Deriving the Capital Asset Pricing Model*

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*This note provides an overview of way to derive the standard Capital Asset Pricing Model (CAPM) and the zero-beta CAPM. The note draws inspiration from the textbook treatments in Huang and Litzenberger (1988), Pennacchi (2008), and Campbell (2017). The note is prepared for use only in the Master's course "Asset Pricing". Please do not cite, circulate, or use for purposes other than this course.

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1. Introduction

The Capital Asset Pricing Model (CAPM) was developed by Treynor (1961), Sharpe (1964), Lintner (1965), and Mossin (1966) independently and represents a first and widely applied asset pricing model. The model is an equilibrium asset pricing model built on the foundation of the mean-variance model of Markowitz (1952). The portfolio selection rules offered by Markowitz (1952) and Tobin (1958) can be viewed from two different angles: (i) as a *normative theory* instructing individual investors about the optimal allocation of wealth among assets or (ii) as a *positive theory* of how an investor actually behaves (Markowitz, 2010). If one takes the latter view, then a natural extension of portfolio theory is to consider the equilibrium asset pricing implications of the rational choices made by all individual investors. The portfolio choices of individual investors represent their demand for risky assets and equilibrium asset pricing relations can then be determined by aggregating demand across investors and equating them with the aggregate supply of assets. This provides the underlying foundation for the CAPM and this note details different approaches to deriving the model and studying its predictions about asset returns. For example, the notion that investor require compensation for *systematic* risk, but not *idiosyncratic* risk, and that an asset's risk premium is linearly related to a single factor, the excess return on the market portfolio of all risky assets, are important implications of the CAPM. In equilibrium, demand equals supply, which in turn implies that the portfolio weights of the mean-variance efficient portfolio of risky assets are those of the market portfolio. A central prediction of the CAPM is therefore that the market portfolio is a mean-variance efficient portfolio.

The classic derivation of the CAPM follows Sharpe (1964) and begins by making a set of assumptions needed to ensure that all investors see the same investment opportunity set and mean-variance frontier. These assumptions are similar to those employed by Markowitz (1952). Specifically, the model assumes that investors are rational, evaluate portfolios using the means and variances of single-period returns (i.e., seeks mean-variance efficiency), have homogeneous beliefs about the means, variances, and covariances of returns, are price takers, face no taxes or transaction costs, can short-sell all assets at will, and, in the basic version of the model due to Sharpe (1964) and Lintner (1965), that investors can borrow or lend unlimited at a given risk-free rate. These assumptions guarantee that the mean-variance efficient frontier is the same for every investor and, by the two-fund separation theorem (Tobin, 1958), that all investors' optimal portfolios are of the same structure: a fraction of wealth is invested in the risk-free asset, the rest in the tangency portfolio. That is, all investors hold risky assets in the same relative proportions to one another. Having reviewed the classic derivation of the CAPM, we turn to an alternative derivation that elegantly illustrates how the CAPM originates directly from the mean-variance model by using the fact that an asset's risk premium is related to the market portfolio.

The standard version of the CAPM requires the existence of a risk-free asset in the economy, but what happens if we relax the assumption that investors can borrow at the risk-free rate? Aggressive investors that would otherwise leverage their position by borrowing at the risk-free rate to move up the security market line will instead choose to hold different risky portfolios with higher expected returns. These portfolio will still be mean-variance efficient and the two-fund separation theorem (Tobin, 1958) for risky assets still describes the frontier portfolios. It follows that the market portfolio must be a combination of these frontier portfolios and therefore must itself be a frontier portfolio. This insight is due to Black (1972) and he describes a version of the CAPM that works without a risk-free rate: the zero-beta CAPM. The model is still an equilibrium model and its central prediction is, like the standard CAPM, that the market portfolio is mean-variance efficient.

The rest of the note progresses as follows. Section 2 outlines the classic derivation found in Sharpe (1964), and further shows how to obtain the CAPM directly from the mean-variance model of Markowitz (1952) by studying the covariance between any portfolio q and a frontier portfolio p. Section 3 presents the derivation of the zero-beta CAPM due to Black (1972) that introduces a zero-covariance portfolio.

