Fama-MacBeth regressions*

Jonas Nygaard Eriksen**

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*This note provides an overview of the Fama-MacBeth two-pass cross-sectional regression methodology. The note is based on Fama and MacBeth (1973) as well as drawing on the excellent textbook treatments in Campbell, Lo and MacKinlay (1997) and Cochrane (2005). The note is prepared for use only in the Master's course "Asset Pricing". Please do not cite, circulate, or use for purposes other than this course.

**CREATES, Department of Economics and Business Economics, Aarhus University, Fuglesangs Allé 4, DK-8210 Aarhus V, Denmark. E-mail: jeriksen@econ.au.dk.

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1. Introduction

One of the most fundamental questions in asset pricing is why different assets earn different average rates of return. A particularly well known and long-standing puzzle is that firms with high book-to-market equity (BE/ME) ratios tend to consistently deliver higher average rates of return than firms with low BE/ME ratios within a given size quintile (Fama and French, 1992). Asset pricing models, albeit only abstractions of reality, have the ability to provide answers to this and other similarly long-standing questions and, in turn, improve our understanding of financial markets. Standard asset pricing theory centers on a single core insight: that excess returns are compensation for bearing risk. Naturally, this notion is imbedded into all existing asset pricing models. They differ, however, in their specification and definition of risk. The Capital Asset Pricing Model (CAPM), for example, defines the market portfolio as the single source of risk in the economy. Other models include additional risk factors or use marginal utility of consumption as the driving force. We will introduce such models later in the course. Finance is a fortunate field that has an abundance of naturally generated data, something that has led the development of theory to go hand-in-hand with empirical analyses. As Campbell (2000) puts it: "*Theorists develop models with testable predictions; empirical researchers document "puzzles" — stylized facts that fail to fit established theories — and this stimulates the development of new theories*." Estimating and evaluating the empirical performance of asset pricing models against a set of stylized facts is therefore an essential part of asset pricing as a discipline.

Engaging in a discussion of methods and approaches to empirically testing and evaluating asset pricing models and their predictions therefore seems worthwhile. In the case of linear models, such as the CAPM, a natural candidate method is the widely applied two-pass cross-sectional regression method suggested in Fama and MacBeth (1973). Their approach is simple and ideal for testing and evaluating linear asset pricing models in the cross-sectional dimension (i.e., on average rates of return), and it provides estimates that can be directly used to test and make inference on model-implied predictions. As an example, it directly allows for a natural way to empirically gauge and test the central prediction of a positive relation between risk (β_{iM}) and expected return implied by the CAPM. Similarly, it allows us to test if β_{iM} is the only source of risk that matter for the determination of expected returns in the cross-section. The implementation of the two-pass regression methodology consists of (i) a set of time-series regressions and (ii) a set of cross-sectional regressions. In the first step, we regress excess portfolio returns onto a constant and risk factors (e.g., the market portfolio in the case of the CAPM) to obtain time series estimates of β_{iM} for each test asset. This step is necessary as β_{iM}s are not directly observable in the market and therefore needs to be estimated. In the second step, we run cross-sectional regression of excess portfolio returns onto the estimated $\widehat{\beta}_{iM}$s to obtain an estimate of the factor's price of risk. The Fama and MacBeth (1973) two-pass cross-sectional regression methodology is featured heavily in empirical work, is straightforward to implement, and will be the workhorse and method of choice in this course.

The rest of the note evolves as follows. Section 2 introduces the notation and sets the stage for empirically testing and evaluating the CAPM. Section 3 outlines the Fama and MacBeth (1973) methodology and discusses its strengths and weaknesses. Section 4 presents an empirical application based on Fama and French (1992, 1993) in which we test the CAPM using size (ME) and value (BE/ME) portfolios.

