CHAPTER 18

Financial Equilibrium with Differential Information

Chapter Outline

18.1 Introduction

The fact that investors often disagree about expected future returns or the evaluation of the risks associated with specific investments is probably the source of the majority of financial trading volume. Yet we have said very little so far about the possibility of such disagreements and, more generally, about differences in investors' information. In fact, two of the equilibrium models we have reviewed have explicitly assumed that investors have identical information sets. In the case of the capital asset pricing model (CAPM), it is assumed that all investors' expectations are summarized by the same vector of expected returns and the same variance–covariance matrix. It is this assumption that gives relevance to the single efficient frontier. Similarly, the assumption of a single representative decision maker in the consumption CAPM (CCAPM) is akin to assuming the existence of a large number of investors endowed with identical preferences and information sets.[1] The rational expectations hypothesis, which is part of the CCAPM, necessarily implies that, at equilibrium, all investors share the same objective views about future returns.

Both the Arbitrage Pricing Theory (APT) and the Martingale pricing models are nonstructural models which, by construction, are agnostic about the background information

[1] Box 10.1 discussed the extent to which this interpretation can be relaxed as far as utility functions are concerned.

Intermediate Financial Theory.

(or preferences) of the investors. In a sense these theories go beyond the homogeneous information assumption, but without being explicit as to the specific implications of such an extension. The Arrow–Debreu model is a structural model equipped to deal, at least implicitly, with heterogeneously informed agents. In particular, it can accommodate general utility representations defined on state-contingent commodities where, in effect, the assumed state probabilities are embedded in the specific form taken by the individual's utility function.[2] Thus, while agents must agree on the relevant states of the world, they could disagree on their probabilities. We did not exploit this degree of generality, however, and typically made our arguments on the basis of time-additive and state-additive utility functions with explicit, investor-homogeneous, state probabilities.

In this chapter, we relax the assumption that all agents in the economy have the same subjective probabilities about states of nature or the same expectations about returns, or that they know the objective probability distributions. In so doing we open a huge and fascinating, yet incomplete, chapter in financial economics, part of which was selectively reviewed in Chapter 2. We will again be very selective in the topics we choose to address under this heading and will concentrate on the issue of market equilibrium with differentially informed traders. This is in keeping with the spirit of this book and enables us to revisit the last important pillar of traditional financial theory left untouched thus far: the efficient market hypothesis.

The import of differential information for understanding financial markets, institutions, and contracts, however, goes much beyond market efficiency. Since Akerlof (1970), asymmetric information—a situation where agents are differentially informed with, moreover, one subgroup having *superior* information—is known potentially to lead to the failure of a market to exist. This *lemons* problem is a relevant one in financial markets: one may be reluctant to purchase a stock from a better-informed intermediary, or, a fortiori, from the primary issuer of a security who may be presumed to have the best information about the exact value of the underlying assets. One may suspect that the issuer would be unwilling to sell at a price lower than the fundamental value of the asset. What is called the *winner's curse* is applicable here: if the transaction is concluded, i.e., if the better-informed owner has agreed to sell, is it not likely that the buyer will have paid too much for the asset? This reasoning might go some way toward explaining the fact that capital raised by firms in equity markets is such a small proportion of total firm financing (on this issue, see Greenwald and Stiglitz, 1993).

Asymmetric information may also explain the phenomenon of credit rationing. The idea here is that it may not be to the advantage of a lender, confronted with a demand for funds larger than he can accommodate, to increase the interest rate he charges as would be

[2] Such preference structures are, strictly speaking, not expected utility.

required to balance supply and demand: in doing so, the lender may alter the pool of applicants in an unfavorable way. Specifically, this possibility depends on the plausible hypothesis that the lender does not know the degree of riskiness of the projects for which borrowers need funds and that, in the context of a debt contract, a higher hurdle rate may eliminate the less profitable, but consequently, also the less risky, projects. It is easy to construct cases where the creditor is worse off lending his funds at a higher rate because at the high rate the pool of borrowers becomes riskier (Stiglitz and Weiss, 1981).

Asymmetric information has also been used to explain the prevalence of debt contracts relative to contingent claims. We have used the argument before (Chapter 9): states of nature are often costly to ascertain and verify for one of the parties in a contract. As a result, when two parties enter into a contract, it may be more efficient to stipulate noncontingent payments most of the time, thus economizing on verification costs. Only states leading to bankruptcy or default are recognized as resulting in different rights and obligations for the parties involved (Townsend, 1979).

These are only a few of the important issues that can be addressed with the asymmetric information assumption. A full review would deserve a whole book in itself. One reason for the need to be selective is that there is a lack of a unifying framework in this literature. It has often proceeded with a set of specific examples rather than more encompassing models. We refer interested readers to Hirshleiffer and Riley (1992) for a broader review of this fascinating and important topic in financial economics.

