Talk 2: $\mathbb{A}^1$

Enumerative geometry: counts alg-geo objects w/ conditions (use an alg closed field).

Arithmetic counts of lines on a cubic surface.

<u>Def</u>: Cubic surface = set of solns of a deg 3 f over K. $\{f(x,y,z)=0\}$

<u>Better</u>: compactify $\hookrightarrow \mathbb{P}^3(k)$

Gives a mfd if f is smooth.

Classical result: $K=\mathbb{C} \rightarrow 27$ lines

<u>Ex</u>: Fermat cubic: $f=\sum_{i=1}^{4} x_i^3$

$$L_1 = \left\{ s\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} + t\begin{bmatrix}0\\0\\1\\-1\end{bmatrix} : s,t \in \mathbb{P}^1 \right\}$$

Choose λ, ω s.t. $\lambda^3 = \omega^3 = -1$, then

$$L_i = \left\{ s \begin{bmatrix} 1 \\ \lambda \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \\ \omega \end{bmatrix} \right\}$$

<u>Modern proof</u>: Look at $Gr(1,3)$

= lines in $\mathbb{P}^3$ parameterizing
$W \subseteq \mathbb{C}^4$, $\dim W = 2$.

Let $\left\{ \begin{array}{c} S \\ \downarrow \\ Gr(1,3) \end{array} \right\}$ be the tautological bundle

and $Sym^3 S^\vee$ = cubic polys on W

$\rightarrow f$ determines an elt in $Sym^3(\mathbb{C}^4)^\vee$

Then the line $\mathbb{P}W$ corresponding to W is in X

<u>iff</u>

$\sigma_f(W) = 0$ $\quad$ $:= f|_W$

section of the bundle

$$\left\{\begin{tikzcd} \mathrm{Sym}^3 S^\vee \arrow[d] \\ Gr(1,3) \end{tikzcd}\right\}$$

Use the <u>Euler class</u>, and choose a section with <u>isolated zeros</u>.

Use degree $\pi_{r-1} S^{r-1} \cong \mathbb{Z}$ to assign a weight.

oriented

$$\begin{tikzcd} \mathbb{R}^r \arrow[r] & V \arrow[d] \\ & M \end{tikzcd}$$

Pick $p \in M$, $\sigma(p) = 0$. Want to define
$\deg_p \sigma \in \mathbb{Z}$.

Choose a chart in $N_\varepsilon(p)$ with only one zero.

Choose local trivialization of V compat w/ orientation $\to V = M \times \mathbb{R}^r$

$$\Longrightarrow e(V) = \sum_T \deg_p(\sigma)$$

where $T = \{p \in M \mid \sigma(p) = 0\}$

X smooth $\Longrightarrow \deg_p(\sigma) = 1$

$$\Longrightarrow |\text{lines on } X| = e(\mathrm{Sym}^3 S^\vee) = 27.$$

Note: Only possibilities are 3, 7, 15, 27

Distinguish hyperbolic & elliptic $\mathbb{R}$-lines

L a real line, $L \in \mathbb{RP}^1$

$\mathrm{Aut}\, L \cong PGL(2, \mathbb{R})$

$$I \mapsto [I] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$I = \left(z \mapsto \frac{az+b}{cz+d} \right)$$

$$\mathrm{Fix}(I) = \{ z^2 + (d-a)z + b = 0 \}$$

$I = 2$ $\mathbb{R}$-pts $\Rightarrow$ Hyperbolic

or

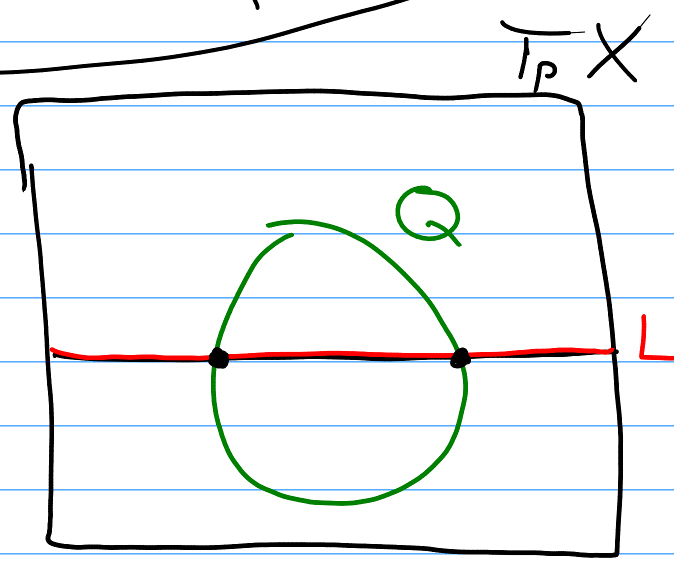
conj. $\mathbb{C}$-pts $\Rightarrow$ elliptic