2. The standard CAPM

This section provides two ways of deriving the standard CAPM and gauging its predictions. The first relies on the classic derivation found in Sharpe (1964) and the second obtains the CAPM directly from the mean-variance model of Markowitz (1952) without imposing any additional assumptions, but simply imposing equilibrium.

2.1. A classic approach

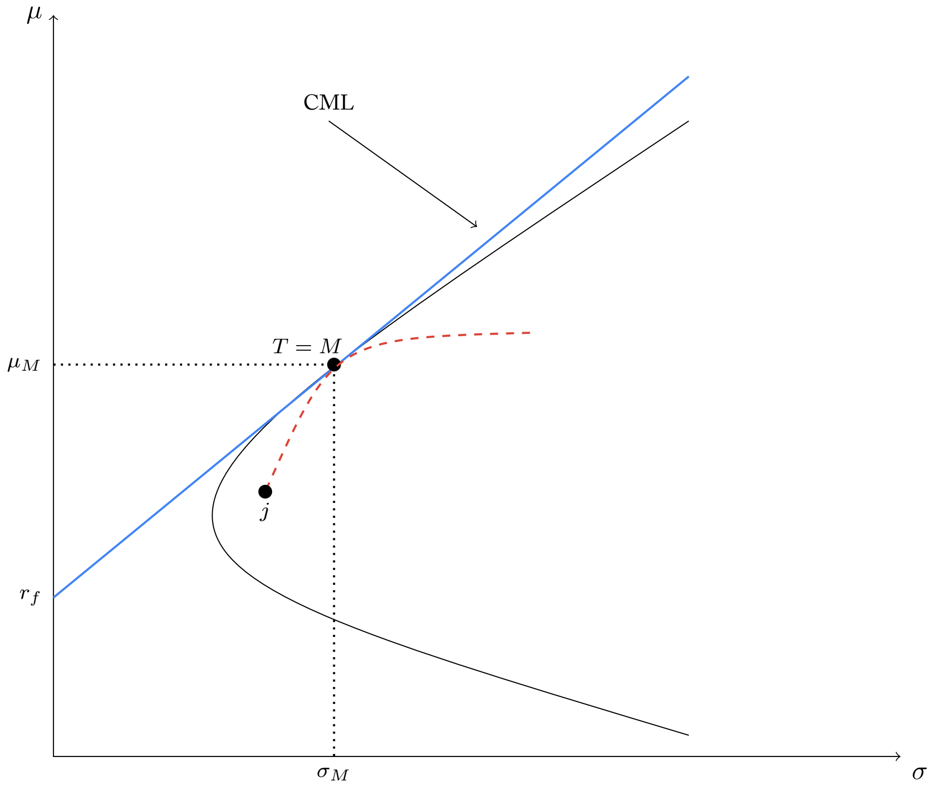
This section details the classic derivation of the CAPM found in Sharpe (1964). Our point of departure is the mean-variance model of Markowitz (1952) and the observation that, in equilibrium, there is a simple linear relationship between the expected return and standard deviation of returns for efficient portfolios. However, so far, we have not uncovered a similar relationship for individual, inefficient, assets. Usually, the (μ,σ) values will lie below the Capital Market Line (CML), reflecting the inefficiency of undiversified holdings. Moreover, such points may be scattered throughout the feasible region, with no consistent relationship between their expected return and total risk. However, as we will show below, there exists a consistent relationship between the expected returns on individual assets and their *systematic risk*.

Figure 1 illustrates the relationship between an individual asset j and the market portfolio M. Importantly, j is part of M per the assumed equilibrium. The red dashed line represents combinations of these assets. In particular, consider a portfolio with a fraction $1-\alpha$ of wealth invested in asset j and a fraction α invested in the market portfolio M

$$\mu_p = \alpha\mu_M + (1-\alpha)\,\mu_j \tag{1}$$

$$\sigma_p^2 = \alpha^2\sigma_M^2 + (1-\alpha)^2\,\sigma_j^2 + 2\alpha\,(1-\alpha)\,\sigma_{jM} \tag{2}$$

