Complex Analysis Problem Set 3

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Contents

1 Problems From Tie

1.1 1

Prove that if f has two Laurent series expansions,

$$f(z) = \sum c_n (z-a)^n \quad \text{and} \quad f(z) = \sum c'_n (z-a)^n$$

then $c_n = c'_n$.

1.1.1 Solution

By taking the difference of two such expansions, it suffices to show that if f is identically zero and $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$ about some point a, then $c_n = 0$ for all n.

Under this assumption, let $D_\varepsilon(a)$ be a disc about a and γ be any contoured contained in its interior. Then for each n, we can apply the formula

$$\begin{aligned} c_n &= \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - a)^{n+1}} \, d\xi \\ &= \frac{1}{2\pi i} \int_\gamma \frac{0}{(\xi - a)^{n+1}} \, d\xi \quad \text{by assumption} \\ &= 0, \end{aligned}$$

which shows that $c_n = 0$ for all n.

■

1.2 2

Find Laurent series expansions of

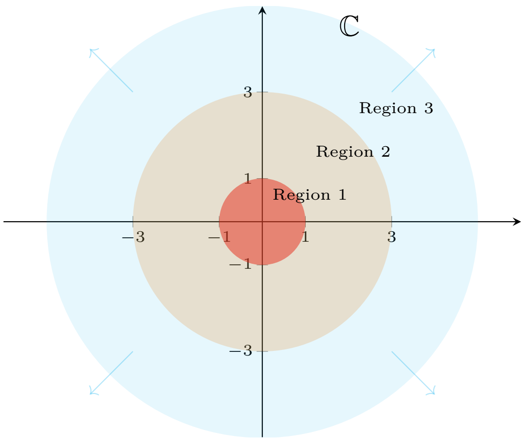
$$\frac{1}{1-z^2} + \frac{1}{3-z}$$

How many such expansions are there? In what domains are each valid?

1.2.1 Solution

Note that f has poles at $z = -1, 1, 3$, all with multiplicity 1, and so there are 3 regions to consider:

1. $|z| < 1$
2. $1 < |z| < 3$
3. $3 < |z|$.



Region 1: Take the following expansion:

$$\begin{aligned} f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\ &= \sum_{n\geq 0} z^{2n} + \frac{1}{3}\left(\frac{1}{1-\frac{3}{z}}\right) \\ &= \sum_{n\geq 0} z^{2n} + \frac{1}{3}\sum_{n\geq 0}\left(\frac{1}{3}\right)^n z^n \\ &= \sum_{n\geq 0} z^{2n} + \sum_{n\geq 0}\left(\frac{1}{3}\right)^{n+1} z^n \end{aligned}$$

.

Noting $\left|z^2\right| < 1$ implies then $|z| < 1$, and that the first term converges for $\left|z^2\right| < 1$ and the second for $\left|\frac{z}{3}\right| < 1 \iff |z| < 3$, this expansion converges to f on the region $|z| < 1$.

Region 2: Take the following expansion:

$$\begin{aligned}
f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\
&= -\frac{1}{z^2}\left(\frac{1}{1-\frac{1}{z^2}}\right) - \frac{1}{3}\left(\frac{1}{1-\frac{z}{3}}\right) \\
&= -\frac{1}{z^2}\sum_{n\geq 0} z^{-2n} + \sum_{n\geq 0}\left(\frac{1}{3}\right)^{n+1} z^n \\
&= -\sum_{n\geq 2}\frac{1}{z^{2n}} + \sum_{n\geq 0}\left(\frac{1}{3}\right)^{n+1} z^n
\end{aligned}$$

.

By construction, the first term converges for $\left|\frac{1}{z^2}\right| < 1 \iff |z| > 1$ and the second for $|z| < 3$.

