

# Complex Analysis Problem Set 3

D. Zack Garza

Tuesday 7<sup>th</sup> April, 2020

## Contents

<b>1</b>	<b>Problems From Tie</b>	<b>2</b>
1.1	1 . . . . .	2
	1.1.1 Solution . . . . .	2
1.2	2 . . . . .	3
	1.2.1 Solution . . . . .	3
1.3	3 . . . . .	4
	1.3.1 Solution . . . . .	5
1.4	4 . . . . .	6
	1.4.1 Solution . . . . .	6
1.5	5 . . . . .	6
	1.5.1 Solution . . . . .	6
1.6	6 . . . . .	7
	1.6.1 Solution Part (a) . . . . .	7
	1.6.2 Solution Part (b) . . . . .	8
1.7	7 . . . . .	8
	1.7.1 Solution . . . . .	8
1.8	8 . . . . .	9
	1.8.1 Solution (Rouche) . . . . .	10
	1.8.2 Solution (Maximum Modulus Principle) . . . . .	10
1.9	9 . . . . .	11
	1.9.1 Solution . . . . .	11
1.10	10 . . . . .	11
1.11	11 . . . . .	13
1.12	12 . . . . .	13
	1.12.1 Solution . . . . .	13
1.13	13 . . . . .	15
1.14	14 . . . . .	15
	1.14.1 Solution . . . . .	15
<b>2</b>	<b>Stein and Shakarchi</b>	<b>15</b>
2.1	S&S 3.8.1 . . . . .	15
2.2	S&S 3.8.2 . . . . .	15
2.3	S&S 3.8.4 . . . . .	16
2.4	S&S 3.8.5 . . . . .	16

---

2.5	S&S 3.8.6	16
2.6	S&S 3.8.7	16
2.7	S&S 3.8.8	16
2.8	S&S 3.8.9	17
2.9	S&S 3.8.10	17
2.10	S&S 3.8.14	17
2.11	S&S 3.8.15	17
2.11.1	a	17
2.11.2	b	17
2.11.3	c	17
2.11.4	d	18
2.12	S&S 3.8.17	18
2.12.1	a	18
2.12.2	b	18
2.13	S&S 3.8.19	18
2.13.1	a	18
2.13.2	b	18

## 1 Problems From Tie

### 1.1 1

Prove that if  $f$  has two Laurent series expansions,

$$f(z) = \sum c_n(z-a)^n \quad \text{and} \quad f(z) = \sum c'_n(z-a)^n$$

then  $c_n = c'_n$ .

#### 1.1.1 Solution

By taking the difference of two such expansions, it suffices to show that if  $f$  is identically zero and  $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$  about some point  $a$ , then  $c_n = 0$  for all  $n$ .

Under this assumption, let  $D_\varepsilon(a)$  be a disc about  $a$  and  $\gamma$  be any contour contained in its interior. Then for each  $n$ , we can apply the formula

$$\begin{aligned}
c_n &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi-a)^{n+1}} d\xi \\
&= \frac{1}{2\pi i} \int_{\gamma} \frac{0}{(\xi-a)^{n+1}} d\xi \quad \text{by assumption} \\
&= 0,
\end{aligned}$$

which shows that  $c_n = 0$  for all  $n$ .

■

## 1.2 2

Find Laurent series expansions of

$$\frac{1}{1-z^2} + \frac{1}{3-z}$$

How many such expansions are there? In what domains are each valid?

## 1.2.1 Solution

Note that  $f$  has poles at  $z = -1, 1, 3$ , all with multiplicity 1, and so there are 3 regions to consider:

1.  $|z| < 1$
2.  $1 < |z| < 3$
3.  $3 < |z|$ .



**Region 1:** Take the following expansion:

$$\begin{aligned} f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\ &= \sum_{n \geq 0} z^{2n} + \frac{1}{3} \left( \frac{1}{1 - \frac{z}{3}} \right) \\ &= \sum_{n \geq 0} z^{2n} + \frac{1}{3} \sum_{n \geq 0} \left( \frac{1}{3} \right)^n z^n \\ &= \sum_{n \geq 0} z^{2n} + \sum_{n \geq 0} \left( \frac{1}{3} \right)^{n+1} z^n \end{aligned}$$

Noting  $|z^2| < 1$  implies then  $|z| < 1$ , and that the first term converges for  $|z^2| < 1$  and the second for  $\left| \frac{z}{3} \right| < 1 \iff |z| < 3$ , this expansion converges to  $f$  on the region  $|z| < 1$ .