2. Notation and testable implications

The Capital Asset Pricing Model (CAPM) will serve as our example throughout this note, but it is important to emphasize that the Fama and MacBeth (1973) two-pass cross-sectional regression methodology can be used to test and evaluate any linear asset pricing model with one or more risk factors. The CAPM is an equilibrium model defined by the linear relation

$$\mathbb{E}\left[\widetilde{r}_i\right] = r_f + \beta_{iM}\left(\mathbb{E}\left[\widetilde{r}_M\right] - r_f\right), \tag{1}$$

where $\mathbb{E}\left[\widetilde{r}_i\right]$ denotes the expected return on asset i, r_f is the risk-free rate, $\mathbb{E}\left[\widetilde{r}_M\right] - r_f$ is the market risk premium, and β_{iM} is a measure of the systematic risk of asset i. We can represent the model in a variety of ways that better facilitates cross-sectional tests and evaluations, but we will favor a specification that relies on the original CAPM definition in (1) and that expresses β_{iM} risk as

$$\mathbb{E}\left[\widetilde{r}_i\right] = r_f + \gamma_M \beta_{iM}, \tag{2}$$

where $\gamma_M = \mathbb{E}\left[\widetilde{r}_M\right] - r_f$ is defined as the risk premium on the market portfolio. β_{iM} can be interpreted as the amount of risk and γ_M as the price of risk (per unit of β_{iM} risk). In empirical applications, we tend to work with the excess return on asset i instead, which prompts us to write the model as

$$\mathbb{E}\left[\widetilde{r}_i\right] - r_f = \gamma_0 + \gamma_M \beta_{iM}, \tag{3}$$

where γ_0 is introduced as a constant that, if the model is true, should be statistically indistinguishable from zero. This is the formulation of the CAPM that we will work with in our empirical applications as it directly allows us to test relevant theoretical predictions implied by the model. It is important to stress that this is nothing more than a different way of representing the model — usually referred to as a *beta-pricing model*. Beta-pricing model representations are common in empirical work, and we will frequently write asset pricing models in this style accordingly. In this representation, β_{iM} is viewed as an input (regressor) and γ_0 and γ_M are the objects of interest (parameters) to estimate. Clearly, if β_{iM} is unobservable in the wild, then we need to estimate it before we can use it as an input variable in the second-pass cross-sectional regressions. This is what the Fama and MacBeth (1973) cleverly incorporates into the estimation and testing procedure that we specify below.

Testable implications The CAPM delivers two central predictions for the formulation in (3) that can be empirically evaluated and tested. Specifically,

1. $\gamma_0 = 0$: β_{iM} is all that matters for the expected excess return on asset i and
2. $\gamma_M > 0$: there is a positive relation between risk and return

The Fama and MacBeth (1973) two-pass regressions offer a simple, yet powerful, way to evaluate these implications. In addition to the outlined testable prediction, the CAPM implies that the estimated value of γ_M should be equal to the market risk premium.[1] The reason is clear: suppose that we use the model of interest to estimate β_{iM} using historical data on the market risk premium and asset i, then we should be able to back out the average market risk premium if β_{iM} is a good description of the riskiness of asset i. More formally, the market portfolio has $\beta_{MM} = 1$ so that $\gamma_M = \mathbb{E}\left[\widetilde{r}_M\right] - r_f$ must be the case if the model is true and satisfies no arbitrage. We can also note that the parameters γ_0 and γ_M can be interpreted as the intercept and slope, respectively, of the Security Market Line (SML) for excess returns. The observations that $\gamma_M = \mathbb{E}\left[\widetilde{r}_M\right] - r_f$ follows directly from this interpretation as well and additionally offers a natural interpretation of the estimates that can be compared to theory.

