$\mathbb{F}_q$ a finite field

$X \to \mathrm{Spec}(\mathbb{F}_q)$ a smooth proj curve

K_X, field of fractions of X

$$\mathbb{Q} \longrightarrow K_X$$

primes or ∞ $\longleftarrow$ Closed pts $x \in X$

$\mathbb{Z}/p\mathbb{Z} \longrightarrow \kappa(x)$, residue field at x

$\mathbb{Z}_p \longrightarrow \mathcal{O}_x$, completed local ring at x

$\mathbb{Q}_p$ or $\mathbb{R} \longrightarrow K_x \approx \kappa(x)((t))$

$\mathbb{A} \qquad\qquad \mathbb{A}_X = \prod_{x \in X}' K_x$

$\mu_{Tam} \longrightarrow \mu_{Tam}$

q, quadratic forms over $\mathbb{Z}$ $\longrightarrow$ $G \to X$ a group scheme

Ex: $G = X \times SL_n$

$$SO_q(\mathbb{Z}/p\mathbb{Z}) \longrightarrow G(K(X))$$

$$\text{Mass} = \sum_{\substack{\text{quadratic} \\ \text{forms}}} \longrightarrow \sum_{\substack{\text{principal} \\ G\text{-bundles} \\ P \text{ on } X}} 1/\mathrm{Aut}(P)$$

Weil conjecture on mass formula $\longrightarrow \sum = q^{D} \prod_{x \in X} \frac{|G(k(x))|}{|k(x)^d|}$

where $d := \dim G_0/k_x$

Let $\mathrm{Bun}_G(X)$ be the moduli stack of G-bundles (an algebraic stack, so almost a variety)

Maps $\mathrm{Spec}(R) \longrightarrow \mathrm{Bun}_G(X)$

$\wr\wr$

G-bundles on $X \times \mathrm{Spec}(R)$

over $\mathrm{Spec}(\mathbb{F}_q)$

We compute

$$\sum 1/|\mathrm{Aut}\, P| = |\mathrm{Bun}_G(X)(\mathbb{F}_q)|$$

For Y an algebraic variety, consider

$$|Y(\mathbb{F}_q)|$$

Let $\overline{Y} = Y \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{\mathbb{F}_q})$

There's a map, the geometric Frobenius

$$\varphi : \overline{Y} \circlearrowleft$$

$$[x_0 : \cdots : x_n] \mapsto [x_0^q : \cdots : x_n^q]$$

So $|Y(\mathbb{F}_q)|$ should be

$$\sum_{i \neq 0} (-1)^i \, \mathrm{Tr}(\varphi \mid \overbrace{H_c^i(\overline{Y})}^{\ell\text{-adic cohom}})$$

Theorem: Grothendieck-Lefschetz Formula

Assume Y smooth, dim n

$$H^i_c(\bar{Y}) \approx H^{2n-i}(\bar{Y})^\vee \quad \left(\text{Poincaré duality}\right)$$

(Not equivariant, so use φ^{-1} instead in trace formula.)

$\mathrm{Bun}_G(X)$ satisfies the G-L trace formula if

$$\frac{\sum 1/\mathrm{Aut}\, P}{q^{\dim(\mathrm{Bun}_G(X))}} = \sum (-1)^i \, \mathrm{Tr}(\varphi^{-1} \mid H^i(\overline{\mathrm{Bun}_G(X)}))$$

the new ideas

Weil conjecture asserts

$$= \prod_{x \in X} |G(k(x))| \Big/ |k(x)|^d$$

$$\mathrm{Bun}_G(\{x\}) = BG_x \text{, a classifying stack}$$

$$\mathrm{Bun}_G(\{x\})(\mathbb{F}_q) = \underbrace{\left\{ \begin{array}{c} \text{Principal } G\text{-bundles} \\ \text{on } \mathrm{Spec}(k(x)) \end{array} \right\}}_{\text{Cat w/ one obj}}$$

has symmetry group $G(k(x))$

$$\underbrace{|\mathrm{Bun}_G(\{x\})(\mathbb{F}_q)|}_{} = |k(x)|^d / |G(k(x))|$$

↳ Satisfies G-L formula

Want

$$\mathrm{Bun}_G(X) = \text{"cts} \prod_{x \in X} \mathrm{Bun}_G(\{x\})\text{"}$$

$$\Rightarrow H^i(\mathrm{Bun}_G(X)) = \text{"cts} \bigotimes_{x \in X} H^i(\mathrm{Bun}_G(\{x\}))\text{"}$$