Topological Hochschild Homology in Arithmetic Geometry

Goals:

- Describe Hochschild and cyclic homology using a classical approach
- Describe the "Topological" counterparts
- Relate these to Algebraic and Arithmetic Geometry

Classical Theory

This only depends on basic homological algebra, but becomes a bit more interesting when objects with a "perfectoid" flavor are input.

Fix a commutative ring k, then for any k- algebra A we can form the **Hochschild complex**

$$HH(A/k) := A \xleftarrow{\partial_1} A^{\otimes_k 2} \xleftarrow{\partial_2} A^{\otimes_k 3} \to \cdots$$

$$\begin{aligned} \partial_1(a_0 \otimes a_1) &= a_0 a_1 - a_1 a_0 \\ \partial_2(a_0 \otimes a_1 \otimes a_2) &= a_0 a_1 \otimes a_2 - a_0 a_1 \otimes a_2 + a_2 a_0 \otimes a_1 \\ &\vdots \end{aligned}$$

Then by definition, the Hochschild homology groups $HH_n(A/k)$ are obtained by taking the homology of this complex.

Example:

- $HH_0(A/k) = A/\langle ab - ba \rangle = A/[A, A] = A$ when A is commutative.
- $HH_1(A/k) = A \otimes_k A/\langle ab \otimes c - a \otimes bc + ac \otimes b \rangle = \Omega^1_{A/k}$
  - Note that the ideal appearing here is the Leibniz rule from differential forms, and so the last object is the module of Kahler differential forms via the identification $adb \mapsto a \otimes b$
- $HH_*(A/k)$ is a commutative graded k-algebra

Using these facts, we get maps

$$\varepsilon_n : \Omega^n_{A/k} \to HH_n$$

by the universal property of the differential forms

$$\Omega^*_{A/k} := \Lambda^*_A \Omega^1_{A/k}$$

Theorem (Hochschild-Kostent-Rosenburg, 60s):
If A is smooth over k, then ε_n are all isomorphisms.

So the philosophy here is that HH_* works as a generalization of differential forms, even if A is not smooth or even not commutative (Connes, Feigen-Tsygen, Loday-Quillen).

Proving HKR: A Lemma

We use $(\,\cdot\,\hat{\otimes}\,\cdot\,)$ to denote the *derived tensor product.*

Lemma: Let $E = A \otimes_k A^{\mathrm{op}}$ be the enveloping algebra of A, then for any flat k- algebra A, up to quasi-isomorphism we have

$$HH(A/k) \simeq A \hat{\otimes}_E A.$$

This works because E is an A bimodule, and so the derived tensor product appearing on the RHS is represented by the Hochschild complex on the left.

> Note that we can do this for non-flat A by replacing the tensor product appearing in E with a derived tensor product, which puts you in the world of simplicial rings.

To prove this lemma, we explicitly calculate the RHS as

$$A \hat{\otimes}_E A = A \hat{\otimes}_E \left[A^{\otimes 2} \xleftarrow{\partial_3'} A^{\otimes 3} \xleftarrow{\partial_4'} \cdots \right]$$

where the complex appearing on the RHS is the bar complex, i.e. a resolution of A by flat E-modules. This is essentially the Hochschild complex, truncating the first term, with a different differential ∂' given by

$$\partial_n'(a_1 \otimes a_2 \otimes \cdots a_n) = \sum_{i=1}^{n} (-1)^{i+1} a_i \otimes (\prod_{j \neq i} a_j)$$

> Note: this may not be quite correct, just a guess.

Because A is a unital algebra, this complex is exact (except at degree zero), which is a purely algebraic result.

It also resolves A. This can be seen because if you use the multiplication map $m : A \otimes A \to A$ and then $\ker m = \operatorname{im} \partial_3'$. Under the flatness assumption, all of the terms in this resolution are flat A-modules.

Finally, when you base change along E down to A, this introduces the "cyclic" part of the multiplication defined in HH earlier, and so this is an on-the-nose identification where the RHS is computing the derived tensor product.

This is because in E, A acts on the left and A^{op} acts on the right, and so taking $A \hat{\otimes}_E$ mods out by the multiplication and thus identifies these two actions and introduces the cyclic part of the boundary maps in HH.

From this we obtain the corollary

$$H_*(A/k) = \mathrm{Tor}_*^E(A, A).$$

Proving HKR

Again take A a smooth k-algebra, we need to show that the graded algebra $HH_*(A/k)$ is the exterior algebra on its degree 1 element.

This is well-known for the graded algebra $\mathrm{Tor}_*^B(C, C)$ when $\ker(B \twoheadrightarrow C)$ is locally generated by a regular sequence. In this case, there is an explicit Koszul complex from which you can read off the fact that these tor groups are just the wedge algebra generated by Tor_1^B.

