Problem Set 2

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Contents

Note on notation: I sometimes use $f_x \coloneqq \dfrac{\partial f}{\partial x}$ to denote partial derivatives, and $\partial_z^n f$ as $f^{(n)}(z)$.

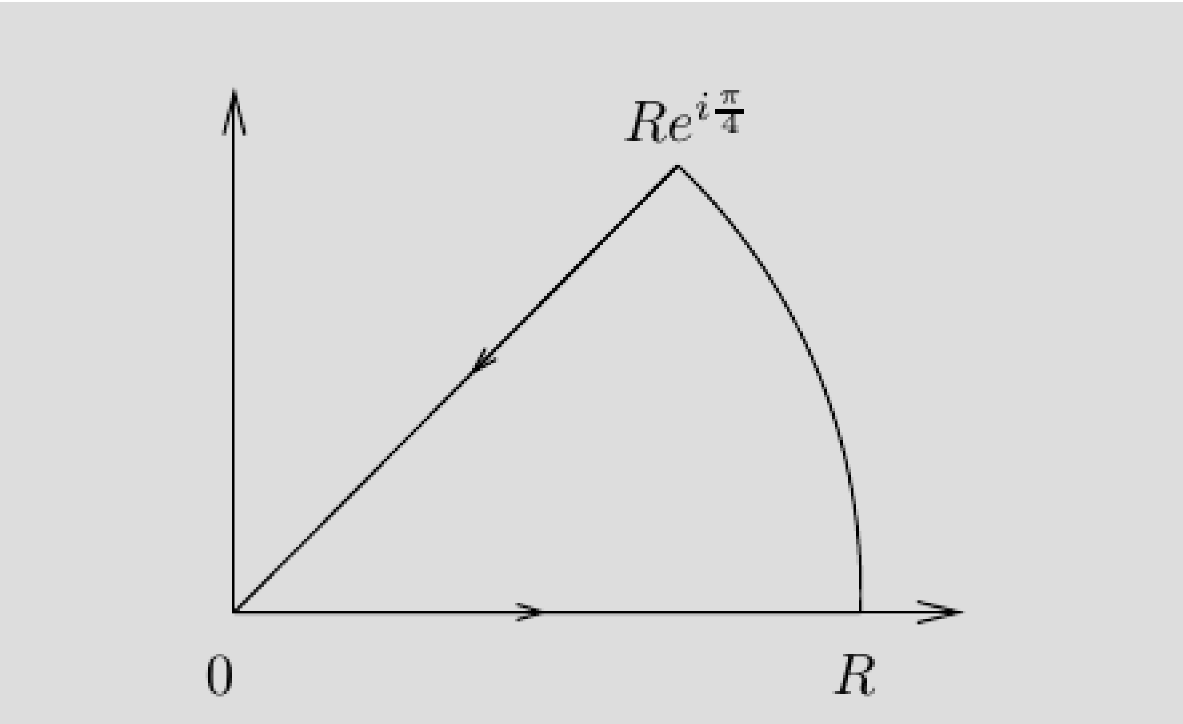
1 Stein And Shakarchi

1.1 2.6.1

Show that

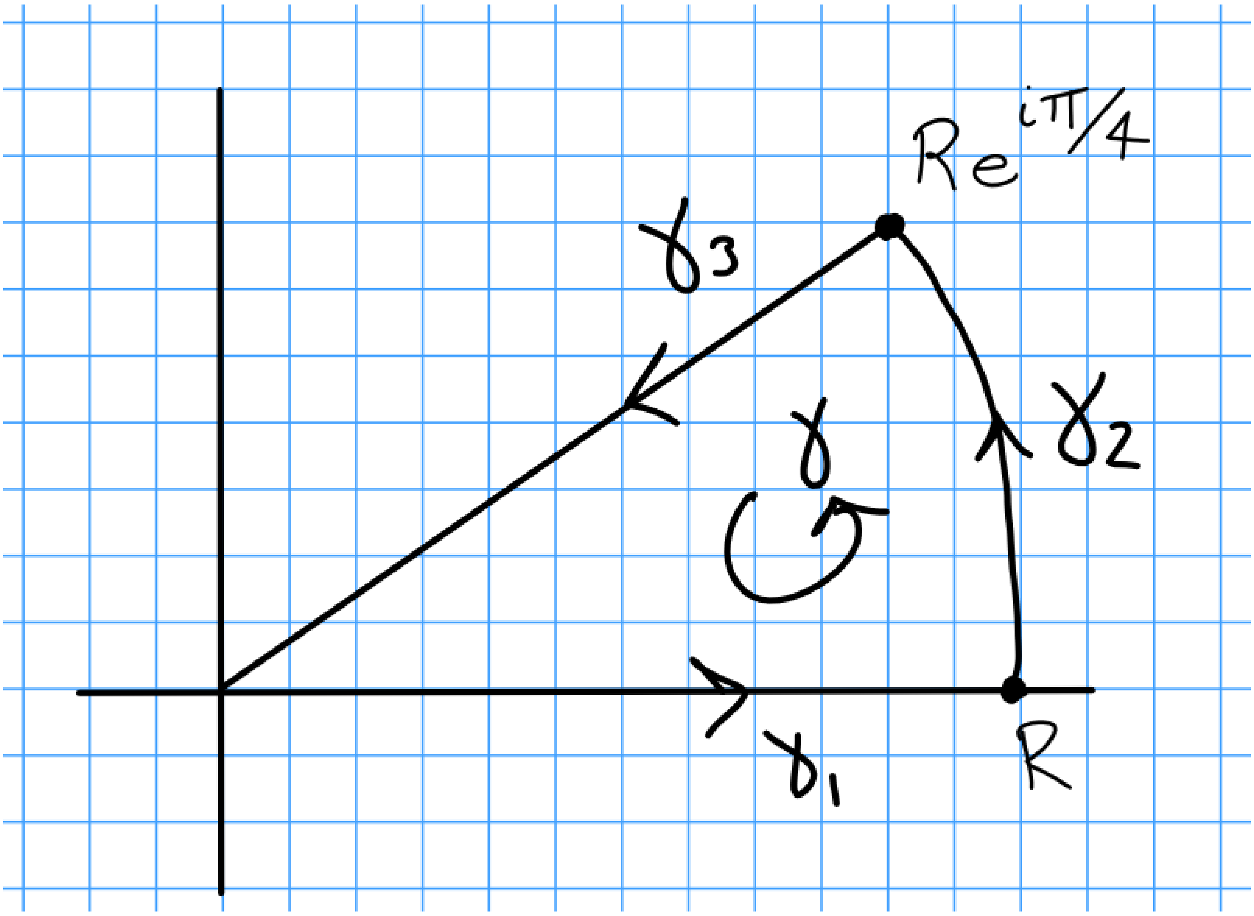
$$\int_0^\infty \sin\left(x^2\right)\,dx = \int_0^\infty \cos\left(x^2\right)\,dx = \frac{\sqrt{2\pi}}{4}.$$

Hint: integrate e^{-x^2} over the following contour, using the fact that $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$:



Solution:

Consider $f(z) = e^{-z^2}$; since f is entire we have $\displaystyle\oint_\gamma f(z)\,dz = 0$ for every closed curve γ. In particular, the integral around the following path $\gamma := \gamma_1 * \gamma_2 * \gamma_3$ (denoting concatenated paths) is zero for any radius R:



Let I_i denote $\displaystyle\int_{\gamma_i} f(z)\,dz$. Then

$$I_1 \overset{R\longrightarrow\infty}{\Longrightarrow} \int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$$

by the hint, using the fact that $g(x) = e^{-x^2}$ is even.

