

VE477

Introduction to Algorithms

Assignment 8

Manuel — UM-JI (Fall 2017)

Reminders

- Write in a neat and legible handwriting or use \LaTeX
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

Ex. 1 — Fast multi-point evaluation and interpolation

Let R be a commutative ring, u_0, \dots, u_{n-1} be n elements in R , and $m_i = X - u_i$, with $0 \leq i < n$, be n degree 1 polynomials in $R[X]$. Without loss of generality we assume n to be a power of 2.

Part I — Fast multi-point evaluation

In order to perform fast multi-point evaluation the set of points $U = \{u_0, \dots, u_n\}$ is recursively split into two halves of equal cardinality.

1. Draw the binary tree resulting from the recursive split of the set U .
2. Denote the depth of the binary tree by k and for all $0 \leq i \leq k$ and $0 \leq j < 2^{k-i}$, define $M_{i,j} = \prod_{l=0}^{2^i-1} m_{j2^i+l}$. Prove that for each i, j

$$\begin{cases} M_{i+1,j} &= M_{i,2j} M_{i,2j+1} \\ M_{0,j} &= m_j. \end{cases} \quad (1.1)$$

3. How do the $M_{i,j}$ relate to the binary tree?
4. Fast multi-point evaluation.
 - a) Write an algorithm that builds the subproduct tree and returns the polynomials $M_{i,j}$ as defined in (1.1).
 - b) Write an recursive algorithm which takes a polynomial P of degree less than $n = 2^k$ as input as well as u_0, \dots, u_{n-1} and the subproducts $M_{i,j}$. It should go down the subproduct tree and return $P(u_0), \dots, P(u_{n-1})$.
5. Correctness and complexity.
 - a) By induction on k , prove the correctness of the previous algorithm.
 - b) Show that the complexity of the algorithm is $\mathcal{O}(M(n) \log n)$ operations in R .

Part II — Fast interpolation

Reusing the notations from part I, let m be the product of all the m_i , i.e. $m = \prod_{i=0}^{n-1} (X - u_i)$.

1. Explain how to perform Lagrange interpolation.
Hint: an element a in R is invertible if there is a b in R such that $ab = e$, with e a unit in R .
2. Let $s_i = \prod_{j \neq i} 1/(u_i - u_j)$. Prove that m' , the derivative of m , is $m' = \sum_{j=0}^{n-1} m/(x - u_j)$ and that $m'(u_i) = 1/s_i$.
3. Devise a divide and conquer algorithm which proceeds from the leaves to the root of the binary tree from part I question 1, in order to return the interpolation of P at the points u_0, \dots, u_{n-1} .
Hint: use the $M_{i,j}$ to apply a recursive approach to Lagrange interpolation.

4. Correctness and complexity.
 - a) By induction on k , prove the correctness of the previous algorithm.
 - b) Prove that computing the s_i in question 2, amounts to $\mathcal{O}(M(n) \log n)$ operations in R .
 - c) Conclude that the interpolation problem can be solved in $\mathcal{O}(M(n) \log n)$ ring operations.
5. Discuss the possibility of pre-computing the subproducts $M_{i,j}$.

Ex. 2 — *Critical thinking*

1. Let G be a group such that for all x, y in G , $(xy)^2 = (yx)^2$, and for any $x \neq e$, $x^2 \neq e$, where e is a unit element. Prove that G is abelian.
2. After passing ve477 two students, s_1 and s_2 , are asked to determine two integers x and y such that $1 < x < y$ and $x + y < 100$. Student s_1 is told that $x + y$, while s_2 is given xy . Remembering the importance of critical thinking they start discussing:

S_2 : "No idea what those two numbers could be. . ."

S_1 : "I'm not surprised, I knew you couldn't know!"

S_2 : "Uhm. . . so now I know. . ."

S_1 : "So do I!"

What about you?
3. A total of n ants are walking at speed 1 m/s on a thin 1 m long cable; if they collide they instantly reverse direction, and if an ant reaches the end of the cable it falls. How long will it take before all the ants have fallen?

Ex. 3 — *Beyond ve477*

Explain what the Swype keyboard is and propose some hints on how it could be implemented.

Ex. 4 — *IDEA survey*

Complete the IDEA survey and get a +5 bonus on the assignments.