CHAPTER 16

Portfolio Management in the Long Run

Chapter Outline

16.1 Introduction 469
16.2 The Myopic Solution 472
16.3 Variations in the Risk-Free Rate 478
16.3.1 The Budget Constraint 479
16.3.2 The Optimality Equation 481
16.3.3 Optimal Portfolio Allocations 482
16.3.4 The Nature of the Risk-Free Asset 484
16.3.5 The Role of Bonds in Investor Portfolios 485
16.4 The Long-Run Behavior of Stock Returns 486
16.4.1 Solving for Optimal Portfolio Proportions in a Mean Reversion Environment 489
16.4.2 Strategic Asset Allocation 491
16.4.3 The Role of Stocks in Investor Portfolios 492
16.5 Background Risk: The Implications of Labor Income for Portfolio Choice 492
16.6 An Important Caveat 501
16.7 Another Background Risk: Real Estate 501
16.8 Conclusions 504
References 505

16.1 Introduction

The canonical portfolio problem (Section 5.1) and the modern portfolio theory (MPT) selection problem embedded in the CAPM are both one-period utility-of-terminal-wealth maximization problems.

As such, the advice to investors implicit in these theories is astonishingly straightforward:

1. *Be well diversified.* Conceptually, this recommendation implies that the risky portion of an investor's portfolio should resemble (be perfectly positively correlated with) the true market portfolio M. In practice, it usually means holding the risky component of invested wealth as a set of stock index funds, each one representing the stock market of a particular major market capitalization country with the relative proportions

dependent upon the relevant *ex ante* variance–covariance matrix estimated from recent historical data.

2. *Be on the capital market line*. That is, the investor should allocate his wealth between risk-free assets and the aforementioned major market portfolio in proportions that are consistent with his subjective risk tolerance. Implicit in this second recommendation is that the investor first estimates her coefficient of relative risk aversion as per Section 4.5 and then solves a joint savings–portfolio allocation problem of the form illustrated in Section 5.6.3. The risk-free rate used in these calculations is customarily a 1-year T-bill rate in the United States or its analogue elsewhere.

But what should the investor do next period after this period's risky portfolio return realization has been observed? Our one-period theory has nothing to say on this score except to invite the investor to repeat the above two-step process, possibly using an updated variance–covariance matrix and an updated risk-free rate. This is what is meant by the investor behaving myopically. Yet we are uneasy about leaving the discussion at this level. Indeed, a number of important considerations seem to be intentionally ignored by following such a set of recommendations.

1. There is some evidence (see Web Note 7.1) that equity return distributions have historically evolved in a pattern that is mean reverting. This is the property that a series of high-return realizations is on average followed by a series of low ones. This variation in conditional returns might reasonably be expected to influence intertemporal portfolio composition. The reader will recall that we first considered the implications of mean reversion in equity returns in Section 7.5.3 and noted there that it has the consequence of reducing long-run equity investment risk.

   In Chapter 7 we also considered the implications of parameter uncertainty; in particular, possible uncertainty in the current mean stock return as well as uncertainty in the future mean returns. In the present chapter we will eschew this important consideration and assume all relevant means, variances and correlations are known and accurately estimated. The qualitative conclusions of this chapter are unlikely to be overturned by adding the sources of uncertainty we choose to ignore.
2. While known *ex ante* relative to the start of a period, the risk-free rate also varies through time (for the period 1928–1985 the standard deviation of the US T-bill rate is 5.67%). From the perspective of a long-term US investor, the one-period T-bill rate no longer represents a truly risk-free return. Can any asset be viewed as risk free from a multiperiod perspective?
3. Investors typically receive labor income, and this fact will likely affect both the quantity of investable savings and the risky/risk-free portfolio composition decision. The latter possibility follows from the observation that labor income may be viewed as the "dividend" on an implicit nontradable human capital asset, whose value may be differentially correlated with various risky assets in the investor's financial wealth

portfolio.[1] If labor income were risk free (tenured professors!), the presence of a high-value risk-free asset in the investor's overall wealth portfolio would likely tilt his security holdings in favor of a greater proportion in risky assets than would otherwise be the case.

4. There are other life cycle considerations: savings for the educational expenses of children, the gradual disappearance of the labor income asset as retirement approaches, and so on. How do these obligations and events impact portfolio choice?
5. There is also the issue of real estate. Not only does real estate (we are thinking of owner-occupied housing for the moment) provide a risk-free service flow, but it is also expensive for an investor to alter his stock of housing. How should real estate figure into an investor's multiperiod investment plan?
6. Other considerations abound. There are substantial taxes and transactions costs associated with rebalancing a portfolio of securities. Taking these costs into account, how frequently should a long-term investor optimally alter his portfolio's composition?

In this chapter, we propose to present some recent research regarding these issues. Our perspective is one in which investors live for many periods. (In the case of private universities, foundations or insurance companies, it is reasonable to postulate an infinite lifetime.) For the moment, we will set aside the issue of real estate and explicit transactions costs, and focus on the problem of a long-lived investor confronted with jointly deciding, on a period-by-period basis, not only how much he should save and consume out of current income, but also the mix of assets, risky and risk free, in which his wealth should be invested.

In its full generality, the problem confronting a multiperiod investor—saver with outside labor income is thus

$$\max_{\{a_t, S_t\}} E\left(\sum_{t=0}^{T} \delta^t U(\widetilde{C}_t)\right) \tag{16.1}$$

$$\begin{aligned}
\text{s.t. } & C_T = S_{T-1}a_{T-1}(1+\tilde{r}_T) + S_{T-1}(1-a_{T-1})(1+r_{f,T}) + \tilde{L}_T,\ t=T \\
& C_t + S_t \le S_{t-1}a_{t-1}(1+\tilde{r}_t) + S_{t-1}(1-a_{t-1})(1+r_{f,t}) + \tilde{L}_t, 1 \le t \le T-1 \\
& C_0 + S_0 \le Y_0 + L_0, t=0
\end{aligned}$$

[1] In particular, the value of an investor's labor income asset is likely to be highly correlated with the return on the stock of the firm with which he is employed. From a wealth management perspective, basic intuition would suggest that the stock of one's employer should not be held in significant amounts.

where $\tilde{L}_t$ denotes the investor's (possibly uncertain) period t labor income, $r_{f,t}$ the period risk-free rate, and $\tilde{r}_t$ represents the period t return on the risky asset which we shall understand to mean a well-diversified stock portfolio.[2]

Equation (16.1) departs from our earlier notation in a number of ways that will be convenient for developments later in this chapter. In particular, C_t and S_t denote, respectively, period t consumption and savings rather than their lower case analogues (as in Chapter 5).

The fact that the risk-free rate is indexed by t admits the possibility that this quantity, though known at the start of a period, can vary from one period to the next. Lastly, a_t will denote the *proportion* of the investor's savings assigned to the risky asset (rather than the absolute amount as before).

All other notation is standard; Eq. (16.1) is simply the multiperiod version of the portfolio problem in Section 5.6.3 augmented by the introduction of labor income. In what follows we will also assume that all risky returns are lognormally distributed and that the investor's $U(C_t)$ is of the power utility constant relative risk aversion (CRRA) class. The latter is needed to make certain that risk aversion is independent of wealth. Although investors have become enormously wealthier over the past 200 years, risk-free rates and the return premium on stocks have not changed markedly, facts otherwise inconsistent with risk aversion dependent on wealth.

In its full generality, Eq. (16.1) is both very difficult to solve and begrudging of intuition. We thus restrict its scope and explore a number of special cases. The natural place to begin is to explore the circumstances under which the myopic solution of Section 5.3 carries over to the dynamic context of problem (16.1).

16.2 The Myopic Solution

With power utility, an investor's optimal savings to wealth ratio will be constant so that the key to a fully myopic decision rule will lie in the constancy of the a ratio. Intuitively, if the same portfolio decisions are to be made, a natural sufficient condition would be to guarantee that the investor is confronted by the same opportunities on a period-by-period basis. Accordingly, we assume the return environment is not changing through time; in other words that $r_{f,t} \equiv r_f$ is constant and $\{\tilde{r}_t\}$ is independently and identically distributed (i.i.d.). These assumptions guarantee that future prospects look the same period after period. Further exploration mandates that $L_t \equiv 0$. (With constant r_f, the value of this asset will

[2] This portfolio might be the market portfolio M but not necessarily. Consider the case in which the investor's labor income is paid by one of the firms in M. It is likely that this particular firm's shares would be underweighted (relative to M) in the investor's portfolio.

otherwise be monotonically declining, which is an implicit change in future wealth.)

We summarize these considerations in the following theorem.

Theorem 16.1 (Merton, 1971) Consider the canonical multiperiod consumption–savings–portfolio allocation problem (16.1); suppose $U(\cdot)$ displays CRRA, r_f is constant, and $\{\tilde{r}_t\}$ is i.i.d. Then the proportion a_t is time invariant.[3]

This is an important result in the following sense. It delineates the conditions under which a pure static portfolio choice analysis may be generalized to a multiperiod context. The optimal portfolio choice—in the sense of the allocation decision between the risk-free and the risky asset—defined in a static one-period context will continue to characterize the optimal portfolio decision in the more natural multiperiod environment. The conditions that are imposed are easy to understand: if the returns on the risky asset were not independently distributed, today's realization of the risky return would provide information about the future return distribution, which would almost surely affect the allocation decision. Suppose, for example, that returns are positively correlated. Then a good realization today would suggest that high returns are more likely again tomorrow. It would be natural to take this into account by, say, increasing the share of the risky asset in the portfolio. (Beware, however, that, as the first sections of Chapter 5 illustrate, without extra assumptions on the shape of the utility function—beyond risk aversion—the more intuitive result may not generally obtain. We will be reminded of this in Chapter 17 where, in particular, the log utility agent will stand out as a benchmark.) The same can be said if the risk-free rate is changing through time. In a period of high-risk-free rates, the riskless asset will be more attractive, all other things equal.

The need for the other assumption—the CRRA utility specification—is a direct consequence of Theorem 5.5. With a utility form other than CRRA, Theorem 5.5 tells us that the share of wealth invested in the risky asset varies with the "initial" wealth level, i.e,, the wealth level carried over from the last period. But in a multiperiod context, the investable wealth, i.e., the savings level, is sure to be changing over time, increasing when realized returns are favorable and decreasing otherwise. With a non-CRRA utility function, optimal portfolio allocations would consistently be affected by these changes.

Now let us illustrate the power of these ideas to evaluate an important practical problem. Consider the problem of an individual investor saving for retirement: at each period he must decide what fraction of his already accumulated wealth should be invested in stocks (understood to mean a well-diversified portfolio of risky assets) and risk-free bonds for the

[3] If the investor's period utility is log, it is possible to relax the independence assumption. This important observation, first made by Samuelson (1969), will be confirmed later on in this chapter.

next investment period. We will maintain the $L_t \equiv 0$ assumption. Popular wisdom in this area can be summarized in the following three assertions:

1. Early in life, the investor should invest nearly all of her wealth in stocks (stocks have historically outperformed risk-free assets over long—20-year—periods), while gradually shifting almost entirely into risk-free instruments as retirement approaches in order to avoid the possibility of a catastrophic loss.
2. If an investor is saving for a target level of wealth (such as, in the United States, college tuition payments for children), he should gradually reduce his holdings in stocks as his wealth approaches the target level in order to minimize the risk of a shortfall due to an unexpected market downturn.
3. Investors who are working and saving from their labor income should rely more heavily on stocks early in their working lives, not only because of the historically higher average returns that stocks provide but also because bad stock market returns, early on, can be offset by increased saving out of labor income in later years.

Following Jagannathan and Kocherlakota (1996), we wish to subject these assertions to the discipline imposed by a rigorous modeling perspective. Let us maintain the assumptions of Theorem 16.1 and hypothesize that the risk-free rate is constant, that stock returns $\{\tilde{r}_t\}$ are i.i.d. (recall the random walk model of Chapter 7), and that the investor's utility function assumes the standard CRRA form.