Region 3: Take the following expansion:

$$\begin{aligned}
f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\
&= -\frac{1}{z^2}\left(\frac{1}{1-\frac{1}{z^2}}\right) - \frac{1}{z}\left(\frac{1}{1-\frac{3}{z}}\right) \\
&= -\frac{1}{z^2}\sum_{n\geq 0}\frac{1}{z^{2n}} - \frac{1}{z}\sum_{n\geq 0} 3^n\frac{1}{z^n} \\
&= -\sum_{n\geq 2}\frac{1}{z^{2n}} - \sum_{n\geq 1}\left(\frac{1}{3}\right)^{n-1}\frac{1}{z^n}.
\end{aligned}$$

Note: in principle, terms could be collected here.

By construction, this converges on $\left\{|z|^2 > 1\right\} \bigcap \{|z| > 3\} = \{|z| > 3\}$.

■

1.3 3

Let P, Q be polynomials with no common zeros. Assume a is a root of Q. Find the principal part of P/Q at $z = a$ in terms of P and Q if a is (1) a simple root, and (2) a double root.

1.3.1 Solution

We'll use the following definition: if $f : \mathbb{C} \longrightarrow \mathbb{C}$ is analytic with Laurent expansion $f(z) = \sum_{k=-\infty}^{\infty} c_k(z-a)^k$ at the point $a \in \mathbb{C}$, then the **principal part** of f at a is given by

$$\sum_{k=-1}^{-\infty} c_k(z-a)^k = c_{-1}(z-a)^{-1} + c_{-2}(z-a)^{-2} + \cdots .$$

Without loss of generality (by performing polynomial long division if necessary), assume that $\deg P < \deg Q$. By the method used in the theorem that proves meromorphic functions are rational, if we let $a_1, \cdots, a_n$ be the finitely many zeros of $Q(z)$, these are the finitely many poles of $P(z)/Q(z)$, and we can write

$$\frac{P(z)}{Q(z)} := f(z) = P_\infty(z) + \sum_{i=1}^{n} P_{a_i}(z)$$

where $P_w(z)$ denotes the principal part of f at the point w.

Note that if w is a pole of order ℓ, we can explicitly write

$$P_w(z) = \frac{\alpha_1}{z-w} + \frac{\alpha_2}{(z-w)^2} + \cdots + \frac{\alpha_\ell}{(z-w)^\ell}$$

for some constants $\alpha_i \in \mathbb{C}$, and thus the first equation expresses f in terms of its partial fraction decomposition.

Thus if a is a simple root of $Q(z)$, it is a simple pole of f, and thus we have $P_a(z) = \frac{\alpha_1}{z-a}$, which consists of a single term. Since we can write $f(z) = P_\infty(z) + P_a(z) + \cdots$ where none of the remaining terms involve a, it follows by definition that $\alpha_1 = \operatorname{Res}(f, a)$ and so

$$P_a(z) = \frac{\operatorname{Res}(f(z), a)}{z-a},$$

where we can use a known formula to express $\operatorname{Res}(f(z), a) = \frac{P(a)}{Q'(a)}$.

Similarly, if now a is a root of multiplicity 2 of $Q(z)$, a is a pole of order 2 of f and $P_a(z) = \frac{\alpha_1}{z-a} + \frac{\alpha_2}{(z-a)^2}$ with precisely two terms. Thus as before, $\alpha_1 = \operatorname{Res}(f(z), a)$, and now $\alpha_2 = \operatorname{Res}((z-a)f(z), a)$, and we have

$$P_a(z) = \frac{\operatorname{Res}(f(z), a)}{z-a} + \frac{\operatorname{Res}((z-a)f(z), a)}{(z-a)^2}.$$

■

1.4 4

Let f be non-constant, analytic in $|z| > 0$, where $f(z_n) = 0$ for infinitely many points z_n with $\lim_{n \longrightarrow \infty} z_n = 0$.
Show that $z = 0$ is an essential singularity for f.

Example: $f(z) = \sin(1/z)$.

1.4.1 Solution

We first note that $z = 0$ is in fact a singularity of f, since the zeros of analytic functions are isolated.

