# Contents

# 1 Computational aspects

In the paper, we consider the problems with infinite variables which have not been amenable to numerical computation. Continuing to work in one-dimensional cases, we relaxed the problems by considering the discrete measures:

$$\mu := \sum_{i=1}^{n_1} p_i \delta_{x_i}, \ \nu := \sum_{j=1}^{n_2} q_j \delta_{y_j}, \ \sigma := \sum_{k=1}^{n_t} r_k \delta_{t_k}, \tag{1}$$

where $p_i, \ q_j, r_k$ are elements of probability vectors $\mathbf{p} \in \mathbb{R}_+^{n_1}$, $\mathbf{q} \in \mathbb{R}_+^{n_2}$, $\mathbf{r} \in \mathbb{R}_+^{n_t}$ such that

$$\left\{ p_i \in \mathbf{p}, q_j \in \mathbf{q}, r_k \in \mathbf{r} \sum_{i=1}^{n_1} p_i = \sum_{j=1}^{n_2} q_j = \sum_{k=1}^{n_t} r_k = 1 \right\}$$

with $\delta_x$,$\delta_y$ and $\delta_t$ are the Dirac at position $x$ and $y$ respectively.

## 1.1 Linear programming formulation of Problem $1'$

Implementing the discrete measures $\mu, \ \nu, \ \sigma$ from Eq.(1) in Problem $\mathbf{1'}$, the cost can be rewritten in a matrix form:

$$\mathbf{C}_{i,j,k} = c(x_i, y_j, t_k) = \frac{x_i^2}{t_k} + \frac{y_i^2}{1 - t_k}.$$

The discretized version of the optimization Problem **1'** then reads

$$\begin{aligned} \underset{\pi \in \Pi(\mathbf{p},\mathbf{q},\mathbf{r})}{\arg\min} \ \langle \mathbf{C}, \pi \rangle &\stackrel{\text{def}}{=} \sum_{i,j,k} c(x_i, y_j, t_k) \cdot \pi_{i,j,k}, \\ \text{s.t.} \ \ r_k &\leq \mathbf{R}, \ \forall k, \end{aligned}$$

in which $\mathbf{R} = \bar{r}\delta_{t_k}$ is the constant flow-rate constraint at the $t_k$ moment. The set of admissible transportation plans $\Pi(\mathbf{p},\mathbf{q},\mathbf{r})$ is defined as

$$\begin{aligned} \Pi(\mathbf{p},\mathbf{q},\mathbf{r}) = \Big\{ \pi \in \mathbb{R}^{n_1 \times n_2 \times n_t} : \ \sum_{j,k} \pi_{i,j,k} = \mathbf{p}, \\ \sum_{i,k} \pi_{i,j,k} = \mathbf{q}, \ \sum_{i,j} \pi_{i,j,k} = \mathbf{r}, \ \pi \succeq 0 \Big\}. \end{aligned}$$

This optimization problem, which has $(n_1 \times n_2 \times n_t)$ variables, $(n_1 + n_2)$ equality marginal constraints on the $x, y$-axis, and $n_t$ inequality constraints on the $t$-axis.

## 1.2 Formulation of the two tolls generalization

For the generalized multimarginal problems, we first consider the case when two tolls located at $\xi_1, \xi_2$ between the two supports that have the same flow-rate constraint. The discretized transportation cost matrix $\mathbf{C}_{i,j,k,l}^{\xi_1,\xi_2} \in \mathbb{R}^{n_1} \times$

$\mathbb{R}^{n_2} \times \mathbb{R}^{t_1} \times \mathbb{R}^{t_2}$ can be defined as

$$\begin{aligned}\mathbf{C}^{\xi_1,\xi_2}_{i,j,k,l} &= c^{\xi_1,\xi_2}(x_i, y_j, t^{(1)}_k, t^{(2)}_l)\\ &= \frac{(\xi_1 - x_i)^2}{t^{(1)}_k} + \frac{(\xi_2 - \xi_1)^2}{t^{(2)}_l - t^{(1)}_k} + \frac{(y_j - \xi_2)^2}{t_f - t^{(2)}_l}.\end{aligned}$$

