Complex Analysis Problem Set 3

D. Zack Garza

Tuesday 7th April, 2020

Contents

1 Problems From Tie

1.1 1

Prove that if f has two Laurent series expansions,

$$f(z) = \sum c_n (z-a)^n \quad \text{and} \quad f(z) = \sum c'_n (z-a)^n$$

then $c_n = c'_n$.

1.1.1 Solution

By taking the difference of two such expansions, it suffices to show that if f is identically zero and $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$ about some point a, then $c_n = 0$ for all n.

Under this assumption, let $D_\varepsilon(a)$ be a disc about a and γ be any contoured contained in its interior. Then for each n, we can apply the formula

$$\begin{aligned} c_n &= \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - a)^{n+1}}\, d\xi \\ &= \frac{1}{2\pi i} \int_\gamma \frac{0}{(\xi - a)^{n+1}}\, d\xi \quad \text{by assumption} \\ &= 0, \end{aligned}$$

which shows that $c_n = 0$ for all n.

■

1.2 2

Find Laurent series expansions of

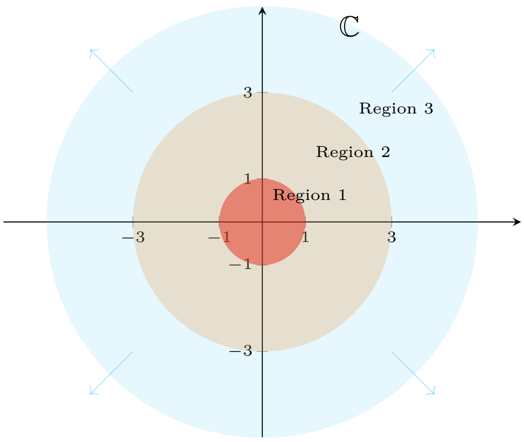
$$\frac{1}{1-z^2}+\frac{1}{3-z}$$

How many such expansions are there? In what domains are each valid?

1.2.1 Solution

Note that f has poles at $z=-1,1,3$, all with multiplicity 1, and so there are 3 regions to consider:

1. $|z|<1$
2. $1<|z|<3$
3. $3<|z|$.



Region 1: Take the following expansion:

$$\begin{aligned} f(z) &= \frac{1}{1-z^2}+\frac{1}{3-z} \\ &= \sum_{n\geq 0} z^{2n} + \frac{1}{3}\left(\frac{1}{1-\frac{3}{z}}\right) \\ &= \sum_{n\geq 0} z^{2n} + \frac{1}{3}\sum_{n\geq 0}\left(\frac{1}{3}\right)^n z^n \\ &= \sum_{n\geq 0} z^{2n} + \sum_{n\geq 0}\left(\frac{1}{3}\right)^{n+1} z^n \end{aligned}$$

.

Noting $\left|z^2\right| < 1$ implies then $|z| < 1$, and that the first term converges for $\left|z^2\right| < 1$ and the second for $\left|\frac{z}{3}\right| < 1 \iff |z| < 3$, this expansion converges to f on the region $|z| < 1$.

Region 2: Take the following expansion:

$$\begin{aligned}
f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\
&= -\frac{1}{z^2}\left(\frac{1}{1-\frac{1}{z^2}}\right) - \frac{1}{3}\left(\frac{1}{1-\frac{z}{3}}\right) \\
&= -\frac{1}{z^2}\sum_{n\geq 0} z^{-2n} + \sum_{n\geq 0}\left(\frac{1}{3}\right)^{n+1} z^n \\
&= -\sum_{n\geq 2}\frac{1}{z^{2n}} + \sum_{n\geq 0}\left(\frac{1}{3}\right)^{n+1} z^n
\end{aligned}$$

.

By construction, the first term converges for $\left|\frac{1}{z^2}\right| < 1 \iff |z| > 1$ and the second for $|z| < 3$.

