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Testing for skewness of regression disturbances

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The importance of testing for symmetry of regression disturbances is discussed and it is argued that it is useful to employ a test that it is robust to non-normality. A suitable large sample test is provided.

1. Introduction

The purpose of this paper is to propose a new procedure for testing the assumption that the disturbances of a linear regression model have a symmetric distribution. The importance of such a test is explained in section 2 and the derivation of the test statistic is outlined section 3. In contrast to a well known check for skewness, the procedure of section 3 is designed to be asymptotically robust to non-normality.

2. Skewness and the behaviour of diagnostic checks

Consider a regression model written as

$$Y_i = x_i'\beta + u_i,\ u_i \text{ iid}(0, \sigma^2), \tag{1}$$

in which the k regressors of x_i are at least weakly exogenous, $i = 1, \ldots, n$. Modern programs for the ordinary least squares (OLS) estimation of equations like (1) produce checks for a large number of specification errors, e.g. autocorrelation, omitted variables and heteroskedasticity. These checks are usually computed as separate test statistics, each of which is asymptotically distributed as a χ^2 variate under the null hypothesis that (1) is not misspecified.

For example, suppose that the assumption of serial independence is to be checked against the alternative that the disturbances u_i are generated by a stationary and invertible autocorrelation process (either autoregressive or moving average) of order p. The autocorrelation test statistic, denoted by T_A, can then be computed by testing (1) against

$$y_i = x_i'\beta + \hat{e}_i'\rho + u_i, \tag{2}$$

in which $\hat{e}_i' = (\hat{u}_{i-1}, \ldots, \hat{u}_{i-p})$ and $\hat{u}_i = y_i - x_i'\hat{\beta}$ is a typical residual from the OLS estimation of (1). It will be assumed that T_A and all other tests described below are in χ^2 form, e.g. $T_A = pF_p$ where F_p

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is the F statistic calculated from the OLS residual sums of squares for (1) and (2); see Breusch and Godfrey (1981). A test for omitted variables can be implemented as a test of (1) against the augmented model

$$y_i = x_i'\beta + w_i'\gamma + u_i, \tag{3}$$

in which w_i is a q-dimensional vector of test variables, e.g. the RESET procedure uses second- and higher-order powers of the predicted values from the OLS estimation of (1) as test variables. Whatever the choice of w_i, the test statistic for $\gamma = 0$ will denoted by T_{OV}. Finally it is assumed that the test for heteroskedasticity is of the type considered by Breusch and Pagan (1979) and by Koenker (1981), involving a regression of the squared OLS residuals $\hat{u}_i^2$ on an intercept term and r test variables with typical observation vector z_i, $i = 1, \ldots, n$. Let Koenker's studentized form of the test for heteroskedasticity be denoted by T_{H}.

As emphasized by Pagan and Hall (1983), applied workers presented with a battery of diagnostic checks such as T_{A}, T_{OV} and T_{H} face the problem of either deciding how best to combine them or how to determine the overall significance level associated with the use of the separate tests. The solutions of these problems is greatly simplified when the tests statistics are asymptotically independent under the null hypothesis of no misspecification. Under mutual asymptotic independence, a joint test can be obtained by adding the individual test statistics with

$$T_{\mathrm{J}} = T_{\mathrm{A}} + T_{\mathrm{OV}} + T_{\mathrm{H}},$$

being asymptotically distributed as $\chi^2(p+q+r)$ on the null hypothesis of correct specification, while if T_{A}, T_{OV} and T_{H} are not combined, the overall significance level of the induced test is

$$\alpha_I = \left[1 - (1-\alpha_A)(1-\alpha_{OV})(1-\alpha_H)\right],$$

in which, e.g. α_{A} is the marginal significance level of T_{A}. The simple forms of T_{J} and α_{I} will be inappropriate if the individual tests are not asymptotically independent and so the conditions for such independence are of considerable practical importance.