To evaluate assertion (1), let us further simplify Eq. (16.1) by abstracting away from the consumption−savings problem. This amounts to assuming that the investor seeks to maximize the utility of his terminal wealth, Y_T in period T, the planned conclusion of his working life. As a result, $S_t = Y_t$ for every period $t < T$ (no intermediate consumption). Under CRRA, we know that the investor would invest the same fraction of his wealth in risky assets every period (disproving the assertion), but it is worthwhile to see how this comes about in a simple multiperiod setting.

Let $\tilde{r}$ denote the (invariant) risky return distribution; the investor solves

$$\max_{\{a_t\}} E\left\{\frac{(\widetilde{Y_T})^{1-\gamma}}{1-\gamma}\right\}$$

$$\begin{aligned}\text{s.t.}\quad & \widetilde{Y_T} = a_{T-1}Y_{T-1}(1+\tilde{r}_T) + (1-a_{T-1})Y_{T-1}(1+r_f), t = T\\ & \widetilde{Y_T} = a_{t-1}Y_{t-1}(1+\tilde{r}_t) + (1-a_{t-1})Y_{t-1}(1+r_f), 1 \le t \le T-1\\ & Y_0 \text{ given}\end{aligned}$$

Problems of this type are most appropriately solved by working backward: first solving for the $T-1$ decision, then solving for the $T-2$ decision conditional on the $T-1$ decision, and so on. In period $T-1$ the investor solves

$$\max_{a_{T-1}} E\left((1-\gamma)^{-1}\left\{[a_{T-1}Y_{T-1}(1+\tilde{r})+(1-a_{T-1})Y_{T-1}(1+r_f)]^{(1-\gamma)}\right\}\right)$$

The solution to this problem, $a_{T-1} \equiv \hat{a}$, satisfies the first-order condition

$$E\{[\hat{a}(1+\tilde{r})+(1-\hat{a})(1+r_f)]^{-\gamma}(\tilde{r}-r_f)\}=0$$

As expected, because of the CRRA assumption, the optimal fraction invested in stocks is independent of the period $T-1$ wealth level. Given this result, we can work backward. In period $T-2$, the investor rebalances his portfolio, knowing that in $T-1$ he will invest the fraction $\hat{a}$ in stocks. As such, this problem becomes

$$\begin{aligned}\max_{a_{T-2}} E\Big((1-\gamma)^{-1}\{[&a_{T-2}Y_{T-2}(1+\tilde{r})+(1-a_{T-2})Y_{T-2}(1+r_f)]\\ &[\hat{a}(1+\tilde{r})+(1-\hat{a})(1+r_f)]\}^{(1-\gamma)}\Big)\end{aligned} \tag{16.2}$$

Because stock returns are i.i.d., this objective function may be written as the product of expectations as per

$$\begin{aligned}&E[\hat{a}(1+\tilde{r})+(1-\hat{a})(1+r_f)]^{(1-\gamma)}.\\ &\max_{\{a_{T-2}\}} E\{(1-\gamma)^{-1}[a_{T-2}Y_{T-2}(1+\tilde{r})+(1-a_{T-2})Y_{T-2}(1+r_f)]^{1-\gamma}\}\end{aligned} \tag{16.3}$$

Written in this way, the structure of the problem is no different from the prior one, and the solution is again $a_{T-2} \equiv \hat{a}$. Repeating the same argument, it must be the case that $a_t = \hat{a}$ in every period, a result that depends critically not only on the CRRA assumption (wealth factors out of the first-order condition) but also on the independence. The risky return realized in any period does not alter our belief about the future return distributions. There is no meaningful difference between the long-run (many periods) and the short-run (one period): agents invest the same fraction in stocks regardless of their portfolio's performance history. Assertion (1) is clearly not generally valid.

To evaluate our second assertion, and following again Jagannathan and Kocherlakota (1996), let us modify the agent's utility function to be of the form

$$U(Y_T)=\begin{cases}\dfrac{(Y_T-\overline{Y})^{1-\gamma}}{1-\gamma} & \text{if } Y_T \geq \overline{Y}\\ -\infty & \text{if } Y_T < \overline{Y}\end{cases}$$

where $\overline{Y}$ is the target level of wealth. Under this formulation, it is absolutely essential that the target be achieved: as long as there exists a positive probability of failing to achieve the target, the investor's expected utility-of-terminal wealth is $-\infty$. Accordingly, we must also require that

$$Y_0(1+r_f)^T > \overline{Y}$$

in other words, that the target can be attained by investing everything in risk-free assets. If such an inequality were not satisfied, then every strategy would yield an expected utility of $-\infty$, with the optimal strategy thus being indeterminate.

A straightforward analysis of this problem yields the following two-step solution:

Step 1. always invest sufficient funds in risk-free assets to achieve the target wealth level with certainty.

Step 2. invest a constant share a^* of any additional wealth in stock, where a^* is time invariant.

By this solution, the investor invests less in stocks than he would in the absence of a target, but since he invests in both stocks and bonds, his wealth will accumulate, on average, more rapidly than it would if invested solely at the risk-free rate, and the stock portion of his wealth will, on average, grow faster. As a result, the investor will typically use proportionally less of his resources to guarantee achievement of the target. And, over time, targeting will tend to *increase* the share of wealth in stocks, again contrary to popular wisdom!

In order to evaluate assertion (3), we must admit savings from labor income into the analysis. Let $\{L_t\}$ denote the stream of savings out of labor income. For simplicity, we assume that the stream of future labor income is fully known at date 0. The investor's problem is now:

$$\max_{\{a_t\}} E\left(\frac{(\widetilde{Y_T})^{1-\gamma}}{1-\gamma}\right) \text{ s.t.}$$

$$\widetilde{Y_T} = L_T + a_{T-1}Y_{T-1}(1+\tilde{r}_T) + (1-a_{T-1})Y_{T-1}(1+r_f), t = T$$

$$\widetilde{Y_{\mathrm{t}}} \le L_t + a_{t-1}Y_{t-1}(1+\tilde{r}_t) + (1-a_{t-1})Y_{t-1}(1+r_f), 1 \le t \le T-1$$

$$Y_0; \{L_t\}_{t=0}^T \text{ given}$$

We again abstract away from the consumption–savings problem and focus on maximizing the expected utility of terminal wealth.

In any period, the investor now has two sources of wealth: financial wealth, Y_t^F, where

$$Y_t^F = L_t + a_{t-1}Y_{t-1}(1+r_t) + (1-a_{t-1})Y_{t-1}(1+r_f)$$

(r_t is the period t realized value of $\tilde{r}$); and labor income wealth, Y_t^L, is measured by the present value of the future stream of labor income. As mentioned, we assume this income stream is risk free with present value,

$$Y_t^L = \frac{L_{t+1}}{(1+r_f)} + \cdots + \frac{L_T}{(1+r_f)^{T-1}}$$

Since the investor continues to have CRRA preferences, he will, in every period, invest a constant fraction of his total wealth $\hat{a}$ in stocks, where $\hat{a}$ depends only upon his CRRA and the characteristics of the return distributions $\tilde{r}$ and r_f, i.e.,

$$A_t = \hat{a}(Y_t^F + Y_t^L)$$

where A_t denotes the *amount* invested in the risky financial asset.

As the investor approaches retirement, his Y_t^L declines. In order to maintain the same fraction of wealth invested in risk-free assets, the fraction of financial wealth invested in stocks,

$$\frac{A_t}{Y_t^F} = \hat{a}\left(1 + \frac{Y_t^L}{Y_t^F}\right)$$

must decline on average. Here at least the assertion has theoretical support, but for a reason different from what is commonly asserted.

In what follows we will consider the impact on portfolio choice of a variety of changes to the myopic context just considered. In particular, we explore the consequences of relaxing the constancy of the risk-free rate and return independence for the aforementioned recommendations. In most (but not all) of the discussion, we will assume an infinitely lived investor ($T = \infty$ in problem (16.1)). Recall that this amounts to postulating that a finitely lived investor is concerned for the welfare of his descendants. In nearly every case it enhances tractability. As a device for tying the discussion together, we will also explore how robust the three investor recommendations just considered are to a more general return environment.

Our first modification admits a variable risk-free rate; the second generalizes the return generating process on the risky asset (no longer i.i.d. but "mean reverting"). Our remarks are largely drawn from a prominent publication (Campbell and Viceira, 2002).

Following the precedents established by these authors, it will prove convenient to log-linearize the investor's budget constraint and optimality conditions. Simple and intuitive expression for optimal portfolio proportions typically results. Some of the underlying deviations are provided in an appendix available on this text's web site; others are simply omitted when they are lengthy and complex and where an attractive intuitive interpretation is available.

In the next section, the risk-free rate is allowed to vary, though in a particularly structured way.

16.3 Variations in the Risk-Free Rate

Following Campbell and Viceira (2002), we specialize Eq. (16.1) to admit a variable risk-free rate. Other assumptions are:

i. $L_t \equiv 0$ for all t; there is no labor income so that all consumption comes from financial wealth alone.
ii. $T = \infty$, i.e., we explore the infinite horizon version of Eq. (16.1); this allows a simplified description of the optimality conditions on portfolio choice.
iii. All relevant return random variables are lognormal with constant variances and covariances. This is an admittedly strong assumption as it mandates that the return on the investor's portfolio has a constant variance and that the constituent assets have constant variances and covariances with the portfolio itself. Thus, the composition of the risky part of the investor's portfolio must itself be invariant. But this will be optimal only if the expected excess returns above the risk-free rate on these same constituent assets are also constant. Expected returns can vary over time, but, in effect, they must move in tandem with the risk-free rate. This assumption is considerably specialized, but it does allow for unambiguous conclusions.
iv. The investor's period utility function is of the Epstein–Zin variety (cf. Section 5.7.3). In this case, the intertemporal optimality condition for Eq. (16.1) when $T = \infty$ and there are multiple risky assets can be expressed as (recall Eqn. 10.3.2)

$$1 = E_t \left\{ \left[\delta \left(\frac{\tilde{C}_{t+1}}{C_t} \right)^{-\frac{1}{\rho}} \right]^{\theta} \left[\frac{1}{\tilde{R}_{P,t+1}} \right]^{1-\theta} \tilde{R}_{i,t+1} \right\} \tag{16.4}$$

where $\tilde{R}_{i,t}$ is the period t gross return on any available asset (risk free or otherwise, including the portfolio itself) and $\tilde{R}_{P,t}$ is the period t overall risky portfolio's gross return. Note that consumption C_t, and the various returns $\tilde{R}_{P,t}$ and $\tilde{R}_{i,t}$ are capitalized. We will henceforth denote the logs of these quantities by their respective lower case counterparts.[4] Equation (16.4) is simply a restatement of Eq. (9.28), where γ is the risk-aversion parameter, ρ is the elasticity of intertemporal substitution, and $\theta = (1-\gamma)/(1-1/\rho)$.

[4] Recall from Chapter 3, Box 31, that $r_t^c - \log \mathrm{R_t}$ is the continuously compounded net rate of return over period t. When met returns are not large, $r_t^c = \log R_t = \log(1 + r_t) \approx r_t$: the net return and its continuously compounded counterpart are essentially the same.

Bearing in mind assumptions (i)−(iv), we now proceed first to the investor's budget constraint and then to his optimality condition. The plan is to log-linearize each expression. This will simplify the necessary development leading to our ultimate goal, Eq. (16.20).

