# **Complex Analysis Problem Set 3**

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## 1 Problems From Tie

## 1.1 1

Prove that if f has two Laurent series expansions,

$$f(z) = \sum c_n(z-a)^n$$
 and  $f(z) = \sum c'_n(z-a)^n$ 

then  $c_n = c'_n$ .

## 1.1.1 Solution

By taking the difference of two such expansions, it suffices to show that if f is identically zero and  $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$  about some point a, then  $c_n = 0$  for all n.

Under this assumption, let  $D_{\varepsilon}(a)$  be a disc about a and  $\gamma$  be any contoured contained in its interior. Then for each n, we can apply the formula

$$c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi - a)^{n+1}} d\xi$$
$$= \frac{1}{2\pi i} \int_{\gamma} \frac{0}{(\xi - a)^{n+1}} d\xi \quad \text{by assumption}$$
$$= 0,$$

which shows that  $c_n = 0$  for all n.

## 1.2 2

Find Laurent series expansions of

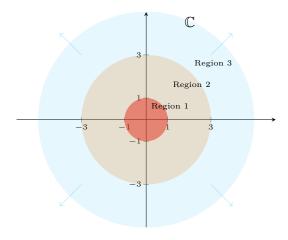
$$\frac{1}{1-z^2} + \frac{1}{3-z}$$

How many such expansions are there? In what domains are each valid?

#### 1.2.1 Solution

Note that f has poles at z = -1, 1, 3, all with multiplicity 1, and so there are 3 regions to consider:

- 1. |z| < 1
- 2. 1 < |z| < 3
- 3. 3 < |z|.



**Region 1:** Take the following expansion:

$$\begin{split} f(z) &= \frac{1}{1 - z^2} + \frac{1}{3 - z} \\ &= \sum_{n \ge 0} z^{2n} + \frac{1}{3} \left( \frac{1}{1 - \frac{3}{z}} \right) \\ &= \sum_{n \ge 0} z^{2n} + \frac{1}{3} \sum_{n \ge 0} \left( \frac{1}{3} \right)^n z^n \\ &= \sum_{n \ge 0} z^{2n} + \sum_{n \ge 0} \left( \frac{1}{3} \right)^{n+1} z^n \end{split}$$

Noting  $\left|z^2\right| < 1$  implies then |z| < 1, and that the first term converges for  $\left|z^2\right| < 1$  and the second for  $\left|\frac{z}{3}\right| < 1 \iff |z| < 3$ , this expansion converges to f on the region |z| < 1.

**Region 2:** Take the following expansion:

$$f(z) = \frac{1}{1 - z^2} + \frac{1}{3 - z}$$

$$= -\frac{1}{z^2} \left( \frac{1}{1 - \frac{1}{z^2}} \right) - \frac{1}{3} \left( \frac{1}{1 - \frac{z}{3}} \right)$$

$$= -\frac{1}{z^2} \sum_{n \ge 0} z^{-2n} + \sum_{n \ge 0} \left( \frac{1}{3} \right)^{n+1} z^n$$

$$= -\sum_{n \ge 2} \frac{1}{z^{2n}} + \sum_{n \ge 0} \left( \frac{1}{3} \right)^{n+1} z^n$$

By construction, the first term converges for  $\left|\frac{1}{z^2}\right| < 1 \iff |z| > 1$  and the second for |z| < 3.

**Region 3:** Take the following expansion:

$$\begin{split} f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\ &= -\frac{1}{z^2} \left( \frac{1}{1-\frac{1}{z^2}} \right) - \frac{1}{z} \left( \frac{1}{1-\frac{3}{z}} \right) \\ &= -\frac{1}{z^2} \sum_{n \geq 0} \frac{1}{z^{2n}} - \frac{1}{z} \sum_{n \geq 0} 3^n \frac{1}{z^n} \\ &= -\sum_{n \geq 2} \frac{1}{z^{2n}} - \sum_{n \geq 1} \left( \frac{1}{3} \right)^{n-1} \frac{1}{z^n}. \end{split}$$

Note: in principle, terms could be collected here.

By construction, this converges on  $\left\{ \left|z\right|^{2}>1\right\} \bigcap \left\{ \left|z\right|>3\right\} =\left\{ \left|z\right|>3\right\} .$ 

1.3 3

Let P, Q be polynomials with no common zeros. Assume a is a root of Q. Find the principal part of P/Q at z=a in terms of P and Q if a is (1) a simple root, and (2) a double root.

1.3.1 Solution

todo

#### 1.4 4

Let f be non-constant, analytic in |z| > 0, where  $f(z_n) = 0$  for infinitely many points  $z_n$  with  $\lim_{n \to \infty} z_n = 0$ .

