Utility theory*

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*This note provides an introduction to choice under uncertainty and expected utility theory. The note draws strong inspiration from the excellent textbook treatments provided in Ingersoll (1987), Huang and Litzenberger (1988), Gollier (2001), and Campbell (2017) as well the survey of Machina (1987). The note is prepared for use only in the Master's course "Asset Pricing". Please do not cite, circulate, or use for purposes other than this course.

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1. Introduction

The ambition of asset pricing theory is to characterize the equilibrium in financial markets, where individuals interact and trade assets with future uncertain payoffs. "Future" and "uncertain" are both essential concepts in asset pricing, but we will — for now — limit our discussion to uncertainty and place time in the background by assuming that all uncertainty is resolved after a single period. This allows us to study the standard theory of choice under uncertainty. Specifically, we will consider individuals' consumption and investment decisions under uncertainty, and their implications for the valuation of financial assets. The purchase of an asset today is seen as a vehicle for saving for future end-of-single-period consumption. The basic idea is that individuals trade financial assets, which differ in their future uncertain payoffs, to smooth consumption over future states of the world. A complete theory of asset demand must therefore specify investors' preferences over different uncertain payoffs. In a nutshell, we need a model of how investors choose between assets with different uncertain payoffs (i.e., with different probability distributions). These single-period developments are central to modern asset pricing, and they form the foundation for the mean-variance model (Markowitz, 1952) and the Capital Asset Pricing Model (CAPM) (Treynor, 1961, Sharpe, 1964, Lintner, 1965, Mossin, 1966).

The expected utility model of preferences over uncertain prospects is the financial economist's canonical model of choice under uncertainty (Machina, 1987). Under this model, individuals allocate their wealth among assets with different future uncertain payoffs by determining the probabilities of all possible payoffs of the asset, assigning a utility index to each possible outcome, and choosing the allocation that maximizes the expected value of the utility index. Consequently, the utility index combines two key components linearly: the preference ordering on the ex post payoffs and the probabilities of the payoffs. More formally, an individual's preferences have an expected utility representation if there exists a function $U(\cdot)$ such that a random payoff L is preferred to another random payoff L^* if and only if

$$\mathbb{E}\left[U(L)\right] \geq \mathbb{E}\left[U(L^*)\right], \tag{1}$$

where $\mathbb{E}[\cdot]$ is the expectation under the individual's probability belief and L is a lottery asset defined as $L = (x,y,\pi)$. In the lottery, the outcome x occurs with probability π and the outcome y occurs with probability $1-\pi$. As such, we can characterize the expected utility of the lottery asset as $\mathbb{E}\left[U(L)\right] = \pi U(x) + (1-\pi)\,\pi U(y)$. An expected utility function is a *cardinal* utility function. This is in contrast to the *ordinal* utility functions often found in standard microeconomic representations. An ordinal utility function $\Psi(\cdot)$ tells us that an individual is indifferent between outcomes x and y if $\Psi(x) = \Psi(y)$ and strictly prefers x over y if $\Psi(x) > \Psi(y)$. An important property of an ordinal utility function is that any strictly increasing function of $\Psi(\cdot)$ will respect the same preference ordering. That is, ordinal utility functions are invariant to monotonically increasing transformations. It defines indifference curves, but there is no way to label the curves so that they have meaningful values. A cardinal utility function $U(\cdot)$, on the other hand, is invariant to linearly increasing transformation, but not to nonlinear transformations. This implies that the preference ordering expressed by $U(\cdot)$ is the same as that displayed by $V(\cdot) = a + bU(\cdot)$ for any $b > 0$. In other words, cardinal utility has no natural units, but given a choice of units, the rate at which cardinal utility increases is meaningful. This facilitates an analysis based on the concept of marginal utility. Marginal utility, measured as the first derivative of the utility function $U'(\cdot)$, is a central component in expected utility theory and in the consumption-based asset pricing models (Rubinstein, 1976, Lucas, 1978, Breeden, 1979).

The objective of this lecture note is to explore the development of the expected utility theory and discuss its applications and implications. We first study the conditions that an individual's preferences must satisfy to be consistent with an expected utility function. We then continue with the development of the link between utility and risk aversion, where we will pay particular attention to how risk aversion can lead to risk premia for particular assets. Modeling investor choices with expected utility functions is a widely applied method in finance. However, empirical and experimental evidence has indicated that individuals may behave in ways inconsistent with the standard formulation of the expected utility model (Allais, 1953, Ellsberg, 1961, Kahneman and Tversky, 1979, Rabin, 2000).[1] We will illustrate and discuss examples of these violations throughout this note. The findings have motivated a search for improved models of investor preferences over uncertain prospects both within and outside the expected utility paradigm. Such models are, however, outside the scope of the course at present.

The rest of the note unfolds as follow. Section 2 discusses the St. Petersburg paradox, which has a central place in the development of utility theory. Section 3 develops the expected utility theorem and discusses its validity. Section 4 introduces the concept of risk aversion and discusses how to measure and quantify it. Section 5 develops the concepts of certainty equivalent and risk premia, and outlines their relation to risk aversion.

2. The St. Petersburg paradox

We take our point of departure by considering a frequently applied criterion for measuring the attractiveness of an asset with future uncertainty payoffs: the expected value. This is not without historic interest. For example, during the development of modern probability theory in the 17th century, mathematicians such as Blaise Pascal and Pierre de Fermat assumed that individuals gauged the attractiveness of a gamble using the expected value of the payoffs (Machina, 1987). Consider a gamble that offers a single random payoff $\widetilde{x}$ at a future date, where the N possible outcomes $(x_1, x_2, \ldots, x_N)$ occur with corresponding probabilities $(\pi_1, \pi_2, \ldots, \pi_N)$ with $\sum_{i=1}^{N} \pi_i = 1$ and $\pi_i \geq 0$. The expected payoff of the gamble is then $\mathbb{E}[\widetilde{x}] = \sum_{i=1}^{N} \pi_i x_i$.

[1] Barberis (2013), Barberis and Thaler (2013), and Hirshleifer (2015) provide recent surveys of this literature. For popular books on behavioral finance and cognitive biases in general, I can recommend Kahneman (2011), Thaler (2015), Lewis (2016), and Kahneman, Sibony and Sunstein (2021).

The fact that individuals are unlikely to value a gamble using expected values was cleverly illustrated by an example posed by Nicholas Bernoulli in 1713 in a letter to mathematician Pierre Raymond de Montmort. Today, this example is known as the *St. Petersburg paradox*, and it is fundamental to the development of utility theory. In a nutshell, Nicholas demonstrated that expected value is insufficient to capture individuals' valuations of gambles. The example goes as follows:

"*Peter tosses a coin and continues to do so until it lands "heads". He agrees to pay Paul two ducats if he gets heads on the very first throw, four ducats if he gets it on the second, eight if on the third, sixteen if on the fourth, and so on, so that on each additional throw the number of ducats he must pay is doubled.*"

If we view Paul's prize from the game as the uncertain payoff of a gamble, how much would Paul then be willing to pay to participate in the gamble under valuation using expected value? If the number of coin flips taken to first arrive at heads is i, then the probability of reaching the ith throw is $\pi_i = \left(\frac{1}{2}\right)^i = 2^{-i}$ and the corresponding payoff is $x_i = 2^i$ so that the expected payoff equals

$$\begin{aligned} \mathbb{E}\left[\widetilde{x}\right] &= \sum_{i=1}^{\infty} \pi_i x_i = \sum_{i=1}^{\infty} 2^{-i} 2^i \qquad (2) \\ &= \frac{1}{2}2 + \frac{1}{4}4 + \frac{1}{8}8 + \frac{1}{16}16 + \cdots \qquad (3) \\ &= \sum_{i=1}^{\infty} 1 = \infty. \qquad (4) \end{aligned}$$

The paradox becomes immediately clear: the expected value of the coin flipping game is infinite, but it seems intuitively clear that most (sane) individuals would only pay a moderate, and finite, amount to participate in the gamble.

