# Complex Analysis Problem Set 3

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Sunday 29th March, 2020

## Contents

# 1 Problems From Tie

## 1.1 1

Prove that if $f$ has two Laurent series expansions,

$$f(z) = \sum c_n (z-a)^n \quad \text{and} \quad f(z) = \sum c_n' (z-a)^n$$

then $c_n = c_n'$.

### 1.1.1 Solution

By taking the difference of two such expansions, it suffices to show that if $f$ is identically zero and $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$ about some point $a$, then $c_n = 0$ for all $n$.

Under this assumption, let $D_\varepsilon(a)$ be a disc about $a$ and $\gamma$ be any contoured contained in its interior. Then for each $n$, we can apply the formula

$$\begin{aligned} c_n &= \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - a)^{n+1}}\, d\xi \\ &= \frac{1}{2\pi i} \int_\gamma \frac{0}{(\xi - a)^{n+1}}\, d\xi \quad \text{by assumption} \\ &= 0, \end{aligned}$$

which shows that $c_n = 0$ for all $n$.

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## 1.2 2

Find Laurent series expansions of

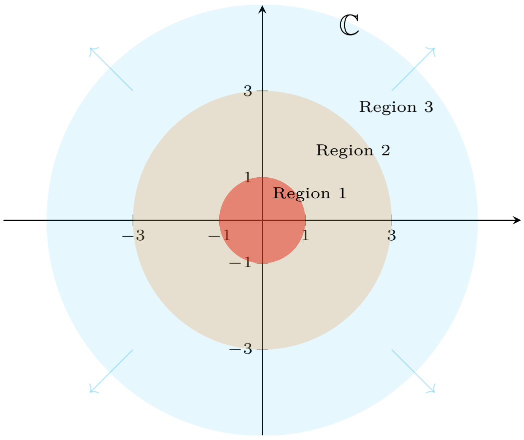
$$\frac{1}{1-z^2}+\frac{1}{3-z}$$

How many such expansions are there? In what domains are each valid?

### 1.2.1 Solution

Note that $f$ has poles at $z=-1,1,3$, all with multiplicity 1, and so there are 3 regions to consider:

1. $|z|<1$
2. $1<|z|<3$
3. $3<|z|$.




**Region 1:** Take the following expansion:

$$\begin{aligned} f(z) &= \frac{1}{1-z^2}+\frac{1}{3-z} \\ &= \sum_{n\geq 0} z^{2n} + \frac{1}{3}\left(\frac{1}{1-\frac{3}{z}}\right) \\ &= \sum_{n\geq 0} z^{2n} + \frac{1}{3}\sum_{n\geq 0}\left(\frac{1}{3}\right)^n z^n \\ &= \sum_{n\geq 0} z^{2n} + \sum_{n\geq 0}\left(\frac{1}{3}\right)^{n+1} z^n \end{aligned}$$

.

Noting $\left|z^2\right|<1$ implies then $|z|<1$, and that the first term converges for $\left|z^2\right|<1$ and the second for $\left|\frac{z}{3}\right|<1 \iff |z|<3$, this expansion converges to $f$ on the region $|z|<1$.

**Region 2:** Take the following expansion:

$$
\begin{aligned}
f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\
&= -\frac{1}{z^2}\left(\frac{1}{1-\frac{1}{z^2}}\right) - \frac{1}{3}\left(\frac{1}{1-\frac{z}{3}}\right) \\
&= -\frac{1}{z^2}\sum_{n\geq 0} z^{-2n} + \sum_{n\geq 0}\left(\frac{1}{3}\right)^{n+1} z^n \\
&= -\sum_{n\geq 2}\frac{1}{z^{2n}} + \sum_{n\geq 0}\left(\frac{1}{3}\right)^{n+1} z^n
\end{aligned}
$$

.

By construction, the first term converges for $\left|\frac{1}{z^2}\right| < 1 \iff |z| > 1$ and the second for $|z| < 3$.

