Problem Sets

D. Zack Garza

Tuesday 5^{th} May, 2020

Contents

1 1.1

1.1 a

If $M \in \mathcal{O}$ and $[\lambda] = \lambda + \Lambda_r$ is any coset of $\mathfrak{h}^\vee/\Lambda_r$, let $M^{[\lambda]}$ be the sum of weight spaces M_μ for which $\mu \in [\lambda]$. Prove that $M^{[\lambda]}$ is a $U(\mathfrak{g})$-submodule of M and that M is the direct sum of finitely many such submodules.

1.2 b

Deduce that the weights of an indecomposable module $M \in \mathcal{O}$ lie in a single coset of $\mathfrak{h}^\vee/\Lambda_r$.

2 1.3*

Show that $M(\lambda)$ has the following property: for any $M \in \mathcal{O}$,

$$\operatorname{Hom}_{U(\mathfrak{g})}(M(\lambda), M) = \operatorname{Hom}_{U(\mathfrak{g})}\left(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda, M\right) \cong \operatorname{Hom}_{U(\mathfrak{b})}\left(\mathbb{C}_\lambda, \operatorname{Res}_{\mathfrak{b}}^{\mathfrak{g}} M\right),$$

where $\operatorname{Res}_{\mathfrak{b}}^{\mathfrak{g}}$ is the restriction functor.

Hint: use the universal mapping property of tensor products.

3 Relevant information (?):

3.1 1

- $\mathfrak{h} \leq \mathfrak{g}$ is the Cartan subalgebra.
  - In finite-dimensional setting: maximal toral
  - Nilpotent subalgebra, i.e. LCS terminates, i.e. $\text{ad}_h = [h, \cdot]$ is a nilpotent operator so $\text{ad}_h^n = 0$ for some n.
  - Self-normalizing, so for a fixed y, $[h, y] \in \mathfrak{h}\ \forall h \in \mathfrak{h} \implies y \in \mathfrak{h}$.
- $\lambda \in \mathfrak{h}^\vee$ is a linear functional $\lambda : \mathfrak{h} \longrightarrow \mathbb{C}$
  - λ is a root relative to $\mathfrak{h}$ if $\lambda \neq 0$ and there is some $g \in \mathfrak{g}$ such that $[hg] = \lambda(h)g$ for all $h \in \mathfrak{h}$.
- $\Phi \subset \mathfrak{h}^\vee$ is the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$.
  - Each $\lambda \in \Phi$ is a root
  - Each root λ has a corresponding root space $\mathfrak{g}_\alpha \coloneqq \left\{ x \in \mathfrak{g} \;\middle|\; [hx] = \lambda(h)x\ \forall h \in \mathfrak{h} \right\}$.
- $\Lambda_r \coloneqq \text{span}_\mathbb{Z} \{\lambda \in \Phi\} \subset \mathbb{C}^n$ is the root lattice.
- $M_\mu \coloneqq \left\{ v \in M \;\middle|\; h \cdot v = \mu(h)v\ \forall h \in \mathfrak{h} \right\}$ is the weight space for μ.
- $M^{[\lambda]} = \sum_{\mu \in [\lambda]} M_\mu$

$M \in \mathcal{O} \implies$

- M is finitely generated as a $U(\mathfrak{g})$-module.
- M is a weight module, so $M = \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda$
- For every $v \in M$, $U(\mathfrak{n}) \cdot v$ is finite-dimensional

3.2 2

$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ where $\mathfrak{b} \leq \mathfrak{g}$ is a fixed Borel subalgebra corresponding to a choice of positive roots, and C_λ is the 1-dimensional $\mathfrak{b}$-module defined for any $\lambda \in \mathfrak{h}^\vee$ by the fact that $\mathfrak{b}/\mathfrak{n} \cong \mathfrak{h}$ and thus $\mathfrak{n} \curvearrowright \mathfrak{h}$ can be taken to be a trivial action. The induction functor is given by $\text{Ind}_\mathfrak{b}^\mathfrak{g}(\,\cdot\,) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\,\cdot\,)$.

The restriction functor is given by $\text{Res}_\mathfrak{b}^\mathfrak{g}(\,\cdot\,) = ?$

Frobenius Reciprocity for groups looks like

$$\begin{aligned}
\hom_{k[G]}(k[G] \otimes_{k[H]} V, W) &\longrightarrow \hom_{k[H]}(V, W) \\
\lambda &\mapsto 1 \otimes (\,\cdot\,) = (v \mapsto \lambda(1 \otimes v)) \\
(g \otimes v \mapsto g \cdot f(v)) &\mapsfrom f.
\end{aligned}$$