These implications will be our focus going forward in this note. Importantly, however, the Fama and MacBeth (1973) cross-sectional regression method is more general and allows us to test a variety of other relevant hypotheses. For example, the implication that β_{iM} is the only thing that matters can also be tested by introducing additional variables into the model

$$\mathbb{E}\left[\widetilde{r}_i\right] - r_f = \gamma_0 + \gamma_M \beta_{iM} + \gamma_X X_i, \tag{4}$$

where the additional implication is $\gamma_X = 0$ for the CAPM to hold true. That is, only β_{iM} should matter when explaining the heterogeneity in observed mean excess returns across the different test assets used in the empirical analysis. For example, the CAPM implies a linear relation between risk and returns. We can test this by added squared betas, i.e., $X_i = \beta_{iM}^2$, where $\gamma_X \neq 0$ would provide evidence for a non-linear relationship.

3. The Fama-MacBeth methodology

The Fama and MacBeth (1973) methodology is a cross-sectional regression method that consists of two-steps. We outline the steps for the CAPM here, but emphasize that they easily generalize to multifactor models. The first step estimates time-series betas β_{iM} from full sample time series regressions. The second step estimates the price of risk γ_M through a series of cross-sectional regressions using the estimated $\widehat{\beta}_{iM}$ from the first step as regressor.

[1] This is true for any traded risk factor and therefore also true for the traded factor models of Fama and French (1993), Fama and French (2015) and Hou, Xue and Zhang (2015).

3.1. First-pass regressions

In the first step of the Fama and MacBeth (1973) two-pass cross-sectional regression approach, we estimate the unobservable β_{iM} for each test asset $i = 1, \ldots, N$ using a single full sample, time series regression of the form

$$\widetilde{r}_{t,i} - \widetilde{r}_{t,f} = \alpha_i + \beta_{iM} \left(\widetilde{r}_{t,M} - \widetilde{r}_{t,f} \right) + \widetilde{\varepsilon}_{t,i}, \tag{5}$$

where $\widetilde{\varepsilon}_{t,i}$ is a zero-mean error term. Although our object of interest is β_{iM}, the constant α_i holds a special place in finance theory and is known as Jensen's alpha (Jensen, 1968). The parameters can be estimated using standard Ordinary Least Squares (OLS). One typically estimates the model using the full sample of available observations under the assumption that β_{iM} is constant over time.[2] Fama and MacBeth (1973) originally used rolling-window betas, but Cochrane (2005) argues that little is gained from this exercise over simply using full sample estimation.[3] We will adopt this simpler approach for convenience.

3.2. Second-pass regressions

In the second step of the Fama and MacBeth (1973) approach, we run cross-sectional regressions for all assets at each point in time $t = 1, \ldots, T$ of the form

$$\widetilde{r}_{t,i} - \widetilde{r}_{t,f} = \gamma_{t,0} + \gamma_{t,M} \widehat{\beta}_{iM} + \eta_{t,i}, \tag{6}$$

where the estimated value $\widehat{\beta}_{iM}$ is obtained from the first-pass time series regressions and $\eta_{t,i}$ is a mean-zero error term. Estimating these T cross-sectional regressions provides us with a time series of estimates of $\{\widehat{\gamma}_{t,0}, \widehat{\gamma}_{t,M}\}$, which can be used to form estimates of $\widehat{\gamma}_0$ and $\widehat{\gamma}_M$ as follows

$$\widehat{\gamma}_j = \frac{1}{T} \sum_{t=1}^{T} \widehat{\gamma}_{t,j}, \tag{7}$$

where $j = \{0, M\}$. Similarly, we can form an estimate of the average pricing error as $\widehat{\eta}_i = \frac{1}{T} \sum_{t=1}^{T} \widehat{\eta}_{t,i}$. However, the most important contribution in Fama and MacBeth (1973) is their suggestion for computing standard errors. Specifically, they suggest that we use the standard deviations of the cross-sectional regression estimates to generate the sampling errors for the estimates. Formally,

$$\sigma^2 \left(\gamma_j \right) = \frac{1}{T^2} \sum_{t=1}^{T} \left(\widehat{\gamma}_{t,j} - \widehat{\gamma}_j \right)^2 \tag{8}$$

[2] This, of course, is an assumption that you might rightfully question. It is more likely, however, to hold true for portfolios (as we consider here), but it is unlikely to hold if considering individual assets.