Construct an involution

$p \in L$

$T_pX \cap X = L \cup Q$

$\{ q \mid T_qX = T_pX \}$



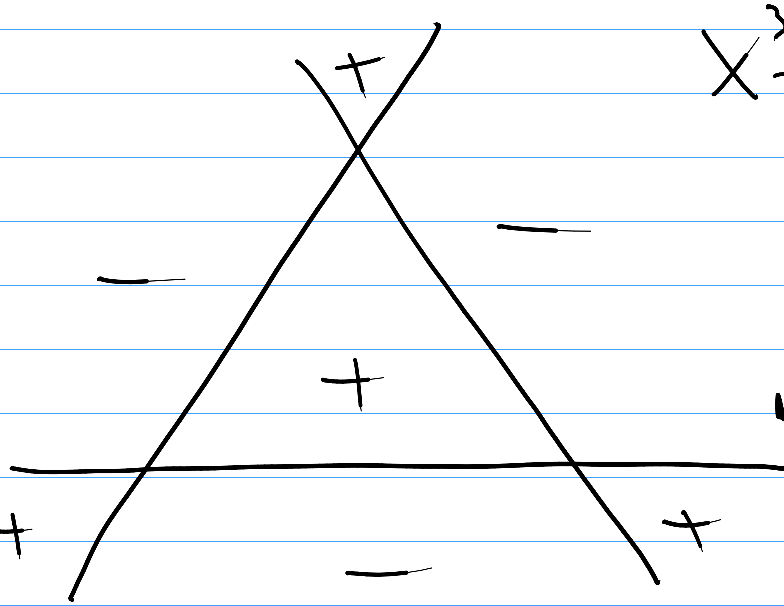
$L \cap Q = \{p, p'\}$

$\longrightarrow I(p) = p'$

Then L is elliptic when I is.

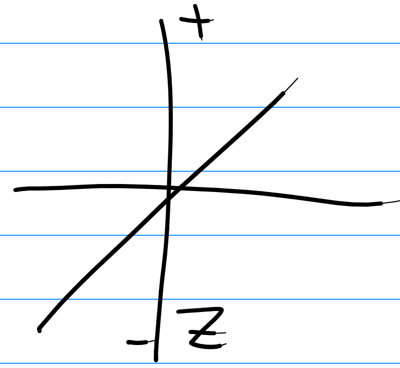
<u>Ex</u>: Fermat cubic surface

$$x^3+y^3+z^3=-1$$



in surface

$+$: $z>0$, $-$: $z<0$



Yields a frame - does it lift?

<u>Thm</u>: $|\text{hyperbolic lines}| - |\text{elliptic lines}| = 3$

Strategy: Results over $\mathbb{R}$ & $\mathbb{C}$ might be realizations of results in $\mathbb{A}_1$-hty.

Use $\mathbb{RP}^n / \mathbb{RP}^{n-1} \cong S^{n-1}$

to define Morel $\deg\left[\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}\right]$

$\longrightarrow GW(k)$

= Grothendieck-Witt group

= group completion of semiring under $\otimes, \oplus$; iso classes of bilinear forms

$$B : V \times V \to K$$

Presentation:

$$\langle a \rangle : K^2 \to k$$

$$(x,y) \mapsto axy$$

$$\langle ab^2 \rangle = \langle a \rangle, \quad \langle a \rangle + \langle b \rangle = \langle a \rangle + \langle b \rangle + \langle ab(a+b) \rangle$$

Ex

$GW(\mathbb{C}) = \mathbb{Z}$ (rank hom.)

$$\begin{tikzcd} GW(\mathbb{R}) \arrow[r, "\mathrm{sig} \times \mathrm{rank}"] \arrow[d, "\cong"'] & \mathbb{Z} \times \mathbb{Z} \arrow[dl, "\cup/"] \\ \mathbb{Z} \times \mathbb{Z} & \end{tikzcd}$$

$$GW(\mathbb{F}_q) \xrightarrow{\mathrm{disc} \times \mathrm{rank}} \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2$$?

There is an Euler class

k a field, $\mathrm{char}(k) \neq 2$, X smooth cubic$/k$

$L \subseteq X$ is a closed pt. of $Gr(1,3,k)$?

$\hookrightarrow L = \{\vec{v}_1 s + \vec{v}_2 t\}$, $k(L) = k(\vec{v}_1, \vec{v}_2)$

$\mathbb{P}^1_{k(L)} \cong L \subseteq X_{k(L)} \subseteq \mathbb{P}^3_{k(L)}$

Given L on X, obtain involution

$$I \in \operatorname{Aut} L \cong PGL(2, k, k(L))$$

$\rightarrow$ $\operatorname{Fix}(I)$ is either 2 $k(L)$-pts or conjugate in $k(L)[\sqrt{D}]$ where

$$D \in k(L)^\times / (k(L)^\times)^2$$

Def: $\operatorname{Type}(L) = \langle D \rangle \in GW(k(L))$

$\rightarrow$ $D = ab - cd$

$\rightarrow$ $\operatorname{Type}(L) = \langle -1 \rangle \deg I$

Thm X smooth cubic

$\longrightarrow$ $$\sum_{\substack{\text{lines} \\ L \text{ of } X}} \operatorname{Tr}_{R(L)/R} \underbrace{\operatorname{type}(L)}_{} = 15\langle 1 \rangle + 12\langle -1 \rangle$$

$R = \mathbb{C}$: Apply rank $\rightarrow$ # lines $= 27$

$R = \mathbb{R}$: Apply sig $\rightarrow$ #hyp $-$ #ellip $= 3$

$R = \mathbb{F}_q$

Stable
hty
gps

$$\left\{ \begin{array}{c} \#\text{ ellip lines } L \text{ with} \\ k(L) = \mathbb{F}_{q^{2n+1}} \end{array} \right\}$$

$$- \left\{ \begin{array}{c} \#\text{ hyperbolic} \\ K(L) = \mathbb{F}_{q^{2n}} \end{array} \right\} \equiv 0 \mod 2$$