

# Complex Analysis Problem Set 3

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## 1 Problems From Tie

### 1.1 1

Prove that if  $f$  has two Laurent series expansions,

$$f(z) = \sum c_n(z-a)^n \quad \text{and} \quad f(z) = \sum c'_n(z-a)^n$$

then  $c_n = c'_n$ .

#### 1.1.1 Solution

By taking the difference of two such expansions, it suffices to show that if  $f$  is identically zero and  $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$  about some point  $a$ , then  $c_n = 0$  for all  $n$ .

Under this assumption, let  $D_\varepsilon(a)$  be a disc about  $a$  and  $\gamma$  be any contour contained in its interior. Then for each  $n$ , we can apply the formula

$$\begin{aligned}
 c_n &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi-a)^{n+1}} d\xi \\
 &= \frac{1}{2\pi i} \int_{\gamma} \frac{0}{(\xi-a)^{n+1}} d\xi \quad \text{by assumption} \\
 &= 0,
 \end{aligned}$$

which shows that  $c_n = 0$  for all  $n$ . ■

## 1.2 2

Find Laurent series expansions of

$$\frac{1}{1-z^2} + \frac{1}{3-z}$$

How many such expansions are there? In what domains are each valid?

## 1.2.1 Solution

Note that  $f$  has poles at  $z = -1, 1, 3$ , all with multiplicity 1, and so there are 3 regions to consider:

1.  $|z| < 1$
2.  $1 < |z| < 3$
3.  $3 < |z|$ .



**Region 1:** Take the following expansion:

$$\begin{aligned} f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\ &= \sum_{n \geq 0} z^{2n} + \frac{1}{3} \left( \frac{1}{1 - \frac{z}{3}} \right) \\ &= \sum_{n \geq 0} z^{2n} + \frac{1}{3} \sum_{n \geq 0} \left( \frac{1}{3} \right)^n z^n \\ &= \sum_{n \geq 0} z^{2n} + \sum_{n \geq 0} \left( \frac{1}{3} \right)^{n+1} z^n \end{aligned}$$

Noting  $|z^2| < 1$  implies then  $|z| < 1$ , and that the first term converges for  $|z^2| < 1$  and the second for  $\left| \frac{z}{3} \right| < 1 \iff |z| < 3$ , this expansion converges to  $f$  on the region  $|z| < 1$ .

**Region 2:** Take the following expansion:

$$\begin{aligned}
 f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\
 &= -\frac{1}{z^2} \left( \frac{1}{1-\frac{1}{z^2}} \right) - \frac{1}{3} \left( \frac{1}{1-\frac{z}{3}} \right) \\
 &= -\frac{1}{z^2} \sum_{n \geq 0} z^{-2n} + \sum_{n \geq 0} \left( \frac{1}{3} \right)^{n+1} z^n \\
 &= -\sum_{n \geq 2} \frac{1}{z^{2n}} + \sum_{n \geq 0} \left( \frac{1}{3} \right)^{n+1} z^n
 \end{aligned}$$

By construction, the first term converges for  $\left| \frac{1}{z^2} \right| < 1 \iff |z| > 1$  and the second for  $|z| < 3$ .

**Region 3:** Take the following expansion:

$$\begin{aligned}
 f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\
 &= -\frac{1}{z^2} \left( \frac{1}{1-\frac{1}{z^2}} \right) - \frac{1}{z} \left( \frac{1}{1-\frac{z}{3}} \right) \\
 &= -\frac{1}{z^2} \sum_{n \geq 0} \frac{1}{z^{2n}} - \frac{1}{z} \sum_{n \geq 0} 3^n \frac{1}{z^n} \\
 &= -\sum_{n \geq 2} \frac{1}{z^{2n}} - \sum_{n \geq 1} \left( \frac{1}{3} \right)^{n-1} \frac{1}{z^n}.
 \end{aligned}$$

Note: in principle, terms could be collected here.

By construction, this converges on  $\{|z|^2 > 1\} \cap \{|z| > 3\} = \{|z| > 3\}$ .

■

### 1.3 3

Let  $P, Q$  be polynomials with no common zeros. Assume  $a$  is a root of  $Q$ . Find the principal part of  $P/Q$  at  $z = a$  in terms of  $P$  and  $Q$  if  $a$  is (1) a simple root, and (2) a double root.

