Lecture notes: Mean Variance Analysis

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Learning objectives: students should

- understand how portfolio variance can be reduced through diversification and the limits of diversification;
- comprehend the trade-off between risk and return, both intuitively and algebraically;
- understand how the efficient frontier of risky assets is obtained and be able to locate 'key' portfolios such as the global minimum-variance portfolio and the tangency portfolio in the mean/standard-deviation diagram.

Reading:

Bodie, Kane, and Marcus (2008) (Chapters 6 and 7)

Supplementary reading:

Grinblatt and Titman (2002) (Chapter 5), Levy and Post (2005) (Chapters 8 and 9)

The foundations of modern portfolio theory can be found in:

Markowitz (1952), Markowitz (1959)

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1 Naïve Diversification

In Lecture 2 we introduced all the statistical concepts needed to study portfolios of risky assets. We saw that a portfolio's variance is **not** just the weighted average of individual assets' variances but also depends on the correlation between asset returns. Correlations between asset returns differ widely. Figures 1 and 2 presents two examples.[1]

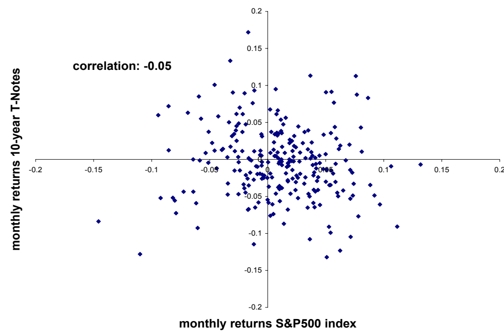


Figure 1: Monthly returns on S&P500 index vs. monthly returns on 10 y T-Notes

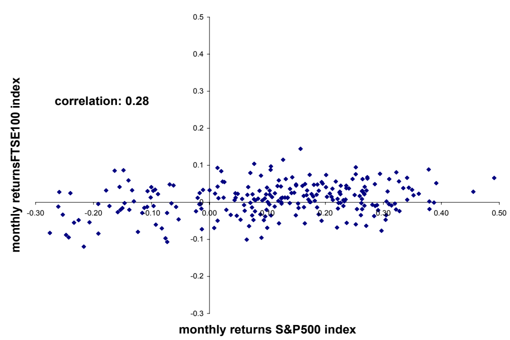


Figure 2: Monthly returns on S&P500 index vs. monthly returns on FTSE100 index

[1]Monthly data from 1984 until September 2004 was used (source: Yahoo Finance).

- If $\rho_{i,j} = +1$, assets i and j are ***perfectly correlated*** and their returns move in exactly the same direction in proportion.

- If $\rho_{i,j} = -1$, assets i and j are ***perfectly negatively correlated*** and their returns move in exactly the opposite direction in proportion.

- If $\rho_{i,j} = 0$, assets i and j are ***uncorrelated*** and on average the movement in the return of one asset is unrelated to the movement in the return of the other asset.

Combining several assets in a portfolio allows an investor to exploit correlations to reduce exposure to *firm specific* risks instead of "betting on one horse". The simplest approach to such ***diversification***, is to throw together randomly selected securities in an equally weighted portfolio. The variance of such a ***naïvly diversified*** portfolio typically has a pattern as depicted in Figure 3.[2]

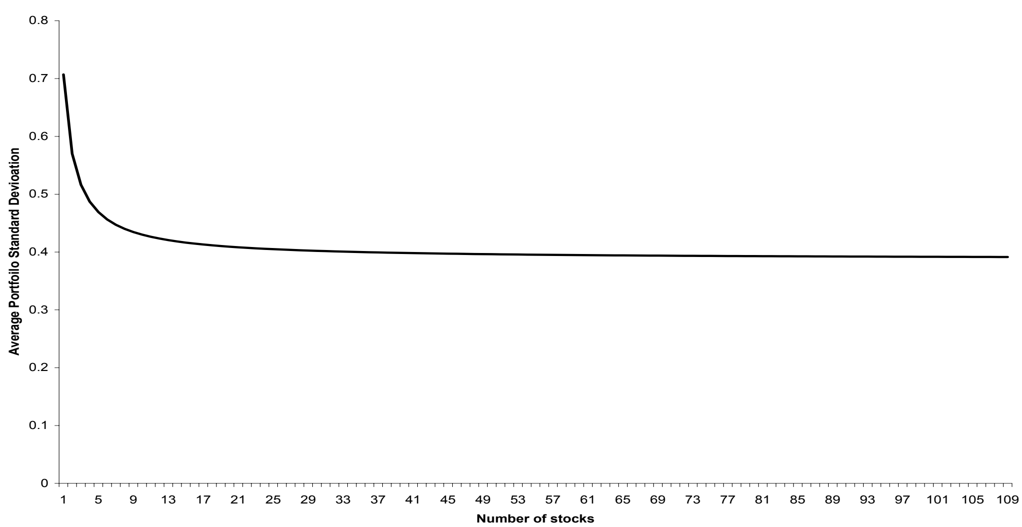


Figure 3: Naïve diversification

[2]How to compute the variance of such a portfolio is explained in Whitmore (1970).

Two important features emerge:

1. Portfolio variance can be significantly reduced through diversification.
2. There typically is a limit to the reduction in portfolio variance that can be achieved with diversification.

In the following, we will move beyond this naïve approach and study how to ***optimally*** pick a portfolio.

2 Portfolios composed of two risky assets (continued)

2.1 An example

Let's consider a portfolio invested in two mutual funds, D invested in bonds ("debt") and E invested in stocks ("equity"). Their descriptive statistics are given in Table 1.

	Debt	Equity
$E[\tilde{r}]$	0.08	0.13
σ	0.12	0.20
$\sigma_{D,E} = 0.0072$		
$\rho_{D,E} = 0.30$		

Table 1: Descriptive statistics for portfolios D and E.

Compute the expected portfolio returns and standard deviations for portfolio weights $w_E = 0, 0.2, 0.5, 0.8, 1$.

Would a risk averse investor chose to invest only in debt or only in equity?

Figure 4 helps us answer these questions (it plots the feasible portfolios without short selling). As can be seen the pure debt portfolio is dominated, e.g., the portfolio with $w_E = 0.2$ has a higher expected return and a lower standard deviation. Thus, no risk averse investor (who has passed this class) would invest in debt only. However, a risk averse investor might choose to invest anywhere on the upper contour of the investment opportunity set, e.g., choosing $w_E = 1$. We will return to this issue later.

We saw that in the example for any degree of risk aversion it was better to allocate part of the investment to equity even though it has a higher return variance than pure debt. The reason for this was that the overall portfolio variance could be reduced by adding another *risky* asset to the portfolio. To understand this better, consider the following situation. You are invested

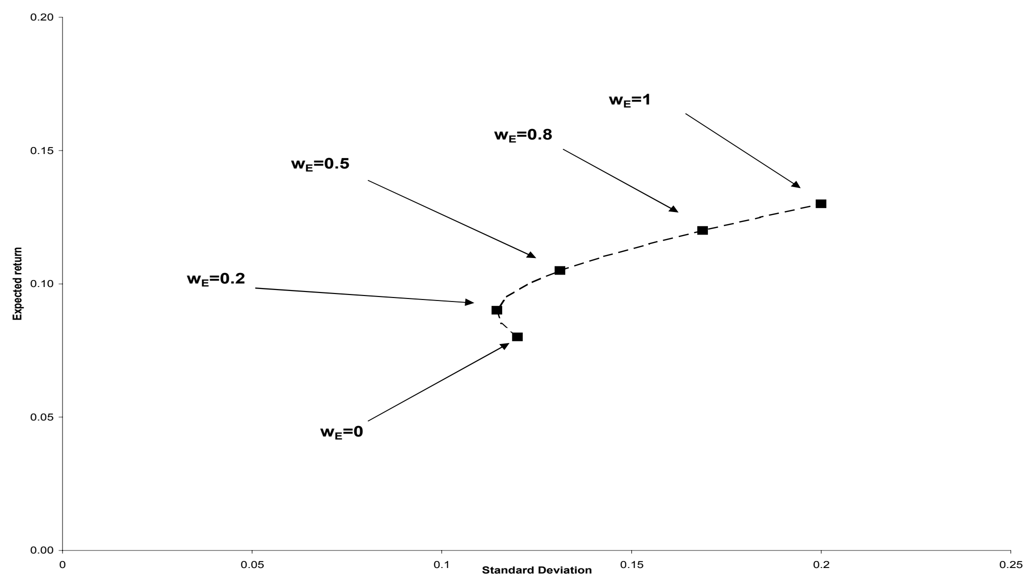


Figure 4: Investment opportunity set (no short sales)

equally in a company that produces suntan lotion and a company that produces umbrellas. If the summer turns out to be sunny, the first company does well and the second poorly. In contrast, if the summer turns out to be rainy, the first company does poorly and the second does well. In other words, their returns are negatively correlated. By investing in both of them, instead of only one of them, obviously you reduce your portfolio risk a lot.