16.3.1 The Budget Constraint

In a model with period consumption exclusively out of financial wealth, the intertemporal budget constraint is of the form

$$Y_{t+1} = (R_{P,t+1})(Y_t - C_t) \tag{16.5}$$

where the risky portfolio P potentially contains many risky assets; equivalently,

$$\frac{Y_t + 1}{Y_t} = (R_{P,t+1})\left(1 - \frac{C_t}{Y_t}\right) \tag{16.6}$$

or, taking the log on both sides of the equation,

$$\Delta y_{t+1} = \log Y_{t+1} - \log Y_t = \log(R_{P,t+1}) + \log(1 - \exp(\log C_t - \log Y_t))$$

Recalling our identification of a lowercase variable with the log of that variable, we have

$$\Delta y_{t+1} = r_{P,t+1} + \log(1 - \exp(\widetilde{c}_t - \widetilde{y}_t))$$

Assuming that the $\log(C_t/Y_t)$ is not too variable (essentially this places us in the $\rho = \gamma = 1$ − the log utility case), then the rightmost term can be approximated around its mean to yield (see Campbell and Viceira, 2001a):

$$\Delta y_{t+1} = k_1 + r_{P,t+1} + \left(1 - \frac{1}{k_2}\right)(\widetilde{c}_t - \widetilde{y}_t) \tag{16.7}$$

where k_1 and $k_2 < 1$ are constants related to $\exp(E(\widetilde{c}_t - \widetilde{y}_t))$.

Speaking somewhat informally in a fashion that would identify the log of a variable with the variable itself, Eq. (16.7) simply states that wealth will be higher next period $(t + 1)$ in a manner that depends on both the portfolio's rate of return $(r_{P,t+1})$ over the next period and on this period's consumption relative to wealth. If c_t greatly exceeds y_t, wealth next period cannot be higher!

We next employ an identity to allow us to rewrite Eq. (16.7) in a more useful way; it is

$$\Delta y_{t+1} = \Delta c_{t+1} + (c_t - y_t) - (c_{t+1} - y_{t+1}) \tag{16.8}$$

where $\Delta c_{t+1} = c_{t+1} - c_t$. Substituting the RHS of Eq. (16.8) into Eq. (16.7) and rearranging terms yields

$$(c_t - y_t) = k_2 k_1 + k_2(r_{P,t+1} - \Delta c_{t+1}) + k_2(c_{t+1} - y_{t+1}) \tag{16.9}$$

Equation (16.9) provides the same information as Eq. (16.7), albeit expressed differently. It states that an investor could infer his (log) consumption–wealth ratio $(c_t - y_t)$ in period t from a knowledge of its corresponding value in period $t+1$, $(c_{t+1} - y_{t+1})$, and his portfolio's return (the growth rate of his wealth) relative to the growth rate of his consumption $(r_{P,t+1} - \Delta c_{t+1})$. (Note that our use of language again informally identifies a variable with its log.)

Equation (16.9) is a simple difference equation that can be solved forward to yield

$$c_t - y_t = \sum_{j=1}^{\infty} (k_2)^j (\widetilde{r}_{P,t+j} - \Delta \widetilde{c}_{t+j}) + \frac{k_2 k_1}{1 - k_2} \tag{16.10}$$

Equation (16.10) also has an attractive intuitive interpretation; a high (above average) consumption–wealth ratio ($(c_t - y_t)$ large and positive), i.e., a burst of consumption must be followed either by high returns on invested wealth or lowered future consumption growth. Otherwise the investor's intertemporal budget constraint cannot be satisfied. But Eq. (16.10) holds *ex ante* relative to time t as well as *ex post*, its current form. Equation (16.11) provides the *ex ante* version:

$$c_t - y_t = E_t \sum_{j=1}^{\infty} (k_2)^j (\widetilde{r}_{P,t+j} - \Delta \widetilde{c}_{t+j}) + \frac{k_2 k_1}{1 - k_2} \tag{16.11}$$

Substituting this expression twice into the RHS of Eq. (16.8), substituting the RHS of Eq. (16.7) for the LHS of Eq. (16.8), and collecting terms yield our final representation for the log-linearized budget constraint equation:

$$c_{t+1} - E_t \widetilde{c}_{t+1} = (E_{t+1} - E_t) \sum_{j=1}^{\infty} (k_2)^j \widetilde{r}_{P,t+1+j} - (E_{t+1} - E_t) \sum_{j=1}^{\infty} (k_2)^j \Delta \widetilde{c}_{t+1+j} \tag{16.12}$$

This equation again has an intuitive interpretation: if consumption in period $t+1$ exceeds its period t expectation ($c_{t+1} > E_t \widetilde{c}_{t+1}$, a positive consumption "surprise"), then this consumption increment must be "financed" either by an upward revision in expected future portfolio returns (the first term on the LHS of Eq. (16.12)) or a downward revision in future consumption growth (as captured by the second term on the LHS of Eq. (16.12)). If it were otherwise, the investor would receive "something for nothing"—as though his budget constraint could be ignored.

Since our focus is on deriving portfolio proportions and returns, it will be useful to be able to eliminate future consumption growth (the Δc_{t+1+j} terms) from the above equation and to replace it with an expression related only to returns. The natural place to look for such an equivalence is the investor's optimality Eq. (16.4), which directly relates the returns on his choice of optimal portfolio to his consumption experience, log-linearized so as to be in harmony with Eq. (16.12).

16.3.2 The Optimality Equation

The log-linearized version of Eq. (16.4) is

$$E_t\,\Delta c_{t+1} = \rho \log \delta + \rho E_t \tilde{r}_{P,t+1} + \frac{\theta}{2\rho}\operatorname{var}_t(\Delta \tilde{c}_{t+1} - \rho \tilde{r}_{P,t+1}) \tag{16.13}$$

where we have specialized Eq. (16.4) somewhat by choosing the ith asset to be the portfolio itself so that $R_{i,t+1} = R_{P,t+1}$. The web appendix provides a derivation of this expression, but it is more important to grasp what it is telling us about an Epstein−Zin investor's optimal behavior: in our partial equilibrium setting where investors take return processes as given, Eq. (16.13) states that an investor's optimal expected consumption growth ($E_t(\Delta c_{t+1})$) is linearly (by the log-linear approximation) related to the time preference parameter δ (an investor with a bigger δ will save more and thus his expected consumption growth will be higher), the portfolio returns he expects to earn ($E_t \tilde{r}_{P,t+1}$), and the miscellaneous effects of uncertainty as captured by the final term $\theta/2\rho\ \operatorname{var}_t(\Delta \tilde{c}_{t+1} - \rho \tilde{r}_{P,t+1})$. A high intertemporal elasticity of substitution ρ means that the investor is willing to experience a steeper consumption growth profile if there are incentives to do so, and thus ρ premultiplies both $\log \delta$ and $E_t \tilde{r}_{P,t+1}$. Lastly, if $\theta > 0$, an increase in the variance of consumption growth relative to portfolio returns leads to a greater expected consumption growth profile. Under this condition, the variance increase elicits greater precautionary savings in period t and thus a greater expected consumption growth rate.

Under assumption (iii) of this section, however, the variance term in Eq. (16.13) is constant, which leads to a much-simplified representation

$$E_t\,\Delta \tilde{c}_{t+1} = k_3 + \rho E_t \tilde{r}_{P,t+1} \tag{16.14}$$

where the constant k_3 incorporates both the constant variance and the time preference term $\rho \log \delta$. Substituting Eq. (16.14) into Eq. (16.11) in the most straightforward way and rearranging terms yield

$$c_t - y_t = (1-\rho)E_t \sum_{j=1}^{\infty} (k_2)^j \tilde{r}_{P,t+j} + \frac{k_2(k_1 - k_2)}{1 - k_2} \tag{16.15}$$

Not surprisingly, Eq. (16.15) suggests that the investor's (log) consumption to wealth ratio (itself a measure of how willing he is to consume out of current wealth) depends linearly on future discounted portfolio returns, negatively if $\rho > 1$ and positively if $\rho < 1$ where ρ is his intertemporal elasticity of substitution. The value of ρ reflects the implied dominance of the substitution over the income effect. If $\rho < 1$, the income effect dominates: if portfolio returns increase, the investor can increase his consumption permanently without diminishing his wealth. If the substitution effect dominates ($\rho > 1$), however, the investor will reduce his current consumption in order to take advantage of the impending higher expected returns. Substituting Eq. (16.15) into Eq. (16.12) yields

$$c_{t+1} - E_t\tilde{c}_{t+1} = \tilde{r}_{P,t+1} - E_t\tilde{r}_{P,t+1} + (1-\rho)(E_{t+1} - E_t)\sum_{j=1}^{\infty}(k_2)^j\tilde{r}_{P,t+1+j} \tag{16.16}$$

an equation that attributes period $t+1$'s consumption surprise to (1) the unexpected contemporaneous component to the overall portfolio's return $\tilde{r}_{P,t+1} - E_t\tilde{r}_{P,t+1}$, plus (2) the revision in expectation of future portfolio returns, $(E_{t+1} - E_t)\sum_{j=1}^{\infty}(k_2)^j\tilde{r}_{P,t+1+j}$. This revision either encourages or reduces consumption, depending on whether, once again, the income or substitution effect dominates. This concludes the background on which the investor's optimal portfolio characterization rests. Note that Eq. (16.16) defines a relationship by which consumption may be replaced—in some other expression of interest—by a set of terms involving portfolio returns alone.

16.3.3 Optimal Portfolio Allocations

So far, we have not employed the assumption that the expected returns on all assets move in tandem with the risk-free rate; indeed the risk-free rate is not explicit in any of expressions (16.2)–(16.14). We address these issues presently.

In an Epstein–Zin context, recall that the risk premium on any risky asset over the safe asset, $E_t\tilde{r}_{P,\,t+1} - r_{f,t+1}$, is given by Eq. (9.32), which is recopied below:

$$E_t\tilde{r}_{t+1} - r_{f,t+1} + \frac{\sigma_t^2}{2} = \frac{\theta\,\mathrm{cov}_t(\tilde{r}_{t+1}, \Delta\tilde{c}_{t+1})}{\rho} + (1-\theta)\,\mathrm{cov}_t(\tilde{r}_{t+1}, \tilde{r}_{P,t+1}) \tag{16.17}$$

where r_{t+1} denotes the return on the stock portfolio, and $r_{P,t+1}$ is the return on the portfolio of all the investor's assets, i.e., including the "risk-free" one. Note that implicit in assumption (iii) is the recognition that all variances and covariances are constant despite the time dependency in notation.

From expression (16.16), we see that the covariance of (log) consumption with any variable (and we have in mind its covariance with the risky return variable of Eq. (16.17)) may be replaced by the covariance of that variable with the portfolio's contemporaneous return plus $(1-\rho)$ times the expectations revisions concerning future portfolio returns. Eliminating consumption from Eq. (16.17) in this way via a judicious insertion of Eq. (16.16) yields

$$E_t\tilde{r}_{t+1} - \tilde{r}_{f,t+1} + \frac{\sigma_t^2}{2} = \gamma\,\mathrm{cov}_t(\tilde{r}_{t+1},\tilde{r}_{P,t+1}) + (\gamma - 1)\mathrm{cov}_t\left(\tilde{r}_{t+1},(E_{t+1} - E_t)\sum_{j=1}^{\infty}(k_2)^j\tilde{r}_{P,t+1+j}\right) \tag{16.18}$$

As noted in Campbell and Viceira (2002), Eqs. (16.16) and (16.18) delineate in an elegant way the consequences of the Epstein and Zin separation of time and risk preferences. In particular, in Eq. (16.16), it is only the elasticity of intertemporal substitution parameter ρ that relates current consumption to future returns (and thus income)—a time preference effect—while, in Eq. (16.18), it is only γ, the risk-aversion coefficient, that appears to influence the risk premium on the risky asset.

