

# A<sup>1</sup> Enumerative Geometry

Enumerative geometry, counts algebro-geometric objects, and in order to actually obtain an invariant number at the end of the day one uses an algebraically closed field  $k$  or  $\mathbb{C}$ . This is essentially because the conditions imposed are polynomial, and polynomials of degree  $n$  over a closed field always have  $n$  roots.

The goal here is to record information about the fields of definition. However, since we may no longer have invariant numbers as solutions to polynomial equations, we replace this with a notion of *weights* to get an “invariance of bilinear form” principle instead. Over characteristic not 2, we can use quadratic forms, which ties to Lurie’s first talk.

## Example: Lines on a Smooth Cubic Surface

| Joint work with Jesse Kass

A **cubic surface**  $X$  consists of the  $\mathbb{C}$  solutions to a polynomial in three variables, i.e.

$$X = \{(x, y, z) \in \mathbb{C}^3 \mid f(x, y, z) = 0\},$$

where  $f$  is degree 3. In general, we want to compactify, so we

view  $X \hookrightarrow \mathbb{CP}^3$  as

$$\mathbb{CP}^3 = \{\mathbf{x} = [w, x, y, z] \neq \mathbf{0} \ni \forall \lambda \in \mathbb{C}^\times, \mathbf{x} = \lambda \mathbf{x}\}$$

and so

$$X = \{[w, x, y, z] \in \mathbb{CP}^3 \ni f(w, x, y, z) = 0\}$$

where  $f$  is homogeneous.

The surface  $X$  is **smooth** if the underlying points form a manifold, or equivalently if the partials don't simultaneously vanish.

### **Theorem (Salmon Cayley 1849):**

If  $X$  is a smooth cubic surface, then  $X$  contains exactly 27 lines.

Example: The Fermat cubic  $f(w, x, y, z) = w^3 + x^3 + y^3 + z^3$ .

We can find one line, given by

$$L = \{[s, -s, t, -t] \ni s, t \in \mathbb{CP}^1\},$$

and in fact this works for any  $\lambda, \omega$  such that  $\lambda^3 = \omega^3 = -1$ , yielding

$$L' = \{[s, \lambda s, t, \omega t] \ni s, t \in \mathbb{CP}^1\}.$$

We can also permute  $s, t$  around to get more lines, and by counting this yields 27 distinct possibilities. (3 choices for  $\lambda$ , 3 choices for  $\omega$ , and  $\frac{1}{2} \binom{4}{2}$  ways to pair them with the  $s, t$  in the original  $L$ .)

There is a proof in the notes that these are the only lines,

which is relatively elementary.

# Modern Proof

We'll use characteristic classes, which we'll later replace by an  $\mathbb{A}^1$  homotopy theory variant.

Let  $\text{Gr}(1, 3)$  be the Grassmannian parameterizing 1-dimensional subspaces of  $\mathbb{CP}^3$ , where the  $\mathbb{C}$  points of this space parameterize 2-dimensional subspaces  $W \subseteq \mathbb{C}^4$ . This is a moduli space of the lines we're looking for.

Let

$$S \rightarrow \text{Gr}(1, 3)$$

be the tautological bundle where the fiber is simply given by  $S_W = W$ . We can also form the bundle

$$(\text{Sym}^3 S)^\vee \rightarrow \text{Gr}(1, 3)$$

where the fiber over the point corresponding to  $W$  is all of the cubic polynomials on  $W$ , i.e.

$$(\text{Sym}^3 S)_W^\vee = (\text{Sym}^3 W)^\vee.$$

Explicitly, we have the following two bundles to work with:

$$\begin{aligned} W &\rightarrow S \rightarrow \text{Gr}(1, 3) \\ (\text{Sym}^3 W)^\vee &\rightarrow (\text{Sym}^3 S)^\vee \rightarrow \text{Gr}(1, 3) \end{aligned}$$

Our chosen  $f$  determines an element of  $(\text{Sym}^3 \mathbb{C}^4)^\vee$ , which is thus a section  $\sigma_f$  of the second bundle above, where

$$\sigma_f(W) = f|_W.$$

We thus have

$$\mathbb{P}W \in X \iff \sigma_f(W) = 0,$$

i.e. the line corresponding to  $W$  is in our surface exactly when this section is zero. We now want to count the zeros of  $\sigma_f$ , which is exactly what the Euler class does.

