Estimating the zero-coupon term structure*

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*This note provides an introduction to selected methods for extracting the zero-coupon term structure from the market prices of traded coupon bonds or equivalent fixed income securities. The note borrows notation from Campbell, Lo and MacKinlay (1997), but draws more broadly on expositions from original articles as cited herein as well as the textbook treatments of Martellini, Priaulet and Priaulet (2003), Veronesi (2010), and Tuckman and Serrat (2011). The note is prepared for use only in the Master's course "Asset Pricing". Please do not cite, circulate, or use for purposes other than this course.

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Contents

1. Introduction

The zero-coupon term structure of interest rates is a fundamental building block in finance that lies at the center of financial decision making. The zero-coupon term structure of interest rates, and its implied discount curve, is traditionally inferred from the market prices of traded Treasury securities, but its measurement is far from straightforward. Estimation relies on a sufficient number of bonds being available and traded with enough liquidity to ensure correct pricing. Nonetheless, having access to a smooth and sensible discount function is important for asset pricing, investment, and corporate finance decisions in many different settings. First, equipped with a full zero-coupon discount curve, investors (and managers) can value any fixed income security delivering known and certain cash flows in the future. Investors can, additionally, compare this theoretical value to observed market prices to hunt for mispricings and arbitrage opportunities in financial markets. If the market price is below (above) the theoretical prediction, then the bond is said to be trading cheap (rich). Second, we may easily imagine situations in which market prices are either unknown, stale, or otherwise unreliable. Examples include various over-the-counter (OTC) products, derivatives, coupon bonds with odd maturities, or securities in illiquid markets where trading only occurs infrequently. In such circumstances, one can use the extracted discount function to fill in the missing and/or sluggish prices. As market participants need to know the fair values of their existing positions, and since they do occasionally trade those illiquid securities, discount functions derived from actively traded and liquid securities are commonly used to perform such valuations. Third, the shape of the zero-coupon term structure on any given day is useful for assessing the market's expectations about future interest rates, and can accordingly guide investment behavior and corporate finance funding decisions. Last, equipped with a full term structure of discount factors, one can obtain implicit curves such as the zero-coupon yield curve, the par yield curve, and the term structure of forward rates.

As in Campbell et al. (1997), we will take our point of departure for pricing any fixed income asset in the prices of zero-coupon (discount) bonds. These securities represent the value today to investors of a \$1 nominal payment to be made n periods in the future (i.e., the time value of money). Consider the classic problem of determining the time t price $P_{c,t}^{n}$ of a coupon-paying bond with n periods to maturity. By the law of one price and no arbitrage, we can view a coupon bond as a portfolio of discount bonds with face values equal to the payments of the coupon bond

$$
\begin{aligned}
P_{c,t}^{n} &= P_t^1 C + P_t^2 C + \cdots + P_t^n (1 + C) && (1) \\
&= C \sum_{i=1}^{n} P_t^i + P_t^n, && (2)
\end{aligned}
$$

where C denotes the *per period* coupon rate and P_t^n is the price of a discount bond with n periods to maturity that pays out a nominal dollar. If a complete zero-coupon term structure exists so that prices for discount bonds are available for all payments dates, then (2) is a no arbitrage pricing relation. It is important to note that the clearest picture of the term structure of yields, and hence the discount curve, is obtained by looking at the yields of zero-coupon bonds with different maturities. However, traded prices P_t^n of discount bonds for long maturities are seldom available as very few zero-coupon bonds are issued and actively traded. If they are, e.g., through Separate Trading of Registered Interest and Principal of Securities (STRIPS), they often lack the liquidity needed for reliable use. We therefore cannot observe a complete zero-coupon term structure, but we can extract it from the market prices of traded coupon-paying bonds. We will introduce the term discount factor and denote it by Z_t^n to distinguish them from the market prices P_t^n of traded discount bonds. Thus, for a complete coupon term structure, we have the following relation

$$P_{c,t}^n = C \sum_{i=1}^{n} Z_t^i + Z_t^n, \tag{3}$$

revealing that discount factors, under no arbitrage, are embedded in the market prices of traded coupon-paying bonds and so can be backed out if market prices $P_{c,t}^n$ are correctly reflecting fair values. This note outlines and illustrates selected methods designed to extract Z_t^n from market prices in different environments. We begin with a simple scenario where the coupon term structure is complete so that the implied zero-coupon term structure can be backed out directly without further assumptions. We then consider the case where the coupon term structure is more-than-complete, meaning that at least one coupon bond matures on each coupon date and multiple coupon bonds mature on some coupon dates. We conclude the note with the empirically relevant scenario in which the coupon term structure is incomplete.

The rest of the note proceeds as follows. Section 2 outlines the bootstrapping technique to estimating the zero-coupon term structure. Section 3 considers a regression-based approach that works well when the term structure is more-than-complete. Section 4 discusses and implements the Nelson and Siegel (1987) continuously compounded yield curve model that works even when the coupon term structure is incomplete. Finally, Section 5 outlines an extension to the Nelson and Siegel (1987) model proposed by Svensson (1994).

2. Bootstrapping

Issued and actively traded (i.e., liquid) discount bonds with long maturities and reliable prices are rare in financial markets. Albeit that all coupon-paying bonds will eventually become zero-coupon bonds when only the last payment remains, we will necessarily have to extract information about discount factors at longer maturities from the market prices of traded coupon bonds, or similar fixed income instruments such as overnight index swaps (OIS). The bootstrap methodology is one such approach that is straightforward to apply when the coupon term structure is complete. The idea rest on the simple no arbitrage argument embedded in the pricing equation in (3). We outline the bootstrap below.

