Here we will choose to restrict our attention to the case in which the financial markets are complete.

In this case there exists exactly one risk-neutral measure, which we denote by $\{\pi_j^{RN}: j = 1, 2, \ldots, N\}$. Since the solution to Eq. (12.12) will be a portfolio in **S** that maximizes the date $t = 1$ expected utility of wealth, the solution procedure can be decomposed into a two-step process:

Step 1 Solve

$$\begin{aligned} &\max EU(\tilde{x}(\theta, 1)) \\ &\text{s.t. } E_{\pi^{RN}}\left(\frac{\tilde{x}(\theta, 1)}{1 + r_f}\right) = V_0 \end{aligned} \tag{12.16}$$

The solution to this problem identifies the feasible uncertain payoff that maximizes the agent's expected utility. But why is the constraint a perfect summary of feasibility? The constraint makes sense first because, under complete markets, every uncertain payoff lies in **S**. Furthermore, in the absence of arbitrage opportunities, every payoff is valued at the present value of its expected payoff computed using the unique risk-neutral probability measure. The essence of the budget constraint is that a feasible payoff be affordable: that its price equals V_0, the agent's initial wealth.

Step 2 Find the portfolio P in **S** such that

$$V_P(\theta_j, 1) = x(\theta_j, 1), \quad j = 1, 2\ldots, J.$$

In step 2 we simply find the precise portfolio allocations of fundamental securities that give rise to the optimal uncertain payoff identified in step 1. The theory is all in step 1; in fact, we have used all of our major results thus far to write the constraint in the indicated form.

Now let us work out a problem, first abstractly and then by a numerical example. Equation (12.16) of step 1 can be written as

$$\max E_\pi U(\tilde{x}(\theta, 1)) - \lambda\left[E_{\pi^{RN}}\left(\frac{\tilde{x}(\theta, 1)}{1 + r_f}\right) - V_0\right] \tag{12.17}$$

where λ denotes the Lagrange multiplier and where we have made explicit the probability distributions with respect to which each of the expectations is being taken.

Equation (12.17) can be rewritten as

$$\max_{x} \sum_{j=1}^{J} \pi_j \left[U(x(\theta_j, 1)) - \lambda \frac{\pi_j^{RN}}{\pi_j} \frac{x(\theta_j, 1)}{(1 + r_f)} \right] + \lambda V_0. \tag{12.18}$$

The necessary first-order conditions, one equation for each state θ_j, are thus

$$U_1(x(\theta_j, 1)) = \frac{\lambda \pi_j^{RN}}{\pi_j(1 + r_f)}, \quad j = 1, 2, \ldots, J. \tag{12.19}$$

from which the optimal asset payoffs may be obtained as per

$$x(\theta_j, 1) = U_1^{-1}\left(\frac{\lambda \pi_j^{RN}}{\pi_j(1 + r_f)}\right), \quad j = 1, 2, \ldots, J. \tag{12.20}$$

with U_1^{-1} representing the inverse of the MU function.

The Lagrange multiplier λ is the remaining unknown. It must satisfy the budget constraint when Eq. (12.20) is substituted for the solution; that is, λ must satisfy

$$E_{\pi^{RN}}\left(\frac{1}{(1 + r_f)} U_1^{-1}\left(\frac{\lambda \pi_j^{RN}}{\pi_j(1 + r_f)}\right)\right) = V_0. \tag{12.21}$$

A value for λ that satisfies Eq. (12.21) may not exist. For all the standard utility functions that we have dealt with, $U(x) = \ln x$ or $\frac{x^{1-\gamma}}{1-\gamma}$ or $e^{-\upsilon x}$, however, it can be shown that such a λ will exist. Let $\hat{\lambda}$ solve Eq. (12.21); the optimal feasible contingent payoff is thus given by

$$x(\theta_j, 1) = U_1^{-1}\left(\frac{\hat{\lambda} \pi_j^{RN}}{\pi_j(1 + r_f)}\right) \tag{12.22}$$

(from Eq. (12.21)). Given this payoff, step 2 involves finding the portfolio of fundamental securities that will give rise to it. This is accomplished by solving the customary system of linear equations.

