Notes for
HUMPHREYS'
GTM 9

Take note: Gau Syu

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Chapter I

Basic Concepts

1 Definition and first examples

Definition 1.1. A **Lie algebra** is a vector space with an skew-symmetric bilinear operation satisfying Jacobi identity.

Example 1.1 (A_l)**.** Let $\dim V = l+1$, the set of endomorphisms of V having trace 0, usually denoted by $\mathfrak{sl}(V)$ or $\mathfrak{sl}(l+1, F)$, is called **special linear algebra**.

$\dim A_l = (l+1)^2 - 1$, basis:

$$\begin{aligned} &h_i = e_{ii} - e_{i+1,i+1} \quad (1 \leqslant i \leqslant l) \\ &e_{ij} \quad (i \neq j) \end{aligned}$$

Example 1.2 (C_l)**.** Let $\dim V = 2l$, f be the symplectic form on V, the set of all endomorphisms x of V satisfying $f(x(v), w) = -f(v, x(w))$, usually denoted by $\mathfrak{sp}(V)$ or $\mathfrak{sp}(2l, F)$, is called the **symplectic algebra**.

$$\begin{aligned} \mathfrak{sp}(2l, F) &= \{x \in \mathfrak{gl}(2l, F) \mid sx = -x^t s\} \qquad s = \left(\begin{smallmatrix} 0 & I_l \\ -I_l & 0 \end{smallmatrix}\right) \\ &= \left\{ \left(\begin{smallmatrix} A & B \\ C & -A^t \end{smallmatrix}\right) \mid A, B, C \in \mathfrak{gl}(l, F), B = B^t, C = C^t \right\} \end{aligned}$$

$\dim C_l = 2l^2 + l$, basis:

$$\begin{aligned} &e_{ij} - e_{l+j,l+i} \quad (1 \leqslant i, j \leqslant l) \\ &e_{i,l+j} + e_{j,l+i} \quad (1 \leqslant i < j \leqslant l) \\ &e_{l+i,j} + e_{l+j,i} \quad (1 \leqslant i < j \leqslant l) \\ &e_{i,l+i} \quad (1 \leqslant i \leqslant l) \\ &e_{l+i,i} \quad (1 \leqslant i \leqslant l) \end{aligned}$$

Example 1.3 (B_l)**.** Let $\dim V = 2l+1$, f be the symmetric bilinear form on V whose matrix is $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}$, the set of all endomorphisms x of V satisfying $f(x(v), w) = -f(v, x(w))$, usually denoted by $\mathfrak{o}(V)$ or $\mathfrak{o}(2l+1, F)$, is called the **orthogonal algebra**.

$$\begin{aligned} \mathfrak{o}(2l+1, F) &= \{x \in \mathfrak{gl}(2l+1, F) \mid sx = -x^t s\} \qquad s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix} \\ &= \left\{ \begin{pmatrix} 0 & b & c \\ -c^t & m & p \\ -b^t & q & -m^t \end{pmatrix} \mid p = -p^t, q = -q^t \right\} \end{aligned}$$

$\dim B_l = 2l^2 + l$, basis:

$$\begin{aligned} &e_{1+i,1+j} - e_{1+l+j,1+l+i} \quad (1 \leqslant i, j \leqslant l) \\ &e_{1+i,1+l+j} - e_{1+j,1+l+i} \quad (1 \leqslant i < j \leqslant l) \\ &e_{1+l+i,1+j} - e_{1+l+j,1+i} \quad (1 \leqslant i < j \leqslant l) \\ &e_{1,1+i} - e_{1+l+i,1} \quad (1 \leqslant i \leqslant l) \\ &e_{1,1+l+i} - e_{1+i,1} \quad (1 \leqslant i \leqslant l) \end{aligned}$$

Example 1.4 (D_l)**.** Let $\dim V = 2l$, f be the symmetric bilinear form on V whose matrix is $s = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$, the set of all endomorphisms x of V satisfying $f(x(v), w) = -f(v, x(w))$, usually denoted by $\mathfrak{o}(V)$ or $\mathfrak{o}(2l, F)$, is also called the **orthogonal algebra**.

$$\begin{aligned}\mathfrak{o}(2l, F) &= \{x \in \mathfrak{gl}(2l, F) \mid sx = -x^t s\} \qquad s = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix} \\ &= \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid A, B, C \in \mathfrak{gl}(l, F), B = -B^t, C = -C^t \right\}\end{aligned}$$

$\dim D_l = 2l^2 - l$, basis:

$$\begin{aligned} &e_{ij} - e_{l+j,l+i} \quad (1 \leqslant i, j \leqslant l) \\ &e_{i,l+j} - e_{j,l+i} \quad (1 \leqslant i < j \leqslant l) \\ &e_{l+i,j} - e_{l+j,i} \quad (1 \leqslant i < j \leqslant l) \end{aligned}$$

Remark. The Lie algebra corresponding to Lie groups $\mathrm{O}(n, F)$ and $\mathrm{SO}(n, F)$ consists of the skew-symmetric $n \times n$ matrices, with the Lie bracket $[,]$ given by the commutator. One Lie algebra corresponds to both groups. It is often denoted by $\mathfrak{o}(n, F)$ or $\mathfrak{so}(n, F)$, and called the **orthogonal Lie algebra** or **special orthogonal Lie algebra**.

Example 1.5. The set of **upper triangular matrices** $\mathfrak{t}(n, F)$; the set of **strictly upper triangular matrices** $\mathfrak{n}(n, F)$; the set of all **diagonal matrices** $\mathfrak{d}(n, F)$.

$$[\mathfrak{d}(n, F), \mathfrak{n}(n, F)] = \mathfrak{n}(n, F) \tag{1.1}$$

$$[\mathfrak{t}(n, F), \mathfrak{t}(n, F)] = \mathfrak{n}(n, F) \tag{1.2}$$

Example 1.6. The only 2−dimensional non-Abelian Lie algebra has a basis x, y with commutation:

$$[x, y] = x$$

- Derivation
  - Inner derivation
  - Jacobi identity is equivalent to say all $\operatorname{ad} x$ are derivations.
  - Adjoint representation
- Structure constants

Exercise 1.1. *Let L be the real vector space $\mathbb{R}^3$. Define $[xy] = x \times y$ (cross product of vectors) for $x, y \in L$, and verify that L is a Lie algebra. Write down the structure constants relative to the usual basis of $\mathbb{R}^3$.*

Solution. Let e_1, e_2, e_3 be the basis of L, then $e_i \times e_j = e_k$ for (ijk) a cycle of (123). To verify that L is a Lie algebra, it suffices to verify the Jacobi identity.

The structure constants are $a_{12}^3 = 1, a_{23}^1 = 1, a_{13}^2 = -1$. □

Exercise 1.2. *Verify that the following equations and those implies by skew-symmetric bilinear define a Lie algebra structure on a three dimensional vector space with basis x, y, z: $[xy] = z, [xz] = y, [yz] = 0$.*

Solution. The structure constants are $a_{12}^3 = 1, a_{23}^1 = 0, a_{13}^2 = 1$. □

Exercise 1.3. *Let* $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ *be an ordered basis for* $\mathfrak{sl}(2, F)$. *Compute the matrices of* $\operatorname{ad} x, \operatorname{ad} h, \operatorname{ad} y$ *relative to this basis.*

Solution. $\operatorname{ad} x = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \operatorname{ad} h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \operatorname{ad} y = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$. ☐

Exercise 1.4. *Find a linear Lie algebra isomorphic to the nonabelian two dimensional algebra constructed in Example 1.6.*

Solution. Consider the adjoint representation $\operatorname{ad} x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \operatorname{ad} y = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$. ☐

Exercise 1.5. *Verify the assertions made in example 1.5, and compute dimension of each algebra, by exhibiting bases.*

Solution. $\dim \mathfrak{t}(n, F) = \frac{l^2+l}{2}$, basis: $e_{ij}(1 \leqslant i \leqslant j \leqslant l)$; $\dim \mathfrak{n}(n, F) = \frac{l^2-l}{2}$, basis: $e_{ij}(1 \leqslant i < j \leqslant l)$; $\dim \mathfrak{d}(n, F) = l$, basis: $e_{ii}(1 \leqslant i \leqslant l)$. ☐

Exercise 1.6. *Let* $x \in \mathfrak{gl}(n, F)$ *have* n *distinct eigenvalues* $a_1, \cdots, a_n$ *in* F. *Prove that the eigenvalues of* $\operatorname{ad} x$ *are precisely the* n^2 *scalars* $a_i - a_j (1 \leqslant i, j \leqslant n)$, *which of course need not be distinct.*

Solution. Choose a basis for F^n so that x is a diagonal matrix whose entries are $a_1, \cdots, a_n$. Then the matrices e_{ij} are eigenvalues of x since $xe_{ij} - e_{ij}x = a_i e_{ij} - a_j e_{ij}$. ☐

Exercise 1.7. *Let* $\mathfrak{s}(n, F)$ *denote the* **scalar matrices** *in* $\mathfrak{gl}(n, F)$. *If* $\operatorname{char} F$ *is* 0 *or else a prime not dividing* n, *prove that* $\mathfrak{gl}(n, F) = \mathfrak{sl}(n, F) \oplus \mathfrak{s}(n, F)$, *with* $[\mathfrak{s}(n, F), \mathfrak{gl}(n, F)] = 0$.

Solution. Choose $x \in \mathfrak{gl}(n, F)$. Let s be the scale $tr(x)/n$. Then $x - s \in \mathfrak{sl}(n, F)$ and $s \in \mathfrak{s}(n, F)$, so $\mathfrak{sl}(n, F)$ and $\mathfrak{s}(n, F)$ generate $\mathfrak{gl}(n, F)$. Since the sum of their dimensions is n^2, $\mathfrak{gl}(n, F) = \mathfrak{sl}(n, F) + \mathfrak{s}(n, F)$ is a direct sum. Since scalar matrices commute with all other matrices, we also get $[\mathfrak{s}(n, F), \mathfrak{gl}(n, F)] = 0$. ☐

Exercise 1.8. *Verify the stated dimension of* D_l.

Exercise 1.9. *When* $\operatorname{char} F = 0$, *show that each classical algebra* $L = A_l, B_l, C_l$ *or* D_l *is equal to* $[LL]$. *(This shows again that each algebra consists of trace* 0 *matrices.)*

Solution. It is sufficient to show $L \subset [L, L]$.

- A_1:

$$\begin{aligned} e_{12} &= \frac{1}{2}[h, e_{12}] \\ e_{21} &= \frac{1}{2}[e_{21}, h] \\ h &= [e_{12}, e_{21}] \end{aligned}$$

- $A_l (l \geqslant 2)$:

$$\begin{aligned} e_{ij} &= [e_{ik}, e_{kj}], \qquad k \neq i, j; i \neq j \\ h_i &= [e_{ij}, e_{ji}], \qquad j \neq i \end{aligned}$$

- $B_l (l \geqslant 2)$:

$$\begin{aligned}
e_{1,l+i+1} - e_{i+1,1} &= [e_{1,j+1} - e_{l+j+1,1}, e_{j+1,l+i+1} - e_{i+1,l+j+1}] \\
e_{1,i+1} - e_{l+i+1,1} &= [e_{1,l+j+1} - e_{j+1,1}, e_{l+j+1,i+1} - e_{l+i+1,j+1}] \\
e_{i+1,i+1} - e_{l+i+1,l+i+1} &= [e_{i+1,1} - e_{1,l+i+1}, e_{1,i+1} - e_{l+i+1,1}] \\
e_{i+1,j+1} - e_{l+i+1,l+j+1} &= [e_{i+1,1} - e_{1,l+i+1}, e_{1,j+1} - e_{l+j+1,1}] \\
e_{i+1,l+j+1} - e_{j+1,l+i+1} &= [e_{i+1,i+1} - e_{l+i+1,l+i+1}, e_{i+1,l+j+1} - e_{j+1,l+i+1}] \\
e_{l+i+1,j+1} - e_{j+l+1,i+1} &= [e_{l+i+1,l+i+1} - e_{i+1,i+1}, e_{l+i+1,j+1} - e_{j+l+1,i+1}]
\end{aligned}$$

where $1 \leqslant i \neq j \leqslant l$.

- $C_l (l \geqslant 3)$:

$$\begin{aligned}
e_{ii} - e_{l+i,l+i} &= [e_{i,l+i}, e_{l+i,i}] \\
e_{ij} - e_{l+j,l+i} &= [e_{ii} - e_{l+i,l+i}, e_{ij} - e_{l+j,l+i}], \qquad i \neq j \\
e_{i,l+j} + e_{j,l+i} &= [e_{ii} - e_{l+i,l+i}, e_{i,l+j} + e_{j,l+i}] \\
e_{l+i,j} + e_{l+j,i} &= [e_{l+i,l+i} - e_{ii}, e_{l+i,j} + e_{l+j,i}]
\end{aligned}$$

- $D_l (l \geqslant 2)$:

$$\begin{aligned}
e_{ii} - e_{l+i,l+i} &= \frac{1}{2}[e_{ij} - e_{l+j,l+i}, e_{ji} - e_{l+i,l+j}] \\
&\quad + \frac{1}{2}[e_{i,l+j} - e_{j,l+i}, e_{l+j,i} - e_{l+i,j}] \\
e_{ij} - e_{l+j,l+i} &= [e_{ii} - e_{l+i,l+i}, e_{ij} - e_{l+j,l+i}] \\
e_{i,l+j} - e_{j,l+i} &= [e_{ii} - e_{l+i,l+i}, e_{i,l+j} - e_{j,l+i}] \\
e_{l+i,j} - e_{l+j,i} &= [e_{l+i,l+i} - e_{ii}, e_{l+i,j} - e_{l+j,i}]
\end{aligned}$$

□

Exercise 1.10. *For small values of l, isomorphisms occur among certain of the classical algebras. Show that A_1, B_1, C_1 are all isomorphic, while D_1 is the one dimensional Lie algebra. Show that B_2 is isomorphic to C_2, D_3 to A_3. What can you say about D_2?*

Solution. The isomorphism of A_1, B_1, C_1 is given as follows:

$$\begin{array}{ccccc}
A_1 & \to & B_1 & \mapsto & C_1 \\
e_{11} - e_{22} & \mapsto & 2(e_{22} - e_{33}) & \mapsto & e_{11} - e_{22} \\
e_{12} & \mapsto & 2(e_{13} - e_{21}) & \mapsto & e_{12} \\
e_{21} & \mapsto & 2(e_{12} - e_{31}) & \mapsto & e_{21}
\end{array}$$

For B_2, C_2 we first calculate the eigenvectors for $h_1 = e_{22} - e_{44}, h_2 = e_{33} - e_{55}$ and $h_1' = e_{11} - e_{33}, h_2' = e_{22} - e_{44}$ respectively. We denote $\lambda = (\lambda(h_1), \lambda(h_2))$ for the eigenvalue of h_1, h_2, λ' is similar. See the following table:

B_2		C_2	
$\alpha = (1,0)$	$e_{21} - e_{14}$	$\alpha' = (-1,1)$	$e_{21} - e_{34}$
$-\alpha = (-1,0)$	$e_{12} - e_{41}$	$-\alpha' = (1,-1)$	$e_{12} - e_{43}$
$\beta = (-1,1)$	$e_{32} - e_{45}$	$\beta' = (2,0)$	e_{13}
$-\beta = (1,-1)$	$e_{23} - e_{54}$	$-\beta' = (-2,0)$	e_{31}
$\alpha + \beta = (0,1)$	$e_{15} - e_{31}$	$\alpha' + \beta' = (1,1)$	$e_{14} + e_{23}$
$-(\alpha + \beta) = (0,-1)$	$e_{13} - e_{51}$	$-(\alpha' + \beta') = (-1,-1)$	$e_{41} + e_{32}$
$2\alpha + \beta = (1,1)$	$e_{25} - e_{34}$	$2\alpha' + \beta' = (0,2)$	e_{24}
$-(2\alpha + \beta) = (-1,-1)$	$e_{43} - e_{52}$	$-(2\alpha' + \beta') = (0,-2)$	e_{42}

We make a linear transformation

$$\tilde{h_1}' = -\frac{1}{2}h_1' + \frac{1}{2}h_2', \tilde{h_2}' = \frac{1}{2}h_1' + \frac{1}{2}h_2'$$

Then $\alpha(h_1) = \alpha'(\tilde{h_1}'), \alpha(h_2) = \alpha'(\tilde{h_2}'), \beta(h_1) = \beta'(\tilde{h_1}'), \beta(h_2) = \beta'(\tilde{h_2}')$. So the isomorphism of B_2, C_2 is given as follows:

$$\begin{array}{ccc}
B_2 & \to & C_2 \\
e_{22} - e_{44} & \mapsto & -\frac{1}{2}(e_{11} - e_{33}) + \frac{1}{2}(e_{22} - e_{44}) \\
e_{33} - e_{55} & \mapsto & \frac{1}{2}(e_{11} - e_{33}) + \frac{1}{2}(e_{22} - e_{44}) \\
e_{12} - e_{41} & \mapsto & \frac{\sqrt{2}}{2}(e_{12} - e_{43}) \\
e_{21} - e_{14} & \mapsto & \frac{\sqrt{2}}{2}(e_{21} - e_{34}) \\
e_{32} - e_{45} & \mapsto & e_{13} \\
e_{23} - e_{54} & \mapsto & e_{31} \\
e_{15} - e_{31} & \mapsto & \frac{\sqrt{2}}{2}(e_{14} + e_{23}) \\
e_{13} - e_{51} & \mapsto & \frac{\sqrt{2}}{2}(e_{32} + e_{41}) \\
e_{25} - e_{34} & \mapsto & e_{24} \\
e_{43} - e_{52} & \mapsto & e_{42}
\end{array}$$

For A_3 and D_3, we calculate the eigenvalues and eigenvectors for $h_1 = e_{11} - e_{22}, h_2 = e_{22} - e_{33}, h_3 = e_{33} - e_{44}$ and $h_1' = e_{11} - e_{44}, h_2' = e_{22} - e_{55}, h_3' = e_{33} - e_{66}$ respectively.

