CHAPTER 17

Financial Structure and Firm Valuation in Incomplete Markets

Chapter Outline

17.1 Introduction

We have thus far motivated the creation of financial markets by the fundamental need of individuals to transfer income across states of nature and across time periods. In Chapter 9 (Section 9.5), we initiated a discussion of the possibility of market failure in financial innovation. There we raised the possibility that coordination problems in the sharing of the benefits and the costs of setting up a new market could result in the failure of a Pareto-improving market to materialize. In reality, however, the bulk of traded securities are issued by firms with the view of raising capital for investment purposes, rather than by private individuals. It is thus legitimate to explore the incentives for security issuance, taking the viewpoint of the corporate sector. This is what we do in this chapter. Doing so involves touching upon a set of fairly wide and not fully understood topics. One of them is the issue of *security design*. This term refers to the various forms financial contracts can take (and to their properties), in particular, in the context of managing the relationship between a firm and its managers on the one hand, and financiers and owners on the other. We will not touch on these incentive issues here but will first focus on the following two questions.

17.1.1 What Securities Should a Firm Issue if the Value of the Firm is to be Maximized?

This question is, of course, central to standard financial theory and is usually resolved under the heading Modigliani–Miller (MM) Theorem (1958). The *MM* Theorem tells us that under a set of appropriate conditions, if markets are complete, the financial decisions of the firm are irrelevant (recall our discussion in Chapter 2). Absent any tax considerations in particular, whether the firm is financed by debt or equity has no impact on its valuation. Here we go one step further and rephrase the question in a context where markets are incomplete and a firm's financing decision modifies the set of available securities. In such a world, the firm's financing decisions are important for individuals as they may affect the possibilities offered to them for transferring income across states. In this context, is it still the case that the firm's financing decisions are irrelevant for its valuation? If not, can we be sure that the interests of the firm's owners as regards the firm's financing decisions coincide with the interests of society at large?

In a second step, we cast the same *security design* issue in the context of intertemporal investment that can be loosely connected with the finance and growth issues touched upon in Chapter 1. Specifically, we raise the following complementary question.

17.1.2 What Securities Should a Firm Issue if it is to Grow as Rapidly as Possible?

We first discuss the connection between the supply of savings and the financial market structure and then consider the problem of a firm wishing to raise capital from the market. The questions raised are important: Is the financial structure relevant for a firm's ability to obtain funds to finance its investments? If so, are the interests of the firm aligned with those of society?

17.2 Financial Structure and Firm Valuation

Our discussion will be phrased in the context of the following simple example. We assume the existence of a unique firm owned by an entrepreneur who wishes only to consume at date $t = 0$; for this entrepreneur, $U'(c_0) > 0$. The assumption of a single entrepreneur circumvents the problem of *shareholder unanimity*: If markets are incomplete, the firm's objective does not need to be the maximization of market value: shareholders cannot reallocate income across all dates and states as they may wish. By definition, there are missing markets. But then shareholders may well have differing preferred payment patterns by the firm—over time and across states—depending on the specificities of their own endowments. One shareholder, for example, may prefer investment project *A* because it implies the firm will flourish and pay high dividends in future circumstances when he himself will otherwise have a low income. Another shareholder may prefer the firm to undertake some other investment project or to pay higher current dividends because her personal circumstances are different. Furthermore, there may be no markets where the two shareholders could insure one another.

The firm's financial structure consists of a finite set of claims against the firm's period 1 output. These securities are assumed to exhaust the returns to the firm in each state of nature. Since the entrepreneur wishes to consume only in period 0, and yet his firm creates consumption goods only in period 1, he will want to sell claims against period 1 output in exchange for consumption in period 0.

The other agents in our economy are agents 1 and 2 of the standard Arrow–Debreu setting of Chapter 9, and we retain the same general assumptions:

1. There are two dates: 0, 1.
2. At date 1, N possible states of nature, indexed $\theta = 1, 2, \ldots, N$, with probabilities π_θ, may be realized. In fact, for nearly all that we wish to illustrate $N = 2$ is sufficient.
3. There is one consumption good.
4. Besides the entrepreneur, there are two consumer-investors, indexed $k = 1, 2$, with preferences given by

$$U_0^k(c_0^k) + \delta^k \sum_{\theta=1}^{N} \pi_\theta U^k(c_0^k) = \alpha c_\theta^k + E \ln c_\theta^k$$

and endowments $e_0^k, (e_\theta^k)_{\theta=1,2,\ldots,N}$. We interpret c_θ^k to be the consumption of agent k if state θ should occur, and c_0^k his period zero consumption. Agents' period utility functions are all assumed to be concave, α is the constant date 0 marginal utility, which, for the moment, we will specify to be 0.1, and the discount factor is unity (there is no time discounting). The endowment matrix for the two agents is assumed to be as given in Table 17.1.

Table 17.1: Endowment matrix

		Date $t = 1$	
	Date $t = 0$	State $\theta = 1$	State $\theta = 2$
Agent $k = 1$	4	1	5
Agent $k = 2$	4	5	1

Table 17.2: Cash flows at date $t = 1$

	$\theta = 1$	$\theta = 2$
Firm	2	2

Each state has probability $\frac{1}{2}$ (equally likely), and consumption in period 0 cannot be stored and carried over into period 1. Keeping matters as simple as possible, let us further assume that the cash flows to the firm are the same in each state of nature, as seen in Table 17.2.