18.2 On the Possibility of an Upward-Sloping Demand Curve

There are plenty of reasons to believe that differences in information and beliefs constitute an important motivation for trading in financial markets. It is extremely difficult to rationalize observed trading volumes in a world of homogeneously informed agents.
The main reason for having neglected what is without doubt an obvious fact is that our equilibrium concept, borrowed from traditional supply and demand analysis (the standard notion of Walrasian equilibrium), must be thoroughly updated once we allow for heterogeneous information.

The intuition is as follows. The Walrasian equilibrium price is necessarily some function of the orders placed by traders. Suppose that traders are heterogeneously informed and that their private information set is a relevant determinant of their orders. The equilibrium price will, therefore, reflect and, in that sense, transmit at least a fraction of the privately held information. In this case, the equilibrium price is not only a signal of relative scarcity, as in a Walrasian world; it also reflects the agents' information. In this context, the price quoted for a commodity or a security may be high because the demand for it is objectively high and/or the supply is low. But it may also be high because a group of investors has private information suggestive that the commodity or security in question will be expensive tomorrow. Of course, this information about the future value of the item is of interest to all. Presumably, except for liquidity reasons, no one will want to sell something today at a low price that will likely be of much higher value tomorrow. This means that when the price quoted on the market is high (in the fiction of standard microeconomics, when the *Walrasian auctioneer* announces a high price), a number of market participants will realize that they have sent in their orders on the basis of information that is probably not shared by the *rest of the market*. Depending on the confidence they place in their own information, they may then want to revise their orders, and to do so in a paradoxical way: because the announced price is higher than they thought it would be, they may want to buy more! Fundamentally, this means that what was thought to be the equilibrium price is not, in fact, an equilibrium.

This is a new situation, and it requires a departure from the Walrasian equilibrium concept. In this chapter, we will develop these ideas with the help of an example. We first illustrate the notion of a rational expectations equilibrium (REE), a concept we have used more informally in preceding chapters (e.g., Chapter 10), in a context where all participants share the same information. We then extend it to encompass situations where agents are heterogeneously informed. We provide an example of a fully revealing REE, which may be deemed to be the formal representation of the notion of an *informationally efficient market*. We conclude by discussing some weaknesses of this equilibrium concept and possible extensions.

18.3 An Illustration of the Concept of REE: Homogeneous Information

Let us consider the joint equilibrium of a spot market for a given commodity and its associated futures market. The context is the familiar now and then, two-date economy. The single commodity is traded at date 1.[3] Viewed from date 0, the date at which producers must make their production decisions, the demand for this commodity, emanating from final users, is stochastic. It can be represented by a linear demand curve shocked by a random term as in

$$D(p, \tilde{\eta}) = a - cp + \tilde{\eta}$$

where $D(\cdot)$ represents the quantity demanded, p is the (spot) price for the commodity in question, a and c are positive constants, and $\tilde{\eta}$ is a stochastic demand-shifting element.[4] This latter quantity is centered at (has mean value) zero, at which point the demand curve assumes its average position, and it is normally distributed with variance σ_η^2. In other words, $h(\tilde{\eta}) = N(0; \sigma_\eta^2)$ where $h(\cdot)$ is the probability density function on $\tilde{\eta}$. See Figure 18.1

[3] The rest of this chapter closely follows Danthine (1978).

[4] Looking forward, the demand for heating oil next winter is stochastic because the severity of the winter is impossible to predict in advance.

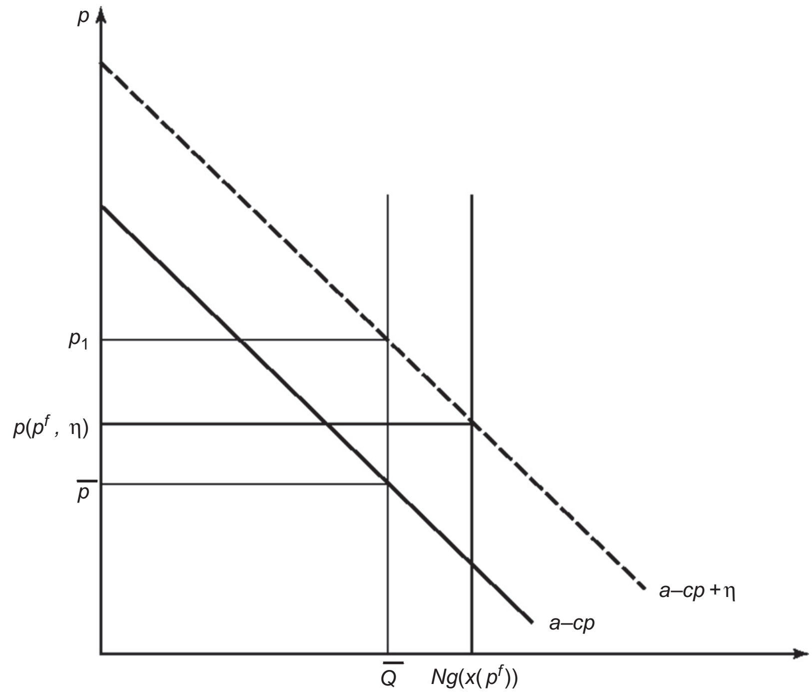

Figure 18.1
Equilibrium with a stochastic demand curve.