#### 1.3.1 Solution

todo

**1.4 4**

Let  $f$  be non-constant, analytic in  $|z| > 0$ , where  $f(z_n) = 0$  for infinitely many points  $z_n$  with  $\lim_{n \rightarrow \infty} z_n = 0$ .

Show that  $z = 0$  is an essential singularity for  $f$ .

Example:  $f(z) = \sin(1/z)$ .

**1.4.1 Solution**

It suffices to show that  $z_0 = 0$  is neither a pole nor a removable singularity, i.e.

1.  $\lim_{z \rightarrow z_0} f(z) \neq \infty$
2.  $|f(z)|$  is not bounded on any neighborhood  $D_\varepsilon(z_0)$ .

The first property follows because if  $f$  is analytic,

**1.5 5**

Show that if  $f$  is entire and  $\lim_{z \rightarrow \infty} f(z) = \infty$ , then  $f$  is a polynomial.

**1.6 6**

Problem : a. Show that

$$\int_0^{2\pi} \log |1 - e^{i\theta}| d\theta = 0$$

b. Show that this identity is equivalent to SS 3.8.9.

**1.7 7**

Let  $0 < a < 4$  and evaluate

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^3} dx$$

**1.8 8**

Prove the fundamental theorem of Algebra using

- a. Rouché's Theorem.
- b. The maximum modulus principle.

### 1.8.1 Solution (Rouche)

We want to show that every  $f \in \mathbb{C}[x]$  has precisely  $n$  roots, and we'll use the following formulation of Rouché's theorem:

**Theorem 1.1 (Rouche).**

If  $f, g$  are holomorphic on  $D(z_0)$  with  $f, g \neq 0$  and  $|f - g| < |f| + |g|$  on  $\partial D(z_0)$ , then  $f$  and  $g$  has the same number of zeros within  $D$ .

We'll also use without proof the fact that the function  $h(z) = z^n$  has precisely  $n$  zeros (counted with multiplicity).

Suppose  $f(z) = a_n z^n + \cdots + a_1 z + a_0$  where  $a_n \neq 0$  and define  $g(z) = a_n z^n$ . Noting that polynomials are entire,  $f, g$  are nonzero by assumption, and

$$\begin{aligned} |f - g| &= |a_{n-1}z^{n-1} + \cdots + a_1z + a_0| \\ &= |a_{n-1}z^{n-1} + \cdots + a_1z + a_0 + a_n z^n - a_n z^n| \\ &\leq |a_{n-1}z^{n-1} + \cdots + a_1z + a_0 + a_n z^n| + |a_n z^n| \quad \text{for } |z| > 1(?) \\ &= |f| + |g| \end{aligned}$$

the conditions of Rouché's theorem apply and  $f$  has precisely  $n$  roots. ■

### 1.8.2 Solution (Maximum Modulus Principle)

Toward a contradiction, suppose  $f$  is non-constant and has no zeros. Then  $g(z) := 1/f(z)$  is non-constant and holomorphic on  $\mathbb{C}$ .

Using the fact that  $\lim_{z \rightarrow \infty} f(z) = \infty$  for any polynomial  $f$ , pick  $r$  large enough such that

$$z \in \mathbb{C} \setminus \overline{D_r}(0) \implies |f(z)| > |f(0)|$$

where  $D_r(0)$  is an open disc of radius  $r$  about  $z = 0$ . Then  $|g(z)| < |g(0)|$  for every such  $z$ .

Noting that  $\overline{D_r}(0)$  is closed and bounded and thus compact by Heine-Borel,  $g$  has a global maximum in the interior  $D_r^\circ$ . But by the maximum modulus principle, this forces  $g$  to be constant, and since  $g = \frac{1}{f}$ , it must also be true that  $f$  is constant. ■

## 1.9 9

Let  $f$  be analytic in a region  $D$  and  $\gamma$  a rectifiable curve in  $D$  with interior in  $D$ . Prove that if  $f(z)$  is real for all  $z \in \gamma$ , then  $f$  is constant.

### 1.9.1 Solution

Since  $f$  is analytic in  $D$  (wlog assuming  $0 \in D$  by translation), take its series expansion  $f(z) = c_0 + c_1 z + \dots$  for  $z \in D$ .

Without loss of generality, suppose that  $\gamma$  is not entirely contained in  $\mathbb{R}$ , so for  $z \in \gamma$  we can write  $z = x + iy$  where  $y \neq 0$ .

