A1 Enumerative Geometry (Arizona Winter School 2019)

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1 Lecture 1: $\mathbb{A}^1$ Enumerative Geometry

Enumerative geometry counts algebro-geometric objects, and in order to actually obtain an invariant number at the end of the day one uses an algebraically closed field k or $\mathbb{C}$. This is essentially because the conditions imposed are polynomial, and polynomials of degree n over a closed field always have n roots.

The goal here is to record information about the fields of definition. However, since we may no longer have invariant numbers as solutions to polynomial equations, we replace this with a notion of *weights* to get an "invariance of bilinear form" principle instead. Over characteristic not 2, we can use quadratic forms, which ties to Lurie's first talk.

1.1 Example: Lines on a Smooth Cubic Surface

Joint work with Jesse Kass

A **cubic surface** X consists of the $\mathbb{C}$ solutions to a polynomial in three variables, i.e.

$$X = \left\{(x, y, z) \in \mathbb{C}^3 \;\middle|\; f(x, y, z) = 0\right\},$$

where f is degree 3. In general, we want to compactify, so we view $X \hookrightarrow \mathbb{CP}^3$ as

$$\mathbb{CP}^3 = \left\{\mathbf{x} = [w, x, y, z] \neq \mathbf{0} \;\middle|\; \forall \lambda \in \mathbb{C}^\times,\ \mathbf{x} = \lambda \mathbf{x}\right\}$$

and so

$$X = \left\{[w, x, y, z] \in \mathbb{CP}^3 \;\middle|\; f(w, x, y, z) = 0\right\}$$

where f is homogeneous.

The surface X is **smooth** if the underlying points form a manifold, or equivalently if the partials don't simultaneously vanish.

Theorem 1.1 ***(Salmon-Cayley, 1849).***
If X is a smooth cubic surface, then X contains exactly 27 lines.

Example 1.1.
The Fermat cubic $f(w, x, y, z) = w^3 + x^3 + y^3 + z^3$.

We can find one line, given by

$$L = \left\{[s, -s, t, -t] \;\middle|\; s, t \in \mathbb{CP}^1\right\},$$

and in fact this works for any λ, ω such that $\lambda^3 = \omega^3 = -1$, yielding

$$L' = \left\{[s, \lambda s, t, \omega t] \;\middle|\; s, t \in \mathbb{CP}^1\right\}.$$

We can also permute s, t around to get more lines, and by counting this yields 27 distinct possibilities: 3 choices for λ, 3 choices for ω, and $\frac{1}{2}\binom{4}{2}$ ways to pair them with the s, t in the original L.

There is a proof in the notes that these are the only lines, which is relatively elementary.

1.1.1 Modern Proof

We'll use characteristic classes, which we'll later replace by an $\mathbb{A}^1$ homotopy theory variant.

Let $\mathrm{Gr}(1,3)$ be the Grassmannian parametrizing 1-dimensional subspaces of $\mathbb{CP}^3$, where the $\mathbb{C}$ points of this space parameterize 2-dimensional subspaces $W \subseteq \mathbb{C}^4$. This is a moduli space of the lines we're looking for.

Let

$$S \longrightarrow \mathrm{Gr}(1,3)$$

be the tautological bundle where the fiber is simply given by $S_W = W$. We can also form the bundle

$$(\mathrm{Sym}^3 S)^\vee \longrightarrow \mathrm{Gr}(1,3)$$

where the fiber over the point corresponding to W is all of the cubic polynomials on W, i.e.

$$(\mathrm{Sym}^3 S)^\vee_W = (\mathrm{Sym}^3 W)^\vee.$$

Explicitly, we have the following two bundles to work with:

$$W \longrightarrow S \longrightarrow \mathrm{Gr}(1,3)$$

$$\left(\mathrm{Sym}^3 W\right)^\vee \longrightarrow \left(\mathrm{Sym}^3 S\right)^\vee \longrightarrow \mathrm{Gr}(1,3)$$

Our chosen f determines an element of $(\mathrm{Sym}^3\mathbb{C}^4)^\vee$, which is thus a section σ_f of the second bundle above, where

$$\sigma_f(W) = f|_W.$$

We thus have

$$\mathbb{P}W \in X \iff \sigma_f(W) = 0,$$

i.e. the line corresponding to W is in our surface exactly when this section is zero. We now want to count the zeros of σ_f, which is exactly what the Euler class does.

To be precise, the Euler class counts the zeros of a section of a properly oriented vector bundle with a given weight. Let $V \longrightarrow M$ be a rank r $\mathbb{R}$- vector bundle over a dimension r real manifold where we assume that V is oriented.

> We choose $\mathbb{R}$ here because $\mathbb{C}$ is slightly too nice and gives us a preferred orientation (which we'll want to track later.)

For any section σ with only isolated zero, we'll assign a weight to each zero which comes from the topological degree function

$$\deg : [S^{r-1}, S^{r-1}] \longrightarrow \mathbb{Z},$$

where we use the brackets to denote homotopy classes of maps.

Definition: Let $p \in M$ where $\sigma(p) = 0$, and define $\deg_p(\sigma)$ in the following way:

Choose local coordinates near p. Since the zeros are isolated, we can choose a ball $B_\varepsilon(p)$ such that $x \in B_\varepsilon(p) - \{p\} \implies \sigma(x) \neq 0$. Choose a local trivialization of the total space V. This allows us to view $\sigma : \mathbb{R}^r \longrightarrow \mathbb{R}^r$ as a real function.

We can choose coordinates such that $p = 0$ in the domain, so $\sigma(0) = 0$, and moreover the image $\sigma(B_\varepsilon(p)) = \mathbb{R} - \{0\}$. We can then form a function

$$\begin{aligned} \bar{\sigma} : \partial B_\varepsilon(p) = S^{r-1} &\longrightarrow S^{r-1} = \partial\sigma(B_\varepsilon(p)) \\ x &\mapsto \frac{\sigma(x)}{\|\sigma(x)\|}, \end{aligned}$$

and so we can take $\deg_p(\sigma) := \deg \bar{\sigma}$.

There is indeterminacy here up to elements of $\mathrm{GL}(r, \mathbb{R})$ which could possibly affect the sign, however, but this can be fixed using the assumption that V is oriented and choosing local trivializations for which the orientations are compatible. This gives us a well-defined local degree of a section at a zero.

