

# Complex Analysis Problem Set 3

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## Contents

<b>1</b>	<b>Problems From Tie</b>	<b>2</b>
1.1	1 . . . . .	2
	1.1.1 Solution . . . . .	2
1.2	2 . . . . .	2
	1.2.1 Solution . . . . .	3
1.3	3 . . . . .	4
	1.3.1 Solution . . . . .	4
1.4	4 . . . . .	5
	1.4.1 Solution . . . . .	5
1.5	5 . . . . .	5
1.6	6 . . . . .	5
1.7	7 . . . . .	5
1.8	8 . . . . .	5
1.9	9 . . . . .	6
1.10	10 . . . . .	6
1.11	11 . . . . .	6
1.12	12 . . . . .	6
1.13	13 . . . . .	6
1.14	14 . . . . .	6
<b>2</b>	<b>Stein and Shakarchi</b>	<b>6</b>
2.1	S&S 3.8.1 . . . . .	6
2.2	S&S 3.8.2 . . . . .	7
2.3	S&S 3.8.4 . . . . .	7
2.4	S&S 3.8.5 . . . . .	7
2.5	S&S 3.8.6 . . . . .	7
2.6	S&S 3.8.7 . . . . .	7
2.7	S&S 3.8.8 . . . . .	8
2.8	S&S 3.8.9 . . . . .	8
2.9	S&S 3.8.10 . . . . .	8
2.10	S&S 3.8.14 . . . . .	8
2.11	S&S 3.8.15 . . . . .	8
	2.11.1 a . . . . .	8
	2.11.2 b . . . . .	8

2.11.3	c . . . . .	9
2.11.4	d . . . . .	9
2.12	S&S 3.8.17 . . . . .	9
2.12.1	a . . . . .	9
2.12.2	b . . . . .	9
2.13	S&S 3.8.19 . . . . .	9
2.13.1	a . . . . .	9
2.13.2	b . . . . .	9

## 1 Problems From Tie

### 1.1 1

Prove that if  $f$  has two Laurent series expansions,

$$f(z) = \sum c_n(z-a)^n \quad \text{and} \quad f(z) = \sum c'_n(z-a)^n$$

then  $c_n = c'_n$ .

#### 1.1.1 Solution

By taking the difference of two such expansions, it suffices to show that if  $f$  is identically zero and  $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$  about some point  $a$ , then  $c_n = 0$  for all  $n$ .

Under this assumption, let  $D_\varepsilon(a)$  be a disc about  $a$  and  $\gamma$  be any contour contained in its interior. Then for each  $n$ , we can apply the formula

$$\begin{aligned}
 c_n &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi-a)^{n+1}} \\
 &= \frac{1}{2\pi i} \int_{\gamma} \frac{0}{(\xi-a)^{n+1}} \quad \text{by assumption} \\
 &= 0,
 \end{aligned}$$

which shows that  $c_n = 0$  for all  $n$ . ■

### 1.2 2

Find Laurent series expansions of

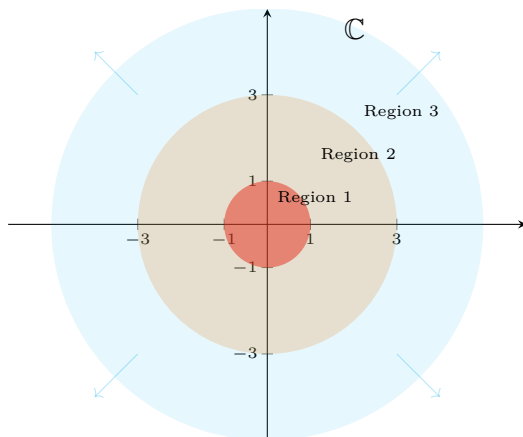
$$\frac{1}{1-z^2} + \frac{1}{3-z}$$

How many such expansions are there? In what domains are each valid?

### 1.2.1 Solution

Note that  $f$  has poles at  $z = -1, 1, 3$ , all with multiplicity 1, and so there are 3 regions to consider:

1.  $|z| < 1$
2.  $1 < |z| < 3$
3.  $3 < |z|$ .



**Region 1:** Take the following expansion:

$$\begin{aligned}
 f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\
 &= \sum_{n \geq 0} z^{2n} + \frac{1}{3} \left( \frac{1}{1-\frac{z}{3}} \right) \\
 &= \sum_{n \geq 0} z^{2n} + \frac{1}{3} \sum_{n \geq 0} \left( \frac{1}{3} \right)^n z^n \\
 &= \sum_{n \geq 0} z^{2n} + \sum_{n \geq 0} \left( \frac{1}{3} \right)^{n+1} z^n
 \end{aligned}$$

Noting that the first term converges for  $|z^2| < 1$  and the second for  $\left| \frac{z}{3} \right| < 1 \iff |z| < 3$ , intersecting these regions yields convergence on  $|z| < 1$ .

