Problem Set 2

D. Zack Garza

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Contents

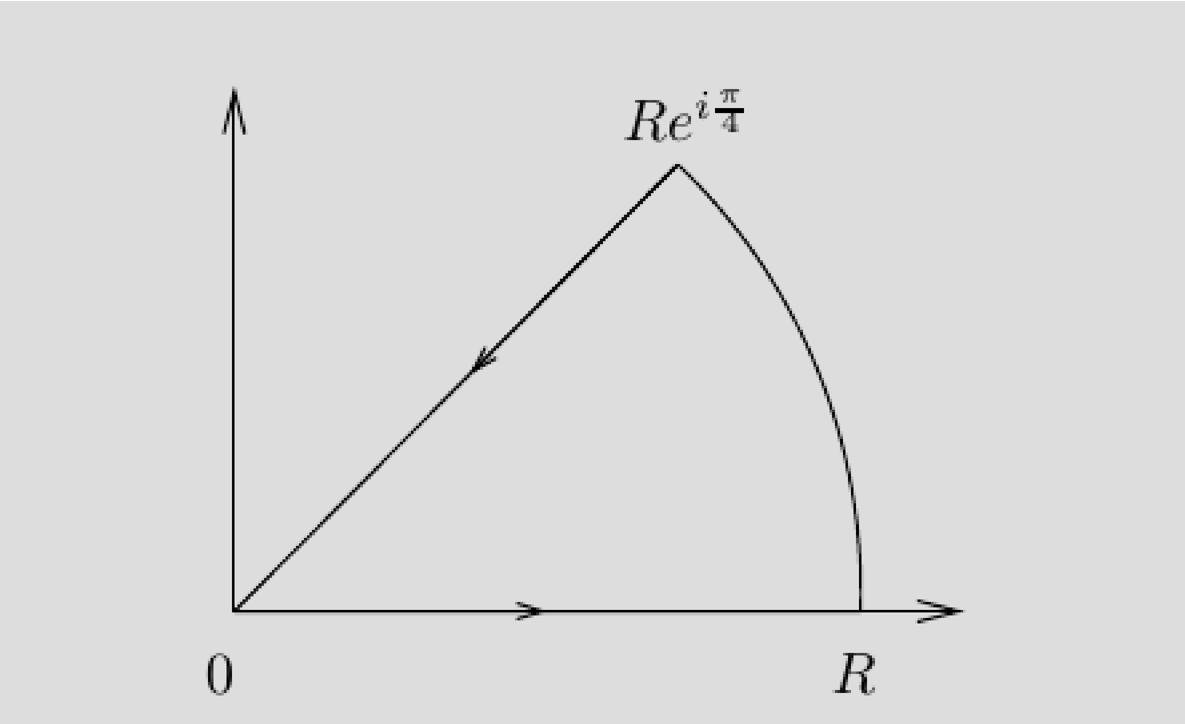
1 Stein And Shakarchi

1.1 2.6.1

Show that

$$\int_0^\infty \sin\left(x^2\right) dx = \int_0^\infty \cos\left(x^2\right) dx = \frac{\sqrt{2\pi}}{4}.$$

Hint: integrate e^{-x^2} over the following contour, using the fact that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$:



1.2 2.6.2

Show that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Hint: use the fact that this integral equals $\frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{ix}-1}{x} dx$, and integrate around an indented semicircle.

1.3 2.6.5

Suppose $f \in C^1_{\mathbb{C}}(\Omega)$ and $T \subset \Omega$ is a triangle with $T^\circ \subset \Omega$. Apply Green's theorem to show that $\displaystyle\int_T f(z)\ dz = 0$.

Assume that f' is continuous and prove Goursat's theorem.

Hint: Green's theorem states

$$\int_T F dx + G dy = \int_{T^\circ} \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx dy.$$

1.4 2.6.6

Suppose that f is holomorphic on a punctured open set $\Omega \setminus \{w_0\}$ and let $T \subset \Omega$ be a triangle containing w_0. Prove that if f is bounded near w_0, then $\displaystyle\int_T f(z)\ dz = 0$.

1.5 2.6.7

Suppose $f : \mathbb{Q} \to \mathbb{C}$ is holomorphic and let $d := \sup\limits_{z,w \in \mathbb{Q}} |f(z) - f(w)|$ be the diameter of the image of f. Show that $2|f'(0)| \leq d$, and that equality holds iff f is linear, so $f(z) = a_1 z + a_2$.

Hint: $2f'(0) = \displaystyle\frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi) - f(-\xi)}{\xi^2}\ d\xi$ whenever $0 < r < 1$.

1.6 2.6.8

Suppose that f is holomorphic on the strip $S = \left\{ x + iy \,\middle|\, x \in \mathbb{R},\ -1 < y < 1 \right\}$ with $|f(z)| \leq A(1 + |z|)^\nu$ for ν some fixed real number. Show that for all $z \in S$, for each integer $n \geq 0$ there exists an $A_n \geq 0$ such that $\left| f^{(n)}(x) \right| \leq A_n (1 + |x|)^\nu$ for all $x \in \mathbb{R}$.

Hint: Use the Cauchy inequalities.