for an illustration. At date 0, the N producers decide on their input level x—the input price is normalized at 1 —knowing that $g(x)$ units of output will then be available after a one-period production lag at date 1. The production process is thus nonstochastic, and the only uncertainty originates from the demand side. Because of the latter feature, the future sale price $\tilde{p}$ is unknown at the time of the input decision.

We shall assume the existence of a futures or forward market[5] that our producers may use for hedging or speculative purposes. Specifically, let $f > 0(<0)$ be the short (long) futures position taken by the representative producer, i.e., the quantity of output sold (bought) for future delivery at the future (or forward) price p^f.

[5] The term *futures market* is normally reserved for a market for future delivery taking place in the context of an organized exchange. A *forward market* refers to private exchanges of similar contracts calling for the future delivery of a commodity or financial instrument. While knowledge of the creditworthiness and honesty of the counterparty is of essence in the case of forward contracts, a futures market is anonymous. The exchange is the relevant counterparty for the two sides in a contract. It protects itself and ensures that both parties' engagements will be fulfilled by demanding initial guarantee deposits as well as issuing daily *margin calls* to the party against whose position the price has moved. In a two-date setting, thus in the absence of interim price changes, the notion of margin calls is not relevant, and it is not possible to distinguish futures from forwards.

Here we shall assume that the good traded in the futures market (i.e., specified as acceptable for delivery in the futures contract) is the same as the commodity exchanged on the spot market. For this reason, arbitrageurs will ensure that, at date 1, the futures and the spot price will be exactly identical. In the language of futures markets, the *basis* is constantly equal to zero, and there is thus no *basis risk*.

Under these conditions, the typical producer's cash flow y is

$$\tilde{y} = \tilde{p}g(x) - x + (p^f - \tilde{p})f$$

which can also be written as

$$\tilde{y} = \tilde{p}(g(x) - f) - x + p^f f$$

It is seen that by setting $f = g(x)$, i.e., by selling forward the totality of his production, the producer can eliminate all his risks. Although this need not be his optimal futures position, the feasibility of shedding all risks explains the separation result that follows (much in the spirit of the CAPM: diversifiable risk is not priced).

Let us assume that producers maximize the expected utility of their future cash flow where $U'(\cdot) > 0$ and $U''(\cdot) < 0$:

$$\max_{\substack{x,f \\ x \geq 0}} EU(\tilde{y})$$

Differentiating with respect to x and f successively, and assuming an interior solution, we obtain the following two first-order conditions (FOCs):

$$x\!: \; E[U_1(\tilde{y})\tilde{p}] = \frac{1}{g_1(x)} EU_1(\tilde{y}) \tag{18.1}$$

$$f\!: \; E[U_1(\tilde{y})\tilde{p}] = p^f EU_1(\tilde{y}) \tag{18.2}$$

which together imply

$$p^f = \frac{1}{g_1(x)} \tag{18.3}$$

Equation (18.3) is remarkable because it says that the optimal input level should be such that the marginal cost of production is set equal to the (known) futures price p^f, the latter replacing the expected spot price as the appropriate production signal. The futures-price-equals-marginal-cost condition is also worth noticing because it implies that, despite the uncertain context in which they operate, producers should not factor in a risk premium when computing their optimal production decision.

For us, a key implication of this result is that, since the supply level will directly depend on the futures price quoted at date 0, the equilibrium spot price at date 1 will be a function of the futures price realized one period earlier. Indeed, writing $x = x(p^f)$ and $g(x) = g(x(p^f))$ to highlight the implications of Eq. (18.3) for the input and output levels, the supply−equals−demand condition for the date 1 spot market reads

$$Ng(x(p^f)) = a - cp + \tilde{\eta}$$

which implicitly defines the equilibrium (date 1) spot price as a function of the date 0 value taken by the futures price, or

$$\tilde{p} = p(p^f, \tilde{\eta}) \tag{18.4}$$

It is clear from Eq. (18.4) that the structure of our problem is such that the probability distribution on $\tilde{p}$ cannot be spelled out independently of the value taken by p^f. Consequently, it would not be meaningful to assume expectations for $\tilde{p}$, on the part of producers or futures market speculators, which would not take account of this fundamental link between the two prices. This observation, which is a first step toward the definition of a rational expectation equilibrium, can be further developed by focusing now on the futures market itself.