Redefine variables $(x_0, x_f, t_1, t_2)$ by $(x_i, y_j, t^{(1)}_k, t^{(2)}_l)$ for the discretized problem, and consider the additional time marginal on the extra $t_2$-axis, the linear optimization problem now reads

$$\begin{aligned}&\underset{\pi\in\Pi(\mathbf{p},\mathbf{q},\mathbf{r}^{(1)},\mathbf{r}^{(2)})}{\arg\min} \quad \langle \mathbf{C}^{\xi_1,\xi_2}, \pi\rangle\\ &\overset{\text{def}}{=} \sum_{i,j,k,l} c_{\xi_1,\xi_2}(x_i, y_j, t^{(1)}_k, t^{(2)}_l)\cdot \pi_{i,j,k,l}\\ &\text{s.t} \;\; r^{(1)}_k,\; r^{(2)}_l \le \mathbf{R},\; \forall k,\; l.\end{aligned} \tag{2}$$

with

$$\Pi = \Big\{ \pi \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{t_1} \times \mathbb{R}^{t_2} : \sum_{j,k,l} \pi = \mathbf{p}, \\ \sum_{i,k,l} \pi = \mathbf{q},\; \sum_{i,j,l} \pi = \mathbf{r^{(1)}},\; \sum_{i,j,k} \pi = \mathbf{r^{(2)}},\; \pi \succeq 0 \Big\}.$$

Furthermore, the element-wise cost in (1.2) can be also weighted by weight $\varepsilon > 0$ and has the formula:

$$\begin{aligned}\mathbf{C}^{\xi_1,\xi_2}_{i,j,k,l} &= c^{\xi_1,\xi_2}(x_i, y_j, t^{(1)}_k, t^{(2)}_l, \varepsilon)\\ &= \varepsilon\cdot\frac{(\xi_1 - x_i)^2}{t^{(1)}_k} + \frac{(\xi_2 - \xi_1)^2}{t^{(2)}_l - t^{(1)}_k} + \varepsilon\cdot\frac{(y_j - \xi_2)^2}{t_f - t^{(2)}_l},\end{aligned}$$

so that with $0 < \varepsilon < 1$, the time of the mass spending between the two tolls is penalized.

## 1.3 Two tolls with one separating mass

The discretization version of the problem can be considered as a combination of the previous two problems. Now we are seeking two transportation plans, the first joint probability distribution $\pi_1(x, y, t_1, t_2)$ which provided the map between a pair of $(p, u)$ and $\pi_2$ between $(q, v)$ with the corresponding cost $c^{\xi_1,\xi_2}$ and $c^{\xi_2}$. The optimization problem then can be formulated as

$$\begin{aligned}&\underset{\substack{\pi_1\in\Pi(\mathbf{p},\mathbf{u},\mathbf{r}^{(1)},\mathbf{r}^{(2)});\\ \pi_2\in\Pi(\mathbf{q},\mathbf{v},\mathbf{r}^{(2)})}}{\arg\min} \quad \langle \mathbf{C}^{\xi_1,\xi_2}, \pi_1\rangle + \langle \mathbf{C}^{\xi_2}, \pi_2\rangle\\ &\text{s.t.} \quad r^{(1)}_k \le \mathbf{R_1},\; \forall\, k.\\ &\qquad (r^{(2)}_l(\pi_1) + r^{(2)}_l(\pi_2)) \le \mathbf{R_2},\; \forall\, l.\end{aligned}$$

## 2 Numerical simulations

All the problems above can be formulated as linear programming problems with linear equality constraints on the marginals and inequality constraints on the flow-rate. Herein, we solve them using the optimization toolbox-CVX [1] with solver MOSEK in its best precision mode. Our code used to conduct all the experiments can be accessed at `https://github.com/dytroshut/OMT-with-Flux-rate-Constraint`. Our approach is for theoretical analysis and illustration purposes and the complexity of the algorithms can be efficiently reduced by considering spares matrices and other specialized optimization tools for linear programming problems. To present a clear view of the transported masses' trajectory for readers in Section 3, we placed the two discrete measures $\mu, \nu$ on two parallel $x, y$-axis and a vertical $t$-axis at the location of the toll, the point-wise trajectories with respect to time are colored in grey.

## References

[1] Michael Grant and Stephen Boyd. CVX: Matlab software for disciplined convex programming, version 2.1, 2014.