VE477
Introduction to Algorithms

Assignment 7

Manuel — UM-JI (Fall 2017)

Reminders

- Write in a neat and legible handwriting or use LaTeX
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

Ex. 1 — *Karger-Stein's Algorithm*

In the lectures, although Karger-Stein's Algorithm was presented (5.28), only a sketch of proof was provided (5.29). In this exercise we want to prove the missing part, i.e. solve the recurrence relation

$$P(t) = 1 - \left(1 - \frac{1}{2}P\left(\frac{t}{\sqrt{2}}\right)\right)^2.$$

1. Prove that the probability of a cut to survive when $n < 6$, at least $1/15$.
2. Using an appropriate change of variable, show that

$$\begin{cases} p_{k+1} & = p_k - \frac{1}{4}p_k^2, \\ p_0 & = 1/15. \end{cases}$$

3. Let $z_k = 4/p_k - 1$.
   a) Prove that

$$\begin{cases} z_{k+1} & = z_k + 1 + \frac{1}{z_k}, \\ z_0 & = 59. \end{cases}$$

   b) Show that for all $k \geq 0$, $k < z_k < 59 + 2k$.
4. Recalling that $t = n/\sqrt{2}$ and noting that the depth of the recursion is $2\log_2 n + \mathcal{O}(1)$, conclude that $P(n) = \Omega(1/\log n)$.

Ex. 2 — *Simplex method*

Explain how the simplex method can be represented and applied from the following perspectives:

1. Tableaux (matrices);
2. Geometric;

For each case show the details of solving example 6.264 (6.16).

Ex. 3 — *Critical thinking*

Is it possible to design a stack supporting `push`, `pop`, and retrieving the minimum element in constant time? Explain.

Ex. 4 — *Farka's lemma*

Prove the following result. Let M be an $m \times n$ matrix and V be an n-vector. Then given an n-vectors x and an m-vector y, exactly one of the following can be true: (i) $Mx \leq 0$ and $V^T x > 0$, or (ii) $M^T y = V$ and $y \geq 0$.