Claim:

$$I_2 \overset{R\longrightarrow\infty}{\Longrightarrow} 0.$$

Parameterize γ_2 as $\gamma_2(t) = Re^{it}$ where $t \in [0, \pi/4]$; then

$$
\begin{aligned}
|I_2| &= \left| \int_{\gamma_2} f(z)\ dz \right| \\
&= \left| \int_0^{\pi/4} f(\gamma_2(t))\ \gamma_2'(t)\ dt \right| \\
&= \left| \int_0^{\pi/4} e^{-\left(Re^{it}\right)^2} iRe^{it}\ dt \right| \\
&\le \int_0^{\pi/4} \left| e^{-\left(Re^{it}\right)^2} iRe^{it} \right|\ dt \\
&\le \int_0^{\pi/4} \left| Re^{-R^2} e^{-\left(e^{it}\right)^2} \right|\ dt \\
&= \left| Re^{-R^2} \right| \int_0^{\pi/4} \left| e^{-e^{2it}} \right|\ dt \\
&= \left| Re^{-R^2} \right| \cdot C \\
&\longrightarrow 0,
\end{aligned}
$$

where C is a constant that does not depend on R.

To compute I_3, parameterize $\gamma_3(t) = te^{i\pi/4}$ with $t \in [0, R]$; then $\gamma_3'(t) = e^{i\pi/4}$.

For notational convenience, let $I_C = \displaystyle\int_0^\infty \cos\left(t^2\right)\ dt$ and $I_S = \displaystyle\int_0^\infty \sin\left(t^2\right)\ dt$.

Note that I_C is the original integral we are looking for.

We then have

$$
\begin{aligned}
-I_3 &= \int_{\gamma_3} f(z)\ dz \\
&= \int_0^R f(\gamma_3(t))\ \gamma_3'(t)\ dt \\
&= \int_0^R e^{-\left(te^{i\pi/4}\right)^2} e^{i\pi/4}\ dt \\
&= e^{i\pi/4} \int_0^R e^{-it^2}\ dt \\
&= \frac{1+i}{\sqrt{2}} \int_0^R \cos(t^2) - i\sin(t^2)\ dt \\
&\overset{R\longrightarrow\infty}{\longrightarrow} \frac{1+i}{\sqrt{2}}(I_C - iI_S) \\
&= \frac{1}{\sqrt{2}}(I_C + I_S) + \frac{i}{\sqrt{2}}(I_C - I_S)
\end{aligned}
$$

.

Since $0 = I_1 + I_2 + I_3$, we can write

$$0 = \frac{1}{2}\sqrt{\pi} - \frac{1}{\sqrt{2}}(I_C + I_S) - \frac{i}{\sqrt{2}}(I_C - I_S).$$

Equating real and imaginary parts, we can conclude that $I_C = I_S$ (since the LHS is zero, and in particular has imaginary part zero), so this yields

$$\frac{1}{2}\sqrt{\pi} = \frac{2}{\sqrt{2}} I_C \implies I_C = \frac{\sqrt{2\pi}}{4}.$$

■

1.2 2.6.2

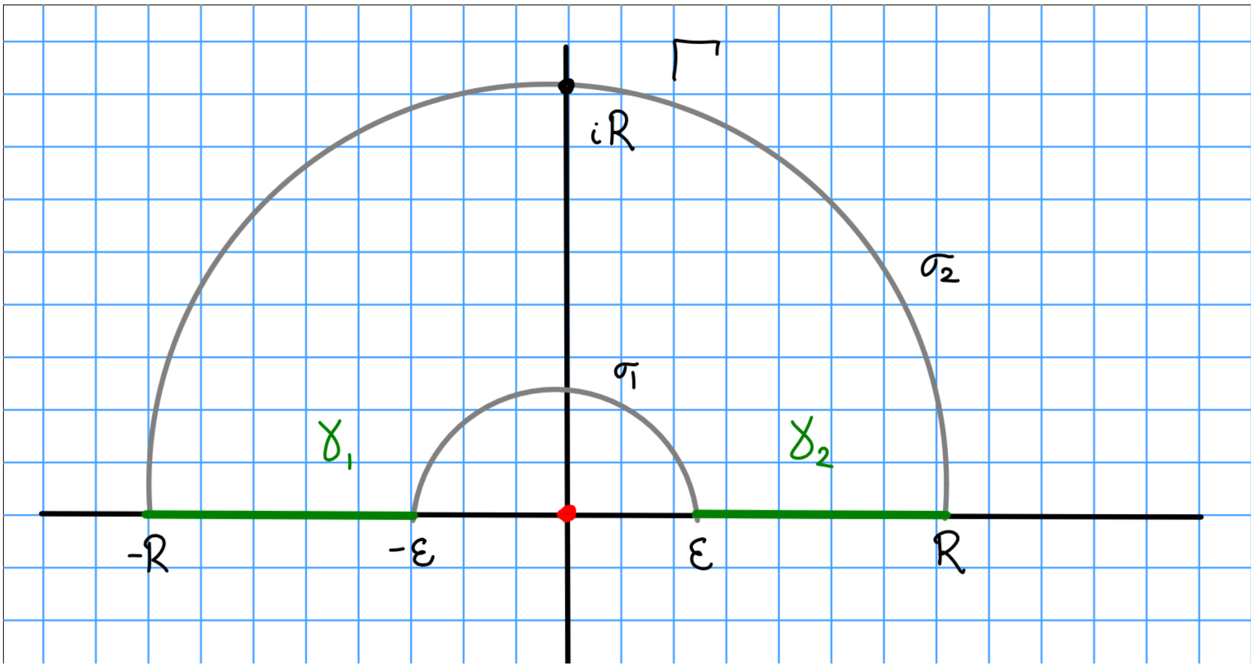
Show that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

> Hint: use the fact that this integral equals $\frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{ix}-1}{x} dx$, and integrate around an indented semicircle.