At least two different financial structures could be written against this output vector:

$F_1 = \{(2,2)\}$—pure equity;[1]
$F_2 = \{(2,0),(0,2)\}$—Arrow–Debreu securities.[2]

From our discussion in Chapter 9, we expect financial structure F_2 to be more desirable to agents 1 and 2, because it better allows them to effect income (consumption) stabilization: F_2 amounts to a complete market structure with the two required Arrow–Debreu securities. Let us compute the value of the firm (what the claims to its output could be sold for) under both financial structures. Note that the existence of either set of securities affords an opportunity to shift consumption between periods. This situation is fundamentally different, in this way, from the pure reallocation examples in the pure exchange economies of Chapter 9.

17.2.1 Financial Structure F_1

Let q^e denote the price (in terms of date 0 consumption) of equity—security $\{(2, 2)\}$—and let z_1, z_2, respectively, be the quantities demanded by agents 1 and 2. In equilibrium, $z_1 + z_2 = 1$ since there is one unit of equity issued; holding z units of equity entitles the owner to a dividend of $2z$ in both states 1 and 2.

Agent 1 solves:

$$\max_{q^e z_1 \leq 4} (0.1)(4 - q^e z_1) + \tfrac{1}{2}[\ln(1 + 2z_1) + \ln(5 + 2z_1)]$$

Agent 2 solves:

$$\max_{q^e z_2 \leq 4} (0.1)(4 - q^e z_2) + \tfrac{1}{2}[\ln(5 + 2z_2) + \ln(1 + 2z_2)]$$

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[1] Equity is risk free here. This is the somewhat unfortunate consequence of our symmetry assumption (same output in the two date $t = 1$ states). The reader may want to check that our message carries over with a state $\theta = 2$ output of 3.

[2] We could assume, equivalently, that the firm issues two units of the two conceivable *pure* Arrow–Debreu securities, $(\{(1, 0), (0, 1)\})$.

Table 17.3: Equilibrium allocation

		$t=1$	
	$t=0$	θ_1	θ_2
Agent 1	$4-3\frac{1}{3}$	$1+1$	$5+1$
Agent 2	$4-3\frac{1}{3}$	$5+1$	$1+1$

Assuming an interior solution, the first-order conditions (FOCs) for agents 1 and 2 are, respectively,

$$z_1\colon \left(\frac{1}{10}\right)q^e = \frac{1}{2}\left[\frac{2}{1+2z_1}\right] + \frac{1}{2}\left[\frac{2}{5+2z_1}\right]$$

$$\frac{q^e}{10} = \left[\frac{1}{1+2z_1} + \frac{1}{5+2z_1}\right]$$

$$z_2\colon \frac{q^e}{10} = \left[\frac{1}{5+2z_2} + \frac{1}{5+2z_2}\right]$$

Clearly, $z_1 = z_2 = \frac{1}{2}$, and $\frac{q^e}{10} = \left[1/(1+1) + 1/(5+1)\right] = \left[1/2 + 1/6\right] = \frac{2}{3}$ or $q^e = \frac{20}{3}$. Thus, $V_{F_1} = q^e = \frac{20}{3} = 6\frac{2}{3}$, and the resulting equilibrium allocation is displayed in Table 17.3.

Agents are thus willing to pay a large proportion of their period 1 consumption in order to increase period 2 consumption. On balance, agents (except the entrepreneur) wish to shift income from the present (where $MU = \alpha = 0.1$) to the future and now there is a device by which they may do so.

Since markets are incomplete in this example, the competitive equilibrium need not be Pareto optimal. That is the case here. There is no way to equate the ratios of the two agents' marginal utilities across the two states: In state 1, the MU ratio is $\frac{1/2}{1/6} = 3$ while it is $\frac{1/2}{1/6} = \frac{1}{3}$ in state 2. A transfer of one unit of consumption from agent 2 to agent 1 in state 1 in exchange for one unit of consumption in the other direction in state 2 would obviously be Pareto improving. Such a transfer cannot, however, be effected with the limited set of financial instruments available. This is the reality of incomplete markets.

Note that our economy is one of three agents: agents 1 and 2, and the original firm owner. From another perspective, the equilibrium allocation under F_1 is not a Pareto optimum because a redistribution of wealth between agents 1 and 2 could be effected, making them both better off in *ex ante* expected utility terms while not reducing the utility of the firm owner (which is, presumably, directly proportional to the price he receives for the firm). In particular, the allocation that dominates the one achieved under F_1 is given in Table 17.4.

Table 17.4: A Pareto-superior allocation

		$t=1$	
	$t=0$	θ_1	θ_2
Agent 1	$\frac{2}{3}$	4	4
Agent 2	$\frac{2}{3}$	4	4
Owner	$6\frac{2}{3}$	0	0

17.2.2 Financial Structure F_2

This is a complete Arrow–Debreu financial structure. It will be notationally clearer here if we deviate from our usual notation and denote the securities as $X = (2, 0)$, $W = (0, 2)$ with prices q_X, q_W, respectively (q_X thus corresponds to the price of two units of the state-1 Arrow–Debreu security, while q_W is the price of two units of the state-2 Arrow–Debreu security), and quantities $z_X^1, z_X^2, z_W^1, z_W^2$. The problems confronting the agents are as follows.