Region 3: Take the following expansion:

$$\begin{aligned}
f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\
&= -\frac{1}{z^2}\left(\frac{1}{1-\frac{1}{z^2}}\right) - \frac{1}{z}\left(\frac{1}{1-\frac{3}{z}}\right) \\
&= -\frac{1}{z^2}\sum_{n\geq 0}\frac{1}{z^{2n}} - \frac{1}{z}\sum_{n\geq 0}3^n\frac{1}{z^n} \\
&= -\sum_{n\geq 2}\frac{1}{z^{2n}} - \sum_{n\geq 1}\left(\frac{1}{3}\right)^{n-1}\frac{1}{z^n}.
\end{aligned}$$

Note: in principle, terms could be collected here.

By construction, this converges on $\left\{|z|^2 > 1\right\} \bigcap \{|z| > 3\} = \{|z| > 3\}$.

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1.3 3

Let P, Q be polynomials with no common zeros. Assume a is a root of Q. Find the principal part of P/Q at $z = a$ in terms of P and Q if a is (1) a simple root, and (2) a double root.

1.3.1 Solution

We'll use the following definition: if $f : \mathbb{C} \longrightarrow \mathbb{C}$ is analytic with Laurent expansion $f(z) = \sum_{k=-\infty}^{\infty} c_k(z-a)^k$ at the point $a \in \mathbb{C}$, then the **principal part** of f at a is given by

$$\sum_{k=-1}^{-\infty} c_k(z-a)^k = c_{-1}(z-a)^{-1} + c_{-2}(z-a)^{-2} + \cdots .$$

Without loss of generality (by performing polynomial long division if necessary), assume that $\deg P < \deg Q$. By the method used in the theorem that proves meromorphic functions are rational, if we let $a_1, \cdots, a_n$ be the finitely many zeros of $Q(z)$, these are the finitely many poles of $P(z)/Q(z)$, and we can write

$$\frac{P(z)}{Q(z)} := f(z) = P_\infty(z) + \sum_{i=1}^{n} P_{a_i}(z)$$

where $P_w(z)$ denotes the principal part of f at the point w.

Note that if w is a pole of order ℓ, we can explicitly write

$$P_w(z) = \frac{\alpha_1}{z-w} + \frac{\alpha_2}{(z-w)^2} + \cdots + \frac{\alpha_\ell}{(z-w)^\ell}$$

for some constants $\alpha_i \in \mathbb{C}$, and thus the first equation expresses f in terms of its partial fraction decomposition.

Thus if a is a simple root of $Q(z)$, it is a simple pole of f, and thus we have $P_a(z) = \frac{\alpha_1}{z-a}$, which consists of a single term. Since we can write $f(z) = P_\infty(z) + P_a(z) + \cdots$ where none of the remaining terms involve a, it follows by definition that $\alpha_1 = \mathrm{Res}(f, a)$ and so

$$P_a(z) = \frac{\mathrm{Res}(f(z), a)}{z-a},$$

where we can use a known formula to express $\mathrm{Res}(f(z), a) = \frac{P(a)}{Q'(a)}$.

Similarly, if now a is a root of multiplicity 2 of $Q(z)$, a is a pole of order 2 of f and $P_a(z) = \frac{\alpha_1}{z-a} + \frac{\alpha_2}{(z-a)^2}$ with precisely two terms. Thus as before, $\alpha_1 = \mathrm{Res}(f(z), a)$, and now $\alpha_2 = \mathrm{Res}((z-a)f(z), a)$, and we have

$$P_a(z) = \frac{\mathrm{Res}(f(z), a)}{z-a} + \frac{\mathrm{Res}((z-a)f(z), a)}{(z-a)^2}.$$

■

1.4 4

Let f be non-constant, analytic in $|z| > 0$, where $f(z_n) = 0$ for infinitely many points z_n with $\lim_{n \longrightarrow \infty} z_n = 0$.
Show that $z = 0$ is an essential singularity for f.

Example: $f(z) = \sin(1/z)$.

1.4.1 Solution

We first note that $z = 0$ is in fact a singularity of f, since the zeros of analytic functions are isolated.

The point $z = 0$ can not be a pole because (by definition) this would force $\lim_{z \longrightarrow 0} |f(z)| = \infty$. Explicitly, this would mean that for every $R > 0$, there would exist a $\delta > 0$ such that $z \in D_\delta(0) \implies |f(z)| > R$.

However, since $z_n \longrightarrow 0$ and $f(z_n) = 0 < R$ for every n, every $D_\delta(0)$ contains a point z_N that violates this condition.