The Euler class, which only depends on the bundle and not the section, is given by

$$e(V) = \sum_{\left\{ p \;\middle|\; \sigma(p)=0 \right\}} \deg_p(\sigma).$$

It can be shown that because X is smooth, the zeros are all simple and so in the complex case, the degrees are all 1. We thus obtain

$$|\{\text{lines on } X\}| = e((\mathrm{Sym}^3 S)^\vee),$$

where the RHS is independent of X and can be computed using the splitting principle and the cohomology of Gr.

1.2 What about $\mathbb{R}$?

Schlafli, 19th century: X can have $3, 7, 15$ or 27 lines. So it's not constant, and thus there's not an invariant number here, but Segre (1942) distinguished between hyperbolic and elliptic lines.

Recall the characterization of elements in $\mathrm{Aut}L$ for $L = \mathbb{RP}^1$ (real lines) as elliptic/hyperbolic: we have $\mathrm{Aut}L \cong \mathrm{PGL}(2, \mathbb{R})$, so pick some I corresponding to a matrix

$$[I] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad z \mapsto \frac{az+b}{cz+d}$$

where the second formulation above shows that there are two fixed points, since solving for $z \mapsto z$ yields a quadratic equation. So we have

$$\mathrm{Fix}(I) = \left\{ z \in \mathbb{C} \;\middle|\; cz^2 + (d-z)z + b = 0 \right\},$$

and we characterize I by the following cases:

- $\mathrm{Fix}(I)$ contains two real points: hyperbolic
- A complex conjugate pair: elliptic

So we'll associate an involution to L, and port over these notions of hyperbolic/elliptic. As we'll see later, for each point on L, there will be a unique other point that has the tangent space, and this involution will swap them.

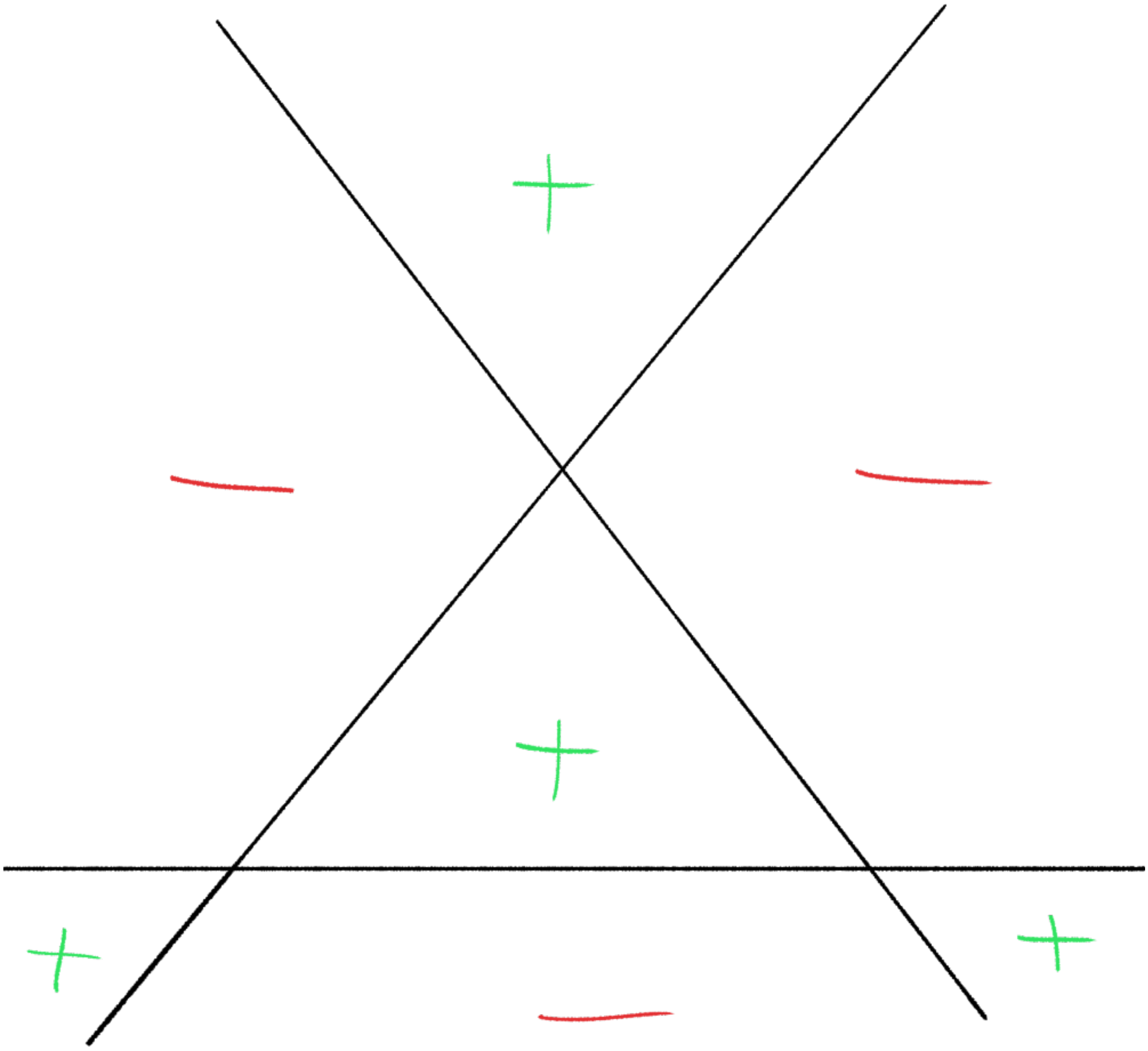
Let $p \in L$, and consider $T_pX \bigcap X$. Since L is in both of the varieties we're intersecting here, and we can apply Bezout's theorem, we know that its complement will some degree 2 variety Q (since the total degree is 3).

So we can write $T_pX \bigcap X = L \bigcup Q$. We know that $L \bigcap Q$ will be the intersection of a degree 1 and a degree 2 curve, which will have 2 points of intersection. At one of these points, say p, Q and L will intersect transversally, and so the tangent vectors T_pQ and T_pL give a 2-dimensional frame, which yields a plane $P \subseteq T_pX$. Since X is smooth, we get equality and $P = T_pX$.