To be precise, the Euler class counts the zeros of a section of a properly oriented vector bundle with a given weight. Let  $V \rightarrow M$  be a rank  $r$   $\mathbb{R}$ - vector bundle over a dimension  $r$  real manifold where we assume that  $V$  is oriented.

We choose  $\mathbb{R}$  here because  $\mathbb{C}$  is slightly too nice and gives us a preferred orientation (which we'll want to track later.)

For any section  $\sigma$  with only isolated zero, we'll assign a weight to each zero which comes from the topological degree function

$$\deg : [S^{r-1}, S^{r-1}] \rightarrow \mathbb{Z},$$

where we use the brackets to denote homotopy classes of maps.

Definition: Let  $p \in M$  where  $\sigma(p) = 0$ , and define  $\deg_p(\sigma)$  in the following way:

Choose local coordinates near  $p$ . Since the zeros are isolated, we can choose a ball  $B_\varepsilon(p)$  such that

$x \in B_\varepsilon(p) - \{p\} \implies \sigma(x) \neq 0$ . Choose a local trivialization of the total space  $V$ . This allows us to view  $\sigma : \mathbb{R}^r \rightarrow \mathbb{R}^r$  as a real function.

We can choose coordinates such that  $p = 0$  in the domain, so  $\sigma(0) = 0$ , and moreover the image  $\sigma(B_\varepsilon(p)) = \mathbb{R} - \{0\}$ . We can then form a function

$$\bar{\sigma} : \partial B_\varepsilon(p) = S^{r-1} \rightarrow S^{r-1} = \partial \sigma(B_\varepsilon(p))$$

$$x \mapsto \frac{\sigma(x)}{\|\sigma(x)\|},$$

and so we can take  $\deg_p(\sigma) := \deg \bar{\sigma}$ .

There is indeterminacy here up to elements of  $\mathrm{GL}(r, \mathbb{R})$  which could possibly effect the sign, however, but this can be fixed using the assumption that  $V$  is oriented and choosing local trivializations for which the orientations are compatible. This gives us a well-defined local degree of a section at a zero.

The Euler class, which only depends on the bundle and not the section, is given by

$$e(V) = \sum_{p \ni \sigma(p)=0} \deg_p(\sigma).$$

It can be shown that because  $X$  is smooth, the zeros are all simple and so in the complex case, the degrees are all 1. We thus obtain

$$|\{\text{lines on } X\}| = e((\mathrm{Sym}^3 S)^\vee),$$

where the RHS is independent of  $X$  and can be computed using the splitting principle and the cohomology of  $\mathrm{Gr}$ .



# What about $\mathbb{R}$ ?

Schlaflfli, 19th century:  $X$  can have 3, 7, 15 or 27 lines. So it's not constant, and thus there's not an invariant number here, but Segre (1942) distinguished between hyperbolic and elliptic lines.

Recall the characterization of elements in  $\text{Aut} L$  for  $L = \mathbb{RP}^1$  (real lines) as elliptic/hyperbolic: we have  $\text{Aut} L \cong \text{PGL}(2, \mathbb{R})$ , so pick some  $I$  corresponding to a matrix

$$[I] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad z \mapsto \frac{az + b}{cz + d}$$

where the second formulation above shows that there are two fixed points, since solving for  $z \mapsto z$  yields a quadratic equation. So we have

$$\text{Fix}(I) = \{z \in \mathbb{C} \mid cz^2 + (d - a)z + b = 0\},$$

and we characterize  $I$  by the following cases:

- $\text{Fix}(I)$  contains two real points: hyperbolic
- A complex conjugate pair: elliptic

So we'll associate an involution to  $L$ , and port over these notions of hyperbolic/elliptic. As we'll see later, for each point on  $L$ , there will be a unique other point that has the tangent space, and this involution will swap them.

