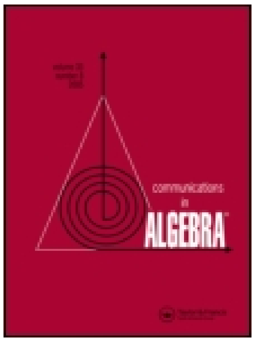

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Exchange Rings, Units and Idempotents

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Abstract

An associative ring R with identity is semiperfect if and only if every element of R is a sum of a unit and an idempotent, and R contains no infinite set of orthogonal idempotents. A ring which contains no infinite set of orthogonal idempotents is an exchange ring if and only if every element is a sum of a unit and an idempotent.

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A module $_RM$ has the *(full) exchange property* if for every module $_RA$ and any two decompositions

$$A = M' \oplus N = \oplus_{i\in I} A_i$$

with $M' \cong M$, there exist submodules $A'_i \subset A_i$ such that

$$A = M' \oplus (\oplus_{i\in I} A'_i).$$

Module $_RM$ has the *finite exchange property* if the above condition is satisfied whenever the index set I is finite.

Warfield [14] called a ring R an *exchange ring* if the left regular module $_RR$ has the finite exchange property and showed that this definition is left-right symmetric. He also proved that a module $_RM$ has the finite exchange property if and only if $end_R M$ is an exchange ring. The first element-wise characterization of exchange rings was given by Monk [8]: a ring R is an exchange ring if and only if for every $a \in R$, there exist $b, c \in R$ such that $bab = b$ and $c(1-a)(1-ba) = 1 - ba$.

Nicholson [9], on the other hand, gave another characterization of exchange rings: a ring R is an exchange ring if and only if idempotents can be lifted modulo every left (equivalently, right) ideal of R. In addition, he called a ring R a *clean ring* if every element of R is a sum of a unit and an idempotent and proved that clean rings are always exchange rings, and the converse is true when idempotents are central in R. The latter result of Nicholson has been extended to a larger class of rings by Yu [17], i.e. for rings whose maximal left (or right) ideals are two-sided ideals, they are exchange rings if and only if they are clean. But, as far as we can determine, it appears to be an open question whether exchange rings are exactly clean rings in general (cf. [17]).

The main aim of this paper is to clarify the relationship of exchange rings, clean rings, potent rings (see definition below) and semiperfect rings.

Following Nicholson [9], a ring R is called a *potent ring* if idempotents can be lifted modulo $J(R)$, the Jacobson radical of R, and every left (equivalently, right) ideal not contained in $J(R)$ contains a nonzero idempotent. We prove: (1) semiperfect rings and unit-regular rings are clean rings. (2) for a ring R which contains no infinite set of orthogonal idempotents, R is exchange if and only if R is potent, if and only if R is clean, if and only if R is semiperfect. As an immediate corollary, we get a new element-wise characterization of semiperfect rings: a ring R is semiperfect if and only if every element of R is a sum of a unit and an idempotent, and R contains no infinite set of orthogonal idempotents. We also give an example which shows that there exists an exchange ring which is not clean.

Throughout, rings are associative with identity and modules are unitary. $J(R)$ always stands for the Jacobson radical of a ring R.

The following proposition was proved by Nicholson [11] (Theorem 4.3) without the assumption that idempotents lift modulo $J(R)$. We include a short argument in the following particular case to make our paper self-contained.

Proposition 1 *Let R be a potent ring. Then R is either semiperfect, or else contains an infinite set of orthogonal idempotents.*

Proof. Assume R contains no infinite set of orthogonal idempotents, then it is easy to see that

$$1 = e_1 + e_2 + \cdots + e_n$$

with primitive orthogonal idempotents $e_i, i = 1, 2, \cdots, n$.

