Problem Set 1

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Contents

1 Problem 1

1.1 a

On the real plane: a circle of radius 1 centered at $(1,0)$.

1.2 b

Let $z = x + iy$. Then

$$
\begin{aligned}
|z-1| = 2|z-2| &\iff |z-1|^2 = 4|z-2|^2 \\
&\iff (x-1)^2 - y^2 = 4((x-2)^2 - y^2) \\
&\iff x^2 - \frac{14}{3}x - y^2 = -5 \\
&\iff \left(x - \frac{14}{6}\right) - y^2 = -5 + \left(\frac{14}{6}\right)^2 = \frac{4}{9} \\
&\iff \left(\frac{x - 14/6}{2/3}\right)^2 - \left(\frac{y}{2/3}\right)^2 = 1,
\end{aligned}
$$

which describes a horizontally shifted hyperbola.

1.3 c.

Equivalently, $z\overline{z} = 1 = |z|^2$, so this is the circle $S^1 = \left\{z \in \mathbb{C} \;\middle|\; |z| = 1\right\}$.

1.4 d.

On the real plane: A vertical line passing through $(3,0)$ and $(3,t)$ for every $t \in \mathbb{R}$.

1.5 e.

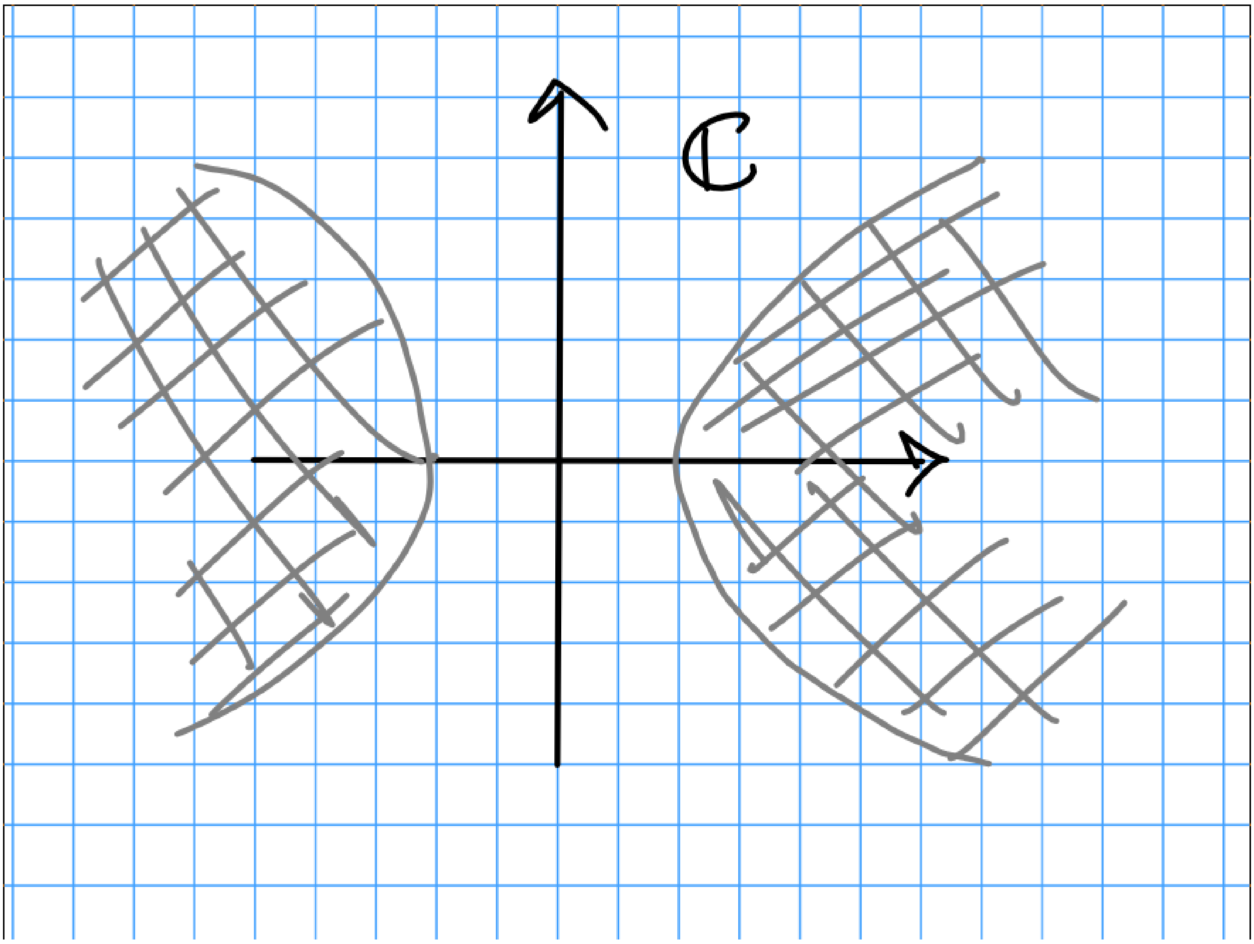
On the real plane: A horizontal line passing through $(0,a)$ and (t,a) for every $t \in \mathbb{R}$.