The point $z = 0$ can not be a pole because (by definition) this would force $\lim_{z \longrightarrow 0} |f(z)| = \infty$. Explicitly, this would mean that for every $R > 0$, there would exist a $\delta > 0$ such that $z \in D_\delta(0) \implies |f(z)| > R$.

However, since $z_n \longrightarrow 0$ and $f(z_n) = 0 < R$ for every n, every $D_\delta(0)$ contains a point z_N that violates this condition.

Similarly, $z = 0$ can not be removable, since the function

$$g(z) = \begin{cases} 0 & z = 0 \\ f(z) & \text{otherwise} \end{cases}$$

defines an analytic continuation of f. However, it is a theorem that the zeros of an analytic function are isolated, whereas every neighborhood of $z = 0$ (which is a zero of g) contains infinitely many distinct zeros of the form z_n, a contradiction.

■

1.5 5

Show that if f is entire and $\lim_{z \longrightarrow \infty} f(z) = \infty$, then f is a polynomial.

1.5.1 Solution

Since f is entire, it is analytic on $\mathbb{C}$, so there is an expansion $f(z) = \sum_{k=0}^{\infty} c_k z^k$ that converges to f everywhere. Let $F(z) = f(1/z)$; then $\lim_{z \longrightarrow 0} F(z) = \infty$ by assumption.

This also implies that since $z = \infty$ is a pole of f, the point $z = 0$ is a pole of F, say of order N.

However, we can expand $F(z) = \sum_{k=0}^{\infty} c_k \frac{1}{z^k}$. Since this is a Laurent expansion for F about $z = 0$, which is a pole of order N, we must in fact have $F(z) = \sum_{k=0}^{N} c_k \frac{1}{z_k}$, i.e. there are only N terms in this expansion.

This implies that $f(z) = \sum_{k=0}^{N} c_k z^k$, which has finitely many terms and is thus a polynomial.

■

1.6 6

a. Show that

$$\int_0^{2\pi} \log \left|1 - e^{i\theta}\right| \, d\theta = 0$$

b. Show that this identity is equivalent to SS 3.8.9:

$$\int_0^1 \log(\sin(\pi x)) \, dx = -\log 2.$$

1.6.1 Solution Part (a)

Let I be the integral in question, then substituting $z = e^{i\theta}$ and $\frac{dz}{iz} = d\theta$ yields

$$I = \int_{S^1} \frac{\log|1 - z|}{iz} \, dz := \Re\left(\int_{S^1} f(z) \, dz\right),$$

where

$$f(z) := \frac{\log(1 - z)}{iz},$$

S^1 denotes the unit circle in $\mathbb{C}$, and since by definition

$$\log_{\mathbb{C}}(z) = \log_{\mathbb{R}}(|z|) + i \arg(z)$$

where the subscripts denote the complex and real logarithms respectively, we have

$$\log_{\mathbb{C}} |1 - z| = \Re(\log_{\mathbb{C}} (1 - z)).$$

So it suffices to show that $\int_{S^1} f(z) \, dz = 0$.

The claim is that $z = 0$ is a removable singularity and thus f is holomorphic in the unit disc. The singularity is removable because we have

$$\begin{aligned}
\lim_{z \longrightarrow 0} f(z) &= \lim_{z \longrightarrow 0} \frac{\log(1 - z)}{iz} \\
&= \lim_{z \longrightarrow 0} \frac{\frac{1}{1-z}}{i} \qquad \text{by L'Hopital's} \\
&= -i,
\end{aligned}$$

so the modified function

$$F(z) = \begin{cases} -i & z = 0 \\ f(z) & \text{otherwise} \end{cases}$$

is holomorphic, making $z = 0$ removable.

Since f is also analytic, the Cauchy-Goursat theorem applies and $\displaystyle\int_{S^1} f = 0$.

1.6.2 Solution Part (b)

No clue how to relate these two!