A_3		D_3	
$\alpha = (1,1,-1)$	e_{13}	$\alpha' = (0,1,1)$	$e_{26} - e_{35}$
$-\alpha = (-1,-1,1)$	e_{31}	$-\alpha' = (0,-1,-1)$	$e_{62} - e_{53}$
$\beta = (-1,1,1)$	e_{24}	$\beta' = (0,1,-1)$	$e_{23} - e_{65}$
$-\beta = (1,-1,-1)$	e_{42}	$-\beta' = (0,-1,1)$	$e_{32} - e_{56}$
$\gamma = (-1,0,-1)$	e_{41}	$\gamma' = (1,-1,0)$	$e_{12} - e_{54}$
$-\gamma = (1,0,1)$	e_{14}	$-\gamma' = (-1,1,0)$	$e_{21} - e_{45}$
$\alpha + \gamma = (0,1,-2)$	e_{43}	$\alpha' + \gamma' = (1,0,1)$	$e_{16} - e_{34}$
$-(\alpha + \gamma) = (0,-1,2)$	e_{34}	$-(\alpha' + \gamma') = (-1,0,-1)$	$e_{61} - e_{43}$
$\beta + \gamma = (-2,1,0)$	e_{21}	$\beta' + \gamma' = (1,0,-1)$	$e_{13} - e_{64}$
$-(\beta + \gamma) = (2,-1,0)$	e_{12}	$-(\beta' + \gamma') = (-1,0,1)$	$e_{31} - e_{46}$
$\alpha + \beta + \gamma = (-1,2,-1)$	e_{23}	$\alpha' + \beta' + \gamma' = (1,1,0)$	$e_{15} - e_{24}$
$-(\alpha + \beta + \gamma) = (1,-2,1)$	e_{32}	$-(\alpha' + \beta' + \gamma') = (-1,-1,0)$	$e_{51} - e_{42}$

We take a linear transformation

$$\tilde{h_1}' = -h_1' + h_3', \tilde{h_2}' = h_1' + h_2', \tilde{h_3}' = -h_1' - h_3'$$

Then $\alpha(h_i) = \alpha'(\tilde{h_i}'), \beta(h_i) = \beta'(\tilde{h_i}'), \gamma(h_i) = \gamma'(\tilde{h_i}'), i = 1, 2, 3$. The isomorphism of A_3 and D_3 can be given as follows:

$$\begin{array}{ccc}
A3 & \to & D3 \\
e_{11} - e_{22} & \mapsto & -(e_{11} - e_{44}) + (e_{33} - e_{66}) \\
e_{22} - e_{33} & \mapsto & (e_{11} - e_{44}) + (e_{22} - e_{55}) \\
e_{33} - e_{44} & \mapsto & -(e_{11} - e_{44}) - (e_{33} - e_{66}) \\
e_{13} & \mapsto & e_{26} - e_{35} \\
e_{31} & \mapsto & e_{62} - e_{53} \\
e_{24} & \mapsto & e_{23} - e_{65} \\
e_{42} & \mapsto & e_{32} - e_{56} \\
e_{41} & \mapsto & e_{12} - e_{54} \\
e_{14} & \mapsto & e_{21} - e_{45} \\
e_{43} & \mapsto & e_{16} - e_{34} \\
e_{34} & \mapsto & e_{61} - e_{43} \\
e_{21} & \mapsto & e_{13} - e_{64} \\
e_{12} & \mapsto & e_{31} - e_{46} \\
e_{23} & \mapsto & e_{15} - e_{24} \\
e_{32} & \mapsto & e_{51} - e_{42}
\end{array}$$

□

Remark. We have

$$\begin{aligned}
&\mathfrak{so}(2) \cong S^1 \\
&\mathfrak{so}(3) \cong \mathfrak{sl}(2) \cong \mathfrak{sp}(1) \\
&\mathfrak{so}(4) \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \\
&\mathfrak{so}(5) \cong \mathfrak{sp}(4) \\
&\mathfrak{so}(6) \cong \mathfrak{sl}(4)
\end{aligned}$$

Remark. When $F = \mathbb{C}$, there exist another isomorphism: $\mathfrak{su}(2, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})$, where the lie structure of $\mathfrak{su}(2, \mathbb{C})$ is given by

$$[e_i, e_j] = \epsilon_{ijk} e_k$$

i.e.

$$\begin{aligned}
[e_1, e_2] &= e_3 \\
[e_2, e_3] &= e_1 \\
[e_3, e_1] &= e_2
\end{aligned}$$

However, $\mathfrak{su}(2, \mathbb{R}) \ncong \mathfrak{sl}(2, \mathbb{R})$.

This example shows that isomorphic Lie algebras over $\mathbb{C}$ may **not** be isomorphic over other fields.

Exercise 1.11. *Verify that the commutator of two derivations of an $F-$algebra is again a derivation, whereas the ordinary product need not be.*

Solution. Let D_1 and D_2 be derivations of an $F-$algebra R, and pick $x, y \in R$. we check that $[D_1, D_2]$ is a derivation:

$$\begin{aligned}
[D_1, D_2](xy) &= D_1(D_2(xy)) - D_2(D_1(xy)) \\
&= D_1(D_2(x)y + xD_2(y)) - (D_2(D_1(x)y + xD_1(y))) \\
&= D_1(D_2(x))y + D_2(x)D_1(y) + D_1(x)D_2(y) + xD_1(D_2(y)) - \\
&\quad (D_2(D_1(x))y + D_1(x)D_2(y) + D_2(x)D_1(y) + xD_2(D_1(y))) \\
&= (D_1(D_2(x)) - D_2(D_1(x)))y + x(D_1(D_2(y)) - D_2(D_1(y))) \\
&= [D_1, D_2](x)y + x[D_1, D_2](y).
\end{aligned}$$

Now consider the $F-$algebra $F[x, y]$ with derivations $\delta = \frac{\partial}{\partial x}$ and $\varepsilon = \frac{\partial}{\partial y}$. Then their product is not a derivation. If it were, then $\varepsilon(a)\delta(b) + \delta(a)\varepsilon(b) = 0$ for all $a, b \in F[x, y]$, but this is false by taking $a = x, b = y$. $\square$

Exercise 1.12. *Let L be a Lie algebra over an algebraically closed field and let $x \in L$. Prove that the subspace of L spanned by the eigenvectors of $\operatorname{ad} x$ is a subalgebra.*

Solution. By definition, it is closed under addition. To see that it closed under the Lie bracket, we need only do so for eigenvectors v and w of $\operatorname{ad} x$. In particular, we have $[x, v] = av$ and $[x, w] = bw$ for some $a, b \in F$. Then

$$[x, [v, w]] = [[x, v], w] - [[x, w], v] = a[v, w] - b[w, v] = (a + b)[v, w]$$

so $[v, w]$ is also an eigenvector of $\operatorname{ad} x$. Hence the subspace of L spanned by eigenvectors of $\operatorname{ad} x$ is a subalgebra of L. $\square$

2 Ideals and homomorphisms

- **Normalizer** $N_L(K)$: the largest Lie subalgebra of L in which K is a Lie ideal.
  - A subalgebra K is called **self-normalizing** if $N_L(K) = K$.
- If a derivation δ is nilpotent, then $e^\delta \in \operatorname{Aut}(L)$.
  - Leibnitz' rule:
  $$\frac{\delta^n}{n!}(xy) = \sum_{i=0}^{n} \frac{\delta^i x}{i!} \frac{\delta^{n-i} y}{(n-i)!}$$

Exercise 2.1. *Prove that the set of all inner derivations is an ideal of* $\operatorname{Der} L$.

Solution. For any $\delta \in \operatorname{Der} L, x \in L$, $[\delta, \operatorname{ad} x] = \operatorname{ad} \delta(x)$ is a inner derivation. $\square$

Exercise 2.2. *Show that $\mathfrak{sl}(n, F)$ is precisely the derived algebra of $\mathfrak{gl}(n, F)$ (cf. Exercise 1.9).*

Solution. It is easy to see

$$[\mathfrak{gl}(n, F), \mathfrak{gl}(n, F)] \subset \mathfrak{sl}(n, F)$$

Conversely, by exercise 1.9,

$$\mathfrak{sl}(n, F) = [\mathfrak{sl}(n, F), \mathfrak{sl}(n, F)] \subset [\mathfrak{gl}(n, F), \mathfrak{gl}(n, F)]$$

$\square$

Exercise 2.3. *Prove that the center of* $\mathfrak{gl}(n,F)$ *equals* $\mathfrak{s}(n,F)$ *(the scalar matrices). Prove that* $\mathfrak{sl}(n,F)$ *has center* 0*, unless* $\operatorname{char} F$ *divides* n*, in which case the center is* $\mathfrak{s}(n,F)$*.*

Solution. Clearly, we have $\mathfrak{s}(n,F) \subset Z(\mathfrak{gl}(n,F))$. Conversely, Let $A = \sum\limits_{i,j} a_{ij}e_{ij} \in Z(\mathfrak{gl}(n,F))$, then for each $e_{kl} \in \mathfrak{gl}(n,F)$,

$$\begin{aligned}
[A, e_{kl}] &= \sum_{i,j} a_{ij}[e_{ij}, e_{kl}] \\
&= \sum_{i,j} a_{ij}(\delta_{jk}e_{il} - \delta_{li}e_{kj}) \\
&= \sum_{i=1}^{n} a_{ik}e_{il} - \sum_{j=1}^{n} a_{lj}e_{kj} \\
&= (a_{kk} - a_{ll})e_{kl} + \sum_{\substack{i=1 \\ i\neq k}}^{n} a_{ik}e_{il} - \sum_{\substack{j=1 \\ j\neq l}}^{n} a_{lj}e_{kj}
\end{aligned}$$

So

$$a_{kk} = a_{ll}, a_{ij} = 0, i \neq j$$

i.e.

$$A \in \mathfrak{s}(n,F)$$

For $\mathfrak{sl}(n,F)$, if $c \in Z(\mathfrak{sl}(n,F))$, $\forall x \in \mathfrak{sl}(n,F), [x,c] = 0$. But we know $\mathfrak{gl}(n,F) = \mathfrak{sl}(n,F) + \mathfrak{s}(n,F)$ and $\mathfrak{s}(n,F)$ is the center of $\mathfrak{gl}(n,F)$. Hence $c \in Z(\mathfrak{gl}(n,F)) = \mathfrak{s}(n,F)$. We have

$$Z(\mathfrak{sl}(n,F)) = \mathfrak{sl}(n,F) \cap \mathfrak{s}(n,F)$$

If $\operatorname{char} F$ does not divide n, each $aI \in \mathfrak{s}(n,F), a \neq 0$ has trace $na \neq 0$, so $aI \notin \mathfrak{sl}(n,F)$. i.e., $Z(\mathfrak{sl}(n,F)) = \mathfrak{sl}(n,F) \cap \mathfrak{s}(n,F) = 0$. Else if $\operatorname{char} F$ divides n, each $aI \in \mathfrak{s}(n,F)$ has trace $na = 0$, in this case $Z(\mathfrak{sl}(n,F)) = \mathfrak{sl}(n,F) \cap \mathfrak{s}(n,F) = \mathfrak{s}(n,F)$. □

Exercise 2.4. *Show that (up to isomorphism) there is a unique Lie algebra over* F *of dimension* 3 *whose derived algebra has dimension* 1 *and lies in* $Z(L)$*.*

Solution. Let L_0 be the $3-$dimensional lie algebra over F with basis (x_0, y_0, z_0) and commutation:

$$[x_0, y_0] = z_0, [x_0, z_0] = [y_0, z_0] = 0.$$

Suppose L be any $3-$dimensional lie algebra over F whose derived algebra has dimension 1 and lies in $Z(L)$. We can take a basis (x, y, z) of L such that $z \in [LL] \subset Z(L)$. By hypothesis, $[x,y] = \lambda z, [x,z] = [y,z] = 0, \lambda \in F$. Then $L \to L_0, x \mapsto x_0, y \mapsto y_0, z \mapsto \lambda z_0$ is a isomorphism. □

Exercise 2.5. *Suppose* $\dim L = 3, L = [LL]$*. Prove that* L *must be simple. [Observe first that any homomorphic image of* L *also equals its derived algebra.] Recover the simplicity of* $\mathfrak{sl}(2,F), \operatorname{char} F \neq 2$*.*

Solution. Let I be a proper ideal of L. It is clear from surjectivity of $L \to L/I$ that $[L/I, L/I] = L/I$. From this, we rule out $\dim I = 2$ because then L/I would have to be Abelian, $[L/I, L/I] = 0 \neq L/I$. Also, $\dim I = 1$ implies that $\dim L/I = 2$, and the only non-Abelian 2−dimensional Lie algebra is described in Example 1.6 and is not equal to its derived algebra. Hence $I = 0$, so L is simple.

Since $[\mathfrak{sl}(2, F), \mathfrak{sl}(2, F)] = \mathfrak{sl}(2, F)$ if $\operatorname{char} F \neq 2$, and $\dim \mathfrak{sl}(2, F) = 3$, we see that $\mathfrak{sl}(2, F)$ is simple for $\operatorname{char} F \neq 2$. $\square$

Exercise 2.6. *Prove that* $\mathfrak{sl}(3, F)$ *is simple, unless* $\operatorname{char} F = 3$ *(cf. Exercise 2.3). [Use the standard basis* $h_1, h_2, e_{ij} (i \neq j)$*. If* $I \neq 0$ *is an ideal, then* I *is the direct sum of eigenspaces for* $\operatorname{ad} h_1$ *or* $\operatorname{ad} h_2$*, compare the eigenvalues of* $\operatorname{ad} h_1, \operatorname{ad} h_2$ *acting on the* e_{ij}*.]*

Solution. Let $I \neq 0$ be an ideal, and $V_0 = \operatorname{span}\{h_1, h_2\}$. It is easy to see that If one of h_1, h_2 is contained in I, then $I = L$.

Let $a \in I$ be an eigenvector of $\operatorname{ad} h_1$ with eigenvalue λ, then by compute the eigenvalues of $\operatorname{ad} h_1$, we see that there exist an $a' \in L$ having eigenvalue $-\lambda$, then $[a, a'] \in I$. On the other hand,

$$\operatorname{ad} h_1([a, a']) = [\lambda a, a'] + [a, -\lambda a'] = 0$$

therefore $[a, a'] \in V_0$. Since I is the direct sum of eigenspaces, $V_0 \subset I$, which implies $I = L$. Hence L is simple. $\square$

Exercise 2.7. *Prove that* $\mathfrak{t}(n, F)$ *and* $\mathfrak{d}(n, F)$ *are self-normalizing subalgebras of* $\mathfrak{gl}(n, F)$*, whereas* $\mathfrak{n}(n, F)$ *has normalizer* $\mathfrak{t}(n, F)$*.*

Solution. Let $a = \sum\limits_{ij} a_{ij} e_{ij} \in \mathfrak{gl}(n, F), [a, \mathfrak{t}(n, F)] \subset \mathfrak{t}(n, F)$. But

$$\begin{aligned}
[a, e_{kk}] &= \sum_{ij} a_{ij} \delta_{jk} e_{ik} - \sum_{ij} a_{ij} \delta_{ki} e_{kj} \\
&= \sum_{i} a_{ik} e_{ik} - \sum_{j} a_{kj} e_{kj} \\
&\subset \mathfrak{t}(n, F)
\end{aligned}$$

It must be $a_{ik} = 0$ for $i > k$, and $a_{kj} = 0$ for $j < k$. Hence $a_{kl} = 0$ for all $k > l$. This implies $a \in \mathfrak{t}(n, F)$, i.e., $\mathfrak{t}(n, F)$ is the self-normalizing subalgebras of $\mathfrak{gl}(n, F)$.

Similarly for $\mathfrak{d}(n, F)$, let $a = \sum_{ij} a_{ij} e_{ij} \in \mathfrak{gl}(n, F), [a, \mathfrak{d}(n, F)] \subset \mathfrak{d}(n, F)$. But

$$\begin{aligned}
[a, e_{kk}] &= \sum_{ij} a_{ij} \delta_{jk} e_{ik} - \sum_{ij} a_{ij} \delta_{ki} e_{kj} \\
&= \sum_{i} a_{ik} e_{ik} - \sum_{j} a_{kj} e_{kj} \\
&\subset \mathfrak{d}(n, F)
\end{aligned}$$

It must be $a_{ik} = 0$ for $i \neq k$, and $a_{kj} = 0$ for $j \neq k$. Hence $a_{kl} = 0$ for all $k \neq l$. This implies $a \in \mathfrak{d}(n, F)$, i.e., $\mathfrak{d}(n, F)$ is the self-normalizing subalgebras of $\mathfrak{gl}(n, F)$. $\square$

Exercise 2.8. *Prove that in each classical linear Lie algebra, the set of diagonal matrices is a self-normalizing subalgebra, when* $\operatorname{char} F = 0$*.*

Exercise 2.9. *Prove the basic theorem for homomorphisms of Lie algebras.*

Exercise 2.10. *Let* σ *be the automorphism of* $\mathfrak{sl}(2, F)$ *defined as*

$$\sigma = \exp \operatorname{ad} x \cdot \exp \operatorname{ad}(-y) \cdot \exp \operatorname{ad} x$$

Verify that $\sigma(x) = -y, \sigma(y) = -x, \sigma(h) = -h$*.*

Solution.

$$\begin{aligned}
\exp \operatorname{ad} x(x) &= x \\
\exp \operatorname{ad} x(h) &= h - 2x \\
\exp \operatorname{ad} x(y) &= y + h - x \\
\exp \operatorname{ad}(-y)(x) &= x + h - y \\
\exp \operatorname{ad}(-y)(h) &= h - 2y \\
\exp \operatorname{ad}(-y)(y) &= y
\end{aligned}$$

$$\begin{aligned}
\sigma(x) &= \exp \operatorname{ad} x \exp \operatorname{ad}(-y)(x) \\
&= \exp \operatorname{ad} x(x + h - y) \\
&= x + h - 2x - y - h + x \\
&= -y \\
\sigma(y) &= \exp \operatorname{ad} x \exp \operatorname{ad}(-y)(y + h - x) \\
&= \exp \operatorname{ad} x(y + h - 2y - x - h + y) \\
&= \exp \operatorname{ad} x(-x) \\
&= -x \\
\sigma(h) &= \exp \operatorname{ad} x \exp \operatorname{ad}(-y)(h - 2x) \\
&= \exp \operatorname{ad} x(h - 2y - 2(x + h - y)) \\
&= \exp \operatorname{ad} x(-h - 2x) \\
&= -h + 2x - 2x = -h
\end{aligned}$$

□

Exercise 2.11. *If $L = \mathfrak{sl}(n, F), g \in \mathrm{GL}(n, F)$, prove that the map of L to itself defined by $x \mapsto -gx^tg^{-1}$ (x^t = transpose of x) belongs to $\operatorname{Aut} L$. When $n = 2, g =$ identity matrix, prove that this automorphism is inner.*

Solution. $g \in \mathrm{GL}(n, F)$ and $\operatorname{Tr}(-gx^tg^{-1}) = -\operatorname{Tr}(x)$, i.e, $\operatorname{Tr}(x) = 0$ if and only if so is $\operatorname{Tr}(-gx^tg^{-1})$. So the map $x \mapsto -gx^tg^{-1}$ is a linear space automorphism of $\mathfrak{sl}(n, F)$. We just verify it is a homomorphism of lie algebras:

$$\begin{aligned}
[-gx^tg^{-1}, -gy^tg^{-1}] &= gx^ty^tg^{-1} - gy^tx^tg^{-1} \\
&= -g((xy)^t - (yx)^t)g^{-1} \\
&= -g[x, y]^tg^{-1}
\end{aligned}$$

When $n = 2, g =$ identity matrix, the automorphism $\sigma \colon x \mapsto -x^t$, i.e.

$$\sigma(x) = -y, \sigma(y) = -x, \sigma(h) = -h$$

So $\sigma = \exp \operatorname{ad} x \exp \operatorname{ad}(-y) \exp \operatorname{ad} x$ is an inner automorphism. □

Remark. Warning: An inner automorphism is not exactly of form exp adx with adx is nilpotent. It can be the composition of elements with this form.