Figure 1: Mean-variance frontiers



where a value of $\alpha = 1$ would indicate a pure investment in the market portfolio and $\alpha = 0$ an investment in j. The resulting frontier is tangent to the CML at the point M. The requirement that curves such as the red dashed line be tangent to the CML can be shown to lead to a relatively simple formula which relates the expected rate of return to various elements of risk for all assets which are included in M. First, the slope (the price of risk) of the CML at $\alpha = 1$ is

$$\left.\frac{d\mu_p}{d\sigma_p}\right|_{\alpha=1} = \frac{\mu_M - r_f}{\sigma_M}, \tag{3}$$

which we know from the mean-variance model of Markowitz (1952). The slope of the tangent line is the Sharpe ratio of the market portfolio. From the chain rule of calculus, we know that

$$\frac{d\mu_p}{d\sigma_p} = \frac{d\mu_p/d\alpha}{d\sigma_p/d\alpha}. \tag{4}$$

We can determine the derivatives with respect to the proportion α as follows

$$\frac{d\mu_p}{d\alpha} = \mu_M - \mu_j \tag{5}$$

$$\frac{d\sigma_p}{d\alpha} = \frac{2\alpha\sigma_M^2 + 2\,(1-\alpha)\,\sigma_j^2 + 2\,(1-2\alpha)\,\sigma_{jM}}{2\sigma_p}, \tag{6}$$

where the latter derivative follows from a straightforward application of the chain rule.[1] Evaluating the last expression (the first is independent of α) at the point $\alpha = 1$ gives use

$$\left.\frac{d\sigma_p}{d\alpha}\right|_{\alpha=1} = \frac{2\sigma_M^2 - 2\sigma_{jM}}{2\sigma_M} = \frac{\sigma_M^2 - \sigma_{jM}}{\sigma_M}, \tag{7}$$

where $\sigma_p = \sigma_M$ for $\alpha = 1$. Inserting our results back into (4) yields

$$\left.\frac{d\mu_p}{d\sigma_p}\right|_{\alpha=1} = \frac{(\mu_M - \mu_j)\,\sigma_M}{\sigma_M^2 - \sigma_{jM}} = \frac{\mu_M - r_f}{\sigma_M}, \tag{8}$$

where the last equality follows from the mean-variance model. Solving for $\mu_M - \mu_j$ gives us

$$\begin{aligned} \mu_M - \mu_j &= \frac{(\mu_M - r_f)\left(\sigma_M^2 - \sigma_{jM}\right)}{\sigma_M^2} && (9)\\ &= (\mu_M - r_f)\left[1 - \frac{\sigma_{jM}}{\sigma_M^2}\right] && (10)\\ &= \mu_M - r_f - \beta_{jM}\,(\mu_M - r_f)\,. && (11) \end{aligned}$$

Finally, re-arrange to obtain the Capital Asset Pricing Model (CAPM)

$$\mu_j = r_f + \beta_{jM}\,(\mu_M - r_f)\,. \tag{12}$$

We see that the expected return on asset j is linearly related to the excess return on the market portfolio and that the risk premium is determined by the systematic risk, β_{jM}, of asset j.

2.2. CAPM in the mean-variance model

We can similarly obtain the CAPM directly from the mean-variance model of Markowitz (1952, 1959). In particular, consider the portfolio choice problem of a von Neumann and Morgenstern (1944) utility maximizing investor for N risky assets and a risk-free asset

$$\begin{aligned} &\min_{\boldsymbol{\omega}} \frac{1}{2}\boldsymbol{\omega}^\top \mathbf{V}\boldsymbol{\omega} && (13)\\ &\text{s.t. } \boldsymbol{\omega}^\top\boldsymbol{\mu} + \left(1 - \boldsymbol{\omega}^\top\mathbf{1}\right) r_f = \mu_p, \end{aligned}$$

where $\boldsymbol{\omega}$ is an $N \times 1$ vector of portfolio weights on risky assets, $\mathbf{V}$ is the covariance matrix of risky asset returns, $\boldsymbol{\mu}$ is a vector of expected risky asset returns, and $\mathbf{1}$ is a vector of ones. The objective is to construct a portfolio with minimum variance for a given target expected return.

[1]In particular, let $\sigma_p = \left[\alpha^2\sigma_M^2 + (1-\alpha)^2\sigma_j^2 + 2\alpha(1-\alpha)\sigma_{jM}\right]^{\frac{1}{2}}$ and define $\sigma_p = h\left(f\left(\alpha\right)\right)$ with first derivative $h'\left(f\left(\alpha\right)\right) f'\left(\alpha\right)$, where $h'\left(f\left(\alpha\right)\right) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sigma_p}$.

