Blackbody Radiation

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Abstract

Comparing Rayleigh-Jeans Law to Planck's Law

1 Limits of Planck's Law

1.1 $\lim_{\lambda \to 0^+} f(x)$

Planck's Law is given by the equation

$$f(\lambda) = \frac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1}$$

Observing the fact that λ occurs in both the numerator and the denominator and that π, h, c, and T are constants, the limit will be analogous to

$$\lim_{\lambda \to 0^+} f(\lambda) = \frac{\lambda^{-5}}{e^{1/(\lambda)} - 1}$$

which is an indeterminate form$\frac{\infty}{\infty}$ as

$$\lim_{\lambda \to 0^+} \lambda^{-5} = \lim_{\lambda \to 0^+} \frac{1}{\lambda^5} = \infty^+$$

and

$$\lim_{\lambda \to 0^+} e^{1/\lambda} = \infty^+$$

Thus L'Hôpital's rule can be applied to $f\lambda$ in order to determine the limit. In order to simplify the calculations, the constants can be factored out by letting the constants $k_1 = 8\pi hc$ and $k_2 = hc/kT$. This leads to the following expression:

$$f(\lambda) = \frac{k_1\lambda^{-5}}{e^{k_2/(\lambda)} - 1}$$

By L'Hôpitals rule, the limit of this function will be the limit of the derivative of both the numerator and the denominator. Letting the value of this limit equal L yields

$$L = \lim_{\lambda \to 0^+} \frac{k_1(-5\lambda^{-6})}{(e^{k_2/(\lambda)})(\frac{-k_2}{\lambda^2})} = \lim_{\lambda \to 0^+} \frac{-5k_1\lambda^2}{-k_2 e^{k_2/(\lambda)}\lambda^6} = \lim_{\lambda \to 0^+} \frac{5k_1\lambda^{-4}}{k_2 e^{k_2/(\lambda)}}$$

which results in a similar indeterminate form. Repeating the application of L'Hôpital's rule:

$$L = \lim_{\lambda \to 0^+} \frac{20k_1\lambda^{-3}}{k_2 e^{k_2/\lambda}}$$

It is clear that repeated applications of L'Hôpital's Rule will tend reduce the exponent of λ in the numerator to zero, while the term $e^{k_2/\lambda}$ in the denominator will persist. Since the term $e^{k_2/(\lambda)} \to \infty$ as $\lambda \to 0^+$ and appears in the denominator, the entire function approaches zero and thus $\lim_{\lambda \to 0^+} f(\lambda) = 0$.

1.2 $\lim_{\lambda \to \infty} f(x)$

By a similar argument to that presented in the previous section, it can be shown that $\lim_{\lambda \to \infty}$ also equals 0. A change of variables simplifies the expression - let $b = \frac{1}{\lambda}$. Thus $\lim_{\lambda \to \infty} f(\lambda) = \lim_{b \to 0} f(b)$, and can be evaluated as such

$$L = \lim_{\lambda \to \infty} \frac{k_1 \lambda^{-5}}{e^{k_2/\lambda} - 1} = \lim_{b \to 0} \frac{k_1 b^5}{e^{bk_2} - 1}$$

Again L'Hôpital's rule can be applied to the indeterminate form $\frac{0}{0}$

$$L = \lim_{b \to 0} \frac{5k_1 b^4}{k_2 e^{bk_2}} = \frac{0}{k_2} = 0$$

Thus $\lim_{\lambda \to \infty} f(\lambda) = 0$

2 $f(\lambda)$ Expressed as a Taylor Polynomial

In the limit that the wavelengths in question are very large, Plank's Law becomes the Rayleigh-Jeans Law. This can be shown by substituting the Taylor polynomial for e^x into Planck's law. Using the variables defined in section one and the fact that $e^x = \sum_{k=0}^{n} \frac{x^k}{k!}$, the function undergoes the following transformation when the first several terms of the Taylor series are substituted in:

$$f(\lambda) = \frac{k_1 \lambda^{-5}}{e^{k_2/\lambda} - 1} = \frac{k_1 \lambda^{-5}}{-1 + \sum_{k=0}^{n} \frac{(k_2/\lambda)^k}{k!}} = \frac{k_1 \lambda^{-5}}{-1 + [1 + (\frac{k_2}{\lambda}) + (\frac{1}{2})(\frac{k_2}{\lambda})^2]}$$