Solution:

We'll proceed by integrating $f(z) = \dfrac{e^{iz}-1}{z}$ along the following contour, which bounds a region in which f is holomorphic:



Let Γ be the curve enclosing the shown region, then

$$0 = \int_\Gamma f = \int_{\gamma_1} f + \int_{\sigma_1} f + \int_{\gamma_2} f + \int_{\sigma_2} f.$$

Note that $\displaystyle\int_{\gamma_1} f + \int_{\gamma_2} f \longrightarrow \int_{-\infty}^{\infty} f(x)\ dx.$

Claim 1: $\displaystyle\int_{\sigma_1} f \longrightarrow 0.$

$$\begin{aligned}
\int_{\sigma_1} f(z)\ dz &= \int_{\sigma_1} \frac{e^{iz}-1}{z}\ dz \\
&= \int_{\sigma_1} \frac{(1+iz+O(z^3))-1}{z}\ dz \quad \text{by expanding } e^{iz} \text{ as a power series} \\
&= \int_{\sigma_1} i - O(z)\ dz \\
&= \int_0^\pi (i+O(\varepsilon)) i\varepsilon e^{i\theta}\ d\theta \qquad \text{writing } z = \varepsilon e^{i\theta} \\
&= \int_0^\pi O(\varepsilon) \\
&\overset{\varepsilon \longrightarrow 0}{\longrightarrow} 0.
\end{aligned}$$

Claim 2: $\displaystyle\int_{\sigma_2} f \longrightarrow -i\pi.$

$$\begin{aligned}
\int_{\sigma_2} f(z)\ dz &= \int_0^\pi \frac{e^{iRe^{i\theta}}-1}{Re^{i\theta}}\left(iRe^{i\theta}\right)\ d\theta \\
&= i\int_0^\pi e^{iRe^{i\theta}} - 1\ d\theta \\
&\le i\int_0^\pi e^{i\Im\left(Re^{i\theta}\right)} - 1\ d\theta \\
&= i\int_0^\pi e^{-R\sin(\theta)} - 1\ d\theta \\
&= i\int_0^\pi e^{-R\left(\theta+O(\theta^3)\right)} - 1\ d\theta \\
&= ie^{-R}\int_0^\pi e^{\theta+O(\theta^3)}\ d\theta - i\int_0^\pi\ d\theta \\
&\overset{R \longrightarrow \infty}{\longrightarrow} 0 - i\int_0^\pi\ d\theta \\
&= -i\pi.
\end{aligned}$$

Taken together, this yields

$$
\begin{aligned}
0 &= \int_{\mathbb{R}} f(x)\ dx - i\pi \\
\implies \int_{\mathbb{R}} \frac{e^{ix}-1}{x} &= i\pi \\
\implies \int_{\mathbb{R}} \frac{\cos(z)-1}{z}\ dz + i\int_{\mathbb{R}} \frac{\sin(z)}{z}\ dz &= i\pi \\
\implies \int_{\mathbb{R}} \frac{\sin(z)}{z}\ dz &= \pi \quad \text{equating imaginary parts} \\
\implies \int_0^{\infty} \frac{\sin(z)}{z}\ dz &= \frac{\pi}{2} \quad \text{since the integrand is even.}
\end{aligned}
$$

■

1.3 2.6.5

Suppose $f \in C^1_{\mathbb{C}}(\Omega)$ and $T \subset \Omega$ is a triangle with $T^{\circ} \subset \Omega$. Apply Green's theorem to show that $\int_T f(z)\ dz = 0$.

Assume that f' is continuous and prove Goursat's theorem.

Hint: Green's theorem states

$$
\int_T F dx + G dy = \int_{T^{\circ}} \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx dy.
$$

Solution: Let T be such a triangle bounding a region R, so $\partial R = T$. Since $f \in C^1_{\mathbb{C}}(\Omega)$, f_x, f_y exist on Ω. Since the partials are continuous, $f'(z)$ exists on Ω, f satisfies the Cauchy-Riemann equations on Ω.

Thus

$$
\frac{\partial f}{\partial \bar{z}} = 0 \implies f_x = \frac{1}{i} f_y \implies i f_x - f_y = 0.
$$

Since the partials are continuous, the conditions for Green's theorem are satisfied and we obtain

$$
\begin{aligned}
\int_T f\ dz &= \int_{\partial R} f(dx + i dy) \\
&= \int_{\partial R} f dx + i f dy \\
&= \iint_R (i f_x - f_y)\ dA \quad \text{by Green's theorem} \\
&= \iint_R 0\ dA \quad \text{by observation above} \\
&= 0.
\end{aligned}
$$

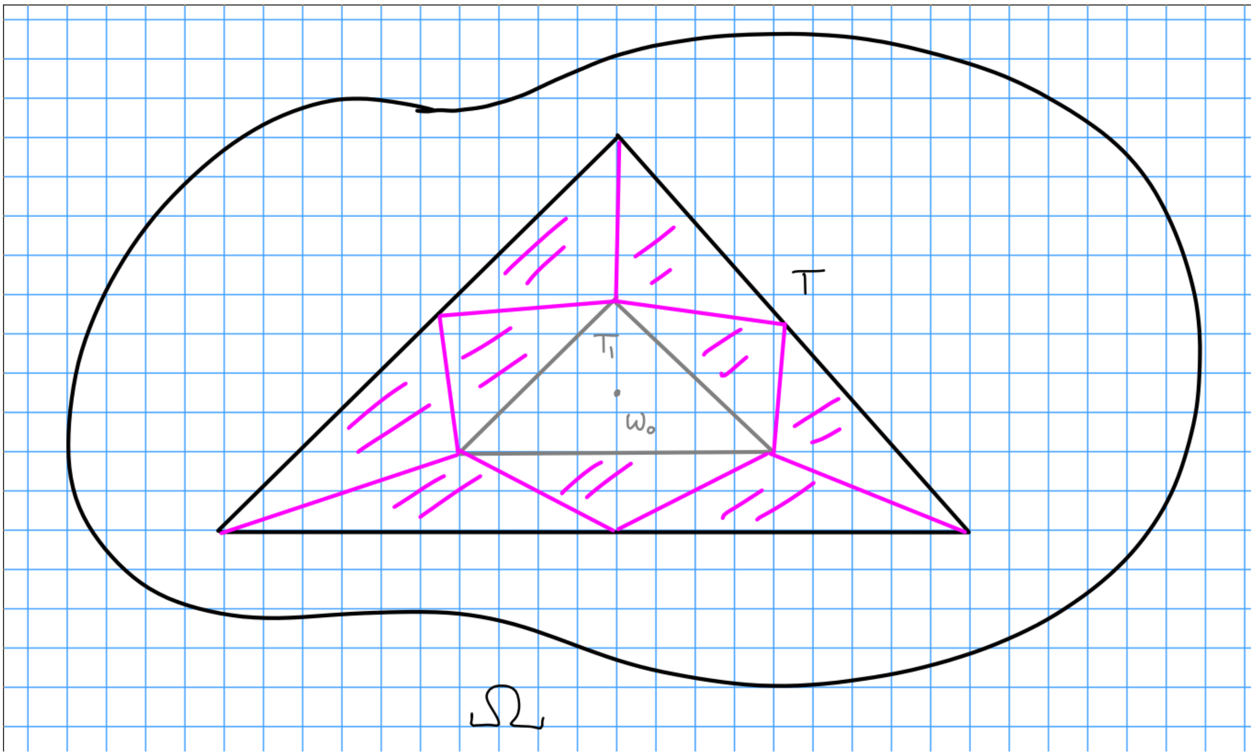
Thus if f is holomorphic with continuous partials on Ω, then $\int_T f = 0$ for all such triangles, which is Goursat's theorem.