Let us assume that, in addition to the N producers, n speculators take positions on the futures market. We define speculators by their exclusive involvement in the futures markets; in particular they have no position in the underlying commodity. Accordingly, their cash flows are simply

$$\tilde{z}_i = (p^f - \tilde{p})b_i$$

where b_i is the futures position ($b_i > 0$ represents a short position; $b_i < 0$ represents a long position) taken by speculator i. Suppose for simplicity that their preferences are represented by a linear mean−variance utility function of their cash flows:

$$W(\tilde{z}_i) = E(\tilde{z}_i) - \frac{\chi}{2}\mathrm{var}(\tilde{z}_i)$$

where χ represents the (Arrow−Pratt) absolute risk-aversion index of the representative speculator. We shall similarly specialize the utility function of producers. The assumption of a linear mean−variance utility representation is, in fact, equivalent to hypothesizing an exponential (constant absolute risk aversion, CARA) utility function such as

$$W(\tilde{z}) = -e^{-x/2\tilde{z}}$$

if the context is such that the argument of the function, z, is normally distributed. This hypothesis will be verified at the equilibrium of our model.

Under these hypotheses, it is easy to verify that the optimal futures position of speculator i is

$$b_i = \frac{p^f - E(\tilde{p}|p^f)}{\chi \ \mathrm{var}(\tilde{p}|p^f)} \tag{18.5}$$

where the conditioning in the expectation and variance operators is made necessary by Eq. (18.4). The form of Eq. (18.5) is not surprising. It implies that the optimal futures position selected by a speculator will have the same sign as the expected difference between the futures price and the expected spot price, i.e., a speculator will be short ($b > 0$) if and only if the futures price at which he sells is larger than the spot price at which he expects to be able to unload his position tomorrow. As to the size of his position, it will be proportional to the expected difference between the two prices, which is indicative of the size of the expected return, and inversely related to the perceived riskiness of the speculation, measured by the product of the variance of the spot price with the Arrow–Pratt coefficient of risk aversion. More risk-averse speculators will assume smaller positions, everything else being the same.

Under a linear mean–variance specification of preferences, the producer's objective function becomes

$$\max_{\substack{x,f \\ x \geq 0}} E(\tilde{p}|p^f)(g(x) - f - x) + p^f f - \frac{\xi}{2}(g(x) - f)^2 \mathrm{var}(\tilde{p}|p^f)$$

where ξ is the absolute risk-aversion measure for producers.

With this specification of the objective function, Eq. (18.2), the FOC with respect to f, becomes

$$f = g(x(p^f)) + \frac{p^f - E(\tilde{p}|p^f)}{\xi \ \mathrm{var}(\tilde{p}|p^f)} \equiv f(p^f) \tag{18.6}$$

which is the second part of the separation result alluded to previously. The optimal futures position of the representative producer consists in selling forward the totality of his production ($g(x)$) and then readjusting by a component that is simply the futures position taken by a speculator with the same degree of risk aversion. To see this, compare the last term in Eq. (18.6) with Eq. (18.5). A producer's actual futures position can be viewed as the sum of these two terms. He may under-hedge, i.e., sell less than his future output at the futures price. This is so if he anticipates paying an insurance premium in the form of a sale price (p^f) lower than the spot price he expects to prevail tomorrow. But he could as well over-hedge and sell forward more than his total future output. That is, if he considers the current futures price to be a high enough price, he may be willing to speculate on it, selling high at the futures price what he hopes to buy low tomorrow on the spot market.

Putting together speculators' and producers' positions, we find that the futures market-clearing condition becomes

$$\sum_{i=1}^{n} b_i + Nf = 0 \text{ or}$$
$$n\left\{\frac{p^f - E(\tilde{p}|p^f)}{\chi \operatorname{var}(\tilde{p}|p^f)}\right\} + N\left\{\frac{p^f - E(\tilde{p}|p^f)}{\xi \operatorname{var}(\tilde{p}|p^f)}\right\} + Ng(x(p^f)) = 0 \tag{18.7}$$

which must be solved for the equilibrium futures price p^f. Equation (18.7) makes clear that the equilibrium futures price p^f is dependent on the expectations held on the future spot price $\tilde{p}$; we have previously emphasized the dependence on p^f of expectations about $\tilde{p}$. This apparently circular reasoning can be resolved under the rational expectations hypothesis, which consists of assuming that individuals have learned to understand the relationship summarized in Eq. (18.4), i.e.,

$$E(\tilde{p}|p^f) = E[p(p^f, \tilde{\eta})|p^f], \ \operatorname{var}(\tilde{p}|p^f) = \operatorname{var}[p(p^f, \tilde{\eta})|p^f] \tag{18.8}$$

Definition 18.1 In the context of this section, an REE is defined as

1. a futures price p^f solving Eq. (18.7) given Eq. (18.8), and the distributional assumption made on η, and
2. a spot price p solving Eq. (18.4) given p^f and the realization of η.