**Region 3:** Take the following expansion:

$$
\begin{aligned}
f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\
&= -\frac{1}{z^2}\left(\frac{1}{1-\frac{1}{z^2}}\right) - \frac{1}{z}\left(\frac{1}{1-\frac{3}{z}}\right) \\
&= -\frac{1}{z^2}\sum_{n\geq 0}\frac{1}{z^{2n}} - \frac{1}{z}\sum_{n\geq 0} 3^n\frac{1}{z^n} \\
&= -\sum_{n\geq 2}\frac{1}{z^{2n}} - \sum_{n\geq 1}\left(\frac{1}{3}\right)^{n-1}\frac{1}{z^n}.
\end{aligned}
$$

Note: in principle, terms could be collected here.

By construction, this converges on $\left\{|z|^2 > 1\right\}\bigcap\{|z| > 3\} = \{|z| > 3\}$.

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## 1.3 3

Let $P, Q$ be polynomials with no common zeros. Assume $a$ is a root of $Q$. Find the principal part of $P/Q$ at $z = a$ in terms of $P$ and $Q$ if $a$ is (1) a simple root, and (2) a double root.

### 1.3.1 Solution

todo

## 1.4 4

Let $f$ be non-constant, analytic in $|z| > 0$, where $f(z_n) = 0$ for infinitely many points $z_n$ with $\lim_{n \longrightarrow \infty} z_n = 0$.
Show that $z = 0$ is an essential singularity for $f$.

Example: $f(z) = \sin(1/z)$.

### 1.4.1 Solution

It suffices to show that $z_0 = 0$ is neither a pole nor a removable singularity, i.e.

1. $\lim_{z \longrightarrow z_0} f(z) \neq \infty$
2. $|f(z)|$ is not bounded on any neighborhood $D_\varepsilon(z_0)$.

The first property follows because if $f$ is analytic,

## 1.5 5

Show that if $f$ is entire and $\lim_{z \longrightarrow \infty} f(z) = \infty$, then $f$ is a polynomial.

## 1.6 6

Problem : a. Show that

$$\int_0^{2\pi} \log \left|1 - e^{i\theta}\right| \, d\theta = 0$$

b. Show that this identity is equivalent to SS 3.8.9.

## 1.7 7

Let $0 < a < 4$ and evaluate

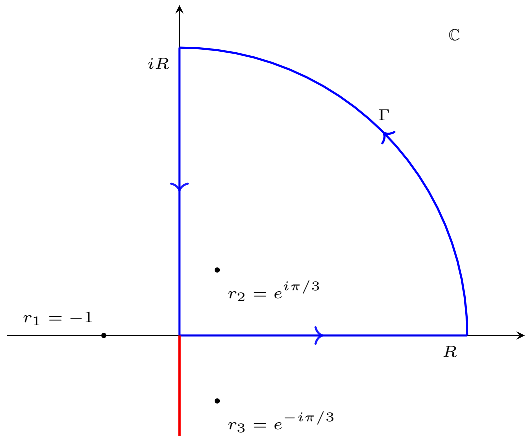
$$\int_0^\infty \frac{x^{\alpha - 1}}{1 + x^3} \, dx$$

### 1.7.1 Solution

We will compute this using a closed contour and the residue theorem, so first note that

$$z^3 + 1 = (z+1)(z - e^{i\pi 3})(z - e^{-i\pi 3}) := (z - r_1)(z - r_2)(z - r_3).$$

Defining $z^\alpha = e^{\alpha \log z}$ for $\alpha \in \mathbb{R}$, we'll take the following contour $\Gamma$ shown in blue along with a branch cut for the logarithm function indicated in red:




Letting

$$f(z) := \frac{z^{\alpha-1}}{z^3+1} := \frac{P(z)}{Q(z)},$$

we find that only $z = r_2$ will contribute a term to $\int_\Gamma f$. Computing the residue here, noting that each pole is simple of order 1, we have

$$\underset{z=r_2}{Res}\, f(z) = \frac{P(r_2)}{Q'(r_2)}.$$

## 1.8 8

Prove the fundamental theorem of Algebra using

a. Rouche's Theorem.
b. The maximum modulus principle.

