Linear CCAPM*

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*The note outlines the derivation of the linear CCAPM from the Euler equation in the consumption-based asset pricing framework. The note is prepared for use only in the Master's course "Empirical Asset Pricing". Please do not cite, circulate, or use for purposes other than this course.

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1. Linear consumption-based factor model

Recall that the following Euler equation holds for any asset i

$$\mathbb{E}_t\left[\delta\frac{u'(c_{t+1})}{u'(c_t)}(R_{it+1}-R_{ft+1})\right]=0. \tag{1}$$

Recall that for any two random variables X and Y, we have that

$$\mathbb{E}[XY]=\mathbb{E}[X]\mathbb{E}[Y]+\mathrm{Cov}[X,Y]. \tag{2}$$

Thus, taking unconditional expectations of the Euler equation and writing in terms of covariances one obtains

$$\mathbb{E}\left[\delta\frac{u'(c_{t+1})}{u'(c_t)}\right]\mathbb{E}\left[R_{it+1}-R_{ft+1}\right]+\mathrm{Cov}\left[\delta\frac{u'(c_{t+1})}{u'(c_t)},(R_{it+1}-R_{ft+1})\right]=0, \tag{3}$$

such that by rewriting one get

$$\mathbb{E}\left[\delta\frac{u'(c_{t+1})}{u'(c_t)}\right]\mathbb{E}\left[R_{it+1}-R_{ft+1}\right]=-\mathrm{Cov}\left[\delta\frac{u'(c_{t+1})}{u'(c_t)},(R_{it+1}-R_{ft+1})\right]. \tag{4}$$

A first-order approximation of $u'(c_{t+1})$ around c_t is

$$u'(c_{t+1})\approx u'(c_t)+u''(c_t)(c_{t+1}-c_t), \tag{5}$$

such that

$$\begin{aligned}\frac{u'(c_{t+1})}{u'(c_t)}&\approx\frac{u'(c_t)+u''(c_t)(c_{t+1}-c_t)}{u'(c_t)}\\&=1+\frac{u''(c_t)(c_{t+1}-c_t)}{u'(c_t)}\\&=1+\frac{c_t}{c_t}\frac{u''(c_t)(c_{t+1}-c_t)}{u'(c_t)}\\&=1+\frac{c_tu''(c_t)}{u'(c_t)}\frac{c_{t+1}-c_t}{c_t}\\&=1-\rho_t\frac{c_{t+1}-c_t}{c_t}\\&=1-\rho_t\tilde{c}_{t+1},\end{aligned} \tag{6}$$

where $\rho_t=-\frac{c_tu''(c_t)}{u'(c_t)}$ is the relative risk aversion by definition and we set $\tilde{c}_{t+1}=\frac{c_{t+1}-c_t}{c_t}=\frac{c_{t+1}}{c_t}-1$ which is the consumption growth rate. Assuming the relative risk aversion is constant[1], $\rho_t = \rho$, and substituting (6) into (4) one gets

$$\mathbb{E}\left[\delta(1-\rho\tilde{c}_{t+1})\right]\mathbb{E}\left[R_{it+1}-R_{ft+1}\right] = -\mathrm{Cov}\left[\delta(1-\rho\tilde{c}_{t+1}),(R_{it+1}-R_{ft+1})\right]. \tag{7}$$

Dividing though by $\mathbb{E}\left[\delta(1-\rho\tilde{c}_{t+1})\right]$ we get

$$\mathbb{E}\left[R_{it+1}-R_{ft+1}\right] = -\frac{\mathrm{Cov}\left[\delta(1-\rho\tilde{c}_{t+1}),R_{it+1}-R_{ft+1}\right]}{\mathbb{E}\left[\delta(1-\rho\tilde{c}_{t+1})\right]}. \tag{8}$$

Now, the δs cancel out and any covariance with a constant (here 1) is equal to zero, such that we get

$$\mathbb{E}\left[R_{it+1}-R_{ft+1}\right] = -\frac{\rho\mathrm{Cov}\left[-\tilde{c}_{t+1},R_{it+1}-R_{ft+1}\right]}{1-\rho\mathbb{E}[\tilde{c}_{t+1}]}. \tag{9}$$

Note that the two minuses go out, multiply $\mathrm{Var}[\tilde{c}_{t+1}]/\mathrm{Var}[\tilde{c}_{t+1}] = 1$ onto the expression on the right hand side, and rewrite to see that

$$\mathbb{E}\left[R_{it+1}-R_{ft+1}\right] = \frac{\rho\mathrm{Var}[\tilde{c}_{t+1}]}{1-\rho\mathbb{E}[\tilde{c}_{t+1}]}\frac{\mathrm{Cov}\left[\tilde{c}_{t+1},R_{it+1}-R_{ft+1}\right]}{\mathrm{Var}[\tilde{c}_{t+1}]}. \tag{10}$$

Finally, note that

$$\frac{\mathrm{Cov}\left[\tilde{c}_{t+1},R_{it+1}-R_{ft+1}\right]}{\mathrm{Var}[\tilde{c}_{t+1}]} = \beta_{ic}, \tag{11}$$

such that by defining

$$\gamma_c = \frac{\rho\mathrm{Var}[\tilde{c}_{t+1}]}{1-\rho\mathbb{E}[\tilde{c}_{t+1}]} \tag{12}$$

we get the β representation

$$\mathbb{E}\left[R_{it+1}-R_{ft+1}\right] = \gamma_c\beta_{ic}. \tag{13}$$

[1]Recall that if one assumes power utility this is indeed consistent with the assumption of constant relative risk aversion.