The first part of the definition indicates that the futures price equilibrates the futures market at date 0 when agents rationally anticipate the effective condition under which the spot market will clear tomorrow and make use of the objective probability distribution on the stochastic parameter $\tilde{\eta}$. Given the supply of the commodity available tomorrow (itself a function of the equilibrium futures price quoted today), and given the particular value taken by $\tilde{\eta}$ (i.e., the final position of the demand curve), the second part specifies that the spot price clears the date 1 spot market.

18.4 Fully Revealing REE: An Example

Let us pursue this example one step further and assume that speculators have access to privileged information in the following sense: Before the futures exchange opens, speculator i, ($i = 1,\ldots, n$), observes some unbiased approximation υ_i to the future realization of the variable $\tilde{\eta}$. The signal υ_i can be viewed as the future η itself plus an error of observation ω_i. The latter is specific to speculator i, but all speculators are similarly imprecise in the information they manage to gather. Thus,

$$\upsilon_i = \eta + \omega_i \text{ where the } \tilde{\omega}_i\text{'s are i.i.d. } N(0; \sigma^2_\omega)$$

across agents and across time periods.

This relationship can be interpreted as follows: η is a summary measure of the mood of consumers or of other conditions affecting demand. Speculators can obtain advanced information as to the particular value of this realization for the relevant period through, for instance, a survey of consumer's intentions or a detailed weather forecast (assuming the latter influences demand). These observations are not without errors, but (regarding these two periods as only one occasion of a multiperiod process where learning has been taking place), speculators are assumed to be sufficiently skilled to avoid systematic biases in their evaluations. In the present model, this advance information is freely available to them.

Under these conditions, Eq. (18.5) becomes

$$b_i = \frac{p^f - E(\tilde{p}|p^f; \upsilon_i)}{\chi \ \mathrm{var}(\tilde{p}|p^f; \upsilon_i)} \equiv b(p^f; \upsilon_i)$$

where we make explicit the fact that both the expected value and the variance of the spot price are affected by the advance piece of information obtained by speculator i. The appendix details how these expectations can actually be computed, but this needs not occupy us for the moment.

Formally, Eq. (18.6) is unchanged, so that the futures market-clearing condition can be written

$$Nf(p^f) + \sum_{i=1}^{n} b(p^f; \upsilon_i) = 0$$

It is clear from this equation that the equilibrium futures price will be affected by the "elements" of information gathered by speculators. In fact, under appropriate regularity conditions, the market-clearing equation implicitly defines a function

$$p^f = l(\upsilon_1, \upsilon_2, \ldots, \upsilon_n) \tag{18.9}$$

that formalizes this link and thus the information content of the equilibrium futures price.

All this implies that there is more than meets the eye in the conditioning on p^f of $E(\tilde{p}|p^f)$ and $\mathrm{var}(\tilde{p}|p^f)$. So far the reasoning for this conditioning was given by Eq. (18.4): a higher p^f stimulates supply from $g(x(p^f))$ and thus affects the equilibrium spot price. Now a higher p^f also indicates high υ_is on average, thus transmitting information about the future realization of $\tilde{\eta}$. The real implications of this link can be understood by reference to Figure 18.1. In the absence of advance information, supply will be geared to the average demand conditions. $\overline{Q}$ represents this average supply level, leading to a spot price $\overline{p}$ under conditions of average demand ($\eta = 0$). If suppliers receive no advance warning of an abnormally high demand level, an above-average realization $\hat{\eta}$ requires a high price p_1 to balance supply and demand. If, on the other hand, speculators' advance information is transmitted to producers via the futures price, supply increases in anticipation of the high demand level and the price increase is mitigated.

We are now in a position to provide a precise answer to the question that has occupied us since Section 18.2: How much information is transmitted by the equilibrium price p^f? It will not be a fully general one. Our model has the nature of an example because it presumes specific functional forms. The result we will obtain certainly stands at one extreme on the spectrum of possible answers; it can be considered as a useful benchmark. In what follows, we will construct, under the additional simplification $g(x) = \alpha x^{\frac{1}{2}}$, a consistent equilibrium in which the futures price is itself a summary of *all the information* there is to obtain, a summary that, in an operational sense, is fully equivalent to the complete list of signals obtained by all speculators. More precisely, we will show that the equilibrium futures price is an invertible (linear) function of $\sum \upsilon_j$ and that, indeed, it clears the futures market given that everyone realizes this property and bases his orders on the information he can thus extract. This result is important because $\sum \upsilon_j$ is a sufficient statistic for the entire vector $(\upsilon_1, \upsilon_2, \ldots, \upsilon_n)$. While we will precisely define the notion of a "sufficient statistic" shortly, here it simply suggests that the sum contains as much relevant information for the problem at hand as the entire vector, in the sense that knowing the sum leads to placing the same market orders as knowing the whole vector. Ours is thus a context where the answer to our question is: *All* the relevant information is aggregated in the equilibrium price and is revealed freely to market participants. The REE is thus *fully revealing*!