Denote $R' = R/J(R)$ and $1' = e'_1 + e'_2 + \cdots + e'_n$. Since R is potent, $R'e'_i$ contains no non-trivial submodule and hence is simple. So R' is Artinian semisimple and R is semiperfect. □

Combining Proposition 1 with the fact that exchange rings are all potent ([9], Proposition 1.9), we have

Corollary 2 *Every exchange ring is either semiperfect, or else contains an infinite set of orthogonal idempotents.* □

It is easy to see that, for any ring R, R contains no infinite set of orthogonal idempotents implies that

$$1 = e_1 + e_2 + \cdots + e_n$$

with e_i's primitive orthogonal idempotents. We do not know if the converse implication hold in general. For exchange rings, the two conditions are equivalent.

Proposition 3 *For an exchange ring R, the following are equivalent:*

(1) R contains no infinite set of orthogonal idempotents;

(2) $1 = e_1 + e_2 + \cdots + e_n$ with e_i's primitive orthogonal idempotents.

Proof. (1) $\Longrightarrow$ (2): Trivial.

(2) $\Longrightarrow$ (1): Primitive idempotents of an exchange ring are local idempotents (i.e. eRe is a local ring) by Warfield [15] (Proposition 1). Assuming (2), we have that R is semiperfect, since

$$1 = e_1 + e_2 + \cdots + e_n$$

with e_iRe_i local. It is well known that semiperfect rings contain no infinite set of orthogonal idempotents. □

The following is Theorem 8 of Yamagata [16]. The original proof there requires a long series of lemmas and propositions.

Corollary 4 *Let R be a ring which is a direct sum of indecomposable right ideals, then the following are equivalent:*

(1) R is right perfect;

(2) every projective right R-module has the exchange property;

(3) the countably generated free module $R_R^{(N)}$ has the finite exchange property.

Proof. (1) $\Longrightarrow$ (2): It is a quite standard argument to show that a ring R has the property in (2) if and only if $R/J(R)$ has the same property and $J(R)$ is right T-nilpotent (see, for example, [13], Theorem 4.1 (2)). Semisimple rings clearly have the property in (2), so do perfect rings.

(2) $\Longrightarrow$ (3): Trivial.

(3) $\Longrightarrow$ (1): It is easy to check that the exchange property and the finite exchange property are inherited by direct summands, so R is an exchange ring. By our Corollary 2 and Proposition 3, R is semiperfect. But the finite exchange property of $R_R^{(N)}$ forces $J(R)$ to be right T-nilpotent (cf. [9], Proposition 2.11), hence R is right perfect. □

Recall that a ring R is called a clean ring if every element of R is a sum of a unit and an idempotent. Nicholson [9] observed that the $n \times n$ matrix ring over an algebraically closed field is clean. We now extend this to unit-regular rings, which answers an question of Nicholson (privately communicated): Is every unit-regular ring a clean ring?

Theorem 5 *Every unit-regular ring is a clean ring.*

Proof. Denote $V = R_R$. Every element of R defines, through left multiplication, a R-homomorphism of the R-module V. Take $L \in R$, since R is regular, we have

$$V = (ImL + kerL) \oplus Y = kerL \oplus A$$

Also, $ImL + kerL$ is projective and ImL is a direct summand. So, $ImL + kerL = ImL \oplus X$ for some $X \subset kerL$. By the unit-regularity of R ([5], Theorem 4.1),

$$kerL \cong cokerL = V/ImL \cong X \oplus Y$$

So $kerL \cong X \oplus Y$ and $ImL \cong A$. Let $h : kerL \to X \oplus Y$ be an isomorphism and extend h to all of V by letting $h(A) = 0$. Let $u = h^{-1} : X \oplus Y \to kerL$ be a partial inverse of h, i.e. $hu \mid_{X\oplus Y} = 1_{X\oplus Y}$. Then $huh = h$ and $e = hu$ is an idempotent, no matter how u is extended to all of V. We show that u can be extended in such a way that $L - hu$ is monic, hence a unit by the regularity of R.

Suppose we have extended u in some way and that for $v \in V, L(v) = hu(v)$. Then by construction of $h, L(v) \in (X \oplus Y) \cap ImL = 0$.