An example of this is the multiplication map $A \otimes_k A \to A$, since $k \to A$ is smooth. This works because locally (after localizing and perhaps some etale extension), this looks like a polynomial algebra and this map looks like $k[t] \otimes_k k[t] \to k[t]$, which has kernel $\langle t \otimes 1 - 1 \otimes t \rangle$, which is a regular sequence. This exactly applies to the corollary above, so we're done.

> Note: something interesting happens here, because we usually look at differential forms arising from the de Rham cochain complex, while here's we've written down a chain complex whose homology gives the terms in this cochain complex. This is what makes HH an enrichment of the de Rham complex.

Cyclic Homology

We have group actions on the Hochschild complex:

$$HH(A/k) := \begin{array}{ccccc} A & \xleftarrow{\partial_1} & A^{\otimes_k 2} & \xleftarrow{\partial_2} & A^{\otimes_k 3} \leftarrow \cdots \\ \mathbb{Z}/1\mathbb{Z} \curvearrowright & & \mathbb{Z}/2\mathbb{Z} \curvearrowright & & \mathbb{Z}/3\mathbb{Z} \curvearrowright \end{array}$$

In general, we have

$$\langle t_n \rangle := \mathbb{Z}/(n+1)\mathbb{Z} \curvearrowright A^{\otimes_k (n+1)}.$$

Note that the label on the generator here is slightly weird, but the reason is that $\mathbb{Z}/n\mathbb{Z}$ is actually acting on terms of homological degree $n-1$. This action is given by a "cyclic right shift",

$$\begin{gathered} t_n : A^{\otimes_k (n+1)} \circlearrowleft \\ a_0 \otimes \cdots \otimes a_{n-1} \otimes a_n \mapsto a_n \otimes a_0 \otimes \cdots \otimes a_{n-1} \end{gathered}$$

We then define a norm

$$\begin{gathered} \mathrm{N} : A^{\otimes_k (n+1)} \circlearrowleft \\ \mathrm{N} := \sum_{i=1}^{n} \left((-1)^n t_n\right)^i \end{gathered}$$

where we think of $(-1)^n t_n$ as a "twisted action", and an "extra degeneracy" map

$$\begin{gathered} s : A^{\otimes_k n} \to A^{\otimes_k (n+1)} \\ \bigotimes_i a_i \mapsto 1 \otimes \bigotimes_i a_i. \end{gathered}$$

These come about because the Hochschild complex is actually the complex associated to some simplicial k-module, which has face maps (going down) and degeneracy maps (going up). The boundary maps are the alternating sums of the face maps here, and the degeneracy maps just involve inserting a 1 somewhere in the tensor product – except this particular map doesn't appear, which is why it's an "extra" degeneracy.

Perhaps the most important map is **Connes operator**,

$$B : A^{\otimes_k n} \xrightarrow{N} A^{\otimes_k n} \xrightarrow{s} A^{\otimes_k (n+1)} \xrightarrow{\mathrm{id} - (-1)^n t_n} A^{\otimes_k (n+1)}$$

Can check that $B^2 = 0$ and $B\partial + \partial B = 0$ (for ∂ the boundary map in the Hochschild complex HH). Thus B gives HH the structure of a cochain complex, in addition to it being a chain complex, which are both compatible. This is called a "mixed complex" or an "algebraic S^1-complex".

We can also think of $B : H(A/k) \to HH(A/k)[-1]$, and B can be said to "refine the de Rham differential", characterized by the commutativity of the following diagram:

$$\begin{tikzcd} HH_n(A/k) \arrow[r, "B"] & HH_{n+1}(A/k) \\ \Omega^n_{A/k} \arrow[u, "\varepsilon_n"] \arrow[r, "\partial_{DR}"] \arrow[ur, phantom, "\circlearrowleft"] & \Omega^{n+1}_{A/k} \arrow[u, "\varepsilon_{n+1}"'] \end{tikzcd}$$

For this reason, we can consider HH a "lift" of the entire structure of the de Rham complex. We can now proceed as if we were constructing de Rham cohomology, but we'll do it with HH instead.

Definitions of Cyclic Homology (and its variants)

First, we'll assemble a bunch of copies of HH into a double complex, rewriting ∂ as b:

$$
\begin{tikzcd}
\ddots & \vdots \arrow[d, "b"'] & \vdots \arrow[d, "b"'] & \vdots \arrow[d, "b"'] & \vdots \arrow[d, "b"'] & \iddots \\
\cdots & A^{\otimes_k 4} & A^{\otimes_k 3} \arrow[l, "B"'] \arrow[d, "b"'] & A^{\otimes_k 2} \arrow[l, "B"'] \arrow[d, "b"'] & A \arrow[l, "B"'] & \cdots \\
\cdots & A^{\otimes_k 3} \arrow[u, leftarrow, "b"] \arrow[d, "b"'] & A^{\otimes_k 2} \arrow[l, "B"'] \arrow[d, "b"'] & A \arrow[l, "B"'] & \cdot & \cdots \\
\cdots & A^{\otimes_k 2} \arrow[d, "b"'] & A \arrow[l, "B"'] & \cdot & \cdot & \cdots \\
\cdots & A & \cdot & \cdot & \cdot & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{tikzcd}
$$

where the condition $Bd = -dB$ from earlier makes this anti-commuting. Note that this extends to infinity in both homological and cohomological degree, so this complex is supported in quadrants 1,2 and 3.