12.7.3 A Numerical Example

Let us choose a utility function from the familiar CRRA class, $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, and consider the market structure of Example 12.2. Markets are complete, and the unique risk-neutral probability measure is as noted.

Since $U_1(x) = x^{-\gamma}, U_1^{-1}(y) = y^{-\frac{1}{\gamma}}$, Eq. (12.20) reduces to

$$x(\theta_j, 1) = \left(\frac{\lambda \pi_j^{RN}}{\pi_j(1+r_f)}\right)^{-\frac{1}{\gamma}} \tag{12.23}$$

from which follows the counterpart to Eq. (12.21):

$$\sum_{j=1}^{J} \pi_j^{RN} \left(\frac{1}{(1+r_f)}\left(\frac{\lambda \pi_j^{RN}}{\pi_j(1+r_f)}\right)^{-\frac{1}{\gamma}}\right) = V_0.$$

Isolating λ gives

$$\hat{\lambda} = \left\{\sum_{j=1}^{J} \pi_j^{RN} \left(\frac{1}{(1+r_f)}\left(\frac{\pi_j^{RN}}{\pi_j(1+r_f)}\right)^{-\frac{1}{\gamma}}\right)\right\}^{\gamma} V_0^{-\gamma}. \tag{12.24}$$

Let us consider some numbers: Assume $\gamma = 3$, $V_0 = 10$, and that (π_1, π_2, π_3), the true probability distribution, takes on the value $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Refer to Example 12.2 where the risk-neutral probability distribution was found to be $(\pi_1^{RN}, \pi_2^{RN}, \pi_3^{RN}) = (0.3, 0.6, 0.1)$. Accordingly, from Eq. (12.24)

$$\hat{\lambda} = 10^{-3}\left\{0.3\left(\frac{1}{(1.1)}\left(\frac{0.3}{\left(\frac{1}{3}\right)(1.1)}\right)^{-\frac{1}{3}}\right)\right.$$

$$\left. +0.6\left(\frac{1}{(1.1)}\left(\frac{0.6}{\left(\frac{1}{3}\right)(1.1)}\right)^{-\frac{1}{3}}\right) +0.1\left(\frac{1}{(1.1)}\left(\frac{0.1}{\left(\frac{1}{3}\right)(1.1)}\right)^{-\frac{1}{3}}\right)\right\}^{3}$$

$$\hat{\lambda} = \left(\frac{1}{1000}\right)\{0.2916 + 0.4629 + 0.14018\}^3 = 0.0007161.$$

The distribution of the state-contingent payoffs follows from Eq. (12.23):

$$x(\theta_j, 1) = \left(\frac{0.0007161 \pi_j^{RN}}{\pi_j(1+r_f)}\right)^{-\frac{1}{\gamma}} = \begin{cases} 11.951 & j = 1 \\ 9.485 & j = 2 \\ 17.236 & j = 3. \end{cases} \tag{12.25}$$

The final step is to convert this payoff to a portfolio structure via the identification:

$$(11.951, 11.485, 17.236) = n_P^b(1.1, 1.1, 1.1) + n_P^1(3, 2, 1) + n_P^2(1, 4, 6) \text{ or}$$

$$11.951 = 1.1n_P^b + 3n_P^1 + n_p^2$$

$$11.485 = 1.1n_P^b + 2n_P^1 + 4n_p^2$$

$$17.236 = 1.1n_P^b + n_P^1 + bn_p^2$$

The solution to this system of equations is

$$n_p^b = 97.08 \text{(invest a lot in the risk-free asset)}$$

$$n_P^1 = -28.192 \text{(short the first stock)}$$

$$n_P^2 = -10.225 \text{(also short the second stock)}$$

Lastly, we confirm that this portfolio is feasible:

Cost of portfolio $= 97.08 + 2(-28.192) + 3(-10.225) = 10 = V_0$, the agent's initial wealth, as required.

Note the computational simplicity of this method: we need only solve a linear system of equations. Using more standard methods would result in a system of three nonlinear equations to solve. Analogous methods are also available to provide bounds in the case of market incompleteness.

