

# Problem Set One

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## Contents

<b>1</b>	<b>Humphreys 1.1</b>	<b>1</b>
1.1	a . . . . .	1
<b>2</b>	<b>Humphreys 1.3*</b>	<b>2</b>

## 1 Humphreys 1.1

### 1.1 a

If  $M \in \mathcal{O}$  and  $[\lambda] = \lambda + \Lambda_r$  is any coset of  $\mathfrak{h}^\vee/\Lambda_r$ , let  $M^{[\lambda]}$  be the sum of weight spaces  $M_\mu$  for which  $\mu \in [\lambda]$ .

**Proposition:**  $M^{[\lambda]}$  is a  $U(\mathfrak{g})$ -submodule of  $M$

*Proof:* It suffices to check that  $\mathfrak{g} \curvearrowright M^{[\lambda]} \subseteq M^{[\lambda]}$ , i.e. this module is closed under the action of  $U(\mathfrak{g})$ . Let  $g \in U(\mathfrak{g})$  and  $m \in M^{[\lambda]}$  be arbitrary. Choose a ordered basis  $\{e_i\}$  for  $\mathfrak{g}$ , then this can be extended to a PBW basis for  $U(\mathfrak{g})$  given by  $\left\{ \prod_i e_i^{t_i} \mid t_i \in \mathbb{Z} \right\}$ . Then take a triangular decomposition  $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})$ . We can then write  $u = \prod_i a_i^{t_i} \prod_j h_j^{t_j} \prod_k b_k^{t_k}$  and consider how each component acts.

First considering how the  $b_k$  act, we compute their weights; we want to show that if  $\mu \in M_\mu$  for some  $\mu \in [\lambda]$ , then  $b_k \curvearrowright \mu \in M_{\mu'}$  for some  $\mu' \in [\lambda]$ .

We know  $h \curvearrowright m = \mu(h)m$  for each  $m \in M_\mu$ . Noting that  $b_k \in g_\alpha$  for some positive root  $\alpha$ , we have  $[hg] = \alpha(h)g$ , and so

$$\begin{aligned} h \curvearrowright (b_k \curvearrowright m) &= b_k \curvearrowright (h \curvearrowright m) + [hb_k] \curvearrowright m \\ &= b_k \curvearrowright (\mu(h)m) + [hb_k] \curvearrowright m \\ &= b_k(\mu(h)m) + \alpha(h)b_k m \\ &= (\mu(h) + \alpha(h))b_k m \\ &\in M_{\mu+\alpha}. \end{aligned}$$

But then  $\mu + \alpha - \mu = \alpha \in \mathbb{Z}\Phi = \Lambda_r$ , so  $\mu$  and  $\mu + \alpha$  are in the same coset  $[\lambda]$ . The same argument shows that  $h \curvearrowright (b_k^t \curvearrowright m)$  is in the weight space  $M_{\mu+t\alpha}$ , which still only differs by an integral number of roots.

But this shows that  $U(\mathfrak{n})$  and  $U(\mathfrak{n}^-)$  leave this space invariant, and  $U(\mathfrak{h})$  acts by scaling, which preserves subspaces. So  $M^{[\lambda]}$  is closed under the action of  $\mathfrak{g}$ . ■

**Proposition:**  $M$  is the direct sum of finitely many submodules of the form  $M^{[\lambda]}$ .

*Proof:*

By axiom 1 for Category  $\mathcal{O}$ ,  $M$  is finitely generated, say by  $\{m_j\}$ . This category is closed under subobjects, so if we write  $M = \bigoplus_{[\lambda]} M^{[\lambda]}$  as a union over all cosets, each  $M^{[\lambda]}$  is finitely generated as well. Since  $m_1$  is in this direct sum, it is in *finitely* many summands by definition of the direct sum,

so for each  $j$ ,  $m_j \in \bigoplus_{k=1}^{R_j} M^{[\lambda_{jk}]}$  for some finite constant  $R_j$  and some coset depending on  $j$  and  $k$ .

But then  $M = \bigoplus_j \bigoplus_k M^{[\lambda_{jk}]}$  is still a finite direct sum, which is what we wanted to show.

**Proposition:** If  $M$  is indecomposable, then all weights of  $M$  lie in a single coset.

*Proof:* By (a), we can write  $M = \bigoplus_{[\lambda_i]} M^{[\lambda_i]}$  for some finite set of  $\lambda_i$ s. If  $M$  is indecomposable, then

there can only be one summand, and so  $M = M^{[\lambda]}$  for exactly 1  $\lambda$ . We can then write  $M = \sum_{\mu \in [\lambda]} M_\mu$ ,

which decomposes  $M$  as a sum of weight spaces. But then if any  $\sigma \in \Pi(M)$  is a weight, it must be one of the  $\mu$  occurring above. So every weight of  $M$  is in the coset  $[\lambda]$ , and in particular they are all in the same coset. ■

## 2 Humphreys 1.3\*

**Proposition:** For any  $M \in \mathcal{O}$ ,  $M(\lambda)$  satisfies the following property:

$$\mathrm{Hom}_{U(\mathfrak{g})}(M(\lambda), M) = \mathrm{Hom}_{U(\mathfrak{g})}(\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda, M) \cong \mathrm{Hom}_{U(\mathfrak{b})}(\mathbb{C}_\lambda, \mathrm{Res}_{\mathfrak{b}}^{\mathfrak{g}} M).$$

