Morrow - THH

i) Classical Hoch./cyclic homology

ii) Topological version

iii) Reln to arith. geom

Classical Theory

Fix comm. base ring k

$$HH_*(A/k) = A \leftarrow A^{\otimes_k 2} \leftarrow A^{\otimes_k 3} \leftarrow \cdots$$

where $a_0 a_1 - a_1 a_0 \leftarrow a_0 \otimes a_1$

$$a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2 + a_2 a_0 \otimes a_1 \leftarrow a_0 \otimes a_1 \otimes a_2$$

Ex

$$HH_0(A/k) = A/\langle ab - ba\rangle = A \quad (A \text{ comm.})$$

$$HH_1(A/k) = A \otimes_k A / \langle ab \otimes c - a \otimes bc + ac \otimes b\rangle$$
$$= \Omega^1_{A/k}$$

$$HH_*(A/k) = \bigoplus_{n\geq 0} H_n(A/k)$$

$\Rightarrow \exists$ maps $\quad \varepsilon_\pi : \Omega^\pi_{A/k} \to HH_\pi$

by universal property of

diff forms $\to$ $\Omega^*_{A/k} := \Lambda^*_A \Omega^1_{A/k}$

<u>Thm</u> If A is smooth, these are <u>isos</u>.

$\to$ HH is a generalization of diff forms

for non-smooth A

<u>Lemma</u>: Let $S = A \otimes_k A^{op}$, then

$$HH(A/k) \simeq A \overset{\mathbb{L}}{\otimes}_S A$$

for flat k-alg A

quasi-iso

Derived tensor prod.

<u>Pf</u>: Take $B = \left[A^{\otimes 2} \leftarrow A^{\otimes 3} \leftarrow \cdots \right]$

resolution of A by flat $A \otimes A^{op}$

WTS: $HH_*(A/k)$ is the exterior algebra on its deg 1 elt

This is well known for $\mathrm{Tor}_*^B(C,C)$ when $B \to C$ has kernel locally generated by a regular sequence.

$\longrightarrow HH_*(A/k) \cong \mathrm{Tor}_*^{A\otimes A^{op}}(A,A)$

Have $\mathbb{Z}/n+1 \circlearrowright A^{\otimes n+1}$

$\quad\shortparallel$

$\langle t_n \rangle$ where $\otimes a_i \mapsto$

Set norm $N := \sum_{i=0}^{n} \underbrace{((-1)^n t_n)^i}_{\text{twist}} : A^{\otimes n+1} \circlearrowleft$

Extra degeneracy

$s: A^{\otimes n} \longrightarrow A^{\otimes n+1}$

$\otimes a_i \to 1 \otimes \otimes a_i$

"Connes operator"

$$B : A^{\otimes n} \xrightarrow[N]{} A^{\otimes n} \xrightarrow[s]{} A^{\otimes n+1} \xrightarrow[id-(-1)^n t_n]{} A^{\otimes n+1}$$

Then $B^2 = 0$, $B\partial = -\partial B$ for $\partial \in HH$

i.e. $B : HH(A/k) \to HH(A/k)[-1]$

This refines the $\partial \in \Omega$

$$\begin{tikzcd} HH_n \arrow[r, "B"] \arrow[dr, phantom, "\circlearrowright"] & HH_{n+1} \\ \Omega^n_{A/k} \arrow[u, "\varepsilon_n"] \arrow[r, "\partial"] & \Omega^{n+1}_{A/k} \arrow[u, "\varepsilon_{n+1}"'] \end{tikzcd}$$

- Periodic cyclic homology : $HP(A/k)$ = Totalization of some $\mathbb{Z}^2$-graded complex (direct product)
- Cyclic homology $HC(A/k)$: Take 1st quad
- Neg. cyc. hom. : Take 2nd quad

Have $0 \to HH \to HC \xrightarrow{S} HC[2] \to 0$

$0 \to HC[-2] \xrightarrow{S} HC^- \to HH \to 0$

$0 \to HC^- \to HP \to HC[2] \to 0$

$\therefore\ HP = \varprojlim \left(\cdots HC[-4] \xrightarrow{S} HC[-2] \to \cdots \right)$

2-periodicity: $HP_n(A/k) \cong HP_{n+2}(A/k)$

Ex

Sp $HH_{2k+1}(A/k) = 0$, then $HP_0(A/k)$ is a filtered ring:

$$F^i HP_0(A/k) := S^n(HC^-(A/k))$$

s.t. $HP_0(A/k)/F^i \cong HC_{2n}(A/k)$

and $gr \cong HH_{2n}(A/k)$