Daniel Bernoulli, a cousin of Nicholas, offered a resolution to the St. Petersburg paradox in 1738 by effectively introducing the concept of expected utility.[2] Daniel's main insight was that an individual's utility (or happiness) from receiving a payoff could differ from the size of the payoff, and that people cared more about the expected utility of an asset's payoffs than the expected payoff itself. Essentially, Daniel hypothesized that individuals' possessed what we today know as a von Neumann-Morgenstern (VNM) expected utility function (von Neumann and Morgenstern, 1944) and that "*the utility resulting from any small increase in wealth will be inversely proportionate to the quantity of goods previously possessed*". The latter is referred to as the individual displaying diminishing marginal utility of wealth, and this is the key to solving the paradox because the utility of prizes will now increase at a rate slower than the decline in probabilities. Diminishing marginal utility refers to the phenomenon that each additional unit of gain leads to an increasingly smaller increase in overall utility. To see this, note that Daniel's resolution

[2] A similar solution was proposed independently by Swiss mathematician Gabriel Cramer in 1728.

implies that the value V of the gamble would be determined as the weighted average of the utility of each of the ex post outcomes

$$V \equiv \mathbb{E}\left[U\left(\widetilde{x}\right)\right] = \sum_{i=1}^{N} \pi_i U\left(x_i\right). \tag{5}$$

To illustrate that this approach can indeed provide a resolution to the paradox, consider a risk averse individual (we will make more clear what we mean by this below) with log utility $U\left(x\right) = \ln\left(x\right)$. The logarithmic function is an increasing and concave function that delivers risk aversion and diminishing marginal utility of wealth. In fact, this is the very utility function used by Daniel himself. Under log utility, the expected utility of the gamble is

$$\begin{aligned} \mathbb{U} &= \sum_{i=1}^{\infty} \pi_i U\left(x_i\right) = \sum_{i=1}^{\infty} 2^{-i} \ln\left(2^i\right) = \sum_{i=1}^{\infty} 2^{-i} i \ln\left(2\right) \\ &= \ln\left(2\right) \underbrace{\sum_{i=1}^{\infty} 2^{-i} i}_{=2} = 2\ln\left(2\right) \cong 1.3863, \end{aligned} \tag{6}$$

implying that a certain payment of $\exp\left(2\ln\left(2\right)\right) = 4$ ducats has the same expected utility as playing the St. Petersburg game. This is orders of magnitude lower than the infinite value implied by the valuation under expected value. We next turn to a more detailed development of the expected utility model and a complete axiomatic treatment of choice under uncertainty.

3. The expected utility model

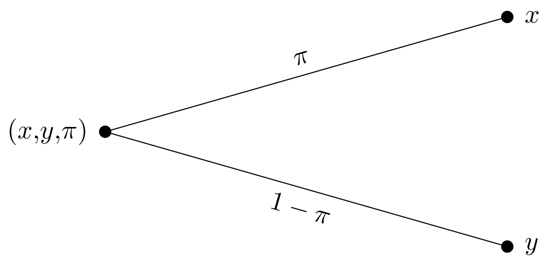
The first complete axiomatic treatment of expected utility is due to mathematicians John von Neumann and Oskar Morgenstern (von Neumann and Morgenstern, 1944), who provided a rigorous basis for individual decision making under uncertainty. Essentially, the pair worked out the conditions under which an individual's preferences could be described by an expected utility function. We will discuss their axioms in a simple context below using lottery assets.

3.1. Lotteries and preference orderings

Define a lottery L as an asset that has an uncertain future payoff. We will consider simple lotteries defined as $L = (x,y,\pi)$ in which the outcome x occurs with probability π and the outcome y occurs with probability $1 - \pi$. The outcomes can be interpreted as different consumption levels (e.g., bundles of consumption goods) enjoyed by the individual in different states of the world, or simply as different monetary gains (or losses).

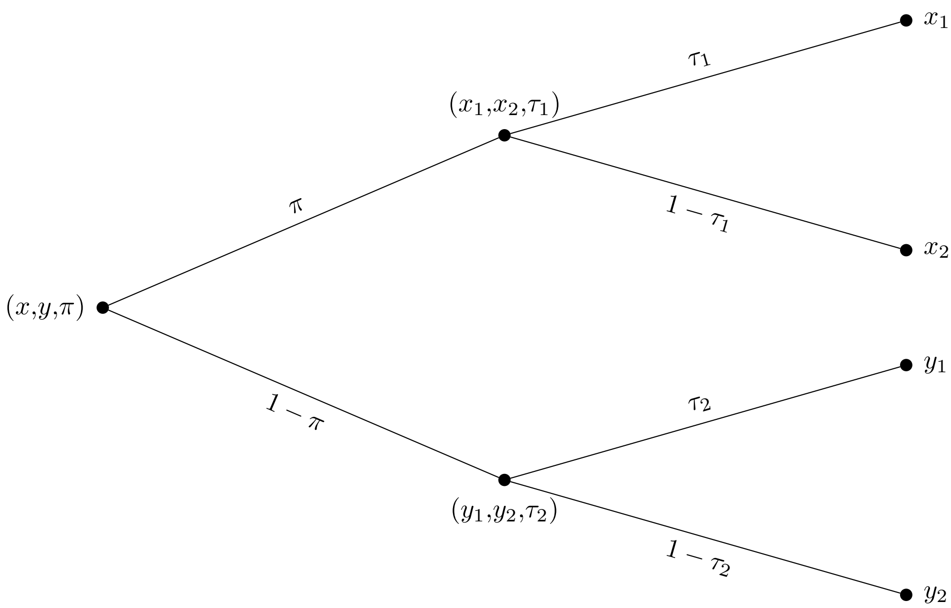
Figure 1 graphically illustrates a simple lottery with two possible outcomes. Although this simple structure may appear restrictive at first, it turns out that this definition of a lottery asset

Figure 1: A simple lottery



is highly flexible and encompasses a large variety of payoff structures. To appreciate this, note that the outcomes x and y may themselves be lotteries. In that case, we refer to the lottery as a compound lottery, and we define it as $L = (x,y,\pi) = ((x_1,x_2,\tau_1),(y_1,y_2,\tau_2),\pi)$. Figure 2 illustrates the structure of a compound lottery in which both x and y are themselves simple lotteries. For example, an individual may obtain lottery x with probability π in the first round, which gives the individual a payoff of x_1 with probability τ_1 or a payoff of x_2 with probability $1-\tau_1$. One can easily imagine how this can be extended to more elaborate structures.

Figure 2: A compound lottery



For simplicity, but without loss of generality, we will assume that different lotteries are characterized through different probabilities for the outcomes. For example, $L^* = (x,y,\pi^*)$ defines another lottery with a different set of probabilities π^* and $1-\pi^*$. We are interested in how a given individual chooses between different lotteries. Put differently, we are interested in determining the individual's preference ordering (ranking) of risky prospects under uncertainty.

Suppose that an individual is offered the choice between two lotteries: $L = (x,y,\pi)$ and $L^* = (x,y,\pi^*)$. To express the individual's choice under uncertainty, we introduce the notion of a preference relation represented by the symbol $\succeq$. For example, $L \succeq L^*$ means that the individual weakly prefers L over L^* (to be precise, that the individual strictly prefers or is indifferent between), $L \succ L^*$ means that the individual strictly prefers L over L^*, and $L \sim L^*$ means that the individual is indifferent between the two lotteries.

3.2. Axioms of expected utility

We are now fully equipped to introduce and appreciate the axiomatic treatment and development of expected utility theory found in von Neumann and Morgenstern (1944). The axioms of expected utility are specified in the box below. Individuals that behave and make decisions in accordance with the axioms are rational economic agents. A rational investor — sometimes referred to as *homo economicus* — has two key attributed: 1) their preferences are described by expected utility and 2) they assign objective probabilities that respect correct statistical procedures to the possible outcomes.

