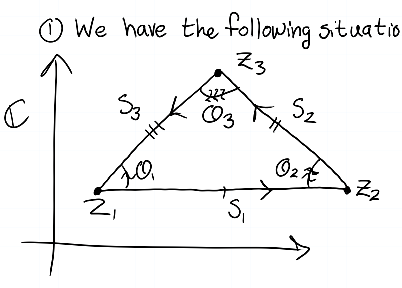
Zack Garza

① We have the following situation:



where

- $\Theta_i \in \mathbb{C}$, $|\Theta_i| = 1$ (So multiplying is a rotation.)
- $S_1 = z_2 - z_1 = \pm(\Theta_1 S_3)$ ①
- $S_2 = z_3 - z_2 = \pm(\Theta_2 S_1)$ ②
- $S_3 = z_1 - z_3 = \pm(\Theta_3 S_2)$ ③

Taking ratios of these equations yields...

$$\begin{cases} ①/② : & \frac{S_1}{S_2} = \frac{\Theta_1}{\Theta_2} \cdot \frac{S_3}{S_1} & (a) \\ ②/③ : & \frac{S_2}{S_3} = \frac{\Theta_2}{\Theta_3} \cdot \frac{S_1}{S_2} & (b) \\ ③/① : & \frac{S_3}{S_1} = \frac{\Theta_3}{\Theta_1} \cdot \frac{S_2}{S_3} & (c) \end{cases}$$

($\Rightarrow$) If $\Delta z_1 z_2 z_3$ is equilateral, then $\Theta_1 = \Theta_2 = \Theta_3 = \xi := e^{\pi i/3}$.

So using equation (b),

$$\frac{S_2}{S_3} = \frac{S_1}{S_2} \iff S_2^2 - S_1 S_3 = 0$$
$$\iff (z_2 - z_3)^2 - (z_2 - z_1)(z_1 - z_3) = 0$$
$$\iff (z_2^2 + z_3^2 - 2z_2 z_3) - (z_2 z_1 - z_2 z_3 - z_1^2 + z_1 z_3) = 0$$
$$\iff z_1^2 + z_2^2 + z_3^2 - (z_1 z_2 + z_2 z_3 + z_3 z_1) = 0. \quad \square$$

($\Leftarrow$) Assuming the last line, we obtain $\frac{S_2}{S_3} = \frac{S_1}{S_2} \Rightarrow \frac{\Theta_2}{\Theta_3} = 1 \Rightarrow \Theta_2 = \Theta_3.$

and since using any of ①,②,③ yields the same calculation, $\frac{S_3}{S_1} = \frac{S_2}{S_3} \Rightarrow \Theta_3 = \Theta_1$,

so $\Theta_1 = \Theta_2 = \Theta_3$, forcing $S_1 = S_2 = S_3$. ■

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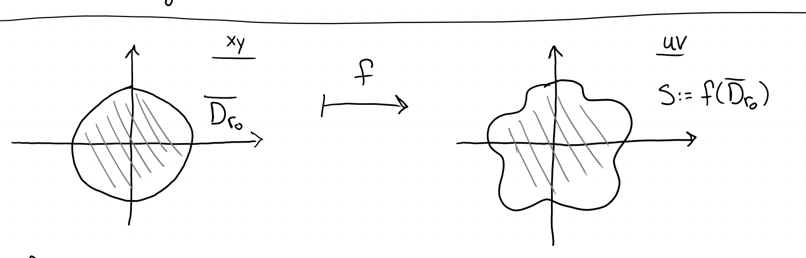
② We have



And by the change of variables theorem, we have
(Uses injectivity)

$$I := \text{Area}(S) = \iint_S 1 \, du\,dv = \iint_{f^{-1}(S)} 1 \cdot \det(J_{u,v}(x,y))\, dx\,dy \quad \text{where } f(x,y) = u(x,y) + iv(x,y).$$

$\leftarrow = D_{r_0}$

We can compute

Cauchy-Riemann

$$\det J_{u,v}(x,y) = \det \begin{bmatrix} \nabla u \\ \nabla v \end{bmatrix} = \det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \det \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix} = u_x^2 + v_x^2$$

$$= |u_x(x,y) + i v_x(x,y)|_{\mathbb{C}}^2 = |f_x(x,y)|_{\mathbb{C}}^2 = |f'(x,y)|^2,$$

so $I = \iint_{D_{r_0}} |f'(x,y)|^2 \, dx\,dy.$

Since f is analytic and $\overline{D_{r_0}}$ is compact, we have uniform convergence of the Taylor expansion and can differentiate term-by-term:

