Differential Equations and Linear Algebra, Spring 2014 Notes

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Contents

Chapter 1

Vector Spaces

Next Exam: April 2nd. Covers 4.6→5.1.
Remember: To show a set spans a space, show that an arbitrary element in that space can be expressed as a linear combination of the vectors in your set. This usually results in a linear system of equations – as long as you can solve this system in terms of your vectors, it will be consistent.

1.1 Bases

1.1.1 Determining a Basis

A set S that forms a basis for a vector space V must satisfy two conditions:

1. S is set of linearly independent vectors.
2. S spans V.

All bases for a given vector space V have the same number of elements, and this is referred to as the *dimension* of V such that dim$[V]$ = the number of elements in the basis.
If the dimension is known, it is equal to the minimum number of vectors needed to span V, as well as the maximum number of linearly independent vectors that a set in V can contain.
Does a set S form a basis for a vector space V?
First, check for linear independence. If dim$[V]$=n and S contains n linearly independent vectors, S is guaranteed to form a basis for V.
This means that given a set of vectors, we only have to shown linear dependence and that the number of elements matches the dimension of the space! So we only need to know their **linear independence** and the space's **dimension**. In other words, n linearly independent vectors in n-dimensional space means, for free, we have a basis.
Note: dim$[P_n]$= $n + 1$.

For the subspaces of a given matrix

Null space:

Suppose we are given a matrix, and want to examine the properties of its null space. Recall that the null space of a matrix A is given by $\{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$, and its dimension is equal to the number of free variables in ref(A).
This is generally found by augmenting the matrix with $\mathbf{0}$ is using ERO to obtain its REF. Once a relationship between the free variables is found, construct a solution set of the form

$$S = \{\mathbf{x} \in \mathbb{R}^n : x = a_1\mathbf{v_1} + a_2\mathbf{v_2} \cdots a_n\mathbf{v_n} \wedge a_1, a_2, \cdots a_n \in R\} = \operatorname{span}\{\text{something}\}$$

At this point, it should be clear what the basis and dimensions are.
Row space:
A basis for the row space of a matrix A is nothing more than the number of nonzero rows in ref(A), which is a subspace of $\mathbb{R}^n$.
Column space:
A basis for the column space of a matrix A consists of the column vectors with leading oness in rref(A).

1.1.2 Extending a Basis

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1.2 Inner Product Spaces (4.11)

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1.2.1 Axioms

4 Axioms of an Inner Product

1. $\langle \mathbf{V_1}, \mathbf{V_1} \rangle >= 0$ and $\langle \mathbf{V_1}, \mathbf{V_1} \rangle = 0 \iff \mathbf{V_1} = 0$
   Check that the scalar result is positive or zero, and show that $\langle \mathbf{A}, \mathbf{A} \rangle = 0$ forces the coefficients to be zero.
2. Commutativity of $\bigodot$
   $\langle \mathbf{V_1}, \mathbf{V_2} \rangle = \langle \mathbf{V_2}, \mathbf{V_1} \rangle$
3. Associativity of scalar multiplication over $\bigodot$
   $\langle c\mathbf{V_1}, \mathbf{V_2} \rangle = c\langle \mathbf{V_1}, \mathbf{V_2} \rangle$
4. Associativity of $\bigoplus$ over $\bigodot$
   $\langle \mathbf{V_1}, \mathbf{V_2} + \mathbf{V_3} \rangle = \langle \mathbf{V_1}, \mathbf{V_2} \rangle + \langle \mathbf{V_1}, \mathbf{V_3} \rangle$

1.2.2 Orthogonality

Two vectors $\mathbf{p}$ and $\mathbf{q}$ are defined to be orthogonal if $\langle \mathbf{p}, \mathbf{q} \rangle = 0$.

1.2.3 The Gram-Schmidt Process

Given a basis

$$S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots \mathbf{v_n}\},$$

the Gram-Schmidt process produces a corresponding orthogonal basis

$$S' = \{\mathbf{u_1}, \mathbf{u_2}, \cdots \mathbf{u_n}\}$$

that spans the same vector space as S.
S' is found using the following pattern:

$$\begin{aligned}
&\mathbf{u_1} = \mathbf{v_1} \\
&\mathbf{u_2} = \mathbf{v_2} - \mathrm{proj}_{\mathbf{u_1}}\mathbf{v_2} \\
&\mathbf{u_3} = \mathbf{v_3} - \mathrm{proj}_{\mathbf{u_1}}\mathbf{v_3} - \mathrm{proj}_{\mathbf{u_2}}\mathbf{v_3}
\end{aligned}$$

where

$$\mathrm{proj}_{\mathbf{u}}\mathbf{v} = (\mathrm{scal}_{\mathbf{u}}\mathbf{v})\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle \mathbf{v}, \mathbf{u}\rangle}{\|\mathbf{u}\|}\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle \mathbf{v}, \mathbf{u}\rangle}{\|\mathbf{u}\|^2}\mathbf{u}$$

is a vector defined as the *orthogonal projection of* $\mathbf{v}$ *onto* $\mathbf{u}$.