### 1.8.1 Solution (Rouche)

We want to show that every $f \in \mathbb{C}[x]$ has precisely $n$ roots, and we'll use the follow formulation of Rouche's theorem:

**Theorem 1.1** ***(Rouche).***
If $f, g$ are holomorphic on $D(z_0)$ with $f, g \neq 0$ and $|f - g| < |f| + |g|$ on $\partial D(z_0)$, then $f$ and $g$ has the same number of zeros within $D$.

We'll also use without proof the fact that the function $h(z) = z^n$ has precisely $n$ zeros (counted with multiplicity).

Suppose $f(z) = a_n z^n + \cdots + a_1 z + a_0$ where $a_n \neq 0$ and define $g(z) = a_n z^n$. Noting that polynomials are entire, $f, g$ are nonzero by assumption, and

$$\begin{aligned}
|f-g| &= \left|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right| \\
&= \left|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 + a_n z^n - a_n z^n\right| \\
&\leq \left|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 + a_n z^n\right| + |a_n z^n| \qquad \text{for } |z| > 1(?) \\
&= |f| + |g|
\end{aligned}$$

the conditions of Rouche's theorem apply and $f$ has precisely $n$ roots.

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### 1.8.2 Solution (Maximum Modulus Principle)

Toward a contradiction, suppose $f$ is non-constant and has *no* zeros. Then $g(z) := 1/f(z)$ is non-constant and holomorphic on $\mathbb{C}$.

Using the fact that $\lim_{z \longrightarrow \infty} f(z) = \infty$ for any polynomial $f$, pick $r$ large enough such that

$$z \in \mathbb{C} \setminus \overline{D_r}(0) \implies |f(z)| > |f(0)|$$

where $D_r(0)$ is an open disc of radius $r$ about $z = 0$. Then $|g(z)| < |g(0)|$ for every such $z$.

Noting that $\overline{D_r}(0)$ is closed and bounded and thus compact by Heine-Borel, $g$ attains a global maximum in the interior $D_r^\circ$. But by the maximum modulus principle, this forces $g$ to be constant, and since $g = \dfrac{1}{f}$, it must also be true that $f$ is constant.

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## 1.9 9

Let $f$ be analytic in a region $D$ and $\gamma$ a rectifiable curve in $D$ with interior in $D$. Prove that if $f(z)$ is real for all $z \in \gamma$, then $f$ is constant.

### 1.9.1 Solution

Since $f$ is analytic in $D$ (wlog assuming $0 \in D$ by translation), take its series expansion $f(z) = c_0 + c_1 z + \cdots$ for $z \in D$.

Without loss of generality, suppose that $\gamma$ is not entirely contained in $\mathbb{R}$, so for $z \in \gamma$ we can write $z = x + iy$ where $y \neq 0$.

Then

$$\begin{aligned}
f(z) &= f(x+iy) \\
&= c_0 + c_1(x+iy) + \cdots \\
&= c_0 + c_1 x + i c_1 y + \cdots \qquad \subset \mathbb{R} \quad \text{by assumption,}
\end{aligned}$$

and so we must have $c_1 y = 0 \implies c_1 = 0$. The same argument applies to further terms in the expansion, so we in fact have $c_i = 0$ for every $i \geq 1$.

But this says $f(z) = c_0$ for an arbitrary $z$, i.e. $f$ is constant.