Similarly, $z = 0$ can not be removable, since the function

$$g(z) = \begin{cases} 0 & z = 0 \\ f(z) & \text{otherwise} \end{cases}$$

defines an analytic continuation of f. However, it is a theorem that the zeros of an analytic function are isolated, whereas every neighborhood of $z = 0$ (which is a zero of g) contains infinitely many distinct zeros of the form z_n, a contradiction.

■

1.5 5

Show that if f is entire and $\lim_{z \longrightarrow \infty} f(z) = \infty$, then f is a polynomial.

1.5.1 Solution

Since f is entire, it is analytic on $\mathbb{C}$, so there is an expansion $f(z) = \sum_{k=0}^{\infty} c_k z^k$ that converges to f everywhere. Let $F(z) = f(1/z)$; then $\lim_{z \longrightarrow 0} F(z) = \infty$ by assumption.

This also implies that since $z = \infty$ is a pole of f, the point $z = 0$ is a pole of F, say of order N.

However, we can expand $F(z) = \sum_{k=0}^{\infty} c_k \frac{1}{z^k}$. Since this is a Laurent expansion for F about $z = 0$, which is a pole of order N, we must in fact have $F(z) = \sum_{k=0}^{N} c_k \frac{1}{z_k}$, i.e. there are only N terms in this expansion.

This implies that $f(z) = \sum_{k=0}^{N} c_k z^k$, which has finitely many terms and is thus a polynomial.

■

1.6 6

a. Show that

$$\int_0^{2\pi} \log \left|1 - e^{i\theta}\right| \, d\theta = 0$$

b. Show that this identity is equivalent to SS 3.8.9:

$$\int_0^1 \log(\sin(\pi x)) \, dx = -\log 2.$$

1.6.1 Solution Part (a)

Let I be the integral in question, then substituting $z = e^{i\theta}$ and $\frac{dz}{iz} = d\theta$ yields

$$I = \int_{S^1} \frac{\log|1-z|}{iz} \, dz := \Re\left(\int_{S^1} f(z) \, dz\right),$$

where

$$f(z) := \frac{\log(1-z)}{iz},$$

S^1 denotes the unit circle in $\mathbb{C}$, and since by definition

$$\log_{\mathbb{C}}(z) = \log_{\mathbb{R}}(|z|) + i\arg(z)$$

where the subscripts denote the complex and real logarithms respectively, we have

$$\log_{\mathbb{C}}|1-z| = \Re(\log_{\mathbb{C}}(1-z)).$$

So it suffices to show that $\int_{S^1} f(z) \, dz = 0$.

The claim is that $z = 0$ is a removable singularity and thus f is holomorphic in the unit disc. The singularity is removable because we have

$$\begin{aligned}
\lim_{z \longrightarrow 0} f(z) &= \lim_{z \longrightarrow 0} \frac{\log(1-z)}{iz} \\
&= \lim_{z \longrightarrow 0} \frac{\frac{1}{1-z}}{i} \qquad \text{by L'Hopital's} \\
&= -i,
\end{aligned}$$

so the modified function

$$F(z) = \begin{cases} -i & z = 0 \\ f(z) & \text{otherwise} \end{cases}$$

is holomorphic, making $z = 0$ removable.

Since f is also analytic, the Cauchy-Goursat theorem applies and $\displaystyle\int_{S^1} f = 0$.

1.6.2 Solution Part (b)

No clue how to relate these two!

1.7 7

Let $0 < a < 4$ and evaluate

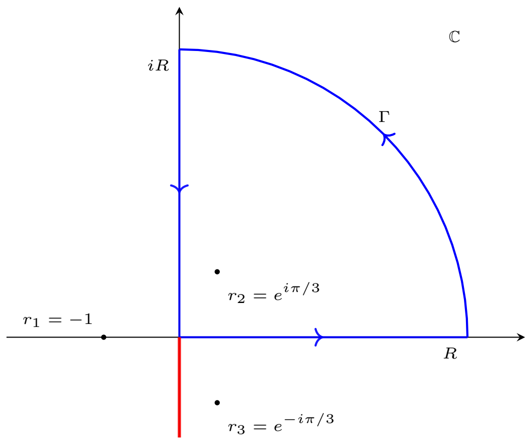
$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^3}\,dx$$