■

1.4 2.6.6

Suppose that f is holomorphic on a punctured open set $\Omega \setminus \{w_0\}$ and let $T \subset \Omega$ be a triangle containing w_0. Prove that if f is bounded near w_0, then $\int_T f(z)\, dz = 0$.

Solution: Let T_1 be a triangle entirely contained in T which still encloses w_0:



We can then triangulate the region between T and T_1, and since f is holomorphic on these shaded regions (since they do not contain w_0), $\int f$ vanishes along each such triangle. Thus

$$\int_T f = \int_{T_1} f.$$

Since T_1 still satisfies the hypotheses applying to T, replace T with T_1 and continue this process inductively to obtain a sequence of triangles T_k for $k \in \mathbb{N}$.

Since f is bounded, say by M, we can obtain the estimate

$$\int_T f = \int_{T_k} f \leq M|T_k| \overset{k \longrightarrow \infty}{\longrightarrow} 0,$$

since the perimeter of T_k evidently goes to zero.

$\blacksquare$

1.5 2.6.7

Suppose $f : \mathbb{D} \longrightarrow \mathbb{C}$ is holomorphic and let $d := \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$ be the diameter of the image of f. Show that $2\big|f'(0)\big| \leq d$, and that equality holds iff f is linear, so $f(z) = a_1 z + a_2$.

Hint: $2f'(0) = \dfrac{1}{2\pi i}\displaystyle\int_{|\xi|=r} \dfrac{f(\xi) - f(-\xi)}{\xi^2}\, d\xi$ whenever $0 < r < 1$.

Solution:

Let $0 < r < 1$ and C_r be a circle of radius r centered at zero. Let $g(\xi) = \dfrac{f(\xi) - f(-\xi)}{\xi^2}$, then as per the hint we have

$$
\begin{aligned}
|2f'(0)| &= \left|\frac{1}{2\pi i}\int_{C_r} g(\xi)\ d\xi\right| \\
&\leq \frac{1}{2\pi}\int_{C_r} |g(\xi)|\ d\xi \\
&\leq \frac{1}{2\pi}\sup_{\xi\in C_r} |g(\xi)|\ |C_r| \\
&\leq \frac{1}{2\pi}\frac{d}{r^2}\cdot 2\pi r \\
&= \frac{d}{r} \\
&\overset{r\longrightarrow 1}{\longrightarrow} d,
\end{aligned}
$$

which yields the desired inequality.

If f is linear, so $f(z) = az + b$, noting that $g(\xi) = \dfrac{2a}{\xi}$ we find

$$
\begin{aligned}
2f'(0) &\leq \frac{1}{2\pi i}\int_{C_r} g(\xi)\ d\xi \\
&= \frac{1}{2\pi i}\int_{C_r} \frac{2a}{\xi}\ d\xi \\
&= \frac{2a}{2\pi i}\int_{C_r} \frac{1}{\xi}\ d\xi \\
&= \frac{2a}{2\pi i} 2\pi i \\
&= 2a,
\end{aligned}
$$

where a direct computation shows that $2f'(0) = 2a$, making this an equality.

$\blacksquare$

1.6 2.6.8

Suppose that f is holomorphic on the strip $S = \left\{x + iy \;\middle|\; x \in \mathbb{R},\ -1 < y < 1\right\}$ with $|f(z)| \leq A(1 + |z|)^{\nu}$ for ν some fixed real number. Show that for all $z \in S$, for each integer $n \geq 0$ there exists an $A_n \geq 0$ such that $\left|f^{(n)}(x)\right| \leq A_n(1 + |x|)^{\nu}$ for all $x \in \mathbb{R}$.

> Hint: Use the Cauchy inequalities.

Let $x \in \mathbb{R}$, $0 < R < 1$, and $C_R = \left\{z \in \mathbb{C} \;\middle|\; |z - x| = R\right\}$ be a circle centered at x. Note that if z is in the region enclosed by C_R, we have

$$|z| \leq |x| + R \implies 1 + |z| \leq 1 + |x| + R$$

by the triangle inequality.

We can then obtain the following estimate

$$\begin{aligned}
\left|f^{(n)}(x)\right| &\leq \frac{n!}{R^n}\|f\|_{C_R} \\
&\leq \frac{n!}{R^n}\ A(1 + |x| + R)^{\nu} \\
&\leq \frac{n!}{R^n}\ A((1 + R)(1 + |x|))^{\nu} \quad \text{since } R, |x| \text{ are positive} \\
&= \frac{n!}{R^n}\ A(1 + R)^{\nu}(1 + |x|)^{\nu} \\
&\overset{R \longrightarrow 1}{\Longrightarrow}\ 2^{\nu}An!(1 + |x|)^{\nu},
\end{aligned}$$

So taking $A_n \coloneqq 2^{\nu}An!$ suffices.

■

1.7 2.6.9

Let $\Omega \subset \mathbb{C}$ be open and bounded and $\phi : \Omega \longrightarrow \Omega$ holomorphic. Prove that if there exists a point $z_0 \in \Omega$ such that $\phi(z_0) = z_0$ and $\phi'(z_0) = 1$, then ϕ is linear.

> Hint: assume $z_0 = 0$ (explain why this can be done) and write $\phi(z) = z + a_n z^n + O(z^{n+1})$ near 0. Let $\phi_k = \phi \circ \phi \circ \cdots \circ \phi$ and prove that $\phi_k(z) = z + ka_n z^n + O(z^{n+1})$. Apply Cauchy's inequalities and let $k \longrightarrow \infty$ to conclude.

Solution: WLOG, suppose $z_0 = 0$, since z can be replaced with $z - z_0$ in what follows. Since ϕ is holomorphic, it is analytic on Ω, so it can be expanded in a power series in a neighborhood of $z_0 = 0$.

Suppose toward a contradiction that ϕ is *not* linear, in which case there is an $n \geq 2$ such that the power series expansion has the form

$$\phi(z) = z + a_n z^n + O(z^{n+1}) \qquad \textbf{where } a_n \neq 0,$$

where we've used the fact that $\phi(0) = 0 \implies a_0 = 0$ and $\phi'(0) = 1 \implies a_1 = 1$.