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## 1.10 10

For $a > 0$, evaluate

$$\int_0^{\pi/2} \frac{d\theta}{a + \sin^2\theta}$$

We have

$$\begin{aligned}
I &:= \int_0^{\pi/2} \frac{1}{1+\sin^2(\theta)}\, d\theta \\
&= \int_{\gamma_1} \frac{1}{a + \left(\frac{z-z^{-1}}{2i}\right)^2} \frac{-i\, dz}{z} \qquad \text{where } \gamma_1 \text{ is } \frac{1}{4} \text{ of the unit circle } S^1 \\
&= -i \int_{\gamma_1} \frac{1}{z} \left( \frac{1}{a + \left(-\frac{1}{4}\right)(z^2 - 2 + z^{-2})} \right) dz \\
&= 4i \int_{\gamma_1} \frac{1}{z} \left( \frac{1}{z^2 - (2+4a) + z^{-2}} \right) dz \\
&= 4i \int_{\gamma_1} \frac{z}{z^4 - (2+4a)z^2 + 1}\, dz \\
&= i \oint_{S^1} \frac{z}{z^4 - (2+4a)z^2 + 1}\, dz \\
&= \frac{i}{2} \oint_{2 \cdot S^1} \frac{1}{u^2 - (2+4a)u + 1}\, du \qquad \text{using } u = z^2, \frac{1}{2}\, du = z\, dz \\
&:= \frac{i}{2} \oint_{2 \cdot S^1} \frac{1}{f_a(u)}\, du \\
&= \frac{i}{2} \cdot 2\pi i \cdot \sum \operatorname{Res}_{u=r_i} \frac{1}{f_a(u)},
\end{aligned}$$

where $2 \cdot S^1$ denotes the contour wrapping around the unit circle twice and $r_i$ denote the poles contained in the region bounded by $S^1$. We can now compute the last integral by the residue theorem.

Factor the denominator as

$$f_a(u) = u^2 - (2+4a)u + 1 = (u - r_1)(u - r_2),$$

where the $r_i$ are given by $(1+2a) \pm 4\sqrt{a^2+a}$ using the quadratic formula. We can then write a partial fraction decomposition

$$\begin{aligned}
\frac{1}{f_a(u)} &:= \frac{1}{u^2-(2+4a)u+1} \\
&= \frac{1}{(u-r_1)(u-r_2)} \\
&= \frac{A}{u-r_1} + \frac{B}{u-r_2} \\
&= \frac{\mathrm{Res}_{u=r_1} 1/f(u)}{u-r_1} + \frac{\mathrm{Res}_{u=r_2} 1/f(u)}{u-r_2} \\
&= \frac{1/f'(r_1)}{u-r_1} + \frac{1/f'(r_2)}{u-r_2} \\
&= -\frac{1}{8\sqrt{a^2+a}(u-r_1)} + \frac{1}{8\sqrt{a^2+a}(u-r_2)}.
\end{aligned}$$

Since $|r_2| = \left|(1+2a)+4\sqrt{a^2+a}\right| > 1$, we find that the only relevant pole inside of $S^1$ is $r_1$. Reading off the residue from the above decomposition, we thus have

$$\begin{aligned}
I &= \frac{i}{2} \cdot 2\pi i \cdot \sum \mathrm{Res}_{u=r_i} \frac{1}{f_a(u)} \\
&= -\pi \cdot \underset{u=r_1}{Res} \frac{1}{f_a(u)} \\
&= \frac{\pi}{8\sqrt{a^2+a}}.
\end{aligned}$$

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Note: I know I'm off by a constant here at least, since $a = 1$ should reduce to $\pi/2\sqrt{2}$.