Figure 5 illustrates the impact that different correlations between the two securities in our initial example have on the investment opportunity set.

2.2 The Global Minimum Variance Portfolio (2 Risky Asset Case)

How much can the portfolio's risk be reduced by tinkering with portfolio weights? In the two-security case this ***global minimum variance portfolio***[3] can easily be derived from the variance equation if there are no short sales restrictions (i.e. portfolio weights can be negative):

[3]We will see later that in the general case with many assets it is useful to distinguish between the ***global*** minimum variance portfolio that leads to the smallest possible variance among ***all*** feasible portfolios and minimum variance portfolios, which have the smallest variance for a ***given*** level of expected return.

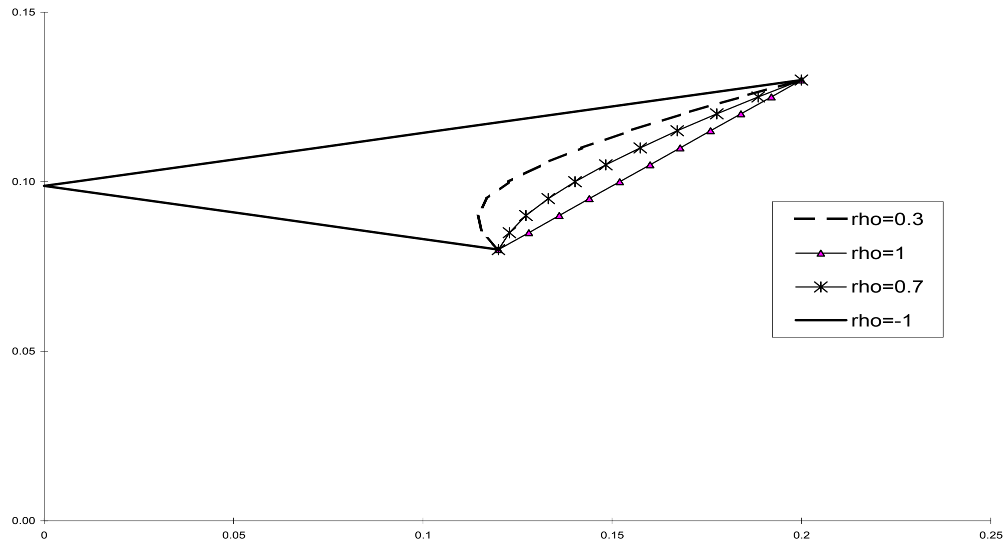


Figure 5: Investment opportunity sets with different correlation coefficients

To minimize the portfolio variance, take the derivative with respect to portfolio weight w_1 of[4]

$$Var[w_1\,\tilde{r}_1 + (1-w_1)\,\tilde{r}_2] = w_1^2\,\sigma_1^2 + (1-w_1)^2\,\sigma_2^2 + 2\,w_1(1-w_1)\,\rho_{1,2}\,\sigma_1\,\sigma_2,$$

and set it equal to zero:

$$\begin{aligned}\frac{d\,Var[w_1\,\tilde{r}_1 + (1-w_1)\,\tilde{r}_2]}{d\,w_1} &= 2\,w_1\,\sigma_1^2 - 2\,(1-w_1)\,\sigma_2^2 + 2\,(1-2\,w_1)\,\rho_{1,2}\,\sigma_1\,\sigma_2 \overset{!}{=} 0 \\ \Leftrightarrow \quad w_1 &= \frac{\sigma_2^2 - \rho_{1,2}\,\sigma_1\,\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\,\rho_{1,2}\,\sigma_1\,\sigma_2}.\end{aligned}$$

To verify that this first-order condition indeed gives us the global minimum variance portfolio, check the second-order condition:[5]

$$\begin{aligned}\frac{d^2 Var[w_1\,\tilde{r}_1 + (1-w_1)\,\tilde{r}_2]}{d\,w_1^2} &= 2\,\left[\sigma_1^2 - 2\,\rho_{1,2}\,\sigma_1\,\sigma_2 + \sigma_2^2\right] \\ &\geq 2\,\left[\sigma_1^2 - 2\,\sigma_1\,\sigma_2 + \sigma_2^2\right] \\ &= 2\,[\sigma_1 - \sigma_2]^2 > 0.\end{aligned}$$

Remark: If the solution has $w_1 \geq 0$, a short sales constraint would not prevent an investor from choosing the global minimum variance portfolio identified above. If there are short sales

[4]Note that $w_2 = 1 - w_1$.

[5]Note that $\rho_{1,2} \leq 1$, thus $-2\,\rho_{1,2} \geq -2$.

restrictions and the solution above is negative, one needs to apply the Kuhn-Tucker conditions from constrained optimization to determine the (constrained) global minimum variance portfolio.

Assignment: determine the global minimum variance portfolio for the example in the beginning of the lecture.

Answer:

$$w_E = \frac{(0.12)^2 - 0.3 \cdot 0.12 \cdot 0.2}{(0.12)^2 + (0.2)^2 - 2 \cdot 0.3 \cdot 0.12 \cdot 0.2} = 0.18$$

3 Portfolio Allocation with Two Risky Assets and a Risk-Free Asset

The textbook quotes an investment professional, claiming that "the really critical decision is how to divvy up your money among stocks, bonds and supersafe investments such as Treasury bills." We can think of this as a portfolio allocation problem with two risky assets and a risk-free asset. Let's return to the example in Section 2.1 and add, as a proxy for a risk-free asset, a T-bill yielding $r_f = 0.05$ (see Table 2).

	Debt	Equity	T-bill
$E[\tilde{r}]$	0.08	0.13	0.05
σ	0.12	0.20	0
$\sigma_{D,E} = 0.0072$			
$\rho_{D,E} = 0.30$			

Table 2: Descriptive statistics.

Figure 6 depicts the investment opportunity set with investment in the two risky portfolios D and E only (how can you tell that short sales are allowed?). By adding the risk-free asset (T-bill), the investment opportunity set can be expanded. The line through the risk-free asset that is tangent to the opportunity set with risky assets only traces the upper bound of investment opportunities achievable through combinations of the risk-free asset and risky assets D and E (make sure you understand why). Thus, an optimal portfolio comprising these three assets will in fact always be a combination of the risk-free asset with the ***tangency portfolio*** P. We proceed to determine the weights of D and E in this portfolio:

The objective is to maximize the slope of the line that connects the risk-free asset with a

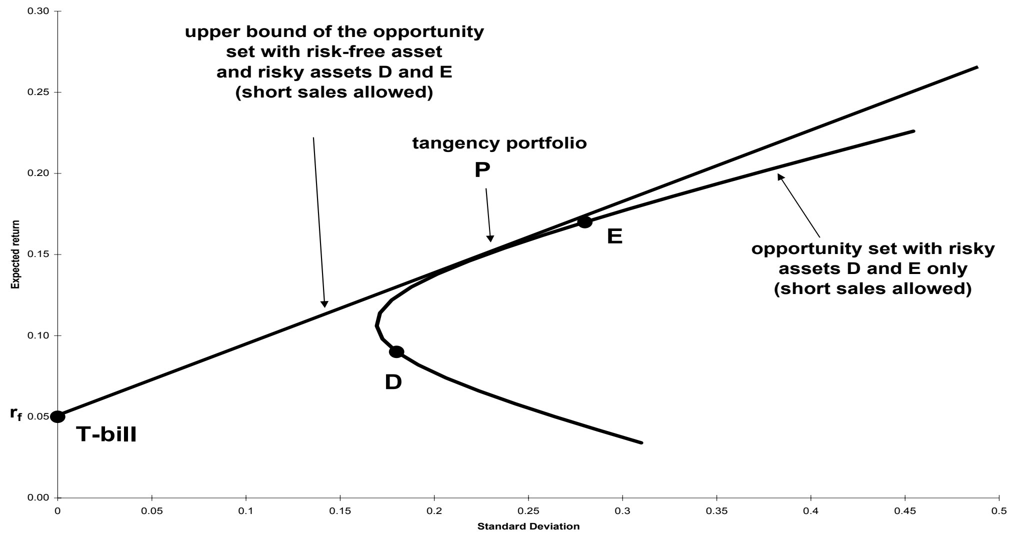


Figure 6: Investment opportunity set with risk-free asset and two risky assets

portfolio made up of the two risky assets D and E. Thus, we solve the following problem:[6]

$$\max_{w_i} S(w_i) = \frac{E[w_i\,\tilde{r}_i + (1-w_i)\,\tilde{r}_j] - r_f}{\sigma[w_i\,\tilde{r}_i + (1-w_i)\,\tilde{r}_j]}.$$