1.7.1 Solution

Let I denote the integral in question. We will compute this using a closed contour and the residue theorem, so first note that

$$z^3 + 1 = (z+1)(z - e^{i\pi 3})(z - e^{-i\pi 3}) := (z - r_1)(z - r_2)(z - r_3).$$

Defining $z^\alpha = e^{\alpha \log z}$ for $\alpha \in \mathbb{R}$, we'll take the following contour Γ shown in blue along with a branch cut for the logarithm function indicated in red:



Letting

$$f(z) := \frac{z^{\alpha-1}}{z^3+1} := \frac{P(z)}{Q(z)},$$

we find that only $z = r_2$ will contribute a term to $\int_\Gamma f$. Noting that each pole is simple of order 1, we have

$$\operatorname{Res}(f(z), z = r_i) = \frac{P(r_i)}{Q'(r_i)} = \frac{r_i^{\alpha-1}}{3r_i^2} = \frac{r_i^{\alpha-3}}{3}$$

We thus have

$$\begin{aligned}\operatorname{Res}(f(z), z = r_2) &= \frac{1}{3} e^{\frac{i\pi(\alpha-3)}{3}} \\ \implies \int_\Gamma f(z)\, dz &= \frac{2\pi i}{3} e^{\frac{i\pi(\alpha-3)}{3}}.\end{aligned}$$

We can now compute the contributions to the integral along the semicircular arc and the portion along the imaginary axis.

Along the arc, Jordan's lemma applies since $\frac{1}{R^3+1} \overset{R\longrightarrow\infty}{\longrightarrow} 0$, and thus this contribution vanishes.

Along the imaginary axis, we can make the following change of variables:

$$\begin{aligned}\int_R^0 f(iy)\, dy &= -\int_0^R \frac{(iy)^{\alpha-1}}{(iy)^3+1}\, dy \\ &= -\frac{1}{i}\int_0^R \frac{t^{\alpha-1}}{t^3+1}\, dt \qquad\qquad (t = iz,\ dt = idz) \\ &= iI,\end{aligned}$$

which is i times the original integral.

We thus have

$$\begin{aligned}\operatorname{Res}(f(z), z = r_2) &= \int_\Gamma f \\ &= \int_0^R f + \int_{C_R} f + \int_{iR}^0 f \\ &\overset{R\longrightarrow\infty}{\longrightarrow} I + 0 + iI \\ &= (1+i)I,\end{aligned}$$

and so

$$I = \frac{\operatorname{Res}(f(z), z = r_2)}{1+i} = \frac{2\pi i}{3(1+i)} e^{\frac{i\pi(\alpha-3)}{3}}.$$

■

Note: this seems to be wrong, because plugging in $a = 1, 2, 3$ doesn't result in a real value.

1.8 8

Prove the fundamental theorem of Algebra using
a. Rouche's Theorem.
b. The maximum modulus principle.

1.8.1 Solution (Rouche)

We want to show that every $f \in \mathbb{C}[x]$ has precisely n roots, and we'll use the follow formulation of Rouche's theorem:

Theorem 1.1 ***(Rouche).***
If f, g are holomorphic on $D(z_0)$ with $f, g \neq 0$ and $|f-g| < |f| + |g|$ on $\partial D(z_0)$, then f and g has the same number of zeros within D.

We'll also use without proof the fact that the function $h(z) = z^n$ has precisely n zeros (counted with multiplicity).

Suppose $f(z) = a_n z^n + \cdots + a_1 z + a_0$ where $a_n \neq 0$ and define

$$g(z) \coloneqq a_n z^n.$$

Noting that polynomials are entire, f, g are nonzero by assumption, and fixing $|z| = R > 1$, we have

$$\begin{aligned}
|f-g| &= \left|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right| \\
&= \left|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 + a_n z^n - a_n z^n\right| \\
&\leq \left|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 + a_n z^n\right| + \left|-a_n z^n\right| \quad \text{by the triangle inequality} \\
&= |f| + |g|
\end{aligned}$$

the conditions of Rouche's theorem apply and f, g have the same number of roots. Since g has precisely n roots, f does as well.

