Theoretical and Experimental Study of Diffraction by a Thin Cone

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A problem of diffraction of acoustical waves by a thin rigid cone is studied. A direct diffraction experiment is used to measure the diffracted field on the surface of a thin cone. The experiment is performed using the MLS (Maximum Length Sequence) method. The cases of axial and non-axial incidence are studied. The wavelength is small compared to the cone length and to the cone local diameter closer to its end. A boundary integral equation is used to describe the field theoretically in the parabolic approximation. The integral equation is solved numerically by iterations. The results of the experiment are compared with the results of the calculation.

1 Introduction

Diffraction by a thin cone attracts considerable attention of researchers. Several different approaches exist to find the diffracted field and its asymptotics. First, there is a traditional asymptotic approach based on ray representation [1]. Second, there is an approach based on the parabolic equation method [2]. Third, there is an approach based on boundary integral equation method for the parabolic equation in Cartesian coordinates [3]. Also, there is an approach based on the Smyshlyaev's formula [4], and another one based on Kontorovich-Lebedev integral representation [5]. The authors are not aware of any work devoted to an experimental study of this problem. In this work, an attempt to fill this gap is made.

A direct diffraction experiment is used to measure the diffracted field on the cone surface. A boundary integral equation in the parabolic approximation [7] is used to validate the results of the experiment.

2 Experiment

2.1 Description

A narrow (the vertex angle is $2\alpha = 5.5°$) 1 meter long duralumin cone (see Fig. 1) is hanged in air. A small (about 1 cm size) microphone is placed on the surface of the cone. The cone is irradiated by a small sound source (Knowles RAB-32257 armature driver, also about 1 cm size) from different directions so the microphone can be shadowed or lit (see Fig. 2).

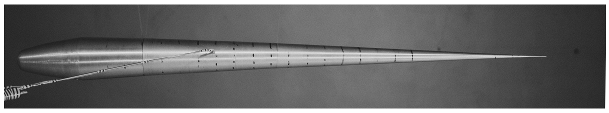

Figure 1: Photo of the duralumin cone with the microphone on its surface.

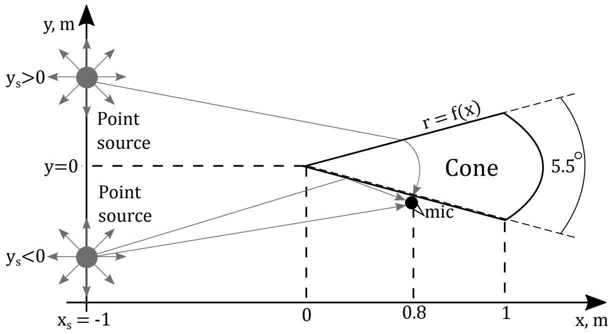

Figure 2: Experiment scheme, view from above.

To conduct the experiment, the MLS method is used [6]. The cone is irradiated with a pseudo-random signal, and the impulse response of the system is found by calculating the correlation between the MLS signal and the microphone output. After some masking in the time domain, a frequency response is calculated by making a Fourier transform of the impulse response. This value can be compared with the theoretical estimations.

3 Theoretical description

3.1 Problem statement

Consider a stationary 3D problem with the time dependence $\exp\{-i\omega t\}$ omitted henceforth. The acoustical field satisfies Helmholtz equation everywhere outside the cone:

$$\Delta\tilde{p} + k^2\tilde{p} = 0, \tag{1}$$

where Δ is the Laplace operator, $\tilde{p}$ is the acoustic pressure, k is the wavenumber.

We use Neumann boundary condition

$$\frac{\partial\tilde{p}}{\partial\vec{n}} = 0, \tag{2}$$

on the surface of the cone. Here $\vec{n}$ is the normal vector to the cone surface. The motivation for such a boundary condition is quite clear. One can study a model problem of wave reflection/transmission at the boundary between air and duraluminium half-spaces. The angle of incidence should be about the cone vertex angle. Fresnel formulae or more exact gas/solid boundary formulae can be used. In both cases one gets the reflection coefficient close to $+1$ due to a high contrast between the medium impedances of air and duraluminium. The accuracy is about 10^{-3}. Such a reflection coefficient corresponds to the Neumann boundary.

The statement of this problem should be supplemented with a radiation condition. This conditions in the limiting absorption form demands that for k having a small positive imaginary part the scattered field should decay as $r \to \infty$.

