$\mathbb{A}^1$ Enumerative Geometry

Enumerative geometry, counts algebro-geometric objects, and in order to actually obtain an invariant number at the end of the day one uses an algebraically closed field k or $\mathbb{C}$. This is essentially because the conditions imposed are polynomial, and polynomials of degree n over a closed field always have n roots.

The goal here is to record information about the fields of definition. However, since we may no longer have invariant numbers as solutions to polynomial equations, we replace this with a notion of *weights* to get an "invariance of bilinear form" principle instead. Over characteristic not 2, we can use quadratic forms, which ties to Lurie's first talk.

Example: Lines on a Smooth Cubic Surface

> Joint work with Jesse Kass

A **cubic surface** X consists of the $\mathbb{C}$ solutions to a polynomial in three variables, i.e.

$$X = \{(x, y, z) \in \mathbb{C}^3 \ \ni \ f(x, y, z) = 0\},$$

where f is degree 3. In general, we want to compactify, so we view $X \hookrightarrow \mathbb{CP}^3$ as

$$\mathbb{CP}^3 = \{\mathbf{x} = [w, x, y, z] \neq \mathbf{0} \ni \forall \lambda \in \mathbb{C}^\times, \mathbf{x} = \lambda \mathbf{x}\}$$

and so

$$X = \{[w, x, y, z] \in \mathbb{CP}^3 \ni f(w, x, y, z) = 0\}$$

where f is homogeneous.

The surface X is **smooth** if the underlying points form a manifold, or equivalently if the partials don't simultaneously vanish.

Theorem (Salmon Cayley 1849):
If X is a smooth cubic surface, then X contains exactly 27 lines.

Example: The Fermat cubic $f(w, x, y, z) = w^3 + x^3 + y^3 + z^3$.

We can find one line, given by

$$L = \{[s, -s, t, -t] \ni s, t \in \mathbb{CP}^1\},$$

and in fact this works for any λ, ω such that $\lambda^3 = \omega^3 = -1$, yielding

$$L' = \{[s, \lambda s, t, \omega t] \ni s, t \in \mathbb{CP}^1\}.$$

We can also permute s, t around to get more lines, and by counting this yields 27 distinct possibilities. (3 choices for λ, 3 choices for ω, and $\frac{1}{2}\binom{4}{2}$ ways to pair them with the s, t in the original L.

> There is a proof in the notes that these are the only lines,

> which is relatively elementary.

Modern Proof

We'll use characteristic classes, which we'll later replace by an $\mathbb{A}^1$ homotopy theory variant.

Let $\mathrm{Gr}(1,3)$ be the Grassmannian parameterizing 1-dimensional subspaces of $\mathbb{CP}^3$, where the $\mathbb{C}$ points of this space parameterize 2-dimensional subspaces $W \subseteq \mathbb{C}^4$. This is a moduli space of the lines we're looking for.

Let

$$S \to \mathrm{Gr}(1,3)$$

be the tautological bundle where the fiber is simply given by $S_W = W$. We can also form the bundle

$$(\mathrm{Sym}^3 S)^\vee \to \mathrm{Gr}(1,3)$$

where the fiber over the point corresponding to W is all of the cubic polynomials on W, i.e.

$$(\mathrm{Sym}^3 S)^\vee_W = (\mathrm{Sym}^3 W)^\vee.$$

Explicitly, we have the following two bundles to work with:

$$W \to S \to \mathrm{Gr}(1,3)$$
$$(\mathrm{Sym}^3 W)^\vee \to (\mathrm{Sym}^3 S)^\vee \to \mathrm{Gr}(1,3)$$

Our chosen f determines an element of $(\mathrm{Sym}^3 \mathbb{C}^4)^\vee$, which is thus a section σ_f of the second bundle above, where

$$\sigma_f(W) = f|_W.$$

We thus have

$$\mathbb{P}W \in X \iff \sigma_f(W) = 0,$$

i.e. the line corresponding to W is in our surface exactly when this section is zero. We now want to count the zeros of σ_f, which is exactly what the Euler class does.

To be precise, the Euler class counts the zeros of a section of a properly oriented vector bundle with a given weight. Let $V \to M$ be a rank r $\mathbb{R}$- vector bundle over a dimension r real manifold where we assume that V is oriented.

> We choose $\mathbb{R}$ here because $\mathbb{C}$ is slightly too nice and gives us a preferred orientation (which we'll want to track later.)