Definition: The **periodic cyclic homology** $HP(A/k)$ is given by the totalization of this complex, obtained by taking the *direct product* along diagonals.

> Note: since this complex has infinite support, we have to distinguish between taking the direct sum and product here. There is also a differential graded algebra structure here which we won't address for now.

The **classical cyclic homology** $HC(A/k)$ is obtained by taking the totalization of just the 1st quadrant, where the origin is taken wherever the bidegree is set to $(0, 0)$ (i.e. take the $x \geq 0$ part).

Finally, we can totalize what's left to define the **negative cyclic homology** $HC^{-}(A/k)$, given by the totalization of the 2nd and 3rd quadrants (i.e. take the $x \leq 0$ part).

> Note that HP and HC^{-} will have degrees extending in both directions, while HC sits in positive degree only.

We can read off some structural properties. Consider the map

$$Sh : HP \to HP[2]$$

given by shifting the entire complex one column to the left (thus excluding one column from positive part).

> Note: Sh is referred to as a "periodicity map". The original lectures use S to denote this map.

Considering the HC part, we obtain another copy of HC, shifted by bidegree $(1, 1)$, while what we lost was a column – i.e. a copy of HH. So there as an exact sequence of complexes

$$0 \to HH \to HC \xrightarrow{Sh} HC[-2] \to 0$$

Considering the HC^{-} part, an exactly analogous dual thing happens – this one is easier to think of as restricting HC^{-} to a single column. What you obtain is a column (of course) corresponding to HH, which kills a copy of HC^{-} shifted by bidegree $(1, 1)$. So this yields another exact sequence

$$0 \to HC^{-}[-2] \xrightarrow{Sh} HC^{-} \to HH \to$$

The most important sequence here is the **norm sequence**

$$0 \to HC^{-} \to HP \to HC[2] \to 0$$

which states an obvious fact: the totalization of the entire complex sits between the totalizations of the left and right pieces respectively.

We can alternatively obtain HP in the following way:

$$HP = \varprojlim (\cdots HC[-4] \xrightarrow{Sh} HC[-2] \xrightarrow{Sh} HC)$$

which says that we can totalize the entire complex by first chopping a column off and totalizing, then chopping off a second column to the left and totalizing, and so on taking the inverse limit.

> Note that for this formula to hold, we had to take direct products instead of direct sums when totalizing.

Looking back at the periodicity maps, we find that this induces 2-periodicity in homology, and so

$$HP_n(A/k) \cong HP_{n+2}(A/k)$$

The moral here: "coarse information about HH yields coarse information about all of HP, HC, HC^{-}."

Example Usage

Suppose $HH_{\text{odd}}(A/k) = 0$, which happens when A is "perfectoid" in flavor.

Then $HP_0(A/k)$ is a complete filtered ring which encodes some useful data.

> Note: The graded direct sum HP_* is actually a graded algebra, so we know that HP_0 will be some ring/k-algebra.

It is filtered by ideals given by

$$F^i HP_0(A/k) := Sh^n(HC_{2i}^{-}(A/k))$$

for some $n \geq 0$, where

$$\frac{HP_0(A/k)}{F^i(HP_0(A/k))} \cong HC_{2i-2}(A/k)$$

and the graded pieces are given by

$$G^i \cong HH_{2i}(A/k).$$

This essentially falls out of looking at the spectral sequence for HP, where page E_2 will only contain elements sitting in even degree, and so this will be true for HP, HC, HC^{-}.

We could also find this by looking at the short exact sequences from earlier, writing the associated long exact sequence, and chasing it backwards noting that HC is only supported in odd degrees.

> Lecture said supported in odd degrees, but maybe even…? Possibly not, due to shift, I don’t know.

Then the map from HC^{-} to HP theories is in fact injective – we can just apply these periodicity map a number of times and land in some ideal in HP_0, and the quotient by this ideal will be some copy of the cyclic homology HC. The graded pieces will be the difference between two cyclic homologies, which is exactly Hochschild homology.

Moral of the story: HP_0 contains a great deal of data about $HPHC, HC^{-}$ as long as we equip it with this filtration and keep track of it.

Later, we’ll feed in perfectoid-ish things, and the result will be some filtered ring which encodes a lot of data about HH. This resulting ring may be identifiable or something completely new!

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> Notes: We can take $A = k[G]$, the group algebra of a group, to recover group homology (provided G is abelian perhaps). In this case, the map B is homotopically trivial, and so HP decomposes into a product of the group homologies – then the periodicity map just shifts these in a periodic way. (?)