**Region 2:** Take the following expansion:

$$\begin{aligned}
 f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\
 &= -\frac{1}{z^2} \left( \frac{1}{1-\frac{1}{z^2}} \right) - \frac{1}{3} \left( \frac{1}{1-\frac{z}{3}} \right) \\
 &= -\frac{1}{z^2} \sum_{n \geq 0} z^{-2n} + \sum_{n \geq 0} \left( \frac{1}{3} \right)^{n+1} z^n \\
 &= -\sum_{n \geq 2} \frac{1}{z^{2n}} + \sum_{n \geq 0} \left( \frac{1}{3} \right)^{n+1} z^n
 \end{aligned}$$

By construction, the first term converges for  $\left| \frac{1}{z^2} \right| < 1 \iff |z| > 1$  and the second for  $|z| < 3$ .

**Region 3:** Take the following expansion:

$$\begin{aligned}
 f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\
 &= -\frac{1}{z^2} \left( \frac{1}{1-\frac{1}{z^2}} \right) - \frac{1}{z} \left( \frac{1}{1-\frac{z}{3}} \right) \\
 &= -\frac{1}{z^2} \sum_{n \geq 0} \frac{1}{z^{2n}} - \frac{1}{z} \sum_{n \geq 0} 3^n \frac{1}{z^n} \\
 &= -\sum_{n \geq 2} \frac{1}{z^{2n}} - \sum_{n \geq 1} \left( \frac{1}{3} \right)^{n-1} \frac{1}{z^n}.
 \end{aligned}$$

Note: in principle, terms could be collected here.

By construction, this converges on  $\{|z|^2 > 1\} \cap \{|z| > 3\} = \{|z| > 3\}$ .

■

### 1.3 3

Let  $P, Q$  be polynomials with no common zeros. Assume  $a$  is a root of  $Q$ . Find the principal part of  $P/Q$  at  $z = a$  in terms of  $P$  and  $Q$  if  $a$  is (1) a simple root, and (2) a double root.

### 1.3.1 Solution

We'll use the following definition: if  $f : \mathbb{C} \rightarrow \mathbb{C}$  is analytic with Laurent expansion  $f(z) = \sum_{k=-\infty}^{\infty} c_k(z-a)^k$  at the point  $a \in \mathbb{C}$ , then the **principal part** of  $f$  at  $a$  is given by

$$\sum_{k=-1}^{-\infty} c_k(z-a)^k = c_{-1}(z-a)^{-1} + c_{-2}(z-a)^{-2} + \dots$$

Without loss of generality (by performing polynomial long division if necessary), assume that  $\deg P < \deg Q$ . By the method used in the theorem that proves meromorphic functions are rational, if we let  $a_1, \dots, a_n$  be the finitely many zeros of  $Q(z)$ , these are the finitely many poles of  $P(z)/Q(z)$ , and we can write

$$\frac{P(z)}{Q(z)} := f(z) = P_{\infty}(z) + \sum_{i=1}^n P_{a_i}(z)$$

where  $P_w(z)$  denotes the principal part of  $f$  at the point  $w$ .

Note that if  $w$  is a pole of order  $\ell$ , we can explicitly write

$$P_w(z) = \frac{\alpha_1}{z-w} + \frac{\alpha_2}{(z-w)^2} + \dots + \frac{\alpha_{\ell}}{(z-w)^{\ell}}$$

for some constants  $\alpha_i \in \mathbb{C}$ , and thus the first equation expresses  $f$  in terms of its partial fraction decomposition.

Thus if  $a$  is a simple root of  $Q(z)$ , it is a simple pole of  $f$ , and thus we have  $P_a(z) = \frac{\alpha_1}{z-a}$ , which consists of a single term. Since we can write  $f(z) = P_{\infty}(z) + P_a(z) + \dots$  where none of the remaining terms involve  $a$ , it follows by definition that  $\alpha_1 = \text{Res}(f, a)$  and so

$$P_a(z) = \frac{\text{Res}(f(z), a)}{z-a},$$

where we can use a known formula to express  $\text{Res}(f(z), a) = \frac{P(a)}{Q'(a)}$ .