Axioms of expected utility

Let $L = (x,y,\pi)$, $L^* = (x,y,\pi^*)$, and $L^{**} = (x,y,\pi^{**})$ denote three different lotteries. The von Neumann and Morgenstern (1944) axioms of expected utility (and rational behavior) can then be stated as

1. **Completeness:** For any two lotteries L^* and L, we must have that either $L^* \succ L$, $L^* \prec L$, or $L^* \sim L$
2. **Transitivity:** If $L^{**} \succeq L^*$ and $L^* \succeq L$, then $L^{**} \succeq L$
3. **Continuity:** If $L^{**} \succeq L^* \succeq L$, there exists some $\lambda \in [0,1]$ such that $L^* \sim \lambda L^{**} + (1-\lambda)\, L$, where $\lambda L^{**} + (1-\lambda)\, L$ denotes a compound lottery
4. **Independence:** For any two lotteries L and L^*, $L^* \succ L$ if and only if for all $\lambda \in (0,1]$ and all L^{**} it holds that $\lambda L^* + (1-\lambda)\, L^{**} \succ \lambda L + (1-\lambda)\, L^{**}$
5. **Dominance:** Let $x \succ y$. Then $L^* \succ L$ if and only if $\pi^* > \pi$

We can give the axioms the following intuitive interpretations. Completeness is a restriction on the preference ordering. In particular, a preference ordering is said to be complete if the lotteries (and their outcomes) can be compared and ranked. Transitivity tells us that individuals make choices consistent with their rankings. Continuity looks complicated, but is intuitively simple. It says that if three lotteries are (weakly) ranked in order of preference, it is always possible to find a compound lottery that mixes the highest-ranked and lowest-ranked lotteries in such a way that the individual is indifferent between this compound lottery and the middle-ranked lottery. These three axioms together imply the existence of an ordinal utility function for choice under certainty. The remaining axioms are used to develop the concept of choice under uncertainty using a cardinal utility function. The independence axiom is critical to this development. The main implication of the independence axiom is as follows: if two lotteries are ranked in order of preference, then the same rank order applies to two compound lotteries that each combines one of the original two lotteries with an arbitrary third lottery using the same probabilities. To fully appreciate the implication and prediction, we first note that L^* is preferred to L by assumption. The choice between $\lambda L^* + (1-\lambda) L^{**}$ and $\lambda L + (1-\lambda) L^{**}$ is then equivalent to a coin toss game that offers L^{**} with probability $1-\lambda$ in both compound lotteries, and a simple lottery between L and L^* with probability λ. In a nutshell, the choice between the two compound lotteries $\lambda L^* + (1-\lambda) L^{**}$ and $\lambda L + (1-\lambda) L^{**}$ is equivalent to being asked ex ante to choose between L^* and L. Thus, the decision should remain unaffected by the presence of the irrelevant third lottery. The axioms further imply that only the utility – or preference orderings – of the final payoffs matter. The way the payoffs are obtained is irrelevant. The independence axiom is not without controversy, however, and much experimental evidence exists that directly contradicts its central predictions (see, e.g., Allais (1953), Ellsberg (1961), Kahneman and Tversky (1979), and Rabin (2000)). We will present selected evidence in the next section and discuss alternative theories of choice under uncertainty at the end of this note. The final axiom, dominance, states that if an individual prefers the outcome x over to the outcome y, then the individual will prefer the lottery offering the highest probability for that outcome.

We are now going to claim that the choice made by an individual facing two (or more) lotteries will be equivalent to the one with the highest expected utility if preferences are characterized by the above axioms of rational decision-making. That is, we will argue that the preference ordering $\succeq$ has an expected utility representation.

Expected utility theorem If the von Neumann-Morgenstern (VNM) axioms are satisfied, then there exists an expected utility function $\mathbb{E}\left[U\left(\cdot\right)\right] \equiv \mathbb{U}\left(\cdot\right)$ defined on the space of lotteries so that

$$L \succeq L^* \Leftrightarrow \mathbb{U}\left(L\right) \geq \mathbb{U}\left(L^*\right) \tag{7}$$

where $\mathbb{U}\left(L\right) = \pi U\left(x\right) + \left(1-\pi\right) U\left(y\right)$ is the VNM expected utility function and $U\left(x\right) = \mathbb{U}\left(x,y,1\right)$ the Bernoulli utility (of money) function. □

A few remarks about the von Neumann and Morgenstern (1944) expected utility function is in order. First, the function is linear in probabilities. Second, the expected utility function is unique up to positive linear transformations, i.e., $\mathbb{V}\left(\cdot\right) = a\mathbb{U}\left(\cdot\right) + b$ with $a > 0$ is also a VNM expected utility function with the same preference ordering. Put differently, preference orderings are preserved under linear transformations that leave the shape of the utility function intact. We can trace this back to the fact that the value of cardinal utility functions has a meaning (up to a scaling) beyond the ranking itself and that *changes* are meaningful for interpretations.

3.3. Behavioral and experimental contradictions

The von Neumann and Morgenstern (1944) expected utility theory is the workhorse of modern asset pricing. Nevertheless, its use is not without controversy and criticism. For example, a large literature provides strong experimental evidence of systematic violations of the independence axiom. Machina (1987) is a classic source for an early summary of the literature. Barberis and Thaler (2013) and Hirshleifer (2015) provide recent updates on the behavioral finance literature. This section illustrates two well know violation: the Allais paradox and framing, and briefly outlines a recent calibration critique of the expected utility framework.

3.3.1. *The Allais paradox*

Allais (1953) provides experimental evidence on a simple choice problem that produces an inconsistency between observed choices and the predictions of expected utility theory. In a nutshell, the Allais paradox is a simple selection of lotteries that, when offered to individuals, leads *most* to violate the independence axiom. Consider a simplified version in which a given individual is offered the following choice

1. $L^1 = (10000, 0, 1)$ versus $L^2 = (15000, 0, 0.9)$

2. $L^3 = (10000, 0, 0.1)$ versus $L^4 = (15000, 0, 0.09)$

The ranking most often observed is one in which the individual chooses $L^1 \succ L^2$ and $L^4 \succ L^3$. The reasons for such choices can be many, but the bottom line is that these choices are problematic for the independence axiom. To see why, note that we can use the structures of compound lotteries to demonstrate that

$$L^3 = \left(L^1, L^0, 0.1\right) \quad \text{and} \quad L^4 = \left(L^2, L^0, 0.1\right), \tag{8}$$

where $L^0 = (0, 0, 1)$ is a newly introduced irrelevant lottery that pays zero with certainty. By the nature of the independence axiom, if the individual chooses $L^1 \succ L^2$, then he should also choose $L^3 \succ L^4$ for his preferences to be consistent with expected utility. You will be asked in an exercise to demonstrate this more formally using the structure of compound lotteries.

An individual that falls victim to the Allais paradox can respond in one of two ways. First, they may realize their mistake, revise their decision making, and learn to avoid the mistake in the future. That is, people can easily be misled once, but are then educated. Second, they can maintain their choice and point to flaws in the expected utility model. For example, one could point out that the axiomatic treatment found in von Neumann and Morgenstern (1944) leaves no role for things like the pleasure of gambling or the notion of regret. The choice of L^4 over L^3 may well be driven by gambling preferences. The choice of L^1 over L^2, conversely, is likely driven by regret considerations. In particular, losing L^2 would lead to substantial regret of not having chosen the sure thing in L^1. Regret is hurtful, and agents therefore choose the sure thing to avoid any anticipated regret (Kahneman and Tversky, 1982, Tversky and Kahneman, 1992). In any case, the second response would mean abandoning the independence axiom.

3.3.2. Framing

Framing is the observation that individuals display a tendency to act inconsistently depending on how the question is framed. Consider the classic example from Tversky and Kahneman (1981) in which a deadly disease is expected to kill 600 individuals. Participants in the study were asked to choose between two alternative treatments. Importantly, one group were presented with a "positive" framing, whereas the other group was presented with a "negative" framing. The treatments and their consequences are sketched below.

Framing	Treatment A	Treatment B
Positive	200 will be saved	$\frac{1}{3}$ probability that 600 people will be saved, and $\frac{2}{3}$ probability that no people will be saved
Negative	400 will die	$\frac{1}{3}$ probability that no one will die, and $\frac{2}{3}$ probability that 600 will die

Individuals presented with the positive framing overwhelmingly choose Treatment A (72%), whereas the majority chose Treatment B in the negative framing (78%). The treatments are clearly identical across the two methods of framing, but individuals choose very differently. In particular, individuals appear to be risk averse in the positive framing in that the majority prefer to save 200 for sure rather than the risky prospect (with equal expected value). Conversely, individuals appear to be risk loving in that the majority opt for the risky prospect rather than the certain death of 400. Importantly, this reversal of preferences happens solely due to the framing of the question. Obviously, this has far reaching implications for optimal policy design. For example, one does not have to think hard to see how scrupulous advertisers, lobbyist, and the like can use this to their advantage.