Then consider

$$\begin{aligned}
\phi_2(z) &= \phi \circ \phi(z) \\
&= \phi(z) + a_n(\phi(z))^n + O(z^{n+1}) \\
&= z + a_n z^n + a_n\Big(z + a_n z^n + O(z^{n+1})\Big)^n + O(z^{n+1}) \\
&= z + a_n z^n + a_n \sum_{k=0}^{n} (a_n z^n)^k z^{n-k} + O(z^{n+1}) \\
&= z + a_n z^n + a_n\Big(z^n + O\Big(z^{n^2}\Big)\Big) + O(z^{n+1}) \\
&= z + 2a_n z^n + O(z^{n+1}).
\end{aligned}$$

Inductively, this yields

$$\phi_k(z) = z + k a_n z^n + O(z^{n+1}),$$

and we can thus compute $\phi_k^{(n)}(z) = n! k a_n + O(z)$ and thus $\phi_k^{(n)}(0) = n! k a_n$.

Taking a circle of fixed radius $R \in (0, 1)$ centered at zero and applying the estimate from Cauchy's inequality, and noting since Ω is bounded and $\phi(\Omega) \subset \Omega$ implies that $|\phi(z)|$ is uniformly bounded by some M, we have

$$\begin{aligned}
|k a_n| &= \frac{1}{n!} \phi_k^{(n)}(0) \\
&\leq \frac{\|\phi_k\|_{C_R}}{R^{n+1}} \\
&\leq \frac{M}{R^{n+1}} \\
\implies k|a_n| &\leq \frac{M}{R^{n+1}} := c_0 \quad \text{where } c_0 \text{ is a fixed constant} \\
\implies \lim_{k \longrightarrow \infty} k|a_n| &\leq c_0,
\end{aligned}$$

which forces $a_n = 0$, a contradiction.

■

1.8 2.6.10

Can every continuous function on $\overline{\mathbb{D}}$ be uniformly approximated by polynomials in the variable z?

> Hint: compare to Weierstrass for the real interval.

Solution: No; consider the function $f(z) = \overline{z}$. This function is known to *not* be holomorphic, but since polynomials are entire, if $P_n(z) \longrightarrow f$ uniformly on the compact set $\overline{\mathbb{D}}$, this would force f to be holomorphic as well.

1.9 2.6.13

Suppose f is analytic, defined on all of $\mathbb{C}$, and for each $z_0 \in \mathbb{C}$ there is at least one coefficient in the expansion $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ is zero. Prove that f is a polynomial.

> Hint: use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.

Solution: Toward a contradiction, suppose that f is not a polynomial, so $f^{(n)}(z) \not\equiv 0$ for any n. Since f is analytic, each $f^{(n)}$ is analytic, and analytic functions vanish for at most countably many values. So for every $n \in \mathbb{N}$, the following set is countable:

$$Z_n := \left\{ z_0 \in \mathbb{C} \;\middle|\; f^{(n)}(z_0) = 0 \right\}.$$

But then we can define the following set:

$$Z := \bigcup_{n \in \mathbb{N}} Z_n,$$

which is a countable union of countable sets and thus countable.

However, by hypothesis, $\mathbb{C} \subset Z$, which is a contradiction. So $f^{(n)}(z) \equiv 0$ for some finite n, making f a polynomial.

■

1.10 2.6.14

Suppose that f is holomorphic in an open set containing $\mathbb{D}$ except for a pole $z_0 \in \partial\mathbb{D}$. Let $\sum_{n=0}^{\infty} a_n z^n$ be the power series expansion of f in $\mathbb{D}$, and show that $\lim \frac{a_n}{a_{n+1}} = z_0$.

Solution: Suppose z_0 is a pole of order p; we can then write

$$f(z) = c(z_0 - z)^{-p} + g(z)$$

for some constant C where $g(z)$ has no poles and is holomorphic on an open set $\Omega \supseteq \mathbb{D}$.

Since g is analytic on Ω, write

$$g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Since $|z_0| = 1$, and in particular is not zero, division by z_0 is justified and we can take the following expansion:

$$
\begin{aligned}
f(z) &= \frac{c}{(z_0 - z)^p} + \sum_{n=0}^{\infty} b_n z^n \\
&= \frac{c}{z_0^p}\left(\frac{1}{1-\left(\frac{z}{z_0}\right)}\right)^p + \sum_{n=0}^{\infty} b_n z^n \\
&= \frac{c}{z_0^p}\sum_{n=0}^{\infty}\binom{p-1+n}{p-1}\left(\frac{z}{z_0}\right)^n + \sum_{n=0}^{\infty} b_n z^n \\
&= \sum_{n=0}^{\infty}\left[\frac{c}{z_0^{p+n}}\binom{p-1+n}{p-1} + b_n\right] z^n.
\end{aligned}
$$

Equating coefficients, we obtain

$$
a_n = \frac{c}{z_0^{p+n}}\binom{p-1+n}{p-1} + b_n.
$$

We can then observe that $\binom{n}{k} / \binom{n+1}{k} \overset{n\longrightarrow\infty}{\longrightarrow} 1$ for any k, which also holds for n replaced by $n+c_0$ for any constant c_0. Moreover, since $g(z)$ is holomorphic, $b_n \longrightarrow 0$, so we can proceed to compute the ratio

$$
\begin{aligned}
\frac{a_n}{a_{n+1}} &= \frac{\frac{c}{z_0^{p+n}}\binom{p-1+n}{p-1} + b_n}{\frac{c}{z_0^{p+n+1}}\binom{p-1+n+1}{p-1} + b_{n+1}} \\
&= \frac{c z_0\binom{n+p-1}{p-1} + b_n z_0^{p+n+1}}{c\binom{(n+p-1)+1}{p-1} + b_{n+1} z_0^{p+n+1}} \\
&\overset{n\longrightarrow\infty}{\longrightarrow} \lim_{n\longrightarrow\infty} \frac{c z_0\binom{n+p-1}{p-1}}{c\binom{(n+p-1)+1}{p-1}} \qquad \text{passing the limit through the quotient and sums respectively} \\
&= z_0 \lim_{n\longrightarrow\infty} \frac{\binom{n+p-1}{p-1}}{\binom{(n+p-1)+1}{p-1}} \\
&= z_0.
\end{aligned}
$$

■

1.11 2.6.15

Suppose f is continuous and nonvanishing on $\overline{\mathbb{D}}$, and holomorphic in $\mathbb{D}$. Prove that if $|z| = 1 \implies |f(z)| = 1$, then f is constant.

> Hint: Extend f to all of $\mathbb{C}$ by $f(z) = 1/\overline{f(1/\overline{z})}$ for any $|z| > 1$, and argue as in the Schwarz reflection principle.