For any section σ with only isolated zero, we'll assign a weight to each zero which comes from the topological degree function

$$\deg : [S^{r-1}, S^{r-1}] \to \mathbb{Z},$$

where we use the brackets to denote homotopy classes of maps.

Definition: Let $p \in M$ where $\sigma(p) = 0$, and define $\deg_p(\sigma)$ in the following way:

Choose local coordinates near p. Since the zeros are isolated, we can choose a ball $B_\varepsilon(p)$ such that $x \in B_\varepsilon(p) - \{p\} \implies \sigma(x) \neq 0$. Choose a local trivialization of the total space V. This allows us to view $\sigma : \mathbb{R}^r \to \mathbb{R}^r$ as a real function.

We can choose coordinates such that $p = 0$ in the domain, so $\sigma(0) = 0$, and moreover the image $\sigma(B_\varepsilon(p)) = \mathbb{R} - \{0\}$. We can then form a function

$$\begin{aligned} \bar{\sigma} : \partial B_\varepsilon(p) = S^{r-1} &\to S^{r-1} = \partial\sigma(B_\varepsilon(p)) \\ x &\mapsto \frac{\sigma(x)}{\|\sigma(x)\|}, \end{aligned}$$

and so we can take $\deg_p(\sigma) := \deg \bar{\sigma}$.

There is indeterminacy here up to elements of $\mathrm{GL}(r, \mathbb{R})$ which could possibly effect the sign, however, but this can be fixed using the assumption that V is oriented and choosing local trivializations for which the orientations are compatible. This gives us a well-defined local degree of a section at a zero.

The Euler class, which only depends on the bundle and not the section, is given by

$$e(V) = \sum_{p \ni \sigma(p)=0} \deg_p(\sigma).$$

It can be shown that because X is smooth, the zeros are all simple and so in the complex case, the degrees are all 1. We thus obtain

$$|\{\text{lines on } X\}| = e((\mathrm{Sym}^3 S)^\vee),$$

where the RHS is independent of X and can be computed using the splitting principle and the cohomology of Gr.

∎

What about $\mathbb{R}$?

Schlafli, 19th century: X can have $3, 7, 15$ or 27 lines. So it's not constant, and thus there's not an invariant number here, but Segre (1942) distinguished between hyperbolic and elliptic lines.

Recall the characterization of elements in $\mathrm{Aut} L$ for $L = \mathbb{RP}^1$ (real lines) as elliptic/hyperbolic: we have $\mathrm{Aut} L \cong \mathrm{PGL}(2, \mathbb{R})$, so pick some I corresponding to a matrix

$$[I] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad z \mapsto \frac{az+b}{cz+d}$$

where the second formulation above shows that there are two fixed points, since solving for $z \mapsto z$ yields a quadratic equation. So we have

$$\mathrm{Fix}(I) = \{z \in \mathbb{C} \ni cz^2 + (d-z)z + b = 0\},$$

and we characterize I by the following cases:

- $\mathrm{Fix}(I)$ contains two real points: hyperbolic
- A complex conjugate pair: elliptic

So we'll associate an involution to L, and port over these notions of hyperbolic/elliptic. As we'll see later, for each point on L, there will be a unique other point that has the tangent space, and this involution will swap them.

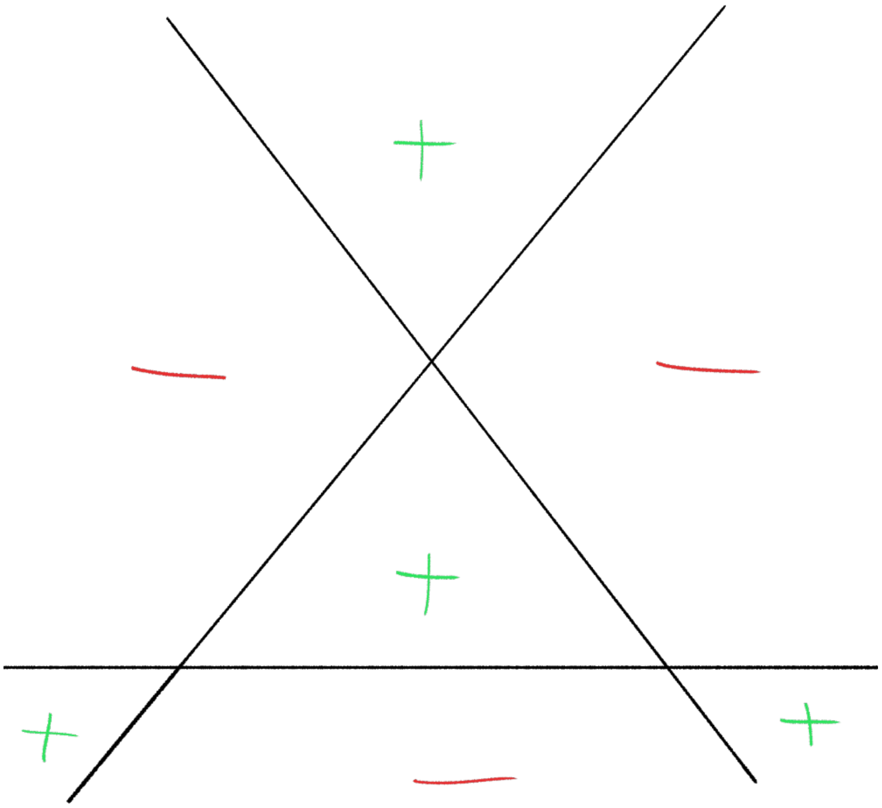
Let $p \in L$, and consider $T_p X \bigcap X$. Since L is in both of the varieties we're intersecting here, and we can apply Bezout's theorem, we know that its complement will some degree 2 variety Q (since the total degree is 3).