2.1. Iterative bootstrapping

Table 1 sets the stage for a general treatment of the bootstrap methodology by presenting data for a collection of traded coupon-paying bonds on a given day. Coupon rates and yields to

Table 1: Observable bond data

Coupon (annualized)	Yield (annualized)	Maturity (years)	Maturity (periods)	Price per face value
1.250%	0.149%	0.5	1	1.0055
4.875%	0.351%	1.0	2	1.0451
4.500%	0.574%	1.5	3	1.0586
4.750%	0.731%	2.0	4	1.0797
3.375%	0.992%	2.5	5	1.0587
3.500%	1.199%	3.0	6	1.0676
2.000%	1.543%	3.5	7	1.0155
2.250%	1.746%	4.0	8	1.0194
2.125%	1.931%	4.5	9	1.0083

maturity are all annualized and we note that the coupon bonds make payments twice a year (semi-annually), which is the standard for U.S. Treasury securities. We assume that the face value of the coupon-paying bonds is a nominal dollar (for simplicity).

If a complete coupon structure is available, then we can use (3) to back out implied zero-coupon discount factors directly without any additional assumptions. We detail the iterative version of the method first. Start with the one-period bond and write up its pricing equation

$$P^1_{c,t} = Z^1_t \left(1 + C^1\right), \tag{4}$$

where C is the *per period* coupon rate. In this case, the coupon bonds pay out semi-annually, so the per period coupon rate is half the annualized rate. Solving for the unknown discount factor

$$Z^1_t = \frac{P^1_{c,t}}{1 + C^1}. \tag{5}$$

Illustrating the approach, we can input the numbers to find a value for Z^1_t, i.e.,

$$Z^1_t = \frac{1.0055}{1 + 0.00625} = 0.99925, \tag{6}$$

where $0.0625 = \frac{0.0125}{2}$ is the per period coupon rate. Having determined Z^1_t, we can continue with the two-period bond and set up the corresponding pricing equation to determine the two-period discount factor Z^2_t

$$P^2_{c,t} = Z^1_t C^2 + Z^2_t \left(1 + C^2\right). \tag{7}$$

Solving for Z_t^2 yields the equation

$$Z_t^2 = \frac{P_{c,t}^2 - Z_t^1 C^2}{1 + C^2}, \tag{8}$$

which when inserting the numbers yields $Z_t^2 = 0.99648$ (I invite you to verify this claim). We can continue this iterative procedure for every maturity n to obtain the full zero-coupon discount curve. In particular, continuing the iteration produces the general formula

$$Z_t^n = \frac{P_{c,t}^n - C^n \sum_{i=1}^{n-1} Z_t^i}{1 + C^n} \tag{9}$$

where $P_{c,t}^n$ denotes the today price of the coupon bond, C^n is the per period coupon rate for the bond with n periods to maturity, and $\sum_{i=1}^{n-1} Z_t^i$ is an annuity using the already extracted discount factors. This iterative procedure is known as bootstrapping (or yield curve stripping). While simple to implement, this approach can be somewhat cumbersome in practice if faced with many maturities.

2.2. Bootstrapping in matrix form

A less tedious line of attack is to view (9) as a system of linear equation that can be solved using matrix algebra. To see this more clearly, let $\mathbf{P}$ and $\mathbf{Z}$ denote the $N \times 1$ and $n \times 1$ vectors of today-available (time t) coupon-bond prices and unknown discount factors, respectively. In particular, let

$$\mathbf{P} = \begin{bmatrix} P_{c,t}^1 \\ P_{c,t}^2 \\ \vdots \\ P_{c,t}^n \end{bmatrix} \tag{10}$$

and let

$$\mathbf{Z} = \begin{bmatrix} Z_t^1 \\ Z_t^2 \\ \vdots \\ Z_t^n \end{bmatrix}, \tag{11}$$

where N denotes the number of bonds available and n denotes the number of different (unique) maturities, which we view as representing future payment dates. Moreover, let $\mathbf{C}$ denote an $N \times n$ matrix of payoffs. To illustrate the structure of the payoff matrix, let CF_i^j denote the payoff (cash flow) of bond $j = 1, \ldots, N$ in period $i = 1, \ldots, n$ such that the payoff matrix $\mathbf{C}$

becomes

$$\mathbf{C} = \begin{bmatrix} CF_1^1 & CF_2^1 & \cdots & CF_n^1 \\ CF_1^2 & CF_2^2 & \cdots & CF_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ CF_1^N & CF_2^N & \cdots & CF_n^N \end{bmatrix}, \tag{12}$$

where CF_i^j consists of the per period coupon rate C prior to maturity and $1 + C$ (per-period coupon plus face value) at maturity. In other words, the matrix $\mathbf{C}$ contains the cash flows of coupon-bonds with face value equal to a nominal dollar. Bonds are down the rows and cash flow streams are across the columns, implying that this matrix is a lower triangular matrix in which all elements above the diagonal are zero. Essentially, CF_i^j elements that are situated above the main diagonal are zero as the bonds have matured.

Having defined the payoff matrix, we turn to an expression for bond prices. In particular, bond prices are defined by the following system of stacked pricing equations

$$\mathbf{P} = \mathbf{C} \times \mathbf{Z}, \tag{13}$$

where $\mathbf{Z}$ is an unknown column vector of zero-coupon discount factors to be determined. To isolate $\mathbf{Z}$, we need to invert the payoff matrix $\mathbf{C}$, which is only possible if $N = n$ so that $\mathbf{C}$ is a square matrix. A unique solution for $\mathbf{Z}$ in (13) will not exist otherwise. For a square and invertible matrix, the unique solution for $\mathbf{Z}$ is found as

$$\mathbf{Z} = \mathbf{C}^{-1} \times \mathbf{P}. \tag{14}$$

Figure 1 illustrates the discount function based on bootstrapping the zero-coupon yield curve from the bond data in Table 1. We note that the discount function looks innocuous enough in the sense that it is reasonably smooth and possess the important property that the curve is declining with maturity. Specifically, we have that the condition

$$1 \geq Z_t^{n_1} \geq Z_t^{n_2} \geq 0, \quad \text{for} \quad n_1 < n_2 \tag{15}$$

is satisfied. It is important to note that the condition in (15) rules out the existence of negative forward rates. However, even when $\mathbf{C}$ is square (i.e., $N = n$) and the discount curve looks reasonable, the interpolation embedded in bootstrapping may provide for undesired shapes of the zero-coupon yield and, especially, the forward rate curves. Put differently, we may encounter undesired non-smoothness properties, something that is clearly visible in Figure 2 for the forward rate series that display a noticeable "kink" around the 3.5-year maturity. We end this section by noting that this non-smoothness comes about from the bootstrap methodology forcing us to extract a zero-coupon term structure that exactly prices the coupon bonds.