If we further recall (assumption (iii)) that variation in portfolio expected returns must be exclusively attributable to variation in the risk-free rate, it follows logically that revisions in expectations of the portfolio expected returns must uniquely follow from revisions of expectations of the risk-free rate:

$$(E_{t+1} - E_t)\sum_{j=1}^{\infty}(k_2)^j\tilde{r}_{P,t+1+j} = (E_{t+1} - E_t)\sum_{j=1}^{\infty}(k_2)^j\tilde{r}_{f,t+1+j} \tag{16.19}$$

In a model with one risky asset (in effect, the risky portfolio whose composition we are a priori holding constant),

$$\mathrm{cov}_t(\tilde{r}_{t+1},\tilde{r}_{p,t+1}) = a_t\sigma_t^2 = a_t\sigma_P^2, t$$

where a_t is, as before, the risky asset proportion in the portfolio.

Substituting both this latter expression and identification (16.19) into Eq. (16.18) and solving for a_t gives the optimal, time-invariant portfolio weight on the risky asset.

$$a_t \equiv a = \frac{1}{\gamma}\left[\frac{E_t\tilde{r}_{t+1} - r_{f,t+1} + \frac{\sigma_t^2}{2}}{\sigma_t^2}\right] + \left(1 - \frac{1}{\gamma}\right)\frac{1}{\sigma_t^2}\mathrm{cov}_t\left(\tilde{r}_{t+1},\ -(E_{t+1} - E_t)\sum_{j=1}^{\infty}(k_2)^j\tilde{r}_{f,t+1+j}\right), \tag{16.20}$$

our first portfolio result. In what follows we offer a set of interpretative comments related to it.

1. The first term in Eq. (16.20) represents the myopic portfolio demand for the risky asset—myopic in the sense that it describes the fraction of wealth invested in the risky portfolio when the investor ignores the possibility of future risk-free rate changes. In

particular, the risky portfolio proportion is inversely related to the investor's CRRA (γ) and positively related to the risk premium. Note, however, that these rate changes are the fundamental feature of this economy in the sense that variances are fixed and all expected risky returns move in tandem with the risk-free rate.

2. The second term in Eq. (16.20) captures the risky asset demand related to its usefulness for hedging intertemporal interest rate risk. The idea is as follows. We may view the risk-free rate as the "base line" return on the investor's wealth with the risky asset providing a premium on some fraction thereof. If expected future risk-free returns are revised downward, $[-(E_{t+1} - E_t)\sum_{j=1}^{\infty}(k_2)^j \tilde{r}_{f,t+1+j}$ increases], the investor's future income (consumption) stream will be reduced unless the risky asset's return increases to compensate. This will be so on average if the covariance term in Eq. (16.20) is positive. It is in this sense that risky asset returns (r_{t+1}) can hedge risk-free interest rate risk. If the covariance term is negative, however, risky asset returns tend only to magnify the consequences of a downward revision in expected future risk-free rates. As such, a long-term investor's holding of risky assets would be correspondingly reduced.

   These remarks have their counterpart in asset price changes: if risk-free rates rise (bond prices fall), the investor would wish for changes in the risky portion of his portfolio to compensate via increased valuations.
3. As the investor becomes progressively more risk averse ($\gamma \mapsto \infty$), she will continue to hold stocks in her portfolio, but only because of their hedging qualities, and not because of any return premium they provide. An analogous myopic investor would hold no risky assets.
4. Note also that the covariance term in Eq. (16.20) depends on changes in expectations concerning the entire course of future interest rates. It thus follows that the investor's portfolio allocations will be much more sensitive to persistent changes in the expected risk-free rate than to transitory ones.

Considering all the complicated formulae that have been developed, the conclusions thus far are relatively modest. An infinitely lived investor principally consumes out of his portfolio's income, and he wishes to maintain a stable consumption series. To the extent that risky equity returns can offset (hedge) variations in the risk-free rate, investors are provided justification for increasing the share of their wealth invested in the high-return risky asset. This leads us to wonder if any asset can serve as a truly risk-free one for the long-term investor.

16.3.4 The Nature of the Risk-Free Asset

Implicit in the above discussion is the question of what asset, if any, best serves as the risk-free one. From the long-term investor's point of view, it clearly cannot be a short-term money market instrument (e.g., a T-bill) because its well-documented rate variation makes uncertain the future reinvestment rates that the investor will receive.

We are reminded at this juncture, however, that it is not the return risk *per se*, but the derived consumption risk that is of concern to investors. Viewed from the consumption perspective, a natural candidate for the risk-free asset is an indexed consol bond that pays (the real monetary equivalent of) one unit of consumption every period. Campbell et al. (1997) show that the (log) return on such a consol is given by

$$r_{c,t+1} = r_{f,t+1} + k_4 - (E_{t+1} - E_t)\sum_{j=1}^{\infty}(k_5)^j \tilde{r}_{f,t+1+j} \tag{16.21}$$

where k_4 is a constant measuring the (constant) risk premium on the consol, and k_5 is another positive constant less than one. Suppose as well that we have an infinitely risk-averse investor ($\gamma = \infty$) so that Eq. (16.20) reduces to

$$a = \frac{1}{\sigma_t^2}\text{cov}_t\left(\tilde{r}_{t+1}, -(E_{t+1} - E_t)\sum_{j=1}^{\infty}(k_2)^j \tilde{r}_{f,t+1+j}\right) \tag{16.22}$$

and that the single risky asset is the consol bond ($r_{t+1} = r_{c,t+1}$). In this case (substituting Eq. (16.21) into Eq. (16.22) and observing that constants do not matter for the computing of covariances), $a \equiv 1$: the highly risk-averse investor will eschew short-term risk-free assets and invest entirely in indexed bonds. This alone will provide him with a risk-free consumption stream, although the value of the asset may change from period to period.

16.3.5 The Role of Bonds in Investor Portfolios

Now that we allow the risk-free rate to vary, let us return to the three life cycle portfolio recommendations mentioned in Section 16.2. Of course, the model—with an infinitely lived investor—is, by construction, not the appropriate one for the life cycle issues of recommendation 2, and, being without labor income, nothing can be said regarding recommendation 3 either. This leaves the first recommendation, which really concerns the portfolio of choice for long-term investors. The single message of this chapter subsection must be that conservative long-term investors should invest the bulk of their wealth in long-term index bonds. If such bonds are not available, then in an environment of low inflation risk, long-term government securities are a reasonable, second best substitute. For persons entering retirement—and likely to be very concerned about significant consumption risk—long-term real bonds should be the investment vehicle of choice.

This is actually a very different recommendation from the static one-period portfolio analysis that would argue for a large fraction of a conservative investor's wealth being assigned to risk-free assets (T-bills). Yet we know that short rate uncertainty, which the long-term investor would experience every time she rolled over her short-term instruments, makes such an investment strategy inadvisable for the long term.

16.4 The Long-Run Behavior of Stock Returns

Should the proportion of an investor's wealth invested in stocks differ systematically for long-term versus short-term investors? In either case, most of the attractiveness of stocks (by stocks we will continue to mean a well-diversified stock portfolio) to investors lies in their high excess returns (recall the equity premium puzzle of Chapter 10). But what about long- versus short-term equity risk, i.e., how does the *ex ante* return variance of an equity portfolio held for many periods compare with its variance in the short run?

The *ex post* historical return experience of equities versus other investments turns out to be quite unexpected in this regard.

From Table 16.1, it is readily apparent that, historically, more than 20-year time horizons, stocks have never yielded investors a negative real annualized return, while for all other

Table 16.1: Minimum and maximum actual annualized real holding period returns for the period, 1802−1997; US securities markets and a variety of investment options[a]

	Maximum Observed Return		Minimum Observed Return
		One-year holding period	
Stocks	66.6%		−38.6%
Bonds	35.1%		−21.9%
T-bills	23.7%		−15.6%
		Two-year holding period	
Stocks	41.0%		−31.6%
Bonds	24.7%		−15.9%
T-bills	21.6%		−15.1%
		Five-year holding period	
Stocks	26.7%		−11.0%
Bonds	17.7%		−10.1%
T-bills	14.9%		−8.2%
		Ten-year holding period	
Stocks	16.9%		−4.1%[b]
Bonds	12.4%		−5.4%
T-bills	11.6%		−5.1%
		Twenty-year holding period	
Stocks	12.6%		1.0%
Bonds	8.8%		−3.1%
T-bills	8.3%		−3.0%
		Thirty-year holding period	
Stocks	10.6%		2.6%
Bonds	7.4%		−2.0%
T-bills	7.6%		−1.8%

[a]*Source*: Siegel (1998), Figure 2−1.
[b]Note that beginning with a 10-year horizon, the minimum observed stock return exceeded the corresponding minimum bill and bond returns.
This table replicates the information in Fig. 7.6.

investment types this has been the case for some sample period. Are stocks in fact less risky than bonds for an appropriately "long run"?

In this section, we propose to explore this issue via an analysis of the following questions:

1. What are the intertemporal equity return patterns in order that the outcomes portrayed in Table 16.1 are pervasive and not just represent the realizations of extremely low-probability events?
2. Given a resolution of (1), what are the implications for the portfolio composition of long-term versus short-term investors?
3. How does a resolution of questions (1) and (2) modify the myopic response to the long-run investment advice of Section 16.2?

It is again impossible to answer these questions in full generality. Following Campbell and Viceira (1999), we elect to examine investor portfolios composed of one risk-free and one risky asset (a diversified portfolio). Otherwise, the context is as follows:

i. The investor is infinitely lived with Epstein–Zin preferences so that Eq. (16.4) remains as the investor's intertemporal optimality condition; furthermore, the investor has no labor income.
ii. The log real risk-free rate is constant from period to period. Under this assumption, all risk-free assets—long or short term—pay the same annualized return. The issues of Section 16.3 thus cannot be addressed.
iii. The equity return generating process builds on the following observations.
   First, note that the cumulative log return over T periods under the benchmark i.i.d. assumption is given by

$$r_{t+1} + r_{t+2} + \cdots + r_{t+T}$$

so that

$$\mathrm{var}(\tilde{r}_{t+1} + \tilde{r}_{t+2} + \cdots + \tilde{r}_{t+T}) = T\ \mathrm{var}(\tilde{r}_{t+1}) > T\ \mathrm{var}(\tilde{r}_{f,t+1})$$

For US data, $\mathrm{var}(r_{t+1}) \approx (0.167)$ (taking the risky asset as the S&P 500 market index) and var $(\tilde{r}_{f,t+1}) = (0.057)^2$ (measuring the risk-free rate as the 1-year T-bill return). With a $T = 20$-year time horizon, the observed range of annualized relative returns given by Table 16.1 is thus extremely unlikely to have arisen from an i.i.d. process. What could be going on?