1.7 7

Let $0 < a < 4$ and evaluate

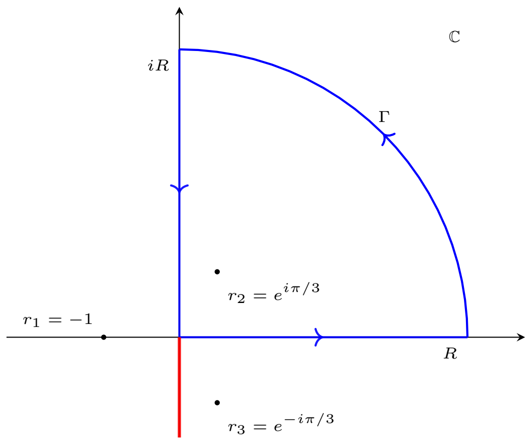
$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^3}\, dx$$

1.7.1 Solution

Let I denote the integral in question. We will compute this using a closed contour and the residue theorem, so first note that

$$z^3 + 1 = (z+1)(z - e^{i\pi 3})(z - e^{-i\pi 3}) := (z - r_1)(z - r_2)(z - r_3).$$

Defining $z^\alpha = e^{\alpha \log z}$ for $\alpha \in \mathbb{R}$, we'll take the following contour Γ shown in blue along with a branch cut for the logarithm function indicated in red:



Letting

$$f(z) := \frac{z^{\alpha-1}}{z^3+1} := \frac{P(z)}{Q(z)},$$

we find that only $z = r_2$ will contribute a term to $\int_\Gamma f$. Noting that each pole is simple of order 1, we have

$$\mathrm{Res}(f(z), z = r_i) = \frac{P(r_i)}{Q'(r_i)} = \frac{r_i^{\alpha-1}}{3r_i^2} = \frac{r_i^{\alpha-3}}{3}$$

We thus have

$$\begin{aligned}\mathrm{Res}(f(z), z = r_2) &= \frac{1}{3} e^{\frac{i\pi(\alpha-3)}{3}} \\ \implies \int_\Gamma f(z)\, dz &= \frac{2\pi i}{3} e^{\frac{i\pi(\alpha-3)}{3}}.\end{aligned}$$

We can now compute the contributions to the integral along the semicircular arc and the portion along the imaginary axis.

Along the arc, Jordan's lemma applies since $\frac{1}{R^3+1} \overset{R\longrightarrow\infty}{\longrightarrow} 0$, and thus this contribution vanishes.

Along the imaginary axis, we can make the following change of variables:

$$\begin{aligned}\int_R^0 f(iy)\, dy &= -\int_0^R \frac{(iy)^{\alpha-1}}{(iy)^3+1}\, dy \\ &= -\frac{1}{i}\int_0^R \frac{t^{\alpha-1}}{t^3+1}\, dt \qquad (t = iz,\ dt = idz) \\ &= iI,\end{aligned}$$

which is i times the original integral.

We thus have

$$\begin{aligned}\mathrm{Res}(f(z), z = r_2) &= \int_\Gamma f \\ &= \int_0^R f + \int_{C_R} f + \int_{iR}^0 f \\ &\overset{R\longrightarrow\infty}{\longrightarrow} I + 0 + iI \\ &= (1+i)I,\end{aligned}$$

and so

$$I = \frac{\mathrm{Res}(f(z), z = r_2)}{1+i} = \frac{2\pi i}{3(1+i)} e^{\frac{i\pi(\alpha-3)}{3}}.$$

■

Note: this seems to be wrong, because plugging in $a = 1, 2, 3$ doesn't result in a real value.

1.8 8

Prove the fundamental theorem of Algebra using

a. Rouche's Theorem.
b. The maximum modulus principle.

1.8.1 Solution (Rouche)

We want to show that every $f \in \mathbb{C}[x]$ has precisely n roots, and we'll use the follow formulation of Rouche's theorem:

Theorem 1.1 ***(Rouche).***
If f, g are holomorphic on $D(z_0)$ with $f, g \neq 0$ and $|f - g| < |f| + |g|$ on $\partial D(z_0)$, then f and g has the same number of zeros within D.

