Problem Set 1

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1 Problem 1

Describe geometrically the sets of points z in the complex plane defined by the following relations:

a. $|z-1|=1$
b. $|z-1|=2|z-2|$
c. $\dfrac{1}{z}=\overline{z}$

d. $\Re(z) = 3$
e. $\Im(z) = a$ with $a \in \mathbb{R}$
f. $\Re(z) > a$ with $a \in \mathbb{R}$
g. $|z - 1| < 2|z - 2|$

2 Problem 2

Prove that

$$|z_1 + z_2| \geq \big|\, |z_1| - |z_2| \,\big|$$

and explain when equality holds.

3 Problem 3

Prove that the equation

$$z^3 + 2z + 4 = 0$$

has its roots outside the unit circle.

Hint: what is the maximum value of the modulus of the first two terms if $|z| \leq 1$.

4 Problem 4

4.1 a

Prove that if $|w_1| = c|w_2|$ where $c > 0$ then

$$\left|w_1 - c^2 w_2\right| = c|w_1 - w_2|.$$

4.2 b

Prove that if $c > 0$, $c \neq 1$, then $\left|\dfrac{z - z_1}{z - z_2}\right| = c$ defines a circle. Find its center and radius.

Hint: use part (a).

5 Problem 5

5.1 a

Let $z, w \in \mathbb{C}$ such that $\overline{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1 \text{ if } |z| < 1, |w| < 1 \text{ with equality when } |z| = 1 \text{ or } |w| = 1.$$

5.2 b

Prove that for a fixed w in the unit disc $\mathbb{D}$, the mapping

$$\begin{aligned} F : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto \frac{w-z}{1-\overline{w}z} \end{aligned}$$

satisfies the following conditions:

1. F is holomorphic and maps $\mathbb{D}$ to itself.
2. $F(0) = w$ and $F(w) = 0$.
3. If $|z| = 1$ then $|F(z)| = 1$.
4. F is a bijection.

> Hint: compute F^2.

6 Problem 6

Use nth roots of unity to show

$$2^{n-1} \sin\left(\frac{\pi}{n}\right) \sin\left(\frac{2\pi}{n}\right) \cdots \sin\left(\frac{(n-1)\pi}{n}\right) = n.$$

> Hint: $1 - \cos(2\theta) = 2\sin^2(\theta)$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.

7 Problem 7

Prove that $f(z) = |z|^2$ has derivative only at $z = 0$ and nowhere else.

8 Problem 8

Let $f(z)$ be analytic in its domain. Prove that f is constant if it satisfies any of the following conditions

a. $|f|$ is constant
b. $\Re(f)$ is constant
c. $\arg(f)$ is constant
d. $\overline{f}$ is constant

How can you generalize (a) and (b)?

9 Problem 9

Show that if f is analytic, $\overline{f}$ is analytic.

10 Problem 10

10.1 a

Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

10.2 b

Use this to show that the logarithm function,

$$\log z = \log r + i\theta \text{ where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi.$$

is holomorphic in the region $S = \left\{ z = re^{i\theta} \;\middle|\; r > 0,\ -\pi < \theta < \pi \right\}$, but is not continuous for $r > 0$.

11 Problem 10

Prove that distinct complex numbers z_1, z_2, z_3 are the vertices of an equilateral triangle iff

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$