For an i.i.d. process, the large relative 20-year variance arises from the possibility of long sequences, respectively, of high and low returns. But if cumulative stock returns are to be less variable than bond returns at long horizons, some aspect of the return generating process must be discouraging these possibilities. That aspect is referred to as "mean reversion": the tendency of high returns today to be followed by low returns tomorrow on an expected basis and vice versa. It is one aspect of the "predictability" of stock returns and is well documented beyond the evidence in Table 16.1.[5]

Campbell and Viceira (1999) statistically model the mean reversion in stock returns in a particular way that facilitates the solution to the associated portfolio allocation problem. In particular, they assume the time variation in log return on the risky asset is captured by

$$r_{t+1} - E_t\tilde{r}_{t+1} = \tilde{u}_{t+1}, \tilde{u}_{t+1} \sim N(0, \sigma_u^2) \tag{16.23}$$

where u_{t+1} captures the unexpected risky return component or "innovation." In addition, the expected premium on this risky asset is modeled as evolving according to

$$E_t\tilde{r}_{t+1} - r_f + \frac{\sigma_u^2}{2} = \tilde{x}_t \tag{16.24}$$

where x_t itself is a random variable following an AR(1) process with mean $\bar{x}$, persistence parameter ϕ, and random innovation $\tilde{\eta}_{t+1} \sim N(0, \sigma_\eta^2)$:[6]

$$\tilde{x}_{t+1} = \bar{x} + \phi(x_t - \bar{x}) + \tilde{\eta}_{t+1} \tag{16.25}$$

[5] We introduced the notion of predictability back in footnote 14 of Chapter 10. Here we review the notion in more detail. Perhaps the most frequently cited predictive variable is $\log(D_t/P_t) = d_t - p_t$, the log of the dividend/price ratio at long horizons. In particular, regressions of the form

$$\tilde{r}_{t,t+k} \equiv \tilde{r}_{t+1} + \cdots + \tilde{r}_{t+k} = \beta_k(d_t - p_t) + \tilde{\varepsilon}_{t,t+k}$$

obtain an R^2 of an order of magnitude of 0.3. In the above expression r_{t+j} denotes the log return on the value weighted index portfolio comprising all NYSE, AMEX, and NASDAQ stocks in month $t+j$, d_t is the log of the sum of all dividends paid on the index over the entire year preceding period t, and P_t denotes the period t value of the index portfolio. See Campbell et al. (1997) for a detailed discussion. More recently, Santos and Veronesi (2006) have studied regressions whereby long horizon excess returns (above the risk-free rate) are predicted by lagged values of the (US data) aggregate labor income/consumption ratio:

$$r_{t+k} = \alpha_1 + \beta_k s_t^W + \varepsilon_{t+k}$$

where $s_t^w = w_t/c_t$; w_t is measured as period t total compensation to employees and c_t denotes consumption of nondurables plus services (quarterly data). For the period 1948–2001, for example, they obtain an adjusted R^2 of 0.42 for $k = 16$ quarters. Returns are computed in a manner identical to Campbell et al. (1997) just mentioned. The basic logic is as follows: when the labor income/consumption ratio is high, investors are less exposed to stock market fluctuations (equity income represents a small fraction of total consumption) and hence demand a lower premium. Stock prices are thus high. Since the s_t^w ratio is stationary (and highly persistent in the data), it will eventually return to its mean value, suggesting a lower future tolerance for risk, a higher risk premium, lower equity prices, and low future returns. Their statistical analysis concludes that the labor income/consumption ratio does indeed move in a direction opposite to long horizon returns. Campbell and Cochrane (1999) are also able to replicate many of the predictability results of the empirical literature using a slow-moving external habit formation model. When it comes down to it predictability would appear to be a direct consequence of mean reversion.

[6] We are reminded that the presence of the $\sigma_u^2/2$ term in Eq. (16.24) follows from the fact that for any lognormal random variable $\tilde{z}$, $\log E_t\tilde{z}_{t+1} = E_t \log \tilde{z}_{t+1} + \frac{1}{2}\text{var}_t \log \tilde{z}_{t+1}$.

The x_t random variable thus moves slowly (depending on ϕ), with a tendency to return to its mean value. Lastly, mean reversion is captured by assuming $\text{cov}(\eta_{t+1}, u_{t+1}) = \sigma_{\eta u} < 0$, which translates, as per below, into a statement about risky return autocorrelations:

$$\begin{aligned}
0 > \sigma_{\eta u} &= \text{cov}(\tilde{u}_{t+1}, \tilde{\eta}_{t+1}) = \text{cov}_t[(\tilde{r}_{t+1} - E_t\tilde{r}_{t+1}), (\tilde{x}_{t+1} - \bar{x} - \phi(x_t - \bar{x}))] \\
&= \text{cov}_t(\tilde{r}_{t+1}, \tilde{x}_{t+1}) \\
&= \text{cov}_t\left(\tilde{r}_{t+1}, E_t\tilde{r}_{t+2} - r_f + \frac{\sigma_u^2}{2}\right) \\
&= \text{cov}_t\left(\tilde{r}_{t+1}, \tilde{r}_{t+2} - \tilde{u}_{t+2} - r_f + \frac{\sigma_u^2}{2}\right) \\
&= \text{cov}_t(\tilde{r}_{t+1}, \tilde{r}_{t+2})
\end{aligned}$$

a high return today reduces expected returns next period. Thus,

$$\text{var}_t(\tilde{r}_{t+1} + \tilde{r}_{t+2}) = 2\ \text{var}_t(\tilde{r}_{t+1}) + 2\ \text{cov}_t(\tilde{r}_{t+1}, \tilde{r}_{t+2}) < 2\ \text{var}_t(r_{t+1})$$

in contrast to the independence case. More generally, for all horizons k,

$$\frac{\text{var}_t(\tilde{r}_{t+1} + \tilde{r}_{t+2} + \cdots + \tilde{r}_{t+k})}{k\ \text{var}_t(\tilde{r}_{t+1})} < 1$$

stocks will appear to be relatively safer to long-term investors.[7] This concludes assumption (iii) and its interpretation. We next explore what this environment implies for optimal investment proportions.

16.4.1 Solving for Optimal Portfolio Proportions in a Mean Reversion Environment

Campbell and Viceira (1999) manipulate the log-linearized version of the optimality condition for Epstein–Zin utility (16.4) in conjunction with Eqs. (16.23)–(16.25) to obtain

$$a_t = a_0 + a_1 x_t \tag{16.26}$$

$$c_t - y_t = b_0 + b_1 x_t + b_2 {x_t}^2 \tag{16.27}$$

where a_t is the (time-varying) wealth proportion invested in the risky asset and Eq. (16.27) describes the behavior of the (log) consumption–wealth ratio. The terms a_0, a_1, b_0, b_1,

[7] In the finance literature, the preceding inequality is often cited as the definition of mean reversion (for an example, see Mukherji (2011)).

b_2, are constants; of special interest are those defining the time-varying risky asset proportion:

$$a_0 = \left(1 - \frac{1}{\gamma}\right)\left[\left(\frac{b_1}{1-\rho}\right) + 2\overline{x}(1-\phi)\left(\frac{b_2}{1-\rho}\right)\right]\left(\frac{-\sigma_{\eta u}}{\sigma_u^2}\right) \text{ and} \tag{16.28}$$

$$a_1 = \frac{1}{\gamma\sigma_u^2} + \left(1 - \frac{1}{\gamma}\right)\left[2\phi\frac{b_2}{(1-\rho)}\right]\left(\frac{-\sigma_{\eta u}}{\sigma_u^2}\right) \tag{16.29}$$

Note that in the case of a myopic investor ($\gamma = 1$) who holds stocks only because of the (conditional) excess returns they offer, $a_0 = 0$ and $a_1 = 1/\sigma_u^2$. This suggests that all the other terms present in the expressions defining a_0 and a_1 must be present to capture some aspect of a nonmyopic investor's *intertemporal hedging demand.*

What is meant by this latter expression in this particular context? With $\overline{x} > 0$, investors are typically long in stocks (the risky asset) in order to capture the excess returns they provide on average. Suppose in some period $t + 1$, stock returns are high, meaning that stock prices rose a lot from t to $t + 1$ (u_{t+1} is large). To keep the discussion less hypothetical, let's identify this event with the big run-up in stock prices in the late 1990s. Under the $\sigma_{\eta u} < 0$ assumption, expected future returns are likely to decline and perhaps even become negative (η_{t+1} is small, possibly negative so that x_{t+1} is small and thus, via Eq. (16.24), so is $E\tilde{r}_{t+1}$). Roughly speaking, this means stock prices are likely to decline—as they did in the 2000–2004 period![8] In anticipation of future price declines, long-term investors would rationally wish to assemble a short position in the risky portfolio, since this is the only way to enhance their wealth in the face of falling prices (r_f is constant by assumption). Most obviously, this is a short position in the risky portfolio itself, since negative returns must be associated with falling prices.

These thoughts are fully captured by Eqs. (16.26) and (16.27). Campbell and Viceira (2002) argue that the empirically relevant case is the one for which $\overline{x} > 0$, $b_1/(1-\rho) > 0$, $b_2/(1-\rho) > 0$, and $\sigma_{\eta u} < 0$. Under these circumstances, $a_0 > 0$, and $a_1 > 0$, for a sufficiently risk-averse investor ($\gamma > 1$). If u_{t+1} is large, then η_{t+1} is likely to be small—let's assume negative—and "large" in absolute value if $|\sigma_{\eta u}|$ is itself large. Via portfolio allocation Eq. (16.26), the optimal $a_t < 0$—a short position in the risky asset.

This distinguishing feature of long-term risk-averse investors is made more striking if we observe that with $\sigma_{\eta u} < 0$, such an investor will maintain a position in the risky asset if average excess returns, $\overline{x} = 0$: even in this case $a_0 > 0$ (provided $\gamma > 1$). Thus, if $x_t = 0$ (no excess returns to the risky asset), the proportion of the investor's wealth in stocks is still positive. In a one-period CAPM investment universe, a mean–variance myopic investor

[8] We have observed a similar pattern thus far in the 21st century. Stocks rose dramatically in value from 2004 until October 2008, and then declined precipitously with the onset of the financial crisis. In 2012 they started again rising in value and, as of this writing, are at a new peak with the S&P 500 index close to a level of 2000.

would invest nothing in stocks under these circumstances. Neither would the myopic expected utility maximizer of Theorem 4.1.

All this is to observe that a risk-averse rational long-term investor will use whatever means are open to him, including shorting stocks, when he (rationally) expects excess future stock returns to be sufficiently negative to warrant it. A major caveat to this line of reasoning, however, is that it cannot illustrate an equilibrium phenomenon: if all investors are rational and equally well informed about the process generating equity returns (16.23)–(16.25), then all will want simultaneously to go long or short. The latter, in particular, is not feasible from an equilibrium perspective.

16.4.2 Strategic Asset Allocation

The expression *strategic asset allocation* is suggestive not only of long-term investing (for which intertemporal hedging is a concern), but also of portfolio weights assigned to broad classes of assets (e.g., "stocks," "long-term bonds"), each well diversified from the perspective of its own kind. This is exactly the setting of this chapter.

Can the considerations of this section, in particular, be conveniently contrasted with those of the preceding chapters? This is captured in Figure 16.1 under the maintained

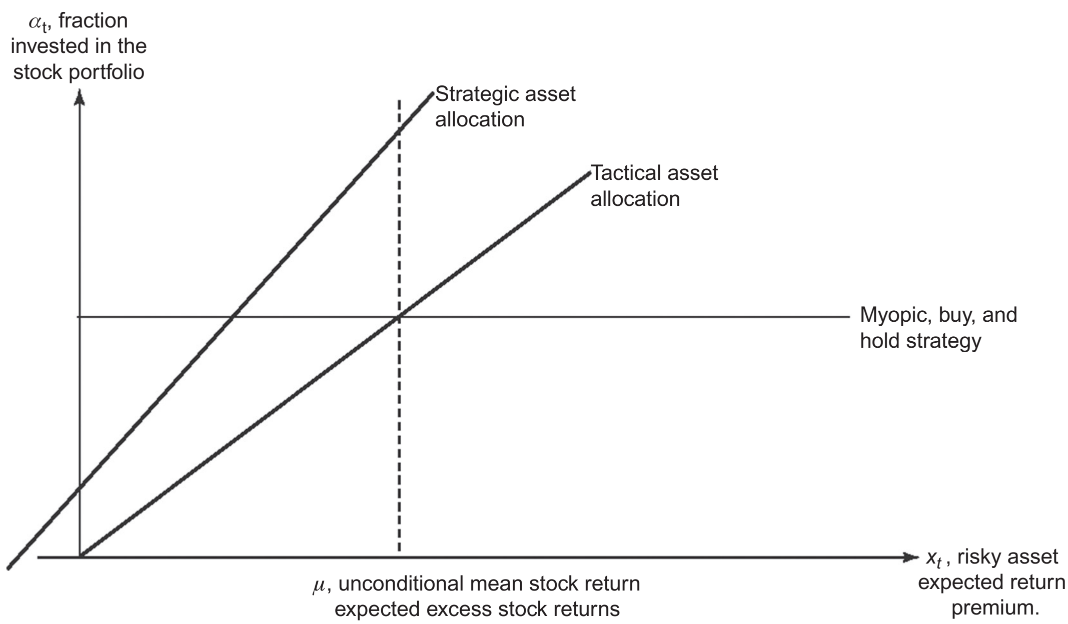

Figure 16.1
Alternative portfolio allocations.[i]
(i) Based on maintained assumptions (i)–(iii), and $\gamma > 1$.

assumptions of this subsection (itself a replica of Figure 4.1 in Campbell and Viceira, 2002). The myopic buy-and-hold strategy assumes a constant excess stock return equal to the true unconditional mean ($E_t\tilde{r}_{t+1} - r_f + \sigma_u^2/2 \equiv \overline{x}$) with the investor solving a portfolio allocation problem as per Theorem 4.1. The line marked "tactical asset allocation" describes the portfolio allocations for an investor who behaves as a one-period investor, conditional on his observation of x_t. Such an investor, by definition, will not take account of long-term hedging opportunities. Consistent with the CAPM recommendation, such an investor will elect $a_t = 0$ when $x_t = 0$ (no premium on the risky asset) but if the future looks good—even for just one period—he will increase his wealth proportions in the risky assets. Again, by construction, if $x_t = \overline{x}$, such an investor will adopt portfolio proportions consistent with the perpetually myopic investor.