Show that z = 0 is an essential singularity for f.

Example:  $f(z) = \sin(1/z)$ .

### 1.4.1 Solution

It suffices to show that  $z_0 = 0$  is neither a pole nor a removable singularity, i.e.

- 1.  $\lim_{z \to z_0} f(z) \neq \infty$
- 2. |f(z)| is not bounded on any neighborhood  $D_{\varepsilon}(z_0)$ .

The first property follows because if f is analytic,

## 1.5 5

Show that if f is entire and  $\lim_{z \to \infty} f(z) = \infty$ , then f is a polynomial.

## 1.6 6

Problem: a. Show that

$$\int_0^{2\pi} \log \left| 1 - e^{i\theta} \right| \, d\theta = 0$$

b. Show that this identity is equivalent to SS 3.8.9.

## 1.7 7

Let 0 < a < 4 and evaluate

$$\int_0^\infty \frac{x^{\alpha - 1}}{1 + x^3} \ dx$$

## 1.8 8

Prove the fundamental theorem of Algebra using

- a. Rouche's Theorem.
- b. The maximum modulus principle.

## 1.9 9

Let f be analytic in a region D and  $\gamma$  a rectifiable curve in D with interior in D. Prove that if f(z) is real for all  $z \in \gamma$ , the f is constant.

## 1.10 10

For a > 0, evaluate

$$\int_0^{\pi/2} \frac{d\theta}{a + \sin^2 \theta}$$

## 1.11 11

Find the number of roots of  $p(z) = 4z^4 - 6z + 3$  in |z| < 1 and 1 < |z| < 2 respectively.

## 1.12 12

Prove that  $z^4 + 2z^3 - 2z + 10$  has exactly one root in each open quadrant.

### **1.12.1 Solution**

Let  $f(z) = z^4 + 2z^3 - 2z + 10$ , and consider the following contour:



By the argument principle, we have

$$\Delta_{\Gamma} \arg f(z) = 2\pi (Z - P),$$

where Z is the number of zeros of f in the region  $\Omega$  enclosed by  $\Gamma$  and P is the number of poles in  $\Omega$ . Since polynomials are holomorphic on  $\mathbb{C}$ , by the argument principle it suffices to show that

- $\bullet$  f does not have any roots on the real or imaginary axes
- f does not vanish on  $\Gamma$ , and
- $\Delta_{\Gamma} \arg f(z) = 1$ , where  $\Delta_{\Gamma}$  denotes the total change in the argument of f over  $\Gamma$ .

It will follow by symmetry that f has exactly one root in each quadrant.

## Claim 1.1.

- f has no roots on the coordinate axes.
- $\Delta_{\gamma_1} \arg f(z) = 0$
- $\Delta_{\gamma_2} \arg f(z) = 2\pi$
- $\Delta_{\gamma_3}$  arg f(z) = 0

Given the claim, we would have

$$\Delta_{\Gamma} \arg f(z) = 2\pi = 2\pi (Z - 0) \implies Z = 1,$$

which is what we wanted to show.

#### **Proof of Claim:**

 $\gamma_2$ : For  $R \gg 0$ , we have  $f(z) \sim z^4$ . Along  $\gamma_2$ , the argument of z ranges from 0 to  $\frac{\pi}{2}$ , and thus the argument of  $z^4$  ranges from 0 to  $4 \cdot \frac{\pi}{2} = 2\pi$ .

 $\gamma_1$ : By cases, for  $z \in \mathbb{R}$ ,

• If |z| > 1, then  $z^3 > z$  and so

$$f(z) = (z^4 + 10) + (2z^3 - 2z) > (z^4 + 10) + (2z - 2z) = z^4 + 10 > 0,$$

so f is strictly positive and does not change argument on  $(1, \infty)$ .

• If  $|z| \le 1$ ,

$$\left| -z^4 - 2z^3 + 2z \right| \le |z|^4 + 2|z|^3 + 2|z| \le 1 + 2 + 2 = 5 < 10$$

$$\implies f(z) = 10 - (-z^4 - 2z^3 + 2z) > 0,$$

so f is strictly positive and does not change argument (0,1).

#### 1.13 13

Prove that for a > 0,  $z \tan z - a$  has only real roots.

#### 1.14 14

Let f be nonzero, analytic on a bounded region  $\Omega$  and continuous on its closure  $\overline{\Omega}$ . Show that if  $|f(z)| \equiv M$  is constant for  $z \in \partial \Omega$ , then  $f(z) \equiv M e^{i\theta}$  for some real constant  $\theta$ .