## 1.11 11

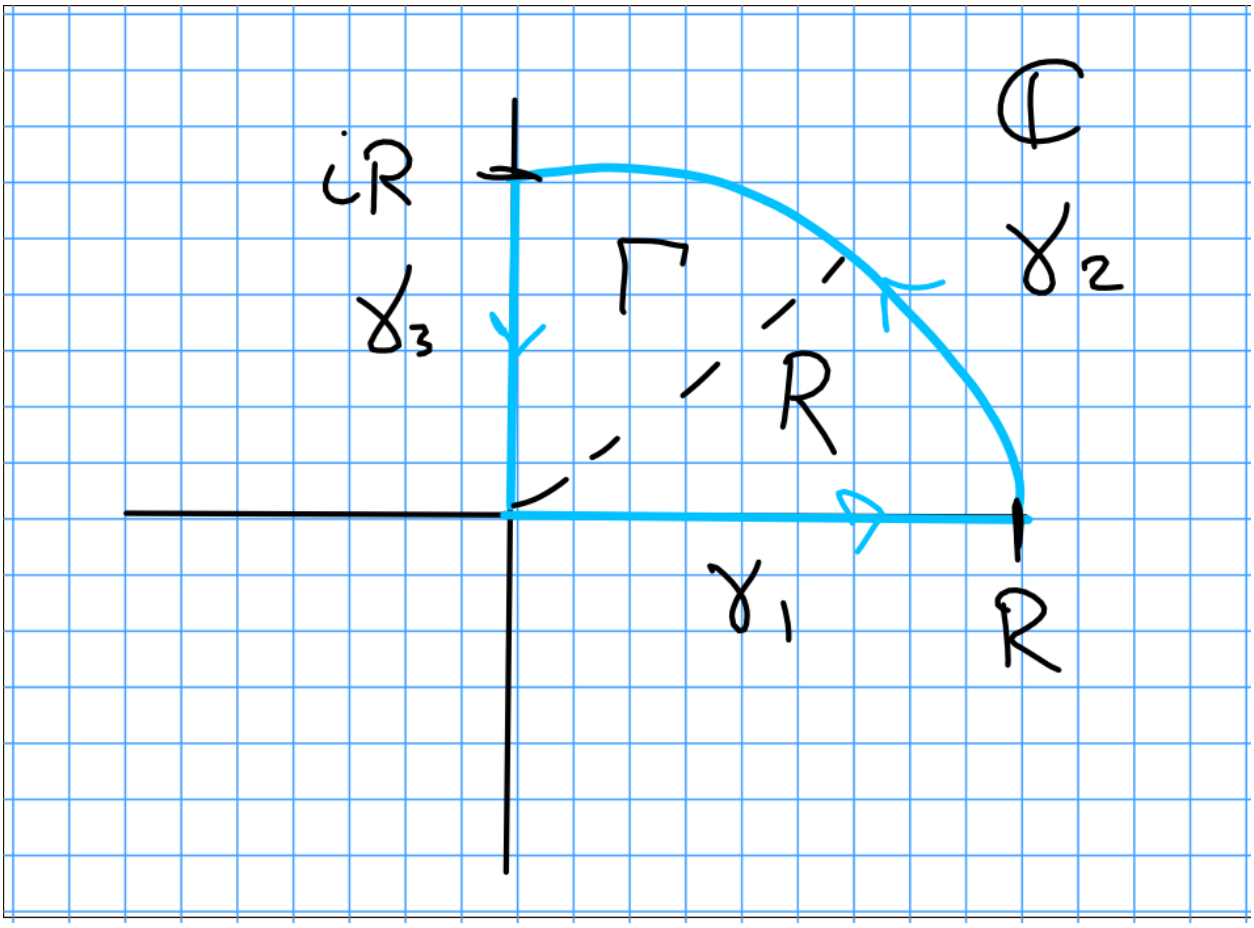
Find the number of roots of $p(z) = 4z^4 - 6z + 3$ in $|z| < 1$ and $1 < |z| < 2$ respectively.

## 1.12 12

Prove that $z^4 + 2z^3 - 2z + 10$ has exactly one root in each open quadrant.

### 1.12.1 Solution

Let $f(z) = z^4 + 2z^3 - 2z + 10$, and consider the following contour:




By the argument principle, we have

$$\Delta_\Gamma \arg f(z) = 2\pi(Z - P),$$

where $Z$ is the number of zeros of $f$ in the region $\Omega$ enclosed by $\Gamma$ and $P$ is the number of poles in $\Omega$.

Since polynomials are holomorphic on $\mathbb{C}$, by the argument principle it suffices to show that

- $f$ does not have any roots on the real or imaginary axes
- $f$ does not vanish on $\Gamma$, and
- $\Delta_\Gamma \arg f(z) = 1$, where $\Delta_\Gamma$ denotes the total change in the argument of $f$ over $\Gamma$.

It will follow by symmetry that $f$ has exactly one root in each quadrant.