Figure 1: Bootstrapped zero-coupon discount curve

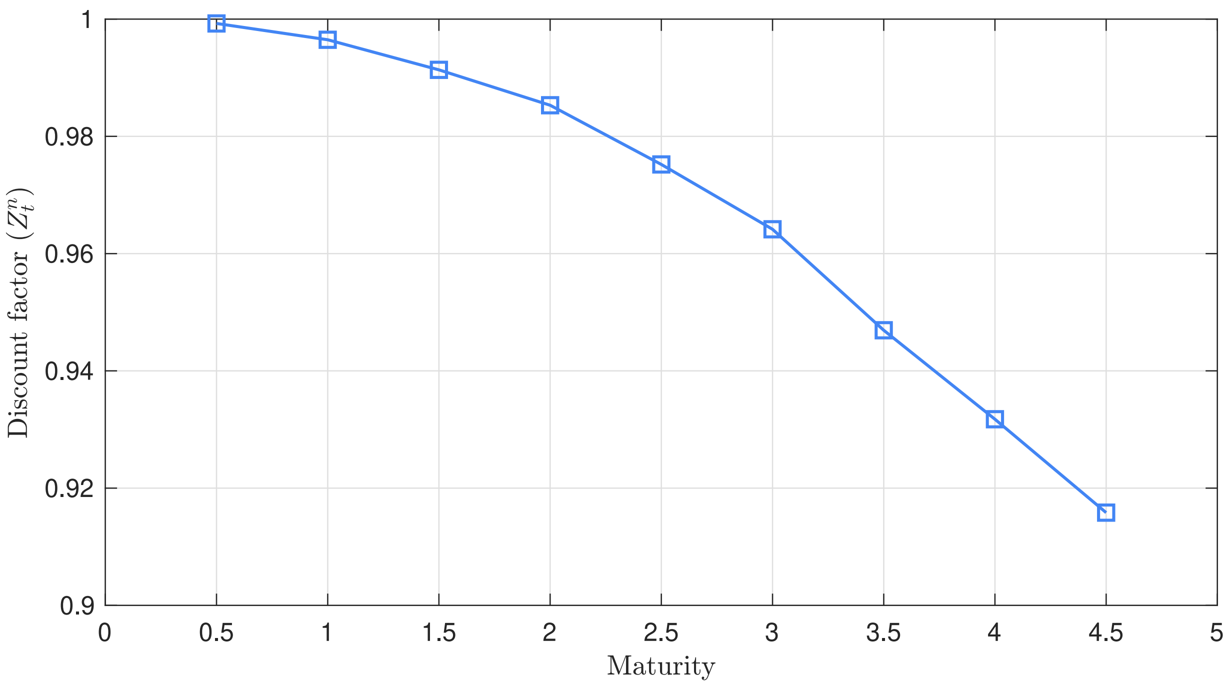
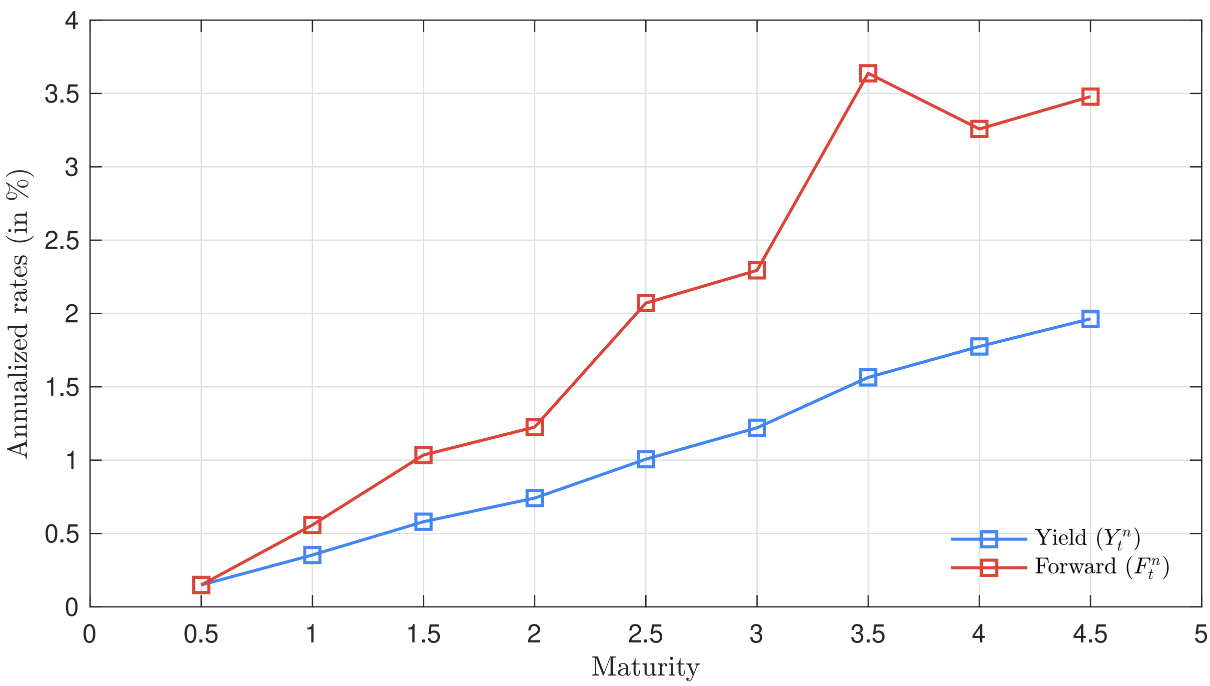


Figure 2: Annualized yields and forward rates (Bootstrap)



2.3. Drawbacks of the bootstrap

Although the bootstrap methodology is useful in many applications, it has the limiting drawback of requiring the number of bonds to equal the number of maturities (i.e., $n = N$) and for payments dates and maturities to be equally and regularly spaced in the time domain. In practice, however, one is often faced with the issue of either having access (i) to a more-than-complete coupon term structure (having observations on multiple bonds with the same maturity $(n < N)$) or (ii) to an incomplete coupon term structure (wanting a discount curve for more maturities than allowed by the number of observable bonds $(n > N)$). We deal with former situation first.