1.6 f.

On the real plane: A right half-plane $H = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; x \geq a, y \in \mathbb{R}\right\}$.

1.7 g.

The two regions "inside" the branches of the hyperbola given in b, i.e.



2 Problem 2

As in the proof of Cauchy-Schwarz, we have

$$|z-w|^2 = |z|^2 + |w|^2 - 2|\overline{z}w| \leq |v|^2 + |w|^2 - 2|v||u| = (|u| + |v|)^2,$$

with equality precisely when $\overline{z}w = |z||w|$ and $z = \lambda w$ for $\lambda \in \mathbb{C}^\times$.

We can check that the additional condition of $\lambda > 0$ is necessary. Letting $w = \lambda z$, we have

$$\begin{aligned} |z+w| &= |z + \lambda z| = |1+\lambda||z| \\ ||z| - |w|| &= ||z| - |\lambda z|| = |1 - |\lambda|||z| \\ &\implies \lambda = |\lambda|. \end{aligned}$$

3 Problem 3

By part 2, we have

$$|z| \leq 1 \implies |f(z)| = \left|z^3 + 2z + 4\right| \geq |z|^3 + 2|z| + 4 \geq 6,$$

so $f(z) = 0$ is not possible for any z in the unit disk.

4 Problem 4

4.1 a

Let w_1, w_2 be fixed and let $c > 0$ be the constant such that $|w_1| = c|w_2|$.

Noting that $|w_1|^2 = c^2|w_2|^2$, we then have

$$\begin{aligned}
\left|w_1 - c^2 w_2\right|^2 &= \left(w_1 - c^2 w_2\right)\overline{(w_1 - c^2 w_2)} \\
&= |w_1|^2 + \left|c^2 w_2\right|^2 - 2\Re(w_1 c^2 \overline{w}_2) \\
&= |w_1|^2 + c^4|w_2|^2 - 2c^2\Re(w_1\overline{w}_2) \\
&= c^2|w_2|^2 + c^4|w_2|^2 - 2c^2\Re(w_1\overline{w}_2) \\
&= c^2\left(|w_2|^2 + c^2|w_2|^2 - 2\Re(w_1\overline{w}_2)\right) \\
&= c^2\left(|w_2|^2 + |w_1|^2 - 2\Re(w_1\overline{w}_2)\right) \\
&= c^2|w_1 - w_2|^2,
\end{aligned}$$

and taking square roots yields the desired inequality.