$$f'(z) = \frac{\partial}{\partial z} \sum_{n=0}^{\infty} c_n z^n = \sum_{n=1}^{\infty} n c_n z^{n-1}$$

$$\Rightarrow |f'(z)|^2 = f(z)\overline{f(z)} = \left(\sum_{n=1}^{\infty} n c_n z^{n-1}\right)\left(\overline{\sum_{n=1}^{\infty} n c_n z^{n-1}}\right)$$
$$= \left(\sum_{n=1}^{\infty} n c_n z^{n-1}\right)\left(\sum_{n=1}^{\infty} n \bar{c}_n \overline{z^{n-1}}\right)$$
$$= \left(\sum_{n=1}^{\infty} n c_n (re^{i\theta})^{n-1}\right)\left(\sum_{n=1}^{\infty} n \bar{c}_n (re^{-i\theta})^{n-1}\right)$$
$$= \sum_{n=1}^{\infty} n c_n r^{n-1} e^{i(n-1)\theta} \cdot \sum_{n=1}^{\infty} n \bar{c}_n r^{n-1} e^{-i(n-1)\theta}$$
$$= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} jk c_j \bar{c}_k r^{(j+k)-2} e^{i(j-k)\theta}, \quad \leftarrow \sum_k a_k \cdot \sum_j c_j = \sum_k \sum_j a_k c_j$$

And so

$$I = \iint_{D_{r_0}} |f'(x,y)|^2 \, dx\,dy = \iint_{D_{r_0}} |f'(r,\theta)| \, r\,dr\,d\theta$$
$$= \int_0^{r_0}\int_0^{2\pi} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} jk c_j \bar{c}_k r^{(j+k)-1} e^{i(j-k)\theta} \, dr\,d\theta$$
$$= \int_0^{r_0} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} jk c_j \bar{c}_k r^{j+k-1} \left(\int_0^{2\pi} e^{i(j-k)\theta}\, d\theta\right) dr$$
$$= \int_0^{r_0} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} jk c_j \bar{c}_k r^{j+k-1} (2\pi \cdot \chi_{(j=k)})\, dr$$
$$= \int_0^{r_0} \sum_{j=1}^{\infty} 2\pi j^2 c_j \bar{c}_j r^{2j-1}\, dr$$
$$= \sum_{j=1}^{\infty} 2\pi j^2 |c_j| \int_0^{r_0} r^{2j-1}\, dr$$
$$= \sum_{j=1}^{\infty} 2\pi j^2 |c_j|^2 \left(\frac{r_0^{2j}}{2j}\right)$$
$$\boxed{= \sum_{j=1}^{\infty} \pi j |c_j|^2 r_0^{2j}. \ ■}$$

Using uniform convergence

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③ Since $\lim_{z\to\infty} (z-a)f(z) = A$, choose $R >> a > 0$ large enough so that

$$r > R \Rightarrow |(z-a)f(z) - A| < \varepsilon \iff (z-a)f(z) = A \pm \varepsilon \iff f(z) = \frac{A \pm \varepsilon}{(z-a)}.$$

Then

$$\int_{\gamma_r} f(z)\,dz = \int_{\gamma_r} \frac{A \pm \varepsilon}{z - a}\, dz = (A \pm \varepsilon)\int_0^{\beta_0} i\left(\frac{re^{i\theta}}{re^{i\theta} - a}\right) d\theta$$

$$\xrightarrow{r\to\infty} i(A \pm \varepsilon)\int_0^{\beta_0} 1 \, d\theta = i(A \pm \varepsilon)\beta_0 \xrightarrow{\varepsilon \to 0} \boxed{iA\beta_0. \ ■}$$

Using the fact that $f(r,\theta)$ is cts and $[0, \beta_0] \subset \mathbb{R}$ is compact.

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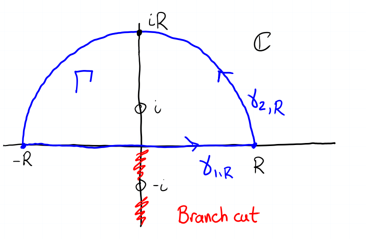
④ Writing $I(b) = \int_0^{\pi/2} (\tan(t))^{ib}\, dt$, we make the substitution

$$\left.\begin{aligned} s = \tan(t) \Rightarrow t = \tan^{-1}(s) \\ dt = \frac{1}{1+s^2}\, ds \end{aligned}\right\} \Rightarrow I(b) = \int_0^{\infty} \frac{s^{ib}}{1+s^2}\, ds = \frac{1}{2}\int_{\gamma_{1,R}} f(z)\,dz \quad \text{where}$$

$$f(z) := \frac{z^{ib}}{1+z^2} := \frac{e^{ib \operatorname{Log} z}}{1+z^2} \quad \text{where we use the branch cut } \mathbb{C} \setminus (-i\infty, 0)$$

and we define the contours



Since $|f(z)| = \frac{|e^{ib}|^{\log z}}{|1+z^2|} = \frac{1}{|1+z^2|} \xrightarrow{R\to\infty} 0,$

the integral along $\gamma_{2,R}$ will vanish and $I(b) = \int_\Gamma f(z)\,dz = 2\pi i \sum_{z_i} \operatorname{Res}(f, z_i).$