Agent 1 solves:

$$\max \left(\frac{1}{10}\right)(4 - q_X z_X^1 - q_W z_W^1) + \left[\frac{1}{2}\ln(1 + 2z_X^1) + \frac{1}{2}\ln(5 + 2z_W^1)\right]$$
$$q_X z_X^1 + q_W z_W^1 \leq 4$$

Agent 2 solves:

$$\max \left(\frac{1}{10}\right)(4 - q_X z_X^2 - q_W z_W^2) + \left[\frac{1}{2}\ln(5 + 2z_X^2) + \frac{1}{2}\ln(5 + 2z_W^2)\right]$$
$$q_X z_X^2 + q_W z_W^2 \leq 4$$

The FOCs are:

$$\text{Agent 1:} \begin{cases} (i)\ \frac{1}{10} q_X = \frac{1}{2}\left(\frac{1}{1 + 2z_X^1}\right)2 \\ (ii)\ \frac{1}{10} q_W = \frac{1}{2}\left(\frac{1}{5 + 2z_W^1}\right)2 \end{cases}$$

$$\text{Agent 2:} \begin{cases} (iii)\ \frac{1}{10} q_X = \frac{1}{2}\left(\frac{1}{1 + 2z_X^2}\right)2 \\ (iv)\ \frac{1}{10} q_W = \frac{1}{2}\left(\frac{1}{5 + 2z_W^2}\right)2 \end{cases}$$

By equation (i): $\frac{1}{10}q_X = \frac{1}{1+2z_X^1} \Rightarrow 1 + 2z_X^1 = \frac{10}{q_X} \Rightarrow z_X^1 = \frac{5}{q_X} - \frac{1}{2}$

By equation (iii): $\frac{1}{10}q_X = \frac{1}{5+2z_X^2} \Rightarrow 5 + 2z_X^2 = \frac{10}{q_X} \Rightarrow z_X^2 = \frac{5}{q_X} - \frac{5}{2}$

With one security of each type issued:

$$z_X^1 + z_X^2 = 1 (z_X^1 \geq 0;\ z_X^2 \geq 0)$$
$$\frac{5}{q_X} - \frac{1}{2} + \frac{5}{q_X} - \frac{5}{2} = 1 \Rightarrow \frac{10}{q_X} = 4 \Rightarrow q_X = \frac{10}{4}$$

Similarly, $q_W = \frac{10}{4}$ (by symmetry) and $V_F = q_X + q_W = \frac{10}{4} + \frac{10}{4} = \frac{20}{4} = 5.$

So we see that V_F *has declined* from $6\frac{2}{3}$ in the F_1 case to 5. Let us further examine this result. Consider the allocations implied by the complete financial structure:

$$z_X^1 = \frac{5}{q_X} - \frac{1}{2} = \frac{5}{5/2} - \frac{1}{2} = 2 - \frac{1}{2} = 1\frac{1}{2}$$

$$z_X^2 = \frac{5}{q_X} - \frac{5}{2} = \frac{5}{5/2} - \frac{5}{2} = 2 - \frac{5}{2} = -\frac{1}{2}$$

$$z_W^1 = -\frac{1}{2},\ z_W^2 = 1\frac{1}{2} \text{by symmetry}$$

Thus, agent 1 wants to short sell security 2 while agent 2 wants to short sell security 1. Of course, in the case of financial structure $F_1(2, 2)$, there was no possibility of short selling since every agent in equilibrium must have the same security holdings. The post-trade allocation is found in Table 17.5. This, unsurprisingly, constitutes a Pareto optimum.[3]

Table 17.5: Post-trade allocation

$t = 0$
Agent 1: $4 - \left(1\frac{1}{2}\right)q_x + \frac{1}{2}q_w = 4 - \frac{3}{2}\left(\frac{10}{4}\right) + \frac{1}{2}\left(\frac{10}{4}\right) = 4 - \frac{10}{4} = 1\frac{1}{2}$
Agent 2 : $4 + \frac{1}{2}q_x - \frac{3}{2}q_w = 4 + \frac{1}{2}\left(\frac{10}{4}\right) - \frac{3}{2}\left(\frac{10}{4}\right) = 4 - \frac{10}{4} = 1\frac{1}{2}$
$t = 1$
*Agent*1: $(1, 5) + 1\frac{1}{2}(2, 0) - \frac{1}{2}(0, 2) = (4, 4)$
Agent 2: $(5, 1) + \left(-\frac{1}{2}\right)(2, 0) + 1\frac{1}{2}(0, 2) = (4, 4)$

3 Note that our example also illustrates the fact that the addition of new securities in a financial market does not necessarily improve the welfare of *all* participants. Indeed, the firm owner is made worse off by the transition from F_1 to F_2.

We have thus reached an important result that we summarize in Propositions 17.1 and 17.2.

Proposition 17.1 When markets are incomplete, the MM Theorem fails to hold and the financial structure of the firm may affect its valuation by the market.

Proposition 17.2 When markets are incomplete, it may not be in the interest of a value-maximizing firm to issue the socially optimal set of securities.

