Algebraic K-Theory

Group completion: Formally adjoin inverses to a comutative monoid.

$$K_0(R) = Gr(\underbrace{\mathcal{P}(R)}_{\substack{\text{iso classes of f.g. projective} \\ R\text{-modules}}})$$

Ex (Number Theory)

$$\begin{array}{ccc} k & \supseteq & \mathcal{O}(k) \\ \downarrow & & \downarrow \\ \mathbb{Q} & \supset & \mathbb{Z} \end{array}$$

a finite extension

$$K_0(\mathcal{O}(k)) = \mathbb{Z} \oplus \underbrace{Cl(k)}_{\text{ideal class gp}}$$

Ex (Topology)

X path connected, $X \simeq Y$, a retract of a finite CW complex. Is $X \simeq$ fin CW complex?

See $\omega(X) \in \widetilde{K}_0(\mathbb{Z}\pi_1 X) = \operatorname{coker}(\mathbb{Z} \to K_0(\mathbb{Z}\pi_1 X))$

"Wall finiteness obstruction"

Def: $K_1 R = Gl(R) / E(R)$

$Gl(R) \to \varinjlim Gl(n, R)$

$E(R) \to$ Elementary matrices

Ex) $K_1(\mathcal{O}(k)) = \mathcal{O}(k)^* = \underbrace{\mu(k)}_{\text{roots of } 1} \oplus \mathbb{Z}^{r_1 + r_2 - 1}$

$r_1 = \# \mathbb{R}$ embeds

$r_2 = \# \mathbb{C}$ pairs of embeds

Ex) When are mfds $M \simeq N$?

Look at h-cobordism W^{n+1}

$M, N \overset{\simeq}{\hookrightarrow} W$, $\partial W = M \amalg N$

Thm $\left\{ \begin{array}{l} \text{Diffeo classes of} \\ \text{h-cob. on } M \end{array} \right\} \longleftrightarrow Wh(\pi, M) = \dfrac{K_1(\mathbb{Z}\pi_1 M)}{\pm g \in \pi_1 M}$?

Whitehead Group

Can check if trivial

Higher K-groups (difficult to construct)

Recover as $\pi_i X$ for some Ω^∞-spaces\spectra

Topological group completion Gr^{Top}

= Classifying space of f.g. proj. modules

(Just paste cells in cat @ compositions)

Ex: $\mathcal{C} = \left\{ \begin{tikzcd} x \arrow[loop above, "f"] \arrow[loop below, "id"] \end{tikzcd}, f^2 = id_x \right\}$, $B\mathcal{C} = \mathbb{RP}^2$

Then $\pi_0 (Gr^{Top} B\mathcal{P}(R)_{iso}) = Gr(\underbrace{\pi_0 B\mathcal{P}(R)_{iso}}_{\mathcal{P}(R)})$

Application: Class number formula $(\zeta_K(s))$

Can be expressed in terms of rank K_i

Kummer-Vandiver Conjecture $\sim$ $K(\mathbb{Z}) = 0$

<u>Thm</u>: $A(x) = \underbrace{\Omega^\infty \Sigma^\infty X_+}_{\text{stable hty of } X} \times Wh(X)$

where $\mathcal{H}(X)$ = stable space of h-cobs

$$\Omega Wh(X) = \mathcal{H}(X)$$

$$\pi_0 \mathcal{H}(M) = Wh(\pi, M)$$

Want to generalize to equivariant $M \circlearrowleft G$