Plugging in the expressions for the expected return and the standard deviation (exercise: derive these), the expression for the slope becomes

$$S(w_i) = \frac{w_i\,\bar{r}_i + (1-w_i)\,\bar{r}_j - r_f}{[w_i^2\,\sigma_i^2 + (1-w_i)^2\,\sigma_j^2 + 2\,w_i(1-w_i)\,\sigma_{ij}]^{1/2}}.$$

The first-order condition is (do this on your own and compare solutions)
Let $Z \equiv w_i^2\,\sigma_i^2 + (1-w_i)^2\,\sigma_j^2 + 2\,w_i(1-w_i)\,\sigma_{ij}$. Then we can rewrite $S(w_i) = \frac{E[w_i\,\tilde{r}_i + (1-w_i)\,\tilde{r}_j] - r_f}{Z^{1/2}}$.
Taking the derivative[7]

$$\frac{d\,S(w_i)}{d\,w_i} = \frac{(\bar{r}_i - \bar{r}_j)\,Z^{1/2} - 1/2\,[w_i\,(\bar{r}_i - \bar{r}_j) + \bar{r}_j - r_f]\,Z^{-1/2}\,\frac{\partial Z}{\partial w_i}}{\left(Z^{1/2}\right)^2}$$

$$= \frac{(\bar{r}_i - \bar{r}_j)\,Z^{1/2} - 1/2\,[w_i\,(\bar{r}_i - \bar{r}_j) + \bar{r}_j - r_f]\,Z^{-1/2}\,[2\,w_i\,(\sigma_i^2 + \sigma_j^2 - 2\sigma_{ij}) + 2(\sigma_{ij} - \sigma_j^2)]}{Z} \tag{1}$$

In the optimum, $\frac{d\,S(w_i)}{d\,w_i} \stackrel{!}{=} 0$. Since we can multiply both sides by Z and still have zero on the

[6] The choice variable is the weight w_i, $i \in \{D, E\}$ (since $w_j = 1 - w_i$, $i \neq j$, $i, j \in \{E, D\}$).

[7] Use the quotient rule!

right side, we need to check only when the numerator of (1) is equal to zero:

$$
\begin{aligned}
& (\bar{r}_i - \bar{r}_j)\, Z^{1/2} - 1/2\, [w_i\, (\bar{r}_i - \bar{r}_j) + \bar{r}_j - r_f]\, Z^{-1/2} \cdot 2\, [w_i\, (\sigma_i^2 + \sigma_j^2 - \sigma_{ij}) + \sigma_{ij} - \sigma_j^2] = 0 \\
& \text{note that } 1/2 \cdot 2 \text{ cancels in } [\cdot]\, Z^{1/2}\, [\cdot] \text{ term; multiply by } Z^{-1/2} \text{ and add } [\cdot]\, [\cdot] \\
\Leftrightarrow \quad & (\bar{r}_i - \bar{r}_j)\, Z = [w_i\, (\bar{r}_i - \bar{r}_j) + \bar{r}_j - r_f]\, [w_i\, (\sigma_i^2 + \sigma_j^2 - 2\sigma_{ij}) + \sigma_{ij} - \sigma_j^2]. \qquad (2)
\end{aligned}
$$

The left-hand-side of (2) is

$$(\bar{r}_i - \bar{r}_j) \Big[\; w_i^2\, \sigma_i^2 \quad +\sigma_j^2 \quad -2\, w_i\, \sigma_j^2 \quad +w_i^2\, \sigma_j^2 \quad +2\, w_i\, \sigma_{ij} \quad -2\, w_i^2\, \sigma_{ij} \Big]$$

The right-hand-side of (2) is

$$
\begin{aligned}
& (\bar{r}_i - \bar{r}_j) \Big[\; w_i^2\, \sigma_i^2 \quad -w_i\, \sigma_j^2 \quad +w_i^2\, \sigma_j^2 \quad +w_i\, \sigma_{ij} \quad -2\, w_i^2\, \sigma_{ij} \Big] \\
+ \quad & (\bar{r}_j - r_f) \Big[\; w_i\, \sigma_i^2 \quad +w_i\, \sigma_j^2 \quad -2\, w_i\, \sigma_{ij} \Big] \quad +(\bar{r}_j - r_f) \Big[\sigma_{ij} \quad -\sigma_j^2 \Big]
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& (\bar{r}_i - \bar{r}_j + \bar{r}_j - r_f)\, \sigma_j^2 - (\bar{r}_j - r_f)\, \sigma_{ij} \\
& = w_i\, \left[(\bar{r}_j - r_f)\, \sigma_i^2 + (\bar{r}_i - \bar{r}_j + \bar{r}_j - r_f)\, \sigma_i^2 - (\bar{r}_i - \bar{r}_i + 2\, \bar{r}_i - 2\, r_f)\, \sigma_{ij} \right] \\
\Leftrightarrow \quad & w_i = \frac{(\bar{r}_i - r_f)\, \sigma_j^2 - (\bar{r}_j - r_f)\, \sigma_{ij}}{(\bar{r}_j - r_f)\, \sigma_i^2 + (\bar{r}_i - r_f)\, \sigma_j^2 - (\bar{r}_j + \bar{r}_i - 2\, r_f)\, \sigma_{ij}}. \qquad (3)
\end{aligned}
$$

Plugging into (3) the values from our example, we get $w_E = 0.6$ and $w_D = 1 - w_E = 0.4$. Thus, any investor faced with these three assets will allocate his or her money among the risk free asset and a portfolio comprising a share 0.6 of the equity fund and share 0.4 of the bond fund.

4 The Sharpe Ratio

A commonly used measure for the risk-return tradeoff offered by a portfolio is the ***Sharpe Ratio*** (also known as reward-to-variability ratio, Sharpe Index or Sharpe Measure).[8] The typical variant used is to divide the average portfolio excess return (over the sample period of historical data) by the standard deviation of returns over that period:[9]

$$(\textbf{Sharpe Ratio}) \qquad \frac{\bar{r}_P - r_f}{\sigma_P}.$$

The Sharpe Ratio gives you the slope of a line connecting the risk-free rate of return with the portfolio of interest in the mean-standard deviation plane. Thus, the objective of finding the greatest possible slope is achieved by selecting the portfolio with the highest Sharpe Ratio.

[8]Sharpe (1966) introduced the measure for the evaluation of mutual fund performance.

[9]If the risk-free return varies over time use its average $\bar{r}_f$.

5 Portfolios with n Assets: Markowitz' Portfolio Selection Model

The previous cases with two risky assets and two risky assets plus a risk-free asset provide us with all the intuition for the general portfolio selection problem. The theory of portfolio choice was developed by Harry Markowitz, a scientific contribution which earned him the 1990 Nobel prize. To start off, we will gather here the assumptions underpinning our analysis:

5.1 Assumptions of Mean-Variance Analysis

1. Investors' preferences can be expressed with a mean-variance utility function. That is, they are only concerned with the expected return and the variance of portfolios over a particular period.
2. Financial markets are frictionless, i.e.
   (a) investors take prices as given
   (b) assets are infinitely divisible
   (c) no transaction costs or taxes
3. one period investment horizon

The formulas for expected return and variance generalize as follows:

$$E[\tilde{r}_P] = \sum_{i=1}^{n} w_i \, E[\tilde{r}_i], \tag{4}$$

$$Var[\tilde{r}_P] = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \, w_j \, Cov[\tilde{r}_i, \tilde{r}_j]. \tag{5}$$

5.2 Minimum Variance Portfolios, the Global Minimum Variance Portfolio, and the Efficient Set

In lecture 2 we learned that the concept of ***mean-variance dominance*** allows us to partially rank portfolios. As we will see this helps us greatly reduce the number of portfolios that we need to consider in an investment decision by identifying those portfolios in the opportunity set of portfolios that no risk-averse mean-variance utility maximizer will choose to hold. We start off by considering a portfolio choice involving risky assets only.