3. Cross-sectional regressions

The cross-sectional regression methodology is designed to handle the situation where the coupon term structure is more-than-complete (Carleton and Cooper, 1976). While not a situation often encountered, it is most commonly found when considering shorter maturities, say up to five years or so. While one could easily revert to a situation in which the bootstrap methodology would be feasible by discarding bonds, this is not a recommended approach. The reason being, of course, that we may potentially be throwing away valuable information. A better approach is to use all available information, but adapt the methodology accordingly.

3.1. Estimating the zero-coupon term structure using regression

To illustrate the cross-sectional regression methodology, suppose that market information on two additional traded coupon bonds becomes available to us. These bonds are highlighted with gray shading in Table 2. The remaining bonds are identical to those in Table 1 from Section 2. In this case, the payoff matrix $\mathbf{C}$ from (12) has $N = 11$ columns, N being the number of bonds,

Table 2: Observable bond data

Coupon (Annualized)	Yield (Annualized)	Maturity (years)	Maturity (periods)	Price per face value
1.250%	0.149%	0.5	1	1.0055
4.875%	0.351%	1.0	2	1.0451
4.500%	0.574%	1.5	3	1.0586
4.250%	0.461%	1.5	3	1.0566
4.750%	0.731%	2.0	4	1.0797
3.375%	0.992%	2.5	5	1.0587
3.500%	1.199%	3.0	6	1.0676
3.850%	1.383%	3.0	6	1.0723
2.000%	1.543%	3.5	7	1.0155
2.250%	1.746%	4.0	8	1.0194
2.125%	1.931%	4.5	9	1.0083

and $n = 9 < N = 11$ rows, where n is the number of unique maturities. Thus, the matrix $\mathbf{C}$ is no longer square and, therefore, non-invertible.

A solution to this issue is to place the estimation of $\mathbf{Z}$ in a linear cross-sectional regression framework. It is a cross-sectional regression because the model is estimated using multiple maturities on a given day. To see how this solves the problem, start by adding an $N \times 1$ vector of pricing errors to the pricing equation in (13)

$$\mathbf{P} = \mathbf{C} \times \mathbf{Z} + \varepsilon. \tag{16}$$

That is, we allow for small discrepancies (pricing errors) between model prices and observable market prices, something that is not permitted in the bootstrap methodology. While this implies that we will no longer price all bonds exactly (which would be the case in the bootstrap method), the benefit is that we are likely to obtain *smoother* curves. The expression in (16) reminds us of a linear regression model and, indeed, the important implication to realize is that we can treat $\mathbf{Z}$ as a constant parameter vector to be estimated using Ordinary Least Squares (OLS). We know that the OLS estimator in matrix form is given by the closed-form solution

$$\mathbf{Z} = \left(\mathbf{C}^{\top}\mathbf{C}\right)^{-1}\mathbf{C}^{\top} \times \mathbf{P}, \tag{17}$$

where it is important to emphasize that the "trick" here is that we need the now square $\mathbf{C}^{\top}\mathbf{C}$ matrix to be invertible rather than the $\mathbf{C}$ matrix itself. This comes about by allowing for small bond mispricings by way of the vector of pricing errors and placing the estimation of discount factors within a regression framework.

Figure 3: Regression-based zero-coupon discount curve

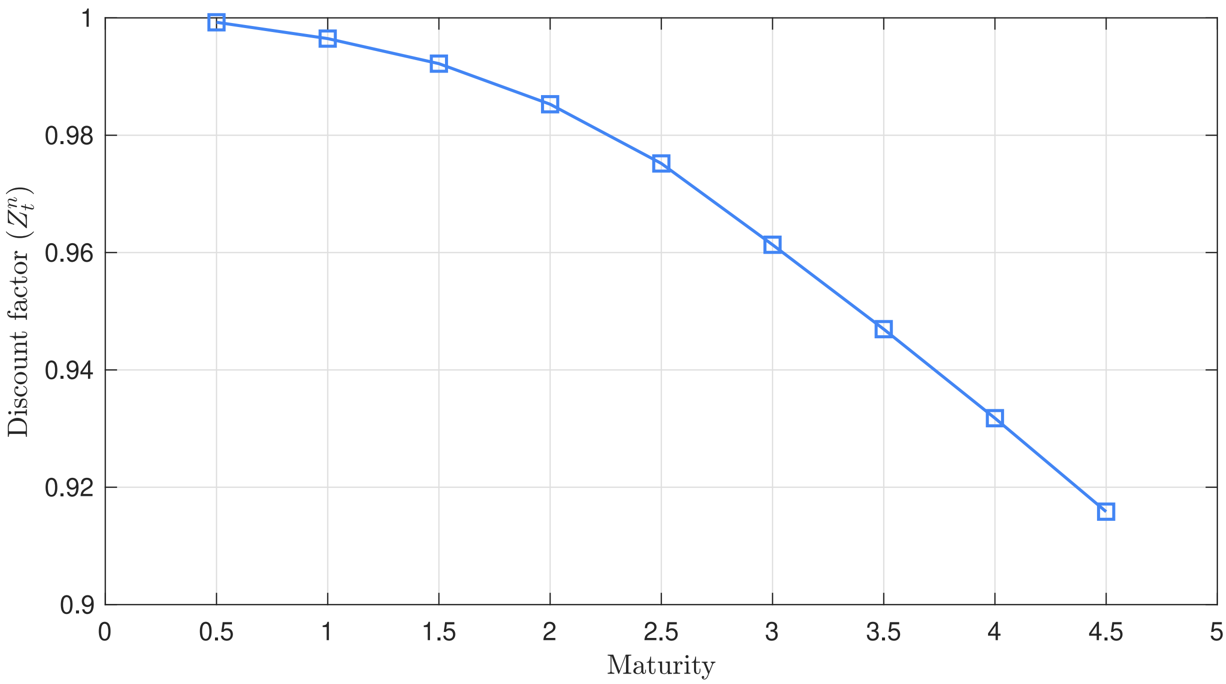


Figure 4: Annualized yields and forward rates (Regression)