This is much simpler than other proofs out there, so I suspect something is slightly wrong but I couldn't sort out what it was.

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1.8.2 Solution (Maximum Modulus Principle)

Toward a contradiction, suppose f is non-constant and has *no* zeros. Then $g(z) \coloneqq 1/f(z)$ is non-constant and holomorphic on $\mathbb{C}$.

Using the fact that $\lim_{z \longrightarrow \infty} f(z) = \infty$ for any polynomial f, pick R large enough such that

$$|z| \geq R \implies |f(z)| > |f(0)|,$$

which inverted yields,

$$|z| \geq R \implies |g(z)| < |g(0)|.$$

Noting that $S_R \coloneqq \{|z| \geq R\}$ is closed (as the complement of the open set $\{|z| < R\}$), bounded (by the argument above), and thus compact by Heine-Borel, g attains a maximum on S_R.

But by the maximum modulus principle, this forces g to be constant, and since $g = \frac{1}{f}$, it must also be true that f is constant.

■

1.9 9

Let f be analytic in a region D and γ a rectifiable curve in D with interior in D. Prove that if $f(z)$ is real for all $z \in \gamma$, then f is constant.

1.9.1 Solution

Without loss of generality, assume $0 \in D$ (by considering the translate $f(z) - w$ if necessary) and γ is not entirely contained in $\mathbb{R}$ (by taking a homotopic curve).

Since f is analytic in D, we can write its Laurent series expansion about $z = 0$:

$$f(z) = c_0 + c_1 z + \cdots \qquad \text{for } z \in D.$$

For $z \in \gamma$ we can write $z = x + iy$ where $y \neq 0$.

Then

$$\begin{aligned} f(z) &= f(x+iy) \\ &= c_0 + c_1(x+iy) + c_2(x+iy)^2 + \cdots \\ &= c_0 + \left(c_1 x + c_2 x^2 + \cdots\right) + i\left(c_1 y + c_2 y^2 + c_2 xy + \cdots\right) \\ &\subset \mathbb{R} \quad \text{by assumption,} \end{aligned}$$

where the second parenthesized term must vanish for all $x + iy \in \gamma$,

But since there is at least one $z \in \gamma$ with $y \neq 0$, this forces $c_1 = c_2 = \cdots = 0$, and thus $f(z) = c_0$ must be constant.

■

1.10 10

For $a > 0$, evaluate

$$\int_0^{\pi/2} \frac{d\theta}{a + \sin^2\theta}$$

We have

$$
\begin{aligned}
I &:= \int_0^{\pi/2} \frac{1}{1+\sin^2(\theta)}\, d\theta \\
&= \int_{\gamma_1} \frac{1}{a+\left(\frac{z-z^{-1}}{2i}\right)^2} \frac{-i\, dz}{z} \qquad \text{where } \gamma_1 \text{ is } \frac{1}{4} \text{ of the unit circle } S^1 \\
&= -i \int_{\gamma_1} \frac{1}{z}\left(\frac{1}{a+\left(-\frac{1}{4}\right)(z^2-2+z^{-2})}\right) dz \\
&= 4i \int_{\gamma_1} \frac{1}{z}\left(\frac{1}{z^2-(2+4a)+z^{-2}}\right) dz \\
&= 4i \int_{\gamma_1} \frac{z}{z^4-(2+4a)z^2+1}\, dz \\
&= i \oint_{S^1} \frac{z}{z^4-(2+4a)z^2+1}\, dz \\
&= \frac{i}{2} \oint_{2\cdot S^1} \frac{1}{u^2-(2+4a)u+1}\, du \qquad \text{using } u=z^2, \frac{1}{2}\, du = z\, dz \\
&:= \frac{i}{2} \oint_{2\cdot S^1} \frac{1}{f_a(u)}\, du \\
&= \frac{i}{2} \cdot 2\pi i \cdot \sum \operatorname{Res}\left(\frac{1}{f_a(u)}, u=r_i\right),
\end{aligned}
$$

where $2 \cdot S^1$ denotes the contour wrapping around the unit circle twice and r_i denote the poles contained in the region bounded by S^1. We can now compute the last integral by the residue theorem.