We'll also use without proof the fact that the function $h(z) = z^n$ has precisely n zeros (counted with multiplicity).

Suppose $f(z) = a_n z^n + \cdots + a_1 z + a_0$ where $a_n \neq 0$ and define $g(z) = a_n z^n$. Noting that polynomials are entire, f, g are nonzero by assumption, and

$$\begin{aligned}
|f - g| &= \left|a_{n-1} z^{n-1} + \cdots + a_1 z + a_0\right| \\
&= \left|a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 + a_n z^n - a_n z^n\right| \\
&\leq \left|a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 + a_n z^n\right| + |a_n z^n| \qquad \text{for } |z| > 1(?) \\
&= |f| + |g|
\end{aligned}$$

the conditions of Rouche's theorem apply and f has precisely n roots.

■

1.8.2 Solution (Maximum Modulus Principle)

Toward a contradiction, suppose f is non-constant and has *no* zeros. Then $g(z) := 1/f(z)$ is non-constant and holomorphic on $\mathbb{C}$.

Using the fact that $\lim_{z \longrightarrow \infty} f(z) = \infty$ for any polynomial f, pick r large enough such that

$$z \in \mathbb{C} \setminus \overline{D_r}(0) \implies |f(z)| > |f(0)|$$

where $D_r(0)$ is an open disc of radius r about $z = 0$. Then $|g(z)| < |g(0)|$ for every such z.

Noting that $\overline{D_r}(0)$ is closed and bounded and thus compact by Heine-Borel, g attains a global maximum in the interior D_r°. But by the maximum modulus principle, this forces g to be constant, and since $g = \dfrac{1}{f}$, it must also be true that f is constant.

■

1.9 9

Let f be analytic in a region D and γ a rectifiable curve in D with interior in D. Prove that if $f(z)$ is real for all $z \in \gamma$, then f is constant.

1.9.1 Solution

Since f is analytic in D (wlog assuming $0 \in D$ by translation), take its series expansion $f(z) = c_0 + c_1 z + \cdots$ for $z \in D$.

Without loss of generality, suppose that γ is not entirely contained in $\mathbb{R}$, so for $z \in \gamma$ we can write $z = x + iy$ where $y \neq 0$.

Then

$$\begin{aligned} f(z) &= f(x+iy) \\ &= c_0 + c_1(x+iy) + \cdots \\ &= c_0 + c_1 x + ic_1 y + \cdots && \subset \mathbb{R} \quad \text{by assumption,} \end{aligned}$$

and so we must have $c_1 y = 0 \implies c_1 = 0$. The same argument applies to further terms in the expansion, so we in fact have $c_i = 0$ for every $i \geq 1$.

But this says $f(z) = c_0$ for an arbitrary z, i.e. f is constant.

■

1.10 10

For $a > 0$, evaluate

$$\int_0^{\pi/2} \frac{d\theta}{a + \sin^2\theta}$$

We have

$$
\begin{aligned}
I &:= \int_0^{\pi/2} \frac{1}{1+\sin^2(\theta)}\, d\theta \\
&= \int_{\gamma_1} \frac{1}{a + \left(\frac{z-z^{-1}}{2i}\right)^2} \frac{-i\, dz}{z} \qquad \text{where } \gamma_1 \text{ is } \frac{1}{4} \text{ of the unit circle } S^1 \\
&= -i \int_{\gamma_1} \frac{1}{z} \left(\frac{1}{a + \left(-\frac{1}{4}\right)(z^2 - 2 + z^{-2})} \right) dz \\
&= 4i \int_{\gamma_1} \frac{1}{z} \left(\frac{1}{z^2 - (2+4a) + z^{-2}} \right) dz \\
&= 4i \int_{\gamma_1} \frac{z}{z^4 - (2+4a)z^2 + 1}\, dz \\
&= i \oint_{S^1} \frac{z}{z^4 - (2+4a)z^2 + 1}\, dz \\
&= \frac{i}{2} \oint_{2\cdot S^1} \frac{1}{u^2 - (2+4a)u + 1}\, du \qquad \text{using } u = z^2, \frac{1}{2}\, du = z\, dz \\
&:= \frac{i}{2} \oint_{2\cdot S^1} \frac{1}{f_a(u)}\, du \\
&= \frac{i}{2} \cdot 2\pi i \cdot \sum \mathrm{Res}_{u=r_i} \frac{1}{f_a(u)},
\end{aligned}
$$