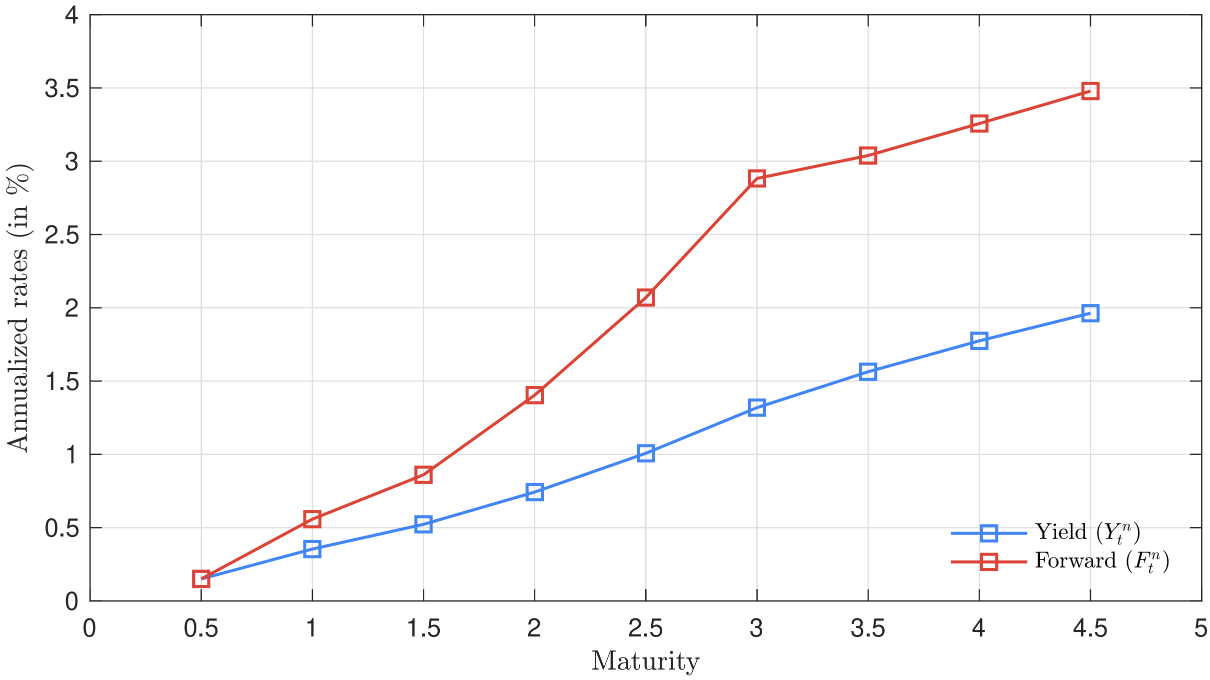


Figure 3 and Figure 4 depict the discount function and the corresponding spot and forward rates obtained from the regression methodology, respectively. Similar to the bootstrap methodology, we see that the discount function and spot rate curve looks somewhat innocuous, whereas the forward rate curve is still subject to a small "kink" (i.e., abrupt slope change) around the 3-year maturity. The forward rate curve, however, is much less ragged compared to its bootstrap counterpart in Figure 2, which is the reward obtained from accepting small pricing errors in the estimation of the zero-coupon term structure (and having more bonds). As such, allowing for small pricing errors can be beneficial as they smooth out kinks and produce more reliable curves overall when the data allows it. As a final note, I emphasize that this is not always the case of course, but it is a potential issue embedded in methods that force us to price the selection of bonds exactly.

Summing up the status so far To sum up the discussion so far, then it is generally true that the shortcomings of a curve fitting technique are least noticeable in the discount function, more noticeable in the yield curve, and particularly visible in the forward rate curve. Smooth curves are difficult to achieve with any technique that forces the discount function to price many bonds exactly such as bootstrapping. Allowing for small deviations in model-implied prices from market prices achieves smoothness from this trade-off, which is illustrated by the cross-sectional regression method. As an alternative method that is capable of producing smooth and flexible yield curves, we will next consider parametric models that allow for pricing errors and assume particular functional forms for the zero-coupon curve that works well even when the coupon term structure is incomplete.

4. The Nelson-Siegel model

This section covers the parametric Nelson and Siegel (1987) continuously compounded zero-coupon yield curve model that works well even when the term structure is incomplete. The model is often applied by Central Banks around the world and has been used to generate a long history of high-frequency yield curve estimates (Gürkaynak, Sack and Wright, 2007). The model originates from a series of concerns regarding prior modeling approaches. While McCulloch (1971, 1975) sensibly suggests to let the prices of discount bonds vary smoothly with maturity using polynomial splines, this is not without issues. First, discount rates may diverge as maturity increases instead of going to zero as required by theory. Forward rates may similarly diverge instead of converging to a fixed long-run value as intuition would require. That is, extrapolated long-term rates may be unbounded and become infinite, which is a serious concern (Shea, 1984). Second, there is no simple way to ensure that forward rates remain positive and that the discount function always declines with maturity. Vasicek and Fong (1982) suggest using exponential splines to remedy the issues by ensuring that forward rates and zero-coupon yields converge to a long-run value as maturity approaches infinity. However, as pointed out by Shea (1985), it is not clear that exponential splines perform better empirically.