Let  $p \in L$ , and consider  $T_p X \cap X$ . Since  $L$  is in both of the varieties we're intersecting here, and we can apply Bezout's theorem, we know that its complement will have some degree 2

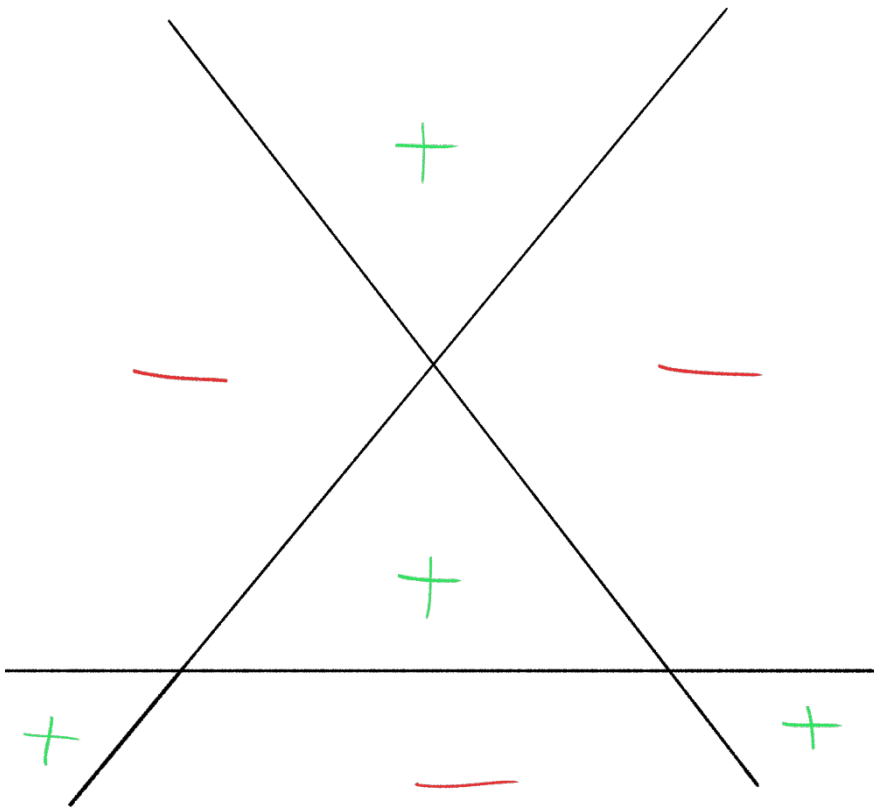
variety  $Q$  (since the total degree is 3).

So we can write  $T_p X \cap X = L \cup Q$ . We know that  $L \cap Q$  will be the intersection of a degree 1 and a degree 2 curve, which will have 2 points of intersection. At one of these points, say  $p$ ,  $Q$  and  $L$  will intersect transversally, and so the tangent vectors  $T_p Q$  and  $T_p L$  give a 2-dimensional frame, which yields a plane  $P \subseteq T_p X$ . Since  $X$  is smooth, we get equality and  $P = T_p X$ .

This also holds for the second point of intersection,  $p'$ , and so we take the involution  $I(p) = p'$  and vice-versa. We then say  $L$  is elliptic/hyperbolic exactly when  $I$  is.

A natural way to see that there should be a distinction between two types of lines is to use spin structures. Consider a physical cubic surface sitting inside  $\mathbb{R}^3$ , and push the tangent plane along a line. There are two things that can happen – one is a twisting by a nontrivial element of  $\pi_1 SO_3(\mathbb{R})$ , the other is no twisting at all.

