Differential Equations and Linear Algebra, Spring 2014 Notes

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Contents

Chapter 1

Vector Spaces

Next Exam: April 2nd. Covers 4.6→5.1.
Remember: To show a set spans a space, show that an arbitrary element in that space can be expressed as a linear combination of the vectors in your set. This usually results in a linear system of equations – as long as you can solve this system in terms of your vectors, it will be consistent.

1.1 Bases

1.1.1 Determining a Basis

A set S that forms a basis for a vector space V must satisfy two conditions:

1. S is set of linearly independent vectors.
2. S spans V.

All bases for a given vector space V have the same number of elements, and this is referred to as the *dimension* of V such that $\dim[V]$ = the number of elements in the basis.
If the dimension is known, it is equal to the minimum number of vectors needed to span V, as well as the maximum number of linearly independent vectors that a set in V can contain.
Does a set S form a basis for a vector space V?
First, check for linear independence. If $\dim[V]=n$ and S contains n linearly independent vectors, S is guaranteed to form a basis for V.
This means that given a set of vectors, we only have to shown linear dependence and that the number of elements matches the dimension of the space! So we only need to know their **linear independence** and the space's **dimension**. In other words, n linearly independent vectors in n-dimensional space means, for free, we have a basis.
Note: $\dim[P_n]= n + 1$.

For the subspaces of a given matrix

Null space:

Suppose we are given a matrix, and want to examine the properties of its null space. Recall that the null space of a matrix A is given by $\{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$, and its dimension is equal to the number of free variables in ref(A).
This is generally found by augmenting the matrix with $\mathbf{0}$ is using ERO to obtain its REF. Once a relationship between the free variables is found, construct a solution set of the form

$$S = \{\mathbf{x} \in \mathbb{R}^n : x = a_1\mathbf{v_1} + a_2\mathbf{v_2} \cdots a_n\mathbf{v_n} \wedge a_1, a_2, \cdots a_n \in R\} = \text{span}\,\{\text{something}\}$$

At this point, it should be clear what the basis and dimensions are.
Row space:
A basis for the row space of a matrix A is nothing more than the number of nonzero rows in ref(A), which is a subspace of $\mathbb{R}^n$.
Column space:
A basis for the column space of a matrix A consists of the column vectors with leading 1s in rref(A).

1.1.2 Extending a Basis

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1.2 Inner Product Spaces (4.11)

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1.2.1 Axioms

4 Axioms of an Inner Product

1. $\langle \mathbf{V_1}, \mathbf{V_1} \rangle >= 0$ and $\langle \mathbf{V_1}, \mathbf{V_1} \rangle = 0 \iff \mathbf{V_1} = 0$

   Check that the scalar result is positive or zero, and show that $\langle \mathbf{A}, \mathbf{A} \rangle = 0$ forces the coefficients to be zero.

2. Commutativity of $\bigodot$

   $\langle \mathbf{V_1}, \mathbf{V_2} \rangle = \langle \mathbf{V_2}, \mathbf{V_1} \rangle$

3. Associativity of scalar multiplication over $\bigodot$

   $\langle c\mathbf{V_1}, \mathbf{V_2} \rangle = c\langle \mathbf{V_1}, \mathbf{V_2} \rangle$

4. Associativity of $\bigoplus$ over $\bigodot$

   $\langle \mathbf{V_1}, \mathbf{V_2} + \mathbf{V_3} \rangle = \langle \mathbf{V_1}, \mathbf{V_2} \rangle + \langle \mathbf{V_1}, \mathbf{V_3} \rangle$

1.2.2 Orthogonality

Two vectors $\mathbf{p}$ and $\mathbf{q}$ are defined to be orthogonal if $\langle \mathbf{p}, \mathbf{q} \rangle = 0$.