Let us proceed and make these assertions precise. Under the assumed technology, $g(x) = \alpha x^{\frac{1}{2}}$, Eqs. (18.3), (18.4), and (18.8) become, respectively,

$$g(x(p^f)) = \frac{\alpha^2}{2} p^f$$

$$p(p^f, \tilde{\eta}) = A - Bp^f + \frac{1}{c}\tilde{\eta}$$

$$\text{with} \quad A = \frac{a}{c}, \; B = \frac{N}{c}\frac{\alpha^2}{2}$$

$$E(\tilde{p}|p^f) = A - Bp^f + \frac{1}{c}E(\tilde{\eta}|p^f)$$

$$\text{var}(\tilde{p}|p^f) = \frac{1}{c^2}\text{var}(\tilde{\eta}|p^f)$$

The informational structure is as follows. Considering the market as a whole, an experiment has been performed consisting of observing the values taken by n independent drawings of some random variable $\overline{\upsilon}$, where $\overline{\upsilon} = \eta + \tilde{w}$ and $\tilde{w}$ is $N(0, \sigma_w^2)$. The results are summarized in the vector $\upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n)$ or, as we shall demonstrate, in the sum of the υ_j's, $\sum \upsilon_j$, which is a *sufficient statistic* for $\upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n)$. The latter expression means that conditioning expectations on $\sum \upsilon_j$ or on $\sum \upsilon_j$ *and* the whole vector of υ yields the same posterior distribution for $\tilde{\eta}$. In other words, the entire vector does not contain any information that is not already present in the sum. Formally, we have Definition 18.2.

Definition 18.2 $(\frac{1}{n})\sum \upsilon_j$ is a sufficient statistic for $\upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n)$ relative to the distribution $h(\eta)$ if and only if $h(\tilde{\eta}| \sum \upsilon_j, \upsilon) = h(\tilde{\eta}| \sum \upsilon_j)$.

Being a function of the observations (see Eq. (18.9)), p^f is itself a statistic used by traders in calibrating their probabilities. The question is: How good a statistic can be? How well can the futures price summarize the information available to the market? As promised, we now display an equilibrium where the price p^f is a sufficient statistic for the information available to the market, i.e., it is invertible for the sufficient statistic $\sum \upsilon_j$. In that case, knowledge of p^f is equivalent to the knowledge of $\sum \upsilon_j$, and farmers' and speculators' expectations coincide. If the futures price has this revealing property, expectations held at equilibrium by all agents must be (see the appendix to this chapter for details)

$$E(\tilde{\eta}|p^f) = E(\tilde{\eta}|\upsilon_j, p^f) = E(\tilde{\eta}| \sum \upsilon_j) = \frac{\sigma_\eta^2}{n\sigma_\eta^2 + \sigma_w^2} \sum \upsilon_j \tag{18.10}$$

$$\mathrm{var}(\tilde{\eta}|p^f) = \mathrm{var}(\tilde{\eta}|\upsilon_j, p^f) = \frac{\sigma_w^2 \sigma_\eta^2}{n\sigma_\eta^2 + \sigma_w^2} \tag{18.11}$$

Equations (18.10) and (18.11) make clear that conditioning on the futures price would, under our hypothesis, be equivalent to conditioning on $\sum \upsilon_j$, the latter being, of course, superior information relative to the single piece of individual information, υ_i, initially obtained by speculator i. Using these expressions for the expectations in Eq. (18.7), one can show after a few tedious manipulations that, as announced, the market-clearing futures price has the form

$$p^f = F + L \sum \upsilon_j \tag{18.12}$$

where

$$F = \frac{(N_\chi + n\xi)A}{(N_\chi + n\xi)(B+1) + N\alpha^2 \xi \chi \dfrac{1}{c^2} \dfrac{\sigma_w^2 \sigma_\eta^2}{n\sigma_\eta^2 + \sigma_w^2}} \quad \text{and}$$

$$L = \frac{1}{c} \frac{\sigma_w^2 \sigma_\eta^2}{n\sigma_\eta^2 + \sigma_w^2} \frac{F}{A}$$

Equation (18.12) shows the equilibrium price p^f to be proportional to $\sum \upsilon_j$ and thus a sufficient statistic as postulated. It satisfies our definition of an equilibrium. It is a market-clearing price, the result of speculators' and farmers' maximizing behavior, and it corresponds to an equilibrium state of expectations. That is, when Eq. (18.12) is the hypothesized functional relationship between p^f and υ, this relationship is indeed realized given that each agent then appropriately extracts the information $\sum \upsilon_j$ from the announcement of the equilibrium price.