[3] See also Fama and French (1992) for a discussion of the impact of rolling versus full sample betas.

where we use T^{-2} because we are computing standard errors for sample means (σ^2/T). The standard errors are robust to cross-sectional correlation, but assumes that the time series of $\gamma_{t,j}$ is i.i.d normal. With an estimate and an associated standard error in hand, we can conduct testing and inference using standard t-tests. In particular, suppose that we consider the following null hypothesis

$$\mathcal{H}_0 : \gamma_j = 0, \tag{9}$$

which we can test using the following test statistic

$$t(\gamma_j) = \frac{\widehat{\gamma}_j}{\sigma(\gamma_j)} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as} \quad T \to \infty. \tag{10}$$

In finite samples, the test statistic is student-t distributed with $T - K - 1$ degrees of freedom, where K denotes the number of factors used in the model. However, it approaches a standard normal distribution for a "large enough" sample size, as indicated in (10). One can naturally entertain many different hypotheses using this framework. Moreover, we can easily extend the method to consider linear asset pricing models with more than a single risk factor, such as the Fama and French (1993), the Fama and French (2015), or the Hou et al. (2015) models, and form estimates for the additional risk prices in the same way.

Obtaining second-pass estimates in a single regression We note that the above procedure is quite different from a traditional cross-sectional regression in which one would simply run a single cross-sectional regression on averages. In the case where we are only interested in the estimates of $\widehat{\gamma}_0$ and $\widehat{\gamma}_M$, then this is in fact a feasible approach. Specifically, we can run a single cross-sectional regression of the form

$$\mathbb{E}[\widetilde{r}_i] - \widetilde{r}_f = \gamma_0 + \gamma_M \widehat{\beta}_{iM} + \eta_i, \tag{11}$$

where $\mathbb{E}[\widetilde{r}_i] - \widetilde{r}_f = \frac{1}{T}\sum_{t=1}^{T} \widetilde{r}_{t,i} - \widetilde{r}_{t,f}$ is the sample average of the excess return to asset i. This will provide us with estimates of $\widehat{\gamma}_0$ and $\widehat{\gamma}_M$ that are numerically identical to those from the full-fledged procedure, but it is very important to note that the usual OLS standard errors will be **very** wrong and are likely to lead to inconsistent inference.

3.3. Drawbacks and issues

Although the Fama and MacBeth (1973) approach is intuitive and straightforward to implement, it does not come without drawbacks and problematic issues. Specifically, using estimated values of β_{iM} as regressors in the second step introduces a so-called errors-in-variables (EIV) problem that originates from $\widehat{\beta}_{iM}$ being measured with error, especially if working with individual assets directly. The effect of this is typically to downward bias the estimate of $\widehat{\gamma}_M$ and $\widehat{\gamma}_0$ upwards (in the case of the CAPM). Generally speaking, the risk premium on the factors are downward biased and any intercept and/or residuals are biased upwards. One way to minimize this problem, proposed by Fama and MacBeth (1973), is to use portfolios instead of individual assets. The insight is that when using portfolios we can expect that some of the individual noise from the first-step regressions to average out and, consequently, reduce the measurement error in $\widehat{\beta}_{iM}$. Note, however, that this is usually not enough to alleviate the EIV issue. Alternatively, one can adjust the standard errors to account for the errors-in-variables problem — but this is outside the scope of this course. There exist two popular approaches to do so. The first is a direct adjustment proposed by Shanken (1992) in which the standard errors are inflated by a scalar adjustment factors. These standard errors are, like the Fama and MacBeth (1973) ones, derived under the assumption of i.i.d normal errors. An alternative approach that jointly accounts for the EIV problem and works under heteroskedasticity and/or autocorrelation is to map the entire system into the Generalized Method of Moment (GMM) approach of Hansen (1982), which would eliminate the errors-in-variables problem as β and γ are then jointly estimated. Additionally, heteroskedasticity and autocorrelation robust inference can be made using, say, Newey and West (1987) HAC standard errors. Kroencke and Thimme (2021) provide a recent overview of ways to deal with the issue and compares their performance in realistic settings.