The solution for the optimal portfolio weights for the N risky assets is given by the expression

$$\boldsymbol{\omega}_p^* = \frac{\mu_p - r_f}{H} \mathbf{V}^{-1} \left(\boldsymbol{\mu} - r_f \mathbf{1}\right), \tag{14}$$

where we clearly see the mean-variance preferences at play. In particular, we see that the optimal portfolio weights $\boldsymbol{\omega}_p^*$ are increasing in expected returns and decreasing in variance. The constants A, B, C, and H defined by the expressions

$$A = \mathbf{1}^\top \mathbf{V}^{-1} \boldsymbol{\mu} \tag{15}$$

$$B = \boldsymbol{\mu}^\top \mathbf{V}^{-1} \boldsymbol{\mu} > 0 \tag{16}$$

$$C = \mathbf{1}^\top \mathbf{V}^{-1} \mathbf{1} > 0 \tag{17}$$

$$H = B - 2Ar_f + Cr_f^2. \tag{18}$$

The mean-variance model has clear predictions for optimal portfolios, but remains silent about inefficient assets not on the portfolio frontier. We do know, however, that there exists a consistent relationship between the expected return on individual assets and their systematic risk. This is true here as well and allows us to consider the covariance $\operatorname{cov}\left[\widetilde{r}_q, \widetilde{r}_p\right] = \sigma_{q,p} = \boldsymbol{\omega}_q^\top \mathbf{V} \boldsymbol{\omega}_p$ between any portfolio q (which is not necessarily on the frontier) and any frontier portfolio p. Using (14), we can write

$$\sigma_{q,p} = \boldsymbol{\omega}_q^\top \mathbf{V} \frac{\mu_p - r_f}{H} \mathbf{V}^{-1} \left(\boldsymbol{\mu} - r_f \mathbf{1}\right) \tag{19}$$

$$= \frac{\mu_p - r_f}{H} \boldsymbol{\omega}_q^\top \left(\boldsymbol{\mu} - r_f \mathbf{1}\right) \tag{20}$$

$$= \frac{\left(\mu_p - r_f\right)\left(\mu_q - r_f\right)}{H}, \tag{21}$$

where we use that the mean-variance model informs us about the portfolios weights of the frontier portfolio p, but not the portfolio q. However, we know that q consists of all risky assets and therefore that $\boldsymbol{\omega}_q^\top \boldsymbol{\mu} = \mu_q$ and that $\boldsymbol{\omega}_q^\top \mathbf{1} = 1$, which we use to arrive at the last equality. Isolating for $u_q - r_f$ gives us

$$\mu_q - r_f = \frac{H \sigma_{q,p}}{\mu_p - r_f} = \frac{\sigma_{q,p}}{\sigma_p^2} \left(\mu_p - r_f\right), \tag{22}$$

where the last equality uses that $H = \left(\mu_p - r_f\right)^2 / \sigma_p^2$ is the squared Sharpe ratio of the frontier portfolio p. The remaining steps in the derivation are to provide a series of arguments that gives us the final equilibrium asset pricing model. Importantly, the relation in (22) is nothing more than a mathematical representation. To turn the expression into an equilibrium model with asset pricing implications, we make the following arguments. First, recall that p was specified as *any* frontier portfolio, so we can easily choose $p = T$, the tangency portfolio. Moreover, in equilibrium, we have that $T = M$, so we can also identify it as the market portfolio such that

$$\mu_q = r_f + \frac{\sigma_{qM}}{\sigma_M^2} \left(\mu_M - r_f\right). \tag{23}$$

The next step is to use that $\beta_{qM} = \frac{\sigma_{qM}}{\sigma_M^2}$ such that we obtain

$$\mu_q = r_f + \beta_{qM} \left(\mu_M - r_f\right). \tag{24}$$

Finally, noting that any risky asset j is itself a feasible portfolio, we obtain the standard CAPM directly from the mean-variance model

$$\mu_j = r_f + \beta_{jM} \left(\mu_M - r_f\right). \tag{25}$$

This is an equilibrium asset pricing model because it requires that the supply equals demand in financial markets so that the tangency portfolio becomes the market portfolio and that the market portfolio is efficient.