18.5 The Efficient Market Hypothesis

The result obtained in Section 18.4 is without doubt extreme. It is interesting, however, as it stands as the paragon of the concept of market efficiency. Here is a formal and precise context in which the valuable pieces of information held by heterogeneously informed market participants are aggregated and freely transmitted to all via the trading process. This outcome is reminiscent of the statements made earlier in the century by the famous liberal economist F. von Hayek who celebrated the virtues of the market as an information aggregator (von Hayek, 1945). It must also correspond to what Fama (1970) intended when introducing the concept of strong form efficiency, defined as a situation where market prices fully reflect all publicly *and* privately held information.

The reader may recall that Fama (1970) also introduced the notions of *weak-form efficiency*, covering situations where market prices fully and instantaneously reflect the information included in historical prices, and of *semi-strong form efficiency* where prices, in addition, reflect all publicly available information (of whatever nature). A securities market equilibrium such as the one described in Chapter 10 under the heading of the CCAPM probably best captures what one can understand as semi-strong efficiency: agents are rational in the sense of being expected utility maximizers, they are homogeneously informed (so that all information is indeed publicly held), and they efficiently use all the relevant information when defining their asset holdings. In the CCAPM, no agent can systematically "beat the market", a largely accepted hallmark of an efficient market equilibrium, provided "beating the market" is appropriately defined in terms of both risk and return.

The concept of a Martingale, first used in Chapters 12 and 13, has long constituted another hallmark of market efficiency. It is useful here to provide a formal definition.

Definition 18.3 A stochastic process $\tilde{x}_t$ is a Martingale with respect to an information set Φ_t if

$$E(\tilde{x}_{t+1}|\Phi_t) = x_t \tag{18.13}$$

It is a short step from this notion of a Martingale to the assertion that one cannot beat the market, which is the case if the current price of a stock is the best predictor of its future price. The latter is likely to be the case if market participants indeed make full use of all available information: In that situation, future price changes can only be unpredictable. An equation like Eq. (18.13) cannot be true exactly for stock prices as stock returns would then be zero on average. It is clear that what could be a Martingale under the previous intuitive reasoning would be a price series normalized to take account of dividends and a normal expected return for holding stock. To get an idea of what this would mean, let us refer to the price equilibrium Eq. (10.2) of the CCAPM

$$U_1(Y_t)q_t^e = \delta E_t\{U_1(Y_{t+1})(q_{t+1}^e + Y_{t+1})\} \tag{18.14}$$

Making the assumption of risk neutrality, one obtains

$$q_t^e = \delta E_t(q_{t+1}^e + Y_{t+1}) \tag{18.15}$$

If we entertain, for a moment, the possibility of a nondividend-paying stock, $Y_t \equiv 0$, then Eq. (18.14) indeed implies that the normalized series $x_t = \delta^t p_t$ satisfies Eq. (18.13) and is thus a Martingale. This normalization implies that the expected return on stockholding is constant and equal to the risk-free rate. In the case of a dividend-paying stock, a similar, but slightly more complicated, normalization yields the same result.

The main points of this discussion are (1) that a pure Martingale process requires adjusting the stock price series to take account of dividends and the existence of a positive normal return and (2) that the Martingale property is a mark of market efficiency only under a strong hypothesis of risk neutrality that includes, as a corollary, the property that expected return to stockholding is constant. The large empirical literature on market efficiency has not always been able to take account appropriately of these qualifications. See LeRoy (1989) for an in-depth survey of this issue.

Our model of the previous section is more ambitious, addressing as it does, the concept of strong form efficiency. Its merit is to underline what it takes for this extreme concept to be descriptive of reality, thus also helping to delineate its limits. Two of these limits deserve mentioning. The first one arises once one attempts, plausibly, to dispense with the hypothesis that speculators are able to obtain their elements of privileged information costlessly. If information is free, it is difficult to see why all speculators would not get all the relevant information, thus reverting to a model of homogeneous information. However, the spirit of our example is that resources are needed to collect information and that speculators are those market participants specializing in this costly search process. Yet why should speculator i expand resources to obtain private information υ_i when the equilibrium price will freely reveal to him the sufficient statistic $\sum \upsilon_j$, which by itself is more informative than the information he could gather at a cost? The very fact that the equilibrium REE price is fully revealing implies that individual speculators have no use for their own piece of information, with the obvious corollary that they will not be prepared to spend a penny to obtain it. On the other hand, if speculators are not endowed with privileged information, there is no way the equilibrium price will be the celebrated information aggregator and transmitter. In turn, if the equilibrium price is not informative, it may well pay for speculators to obtain valuable private information. We are thus trapped in a vicious circle that results in the nonexistence of equilibrium, an outcome Grossman and Stiglitz (1980) have logically dubbed “the impossibility of informationally efficient markets.”