A final issue with the Fama and MacBeth (1973) method is that the risk premia estimates depend on whether you include control variables, and which control variables that you include. Specifically, estimates of risk premia in linear asset pricing models are biased if some priced factors are omitted. Giglio and Xiu (2021) provide a discussion of how to deal with this issue. Overall, while the two-pass cross-sectional regression method is easy to implement and heavily used in applied work, there are several issues that one should be keenly aware off in empirical work and accounting for these issues is increasingly important.

4. Empirical application

This section provides an empirical illustration of the Fama and MacBeth (1973) two-pass cross-sectional regression method using the CAPM as the asset pricing model of choice. We wish to study the ability of the CAPM to explain the cross-section of expected returns to a set of well known and widely studied portfolios sorted on size (ME) and book-to-market equity (BE/ME) ratios. Recall that a standard way to judge whether a given asset pricing model is a good abstraction of reality is to assess its ability to reproduce (or explain) stylized facts and anomalies. The portfolios originate from the seminal work by Banz (1981) in which smaller ME stocks tend to outperform bigger ME stocks and Fama and French (1992, 1993) in which firms with high book-to-market equity (BE/ME) ratios (value firms) earn higher average rates of return relative to firms with low book-to-market-equity (BE/ME) ratios (growth firms).

Panel A of Table 1 reports annualized mean excess returns for these 25 portfolios for the period 1963:07 to 2019:12. A number of important observations emerge. First, the excess

Table 1: Descriptive statistics for the 25 size-value sorted portfolios

	Book-to-market equity (BE/ME)				
Size (ME)	Low	2	3	4	High
	Panel A: Mean annualized returns				
Small	3.19	9.56	9.16	11.76	12.50
2	6.12	9.40	10.49	10.66	11.53
3	6.44	9.31	9.24	10.33	11.97
4	7.67	7.60	8.31	10.02	10.30
Large	6.43	6.28	6.98	6.18	7.60
	Panel B: Time series betas				
Small	1.42	1.24	1.11	1.03	1.06
2	1.40	1.18	1.07	1.02	1.13
3	1.32	1.13	1.01	0.98	1.06
4	1.22	1.08	1.00	0.97	1.08
Large	0.98	0.93	0.85	0.89	0.95

returns are increasing with the book-to-market equity ratio within each size quintile. This is the so-called value premium: high BE/ME (value) firms, on average, earn a higher rate of return than low BE/ME (growth) firms (Fama and French, 1992). Second, for most book-to-market equity quintiles, excess returns are decreasing in firm size. This is the so-called size premium: small firms, on average, earn a higher rate of return than large firms (Banz, 1981). These portfolios have a special significance in empirical asset pricing studies and has long been the standard set of portfolios used to evaluate asset pricing models. This is changing, however, and future work should make use of the abundance of portfolio sorts available for empirical work.

4.1. Time series betas

We begin our analysis by estimating first-pass, time series betas using the time series regression in (5) using the full sample of available observations. In the regression, $\widetilde{r}_{t,i}$ refers to the return on one of the $i = 1, \ldots, 25$ size-value sorted portfolios. The estimates of $\widehat{\beta}_{iM}$ are presented in Panel B of Table 1. If the CAPM is to be considered a valid model, then there should be a clear and positive relation between risk (β_{iM}) and expected returns. That is, a higher $\widehat{\beta}_{iM}$ should translate into a higher expected return. Yet, such a relation is hard to identify. In fact, while excess returns are increasing with book-to-market equity ratios, betas are declining. This is particularly evident for growth and value firms in the smallest size quintile. In all fairness, however, things are looking somewhat better when studying size within each book-to-market equity ratio quintile, but there is no clear indication of the strictly positive risk-reward relation implied by the CAPM.