3. Black's version of the CAPM

The standard CAPM relies on the investor having unrestricted access to borrowing and lending at the same risk-free rate. What if we relax the assumption that investors can borrow at the risk-free interest rate? Or what if a riskless asset does not exist in the economy? Black (1972) is an important contribution in this area and he shows that a similar equilibrium asset pricing relation exists even in an economy without a risk-free rate. Here we outline his zero-beta CAPM. The Black (1972) version of the model builds on the following observation: the separation theorem for N risky assets implies that all frontier portfolios

$$\boldsymbol{\omega}_p^* = \mathbf{g} + \mathbf{h}\mu_p \tag{26}$$

with

$$\mathbf{g} = \frac{1}{D}\left[B\mathbf{V}^{-1}\mathbf{1} - A\mathbf{V}^{-1}\boldsymbol{\mu}\right] \tag{27}$$

$$\mathbf{h} = \frac{1}{D}\left[C\mathbf{V}^{-1}\boldsymbol{\mu} - A\mathbf{V}^{-1}\mathbf{1}\right] \tag{28}$$

and $D = BC - A^2$ can be linearly combined to create a new portfolio that itself is also a frontier portfolio. That is, a linear combination of frontier portfolios is also a frontier portfolio. Moreover, a linear combination of efficient portfolios is also an efficient portfolio. Again, these observations come directly from the mean-variance model of Markowitz (1952). Suppose that there are $i = 1, 2, \ldots, I$ investors in the economy who own a proportion W_i of the economy's total wealth. Each investor i chooses an efficient frontier portfolio $\boldsymbol{\omega}_{i,p}^* = \mathbf{g} + \mathbf{h}\mu_{i,p}$, where $\mu_{i,p}$

is the rate of return required by investor i (and reflects her risk aversion). The portfolio weights of the market portfolio is then characterized as

$$\boldsymbol{\omega}_M = \sum_{i=1}^{I} W_i \boldsymbol{\omega}_{i,p}^{*} \tag{29}$$

$$= \sum_{i=1}^{I} W_i \left(\mathbf{g} + \mathbf{h}\mu_{i,p}\right) \tag{30}$$

$$= \mathbf{g}\sum_{i=1}^{I} W_i + \mathbf{h}\sum_{i=1}^{I} W_i \mu_{i,p} \tag{31}$$

$$= \mathbf{g} + \mathbf{h}\mu_M, \tag{32}$$

where $\mu_M = \sum_{i=1}^{I} W_i \mu_{i,p}$ and the last equality follows from the fact that $\sum_{i=1}^{I} W_i = 1$ so that the proportions sum to the total wealth in the economy. The important implication of (32) is that the market portfolio, being a linear combination of efficient portfolios, must itself be an efficient frontier portfolio. Similar to the standard CAPM, this is a central and critical prediction of the equilibrium model.

The essential element to developing the Black (1972) version of the CAPM without a risk-free asset is to consider the proposition:

Proposition For any frontier portfolio p, except for the minimum variance portfolio, there exists a unique frontier portfolio with which p has zero covariance.

We will refer to this portfolio as the zero-covariance portfolio relative to p and denote its vector of portfolio weights by $ZC(p)$. We also note that the exclusion of the minimum variance portfolio (MVP) is needed as the MVP can be shown to have a constant covariance of $\frac{1}{C}$ with *all* assets. The proposition implies that

$$\operatorname{cov}\left[\widetilde{r}_p, \widetilde{r}_{ZC(p)}\right] = \sigma_{p,ZC(p)} = \frac{C}{D}\left[\mu_p - \frac{A}{C}\right]\left[\mu_{ZC(p)} - \frac{A}{C}\right] + \frac{1}{C} = 0, \tag{33}$$

which holds for any frontier portfolio p, except the MVP. This is clearly evident as the formula reduces to $\frac{1}{C}$ for $ZC(p) = \frac{A}{C}$ for *any* asset, not just efficient ones. The proposition then implies that the expected return on the zero-covariance portfolio is

$$\mu_{ZC(p)} = \frac{A}{C} - \frac{\frac{D}{C^2}}{\mu_p - \frac{A}{C}} \tag{34}$$

where the constants A, C, and D are defined as above. Next, we want to identify a pricing relation for individual assets and consider again the covariance between a frontier portfolio p