Factor the denominator as

$$
f_a(u) = u^2 - (2+4a)u + 1 = (u-r_1)(u-r_2),
$$

where the r_i are given by $(1+2a) \pm 4\sqrt{a^2+a}$ using the quadratic formula. We can then write a partial fraction decomposition

$$
\begin{aligned}
\frac{1}{f_a(u)} &:= \frac{1}{u^2-(2+4a)u+1} \\
&= \frac{1}{(u-r_1)(u-r_2)} \\
&= \frac{A}{u-r_1} + \frac{B}{u-r_2} \\
&= \frac{\operatorname{Res}_{u=r_1} 1/f(u)}{u-r_1} + \frac{\operatorname{Res}_{u=r_2} 1/f(u)}{u-r_2} \\
&= \frac{1/f'(r_1)}{u-r_1} + \frac{1/f'(r_2)}{u-r_2} \\
&= -\frac{1}{8\sqrt{a^2+a}(u-r_1)} + \frac{1}{8\sqrt{a^2+a}(u-r_2)}.
\end{aligned}
$$

Since $|r_2| = \left|(1+2a)+4\sqrt{a^2+a}\right| > 1$, we find that the only relevant pole inside of S^1 is r_1. Reading off the residue from the above decomposition, we thus have

$$\begin{aligned} I &= \frac{i}{2} \cdot 2\pi i \cdot \sum \text{Res}_{u=r_i} \frac{1}{f_a(u)} \\ &= -\pi \cdot \text{Res}_{u=r_1} \frac{1}{f_a(u)} \\ &= \frac{\pi}{8\sqrt{a^2+a}}. \end{aligned}$$

■

Note: I know I'm off by a constant here at least, since $a = 1$ should reduce to $\pi/2\sqrt{2}$.

1.11 11

Find the number of roots of $p(z) = 4z^4 - 6z + 3$ in $|z| < 1$ and $1 < |z| < 2$ respectively.

For $|z| < 1$, take $f(z) = -6z$ and $g(z) = z^4 + 3$, noting that $f + g = p$. Using the maximum modulus principal, we know that the max/mins of f, g occur on $|z| = 1$, on which we have

$$|g(z)| = 4 < 6 = |f(z)|,$$

so Rouche's theorem applies and both p and f have the same number of zeros. Since f clearly has **one** zero, p has one zero in this region.

Now consider $|z| < 2$ and set $f(z) = z^4$ and $g(z) = -6z + 3$. By a similar argument, we have

$$|g(z)| = 15 < 16 = |f|$$

on $|z| = 2$, and thus f and p have the same number of zeros in this region. Since f has **four** zeros here, so does p.

Thus p has $4 - 1 =$ **3 zeros** on $1 \leq |z| \leq 2$.

■

1.12 12

Prove that $z^4 + 2z^3 - 2z + 10$ has exactly one root in each open quadrant.

1.12.1 Solution

Let $f(z) = z^4 + 2z^3 - 2z + 10$, and consider the following contour:

Image

By the argument principle, we have

$$\Delta_\Gamma \arg f(z) = 2\pi(Z - P),$$

where Z is the number of zeros of f in the region Ω enclosed by Γ and P is the number of poles in Ω.

Since polynomials are holomorphic on $\mathbb{C}$, by the argument principle it suffices to show that

- f does not have any roots on the real or imaginary axes
- f does not vanish on Γ, and
- $\Delta_\Gamma \arg f(z) = 1$, where Δ_Γ denotes the total change in the argument of f over Γ.

It will follow by symmetry that f has exactly one root in each quadrant.

Claim 1.2.

- f has no roots on the coordinate axes.
- $\Delta_{\gamma_1} \arg f(z) = 0$
- $\Delta_{\gamma_2} \arg f(z) = 2\pi$
- $\Delta_{\gamma_3} \arg f(z) = 0$

Given the claim, we would have

$$\Delta_\Gamma \arg f(z) = 2\pi = 2\pi(Z - 0) \implies Z = 1,$$

which is what we wanted to show.

Proof of Claim:

γ_2: For $R \gg 0$, we have $f(z) \sim z^4$. Along γ_2, the argument of z ranges from 0 to $\frac{\pi}{2}$, and thus the argument of z^4 ranges from 0 to $4 \cdot \frac{\pi}{2} = 2\pi$.