We can make a similar observation in a simple setting using monetary lotteries. Consider an example taken from Kahneman and Tversky (1979) in which individuals are presented with the following lottery choices

1. In addition to whatever you own, you have been given $1,000, and are asked to choose between
$$L^A = (1000, 0, 0.5) \quad \text{and} \quad L^B = (500, 0, 1) \tag{9}$$

2. In addition to whatever you own, you have been given $2,000, and are asked to choose between
$$L^C = (-1000, 0, 0.5) \quad \text{and} \quad L^D = (-500, 0, 1) \tag{10}$$

In this example, the lottery is framed over gains relative to the initial payment in the first set and over losses relative to the initial payment in the second set of lotteries. We will refer to the initial payment as a reference level below, and note that the reference level differs in the two cases so that the final monetary payoffs happen with equal probability in both cases. This is a central element in the structure of the gamble as expected utility theory explicitly tells us that individuals care only about the final monetary payoffs, not how they are obtained. When individuals are presented with the above lotteries, the majority makes the following choice: $L^B \succ L^A$ and $L^C \succ L^D$. This is inconsistent with expected utility as the pair L^B and L^D and the pair L^A and L^C are equivalent in terms of monetary payments once the initial payment – the reference level – is taken into account. Apparently, framing matters greatly, and we see that individuals are risk averse over gains and risk loving over losses — with respect to the initial payment that serves as a reference level. We discuss a model of choice that can explain such behavior in Section 6.

3.3.3. Rabin's calibration paradox

Rabin (2000) criticizes expected utility on the ground that it cannot explain observed aversion to small gambles without simultaneously implying ridiculous aversion to large gambles. In a nutshell, he argues that plausible risk aversion at small stakes implies implausible risk aversion at large stakes. Put differently, expected utility theory cannot explain, with plausible levels of aversion to large risks, the degree to which people avoid small gambles. The perhaps most well known statement of Rabin's critique is as follows. Suppose that, for any initial wealth, an individual rejects a gamble in which she loses \$100 or gains \$110 with probability one-half. Then she will turn down all 50/50 bets of losing \$1000 or gaining *any* sum of money. This seems absurd.

4. Risk aversion

We will assume the existence of a cardinal utility function and ask what it means to say that an individual is risk averse. As discussed in the previous section, Daniel Bernoulli proposed that utility functions should display diminishing marginal utility, i.e., that $U(Y)$ should be an increasing but concave function of wealth Y. He recognized that this concavity implies that an individual will be risk averse. We will build our notion of risk aversion around the simple observation that individuals want to avoid risk and that individuals have a preference for smoothing their consumption stream across states of nature. Put differently, individuals prefer a similar consumption level irregardless of the state of nature that materializes. An individual can therefore be said to be risk averse if that individual is unwilling to accept a "fair" lottery $\widetilde{h}$, where a fair, or pure risk, lottery is defined as one that has an expected value of zero. For example, suppose that we consider a pure risk gamble with random payoff

$$\widetilde{h} = \begin{cases} +h & \text{with probability } \frac{1}{2} \\ -h & \text{with probability } \frac{1}{2}. \end{cases} \tag{11}$$

Would a VMN expected utility maximizer whose current wealth equals Y accept the lottery if offered? That is, would such an individual place a positive valuation on the pure risk gamble? The expected utility of accepting the gamble is equal to $\mathbb{E}\left[U\left(Y+\widetilde{h}\right)\right]$. The expected utility of rejecting the lottery is equal to $\mathbb{E}\left[U\left(Y\right)\right]=U\left(Y\right)$, which is interpreted as the "sure thing". A rejection of the pure risk gamble must therefore imply that

$$U\left(Y\right)>\mathbb{E}\left[U\left(Y+\widetilde{h}\right)\right]=\frac{1}{2}U\left(Y+h\right)+\frac{1}{2}U\left(Y-h\right), \tag{12}$$

which states that the individual prefers the sure thing over the risky gamble per the expected utility theorem. For example, we can view the sure thing as a degenerate lottery that pays Y with certainty, i.e., $L=(Y,0,1)$, and the pure risk gamble as the lottery $L^{*}=\left(Y+h,Y-h,\frac{1}{2}\right)$. The above rejection implies that the individual $L\succ L^{*}$ since $\mathbb{E}\left[U\left(L\right)\right]>\mathbb{E}\left[U\left(L^{*}\right)\right]$. We are now going to argue that the rejection of the pure risk gamble is equivalent to having a concave utility function.

1. We begin by arguing that the rejection implies a concave utility function. Note that $U\left(Y\right)=U\left(Y+\frac{1}{2}h-\frac{1}{2}h\right)$ since the gamble is fair. We can then write (12) as

$$U\left(Y+\tfrac{1}{2}h-\tfrac{1}{2}h\right)>\frac{1}{2}U\left(Y+h\right)+\frac{1}{2}U\left(Y-h\right). \tag{13}$$

   This turns out to be useful because this is the very definition of a concave function. A utility function is said to be concave if a line joining to the two points $U\left(Y+h\right)$ and $U\left(Y-h\right)$ lies below $U\left(Y\right)$ for all wealth levels Y.

2. We then argue that concavity implies an unwillingness to accept a fair gamble using the famous Jensen's Inequality, which is derived by Danish mathematician and telephone engineer Johan Jensen. If $U\left(Y\right)$ is a concave function, then Jensen's Inequality for concave functions immediately provides us with the result

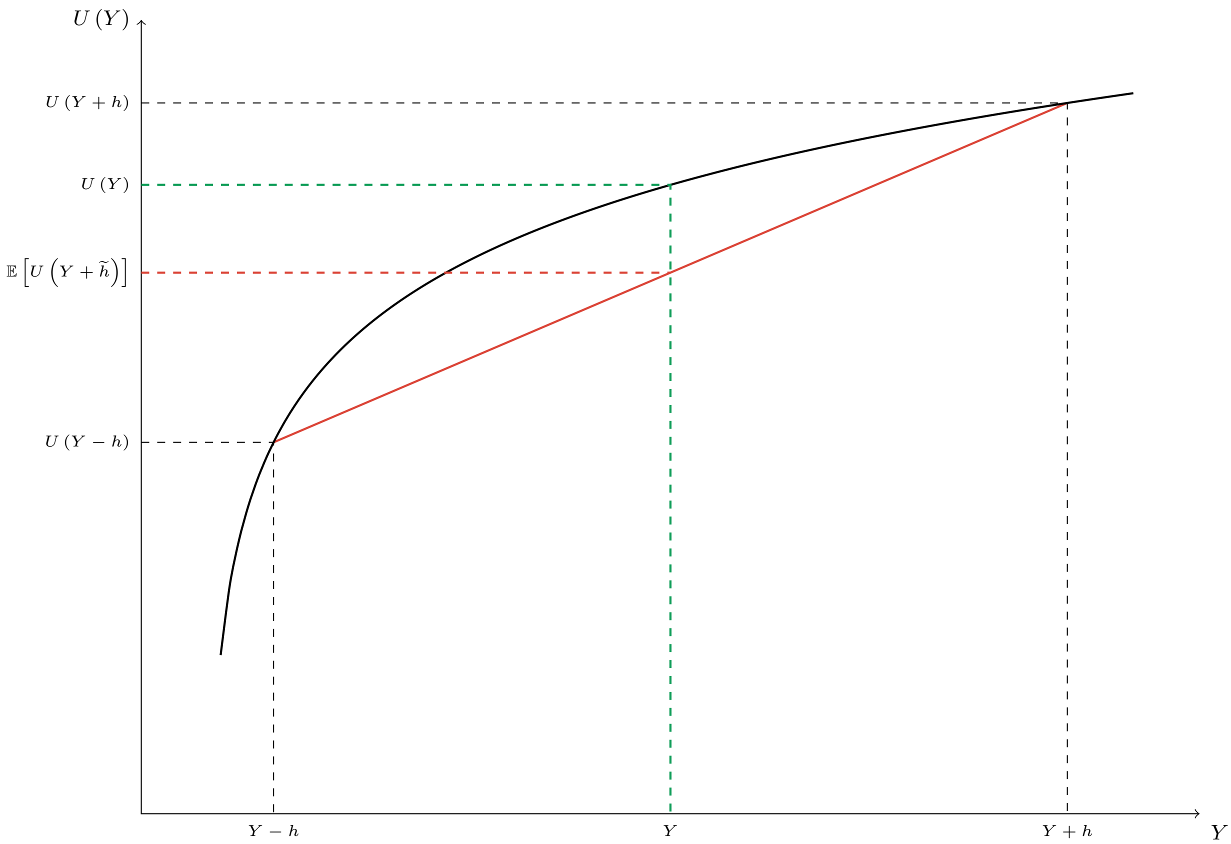
$$U\left(Y\right)>\mathbb{E}\left[U\left(Y+\widetilde{h}\right)\right], \tag{14}$$

   implying that an individual with a concave utility function will reject a pure risk gamble.

