Differential Equations and Linear Algebra Spring 2014 Notes

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Contents

Chapter 1

Vector Spaces

Next Exam: April 2nd. Covers 4.6→5.1.

1.1 Bases

1.1.1 Determining a Basis

A set S that forms a basis for a vector space V must satisfy two conditions:

1. S is set of linearly independent vectors.
2. S spans V.

Does a set S form a basis for a vector space V?
First, check for linear independence. If dim[V]=n and S contains n linearly independent vectors, S is guaranteed to form a basis for V.
Note: dim[P_n]= $n + 1$.
Example 1
Determine a basis for

$$S = \left\{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R} \wedge a_0 - a_1 - 2a_2 = 0\right\}.$$

Let $a_2 = t, a_1 = s, a_0 = s + 2t$, then

$$\begin{aligned} S &= \left\{(s + 2t) + (sx + tx^2) \mid s, t \in \mathbb{R}\right\} \\ &= \left\{(s + sx) + (2t + tx^2) \mid s, t \in \mathbb{R}\right\} \\ &= \left\{s(1 + x) + t(2 + x^2) \mid s, t \in \mathbb{R}\right\} \\ &= \text{span}\left\{(1 + x), (2 + x^2)\right\} \end{aligned}$$

and a basis for S is

$$\left\{(1 + x), (2 + x^2)\right\}$$

1.2 Inner Product Spaces (4.11)

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1.2.1 Axioms

4 Axioms of an Inner Product

1. $V_1 \cdot V_1 >= 0$ and $V_1 \cdot V_1$ iff $V_1 = 0$
   Check that the scalar result is positive or zero.
   Show that $\langle A, A \rangle = 0$ forces the coefficients to be zero.
2. $V_1 \cdot V_2 = V_2 \cdot V_1$
3. $(cV_1) \cdot V_2 = c(V_1 \cdot V_2)$
4. $V_1 \cdot (V_2 + V_3) = V_1 V_2 + V_1 V_3$

1.2.2 Orthogonality

$\langle p, q \rangle = 0 \Rightarrow$ Orthogonality.

1.2.3 The Gram-Schmidt Procedure

Given a set of vectors

$$S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots \mathbf{v_n}\},$$

the Gram-Schmidt procedure produces a corresponding orthogonal set

$$S' = \{\mathbf{u_1}, \mathbf{u_2}, \cdots \mathbf{u_n}\}$$

the is a basis for the same vector space as S.
Given the set S, S' is found using the following pattern:

$$\begin{aligned}
\mathbf{u_1} &= \mathbf{v_1} \\
\mathbf{u_2} &= \mathbf{v_2} - \mathrm{proj}_{\mathbf{u_1}} \mathbf{v_2} \\
\mathbf{u_3} &= \mathbf{v_3} - \mathrm{proj}_{\mathbf{u_1}} \mathbf{v_3} - \mathrm{proj}_{\mathbf{u_2}} \mathbf{v_3}
\end{aligned}$$

where

$$\mathrm{proj}_{\mathbf{u}} \mathbf{v} = (\mathrm{scal}_{\mathbf{u}} \mathbf{v}) \frac{\mathbf{u}}{\mathbf{u}}$$

A few definitions are needed:

$$\mathrm{scal}_{u1}$$

1.2.4 Examples

1.

Let $A, B.C \in M_2(\mathbb{R})$. Define $\langle A, B \rangle = a_{11}b_{11} + 2a_{12}b12 + 3a_{21}b_{21}$. Does this define an inner product on $M_2(\mathbb{R})$?

2.

Instead, let $\langle A, B \rangle = a_{11} + b_{22}$. Does this define an inner product on $M_2(\mathbb{R})$?

3.

Let $p = a_0 + a_1x + a_2x^2$ and $q = b_0 + b_1x + b_2x^2$.
Define $\langle p, q\rangle = \sum_{i=0}^{2}(i+1)a_ib_i$. Does this define an inner product on P_2?

4.

Let $f, g \in C((-\infty, \infty))$. Define

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx.$$

1.3 Gram