**Region 2:** Take the following expansion:

$$\begin{aligned}
f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\
&= -\frac{1}{z^2} \left( \frac{1}{1-\frac{1}{z^2}} \right) - \frac{1}{3} \left( \frac{1}{1-\frac{z}{3}} \right) \\
&= -\frac{1}{z^2} \sum_{n \geq 0} z^{-2n} + \sum_{n \geq 0} \left( \frac{1}{3} \right)^{n+1} z^n \\
&= -\sum_{n \geq 2} \frac{1}{z^{2n}} + \sum_{n \geq 0} \left( \frac{1}{3} \right)^{n+1} z^n
\end{aligned}$$

By construction, the first term converges for  $|1|z^2| < 1 \iff |z| > 1$  and the second for  $|z| < 3$ .

**Region 3:** Take the following expansion:

$$\begin{aligned}
f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\
&= -\frac{1}{z^2} \left( \frac{1}{1-\frac{1}{z^2}} \right) - \frac{1}{z} \left( \frac{1}{1-\frac{3}{z}} \right) \\
&= -\frac{1}{z^2} \sum_{n \geq 0} \frac{1}{z^{2n}} - \frac{1}{z} \sum_{n \geq 0} 3^n \frac{1}{z^n} \\
&= -\sum_{n \geq 2} \frac{1}{z^{2n}} - \sum_{n \geq 1} 3^{n-1} \frac{1}{z^n}
\end{aligned}$$

Note: in principle, terms could be collected here.

By construction, this converges on  $\{|z|^2 > 1\} \cap \{|z| > 3\} = \{|z| > 3\}$ .

### 1.3 3

Let  $P, Q$  be polynomials with no common zeros. Assume  $a$  is a root of  $Q$ . Find the principal part of  $P/Q$  at  $z = a$  in terms of  $P$  and  $Q$  if  $a$  is (1) a simple root, and (2) a double root.

#### 1.3.1 Solution

todo

## 1.4 4

Let  $f$  be non-constant, analytic in  $|z| > 0$ , where  $f(z_n) = 0$  for infinitely many points  $z_n$  with  $\lim_{n \rightarrow \infty} z_n = 0$ .

Show that  $z = 0$  is an essential singularity for  $f$ .

Example:  $f(z) = \sin(1/z)$ .

### 1.4.1 Solution

It suffices to show that  $z_0 = 0$  is neither a pole nor a removable singularity, i.e.

1.  $\lim_{z \rightarrow z_0} f(z) \neq \infty$
2.  $|f(z)|$  is not bounded on any neighborhood  $D_\varepsilon(z_0)$ .

The first property follows because if  $f$  is analytic,

## 1.5 5

Show that if  $f$  is entire and  $\lim_{z \rightarrow \infty} f(z) = \infty$ , then  $f$  is a polynomial.

## 1.6 6

- a. Show that

$$\int_0^{2\pi} \log |1 - e^{i\theta}| d\theta = 0$$

- b. Show that this identity is equivalent to SS 3.8.9.

## 1.7 7

Let  $0 < a < 4$  and evaluate

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^3} dx$$

## 1.8 8

Prove the fundamental theorem of Algebra using

- a. Rouché's Theorem.
- b. The maximum modulus principle.

### 1.9 9

Let  $f$  be analytic in a region  $D$  and  $\gamma$  a rectifiable curve in  $D$  with interior in  $D$ . Prove that if  $f(z)$  is real for all  $z \in \gamma$ , the  $f$  is constant.

### 1.10 10

For  $a > 0$ , evaluate

$$\int_0^{\pi/2} \frac{d\theta}{a + \sin^2 \theta}$$

### 1.11 11

Find the number of roots of  $p(z) = 4z^4 - 6z + 3$  in  $|z| < 1$  and  $1 < |z| < 2$  respectively.

### 1.12 12

Prove that  $z^4 + 2z^3 - 2z + 10$  has exactly one root in each open quadrant.

### 1.13 13

Prove that for  $a > 0$ ,  $z \tan z - a$  has only real roots.

### 1.14 14

Let  $f$  be nonzero, analytic on a bounded region  $\Omega$  and continuous on its closure  $\bar{\Omega}$ . Show that if  $|f(z)| \equiv M$  is constant for  $z \in \partial\Omega$ , then  $f(z) \equiv Me^{i\theta}$  for some real constant  $\theta$ .

## 2 Stein and Shakarchi

### 2.1 S&S 3.8.1

Using Euler's formula

$$\sin(\pi z) = \frac{1}{2i}(e^{i\pi z} - e^{-i\pi z})$$

show that the complex zeros of  $\sin(\pi z)$  are exactly the integers, each of order one.