Exercise 2.12. *Let L be an orthogonal Lie algebra (type B_l or D_l). If g is an* **orthogonal matrix**, *in the sense that g is invertible and $g^tsg = s$, prove that $x \mapsto gxg^{-1}$ defines an automorphism of L.*

Solution. $x \in B_l$ or D_l, $sx = -x^t s$. Hence

$$\begin{aligned} sgxg^{-1} &= (g^{-1})^t sxg^{-1} \\ &= -(g^{-1})^t x^t sg^{-1} \\ &= -(g^{-1})^t x^t g^t s \\ &= -(gxg^{-1})^t s \end{aligned}$$

So the map $x \mapsto gxg^{-1}$ is a linear automorphism of B_l or C_l. We just verify it is a homomorphism of lie algebras:

$$[gxg^{-1}, gyg^{-1}] = gxyg^{-1} - gyxg^{-1} = g[x, y]g^{-1}$$

$\square$

3 Solvable and nilpotent Lie algebras

Exercise 3.1. *Let I be an ideal of L. Then each member of the derived series or descending central series of I is also an ideal of L.*

Solution. For the derived series, it is enough to show that $[I, I]$ is an ideal by induction. Pick $x, y \in I$ and $z \in L$. Then $[z, [x, y]] = -[y, [z, x]] - [x, [y, z]] \in [I, I]$ since $[z, x], [y, z] \in I$. For the descending central series of I, we have seen that $I^1 = I^{(1)}$ is an ideal. So by induction, suppose that I^k is an ideal. Then pick $x \in I, y \in I^k, z \in L$. We have $[z, [x, y]] = -[y, [z, x]] - [x, [y, z]] \in I^{k+1}$ since $[z, x] \in I$ and $[y, z] \in I^k$. So I^{k+1} is also an ideal. $\square$

Exercise 3.2. *Prove that L is solvable if and only if there exists a chain of subalgebras $L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_k = 0$ such that L_{i+1} is an ideal of L_i and such that each quotient L_i/L_{i+1} is abelian.*

Solution. If L is solvable, take $L_i = L^{(i)}$. Then $[L_i, L_i]$ is an ideal of L_i by the Jacobi identity, and $L_i/[L_i, L_i]$ is abelian. Conversely, given such a chain of subalgebras, we see by induction that $L^(i) \subset L_i$ because $[L_i, L_i]$ is the smallest ideal I for which L_i/I is abelian. $\square$

Exercise 3.3. *Let* $\operatorname{char} F = 2$. *Prove that* $\mathfrak{sl}(2, F)$ *is nilpotent.*

Solution. Let (x, h, y) be the standard basis for $\mathfrak{sl}(2, F)$, then $[hx] = 2x = 0, [xy] = h, [hy] = -2y = 0$. Hence $[\mathfrak{sl}(2, F), \mathfrak{sl}(2, F)] = Fh$, then $\mathfrak{sl}(2, F)$ is nilpotent. $\square$

Exercise 3.4. *Prove that L is solvable (resp. nilpotent) if and only if* $\operatorname{ad} L$ *is solvable (resp. nilpotent).*

Solution. $\operatorname{ad} L$ is a homomorphic image of L, moreover, $\operatorname{ad} L \cong L/Z(L)$. $\square$

Exercise 3.5. *Prove that the non-abelian two dimensional algebra constructed in Example 1.6 is solvable but not nilpotent. Do the same for the algebra in Exercise 1.2.*

Solution. The 2−dimensional non-abelian Lie algebra L has basis $\{x, y\}$ such that $[x, y] = x$. Then it is clear that L^i is the subspace spanned by x for $i > 0$, so L is not nilpotent. However, $L^{(1)} = \langle x\rangle$, so $L^{(2)} = 0$ and hence L is solvable.

The Lie algebra L of Exercise 1.2 has a basis $\{x, y, z\}$ such that $[x, y] = z, [x, z] = y$ and $[y, z] = 0$. Then $L^i = \langle y, z\rangle$ for $i > 0$, so L is not nilpotent. However, $L^{(1)} = \langle y, z\rangle$ and $L^{(2)} = 0$, so L is solvable. □

Exercise 3.6. *Prove that the sum of two nilpotent ideals of a Lie algebra L is again a nilpotent ideal. Therefore, L possesses a unique maximal nilpotent ideal. Determine this ideal for each algebra in Exercise 3.5.*

Solution. Let I, J are nilpotent ideals. We can deduce by induction that

$$(I + J)^n \subset \sum_{k=0}^{n} I^k \cap J^{n-k}$$

where $I^0 = J^0 = L$. Then $I + J$ is clear a nilpotent ideal.

Taking the sum of all nilpotent ideals of L gives a unique maximal nilpotent ideal.

In the 2−dimensional algebra of Example 1.6, the maximal nilpotent ideal can have dimension at most 1 since it is not itself nilpotent, hence is the subspace spanned by x. Similarly, the unique maximal nilpotent ideal of the Lie algebra of Exercise 1.2 is the subspace spanned by y and z. □

Exercise 3.7. *Let L be nilpotent, K a proper subalgebra of L. Prove that $N_L(K)$ includes K properly.*

Solution. Since K is a subalgebra of L, $\operatorname{ad} K$ acts on L/K (quotient taken as vector spaces). Since K is a proper subalgebra of L, we have $L/K \neq 0$, so there exists a vector $v \notin K$ such that $[K, v] \subset K$. In particular, $v \in N_L(K)$, so $N_L(K)$ properly contains K. □

Exercise 3.8. *Let L be nilpotent. Prove that L has an ideal of codimension* 1.

Solution. If $\dim L = 1$, then 0 is a codimension 1 ideal. So suppose $\dim L > 1$. If L is abelian, then we take any codimension 1 subspace. Otherwise, $0 < \dim L/[L, L] < \dim L$ since L is nilpotent. Since $L = [L, L]$ is abelian, it has a codimension 1 ideal I. By the dimension formula for vector spaces the inverse image of I has codimension 1 in L. □

Exercise 3.9. *Prove that every nilpotent Lie algebra L has an outer derivation, as follows: Write $L = K + Fx$ for some ideal K of codimension one (Exercise 3.8). Then $C_L(K) \neq 0$ (why?). Choose n so that $C_L(K) \subset L^n, C_L(K) \not\subset L^{n+1}$, and let $z \in C_L(K) - L^{n+1}$. Then the linear map δ sending K to 0, x to z, is an outer derivation.*

Solution. If $K = 0$, then $C_L(K) = L$. Otherwise, K is nonzero and nilpotent since it is a subalgebra of L, and hence $Z(K) \neq 0$. Since $Z(K) \subset C_L(K)$, we conclude that $C_L(K) \neq 0$.

For all $k_1 + \lambda_1 x, k_2 + \lambda_2 x \in L, [k_1 + \lambda_1 x, k_2 + \lambda_2 x] \in K$, so $\delta([k_1 + \lambda_1 x, k_2 + \lambda_2 x]) = 0$. In the other hand,

$$\begin{aligned}
&[\delta(k_1 + \lambda_1 x), k_2 + \lambda_2 x] + [k_1 + \lambda_1 x, \delta(k_2 + \lambda_2 x)] \\
&= [\lambda_1 z, k_2 + \lambda_2 x] + [k_1 + \lambda_1 x, \lambda_2 z] \\
&= \lambda_1\lambda_2[z, x] + \lambda_1\lambda_2[x, z] \\
&= 0
\end{aligned}$$

We conclude that δ is a derivation. If δ is a inner derivation, $\delta = \operatorname{ad} y$, then $[y, K] = \delta(K) = 0$, so $y \in C_L(K) \in L^n$. Then we have $[y, x] \subset L^{n+1}$. But $[y, x] = \delta(x) = z \notin L^{n+1}$. This is a contradiction. So δ is a outer derivation. $\square$

Exercise 3.10. *Let L be a Lie algebra, K an ideal of L such that $L = K$ is nilpotent and such that $\operatorname{ad} x|_K$ is nilpotent for all $x \in L$. Prove that L is nilpotent.*

Solution. If L/K is nilpotent, say $(L/K)^n = 0$, then we know that $L^n \subset K$. The fact that $\operatorname{ad} x|_K$ is nilpotent for all $x \in L$ then implies that $\operatorname{ad} x|_{L^n}$ is nilpotent for all $x \in L$, and hence L is nilpotent by Engel's theorem. $\square$

Chapter II
Semisimple Lie Algebras

4 Theorems of Lie and Cartan

Exercise 4.1. *Let* $L = \mathfrak{sl}(V)$*. Use Lie's Theorem to prove that* $\operatorname{Rad} L = Z(L)$*, conclude that* L *is semisimple (cf. Exercise 2.3).*

Solution. Observe that $\operatorname{Rad} L$ lies in each maximal solvable subalgebra B of L. Select a basis of V so that $B = L \cap \mathfrak{t}(n, F)$, and notice that the transpose of B is also a maximal solvable subalgebra of L. Conclude that $\operatorname{Rad} L \subset L \cap \mathfrak{d}(n, F)$, then that $\operatorname{Rad} L = Z(L)$. □

Exercise 4.2. *Show that the proof of Theorem 4.1 still goes through in prime characteristic, provided* $\dim V$ *is less than* $\operatorname{char} F$*.*

Solution. The only part in which $\operatorname{char} F = 0$ is used in the proof of Theorem 4.1 is to show that $n\lambda([x, y]) = 0$ implies $\lambda([x, y]) = 0$. Here $n < \dim V$, so using our relaxed condition, this implication still holds. □

Exercise 4.3. *This exercise illustrates the failure of Lie's Theorem when* F *is allowed to have prime characteristic* p*. Consider the* $p \times p$ *matrices:*

$$x = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad y = \operatorname{diag}(0, 1, 2, 3, \cdots, p - 1)$$

Check that $[x, y] = x$*, hence that* x *and* y *span a two dimensional solvable subalgebra* L *of* $\mathfrak{gl}(p, F)$*. Verify that* x, y *have no common eigenvector.*

Solution. $[x, y] = x$. However, the eigenvectors of y are the standard basis vectors, and none of these are eigenvectors for x since it operates by shifting entries. □

Exercise 4.4. *When* $p = 2$*, Exercise 3.3 show that a solvable Lie algebra of endomorphisms over a field of prime characteristic* p *need not have derived algebra consisting of nilpotent endomorphisms (cf. Corollary C of Theorem 4.1). For arbitrary* p*, construct a counterexample to Corollary C as follows: Start with* $L \subset \mathfrak{gl}(p, F)$ *as in Exercise 4.3. Form the vector space direct sum* $M = L + F^p$*, and make* M *a Lie algebra by decreeing that* F^p *is abelian, while* L *has its usual product and acts on* F^p *in the given way. Verify that* M *is solvable, but that its derived algebra (*$= Fx + F^p$*) fails to be nilpotent.*

Exercise 4.5. *If* $x, y \in \operatorname{End} V$ *commute, prove that* $(x + y)_s = x_s + y_s$*, and* $(x + y)_n = x_n + y_n$*. Show by example that this can fail if* x, y *fail to commute. [Hint: Show first that* x, y *semisimple (resp. nilpotent) implies* $x + y$ *semisimple (resp. nilpotent).]*

Solution. For a counterexample when x and y do not commute, take $x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Then both x and y are nilpotent, but $x + y$ is not nilpotent because its eigenvalues are ± 1. □

Exercise 4.6. *Check formula*

$$(\delta - (a+b).1)^n(xy) = \sum_{i=0}^{n} \binom{n}{i} ((\delta - a.1)^{n-i}x) \cdot ((\delta - b.1)^i y)$$

Exercise 4.7. *Prove the converse of Theorem 4.3.*

Solution. The converse of Theorem 4.3 says that if L is a solvable subalgebra of $\mathfrak{gl}(V)$ where $\dim V < 1$, then $\mathrm{Tr}(xy) = 0$ for all $x \in [L, L]$ and $y \in L$. By Lie's theorem, we may choose a basis for V such that L consists of upper triangular matrices. Then $x \in [L, L]$ is a strictly upper triangular matrix, and hence so is yx. Finally, $tr(xy) = tr(yx) = 0$, so we are done. □

Exercise 4.8. *Note that it suffices to check the hypothesis of Theorem 4.3 (or its corollary) for x, y ranging over a basis of $[L, L]$, resp.L. For the example given in Exercise 1.2, verify solvability by using Cartan's criterion.*

Solution. In the example given in Exercise 1.2,

$$\operatorname{ad} x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \operatorname{ad} y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \operatorname{ad} z = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Also, $[L, L]$ is spanned by y and z. Hence $\mathrm{Tr}(\operatorname{ad} x \operatorname{ad} y) = \mathrm{Tr}(\operatorname{ad} x \operatorname{ad} z) = \mathrm{Tr}(\operatorname{ad} y \operatorname{ad} z) = 0$, so L is solvable. □

5 Killing form

Exercise 5.1. *Prove that if L is nilpotent, the Killing form of L is identically zero.*

Solution. Pick $x, y \in L$. Then $\operatorname{ad}([x, y])$ is nilpotent, and hence $\mathrm{Tr}(\operatorname{ad}([x, y])) = 0$. This implies $\mathrm{Tr}(\operatorname{ad} x \operatorname{ad} y) = -\mathrm{Tr}(\operatorname{ad} y \operatorname{ad} x)$, but $\mathrm{Tr}(\operatorname{ad} x \operatorname{ad} y) = \mathrm{Tr}(\operatorname{ad} y \operatorname{ad} x)$ then gives that $\mathrm{Tr}(\operatorname{ad} x \operatorname{ad} y) = 0$, so the Killing form of L is identically zero. □

Exercise 5.2. *Prove that L is solvable if and only if $[LL]$ lies in the radical of the Killing form.*

Solution. If L is solvable, then $[L, L]$ lies in the radical of the Killing form by the corollary to Theorem 4.3. The converse is Exercise 4.7. □

Exercise 5.3. *Let L be the two dimensional non-abelian Lie algebra of Exercise 1.6, which is solvable. Prove that L has nontrivial Killing form.*

Solution. The image of the adjoint representation of L is a subalgebra of $\mathfrak{gl}(2, F)$ with basis elements $\operatorname{ad} x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \operatorname{ad} y = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. So $\mathrm{Tr}(\operatorname{ad} x \operatorname{ad} x) = 1$, and hence the Killing form of L is nontrivial. □

Exercise 5.4. *Let L be the three dimensional solvable Lie algebra of Exercise 1.2. Compute the radical of its Killing form.*

Solution. We compute the matrix of Killing Form κ relative to the basis (x, y, z):

$$\kappa = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let $s = ax + by + cz$ be any element in the radical S of κ. Then

$$(a, b, c) \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

So $a = 0$, and we see that S is spanned by y and z. $\square$

Exercise 5.5. *Let $L = \mathfrak{sl}(2, F)$. Compute the basis of L dual to the standard basis, relative to the Killing form.*

Solution. The matrix of the Killing form relative to the basis (x, h, y) is

$$\begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

The basis of L dual to the standard basis is $(\frac{1}{4}y, \frac{1}{8}h, \frac{1}{4}x)$. $\square$

Exercise 5.6. *Let $\operatorname{char} F = p \neq 0$. Prove that L is semisimple if its Killing form is nondegenerate. Show by example that the converse fails. [Look at $\mathfrak{sl}(3, F)$ modulo its center, when $\operatorname{char} F = 3$.]*

Solution. If $\operatorname{Rad}(L) \neq 0$, the last nonzero term I in its derived series is a abelian subalgebra of L, and by Exercise 3.1, I is a ideal of L. In another words, L has a nonzero abelian ideal. It is suffice to prove any abelian ideal of L is zero.

Let S be the radical of the Killing form, which is nondegenerate. So $S = 0$. To prove that L is semisimple, it will suffice to prove that every abelian ideal I of L is included in S. Suppose $x \in I, y \in L$. Then $\operatorname{ad} x \operatorname{ad} y$ maps $L \to L \to I$, and $(\operatorname{ad} x \operatorname{ad} y)^2$ maps L into $[II] = 0$. This means that $\operatorname{ad} x \operatorname{ad} y$ is nilpotent, hence that $0 = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y) = \kappa(x, y)$, so $I \subset S = 0$. $\square$

Exercise 5.7. *Relative to the standard basis of $\mathfrak{sl}(3, F)$, compute the determinant of κ. Which primes divide it?*

Solution. We write down the matrix of $\operatorname{ad} x$ relative to basis

$$\{e_{11} - e_{22}, e_{22} - e_{33}, e_{12}, e_{13}, e_{21}, e_{23}, e_{31}, e_{33}\}$$

when x runs over this basis.

$$\operatorname{ad}(e_{11} - e_{22}) = \operatorname{diag}(0, 0, 2, 1, -2, -1, -1, 1)$$
$$\operatorname{ad}(e_{22} - e_{33}) = \operatorname{diag}(0, 0, -1, 1, 1, 2, -1, -2)$$

$$\operatorname{ad} e_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\operatorname{ad} e_{21} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\operatorname{ad} e_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\operatorname{ad} e_{31} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\operatorname{ad} e_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\operatorname{ad} e_{32} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix of the Killing form relative to this basis is

$$\kappa = \begin{pmatrix} 12 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \end{pmatrix}$$

Its determinant is $\det(\kappa) = 2^8 3^9$, so prime 2 and 3 divide the determinant of κ. □

Exercise 5.8. *Let $L = L_1 \oplus \cdots \oplus L_t$ be the decomposition of a semisimple Lie algebra L into its simple ideals. Show that the semisimple and nilpotent parts of $x \in L$ are the sums of the semisimple and nilpotent parts in the various L_i of the components of x.*

Solution. Write $x = x_1 + \cdots + x_t$ where $x_i \in L_i$. We can decompose each x_i as $x_{i,s} + x_{i,n}$ where $x_{i,s}$ is semisimple and $x_{i,n}$ is nilpotent. Note that $\operatorname{ad} x_i$ and $\operatorname{ad} x_j$ commute since $[x_i, x_j] = 0$. Hence $\operatorname{ad} x_{i,s}$ and $\operatorname{ad} x_{j,s}$ commute, as well as $\operatorname{ad} x_{i,n}$ and $\operatorname{ad} x_{j,n}$. This means that $x_{1,s} + \cdots + x_{t,s}$ is semisimple and that $x_{1,n} + \cdots + x_{t,n}$ is nilpotent, so by uniqueness of Jordan-Chevalley decomposition, we conclude that $x_s = x_{1,s} + \cdots + x_{t,s}$ and that $x_n = x_{1,n} + \cdots + x_{t,n}$. □

6 Complete reducibility of representations

Exercise 6.1. *Using the standard basis for $L = \mathfrak{sl}(2, F)$, write down the Casimir element of the adjoint representation of L (cf. Exercise 5.5). Do the same thing for the usual ($3-$dimensional) representation of $\mathfrak{sl}(3, F)$, first computing dual bases relative to the trace form.*

Solution. For the adjoint representation of $L = \mathfrak{sl}(2, F)$, The matrix of β respect to basis (x, h, y) is

$$\begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

we can deduce the dual basis of (x, h, y) is $(\frac{1}{4}y, \frac{1}{8}h, \frac{1}{4}x)$. So the Casimir element of this representation is

$$c_{\text{ad}} = \frac{1}{4} \operatorname{ad} x \operatorname{ad} y + \frac{1}{8} \operatorname{ad} h \operatorname{ad} h + \frac{1}{4} \operatorname{ad} y \operatorname{ad} x$$

For the usual representation of $L = \mathfrak{sl}(3, F)$, The matrix of β respect to basis

$$\{e_{11} - e_{22}, e_{22} - e_{33}, e_{12}, e_{13}, e_{21}, e_{23}, e_{31}, e_{33}\}$$

is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

We can deduce the dual basis is

$$\frac{2}{3}e_{11} - \frac{1}{3}e_{22} - \frac{1}{3}e_{33}, \frac{1}{3}e_{11} + \frac{1}{3}e_{22} - \frac{2}{3}e_{33}, e_{21}, e_{31}, e_{32}, e_{12}, e_{13}, e_{23}$$

So

$$c_\varphi = \sum_x xx' = \begin{pmatrix} \frac{8}{3} & 0 & 0 \\ 0 & \frac{8}{3} & 0 \\ 0 & 0 & \frac{8}{3} \end{pmatrix}$$

□

Exercise 6.2. *Let V be an $L-$module. Prove that V is a direct sum of irreducible submodules if and only if each $L-$submodule of V possesses a complement.*

Solution. If each $L-$submodule of V possesses a complement, then we can write V as a direct sum of irreducible submodules by induction on $\dim V$.