The two arguments highlight that (i) risk aversion implies that the individual must have a concave utility function and (ii) that having a concave utility function implies an unwillingness to accept a fair gamble. The two concepts are therefore intimately linked.

Figure 3 illustrates the rejection of the fair gamble, and the imbedded preference for smoothness, graphically. The solid black line depicts the concave utility function, the dashed green line indicates the utility of the sure thing, and the solid red line provides the linear points between winning and losing the gamble. The point at which it intersects with current wealth defines the expected utility of accepting the gamble as indicated by the dashed red line. The rejection is then immediately visible. The central element to understand this lies in the diminishing marginal utility of wealth. In a nutshell, the increase in utility from winning the lottery is much smaller than the corresponding decline in utility from losing the lottery.

Figure 3: Illustrating a preference for smoothness



4.1. Risk aversion coefficients

The rejection of the pure risk gamble implies that the utility function of the individual is concave. Concavity is equivalent to having non-increasing marginal utility, i.e., $U'(Y_1) \leq U'(Y_2)$ for $Y_1 \geq Y_2$, so that marginal utility is always weakly decreasing in wealth. This is referred to as diminishing marginal value of wealth. We will further assume that $U'(Y) > 0$, which implies that the individual is non-satiated in wealth and always prefer more to less.

The concavity of a risk averse agent's utility function restricts the sign of the second derivative of the utility function. In particular, concavity implies that $U''(Y) < 0$. The second derivative is a measure of the curvature of the function. From Figure 3, it is clearly evident that a larger curvature is equivalent to higher risk aversion (you can verify this yourself). A natural question to ask is therefore whether $U''(Y)$ is sufficient to quantify risk aversion. It turns out, however, that while $U''(Y)$ is necessary and sufficient to indicate whether the individual is risk averse, it is insufficient to quantify risk aversion. The following example illustrates why. Suppose that we attempt to measure risk aversion using the absolute value of $U''(Y)$ so that Investor A is more risk averse than Investor B for all wealth levels Y if and only if $|U''_A(Y)| \geq |U''_B(Y)|$. To see why this is insufficient, recall that VNM expected utility is cardinal and invariant to positive linear transformations, i.e., $\overline{U}_A(Y) = a + bU_A(Y)$ for $b > 0$ is also a VNM expected utility function that respect the same preference ordering. As such, $\overline{U}_A(Y)$ and $U_A(Y)$ must provide identical rankings, preferences, and levels of risk aversion. Yet, one can easily demonstrate that $\left|\overline{U}''_A(Y)\right| \geq |U''_A(Y)|$ for $b \geq 1$, and vice versa for $b < 1$. This is problematic as Investor A is now seemingly more risk averse than himself. This is nonsense, of course, and forcefully demonstrates that $U''(Y)$ is insufficient to quantify risk aversion. The example instead highlights that risk aversion measures need to be invariant to positive linear transformations.

Pratt (1964) and Arrow (1971) introduce two such widely used measures. First, the Pratt-Arrow coefficient of absolute risk aversion is defined as

$$R_A(Y) \equiv -\frac{U''(Y)}{U'(Y)}, \tag{15}$$

where the negative sign in front of the fraction reflects the observation that $U''(Y) < 0$ for a strictly risk averse individual. Note that the scaling of $U''(Y)$ with $U'(Y)$ avoids dependence on the units of measurement for utility, which leads to a risk aversion measure that is invariant to positive linear transformations. Note, however, that the function depends critically on the initial level of wealth Y.

Second, the Pratt-Arrow coefficient of relative risk aversion is defined as

$$R_R(Y) \equiv -Y \cdot \frac{U''(Y)}{U'(Y)}, \tag{16}$$

where we multiply with the initial level of wealth Y. In general, the higher values of $R_A(Y)$ and $R_R(Y)$, the more risk averse the individual is. By extension, we have that a risk-neutral investor would have $R_A(Y) = R_R(Y) = 0$. The intuition is straightforward: a risk-neutral individual have a linear utility function, e.g., $U(Y) = d + cY$, and so $U''(Y) = 0$.

The Pratt-Arrow coefficients of risk aversion have implications for the behavior of the individual. For example, depending on the shape of the utility function, the investor can exhibit either (i) increasing absolute risk aversion (IARA), (ii) constant absolute risk aversion (CARA), or (iii) decreasing absolute risk aversion (DARA) with respect to $R_A(Y)$. IARA is often viewed as unrealistic as it implies that individuals get increasingly risk averse the more wealth they accumulate. Imagine that the individual is faced with allocating her wealth between a risk-free asset and a risky asset. IARA would then imply that the individual *decreases* the money amount invested in the risky asset as wealth increases. CARA implies that risk aversion is independent of wealth so that the individual always invest the same absolute amount of wealth in the risky asset for any wealth level. Last, DARA implies that risk aversion is decreasing in wealth so that richer individuals also invest a larger amount of wealth in the risky asset. As such, DARA is viewed as the more realistic behavior for the average individual.

The same arguments can be made for the Pratt-Arrow coefficient of relative risk aversion. In that case, the investor can exhibit either (i) increasing relative risk aversion (IRRA), (ii) constant relative risk aversion (CRRA), or (iii) decreasing relative risk aversion (DRRA) with respect to $R_R(Y)$. While absolute risk aversion speaks about the amount of money that the individual allocates to the risky asset, relative risk aversion speaks about the fraction of wealth allocated to the risky assets. An IRRA investor will allocate a smaller fraction as wealth increases, a CRRA investor will always invest the same fraction of wealth in the risky asset, and a DRRA investor will invest a larger and larger fraction in the risky asset as wealth increases. We often view CRRA as a desirable description of individual behavior, and note that it implies DARA. This is important in applications of portfolio selection that we will address later in the course.

4.2. The odds of a bet

The Pratt-Arrow coefficients are useful in many applications in asset pricing. We begin by showing their usefulness in approximating the odds required to make a risk averse individual indifferent between the sure thing and a pure risk gamble. To be more precise, consider an investor with current wealth Y who is offered, at no charge, the lottery $L = (h, -h, \pi)$. We already know that a risk averse individual will reject the pure risk gamble for $\pi = \frac{1}{2}$. More generally, we can say that any investor will accept the lottery if π is high enough (and clearly if $\pi = 1$) and reject it if π is too small (especially if $\pi = 0$). Moreover, the willingness to accept the gamble may, in addition to the probability of winning the gamble π, also depend on the individual's current wealth. We have the following proposition.

Proposition Let $\pi = \pi(Y,h)$ be the probability at which the investor is indifferent between accepting and rejecting the lottery. Then

$$\pi(Y,h) \cong \frac{1}{2} + \frac{1}{4} h R_A(Y) \tag{17}$$

The proposition states the odds required are increasing in the Pratt-Arrow coefficient of absolute risk aversion $R_A(Y)$ and the size of the gamble h. This is intuitive. The more risk averse an individual is, the more favorable odds are required. Similarly, the higher the amount at risk, the more favorable odds are required. The proof of the proposition goes as follows. By definition, the indifference probability $\pi(Y,h)$ must be the solution to the equation

$$U(Y) = \pi(Y,h) U(Y+h) + [1 - \pi(Y,h)] U(Y-h). \tag{18}$$

Take a second-order Taylor expansion around $h = 0$ (a small bet) for winning and losing the bet, respectively, to obtain

$$U(Y+h) \cong U(Y) + hU'(Y) + \frac{h^2}{2}U''(Y) \tag{19}$$

$$U(Y-h) \cong U(Y) - hU'(Y) + \frac{h^2}{2}U''(Y). \tag{20}$$

Substituting the approximations into (18) gives us

$$\begin{aligned} U(Y) \cong\ & \pi(Y,h)\left\{U(Y) + hU'(Y) + \frac{h^2}{2}U''(Y)\right\} \\ & + [1-\pi(Y,h)]\left\{U(Y) - hU'(Y) + \frac{h^2}{2}U''(Y)\right\}. \end{aligned} \tag{21}$$

Collecting terms, and using that the probabilities sum to one, gives us

$$U(Y) \cong U(Y) + (2\pi(Y,h) - 1)[hU'(Y)] + \frac{h^2}{2}U''(Y). \tag{22}$$

Solving for $\pi(Y,h)$ gives us the desired result

$$\pi(Y,h) \cong \frac{1}{2} + \frac{1}{4}h\left[-\frac{U''(Y)}{U'(Y)}\right] \cong \frac{1}{2} + \frac{1}{4}hR_A(Y). \tag{23}$$

We note that the odds will vary as a function of initial wealth according to $R_A(Y)$. Put differently, the odds required can be increasing, decreasing, or constant over wealth depending on the choice of utility function. We note that risk-neutral individuals always require fair odds for the lottery as $R_A(Y) = 0$ for a linear utility function as discussed above.