4.1. The functional form

Nelson and Siegel (1987) address these issues by assuming a functional form for the forward rate curve.[1] For example, they let the instantaneous log forward rate $f_t^n(0) = \lim_{m\to 0} f_t^n = \lim_{m\to 0} p_t^{n-m} - p_t^n = -\partial p_t/\partial n$ be expressed as

$$f_t^n(0) = \theta_0 + \theta_1 e^{-\frac{n}{\lambda}} + \theta_2 (n/\lambda) e^{-\frac{n}{\lambda}}, \tag{18}$$

which can be viewed as a constant plus a so-called Laguerre function. A Laguerre function is a polynomial times an exponential decay term and is a popular mathematical approximation function. Alternatively, (18) can similar be obtained as the solution to a second-order differential equation with real and equal roots. Integrating (18) from 0 to n and dividing by n results in the following expression for the per period log zero-coupon yield

$$y_t^n = \theta_0 + \theta_1 \frac{1-e^{-\frac{n}{\lambda}}}{\frac{n}{\lambda}} + \theta_2 \left[\frac{1-e^{-\frac{n}{\lambda}}}{\frac{n}{\lambda}} - e^{-\frac{n}{\lambda}}\right], \tag{19}$$

which can generate a broad range of yield curve shapes, including upward-sloping, downward-sloping (inverted), and hump-shaped curves of many varieties.[2] Equipped with log zero-coupon

[1] See also Siegel and Nelson (1988) for further work on estimating the zero-coupon term structure.

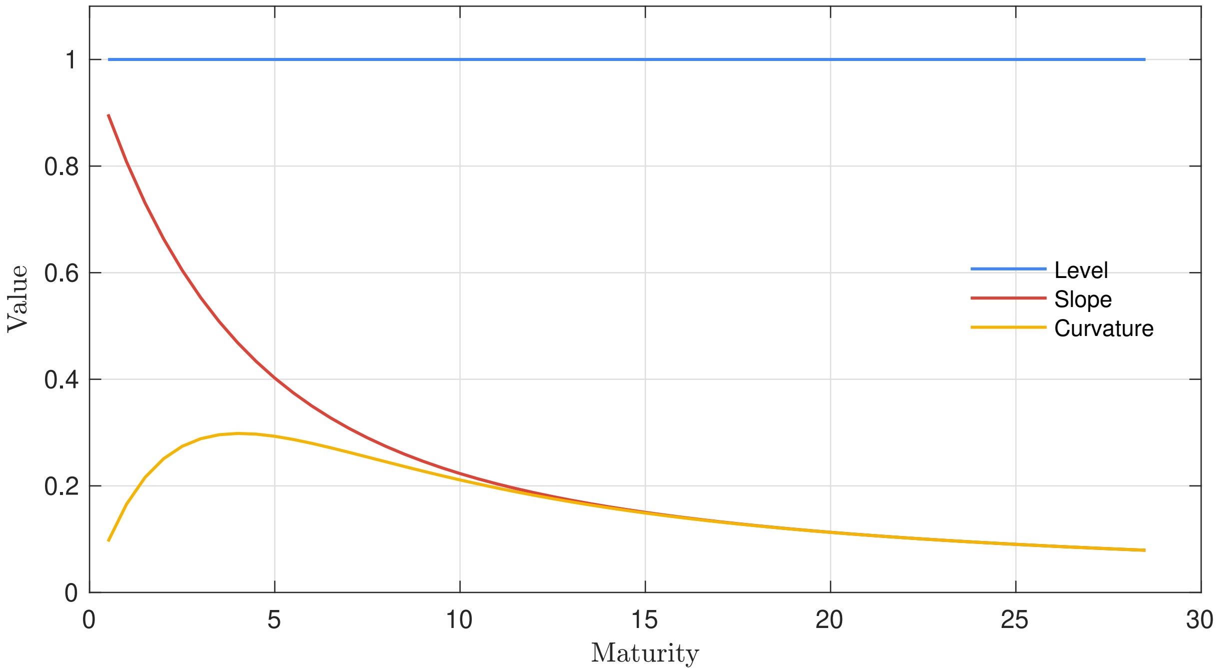
[2] The relation between instantaneous (a very small time period) forward rates and log yields here, i.e., $y_t^n = \frac{1}{n}\int_0^n f_t^x dx$, is equivalent to the discrete counterpart $y_t^n = \frac{1}{n}\sum_{i=0}^{n-1} f_t^{n-i}$.

yields, we can determine the zero-coupon discount factors using the standard relation

$$Z_t^n = e^{-n y_t^n}, \tag{20}$$

which can then be used to compute prices of fixed income securities for any future maturity n.

Figure 5: Nelson-Siegel loadings



The Nelson and Siegel (1987) model has the very attractive feature that the involved parameters can be given an economic interpretation. This is important as it allows the model-user to make informed assessments of the particular shape of the fitted curve. Diebold and Li (2006), among others, show what we can think of θ_0, θ_1, and θ_2 as being linked to level shifts, slope shifts, and curvature changes in the yield curve as illustrated by Figure 5.[3] The level factor is constant across maturities. The slope factor $\frac{1-e^{-\frac{n}{\lambda}}}{\frac{n}{\lambda}}$ is largest for short maturities and declines exponentially toward zero as maturity increases. Beginning at zero for short maturities, the curvature factor $\left[\frac{1-e^{-\frac{n}{\lambda}}}{\frac{n}{\lambda}} - e^{-\frac{n}{\lambda}}\right]$ reaches a maximum at the middle of the maturity spectrum and then decreases to zero for longer maturities. Thus, the parameters θ_0, θ_1, and θ_2 measures the loadings on these factors for determining the shape of of the yield curve. In particular,

- θ_0 is interpreted as the long-run level of interest rates as $n \to \infty$
- θ_1 governs the short-term (hump) component that starts at 1 and decays monotonically and quickly to 0
- θ_2 governs the medium-term (hump) component that starts at 0, increases, then decays to zero

[3] If we fix λ a priori and treat the then known functions attached to the θs as loadings.

- λ is a scale parameter that measures the rate at which the short-term and medium-term components decay to zero and governs when the curvature factor reaches its maximum

Understanding the economic implications of the estimated parameters is important as it allows the model-user to judge whether the obtained estimates are reasonable or not, and whether the estimates align with an initial inspection of yields.