The Meixner's condition of absence of a source at the tip should also be satisfied by the field. As usually, this is a condition of local integrability of the combination $|\tilde{p}|^2 + |\nabla\tilde{p}|^2$ (see [4]).

The cone is narrow, so most of the waves are supposed to be scattered under small angles. This means that the parabolic approximation can be used to describe the diffraction process [7]. By following the standard procedure, the field can be represented as

$$\tilde{p}(x, r, \varphi) = \exp\{ikx\}\, p(x, r, \varphi), \tag{3}$$

where p is a slow function of x, (x, r, φ) are cylindrical coordinates (see Fig. 3). Substituting (3) into (1) and neglecting the small terms, get the approximate equation for p:

$$\left(\frac{\partial}{\partial x} + \frac{1}{2ik}\Delta_\perp\right) p = 0, \tag{4}$$

where

$$\Delta_\perp = \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right). \tag{5}$$

This is the parabolic equation of the diffraction theory.

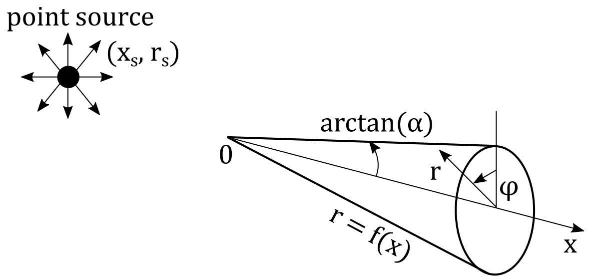

Figure 3: Cylindrical coordinates and the cone

The field p is traditionally called the attenuation function. It can be represented as a sum of the incident and the scattered terms:

$$p = p^{\text{in}} + p^{\text{sc}}. \tag{6}$$

As use a point source in the experiment, p^{in} is equal to the Green's function of the parabolic equation. Denote the points of space by $\vec{r} = (x, r, \varphi)$ and let the point source be located at $\vec{r_s} = (x_s, r_s, \varphi_s)$. The expression for the Green's function is well known:

$$p^{\text{in}} = \frac{k}{2\pi i(x - x_s)} \exp\left[\frac{ik}{2}\frac{(\Delta r)^2}{x - x_s}\right], \quad x > x_s, \tag{7}$$

where Δr is the distance between the projections of $\vec{r}$ and $\vec{r_s}$ onto the transversal plane:

$$(\Delta r)^2 = r^2 + r_s^2 - 2rr_s\cos(\varphi - \varphi_s). \tag{8}$$

Note that p^{in} is equal to zero if $x < x_s$.

Unlike the Helmholtz equation, the parabolic equation describes only waves travelling in the positive direction. So there should be no scattered waves at $x < 0$. This means that the initial condition should be formulated:

$$p^{sc}(x, r, \varphi) = 0 \quad \text{for} \quad x < 0. \tag{9}$$

For the details, see [7].

3.2 Calculation of the diffracted field

In [7], a boundary integral equation is derived for the problem of diffraction by a body of revolution. The derived equation can be applied to bodies having their local radii depending on x as $r = f(x)$, where $f(x)$ is an arbitrary smooth function. Indeed, $f(x)$ is a linear function for the cone case.

Let $P(x, \varphi)$ be the total field on the surface, P^{in} be the incident field on the surface, i.e.

$$P^{\text{in}}(x, \varphi) \equiv p^{\text{in}}(x, f(x), \varphi), \tag{10}$$

$$P(x,\varphi) \equiv p(x, f(x), \varphi). \tag{11}$$

Denote by x_*, φ_* the observation point coordinates on the cone surface. Then the following integral equation is satisfied:

$$P(x_*, \varphi_*) = \int_0^{2\pi}\int_0^{x_*} K(x_*, \varphi_*, x, \varphi) P(x,\varphi)dx d\varphi + 2P^{\text{in}}(x_*, \varphi_*), \tag{12}$$

$$K(x_*, \varphi_*, x, \varphi) = \frac{ikf(x)}{2\pi}\times \left[\frac{\dot f(x)}{x_* - x} + \frac{f(x) - f(x_*)\cos(\varphi - \varphi_*)}{(x_* - x)^2}\right]\exp(\zeta),$$
$$\zeta = \frac{ik}{2}\frac{f^2(x_*) + f^2(x) - 2f(x_*)f(x)\cos(\varphi-\varphi_*)}{x_* - x}, \tag{13}$$

The scheme of derivation of (12) is similar to that for the Helmholtz equation. Namely, a Green's formula is derived for the parabolic equation, and this formula is applied to the scattered field and the Green's function of the equation.