Similarly, if now  $a$  is a root of multiplicity 2 of  $Q(z)$ ,  $a$  is a pole of order 2 of  $f$  and  $P_a(z) = \frac{\alpha_1}{z-a} + \frac{\alpha_2}{(z-a)^2}$  with precisely two terms. Thus as before,  $\alpha_1 = \text{Res}(f(z), a)$ , and now  $\alpha_2 = \text{Res}((z-a)f(z), a)$ , and we have

$$P_a(z) = \frac{\text{Res}(f(z), a)}{z-a} + \frac{\text{Res}((z-a)f(z), a)}{(z-a)^2}.$$

■

## 1.4 4

Let  $f$  be non-constant, analytic in  $|z| > 0$ , where  $f(z_n) = 0$  for infinitely many points  $z_n$  with  $\lim_{n \rightarrow \infty} z_n = 0$ .

Show that  $z = 0$  is an essential singularity for  $f$ .

Example:  $f(z) = \sin(1/z)$ .

## 1.4.1 Solution

We first note that  $z = 0$  is in fact a singularity of  $f$ , since the zeros of analytic functions are isolated.

The point  $z = 0$  can not be a pole because (by definition) this would force  $\lim_{z \rightarrow 0} |f(z)| = \infty$ . Explicitly, this would mean that for every  $R > 0$ , there would exist a  $\delta > 0$  such that  $z \in D_\delta(0) \implies |f(z)| > R$ .

However, since  $z_n \rightarrow 0$  and  $f(z_n) = 0 < R$  for every  $n$ , every  $D_\delta(0)$  contains a point  $z_N$  that violates this condition.

Similarly,  $z = 0$  can not be removable, since the function

$$g(z) = \begin{cases} 0 & z = 0 \\ f(z) & \text{otherwise} \end{cases}$$

defines an analytic continuation of  $f$ . However, it is a theorem that the zeros of an analytic function are isolated, whereas every neighborhood of  $z = 0$  (which is a zero of  $g$ ) contains infinitely many distinct zeros of the form  $z_n$ , a contradiction. ■

## 1.5 5

Show that if  $f$  is entire and  $\lim_{z \rightarrow \infty} f(z) = \infty$ , then  $f$  is a polynomial.

## 1.5.1 Solution

Since  $f$  is entire, it is analytic on  $\mathbb{C}$ , so there is an expansion  $f(z) = \sum_{k=0}^{\infty} c_k z^k$  that converges to  $f$  everywhere. Let  $F(z) = f(1/z)$ ; then  $\lim_{z \rightarrow 0} F(z) = \infty$  by assumption.

This also implies that since  $z = \infty$  is a pole of  $f$ , the point  $z = 0$  is a pole of  $F$ , say of order  $N$ .

However, we can expand  $F(z) = \sum_{k=0}^{\infty} c_k \frac{1}{z^k}$ . Since this is a Laurent expansion for  $F$  about  $z = 0$ ,

which is a pole of order  $N$ , we must in fact have  $F(z) = \sum_{k=0}^N c_k \frac{1}{z^k}$ , i.e. there are only  $N$  terms in this expansion.

This implies that  $f(z) = \sum_{k=0}^N c_k z^k$ , which has finitely many terms and is thus a polynomial. ■

## 1.6 6

a. Show that

$$\int_0^{2\pi} \log |1 - e^{i\theta}| d\theta = 0$$

b. Show that this identity is equivalent to SS 3.8.9:

$$\int_0^1 \log(\sin(\pi x)) dx = -\log 2.$$

### 1.6.1 Solution Part (a)

Let  $I$  be the integral in question, then substituting  $z = e^{i\theta}$  and  $\frac{dz}{iz} = d\theta$  yields

$$I = \int_{S^1} \frac{\log |1 - z|}{iz} dz := \Re \left( \int_{S^1} f(z) dz \right),$$

where

$$f(z) := \frac{\log(1 - z)}{iz},$$

$S^1$  denotes the unit circle in  $\mathbb{C}$ , and since by definition

$$\log_{\mathbb{C}}(z) = \log_{\mathbb{R}}(|z|) + i \arg(z)$$

where the subscripts denote the complex and real logarithms respectively, we have

$$\log_{\mathbb{C}} |1 - z| = \Re(\log_{\mathbb{C}}(1 - z)).$$

So it suffices to show that  $\int_{S^1} f(z) dz = 0$ .

