Machine Learning for Signal Processing and Pattern Classification

DE-CONVOLUTION

SIMRAN

De-convolution refers to the problem of finding the input to an LTI system when the output signal is known. Here, we assume that the impulse response of the system is known. The output $y(n)$, is given by

$$y(n) = h(0)x(n) + h(1)x(n-1) + ... + h(N)x(n-N) \tag{1}$$

Where $x(n)$ is the input signal and $h(n)$ is the impulse response. (1) can be written as $y = \mathbf{H}x$ where **H** is a matrix of the form

$$\mathbf{H}=\begin{bmatrix} h(0) & & & \\ h(1) & h(0) & & \\ h(2) & h(1) & h(0) & \\ \vdots & & & \ddots \end{bmatrix}$$

These matrix are constant valued along its diagonals and are called Toeplitz matrices.

It may be expected that x can be obtained from y by solving the linear system $y = \mathbf{H}x$. In some situations, this is possible.

For example, Fig. 1 illustrates an input signal, $x(n)$, an impulse response, $h(n)$, and the output signal, $y(n)$. When we attempt to obtain x by solving $y = \mathbf{H}x$ in Matlab, we receive the warning message: `Matrix is singular to working precision' and we obtain a vector of all NaN (not a number).

$$\min_x \|y - \mathbf{H}x\|_2^2 + \lambda \|x\|_2^2 \tag{2}$$

Where $\lambda > 0$ is a parameter to be specified. Minimizing $\|y - \mathbf{H}x\|_2^2$ forces x to be consistent with the output signal y. Minimizing $\|x\|_2^2$ forces x to have low energy. Minimizing the sum in (2) forces x to be consistent with y and to have low energy (as far as possible, depending on λ).

Using LS formulation the signal x minimizing (2) is given by,

$$x = (\mathbf{H}^T\mathbf{H} + \lambda\mathbf{I})^{-1}\mathbf{H}^T y \tag{3}$$

In practice, the available data is also noisy. In this case, the data y is given by $y = \mathbf{H}x + w$ where w is the noise. The noise is often modeled as an additive white Gaussian random signal.

To improve the deconvolution result in the presence of noise, we can minimize the energy of the derivative (or second-order derivative) of x instead. As in the smoothing example above, minimizing the energy of the second-order derivative forces x to be smooth. In order that x is consistent with the data y and is also smooth, we solve the problem:

$$\min_x \|y - \mathbf{H}x\|_2^2 + \lambda \|\mathbf{D}x\|_2^2 \qquad (4)$$

where **D** is the second-order difference matrix . The signal x minimizing (4) is given by,

$$x = (\mathbf{H}^T\mathbf{H} + \lambda\mathbf{D}^T\mathbf{D})^{-1}\mathbf{H}^T y \qquad (5)$$