Risky assets only

Applying the formulae (4) and (5) to compute the combinations of expected return and standard deviation for all possible portfolio weights we obtain the set of feasible portfolios (the

opportunity set). Typically, this opportunity set will look like the grey shaded area in Figure 7. Instead of applying the concept of mean-variance dominance directly it is more convenient to do an intermediary step: identify for each level of expected return the portfolio that has the minimum variance. In Figure 7 this is illustrated for the two return levels $\bar{r}_1$ and $\bar{r}_2$. These portfolios are called ***minimum variance portfolios***. Note that for any portfolio standard deviation greater than σ_g there exist two minimum variance portfolios so that the set of minimum variance portfolios forms an envelope around the feasible set (see Figure 8). Now we can easily apply the concept of mean-variance dominance. Clearly, there is always a minimum variance portfolio that dominates any other feasible portfolio since this minimum variance portfolio offers the same expected return but has a lower variance. Thus, we can eliminate from our list of portfolios to consider all portfolios which are not in the set of minimum variance portfolios. However, we can go further still. As we have already noted, for any portfolio standard deviation greater than σ_g there exist two minimum variance portfolios. Thus, the portfolio with the higher expected return for a given standard deviation dominates and we can cross the other one off from our list. This leaves us with the upper contour of the opportunity set, starting at the minimum variance portfolio with standard deviation σ_g. The resulting set is called the ***efficient set*** or ***efficient frontier***. All the portfolios that it contains mean-variance dominate the other feasible portfolios and thus are the most efficient portfolios. The portfolio with standard deviation σ_g is called ***global minimum variance portfolio*** since it is the minimum variance portfolio with the smallest possible variance in the set of feasible portfolios. There exists no other portfolio with a lower variance or a higher expected return, and therefore it is part of the efficient set (see Figure 8).

Risky assets and a risk-free asset

If the investor has access to a risk-free asset in addition to the risky assets, the above exercise is done in exactly the same way. Construct the set of feasible portfolios, which has now expanded since it also includes combinations of risky assets with the risk-free asset, i.e. portfolios that lie on a line connecting a risky asset with r_f (compare the hatched area with the grey shaded area in Figure 9). The minimum variance set is again found by looking for the portfolio with the smallest variance for each level of expected return. The set is given by the envelope of the set of feasible portfolios in Figure 9. The global minimum variance portfolio now is the risk-free asset with a variance of zero (since it is risk-free) and the efficient set is formed by the ray extending from r_f which has the steepest slope and is tangent to the set formed by the risky assets. In other words, the efficient set is generated simply by combinations of two assets: the risk-free asset with the ***tangency portfolio***.

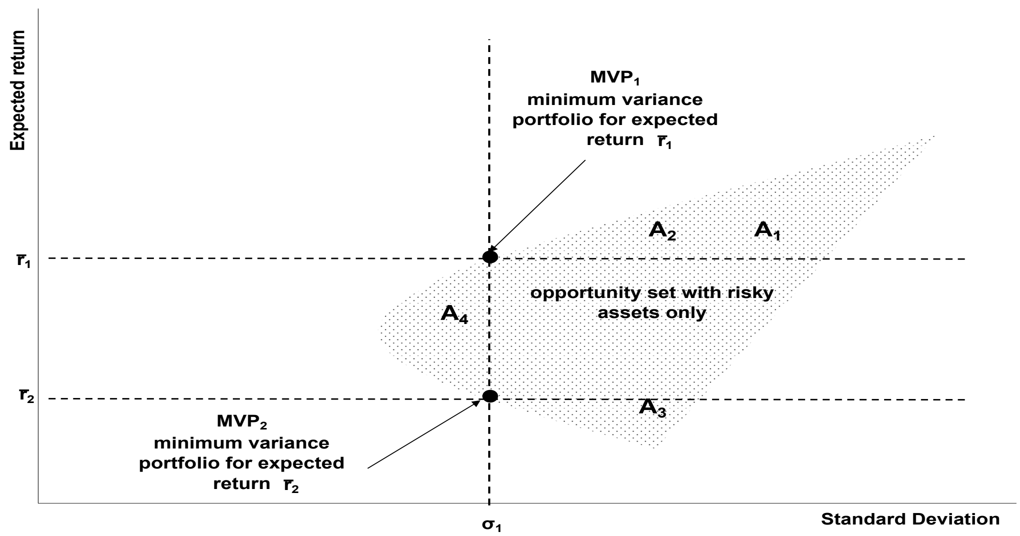


Figure 7: Identifying minimum variance portfolios

5.3 Optimal portfolio choice

Optimal portfolio choice with n assets follows the same steps as in we have already seen in the three security case (although this typically requires the use of a computer):

1. Identify the available risk-return combinations offered by the $n-1$ risky assets (opportunity set of risky assets) [panel (a) in Figure 10].

   Note: If there is no risk-free asset, the optimal portfolio is given by the tangency point of the investor's indifference curves and the ***efficient frontier of risky assets*** [panel (b) in Figure 10].

2. Determine the ***tangency portfolio*** and construct the ***capital allocation line***: draw a line with intercept r_f tangent to the efficient frontier of risky assets [panel (c) in Figure 10].

3. The investor's optimal portfolio choice is given by the tangency point of the investor's indifference curves and the capital allocation line [panel (d) in Figure 10].

Remarks:

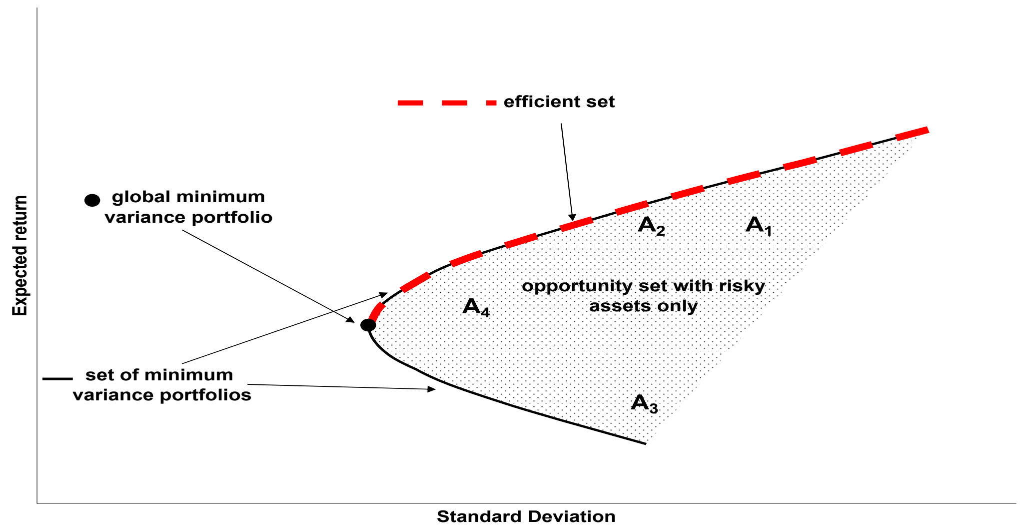


Figure 8: Minimum variance portfolios and the efficient set

- The Capital Allocation Line is the efficient frontier of investment opportunities offered by combinations of the risk-free asset and risky assets.
- Short sales restrictions can also be incorporated when constructing the efficient frontier for risky assets (see Excel application below).

