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Space of Vector Bundles on Spheres

Topology of moduli spaces of vec. bundles over $\mathbb{CP}^1$

I. $S^1 = \{ (x-a)^2 + (x-b)^2 = r^2 \mid a, b \in \mathbb{R},\ r > 0 \}$

Notion of closeness if parameters perturbed

Orbifolds $\sim$ stacks, both moduli spaces

$\mathcal{O}(m) =$ Line bundles over $\mathbb{CP}^1$, one for each $m \in \mathbb{Z}$

Any rank n vector bundle on $\mathbb{CP}^1$ splits as $\bigoplus_{i=1}^{n} \mathcal{O}(m_i)$

Moduli space of v.b.s of dim n over X

with basis:

$$\begin{array}{c} \mathbb{C}^n \times X \\ \downarrow \\ X \longrightarrow \{*\} \end{array}$$

without: $\{*\}/Gl(n, \mathbb{C}) = \mathcal{M}$

Take hty quotient $= BGl(n)$

$\dim = 0 - n^2 = -n^2$

$$H^*(\mathcal{M}) = H^*(BGl(n)) = \mathbb{Z}[\underbrace{c_1 \cdots c_n}_{\text{Chern classes}}], \quad |c_i| = i$$

Aside: How many finite sets are there

$$\sum_{n\geq 0} \infty \longrightarrow \underbrace{\sum_{n\geq 0} 1}_{\text{identify sets of same size}} \longrightarrow \underbrace{\sum_{n\geq 0} 1/n!}_{\text{Divide by \# automorphisms}} = e$$

How to study high-dim: look at $\mathbb{Q}$-curves
dim 1 or codim 1

$X = \mathbb{CP}^1$, want to look at $H^*(\mathcal{M})$

Bott periodicity: A correspondence

$$\text{V.b.s} \longrightarrow \text{V.b.s over } S^n$$

H^* known

$$\mathbb{Z} \times BGl(\infty) \xrightarrow{\sim} \Omega^2 BGl(\infty)$$

Based

Chow rings: cycles mod "algebraic" homotopy

Chow: given $Z \subseteq X$ closed

$X \subseteq Y$ open

There is a LES

$$\cdots \to CH_i(Z) \to CH_i(X) \to CH_i(Y) \to 0$$