Long-term "strategic" investors, with rational expectations vis-à-vis the return generating process (i.e., they know and fully take account of Eqs. (16.23)−(16.25)) will always elect to hold a greater proportion of their wealth in the risky portfolio than will be the case for the "tactical" asset allocator. In itself this is not entirely surprising, for only he is able to illustrate the "hedging demand." But this demand is present in a very strong way; in particular, even if excess returns are zero, the strategic investor holds a positive wealth fraction in risky assets ($a_0 > 0$). Note also that the slope of the strategic asset allocation line exceeds that of the tactical asset allocation line. In the context of Eqs. (16.23)−(16.25), this is a reflection of the fact that $\phi = 0$ for the tactical asset allocator.

16.4.3 The Role of Stocks in Investor Portfolios

Suppose stocks are less risky in the long run because of mean reversion in stock returns. But does this necessarily imply a 100% stock allocation in perpetuity for long-term investors? Under the assumptions of Campbell and Viceira (2002), this is clearly not the case: long-term investors should be prepared to take advantage of mean reversion by *timing the market* in a manner illustrated in Figure 16.1. But this in turn presumes the ability of investors to short stocks when their realized returns have recently been very high. Especially for small investors, shorting securities may entail prohibitive transactions costs. Even more significantly, this cannot represent an equilibrium outcome for all investors.

16.5 Background Risk: The Implications of Labor Income for Portfolio Choice

Background risks refer to uncertainties in the components of an investor's income not directly related to his tradable financial wealth and, in particular, his stock−bond portfolio allocation. Labor income risk is a principal component of background risk; variations in proprietary income (income from privately owned businesses) and in the value of owner-occupied real estate are the others.

In this section, we explore the significance of labor income risk for portfolio choice. It is a large topic and one that must be dealt with using models of varying complexity. The basic insight we seek to develop is as follows: an investor's labor income stream constitutes an element of his wealth portfolio. The desirability of the risky asset in the investor's portfolio will therefore depend not only upon its excess return (above the risk-free rate) relative to its variance (risk), but also the extent to which it can be used to hedge variations in the investor's labor income. Measuring how the proportion of an investor's financial wealth invested in the risky asset depends on its hedging attributes in the above sense is the principal focus of this section. Fortunately, it is possible to capture the basic insights in a very simple framework. As discussed in Campbell and Viceira (2002), that framework makes a number of assumptions:

i. The investor has a one-period horizon, investing his wealth to enhance his consumption tomorrow (as such, the focus is on the portfolio allocation decision exclusively; there is no $t = 0$ simultaneous consumption−savings decision).
ii. The investor receives labor income L_{t+1}, tomorrow, which for analytical simplicity is assumed to be lognormally distributed: $\log \tilde{L}_{t+1} \equiv \tilde{\ell}_{t+1} \sim N(l, \sigma_\ell^2)$.
iii. There is one risk-free and one risky asset (a presumed-to-be well diversified portfolio). Following our customary notation, ($r_f = \log(R_f)$ and $\tilde{r}_{t+1} = \log(\tilde{R}_{t+1})$. Furthermore, $\tilde{r}_{t+1} - E_t\tilde{r}_{t+1} = \tilde{u}_{t+1}$ where $\tilde{u}_{t+1} \sim N(0, \sigma_u^2)$. The possibility is admitted that the risky asset return is correlated with labor income in the sense that $\mathrm{cov}(\tilde{\ell}_{t+1}, \tilde{r}_{t+1}) = \sigma_{\ell u} \neq 0$.
iv. The investor's period $t+1$ utility function is of the CRRA-power utility type, with coefficient of relative risk aversion γ.

Since there is no labor−leisure choice, this model is implicitly one of fixed labor supply in conjunction with a random wage.

Accordingly, the investor solves the following problem:

$$\max_{\alpha_t} E_t\left[\delta \frac{\tilde{C}_{t+1}^{1-\gamma}}{1-\gamma}\right] \tag{16.30}$$
$$\text{s.t. } C_{t+1} = Y_t R_{P,t+1} + L_{t+1}$$

where

$$R_{P,t+1} = a_t(R_{t+1} - R_f) + R_f \tag{16.31}$$

a_t represents the fraction of the investor's wealth assigned to the risky portfolio, and P denotes his overall wealth portfolio. As in nearly all of our problems to date, insights can be neatly obtained only if approximations are employed that take advantage of the lognormal setup. In particular, we first need to modify the portfolio return expression (16.31).[9] Since

$$R_{P,t+1} = a_t R_{t+1} + (1 - a_t) R_f,$$
$$\frac{R_{P,t+1}}{R_f} = 1 + a_t \left(\frac{R_{t+1}}{R_f} - 1 \right)$$

Taking the log on both sides of this equation yields

$$r_{P,t+1} - r_f = \log\left[1 + a_t(\exp(r_{t+1} - r_f - 1))\right] \tag{16.32}$$

The RHS of this equation can be approximated using a second-order Taylor expansion around $r_{P,t+1} - r_{\mathrm{f}} = 0$, where the function to be approximated is

$$g_t(r_{P,t+1} - r_f) = \log\left[1 + a_t(\exp(r_{t+1} - r_f) - 1)\right]$$

By Taylor's theorem

$$g_t(r_{P,t+1} - r_f) \approx g_t(0) + g_t'(0)(r_{t+1} - r_f) + \frac{1}{2} g''_t(0)(r_{t+1} - r_f)^2$$

Clearly, $g_t(0) \equiv 0$; straightforward calculations (simple calculus) yield $g_t'(0) = a_t$, and $g''_t(0) = a_t(1 - a_t)$. Substituting into the Taylor expansion the indicated coefficient values, for the RHS of Eq. (16.32) yields

$$r_{P,t+1} - r_f = a_t(r_{t+1} - r_f) + \frac{1}{2} a_t(1 - a_t)\sigma_t^2$$

where $(r_{t+1} - r_f)^2$ is replaced by its conditional expectation. By the special form of the risky return generating process, $\sigma_t{}^2 = \sigma_u{}^2$, which yields

$$r_{P,t+1} = a_t(r_{t+1} - r_f) + r_f + \frac{1}{2} a_t(1 - a_t)\sigma_u^2 \tag{16.33}$$

We next modify the budget constraint to problem (16.30).

$$\frac{C_{t+1}}{L_{t+1}} = \frac{Y_t}{L_{t+1}}(R_{P,t+1}) + 1$$

or taking the log on both sides of the equation,

$$\begin{aligned} c_{t+1} - \ell_{t+1} &= \log[\exp(y_t + r_{P,t+1} - \ell_{t+1}) + 1] \\ &\approx k + \xi(y_t + r_{P,t+1} - \ell_{t+1}), \end{aligned} \tag{16.34}$$

[9] The derivation to follow is presented in greater detail in Campbell and Viceira (2001b).

where k and ξ, $0 < \xi < 1$, are constants of approximation. Adding log labor income—ℓ_{t+1}—to both sides of the equation yields

$$c_{t+1} = k + \xi(y_t + r_{P,t+1}) + (1-\xi)\ell_{t+1}, 1 > \xi > 0 \tag{16.35}$$

In other words, (log-) end of period consumption is a constant plus a weighted average of (log-) end-of-period financial wealth and (log-) labor income, with the weights ξ, $1-\xi$ serving to describe the respective elasticities of consumption with respect to these individual wealth components.

So far, nothing has been said regarding optimality. Equation (16.30) is a one-period optimization problem. The first-order necessary and sufficient condition for this problem with respect to a_t, the proportion of financial wealth invested in the risky portfolio, is given by

$$E_t[\delta(\tilde{C}_{t+1})^{-\gamma}(\tilde{R}_{t+1})] = E_t[\delta(\tilde{C}_{t+1})^{-\gamma}(R_f)] \tag{16.36}$$

In log-linear form, Eq. (16.36) has the familiar form:

$$E_t(\tilde{r}_{t+1} - r_f) + \frac{1}{2}\sigma_t^2 = \gamma \operatorname{cov}_t(\tilde{r}_{t+1}, \tilde{c}_{t+1})$$

Substituting the expression in Eq. (16.35) for c_{t+1} yields

$$E_t(\tilde{r}_{t+1} - r_f) + \frac{1}{2}\sigma_t^2 = \gamma \operatorname{cov}_t(\tilde{\mathrm{r}}_{t+1}, k + \xi(y_t + \tilde{r}_{P,t+1}) + (1-\xi)\tilde{\ell}_{t+1})$$

After substituting Eq. (16.33) for $r_{P,t+1}$, we are left with

$$E_t(\tilde{r}_{t+1} - r_f) + \frac{1}{2}\sigma_t^2 = \gamma\left[\xi a_t \sigma_t^2 + (1-\xi)\operatorname{cov}_t(\tilde{\ell}_{t+1}, \tilde{r}_{P,t+1})\right]$$

from which we can solve directly for a_t.

Recall that our objective was to explore how the hedging (with respect to labor income) features of risky securities influence the proportion of financial wealth invested in the risky asset. Accordingly, it is convenient first to simplify the expression via the following identifications: let

i. $\mu = E_t(\tilde{r}_{t+1} - r_f)$;
ii. $\sigma_t^2 = \sigma_u^2$, since $r_{t+1} - E_t\tilde{r}_{t+1} = u_{t+1}$;
iii. $\operatorname{cov}(\tilde{\ell}_{t+1}, \tilde{r}_{t+1}) = \operatorname{cov}(\tilde{\ell}_{t+1}, \tilde{r}_{t+1} - E_t\tilde{r}_{t+1}) = \operatorname{cov}(\tilde{\ell}_{t+1}, \tilde{u}_{t+1}) = \sigma_{\ell u}$.

With these substitutions, the above expression reduces to

$$\mu + \frac{1}{2}\sigma_u^2 = \gamma\left[\xi a_t \sigma_u^2 + (1-\xi)\sigma_{\ell u}\right].$$

Straightforwardly solving for a_t yields

$$a_t = \frac{1}{\xi}\left(\frac{\mu + \frac{\sigma_u^2}{2}}{\gamma\sigma_u^2}\right) + \left(1 - \frac{1}{\xi}\right)\frac{\sigma_{\ell u}}{\sigma_u^2}, \tag{16.37}$$

an expression with an attractive interpretation. The first term on the RHS of Eq. (16.37) represents the fraction in the risky asset if labor income is uncorrelated with the risky asset return ($\sigma_{\ell u} = 0$). It is positively related to the adjusted return premium ($\mu + \sigma_u^2/2$) and inversely related to the investor's risk-aversion coefficient γ. The second term represents the hedging component: if $\sigma_{\ell u} < 0$, then since $\xi < 1$, demand for the risky asset is enhanced since it can be employed to diversify away some of the investor's labor income risk. Or, to express the same idea from a slightly different perspective, if the investor's labor income has a "suitable" statistical pattern vis-à-vis the stock market, he can reasonably take on greater financial risk.