4.2. Cross-sectional risk prices

The next step in the analysis is to estimate the price of risk for the risk factor, the market portfolio, by way of running the T cross-sectional regressions in (6), one for each period $t = 1, \ldots, T$, to obtain a time series of cross-sectional risk price estimates $\{\gamma_{t,j}\}$ from which we can compute means and standard errors using the formulas in (7) and (8), respectively. Finally, we can conduct standard inference and hypothesis testing using (10).

Table 2: Fama-MacBeth estimates of the CAPM on 25 size-value portfolios

	γ_0	γ_M	R^2 (%)
Estimate	1.11	-0.35	7.38
s.e.	(0.37)	(0.40)	
t-stat	[2.99]	[-0.87]	

Table 2 presents the results from the second-pass regression. Several important observations emerge. First, the risk price γ_M, although insignificant (t-stat of -0.87), is negative. This is a strong contradiction of the positive risk-reward trade-off implied by the CAPM ($\gamma_M > 0$) and inconsistent with the observed positive equity premium of 0.54% per month (recall that $\gamma_M = \mathbb{E}\left[\widetilde{r}_M\right] - r_f$). The results here, in contrast, points to a negative risk-reward trade-off — taking on more risk leads to lower expected returns. Second, the intercept γ_0 is large in economic terms and statistically significant (t-stat of 2.99). Under the CAPM, $\gamma_0 = 0$, which is clearly not the case here and prompts us to conclude that β_{iM} may not be the only source of risk relevant to investors. Last, we note that the CAPM is able to explain only a small fraction, about 7%, of the variation in expected portfolio returns when evaluated using 25 portfolios sorted on size and book-to-market equity ratios. This does not conspire confidence in the model as, if the CAPM is to be considered a valid model, it should explain a substantially larger fraction of the cross-sectional variation in expected returns.