Simplifying this yields the following expression for energy density

$$f(\lambda) = \frac{k_1 \lambda^{-5}}{(\frac{k_2}{\lambda}) + (\frac{1}{2})(\frac{k_2}{\lambda})^2} = \frac{k_1}{\lambda^5 (\frac{k_2}{\lambda}) + (\frac{1}{2})(\frac{k_2}{\lambda})^2} = \frac{k_1}{(k_2)(\lambda^4 + \frac{k_2 \lambda^3}{2})}$$

Expanding the constants k_1 and k_2 and evaluating $\frac{k1}{k2}$:

$$\frac{k_1}{k_2} = \frac{8\pi hc}{\frac{hc}{kT}} = 8\pi kT$$

Substituting this into the previous equation:

$$f(\lambda) = \frac{8\pi kT}{\lambda^4 + \frac{k_2 \lambda^3}{2}}$$

This expression strongly resembles the Rayleigh-Jeans Law and in the limit that λ is very large, the term λ^4 in the denominator dominates the expression and thus

$$f(\lambda) = \frac{8\pi kT}{\lambda^4 + \frac{k_2 \lambda^3}{2}} \approx \frac{8\pi kT}{\lambda^4}$$

Which is identical to Rayleigh-Jean's Law.

3 Comparing Graphs

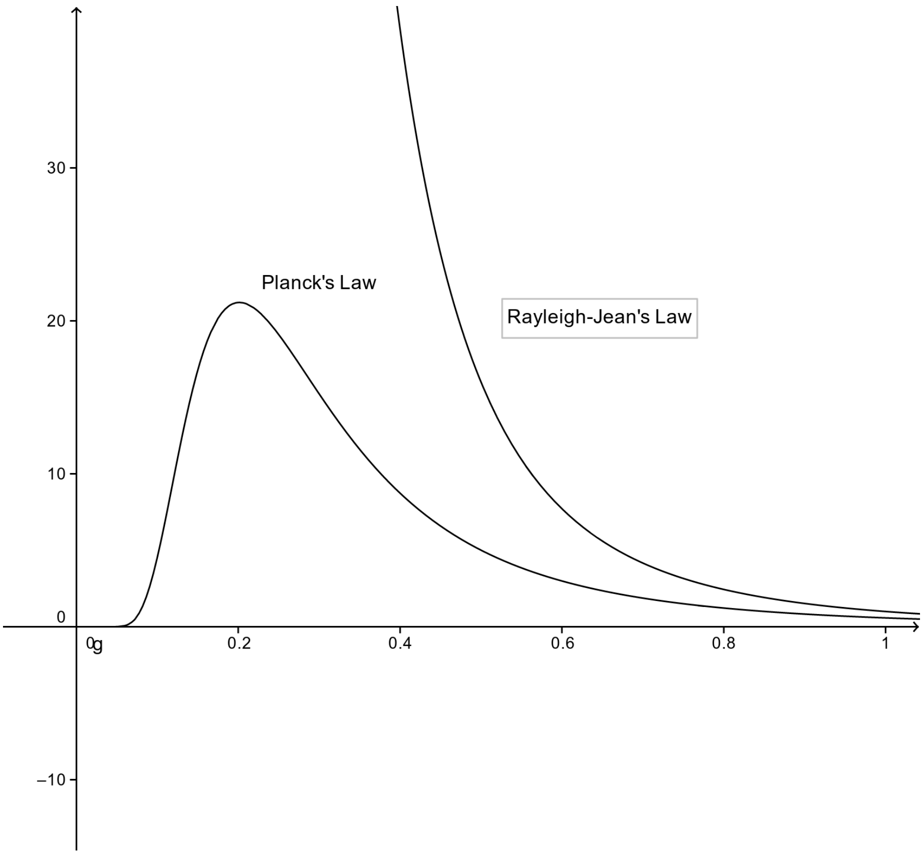

Figure 1: General form of Planck's Law compared to Rayleigh-Jean's Law

As confirmed in the previous section, for large values of λ, both functions tend to behave similarly. However, at very small wavelengths (on the order of micrometers), the graphs exhibit differing behaviors. Rayleigh-Jean's model diverge toward infinity as wavelength approaches 0, while Planck's model diverges slowly toward a local maximum before converging to 0. Planck's model also follows a more gradual curve, as opposed to the steep slope exhibited by Rayleigh'Jean's Law.