**Claim 1.2.**

- $f$ has no roots on the coordinate axes.
- $\Delta_{\gamma_1} \arg f(z) = 0$
- $\Delta_{\gamma_2} \arg f(z) = 2\pi$
- $\Delta_{\gamma_3} \arg f(z) = 0$

Given the claim, we would have

$$\Delta_\Gamma \arg f(z) = 2\pi = 2\pi(Z - 0) \implies Z = 1,$$

which is what we wanted to show.

**Proof of Claim**:

$\gamma_2$: For $R \gg 0$, we have $f(z) \sim z^4$. Along $\gamma_2$, the argument of $z$ ranges from 0 to $\frac{\pi}{2}$, and thus the argument of $z^4$ ranges from 0 to $4 \cdot \frac{\pi}{2} = 2\pi$.

$\gamma_1$: By cases, for $z \in \mathbb{R}$,

- If $|z| > 1$, then $z^3 > z$ and so

$$\begin{aligned} f(z) &= (z^4 + 10) + (2z^3 - 2z) \\ &> (z^4 + 10) + (2z - 2z) \\ &= z^4 + 10 \\ &> 0, \end{aligned}$$

  so $f$ is strictly positive and does not change argument on $(\pm 1, \pm\infty)$ or $i \cdot (\pm 1, \pm\infty)$.

- If $|z| \leq 1$,

$$\begin{aligned} \left| -z^4 - 2z^3 + 2z \right| &\leq |z|^4 + 2|z|^3 + 2|z| \\ &\leq 1 + 2 + 2 \\ &= 5 \\ &< 10 \\ \implies f(z) &= 10 - (-z^4 - 2z^3 + 2z) > 0, \end{aligned}$$

  so $f$ is strictly positive and does not change argument $(0, \pm 1)$ or $i \cdot (0, \pm 1)$.

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## 1.13 13

Prove that for $a > 0$, $z \tan z - a$ has only real roots.

## 1.14 14

Let $f$ be nonzero, analytic on a bounded region $\Omega$ and continuous on its closure $\overline{\Omega}$. Show that if $|f(z)| \equiv M$ is constant for $z \in \partial\Omega$, then $f(z) \equiv Me^{i\theta}$ for some real constant $\theta$.

### 1.14.1 Solution

By the maximum modulus principle applied to $f$ in $\overline{\Omega}$, we know that $\max|f| = M$. Similarly, the maximum modulus principle applied to $\frac{1}{f}$ in $\overline{\Omega^c}$ since $f$ is nonzero in $\Omega$, and we can conclude that $\min|f| = M$ as well. Thus $|f| = M$ is constant on $\overline{\Omega}$.

So consider the function $g(z) = |f(z)|$; from the above observation, we find that $g(\overline{\Omega}) = \{M\}$. Letting $S_M$ be the circle of radius $M$, this implies that $f(\Omega) \subseteq S_M$. In particular, $S_M \subset \mathbb{C}$ is a closed set.

However, by the open mapping theorem, $f(\Omega) \subset \mathbb{C}$ must be an open set. A basis for the topology on $\mathbb{C}$ is given by open discs, so in particular, the open sets of $\mathbb{C}$ have real dimension either zero or two. Since $S_M$ has real dimension 1, $f(\Omega)$ must have dimension zero and is thus a collection of points. Since $f$ is continuous, the image can only be one point, i.e. $f(\Omega) = \{\mathrm{pt}\} \in S_M$. So $f$ is constant.

■

# 2 Stein and Shakarchi

## 2.1 S&S 3.8.1

Using Euler's formula

$$\sin(\pi z) = \frac{1}{2i}(e^{i\pi z} - e^{-i\pi z})$$

show that the complex zeros of $\sin(\pi z)$ are exactly the integers, each of order one.

Calculate the residue of $\frac{1}{\sin(\pi z)}$ at $z = n \in \mathbb{Z}$.