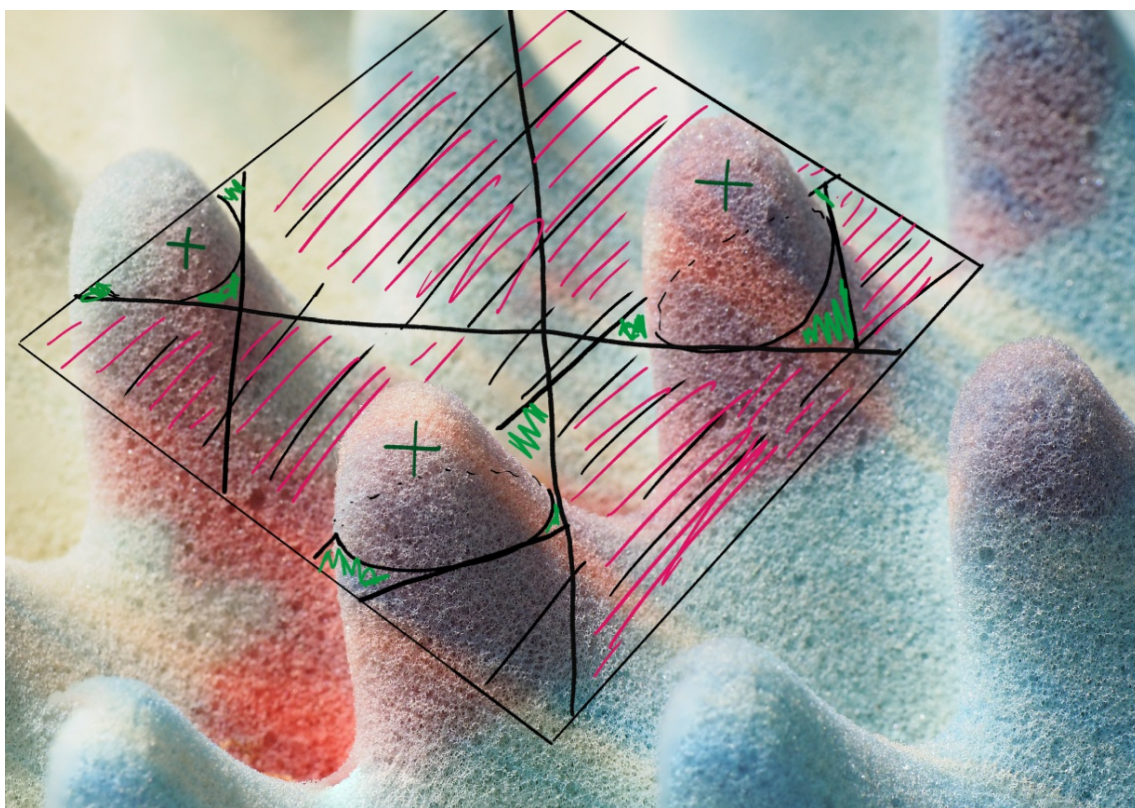
Example: Look at the Fermat cubic surface  $x^3 + y^3 + z^3 = -1$



Interpretation of this image:  $X \subset \mathbb{R}^3$  is a surface, which has 3 lines that are contained in a plane. We view  $X$  from above this plane, marking a plus/minus to denote the relative height of the surface within each bounded region. Plus denotes part of the surface that bubbles up over the plane, having positive height/ $z$  coordinates, etc.

This took me a while to visualize – what worked for me was thinking about “egg crate” padding:





After thinking about what physically happens as you push a plane around, it becomes clear that these three lines are all hyperbolic. Note that this question is the same as asking if a path in the frame bundle lifts.

Although the number of lines isn't a constant, we can take a "signature" sort of formula to obtain an invariant. In this case, the number hyperbolic lines minus the number of elliptic lines *is* constant. In this case, the constant is 3.