Then

$$\begin{aligned} f(z) &= f(x + iy) \\ &= c_0 + c_1(x + iy) + \dots \\ &= c_0 + c_1 x + ic_1 y + \dots \end{aligned} \quad \subset \mathbb{R} \text{ by assumption,}$$

and so we must have  $c_1 y = 0 \implies c_1 = 0$ . The same argument applies to further terms in the expansion, so we in fact have  $c_i = 0$  for every  $i \geq 1$ .

But this says  $f(z) = c_0$  for an arbitrary  $z$ , i.e.  $f$  is constant. ■

## 1.10 10

For  $a > 0$ , evaluate

$$\int_0^{\pi/2} \frac{d\theta}{a + \sin^2 \theta}$$

We have

$$\begin{aligned} I &:= \int_0^{\pi/2} \frac{1}{1 + \sin^2(\theta)} d\theta \\ &= \int_{\gamma_1} \frac{1}{a + \left(\frac{z - z^{-1}}{2i}\right)^2} \frac{-i dz}{z} \quad \text{where } \gamma_1 \text{ is } \frac{1}{4} \text{ of the unit circle } S^1 \\ &= -i \int_{\gamma_1} \frac{1}{z} \left( \frac{1}{a + \left(-\frac{1}{4}\right)(z^2 - 2 + z^{-2})} \right) dz \\ &= 4i \int_{\gamma_1} \frac{1}{z} \left( \frac{1}{z^2 - (2 + 4a) + z^{-2}} \right) dz \\ &= 4i \int_{\gamma_1} \frac{z}{z^4 - (2 + 4a)z^2 + 1} dz \\ &= i \oint_{S^1} \frac{z}{z^4 - (2 + 4a)z^2 + 1} dz \\ &= \frac{i}{2} \oint_{2 \cdot S^1} \frac{1}{u^2 - (2 + 4a)u + 1} du \quad \text{using } u = z^2, \frac{1}{2} du = z dz \\ &:= \frac{i}{2} \oint_{2 \cdot S^1} \frac{1}{f_a(u)} du \\ &= \frac{i}{2} \cdot 2\pi i \cdot \sum \text{Res}_{u=r_i} \frac{1}{f_a(u)}, \end{aligned}$$

where  $2 \cdot S^1$  denotes the contour wrapping around the unit circle twice and  $r_i$  denote the poles contained in the region bounded by  $S^1$ . We can now compute the last integral by the residue theorem.

Factor the denominator as

$$f_a(u) = u^2 - (2 + 4a)u + 1 = (u - r_1)(u - r_2),$$

where the  $r_i$  are given by  $(1 + 2a) \pm 4\sqrt{a^2 + a}$  using the quadratic formula. We can then write a partial fraction decomposition

$$\begin{aligned} \frac{1}{f_a(u)} &:= \frac{1}{u^2 - (2 + 4a)u + 1} \\ &= \frac{1}{(u - r_1)(u - r_2)} \\ &= \frac{A}{u - r_1} + \frac{B}{u - r_2} \\ &= \frac{\operatorname{Res}_{u=r_1} 1/f(u)}{u - r_1} + \frac{\operatorname{Res}_{u=r_2} 1/f(u)}{u - r_2} \\ &= \frac{1/f'(r_1)}{u - r_1} + \frac{1/f'(r_2)}{u - r_2} \\ &= -\frac{1}{8\sqrt{a^2 + a}(u - r_1)} + \frac{1}{8\sqrt{a^2 + a}(u - r_2)}. \end{aligned}$$

Since  $|r_2| = |(1 + 2a) + 4\sqrt{a^2 + a}| > 1$ , we find that the only relevant pole inside of  $S^1$  is  $r_1$ . Reading off the residue from the above decomposition, we thus have

$$\begin{aligned} I &= \frac{i}{2} \cdot 2\pi i \cdot \sum \operatorname{Res}_{u=r_i} \frac{1}{f_a(u)} \\ &= -\pi \cdot \operatorname{Res}_{u=r_1} \frac{1}{f_a(u)} \\ &= \frac{\pi}{8\sqrt{a^2 + a}}. \end{aligned}$$

■

Note: I know I'm off by a constant here at least, since  $a = 1$  should reduce to  $\pi/2\sqrt{2}$ .

## 1.11 11

Find the number of roots of  $p(z) = 4z^4 - 6z + 3$  in  $|z| < 1$  and  $1 < |z| < 2$  respectively.