So we can write $T_pX \bigcap X = L \bigcup Q$. We know that $L \bigcap Q$ will be the intersection of a degree 1 and a degree 2 curve, which will have 2 points of intersection. At one of these points, say p, Q and L will intersect transversally, and so the tangent vectors T_pQ and T_pL give a 2-dimensional frame, which yields a plane $P \subseteq T_pX$. Since X is smooth, we get equality and $P = T_pX$.

This also holds for the second point of intersection, p', and so we take the involution $I(p) = p'$ and vice-versa. We then say L is elliptic/hyperbolic exactly when I is.

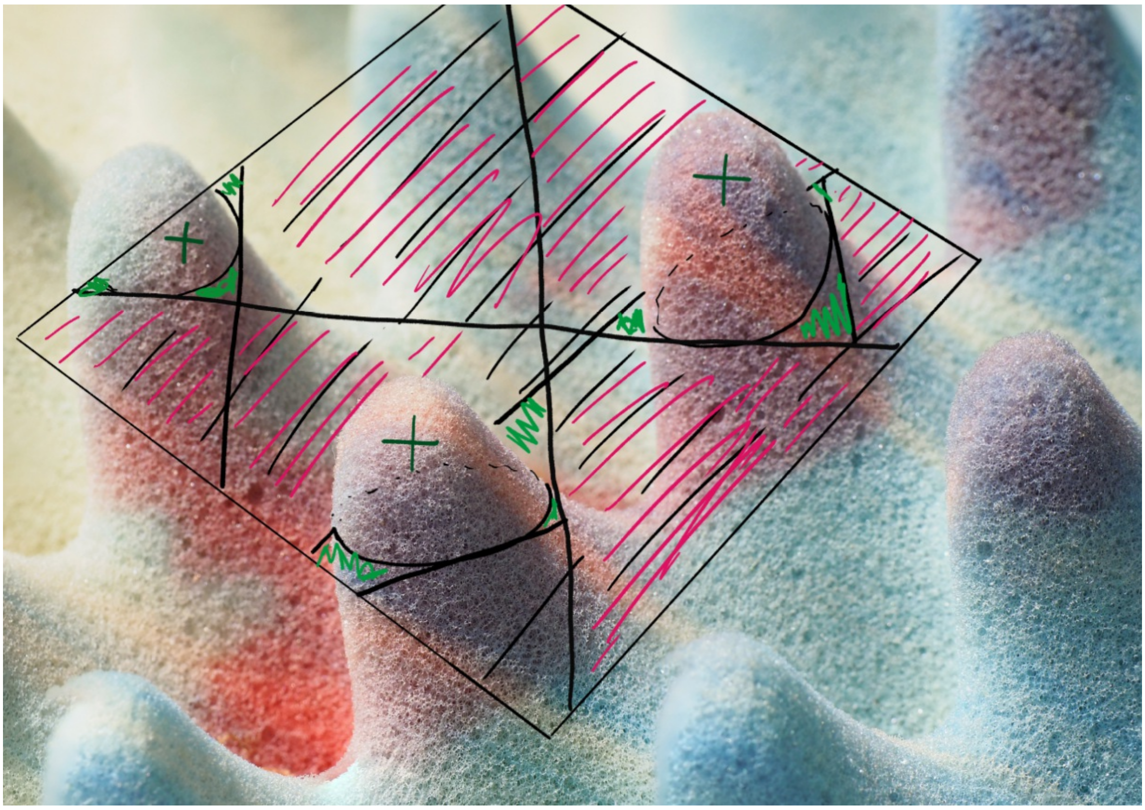
A natural way to see that there should be a distinction between two types of lines is to use spin structures. Consider a physical cubic surface sitting inside $\mathbb{R}^3$, and push the tangent plane alone a line. There are two things that can happen – one is a twisting by a nontrivial element of $\pi_1 SO_3(\mathbb{R})$, the other is no twisting at all.