This also holds for the second point of intersection, p', and so we take the involution $I(p) = p'$ and vice-versa. We then say L is elliptic/hyperbolic exactly when I is.

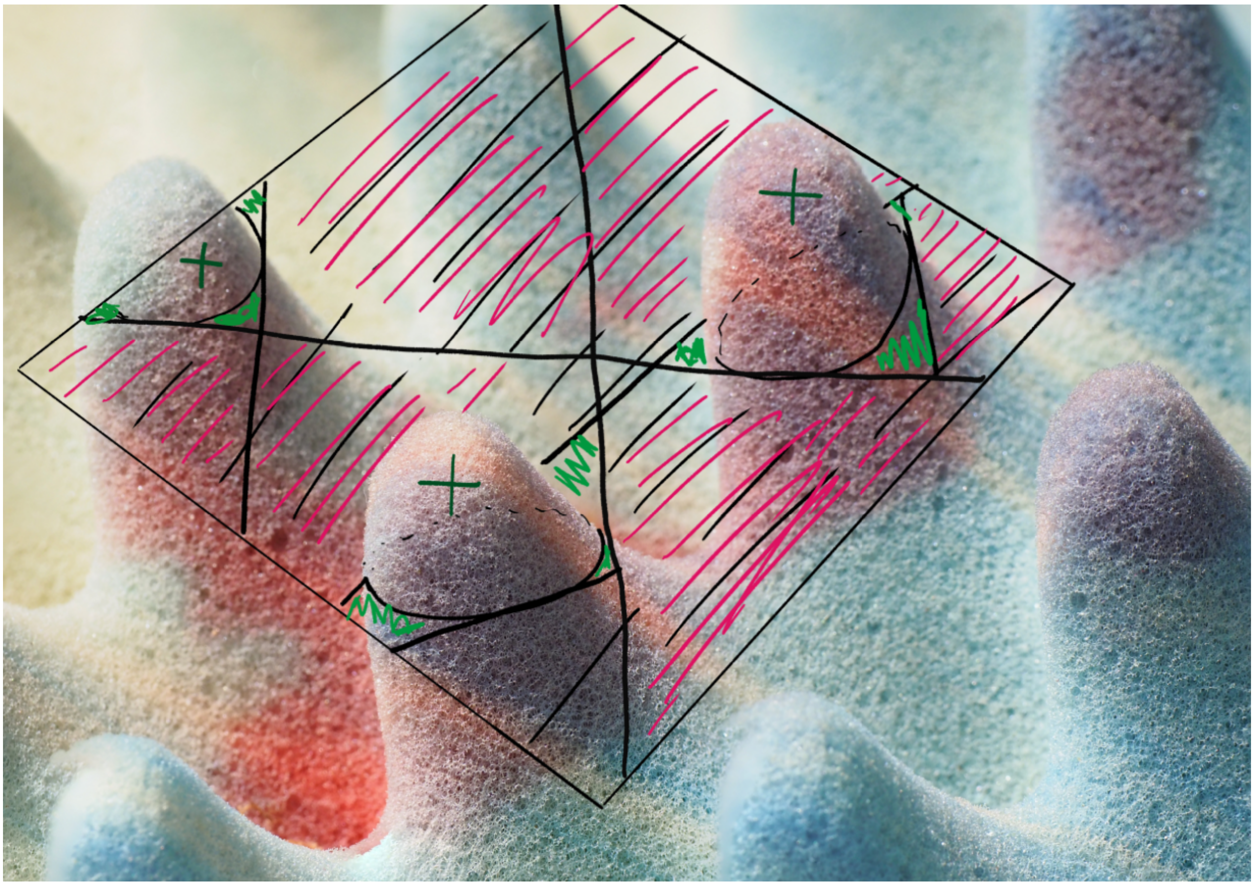
A natural way to see that there should be a distinction between two types of lines is to use spin structures. Consider a physical cubic surface sitting inside $\mathbb{R}^3$, and push the tangent plane alone a line. There are two things that can happen – one is a twisting by a nontrivial element of $\pi_1 SO_3(\mathbb{R})$, the other is no twisting at all.

Example 1.2.
Look at the Fermat cubic surface $x^3 + y^3 + z^3 = -1$



Interpretation of this image: $X \subset \mathbb{R}^3$ is a surface, which has 3 lines that are contained in a plane. We this view X from above this plane, marking a plus/minus to denote the relative height of the surface within each bounded region. Plus denotes part of the surface that bubbles up over the plane, having positive height/z coordinates, etc.

(DZG) This took me a while to visualize – what worked for me was thinking about "egg crate" padding:



After thinking about what physically happens as you push a plane around, it becomes clear that these three lines are all hyperbolic. Note that this question is the same as asking if a path in the frame bundle lifts.

Although the number of lines isn't a constant, we can take a "signature" sort of formula to obtain an invariant. In this case, the number hyperbolic lines minus the number of elliptic lines *is* constant. In this case, the constant is 3.

General mantra for $\mathbb{A}^1$ homotopy theory: if you have a result that works over $\mathbb{C}$ and $\mathbb{R}$, it may be a result in $\mathbb{A}^1$ theory that has realizations recovering the original results.

1.3 $\mathbb{A}^1$ Homotopy Theory

This will allow us to do with schemes much of what we can do in **Top**. Smooth schemes behave like manifolds, where there are balls around points. The convention here will be that we're working over smooth schemes, denoted Sm_k where k is a field.

> Remark: in my notation I use $\mathbb{RP}^n, \mathbb{CP}^n$, and $\mathbb{P}^n(k)$ to denote various projective spaces. I'll adopt Kirsten's convention here and just denote $\mathbb{P}^n(k)$ as $\mathbb{P}^n$.

We'll get spheres from $S^n_{\mathbb{A}} := \mathbb{P}^n/\mathbb{P}^{n-1}$. One nice result due to Morel is that there is a degree map

$$[S^n_{\mathbb{A}}, S^n_{\mathbb{A}}] \longrightarrow GW(k),$$

where the target is not the integers in this case, but rather a group of bilinear forms that are quadratic in characteristic not equal to 2. It is the Grothendieck-Witt group, whose elements are formal difference of bilinear forms.