1.7 2.6.9

Let $\Omega \subset \mathbb{C}$ be open and bounded and $\phi : \Omega \to \Omega$ holomorphic. Prove that if there exists a point $z_0 \in \Omega$ such that $\phi(z_0) = z_0$ and $\phi'(z_0) = 1$, then ϕ is linear.

Hint: assume $z_0 = 0$ (explain why this can be done) and write $\phi(z) = z + a_n z^n + O(z^{n+1})$ near 0. Let $\phi_k = \phi \circ \phi \circ \cdots \circ \phi$ and prove that $\phi_k(z) = z + k a_n z^n + O(z^{n+1})$. Apply Cauchy's inequalities and let $k \to \infty$ to conclude.

1.8 2.6.10

Can every continuous function on $\overline{\mathbb{Q}}$ be uniformly approximated by polynomials in the variable z?

Hint: compare to Weierstrass for the real interval.

1.9 2.6.13

Suppose f is analytic, defined on all of $\mathbb{C}$, and for each $z_0 \in \mathbb{C}$ there is at least one coefficient in the expansion $f(z) = \sum c_n(z - z_0)^n$ is zero. Prove that f is a polynomial.

> Hint: use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.

1.10 2.6.14

Suppose that f is holomorphic in an open set containing $\mathbb{Q}$ except for a pole $z_0 \in \partial\mathbb{Q}$. Let $\sum a_n z^n$ be the power series expansion of f in $\mathbb{Q}$, and show that $\lim \dfrac{a_n}{a_{n+1}} = z_0$.

1.11 2.6.15

Suppose f is continuous, nonvanishing on $\overline{\mathbb{Q}}$, and holomorphic in $\mathbb{Q}$. Prove that if $|z| = 1 \implies |f(z)| = 1$, then f is constant.

> Hint: Extend f to all of $\mathbb{C}$ by $f(z) = 1/\overline{f(1/\overline{z})}$ for any $|z| > 1$, and argue as in the Schwarz reflection principle.

2 Additional Problems

2.1 Problem 1

Proposition: $L = \lim |a_{n+1}|/|a_n| \implies L = \lim \sqrt[n]{a_n}$

2.2 Problem 2

Proposition: If f is a power series centered at the origin, then f has a power series expansion about any point in its domain.

2.3 Problem 3

2.3.1 a

Proposition: $\sum nz^n$ does not converge for any $|z| \leq 1$.

2.3.2 b

Proposition: $\sum z^n/n^2$ converges for every $|z| \leq 1$.

2.3.3 c

Proposition: $\sum z^n/n$ converges for every $|z| \leq 1$ except $z = 1$.

2.4 Problem 4

Proposition: Let γ denote a circle centered at the origin of radius r with positive orientation. Then if $|\alpha| \leq r \leq |\beta|$,

$$\int_{\gamma} \frac{dz}{(z-\alpha)(z-\beta)} = \frac{2\pi i}{\alpha - \beta}.$$

2.5 Problem 5

Proposition: Suppose x is continuous in the region $(x, y) \in [x_0, \infty) \times i[0, b] \subset \mathbb{R} \oplus i\mathbb{R}$, and $\lim_{x \to \infty} f(x+iy) = A$ independent of y. Let $\gamma = \left\{ z = x + it \;\middle|\; 0 \leq t \leq b \right\}$, then

$$\lim_{x \to +\infty} \int_{\gamma_x} f(z) dz = iAb.$$

2.6 Problem 6

Show that there exists a function f that is holomorphic on $0 < |z| < 1$ with $\int_{\partial D_r(0)} = 0$ for all $r < 1$ but f is not holomorphic at $z = 0$.

2.7 Problem 7

Let f be analytic on Ω and $f'(z_0) \neq 0$ for some $z_0 \in \Omega$. Show that if C is a circle centered at z_0 of sufficiently small radius, then

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{dz}{f(z) - f(z_0)}.$$

2.8 Problem 8

Let $u, v \in C^1(\mathbb{R}^2)$. Show that $f = u + iv$ has derivative $f'(z_0) = x_0 + iy_0$ iff

$$\lim_{r \to 0} \frac{1}{\pi r^2} \int_{|z - z_0| = r} f(z) dz = 0.$$

2.9 Problem 9

Let γ be piecewise smooth with interior Ω_1 and exterior Ω_2. Assume f' exists on an open set containing γ and Ω_2. Show that if $\lim_{z \to \infty} f(z) = A$, then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1 \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

2.10 Problem 10

Let f be bounded and analytic and $a \neq b \in \mathbb{C}$ be fixed, then the following limit exists:

$$\lim_{R\to\infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

Conclude that f must be constant.

2.11 Problem 11

Suppose f is entire and $\frac{f(z)}{z} \overset{z\to\infty}{\to} 0$. Show that f is constant.

2.12 Problem 12

Let f be analytic on Ω and γ a closed curve in Ω. Show that for any $z_0 \in \Omega \setminus \gamma$,

$$\int_\gamma \frac{f'(z)}{(z-z_0)} dz = \int_\gamma \frac{f(z)}{(z-z_0)^2} dz.$$

2.13 Problem 13

Compute

$$\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z}.$$

Use this to show that

$$\int_0^{2\pi} \cos^{2n} \theta d\theta = 2\pi \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$