For our case $r = f(x) = x\tan\alpha \approx x\alpha$, where 2α is the vertex angle of the cone.

Note that equation (12) is of Volterra type with respect to variable x, and its kernel is of difference type with respect to variable φ. This means that it can be solved by iterations.

Let us solve the equation numerically. Use the method of approximation of the unknown function by piecewise-linear basis functions [8]. The following basis functions $N_i(x)$ are used:

$$N_i(x) = \begin{cases} 0, x_{i-1} > x, \\ (x - x_{i-1})/(x_i - x_{i-1}), x_i > x > x_{i-1}, \\ (x_{i+1} - x)/(x_{i+1} - x_i), x_{i+1} > x > x_i, \\ 0, x > x_{i+1}. \end{cases} \tag{14}$$

The solution P is sought in the form:

$$P(x,\varphi) \approx \sum_{i=1}^{M} P_i(\varphi) N_i(x), \tag{15}$$

where $P_i(\varphi)$ are some unknown functions. We substitute (15) into (12) and get the following system of linear integral equations:

$$P_i(\varphi_*) = \sum_{j=1}^{M}\int_0^{2\pi} K_{ij}(\varphi_*, \varphi)P_j(\varphi)d\varphi + 2P_i^{\text{in}}(\varphi_*), \quad i = 1...M, \tag{16}$$

where

$$P_i^{\text{in}}(\varphi_*) \equiv P^{\text{in}}(x_i, \varphi_*), \tag{17}$$

$$K_{ij}(\varphi_*, \varphi) \equiv \int_0^{x_i} K(x_i, \varphi_*, x, \varphi) N_j(x) dx, \tag{18}$$

One can notice that the integrand in (18) oscillates near the point $x = x_i$, which can be a source of numerical errors. To address this issue, the contour of integration is slightly shifted to the upper complex half-plane, making the kernel K exponentially decaying.

Then, the system (16) is discretized using the finite difference method and is solved by iterations.

4 Experimental results and modeling

4.1 *Axial incidence*

In the case of axial incidence ($y_s = 0$) we observed almost no diffraction field, i. e. the field value measured on the surface of the cone was close to the free field value. This result may look a little surprising, but it can be explained in terms of the Fresnel zone approach. We assume that diffraction by an obstacle is well developed if the obstacle is covered by several Fresnel zones corresponding to the given positions of the source and the receiver. The difference of path lengths between the shortest ray and the ray diffracted by the cone can be easily estimated (see Fig. 4):

$$L_1 - L_0 = \Delta l(1 - \cos\alpha) \approx \Delta l\alpha^2/2, \tag{19}$$

The first Fresnel zone size Δl is found by equating the latter to $\lambda/2$:

$$\Delta l = \lambda/\alpha^2. \tag{20}$$

We used frequencies from the range 2–10 kHz. Thus, for the highest frequency the size of the first Fresnel zone Δl is around 15 meters, which is 15 times bigger than the total length of the cone. That is why we see no diffraction in the case of axial incidence. The same result is provided by the solution of equation (16). Also, a similar result for the case of elliptical cone is obtained in [11].

4.2 *Non-axial incidence*

Using a calculation similar to (19), one can show that diffraction process becomes more noticeable when the source is shifted from the axis of the cone. The experiment confirms this reasoning. The results of the experiment for shifts $y_s =$

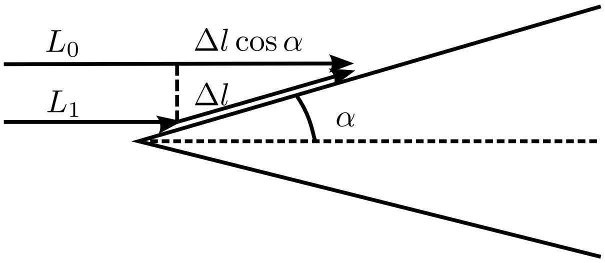

Figure 4: Illustration to calculation of the Fresnel zone size (19)