In our example the issuing of the *right* set of securities by the firm leads to completing the market and making a Pareto optimal allocation attainable. The impact of the financial decision of the firm on the set of markets available to individuals in the economy places us outside the realm of the MM Theorem; indeed, the value of the firm is not left unaffected by the choice of financing. Moreover, it appears that it is not, in this situation, in the private interest of the firm's owner to issue the socially optimal set of securities. Our example thus suggests that there is no reason to necessarily expect that value-maximizing firms will issue the set of securities society would find preferable.[4]

17.3 Arrow–Debreu and Modigliani–Miller

In order to understand why V_F declines when the firm issues the richer set of securities, it is useful to draw on our work on Arrow–Debreu pricing (Chapter 9). Think of the economy under financial structure F_2. This is a complete Arrow–Debreu structure in which we can use the information on equilibrium endowments to recompute the pure Arrow–Debreu prices as per Eq. (17.1),

$$q_\theta = \frac{\delta \pi_\theta \frac{\partial U^k}{\partial c^k_\theta}}{\frac{\partial U^k_0}{\partial c^k_0}},\ \theta = 1, 2 \tag{17.1}$$

which, in our example, given the equilibrium allocation (four units of commodity in each state for both agents) reduces to

$$q_\theta = \frac{1\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)}{0.1} = \frac{5}{4},\ \theta = 1, 2$$

which corresponds, of course, to

$$q_X = q_W = \frac{10}{4}$$

and to $V_F = 5$.

[4] The reader may object that our example is just that, an example. Because it helps us reach results of a negative nature, this example is, however, a fully general *counterexample*, ruling out the proposition that the MM Theorem continues to hold and that firms' financial structure decisions will always align with the social interest.

This Arrow−Debreu complete markets equilibrium is unique: this is generically the case in an economy such as ours, implying there are no other allocations satisfying the required conditions and no other possible prices for the Arrow−Debreu securities. This implies the MM proposition as the following reasoning illustrates. In our example, the firm is a mechanism to produce two units of output in date 1, both in states 1 and 2. Given that the date 0 price of one unit of the good in state 1 at date 1 is $\frac{5}{4}$ and the price of one unit of the good in state 2 at date 1 is $\frac{5}{4}$ as well, it must of necessity be that the price (value) of the firm is four times $\frac{5}{4}$, i.e., 5. In other words, absent any romantic love for this firm, no one will pay more than five units of the current consumption good (which is the numeraire) for the title of ownership to this production mechanism, knowing that the same bundle of goods can be obtained for five units of the numeraire by purchasing two units of each Arrow−Debreu security. The converse reasoning guarantees that the firm will also not sell for less. The value of the firm is thus given by its fundamentals and is independent of the specific set of securities the entrepreneur choses to issue: This is the essence of the MM Theorem!

Now let us try to understand how this reasoning is affected when markets are incomplete and why, in particular, the value of the firm is higher in that context. The intuition is as follows. In the incomplete market environment of financial structure F_1, security $\{(2, 2)\}$ is desirable for two reasons: to transfer income across time *and* to reduce date 1 consumption risk. In this terminology, the firm in the incomplete market environment is more than a mechanism to produce two units of output in either states of nature in date 1. The security issued by the entrepreneur is also the only available vehicle to reduce second-period consumption risk. Individual consumers are willing to pay something, i.e., to sacrifice current consumption, to achieve such risk reduction. To see that trading of security $\{(2, 2)\}$ provides some risk reduction in the former environment, we need only compare the range of date 1 utilities across states after trade and before trade for agent 1 (agent 2 is symmetric). See Table 17.6.

The premium paid for the equity security, over and above the value of the firm in complete markets, thus originates in the dual role it plays as a mechanism for consumption risk smoothing and as a title to two units of output in each future state. A question remains: Given that the entrepreneur, by his activity and security issuance, plays this dual role, why can't he reap the corresponding rewards independently of the security structure he chooses to issue? In other words, why is it that his incentives are distorted away from the socially optimal financial structure? To understand this, note that if any amount of Arrow−Debreu-like securities, such as in $F_2 = \{(2, 0), (0, 2)\}$ is issued, no matter how

Table 17.6: Agent 1 state utilities under F_1

	Before Trade	$\{(2, 2)\}$; $z^1 = 0.5$ (Equilibrium Allocation)
State 1	$U^1(c_1^1) = \ln 1 = 0$	$U^1(c_1^1) = \ln 2 = 0.693$
State 2	$U^1(c_2^1) = \ln 5 = 1.609$	$U^1(c_2^1) = \ln 6 = 1.792$
	Difference = 1.609	Difference = 1.099

Table 17.7: Allocation when the two agents trade Arrow–Debreu securities among themselves

	$t=0$	$t=1$ θ_1	$t=1$ θ_2
Agent 1	4	3	3
Agent 2	4	3	3

small, the market for such securities has effectively been created. With no further trading restrictions, the agents can then supply additional amounts of these securities to one another. This has the effect of empowering them to trade, entirely independently of the magnitude of the firm's security issuance, to the following endowment allocation (Table 17.7).

In effect, investors can eliminate all *second-period* endowment uncertainty *themselves*. Once this has been accomplished and markets are effectively completed (because there is no further demand for across-state income redistribution, it is irrelevant to the investor whether the firm issues {(2, 2)} or {(2, 0), (0, 2)}, since either package is equally appropriate for transferring income *across time periods*. Were {(2, 0), (0, 2)} to be the package of securities issued, each agent would buy equal amounts of (2, 0), and (0, 2), and effectively repackage them as (2, 2). To do otherwise would be to reintroduce date 1 endowment uncertainty. Thus, the relative value of the firm under either financial structure, {(2, 2)} or {(2, 0), (0, 2)}, is determined solely by whether the security (2, 2) is worth more to the investors in the environment of period 2 endowment uncertainty or when all risk has been eliminated as in the environment noted previously.