Write $ImL = K \oplus Z$, where $K = ImL \cap kerL$ (which is possible since all these modules are finitely generated). Now let $\phi : ImL \to A$ be an isomorphism. Define $u : V \to V$ by $u(x+y+k+z) = h^{-1}(x+y)+k+\phi(k+z)$, for $x \in X, y \in Y, k \in K, z \in Z$. This is monic since $u(x + y + k + z) = 0$ implies $h^{-1}(x + y) + k + \phi(k + z) = 0$ giving $h^{-1}(x + y) + k = 0$ and $\phi(k+z) = 0$; thus $k+z = 0, k = 0$ and, finally, $x+y = 0$, by various direct sum decompositions.

Next, if $L(v) = hu(v), v = x + y + k + z$, as above, we have, as already established, $L(v) = 0$ and $u(v) \in A$. Now $u(v) = h^{-1}(x+y)+k+\phi(k+z) \in A$ and $\phi(k + z) \in A$. Hence, $h^{-1}(x + y) + k \in A \cap kerL = 0$. Thus, $k \in kerL \cap A = 0$. It follows that $x + y = 0$ as well. Hence $v = z \in Z$. But, also, $L(z) = 0$ so $z \in InL \cap kerL = K$, so $z = 0$. $\square$

The following Corollary 6 removes the " algebraically closed " requirement in Nicholson's observation above.

Corollary 6 *Let D be a division ring, then the $n \times n$ matrix ring $M_n(D)$ over D is clean for $n \geq 1$.* □

The following proposition has been observed by Nicholson [9] (p272.) without proof. In fact, it appears that no proof is available in literature. We include a proof for the convenience of reader.

Proposition 7 *A ring R is clean if and only if $R/J(R)$ is clean and idempotents can be lifted modulo $J(R)$.*

Proof. One direction is trivial, for clean rings are exchange, whence idempotents can be lifted modulo $J(R)$, and units and idempotents are mapped to units and idempotents by the canonical map from R to $R/J(R)$, respectively.

For the other direction, let $x \in R$ and $\overline{x} = \overline{e} + \overline{u}$, where $e^2 - e \in J(R)$ and $\overline{uv} = \overline{vu} = \overline{1}$ for some $v \in R$. Since idempotents can be lifted modulo $J(R)$, we may assume that $e^2 = e$. Also, we can assume that u is a unit in R, because $uv = 1 + r_1, vu = 1 + r_2$ for some $r_1, r_2 \in J(R)$. Then $x = e + u + r$ for some $r \in J(R)$. But $J(R) = \{r \in R | r + a$ is a unit for every unit $a \in R\}$ (To see this, note that $(r + a)^{-1} = (1 + a^{-1}r)^{-1}a^{-1}$.), so we get x written as a sum of a unit and an idempotent, which completes the proof. □

Call a ring R *semi-unit-regular* if idempotents lift modulo $J(R)$ and $R/J(R)$ is unit-regular. Semi-unit-regular rings are exchange rings by Corollary 2.4 and Proposition 1.6 of Nicholson [9].

Theorem 8 *For a semi-unit-regular ring R, $M_n(R)$ is clean for all $n \geq 1$.*

Proof. $M_n(R)$ is an exchange ring, since exchange rings are Morita invariant ([14], Theorem 2). So idempotents can be lifted modulo $J(M_n(R)) =$

$M_n(J)$ where $J = J(R)$. Now the result follows from Theorem 5 and Proposition 7, since $M_n(R)/J(M_n(R)) \cong M_n(R/J)$ and R/J is a unit-regular ring. □

In Goodearl [5], a ring R is called strongly regular if for every $x \in R$, there exists an element $y \in R$ such that $x = x^2y$ and strongly regular rings are unit-regular (cf. [5]). For a ring R, let $X = Spec\mathbf{B}(R)$ where $\mathbf{B}(R)$ is the Boolean algebra of central idempotents of R. For $x \in X$, the Pierce stalk of R at x is defined to be $R_x = R/Rx$. We note that the following classes of rings are all contained in the class of semi-unit-regular rings:(1) local rings; (2) rings which are strongly regular modulo their Jacobson radical and idempotents lift modulo Jacobson radical; (3) rings whose Pierce stalks are local and whose Jacobson radical factor ring is strongly regular (see Theorem 3.4 of Burgess and Stephenson [2]).