4 Maximum Value of $f(\lambda)$ Under Planck's Law

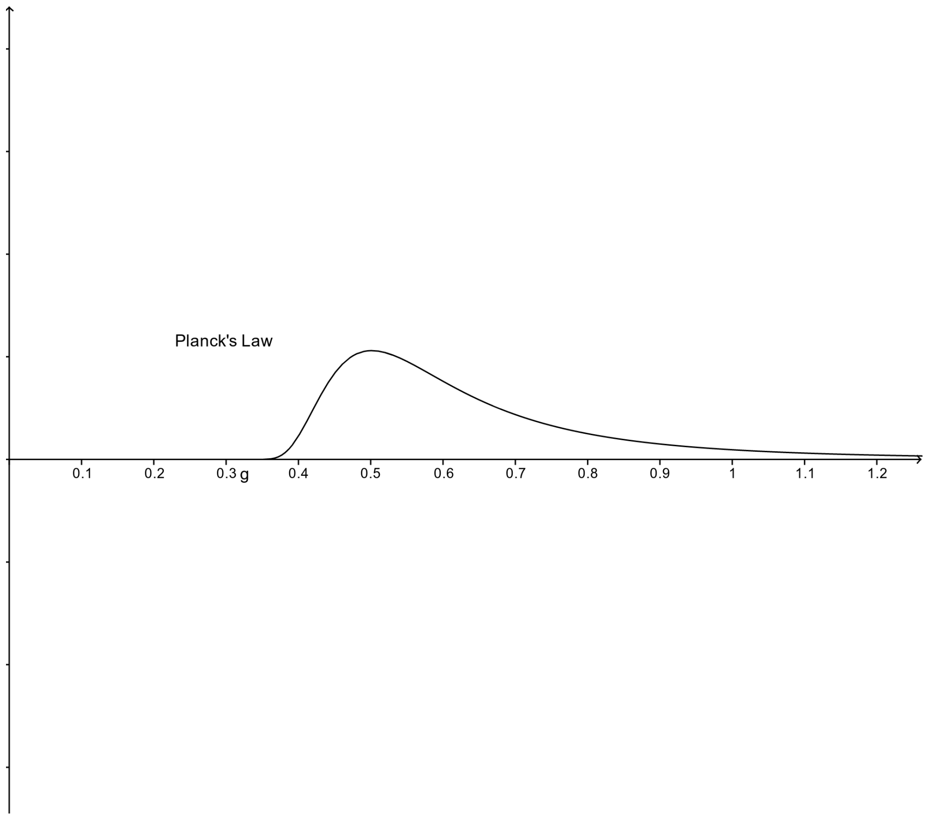

Figure 2: Planck's Law at 5700 K

Reorienting the previous graph shows that the maximum amount of energy density occurs at about .5 μm