## 2.2 S&S 3.8.2

Evaluate the integral

$$\int_{\mathbb{R}} \frac{dx}{1 + x^4}$$

What are the poles of the integrand?

## 2.3 S&S 3.8.4

Show that

$$\int_{\mathbb{R}} \frac{x \sin x}{x^2 + a^2} = \frac{\pi e^{-a}}{a} \qquad a > 0$$

## 2.4 S&S 3.8.5

Show that for $\xi \in \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{e^{2\pi i x \xi}}{(1+x^2)^2} = \frac{\pi}{2}(1+2\pi|\xi|)e^{-2\pi|\xi|}$$

## 2.5 S&S 3.8.6

Show that

$$\int_{\mathbb{R}} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)\pi}{2 \cdot 4 \cdots (2n}$$

## 2.6 S&S 3.8.7

Show that for $a > 1$,

$$\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{3/2}}$$

## 2.7 S&S 3.8.8

Show that if $a, b \in \mathbb{R}$ with $a > |b|$ then

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi a}{\sqrt{a^2-b^2}}$$

## 2.8 S&S 3.8.9

Show that

$$\int_0^1 \log(\sin \pi x)\ dx = -\log 2$$

## 2.9 S&S 3.8.10

Show that if $a > 0$

$$\int_0^\infty \frac{\log x}{x^2+a^2}\ dx = \frac{\pi \log a}{2a}$$

## 2.10 S&S 3.8.14

Prove that if $f$ is entire and injective, then $f(z) = az + b$ with $a, b \in \mathbb{C}$ with $a \neq 0$.

Hint: apply the Casorati-Weierstrass theorem to $f(1/z)$.

## 2.11 S&S 3.8.15

Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

### 2.11.1 a

If $f$ is entire and for all $R > 0$, there are constants $A, B > 0$ such that $\sup_{|z|=R} |f(z)| \leq AR^k + B$, then $f$ is a polynomial of degree less than $k$.

### 2.11.2 b

Show that if $f$ is holomorphic on the unit disk, is bounded, and converges to zero uniformly in the sector $\theta \leq \arg z \leq \phi$ as $|z| \longrightarrow 1$, then $f \equiv 0$.

### 2.11.3 c

Let $w_1, \cdots, w_n$ be points on $S^1 \subset \mathbb{C}$. Show that there exists a point $z \in S^1$ such that

$$\prod_{i=1}^{n} |z - w_i| \geq 1.$$

Conclude that there exists a point $w \in S^1$ such that

$$\prod_{i=1}^{n} |w - w_i| = 1.$$

### 2.11.4 d

Show that if $f$ is entire and $\Re(f)$ is bounded, then $f$ is constant.

## 2.12 S&S 3.8.17

Let $f$ be non-constant, and holomorphic in an open set containing the open unit disc.

### 2.12.1 a

Show that $|z| = 1 \implies |f(z)| = 1$, then the image of $f$ contains the unit disc.

Hint: Show that $f(z) = w_0$ has a root for every $w_0 \in \mathbb{D}$, for which it suffices to show that $f(z) = 0$ has a root, then use the maximum modulus principle.

### 2.12.2 b

Show that if $|z| \geq 1 \implies |f(z)| = 1$ **and** there exists a point $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$, then the image of $f$ contains the unit disc.

## 2.13 S&S 3.8.19

Prove the maximum modulus principle for harmonic functions; i.e.,

### 2.13.1 a

If $u$ is a non-constant real-valued harmonic function on $\Omega$, then $u$ can not attain its extrema on $\Omega$.

### 2.13.2 b

Suppose $\Omega$ has compact closure $\overline{\Omega}$, then if $u$ is harmonic on $\Omega$ and continuous on $\overline{\Omega}$, then

$$\sup_{z\in\Omega} |u(z)| \leq \sup_{z\in\overline{\Omega}-\Omega} |u(z)|$$

Hint: to prove (a), assume that $u$ attains a local maximum at $z_0$, and let $f$ be holomorphic near $z_0$ with $u = \Re(f)$, then show that $f$ is not open. Part (b) is a direct consequence.