Thus the group itself is the group completion of nondegenerate symmetric isomorphism classes of bilinear forms $V^2 \longrightarrow k$ where V is a finite-dimensional k-vector space.

The group structure arises because if we have two bilinear forms B, B' on vector spaces V, W respectively, then we can define a new form on $V \oplus W$ by working in components and declaring orthogonality between any of the factors. We then take formal differences of these, and inherit a ring structure from the tensor product of forms.

Bilinear forms over fields can all be diagonalized, although in characteristic 2, this only holds in a stable sense.

1.4 The Grothendieck-Witt Group

Since we can diagonalize, the group $GW(k)$ has a presentation coming from the one dimensional forms. Any of these work as a generator, so we have

- Generators: $\langle a \rangle$ where $a \in k^\times$, corresponding to the form

$$\begin{aligned}\langle a \rangle : k^2 &\longrightarrow k \\ (x, y) &\mapsto axy.\end{aligned}$$

- Relations: if we change the basis of k using a multiplication by $b \in k^\times$, we get $\left\langle ab^2 \right\rangle = \langle a \rangle$. > This means that $a \in k^\times/(k^\times)^2$
  - We also get $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a+b) \rangle$

There are many concrete computations of this known for global fields, local fields, finite fields, function fields, etc.

Example 1.3 (The Complex Numbers)**.**
Computing $GW(\mathbb{C})$: The generators are in bijection with $k^\times/(k^\times)^2$, but since every element of $\mathbb{C}$ is a square, so there's only one element here. We thus obtain

$$\begin{aligned}GW(\mathbb{C}) &\xrightarrow{\cong} \mathbb{Z} \\ \beta &\mapsto \dim V\end{aligned}$$

which is realized by taking the rank.

Example 1.4 (The Reals)**.**
Computing $GW(\mathbb{R})$: We still have the rank, but now we can also take the signature, so we have

$$GW(\mathbb{R}) \xrightarrow{\text{rank} \times \text{signature}} \mathbb{Z}^2,$$

although a minor parity issue crops up here that can be fixed without damaging the isomorphism type.

Example 1.5 (Finite Fields)**.**
Computing $GW(\mathbb{F}_q)$: We can make a matrix out of how β acts on basis elements and take the determinant of it to obtain an invariant called the *discriminant*, and so

$$GW(\mathbb{F}_q) \xrightarrow{\text{discriminant}\times\text{rank}} \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2 \times \mathbb{Z}$$

Note that the quotient is needed because we can change basis in $\mathbb{F}_q$, which amounts to conjugating by a matrix A, and so this discriminant is only well-defined up to squares.

1.5 Euler Class

There is an Euler class in this setting,

$$e(V) = \sum_{p \,\middle|\, \sigma(p)=0} \deg_p(\sigma).$$

Letting X be a smooth cubic surface over k, then a line $L \subset X$ will be a closed point of the Grassmannian $\mathrm{Gr}(1,3)$, so we can think of it as points of the form

$$L = \left\{ [a,b,c,d]s + [a',b',c',d']t \,\middle|\, s,t \in \mathbb{P}^1(k(L) \right\}$$

where the extension field $k(L) = k(a,b,c,d,a',b',c',d')$ is obtained by adjoining the coefficients to k.

DZG: I think these are always separable, mentioned later in the talk.

We thus get

$$\mathbb{P}^1(k(L)) \cong L \underset{\substack{\text{closed}\\ \text{subscheme}}}{\subseteq} X_{k(L)} \subseteq \mathbb{P}^3(k(L)).$$

Given such a line $L \subseteq X$, similar to the real setting, we obtain an involution $I \in \mathrm{Aut} L \cong PGL(2,k(L))$ after choosing coordinates. We also find that $\mathrm{Fix}(L)$ again falls into two cases:

- $2k(L)$ points, or
- 2 conjugate points in some quadratic extension $k(L)[\sqrt{D}]$ where $D \in k(L)^\times/(k(L)^\times)^2$. These correspond to the oddities in the tangent plane in the real case.

We then define

$$\mathrm{Type}(L) = \langle D \rangle \in GW(k(L)),$$

or equivalently $D = ab - cd$ when $I = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, in which case $\mathrm{Type}(L) = \langle -1 \rangle \deg I$.

1.6 An Analogous Trace Formula

Theorem 1.2.
Supposing X is a smooth cubic surface over k of characteristic not equal to 2, we then have

$$\sum_{L\in X} \mathrm{Tr}_{k(L)/k}\mathrm{Type}(L) = \text{One fixed quadratic form} = 15\langle 1\rangle + 12\langle -1\rangle$$

where the trace/transfer maps are defined as

$$\mathrm{Tr}_{k(L)/k} : GW(k(L)) \longrightarrow GW(k)$$
$$(V^2 \xrightarrow{\beta} k(L)) \mapsto (V^2 \xrightarrow{\beta} k) \circ \mathrm{Trace}_{\mathrm{Galois}}$$

where $\mathrm{Trace}_{\mathrm{Galois}}$ comes from summing the conjugates. Note that we can do this because we can view V as a vector space over either k or $k(L)$, so we end up with a quadratic form over k.

Note: we have a well-defined map in the other direction, since the GW is a stable homotopy group of spheres.

Example 1.6 (Complex Numbers)**.**
Let $k = \mathbb{C}$, then apply rank to get $15 + 12 = 27$ on the RHS, while since every element is a square, the Type is just 1, so we get 27 total.

Example 1.7 (Reals)**.**
Let $k = \mathbb{R}$, apply signature. If L is defined over C, so the type is 1, and we're just left with the trace of $\mathbb{C}/\mathbb{R}$ – but this contributes a $+1$ and -1, so there is no contribution. What's left are the lines of $\mathbb{R}$, and since we set it up so type 1 lines are hyperbolic, we just get the trace $15 - 12 = 3$.

Example 1.8 (Finite Fields)**.**
Let $k = \mathbb{F}_q$. We can define lines in $\mathbb{F}_q^n$, and the "begin a square" partitions $(\mathbb{F}_q^n)^\times$ into two disjoint subsets, we can assign types and we let squares be the hyperbolic elements.