4.2 b

By letting $w_1 = z - z_1$ and $w_2 = z - z_2$, we can use part (a) to write

$$\begin{aligned}
&\left|(z - z_1) - c^2(z - z_2)\right| = c|(z - z_1) - (z - z_2)| \\
\implies &\left|(1 - c^2)z - (z_2 - c^2 z_2)\right| = c|z_2 - z_1| \\
\implies &\left|z - \frac{z_1 - c^2 z_2}{1 - c^2}\right| = c\left|\frac{z_2 - z_1}{1 - c^2}\right| \\
\implies &|z - z_3| = r_3,
\end{aligned}$$

which describes a circle of radius r_3 centered at z_3 as defined above. (Note that we've used the fact that $c \neq 1$ to divide $c^2 - 1$.)

5 Problem 5

5.1 a

$$\begin{aligned} 0 < (1-|w|^2)(1-|z|^2) = 1 + |w|^2|z|^2 - |w|^2 - |z|^2 \\ \implies |w|^2 + |z|^2 < 1 + |w|^2|z|^2 \\ \implies |w|^2 + |z|^2 - 2\Re(\overline{w}z) < 1 + |w|^2|z|^2 - 2\Re(\overline{w}z) \\ \implies |w-z|^2 < |1-\overline{w}z|^2, \end{aligned}$$

and taking square roots yields the desired inequality.

If $|w| = |z| = 1$, we have

$$\begin{aligned} |w-z|^2 - |1-\overline{w}z|^2 &= \Big(|w|^2 + |z|^2 - 2\Re(\overline{w}z)\Big) - \Big(|1|^2 + |\overline{w}z|^2 - 2\Re(1 \cdot \overline{w}z)\Big) \\ &= |w|^2 + |z|^2 - 2\Re(\overline{w}z) - 1 - |\overline{w}|^2|z|^2 + 2\Re(\overline{w}z) \\ &= |w|^2 + |z|^2 - 1 - |w|^2|z|^2 \\ &= \begin{cases} 1 + |z|^2 - 1 - |z|^2 = 0 & \text{if } |w| = 1 \\ 1 + |w|^2 - 1 - |w|^2 = 0 & \text{if } |z| = 1 \end{cases}, \end{aligned}$$

thus the original two terms are equal and their ratio is 1.

5.2 b

5.2.1 1

By part (a), if $z \in \mathbb{Q}$ then $|z| \leq 1$ and thus $|F(z)| = \left|\dfrac{w-z}{1-\overline{w}z}\right|$ł1 and thus $F(z) \in \mathbb{Q}$.

5.2.2 2

We have

$$\begin{aligned} F(0) &= \frac{w-0}{1-\overline{w}0} = w \\ F(w) &= \frac{w-w}{1-\overline{w}w} = 0. \end{aligned}$$

5.2.3 3

If $|z| = 1$, then part (a) applies and $|F(z)| = 1$.

5.2.4 4

We have

$$
\begin{aligned}
(F \circ F)(z) &= \frac{w - F(z)}{1 - \overline{w}F(z)} \\
&= \frac{w - \frac{w-z}{1-\overline{w}z}}{1 - \overline{w}\frac{w-z}{1-\overline{w}z}} \\
&= \frac{w(1-\overline{w}z) - (w-z)}{(1-\overline{w}z) - \overline{w}(w-z)} \\
&= \frac{w - w\overline{w}z - w + z}{1 - \overline{w}z - \overline{w}w + \overline{w}z} \\
&= \frac{z + w\overline{w}z}{1 - w\overline{w}} \\
&= \frac{z(1-|w|^2)}{1-|w|^2} \\
&= z,
\end{aligned}
$$

so F is an involution and thus F is invertible with $F^{-1} = F$ and F is a bijection.

6 Problem 6

Let $\zeta_n := e^{2\pi i/n}$ denote a primitive nth root of unity, and

$$
\Phi_n(x) = \prod_{j=1}^{n}(x - \zeta_n^j) = \frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \cdots + x + 1
$$

denote the nth cyclotomic polynomial.