Conversely, suppose that V is a direct sum of irreducible submodules $V_1 \oplus \cdots \oplus V_r$, and let W be a proper $L-$submodule of V . The map $V \to V/W$ is surjective, and hence there is some i such that $V_i \to V/W$ is a nonzero map. Since V_i is irreducible, it must be injective, which means that $V_i \cap W = 0$. By induction on codimension, $V_i + W$ has a direct sum complement W''. Set $W' = W'' + V_i$. Then $W \cap (V_i + W'') = 0$ and $V = W \oplus W'$. □

Exercise 6.3. *If L is solvable, every irreducible representation of L is one dimensional.*

Solution. Let V be a representation of L. By Lie's theorem, there is a basis for V such that L acts by upper triangular matrices. Then the subspace of V spanned by the first basis vector is invariant under L. Hence if V is irreducible, it must be $1-$dimensional. □

Exercise 6.4. *Use Weyl's Theorem to give another proof that for L semisimple,* $\operatorname{ad} L = \operatorname{Der} L$ *(Theorem 5.3). [If $\delta \in \operatorname{Der} L$, make the direct sum $F + L$ into an $L-$module via the rule $x.(a, y) = (0, a\delta(x) + [xy])$. Then consider a complement to the submodule L.]*

Solution. Clearly, L is a submodule of $F + L$. By Weyl's Theorem, it has a complement of dimension 1. Let $(a_0, x_0), a_0 \neq 0$ be its basis. Then L acts on it trivially. Hence

$$0 = x.(a_0, x_0) = (0, a_0\delta(x) + [x, x_0])$$

i.e.

$$\delta(x) = [\frac{1}{a_0}x_0, x] = \operatorname{ad} \frac{1}{a_0}x_0(x)$$

So $\delta \in \operatorname{Int} L$. □

Exercise 6.5. *A Lie algebra L for which* $\operatorname{Rad} L = Z(L)$ *is called* **reductive**. *(Examples: L abelian, L semisimple, $L = \mathfrak{gl}(n, F)$.)*

1. *If L is reductive, then L is a completely reducible* $\operatorname{ad} L-$*module. [If* $\operatorname{ad} L \neq 0$*, use Weyl's Theorem.] In particular, L is the direct sum of $Z(L)$ and $[LL]$, with $[LL]$ semisimple.*

2. *If L is a classical linear Lie algebra, then L is semisimple. [Cf. Exercise 1.9.]*

3. *If L is a completely reducible* $\operatorname{ad} L-$*module, then L is reductive.*

4. *If L is reductive, then all finite dimensional representations of L in which $Z(L)$ is represented by semisimple endomorphisms are completely reducible.*

Solution. 1. Let L be reductive. If L is abelian, then it is clearly completely reducible as an $\operatorname{ad} L-$module ($\operatorname{ad} L = 0$). So assume $\operatorname{ad} L \neq 0$. Since $\operatorname{ad} L \cong L/Z(L) = L/\operatorname{Rad} L$, we see that $\operatorname{ad} L$ is semisimple. So by Weyl's theorem, L is a completely reducible $\operatorname{ad} L-$module.

$L/Z(L)$ is semisimple, so $[LL]/Z(L) \cong [L/Z(L), L/Z(L)] \cong L/Z(L)$, hence

$$L = Z(L) + [LL]$$

On the other hand, $Z(L)$ is a $\operatorname{ad} L-$submodule of L and L is a completely reducible $\operatorname{ad} L-$module. So $Z(L)$ has a component M in L.

$$L = M \oplus Z(L)$$

where M is a ideal of L.

$$[LL] \subset [M \oplus Z(L), M \oplus Z(L)] \subset [M, M] \subset M$$

We conclude that

$$L = [LL] \oplus Z(L)$$

Hence $[LL] \cong L/Z(L)$ is semisimple.

2. If L is a classical linear Lie algebra, by Exercise 4.1, $\operatorname{Rad} L = Z(L)$. And by Exercise 1.9, $Z(L) = 0$, so $L = [LL]$ is semisimple.

3. L is a completely reducible $\operatorname{ad} L-$module. Clearly $Z(L)$ is a submodule. So

$$L = Z(L) \oplus M$$

where M is a direct sum of some simple ideal of L. So M is semisimple. $L/Z(L) \cong M$ is semisimple. Hence $0 = \operatorname{Rad}(L/Z(L)) = \operatorname{Rad} L/Z(L)$. Hence $\operatorname{Rad} L \subset Z(L)$.

On the other hand, $Z(L) \subset \operatorname{Rad} L$ is clearly. We conclude that $\operatorname{Rad} L = Z(L)$, L is reductive.

4. let L be reductive and let $\varphi\colon L \to \mathfrak{gl}(V)$ be a finite-dimensional representation of L in which $Z(L)$ is represented by semisimple endomorphisms. Since $Z(L)$ is abelian, we may simultaneously diagonalize the elements of $\varphi(Z(L))$ to get an eigenspace decomposition of V . Since $[L, L]$ commutes with $Z(L)$, each eigenspace is invariant under $[L, L]$. On each such eigenspace W, each element of $\varphi(Z(L))$ acts as a scalar, so $L-$submodules of W coincide with $[L, L]-$submodules of W. We conclude complete reducibility from Weyl's theorem for semisimple Lie algebras.

$\square$

Exercise 6.6. *Let L be a simple Lie algebra. Let $\beta(x, y)$ and $\gamma(x, y)$ be two symmetric associative bilinear forms on L. If β, γ are nondegenerate, prove that β and γ are proportional. [Use Schur's Lemma.]*

Solution. L is a irreducible $L-$module by ad, and L^* is a $L-$module, We can define a linear map $\phi\colon L \to L^*; x \mapsto \beta_x$, where $\beta_x \in L^*$ defined by $\beta_x(y) = \beta(x, y)$. Then it is easy to check that ϕ is a module homomorphism of $L-$module.

Similarly, we can define a linear map $\psi\colon L^* \to L; f \to x_f$, where x_f defined by $f(z) = \gamma(x_f, z)$ for all $z \in L$. This x_f exists because γ is non-degenerate. Then ψ is also a homomorphism of $L-$modules.

So $\psi \circ \phi$ is a homomorphism from L to L, i.e, $\psi \circ \phi$ is a endomorphism of L which commutative with all $\operatorname{ad} x, x \in L$, and L is a irreducible $L-$module. By Schur's lemma we have

$$\psi \circ \phi = \lambda \operatorname{id}$$

So

$$\beta(x, y) = \beta_x(y) = \gamma(x_{\beta_x}, y) = \gamma(\psi \circ \phi(x), y) = \gamma(\lambda x, y) = \lambda\gamma(x, y) \quad \forall x, y \in L$$

□

Exercise 6.7. *It will be seen later on that $\mathfrak{sl}(n, F)$ is actually simple. Assuming this and using Exercise 6.6, prove that the Killing form κ on $\mathfrak{sl}(n, F)$ is related to the ordinary trace form by $\kappa(x, y) = 2n \operatorname{Tr}(xy)$.*

Solution. Clearly $\operatorname{Tr}(xy)$ is a nonzero symmetric associative bilinear form on $\mathfrak{sl}(n, F)$, its radical is a ideal of $\mathfrak{sl}(n, F)$, hence is equal to 0, and $\operatorname{Tr}(xy)$ is nondegenerate. By Exercise 6.6, $\kappa(x, y) = \lambda \operatorname{Tr}(xy)$.

We can only compute it for $x = y = e_{11} - e_{22}$. In this case, $\operatorname{Tr}(xy) = 2$. The matrix of $\operatorname{ad}(e_{11} - e_{22})$ relative to the standard basis of $\mathfrak{sl}(n, F)$ is a diagonal matrix

$$\operatorname{diag}(\underbrace{0, \cdots, 0}_{n-1}, 2, -2, \underbrace{1, \cdots, 1}_{2n-4}, \underbrace{-1, \cdots, -1}_{2n-4}, 0, \cdots, 0)$$

Hence $\kappa(x, y) = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y) = 4 + 4 + 2(2n - 4) = 4n = 2n \operatorname{Tr}(xy)$. □

Exercise 6.8. *If L is a Lie algebra, then L acts (via* ad*) on $(L \otimes L)^*$, which may be identified with the space of all bilinear forms β on L. Prove that β is associative if and only if $L.\beta = 0$.*

Solution. By definition,

$$\begin{aligned} z.\beta(x \otimes y) &= -\beta(z.(x \otimes y)) \\ &= -\beta(z.x \otimes y + x \otimes z.y) \\ &= \beta([x, z] \otimes y) - \beta(x \otimes [z, y]) \end{aligned}$$

Hence

$$L.\beta = 0 \iff \beta([x, z] \otimes y) = \beta(x \otimes [z, y]), \forall x, y, z \in L$$

which means β is associative. □

Exercise 6.9. *Let L' be a semisimple subalgebra of a semisimple Lie algebra L. If $x \in L'$, its Jordan decomposition in L' is also its Jordan decomposition in L.*

Solution. This follows from the corollary to Theorem 6.4 by using the adjoint representation of L and noting that it is injective. □

7 Representations of $\mathfrak{sl}(2,F)$

In these exercises, $L = \mathfrak{sl}(2,F)$.

Exercise 7.1. *Use Lie's Theorem to prove the existence of a maximal vector in an arbitrary finite dimensional $L-$module. [Look at the subalgebra B spanned by h and x.]*

Solution. $\phi\colon L \to \mathfrak{gl}(V)$ is a representation. Let B be the subalgebra of L spanned by h and x. Then $\phi(B)$ is a solvable subalgebra of $\mathfrak{gl}(V)$. And $\phi(x)$ is a nilpotent endomorphism of V. By Lie's theorem, there is a common eigenvector v for B. So $h.v = \lambda v, x.v = 0$, v is a maximal vector. □

Exercise 7.2. *$M = \mathfrak{sl}(3,F)$ contains a copy of L in its upper left-hand 2×2 position. Write M as direct sum of irreducible $L-$submodules (M viewed as $L-$module via the adjoint representation): $V(0) \oplus V(1) \oplus V(1) \oplus V(2)$.*

Solution. Let $h = e_{11} - e_{22}, x = e_{12}, y = e_{21}$.

First, we know $\operatorname{ad} h.e_{12} = 2e_{12}, \operatorname{ad} x.e_{12} = 0$. So e_{12} is a maximal vector with highest weight 2. It can generate a irreducible module isomorphic to $V(2)$. Let

$$v_0 = e_{12}, v_1 = [e_{21}, e_{12}] = -(e_{11} - e_{22}), v_2 = [e_{21}, -(e_{11} - e_{22})] = -e_{21}.$$

So $V(2) \cong \operatorname{span}\{e_{12}, e_{11} - e_{22}, e_{21}\}$.

$\operatorname{ad} h.e_{13} = e_{13}, \operatorname{ad} x.e_{13} = 0$. So e_{13} is a maximal vector with weight 1. It can generate a irreducible module isomorphic to $V(1)$.

$$[e_{21}, e_{13}] = e_{23}.$$

We have $V(1) \cong \operatorname{span}\{e_{13}, e_{23}\}$.

$\operatorname{ad} h.e_{32} = e_{32}, \operatorname{ad} x.e_{32} = 0$. So e_{32} is a maximal vector with weight 1. It can generate a irreducible module isomorphic to $V(1)$.

$$[e_{21}, e_{32}] = -e_{31}.$$

We have another $V(1) \cong \operatorname{span}\{e_{31}, e_{32}\}$.

At last, we have a $1-$dimensional irreducible submodule of $V(0) \cong span\{e_{22} - e_{33}\}$.

Then

$$M \cong V(0) \oplus V(1) \oplus V(1) \oplus V(2),$$

□

Exercise 7.3. *Verify that formulas (a)-(c) of Lemma 7.2 do define an irreducible representation of L. [Hint: To show that they define a representation, it suffices to show that the matrices corresponding to x, y, h satisfy the same structural equations as x, y, h.]*

Solution.

$$\begin{aligned}
[h,x].v_i &= 2x.v_i = 2(\lambda - i + 1)v_{i-1} \\
h.x.v_i - x.h.v_i &= (\lambda - i + 1)h.v_{i-1} - (\lambda - 2i)x.v_i \\
&= (\lambda - i + 1)(\lambda - 2i + 2)v_{i-1} - (\lambda - 2i)(\lambda - i + 1)v_{i-1} \\
&= 2(\lambda - i + 1)v_{i-1} \\
[h,y].v_i &= -2y.v_i = -2(i+1)v_{i+1} \\
h.y.v_i - y.h.v_i &= (i+1)h.v_{i-1} - (\lambda - 2i)y.v_i \\
&= (i+1)(\lambda - 2i - 2)v_{i+1} - (\lambda - 2i)(i+1)v_{i+1} \\
&= -2(i+1)v_{i+1} \\
[x,y].v_i &= hv_i = (\lambda - 2i)v_i \\
x.y.v_i - y.x.v_i &= (i+1)x.v_{i+1} - (\lambda - i + 1)y.v_{i-1} \\
&= (i+1)(\lambda - i)v_i - (\lambda - i + 1)iv_i \\
&= (\lambda - 2i)v_i
\end{aligned}$$

□

Exercise 7.4. *The irreducible representation of L of highest weight m can also be realized "naturally", as follows. Let X, Y be a basis for the two dimensional vector space F^2, on which L acts as usual. Let $\mathscr{R} = F[X,Y]$ be the polynomial algebra in two variables, and extend the action of L to R by the derivation rule: $z.fg = (z.f)g + f(z.g)$, for $z \in L, f, g \in \mathscr{R}$. Show that this extension is well defined and that $\mathscr{R}$ becomes an $L-$module. Then show that the subspace of homogeneous polynomials of degree m, with basis $X^m, X^{m-1}Y, \cdots, XY^{m-1}, Y^m$, is invariant under L and irreducible of highest weight m.*

Exercise 7.5. *Suppose* $\operatorname{char} F = p > 0, L = \mathfrak{sl}(2,F)$*. Prove that the representation $V(m)$ of L constructed as in Exercise 7.3 or 7.4 is irreducible so long as the highest weight m is strictly less than p, but reducible when $m = p$.*

Solution. When $m < p$, conditions (a)-(c) of Lemma 7.2 still imply the irreducibility of $V(m)$. However, when $m = p$, the submodule spanned by $\{v_0, \cdots, v_{m-1}\}$ is invariant under L, so $V(m)$ is reducible. □

Exercise 7.6. *Decompose the tensor product of the two $L-$modules $V(3), V(7)$ into the sum of irreducible submodules: $V(4) \oplus V(6) \oplus V(8) \oplus V(10)$. Try to develop a general formula for the decomposition of $V(m) \otimes V(n)$.*

Solution. In general, for $V = V(m) \otimes V(n)$. We suppose $m \geqslant n$. $u_i, i = 0, \cdots, m$ is the basis of $V(m)$ and $v_j, j = 1, \cdots, n$ is the basis of $V(n)$.

$$h.(u_i \otimes v_j) = (m + n - 2(i+j))u_i \otimes v_j$$

Hence

$$V_{m+n-2k} = \operatorname{span}\{u_i \otimes v_j, i + j = k\}$$

For $k = 0, \cdots, m$, suppose $w = \sum_{i=0}^{k} \lambda_i u_i \otimes v_{k-i} \in V_{m+n-2k}$ is a maximal vector. Then

$$\begin{aligned}
x.w &= \sum_{i=0}^{k} \lambda_i((x.u_i) \otimes v_{k-i} + u_i \otimes (x.v_{k-i})) \\
&= \sum_{i=1}^{k} \lambda_i(m-i+1)u_{i-1} \otimes v_{k-i} + \sum_{i=0}^{k-1} \lambda_i(n-k+i+1)u_i \otimes v_{k-i-1} \\
&= \sum_{i=1}^{k} (\lambda_i(m-i+1) + \lambda_{i-1}(n-k+i))u_{i-1} \otimes v_{k-i} \\
&= 0
\end{aligned}$$

Therefore

$$\lambda_i(m-i+1) + \lambda_{i-1}(n-k+i) = 0$$

We conclude that

$$\lambda_i = (-1)^i \frac{(n-k+i)!(m-i)!}{(n-k)!m!} \lambda_0$$

Let $\lambda_0 = 1$, then $w = \sum_{i=0}^{k} \lambda_i u_i \otimes v_{k-i}$ is a maximal vector with weight $m+n-2k$. It generates a irreducible submodule of V isomorphic to $V(m+n-2k)$. So

$$\bigoplus_{k=0}^{n} V(m+n-2k) \subset V(m) \otimes V(n).$$

Compare the dimensional of two sides, we have the decomposition $V(m) \otimes V(n) \cong V(m-n) \oplus V(m-n+2) \oplus \cdots \oplus V(m+n)$. □

Solution. To find a decomposition of $V(m) \otimes V(n)$, it is enough to count the dimensions of eigenspaces of h. In particular, note that since $h.(v \otimes w) = h.v \otimes w + v \otimes h.w$, if $v \in V_\lambda \subset V(m)$ and $w \in V_\mu \subset V(n)$, then $v \otimes w \in V_{\lambda+\mu} \subset V(m) \otimes V(n)$.

Hence the dimension of V_λ in $V(m) \otimes V(n)$ is the number of ways to write λ as a sum of elements from the two sets $\{m, m-2, \cdots, -m\}$ and $\{n, n-2, \cdots, -n\}$. Without loss of generality, assume that $m \geqslant n$. Then λ can be written as such a sum $\min(\frac{m+n-|\lambda|}{2} + 1, \frac{m+n-(m-n)}{2} + 1)$ ways if $m+n-|\lambda|$ is even and 0 ways otherwise.

Hence we have the decomposition $V(m) \otimes V(n) \cong V(m-n) \oplus V(m-n+2) \oplus \cdots \oplus V(m+n)$. □

Exercise 7.7. *In this exercise we construct certain infinite dimensional $L-$modules. Let $\lambda \in F$ be an arbitrary scalar. Let $Z(\lambda)$ be a vector space over F with countably infinite basis $(v_0, v_1, v_2, \cdots)$.*

1. *Prove that formulas (a)-(c) of Lemma 7.2 define an $L-$module structure on $Z(\lambda)$, and that every nonzero $L-$submodule of $Z(\lambda)$ contains at least one maximal vector.*

2. *Suppose $\lambda + 1 = i$ is a nonnegative integer. Prove that v_i is a maximal vector (e.g., $\lambda = -1, i = 0$). This induces an $L-$module homomorphism $Z(\mu) \xrightarrow{\phi} Z(\lambda), \mu = \lambda - 2i$, sending v_0 to v_i. Show that ϕ is a monomorphism, and that $\operatorname{im}\phi, Z(\lambda)/\operatorname{im}\phi$ are both irreducible $L-$modules (but $Z(\lambda)$ fails to be completely reducible when $i > 0$).*

3. *Suppose $\lambda + 1$ is not a nonnegative integer. Prove that $Z(\lambda)$ is irreducible.*

Solution. 1. We need to verify that $[xy] = h, [hx] = 2x, [hy] = -2y$ as linear transformations on $Z(\lambda)$.