The claim is that  $z = 0$  is a removable singularity and thus  $f$  is holomorphic in the unit disc. The singularity is removable because we have

$$\begin{aligned} \lim_{z \rightarrow 0} f(z) &= \lim_{z \rightarrow 0} \frac{\log(1 - z)}{iz} \\ &= \lim_{z \rightarrow 0} \frac{\frac{1}{1-z}}{i} \quad \text{by L'Hopital's} \\ &= -i, \end{aligned}$$

so the modified function

$$F(z) = \begin{cases} -i & z = 0 \\ f(z) & \text{otherwise} \end{cases}$$

is holomorphic, making  $z = 0$  removable.

Since  $f$  is also analytic, the Cauchy-Goursat theorem applies and  $\int_{S^1} f = 0$ .

### 1.6.2 Solution Part (b)

No clue how to relate these two!

## 1.7 7

Let  $0 < a < 4$  and evaluate

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^3} dx$$

### 1.7.1 Solution

Let  $I$  denote the integral in question. We will compute this using a closed contour and the residue theorem, so first note that

$$z^3 + 1 = (z + 1)(z - e^{i\pi/3})(z - e^{-i\pi/3}) := (z - r_1)(z - r_2)(z - r_3).$$

Defining  $z^\alpha = e^{\alpha \log z}$  for  $\alpha \in \mathbb{R}$ , we'll take the following contour  $\Gamma$  shown in blue along with a branch cut for the logarithm function indicated in red:



Letting

$$f(z) := \frac{z^{\alpha-1}}{z^3 + 1} := \frac{P(z)}{Q(z)},$$



we find that only  $z = r_2$  will contribute a term to  $\int_{\Gamma} f$ . Noting that each pole is simple of order 1, we have

$$\text{Res}(f(z), z = r_i) = \frac{P(r_i)}{Q'(r_i)} = \frac{r_i^{\alpha-1}}{3r_i^2} = \frac{r_i^{\alpha-3}}{3}$$

We thus have

$$\begin{aligned} \text{Res}(f(z), z = r_2) &= \frac{1}{3} e^{\frac{i\pi(\alpha-3)}{3}} \\ \implies \int_{\Gamma} f(z) dz &= \frac{2\pi i}{3} e^{\frac{i\pi(\alpha-3)}{3}}. \end{aligned}$$

We can now compute the contributions to the integral along the semicircular arc and the portion along the imaginary axis.

Along the arc, Jordan's lemma applies since  $\frac{1}{R^3 + 1} \xrightarrow{R \rightarrow \infty} 0$ , and thus this contribution vanishes.

Along the imaginary axis, we can make the following change of variables:

$$\begin{aligned} \int_R^0 f(iy) dy &= - \int_0^R \frac{(iy)^{\alpha-1}}{(iy)^3 + 1} dy \\ &= -\frac{1}{i} \int_0^R \frac{t^{\alpha-1}}{t^3 + 1} dt \quad (t = iz, dt = idz) \\ &= iI, \end{aligned}$$

which is  $i$  times the original integral.

We thus have

$$\begin{aligned} \text{Res}(f(z), z = r_2) &= \int_{\Gamma} f \\ &= \int_0^R f + \int_{C_R} f + \int_{iR}^0 f \\ &\xrightarrow{R \rightarrow \infty} I + 0 + iI \\ &= (1 + i)I, \end{aligned}$$

and so

$$I = \frac{\text{Res}(f(z), z = r_2)}{1 + i} = \frac{2\pi i}{3(1 + i)} e^{\frac{i\pi(\alpha-3)}{3}}.$$

■

Note: this seems to be wrong, because plugging in  $a = 1, 2, 3$  doesn't result in a real value.

Prove the fundamental theorem of Algebra using

- Rouche's Theorem.
- The maximum modulus principle.

### 1.8.1 Solution (Rouche)

We want to show that every  $f \in \mathbb{C}[x]$  has precisely  $n$  roots, and we'll use the following formulation of Rouché's theorem:

**Theorem 1.1 (Rouche).**

If  $f, g$  are holomorphic on  $D(z_0)$  with  $f, g \neq 0$  and  $|f - g| < |f| + |g|$  on  $\partial D(z_0)$ , then  $f$  and  $g$  has the same number of zeros within  $D$ .

We'll also use without proof the fact that the function  $h(z) = z^n$  has precisely  $n$  zeros (counted with multiplicity).