The next result is a new element-wise characterization of semiperfect rings.

Theorem 9 *A ring R is semiperfect if an only if R is clean and contains no infinite set of orthogonal idempotents.*

Proof. Sufficiency is clear from Proposition 1 since clean rings are exchange rings.

For necessity, we may assume $J(R) = 0$ since idempotents can be lifted modulo $J(R)$ in a semiperfect ring. Noticing the fact that direct product of clean rings is clean, it follows immediately from Corollary 6 that semisimple Artinian rings are clean. □

For any positive integer n, let $U_n(R)$ denote the set of elements of R which can be written as a sum of no more than n units of R. A ring R is called *generated by its units* if $R = \bigcup_{n=1}^{\infty} U_n(R)$. Rings generated by their units has been studied by several authors (cf. [12], [7], [4], etc.). The best results on this topic so far we know of are: (1) every unit regular ring in which 2 is invertible is generated by its units (Ehrlich [3]); (2) every strongly π-regular ring (see definition below) in which 2 is invertible is generated by its units (Fisher and Snider [4], Theorem 3). Our next result subsumes and sharpens all these in Theorem 11. Call an element $t \in R$ a square root of 1 if $t^2 = 1$.

Proposition 10 *Let R be a ring in which 2 is invertible, then R is clean if and only if every element of R is a sum of a unit and a square root of 1.*

Proof. Suppose R is clean and $x \in R$, then $(x+1)/2 = e + u$ with $e^2 = e$ and u a unit in R. So $x = (2e-1) + 2u$ with $(2e-1)^2 = 1$ and $2u$ a unit in R.

Conversely, if every element of R is a sum of a square root of 1 and a unit, then given $x \in R$ we have $2x - 1 = t + u$ with $t^2 = 1$ and u a unit of R. So $x = (t+1)/2 + u/2$ with $((t+1)/2)^2 = (t+1)/2$ and $u/2$ a unit of R. □

Call a ring R *strongly π-regular* if the descending chain $Ra \supseteq Ra^2 \supseteq \cdots \supseteq Ra^n \supseteq \cdots$ terminates for every $a \in R$. It is well known that this definition is left-right symmetric. By a result of Burgess and Menal ([1], Proposition 2.6 (iii).), all strongly π-regular rings are clean rings. But even commutative clean rings need not to be strongly π-regular (see [9],example 1.7). So the following Theorem 11 is a nontrivial generalization and improvement of Theorem 3 of [4].

Theorem 11 *Every element of a clean ring (e.g. unit-regular, or strongly π-regular) in which 2 is invertible is a sum of a unit and a square root of 1. In particular, $R = U_2(R)$.* □

From this approach, we now know that there exists an exchange ring which is not clean. Bergman ([6], Example 1) has constructed a directly finite regular ring R with 2 invertible in R and R is not generated by its units. Since regular rings are exchange (Warfield [14], Theorem 3), this ring is an exchange ring which is not clean.

Now the picture of relationships among all these classes of rings is as follows: *semiperfect* $\subsetneq$ *clean* $\subsetneq$ *exchange* $\subsetneq$ *potent*. All the containments are proper: Nicholson [9] example 1.7 justifies the first and third one, while the example of Bergman above justifies the second. However, when the ring contains no infinite set of orthogonal idempotents, they all collapse to one.

Corollary 12 *For a ring R containing no infinite set of orthogonal idempotents, the following are equivalent:*

(1) R is semiperfect;

(2) R is clean;

(3) R is exchange;

(4) R is potent. □

Rings of finite uniform (or, Goldie) dimensions contain no infinite set of orthogonal idempotents by definition, so Corollary 12 particularly applies to these rings.