γ_1: By cases, for $z \in \mathbb{R}$,

- If $|z| > 1$, then $z^3 > z$ and so

$$\begin{aligned} f(z) &= (z^4 + 10) + (2z^3 - 2z) \\ &> (z^4 + 10) + (2z - 2z) \\ &= z^4 + 10 \\ &> 0, \end{aligned}$$

  so f is strictly positive and does not change argument on $(\pm 1, \pm\infty)$ or $i \cdot (\pm 1, \pm\infty)$.

- If $|z| \leq 1$,

$$\begin{aligned} \left| -z^4 - 2z^3 + 2z \right| &\leq |z|^4 + 2|z|^3 + 2|z| \\ &\leq 1 + 2 + 2 \\ &= 5 \\ &< 10 \\ \implies f(z) &= 10 - (-z^4 - 2z^3 + 2z) > 0, \end{aligned}$$

  so f is strictly positive and does not change argument $(0, \pm 1)$ or $i \cdot (0, \pm 1)$.

■

1.13 13

Prove that for $a > 0$, $z \tan z - a$ has only real roots.

1.13.1 Solution

We can extend Rouche's theorem in the following way: if $f = g + h$ with $|g| > |h|$ on γ then $Z_f - P_f = Z_g - P_g$, where Z, P denote the number of zeros and poles respectively.

So we proceed by explicitly counting the number of real roots Z_f of $f(z) = z\tan(z) - a$ on a certain arbitrary real interval, then extend that interval to a rectangle in $\mathbb{C}$ and apply Rouche to show that there are still Z_f zeros within the rectangle. This will imply that the only roots on that region are real, and in the limit as the length of the interval goes to infinity, this will remain true (since any potential root must fall within such a bounded rectangle).

Fix some parameter $N \in \mathbb{Q}$ (to be determined) and consider the interval $[-N\pi - \varepsilon, N\pi + \varepsilon]$ for some $0 < \varepsilon \ll 1$. In this interval, we can compute $\sin(x) = 0 \iff x = 2k\pi$, yielding $2N + 1$ zeros (including $x = 0$), and thus $x \tan(x)$ has exactly $2N + 1$ zeros here.

We can also compute $\cos(x) = 0 \iff x = (2k + 1)\pi$, yielding $2N$ zeros and thus $2N$ poles of $x \tan(x)$.

Thus letting $\tilde{Z}_f, \tilde{P}_f$ denote the number of real zeros/poles of f, we have

$$\tilde{Z}_f - \tilde{P}_f = (2N + 1) - 2N = 1.$$

> I couldn't get the inequalities to work! :(

1.14 14

Let f be nonzero, analytic on a bounded region Ω and continuous on its closure $\overline{\Omega}$.
Show that if $|f(z)| \equiv M$ is constant for $z \in \partial\Omega$, then $f(z) \equiv Me^{i\theta}$ for some real constant θ.

1.14.1 Solution

By the maximum modulus principle applied to f in $\overline{\Omega}$, we know that $\max |f| = M$. Similarly, the maximum modulus principle applied to $\dfrac{1}{f}$ in $\overline{\Omega^c}$ since f is nonzero in Ω, and we can conclude that $\min |f| = M$ as well. Thus $|f| = M$ is constant on $\overline{\Omega}$.

So consider the function $g(z) = |f(z)|$; from the above observation, we find that $g(\overline{\Omega}) = \{M\}$. Letting S_M be the circle of radius M, this implies that $f(\Omega) \subseteq S_M$. In particular, $S_M \subset \mathbb{C}$ is a closed set.

However, by the open mapping theorem, $f(\Omega) \subset \mathbb{C}$ must be an open set. A basis for the topology on $\mathbb{C}$ is given by open discs, so in particular, the open sets of $\mathbb{C}$ have real dimension either zero or two. Since S_M has real dimension 1, $f(\Omega)$ must have dimension zero and is thus a collection of points. Since f is continuous, the image can only be one point, i.e. $f(\Omega) = \{\text{pt}\} \in S_M$. So f is constant.