where $2 \cdot S^1$ denotes the contour wrapping around the unit circle twice and r_i denote the poles contained in the region bounded by S^1. We can now compute the last integral by the residue theorem.

Factor the denominator as

$$
f_a(u) = u^2 - (2+4a)u + 1 = (u - r_1)(u - r_2),
$$

where the r_i are given by $(1+2a) \pm 4\sqrt{a^2+a}$ using the quadratic formula. We can then write a partial fraction decomposition

$$
\begin{aligned}
\frac{1}{f_a(u)} &:= \frac{1}{u^2 - (2+4a)u + 1} \\
&= \frac{1}{(u-r_1)(u-r_2)} \\
&= \frac{A}{u-r_1} + \frac{B}{u-r_2} \\
&= \frac{\mathrm{Res}_{u=r_1} 1/f(u)}{u-r_1} + \frac{\mathrm{Res}_{u=r_2} 1/f(u)}{u-r_2} \\
&= \frac{1/f'(r_1)}{u-r_1} + \frac{1/f'(r_2)}{u-r_2} \\
&= -\frac{1}{8\sqrt{a^2+a}(u-r_1)} + \frac{1}{8\sqrt{a^2+a}(u-r_2)}.
\end{aligned}
$$

Since $|r_2| = \left|(1+2a) + 4\sqrt{a^2+a}\right| > 1$, we find that the only relevant pole inside of S^1 is r_1. Reading off the residue from the above decomposition, we thus have

$$\begin{aligned} I &= \frac{i}{2} \cdot 2\pi i \cdot \sum \operatorname{Res}_{u=r_i} \frac{1}{f_a(u)} \\ &= -\pi \cdot \operatorname{Res}_{u=r_1} \frac{1}{f_a(u)} \\ &= \frac{\pi}{8\sqrt{a^2+a}}. \end{aligned}$$

■

Note: I know I'm off by a constant here at least, since $a = 1$ should reduce to $\pi/2\sqrt{2}$.

1.11 11

Find the number of roots of $p(z) = 4z^4 - 6z + 3$ in $|z| < 1$ and $1 < |z| < 2$ respectively.

For $|z| < 1$, take $f(z) = -6z$ and $g(z) = z^4 + 3$, noting that $f + g = p$. Using the maximum modulus principal, we know that the max/mins of f, g occur on $|z| = 1$, on which we have

$$|g(z)| = 4 < 6 = |f(z)|,$$

so Rouche's theorem applies and both p and f have the same number of zeros. Since f clearly has **one** zero, p has one zero in this region.

Now consider $|z| < 2$ and set $f(z) = z^4$ and $g(z) = -6z + 3$. By a similar argument, we have

$$|g(z)| = 15 < 16 = |f|$$

on $|z| = 2$, and thus f and p have the same number of zeros in this region. Since f has **four** zeros here, so does p.

Thus p has $4 - 1 =$ **3 zeros** on $1 \leq |z| \leq 2$.

■

1.12 12

Prove that $z^4 + 2z^3 - 2z + 10$ has exactly one root in each open quadrant.

1.12.1 Solution

Let $f(z) = z^4 + 2z^3 - 2z + 10$, and consider the following contour:

Image

By the argument principle, we have

$$\Delta_\Gamma \arg f(z) = 2\pi(Z - P),$$

where Z is the number of zeros of f in the region Ω enclosed by Γ and P is the number of poles in Ω.