We thus get

$$\left\{ \begin{array}{c} \text{Elliptic lines } L \\ \text{with } k(L) = \mathbb{F}_{\text{odd}} \end{array} \right\} - \left\{ \begin{array}{c} \text{Hyperbolic lines } L \\ \text{with } k(L) = \mathbb{F}_{\text{even}} \end{array} \right\} \equiv 0 \mod 2$$

which follows from computing the discriminant of the given form.

2 Lecture 2: User's Guide to $\mathbb{A}^1$ Homotopy Theory

Particularly, arithmetically enriching enumerative results. The first part of this talk focuses on setting up the correct category for this theory.

2.1 Adding Colimits

Recall from last time that we wanted to form a space analogous to a sphere, given by $\mathbb{P}^n/\mathbb{P}^{n-1}$, which we get from a **colimit**

$$\begin{tikzcd} \mathbb{P}^{n-1} \arrow[r] \arrow[d] & \mathbb{P}^n \arrow[d] \\ \{\mathrm{pt}\} \arrow[r] & \mathbb{P}^n/\mathbb{P}^{n-1} \end{tikzcd}$$

which is the same as requiring that for maps from the total space into the quotient, maps coming from the quotiented space and maps coming from the point agree when the compositions are taken.

Example: Another example of a colimit is the union, which is given by

$$\begin{tikzcd} U \bigcap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & U \bigcup V \end{tikzcd}$$

These correspond to crushing and gluing operations, which we can do with topological spaces and would like to do with schemes as well. We'd also like smooth schemes to behave like smooth manifolds, in the sense that we can take an open ball around each point. This is part of what $\mathbb{A}^1$ homotopy theory buys us.

We want colimits, so we add them: let Sm_k to be the category of smooth schemes over k. There is a Yoneda embedding

$$\begin{aligned} \mathrm{Sm}_k &\xrightarrow{Y} \mathrm{Func}({\mathrm{Sm}_k}^{\mathrm{op}}, \mathrm{Sset}) = \mathrm{PreSh}(\mathrm{Sm}_k) \\ X &\mapsto \hom(\,\cdot\,, X). \end{aligned}$$

where one might normally require the target to be sets, but since we'd like homotopy colimits and to be able to do things analogous to fibrant/cofibrant replacements, we aim for simplicial sets instead which can essentially be regarded as topological spaces. We can also identify the target with presheaves on the category of smooth schemes.

We're building a category for a homotopy theory, which means we need either

- A simplicial model category, or
- An ∞-category

Both have notions of fibrations, cofibrations, an associated homotopy category, weak equivalences, etc, and $\mathrm{PreSh}(\mathrm{Sm}_k)$ has this structure.

2.2 Preserving Old Colimits: Picking a Topology

This construction is essentially "freely adding colimits". Since Sm_k had colimits (e.g. the union/intersection of open sets), we want Y to preserve these. We fix this be forcing certain maps to be equivalences using **Bousfield localization**.

This is carried out by looking at open covers

$$U = \coprod_\alpha U_\alpha \longrightarrow X$$

and making a simplicial object out of this map and forcing a weak equivalence

$$\cos_x^0 \coprod_\alpha U_\alpha \xrightarrow{\simeq} X$$

(DZG) Note: this may be the Cech nerve, not entirely sure. Pretty sure I got the notation wrong though.

This gives us a localization functor

$$\mathrm{PreSh}(\mathrm{Sm}_k) \xrightarrow{L_\tau} \mathrm{Sh}_k$$

for τ a Grothendieck topology which declares certain classes of maps to be covers. We have some choices of topology here, roughly in order of increasing number of open sets:

- Zariski (on schemes)
- Nisnevich
- Etale

Definition 2.0.1 (Etale)**.**
A map $f \in \hom(X, Y) \in \mathrm{Sm}_k$ (not necessarily smooth) is **etale** at a point $x \in X$ if the induced map on tangent/cotangent spaces is an isomorphism:

$$T_x X \xrightarrow{f^*} T_{f(x)} Y$$

Definition 2.0.2 (Etale Covers)**.**
A map $f : \coprod_\alpha U_\alpha \longrightarrow X$ is an **etale cover** if it is etale and surjective.

Definition 2.0.3 (Nisnevich Covers)**.**
A map $f : \coprod_\alpha U_\alpha \longrightarrow X$ is a **Nisnevich cover** if it is an etale cover and $x \in X \implies \exists u \in U \mid f^* : k(x) \xrightarrow{\cong} k(u)$.

This topology has a few nice properties:

- Smooth schemes have etale maps into $\mathbb{A}^n$, inclusions/closed immersions $Z \hookrightarrow X$ induce maps $\mathbb{A}^d \hookrightarrow \mathbb{A}^n$
- Satisfies descent for K-theory
- The cohomological dimension equals the Krull dimension
- Cohomology can be computed Cech complexes
- More listed in Voevodsky's original paper

2.3 Contracting the Affine Line

The last step is forcing $\mathbb{A}^1$ to be contractible, i.e. $\mathbb{A}^1 \times X \simeq X$, which will come from another localization $L_{\mathbb{A}}$. This composition will land us in the homotopy theory we want:

$$\mathrm{Sm}_k \xrightarrow{Y} \mathrm{PreSh}_k \xrightarrow{L_\tau} \mathrm{Sh}_k \xrightarrow{L_{\mathbb{A}}} \mathrm{Spc}_k$$

where τ is the choice of the Nisnevich topology, and so we'll call Spc_k our $\mathbb{A}^1$ homotopy theory.

2.4 Making Spheres

Given two pointed spaces X, Y, we have

$$X \wedge Y = \frac{X \times Y}{(X \times \{\mathrm{pt}\}) \bigcup (\{\mathrm{pt}\} \times Y)}$$

In topology, we have $S^m \wedge S^n = S^{m+n}$. In $\mathbb{A}^1$ homotopy theory, we have functors to simplicial sets, and so we can take constant functors, and in particular any element space living in simplicial sets is in our new homotopy theory as well. So we have S^1, we can also take $\mathbb{G}_m = \mathbb{A}^1 - \{0\}$, and so we have spheres

$$S^{p+q,q} = (S^1)^{\wedge p} \wedge (\mathbb{G}_m)^{\wedge q}.$$

Some of these end up being familiar spaces. For example, we can look at the colimit

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \simeq \{\mathrm{pt}\} \\ \big\downarrow & & \big\downarrow \\ \{\mathrm{pt}\} \simeq \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

which follows from the fact that $\mathbb{P}^1 = \mathbb{A}^1 \bigcup \{\infty\}$ (yielding the top-right copy of $\mathbb{A}^1$), and we can take a neighborhood around the point at ∞ to obtain the bottom-left copy – these intersect in $\mathbb{G}_m$.