It is perhaps even more striking to explore further the case where $\sigma_{\ell u} = 0$: since $\xi < 1$, even in this case, the optimal fraction invested in the risky portfolio is

$$a_t = \frac{1}{\xi}\left(\frac{\mu + \frac{\sigma_u^2}{2}}{\gamma\sigma_u^2}\right) > \left(\frac{\mu + \frac{\sigma_u^2}{2}}{\gamma\sigma_u^2}\right),$$

where the rightmost ratio represents the fraction the investor places in the risky portfolio were there to be no labor income at all. If $\sigma_{\ell u} = 0$, then at least one of the following is true: $\mathrm{corr}(u, \ell) = 0$ or $\sigma_\ell = 0$, and each leads to a slightly different interpretation of the optimal a_t. First, if $\sigma_\ell > 0$ (there is variation in labor income), then the independence of labor and equity income allows for a good deal of overall risk reduction, thereby implying a higher optimal risky asset portfolio weight. If $\sigma_\ell = 0$—labor income is constant—then human capital wealth is a nontradable risk-free asset in the investor's overall wealth portfolio. Ceteris paribus, this also allows the investor to rebalance his portfolio in favor of a greater fraction held in risky assets. If, alternatively $\sigma_{\ell u} > 0$—a situation in which the investor's income is closely tied to the behavior of the stock market—then the investor should correspondingly reduce his position in risky equities. In fact, if the investor's coefficient of relative risk aversion is sufficiently high and $\sigma_{\ell u}$ large and positive (say, if the investor's portfolio contained a large position in his own firm's stock) then $a_t < 0$, i.e., the investor should hold a short position in the overall equity market.

These remarks formalize, though in a very simple context, the idea that an investor's wage income stream represents an asset and that its statistical covariance with the equity portion of his portfolio should matter for his overall asset allocation. To the extent that variations in stock returns are offset by variations in the investor's wage income, stocks are effectively less risky (so also is wage income less risky) and he can comfortably hold more of them. The reader may be suspicious, however, of the one-period setting. We remedy this next.

Viceira (2001) extends these observations to a multiperiod infinite horizon setting by adopting a number of special features. There is a representative investor–worker who saves for retirement and who must take account in his portfolio allocation decisions of the expected length of his retirement period. In any period, there is a probability π^r that the investor will retire; his probability of remaining employed and continuing to receive labor income is $\pi^e = 1 - \pi^r$, with constant probability period by period. With this structure of uncertainty, the expected number of periods until an investor's retirement period is $1/\pi^r$. Once retired (zero labor income), the period constant probability of death is π^d; in a like manner the expected length of his retirement is $1/\pi^d$. Viceira (2001) also assumes that labor income is growing in the manner of

$$\Delta\ell_{t+1} = \log L_{t+1} - \log L_t = g + \tilde{\varepsilon}_{t+1} \tag{16.38}$$

where $g > 0$ and $\tilde{\varepsilon}_{t+1} \sim N(0, \sigma_\varepsilon^2)$. In expression (16.38), g represents the mean growth in labor income (for the United States this figure is approximately 2%) while $\tilde{\varepsilon}_t$ denotes random variations about the mean. The return on the risky asset is assumed to follow the same hypothetical process as in the prior example. In this case,

$$\sigma_{u\ell} = \text{cov}_t(\tilde{r}_{t+1}, \Delta\tilde{\ell}_{t+1}) = \text{cov}_t(\tilde{u}_{t+1}, \tilde{\varepsilon}_{t+1}) = \sigma_{u\varepsilon}$$

With an identical asset structure as in the previous model, the investor's problem appears deceptively similar to Eq. (16.30):

$$\begin{gathered} \max E_t\left(\sum_{i=0}^{\infty}\delta^i \frac{\tilde{C}_{t+i}^{1-\gamma}}{1-\gamma}\right) \\ \text{s.t. } Y_{t+1} = (Y_t + L_t - C_t)R_{P,t+1} \end{gathered} \tag{16.39}$$

The notation in Eq. (16.39) is identical to that of the previous model. Depending on whether an agent is employed or retired, however, the first-order optimality condition will be different, reflecting the investor's differing probability structure looking forward. In Eqs. (16.40) and (16.41) to follow, the (not log) consumption is superscripted by e or r, depending on its enjoyment in the investor's period of employment or retirement, respectively. If the investor is retired, for any asset i (the portfolio P, or the risk-free asset):

$$1 = E_t\left[(1-\pi^d)\delta\left(\frac{\tilde{C}_{t+1}^r}{C_t^r}\right)^{-\gamma}\tilde{R}_{i,t+1}\right] \tag{16.40}$$

The interpretation of Eq. (16.40) is more or less customary: the investor trades off the marginal utility lost in period t by investing one more consumption unit against the expected utility gain in period $t+1$ for having done so. The expectation is adjusted by the probability $(1 - \pi^d)$ that the investor is, in fact, still living in the next period. Analytically, its influence on the optimality condition is the same as a reduction in his subjective discount factor δ. If the investor is employed, but with positive probability of retirement and subsequent death, then each asset i satisfies:

$$1 = E_t\left\{\left[\pi^e\delta\left(\frac{\tilde{C}^e_{t+1}}{C^e_t}\right)^{-\gamma} + (1-\pi^e)(1-\pi^d)\delta\left(\frac{\tilde{C}^r_{t+1}}{C^r_t}\right)^{-\gamma}\right](\tilde{R}_{i,t+1})\right\} \tag{16.41}$$

Equation (16.41)'s interpretation is analogous to Eq. (16.40) except that the investor must consider the likelihood of his two possible states next period: either he is employed (probability π^e) or retired and still living (probability $(1-\pi^e)(1-\pi^d)$). Whether employed or retired, these equations implicitly characterize the investor's optimal risk-free–risky portfolio proportions as those for which his expected utility gain to a marginal dollar invested in either one is the same.

Viceira (2001) log-linearizes these equations and their associated budget constraints to obtain the following expressions for log consumption and the optimal risky portfolio weight in both retirement and employment; for a retired investor:

$$c^r_t = b^r_0 + b^r_1 y_t \quad \text{and} \tag{16.42}$$

$$a^r = \frac{\mu + \frac{\sigma_u^2}{2}}{\gamma b^r_1 \sigma_u^2} \tag{16.43}$$

where $b^r_1 = 1$ and b^r_0 is a complicated (in terms of the model's parameters) constant of no immediate concern; for an employed investor, the corresponding expressions are

$$c^e_t = b^e_0 + b^e_1 y_t + (1 - b^e_1)\ell_t \tag{16.44}$$

$$a^e = \frac{\mu + \frac{\sigma_u^2}{2}}{\gamma \bar{b}_1 \sigma_u^2} - \left(\frac{\pi^e(1-b^e_1)}{\bar{b}_1}\right)\frac{\sigma_{eu}}{\sigma_u^2} \tag{16.45}$$

with $0 < b^e_1 < 1$, $\bar{b}_1 = \pi^e b^e_1 + (1-\pi^e)b^r_1$, and b^e_0, again, a complex constant whose precise form is not relevant for the discussion.

These formulas allow a number of observations:

1. Since $b^r_1 > b^e_1$, (log) consumption is more sensitive to (log) wealth changes for the retired (Eq. (16.42)) as compared with the employed (Eq. (16.44)). This is not surprising as the employed can hedge this risk via his labor income. The retired cannot.

2. As in the prior model with labor income, there are two terms that together comprise the optimal risky asset proportions for the employed, a^e. The first $(\mu + (\sigma_u^2/2))/\gamma \bar{b}_1 \sigma_u^2$ reflects the proportion when labor income is independent of risky returns ($\sigma_{\varepsilon u} = 0$). The second, $-(\pi^e(1-b_1^e)/\bar{b}_1)\sigma_{\varepsilon u}/\sigma_u^2$ accounts for the hedging component. If $\sigma_{\varepsilon u} < 0$, then the hedge that labor income provides to the risky component of the investor's portfolio is more powerful: the optimal a^e is thus higher, while the opposite is true if $\sigma_{\varepsilon u} > 0$. With a longer expected working life (greater π^e), the optimal hedging component is also higher: the present value of the gains to diversification provided by labor income variation correspondingly increases. Note also that the hedging feature is very strong in the sense that even if the mean equity premium, $\mu = 0$, the investor will retain a positive allocation in risky assets purely for their diversification effect vis-à-vis labor income.
3. Let us next separate the hedging effect by assuming $\sigma_\varepsilon = 0$ (and thus $\sigma_{\varepsilon u} = 0$). Since $b_1^e < b_1^r$, $\bar{b}_1 < b_1^r$,

$$a^e = \frac{\mu + \frac{\sigma_u^2}{2}}{\gamma \bar{b}_1 \sigma_u^2} > \frac{\mu + \frac{\sigma_u^2}{2}}{\gamma \bar{b}_1^r \sigma_u^2} = a^r$$

for any level of risk aversion γ: even if labor income provides no hedging services, the employed investor will hold a greater fraction of his wealth in the risky portfolio than will the retired investor. This is the labor income wealth effect. Ceteris paribus, a riskless labor income stream contributes a valuable riskless asset, and its presence allows the investor to tilt the financial component of his wealth in favor of a greater proportion in stocks. It also enhances his average consumption suggesting less aversion to risk. If $\sigma_{\varepsilon u} = 0$ because $\rho_{\varepsilon u} = 0$, $a^e > a^r$ can be justified on the basis of diversification alone. This latter comment is strengthened (weakened) when greater (lesser) diversification is possible: $a_{\varepsilon u} < 0$ ($\sigma_{\varepsilon u} > 0$).

Before summarizing these thoughts, we return to a consideration of the initial three life cycle portfolio recommendations. Strictly speaking, Eq. (16.39) is not a life cycle model. Life cycle considerations can be dealt with to a good approximation, however, if we progressively resolve Eq. (16.39) for a variety of choices of π^e, π^d. If π^d is increased, the expected length of the period of retirement falls. If π^r is increased (π^e decreased), it is as if the investor's expected time to retirement was declining as he "aged." Campbell and Viceira (2002) present the results of such an exercise, which we report in Table 16.2 for a selection of realistic risk-aversion parameters.

Here we find more general theoretical support for at least the first and third of our original empirical observations. As investors approach retirement, the fraction of their wealth invested in the risky portfolio does decline strongly. Note also that it is optimal for mildly risk-averse young investors ($\gamma = 2$) to short dramatically the risk-free asset to buy more of the risky one in order to "capture" the return supplement inherent in the equity premium. In

Table 16.2: Optimal percentage allocation to stocks[a,b]

	Employed				Retired
	Expected time to retirement (years)				
	35	25	10	5	
	Panel A: $\text{corr}(\tilde{r}_{P,t+1}, \Delta\tilde{l}_{t+1}) = 0$				
$\gamma = 2$	184	156	114	97	80
$\gamma = 5$	62	55	42	37	32
	Panel B: $\text{corr}(\tilde{r}_{P,t+1}, \Delta\tilde{l}_{t+1}) = 0.35$				
$\gamma = 2$	155	136	116	93	80
$\gamma = 5$	42	39	35	33	32

[a] $\tilde{r}_f = 0.02$, $E\tilde{r}_{P,\,t+1} - \tilde{r}_{f,t+1} = \mu = 0.04$, $\sigma_u = 0.157$, $g = 0.03$, $\sigma_\varepsilon = 0.10$.
[b] Table 16.2 is a subset of Table 6.1 in Campbell and Viceira (2002).

actual practice, however, such a leverage level is unlikely to be feasible for young investors without a high level of collateral assets. However, the "pull of the premium" is so strong that even retired persons with $\gamma = 5$ (the upper bound for which there is empirical support) will retain roughly one-third of their wealth in the risky equity index. In this sense, the latter aspect of the first empirical assertion is not borne out, at least for this basic preference specification.