A projective module is called *semiperfect* if every homomorphic image of it has a projective cover. A module is called *finite-dimensional* if it contains no infinite direct sum of nonzero submodules. Nicholson [10] proved that if a finite-dimensional module M is either injective or semiperfect, then its endomorphism ring $endM$ is a semiperfect ring (Corollary 3.4, p.1118.). Since injective modules and semiperfect modules both have the exchange property (for the latter, see Stock [13] comments after the proof of Theorem 3.1, p.443.), we conclude by generalising and sharpening the above result of Nicholson to the following

Theorem 13 *For a finite-dimensional module M, the following are equivalent:*

(1) M has the finite exchange property;

(2) M has the full exchange property;

(3) $endM$ is a semiperfect ring.

Proof. (1) $\Longrightarrow$ (2): Zimmermann-Huisgen and Zimmermann [18] proved that for a module with an indecomposable decomposition, the finite exchange property implies the full exchange property (Corollary 6). A finite-dimensional module has an indecomposable decomposition.

(2) $\Longrightarrow$ (3): We know that $endM$ is an exchange ring by Warfield [14] (Theorem 2), and $endM$ contains no infinite set of orthogonal idempotents since M is finite-dimensional. Now the proof follows from Corollary 12.

(3) $\Longrightarrow$ (1): A module has the finite exchange property if and only if its endomorphism ring is an exchange ring ([14], Theorem 2.) and semiperfect rings are exchange rings ([14], Theorem 3.). □

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References

[1] W. D. Burgess and P. Menal, On strongly π-regular rings and homomorphisms into them, *Comm. Alg.* **16** (1988), 1701-1725.

[2] W. D. Burgess and W. Stephenson, Rings all of whose Pierce stalks are local, *Canad. Math. Bull.* **22** (1979), 159-164.

[3] G. Ehrlich, Unit-regular rings, *Portugal. Math.* **27** (1968), 209-212.

[4] J. W. Fisher and R. L. Snider, Rings generated by their units, *J. Alg.* **42** (1976), 363-368.

[5] K. R. Goodearl, Von Neumann Regular Rings, *Pitman* (London, San Francisco, Melbourne), 1979.

[6] D. Handelman, Perspectivity and cancellation in regular rings, *J. Alg.* **48** (1977), 1-16.

[7] M. Henriksen, Two classes of rings generated by their units, *J. Alg.* **31** (1974), 182-193.

[8] G. S. Monk, A characterization of exchange rings, *Proc. Amer. Math. Soc.* **35** (1972), 344-353.

[9] W. K. Nicholson, Lifting idempotents and exchange rings, *Trans. Amer. Math. Soc.***229**(1977), 269-278.

[10] W. K. Nicholson, Semiregular modules and rings, *Canad. Math. J.* **28** (1976), 1105-1120.

[11] W. K. Nicholson, I-rings, *Trans. Amer. Math. Soc.* **207** (1975), 361-373.

[12] R. Raphael, Rings which are generated by their units, *J. Alg.* **28** (1974), 199-205.

[13] J. Stock, On rings whose projective modules have the exchange property, *J. Alg.* **103** (1986), 437-453.

[14] R. B. Warfield, Exchange rings and decompositions of modules, *Math. Ann.*199 (1972), 31-36.

[15] R. B. Warfield, A Krull-Schmidt theorem for infinite sums of modules, *Proc. Amer. Math. Soc.* **22** (1969), 460-465.

[16] K. Yamagata, On projective modules with the exchange property, *Sci. Rep. Tokyo Koiku Daigaku* **12 Sec A.** (1974), 149-158.

[17] H. P. Yu, On quasi-duo rings, *Glasgow Math. J.* (to appear).

[18] B. Zimmermann-Huisgen and W. Zimmermann, Classes of modules with the exchange property, *J. Alg.* **88** (1984), 416-434.

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