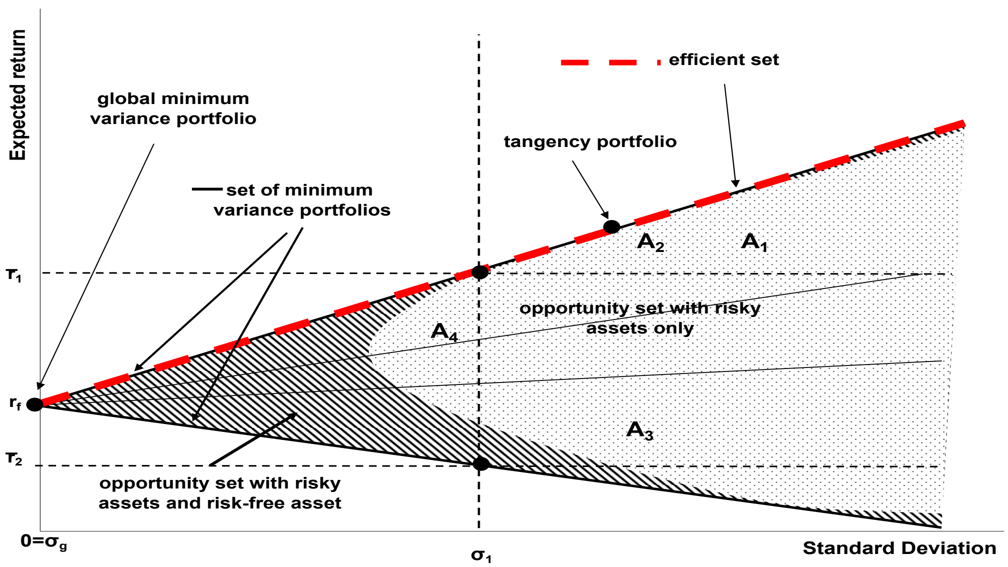
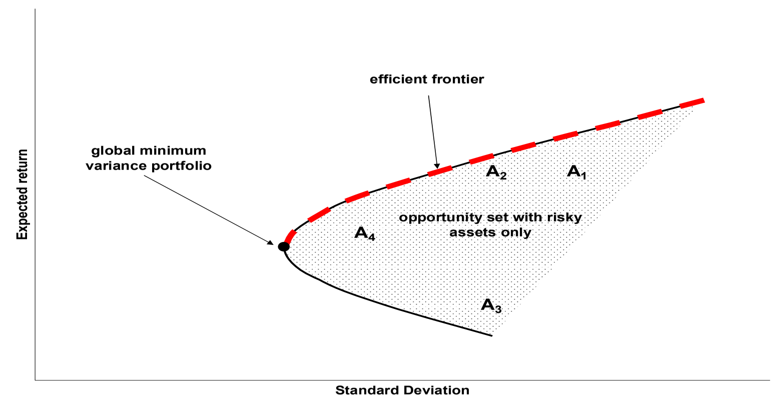
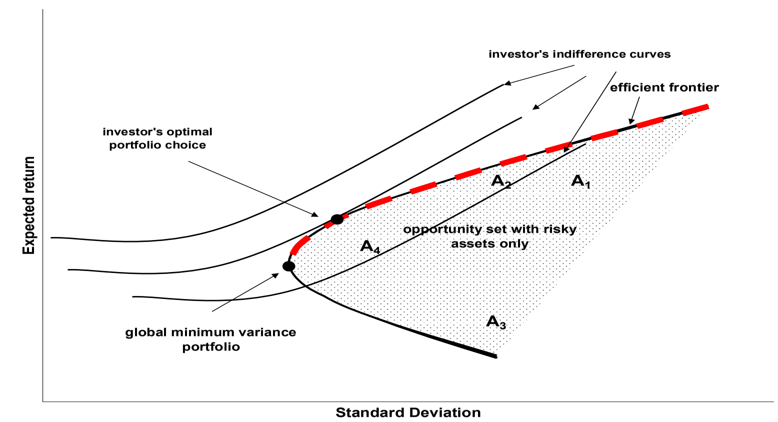


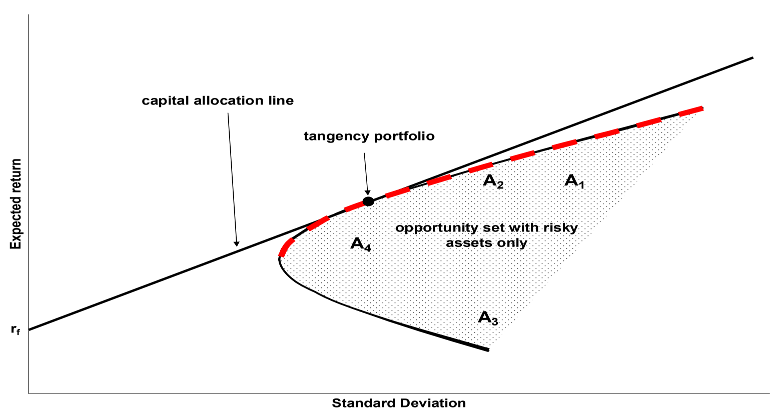
Figure 9: Minimum variance portfolios and the efficient set with a risk-free asset



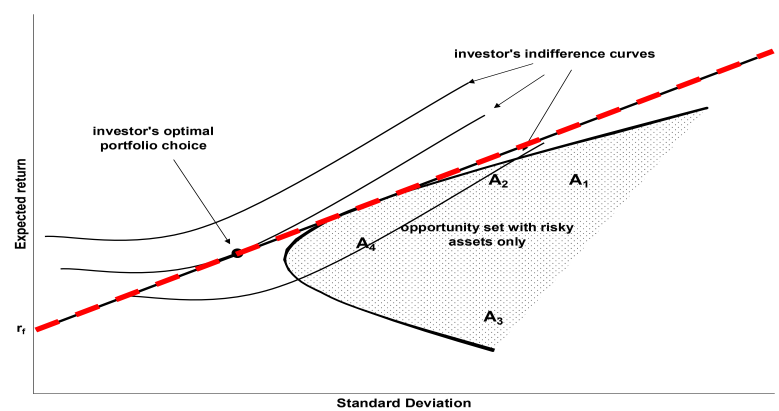
(a) Step 1



(b) Step 2/3 (if no risk-free asset available)



(c) Step 2 (if risk-free asset available)



(d) Step 3 (if risk-free asset available)

Figure 10: Portfolio choice with n assets

5.4 Two fund separation

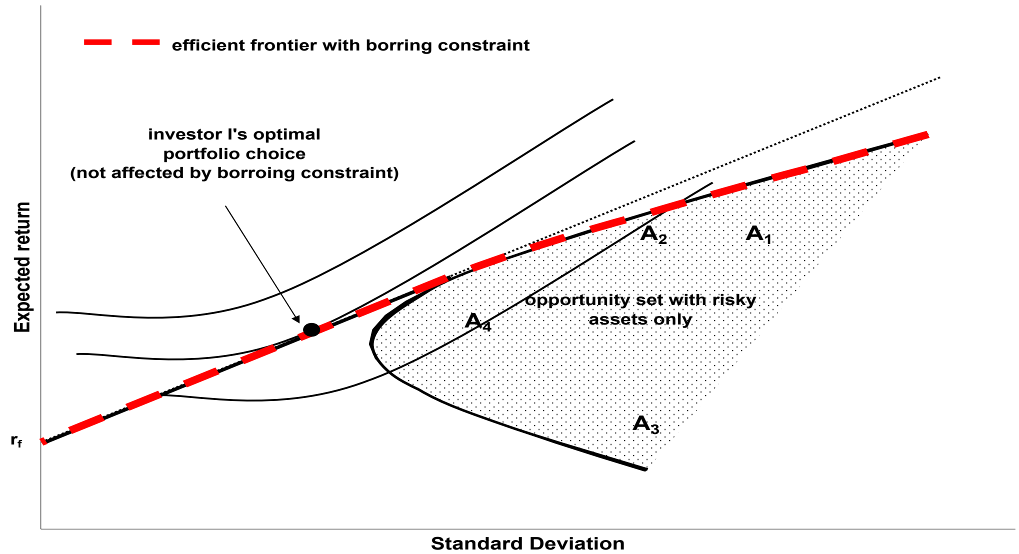
As in the three-asset case, ultimately all investors hold only two different types of funds in their portfolios, the risk-free asset and the tangency portfolio. This is called the ***two fund separation*** property.

The two fund separation property means that it is possible to split the portfolio choice problem into two parts. The first involves determining the tangency portfolio and does not require any knowledge about an investor's preferences. The second involves finding the optimal mix of this portfolio of risky assets with the risk-free asset by locating the tangency point of the investor's indifference curves and the capital allocation line.

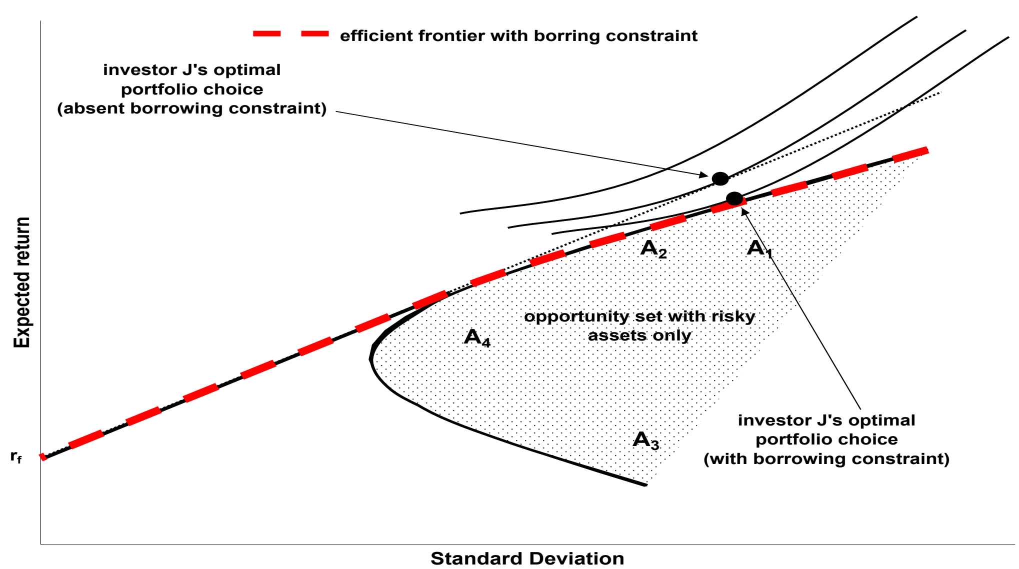
The portfolio choice at which one arrives depends on the data that are used as inputs. These inputs are generated by the investor's (or the financial advisor's) ***security analysis***. For a company's stock this process may involve looking at historical data, examining the company's income statements, reading related news items, and using judgement on how to aggregate all this into statistical figures. As a consequence, investors will hold different opinions as to what are the appropriate descriptive statistics for a financial asset and therefore will arrive at different efficient frontiers and tangency portfolios. Different constraints, such as different tax rates, access to borrowing at different rates, or short sales constraints also lead to different efficient frontiers for different types of investors.