Said otherwise, once the markets have been completed, the value of the firm is fixed at 5 as we have seen before, and there is nothing the entrepreneur can do to appropriate the extra insurance premium. If investors can eliminate all the risk themselves (via short selling), there is no premium to be paid to the firm, in terms of value enhancement, for doing so. This is confirmed if we examine the value of the firm when security {(2, 2)} is issued *after* the agents have traded among themselves to equal second-period allocation (3, 3). In this case $V_F = 5$ also.

There is another lesson to be gleaned from this example and that leads us back to the CAPM. One implication of the CAPM was that securities could not be priced in isolation: their prices and rates of return depended on their interactions with other securities as measured by the covariance. This example follows in that tradition by confirming that the value of the securities issued by the firm is not independent of the other securities available on the market or which the investors can themselves create.

17.4 On the Role of Short Selling

From another perspective (as noted in Allen and Gale, 1994), short selling expands the supply of securities and provides additional opportunities for risk sharing, but in such a way that the benefits are not internalized by the innovating firm. When deciding what securities to issue, however, the firm only takes into account the impact of the security issuance on its own value; in other words, it only considers those benefits it can internalize. Thus, in an incomplete market setting, the firm may not issue the socially optimal package of securities.

It is interesting to consider the consequence of forbidding or making it impossible for investors to increase the supply of securities (2, 0) and (0, 2) via short selling. Accordingly, let us impose a no-short-selling condition (by requiring that all holdings of all securities by all agents are positive). Agent 1 wants to short sell (0, 2); agent 2 wants to short sell (2, 0). We thus know that the constrained optimum will have (simply setting $z = 0$ wherever the unconstrained optimum had a negative z and anticipating the market clearing condition):

$$z_X^2 = 0 \; z_W^1 = 0$$
$$z_X^1 = 1 \; z_W^2 = 1$$

$$\frac{1}{10} q_X = MU_1 = \frac{1}{2}\left(\frac{1}{1+2(1)}\right)2 = \frac{1}{3}$$
$$\frac{1}{10} q_W = MU_2 = \frac{1}{2}\left(\frac{1}{1+2(1)}\right)2 = \frac{1}{3}$$

$$q_X = \frac{10}{3}, q_W = \frac{10}{3}$$
$$V_F = \frac{20}{3} = 6\frac{2}{3}$$

which coincides with the valuations when the security (2, 2) was issued.

The fact that V_F increases when short sales are prohibited is not surprising since it reduces the supply of securities (2, 0) and (0, 2). With demand unchanged, both q_X and q_W increase, and with it, V_F. In some sense, now the firm has a monopoly in the issuance of (2, 0) and (0, 2), and that monopoly position has value. All this is in keeping with the general reasoning developed previously. While it is, therefore, not surprising that the value of the firm has risen with the imposition of the short sales constraint, the fact that its value has returned precisely to what it was when it issued {(2, 2)} is striking and possibly somewhat of a coincidence.

Is the ruling out of short selling realistic? In practice, short selling on the US stock exchanges is costly, and only a very limited amount of it occurs. The reason for this is that the short seller must deposit as collateral with the lending institution, as much as 100% of the value of the securities he borrows to short sell. Under current practice in the United States, the interest on this deposit is less than the T-bill rate even for the largest participants, and for small investors it is near zero. There are other exchange-imposed restrictions on short selling. On the NYSE, for example, investors are forbidden to short sell on a down-tick in the stock's price.[5]

17.5 Financing and Growth

Now we must consider our second set of issues, which we may somewhat more generally characterize as follows: How does the degree of completeness in the securities markets affect the level of capital accumulation? This is a large topic, touched upon in our introductory chapter, for which there is little existing theory. Once again we pursue our discussion in the context of examples.

Example 17.1 Our first example serves to illustrate the fact that, although a more complete set of markets is unambiguously good for welfare, it is not necessarily so for growth. Consider the following setup. Agents own firms (have access to a productive technology) while also being able to trade state-contingent claims with one another (net supply is zero). We retain the two-agent, two-period setting. Agents have state-contingent consumption endowments in the second period. They also have access to a productive technology which, for every k units of period one consumption foregone, produces $\sqrt{k}$ in date 1 in either state of nature[6] (Table 17.8).

The agent endowments are given in Table 17.9, and the agent preference orderings are now (identically) given by

$$EU(c_0, c_\theta) = \ln(c_0) + \frac{1}{2}\ln(c_1) + \frac{1}{2}\ln(c_2)$$

Table 17.8: The return from investing k units

$t = 1$	
θ_1 $\sqrt{k}$	θ_2 $\sqrt{k}$

[5] Brokers must obtain permission from clients to borrow their shares and relend them to a short seller. In the early part of 2000, a number of high-technology firms in the United States asked their shareholders to deny such permission as it was argued short sellers were depressing prices! Of course, if a stock's price begins rising, short sellers may have to enter the market to buy shares to cover their short position. This boosts the share price even further.

[6] Such a technology may not look very interesting at first sight! But, at the margin, agents may be very grateful for the opportunity it provides to smooth consumption across time periods.

Table 17.9: Agent endowments

		$t=1$	
	$t=0$	θ_1	θ_2
Agent 1	3	5	1
Agent 2	3	1	5
$\text{Prob}(\theta_1) = \text{Prob}(\theta_2) = \frac{1}{2}$			

In this context, we compute the agents' optimal savings levels under two alternative financial structures. In one case, there is a complete set of contingent claims, and in the other, the productive technology is the only possibility for redistributing purchasing power across states (as well as across time) among the two agents.