5 Variation of Radiation with Temperature

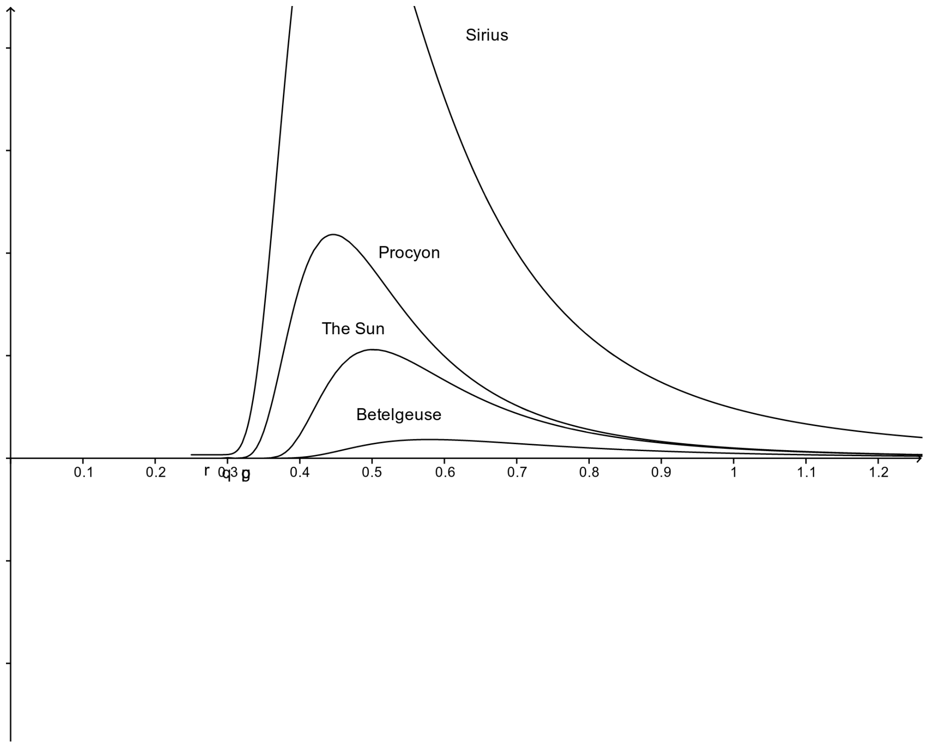

Figure 3: Planck's Law Model of Star Radiation

As the temperature increases, the local maximum increases as well. This corresponds to specific wavelengths of light being emitted more strongly than others. Additionally, the maximum shifts to the left at increased temperature, and the total area under the curve increases, denoting a greater total amount of energy being radiated.

5.1 Interpretations

Why Sirius is a Blue Star and Betelgeuse is a Red Star

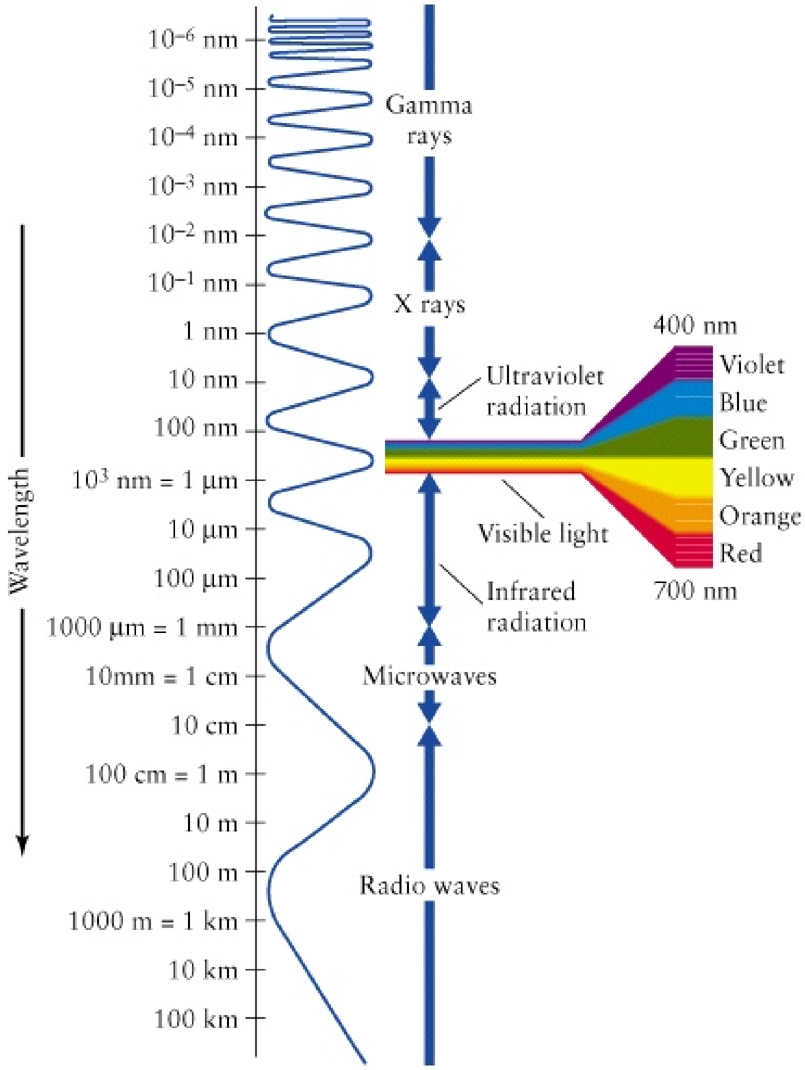

Figure 4: Relation between wavelength and visible color

Decreasing wavelengths correspond to higher energy, which in turn corresponds to a shift toward the ultraviolet in the star's visible radiation. Since Sirius has a higher temperature, its radiation peaks closer in the visible spectrum to blue. Conversely, Betelgeuse has a lower temperature and predictably emits wavelengths that peak in the red portion of the spectrum.