5.5 Portfolio choice with borrowing constraints

If an investor cannot borrow funds at the risk-free rate the efficient frontier of investment opportunities offered by combinations of the risk-free asset and risky assets no longer is a straight line. Rather it is given by the line connecting the risk-free asset and the tangency portfolio and then continues along the upper contour of the opportunity set with risky assets only (see Figure 11). Inability to borrow at the risk-free rate poses no constraint on the portfolio choice of investor I (panel (a)) however does restrict investor J's portfolio choice (panel (b)).



(a) Investor I



(b) Investor J

Figure 11: Portfolio choices with/without borrowing constraints

5.6 Different borrowing and lending rates

If an investor faces a higher borrowing rate, r_f^B, than the available risk-free lending rate, r_f^L, the efficient frontier of investment opportunities offered by combinations of the risk-free asset and risky assets is characterized by two tangency portfolios (see Figure 12). Tangency portfolio P_1 is derived using the risk-free lending rate, tangency portfolio P_2 is derived using the borrowing rate. In between the two tangency portfolios, the efficient frontier follows the upper contour of the opportunity set with risky assets only.

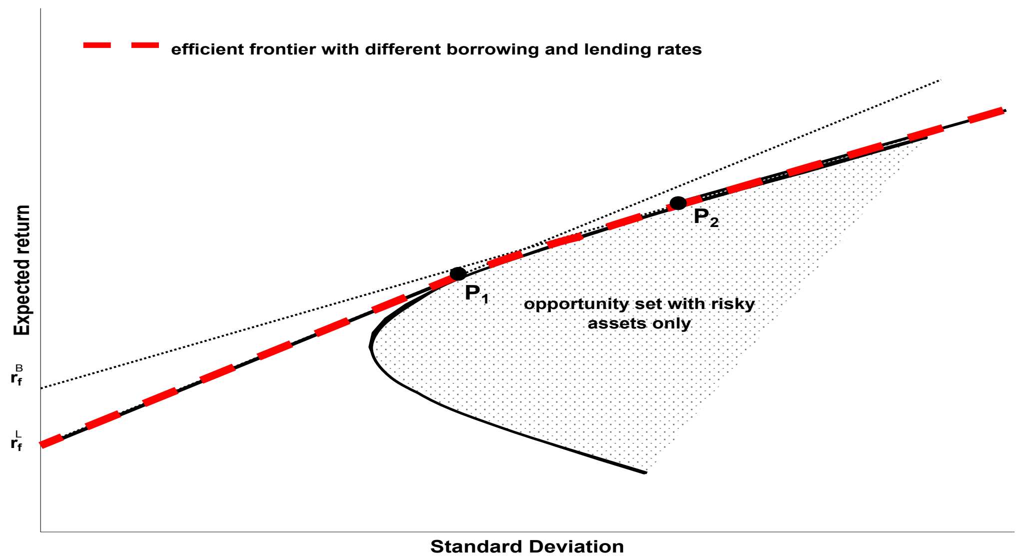


Figure 12: Portfolio choices with different borrowing and lending rates

5.7 Implementing Mean-Variance Analysis

In practice portfolio optimization is done using computer programs, such as Excel. (An excel application is available from the textbook's website):

The goal is to produce the graph of the efficient frontier shown in Figure 13. As a first step one needs to input the descriptive statistics for the assets to be considered. Panel (a) in Figure 14 shows this. A good starting point for the optimization is to start off with an equally weighted portfolio (as in panel (b) in Figure 14). The portfolio has an expected return of 16.5 percent and a portfolio variance of 314.77. The "Solver" is instructed to minimize the portfolio variance (cell B93) by varying the portfolio weights (cells A85 to A91) subject to the constraints that all portfolio weights are nonnegative (short sales constraint) and that they sum up to one. The result of the optimization (panel (a) in Figure 15) shows that for a given return of 16.5 percent the variance can be reduced to 297.46 compared to an equally weighted portfolio. Repeating this step for a range of different target expected returns (panel (b) in Figure 15), one can plot the efficient frontier (Figure 13). Note that the procedure can be carried out without a short sales constraint in the solver as well.

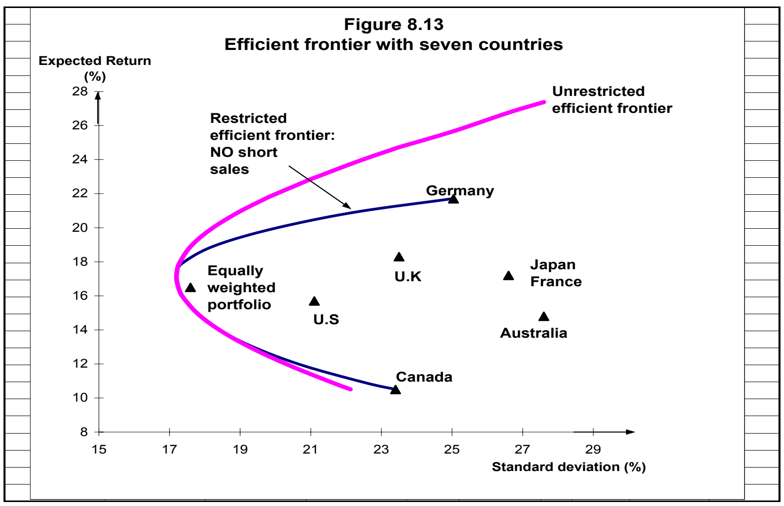


Figure 13: Efficient frontier

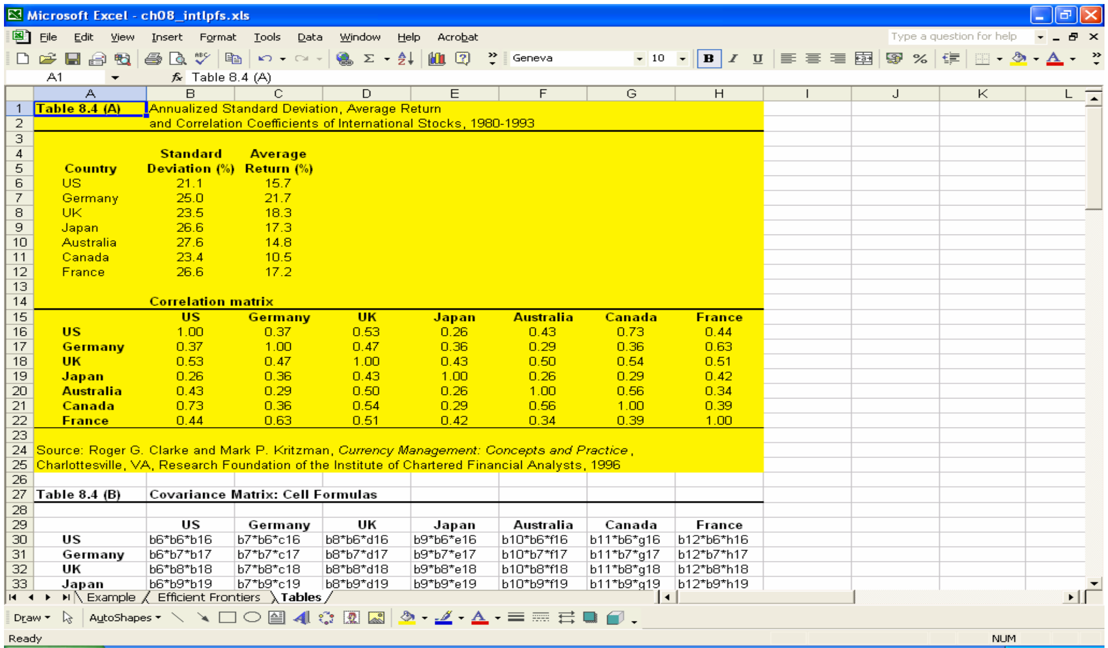


Table 8.4 (A) Annualized Standard Deviation, Average Return and Correlation Coefficients of International Stocks, 1980-1993