It suffices to check these on the basis $(v_0, v_1, v_2, \cdots)$. Given v_i, we have

$$\begin{aligned}
[xy].v_i &= (i+1)x.v_{i+1} - (\lambda - i + 1)y.v_{i-1} \\
&= (i+1)(\lambda - i)v_i - (\lambda - i + 1)iv_i \\
&= (\lambda - 2i)v_i = h.v_i; \\
[hx].v_i &= (\lambda - i + 1)h.v_{i-1} - (\lambda - 2i)x.v_i \\
&= (\lambda - i + 1)(\lambda - 2i + 2)v_{i-1} - (\lambda - 2i)(\lambda - i + 1)v_{i-1} \\
&= 2(\lambda - i + 1)v_{i-1} = 2x.v_i; \\
[hy].v_i &= (i+1)h.v_{i+1} - (\lambda - 2i)y.v_i \\
&= (i+1)(\lambda - 2i - 2)v_{i+1} - (\lambda - 2i)(i+1)v_{i+1} \\
&= -2(i+1)v_{i+1} = -2y.v_i;
\end{aligned}$$

So $Z(\lambda)$ is an $L-$module.

Let U be an arbitrary nonzero $L-$submodule of $Z(\lambda)$. For any nonzero $v \in U$ write v as a linear combination of basis: $v = \sum\limits_{i \in I} a_i v_i$, where all $a_i \neq 0$. We have

$$h.v = \sum_{i \in I} a_i(\lambda - 2i)v_i \in U$$

This implies all $v_i \in U$. So

$$U = \mathrm{span}\{v_j, j \in J\}$$

Let $k = \min J$, then $v_k \in U$, and $v_{k-1} \notin U$, so $x.v_k = (\lambda - k + 1)v_{k-1} = 0$. We conclude that v_k is a maximal vector in U.

2. $x.v_i = (\lambda - i + 1)v_{i-1} = 0$, hence v_i is a maximal vector.

To see ϕ is a monomorphism, it suffices to show it is injective on basis. In deed,

$$\phi(v_k) = \binom{k+i}{i} v_{k+i}$$

We prove this by induction on k

When $k = 0$, $\phi(v_0) = v_i$. If we already have $\phi(v_{k-1}) = \binom{k-1+i}{i} v_{k-1+i}$, then

$$\begin{aligned}
\phi(v_k) &= \phi(\frac{1}{k}y.v_{k-1}) = \frac{1}{k}y.\phi(v_{k-1}) \\
&= \frac{1}{k}\binom{k-1+i}{i}y.v_{k-1+i} = \frac{k+i}{k}\binom{k-1+i}{i}v_{k+i} \\
&= \binom{k+i}{i}v_{k+i}
\end{aligned}$$

$\operatorname{im}\phi \cong= Z(\mu)$ is a submodule of $Z(\lambda)$ and by (1) it has a maximal vector of form v_s. But

$$x.v_s = (\mu - s + 1) = -(i+s)v_{s-1} = 0$$

From $i+s>0$, we have $v_{s-1}=0$. So v_0 is the unique maximal vector in $Z(\mu)$ and $Z(\mu)$ is irreducible. $Z(\lambda)/\operatorname{im}\phi \cong V(i-1)$ is a irreducible module.

Next we show $Z(\lambda)$ is not completely reducible. If $Z(\lambda)$ is completely reducible, then we have an $L-$module decomposition $Z(\lambda) = \operatorname{im}\phi \oplus V$. Then there exists a $w \in \operatorname{im}\phi$ such that $v_0 + w \in V$. But

$$y^i.(v_0+w) = v_i + y^i.w \in \operatorname{im}\phi$$

which contradicts with the fact that V is an $L-$module.

3. If $Z(\lambda)$ reducible, it has a proper nonzero submodule U. By (1) U has a maximal vector v_k with $k>0$.
$$x.v_k = (\lambda - k + 1)v_{k-1} = 0$$
Hence $\lambda + 1 = k$ is a positive integer. We get a contradiction.

□

8 Root space decomposition

Exercise 8.1. *If L is a classical linear Lie algebra of type A_l, B_l, C_l or D_l, prove that the set of all diagonal matrices in L is a maximal toral subalgebra, of dimension l (Cf. Exercise 2.8.)*

Solution. Since a toral subalgebra is abelian, any toral subalgebra containing the set of diagonal matrices in L must contain only diagonal matrices because commuting semisimple matrices are simultaneously diagonalizable. Its dimension can be immediately verified in the four cases to be l. □

Exercise 8.2. *For each algebra in Exercise 8.1, determine the roots and root spaces. How are the various h_α expressed in terms of the basis for H given in section 1?*

Exercise 8.3. *If L is of classical type, compute explicitly the restriction of the Killing form to the maximal toral subalgebra described in Exercise 8.1.*

Exercise 8.4. *If $L = \mathfrak{sl}(2,F)$, prove that each maximal toral subalgebra is one dimensional.*

Solution. $\mathfrak{h}$ is a maximal toral subalgebra of L, $L = \mathfrak{h} \oplus \sum_{\alpha\in\Phi} L_\alpha, \dim L_\alpha = 1 . \alpha \in \Phi$ then $-\alpha \in \Phi$. This implies $\operatorname{Card}(\Phi)$ is even and nonzero. So $\dim \mathfrak{h} = 1$. □

Exercise 8.5. *If L is semisimple, H a maximal toral subalgebra, prove that H is self-normalizing (i.e., $H = N_L(H)$).*

Solution. $L = H \oplus \sum_{\alpha\in\Phi} L_\alpha$. For $x \in N_L(H)$, $x = h_0 + \sum_{\alpha\in\Phi} x_\alpha, x_\alpha \in L_\alpha$. Choose $h \in H$ such that $\alpha(h) \neq 0, \forall \alpha \in \Phi$, then

$$[h,x] = \sum_{\alpha\in\Phi} \alpha(h) x_\alpha \in H$$

Hence $x_\alpha = 0, \forall\alpha\in\Phi$. $x = h_0 \in H$. i.e, $N_L(H) = H$. □

Exercise 8.6. *Compute the basis of $\mathfrak{sl}(n,F)$ which is dual (via the Killing form) to the standard basis. (Cf. Exercise 5.5.)*

Solution. The dual of $e_{ij}(i \neq j)$ via the Killing form is e_{ji}, the dual of h_i via the Killing form is $e_{ii} - \frac{1}{n}I_n$ $\square$

Exercise 8.7. *Let L be semisimple, H a maximal toral subalgebra. If $h \in H$, prove that $C_L(h)$ is reductive (in the sense of Exercise 6.5). Prove that H contains elements h for which $C_L(h) = H$; for which h in $\mathfrak{sl}(n,F)$ is this true ?*

Solution. L is semisimple. We have a decomposition $L = H \oplus \sum\limits_{\alpha\in\Phi} L_\alpha$

$$\begin{aligned} & x = h_0 + \sum_{\alpha\in\Phi} x_\alpha \in C_L(h) \\ \iff & [h,x] = \sum_{\alpha\in\Phi} \alpha(h) x_\alpha = 0 \\ \iff & \alpha(h) = 0 \quad \text{or} \quad x_\alpha = 0 \end{aligned}$$

Hence

$$C_L(h) = H \oplus \sum_{\substack{\alpha\in\Phi \\ \alpha(h)=0}} L_\alpha$$

Denote $\Phi_h = \{\alpha \in \Phi \mid \alpha(h) = 0\}$. Now we claim that

$$Z(C_L(h)) = \{h' \in H \mid \alpha(h') = 0, \forall \alpha \in \Phi_h\}$$

Let $x = h_0 + \sum\limits_{\alpha\in\Phi_h} x_\alpha \in Z(C_L(h))$. We can find a $h' \in H$ such that $\alpha(h') \neq 0, \forall \alpha \in \Phi_h$. Then $[h', x] = \sum\limits_{\alpha\in\Phi_h} \alpha(h') x_\alpha = 0$. It implies $x_\alpha = 0$. We have $x = h_0 \in H$. Next we take $0 \neq x_\alpha \in L_\alpha, \forall \alpha \in \Phi_h$, then $[x, x_\alpha] = \alpha(h_0) x_\alpha = 0$. Hence $\alpha(x) = \alpha(h_0) = 0, \forall \alpha \in \Phi_h$.

Next we show $Z(C_L(h)) = \mathrm{Rad}(C_L(h))$. Clearly $Z(C_L(h))$ is a solvable ideal of $C_L(h)$, it is enough to show it is a maximal solvable ideal.

If $x = h_0 + \sum\limits_{\alpha\in\Phi_h} x_\alpha \in \mathrm{Rad}(C_L(h)) \setminus Z(C_L(h))$. We have a $h' \in H$ such that $\alpha(h') \neq 0$ and $\alpha(h') \neq \beta(h'), \forall \alpha \neq \beta \in \Phi_h$. Then $[h', x] = \sum\limits_{\alpha\in\Phi_h} \alpha(h') x_\alpha \in \mathrm{Rad}(C_L(h))$. Hence $h_0, x_\alpha \in \mathrm{Rad}(C_L(h)), \alpha \in \Phi_h$. If there is a $\alpha \in \Phi_h$ such that $x_\alpha \neq 0$, then $h_\alpha = [x_\alpha, y_\alpha] \in \mathrm{Rad}(C_L(h)), 2y_\alpha = -[h_\alpha, y_\alpha] \in \mathrm{Rad}(C_L(h))$. Hence $\mathfrak{sl}(2,F) \cong S_\alpha \subset \mathrm{Rad}(C_L(h))$ which contradict with the solvability of $\mathrm{Rad}(C_L(h))$.

Now we get $x = h_0 \in \mathrm{Rad}(C_L(h)) \setminus Z(C_L(h))$. So there is a $\alpha \in \Phi_h$ such that $\alpha(h_0) \neq 0$. Then $[x, x_\alpha] = \alpha(h_0) x_\alpha \in \mathrm{Rad}(C_L(h)), [x, y_\alpha] = -\alpha(h_0) y_\alpha \in \mathrm{Rad}(C_L(h))$. We also have $S_\alpha \subset \mathrm{Rad}(C_L(h))$ which contradict with the solvability of $\mathrm{Rad}(C_L(h))$.

All of the above show that $Z(C_L(h)) = \mathrm{Rad}(C_L(h))$. i.e., $C_L(h)$ is reductive. We know there is a $h \in H, \alpha(h) \neq 0, \forall \alpha \in \Phi$. In this case, $C_L(h) = H$.

In $\mathfrak{sl}(n,F)$, for $e_{ij}(i \neq j) \in L_\alpha$ and $h = \sum a_k h_k \in H$, we have

$$[h, e_{ij}] = \alpha(h) e_{ij} = \begin{cases} (4a_i - a_{i+1} - a_{i-1}) e_{i,i+1} & j = i+1 \\ (a_{j+1} + a_{j-1} - 4a_j) e_{j+1,j} & i = j+1 \\ (a_i - a_j - a_{i-1} + a_{j-1}) e_{ij} & |i-j| > 1 \end{cases}$$

Then these h for which $C_L(h) = H$ is these satisfying

$$\begin{cases} 4a_i - a_{i+1} - a_{i-1} \neq 0 & 1 \leqslant i \leqslant n \\ a_i - a_j - a_{i-1} + a_{j-1} \neq 0 & |i-j| > 1 \end{cases}$$

□

Exercise 8.8. *For $\mathfrak{sl}(n, F)$ (and other classical algebras), calculate explicitly the root strings and Cartan integers. In particular, prove that all Cartan integers $2\frac{(\alpha,\beta)}{(\beta,\beta)}, \alpha \neq \pm\beta$, for $\mathfrak{sl}(n, F)$ are $0, \pm 1$.*

Exercise 8.9. *Prove that every three dimensional semisimple Lie algebra has the same root system as $\mathfrak{sl}(2, F)$, hence is isomorphic to $\mathfrak{sl}(2, F)$.*

Solution. This is a direct consequence of Proposition 8.3(f). □

Exercise 8.10. *Prove that no four, five or seven dimensional semisimple Lie algebras exist.*

Solution. Let L is a semisimple Lie algebra with a maximal toral subalgebra H. We have $L = H \oplus \sum\limits_{\alpha \in \Phi} L_\alpha$. Since $\alpha \in \Phi$ implies $-\alpha \in \Phi$, $\sum\limits_{\alpha \in \Phi} L_\alpha$ has dimensional $2k$ with $k > 1$. Therefore

$$\dim H = \dim L - \dim(\sum_{\alpha \in \Phi} L_\alpha) = \dim L - 2k$$

In the other hands, $\Phi = \{\pm\alpha_1, \cdots, \pm\alpha_k\}$ span H^*, then

$$\dim H = \dim H^* \leqslant k$$

We conclude

$$\frac{\dim L}{3} \leqslant k < \frac{\dim L}{2}$$

If $\dim L = 4$, we can not find a integer k satisfying it.

If $\dim L = 5, k = 2$. Then $\dim H = 1$, i.e, Φ spans a $1-$dimensional space. $\alpha_2 = m\alpha_1$ with $m = \pm 1$. We get a contradiction.

If $\dim L = 7, k = 3$. Then $\dim H = 1$. We can deduce a contradiction as the case $\dim L = 5$. Hence, there is no four, five or seven dimensional semisimple Lie algebra. □

Exercise 8.11. *If $(\alpha, \beta) > 0$, and $\alpha \neq \pm\beta$, prove that $\alpha - \beta \in \Phi(\alpha, \beta \in \Phi)$. Is the converse true?*

Solution. We have $\beta - \frac{2(\alpha,\beta)}{(\beta,\beta)}\alpha \in \Phi$ since $\alpha, \beta \in \Phi$.

Let the β string through α is $\alpha - r\beta, \cdots, \alpha, \cdots, \alpha + q\beta$. We have $r > 0$ since $(\alpha, \beta) > 0$. Hence $\alpha - \beta$ appeals in the string. It is a root. □

Chapter III
Root System

9 Axiomatics

Unless otherwise specified, Φ denotes a root system in E, with Weyl group $\mathcal{W}$.

Exercise 9.1. *Let E' be a subspace of E. If a reflection σ_α leaves E' invariant, prove that either $\alpha \in E'$ or else $E' \subset P_\alpha$.*

Solution. Suppose $E' \not\subset P_\alpha$. Let $\lambda \in E' \setminus P_\alpha$, then $\sigma_\alpha(\lambda) = \lambda - <\lambda, \alpha> \alpha \in E'$. Since $\lambda \notin P_\alpha$, $<\lambda, \alpha> \neq 0$. Hence $\alpha \in E'$. □

Exercise 9.2. *Prove that $\Phi^\vee$ is a root system in E, whose Weyl group is naturally isomorphic to $\mathcal{W}$; show also that $<\alpha^\vee, \beta^\vee> = <\beta, \alpha>$, and draw a picture of $\Phi^\vee$ in the cases A_1, A_2, B_2, G_2.*

Solution. By definition, $\Phi^\vee$ consists of $\alpha^\vee = 2\alpha/(\alpha,\alpha)$ for $\alpha \in \Phi$. If Φ is finite, spans E, and does not contain 0, it is obvious that $\Phi^\vee$ also satisfies these conditions. Since $(c\alpha)^\vee = c(\alpha)^\vee$ for $c \in \mathbb{R}$, it follows that the only multiples of $\alpha^\vee$ in $\Phi^\vee$ are $\pm\alpha^\vee$. Also, for $\alpha, \beta \in \Phi$,

$$\sigma_{\alpha^\vee}\beta^\vee = \beta^\vee - \frac{2(\beta^\vee, \alpha^\vee)}{(\alpha^\vee, \alpha^\vee)}\alpha^\vee = \frac{2\beta}{(\beta,\beta)} - \frac{4(\beta,\alpha)}{(\beta,\beta)(\alpha,\alpha)}\alpha$$

and

$$(\sigma_\alpha(\beta))^\vee = \left(\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha\right)^\vee = \frac{2\beta - 4\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha}{(\beta,\beta) - 4\frac{(\beta,\alpha)^2}{(\alpha,\alpha)} + 44\frac{(\beta,\alpha)^2}{(\alpha,\alpha)}} = \frac{2\beta}{(\beta,\beta)} - \frac{4(\beta,\alpha)}{(\beta,\beta)(\alpha,\alpha)}\alpha$$

so $\sigma_{\alpha^\vee}\beta^\vee = (\sigma_\alpha(\beta))^\vee$, and hence $\Phi^\vee$ is invariant under $\sigma_{\alpha^\vee}$. Finally,

$$<\alpha^\vee, \beta^\vee> = \frac{2(\alpha^\vee, \beta^\vee)}{(\beta^\vee, \beta^\vee)} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} = <\beta, \alpha> \in \mathbb{Z}$$

so all of the axioms for a root system are satisfied for $\Phi^\vee$.

Finally, by the above calculations, the bijection $\alpha \mapsto \alpha^\vee$ induces an isomorphism $\sigma_\alpha \mapsto \sigma_{\alpha^\vee}$ (thinking of the Weyl groups as subgroups of the symmetric group on Φ and $\Phi^\vee$). □

Exercise 9.3. *In Table 1, show that the order of $\sigma_\alpha\sigma_\beta$ in $\mathcal{W}$ is (respectively) $2, 3, 4, 6$ when $\theta = \pi/2, \pi/3$ (or $2\pi/3$), $\pi/4$ (or $3\pi/4$), $\pi/6$ (or $5\pi/6$). [Note that $\sigma_\alpha\sigma_\beta$ = rotation through 2θ.]*

Exercise 9.4. *Prove that the respective Weyl groups of $A_1 \times A_1, A_2, B_2, G_2$ are dihedral of order $4, 6, 8, 12$. If Φ is any root system of rank 2, prove that its Weyl group must be one of these.*

Solution. σ_α and $\sigma_\alpha\sigma_\beta$ generate the Weyl group, then the conclusions follow from Exercise 9.3. □

Solution. There are only two reflections for the Weyl group of $A_1 \times A_1$, and they commute with each other, so its Weyl group is isomorphic to $Z/2 \times Z/2$, which is the dihedral group of order 4.

Picking alternating chambers of A_2, draw a regular triangle. The reflections of A_2 are symmetries of this triangle, so its Weyl group is $\mathscr{S}_3$, the dihedral group of order 6.

Drawing a square with vertices on the diagonal vectors of B_2, we see that all reflections preserve this square, and since the symmetries of the square are generated by reflections, the Weyl group of B_2 is D_4, the dihedral group of order 8.

Finally, the reflections of G_2 preserve a regular hexagon whose vertices are on the short vectors. So the Weyl group of G_2 is a subgroup of D_6, but since the reflections generate it, we get the whole group.