Noting that

- $\Phi_n(1) = n$,
- $\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$, and
- $\sum_{j=1}^{m} j = \frac{m(m-1)}{2}$,

we have

$$\begin{aligned}
\prod_{j=1}^{n-1} \sin\left(\frac{j\pi}{n}\right) &= \prod_{j=1}^{n-1} \frac{1}{2i}\left(e^{ij\pi/n} - e^{-ij\pi/n}\right) \\
&= \left(\frac{1}{2i}\right)^{n-1} \prod_{j=1}^{n-1} e^{ij\pi/n} \prod_{j=1}^{n-1}\left(1 - e^{-2ij\pi/n}\right) \\
&= \left(\frac{1}{2i}\right)^{n-1} \prod_{j=1}^{n-1} e^{ij\pi/n} \prod_{j=1}^{n-1}\left(1 - \zeta_n^j\right) \\
&= \left(\frac{1}{2i}\right)^{n-1} \exp\left(\sum_{j=1}^{n-1} \frac{ij\pi}{n}\right) \Phi_n(1) \\
&= \left(\frac{1}{2i}\right)^{n-1} \exp\left(\frac{i\pi}{n}\sum_{j=1}^{n-1} j\right) \Phi_n(1) \\
&= \left(\frac{1}{2i}\right)^{n-1} \exp\left(\frac{(n-1)i\pi}{2}\right) \Phi_n(1) \\
&= \left(\frac{1}{2i}\right)^{n-1} \left(e^{i\pi/2}\right)^{n-1} \Phi_n(1) \\
&= \left(\frac{1}{2i}\right)^{n-1} i^{n-1} \Phi_n(1) \\
&= \left(\frac{1}{2}\right)^{n-1} \Phi_n(1) \\
&= \frac{n}{2^{n-1}}.
\end{aligned}$$

■

7 Problem 7

Write $f(z) = |z|^2 = z\overline{z}$, then decompose $f(z) = h(z)g(z)$ where $h(z) = z$ and $g(z) = \overline{z}$. We can then apply the product rule:

$$f'(z) = (h(z)g(z))' = h'(z)g(z) + g'(z)h(z),$$

however, $g(z) = \overline{z}$ is not complex-differentiable, so $g'(z)$ and thus $f'(z)$ do not exist.

8 Problem 8

Note that if $f = u + iv$ is analytic, then f satisfies the Cauchy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$.

8.1 a

Supposing $|f| = c_0$, we have $u^2(x, y) + v^2(x, y) = c_0$ for every $z = x + iy$ in the domain of f. (Note: we'll immediately drop the (x, y) from the notation and just write u, v.)

Differentiating with respect to x, we obtain

$$2uu_x + 2vv_x = 0 \implies uu_x + vv_x = 0.$$

Differentiating with respect to y yields

$$2uu_y + 2vv_x = 0 \implies uu_y + vv_x = 0.$$

First consider v. Substituting in the relations from Cauchy-Riemann to collect terms yields the system of equations

$$\begin{aligned} uv_y + vv_x &= 0 \\ -uv_x + vv_y &= 0. \end{aligned}$$

We can rewrite this as the matrix equation

$$\begin{bmatrix} u & v \\ v & -u \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The determinant of this matrix is $-u^2 - v^2 = -(u^2 + v^2) = c_0^2$ by assumption, which is nonzero, and thus this homogeneous system has only the trivial solution $v_x = v_y = 0$. Thus $v(x, y)$ is a constant function.

A nearly identical argument shows that $u(x, y)$ is constant, and thus $f(z) = u + iv$ is constant as well.

8.2 b

Again writing $f = u + iv$, if u is constant then $0 = u_x = v_y$ and $0 = u_y = -v_x$, so v is constant and thus f is constant.

8.3 c

Suppose $\arg f = c_0$ for some constant, then writing $f = u + iv$ we have $\arg f = \tan^{-1}\left(\frac{u}{v}\right) = c_0$, so $\frac{u}{v} = \tan(c_0) := C$ for some other constant, and thus $u = Cv$.