$\blacksquare$

2 Stein and Shakarchi

2.1 S&S 3.8.1

Using Euler's formula

$$\sin(\pi z) = \frac{1}{2i}(e^{i\pi z} - e^{-i\pi z})$$

show that the complex zeros of $\sin(\pi z)$ are exactly the integers, each of order one. Calculate the residue of $\dfrac{1}{\sin(\pi z)}$ at $z = n \in \mathbb{Z}$.

2.2 S&S 3.8.2

Evaluate the integral

$$\int_{\mathbb{R}} \frac{dx}{1+x^4}$$

What are the poles of the integrand?

2.3 S&S 3.8.4

Show that

$$\int_{\mathbb{R}} \frac{x \sin x}{x^2 + a^2} = \frac{\pi e^{-a}}{a} \qquad a > 0$$

2.4 S&S 3.8.5

Show that for $\xi \in \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{e^{2\pi i x \xi}}{(1+x^2)^2} = \frac{\pi}{2}(1 + 2\pi|\xi|)e^{-2\pi|\xi|}$$

2.5 S&S 3.8.6

Show that

$$\int_{\mathbb{R}} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)\pi}{2 \cdot 4 \cdots (2n}$$

2.6 S&S 3.8.7

Show that for $a > 1$,

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos\theta)^2} = \frac{2\pi a}{(a^2 - 1)^{3/2}}$$

2.7 S&S 3.8.8

Show that if $a, b \in \mathbb{R}$ with $a > |b|$ then

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi a}{\sqrt{a^2 - b^2}}$$

2.8 S&S 3.8.9

Show that

$$\int_0^1 \log(\sin \pi x)\ dx = -\log 2$$

2.9 S&S 3.8.10

Show that if $a > 0$

$$\int_0^\infty \frac{\log x}{x^2 + a^2}\ dx = \frac{\pi \log a}{2a}$$

2.10 S&S 3.8.14

Prove that if f is entire and injective, then $f(z) = az + b$ with $a, b \in \mathbb{C}$ with $a \neq 0$.

Hint: apply the Casorati-Weierstrass theorem to $f(1/z)$.

2.11 S&S 3.8.15

Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

2.11.1 a

If f is entire and for all $R > 0$, there are constants $A, B > 0$ such that $\sup_{|z|=R} |f(z)| \leq AR^k + B$, then f is a polynomial of degree less than k.

2.11.2 b

Show that if f is holomorphic on the unit disk, is bounded, and converges to zero uniformly in the sector $\theta \leq \arg z \leq \phi$ as $|z| \longrightarrow 1$, then $f \equiv 0$.

2.11.3 c

Let $w_1, \cdots, w_n$ be points on $S^1 \subset \mathbb{C}$. Show that there exists a point $z \in S^1$ such that

$$\prod_{i=1}^{n} |z - w_i| \geq 1.$$

Conclude that there exists a point $w \in S^1$ such that

$$\prod_{i=1}^{n} |w - w_i| = 1.$$

2.11.4 d

Show that if f is entire and $\Re(f)$ is bounded, then f is constant.

2.12 S&S 3.8.17

Let f be non-constant, and holomorphic in an open set containing the open unit disc.

2.12.1 a

Show that $|z| = 1 \implies |f(z)| = 1$, then the image of f contains the unit disc.

> Hint: Show that $f(z) = w_0$ has a root for every $w_0 \in \mathbb{D}$, for which it suffices to show that $f(z) = 0$ has a root, then use the maximum modulus principle.

2.12.2 b

Show that if $|z| \geq 1 \implies |f(z)| = 1$ **and** there exists a point $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$, then the image of f contains the unit disc.

2.13 S&S 3.8.19

Prove the maximum modulus principle for harmonic functions; i.e.,

2.13.1 a

If u is a non-constant real-valued harmonic function on Ω, then u can not attain its extrema on Ω.

2.13.2 b

Suppose Ω has compact closure $\overline{\Omega}$, then if u is harmonic on Ω and continuous on $\overline{\Omega}$, then

$$\sup_{z\in\Omega} |u(z)| \leq \sup_{z\in\overline{\Omega}-\Omega} |u(z)|$$

Hint: to prove (a), assume that u attains a local maximum at z_0, and let f be holomorphic near z_0 with $u = \Re(f)$, then show that f is not open. Part (b) is a direct consequence.