General mantra for  $\mathbb{A}^1$  homotopy theory: if you have a result that works over  $\mathbb{C}$  and  $\mathbb{R}$ , it may be a result in  $\mathbb{A}^1$  theory that has realizations recovering the original results.

# $\mathbb{A}^1$ Homotopy Theory

This will allow us to do with schemes much of what we can do in

**Top.** Smooth schemes behave like manifolds, where there are balls around points. The convention here will be that we're working over smooth schemes, denoted  $\mathbf{Sm}_k$  where  $k$  is a field.

Remark: in my notation I use  $\mathbb{RP}^n$ ,  $\mathbb{CP}^n$ , and  $\mathbb{P}^n(k)$  to denote various projective spaces. I'll adopt Kirsten's convention here and just denote  $\mathbb{P}^n(k)$  as  $\mathbb{P}^n$ .

We'll get spheres from  $S_{\mathbb{A}}^n := \mathbb{P}^n / \mathbb{P}^{n-1} \dots$  One nice result due to Morel is that there is a degree map

$$[S_{\mathbb{A}}^n, S_A A^n] \rightarrow GW(k),$$

where the target is not the integers in this case, but rather a group of bilinear forms that are quadratic in characteristic not equal to 2. It is the Grothendieck-Witt group, whose elements are formal difference of bilinear forms.

Thus the group itself is the group completion of nondegenerate symmetric isomorphism classes of bilinear forms  $V^2 \rightarrow k$  where  $V$  is a finite-dimensional  $k$ -vector space.

The group structure arises because if we have two bilinear forms  $B, B'$  on vector spaces  $V, W$  respectively, then we can define a new form on  $V \oplus W$  by working in components and declaring orthogonality between any of the factors. We then take formal differences of these, and inherit a ring structure from the tensor product of forms.

Bilinear forms over fields can all be diagonalized, although in characteristic 2, this only holds in a stable sense.

# The Grothendieck-Witt Group

Since we can diagonalize, the group  $GW(k)$  has a presentation coming from the one dimensional forms. Any of these work as a generator, so we have

- Generators:  $\langle a \rangle$  where  $a \in k^\times$ , corresponding to the form

$$\begin{aligned} \langle a \rangle: k^2 &\rightarrow k \\ (x, y) &\mapsto axy \end{aligned}$$

- Relations: if we change the basis of  $k$  using a multiplication by  $b \in k^\times$ , we get  $\langle ab^2 \rangle = \langle a \rangle$ . (This means that  $a \in k^\times / (k^\times)^2$ )
  - We also get
 
$$\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$$

There are many concrete computations of this known for global fields, local fields, finite fields, function fields, etc.

Example: compute  $GW(\mathbb{C})$ .

The generators are in bijection with  $k^\times / (k^\times)^2$ , but since every element of  $\mathbb{C}$  is a square, so there's only one element here. We thus obtain

$$\begin{aligned} GW(\mathbb{C}) &\xrightarrow{\cong} \mathbb{Z} \\ \beta &\mapsto \dim V \end{aligned}$$

which is realized by taking the rank.

Example:  $GW(\mathbb{R})$

We still have the rank, but now we can also take the signature, so we have

$$GW(\mathbb{R}) \xrightarrow{\text{rank} \times \text{signature}} \mathbb{Z}^2,$$

although a minor parity issue crops up here that can be fixed without damaging the isomorphism type.

Example:  $GW(\mathbb{F}_q)$

We can make a matrix out of how  $\beta$  acts on basis elements and take the determinant of it to obtain an invariant called the *discriminant*, and so

$$GW(\mathbb{F}_q) \xrightarrow{\text{discriminant} \times \text{rank}} \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2 \times \mathbb{Z}$$

Note that the quotient is needed because we can change basis in  $\mathbb{F}_q$ , which amounts to conjugating by a matrix  $A$ , and so this discriminant is only well-defined up to squares.