17.5.1 No Contingent Claims Markets

Each agent acts autonomously and solves

$$\max_k \ \ln(3-k) + \frac{1}{2}\ln(5+\sqrt{k}) + \frac{1}{2}(1+\sqrt{k})$$

Assuming an interior solution, the optimal level of savings k^* solves

$$-\frac{1}{3-k^*} + \left\{\frac{1}{2}\left(\frac{1}{5+\sqrt{k^*}}\right)\frac{1}{2}(k^*)^{-\frac{1}{2}} + \frac{1}{2}\left(\frac{1}{1+\sqrt{k^*}}\right)\frac{1}{2}(k^*)^{-\frac{1}{2}}\right\} = 0$$

which, after several simplifications, yields

$$3(k^*)^{\frac{3}{2}} + 15k^* + 7\sqrt{k^*} - 9 = 0$$

The solution to this equation is $k^* = 0.31$. With two agents in the economy, *economy-wide savings* are 0.62. Let us now compare this result with the case in which the agents also have access to contingent claims markets.

17.5.2 Contingent Claims Trading

Let q_1 be the price of a security that pays one unit of consumption if state 1 occurs, and let q_2 be the price of a security that pays one unit of consumption if state 2 occurs. Similarly, let $z_1^1, z_2^1, z_1^2, z_2^2$ denote, respectively, the quantities of these securities demanded by agents 1 and 2. These agents continue to have simultaneous access to the technology.

Agent 1 solves

$$\max_{k_1, z_1^1, z_2^2} \ln(3 - k_1 - q_1 z_1^1 - q_2 z_2^1) + \frac{1}{2}\ln(5 + \sqrt{k_1} + z_1^1) + \frac{1}{2}\ln(1 + \sqrt{k_1} + z_2^1)$$

Agent 2's problem is essentially the same:

$$\max_{k_2, z_1^2, z_2^2} \ln(3 - k_2 - q_1 z_1^2 - q_2 z_2^2) + \frac{1}{2}\ln(1 + \sqrt{k_2} + z_1^2) + \frac{1}{2}\ln(5 + \sqrt{k_2} + z_2^2)$$

By symmetry, in equilibrium

$$\begin{gathered} k_1 = k_2;\ q_1 = q_2; \\ z_1^1 = z_2^2 = -z_1^2, z_2^1 = z_1^2 = -z_2^2 \end{gathered}$$

Using these facts and the FOCs (see the Appendix), it can be directly shown that

$$-2 = z_1^1$$

and, it then follows that $k_1 = 0.16$. Thus, *total savings* $= k_1 + k_2 = 2k_1 = 0.32$.

Savings have thus been substantially reduced. This result also generalizes to situations of more general preference orderings, and to the case where the uncertainty in the states is in the form of uncertainty in the production technology rather than in the investor endowments. The explanation for this phenomenon is relatively straightforward, and it parallels the mechanism at work in the previous sections. With the opening of contingent claims markets, the agents can eliminate all second-period risk. In the absence of such markets, it is real investment that alone must provide for any risk reduction as well as for income transference across time periods—a dual role. In a situation of greater uncertainty, resulting from the absence of contingent claims markets, more is saved and the extra savings take, necessarily, the form of productive capital: there is a precautionary demand for capital. Jappelli and Pagano (1994) found traces of a similar behavior in Italy prior to recent measures of financial deregulation.

Example 17.2 This result also suggests that if firms want to raise capital in order to invest for date 1 output, it may not be value maximizing to issue a more complete set of securities, an intuition we confirm in our second example.

Consider a firm with access to a technology with the output pattern found in Table 17.10.

Investor endowments are given in Table 17.11. Their preference orderings are both of the form

$$EU(c_1, c_\theta) = \frac{1}{12}c_0 + \frac{1}{2}\ln(c_1) + \frac{1}{2}\ln(c_2)$$

Table 17.10: The firm's technology

	$t=1$	
$t=0$	θ_1	θ_2
$-k$	$\sqrt{k}$	$\sqrt{k}$

Table 17.11: Investor endowments

		$t=1$	
	$t=0$	θ_1	θ_2
Agent 1	12	$\frac{1}{2}$	10
Agent 2	12	10	$\frac{1}{2}$

17.5.3 Incomplete Markets

Suppose a security of the form (1, 1) is traded, at a price q; agents 1 and 2 demand, respectively, z_1 and z_2. The agent maximization problems that define their demand are as follows:

Agent 1:

$$\max_{qz_1 \leq 12} \left(\frac{1}{12}\right)(12 - qz_1) + \frac{1}{2}\ln\left(\frac{1}{2} + z_1\right) + \frac{1}{2}\ln(10 + z_1)$$

Agent 2:

$$\max_{qz_2 \leq 12} \left(\frac{1}{12}\right)(12 - qz_2) + \frac{1}{2}\ln(10 + z_2) + \frac{1}{2}\ln\left(\frac{1}{2} + z_2\right)$$

It is obvious that $z_1 = z_2$ at equilibrium. The FOCs are (again assuming an interior solution):

$$\text{Agent 1: } \frac{q}{12} = \frac{1}{2}\frac{1}{(\frac{1}{2} + z_1)} + \frac{1}{2}\frac{1}{(10 + z_1)}$$

$$\text{Agent 2: } \frac{q}{12} = \frac{1}{2}\frac{1}{(10 + z_2)} + \frac{1}{2}\frac{1}{(\frac{1}{2} + z_2)}$$