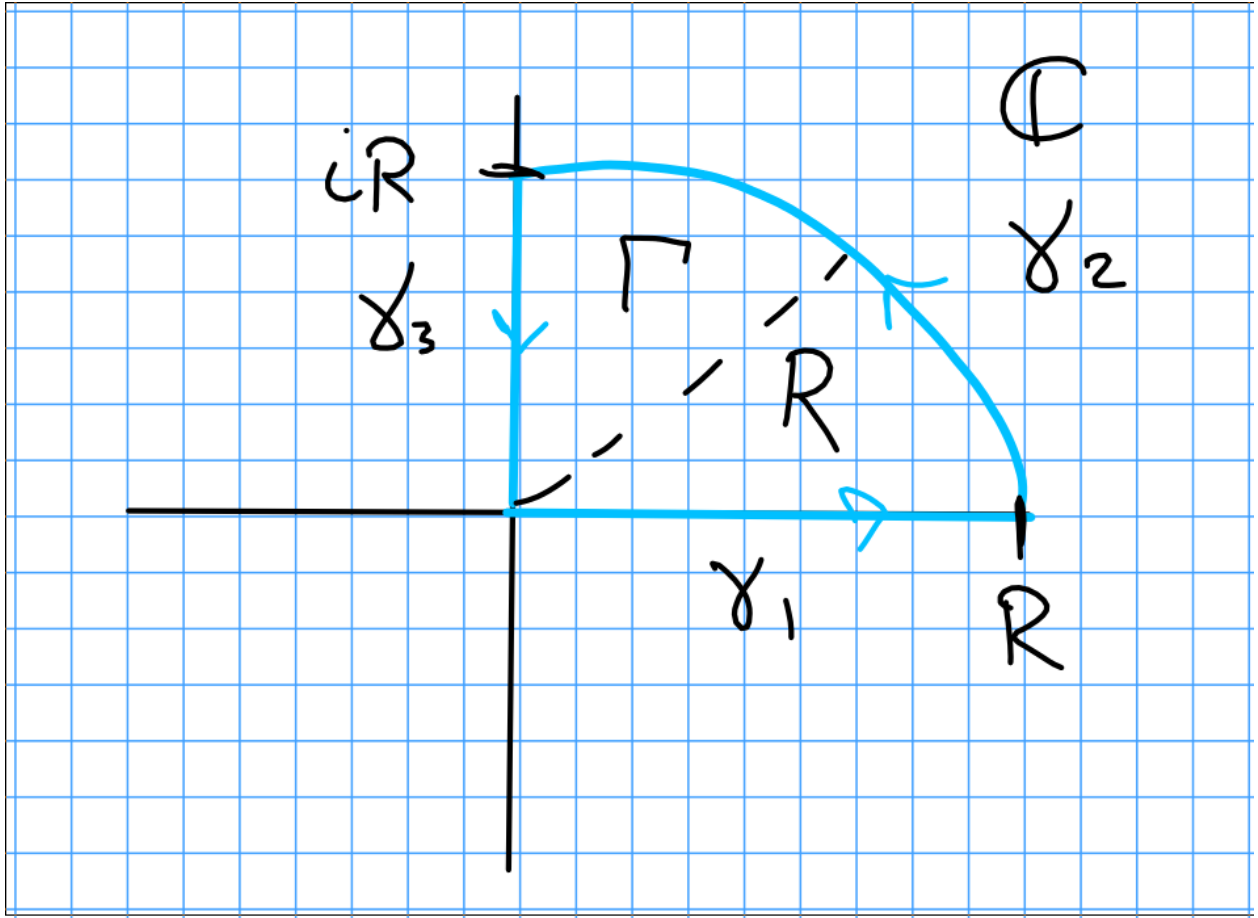
## 1.12 12

Prove that  $z^4 + 2z^3 - 2z + 10$  has exactly one root in each open quadrant.



## 1.12.1 Solution

Let  $f(z) = z^4 + 2z^3 - 2z + 10$ , and consider the following contour:



By the argument principle, we have

$$\Delta_{\Gamma} \arg f(z) = 2\pi(Z - P),$$

where  $Z$  is the number of zeros of  $f$  in the region  $\Omega$  enclosed by  $\Gamma$  and  $P$  is the number of poles in  $\Omega$ .

Since polynomials are holomorphic on  $\mathbb{C}$ , by the argument principle it suffices to show that

- $f$  does not have any roots on the real or imaginary axes
- $f$  does not vanish on  $\Gamma$ , and
- $\Delta_{\Gamma} \arg f(z) = 1$ , where  $\Delta_{\Gamma}$  denotes the total change in the argument of  $f$  over  $\Gamma$ .