12.8 Conclusions

Under the procedure of risk-neutral valuation, we construct a new probability distribution—the risk-neutral probabilities—under which all assets may be valued at their expected payoff discounted at the risk-free rate. More formally it would be said that we undertake a *transformation of measure* by which all assets are then expected to earn the risk-free rate. The key to our ability to find such a measure is that the financial markets exhibit no arbitrage opportunities.

Our setting was the standard Arrow–Debreu two-period equilibrium and we observed the intimate relationship between the risk-neutral probabilities and the relative prices of state claims. Here the practical applicability of the idea is limited. Applying these ideas to the real-world would, after all, require a denumeration of all future states of nature and the contingent payoffs to all securities in order to compute the relevant risk-neutral probabilities, something for which there would be no general agreement.

Even so, this particular way of approaching the optimal portfolio problem was shown to be a source of useful insights. In more restrictive settings, it is also practically powerful and, as noted in Chapter 11, lies behind all modern derivatives pricing.

References

Harrison, M., Kreps, D., 1979. Martingales and multi-period securities market. J. Econ. Theor. 20, 381–408.
Pliska, S.R., 1997. Introduction to Mathematical Finance: Discrete Time Models. Basil Blackwell, Malden, MA.

Appendix 12.1 Finding the Stock and Bond Economy That Is Directly Analogous to the Arrow–Debreu Economy in Which Only State Claims Are Traded

The Arrow–Debreu economy is summarized in Table 12.6. We wish to price the stock and bond with the payoff structures in Table A12.1.

In order for the economy in which the stock and bond are traded to be equivalent to the Arrow–Debreu economy where state claims are traded, we need the former to imply the same effective endowment structure. This is accomplished as shown on the next page

Agent 1: Let his endowments of the stock and bond be denoted by $\hat{z}_1^e$ and $\hat{z}_1^b$, then,

$$\text{In state } \theta_1\text{: } \hat{z}_1^b + \hat{z}_1^e = 1$$

$$\text{In state } \theta_2\text{: } \hat{z}_1^b + 3\hat{z}_1^e = 2$$

$$\text{Solution: } \hat{z}_1^e + \hat{z}_1^b = \frac{1}{2}\text{(half a share and half a bond)}$$

Agent 2: Let his endowments of the stock and bond be denoted by $\hat{z}_2^e$ and $\hat{z}_2^b$, then,

$$\text{In state } \theta_1\text{: } \hat{z}_2^b + \hat{z}_2^e = 4$$

$$\text{In state } \theta_2\text{: } \hat{z}_2^b + 3\hat{z}_2^e = 6$$

$$\text{Solution: } \hat{z}_2^e = \hat{z}_2^b = 3$$

Table A12.1: Payoff structure

$t=0$	$t=1$	
	θ_1	θ_2
$-q_e\,(0)$	1	3
$-q_b\,(0)$	1	1

With these endowments the decision problems of the agent become:

Agent 1:

$$\max_{z_1^e + z_1^b} \frac{1}{2}\left(10 + \frac{1}{2}q^e + \frac{1}{2}q^b - z_1^b q^b\right) + 0.9\left(\frac{1}{3}\ln(z_1^e + z_1^b) + \frac{2}{3}\ln(3z_1^e + z_1^b)\right)$$

Agent 2:

$$\max_{z_2^e + z_2^b} \frac{1}{2}(5 + q^e + 3q^b - z_2^e q^e - z_2^b q^b) + 0.9\left(\frac{1}{3}\ln(z_2^e + z_2^b) + \frac{2}{3}\ln(3z_2^e + z_2^b)\right)$$

The FOCs are

$$z_1^e\!: \frac{1}{2}q^e = 0.9\left(\frac{1}{3}\left(\frac{1}{z_1^e + z_1^b}\right) + \frac{2}{3}\left(\frac{1}{3z_1^e + z_1^b}\right)(3)\right)$$

$$z_1^b\!: \frac{1}{2}q^b = 0.9\left(\frac{1}{3}\left(\frac{1}{z_1^e + z_1^b}\right) + \frac{2}{3}\left(\frac{1}{3z_1^e + z_1^b}\right)\right)$$

$$z_2^e\!: \frac{1}{2}q^e = 0.9\left(\frac{1}{3}\left(\frac{1}{z_2^e + z_2^b}\right) + \frac{2}{3}\left(\frac{1}{3z_2^e + z_2^b}\right)(3)\right)$$

$$z_2^b\!: \frac{1}{2}q^b = 0.9\left(\frac{1}{3}\left(\frac{1}{z_2^e + z_2^b}\right) + \frac{2}{3}\left(\frac{1}{3z_2^e + z_2^b}\right)\right)$$