We can obtain a similar result for the coefficient of relative risk aversion. Consider a lottery $L = (h, \text{-}h, \pi)$ in which the amount at risk is now a proportion θ of the investor's wealth, i.e., $h = \theta Y$. Following steps similar to above, we can show that

$$\pi(Y,\theta) \cong \frac{1}{2} + \frac{1}{4}\theta R_R(Y), \tag{24}$$

so that the odds are increasing in the coefficient of relative risk aversion and the proportion of wealth at stake.

5. Certainty equivalent and risk premia

We have defined and explored risk aversion in terms of an individual's utility function. This section considers how risk aversion can be quantified by introducing two fundamental concepts: certainty equivalent and risk premia. Consider some lottery $L = (Z_1, Z_2, \pi)$, which is not necessarily fair (i.e., may have some expected value different from zero). Intuition tells us that a risk averse individual is willing to pay less than the expected value of the lottery (i.e., from the

St. Petersburg paradox) since $U\left(Y+\mathbb{E}\left[\widetilde{Z}\right]\right)>\mathbb{E}\left[U\left(Y+\widetilde{Z}\right)\right]$. The fundamental question is then how much the investor is willing to pay (her certainty equivalent).

To determine the certainty equivalent, we first introduce the individual's risk premium for the gamble $\widetilde{Z}$. Let $\Pi\left(Y,\widetilde{Z}\right)$ denote the individual's risk premium, which we can interpret as the maximum insurance payment that an investor would pay to avoid the lottery risk. Pratt (1964) defines this risk premium as the quantity that satisfies

$$U\left(Y+\mathbb{E}\left[\widetilde{Z}\right]-\Pi\left(Y,\widetilde{Z}\right)\right)=\mathbb{E}\left[U\left(Y+\widetilde{Z}\right)\right], \tag{25}$$

where $\mathbb{E}\left[\widetilde{Z}\right]-\Pi\left(Y,\widetilde{Z}\right)=CE\left(Y,\widetilde{Z}\right)$ is then the investor's certainty equivalent. $CE\left(Y,\widetilde{Z}\right)$ is defined as the maximal amount of money that an individual is willing to pay to acquire the lottery. We can note, since utility is an increasing and concave function of wealth, that $\Pi\left(Y,\widetilde{Z}\right)$ must be positive so that $CE\left(Y,\widetilde{Z}\right)<\mathbb{E}\left[\widetilde{Z}\right]$. Note that for a fair pure risk gamble with $\mathbb{E}\left[\widetilde{Z}\right]=0$, this implies that $CE\left(Y,\widetilde{Z}\right)=-\Pi\left(Y,\widetilde{Z}\right)$ and that the individual would accept a reduction in wealth if the lottery could be avoided.

Figure 4 provides a graphical illustration for a gamble with $\mathbb{E}\left[\widetilde{Z}\right]>0$ and where $Z_2>Z_1>0$. The figure illustrates the graphical identification of the certainty equivalent as the point where the expected utility intersects with the utility function of the individual. At this point, the expected utility from accepting the gamble equals the utility for a sure, but lower, consumption level that reflects how much the individual is willing to forego to avoid the lottery. I will encourage you to construct a similar figure for a pure risk lottery where $\mathbb{E}\left[\widetilde{Z}\right]=0$ and $Z_2=-Z_1$. In that case, the fact that $CE\left(Y,\widetilde{Z}\right)=-\Pi\left(Y,\widetilde{Z}\right)$ should be visible in the figure.

5.1. Pratt's approximation for the risk premium

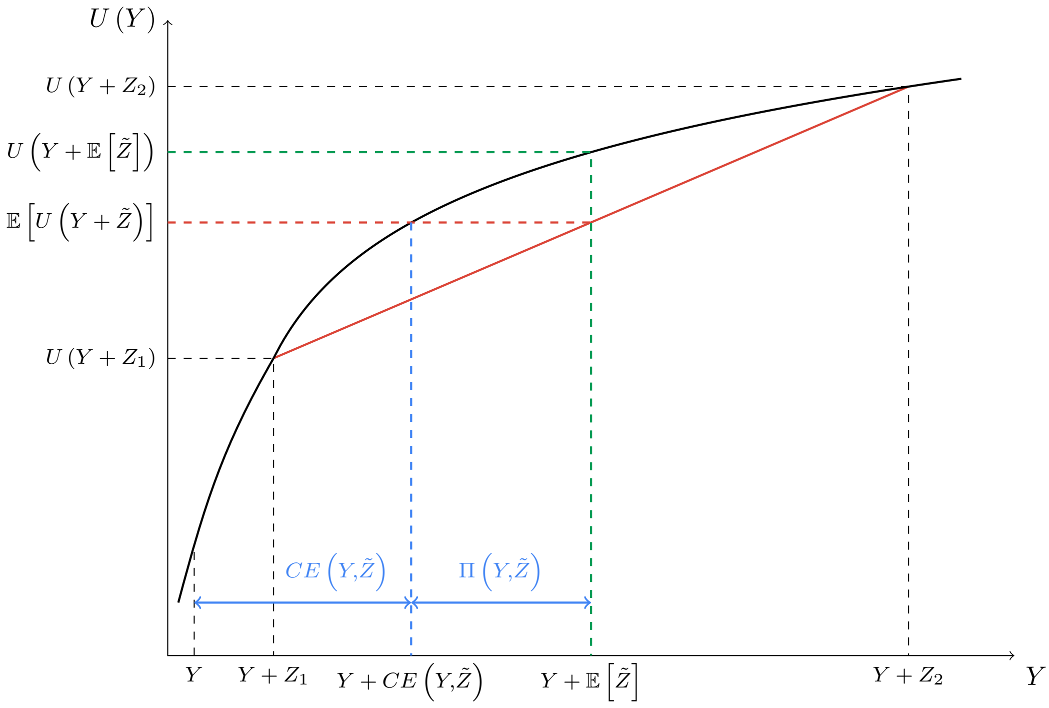
Consider a risk averse individual that is offered a fair lottery $\widetilde{Z}$. The Pratt (1964) risk premium is then defined by the expression

$$U\left(Y-\Pi\left(Y,\widetilde{Z}\right)\right)=\mathbb{E}\left[U\left(Y+\widetilde{Z}\right)\right], \tag{26}$$

since $E\left[\widetilde{Z}\right]=0$. We will further assume that $\widetilde{Z}$ is "small". The assumption of $\widetilde{Z}$ being "small" and fair ensures that we can apply a Taylor series expansion to study the risk premium around the point $\widetilde{Z}=0$ and $\Pi\left(Y,\widetilde{Z}\right)=0$. First, take a first-order Taylor expansion around the left-hand side of (26) around $\Pi\left(Y,\widetilde{Z}\right)=0$ to obtain

$$U\left(Y-\Pi\left(Y,\widetilde{Z}\right)\right)\cong U\left(Y\right)-\Pi\left(Y,\widetilde{Z}\right)U'\left(Y\right) \tag{27}$$

Figure 4: Certainty equivalent and risk premium



and, then, take a second-order Taylor expansion around $\widetilde{Z}=0$ to obtain

$$\begin{aligned}\mathbb{E}\left[U\left(Y+\widetilde{Z}\right)\right] &\cong \mathbb{E}\left[U(Y)+\widetilde{Z}U'(Y)+\frac{1}{2}\widetilde{Z}^2U''(Y)\right]\\ &= U(Y)+\frac{1}{2}\sigma_{\widetilde{Z}}^2U''(Y),\end{aligned} \tag{28}$$

where $\sigma_{\widetilde{Z}}^2=\mathbb{E}\left[\widetilde{Z}^2\right]$ is the lottery's variance. Equating (27) and (28) yields the Pratt (1964) risk premium approximation

$$\Pi\left(Y,\widetilde{Z}\right)\cong\frac{1}{2}\sigma_{\widetilde{Z}}^2\left[-\frac{U''(Y)}{U'(Y)}\right]=\frac{1}{2}\sigma_{\widetilde{Z}}^2R_A(Y), \tag{29}$$

where $R_A(Y)$ is the Arrow-Pratt measure of absolute risk aversion. We note that $\Pi\left(Y,\widetilde{Z}\right)$ is increasing in the uncertainty of the lottery $\sigma_{\widetilde{Z}}^2$ and in the investor's coefficient of absolute risk aversion. This seems intuitive.