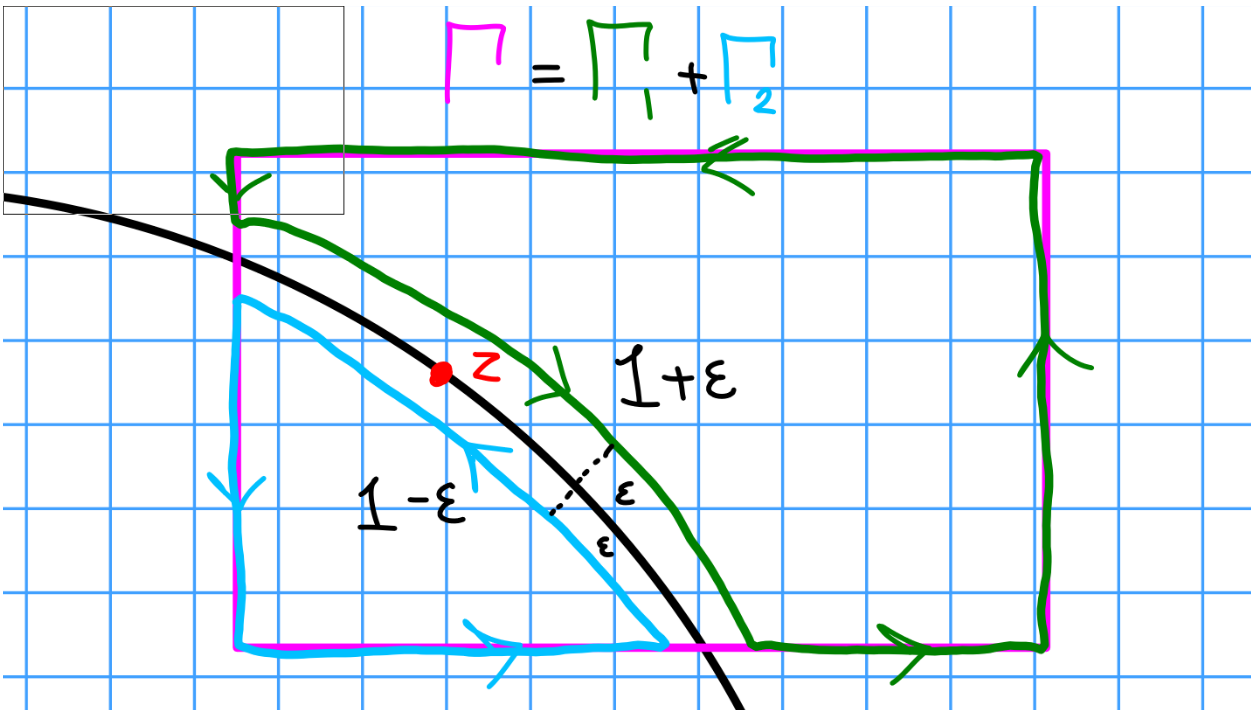
Solution: Since f is continuous on a compact set, it is bounded. We then make the following extension:

$$\begin{aligned} F : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\mapsto \begin{cases} f(z) & z \in \overline{\mathbb{D}} \\ 1/\overline{f(1/\overline{z})} & z \in \mathbb{C} \setminus \overline{\mathbb{D}} \end{cases}. \end{aligned}$$

We will now argue that F is bounded and entire, and thus constant by Liouville's theorem.

Since F is holomorphic on $\mathbb{D}^\circ$ and $(\mathbb{D}^\circ)^c$ by construction, it remains to show that F is also holomorphic on $\partial\mathbb{D}$. Let $z \in \partial\mathbb{D}$ be a fixed point; then by Morera's theorem, F will be holomorphic at z if the contour integral around every rectangle containing z vanishes.

Let Γ be such a rectangle, and let Γ_1, Γ_2 be two contours that run along Γ in the interior and exterior of the disc respectively, with circular arcs at radii $1 \pm \varepsilon$:



However, since F is known to be holomorphic away from $\partial\mathbb{D}$,

$$\int_\Gamma F = \lim_{\varepsilon \longrightarrow 0} \left(\int_{\Gamma_1} F + \int_{\Gamma_2} F \right) = \lim_{\varepsilon \longrightarrow 0} (0+0) = 0.$$

■

2 Additional Problems

2.1 Problem 1

Proposition: $L = \lim |a_{n+1}|/|a_n| \implies L = \lim \sqrt[n]{a_n}$

Proof:

Following a proof by Pete.

WLOG, we'll take all terms to be positive and drop absolute values.

Since the ratios converge to L, we have

$$L = \lim \frac{a_{n+1}}{a_n} = \liminf \frac{a_{n+1}}{a_n}.$$

Let ε be arbitrary and let $c_0 = L + \varepsilon$, and let $R = \lim_{n \longrightarrow \infty} \sqrt[n]{a_n}$. Since the above limit exists, for n large enough, that $\frac{a_{n+1}}{a_n} \leq c_0$. Then

$$\begin{aligned} a_{n+1} &\leq c_0 a_n \\ \implies a_{n+k} &\leq c_0^k a_n \\ \implies a_{n+k} &\leq c_0^{n+k} \left(\frac{a_n}{c_0^n} \right) \\ \implies (a_{n+k})^{\frac{1}{n+k}} &\leq c_0 \left(\frac{a_n}{c_0^n} \right)^{\frac{1}{n+k}} \\ \implies R = \lim_{n \longrightarrow \infty} (a_n)^{\frac{1}{n}} = \lim_{k \longrightarrow \infty} (a_{n+k})^{\frac{1}{n+k}} &\leq \lim_{k \longrightarrow \infty} c_0 \left(\frac{a_n}{c_0^n} \right)^{\frac{1}{n+k}} = c_0 = L + \varepsilon \end{aligned}$$

.

So $R \leq L + \varepsilon$ for every $\varepsilon > 0$, and thus $R = L$.

■

2.2 Problem 2

Proposition: If f is a power series centered at the origin, then f has a power series expansion about any point in its domain.

Proof: Let D_R be the disc of convergence for f and let $z_0 \in D_R$ and take a disc $z_0 \in D \subset D_R$. We can then find a power series expansion about z_0, namely

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)}{\xi - z}\ d\xi \\
&= \frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)}{\xi - z_0 + z_0 - z}\ d\xi \\
&= \frac{1}{2\pi i}\int_{\partial D} f(\xi)\Big(\frac{1}{(\xi - z_0) - (z - z_0)}\Big)\ d\xi \\
&= \frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)}{\xi - z_0}\left(\frac{1}{1 - \left(\frac{z-z_0}{\xi-z_0}\right)}\right)\ d\xi \\
&= \frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)}{\xi - z_0}\sum_{j=0}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^j\ d\xi \\
&= \frac{1}{2\pi i}\int_{\partial D} \sum_{j=0}^{\infty}\frac{f(\xi)}{(\xi - z_0)^{j+1}}(z - z_0)^j\ d\xi \\
&= \frac{1}{2\pi i}\sum_{j=0}^{\infty}\int_{\partial D} \frac{f(\xi)}{(\xi - z_0)^{j+1}}(z - z_0)^j\ d\xi \quad \text{since the series converges uniformly on } D \\
&= \sum_{j=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)}{(\xi - z_0)^{j+1}}\ d\xi\right)(z - z_0)^j \\
&:= \sum_{j=0}^{\infty} a_j (z - z_0)^j.
\end{aligned}
$$

■

2.3 Problem 3

2.3.1 a

Proposition: $\sum nz^n$ does not converge for any $|z| = 1$.