Country	Standard Deviation (%)	Average Return (%)
US	21.1	15.7
Germany	25.0	21.7
UK	23.5	18.3
Japan	26.6	17.3
Australia	27.6	14.8
Canada	23.4	10.5
France	26.6	17.2

Correlation matrix

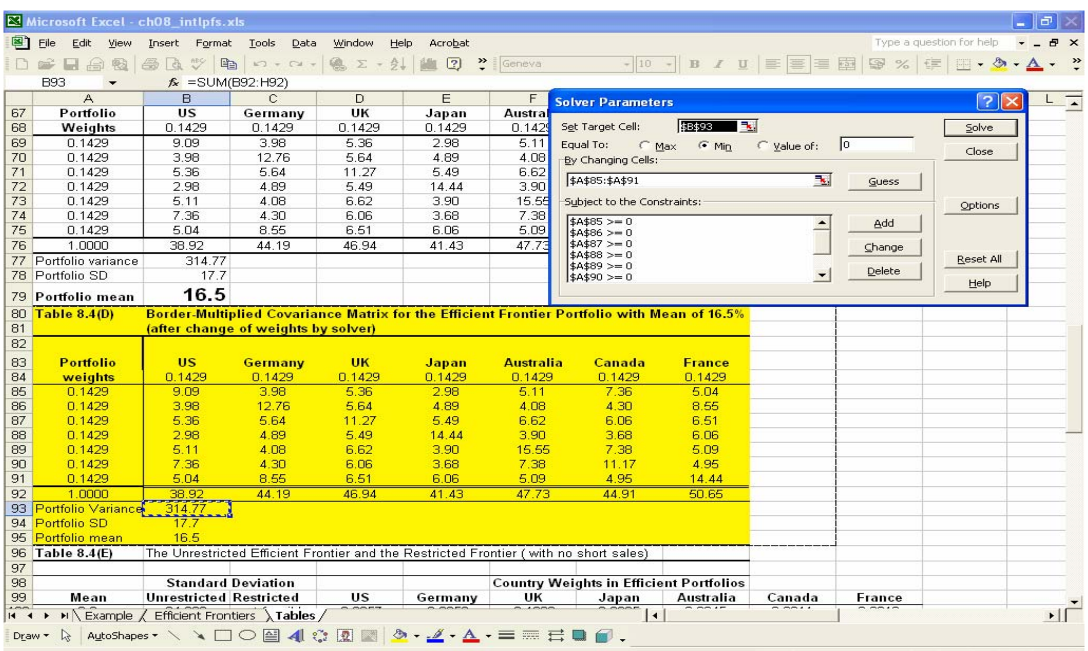
	US	Germany	UK	Japan	Australia	Canada	France
US	1.00	0.37	0.53	0.26	0.43	0.73	0.44
Germany	0.37	1.00	0.47	0.36	0.29	0.36	0.63
UK	0.53	0.47	1.00	0.43	0.50	0.54	0.51
Japan	0.26	0.36	0.43	1.00	0.26	0.29	0.42
Australia	0.43	0.29	0.50	0.26	1.00	0.56	0.34
Canada	0.73	0.36	0.54	0.29	0.56	1.00	0.39
France	0.44	0.63	0.51	0.42	0.34	0.39	1.00

Source: Roger G. Clarke and Mark P. Kritzman, *Currency Management: Concepts and Practice*, Charlottesville, VA, Research Foundation of the Institute of Chartered Financial Analysts, 1996

Table 8.4 (B) Covariance Matrix: Cell Formulas

	US	Germany	UK	Japan	Australia	Canada	France
US	b6*b6*b16	b7*b6*c16	b8*b6*d16	b9*b6*e16	b10*b6*f16	b11*b6*g16	b12*b6*h16
Germany	b6*b7*b17	b7*b7*c17	b8*b7*d17	b9*b7*e17	b10*b7*f17	b11*b7*g17	b12*b7*h17
UK	b6*b8*b18	b7*b8*c18	b8*b8*d18	b9*b8*e18	b10*b8*f18	b11*b8*g18	b12*b8*h18
Japan	b6*b9*b19	b7*b9*c19	b8*b9*d19	b9*b9*e19	b10*b9*f19	b11*b9*g19	b12*b9*h19

(a) Descriptive statistics



Portfolio Weights	US	Germany	UK	Japan	Austral
	0.1429	0.1429	0.1429	0.1429	0.1429
0.1429	9.09	3.98	5.36	2.98	5.11
0.1429	3.98	12.76	5.64	4.89	4.08
0.1429	5.36	5.64	11.27	5.49	6.62
0.1429	2.98	4.89	5.49	14.44	3.90
0.1429	5.11	4.08	6.62	3.90	15.55
0.1429	7.36	4.30	6.06	3.68	7.38
0.1429	5.04	8.55	6.51	6.06	5.09
1.0000	38.92	44.19	46.94	41.43	47.73
Portfolio variance	314.77				
Portfolio SD	17.7				
Portfolio mean	**16.5**				

Solver Parameters

Set Target Cell: $B$93

Equal To: Max Min Value of: 0

By Changing Cells: $A$85:$A$91

Subject to the Constraints:

$A$85 >= 0
$A$86 >= 0
$A$87 >= 0
$A$88 >= 0
$A$89 >= 0
$A$90 >= 0

Solve Close Guess Options Add Change Reset All Delete Help

Table 8.4(D) Border-Multiplied Covariance Matrix for the Efficient Frontier Portfolio with Mean of 16.5% (after change of weights by solver)

Portfolio weights	US	Germany	UK	Japan	Australia	Canada	France
	0.1429	0.1429	0.1429	0.1429	0.1429	0.1429	0.1429
0.1429	9.09	3.98	5.36	2.98	5.11	7.36	5.04
0.1429	3.98	12.76	5.64	4.89	4.08	4.30	8.55
0.1429	5.36	5.64	11.27	5.49	6.62	6.06	6.51
0.1429	2.98	4.89	5.49	14.44	3.90	3.68	6.06
0.1429	5.11	4.08	6.62	3.90	15.55	7.38	5.09
0.1429	7.36	4.30	6.06	3.68	7.38	11.17	4.95
0.1429	5.04	8.55	6.51	6.06	5.09	4.95	14.44
1.0000	38.92	44.19	46.94	41.43	47.73	44.91	50.65
Portfolio Variance	314.77						
Portfolio SD	17.7						
Portfolio mean	16.5						

Table 8.4(E) The Unrestricted Efficient Frontier and the Restricted Frontier (with no short sales)

Mean	Standard Deviation Unrestricted	Restricted	US	Germany	Country Weights in Efficient Portfolios UK	Japan	Australia	Canada	France

(b) Equally weighted portfolio

Figure 14: Application of Mean Variance Analysis I

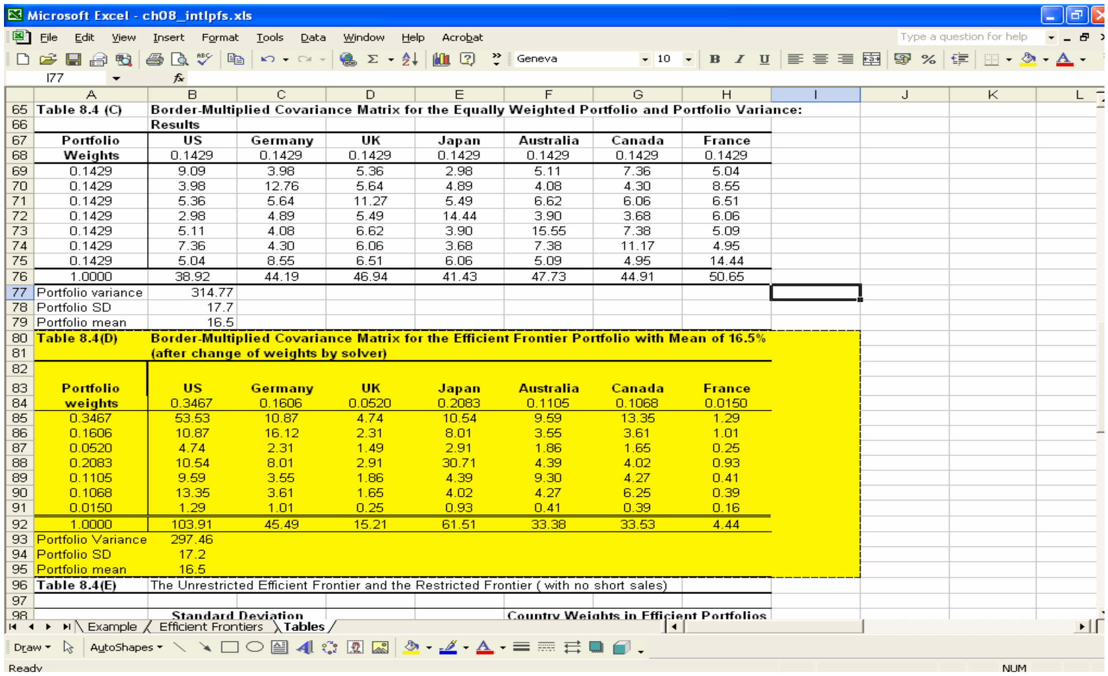


Table 8.4 (C) Border-Multiplied Covariance Matrix for the Equally Weighted Portfolio and Portfolio Variance: Results