So $\mathbb{P}^1$ is the colimit of maps from $\mathbb{G}_m$ to a point, so we can conclude that

$$\mathbb{P}^1 \simeq \Sigma \mathbb{G}_m = S^1 \wedge \mathbb{G}_m.$$

We can also show $\mathbb{A}^n - \{0\} \simeq (S^1)^{\wedge n-1} \wedge (\mathbb{G}_m)^{\wedge n}$. This will rely on a general fact about the colimit of $X \times Y$ with its projections is a suspension, given by

$$\begin{array}{ccc} X \times Y & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & \therefore \Sigma X \wedge Y \end{array}$$

and so we can proceed by induction on the following diagram:

$$\begin{array}{ccc} (\mathbb{A}^{n-1} - \{0\}) \times (\mathbb{A}^1 - \{0\}) & \longrightarrow & (\mathbb{A}^{n-1} - \{0\}) \times \mathbb{A}^1 \\ \big\downarrow & & \big\downarrow \\ \mathbb{A}^n \times (\mathbb{A}^1 - \{0\}) & \longrightarrow & \mathbb{A}^n - \{0\} \end{array}$$

We also have $\mathbb{P}^n/\mathbb{P}^{n-1} \simeq (S^1)^{\wedge n} \wedge (\mathbb{G}_m)^{\wedge n}$. This can be show because $\mathbb{P}^n/\mathbb{P}^{n-1} \simeq \mathbb{P}^n/\mathbb{P}^n - \{0\}$ because $\mathbb{A}^1$ is trivial and we can homotop the embedded $\mathbb{P}^{n-1}$ down to the origin, giving a line bundle over $\mathbb{P}^{n-1}$. We can then cut out the copy of $\mathbb{P}^{n-1}$ at infinity, yielding $\mathbb{A}^n/\mathbb{A}^n - \{0\} \simeq \{\mathrm{pt}\}/\mathbb{A}^n - \{0\} = \Sigma(\mathbb{A}^n - \{0\})$, where the last equality comes from looking at a similar colimit diagram as earlier.

2.5 Thom Spaces

These can be made out of vector bundles, which will prove to be useful in viewing smooth schemes like manifolds. Let $V \longrightarrow X$ be an algebraic vector bundle. Then the Thom space

$$\mathrm{Th}(V) = \frac{V}{V - X} \simeq \frac{\mathbb{P}(V \oplus \mathcal{O})}{\mathbb{P}V}$$

where X here corresponds to the zero section, $\mathcal{O}$ is the trivial line bundle, and $\mathbb{P}V$ is the projectivization of V where the coordinate is zero.

Note: If this was a virtual vector bundle, we could make a Thom spectrum.

The next theorem gives us neighborhoods around points

Theorem 2.1 ***(The Purity Theorem).***
Let $Z \hookrightarrow X$ be a closed immersion in Sm_k. Consider $\dfrac{X}{X - Z}$, in topology we could take a tubular neighborhood around Z and view this as a neighborhood mod its boundary. This is equivalent to $\mathrm{Th}(N_Z X)$, the Thom space of the normal bundle of Z in X.

Example 2.1.
Let $Z = \mathrm{Spec}\ (k)$ and $X \in \mathrm{Sm}_k$, then let U be a Zariski open neighborhood of z. Then $U/U - Z \simeq \mathbb{P}^n/\mathbb{P}^{n-1}$ since the Thom space is just a vector space here. So this produces a sphere around z.

Example 2.2.
Replace Spec k with Spec $(k(z))$, this yields $\mathbb{P}^n_{k(z)}/\mathbb{P}^{n-1}_{k(z)} \simeq \mathbb{P}^n/\mathbb{P}^{n-1} \wedge (\mathrm{Spec}\ (k(z) \coprod \{\mathrm{pt}\})$.

(DZG) Note: video says "disjoint basepoint" here and uses different notation, so what I've written may not be correct.

Compare to manifolds: if $z \in U$ a small ball, then $\Sigma \partial U \simeq U/U - z$. So if we wanted to look at maps between boundaries, we could suspend and take degrees.

2.6 The Grothendieck-Witt Group

Recall that the target of the degree map was $GW(k)$; we'll also talk a bit about Milnor K-theory $K_*^M(k)$.

From yesterday, we defined $GW(k)$ as the isomorphism classes of symmetric nondegenerate bilinear forms over k, which had a generators

$$\begin{aligned} \langle a \rangle, &\quad a \in k^\times \\ \langle a \rangle &: k^2 \longrightarrow k \\ (x, y) &\mapsto axy. \end{aligned}$$

and relations

$$\begin{aligned} \left\langle ab^2 \right\rangle &= \langle a \rangle && (b \neq 0) \\ \langle a \rangle \otimes \langle b \rangle &= \langle ab \rangle \\ \langle a \rangle + \langle b \rangle &= \langle a + b \rangle + \langle ab(a+b) \rangle && (a + b \neq 0) \end{aligned}$$

which follows because we're in $k^\times/(k^\times)^2$. Note that the last relation is very important.

These relations imply a special relation concerning a **hyperbolic form,** which is given by

$$h \coloneqq \langle 1 \rangle + \langle -1 \rangle = \langle a \rangle + \langle -a \rangle$$

for any a.

We'll look at invariants on bilinear forms – for many common fields, there are algorithms to determine equality of sums of generators, and thus in GW there are many tools to work with. Some of these tools are invariants arising from the Milnor conjecture, which involves this group and is a huge achievement in $\mathbb{A}^1$ homotopy theory.

We have a **rank homomorphism:**

$$\begin{aligned} \mathrm{rank} : GW(k) &\longrightarrow \mathbb{Z} \\ (B : V^2 \longrightarrow k) &\mapsto \dim V \end{aligned}$$

and the **fundamental ideal** is defined as $I \coloneqq \ker \mathrm{rank}$. This yields a filtration

$$GW(k) \supseteq I \supseteq I^2 \supseteq \cdots$$

where the associated graded are etale cohomology groups and (by the Milnor conjecture) Milnor K-theory groups.