It will follow by symmetry that  $f$  has exactly one root in each quadrant.

**Claim 1.2.**

- $f$  has no roots on the coordinate axes.
- $\Delta_{\gamma_1} \arg f(z) = 0$
- $\Delta_{\gamma_2} \arg f(z) = 2\pi$

$$\bullet \Delta_{\gamma_3} \arg f(z) = 0$$

Given the claim, we would have

$$\Delta_{\Gamma} \arg f(z) = 2\pi = 2\pi(Z - 0) \implies Z = 1,$$

which is what we wanted to show.

**Proof of Claim:**

$\gamma_2$ : For  $R \gg 0$ , we have  $f(z) \sim z^4$ . Along  $\gamma_2$ , the argument of  $z$  ranges from 0 to  $\frac{\pi}{2}$ , and thus the argument of  $z^4$  ranges from 0 to  $4 \cdot \frac{\pi}{2} = 2\pi$ .

$\gamma_1$ : By cases, for  $z \in \mathbb{R}$ ,

- If  $|z| > 1$ , then  $z^3 > z$  and so

$$\begin{aligned} f(z) &= (z^4 + 10) + (2z^3 - 2z) \\ &> (z^4 + 10) + (2z - 2z) \\ &= z^4 + 10 \\ &> 0, \end{aligned}$$

so  $f$  is strictly positive and does not change argument on  $(\pm 1, \pm\infty)$  or  $i \cdot (\pm 1, \pm\infty)$ .

- If  $|z| \leq 1$ ,

$$\begin{aligned} \left| -z^4 - 2z^3 + 2z \right| &\leq |z|^4 + 2|z|^3 + 2|z| \\ &\leq 1 + 2 + 2 \\ &= 5 \\ &< 10 \end{aligned}$$

$$\implies f(z) = 10 - (-z^4 - 2z^3 + 2z) > 0,$$

so  $f$  is strictly positive and does not change argument  $(0, \pm 1)$  or  $i \cdot (0, \pm 1)$ . ■

### 1.13 13

Prove that for  $a > 0$ ,  $z \tan z - a$  has only real roots.

**1.14 14**

Let  $f$  be nonzero, analytic on a bounded region  $\Omega$  and continuous on its closure  $\bar{\Omega}$ . Show that if  $|f(z)| \equiv M$  is constant for  $z \in \partial\Omega$ , then  $f(z) \equiv Me^{i\theta}$  for some real constant  $\theta$ .

**1.14.1 Solution**

By the maximum modulus principle applied to  $f$  in  $\bar{\Omega}$ , we know that  $\max |f| = M$ . Similarly, the maximum modulus principle applied to  $\frac{1}{f}$  in  $\bar{\Omega}^c$  since  $f$  is nonzero in  $\Omega$ , and we can conclude that  $\min |f| = M$  as well. Thus  $|f| = M$  is constant on  $\bar{\Omega}$ .

So consider the function  $g(z) = |f(z)|$ ; from the above observation, we find that  $g(\bar{\Omega}) = \{M\}$ . Letting  $S_M$  be the circle of radius  $M$ , this implies that  $f(\Omega) \subseteq S_M$ . In particular,  $S_M \subset \mathbb{C}$  is a closed set.

However, by the open mapping theorem,  $f(\Omega) \subset \mathbb{C}$  must be an open set. A basis for the topology on  $\mathbb{C}$  is given by open discs, so in particular, the open sets of  $\mathbb{C}$  have real dimension either zero or two. Since  $S_M$  has real dimension 1,  $f(\Omega)$  must have dimension zero and is thus a collection of points. Since  $f$  is continuous, the image can only be one point, i.e.  $f(\Omega) = \{\text{pt}\} \in S_M$ . So  $f$  is constant. ■

**2 Stein and Shakarchi****2.1 S&S 3.8.1**

Using Euler's formula

$$\sin(\pi z) = \frac{1}{2i}(e^{i\pi z} - e^{-i\pi z})$$

show that the complex zeros of  $\sin(\pi z)$  are exactly the integers, each of order one.

Calculate the residue of  $\frac{1}{\sin(\pi z)}$  at  $z = n \in \mathbb{Z}$ .

**2.2 S&S 3.8.2**

Evaluate the integral

$$\int_{\mathbb{R}} \frac{dx}{1+x^4}$$

What are the poles of the integrand?