Example: Look at the Fermat cubic surface $x^3 + y^3 + z^3 = -1$



Interpretation of this image: $X \subset \mathbb{R}^3$ is a surface, which has 3 lines that are contained in a plane. We this view X from above this plane, marking a plus/minus to denote the relative height of the surface within each bounded region. Plus denotes part of the surface that bubbles up over the plane, having positive height/z coordinates, etc.

> This took me a while to visualize – what worked for me was thinking about “egg crate” padding:



After thinking about what physically happens as you push a plane around, it becomes clear that these three lines are all hyperbolic. Note that this question is the same as asking if a path in the frame bundle lifts.

Although the number of lines isn't a constant, we can take a "signature" sort of formula to obtain an invariant. In this case, the number hyperbolic lines minus the number of elliptic lines *is* constant. In this case, the constant is 3.

General mantra for $\mathbb{A}^1$ homotopy theory: if you have a result that works over $\mathbb{C}$ and $\mathbb{R}$, it may be a result in $\mathbb{A}^1$ theory that has realizations recovering the original results.

$\mathbb{A}^1$ Homotopy Theory

This will allow us to do with schemes much of what we can do in

$\mathbf{T}op$. Smooth schemes behave like manifolds, where there are balls around points. The convention here will be that we're working over smooth schemes, denoted Sm_k where k is a field.

> Remark: in my notation I use $\mathbb{RP}^n, \mathbb{CP}^n$, and $\mathbb{P}^n(k)$ to denote various projective spaces. I'll adopt Kirsten's convention here and just denote $\mathbb{P}^n(k)$ as $\mathbb{P}^n$.

We'll get spheres from $S^n_{\mathbb{A}} := \mathbb{P}^n/\mathbb{P}^{n-1}$... One nice result due to Morel is that there is a degree map

$$[S^n_{\mathbb{A}}, S_A A^n] \to GW(k),$$

where the target is not the integers in this case, but rather a group of bilinear forms that are quadratic in characteristic not equal to 2. It is the Grothendieck-Witt group, whose elements are formal difference of bilinear forms.

Thus the group itself is the group completion of nondegenerate symmetric isomorphism classes of bilinear forms $V^2 \to k$ where V is a finite-dimensional k-vector space.

The group structure arises because if we have two bilinear forms B, B' on vector spaces V, W respectively, then we can define a new form on $V \oplus W$ by working in components and declaring orthogonality between any of the factors. We then take formal differences of these, and inherit a ring structure from the tensor product of forms.

Bilinear forms over fields can all be diagonalized, although in characteristic 2, this only holds in a stable sense.

The Grothendieck-Witt Group

Since we can diagonlize, the group $GW(k)$ has a presentation coming from the one dimensional forms. Any of these work as a generator, so we have

- Generators: $< a >$ where $a \in k^{\times}$, corresponding to the form

$$\begin{gathered} < a >: k^2 \to k \\ (x, y) \mapsto axy \end{gathered}$$

- Relations: if we change the basis of k using a multiplication by $b \in k^{\times}$, we get $< ab^2 >=< a >$. (This means that $a \in k^{\times}/(k^{\times})^2$
  - We also get $< a > + < b >=< a + b > + < ab(a + b) >$

There are many concrete computations of this known for global fields, local fields, finite fields, function fields, etc.

Example: compute $GW(\mathbb{C})$.

The generators are in bijection with $k^{\times}/(k^{\times})^2$, but since every element of $\mathbb{C}$ is a square, so there's only one element here. We thus obtain

$$\begin{gathered} GW(\mathbb{C}) \xrightarrow{\cong} \mathbb{Z} \\ \beta \mapsto \dim V \end{gathered}$$

which is realized by taking the rank.