and any portfolio q not necessarily on the frontier

$$
\begin{aligned}
\sigma_{p,q} &= \boldsymbol{\omega}_p^\top \mathbf{V} \boldsymbol{\omega}_q \\
&= \left[\lambda \mathbf{V}^{-1}\boldsymbol{\mu} + \gamma \mathbf{V}^{-1}\mathbf{1}\right]^\top \mathbf{V}\boldsymbol{\omega}_q \\
&= \lambda \boldsymbol{\mu}^\top \mathbf{V}^{-1}\mathbf{V}\boldsymbol{\omega}_q + \gamma \mathbf{1}^\top \mathbf{V}^{-1}\mathbf{V}\boldsymbol{\omega}_q \\
&= \lambda \mu_q + \gamma
\end{aligned} \tag{35}
$$

where we use that the solution to the portfolio choice problem for N risky assets in (26) can equivalently be written as $\boldsymbol{\omega}_p^* = \lambda \mathbf{V}^{-1}\boldsymbol{\mu} + \gamma \mathbf{V}^{-1}\mathbf{1}$. Inserting the expression for λ and γ gives us (see the slides and notes on the mean-variance model)

$$
\sigma_{p,q} = \lambda \mu_q + \gamma = \frac{C\mu_p - A}{D}\mu_q + \frac{B - A\mu_p}{D} \tag{36}
$$

Solving for μ_q gives us the expression

$$
\mu_q = \frac{A\mu_p - B}{C\mu_p - A} + \sigma_{p,q}\frac{D}{C\mu_p - A}. \tag{37}
$$

The objective is then to obtain a CAPM expression from this equation. We already have the covariance term, so we start by introducing the variance term to get a CAPM beta. We do so by focusing on the second term and multiply and divide with $\sigma_p^2 = \frac{C}{D}\left[\mu_p - \frac{A}{C}\right]^2 + \frac{1}{C} = \frac{1}{C} + \frac{(\mu_p - A/C)^2}{D/C}$ so that

$$
\mu_q = \frac{A\mu_p - B}{C\mu_p - A} + \frac{\sigma_{p,q}}{\sigma_p^2}\left[\frac{1}{C} + \frac{(\mu_p - A/C)^2}{D/C}\right]\frac{D}{C\mu_p - A}. \tag{38}
$$

We can then focus on the first term and use that $D = BC - A^2$ implies that $B = (D + A)^2/C$ so that

$$
\mu_q = \frac{A}{C} - \frac{\frac{D}{C^2}}{\mu_p - \frac{A}{C}} + \frac{\sigma_{p,q}}{\sigma_p^2}\left[\frac{1}{C} + \frac{\left(\mu_p - \frac{A}{C}\right)^2}{\frac{D}{C}}\right]\frac{D}{C\mu_p - A}. \tag{39}
$$

We then realize that the first term is equal to the expected return on the zero-covariance portfolio and that the ratio of the covariance to the variance is equal to the CAPM beta. Finally, multiplying out the last expression yields

$$
\mu_q = \mu_{ZC(p)} + \beta_{qp}\left(\mu_p - \frac{A}{C} + \frac{\frac{D}{C^2}}{\mu_p - \frac{A}{C}}\right). \tag{40}
$$

Again, the we see that we have the negative of the expected return on the zero-covariance portfolio in parenthesis, which then yields the mathematical expression for the price relation we are after

$$\mu_q = \mu_{ZC(p)} + \beta_{qp} \left(\mu_p - \mu_{ZC(p)} \right). \tag{41}$$

The last step is to use a series argument analogous to those in the standard CAPM in that the above expression is a purely mathematical expression that holds for all p. However, since it holds for all p, we can choose the market portfolio (in Black's sense), which delivers the zero-beta CAPM for any asset j

$$\mu_j = \mu_{ZC(M)} + \beta_{jM} \left(\mu_M - \mu_{ZC(M)} \right). \tag{42}$$

This relationship is identical to the previous CAPM result in (25), except that $\mu_{ZC(p)}$ replaces r_f. That is, in the zero-beta CAPM, we measure the excess return relative to the zero-beta rate instead. The zero-beta rate is the return on an asset with $\beta_{jM} = 0$ (hence the name) and thus mimicks a risk-free asset.

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