Suppose  $f(z) = a_n z^n + \cdots + a_1 z + a_0$  where  $a_n \neq 0$  and define

$$g(z) := a_n z^n.$$

Noting that polynomials are entire,  $f, g$  are nonzero by assumption, and fixing  $|z| = R > 1$ , we have

$$\begin{aligned} |f - g| &= |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| \\ &= |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 + a_n z^n - a_n z^n| \\ &\leq |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 + a_n z^n| + |-a_n z^n| \quad \text{by the triangle inequality} \\ &= |f| + |g| \end{aligned}$$

the conditions of Rouché's theorem apply and  $f, g$  have the same number of roots. Since  $g$  has precisely  $n$  roots,  $f$  does as well.

This is much simpler than other proofs out there, so I suspect something is slightly wrong but I couldn't sort out what it was.



### 1.8.2 Solution (Maximum Modulus Principle)

Toward a contradiction, suppose  $f$  is non-constant and has no zeros. Then  $g(z) := 1/f(z)$  is non-constant and holomorphic on  $\mathbb{C}$ .

Using the fact that  $\lim_{z \rightarrow \infty} f(z) = \infty$  for any polynomial  $f$ , pick  $R$  large enough such that

$$|z| > R \implies |f(z)| > |f(0)|$$

where  $D_R(0)$  is an open disc of radius  $r$  about  $z = 0$ .

Then by inverting,

$$|z| > R \implies |g(z)| < |g(0)|.$$

Noting that  $\overline{D_R}(0)$  is closed, bounded (by the argument above), and thus compact by Heine-Borel,  $g$  attains a maximum in the interior  $D_r^\circ$ . But by the maximum modulus principle, this forces  $g$  to be constant, and since  $g = \frac{1}{f}$ , it must also be true that  $f$  is constant. ■

## 1.9 9

Let  $f$  be analytic in a region  $D$  and  $\gamma$  a rectifiable curve in  $D$  with interior in  $D$ . Prove that if  $f(z)$  is real for all  $z \in \gamma$ , then  $f$  is constant.

### 1.9.1 Solution

Since  $f$  is analytic in  $D$  (wlog assuming  $0 \in D$  by translation), take its series expansion  $f(z) = c_0 + c_1z + \cdots$  for  $z \in D$ .

Without loss of generality, suppose that  $\gamma$  is not entirely contained in  $\mathbb{R}$ , so for  $z \in \gamma$  we can write  $z = x + iy$  where  $y \neq 0$ .

Then

$$\begin{aligned} f(z) &= f(x + iy) \\ &= c_0 + c_1(x + iy) + \cdots \\ &= c_0 + c_1x + ic_1y + \cdots \end{aligned} \quad \subset \mathbb{R} \quad \text{by assumption,}$$

and so we must have  $c_1y = 0 \implies c_1 = 0$ . The same argument applies to further terms in the expansion, so we in fact have  $c_i = 0$  for every  $i \geq 1$ .

But this says  $f(z) = c_0$  for an arbitrary  $z$ , i.e.  $f$  is constant. ■

## 1.10 10

For  $a > 0$ , evaluate

$$\int_0^{\pi/2} \frac{d\theta}{a + \sin^2 \theta}$$

We have

$$\begin{aligned}
I &:= \int_0^{\pi/2} \frac{1}{1 + \sin^2(\theta)} d\theta \\
&= \int_{\gamma_1} \frac{1}{a + \left(\frac{z-z^{-1}}{2i}\right)^2} \frac{-i dz}{z} \quad \text{where } \gamma_1 \text{ is } \frac{1}{4} \text{ of the unit circle } S^1 \\
&= -i \int_{\gamma_1} \frac{1}{z} \left( \frac{1}{a + \left(-\frac{1}{4}\right)(z^2 - 2 + z^{-2})} \right) dz \\
&= 4i \int_{\gamma_1} \frac{1}{z} \left( \frac{1}{z^2 - (2 + 4a) + z^{-2}} \right) dz \\
&= 4i \int_{\gamma_1} \frac{z}{z^4 - (2 + 4a)z^2 + 1} dz \\
&= i \oint_{S^1} \frac{z}{z^4 - (2 + 4a)z^2 + 1} dz \\
&= \frac{i}{2} \oint_{2 \cdot S^1} \frac{1}{u^2 - (2 + 4a)u + 1} du \quad \text{using } u = z^2, \frac{1}{2} du = z dz \\
&:= \frac{i}{2} \oint_{2 \cdot S^1} \frac{1}{f_a(u)} du \\
&= \frac{i}{2} \cdot 2\pi i \cdot \sum \text{Res}_{u=r_i} \frac{1}{f_a(u)},
\end{aligned}$$

where  $2 \cdot S^1$  denotes the contour wrapping around the unit circle twice and  $r_i$  denote the poles contained in the region bounded by  $S^1$ . We can now compute the last integral by the residue theorem.