The approximation in (29) also reveals that the concavity of the utility function, $U''(Y)$, is insufficient to quantify the risk premium an individual requires for holding the lottery asset (it is, however, still sufficient – and necessary – to indicate whether an individual is risk averse in the first place). The first derivative, $U'(Y)$, which measures marginal utility of wealth, is also needed to determine the risk premium. Intuitively, a given individual may well be very risk averse ($|U''(Y)|$ is large), but may simultaneously be unwilling to pay a large premium if he is poor and his marginal utility is high ($U'(Y)$ is large). This is equivalent to the discussion above that $U''(Y)$ is insufficient to measure the degree of risk aversion.

We can formalize this intuition using an example. Suppose that a given individual has a utility function described by a negative exponential, i.e.,

$$U(Y) = -e^{-aY}, \quad a > 0, \tag{30}$$

where a is a Pratt-Arrow coefficient of absolute risk aversion. To see this, note that $U'(Y) = ae^{-aY} > 0$ and $U''(Y) = -a^2e^{-aY} < 0$ so that $R_A(Y) = a$. We not that both $U'(Y)$ and $U''(Y)$ goes towards zero as wealth Y tends to infinity. At the limit, when $Y \to \infty$, the utility function becomes flat and concavity disappears. At first glance, this would imply that an extremely rich individual would be willing to pay very little for insurance against a random gamble $\widetilde{Z}$, and surely less than a very poor person with the same utility function. However, this is not the case here because the marginal utility of wealth is also very small and neutralizes the effect of smaller concavity. This leads to the utility function exhibiting constant absolute risk aversion, implying that an individual with this utility function would always pay the same insurance premium to avoid the random gamble $\widetilde{Z}$. This example therefore highlights the importance of the choice of utility function for specifying investor behavior in our models of asset prices.

5.2. Arrow's approximation for the risk premium

The definition of the risk premium employed by Pratt (1964) is related to the insurance literature because it can be interpreted as the payment that an individual is willing to make to insure against a particular risk. However, the field of finance is often accustomed to understand a risk premium as a rate of return. In this context, an asset's risk premium is defined as its expected rate of return in excess of the risk-free rate of return. Arrow (1971) builds on this alternative definition to derive an approximation of the risk premium that is equivalent to that of Pratt (1964). As above, consider a fair gamble in which $\widetilde{Z}$ is "small". Arrow (1971) asks the questions: by how much should the expected value (return) on the asset change to make the individual indifferent between the sure thing and the gamble? Put differently, by how much should we change the probability of winning? The factor needed to adjust probabilities will be denoted $(1+\theta)$, where θ is to be interpreted as a risk premium. As such, the premium θ that makes the individual indifferent between accepting and rejecting the lottery satisfies the relation

$$U(Y) = \frac{1}{2}(1+\theta)\,U(Y+Z) + \frac{1}{2}(1-\theta)\,U(Y-Z). \tag{31}$$

Next, take a second-order Taylor expansion around $Z = 0$ for the utility of winning and losing the lottery, respectively, to obtain

$$U(Y) \cong \frac{1}{2}(1+\theta)\left[U(Y) + ZU'(Y) + \frac{1}{2}Z^2U''(Y)\right] \tag{32}$$

$$+\frac{1}{2}(1-\theta)\left[U(Y) - ZU'(Y) + \frac{1}{2}Z^2U''(Y)\right] \tag{33}$$

and collect terms to obtain

$$U(Y) \cong U(Y) + \theta ZU'(Y) + \frac{1}{2}Z^2U''(Y). \tag{34}$$

Last, solve for the risk premium θ, which is the fraction of Z one is willing to forego to avoid the lottery,

$$\theta \cong \frac{1}{2}ZR_A(Y). \tag{35}$$

As before, we note that it is a function of the coefficient of absolute risk aversion and the size of the gamble. A few comments are in order. First, the Arrow (1971) risk premium θ is related to the indifference probability $\pi(Y,h) \cong \frac{1}{2} + \frac{1}{4}ZR_A(Y)$ from (24). In particular, $\pi(Y,h) = \frac{1}{2}(1+\theta) = \frac{1}{2}\left(1 + \frac{1}{2}ZR_A(Y)\right) = \frac{1}{2} + \frac{1}{4}ZR_A(Y)$. Second, the Arrow (1971) risk premium θ is defined in terms of a probability, whereas the Pratt (1964) risk premium, Π is defined in terms of a monetary outcome. We can therefore multiply θ with the monetary payment Z

$$Z\theta = \Pi\left(Y,\widetilde{Z}\right) \cong \frac{1}{2}Z^2R_A(Y), \tag{36}$$

which reveals that Pratt's and Arrow's approximations and measures of risk premia are indeed equivalent. As such, the measures – which were developed independently – arrive at the same conclusion and make use of the same risk aversion coefficient.

The results of this section therefore illustrates how risk aversion depends on the shape of an individual's utility function. Moreover, it demonstrates that a risk premium, equal to either the payment an individual would make to avoid a risk or the individual's required excess rate of return, is proportional to the individual's Pratt-Arrow coefficient of absolute risk aversion.

6. Prospect theory

An essential component of any asset pricing model is an assumption about investor preferences, i.e., an assumption about how investors choose between risky gambles. The expected utility paradigm is the standard workhorse of asset pricing, but its predictions are not without controversy as demonstrated in Section 3.3. In a nutshell, a large body of experimental work has demonstrated that individuals systematically violate the von Neumann and Morgenstern (1944) axioms when choosing between risky gambles. Consequently, numerous alternative theories of choice under uncertainty has been proposed in the literature. However, none is as developed and successful in capturing the experimental evidence on risk taking as prospect theory (Kahneman and Tversky, 1979, Tversky and Kahneman, 1992).[3] Barberis (2013) provides an excellent review of theory and its progress over the last thirty years.

The original version of prospect theory appears in Kahneman and Tversky (1979). Although this version contains all the fundamental insights, it did come with certain limitations. For example, it could only be applied to gambles with at most two nonzero outcomes. A modified version, known as cumulative prospect theory, is presented in Tversky and Kahneman (1992) that resolves these issues. This is the version most refer to when they say prospect theory, and we will do the same here.

Consider a risky gamble with ordered outcomes $\{x_1,x_2,\ldots,x_N\}$ with objective probabilities $\{\pi_1,\pi_2,\ldots,\pi_N\}$. Under the expected utility framework, an individual would value this gamble as

$$\sum_{i=1}^{N} \pi_i U\left(x_i\right), \tag{37}$$

where $U\left(\cdot\right)$ is an increasing and concave utility function. Under cumulative prospect theory, we have a similar valuation, but yet vastly different. In particular, an individual under prospect theory would value the gamble as

$$\sum_{i=1}^{N} \delta_i V\left(x_i\right), \tag{38}$$

where δ_i are so-called decision weights and $V\left(\cdot\right)$ is a value function with $V\left(0\right) = 0$. We will discuss these in more detail below. There are four central elements to prospect theory: (i) reference dependence, (ii) loss aversion, (iii) changing behavior relative to the reference point, and (iv) probability weighting through decision weights.

First, reference dependence is central to prospect theory, and it represents a material deviation from expected utility theory. Under expected utility, individuals derive utility from absolute levels of wealth. In prospect theory, conversely, individuals derive utility from gains and losses relative to some reference point (e.g., current wealth, expected wealth, or zero). Kahneman and Tversky (1979) illustrate this effect using the framing example above in which individuals ignored the absolute wealth levels and judged the gamble relative to the initial payment (the reference point). Moreover, they argue that our perceptual system works in a similar way in general: homo sapiens are more attuned to changes rather than absolute levels. Examples include changes to brightness, loudness, and temperatures. The value function models this by having $V\left(0\right) = 0$ at the reference point.

[3]Daniel Kahneman won the Nobel Prize in economic sciences in 2002, and this work was surely a contributing factor. Amos Tversky would likely have shared the price, but he sadly passed away in 1996.