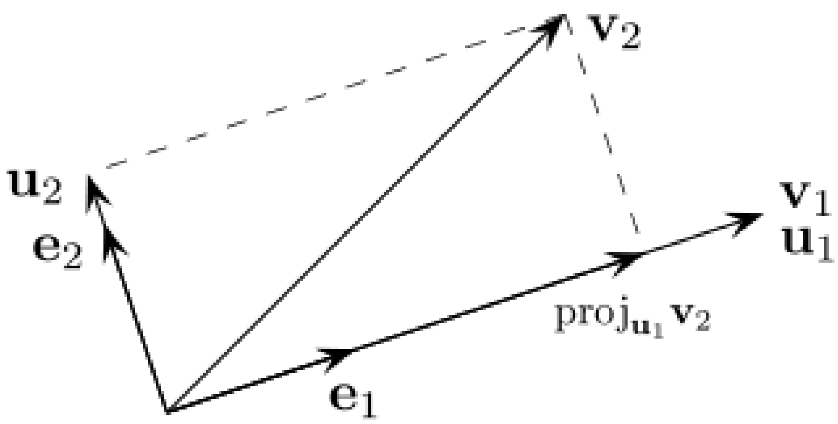

Figure 1.1: Orthogonal projection of $\mathbf{v_2}$ onto $\mathbf{u_1}$

The orthogonal set S' can then be transformed into an orthonormal set S'' by simply dividing the vectors $s \in S'$ by their magnitudes. The usual definition of a vector's magnitude is

$$\|\mathbf{a}\| = \sqrt{\langle \mathbf{a}, \mathbf{a}\rangle} \text{ and } \|\mathbf{a}\|^2 = \langle \mathbf{a}, \mathbf{a}\rangle$$

As a final check, all vectors in S' should be orthogonal to each other, such that

$$\langle \mathbf{v_i}, \mathbf{v_j}\rangle = 0 \text{ when } i \neq j$$

and all vectors in S'' should be orthonormal, such that

$$\langle \mathbf{v_i}, \mathbf{v_j} \rangle = \delta_{ij}$$

1.2.4 Types of Questions

1. *Is a given set of vectors an orthogonal set?*

   This is only the case if the inner product of every vector with every other vector is zero.

2. *Show a set of vectors is orthonormal (or force it to be)*

   Take the norm of the vector (i.e., the square root of its own inner product). If it is 1, it is a unit vector. Otherwise, divide the vector by its norm to create a an orthonormal set.

1.3 Dimension Counting: The Rank-Nullity Theorem

Let A be an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n \text{ (Number of columns)}$$

Similarly,

$$\dim[\text{rowspace}(A)] = \dim[\text{colspace}(A)] = \text{rank}(A)$$

However, the row space and column space are subspaces of different vector spaces.

Chapter 2

Linear Transformations

Definition: A mapping $T : V \mapsto W$ is said to be a *linear transformation* if the following properties hold:

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{u})$$
$$T(c\mathbf{v}) = cT(\mathbf{v})$$

Definition: Let $T : V \mapsto W$ be a linear transformation from one vector space to another, then

$$\text{kernel}(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}_w\}$$

The kernel of T is a subspace of V, and every vector in it is mapped to the zero vector in W. Defining A as a linear transformation matrix, the kernel is identical to the solution set to $A\mathbf{x} = \mathbf{0}$.
Definition: Similarly, the subset of W consisting of all transformed vectors is denoted the range of T and is defined as

$$\text{Rng}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$$

The range of T is instead a subspace of W into which the transformations are mapped.
For any matrix transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ given by an $m \times n$ matrix,

$$\text{Ker}(T) = \text{nullspace}(A)$$
$$\text{Rng}(T) = \text{colspace}(A).$$

Theorem: The General Rank-Nullity Theorem

$$\dim[\text{Ker}(T)] + \dim[\text{Rng}(T)] = \dim[V].$$

Notes:
It is common to want to know the range and kernel of a specific linear transformation T. T can be given in many ways, but a general strategy for deducing these properties involves:

1. Express an arbitrary vector in V as a linear combination of its basis vectors, and set it equal to an arbitrary vector in W.

2. Use the linear properties of T to make a substitution from known transformations
3. Find a restriction or relation given by the constants of the initial linear combination.

Definition: A linear transformation is said to be

1. **one-to-one** if whenever $\mathbf{v_1} \neq \mathbf{v_2}, T(\mathbf{v_1}) \neq T(\mathbf{v_2})$.
2. **onto** if every $w \in W$ is the image under T of at least one vector in V.