The fact that the Weyl group of every rank 2 root system must be dihedral of order $4, 6, 8$ or 12 follows from the possibilities allowed in Table 1. $\square$

Exercise 9.5. *Show by example that $\alpha - \beta$ may be a root even when $(\alpha, \beta) \leqslant 0$ (cf. Lemma 9.4).*

Solution. This can be seen in the root system G_2. Using the labels in Figure 9.1, we have that α and $\alpha + \beta$ form an obtuse angle, i.e., $(\alpha, \alpha + \beta) \leqslant 0$, but that β is a root. $\square$

Exercise 9.6. *Prove that $\mathcal{W}$ is a normal subgroup of* $\operatorname{Aut} \Phi$ *(= group of all isomorphisms of Φ onto itself).*

Solution. Any element of $\mathcal{W}$ can be written $\sigma_{\alpha_1} \cdots \sigma_{\alpha_r}$ for $\alpha_i \in \Phi$. Then for $\tau \in \operatorname{Aut} \Phi$, we have

$$\tau(\sigma_{\alpha_1} \cdots \sigma_{\alpha_r})\tau^{-1} = (\tau\sigma_{\alpha_1}\tau^{-1}) \cdots (\tau\sigma_{\alpha_r}\tau^{-1}) = \sigma_{\tau(\alpha_1)} \cdots \sigma_{\tau(\alpha_r)} \in \mathcal{W};$$

where the last equality is Lemma 9.2. $\square$

Exercise 9.7. *Let $\alpha, \beta \in \Phi$ span a subspace E' of E. Prove that $E' \cap \Phi$ is a root system in E'. Prove similarly that $\Phi \cap (\mathbb{Z}\alpha + \mathbb{Z}\beta)$ is a root system in E' (must this coincide with $E' \cap \Phi$?). More generally, let Φ' be a nonempty subset of Φ such that $\Phi' = -\Phi'$, and such that $\alpha, \beta \in \Phi', \alpha + \beta \in \Phi$ implies $\alpha + \beta \in \Phi'$. Prove that Φ' is a root system in the subspace of E it spans. [Use Table 1].*

Exercise 9.8. *Compute root strings in G_2 to verify the relation $r - q = <\beta, \alpha>$.*

Exercise 9.9. *Let Φ be a set of vectors in a euclidean space E, satisfying only (R1), (R3), (R4). Prove that the only possible multiples of $\alpha \in \Phi$ which can be in Φ are $\pm\frac{1}{2}\alpha, \pm\alpha, \pm 2\alpha$. Verify that $\{\alpha \in \Phi \mid 2\alpha \notin \Phi\}$ is a root system.*

Exercise 9.10. *Let $\alpha, \beta \in \Phi$. Let the $\alpha-$string through β be $\beta - r\alpha, \cdots, \beta + q\alpha$, and let the $\beta-$string through α be $\alpha - r'\beta, \cdots \alpha + q'\beta$. Prove that $\frac{q(r+1)}{(\beta,\beta)} = \frac{q'(r'+1)}{(\alpha,\alpha)}$.*

Exercise 9.11. *Let c be a positive real number. If Φ possesses any roots of squared length c, prove that the set of all such roots is a root system in the subspace of E it spans. Describe the possibilities occurring in Figure 1.*

10 Simple roots and Weyl group

Exercise 10.1. *Let $\Phi^\vee$ be the dual system of Φ, $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$. Prove that $\Delta^\vee$ is a base of $\Phi^\vee$.[Compare Weyl chambers of Φ and $\Phi^\vee$.]*

Exercise 10.2. *If Δ is a base of Φ, prove that the set $(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Phi(\alpha \neq \beta \in \Delta)$ is a root system of rank 2 in the subspace of E spanned by α, β (cf. Exercise 9.7). Generalize to an arbitrary subset of Δ.*

Exercise 10.3. *Prove that each root system of rank 2 is isomorphic to one of those listed in (9.3).*

Exercise 10.4. *Verify the Corollary of Lemma 10.2A directly for G_2.*

Solution. $a+b+b+b+a, a+b+b+b, a+b+b, a+b, a, b$. □

Exercise 10.5. *If $\sigma \in \mathcal{W}$ can be written as a product of t simple reflections, prove that t has the same parity as $l(\sigma)$.*

Solution. It is enough to prove that if the identity is written as t simple reflections, then t is even. To see this, first note that the number of negative roots in the set $\sigma_1 \cdots \sigma_t(\Delta)$ has the same parity of t. This follows by induction since each $\sigma_i = \sigma_{\alpha_i}$ fixes the sign of α_j if $\alpha_j \neq \alpha_i$, and changes the sign of α_i. So if $1 = \sigma_1 \cdots \sigma_t$, then t is even because Δ has no negative roots. □

Exercise 10.6. *Define a function $sn\colon \mathcal{W} \to \{\pm 1\}$ by $sn(\sigma) = (-1)^{l(\sigma)}$. Prove that sn is a homomorphism (cf. the case A_2, where $\mathcal{W}$ is isomorphic to the symmetric group $\mathscr{S}_3$).*

Solution. This is immediate from Exercise 10.5: given $\sigma, \tau \in \mathcal{W}$, we have that $l(\sigma\tau) = l(\sigma) + l(\tau) (\bmod 2)$. □

Exercise 10.7. *Prove that the intersection of "positive" open half-spaces associated with any basis $\gamma_1, \cdots, \gamma_l$ of E is nonvoid. [If δ_i is the projection of γ_i on the orthogonal complement of the subspace spanned by all basis vectors except γ_i, consider $\gamma = \sum r_i\delta_i$ when all $r_i > 0$.]*

Solution. $(\gamma, \delta_j) = \sum r_i(\delta_i, \delta_j) = (\gamma_j, \delta_j) > 0$. □

Exercise 10.8. *Let Δ be a base of Φ, $\alpha \neq \beta$ simple roots, $\Phi_{\alpha\beta}$ the rank 2 root system in $E_{\alpha\beta} = \mathbb{R}\alpha + \mathbb{R}\beta$ (see Exercise 10.2 above). The Weyl group $\mathcal{W}_{\alpha\beta}$ of $\Phi_{\alpha\beta}$ is generated by the restrictions τ_α, τ_β to $E_{\alpha\beta}$ of $\sigma_\alpha, \sigma_\beta$, and $\mathcal{W}_{\alpha\beta}$ may be viewed as a subgroup of $\mathcal{W}$. Prove that the "length" of an element of $\mathcal{W}_{\alpha\beta}$ (relative to τ_α, τ_β) coincides with the length of the corresponding element of $\mathcal{W}$.*

Exercise 10.9. *Prove that there is a unique element σ in $\mathcal{W}$ sending Φ^+ to Φ^- (relative to Δ). Prove that any reduced expression for σ must involve all $\sigma_\alpha (\alpha \in \Delta)$. Discuss $l(\sigma)$.*

Solution. Note that $-\Delta = \{-\alpha \mid \alpha \in \Delta\}$ is also a base for Φ, so since $\mathcal{W}$ acts transitively on bases of Φ (Theorem 10.3), there is a $\sigma \in \mathcal{W}$ such that $\sigma(\Delta) = -\Delta$. Then σ necessarily takes positive roots of Φ to negative roots of Φ (relative to Δ). If $\tau \in \mathcal{W}$ also has this property, then $\sigma\tau$ takes a positive base to another positive base. By definition, two bases can be positive with respect to Δ only if they are equal, so since $\mathcal{W}$ acts simply transitively on bases, $\sigma\tau = 1$, so $\tau = \sigma$ because σ has order 2 (for the same reason just discussed). Hence σ is unique.

Let $\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$ be a reduced expression for $\sigma(\alpha_i \in \Delta)$. Suppose $\beta \in \Delta$ is not in this expression. Since $\sigma_\alpha(\beta) = \beta - <\beta, \alpha> \alpha$, it is clear that $\sigma_{\alpha_s} \cdots \sigma_{\alpha_t}$ cannot take β to another simple root. Since each σ_{α_i} permutes $\Phi^+ \setminus \{\alpha_i\}$ (Lemma 10.2B), $\sigma(\beta) \notin \Phi^-$, which is a contradiction. Hence a reduced expression for σ must involve all $\sigma_\alpha(\alpha \in \Delta)$.

Since $\sigma(\Phi^+) = \Phi^-$, $l(\sigma) = n(\sigma) = \#(\Phi^+) = \#(\Phi)/2$. $\square$

Exercise 10.10. *Given $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ in Φ, let $\lambda = \sum_{i=1}^{l} k_i\alpha_i$ ($k_i \in \mathbb{Z}$, all $k_i \geqslant 0$ or all $k_i \leqslant 0$). Prove that either λ is a multiple (possibly 0) of a root, or else there exists $\sigma \in \mathcal{W}$ such that $\sigma\lambda = \sum_{i=1}^{l} k_i'\alpha_i$, with some $k_i' > 0$ and some $k_i' < 0$.*

Solution. If λ is not a multiple of any root, then the hyperplane P_λ orthogonal to λ is not included in $\bigcup_{\alpha\in\Phi} P_\alpha$. Take $\mu \in P_\lambda - \bigcup_{\alpha\in\Phi} P_\alpha$. Then find $\sigma \in \mathcal{W}$ for which all $(\alpha, \sigma\mu) > 0$. It follows that $0 = (\lambda, \mu) = (\sigma\lambda, \sigma\mu) = \sum_{i=1}^{l} k_i(\alpha_i, \sigma\mu)$. $\square$

Exercise 10.11. *Let Φ be irreducible. Prove that $\Phi^\vee$ is also irreducible. If Φ has all roots of equal length, so does $\Phi^\vee$ (and then $\Phi^\vee$ is isomorphic to Φ). On the other hand, if Φ has two root lengths, then so does $\Phi^\vee$; but if α is long, then $\alpha^\vee$ is short (and vice versa). Use this fact to prove that Φ has a unique maximal short root (relative to the partial order $\prec$ defined by Δ).*

Solution. Since $({}^\vee\Phi^\vee) = \Phi$, to prove that Φ irreducible implies $\Phi^\vee$ irreducible, it is enough to prove that if Φ is reducible, then so is $\Phi^\vee$. But this is obvious because $(\alpha^\vee, \beta^\vee) = 0$ if and only if $(\alpha, \beta) = 0$.

Also, if all roots of Φ have the same length, then (α, α) is a constant C for $\alpha \in \Phi$, so $\beta^\vee = \frac{2}{C}\beta$ for all $\beta \in \Phi$, which means all root lengths are the same in $\Phi^\vee$. Multiplication by this nonzero scalar gives an isomorphism between Φ and $\Phi^\vee$.

If instead Φ has 2 root lengths, then so does $\Phi^\vee$. This must hold because if $\Phi^\vee$ had one root length, then so would $({}^\vee\Phi^\vee) = \Phi$. Since the length of $\beta^\vee$ gets shorter the longer β is, it is clear that short roots of Φ correspond to long roots of $\Phi^\vee$, and vice versa. Finally, a maximal root of $\Phi^\vee$ is long (Lemma 10.4D), so corresponds to a maximal short root of Φ (heights are preserved in passing to duals). $\square$

Exercise 10.12. *Let $\lambda \in \mathfrak{C}(\Delta)$. If $\sigma\lambda = \lambda$ for some $\sigma \in \mathcal{W}$, then $\sigma = 1$.*

Solution. Note that $\mathcal{W}$ sends chambers of Φ to other chambers, so if $\sigma\lambda = \lambda$, then σ fixes a chamber of Φ. Then the exercise is a consequence of Theorem 10.3(e) and the fact that the set of chambers of Φ and the set of bases of Φ are isomorphic as $\mathcal{W}-$sets (10.1). $\square$

Exercise 10.13. *The only reflections in $\mathcal{W}$ are those of the form $\sigma_\alpha(\alpha \in \Phi)$. [A vector in the reflecting hyperplane would, if orthogonal to no root, be fixed only by the identity in $\mathcal{W}$.]*

Solution. Let $\tau \in \mathcal{W}$ be some reflection. If the reflecting hyperplane of τ is not orthogonal to a root of Φ, then let γ be a vector in this hyperplane. Then γ is contained in $\mathfrak{C}(\Delta)$ for some base Δ. By Exercise 10.12, the only element of $\mathcal{W}$ fixing γ is the identity, so $\tau = 1$. $\square$

Exercise 10.14. *Prove that each point of E is $\mathcal{W}-$conjugate to a point in the closure of the fundamental Weyl chamber relative to a base Δ. [Enlarge the partial order on E by defining $\mu \prec \lambda$ iff $\lambda - \mu$ is a nonnegative $\mathbb{R}-$linear combination of simple roots. If $\mu \in E$, choose $\sigma \in \mathcal{W}$ for which $\lambda = \sigma\mu$ is maximal in this partial order.]*

Solution. This is similar to the proof of Theorem 10.3(a). Here we replace a regular element γ with any point of E. The difference is that $(\sigma(\gamma), \alpha)$ may be 0 for some α. This will imply only that $(\sigma(\gamma), \alpha) \geqslant 0$ for all $\alpha \in \Delta$, which is to say that $\sigma(\gamma) \in \overline{\mathfrak{C}(\Delta)}$. □

11 Classification

Exercise 11.1. *Verify the Cartan matrices (Table 1).*

Exercise 11.2. *Calculate the determinants of the Cartan matrices (using induction on l for types $A_l - D_l$), which are as follows:*

$$A_l : l+1;\ B_l : 2;\ C_l : 2;\ D_l : 4;\ E_6 : 3;\ E_7 : 2;\ E_8, F_4, G_2 : 1$$

Solution. By expanding along the first row, we get that $\det A_l = 2\det A_{l-1} - \det A_{l-2}$ (the second matrix needs to again be expanded along the first column). Similar relations hold for B_l, C_l and D_l.

By inspection, $\det A_1 = 2$ and $\det A_2 = 3$, so by induction, $\det A_l = l+1$.

Also, $\det B_1 = \det B_2 = 2$, and $\det C_1 = \det C_2 = 2$, so $\det B_l = 2, \det C_l = 2$.

Furthermore $D_2 = A_1 \times A_1$ and $D_3 = A_3$, so $\det D_2 = \det D_3 = 4$, which gives $\det D_l = 4$.

The determinants of E_6, E_7, E_8, F_4 and G_2 can all be done via row reduction. □

Exercise 11.3. *Use the algorithm of (11.1) to write down all roots for G_2. Do the same for C_3 :*

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

Solution. Using the algorithm of (11.1), we start with simple roots for G_2, a short root α, and a long root β. We know that $<\alpha, \beta>= -1$ and $<\beta, \alpha>= -3$. This means that we have a root string

$$\{\beta, \beta+\alpha, \beta+2\alpha, \beta+3\alpha\}$$

Also, if $2\beta + k\alpha$ is to be a root, we need $r - q =<\beta + k\alpha, \beta>= 2 - k$ to be negative, i.e., $k \geqslant 3$. This means $2\beta + 3\alpha$ is also a root, and we have listed all positive roots of G_2 (cf. p. 44).

In the case of C_3, we have 3 simple roots, α and β (which are short), and γ (which is long). We immediately see that $\{\alpha+\beta, \gamma+\beta, \gamma+2\beta\}$ are roots. Since $<\alpha+\beta, \gamma>= -1$, we also get that $\alpha+\beta+\gamma$ is a root. Also, $<\gamma+2\beta, \alpha>= -2$, so $\gamma+2\beta+\alpha$ and $\gamma+2\beta+2\alpha$ are also roots. All other combinations of roots result in nonnegative brackets $<,>$, so these are all of the positive roots of C_3. To summarize, the positive roots are:

$$\{\alpha, \beta, \gamma, \alpha+\beta, \beta+\gamma, 2\beta+\gamma, \alpha+\beta+\gamma, \alpha+2\beta+\gamma, 2\alpha+2\beta+\gamma\}$$

□

Exercise 11.4. *Prove that the Weyl group of a root system Φ is isomorphic to the direct product of the respective Weyl groups of its irreducible components.*

Solution. By induction on the number of components, we need only show this in the case that Φ is partitioned into two orthogonal components Φ_1 and Φ_2. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be their respective Weyl groups, and let $\mathcal{W}$ be the Weyl group of Φ.

Then $\mathcal{W}_1$ and $\mathcal{W}_2$ are commuting subgroups of $\mathcal{W}$ because of the orthogonality condition. Since $\mathcal{W}$ is generated by reflections in both $\mathcal{W}_1$ and $\mathcal{W}_2$, and $\mathcal{W}_1 \cap \mathcal{W}_2 = 1$, we see that $\mathcal{W} = \mathcal{W}1 \times \mathcal{W}2$. $\square$

Exercise 11.5. *Prove that each irreducible root system is isomorphic to its dual, except that B_l, C_l are dual to each other.*

Solution. Since $< \beta^\vee, \alpha^\vee >=< \alpha, \beta >$ Exercise 9.2, the Cartan matrix of $\Phi^\vee$ is the transpose of the Cartan matrix of Φ. In terms of Dynkin diagrams, this corresponds to reversing the directions of the arrows. In the case of A_l, D_l, E_6, E_7 and E_8, nothing happens, so they are self-dual. In the cases of F_4 and G_2, one can find an isomorphism to their duals by reordering the simple roots. Finally, B_l and C_l become one another under this correspondence, so they are dual to one another. $\square$

Exercise 11.6. *Prove that an inclusion of one Dynkin diagram in another (e.g., E_6 in E_7 or E_7 in E_8) induces an inclusion of the corresponding root systems.*

Solution. An inclusion of Dynkin diagrams $D_1 \hookrightarrow D_2$ corresponds to the Cartan matrix of D_1 being a submatrix of the Cartan matrix of D_2. $\square$

12 Construction of root systems and automorphisms

Definition 12.1. Let $\Gamma = \{\sigma \in \operatorname{Aut}\Phi \mid \sigma(\Delta) = \Delta\}$, then $\operatorname{Aut}\Phi = \mathcal{W} \rtimes \Gamma$. This Γ is usually viewed as the group of **diagram automorphisms** or **graph automorphisms**.

Example 12.1 ($A_l(l \geqslant 1)$). $E = \operatorname{span}\{\varepsilon_1 + \cdots + \varepsilon_{l+1}\}^\perp \subset \mathbb{R}^{l+1}, I' = I \cap E$.
$\Phi = \{\alpha \in I' \mid (\alpha, \alpha) = 2\} = \{\varepsilon_i - \varepsilon_j, i \neq j\}$, $\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1}, 1 \leqslant i \leqslant l\}$.
Weyl group: $\mathcal{W} \cong \mathscr{S}_{l+1}$ by $\sigma_{\alpha_i} \mapsto (i, i+1)$. $\Gamma \cong \mathbb{Z}/2$ when $l \geqslant 2$.

Example 12.2 ($B_l(l \geqslant 2)$). $E = \mathbb{R}^l$. $\Phi = \{\alpha \in I \mid (\alpha, \alpha) = 1, 2\} = \{\pm\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j), i \neq j\}$,
$\Delta = \{\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_l\}$.
Weyl group: $\mathcal{W} \cong (\mathbb{Z}/2)^l \rtimes \mathscr{S}_l$ by corresponding σ_{ε_i} to sign changes. $\Gamma = 1$.