The asymptotic independence of T_{A} and T_{OV} requires that the (k + q) variables of (x_i', w_i') be strictly exogenous. If this condition is not satisfied, a valid large sample joint test against autocorrelation and omitted variables can be derived by testing (1) against the extended model

$$Y_i = x_i'\beta + \hat{e}_i'\rho + w_i'\gamma + u_i, \tag{4}$$

to obtain a test statistic T_{AOV}. For the heteroskedasticity test to be asymptotically independent of T_{A} and T_{OV}, it is necessary and sufficient that $\mathrm{E}(u_i^3) = 0$; see Pagan and Hall (1983, Table I, p. 202). If the disturbances are symmetric, then a joint test can be based upon either $(T_{\mathrm{A}} + T_{\mathrm{OV}} + T_{\mathrm{H}})$ or $(T_{\mathrm{AOV}} + T_{\mathrm{H}})$, depending upon whether or not T_{A} and T_{OV} are asymptotically independent. Thus the symmetry of the distribution of the disturbances plays a crucial role in determining the behaviour of the misspecification checks since if $\mathrm{E}(u_i^3) \neq 0$ neither T_{J} nor α_{I} will be valid.

Jarque and Bera (1987) include a test for symmetry in their widely used check for normality, but their test of $\mathrm{E}(u_i^3) = 0$ is only valid under the assumption that the u_i are normally distributed. The Jarque-Bera statistic checks the significance of $\hat{m}_3 = n^{-1}\Sigma\hat{u}_i^3$. If the disturbances are non-normal (with a symmetric distribution) then the estimator for the variance of $\hat{m}_3$ obtained by Jarque and Bera will be inappropriate and the test statistic will not have the asymptotic $\chi^2(1)$ sampling distribution when $\mathrm{E}(u_i^3) = 0$. Thus the Jarque-Bera check may lead to the rejection of the true assumption of symmetry simply as a result of non-normality. (The conventional theory concerning the consistency and asymptotic normality of the OLS estimators does not, of course, require normality of the regression disturbances.) There is therefore a need for a test for skewness which remains robust in the presence of non-normality. We shall now derive such a test.

3. The test statistic

In order to derive a large sample test, it is assumed that

(i) The regressors of x_i are such that $n^{-1}\Sigma x_i$ tends to a finite vector μ_x and $n^{-1}\Sigma x_i x_i'$ tends to a finite positive definite matrix M_{xx}; and

(ii) $\{u_i\}$ is a sequence of iid random variables with $\mathrm{E}(u_i)=0$, $\mathrm{E}(u_i^2)=\sigma^2$ and has bounded moments up to order 6.

The starting point for the analysis is a first order Taylor series expansion of $n^{1/2}\hat{m}_3 = n^{-1/2}\Sigma\hat{u}_i^3$ about the true regression parameter vector, which is

$$n^{-1/2}\sum\hat{u}_i^3 = n^{-1/2}\sum u_i^3 - 3\left[n^{-1}\sum u_i^2 x_i'\right]\left[n^{1/2}(\hat{\beta}-\beta)\right] + o_p(1), \tag{5}$$

in which $o_p(1)$ denotes a term that tends to zero as $n\to\infty$.

Under H_0: $\mathrm{E}(u_i^3)=0$ and standard regularity conditions, $n^{-1/2}\Sigma u_i^3$ and $n^{1/2}(\hat{\beta}-\beta)$ are both asymptotically normally distributed with zero means and finite variances. It follows that $n^{-1/2}\Sigma\hat{u}_i^3$ is also asymptotically normally distributed with zero mean. The derivation of a suitable test of the significance of a sample value of $n^{-1/2}\Sigma\hat{u}_i^3$ therefore only requires a consistent estimator of the variance in the limiting distribution which is denoted by $\mathrm{avar}(n^{1/2}\hat{m}_3)$. It can be shown that an appropriate estimator of this variance is

$$\hat{v} = n^{-1}\sum\hat{u}_i^6 + 9\hat{\sigma}^6(S/n) - 6\hat{\sigma}^2(S/n)\left(n^{-1}\sum\hat{u}_i^4\right), \tag{6}$$