We conclude this section by summarizing the basic points.

1. Riskless labor income creates a tilt in investor portfolios toward risky equities. This is not surprising, for the labor income stream in this case contributes a risk-free asset in the investor's portfolio. There are two effects going on. One is a wealth effect: ceteris paribus, an investor with a labor income stream is wealthier than an investor without one, and with CRRA utility some of that additional wealth will be assigned to equities. This is complemented by a pure portfolio effect: the risk-free asset alters overall portfolio proportions in a way that is manifest as an increased share of financial wealth in risky assets.
2. These same effects are strengthened by the hedging effect if $\sigma_{\varepsilon u} \leq 0$ (effectively, this means $\sigma_{r\ell} \leq 0$). Stocks and risky labor income covary in a way that each reduces the effective risk of the other. Only if $\sigma_{\varepsilon u}$ is large and positive will the presence of labor income risk reduce the fraction of financial wealth invested in risky assets.
3. Although not discussed explicitly, the ability of an investor to adjust her labor supply—and thus her labor income—only enhances these effects. In this case, the investor can elect not only to save more but also to work more if she experiences an unfavorably risky return realization. Her ability to hedge averse risky return realizations is thus enriched, and stocks appear effectively less risky.

16.6 An Important Caveat

We again acknowledge the concerns of Chapter 7. The accuracy and usefulness of the notions developed in the preceding sections, especially as regards applications of the formulas for practical portfolio allocations, should not be overemphasized. Their usefulness depends in every case on the accuracy of the forecast means, variances, and covariances, which represent the inputs to them: garbage in; garbage out still applies! Unfortunately, these quantities—especially expected risky returns—have been notoriously difficult to forecast accurately, even 1 year in advance. Errors in these estimates can have substantial significance for risky portfolio proportions, as these are generally computed using a formula of the generic form

$$\mathbf{a}_t = \frac{1}{\gamma}\sum{}^{-1}(E_t\mathbf{r}_{t+1} - \mathbf{r}_{f,t+1}\mathbf{1})$$

where boldface letters represent vectors and Σ^{-1} is a matrix of "large" numbers. Errors in $E_t\mathbf{r}_{t+1}$, the return vector forecasts, are magnified accordingly in the portfolio proportion choice.

In a recent paper, Garlappi et al. (2009) evaluate a number of complex portfolio strategies against a simple equal-portfolio-weights-buy-and-hold strategy. Using the same data set as Campbell and Viceira (2002) use, the equal weighting strategy tends to dominate all the others, simply because, under this strategy, the forecast return errors (which tend to be large) do not affect the portfolio's makeup.

16.7 Another Background Risk: Real Estate

In this final section, we explore the impact of real estate holdings on an investor's optimal stock–bond allocations. As before, our analysis will be guided by two main principles: (1) all assets—including human capital wealth—should be explicitly considered as components of the investor's overall wealth portfolio and (2) it is the correlation structure of cash flows from these various income sources that will be paramount for the stock–bond portfolio proportions. Residential real estate is important because it represents roughly half of the US aggregate wealth, and it is not typically included in empirical stand-ins for the US market portfolio M.

Residential real estate also has features that make it distinct from pure financial assets. In particular, it provides a stream of housing services that are inseparable from the house itself. Houses are indivisible assets: one may acquire a small house but not one-half of a house. Such indivisibilities effectively place minimum bounds on the amount of real estate that can be acquired. Furthermore, houses cannot be sold without paying a substantial transactions fee, variously estimated to be between 8% and 10% of the value of the unit being exchanged. As the purchase of a home is typically a leveraged transaction, most lenders require minimum "down payments" or equity investments by the purchaser in the house. Lastly, investors may be forced to sell their houses for totally exogenous reasons, such as a job transfer to a new location.

Cocco (2005) studies the stock–bond portfolio allocation problem in the context of a model with the above features, which is otherwise very similar to the ones considered thus far in this chapter. Recall that our perspective is one of partial equilibrium where, in this section, we seek to understand how the ownership of real estate influences an investor's other asset holdings, given assumed return processes on the various assets. In the following, we highlight certain aspects of Cocco's (2005) modeling of the investor's problem.

The investor's objective function, in particular, is

$$\max_{\{S_t,B_t,D_t,FC_t\}} E\left\{\sum_{t=0}^{T}\beta^t\frac{(\tilde{C}_t^{\theta}\tilde{H}_t^{1-\theta})^{1-\gamma}}{1-\gamma}+\beta^T\frac{(\tilde{Y}_{T+1})^{1-\gamma}}{1-\gamma}\right\}$$

where, as before, C_t is his period t (nondurable) consumption (not logged; in fact, no variables will be logged in the problem description), H_t denotes period t housing services (presumed proportional to housing stock with a proportionality constant of one), and Y_t the investor's period t wealth. Under this formulation, nondurable consumption and housing services complement one another with the parameter θ describing the relative preference of one to the other.[10] Investor risk sensitivity to variations in the nondurable consumption–housing joint consumption decision is captured by γ (the investor displays CRRA with respect to the composite consumption product). The rightmost term, $((Y_{T+1})^{1-\gamma})/1-\gamma$, is to be interpreted as a bequest function: the representative investor receives utility from nonconsumed terminal wealth, which is presumed to be bequeathed to the next generation, with the same risk preference structure applying to this quantity as well. In order to capture the idea that houses are indivisible assets, Cocco (2005) imposes a minimum size constraint

$$H_t \geq H_{\min}$$

to capture the fact that transactions costs are involved in changing one's stock of housing, the agent is assumed to receive only

$$(1-\lambda)P_tH_{t-1}$$

if he sells his housing stock H_{t-1}, in period t for a price P_t. In his calibration, λ—the magnitude of the transaction cost—is fixed at 0.08, a level for which there is substantial empirical support in US data. Note the apparent motivation for a bequest motive: given the minimum housing stock constraint, an investor in the final period of his life would otherwise own a significant level of housing stock for which the disposition at his death would be ambiguous.[11]

[10] The idea is simply that an investor will "enjoy his dinner more if he eats it in a warm and spacious house."

[11] An alternative device for dealing with this modeling feature would be to allow explicitly for reverse mortgages.

Let $\tilde{R}_t$, R_f, and R_D denote the gross random exogenous return on equity, (constant) risk-free rate, and the (constant) mortgage interest rate. If the investor elects not to alter his stock of housing in period t relative to $t-1$, his budget constraint for that period is

$$S_t + B_t = \tilde{R}_t S_{t-1} + R_f B_t - R_D D^M_{t-1} + L_t - C_t - \chi^{FC}_t F - \Omega P_t H_{t-1} + D^M_t = Y_t \tag{16.46}$$

where the notation is suggestive: S_t and B_t denote his period t stock and bond holdings, respectively, D^M_t the level of period t mortgage debt, Ω is a parameter measuring the maintenance cost of home ownership, and F is a fixed cost of participating in the financial markets. The indicator function χ^{FC}_t assumes values $\chi^{FC}_t = 1$, if the investor alters his stock or bondholdings relative to period $t-1$ and 0 otherwise. This device is meant to capture the cost of participating in the securities markets. In the event the investor wishes to alter his stock of housing in period t, his budget constraint is modified in the to-be-expected way (most of it is unchanged except for the addition of the costs of trading houses):

$$S_t + B_t = Y_t + (1-\lambda)P_t H_{t-1} - P_t H_t \text{ and} \tag{16.47}$$

$$D_t \leq (1-d)P_t H_t \tag{16.48}$$

The additional terms in Eq. (16.47) relative to Eq. (16.46) are simply the net proceeds from the sale of the "old" house, $(1-\lambda)P_t H_{t-1}$, less the costs of the "new" one $P_t H_t$. Constraint (16.48) reflects the down payment equity requirement and the consequent limits to mortgage debt. (In his simulation Cocco, 2005, chooses $d = 0.15$.)

Cocco (2005) numerically solves the above problem given various assumptions on the return and house price processes that are calibrated to historical data. In particular, he allows for house prices and aggregate labor income shocks to be perfectly positively correlated and for labor income to have both random and determinate components.[12]

[12] In particular, Cocco (2005) assumes

$$\tilde{L}_t = \begin{cases} f(t) + \tilde{u}_t, & t \leq T \\ f(t) & t > T \end{cases}$$

where T is the retirement date and the deterministic component $f(t)$ is chosen to replicate the hump shape earnings pattern typically observed. The random component $\tilde{u}_t$ has aggregate ($\tilde{\eta}_t$) and idiosyncratic components ($\tilde{\omega}_t$) where

$$\tilde{u}_t = \tilde{\eta}_t + \tilde{\omega}_t \text{ and} \tag{16.49}$$

$$\tilde{\eta}_t = \kappa_\eta \tilde{P}_t \tag{16.50}$$

where P_t is the log of the average house price. In addition, he assumes $R_f = 1.02$ is fixed for the period [0, T], as is the mortgage rate $R_D = 1.04$. The return on equity follows

$$r_t = \log(\tilde{R}_t) = E \log \tilde{R} + \tilde{l}_t$$

with $\tilde{l}_t \sim N(0; \sigma^2_l)$, $\sigma_l = 0.1674$, and $E \log \tilde{R} = 0.10$.

Cocco (2005) uses his model to comment upon a number of outstanding financial puzzles, of which we will review three: (1) considering the magnitude of the equity premium and the mean reversion in equity returns, why do all investors not hold at least some of their wealth as a well-diversified equity portfolio? Simulations of the model reveal that the minimum housing level $H_{\min}$ (which is calculated at US$20,000US$) in conjunction with the down payment requirement make it nonoptimal for lower labor income investors to pay the fixed costs of entering the equity markets. This is particularly the case for younger investors who remain liquidity constrained. (2) While the material in Section 16.5 suggests that the investors' portfolio share invested in stocks should decrease in later life (as the value of labor income wealth declines), the empirical literature finds that for most investors the portfolio share invested in stocks is increasing over their life cycle. Cocco's (2005) model implies that the share in equity investments increases over the life cycle. As noted above, early in life, housing investments keep investors' liquid assets low, and they choose not to participate in the markets. More surprisingly, he notes that the presence of housing can prevent a decline in the share invested in stocks as investors age: as housing wealth increases, investors are more willing to accept equity risk as that risk is not highly correlated with this component. Lastly, (3) Cocco deals with the cross-sectional observation that the extent of leveraged mortgage debt is highly positively correlated with equity asset holdings. His model is able to replicate this phenomenon as well because of the consumption dimension of housing: investors with more human capital acquire more expensive houses and thus borrow more. Simultaneously, the relatively less risky human capital component induces a further tilt toward stock in high labor income investor portfolios.

16.8 Conclusions

The analysis in this chapter has brought us conceptually to the state of the art in MPT. It is distinguished by (1) the comprehensive array of asset classes that must be explicitly considered in order properly to understand an investor's *financial* asset allocations. Labor income (human capital wealth) and residential real estate are two principal cases in point. To some extent, these two asset classes provide conflicting influences on an investor's stock—bond allocations. On the one hand, as relatively riskless human capital diminishes as an investor ages, then ceteris paribus, his financial wealth allocation to stocks should fall. On the other hand, if his personal residence has dramatically increased in value over the investor's working years, this fact argues for increased equity holdings given the low correlation between equity and real estate returns. Which effectively dominates is unclear. (2) Long-run portfolio analysis is distinguished by its consideration of security return paths beyond the standard one-period-ahead mean, variance, and covariance characterization. Mean reversion in stock returns suggests intertemporal hedging opportunities, as does the long-run variation in the risk-free rate.

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