Tamagawa Numbers (in the Function Field Case)

Sourced from a q. in number theory (today's focus).

Quadratic Forms -

When are two equivalent via a linear change of variables?

x^2+y^2 x^2-y^2

P.D.   N.D.

Can take over any ring R

Two methods for invariants

- Take $R = \mathbb{R}$
- Reduce mod n

Is this all you need...?

Def. Let all forms be P.D.; two forms q, q' are <u>in the same genus</u> iff they are equivalent mod N $\forall N > 0$.

Let q be a $\mathbb{Z}$-form, $R \in \mathrm{CommRings}$.

$$O_q(R) = \{ A \in GL(n,R) \mid q \circ A = q \}$$

$$\mathrm{Mass}(q) = \sum_{\substack{q' \text{ of genus} \\ \text{equal to } q}} 1 / |O_q(\mathbb{Z})|$$

Defn Unimodular iff nondegenerate mod p $\forall p \in \mathbb{P}$.

Ex: $x^2 + y^2 \equiv (x+y)^2 \bmod 2$
$\Rightarrow$ degenerate.

Also have

$$\mathrm{Mass}(q) = \zeta\left(\tfrac{n}{2}\right) \cdot \frac{\zeta(2)\cdot\zeta(4)\cdots\zeta(n-2)}{\mathrm{vol}\, S^1 \cdot \mathrm{vol}\, S^2 \cdots \mathrm{vol}\, S^{n+1}}$$

$n=8$: $\mathrm{RHS} = 1 \Big/ \underbrace{2^{14}\, 3^{5}\, 5^{2}\, 7}$

Weil group of Exceptional group of lie type $\underline{\underline{E8}}$ — $W(E8)$

$\Rightarrow$ Only one unimodular form in 8 vars!

Attempt to prove all q.f.s of same genus are equivalent.

Let q, q' be of genus g and equiv., so

$q = q' \circ A_N$ w/ $A_N \in GL(n, \mathbb{Z}/n\mathbb{Z})$

Let $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/N\mathbb{Z} = \prod_{p \in P} \mathbb{Z}_p$

and $\{A_N\} = A \in GL(n, \hat{\mathbb{Z}})$

Then $q = q' \circ A \implies q \sim q'$ over all $\mathbb{Z}_p$

$\implies q \sim q'$ over $\mathbb{Q}_p = \mathbb{Z}[1/p]$

Hasse Minkowski

$\implies q = q' \circ B$ where $B \in GL(n, \mathbb{Q})$

$= B^{-1} A$

Let $A^{Fin} = \hat{\mathbb{Z}} \otimes \mathbb{Q} = \prod \mathbb{Q}_p$ (Finite Adeles)

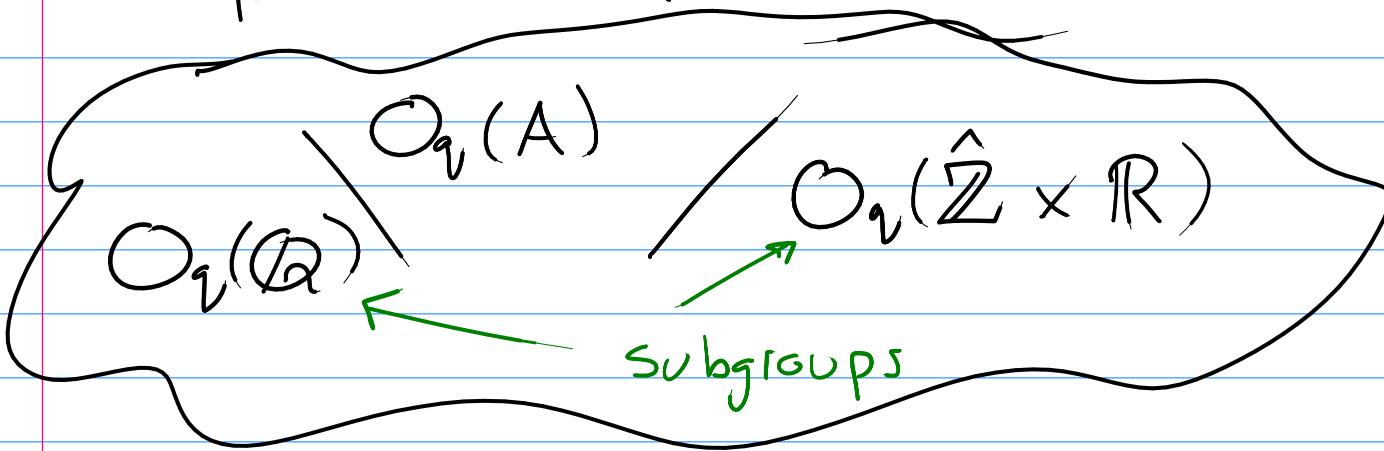
$B^{-1} \circ A \in O_q(A^{Fin})$ ~ ambiguous, so take cosets

$\to \in O_q(\mathbb{Q}) \backslash O_q(A^{Fin}) / O_q(\hat{\mathbb{Z}})$

double coset

Count these; in bijection w/ equiv classes of q.F.s.

Drop A^{fin} to A



where A is a locally compact group, so it has a Haar measure μ that is left-invar.

Look at

$$S = \mu\left(O_q(A) / O_q(\mathbb{Q}) \right) \Big/ \mu\left(O_q(\hat{\mathbb{Z}} \times \mathbb{R})\right)$$

Not the # of orbits, since there is not a <u>free</u> action. But counts w/multiplicity.

$$\boxed{\longrightarrow S = \mathrm{Mass}(q).}$$

Look at special orthog. $SO_q(\mathbb{A})$

Has a canonical Haar measure (generally only defined up to scalar mult) called the <u>Tamagawa Measure</u>.

$$\mathbb{Z}^k \operatorname{Mass}(q) = \mu\left(\frac{SO_q\mathbb{Q} \,/\, SO_q\mathbb{A}}{SO_q\hat{\mathbb{Z}} \times \mathbb{R}}\right)$$

$SO_q(\mathbb{A})$ is a smooth mfd, can take top form

but <u>also</u> a linear algebraic gp., so do AE over $\mathbb{Q}$

Let $V_{\mathbb{R}} = \left\{ \begin{array}{l} \text{translation-invariant} \\ \text{top forms on } SO_q(\mathbb{R}) \end{array} \right\}$

$\mu_{\omega,\mathbb{R}}$

$\cup|$

determines measures $\omega \longrightarrow$ $V_{\mathbb{Q}} = \{\text{algebraic top forms}\}$ Vector space

$\cap$

$SO_q(\mathbb{Q}_p)$ p-adic analytic Lie group

$\mu_{q\omega,\mathbb{Q}_p}$

Yields the Tamagawa measure

$$\mu_{Tam} := \prod \mu_{q_w, \mathbb{Q}_p} \times \mu_{\omega, \mathbb{R}}$$

Invariant under scaling

$$\omega \mapsto 5\omega \qquad \underbrace{\mapsto \tfrac{1}{5}\mu} \qquad \underbrace{\mapsto 5\mu}$$

Mass formula (Tamagawa-Weil version)

$$\mu_{Tam}\left(SO_q(\mathbb{A}) / SO_q(\mathbb{Q}) \right) = 2$$

because SO_q is not simply connected & has double cover $Spin_q$.

Conjecture

G simply connected, semi-simple alg. group over $\mathbb{Q}$. Then

$$\mu_{Tam}\left(G(\mathbb{A}) / G(\mathbb{Q}) \right) = 1$$

Rest of the week - Function field analog.