in which $\hat{\sigma}^2$ is the residual variance, $n^{-1}\Sigma\hat{u}_i^2$, and S is the explained sum of squares (about the origin) from the OLS regression of ι, a vector of ones, on the regressor matrix X which has typical row x_i'. This variance estimator, $\hat{v}$, is guaranteed non-negative since it can be expressed as

$$\hat{v} = (S/n)\left[n^{-1}\sum\hat{w}_i^2\right] + (1-S/n)\left[n^{-1}\sum\hat{u}_i^6\right], \tag{7}$$

where $0\le S/n\le 1$ and $\hat{w}_i = \hat{u}_i^3 - 3\hat{\sigma}^2\hat{u}_i$. In many cases, however, x_i will include a term equal to 1, corresponding to an intercept parameter in β, and so $S=n$ yielding the simplified variance estimator

$$\hat{v} = n^{-1}\sum\hat{u}_i^6 + 9\hat{\sigma}^6 - 6\hat{\sigma}^2\left(n^{-1}\sum\hat{u}_i^4\right). \tag{8}$$

It is important to note that the estimators detailed in (7) and (8) are derived assuming symmetry, but not normality. If the errors, u_i, are normally distributed then $n^{-1}\Sigma u_i^4$ tends to $3\sigma^4$ and $n^{-1}\Sigma u_i^6$ tends to $15\sigma^6$ and $\hat{v}$, in (6) and (7), reduces to $\hat{v}=15\sigma^6-9\hat{\sigma}^6(S/n)$ or $\hat{v}=6\hat{\sigma}^6$ in the case of (8). The use of $6\hat{\sigma}^6$ (as in the Jarque–Bera criterion) will, however, usually provide an inconsistent estimator of $\mathrm{avar}(n^{1/2}\hat{m}_3)$ and hence an invalid test for skewness. [Note that the Jarque–Vera test will be invalid under normality if there is no intercept term in (1).] A valid and non-normality robust test for skewness can be obtained from OLS residuals by comparing

$$T_s = n^{-1}\left(\sum\hat{u}_i^3\right)^2/\hat{v} \tag{9}$$

to a prespecified critical value in the right hand tail of the $\chi^2(1)$ distribution with $\hat{v}$ being given by (7), which reduces to (8) when the initial model (1) is estimated with an intercept term. Alternatively, if the initial model does contain an intercept term (so that $n^{-1}\Sigma\hat{u}_i = 0$), the (robust) test statistic, T_S, can be computed as *the sample size (n) minus the residual sum of squares* following an OLS regression of 1 on $\hat{w}_i = \hat{u}_i^3 - 3\hat{\sigma}^2\hat{u}_i$.

Further consideration of the Taylor series expansion (5), in particular noting that $E(u_i^2) = \sigma^2$ and $\hat{\beta} - \beta = (X'X/n)^{-1}n^{-1}\Sigma x_i u_i$, yields

$$n^{-1/2}\sum \hat{u}_i^3 = n^{-1/2}\sum\left[u_i^3 - 3\sigma^2\iota' X(X'X)^{-1}x_i u_i\right] + o_p(1). \tag{10}$$

This gives rise to an alternative robust variance estimator as

$$\hat{v}^* = n^{-1}\sum \hat{z}_i^2, \tag{11}$$

where $\hat{z}_i = \hat{u}_i^3 - 3\hat{\sigma}^2\iota' X(X'X)^{-1}x_i\hat{u}_i$ and $n^{-1}\Sigma\hat{z}_i = n^{-1}\Sigma\hat{u}_i^3$. Using this variance estimator produces the test statistic $T_S^* = n^{-1}(\Sigma\hat{u}_i^3)^2/\hat{v}^*$, which can be computed as *the sample size (n) minus the residual sum of squares* following an OLS regression of 1 on $\hat{z}_i$. This is, in fact, identically equivalent to T_S when the initial model (1) contains an intercept term since, then, $\hat{z}_i = \hat{w}_i$ and S = n; differences between T_S and T_S^* will only arise when an intercept term does not appear in (1).

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