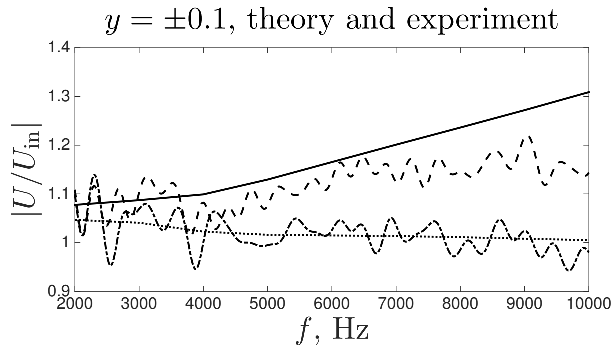

Figure 5: Comparison of experiment and theory for $y = \pm 0.1$ m. Theory (solid line) and experiment (dashed line) for $y = -0.1$ m, theory (dotted line) and experiment (dash-dotted line) for $y = 0.1$ m.

$\pm 0.5, \pm 0.3, \pm 0.1$ meters and $x_s = -1$ m compared with the results of the solving equation (16) are shown in Fig. 5, Fig. 6, Fig. 7, respectively. The receiver has been placed on the surface of the cone with its x-coordinate equal to 0.8 m, as it is shown in Fig. 2. We present the absolute values $|P/P_{\text{in}}|$ to suppress the geometrical attenuation of the field. The value P/P_{in} can be considered as an approximation to the field of the plane wave diffraction problem.

The following conclusions can be made from the experiment. First, the absolute value of the field in the lit zone ($y_s < 0$) is bigger than in the shadow ($y_s > 0$). Second, the value $|P/P_{\text{in}}|$ tends to 2 in the lit zone as frequency grows (this corresponds to the case of reflection from hard wall), and tends to 0 in shadow.

The authors associate the theory / experiment discrepancies with positioning errors, electrical noises, and imperfections of the source and the microphone. Besides, for some wavenumbers the Neumann condition might fail due to interference of the surface waves excited in the cone with the incident spherical waves from the source. Also note that the accuracy of the parabolic approximation decreases as $|y_s|$ grows, since the diffraction process cannot be considered paraxial. In the upcoming papers the authors are going to pay more attention to the problems mentioned above and to provide comparisons with some other existing theoretical approaches [1, 2, 4, 5, 10].

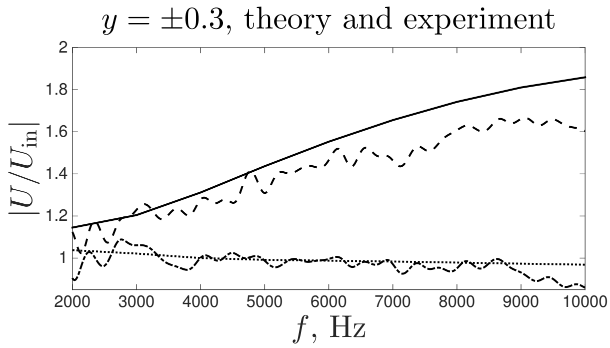

Figure 6: Comparison of experiment and theory for $y = \pm 0.3$ m. Theory (solid line) and experiment (dashed line) for $y = -0.3$ m, theory (dotted line) and experiment (dash-dotted line) for $y = 0.3$ m.

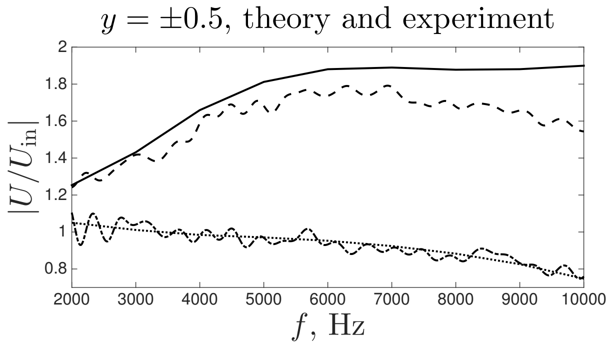

Figure 7: Comparison of experiment and theory for $y = \pm 0.5$ m. Theory (solid line) and experiment (dashed line) for $y = -0.5$ m, theory (dotted line) and experiment (dash-dotted line) for $y = 0.5$ m.

5 Conclusion

In this work, a direct acoustical diffraction experiment has been held to study diffraction by a narrow rigid cone. The results of the experiment for axial incidence show that there is almost no diffracted field. For non-axial incidence (if the angle of incidence is small enough) the results agree with the solution of boundary integral equation for the parabolic approximation.

Acknowledgements

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