Portfolio Weights	US	Germany	UK	Japan	Australia	Canada	France
	0.1429	0.1429	0.1429	0.1429	0.1429	0.1429	0.1429
0.1429	9.09	3.98	5.36	2.98	5.11	7.36	5.04
0.1429	3.98	12.76	5.64	4.89	4.08	4.30	8.55
0.1429	5.36	5.64	11.27	5.49	6.62	6.06	6.51
0.1429	2.98	4.89	5.49	14.44	3.90	3.68	6.06
0.1429	5.11	4.08	6.62	3.90	15.55	7.38	5.09
0.1429	7.36	4.30	6.06	3.68	7.38	11.17	4.95
0.1429	5.04	8.55	6.51	6.06	5.09	4.95	14.44
1.0000	38.92	44.19	46.94	41.43	47.73	44.91	50.65
Portfolio variance	314.77						
Portfolio SD	17.7						
Portfolio mean	16.5						

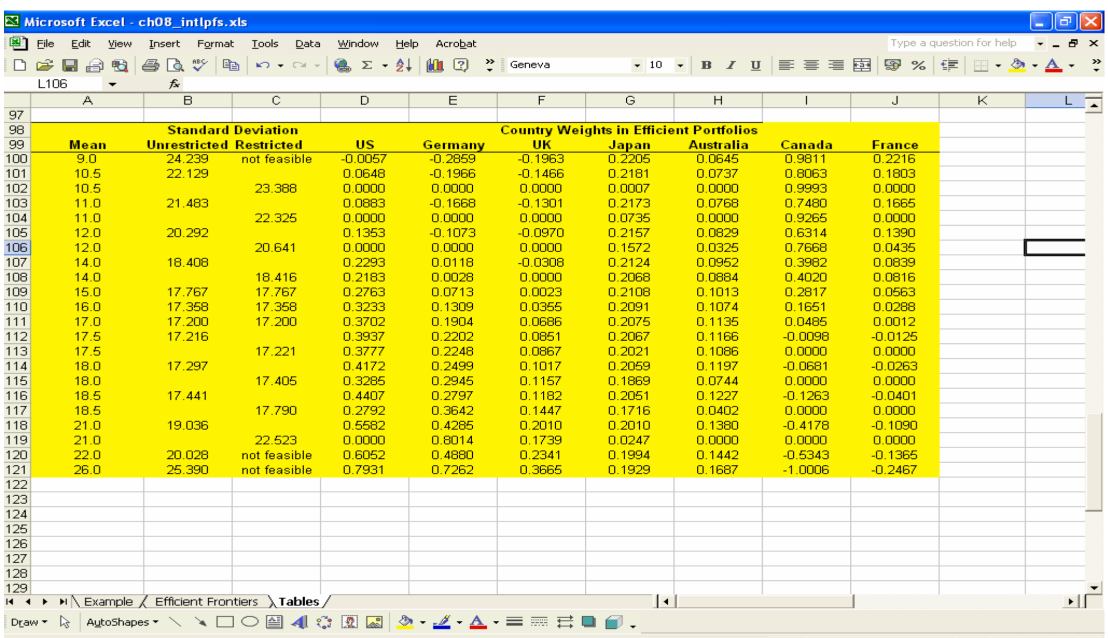
Table 8.4(D) Border-Multiplied Covariance Matrix for the Efficient Frontier Portfolio with Mean of 16.5% (after change of weights by solver)

Portfolio weights	US	Germany	UK	Japan	Australia	Canada	France
	0.3467	0.1606	0.0520	0.2083	0.1105	0.1068	0.0150
0.3467	53.53	10.87	4.74	10.54	9.59	13.35	1.29
0.1606	10.87	16.12	2.31	8.01	3.55	3.61	1.01
0.0520	4.74	2.31	1.49	2.91	1.86	1.65	0.25
0.2083	10.54	8.01	2.91	30.71	4.39	4.02	0.93
0.1105	9.59	3.55	1.86	4.39	9.30	4.27	0.41
0.1068	13.35	3.61	1.65	4.02	4.27	6.25	0.39
0.0150	1.29	1.01	0.25	0.93	0.41	0.39	0.16
1.0000	103.91	45.49	15.21	61.51	33.38	33.53	4.44
Portfolio Variance	297.46						
Portfolio SD	17.2						
Portfolio mean	16.5						

Table 8.4(E) The Unrestricted Efficient Frontier and the Restricted Frontier (with no short sales)

Standard Deviation | Country Weights in Efficient Portfolios

(a) Minimum variance portfolio for target return 16.5 percent



	Standard Deviation		Country Weights in Efficient Portfolios						
Mean	Unrestricted	Restricted	US	Germany	UK	Japan	Australia	Canada	France
9.0	24.239	not feasible	-0.0057	-0.2859	-0.1963	0.2205	0.0645	0.9811	0.2216
10.5	22.129		0.0648	-0.1966	-0.1466	0.2181	0.0737	0.8063	0.1803
10.5		23.388	0.0000	0.0000	0.0000	0.0007	0.0000	0.9993	0.0000
11.0	21.483		0.0883	-0.1668	-0.1301	0.2173	0.0768	0.7480	0.1665
11.0		22.325	0.0000	0.0000	0.0000	0.0735	0.0000	0.9265	0.0000
12.0	20.292		0.1353	-0.1073	-0.0970	0.2157	0.0829	0.6314	0.1390
12.0		20.641	0.0000	0.0000	0.0000	0.1572	0.0325	0.7668	0.0435
14.0	18.408		0.2293	0.0118	-0.0308	0.2124	0.0952	0.3982	0.0839
14.0		18.416	0.2183	0.0028	0.0000	0.2068	0.0884	0.4020	0.0816
15.0	17.767	17.767	0.2763	0.0713	0.0023	0.2108	0.1013	0.2817	0.0563
16.0	17.358	17.358	0.3233	0.1309	0.0355	0.2091	0.1074	0.1651	0.0288
17.0	17.200	17.200	0.3702	0.1904	0.0686	0.2075	0.1135	0.0485	0.0012
17.5	17.216		0.3937	0.2202	0.0851	0.2067	0.1166	-0.0098	-0.0125
17.5		17.221	0.3777	0.2248	0.0867	0.2021	0.1086	0.0000	0.0000
18.0	17.297		0.4172	0.2499	0.1017	0.2059	0.1197	-0.0681	-0.0263
18.0		17.405	0.3285	0.2945	0.1157	0.1869	0.0744	0.0000	0.0000
18.5	17.441		0.4407	0.2797	0.1182	0.2051	0.1227	-0.1263	-0.0401
18.5		17.790	0.2792	0.3642	0.1447	0.1716	0.0402	0.0000	0.0000
21.0	19.036		0.5582	0.4285	0.2010	0.2010	0.1380	-0.4178	-0.1090
21.0		22.523	0.0000	0.8014	0.1739	0.0247	0.0000	0.0000	0.0000
22.0	20.028	not feasible	0.6052	0.4880	0.2341	0.1994	0.1442	-0.5343	-0.1365
26.0	25.390	not feasible	0.7931	0.7262	0.3665	0.1929	0.1687	-1.0006	-0.2467

(b) Efficient frontier

Figure 15: Application of Mean Variance Analysis II

Problems in implementing Mean-Variance Analysis

One problem in implementing Mean Variance Analysis is that it is data intensive. For example, there are more than 2,600 companies listed on the London Stock Exchange (LSE), more than 2,800 on the New York Stock Exchange, ... Thus, to determine the investment opportunity set, restricting ourselves say to the LSE, it is necessary to estimate 2,600 means and variances as well as 3,378,700 covariances!

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