Proof:

$$a_n = nz^n \implies |a_n| = n|z|^n = n \overset{n\longrightarrow\infty}{\not\longrightarrow} 0.$$

2.3.2 b

Proposition: $\sum_n z^n/n^2$ converges for every $|z| \leq 1$.

Proof:

$$\left|\sum_n \frac{z^n}{n^2}\right| \leq \sum_n \left|\frac{z^n}{n^2}\right| \leq \sum_n \left|\frac{1}{n^2}\right| = \frac{\pi^2}{6} < \infty.$$

2.3.3 c

Proposition: $\sum z^n/n$ converges for every $|z| \leq 1$ except $z = 1$.

Proof: By Abel's test, since $\left\{a_n = \frac{1}{n}\right\} \searrow 0$ and this sequence is bounded by 1, $f(z) = \sum a_n z^n$ converges for $z \in \overline{\mathbb{D}} \setminus \{1\}$. To see that $f(1)$ diverges, we just note that $f(1) = \sum \frac{1}{n}$ is the harmonic series, which is known to diverge.

2.4 Problem 4

Proposition: Let γ denote a circle centered at the origin of radius r with positive orientation. Then if $|\alpha| \leq r \leq |\beta|$,

$$\int_\gamma \frac{dz}{(z-\alpha)(z-\beta)} = \frac{2\pi i}{\alpha - \beta}.$$

Proof: By partial fraction decomposition, we have

$$\begin{aligned}
\int_\gamma \frac{1}{(z-\alpha)(z-\beta)}\, dz &= \frac{1}{\alpha-\beta}\left(\int_\gamma \frac{1}{z-\alpha}\, dz - \int_\gamma \frac{1}{z-\beta}\, dz\right) \\
&= \frac{1}{\alpha-\beta}\int_\gamma \frac{1}{z-\alpha}\, dz \quad \text{since } \beta > r \text{ and thus } g(z) := \frac{1}{z-\beta} \text{ is holomorphic in } D_r \\
&= \frac{1}{\alpha-\beta}\int_\gamma \frac{1}{w}\, dw \\
&= \frac{2\pi i}{\alpha-\beta}.
\end{aligned}$$

2.5 Problem 5

Proposition: Suppose f is continuous for $x \geq x_0, 0 \leq y \leq b$ and $\lim_{x \longrightarrow \infty} f(x+iy) = A$ independent of y. Let $\gamma_x := \left\{z \;\middle|\; z = x + it,\ 0 \leq t \leq b\right\}$, then

$$\lim_{x \to +\infty} \int_{\gamma_x} f(z) dz = iAb.$$

Proof:

> ??? Seems like this should involve integrating over the rectangular contour $[x_0, R] \times i[0, b]$ and taking $R \longrightarrow \infty$, but it's not clear what the integral over the whole contour is or even any of the edges.

2.6 Problem 6

Proposition: There exists a function f that is holomorphic on $0 < |z| < 1$ with $\int_{\partial D_r(0)} = 0$ for all $r < 1$ but f is not holomorphic at $z = 0$.

Proof: Since holomorphic functions are continuous, it suffices to find a function that is discontinuous at zero for which the contour integral vanishes. To this end, take $f(z) = \frac{1}{z^2}$, which is clearly discontinuous at zero, and

$$\int_\gamma f = \int_0^{2\pi} \frac{1}{re^{it}}\,dt = 0.$$

2.7 Problem 7

Let f be analytic on Ω and $f'(z_0) \neq 0$ for some $z_0 \in \Omega$. Show that if C is a circle centered at z_0 of sufficiently small radius, then

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{dz}{f(z) - f(z_0)}.$$

Proof:

We'll use the formula

$$f'(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - a)^2}\,d\xi.$$

Note that by the inverse function theorem, we have

$$(f^{-1})'(f(z))f'(z) = 1 \implies \frac{1}{f'(z)} = (f^{-1})'(f(z)).$$

Applying Cauchy's Integral Formula, we obtain

$$\begin{aligned}
\frac{1}{f'(z_0)} &= (f^{-1})'(f(z_0)) \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f^{-1}(\xi)}{(\xi - f(z_0))^2}\,d\xi \\
&= \frac{1}{2\pi i}\int_\gamma \frac{wf'(w)}{(f(w) - f(z_0))^2}\,dw \qquad (\xi = f(w),\ d\xi = f'(w)\,dw) \\
\implies \frac{2\pi i}{f'(z_0)} &= \int_\gamma \frac{wf'(w)}{(f(w) - f(z_0))^2}\,dw =_{??} \int_\gamma \frac{1}{f(w) - f(z_0)}\,dw.
\end{aligned}$$

It's not clear to me how to prove the last equality! It seems like this is where we would use the fact that f is analytic, but expanding in a power series doesn't seem to help remove the square in the denominator.

2.8 Problem 8

Proposition: Let $u, v \in C^1(\mathbb{R}^2)$. Then $f = u + iv$ has derivative $f'(z_0) = x_0 + iy_0$ iff

$$\lim_{r \to 0} \frac{1}{\pi r^2} \int_{|z-z_0|=r} f(z) dz = 0.$$

Proof: Let γ_r be a circle of radius r about z_0 and D_r be the region it bounds. Then

$$\begin{aligned}
\int_{\gamma_r} f\ dz &= \int_{\gamma_r} u\ dx - v\ dy + i \int_{\gamma_r} v\ dx + u\ dy \\
&= \iint_{D_r} (v_x - u_y)\ dA - i \iint_{D_r} (u_x + v_y) \\
&= \pi r^2((v_x - u_y)(w_1) + i(u_x + v_y)(w_2)) \quad \text{for some } w_i \in D_r \text{ by the MVT} \\
\implies \frac{1}{\pi r^2} \int_{\gamma_r} f\ dz &= (v_x - u_y)(w_1) + i(u_x + v_y)(w_2) \\
&\overset{r \longrightarrow 0}{\longrightarrow} (v_x - u_y)(z_0) + i(u_x + v_y)(z_0),
\end{aligned}$$

and this limit is zero iff the Cauchy-Riemann equations hold at z_0 iff $f'(z_0)$ exists.