Example: $GW(\mathbb{R})$
We still have the rank, but now we can also take the signature, so we have

$$GW(\mathbb{R}) \xrightarrow{\text{rank}\times\text{signature}} \mathbb{Z}^2,$$

although a minor parity issue crops up here that can be fixed without damaging the isomorphism type.

Example: $GW(\mathbb{F}_q)$
We can make a matrix out of how β acts on basis elements and take the determinant of it to obtain an invariant called the *discriminant*, and so

$$GW(\mathbb{F}_q) \xrightarrow{\text{discriminant}\times\text{rank}} \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2 \times \mathbb{Z}$$

Note that the quotient is needed because we can change basis in $\mathbb{F}_q$, which amounts to conjugating by a matrix A, and so this discriminant is only well-defined up to squares.

Euler Class

There is an Euler class in this setting,

$$e(V) = \sum_{p \ni \sigma(p)=0} \deg_p(\sigma).$$

Letting X be a smooth cubic surface over k, then a line $L \subset X$ will be a closed point of the Grassmannian $\mathrm{Gr}(1,3)$, so we can think of it as points of the form

$$L = \{[a,b,c,d]s + [a',b',c',d']t \ \ni \ s,t \in \mathbb{P}^1(k(L)\}$$

where the extension field $k(L) = k(a, b, c, d, a', b', c', d')$ is obtained by adjoining the coefficients to k.

> I think these are always separable, mentioned later in the talk.

We thus get

$$\mathbb{P}^1(k(L)) \cong L \underset{\substack{\text{closed} \\ \text{subscheme}}}{\subseteq} X_{k(L)} \subseteq \mathbb{P}^3(k(L)).$$

Given such a line $L \subseteq X$, similar to the real setting, we obtain an involution $I \in \mathrm{Aut} L \cong PGL(2, k(L))$ after choosing coordinates. We also find that $\mathrm{Fix}(L)$ again falls into two cases:

- $2k(L)$ points, or
- 2 conjugate points in some quadratic extension $k(L)[\sqrt{D}]$ where $D \in k(L)^\times/(k(L)^\times)^2$. These correspond to the oddities in the tangent plane in the real case.

We then define

$$\mathrm{Type}(L) =< D >\in GW(k(L)),$$

or equivalently $D = ab - cd$ when $I = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, in which case $\mathrm{Type}(L) =< -1 > \deg I$.

An Analogous Trace Formula

Theorem:

Supposing X is a smooth cubic surface over k of characteristic not equal to 2, we then have

$$\sum_{L \in X} \mathrm{Tr}_{k(L)/k} \mathrm{Type}(L) = \text{One fixed quadratic form} = 15 < 1$$

where the trace/transfer maps are defined as

$$\begin{gathered} \mathrm{Tr}_{k(L)/k} : GW(k(L)) \to GW(k) \\ (V^2 \xrightarrow{\beta} k(L)) \mapsto (V^2 \xrightarrow{\beta} k) \circ \mathrm{Trace}_{\mathrm{Galois}} \end{gathered}$$

where $\mathrm{Trace}_{\mathrm{Galois}}$ comes from summing the conjugates. Note that we can do this because we can view V as a vector space over either k or $k(L)$, so we end up with a quadratic form over k.

■

> Note: we have a well-defined map in the other direction, since the GW is a stable homotopy group of spheres.

Example: let $k = \mathbb{C}$, then apply rank to get $15 + 12 = 27$ on the RHS, while since every element is a square, the Type is just 1, so we get 27 total.

Example: let $k = \mathbb{R}$, apply signature. If L is defined over C, so the type is 1, and we're just left with the trace of $\mathbb{C}/\mathbb{R}$ – but this contributes a $+1$ and -1, so there is no contribution. What's left are the lines of $\mathbb{R}$, and since we set it up so type 1 lines are hyperbolic, we just get the trace $15 - 12 = 3$.

Example: let $k = \mathbb{F}^q$. We can define lines in $\mathbb{F}_q^n$, and the "begin a square" partitions $\left(\mathbb{F}_q^n\right)^\times$ into two disjoint subsets, we can assign types and we let squares be the hyperbolic elements.

We thus get

$$\begin{pmatrix} \text{elliptic lines } L \\ \text{with } k(L) = \mathbb{F}_{\text{odd}} \end{pmatrix} - \begin{pmatrix} \text{hyperbolic lines } L \\ \text{with } k(L) = \mathbb{F}_{\text{even}} \end{pmatrix} = 0 \mod 2$$

which follows from computing the discriminant of the given form.

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