In order for the technological constraint to be satisfied, it must also be that

$$[q(z_1+z_2)]^{\frac{1}{2}} = z_1 + z_2, \text{ or}$$
$$q = z_1 + z_2 = 2z_1 \text{ as noted earlier}$$

Substituting for q in the first agent's FOC gives

$$\frac{2z_1}{12} = \frac{1}{2}\frac{1}{\left(\frac{1}{2} + z_1\right)} + \frac{1}{2}\frac{1}{(10 + z_1)} \quad \text{or}$$
$$0 = z_1^3 + 10.5z_1^2 - z_1 - 31.5$$

Trial and error gives $z_1 = 1.65$. Thus, $q = 3.3$ and total investment is $q = z_1 + z_2 = (3.3)(3.3) = 10.89 = V_F$; date 1 output in each state is thus $\sqrt{10.89} = 3.3$.

17.5.4 Complete Contingent Claims

Now suppose securities $R = (1, 0)$ and $S = (0, 1)$ are traded at prices q_R and q_S and denote quantities demanded, respectively, as $z_R^1, z_R^2, z_S^1, z_S^2$. The no-short sales assumption is retained. With this assumption, agent 1 buys only R, while agent 2 buys only security S. Each agent thus prepares himself for his worst possibility.

Agent 1:

$$\max_{0 \leq q_R z_R^1} \left(\frac{1}{12}\right)(12 - q_R z_R^1) + \frac{1}{2}\ln\left(\frac{1}{2} + z_R^1\right) + \frac{1}{2}\ln(10)$$

Agent 2:

$$\max_{0 \leq q_S z_S^2} \left(\frac{1}{12}\right)(12 - q_S z_S^2) + \frac{1}{2}\ln(10) + \frac{1}{2}\ln\left(\frac{1}{2} + z_S^2\right)$$

The FOCs are thus

$$\text{Agent 1:} \frac{q_R}{12} = \frac{1}{2}\frac{1}{(\frac{1}{2} + z_R^1)}$$

$$\text{Agent 2:} \frac{q_S}{12} = \frac{1}{2}\frac{1}{(\frac{1}{2} + z_S^2)}$$

Clearly, $q_R = q_S$ by symmetry, and $z_R^1 = z_S^2$; by the technological constraints:

$$(q_R z_R^1 + q_S z_S^2)^{\frac{1}{2}} = \left(\frac{z_R^1 + z_S^2}{2}\right) \text{ or}$$
$$q_R = \frac{z_R^1}{2}$$

Solving for z_R^1:

$$\frac{q_R}{12} = \frac{z_R^1}{24} = \frac{1}{2}\frac{1}{(\frac{1}{2} + z_R^1)} = \frac{1}{1 + 2z_R^1}$$

$$z_R^1(1 + 2z_R^1) = 24$$
$$z_R^1 = \frac{-1 \pm \sqrt{1 - 4(2)(-24)}}{4} = \frac{-1 \pm \sqrt{1 + 192}}{4} = \frac{-1 \pm 13.892}{4}$$

(taking positive root)

$$z_R^1 = 3.223$$
$$z_S^2 = 3.223$$
$$q_R = 1.61, \text{ and}$$

$$q_R(z_R^1 + z_S^2) = 1.61(6.446) = 10.378 = V_F$$

As suspected, this is less than what the firm could raise issuing only (1, 1).

Much in the spirit of our discussion of Section 17.2, this example illustrates the fact that, for a firm wishing to maximize the amount of capital levied from the market, it may not be a good strategy to propose contracts leading to a (more) complete set of markets. This is another example of the failure of the MM Theorem in a situation of incomplete markets, and the reasoning is the same as before: in incomplete markets, the firm's value is not necessarily equal to the value, computed at Arrow–Debreu prices, of the portfolio of goods it delivers in future date states. This is because the security it issues may, in addition, be valued by market participants for its unintended role as an insurance mechanism, a role that disappears if markets are complete. In the growth context of our last examples, this may mean that more savings will be forthcoming when markets are incomplete, a fact that may lead a firm wishing to raise capital from the markets to refrain from issuing the optimal set of securities.

17.6 Conclusions

We have reached a number of conclusions in this chapter.

1. In an incomplete market context, it may not be value maximizing for firms to offer the socially optimal (complete) set of securities. This follows from the fact that, in a production setting, securities can be used not only to handle risk reduction but also to transfer income across dates. The value of a security will depend upon its usefulness in accomplishing these alternative tasks.
2. The value of securities issued by the firm is not independent of the supply of similar securities issued by other market participants. To the extent that others can increase the supply of a security initially issued by the firm (via short selling), its value will be reduced.
3. Finally, welfare is promoted by the issuance of a more complete set of markets, but growth may not be.[7] As a result, it may not be in the best interest of a firm aiming at maximizing the amount of capital it wants to raise, to issue the most socially desirable set of securities.

All these results show that if markets are incomplete, the link between private interests and social optimality is considerably weakened. Here lies the intellectual foundation for financial market regulation and supervision.

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[7] The statement regarding welfare is strictly true only when financial innovation achieves full market completeness. Hart (1975) shows that it is possible that everyone is made worse off when the markets become more complete but not fully complete (say, going from 9 to 10 linearly independent securities when 15 would be needed to make the markets complete).