#### 2 Stein and Shakarchi

#### 2.1 S&S 3.8.1

Using Euler's formula

$$\sin(\pi z) = \frac{1}{2i} (e^{i\pi z} - e^{-i\pi z})$$

show that the complex zeros of  $\sin(\pi z)$  are exactly the integers, each of order one. Calculate the residue of  $\frac{1}{\sin(\pi z)}$  at  $z=n\in\mathbb{Z}$ .

## 2.2 S&S 3.8.2

Evaluate the integral

$$\int_{\mathbb{R}} \frac{dx}{1+x^4}$$

What are the poles of the integrand?

## 2.3 S&S 3.8.4

Show that

$$\int_{\mathbb{R}} \frac{x \sin x}{x^2 + a^2} = \frac{\pi e^{-a}}{a} \quad a > 0$$

#### 2.4 S&S 3.8.5

Show that for  $\xi \in \mathbb{R}$ ,

$$\int_{\mathbb{R}} \frac{e^{2\pi i x \xi}}{(1+x^2)^2} = \frac{\pi}{2} (1 + 2\pi |\xi|) e^{-2\pi |\xi|}$$

## 2.5 S&S 3.8.6

Show that

$$\int_{\mathbb{R}} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot \dots \cdot (2n-1)\pi}{2 \cdot 4 \cdot \dots \cdot (2n-1)\pi}$$

## 2.6 S&S 3.8.7

Show that for a > 1,

$$\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{3/2}}$$

#### 2.7 S&S 3.8.8

Show that if  $a, b \in \mathbb{R}$  with a > |b| then

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi a}{\sqrt{a^2 - b^2}}$$

#### 2.8 S&S 3.8.9

Show that

$$\int_0^1 \log(\sin \pi x) \ dx = -\log 2$$

#### 2.9 S&S 3.8.10

Show that if a > 0

$$\int_0^\infty \frac{\log x}{x^2 + a^2} \ dx = \frac{\pi \log a}{2a}$$

#### 2.10 S&S 3.8.14

Prove that if f is entire and injective, then f(z) = az + b with  $a, b \in \mathbb{C}$  with  $a \neq 0$ .

Hint: apply the Casorati-Weierstrass theorem to f(1/z).

#### 2.11 S&S 3.8.15

Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

#### 2.11.1 a

If f is entire and for all R > 0, there are constants A, B > 0 such that  $\sup_{|z|=R} |f(z)| \le AR^k + B$ , then f is a polynomial of degree less than k.

#### 2.11.2 b

Show that if f is holomorphic on the unit disk, is bounded, and converges to zero uniformly in the sector  $\theta \leq \arg z \leq \phi$  as  $|z| \longrightarrow 1$ , then  $f \equiv 0$ .

#### 2.11.3 c

Let  $w_1, \dots, w_n$  be points on  $S^1 \subset \mathbb{C}$ . Show that there exists a point  $z \in S^1$  such that

$$\prod_{i=1}^{n} |z - w_i| \ge 1.$$

Conclude that there exists a point  $w \in S^1$  such that

$$\prod_{i=1}^{n} |w - w_i| = 1.$$

#### 2.11.4 d

Show that if f is entire and  $\Re(f)$  is bounded, then f is constant.

#### 2.12 S&S 3.8.17

Let f be non-constant, and holomorphic in an open set containing the open unit disc.

#### 2.12.1 a

Show that  $|z|=1 \implies |f(z)|=1$ , then the image of f contains the unit disc.

Hint: Show that  $f(z) = w_0$  has a root for every  $w_0 \in \mathbb{D}$ , for which it suffices to show that f(z) = 0 has a root, then use the maximum modulus principle.

### 2.12.2 b

Show that if  $|z| \ge 1 \implies |f(z)| = 1$  and there exists a point  $z_0 \in \mathbb{D}$  such that  $|f(z_0)| < 1$ , then the image of f contains the unit disc.

#### 2.13 S&S 3.8.19

Prove the maximum modulus principle for harmonic functions; i.e.,

#### 2.13.1 a

If u is a non-constant real-valued harmonic function on  $\Omega$ , then u can not attain its extrema on  $\Omega$ .

#### 2.13.2 b

Suppose  $\Omega$  has compact closure  $\overline{\Omega}$ , then if u is harmonic on  $\Omega$  and continuous on  $\overline{\Omega}$ , then

$$\sup_{z\in\Omega}|u(z)|\leq \sup_{z\in\overline{\Omega}-\Omega}|u(z)|$$

Hint: to prove (a), assume that u attains a local maximum at  $z_0$ , and let f be holomorphic near  $z_0$  with  $u = \Re(f)$ , then show that f is not open. Part (b) is a direct consequence.