2.7 Milnor K-Theory

We define **Milnor K-theory** as

$$K_i^M \coloneqq \frac{\bigoplus_{i=1}^\infty (k^\times)^{\otimes i}}{\langle a \otimes (1-a) \rangle}$$

which is tensor algebra on $k^\times$, modded out by the Steinberg relation.

Theorem 2.2 ***(The Milnor Conjecture (Voevodsky)).***
There is a map

$$\begin{aligned} K_n^M &\longrightarrow I^n/I^{n+1} \\ \bigotimes_{i=1}^{n} a_i &\mapsto \prod_{i=1}^{n} \left(\langle 1\rangle - \langle a_i\rangle\right). \end{aligned}$$

We can also look at the Kummer map coming from the short exact sequence

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \overline{k^\times} \longrightarrow \overline{k^\times} \longrightarrow 1$$

which lets us make a map

$$k^\times \longrightarrow H^1_{\text{et}}(k;\ \mathbb{Z}/2\mathbb{Z})$$

where we can use that fact that $k^\times \cong K_1^M$ to land in Milnor K-theory,

$$K_1^M \longrightarrow H^1_{\text{et}}(k;\ \mathbb{Z}/2\mathbb{Z}),$$

where we can use the cup product to lift this to a map to the nth graded piece

$$K_n^M \longrightarrow H^n_{\text{et}}(k;\ \mathbb{Z}/2\mathbb{Z}).$$

Fitting all of this together, we get maps

$$sI^n/I^{n+1} \leftarrow K_n^M \longrightarrow H^n_{\text{et}}(k;\ \mathbb{Z}/2\mathbb{Z}),$$

and the Milnor conjecture states that these are isomorphisms.

In other words, the associated graded of this filtration is the etale cohomology or Milnor K-theory, and if you have a field for which the nth etale cohomology in $\mathbb{Z}/2\mathbb{Z}$ coefficients doesn't vanish, then there is a nontrivial piece in the associated graded.

2.8 Grothendieck-Witt Group Invariants

This lets us view maps $I^n \longrightarrow I^n/I^{n+1}$ as invariants on $GW(k)$.

- For $n = 0$, this is the rank homomorphism.
- For $n = 1$ we get the discriminant, which is the determinant of the linear map associated to the bilinear form obtained after choosing a basis.
- For $n = 2$ this is the Hasse-Witt invariant (see written notes)
- For $n = 3$ this is the Arason invariant

For higher n these invariants don't have names, but for various fields, the lower degrees form a complete invariant – for example, for finite fields, one needs only check $n = 0, 1$, while $\mathbb{Q}$ requires $n = 0, 1, 2$.

The Grothendieck-Witt group is the 0th graded piece of Milnor-Witt K-theory, $M_*^{MW}(k)$, which is also a homotopy group of spheres in $\mathbb{A}^1$ homotopy theory (due to Hopkins and Morel).

This group has generators

$$\begin{aligned} &\langle a \rangle, \quad a \in k^\times, && \deg a = 1 \\ &\eta, && \deg \eta = -1 \end{aligned}$$

and relations

$$\begin{aligned} \eta \langle a \rangle &= \langle a \rangle \eta \\ \langle a \rangle \langle 1 - a \rangle &= 0 \\ \langle ab \rangle &= \langle a \rangle + \langle b \rangle + \eta \langle a \rangle \langle b \rangle \\ \eta h &= 0 \end{aligned}$$

where h is the same as earlier, but since it's in the wrong group, we need to define this using the isomorphism

$$\begin{aligned} GW(k) &\cong K_0^{MW}(k) \\ \langle a \rangle &\mapsto 1 + n \langle a \rangle \\ h := \langle 1 \rangle + \langle -1 \rangle &\mapsto 2 + \eta \langle -1 \rangle := h. \end{aligned}$$

2.9 Degree Theorem

This theorem says that η corresponds to a Hopf map.

Theorem (Morel):

$$[(S^1)^{\wedge n} \wedge (\mathbb{G}_m)^{\wedge j},\ (S^1)^{\wedge n} \wedge (\mathbb{G}_m)^{\wedge r}] \cong K_{r-j}^{MW}$$

where the square brackets correspond to homotopy classes of maps.

In particular, when $j = r = n$, we obtain

$$[\mathbb{P}^n/\mathbb{P}^{n-1},\ \mathbb{P}^n/\mathbb{P}^{n-1}] \cong GW(k)$$

This is a fantastic theorem, which we will see again later when doing oriented Chow groups.

A nice consequence is that if we let $k = \mathbb{R}$, the degrees behave nicely, characterized by the commutativity of this diagram:

$$\begin{tikzcd} {[S^{2n},\ S^{2n}]} \arrow[d, "\deg"] & {[\mathbb{P}^n/\mathbb{P}^{n-1},\ \mathbb{P}^n/\mathbb{P}^{n-1}]} \arrow[l, "\mathbb{C}\text{-points}"'] \arrow[r, "\mathbb{R}\text{-points}"] \arrow[d, "\deg"] & {[S^n,\ S^n]} \arrow[d, "\deg"] \\ \mathbb{Z} & GW(k) \arrow[l, "\text{rank}"] \arrow[r, "\text{signature}"'] & \mathbb{Z} \end{tikzcd}$$

where the edge degree maps are just the topological degree of maps between spheres and the middle is the $\mathbb{A}^1$ degree. The signature is the usual difference in ± 1s occurring after diagonalization. Thus

$GW(k)$ lets us simultaneously read off the real and complex degrees of maps between schemes over $\mathbb{R}$.

So these homotopy groups are actually homotopy sheaves (not just global sections of sheaves), where we can form a sheaf by taking smash with U^+ and sheafifying. Thus $GW(k), K_*^{MW}(k)$, and $K_*^M(k)$ are all global sections of sheaves.

> DZG: Not sure what U^+ is here.

2.10 Producing a Sheaf

There is a procedure in Morel's book for producing an unramified sheaf K_*^{MW} from the values on fields, i.e. $K_*^{MW}(E)$ for some $E \supset k$ of finite type. It proceeds as follows:

We want to know what the sections are on some scheme Y, so we look at its function field and check K_*^{MW} on it to see which sections are defined over all of Y and not over the generic point. This produces the additional data of boundary/residue maps that determine when sections extend globally.