Factor the denominator as

$$f_a(u) = u^2 - (2 + 4a)u + 1 = (u - r_1)(u - r_2),$$

where the  $r_i$  are given by  $(1 + 2a) \pm 4\sqrt{a^2 + a}$  using the quadratic formula. We can then write a partial fraction decomposition

$$\begin{aligned}
\frac{1}{f_a(u)} &:= \frac{1}{u^2 - (2 + 4a)u + 1} \\
&= \frac{1}{(u - r_1)(u - r_2)} \\
&= \frac{A}{u - r_1} + \frac{B}{u - r_2} \\
&= \frac{\text{Res}_{u=r_1} 1/f(u)}{u - r_1} + \frac{\text{Res}_{u=r_2} 1/f(u)}{u - r_2} \\
&= \frac{1/f'(r_1)}{u - r_1} + \frac{1/f'(r_2)}{u - r_2} \\
&= -\frac{1}{8\sqrt{a^2 + a}(u - r_1)} + \frac{1}{8\sqrt{a^2 + a}(u - r_2)}.
\end{aligned}$$

Since  $|r_2| = |(1 + 2a) + 4\sqrt{a^2 + a}| > 1$ , we find that the only relevant pole inside of  $S^1$  is  $r_1$ . Reading

off the residue from the above decomposition, we thus have

$$\begin{aligned}
 I &= \frac{i}{2} \cdot 2\pi i \cdot \sum \operatorname{Res}_{u=r_i} \frac{1}{f_a(u)} \\
 &= -\pi \cdot \operatorname{Res}_{u=r_1} \frac{1}{f_a(u)} \\
 &= \frac{\pi}{8\sqrt{a^2 + a}}.
 \end{aligned}$$

■

Note: I know I'm off by a constant here at least, since  $a = 1$  should reduce to  $\pi/2\sqrt{2}$ .

## 1.11 11

Find the number of roots of  $p(z) = 4z^4 - 6z + 3$  in  $|z| < 1$  and  $1 < |z| < 2$  respectively.

For  $|z| < 1$ , take  $f(z) = -6z$  and  $g(z) = z^4 + 3$ , noting that  $f + g = p$ . Using the maximum modulus principal, we know that the max/mins of  $f, g$  occur on  $|z| = 1$ , on which we have

$$|g(z)| = 4 < 6 = |f(z)|,$$

so Rouché's theorem applies and both  $p$  and  $f$  have the same number of zeros. Since  $f$  clearly has **one** zero,  $p$  has one zero in this region.

Now consider  $|z| < 2$  and set  $f(z) = z^4$  and  $g(z) = -6z + 3$ . By a similar argument, we have

$$|g(z)| = 15 < 16 = |f|$$

on  $|z| = 2$ , and thus  $f$  and  $p$  have the same number of zeros in this region. Since  $f$  has **four** zeros here, so does  $p$ .

Thus  $p$  has  $4 - 1 = \mathbf{3}$  zeros on  $1 \leq |z| \leq 2$ .

■

## 1.12 12

Prove that  $z^4 + 2z^3 - 2z + 10$  has exactly one root in each open quadrant.

### 1.12.1 Solution

Let  $f(z) = z^4 + 2z^3 - 2z + 10$ , and consider the following contour:

Image

By the argument principle, we have

$$\Delta_{\Gamma} \arg f(z) = 2\pi(Z - P),$$

where  $Z$  is the number of zeros of  $f$  in the region  $\Omega$  enclosed by  $\Gamma$  and  $P$  is the number of poles in  $\Omega$ . Since polynomials are holomorphic on  $\mathbb{C}$ , by the argument principle it suffices to show that

- $f$  does not have any roots on the real or imaginary axes
- $f$  does not vanish on  $\Gamma$ , and
- $\Delta_{\Gamma} \arg f(z) = 1$ , where  $\Delta_{\Gamma}$  denotes the total change in the argument of  $f$  over  $\Gamma$ .