Calculate the residue of  $\frac{1}{\sin(\pi z)}$  at  $z = n \in \mathbb{Z}$ .

## 2.2 S&S 3.8.2

Evaluate the integral

$$\int_{\mathbb{R}} \frac{dx}{1+x^4}$$

What are the poles of the integrand?

## 2.3 S&S 3.8.4

Show that

$$\int_{\mathbb{R}} \frac{x \sin x}{x^2 + a^2} = \frac{\pi e^{-a}}{a} \quad a > 0$$

## 2.4 S&S 3.8.5

Show that for  $\xi \in \mathbb{R}$ ,

$$\int_{\mathbb{R}} \frac{e^{2\pi i x \xi}}{(1+x^2)^2} = \frac{\pi}{2} (1 + 2\pi|\xi|) e^{-2\pi|\xi|}$$

## 2.5 S&S 3.8.6

Show that

$$\int_{\mathbb{R}} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdots (2n-1)\pi}{2 \cdot 4 \cdots (2n)}$$

## 2.6 S&S 3.8.7

Show that for  $a > 1$ ,

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos \theta)^2} = \frac{2\pi a}{(a^2 - 1)^{3/2}}$$

## 2.7 S&S 3.8.8

Show that if  $a, b \in \mathbb{R}$  with  $a > |b|$  then

$$\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \frac{2\pi a}{\sqrt{a^2 - b^2}}$$

## 2.8 S&S 3.8.9

Show that

$$\int_0^1 \log(\sin \pi x) dx = -\log 2$$

## 2.9 S&S 3.8.10

Show that if  $a > 0$

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx = \frac{\pi \log a}{2a}$$

## 2.10 S&S 3.8.14

Prove that if  $f$  is entire and injective, then  $f(z) = az + b$  with  $a, b \in \mathbb{C}$  with  $a \neq 0$ .

Hint: apply the Casorati-Weierstrass theorem to  $f(1/z)$ .

## 2.11 S&S 3.8.15

Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

### 2.11.1 a

If  $f$  is entire and for all  $R > 0$ , there are constants  $A, B > 0$  such that  $\sup_{|z|=R} |f(z)| \leq AR^k + B$ , then  $f$  is a polynomial of degree less than  $k$ .

### 2.11.2 b

Show that if  $f$  is holomorphic on the unit disk, is bounded, and converges to zero uniformly in the sector  $\theta \leq \arg z \leq \phi$  as  $|z| \rightarrow 1$ , then  $f \equiv 0$ .



### 2.11.3 c

Let  $w_1, \dots, w_n$  be points on  $S^1 \subset \mathbb{C}$ . Show that there exists a point  $z \in S^1$  such that

$$\prod_{i=1}^n |z - w_i| \geq 1.$$

Conclude that there exists a point  $w \in S^1$  such that

$$\prod_{i=1}^n |w - w_i| = 1.$$

### 2.11.4 d

Show that if  $f$  is entire and  $\Re(f)$  is bounded, then  $f$  is constant.

## 2.12 S&S 3.8.17

Let  $f$  be non-constant, and holomorphic in an open set containing the open unit disc.

### 2.12.1 a

Show that  $|z| = 1 \implies |f(z)| = 1$ , then the image of  $f$  contains the unit disc.

Hint: Show that  $f(z) = w_0$  has a root for every  $w_0 \in \mathbb{D}$ , for which it suffices to show that  $f(z) = 0$  has a root, then use the maximum modulus principle.

### 2.12.2 b

Show that if  $|z| \geq 1 \implies |f(z)| = 1$  **and** there exists a point  $z_0 \in \mathbb{D}$  such that  $|f(z_0)| < 1$ , then the image of  $f$  contains the unit disc.

## 2.13 S&S 3.8.19

Prove the maximum modulus principle for harmonic functions; i.e.,

### 2.13.1 a

If  $u$  is a non-constant real-valued harmonic function on  $\Omega$ , then  $u$  can not attain its extrema on  $\Omega$ .

### 2.13.2 b

Suppose  $\Omega$  has compact closure  $\overline{\Omega}$ , then if  $u$  is harmonic on  $\Omega$  and continuous on  $\overline{\Omega}$ , then

$$\sup_{z \in \Omega} |u(z)| \leq \sup_{z \in \overline{\Omega} - \Omega} |u(z)|$$

Hint: to prove (a), assume that  $u$  attains a local maximum at  $z_0$ , and let  $f$  be holomorphic near  $z_0$  with  $u = \Re(f)$ , then show that  $f$  is not open. Part (b) is a direct consequence.