So let $V : E \longrightarrow \mathbb{Z} \bigcup \{\infty\}$ be a valuation and $\mathcal{O}_V = \left\{ e \in E \;\middle|\; V(e) \geq 0 \right\}$ and choose a uniformizer π such that $v(\pi) = 1$. Then form the residue field $k(V) \coloneqq \mathcal{O}_V / \langle \pi \rangle$. Then this residue map plus the sections will allow us to define a sheaf, so define the **residue homomorphism**

$$\begin{aligned}
\partial_V^\pi : K_*^{MW}(E) &\longrightarrow K_*^{MW}(k(V))[-1] \\
\langle \pi \rangle \langle a_1 \rangle \cdots \langle a_n \rangle &\mapsto \langle \overline{a}_1 \rangle \cdots \langle \overline{a}_n \rangle \\
\langle a_1 \rangle \cdots \langle a_n \rangle &\mapsto 0 \\
\langle \eta \rangle &\mapsto 0
\end{aligned}$$

where $a_i \in \mathcal{O}_V^\times$ and $\overline{a}_i$ is the reduction.

> Note that it is true that $\partial_V^\pi \eta = \eta \partial_V^\pi$, and this is part of what uniquely defines this map. However, we can't have $\eta \mapsto \eta$, because this does not decrease the degree by 1.

So then the sections on $\mathcal{O}$ are given by

$$K_*^{MW}(\mathcal{O}_V) \coloneqq \ker \partial_V^\pi$$

and there is a procedure for making a sheaf from this denoted K_*^{MW}.

The fact this is a stable homotopy sheaf provides some transfers (seen last time), which we'll start with next time.

3 Lecture 3: Transfers

3.1 Defining Some Transfers

Recall that we have the sheaves K_*^{MW}, GW, and the sheaf property means that an inclusion $K \hookrightarrow L$ induces a map Spec $(L) \longrightarrow$ Spec (K). We can take $GW(\text{Spec } (L) \longrightarrow \text{Spec } (K))$, and this is exactly the restriction/base change given by $\cdot \otimes_k L$ of bilinear forms.

We also saw that these were stable homotopy sheaves, so there should be transfers, and we want to use them for field extensions. Let $K \subset L$ be a finite extension of finite-type schemes over k. This leads to transfer maps

$$\mathrm{Tr}_{L/K} : GW(K) \longrightarrow GW(L).$$

There is also a geometric transfer (which is the prettiest!) which we'll define momentarily, given by with multiplication by one of those brackets to define a cohomological transfer. The geometric transfer will depend on a sequence of generators, while, while this choice can be removed for the cohomological transfer. If you use the twisting data you can get an absolute transfer.

In the case where $K \subset L$ is separable, there is a canonical way to explicitly untwist, and the absolute and cohomological transfers agree. For these two, we took

$$\mathrm{Tr}(B : V^2 \longrightarrow L) = V^2 \xrightarrow{B} L \xrightarrow{\mathrm{Tr}_{W^k}}$$

where we now view V as a k-vector space, and Tr_{W^k} is the trace from Galois theory, the sum of the Galois conjugates.

We'll show that we have this structure for the geometric transfer. If $L = K[z]/\langle f\rangle$, so we've chosen some generator, then we get an induced map Spec $(L) \overset{z}{\hookrightarrow} \mathbb{P}^1_K$. Since this is a closed immersion corresponding to z, we can form a backwards map

$$\mathbb{P}^1_K \longrightarrow \frac{\mathbb{P}^1_K}{\mathbb{P}^1_K - \{z\}} \simeq \mathbb{P}^1_L$$

by crushing everything but z, where the last equivalence was seen in the previous lecture. But now we can take $K_1^{MW}(\mathbb{P}^1_K \longrightarrow \mathbb{P}^1_L)$, which is a map

$$\mathrm{Tr}^{\mathrm{geom}}_{L/K} : GW(L) \longrightarrow GW(K)$$

So we have some transfers.

3.2 Bilinear Forms on Chow Groups

The finale of this morning was going to be adding bilinear forms to Chow groups for the purposes of having a tool in enumerative geometry. So let $X \in \mathrm{Sm}_k$ and $X^{(i)}$ codimension i reduced, irreducible subschemes of X. Then

$$CH^i(X) = \frac{\bigoplus_{X^{(i)}} \mathbb{Z}}{\sim}$$

where $\sim$ is rational equivalence, the equivalence relation generated by taking subvarieties of $V \subset X \times \mathbb{P}^1$ and equating the fibers and the endpoints $V_{\{1\}} \sim V_{\{0\}}$, i.e. $V \bigcap (X \times \{1\}) \sim V \bigcap (X \times \{0\})$.

These are useful in enumerative geometry – there are Chern classes, pushforwards, pullbacks, a ring structure, etc. This ring structure lets us do intersection theory, providing some machinery to help with enumerative questions.

The ith Chow group, in addition to being a motivic homology group, also has a nice formula due to Bloch that applies in the case of smooth schemes: $CH^i(X) \cong H^i(X; K_i^M)$ where the RHS is the Nisnevich cohomology of X with coefficients in Milnor K-theory.

Oriented Chow groups (AKA Chow-Witt groups) which are the original Chow groups together with a bilinear form. By Borge and Morel, motivated by the Bloch formula above, these can be defined as

$$\widetilde{CH}^i(X) \coloneqq H^i(X; K_i^{MW}).$$

This can be computed by a complex (as in Morel's book):

$$\cdots \longrightarrow \bigoplus_{z \in X^{(ii1)}} K_1^{MW}(K(z), \det_{k(z)} T_z X) \longrightarrow \bigoplus_{z \in X^{(i)}} GW(K(z), \det_{k(z)} T_z X) \longrightarrow \bigoplus_{z \in X^{(i+1)}} K_{-1}^{MW}(K(z), \det_{k(z)} T_z X) \longrightarrow$$

where $k(z)$ is the function field, and since z has a generic point, we can take the highest wedge power of the tangent space of X at z to yield the determinant term, which serves as an added twist. This explains why elements of the oriented Chow are formal combinations of codimension i subvarieties $z \in X^{(i)}$ and a bilinear form over $k(z)$, $B \in GW(k(z))$.