The value function also captures "loss aversion", which is the idea that people are much more sensitive to losses than to gains of the same magnitude. That is, individuals feels losses much more acutely than they do gains. This is true even for small losses. This idea is embedded in the value function by making it steeper in the region of losses relative to the region of gains. As such, prospect theory is a particular shape of the utility function. Figure 5 plots an example of the value function. The x-axis represents the dollar gain or loss relative to the reference level, and the y-axis represents the value assigned to the different payoffs. Importantly, we notice that the value placed on a \$100 gain is smaller in absolute magnitude than a loss of \$100. This is loss aversion. Kahneman and Tversky (1979) infer loss aversion from the observation that most individuals turn down the gamble $L = (-100{,}110{,}0.5)$. As also pointed out by Rabin (2000), this is hard to reconcile within the expected utility model. Under loss aversion, however, this is more easily explained because the pain of losing \$100 far outweights the pleasure of winning \$110. That is, the gamble is unappealing for individuals displaying loss aversion.

Figure 5: The value function in prospect theory

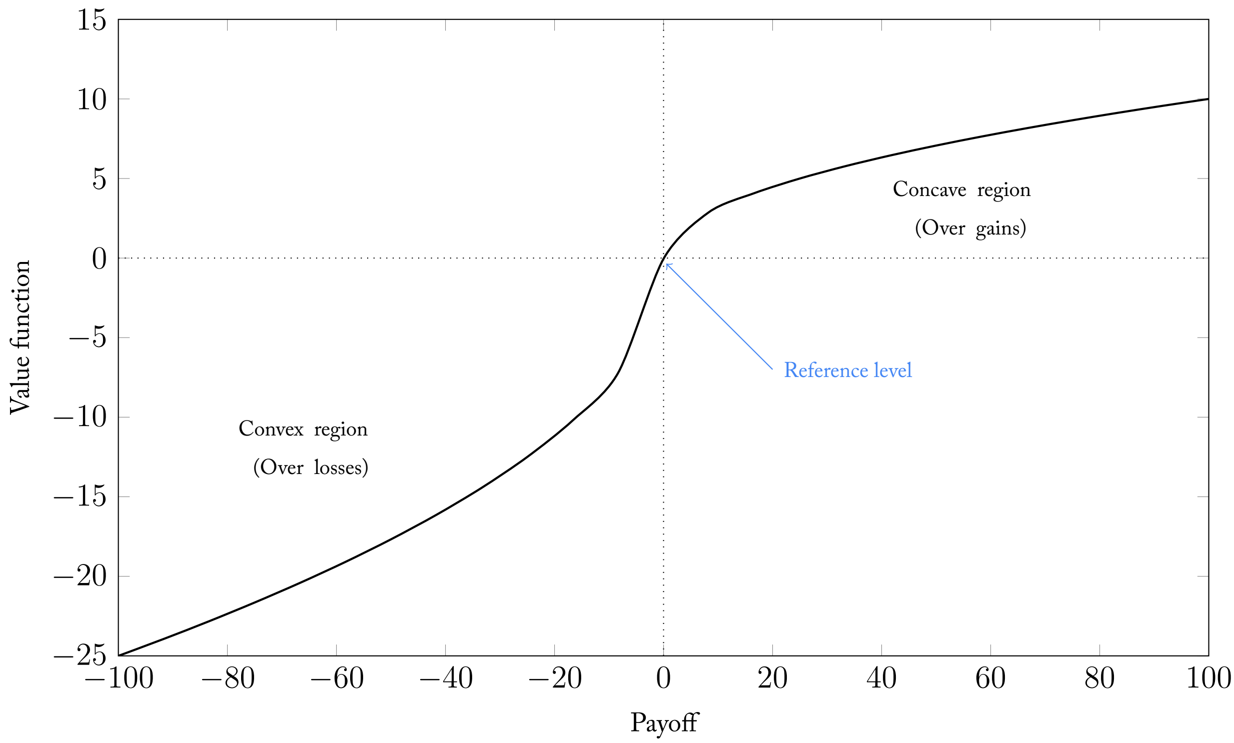
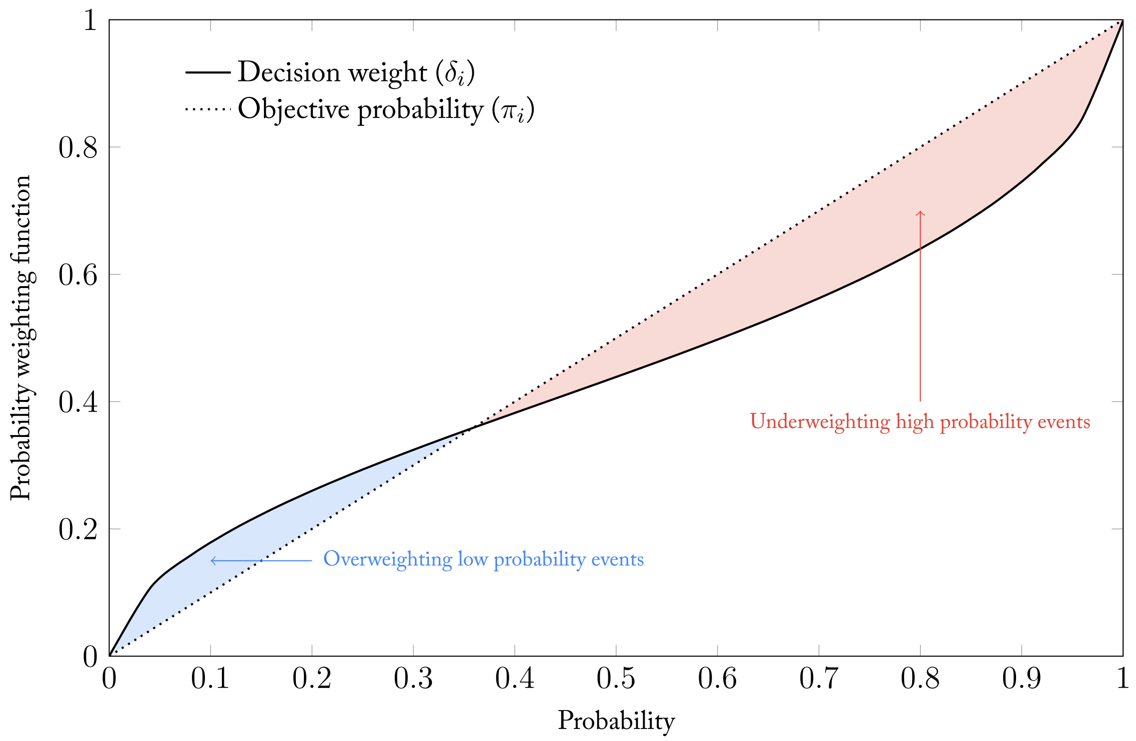


Figure 5 further illustrates that the value function is concave over gains, but convex over losses. This is a critical element of prospect theory and implies that individuals behave differently over gains and losses, respectively. In particular, the concavity over gains captures the observation that individuals tend to be risk averse over gains. This is consistent with the expected utility model in which individuals prefer the a certain gain of, say, \$500 over a 50/50 gamble of winning \$1,000 or nothing (as in the framing example). In contrast, individuals tend to be risk seeking over losses. They prefer a 50/50 chance of losing \$1,000 or nothing to losing \$500 for sure.

Clearly, this is also related to loss aversion, and motivates a convex shape over the domain of losses. This is inconsistent with the expected utility model, and gives prospect theory its unique flavor. In the expected utility model, individuals are globally risk averse because they judge risky prospect from absolute levels, not relative to some reference point.

Figure 6: The probability weighting function



The fourth and final component of prospect theory is the probability weighting. Importantly, in prospect theory, individuals do not weight outcomes by their objective probabilities π as in the expected utility model, but rather by transformed probabilities referred to as decision weights δ. The decision weights are computed using some weighting function $\delta = w(\cdot)$ that takes the objective probabilities as input. The specification of this weighting function is part of the practical implementation of the theory. Figure 6 provides an example of the decision weights (solid line) relative to the objective probabilities (dotted 45-degree line) taken from Tversky and Kahneman (1992). The comparison highlights that individuals overweights low probabilities and underweights high probabilities in prospect theory – relative to the benchmark objective probabilities from the expected utility framework. That is, the key feature is that individuals overweight the tails of any distribution and, consequently, overweight extreme outcomes that are unlike to occur. Kahneman and Tversky (1979) infer this observation from the odd fact that people enjoy both lotteries *and* insurance. Such behavior is difficult to square within expected utility. Importantly, and as emphasized by Tversky and Kahneman (1992), the transformed probabilities δ do not represent erroneous beliefs. Instead, they should be interpreted as decision weights. In prospect theory, individuals are perfectly capable of understanding objective probabilities, but simply apply different weights when judging risky gambles.

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