Since polynomials are holomorphic on $\mathbb{C}$, by the argument principle it suffices to show that

- f does not have any roots on the real or imaginary axes
- f does not vanish on Γ, and
- $\Delta_\Gamma \arg f(z) = 1$, where Δ_Γ denotes the total change in the argument of f over Γ.

It will follow by symmetry that f has exactly one root in each quadrant.

Claim 1.2.

- f has no roots on the coordinate axes.
- $\Delta_{\gamma_1} \arg f(z) = 0$
- $\Delta_{\gamma_2} \arg f(z) = 2\pi$
- $\Delta_{\gamma_3} \arg f(z) = 0$

Given the claim, we would have

$$\Delta_\Gamma \arg f(z) = 2\pi = 2\pi(Z - 0) \implies Z = 1,$$

which is what we wanted to show.

Proof of Claim:

γ_2: For $R \gg 0$, we have $f(z) \sim z^4$. Along γ_2, the argument of z ranges from 0 to $\frac{\pi}{2}$, and thus the argument of z^4 ranges from 0 to $4 \cdot \frac{\pi}{2} = 2\pi$.

γ_1: By cases, for $z \in \mathbb{R}$,

- If $|z| > 1$, then $z^3 > z$ and so

$$\begin{aligned} f(z) &= (z^4 + 10) + (2z^3 - 2z) \\ &> (z^4 + 10) + (2z - 2z) \\ &= z^4 + 10 \\ &> 0, \end{aligned}$$

  so f is strictly positive and does not change argument on $(\pm 1, \pm\infty)$ or $i \cdot (\pm 1, \pm\infty)$.

- If $|z| \le 1$,

$$\begin{aligned} \left| -z^4 - 2z^3 + 2z \right| &\le |z|^4 + 2|z|^3 + 2|z| \\ &\le 1 + 2 + 2 \\ &= 5 \\ &< 10 \\ \implies f(z) &= 10 - (-z^4 - 2z^3 + 2z) > 0, \end{aligned}$$

  so f is strictly positive and does not change argument $(0, \pm 1)$ or $i \cdot (0, \pm 1)$.

■

1.13 13

Prove that for $a > 0$, $z \tan z - a$ has only real roots.

1.14 14

Let f be nonzero, analytic on a bounded region Ω and continuous on its closure $\overline{\Omega}$. Show that if $|f(z)| \equiv M$ is constant for $z \in \partial\Omega$, then $f(z) \equiv Me^{i\theta}$ for some real constant θ.

1.14.1 Solution

By the maximum modulus principle applied to f in $\overline{\Omega}$, we know that $\max|f| = M$. Similarly, the maximum modulus principle applied to $\dfrac{1}{f}$ in $\overline{\Omega^c}$ since f is nonzero in Ω, and we can conclude that $\min|f| = M$ as well. Thus $|f| = M$ is constant on $\overline{\Omega}$.

So consider the function $g(z) = |f(z)|$; from the above observation, we find that $g(\overline{\Omega}) = \{M\}$. Letting S_M be the circle of radius M, this implies that $f(\Omega) \subseteq S_M$. In particular, $S_M \subset \mathbb{C}$ is a closed set.

However, by the open mapping theorem, $f(\Omega) \subset \mathbb{C}$ must be an open set. A basis for the topology on $\mathbb{C}$ is given by open discs, so in particular, the open sets of $\mathbb{C}$ have real dimension either zero or two. Since S_M has real dimension 1, $f(\Omega)$ must have dimension zero and is thus a collection of points. Since f is continuous, the image can only be one point, i.e. $f(\Omega) = \{\text{pt}\} \in S_M$. So f is constant.

■

2 Stein and Shakarchi

2.1 S&S 3.8.1

Using Euler's formula

$$\sin(\pi z) = \frac{1}{2i}(e^{i\pi z} - e^{-i\pi z})$$

show that the complex zeros of $\sin(\pi z)$ are exactly the integers, each of order one. Calculate the residue of $\dfrac{1}{\sin(\pi z)}$ at $z = n \in \mathbb{Z}$.