■

2.9 Problem 9

Proposition: Let γ be piecewise smooth with interior Ω_1 and exterior Ω_2. Assume f' exists on an open set containing γ and Ω_2. Show that if $\lim_{z \longrightarrow \infty} f(z) = A$, then

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1 \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

Proof:

Idea: take z small enough such that $\frac{1}{z} \in \Omega_2$, then f is holomorphic for such z since f' exists, so take a Laurent expansion $f(z) = \sum_{j=0}^{\infty} a_j z^{-j}$. We can deduce that $a_0 = A$, and integrate term by term:

$$
\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\ d\xi &= \frac{1}{2\pi i}\int_\gamma \sum_{j=0}^{\infty} \frac{a_j}{z^j(\xi - z)}\ d\xi \\
&= \frac{1}{2\pi i}\sum_{j=0}^{\infty}\int_\gamma \frac{a_j z^{-j}}{\xi - z}\ d\xi \\
&:= \frac{1}{2\pi i}\sum_{j=0}^{\infty}\int_\gamma \frac{g_j(\xi)}{\xi - z}\ d\xi \\
&= -\frac{1}{2\pi i}\sum_{j=1}^{\infty} 2\pi i\ g_j(z) \\
&= -\sum_{j=1}^{\infty} a_j z^{-j} \\
&= -\left(\sum_{j=0}^{\infty} a_j z^{-j} - a_0\right) \\
&= -f(z) + A.
\end{aligned}
$$

Note: pretty sure this is not right. Also not sure how to compute for $z \in \Omega_1$ without using the residue theorem.

2.10 Problem 10

Proposition: Let f be bounded and analytic and $a \neq b \in \mathbb{C}$ be fixed, then the following limit exists:

$$
\lim_{R\to\infty}\int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.
$$

Then f must be constant.

Proof:

Then take R large enough such that a, b are contained in a disc of radius R and let γ be its boundary. By partial fractions we have

$$
\begin{aligned}
\int_\gamma \frac{f}{(z-a)(z-b)} &= \frac{1}{b-a}\left(\int_\gamma \frac{f(z)}{z-b} - \int_\gamma \frac{f(z)}{z-a}\right) \\
&= \frac{1}{b-a}(2\pi i f(b) + 2\pi i f(a)) \\
&= \frac{2\pi i(f(b) - f(a))}{b-a} \\
&\xrightarrow{R\to\infty} \frac{2\pi i(f(b) - f(a))}{b-a}
\end{aligned}
$$

since this is a constant independent of R.

2.11 Problem 11

Proposition: Suppose f is entire and $\dfrac{f(z)}{z} \overset{z\longrightarrow\infty}{\longrightarrow} 0$. Then f must be constant.

Proof: Since f is entire, it is in fact *equal* to its Taylor series expansion, and thus

$$\begin{aligned}
z^{-1}f(z) &= z^{-1}\sum_{j=0}^{\infty}\frac{\partial_z^n f}{n!}z^j \\
&= \sum_{j=0}^{\infty}\frac{\partial_z^n f(z)}{n!}z^{j-1} \\
&= z^{-1}f(z) + f'(z) + O(z) \\
&\overset{z\longrightarrow\infty}{\longrightarrow} f'(z).
\end{aligned}$$

Since $z^{-1}f(z) \longrightarrow 0$ by assumption, we must have $f'(z) \equiv 0$, which forces f to be constant.

2.12 Problem 12

Proposition: Let f be analytic on Ω and γ a closed curve in Ω. Then for any $z_0 \in \Omega \setminus \gamma$,

$$\int_\gamma \frac{f'(z)}{z-z_0}dz = \int_\gamma \frac{f(z)}{(z-z_0)^2}dz.$$

Proof: Since f is analytic, it uniformly converges to its power series on Ω and we can integrate term-by-term. So write $f(z) = \sum_{j\in\mathbb{N}} a_j(z-z_0)^j$.

We can first compute the LHS:

$$\begin{aligned}
(z-z_0)^{-1}f'(z) &= (z-z_0)^{-1}\sum_{j\in\mathbb{N}} a_j(z-z_0)^j \\
&= (z-z_0)^{-1}\sum_{j\in\mathbb{N}} a_{j+1}(z-z_0)^j \\
&= \sum_{j\in\mathbb{N}} a_{j+1}(z-z_0)^{j-1} \\
&= \frac{a_1}{z-z_0} + 2a_2 + O(z-z_0) \\
\implies \int_\gamma \frac{f'(z)}{z-z_0} &= \int_\gamma \frac{a_1}{z-z_0} + O(1) + O(z-z_0)\ dz \\
&= \int_\gamma \frac{a_1}{z-z_0}\ dz \\
&= 2\pi i\ a_1.
\end{aligned}$$

And similarly the RHS:

$$
\begin{aligned}
(z-z_0)^{-2}f(z) &= (z-z_0)^{-2}\sum_{j=0}^{\infty} a_j (z-z_0)^j \\
&= \sum_{j=0}^{\infty} a_j (z-z_0)^{j-2} \\
&= \frac{a_1}{z-z_0} + O((z-z_0)^{-2}) + O(z-z_0) \\
\implies \int_\gamma \frac{f(z)}{(z-z_0)^2}\, dz &= \int_\gamma \frac{a_1}{z-z_0} + O((z-z_0)^{-2}) + O(z-z_0)\, dz \\
&= \int_\gamma \frac{a_1}{z-z_0} \\
&= 2\pi i\ a_1.
\end{aligned}
$$

■

2.13 Problem 13

Compute

$$\int_{|z|=1} \left(z+\frac{1}{z}\right)^{2n} \frac{dz}{z}.$$

Use this to show that

$$\int_0^{2\pi} \cos^{2n}\theta d\theta = 2\pi \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}.$$

Solution:

Using the Binomial theorem, we can expand

$$
\begin{aligned}
\int_\gamma z^{-1}\left(z+z^{-1}\right)^{2n}\, dz &= \int_\gamma z^{-1}\sum_{j=0}^{\infty}\binom{2n}{j} z^j z^{2n-j}\, dz \\
&= \int_\gamma \sum_{j=0}^{\infty}\binom{2n}{j} z^{2n-2j-1}\, dz \\
&= \int_\gamma \binom{2n}{n} z^{-1} + O(z^{-3}) + O(z)\, dz \\
&= 2\pi i\binom{2n}{n}.
\end{aligned}
$$

Now writing $\cos(\theta) = \dfrac{1}{2}\left(z + z^{-1}\right)$ where

$$z = ei\theta \implies dz = iz\ d\theta \implies d\theta = -iz\ dz,$$

letting γ denote the unit circle, we have

$$\begin{aligned}
\int_{\theta}^{2\pi} \cos^{2n}(\theta)\ d\theta &= -\int_{\gamma} \left(\frac{1}{2}\left(z + z^{-1}\right)\right)^{2n} iz\ dz \\
&= -\frac{i}{2^n} \int_{\gamma} \left(z + z^{-1}\right)^{2n} \frac{dz}{z} \\
&= -\frac{i}{2^n} \left(2\pi i \binom{2n}{n}\right) \\
&= \frac{\pi}{2^{n-1}} \binom{2n}{n}.
\end{aligned}$$

I assume this is equal to the given expression!

■