First taking partial derivatives yields

$$
\begin{aligned}
u_x &= Cv_x \\
u_y &= Cv_y.
\end{aligned}
$$

Substituting in Cauchy-Riemann yields the system

$$
\begin{aligned}
v_y = Cv_x &\implies -Cv_x + v_y = 0 \\
-v_x = Cv_y &\implies v_x + Cv_y = 0,
\end{aligned}
$$

which can be written

$$
\begin{bmatrix} -C & 1 \\ 1 & C \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \mathbf{0},
$$

where the relevant determinant is $-C^2 - 1 = -(C^2 + 1) \neq 0$, which forces $v_x = v_y = 0$ and thus v is constant. A similar argument shows u is constant, so f itself is constant.

8.4 d

If $f = u + iv$, then $\overline{f} = u - iv$. If $\overline{f}$ is analytic, then $u_x = -v_y$ and $u_y = v_x$; but then applying Cauchy-Riemann to the original f yields e.g. $u_x = v_y$ and thus $v_y = -v_y$. This forces $v_y = 0$ and similarly $v_x = 0$, so v is constant. The same argument works for u, making f constant.

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9 Problem 9

Let $g(z) = \overline{f\overline{z}}$; then if we write $f(x, y) = u(x, y) + iv(x, y)$ we have

$$
g(x, y) = u(x, -y) - iv(x, -y) := a(x, y) + ib(x, y)
$$

where we take

$$
\begin{aligned}
a(x, y) = u(x, -y) &\implies a_x = u_x, \quad a_y = -u_y \\
b(x, y) = -v(x, -y) &\implies b_x = -v_x, \quad b_y = -(-v_y) = v_y.
\end{aligned}
$$

By assumption, Cauchy-Riemann holds for f, so $u_x = v_y$ and $u_y = -v_x$, so

$$
\begin{aligned}
a_x &= u_x = v_y = b_y \\
a_y &= -u_y = v_x = -b_x,
\end{aligned}
$$

which are exactly the Cauchy-Riemann equations for g. Thus g is analytic.

10 Problem 10

10.1 a

We first write $z = re^{i\theta}$ and $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ and compute $f'(z) = f'(re^{i\theta})$ in two ways: first holding θ constant, and then holding r constant.

Holding θ constant yields

$$\begin{aligned} f'(re^{i\theta}) &= \lim_{r\to r_0} \frac{f(r_0e^{i\theta}) - f(re^{i\theta})}{r_0e^{i\theta} - re^{i\theta}} \\ &= \lim_{r\to r_0} \frac{1}{e^{i\theta}} \frac{u(r,\theta) - u(r_0,\theta) + i(v(r,\theta) - v(r,\theta_0))}{r_0 - r} \\ &= e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right). \end{aligned}$$

Similarly, holding r constant yields

$$\begin{aligned} f'(re^{i\theta}) &= \lim_{\theta\to\theta_0} \frac{f(re^{i\theta}) - f(re^{i\theta_0})}{re^{i\theta} - re^{i\theta_0}} \\ &= \lim_{\theta\to\theta_0} \frac{1}{r} \frac{u(r,\theta) - u(r,\theta_0) + i(v(r,\theta) - v(r,\theta_0))}{e^{i\theta} - e^{i\theta_0}} \\ &= \lim_{\theta\to\theta_0} \frac{1}{r} \frac{u(r,\theta) - u(r,\theta_0) + i(v(r,\theta) - v(r,\theta_0))}{\theta - \theta_0}\left(\frac{\theta-\theta_0}{e^{i\theta} - e^{i\theta_0}}\right) \\ &= \frac{1}{r}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right) \lim_{\theta\to\theta_0}\left(\frac{e^{i\theta} - e^{i\theta_0}}{\theta-\theta_0}\right)^{-1} \\ &= \frac{1}{r}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\frac{1}{ie^{i\theta}} \\ &= -\frac{1}{r}e^{-i\theta}\left(-\frac{\partial v}{\partial \theta} + i\frac{\partial u}{\partial \theta}\right), \end{aligned}$$

where in the last equality we use the fact that the ratio is precisely the complex derivative of $g(r,\theta) = e^{i\theta}$.