*Proof:*

Noting that

- $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ ,
- $\mathrm{Res}_{\mathfrak{b}}^{\mathfrak{g}} M$  is an identification of the  $\mathfrak{g}$ -module  $M$  has a  $\mathfrak{b}$ -module by restricting the action of  $\mathfrak{g}$ ,

consider the following two maps:

$$\begin{aligned} F : \mathrm{hom}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda, M) &\longrightarrow \mathrm{hom}_{U(\mathfrak{b})}(\mathbb{C}_\lambda, M) \\ \phi &\mapsto (F\phi : z \mapsto \phi(1 \otimes z)), \end{aligned}$$

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and using the action of  $\mathfrak{g}$  on  $M$ ,

$$\begin{aligned} G : \text{hom}_{U(\mathfrak{b})}(\mathbb{C}_\lambda, M) &\longrightarrow \text{hom}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda, M) \\ \psi &\mapsto (G\psi : g \otimes v \mapsto g \curvearrowright \psi(v)). \end{aligned}$$

Note that the maps  $G\psi$  are defined on ordered pairs, but are clearly bilinear and thus uniquely extend to maps on the tensor product.

It suffices to show that these maps are well-defined and mutually inverse.

To see that  $F$  is well-defined, let  $\phi : U(\mathfrak{g}) \otimes C_\lambda \longrightarrow M$  be fixed; we will show that the set map  $F\phi : \mathbb{C}_\lambda \longrightarrow M$  is  $U(\mathfrak{b})$ -linear. Let  $b \in U(\mathfrak{b})$ , then

$$\begin{aligned} b \curvearrowright F\phi(v) &:= b \curvearrowright (z \mapsto \phi(1 \otimes z))(v) \\ &:= b \curvearrowright \phi(1 \otimes v) \\ &= \phi(b \curvearrowright (1 \otimes v)) \quad \text{since } \phi \text{ is } U(\mathfrak{g})\text{-linear and } b \in U(\mathfrak{g}) \\ &= \phi((b \curvearrowright 1) \otimes v) \quad \text{by the definition/construction of } M(\lambda) \text{ as a } U(\mathfrak{g})\text{-module.} \\ &= \phi(1 \otimes (b \curvearrowright v)) \quad \text{since } \mathbb{C}_\lambda \text{ is a } \mathfrak{b}\text{-module and the tensor is over } U(\mathfrak{b}) \\ &:= (z \mapsto \phi(1 \otimes z))(b \curvearrowright v) \\ &:= F\phi(b \curvearrowright v). \end{aligned}$$

To see that  $G$  is well-defined, let  $\psi : C_\lambda \longrightarrow M$  be fixed; we will show that the set map  $G\psi : U(\mathfrak{g}) \otimes C_\lambda \longrightarrow M$  is  $U(\mathfrak{g})$ -linear. Let  $u \in U(\mathfrak{g})$ , then

$$\begin{aligned} u \curvearrowright G\psi(g \otimes v) &:= u \curvearrowright (g \otimes v \mapsto g \curvearrowright \psi(v))(g \otimes v) \\ &:= u \curvearrowright (g \curvearrowright \psi(v)) \\ &= (ug) \curvearrowright \psi(v) \quad \text{since } M \text{ is a } \mathfrak{g}\text{-module with a well-defined action.} \\ &:= (g \otimes v \mapsto g \curvearrowright \psi(v))(ug \otimes v) \\ &:= G\psi(ug \otimes v). \end{aligned}$$

To see that  $FG$  is the identity, let  $\phi$  be defined as above and fix  $g_0 \otimes v_0 \in U(\mathfrak{g}) \otimes \mathbb{C}_\lambda$ . Then

$$\begin{aligned} GF\phi(g_0 \otimes v_0) &= G(v \mapsto \phi(1 \otimes v))(g_0 \otimes v_0) \\ &:= G(f) \quad \text{for notational convenience} \\ &:= G(g \otimes v \mapsto g \curvearrowright f(v))(g_0 \otimes v_0) \\ &= g_0 \curvearrowright f(v_0) \\ &= g_0 \curvearrowright \phi(1 \otimes v_0) \\ &= \phi(g \curvearrowright (1 \otimes v_0)) \quad \text{since } g_0 \in \mathfrak{g} \text{ and } \phi \text{ thus commutes with the } \mathfrak{g}\text{-action by definition} \\ &= \phi(g_0 \curvearrowright 1 \otimes v_0) \quad \text{by definition of the action on } U(\mathfrak{g}) \otimes C_\lambda \text{ as a } U(\mathfrak{g}) \text{ module} \\ &:= \phi(g_0 \otimes v_0) \\ &\quad . \end{aligned}$$

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To see that  $GF := G \circ F$  is the identity, let  $\psi$  be defined as above and fix  $z_0 \in \mathbb{C}_\lambda$ . Then

$$\begin{aligned}
FG\psi(z_0) &= F(g \otimes v \longrightarrow g \curvearrowright \psi(v))(z_0) \\
&:= F(\lambda)(z_0) \quad \text{for notational convenience} \\
&= (v \mapsto \lambda(1 \otimes v))(z_0) \\
&= \lambda(1 \otimes z_0) \\
&:= 1 \curvearrowright \psi(z_0) \\
&= \psi(z_0).
\end{aligned}$$

■