Appendix: Details of the Solution of the Contingent Claims Trade Case of Section 17.5

Agent 1 solves:

$$\max_{k_1, z_1^1, z_2^1} \ln(3 - k_1 - q_1 z_1^1 - q_2 z_2^1) + \frac{1}{2}\ln(5 + \sqrt{k_1} + z_1^1) + \frac{1}{2}\ln(1 + \sqrt{k_1} + z_2^1)$$

$$k_1: \frac{-1}{3 - k_1 - q_1 z_1^1 - q_2 z_2^1} + \frac{1}{2}\left(\frac{1}{5 + \sqrt{k_1} + z_1^1}\right)\frac{1}{2}k_1^{-\frac{1}{2}} + \frac{1}{2}\left(\frac{1}{1 + \sqrt{k_1} + z_2^1}\right)\frac{1}{2}k_1^{-\frac{1}{2}} = 0 \quad (17.2)$$

$$z_1^1: \frac{-q_1}{3 - k_1 - q_1 z_1^1 - q_2 z_2^1} + \frac{1}{2}\left(\frac{1}{5 + \sqrt{k_1} + z_1^1}\right) = 0 \quad (17.3)$$

$$z_2^1: \frac{-q_2}{3 - k_1 - q_1 z_1^1 - q_2 z_2^1} + \frac{1}{2}\left(\frac{1}{1 + \sqrt{k_1} + z_2^1}\right) = 0 \quad (17.4)$$

Agent 2's problem and FOC are essentially the same:

$$\max_{k_2, z_1^2, z_2^2} \ln(3 - k_2 - q_1 z_1^2 - q_2 z_2^2) + \frac{1}{2}\ln(1 + \sqrt{k_2} + z_1^2) + \frac{1}{2}\ln(5 + \sqrt{k_2} + z_2^2)$$

$$k_2: \frac{-1}{3 - k_2 - q_1 z_1^2 - q_2 z_2^2} + \frac{1}{2}\left(\frac{1}{1 + \sqrt{k_2} + z_1^2}\right)\frac{1}{2}k_2^{-\frac{1}{2}} + \frac{1}{2}\left(\frac{1}{5 + \sqrt{k_2} + z_2^2}\right)\frac{1}{2}k_2^{-\frac{1}{2}} = 0 \quad (17.5)$$

$$z_1^2: \frac{-q_1}{3 - k_2 - q_1 z_1^2 - q_2 z_2^2} + \frac{1}{2}\left(\frac{1}{1 + \sqrt{k_2} + z_1^2}\right) = 0 \quad (17.6)$$

$$z_2^2: \frac{-q_2}{3 - k_2 - q_1 z_1^2 - q_2 z_2^2} + \frac{1}{2}\left(\frac{1}{5 + \sqrt{k_2} + z_2^2}\right) = 0 \quad (17.7)$$

By symmetry, in equilibrium

$$k_1 = k_2;\ q_1 = q_2;$$
$$z_1^1 = z_2^2 = -z_1^2,\ z_2^1 = z_1^2 - z_2^2$$

By Eqs. (17.3) and (17.6), using the fact that $z_1^1 + z_2^1 = z_2^2 + z_1^2$:

$$\frac{1}{5 + \sqrt{k_1} + z_1^1} = \frac{1}{1 + \sqrt{k_2} + z_1^2}$$

Eqs. (17.4) and (17.7):

$$\frac{1}{1+\sqrt{k_1}+z_2^1}=\frac{1}{5+\sqrt{k_2}+z_2^2}$$

The equations defining k_1 and z_1^1 are thus reduced to

$$k_1\colon \frac{1}{3-k_1-q_1z_1^2-q_2z_2^2}+\frac{1}{4}\frac{1}{\sqrt{k_1}}\left(\frac{1}{5+\sqrt{k_1}+z_1^1}\right)+\frac{1}{4}\frac{1}{\sqrt{k_1}}\left(\frac{1}{1+\sqrt{k_1}-z_1^1}\right)=0 \qquad (17.8)$$

$$z_1^1\colon \frac{1}{5+\sqrt{k_1}+z_1^1}=\frac{1}{1+\sqrt{k_1}-z_1^1} \qquad (17.9)$$

Solving for k_1, z_1^1, yields from Eq. (17.8)

$$1+\sqrt{k_1}-z_1^1=5+\sqrt{k_1}+z_1^1$$
$$-4=2z_1^1$$
$$-2=z_1^1$$

Substituting this value into Eq. (17.8) gives

$$\frac{1}{3-k_1}=\frac{1}{4}\frac{1}{\sqrt{k_1}}\left\{\frac{1}{5+\sqrt{k_1}-2}+\frac{1}{1+\sqrt{k_1}+2}\right\}$$
$$\frac{1}{3-k_1}=\frac{1}{4}\frac{1}{\sqrt{k_1}}\left\{\frac{1}{3+\sqrt{k_1}}\right\}$$

$$4\sqrt{k_1}\left\{3+\sqrt{k_1}\right\}=2(3-k_1),\quad \text{or simplifying}$$
$$-6+12\sqrt{k_1}+6k_1=0$$
$$-1+2\sqrt{k_1}+k_1=0$$

Let $X=\sqrt{k_1}$

$$X=\frac{-2\pm\sqrt{4-4(1)(-1)}}{2}=\frac{-2\pm\sqrt{8}}{2}$$

$$X=-1+\sqrt{2}=-1+1.4$$
$$X=0.4$$

$k_1=0.16$ and total savings $=k_1+k_2=2k_1=0.32$