It will follow by symmetry that  $f$  has exactly one root in each quadrant.

**Claim 1.2.**

- $f$  has no roots on the coordinate axes.
- $\Delta_{\gamma_1} \arg f(z) = 0$
- $\Delta_{\gamma_2} \arg f(z) = 2\pi$
- $\Delta_{\gamma_3} \arg f(z) = 0$

Given the claim, we would have

$$\Delta_{\Gamma} \arg f(z) = 2\pi = 2\pi(Z - 0) \implies Z = 1,$$

which is what we wanted to show.

**Proof of Claim:**

$\gamma_2$ : For  $R \gg 0$ , we have  $f(z) \sim z^4$ . Along  $\gamma_2$ , the argument of  $z$  ranges from 0 to  $\frac{\pi}{2}$ , and thus the argument of  $z^4$  ranges from 0 to  $4 \cdot \frac{\pi}{2} = 2\pi$ .

$\gamma_1$ : By cases, for  $z \in \mathbb{R}$ ,

- If  $|z| > 1$ , then  $z^3 > z$  and so

$$\begin{aligned} f(z) &= (z^4 + 10) + (2z^3 - 2z) \\ &> (z^4 + 10) + (2z - 2z) \\ &= z^4 + 10 \\ &> 0, \end{aligned}$$

so  $f$  is strictly positive and does not change argument on  $(\pm 1, \pm\infty)$  or  $i \cdot (\pm 1, \pm\infty)$ .

- If  $|z| \leq 1$ ,

$$\begin{aligned} \left| -z^4 - 2z^3 + 2z \right| &\leq |z|^4 + 2|z|^3 + 2|z| \\ &\leq 1 + 2 + 2 \\ &= 5 \\ &< 10 \\ \implies f(z) &= 10 - (-z^4 - 2z^3 + 2z) > 0, \end{aligned}$$

so  $f$  is strictly positive and does not change argument  $(0, \pm 1)$  or  $i \cdot (0, \pm 1)$ .

■

**1.13 13**

Prove that for  $a > 0$ ,  $z \tan z - a$  has only real roots.

**1.14 14**

Let  $f$  be nonzero, analytic on a bounded region  $\Omega$  and continuous on its closure  $\bar{\Omega}$ . Show that if  $|f(z)| \equiv M$  is constant for  $z \in \partial\Omega$ , then  $f(z) \equiv Me^{i\theta}$  for some real constant  $\theta$ .

**1.14.1 Solution**

By the maximum modulus principle applied to  $f$  in  $\bar{\Omega}$ , we know that  $\max |f| = M$ . Similarly, the maximum modulus principle applied to  $\frac{1}{f}$  in  $\bar{\Omega}^c$  since  $f$  is nonzero in  $\Omega$ , and we can conclude that  $\min |f| = M$  as well. Thus  $|f| = M$  is constant on  $\bar{\Omega}$ .

So consider the function  $g(z) = |f(z)|$ ; from the above observation, we find that  $g(\bar{\Omega}) = \{M\}$ . Letting  $S_M$  be the circle of radius  $M$ , this implies that  $f(\Omega) \subseteq S_M$ . In particular,  $S_M \subset \mathbb{C}$  is a closed set.

However, by the open mapping theorem,  $f(\Omega) \subset \mathbb{C}$  must be an open set. A basis for the topology on  $\mathbb{C}$  is given by open discs, so in particular, the open sets of  $\mathbb{C}$  have real dimension either zero or two. Since  $S_M$  has real dimension 1,  $f(\Omega)$  must have dimension zero and is thus a collection of points. Since  $f$  is continuous, the image can only be one point, i.e.  $f(\Omega) = \{\text{pt}\} \in S_M$ . So  $f$  is constant. ■

**2 Stein and Shakarchi****2.1 S&S 3.8.1**

Using Euler's formula

$$\sin(\pi z) = \frac{1}{2i}(e^{i\pi z} - e^{-i\pi z})$$

show that the complex zeros of  $\sin(\pi z)$  are exactly the integers, each of order one.

Calculate the residue of  $\frac{1}{\sin(\pi z)}$  at  $z = n \in \mathbb{Z}$ .

**2.2 S&S 3.8.2**

Evaluate the integral

$$\int_{\mathbb{R}} \frac{dx}{1+x^4}$$

What are the poles of the integrand?