Since $f(z) = \frac{z^{ib}}{(z+i)(z-i)}$, Γ only encloses the pole $z_i = i$, and

$$2\pi i \operatorname{Res}(f, i) = 2\pi i \cdot \lim_{z\to i} \frac{z^{ib}}{(z+i)} = \frac{2\pi i \, i^{ib}}{2i} = \pi e^{ib \operatorname{Log} i} = \pi e^{ib(\ln|i| + i\operatorname{Arg}(i))}$$
$$= \pi e^{ib(0 + i\pi)} = \pi e^{-b\pi} \quad ■$$

$$\boxed{\therefore I(b) = \frac{\pi e^{b\pi}}{2} \quad ■}$$

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⑤ • On $D_1(z_0 = 0)$, write $f(z) = 2z^5 + 8z - 1 = (2z^5 - 1) + 8z := p(z) + q(z)$.

Then on $\partial D_1(0)$, $|z| = 1 \Rightarrow |p(z)| = |2z^5 - 1| \leq |2z^5| + |1| = 2 + 1 = 3 < 8 = |8z| = |q(z)|$,
so $f(z)$ and $q(z)$ have the same number of zeros in $D_1(0)^\circ$, and $q(z) = 8z$ has just one zero $z_0 = 0$.

• On $D_2(0)$, write $f(z) = (8z - 1) + 2z^5 := p(z) + q(z)$, then

$|z| = 2 \Rightarrow |p(z)| = |8z - 1| \leq |8z| + 1 = 17 < 64 = |2z^5| = |q(z)|$,

and $q(z) = 2z^5$ has five zeros in $D_2(0)$, thus so does $f(z)$.

(All using Rouché's theorem, using the fact that polynomials are entire.)

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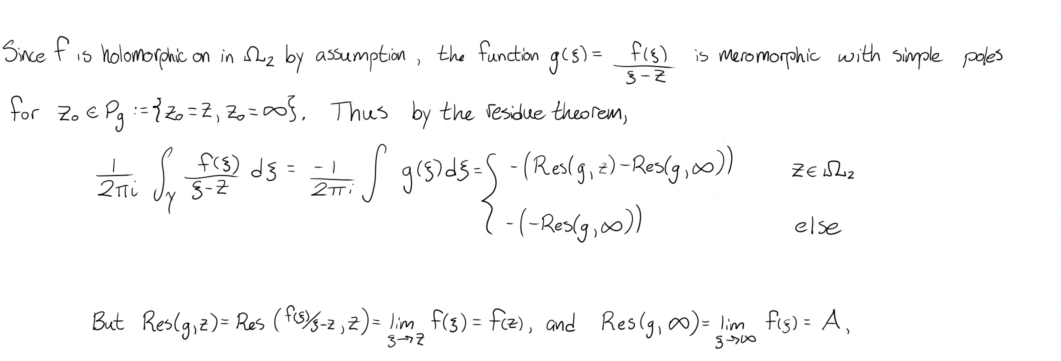
⑥ By inverting through the Riemann sphere, we can view $\gamma = \partial\Omega_2$ with reversed orientation:



Since f is holomorphic on in Ω_2 by assumption, the function $g(\xi) = \frac{f(\xi)}{\xi - z}$ is meromorphic with simple poles for $z_0 \in P_g := \{z_0 = z, z_0 = \infty\}$. Thus by the residue theorem,

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi = \frac{-1}{2\pi i}\int g(\xi)\,d\xi = \begin{cases} -(\operatorname{Res}(g,z) - \operatorname{Res}(g,\infty)) & z \in \Omega_2 \\ -(-\operatorname{Res}(g,\infty)) & \text{else} \end{cases}$$

But $\operatorname{Res}(g,z) = \operatorname{Res}\left(f(\xi)/\xi - z, z\right) = \lim_{\xi\to z} f(\xi) = f(z)$, and $\operatorname{Res}(g,\infty) = \lim_{\xi\to\infty} f(\xi) = A$,

so we obtain
$$\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi = \begin{cases} -(f(z) - A) = -f(z) + A, & z \in \Omega_2 \\ -(-A) = A, & z \in \Omega_1. \end{cases} \quad ■$$