Since these securities span the space and since the period 1 and period 2 endowments are the same, the real consumption allocations must be the same as in the Arrow–Debreu economy:

$$c_1^1 = c_1^2 = 2.5$$
$$c_2^1 = c_2^2 = 4$$

Thus,

$$q^e = 2(0.9)\left\{\frac{1}{3}\left(\frac{1}{2.5}\right) + \frac{2}{3}\left(\frac{1}{4}\right)3\right\} = 1.14$$

$$q^b = 2(0.9)\left\{\frac{1}{3}\left(\frac{1}{2.5}\right) + \frac{2}{3}\left(\frac{1}{4}\right)\right\} = 0.54,$$

as computed previously.

To compute the corresponding security holding, observe that:

Agent 1:

$$\begin{aligned} z_1^e + z_1^b &= 2.5 \\ 3z_1^e + z_1^b &= 4 \end{aligned} \Rightarrow \begin{aligned} z_1^e &= 0.75 \\ z_1^b &= 1.75 \end{aligned}$$

Agent 2: (same holdings)

$$z_2^e = 0.75$$
$$z_2^b = 1.75$$

Supply must equal demand in equilibrium:

$$\hat{z}_1^e + \hat{z}_2^e = \frac{1}{2} + 1 = 1.5 = z_1^e + z_2^e$$

$$\hat{z}_1^b + \hat{z}_2^b = \frac{1}{2} + 3 = 3.5 = z_1^b + z_2^b$$

The period zero consumptions are identical to the earlier calculation as well.

Appendix 12.2 Proof of the Second Part of Proposition 12.6

Define $\hat{U}(x, \theta_j) = x\left\{\frac{\pi_j^{RN}}{\pi_j(1+r_f)}\right\}$, where $\{\pi_j\colon j = 1,2,\ldots,J\}$ are the true objective state probabilities. This is a state-dependent utility function that is linear in wealth. We will show that for this function, Eq. (12.13), indeed, has a solution. Consider an arbitrary allocation of wealth to the various fundamental assets $\{n_P^i\colon j = 1, 2, \ldots, J\}$ and let P denote that portfolio. Fix the wealth at any level V_0, arbitrary. We next compute the expected utility associated with this portfolio, taking advantage of representation (12.14):

$$\begin{aligned}
E\hat{U}(\tilde{V}_P(\theta, 1)) &= E\hat{U}\left\{(1+r_f)\left[V_0 + \sum_{i=1}^{N} n_P^i\left(\frac{\tilde{q}_i^e(\theta, 1)}{(1+r_f)} - q_i^e(0)\right)\right]\right\} \\
&= \sum_{j=1}^{J} \pi_j(1+r_f)\left\{V_0 + \sum_{i=1}^{N} n_P^i\left(\frac{q_i^e(\theta_j, 1)}{(1+r_f)} - q_i^e(0)\right)\right\}\frac{\pi_j^{RN}}{\pi_j(1+r_f)} \\
&= \sum_{j=1}^{J} \pi_j^{RN}\left\{V_0 + \sum_{i=1}^{N} n_P^i\left(\frac{q_i^e(\theta_j, 1)}{(1+r_f)} - q_i^e(0)\right)\right\} \\
&= \sum_{j=1}^{J} \pi_j^{RN} V_0 + \sum_{j=1}^{J} \pi_j^{RN} \sum_{i=1}^{N} n_P^i\left(\frac{q_i^e(\theta_j, 1)}{(1+r_f)} - q_i^e(0)\right) \\
&= V_0 + \sum_{i=1}^{N} n_P^i\left(\sum_{j=1}^{J} \pi_j^{RN}\left(\frac{q_i^e(\theta_j, 1)}{(1+r_f)} - q_i^e(0)\right)\right) \\
&= V_0
\end{aligned}$$

in other words, with this utility function, every trading strategy has the same value. Thus, problem (12.12) has, trivially, a solution.