## Euler Class

There is an Euler class in this setting,

$$e(V) = \sum_{p \ni \sigma(p)=0} \deg_p(\sigma).$$

Letting  $X$  be a smooth cubic surface over  $k$ , then a line  $L \subset X$  will be a closed point of the Grassmannian  $\text{Gr}(1, 3)$ , so we can think of it as points of the form

$$L = \{[a, b, c, d]s + [a', b', c', d']t \ni s, t \in \mathbb{P}^1(k(L)\}$$

where the extension field  $k(L) = k(a, b, c, d, a', b', c', d')$  is obtained by adjoining the coefficients to  $k$ .

I think these are always separable, mentioned later in the talk.

We thus get

$$\mathbb{P}^1(k(L)) \cong L \subseteq_{\substack{\text{closed} \\ \text{subscheme}}} X_{k(L)} \subseteq \mathbb{P}^3(k(L)).$$

Given such a line  $L \subseteq X$ , similar to the real setting, we obtain an involution  $I \in \text{Aut} L \cong \text{PGL}(2, k(L))$  after choosing coordinates. We also find that  $\text{Fix}(L)$  again falls into two cases:

- $2k(L)$  points, or
- 2 conjugate points in some quadratic extension  $k(L)[\sqrt{D}]$  where  $D \in k(L)^\times / (k(L)^\times)^2$ . These correspond to the oddities in the tangent plane in the real case.

We then define

$$\text{Type}(L) = \langle D \rangle \in \text{GW}(k(L)),$$

or equivalently  $D = ab - cd$  when  $I = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , in which case  $\text{Type}(L) = \langle -1 \rangle \deg I$ .

### Theorem:

Supposing  $X$  is a smooth cubic surface over  $k$  of characteristic not equal to 2, we then have

$$\sum_{L \in X} \text{Tr}_{k(L)/k} \text{Type}(L) = \text{One fixed quadratic form} = 15 < 1$$

where the trace/transfer maps are defined as

$$\begin{aligned} \mathrm{Tr}_{k(L)/k} : GW(k(L)) &\rightarrow GW(k) \\ (V^2 \xrightarrow{\beta} k(L)) &\mapsto (V^2 \xrightarrow{\beta} k) \circ \mathrm{Trace}_{\mathrm{Galois}} \end{aligned}$$

where  $\mathrm{Trace}_{\mathrm{Galois}}$  comes from summing the conjugates. Note that we can do this because we can view  $V$  as a vector space over either  $k$  or  $k(L)$ , so we end up with a quadratic form over  $k$ . ■

Note: we have a well-defined map in the other direction, since the  $GW$  is a stable homotopy group of spheres.

Example: let  $k = \mathbb{C}$ , then apply rank to get  $15 + 12 = 27$  on the RHS, while since every element is a square, the Type is just 1, so we get 27 total.

Example: let  $k = \mathbb{R}$ , apply signature. If  $L$  is defined over  $\mathbb{C}$ , so the type is 1, and we're just left with the trace of  $\mathbb{C}/\mathbb{R}$  – but this contributes a  $+1$  and  $-1$ , so there is no contribution. What's left are the lines of  $\mathbb{R}$ , and since we set it up so type 1 lines are hyperbolic, we just get the trace  $15 - 12 = 3$ .

Example: let  $k = \mathbb{F}^q$ . We can define lines in  $\mathbb{F}_q^n$ , and the "begin a square" partitions  $(\mathbb{F}_q^n)^\times$  into two disjoint subsets, we can assign types and we let squares be the hyperbolic elements.

We thus get

$$\left( \begin{array}{c} \text{elliptic lines } L \\ \text{with } k(L) = \mathbb{F}_{\text{odd}} \end{array} \right) - \left( \begin{array}{c} \text{hyperbolic lines } L \\ \text{with } k(L) = \mathbb{F}_{\text{even}} \end{array} \right) = 0 \pmod{2}$$

which follows from computing the discriminant of the given form.