Example 12.3 ($C_l(l \geqslant 3)$). $E = \mathbb{R}^l$. C_l is dual to B_l, hence
$\Phi = \{\pm 2\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j), i \neq j\}$,
$\Delta = \{\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{l-1} - \varepsilon_l, 2\varepsilon_l\}$.
Weyl group: $\mathcal{W} \cong (\mathbb{Z}/2)^l \rtimes \mathscr{S}_l$ same as B_l. $\Gamma = 1$.

Example 12.4 ($D_l(l \geqslant 4)$). $E = \mathbb{R}^l$. $\Phi = \{\alpha \in I \mid (\alpha, \alpha) = 2\} = \{\pm(\varepsilon_i \pm \varepsilon_j), i \neq j\}$,
$\Delta = \{\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_{l-1} + \varepsilon_l\}$.
Weyl group: $\mathcal{W} \cong (\mathbb{Z}/2)^{l-1} \rtimes \mathscr{S}_l$ where $\sigma_{\varepsilon_i+\varepsilon_j}\sigma_{\varepsilon_i-\varepsilon_j}$ is corresponding to the sign change of i, j position.
$\Gamma = \mathscr{S}_3$ when $l = 4$, and $\mathbb{Z}/2$ when $l > 4$.

Example 12.5 ($E_6.E_7, E_8$)**.** It suffices to construct E_8.

$E = \mathbb{R}^8, I' = I + \mathbb{Z}((\varepsilon_1 + \cdots + \varepsilon_8)/2)$, I'' = subgroup of I' consisting of all elements $\sum c_i\varepsilon_i + \frac{c}{2}(\varepsilon_1 + \cdots + \varepsilon_8)$ for which $\sum c_i$ is an even integer.

$\Phi = \{\alpha \in I'' \mid (\alpha, \alpha) = 2\} = \{\pm(\varepsilon_i \pm \varepsilon_j), i \neq j\} \cup \{\frac{1}{2}\sum(-1)^{k_i}\varepsilon_i\}$ (where the $k_i = 0, 1$, add up to an even integer).

$\Delta = \{\frac{1}{2}(\varepsilon_1 + \varepsilon_8 - (\varepsilon_2 + \cdots + \varepsilon_7)), \varepsilon_1 + \varepsilon_2, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \varepsilon_4 - \varepsilon_3, \varepsilon_5 - \varepsilon_4, \varepsilon_6 - \varepsilon_5, \varepsilon_7 - \varepsilon_6\}$.

Weyl group has order $2^{14}3^5 5^2 7 = 696729600$.

Example 12.6 (F_4)**.** $E = \mathbb{R}^4, I' = I + \mathbb{Z}((\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2)$.

$\Phi = \{\alpha \in I' \mid (\alpha, \alpha) = 1, 2\} = \{\pm\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j), i \neq j\} \cup \{\pm\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$.

$\Delta = \{\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_4, \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\}$.

Weyl group has order 1152.

Example 12.7 (G_2)**.** $E = \text{span}\{\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}^\perp \subset \mathbb{R}^3, I' = I \cap E$.

$\Phi = \{\alpha \in I' \mid (\alpha, \alpha) = 2, 6\} = \pm\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_3, 2\varepsilon_1 - \varepsilon_2 - \varepsilon_3, 2\varepsilon_2 - \varepsilon_1 - \varepsilon_3, 2\varepsilon_3 - \varepsilon_1 - \varepsilon_2\}$.

$\Delta = \{\varepsilon_1 - \varepsilon_2, -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}$.

Weyl group: $\mathcal{W} \cong D_6$.

Exercise 12.1. *Verify the details of the constructions in (12.1).*

Exercise 12.2. *Verify Table 2.*

Type	*Long*	*Short*
A_l	$\alpha_1 + \alpha_2 + \cdots + \alpha_l$	
B_l	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_l$	$\alpha_1 + \alpha_2 + \cdots + \alpha_l$
C_l	$2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l$	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l$
D_l	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l$	
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	
E_8	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$	
F_4	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$
G_2	$3\alpha_1 + 2\alpha_2$	$2\alpha_1 + \alpha_2$

Exercise 12.3. *Let $\Phi \subset E$ satisfy (R1), (R3), (R4), but not (R2), cf. Exercise 9.9. Suppose moreover that Φ is irreducible, in the sense of Section 11. Prove that Φ is the union of root systems of type B_n, C_n in E ($n = \dim E$), where the long roots of B_n are also the short roots of C_n. (This is called the non-reduced root system of type BC_n in the literature.)*

Exercise 12.4. *Prove that the long roots in G_2 form a root system in E of type A_2.*

Solution. Let α be a short simple root of G_2, and let β be a long simple root. The long positive roots of G_2 are $\{\beta, 3\alpha + \beta, 3\alpha + 2\beta\}$ Exercise 11.3. It is clear from this description that the long roots form a root system, and that $\{\beta, -3\alpha - 2\beta\}$ forms a base. Using the Cartan matrix for G_2, one deduces that the Cartan matrix for this base is the same as that of A_2. □

Exercise 12.5. *In constructing C_l, would it be correct to characterize Φ as the set of all vectors in I of squared length 2 or 4? Explain.*

Solution. No, this would give vectors such as $\pm 4\varepsilon_i$. But ignoring that problem, one would also have vectors like $2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$. The resulting set of vectors would be much larger than Φ. $\square$

Exercise 12.6. *Prove that the map $\alpha \mapsto -\alpha$ is an automorphism of Φ. Try to decide for which irreducible Φ this belongs to the Weyl group.*

Solution. It is immediate that $\alpha \mapsto -\alpha$ is an automorphism of Φ since

$$< -\alpha, -\beta > = \frac{2(-\alpha, -\beta)}{(-\beta, -\beta)} = \frac{2(\alpha, \beta)}{(\beta, \beta)} = < \alpha, \beta > .$$

Since $\mathrm{Aut}\,\Phi$ is a semidirect product of $\mathcal{W}$ and the subgroup Γ, it is immediate that $\alpha \mapsto -\alpha$ is an element of $\mathcal{W}$ for B_l, C_l, E_7, E_8, F_4 and G_2 by Table 12.1.

For A_l, $\alpha \mapsto -\alpha$ is not an element of $\mathcal{W}$ if $l > 1$. To see this, we use the description of A_l as the set of vectors $\{\varepsilon_i - \varepsilon_j, i \neq j\}$ in $\mathbb{R}^{l+1}$. Then the reflections σ_{α_i} where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ permutes the vectors ε_i and ε_{i+1} and leaves the other standard vectors fixed. From this, it is clear that, for example, one cannot send $(\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3)$ to $(\varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_1)$ because this would require a permutation which swapped 1 with 2 and swapped 1 with 3. Of course it is clear that A_1 has $\alpha \mapsto -\alpha$ as an element of its Weyl group.

Similarly, D_l for $l \geqslant 4$ does not have $\alpha \mapsto -\alpha$ in its Weyl group for l odd, but does for l even. It can be described as the set of vectors $\{\pm\varepsilon_i \pm \varepsilon_j i \neq j\}$ in $\mathbb{R}^l$, and its Weyl group consists of permutations of the ε_i along the sign changes that involve an even number of sign changes.

Not sure about E_6. $\square$

Exercise 12.7. *Describe* $\mathrm{Aut}\,\Phi$ *when* Φ *is not irreducible*

Solution. Write $\Phi = \Phi_1 \cup \cdots \cup \Phi_r$ (disjoint) where the Φ_i are irreducible root systems and $(\Phi_i, \Phi_j) = 0$. Any automorphism σ of Φ must satisfy $\sigma(\Phi_i) \subset \Phi_j$ because irreducibility is an invariant of isomorphism of root systems, i.e., if $\sigma(\Phi_i)$ were contained in two or more components of Φ, then we could write it as a disjoint union of pairwise orthogonal sets. Since Φ is finite, it follows from a counting argument that $\sigma(\Phi_i) = \Phi_j$. Let S be the subgroup of permutations σ of $\{1, \cdots .r\}$ such that Φ_i is isomorphic to $\Phi_{\sigma(i)}$. Then $\mathrm{Aut}\,\Phi$ is the semidirect product of S with $\mathrm{Aut}\,\Phi_1 \times \cdots \times \mathrm{Aut}\,\Phi_r$. $\square$

13 Abstract theory of weights

Example 13.1 ($\mathfrak{sl}(3, F)$)**.** $h_1 = e_{11} - e_{22}, h_2 = e_{22} - e_{33}, e_{12}, e_{13}, e_{23}, e_{21}, e_{31}, e_{32}$.

$$\mathrm{ad}\, h_1 = \mathrm{diag}(0, 0, 2, 1, -1, -2, -1, 1), \mathrm{ad}\, h_2 = \mathrm{diag}(0, 0, -1, 1, 2, 1, -1, -2)$$

$$\alpha_1 = (2, -1), \alpha_1 + \alpha_2 = (1, 1), \alpha_2 = (-1, 2)$$

The Cartan matrix is $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

$$\alpha_1 = 2\lambda_1 - \lambda_2, \alpha_2 = -\lambda_1 + 2\lambda_2;$$
$$\lambda_1 = \frac{1}{3}(2\alpha_1 + \alpha_2), \lambda_2 = \frac{1}{3}(\alpha_1 + 2\alpha_2)$$

$\text{ord}(\lambda_1) = \text{ord}(\lambda_2) = 3$, the fundamental group is Cycle(3).

$$\delta = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_1 + \alpha_2) = (1, 1) = \lambda_1 + \lambda_2$$

The standard set with highest weight must be like the follow:

$$\Pi = \{m, m - 2, , \cdots , -m\}$$

with highest weight m.

Exercise 13.1. *Let $\Phi = \Phi_1 \cup \cdots \cup \Phi_t$ be the decomposition of Φ into its irreducible components, with $\Delta = \Delta_1 \cup \cdots \cup \Delta_t$. Prove that Λ decomposes into a direct sum $\Lambda_1 \oplus \cdots \oplus \Lambda_t$; what about Λ^+?*

Solution. Given any element $\lambda \in \Lambda$, let λ_i be defined by $< \lambda_i, \alpha >=< \lambda, \alpha >$ if $\alpha \in \Phi_i$ and 0 otherwise. In other words, let λ_i be the orthgonal projection of λ onto the subspace spanned by Φ_i. Then $\lambda = \lambda_1 + \cdots + \lambda_t$, so $\Lambda = \Lambda_1 + \cdots + \Lambda_t$. It is clear that $\Lambda_i \cap \Lambda_j = 0$if $i \neq j$, so the sum is direct.

However, we cannot say the same thing about Λ^+. For example, consider the root system $A_1 \times A_1$ with base $\{(1, 0), (0, 1)\}$. Write $A_1 \times A_1 = \Phi_1 \cup \Phi_2$ where $\Phi_1 = \{(\pm 1, 0)\}$ and $\Phi_2 = \{(0, \pm 1)\}$. Then $(2, -1) \in \Lambda^+$, but $(0, -1) \notin \Lambda_2^+$, so we do not have Λ^+ generated by Λ_1^+ and Λ_2^+. □

Exercise 13.2. *Show by example (e.g., for A_2) that $\lambda \notin \Lambda^+, \alpha \in \Delta, \lambda - \alpha \in \Lambda^+$ is possible.*

Solution. In A_2, $\lambda = 3\lambda_1 - \lambda_2 \notin \Lambda^+, \alpha = 2\lambda_1 - \lambda_2 \in \Delta, \lambda - \alpha = \lambda_1 \in \Lambda^+$. □

Exercise 13.3. *Verify some of the data in Table 1, e.g., for F_4.*

Exercise 13.4. *Using Table 1, show that the fundamental group of A_l is cyclic of order $l + 1$, while that of D_l is isomorphic to $\mathbb{Z}/4$ (l odd), or $\mathbb{Z}/2 \times \mathbb{Z}/2$ (l even). (It is easy to remember which is which, since $A_3 = D_3$.)*

Solution. The first is clear because the element λ_i listed for A_l in Table 1 does not have order smaller than $l + 1$. If it did, then $\frac{j(l-i+1)}{l+1}$ would be an integer for $j < l + 1$ which is not true for $i = 1$, for example. Since the determinant of the Cartan matrix for A_l is $l + 1$, we conclude that its fundamental group is $\mathbb{Z}/(l + 1)$.

The same considerations show that if l is even, then every element λ_i listed for D_l has order 2, whereas if l is odd, then λ_l has order 4. Hence the respective fundamental groups are $\mathbb{Z}/2 \times \mathbb{Z}/2$ for l even, and $\mathbb{Z}/4$ for l odd since the Cartan matrix for D_l has determinant 4. □

Exercise 13.5. *If Λ' is any subgroup of Λ which includes Λ_r, prove that Λ' is $\mathcal{W}-$invariant. Therefore, we obtain a homomorphism ϕ: $\text{Aut}\,\Phi/\mathcal{W} \to \text{Aut}(\Lambda/\Lambda_r)$. Prove that ϕ is injective, then deduce that $-1 \in \mathcal{W}$ if and only if $\Lambda_r \supset 2\Lambda$ (cf. Exercise 12.6). Show that $-1 \in \mathcal{W}$ for precisely the irreducible root systems A_1, B_l, C_l, D_l (l even), E_7, E_8, F_4, G_2.*

Solution. $\sigma_I \lambda_j = \lambda_j - \delta_{ij}\alpha_i$, hence $\sigma(\lambda + \Lambda_r) = \lambda + \Lambda_r$. Then Λ' is $\mathcal{W}-$invariant since $\Lambda' = \coprod_{\lambda} \lambda + \Lambda_r$.

$$\ker \phi = \{\sigma \mid \sigma(\lambda + \Lambda_r) = \lambda + \Lambda_r, \forall \lambda \in \Lambda\} \quad \text{Aut}\,\Phi/\mathcal{W} = \Gamma$$

□

Exercise 13.6. *Prove that the roots in Φ which are dominant weights are precisely the highest long root and (if two root lengths occur) the highest short root (cf. (10.4) and Exercise 10.11), when Φ is irreducible.*

Solution. By Lemma 10.4C, roots of same length are $\mathcal{W}-$conjugate to exactly one dominant weight. By Lemma 13.2A, the dominant weight is maximal among its $\mathcal{W}-$orbit. As $\mathcal{W}$ permutes the roots of same length, the conclusion follows. □

Exercise 13.7. *If $\varepsilon_1, \cdots, \varepsilon_l$ is an* **obtuse** *basis of the euclidean space E (i.e., all $(\varepsilon_i, \varepsilon_j) \leqslant 0$ for $i \neq j$), prove that the dual basis is* **acute** *(i.e., all $(\varepsilon_i^*, \varepsilon_j^*) \geqslant 0$ for $i \neq j$). [Reduce to the case $l = 2$.]*

Solution. In the case $l = 2$, this is obvious. Let's show this by induction on l. Then, for an euclidean space E with obtuse basis $\varepsilon_1, \cdots, \varepsilon_{l+1}$. Then let E' be the subspace spanned by $\varepsilon_1, \cdots, \varepsilon_l$. Let $\varepsilon_1', \cdots, \varepsilon_l'$ be the dual basis in E' which, by the conduction hypothesis, is acute. Then the dual basis of $\varepsilon_1, \cdots, \varepsilon_{l+1}$ satisfies

$$\varepsilon_i^* = \varepsilon_i' - (\varepsilon_i', \varepsilon_{l+1})\varepsilon_{l+1}^*, \qquad i = 1, \cdots, l.$$

Therefore $(\varepsilon_i^*, \varepsilon_j^*) = (\varepsilon_i', \varepsilon_j') \geqslant 0$ for $1 \leqslant i, j \leqslant l$. On the other hand, since ε_i' are linear combinations of $\varepsilon_1, \cdots, \varepsilon_l$ with nonnegative coefficients, we conclude that $(\varepsilon_i^*, \varepsilon_{l+1}^*) \geqslant 0$. Note that this proof actually works for more strict result. □

Exercise 13.8. *Let Φ be irreducible. Without using the data in Table 1, prove that each λ_i is of the form $\sum_j q_{ij}\alpha_j$, where all q_{ij} are positive rational numbers. [Deduce from Exercise 13.7 that all q_{ij} are nonnegative. From $(\lambda_i, \lambda_i) > 0$, show that $q_{ii} > 0$. Then show that if $q_{ij} > 0$ and $(\alpha_j, \alpha_k) < 0$, then $q_{ik} > 0$.]*

Solution. By Lemma 10.1, α_i form an obtuse basis. Therefore $(\lambda_i, \lambda_j) \geqslant 0$. Since

$$(\lambda_i, \lambda_j) = (\sum_k q_{ik}\alpha_k, \lambda_j) = \sum_k q_{ik}(\alpha_k, \lambda_j) = \frac{q_{ij}}{2}(\alpha_j, \alpha_j),$$

We have $q_{ij} \geqslant 0$. In particular, since $(\lambda_i, \lambda_i) > 0$, $q_{ii} > 0$. Since Φ is irreducible, the Coxeter graph is connected. Thus for any α_j, there exists some α_k such that $(\alpha_j, \alpha_k) < 0$. Follow a similar reasoning of Exercise 13.7, we see that all (λ_i, λ_j) are positive. □

Exercise 13.9. *Let $\lambda \in \Lambda^+$. Prove that $\sigma(\lambda + \delta) - \delta$ is dominant only for $\sigma = 1$.*

Solution. Since δ is strongly dominant, we have

$$\delta \succ \sigma(\delta), \qquad \delta \succ \sigma^{-1}(\delta),$$

and the equality holds only for $\sigma = 1$. Hence $(\sigma(\lambda + \delta) - \delta) \prec (\lambda + \delta - \sigma^{-1}(\delta))$ and the equality holds only for $\sigma = 1$. □

Exercise 13.10. *If $\lambda \in \Lambda^+$, prove that the set Π consisting of all dominant weights $\mu \prec \lambda$ and their $\mathcal{W}-$conjugates is saturated, as asserted in (13.4).*

Exercise 13.11. *Prove that each subset of Λ is contained in a unique smallest saturated set, which is finite if the subset in question is finite.*

Exercise 13.12. *For the root system of type A_2, write down the effect of each element of the Weyl group on each of λ_1, λ_2. Using this data, determine which weights belong to the saturated set having highest weight $\lambda_1 + 3\lambda_2$. Do the same for type G_2 and highest weight $\lambda_1 + 2\lambda_2$.*

Exercise 13.13. *Call $\lambda \in \Lambda^+$* **minimal** *if $\mu \in \Lambda^+, \mu \prec \lambda$ implies that $\mu = \lambda$. Show that each coset of Λ_r in Λ contains precisely one minimal Λ. Prove that λ is minimal if and only if the $\mathcal{W}-$orbit of λ is saturated (with highest weight λ), if and only if $\lambda \in \Lambda^+$ and $< \lambda, \alpha >= 0, 1, -1$ for all roots α. Determine (using Table 1) the nonzero minimal λ for each irreducible Φ, as follows:*

$$
\begin{aligned}
&A_l\colon \lambda_1, \cdots, \lambda_l \\
&B_l\colon \lambda_l \\
&C_l\colon \lambda_1 \\
&D_l\colon \lambda_1, \lambda_{l-1}, \lambda_l \\
&E_6\colon \lambda_1, \lambda_6 \\
&E_7\colon \lambda_7
\end{aligned}
$$

Chapter IV
Isomorphism and Conjugacy Theorem

14 Isomorphism theorem

Exercise 14.1. *Generalize Theorem 14.2 to the case: L semisimple.*

Solution. As the simple decompositions of semisimple Lie algebras compatible preserve maximal toral subalgebras and are compatible with the rreducible decompositions of root systems, then conclusion follows. □

Exercise 14.2. *Let $L = \mathfrak{sl}(2, F)$. If H, H' are any two maximal toral subalgebras of L, prove that there exists an automorphism of L mapping H onto H'.*

Solution. Any maximal toral subalgebra of $\mathfrak{sl}(2, F)$ has dimension 1. Therefore the corresponding root system has rank 1. Since all rank 1 root systems are isomorphic, the conclusion follows. □

Exercise 14.3. *Prove that the subspace M of $L \times L'$ introduced in the proof of Theorem 14.2 will actually equal D, if x and x' are chosen carefully.*

Exercise 14.4. *Let σ be as in Proposition 14.3. Is it necessarily true that $\sigma(x_\alpha) = -y_\alpha$ for nonsimple α, where $[x_\alpha, y_\alpha] = h_\alpha$?*

Exercise 14.5. *Consider the simple algebra $\mathfrak{sl}(3, F)$ of type A_2. Show that the subgroup of $\operatorname{Int} L$ generated by the automorphisms τ_α in (14.3) is strictly larger than the Weyl group (here $\mathscr{S}_3$). [View $\operatorname{Int} L$ as a matrix group and compute τ_α^2 explicitly.]*

Exercise 14.6. *Use Theorem 14.2 to construct a subgroup $\Gamma(L)$ of $\operatorname{Aut} L$ isomorphic to the group of all graph automorphisms (12.2) of Φ.*

Exercise 14.7. *For each classical algebra (1.2), show how to choose elements $h_\alpha \in H$ corresponding to a base of Φ (cf. Exercise 8.2).*

15 Cartan subalgebras

Cartan subalgebra will be abbreviated as CSA.