4.2. Estimating the Nelson-Siegel model

Table 3 provides coupon, yield, maturity, and price data for a selection of coupon bonds that we will use to illustrate the estimation and mechanisms of the parametric yield curve models. We

Table 3: Observable bond data

Coupon (Annualized)	Yield (Annualized)	Maturity (years)	Maturity (periods)	Price per face value
5.500%	5.00%	0.5	1	1.0024
5.875%	4.95%	1.0	2	1.0089
5.625%	4.88%	1.5	3	1.0107
5.250%	4.89%	2.5	5	1.0084
5.875%	4.98%	3.5	7	1.0284
6.500%	5.10%	5.5	11	1.0664
5.500%	5.15%	7.0	14	1.0204
6.500%	5.25%	9.0	18	1.0887
8.250%	5.48%	14.0	28	1.2683
6.125%	5.65%	28.5	57	1.0669

have observations on a total of ten coupon-paying bonds spanning a maturity range from one to 57 periods (28.5 years). Importantly, we note that the maturities are not evenly spaced in time and that the maturity range far exceeds the number of available bonds, i.e., the coupon term structure is incomplete. Thus, the situation is ill-suited (infeasible, in fact) for the bootstrap and the cross-sectional regression methodologies discussed above.

The Nelson and Siegel (1987) model can be estimated using non-linear least squares (NLS), something that is relatively straightforward to implement for this model in Matlab (and other statistical softwares).[4] In the following, we will collect the parameters of the model in the parameter vector $\boldsymbol{\beta} = (\theta_0,\theta_1,\theta_2,\lambda)$. The estimation procedure for the Nelson-Siegel model can then be summarized by the following steps:

1. Choose a set of sensible starting values for $\boldsymbol{\beta} = (\theta_0,\theta_1,\theta_2,\lambda)$.

2. Compute model-implied yields and discount factors using (19) and (20).

[4] See the implementation in the accompanying Matlab script for the details.

3. Compute model-implied bond prices using (13), i.e.,

$$\mathbf{P}^{\text{NS}} = \mathbf{C} \times \mathbf{Z}^{\text{NS}} \tag{21}$$

where we add the superscript NS to emphasize the dependence on the model.

4. Estimate model parameters using NLS by minimizing the weighted sum of squared errors between the model-implied prices and market prices

$$\boldsymbol{\beta}^{\star} = \arg\min_{\boldsymbol{\beta}} \sum_{i=1}^{N} \left[\omega_i \left(P_c^{i,\text{Market}} - P_c^{i,\text{NS}}\right)\right]^2 \tag{22}$$

using, as an example, Matlab's *fminsearch*, *fminunc*, or *fmincon* functions, by varying the parameter vector $\boldsymbol{\beta}$ while, potentially, imposing sensible restrictions such as $\theta_0 \geq 0$ (long-term rates should be positive). We will rely on the *fminsearch* function.

There are several choices for the weight ω_i. If we wish to pay equal attention to all bonds, we can set $\omega_i = 1$. Following Gürkaynak et al. (2007), one can also set the weight equal to the inverse of the (Macaulay) duration, i.e., $\omega_i = \left(D_{c,t}^{n}\right)^{-1}$. This may improve fit because weighting prices by inverse durations, to a first approximation, converts the pricing errors into yield fitting errors. We will consider the $\omega_i = 1$ case in this note (and accompanying Matlab live script) and entertain the inverse duration weighting scheme in an upcoming exercise.

With a complete estimated discount curve in hand, we can subsequently use it to price any fixed income security with known (and certain) payments at any points in time. In fact, having estimated the parameters of the model on a given day, we can compute discount factors for *any* maturity imaginable. This is what ultimately separates the Nelson and Siegel (1987) model from bootstrapping and regression, which are only able to estimate discount factors for coupon payment dates on available bonds.

Figure 6 illustrates the discount curve and Figure 7 illustrates the resulting log spot and forward rate curves. We emphasize that the resulting discount curves and yield and forward rate curves are smooth and flexible. This is exactly what we want to any method designed to extract the zero-coupon term structure. The smoothness originates from allowing for small pricing errors as we do not require bonds to be priced exactly. The flexibility comes from the functional form assumed for the instantaneous forward rates. The Nelson and Siegel (1987) model, while flexible, can only accommodate one "hump" in the forward rate curve. At times, the curve is so non-linear that a single "hump" is not enough. For instance, the yield curve often needs two humps: one at short maturities associated with monetary policy expectations and another at long maturities to capture convexity effects (Gürkaynak et al., 2007). We next turn to an extension that allows for two "humps" in the forward rate curve.