**2.3 S&S 3.8.4**

Show that

$$\int_{\mathbb{R}} \frac{x \sin x}{x^2 + a^2} = \frac{\pi e^{-a}}{a} \quad a > 0$$

**2.4 S&S 3.8.5**

Show that for  $\xi \in \mathbb{R}$ ,

$$\int_{\mathbb{R}} \frac{e^{2\pi i x \xi}}{(1 + x^2)^2} = \frac{\pi}{2} (1 + 2\pi|\xi|) e^{-2\pi|\xi|}$$

**2.5 S&S 3.8.6**

Show that

$$\int_{\mathbb{R}} \frac{dx}{(1 + x^2)^{n+1}} = \frac{1 \cdot 3 \cdots (2n - 1)\pi}{2 \cdot 4 \cdots (2n)}$$

**2.6 S&S 3.8.7**

Show that for  $a > 1$ ,

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos \theta)^2} = \frac{2\pi a}{(a^2 - 1)^{3/2}}$$

**2.7 S&S 3.8.8**

Show that if  $a, b \in \mathbb{R}$  with  $a > |b|$  then

$$\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \frac{2\pi a}{\sqrt{a^2 - b^2}}$$



**2.8 S&S 3.8.9**

Show that

$$\int_0^1 \log(\sin \pi x) \, dx = -\log 2$$

**2.9 S&S 3.8.10**

Show that if  $a > 0$

$$\int_0^\infty \frac{\log x}{x^2 + a^2} \, dx = \frac{\pi \log a}{2a}$$

**2.10 S&S 3.8.14**

Prove that if  $f$  is entire and injective, then  $f(z) = az + b$  with  $a, b \in \mathbb{C}$  with  $a \neq 0$ .

Hint: apply the Casorati-Weierstrass theorem to  $f(1/z)$ .

**2.11 S&S 3.8.15**

Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

**2.11.1 a**

If  $f$  is entire and for all  $R > 0$ , there are constants  $A, B > 0$  such that  $\sup_{|z|=R} |f(z)| \leq AR^k + B$ , then  $f$  is a polynomial of degree less than  $k$ .

**2.11.2 b**

Show that if  $f$  is holomorphic on the unit disk, is bounded, and converges to zero uniformly in the sector  $\theta \leq \arg z \leq \phi$  as  $|z| \rightarrow 1$ , then  $f \equiv 0$ .

**2.11.3 c**

Let  $w_1, \dots, w_n$  be points on  $S^1 \subset \mathbb{C}$ . Show that there exists a point  $z \in S^1$  such that

$$\prod_{i=1}^n |z - w_i| \geq 1.$$

Conclude that there exists a point  $w \in S^1$  such that

$$\prod_{i=1}^n |w - w_i| = 1.$$

**2.11.4 d**

Show that if  $f$  is entire and  $\Re(f)$  is bounded, then  $f$  is constant.

**2.12 S&S 3.8.17**

Let  $f$  be non-constant, and holomorphic in an open set containing the open unit disc.

**2.12.1 a**

Show that  $|z| = 1 \implies |f(z)| = 1$ , then the image of  $f$  contains the unit disc.

Hint: Show that  $f(z) = w_0$  has a root for every  $w_0 \in \mathbb{D}$ , for which it suffices to show that  $f(z) = 0$  has a root, then use the maximum modulus principle.

**2.12.2 b**

Show that if  $|z| \geq 1 \implies |f(z)| = 1$  **and** there exists a point  $z_0 \in \mathbb{D}$  such that  $|f(z_0)| < 1$ , then the image of  $f$  contains the unit disc.

**2.13 S&S 3.8.19**

Prove the maximum modulus principle for harmonic functions; i.e.,

**2.13.1 a**

If  $u$  is a non-constant real-valued harmonic function on  $\Omega$ , then  $u$  can not attain its extrema on  $\Omega$ .

**2.13.2 b**

Suppose  $\Omega$  has compact closure  $\overline{\Omega}$ , then if  $u$  is harmonic on  $\Omega$  and continuous on  $\overline{\Omega}$ , then

$$\sup_{z \in \Omega} |u(z)| \leq \sup_{z \in \overline{\Omega} - \Omega} |u(z)|$$

Hint: to prove (a), assume that  $u$  attains a local maximum at  $z_0$ , and let  $f$  be holomorphic near  $z_0$  with  $u = \Re(f)$ , then show that  $f$  is not open. Part (b) is a direct consequence.