We can now equate real and imaginary parts to obtain

$$\begin{aligned} e^{-i\theta}u_r = -e^{i\theta}\frac{1}{r}v_\theta &\implies u_r = \frac{1}{r}v_\theta \\ e^{-i\theta}v_r = -\frac{1}{r}e^{-i\theta}u_\theta &\implies v_r = -\frac{1}{r}u_\theta. \end{aligned}$$

10.2 2

To see that $f(z) = f(e^{-\theta}) = \log(r) + i\theta$ is holomorphic, we can check that it satisfies the Cauchy-Riemann equations.

We have $f(e^{-\theta}) = u(r,\theta) + iv(r,\theta)$ where

- $u(r,\theta) = \log(r) \implies u_r = \dfrac{1}{r}, u_\theta = 0,$

- $v(r,\theta) = \theta \implies v_r = 0, v_\theta = 1$,

and so indeed $u_r = \dfrac{1}{r} = \dfrac{1}{r} v_\theta$ and $v_r = 0 = -\dfrac{1}{r} u_\theta$ for $r \neq 0$.

To see that f is not continuous, we use the limit definition of continuity: let $z = re^{i\theta}$ and note that we also have $z = re^{i(\theta+2\pi)}$. Thus the sequence $z_n = re^{i(\theta+1/n)} \to z$

11 Problem 11

$\implies$: Suppose that z_1, z_2, z_3 are the vertices of an equilateral triangle Δ, then every interior angle of Δ is $\pi/3$. WLOG, orient Δ clockwise so that the sides are given by $s_1 = z_2 - z_1, s_2 = z_3 - z_2, s_3 = z_1 - z_3$. Let $\zeta = e^{i\pi/3}$ denote a rotation by $\pi/3$, then up to translations we have

$$\begin{aligned} \zeta s_1 &= -s_3 \\ \zeta s_3 &= -s_2, \end{aligned}$$

and by dividing equations we have

$$\begin{aligned} \frac{s_1}{s_3} = \frac{s_3}{s_2} &\implies s_1 s_2 - s_3^2 = 0 \\ &\implies (z_2 - z_1)(z_3 - z_2) - (z_1 - z_3)^2 = 0 \\ &\implies z_2 z_3 - z_2^2 - z_1 z_3 + z_1 z_2 - (z_1^2 + z_3^2 - 2z_1 z_3) = 0 \\ &\implies z_1 z_2 + z_2 z_3 + z_3 z_1 = z_1^2 + z_2^2 + z_3^2. \end{aligned}$$

$\impliedby$: Reversing the above calculation, we have

$$\frac{s_1}{s_3} = \frac{s_3}{s_2}.$$

Moreover, we find that

$$\begin{aligned} &z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1 \\ &\implies z_1^2 + z_2^2 - z_1 z_2 = -z_3^2 + z_2 z_3 + z_3 z_1 \\ &\implies z_1^2 + z_2^2 - 2z_1 z_2 = -z_3^2 + z_2 z_3 + z_3 z_1 - z_1 z_2 \\ &\implies (z_1 - z_2)^2 = (z_3 - z_2)(z_1 - z_3) \\ &\implies \frac{s_1}{s_2} = \frac{s_3}{s_1} \end{aligned}$$

and thus

$$\frac{s_1}{s_3} = \frac{s_3}{s_2} = \frac{s_1}{s_2} = c$$

for some constant c.

Since z_1, z_2, z_3 form some triangle, there are (a priori distinct) angles such that

$$\begin{aligned} s_1 &= \zeta_1 s_3 \\ s_2 &= \zeta_2 s_1 \\ s_3 &= \zeta_3 s_2, \end{aligned}$$

and so

$$\begin{aligned} \zeta_1 &= \frac{s_1}{s_3} = c \\ \zeta_2 &= \frac{s_2}{s_1} = c \\ \zeta_3 &= \frac{s_3}{s_2} = c, \end{aligned}$$

which shows that the 3 interior angles must be equal, yielding an equilateral triangle.