**2.3 S&S 3.8.4**

Show that

$$\int_{\mathbb{R}} \frac{x \sin x}{x^2 + a^2} = \frac{\pi e^{-a}}{a} \quad a > 0$$

**2.4 S&S 3.8.5**

Show that for  $\xi \in \mathbb{R}$ ,

$$\int_{\mathbb{R}} \frac{e^{2\pi i x \xi}}{(1 + x^2)^2} = \frac{\pi}{2} (1 + 2\pi|\xi|) e^{-2\pi|\xi|}$$

**2.5 S&S 3.8.6**

Show that

$$\int_{\mathbb{R}} \frac{dx}{(1 + x^2)^{n+1}} = \frac{1 \cdot 3 \cdots (2n - 1)\pi}{2 \cdot 4 \cdots (2n)}$$

**2.6 S&S 3.8.7**

Show that for  $a > 1$ ,

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos \theta)^2} = \frac{2\pi a}{(a^2 - 1)^{3/2}}$$

**2.7 S&S 3.8.8**

Show that if  $a, b \in \mathbb{R}$  with  $a > |b|$  then

$$\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \frac{2\pi a}{\sqrt{a^2 - b^2}}$$

**2.8 S&S 3.8.9**

Show that

$$\int_0^1 \log(\sin \pi x) dx = -\log 2$$

**2.9 S&S 3.8.10**

Show that if  $a > 0$

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx = \frac{\pi \log a}{2a}$$

**2.10 S&S 3.8.14**

Prove that if  $f$  is entire and injective, then  $f(z) = az + b$  with  $a, b \in \mathbb{C}$  with  $a \neq 0$ .

Hint: apply the Casorati-Weierstrass theorem to  $f(1/z)$ .

**2.11 S&S 3.8.15**

Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

**2.11.1 a**

If  $f$  is entire and for all  $R > 0$ , there are constants  $A, B > 0$  such that  $\sup_{|z|=R} |f(z)| \leq AR^k + B$ , then  $f$  is a polynomial of degree less than  $k$ .

**2.11.2 b**

Show that if  $f$  is holomorphic on the unit disk, is bounded, and converges to zero uniformly in the sector  $\theta \leq \arg z \leq \phi$  as  $|z| \rightarrow 1$ , then  $f \equiv 0$ .

**2.11.3 c**

Let  $w_1, \dots, w_n$  be points on  $S^1 \subset \mathbb{C}$ . Show that there exists a point  $z \in S^1$  such that

$$\prod_{i=1}^n |z - w_i| \geq 1.$$

Conclude that there exists a point  $w \in S^1$  such that

$$\prod_{i=1}^n |w - w_i| = 1.$$

**2.11.4 d**

Show that if  $f$  is entire and  $\Re(f)$  is bounded, then  $f$  is constant.

**2.12 S&S 3.8.17**

Let  $f$  be non-constant, and holomorphic in an open set containing the open unit disc.

**2.12.1 a**

Show that  $|z| = 1 \implies |f(z)| = 1$ , then the image of  $f$  contains the unit disc.

Hint: Show that  $f(z) = w_0$  has a root for every  $w_0 \in \mathbb{D}$ , for which it suffices to show that  $f(z) = 0$  has a root, then use the maximum modulus principle.

**2.12.2 b**

Show that if  $|z| \geq 1 \implies |f(z)| = 1$  **and** there exists a point  $z_0 \in \mathbb{D}$  such that  $|f(z_0)| < 1$ , then the image of  $f$  contains the unit disc.

**2.13 S&S 3.8.19**

Prove the maximum modulus principle for harmonic functions; i.e.,

**2.13.1 a**

If  $u$  is a non-constant real-valued harmonic function on  $\Omega$ , then  $u$  can not attain its extrema on  $\Omega$ .

**2.13.2 b**

Suppose  $\Omega$  has compact closure  $\overline{\Omega}$ , then if  $u$  is harmonic on  $\Omega$  and continuous on  $\overline{\Omega}$ , then

$$\sup_{z \in \Omega} |u(z)| \leq \sup_{z \in \overline{\Omega} - \Omega} |u(z)|$$

Hint: to prove (a), assume that  $u$  attains a local maximum at  $z_0$ , and let  $f$  be holomorphic near  $z_0$  with  $u = \Re(f)$ , then show that  $f$  is not open. Part (b) is a direct consequence.