2.2 S&S 3.8.2

Evaluate the integral

$$\int_{\mathbb{R}} \frac{dx}{1 + x^4}$$

What are the poles of the integrand?

2.3 S&S 3.8.4

Show that

$$\int_{\mathbb{R}} \frac{x \sin x}{x^2 + a^2} = \frac{\pi e^{-a}}{a} \qquad a > 0$$

2.4 S&S 3.8.5

Show that for $\xi \in \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{e^{2\pi i x \xi}}{(1 + x^2)^2} = \frac{\pi}{2}(1 + 2\pi|\xi|)e^{-2\pi|\xi|}$$

2.5 S&S 3.8.6

Show that

$$\int_{\mathbb{R}} \frac{dx}{(1 + x^2)^{n+1}} = \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)\pi}{2 \cdot 4 \cdots (2n}$$

2.6 S&S 3.8.7

Show that for $a > 1$,

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos\theta)^2} = \frac{2\pi a}{(a^2 - 1)^{3/2}}$$

2.7 S&S 3.8.8

Show that if $a, b \in \mathbb{R}$ with $a > |b|$ then

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi a}{\sqrt{a^2 - b^2}}$$

2.8 S&S 3.8.9

Show that

$$\int_0^1 \log(\sin \pi x)\ dx = -\log 2$$

2.9 S&S 3.8.10

Show that if $a > 0$

$$\int_0^\infty \frac{\log x}{x^2 + a^2}\ dx = \frac{\pi \log a}{2a}$$

2.10 S&S 3.8.14

Prove that if f is entire and injective, then $f(z) = az + b$ with $a, b \in \mathbb{C}$ with $a \neq 0$.

Hint: apply the Casorati-Weierstrass theorem to $f(1/z)$.

2.11 S&S 3.8.15

Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

2.11.1 a

If f is entire and for all $R > 0$, there are constants $A, B > 0$ such that $\sup_{|z|=R} |f(z)| \leq AR^k + B$, then f is a polynomial of degree less than k.

2.11.2 b

Show that if f is holomorphic on the unit disk, is bounded, and converges to zero uniformly in the sector $\theta \leq \arg z \leq \phi$ as $|z| \longrightarrow 1$, then $f \equiv 0$.

2.11.3 c

Let $w_1, \cdots, w_n$ be points on $S^1 \subset \mathbb{C}$. Show that there exists a point $z \in S^1$ such that

$$\prod_{i=1}^n |z - w_i| \geq 1.$$

Conclude that there exists a point $w \in S^1$ such that

$$\prod_{i=1}^n |w - w_i| = 1.$$

2.11.4 d

Show that if f is entire and $\Re(f)$ is bounded, then f is constant.

2.12 S&S 3.8.17

Let f be non-constant, and holomorphic in an open set containing the open unit disc.

2.12.1 a

Show that $|z| = 1 \implies |f(z)| = 1$, then the image of f contains the unit disc.

> Hint: Show that $f(z) = w_0$ has a root for every $w_0 \in \mathbb{D}$, for which it suffices to show that $f(z) = 0$ has a root, then use the maximum modulus principle.

2.12.2 b

Show that if $|z| \geq 1 \implies |f(z)| = 1$ **and** there exists a point $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$, then the image of f contains the unit disc.

2.13 S&S 3.8.19

Prove the maximum modulus principle for harmonic functions; i.e.,

2.13.1 a

If u is a non-constant real-valued harmonic function on Ω, then u can not attain its extrema on Ω.

2.13.2 b

Suppose Ω has compact closure $\overline{\Omega}$, then if u is harmonic on Ω and continuous on $\overline{\Omega}$, then

$$\sup_{z \in \Omega} |u(z)| \leq \sup_{z \in \overline{\Omega} - \Omega} |u(z)|$$

> Hint: to prove (a), assume that u attains a local maximum at z_0, and let f be holomorphic near z_0 with $u = \Re(f)$, then show that f is not open. Part (b) is a direct consequence.