1.2.3 The Gram-Schmidt Process

Given a basis

$$S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots \mathbf{v_n}\},$$

the Gram-Schmidt process produces a corresponding orthogonal basis

$$S' = \{\mathbf{u_1}, \mathbf{u_2}, \cdots \mathbf{u_n}\}$$

that spans the same vector space as S.
S' is found using the following pattern:

$$\begin{aligned}
\mathbf{u_1} &= \mathbf{v_1} \\
\mathbf{u_2} &= \mathbf{v_2} - \mathrm{proj}_{\mathbf{u_1}} \mathbf{v_2} \\
\mathbf{u_3} &= \mathbf{v_3} - \mathrm{proj}_{\mathbf{u_1}} \mathbf{v_3} - \mathrm{proj}_{\mathbf{u_2}} \mathbf{v_3}
\end{aligned}$$

where

$$\mathrm{proj}_{\mathbf{u}} \mathbf{v} = (\mathrm{scal}_{\mathbf{u}} \mathbf{v}) \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{u}\|} \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{u}\|^2} \mathbf{u}$$

is a vector defined as the *orthogonal projection of* $\mathbf{v}$ *onto* $\mathbf{u}$.

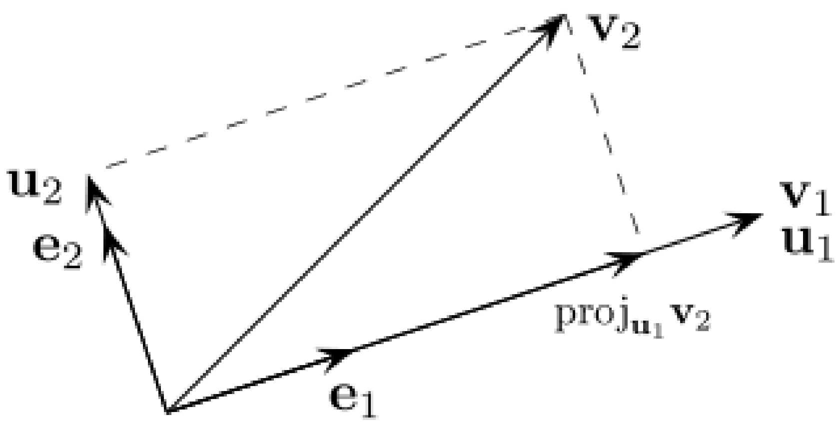

Figure 1.1: Orthogonal projection of $\mathbf{v_2}$ onto $\mathbf{u_1}$

The orthogonal set S' can then be transformed into an orthonormal set S'' by simply dividing the vectors $s \in S'$ by their magnitudes. The usual definition of a vector's magnitude is

$$\|\mathbf{a}\| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle} \text{ and } \|\mathbf{a}\|^2 = \langle \mathbf{a}, \mathbf{a} \rangle$$

As a final check, all vectors in S' should be orthogonal to each other, such that

$$\langle \mathbf{v_i}, \mathbf{v_j} \rangle = 0 \text{ when } i \neq j$$

and all vectors in S'' should be orthonormal, such that

$$\langle \mathbf{v_i}, \mathbf{v_j} \rangle = \delta_{ij}$$

1.2.4 Types of Questions

1. *Is a given set of vectors an orthogonal set?*

   This is only the case if the inner product of every vector with every other vector is zero.

2. *Show a set of vectors is orthonormal (or force it to be)*

   Take the norm of the vector (i.e., the square root of its own inner product). If it is 1, it is a unit vector. Otherwise, divide the vector by its norm to create a an orthonormal set.

1.3 Dimension Counting: The Rank-Nullity Theorem

Let A be an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n \text{ (Number of columns)}$$

Similarly,

$$\dim[\text{rowspace}(A)] = \dim[\text{colspace}(A)] = \text{rank}(A)$$

However, the row space and column space are subspaces of different vector spaces.

Chapter 2

Linear Transformations

Definition: A mapping $T : V \mapsto W$ is said to be a *linear transformation* if the following properties hold:

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T(\mathbf{u}) + T(\mathbf{u}) \\ T(c\mathbf{v}) &= cT(\mathbf{v}) \end{aligned}$$

Definition: Let $T : V \mapsto W$ be a linear transformation from one vector space to another, then

$$\text{kernel}(T) = \{\mathbf{v} \in V | T(\mathbf{v}) = \mathbf{0}_w\}$$