Exercise 15.1. *A semisimple element of $\mathfrak{sl}(n, F)$ is regular if and only if its eigenvalues are all distinct (i.e., if and only if its minimal and characteristic polynomials coincide).*

Exercise 15.2. *Let L be semisimple ($\operatorname{char} F = 0$). Deduce from Exercise 8.7 that the only solvable Engel subalgebras of L are the CSA's.*

Exercise 15.3. *Let L be semisimple ($\operatorname{char} F = 0$), $x \in L$ semisimple. Prove that x is regular if and only if x lies in exactly one CSA.*

Exercise 15.4. *Let H be a CSA of a Lie algebra L. Prove that H is maximal nilpotent, i.e., not properly included in any nilpotent subalgebra of L. Show that the converse is false.*

Exercise 15.5. *Show how to carry out the proof of Lemma A of (15.2) if the field F is only required to be of cardinality exceeding $\dim L$.*

Exercise 15.6. *Let L be semisimple ($\operatorname{char} F = 0$), L' a semisimple subalgebra. Prove that each CSA of L' lies in some CSA of L. [Cf. Exercise 6.9.]*

16 Conjugacy theorems

Exercise 16.1. *Prove that $\mathscr{E}(L)$ has order one if and only if L is nilpotent.*

Exercise 16.2. *Let L be semisimple, H a CSA, Δ a base of Φ. Prove that any subalgebra of L consisting of nilpotent elements, and maximal with respect to this property, is conjugate under $\mathscr{E}(L)$ to $N(\Delta)$, the derived algebra of $B(\Delta)$.*

Exercise 16.3. *Let Ψ be a set of roots which is* **closed root set***[closed] ($\alpha, \beta \in \Psi, \alpha+\beta \in \Phi$ implies $\alpha + \beta \in \Psi$) and satisfies $\Psi \cap -\Psi = \varnothing$. Prove that Ψ is included in the set of positive roots relative to some base of Φ. [Use Exercise 16.2.] (This exercise belongs to the theory of root systems, but is easier to do using Lie algebras.)*

Exercise 16.4. *How does the proof of Theorem 16.4 simplify in case $L = \mathfrak{sl}(2, F)$?*

Exercise 16.5. *Let L be semisimple. If a semisimple element of L is regular, then it lies in only finitely many Borel subalgebras. (The converse is also true, but harder to prove, and suggests a notion of "regular" for elements of L which are not necessarily semisimple.)*

Chapter V

Existence Theorem

17 Universal enveloping algebras

Exercise 17.1. *Prove that if* $\dim L < \infty$*, then* $\mathfrak{U}(L)$ *has no zero divisors. [Hint: Use the fact that the associated graded algebra* $\mathfrak{G}$ *is isomorphic to a polynomial algebra.]*

Solution. Let A be a filtered algebra and B its associated graded algebra ($B_i = A_i/A_{i-1}$). Then, if B is integral, then so is A. Otherwise, let $xy = 0$ in A with $x \in A_n \setminus A_{n-1}$, $y \in A_m \setminus A_{m-1}$ and $\bar{x}, \bar{y}$ their images in B. Then $\bar{x}\bar{y} = 0$. But B is integral, so either $\bar{x} = 0$ or $\bar{y} = 0$, which means either $x \in A_{n-1}$ or $y \in A_{m-1}$, a contradiction. In our case, $\mathfrak{G}$ is isomorphic to a polynomial algebra, hence integral. □

Exercise 17.2. *Let L be the two dimensional nonabelian Lie algebra (1.4), with* $[x, y] = x$*. Prove directly that* $i\colon L \to \mathfrak{U}(L)$ *is injective (i.e., that* $J \cap L = 0$*).*

Solution. In this case, J is generated by $x \otimes y - y \otimes x - x$. Therefore, $J \cap L = 0$. □

Exercise 17.3. *If* $x \in L$*, extend* $\operatorname{ad} x$ *to an endomorphism of* $\mathfrak{U}(L)$ *by defining* $\operatorname{ad} x(y) = xy - yx (y \in \mathfrak{U}(L))$*. If* $\dim L < \infty$*, prove that each element of* $\mathfrak{U}(L)$ *lies in a finite dimensional L-submodule. [If* $x, x_1, \cdots, x_m \in L$*, verify that*

$$\operatorname{ad} x(x_1 \cdots x_m) = \sum_{i=1}^{m} x_1 x_2 \cdots \operatorname{ad} x(x_i) \cdots x_m.$$

Solution. Direct computation shows

$$\begin{aligned}
\sum_{i=1}^{m} x_1 x_2 \cdots \operatorname{ad} x(x_i) \cdots x_m &= \sum_{i=1}^{m} x_1 x_2 \cdots (x x_i - x_i x) \cdots x_m \\
&= x x_1 x_2 \cdots x_m + \sum_{i=2}^{m} x_1 \cdots x x_i \cdots x_m \\
&\quad - \sum_{i=1}^{m-1} x_1 \cdots x_i x \cdots x_m - x_1 x_2 \cdots x_m x \\
&= x x_1 x_2 \cdots x_m - x_1 x_2 \cdots x_m x \\
&= \operatorname{ad} x(x_1 x_2 \cdots x_m).
\end{aligned}$$

□

Exercise 17.4. *If L is a free Lie algebra on a set X, prove that* $\mathfrak{U}(L)$ *is isomorphic to the tensor algebra on a vector space having X as basis.*

Solution. Let V be the vector space spanned by X and $\mathfrak{T}(V)$ the tensor algebra of V. Consider the following diagram where v, l, i, j are the canonical inclusions and other are constructed as follows.

$$\begin{tikzcd}
X \arrow[r, "l"] \arrow[d, "v"'] & L \arrow[r, "i"] \arrow[d, "\lambda"] & \mathfrak{U}(L) \arrow[dl, "\phi"'] \\
V \arrow[ur, "\tau"] \arrow[r, "j"'] & \mathfrak{T}(V) \arrow[ur, bend right, "\psi"'] &
\end{tikzcd}$$

1. By the universal property of V, there exists a unique linear map $\tau\colon V \to L$ such that $\tau \circ v = l$;

2. By the universal property of L, there exists a unique Lie algebra homomorphism $\lambda\colon L \to \mathfrak{T}(V)$ such that $\lambda \circ \tau = j$;

3. By the universal property of $\mathfrak{U}(L)$, there exists a unique algebra homomorphism $\phi\colon \mathfrak{U}(L) \to \mathfrak{T}(V)$ such that $\phi \circ i = \lambda$;

4. By the universal property of $\mathfrak{T}(V)$, there exists a unique algebra homomorphism $\psi\colon \mathfrak{T}(V) \to \mathfrak{U}(L)$ such that $\psi \circ j = i \circ \tau$.

By the universal property of $\mathfrak{U}(L)$ and $\mathfrak{T}(V)$, one can see that $\phi \circ \psi = \mathrm{id}$ and $\psi \circ \phi = \mathrm{id}$. Therefore, they are isomorphic. □

Exercise 17.5. *Describe the free Lie algebra on a set $X = \{x\}$.*

Solution. Let V be the vector space spanned by X and $\mathfrak{T}(V)$ the tensor algebra of V. In this case, we have $\mathfrak{T}(V) \cong F[x]$. The free Lie algebra L generated by X is the Lie subalgebra of $\mathfrak{T}(V)$ generated by X. Thus $L = V$ equipped with trivial bracket. □

Exercise 17.6. *How is the PBW theorem used in the construction of free Lie algebras?*

Solution. Let L be the free Lie algebra generated by X. Let M be an arbitrary Lie algebra and $f\colon X \to M$ a map. Then consider the following diagram.

$$\begin{tikzcd}
X \arrow[r, "l"] \arrow[dr, "f"'] & L \arrow[r, "\lambda"] \arrow[d, dashed, "f'"] & \mathfrak{T}(V) \arrow[d, "g"] \\
& M \arrow[r, "i"] & \mathfrak{U}(M)
\end{tikzcd}$$

By the universal property of $\mathfrak{T}(V)$, there exists a unique algebra homomorphism g making the diagram commute. Restrict g to L, we get a Lie algebra homomorphism f' fitting the commutative diagram. Since $\lambda\colon L \to \mathfrak{T}(V)$ is injective, to guarantee the uniqueness of f', one need the fact that $i\colon M \to \mathfrak{U}(M)$ is injective, which follows from PBW. □

18 Generators and relations

Exercise 18.1. *Using the representation of L_0 on V (Proposition 18.2), prove that the algebras X, Y described in Theorem 18.2 are (respectively) free Lie algebras on the sets of x_i, y_i.*

Solution. For Y: the restriction of the action of L_0 on V to Y is isomorphic to its left multiplication. In this way, we can identify Y as the Lie subalgebra of the tensor algebra V, which is generated by $v_1, \cdots, v_l$. Thus Y is the free Lie algebra on $y_1, \cdots, y_l$.

By the duality of X and Y, the conclusion follows. □

Exercise 18.2. *When* $\operatorname{rank}\Phi = 1$*, the relations $(S_{ij}^+), (S_{ij})$ are vacuous, so $L_0 = L \cong \mathfrak{sl}(2, F)$.By suitably modifying the basis of V in (18.2), show that V is isomorphic to the module $Z(0)$ constructed in Exercise 7.7.*

Solution. In this case, $V = F[v]$. So, we put $w_i = \frac{1}{i!}v^i$. Then these w_i form a basis of V. Direct computation shows:

$$\begin{aligned}
h.w_i &= \frac{1}{i!}h.v^i = \frac{-2i}{i!}v^i = -2iw_i, \\
y.w_i &= \frac{1}{i!}y.v^i = \frac{1}{i!}v^{i+1} = (i+1)w_{i+1}, \\
x.w_i &= \frac{1}{i!}x.v^i = \frac{1}{i!}(vx.v^{i-1} - 2(i-1)v^{i-1}) \\
&= \frac{-i(i-1)}{i!}v^{i-1} = -(i-1)w_{i-1}.
\end{aligned}$$

Therefore $v^i \mapsto w_i$ gives the isomorphism $V \cong Z(0)$. □

Exercise 18.3. *Prove that the ideal K of L_0 in (18.3) lies in every ideal of L_0 having finite codimension (i.e., L is the largest finite dimensional quotient of L_0).*

Solution. Let I be an ideal of L_0 having finite codimension. If I doesn't contain some x_{ij}. Then, by apply $\operatorname{ad} x_i$ to the image of x_j in L_0/I, we get an infinite dimensional space, which is a contradiction. □

Exercise 18.4. *Prove that each inclusion of Dynkin diagrams (e.g. $E_6 \subset E_7 \subset E_8$) induces a natural inclusion of the corresponding semisimple Lie algebras.*

Solution. By Exercise 11.6, each inclusion of Dynkin diagrams induces an inclusion of the corresponding root systems $f\colon \Phi \to \Phi'$ and hence an isomorphism from Φ to its image. Then, the Serre theorem tells that this induces an isomorphism from the semisimple Lie algebra corresponding to Φ to that corresponding to $f(\Phi)$, which is a subalgebra of the semisimple Lie algebra corresponding to Φ'. This gives an inclusion of Lie algebras. □

19 The simple algebras

Chapter VI

Representation Theory

20 Weights and maximal vectors

Exercise 20.1. *If V is an arbitrary $L-$module, then the sum of its weight spaces is direct.*

Solution. Let λ, μ be two distinct weights and $v \in V_\lambda \cap V_\mu$. Then, for all $h \in H$, we have $h.v = \lambda(h)v = \mu(h)v$. Since $\lambda \neq \mu$, there exists some $h \in H$ such that $\lambda(h) \neq \mu(h)$. Thus $v = 0$. $\square$

Exercise 20.2. *(a) If V is an irreducible $L-$module having at least one (nonzero) weight space, prove that V is the direct sum of its weight spaces.*

(b) Let V be an irreducible $L-$module. Then V has a (nonzero) weight space if and only if $\mathfrak{U}(H).v$ is finite dimensional for all $v \in V$, or if and only if $\mathfrak{A}.v$ is finite dimensional for all $v \in V$ (where $\mathfrak{A}$ = subalgebra with 1 generated by an arbitrary $h \in H$ in $\mathfrak{U}(H)$).

(c) Let $L = \mathfrak{sl}(2, F)$, with standard basis (x, y, h). Show that $1-x$ is not invertible in $\mathfrak{U}(L)$, hence lies in a maximal left ideal I of $\mathfrak{U}(L)$. Set $V = \mathfrak{U}(L)/I$, so V is an irreducible $L-$module. Prove that the images of $1, h, h^2, \cdots$ are all linearly independent in V (so $\dim V = \infty$), using the fact that

$$(x-1)^r h^s \equiv \begin{cases} 0 \bmod I, & r > s \\ (-2)^r r! \cdot 1 \bmod I, & r = s. \end{cases}$$

Conclude that V has no (nonzero) weight space.

Solution. (a) follows from Lemma 20.1(b).

(b): Let $v \in V$ and write $v = \sum v_\lambda$ where λ are some weights and $v_\lambda \in V_\lambda$. Then $\mathfrak{A}.v \subset \bigoplus V_\lambda$ and hence is finite dimensional.

Conversely, if $\mathfrak{A}.v$ is finite dimensional for all $v \in V$, then there exists n such that $v, h.v, \cdots, h^{n-1}.v$ are linearly independent while $v, h.v, \cdots, h^n.v$ are not. If $n = 1$, then v is an eigenvector of h and hence H and therefore V has a (nonzero) weight space. If $n > 1$, let $f(x)$ be the monic polynomial of degree n such that $f(h).v = 0$ and α a root of $f(x)$. Write $f(x) = (x-\alpha)g(x)$ and let $v' = h.v - \alpha v$. Then we have

$$g(h).v' = g(h).((h-\alpha).v) = f(h).v = 0.$$

Repeat this process, we sill finally find a v_0 such that $v_0, h.v_0$ are linearly dependent. As we have shown this implies the existence of (nonzero) weight space. $\square$

Exercise 20.3. *Describe weights and maximal vectors for the natural representations of the linear Lie algebras of types $A_l - D_l$ described in (1.2).*

Exercise 20.4. *Let $L = \mathfrak{sl}(2, F), \lambda \in H^*$. Prove that the module $Z(\lambda)$ for $\lambda = \lambda(h)$ constructed in Exercise 7.7 is isomorphic to the module $Z(\lambda)$ constructed in (20.3). Deduce that $\dim V(\lambda) < \infty$ if and only if $\lambda(h)$ is a nonnegative integer.*

Solution. Let $w_i = \frac{y^i}{i!} \otimes v^+$. Then direct computation shows

$$\begin{aligned}
yw_i &= (i+1)w_{i+1};\\
hw_i &= \frac{hy^i}{i!} \otimes v^+\\
&= \frac{([h,y]+yh)y^{i-1}}{i!} \otimes v^+\\
&= \frac{1}{i!}(-2y^i \otimes v^+ + y(hy^{i-1} \otimes v^+))\\
&= \frac{1}{i!}(-2y^i \otimes v^+ + (i-1)!yhw_{i-1})\\
&= \frac{1}{i!}(-2y^i \otimes v^+ + y(-2y^{i-1} \otimes v^+ + (i-2)!yhw_{i-2}))\\
\cdots & \quad \cdots\\
&= \frac{1}{i!}(-2iy^i \otimes v^+ + y^i h \otimes v^+)\\
&= \frac{1}{i!}(-2iy^i \otimes v^+ + y^i \otimes \lambda v^+)\\
&= (\lambda - 2i)w_i;\\
xw_i &= \frac{xy^i}{i!} \otimes v^+\\
&= \frac{([x,y]+yx)y^{i-1}}{i!} \otimes v^+\\
&= \frac{1}{i}(hw_{i-1} + yxw_{i-1})\\
&= \frac{1}{i}((\lambda - 2i + 2)w_{i-1} + y\frac{1}{i-1}(hw_{i-2} + yxw_{i-2}))\\
&= \cdots \quad \cdots\\
&= \frac{1}{i}((\lambda - 2i + 2)w_{i-1} + (\lambda - 2(i-2))w_{i-1} + \cdots + \lambda w_{i-1} + \frac{1}{(i-1)!}y^i xw_0)\\
&= (\lambda - i + 1)w_{i-1}.
\end{aligned}$$

Therefore $\frac{y^i}{i!} \otimes v^+ \mapsto w_i$ gives the required isomorphism. □

Exercise 20.5. *If $\mu \in H^*$, define $\mathcal{P}(\mu)$ to be the number of distinct sets of nonnegative integers $k_\alpha (\alpha \succ 0)$ for which $\mu = \sum_{\alpha \succ 0} k_\alpha \alpha$. Prove that* $\dim Z(\lambda)_\mu = \mathcal{P}(\lambda - \mu)$, *by describing a basis for $Z(\lambda)_\mu$.*

Solution. Let $\lambda - \mu = \sum_{\alpha \succ 0} k_\alpha \alpha$. By Theorem 20.2(b), $(\prod_{\alpha \succ 0} y_\alpha^{k_\alpha}).v^+$ spans $Z(\lambda)_\mu$. By PBW, they are linearly independent. □

Exercise 20.6. *Prove that the left ideal $I(\lambda)$ introduced in (20.3) is already generated by the elements $x_\alpha, h_\alpha - \lambda(h_\alpha).1$ for α simple.*

Solution. By Serre relations, we can construct all $x_\alpha, h_\alpha - \lambda(h_\alpha).1$ from those for simple α. □

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