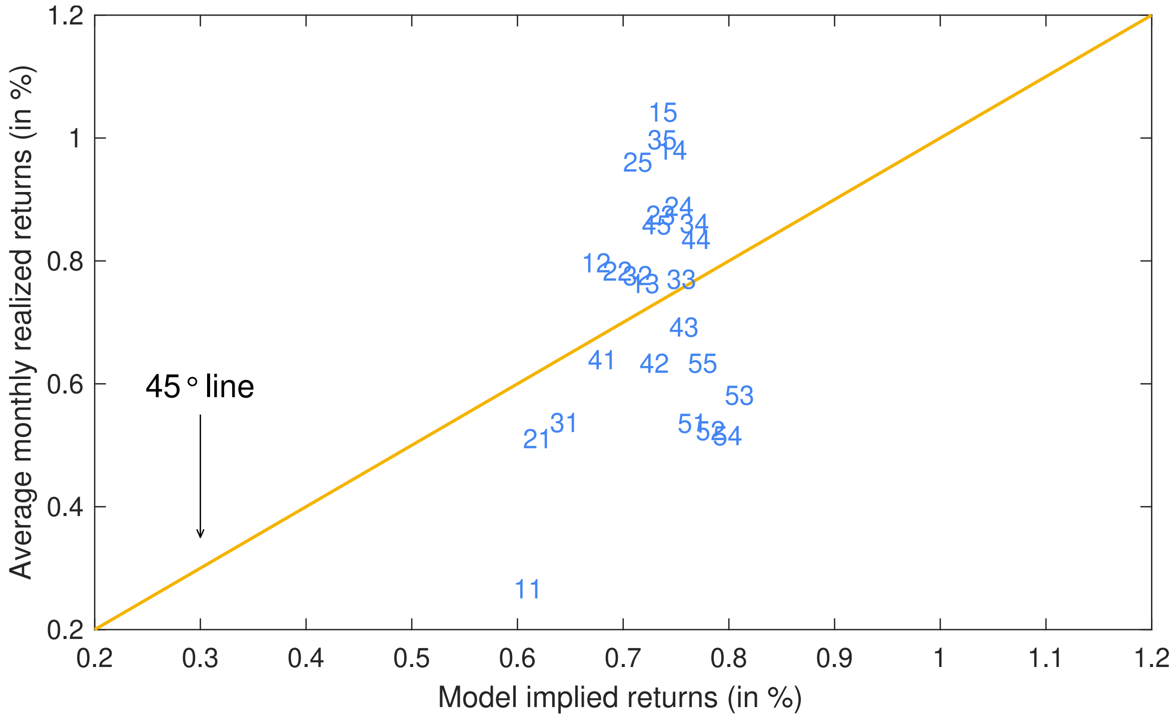
4.3. Pricing error plot

The above results indicate that the CAPM is not a good model for explaining the cross-section of expected returns on portfolios sorted on size (ME) and book-to-market equity (BE/ME) ratios. As a final component of the empirical illustration, we can compare the realized average excess returns to those implied by our estimated asset pricing model: the CAPM. This corresponds to using the estimated $\widehat{\beta}_{iM}$ values and the estimates of $\widehat{\gamma}_0$ and $\widehat{\gamma}_M$ to calculate model-implied (fitted) excess returns. Formally,

$$\text{fit}_i = \widehat{\gamma}_0 + \widehat{\gamma}_M \widehat{\beta}_{iM}. \tag{12}$$

Figure 1 gauges the performance of the model by plotting realized average excess returns against model-implied excess returns to graphically assess the ability of the model to reproduce the cross-sectional variation in expected returns. If the CAPM is a good description of reality, then all

Figure 1: Pricing error plot for 25 size-value portfolios



points should lie on the yellow 45° degree line. As Figure 1 highlights, this is clearly not the case. Specifically, while there is large variation in realized average excess returns, the CAPM-implied excess returns are nearly identical and so the model is unable to reproduce the spread in observed mean excess returns. We conclude that the CAPM is a poor description of reality and, specifically, that is unable to explain the returns to portfolios sorted on size and book-to-market equity ratios.

References

Banz, R.W., 1981. The relationship between return and market value of common stocks. Journal of Financial Economics 9, 3–18.

Campbell, J.Y., 2000. Asset pricing at the millennium. Journal of Finance 55, 1515–1567.

Campbell, J.Y., Lo, A.W., MacKinlay, A.C., 1997. The econometrics of financial markets. Princeton University Press.

Cochrane, J.H., 2005. Asset Pricing. Revised ed., Princeton University Press.

Fama, E.F., French, K.R., 1992. The cross-section of expected stock returns. Journal of Finance 47, 427–465.

Fama, E.F., French, K.R., 1993. Common risk factors in the returns on stocks and bonds. Journal of Financial Economics 33, 3–56.

Fama, E.F., French, K.R., 2015. A five-factor asset pricing model. Journal of Financial Economics 116, 1–22.

Fama, E.F., MacBeth, J.D., 1973. Risk, return, and equilibrium: Empirical tests. Journal of Political Economy 81, 607–636.

Giglio, S., Xiu, D., 2021. Asset pricing with omitted factors. Journal of Political Economy 129, 1947–1990.

Hansen, L.P., 1982. Large sample properties of generalized method of moments estimators. Econometrica 50, 1029–1054.

Hou, K., Xue, C., Zhang, L., 2015. Digesting anomalies: An investment approach. Review of Financial Studies 28, 650–705.

Jensen, M.C., 1968. The performance of mutual funds in the period 1945–1964. Journal of Finance 23, 389–416.

Kroencke, T.A., Thimme, J., 2021. A skeptical appraisal of robust asset pricing tests. Working paper, University of Neuchâtel.

Newey, W.K., West, K.D., 1987. A simple positive-definite, heteroskedasticity and autocorrelation consistent covariance matrix. Econometrica 55, 703–708.

Shanken, J., 1992. On the estimation of beta-pricing models. Review of Financial Studies 5, 1–33.