Theorem: A given linear transformation is:

1. one-to-one $\iff$ $\mathrm{Ker}(T) = \{0\}$
2. onto $\iff$ $\mathrm{Rng}(T) = W$.

Corollary:

1. If T is one-to-one, then $\dim[V] \leq \dim[W]$.
2. If T is onto, then $\dim[V] \geq \dim[W]$.
3. If T is both, then $\dim[V] = \dim[W]$, and a transformation T^{-1} exists, and the two spaces are isomorphic.
4. If $\dim[V] = \dim[W]$, T is one-to-one $\iff$ T is onto.

These are often useful in the contrapositive.
Definition: Inverse Functions. $T_2 = T_1^{-1}$ means that the following relationship holds:

$$(T_1T_2)\mathbf{v} = (T_2T_1)\mathbf{v} = \mathbf{v}$$

Definition: A map between vector spaces is said to be an *isomorphism* if it is an invertible, linear map.

Showing a linear transformation is one-to-one: Simply show that the kernel only contains $\mathbf{0}$.
Showing a linear transformation is onto: Use the generalized rank nullity theorem. If $\dim[\mathrm{Rng}(T)] = \dim[V]$, then $\mathrm{Rng}(T) = V$ and T is onto. Useful fact: If W is a subspace of V with the same dimension, W and V are the same thing.
Showing an inverse map exists: Show the map is one-to-one and onto. In other words, that the kernel is empty and the dimension of the range is the dimension of the codomain.

2.1 Examples

1. Show that if T_1 and T_2 are both injective, so is the composition (T_2T_1).

   Solution: Goal: Show the kernel of the composition is the singleton set containing the zero vector.

(a) Suppose we have $v_1 \in \text{Ker}(T_2T_1)$.

(b) Then $(T_2T_1)(v_1) = 0 \Rightarrow T_2(T_1(v_1)) = 0$.

(c) Since T_2 is onto, $T_1(v_1) = 0$.

(d) Use assumptions to tracing this back, showing the only vector in the kernel is $\mathbf{0}$.

2. Show that if the two maps are both surjective, so is the composition.

   Solution: Goal: Show for every vector in V_3, there has to exist a vector in V_1 that can get you there!

Chapter 3

Eigenthings

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Chapter 4

Applications to Differential Equations

4.1 Linear Equations of Order n

The standard form of such equations is

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y'' + a_{n-1} y' + y = F(x).$$

All solutions will be the sum of the solution to the associated homogeneous equation and a single particular solution.
In the homogeneous case, examine the discriminant of the characteristic polynomial. Three cases arise:

1. $D > 0 \Rightarrow 2$ Real solutions, $c_1 e^{r_1 x} + c_2 e^{r_2 x}$
2. $D = 0 \Rightarrow 1$ Real, 1 Complex, $(c_1 + c_2 x)e^{r_1 x}$
3. $D < 0 \Rightarrow 2$ Complex, $e^{ax}(c_1 \cos bx + c_2 \sin bx)$

That is, every real root contributes a term of ce^{rx}, while a multiplicity of m multiplies the solution by a polynomial in x of degree $m - 1$.
Every pair of complex roots contributes a term $ce^{r}(a \cos \omega x + b \sin \omega x)$, where r is the real part of the roots and ω is the complex part.
In the nonhomogeneous case, assume a solution in the most general form of $F(x)$, and substitute it into the equation to solve for constant terms. For example,

1. $F(x) = P(x^n) \Rightarrow y_p = a + bx + cx^2 + \cdots + (n+1)x^n$
2. $F(x) = e^x \Rightarrow y_p = Ae^x$
3. $F(x) = A\cos(\omega x) \Rightarrow y_p = a\cos(\omega x) + b\sin(\omega x)$

4.1.1 Annihilators

Use to reduce a nonhomogeneous equation to a homogeneous one as a polynomial in the operator D.

1. $(D - a) \Rightarrow e^{ax}$

2. $(D-a)^{k+1} \Rightarrow x^k e^{ax}, x^{k-1}e^{ax}, \cdots, e^{ax}$

3. $D^{k+1} \Rightarrow x^k, x^{k-1}, \cdots, C$

4. $D^2 - 2aD + a^2 + b^2 \Rightarrow e^{ax}\cos(bx), e^{ax}\sin(bx)$

5. $(D^2-2aD+a^2+b^2)^{k+1} \Rightarrow x^k e^{ax}\cos(bx), x^{k-1}e^{ax}\cos(bx), x^k e^{ax}\sin(bx), x^{k-1}e^{ax}\sin(bx), \cdots$

4.1.2 Complex Solutions

$F(x)$ of the form $e^{ax} sin(kx)$ can be rewritten as $e^{(a+ki)x}$