Figure 6: Nelson-Siegel-based zero-coupon discount curve

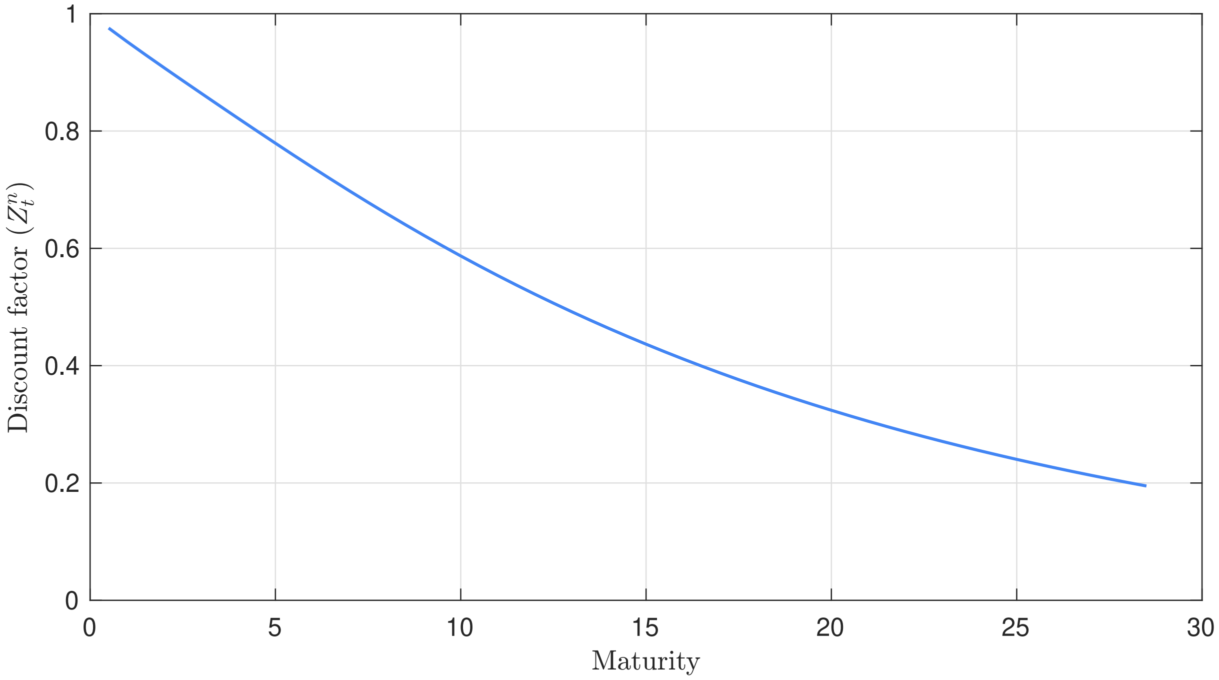
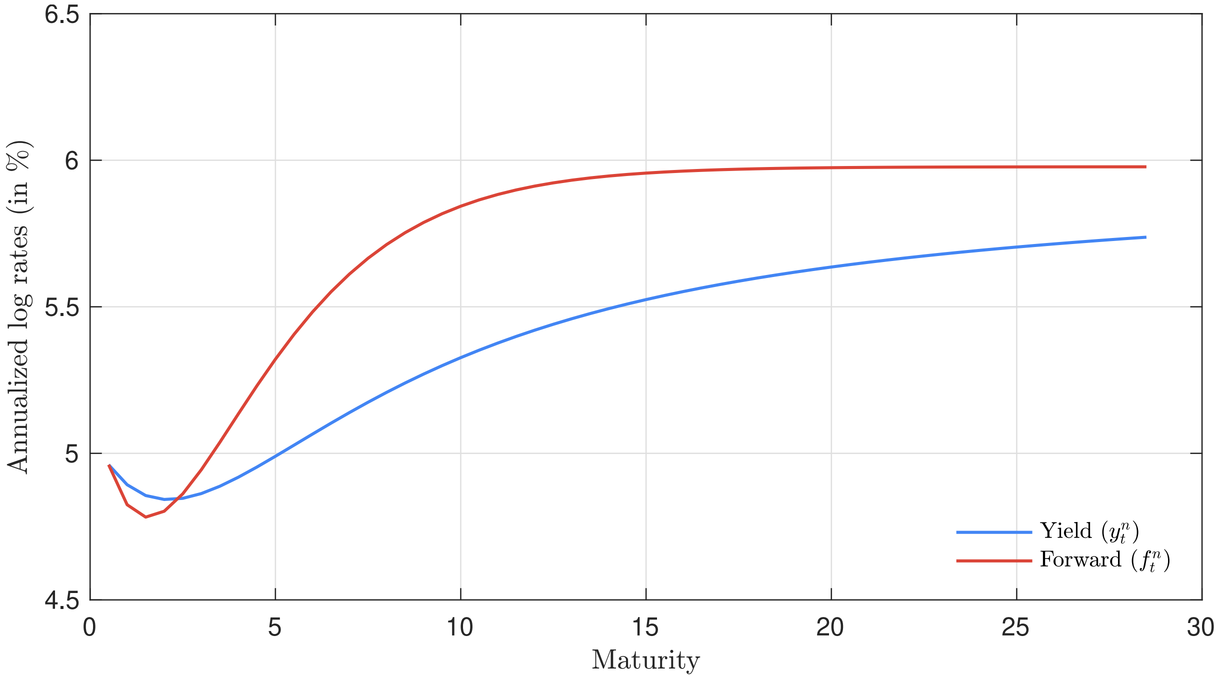


Figure 7: Annualized log yields and forward rates (Nelson-Siegel)



5. The Nelson-Siegel-Svensson model

Although the Nelson and Siegel (1987) model usually works quite well, it lacks the flexibility to match term structures that are *highly* non-linear and/or contains multiple "humps". Svensson (1994) extends the Nelson-Siegel model by adding an additional term to the instantaneous forward rate

$$f_t^n(0) = \theta_0 + \theta_1 e^{-\frac{n}{\lambda_1}} + \theta_2 (n/\lambda_1)\, e^{-\frac{n}{\lambda_1}} + \theta_3 (n/\lambda_2)\, e^{-\frac{n}{\lambda_2}}, \tag{23}$$

which results in the following extended version of the log yield expression

$$y_t^n = \theta_0 + \theta_1 \frac{1 - e^{-\frac{n}{\lambda_1}}}{\frac{n}{\lambda_1}} + \theta_2 \left[\frac{1 - e^{-\frac{n}{\lambda_1}}}{\frac{n}{\lambda_1}} - e^{-\frac{n}{\lambda_1}} \right] + \theta_3 \left[\frac{1 - e^{-\frac{n}{\lambda_2}}}{\frac{n}{\lambda_2}} - e^{-\frac{n}{\lambda_2}} \right], \tag{24}$$

where the new parameters θ_3 and λ_2 are interpreted similar to the parameters θ_2 and λ above. We will refer to this as the Nelson–Siegel–Svensson (NSS) functional form. With this function, instantaneous forward rates begin at horizon zero at the level $\theta_0 + \theta_1$ and eventually asymptote to the level θ_0. Two "humps" are allowed for in the forward curve, the locations of which are determined by the parameters λ_1 and λ_2. The estimation of the model parameters is identical to the Nelson-Siegel model, except that the parameter vector now consists of 6 variables, i.e., $\boldsymbol{\beta} = (\theta_0, \theta_1, \theta_2, \theta_3, \lambda_1, \lambda_2)$. As for the Nelson-Siegel model, we can either pay equal attention to all bonds or use the inverse of the (Macaulay) duration as the weight in the estimation procedure.

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