Another limitation of the conceptual setup of Section 18.4 resides in the fact that the hypotheses required for the equilibrium price to be fully revealing are numerous and particularly severe. The rational expectations hypothesis includes, as always, the assumption that market participants understand the environment in which they operate. This segment of the hypothesis is particularly demanding in the context of our model, and it is crucial for agents to be able to extract sufficient statistics from the equilibrium futures price. By that we mean that, for individual agents to be in position to read all the information concealed in the equilibrium price, they need to know exactly the number of uninformed and informed agents and their respective degrees of risk aversion, which must be identical within each agent class. The information held by the various speculators must have identical precision (i.e., an error term with the same variance), and none of the market participants can be motivated by liquidity considerations. All in all, these requirements are simply too strong to be plausibly met in real-life situations. Although the real-life complications may be partly compensated for by the fact that trading is done on a repeated, almost continuous basis, it is more reasonable to assume that the fully revealing equilibrium is the exception rather that the rule. See the recent paper by Vives (2014) on this score.

The more normal situation is certainly one where some, but not all, information is aggregated and transmitted by the equilibrium price. In such an equilibrium, the incentives to collect information remain, although if the price is too good a transmitter, they may be significantly reduced. The nonexistence-of-equilibrium problem uncovered by Grossman and Stiglitz is then more a curiosum than a real source of worry. Equilibria with partial transmission of information have been described in the literature under the heading *noisy rational expectation equilibrium*. The apparatus is quite a bit messier than the one in the reference case discussed in Section 18.4, and we will not explore it further (see Hellwig, 1980 for a first step in this direction). Suffice it to say that this class of models serves as the basis for the branch of financial economics known as *market microstructure* which strives to explain the specific forms and rules underlying asset trading in a competitive market environment. The reader is referred to O’Hara (1997) for a broad coverage of these topics.

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Appendix: Bayesian Updating with the Normal Distribution

Theorem A18.1 If we assume $\tilde{x}$ and $\tilde{y}$ are two normally distributed vectors with

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \sim N\left(\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}, V \right)$$

with matrix of variances and covariances

$$V = \begin{pmatrix} V_{xx} & V_{xy} \\ V_{xy} & V_{yy} \end{pmatrix}$$

then the distribution of $\tilde{x}$ conditional on the observation $\tilde{y} = y^0$ is normal with mean $\overline{x} + V_{xx}V_{yy}^{-1}\left(y^0 - \overline{y}\right)$ and covariance matrix $V_{xx} - V_{xy}V_{yy}^{-1}V_{xy}$.

Applications

Let $\tilde{v}_i = \tilde{\eta} + \tilde{\omega}_i$.

$$\text{If} \begin{pmatrix} \tilde{\eta} \\ \tilde{\omega}_i \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_\eta^2 & \sigma_\eta^2 \\ \sigma_\eta^2 & \sigma_\eta^2 + \sigma_\omega^2 \end{pmatrix} \right), \text{then}$$

$$E(\tilde{\eta}|\upsilon_i) = 0 + \frac{\sigma_\eta^2}{\sigma_\eta^2 + \sigma_\omega^2}\upsilon_i$$

$$V(\tilde{\eta}|\upsilon_i) = \sigma_\eta^2 - \frac{\sigma_\eta^4}{\sigma_\eta^2 + \sigma_\omega^2} = \frac{\sigma_\eta^2\sigma_\omega^2}{\sigma_\eta^2 + \sigma_\omega^2}$$

$$\text{If}\begin{pmatrix} \tilde{\eta} \\ \sum \tilde{\upsilon}_i \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_\eta^2 & \sigma_\eta^2 \\ n\sigma_\eta^2 & n^2\sigma_\eta^2 + n\sigma_\omega^2 \end{pmatrix}\right), \text{then}$$

$$E\left(\tilde{\eta}| \sum \upsilon_i\right) = 0 + \frac{n\sigma_\eta^2}{n^2\sigma_\eta^2 + n\sigma_\omega^2}\left(\sum \upsilon_i\right) = \frac{\sigma_\eta^2}{n\sigma_\eta^2 + \sigma_\omega^2}\left(\sum \upsilon_i\right)$$

$$\text{var}\left(\tilde{\eta}| \sum \upsilon_i\right) = \sigma_\eta^2 - n\sigma_\eta^2 \frac{1}{n^2\sigma_\eta^2 + n\sigma_\omega^2} n\sigma_\eta^2 = \frac{\sigma_\eta^2\sigma_\omega^2}{n\sigma_\eta^2 + \sigma_\omega^2}$$