Qualifying Exam

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Sunday 5th July, 2020

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1 Question 1 (UGA 2016 #5)

Let $(X, \mathcal{M}, \mu)$ be a measure space. For $f \in L^1(\mu)$ and $\lambda > 0$, define

$$\varphi(\lambda) = \mu(\{x \in X | f(x) > \lambda\}) \quad \text{and} \quad \psi(\lambda) = \mu(\{x \in X | f(x) < -\lambda\})$$

Show that φ, ψ are Borel measurable and

$$\int_X |f| \, d\mu = \int_0^\infty [\varphi(\lambda) + \psi(\lambda)] \, d\lambda$$

2 Question 2 (NUS 1970 #5)

(a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function. If $f'(-1) < 2$ and $f'(1) > 2$, show that there exists $x_0 \in (i1, 1)$ such that $f'(x_0) = 2$.

Hint: consider the function $f(x)-2x$ and recall the proof of Rolle's theorem.)

(b) Let $f:(-1,1)\longrightarrow\mathbb{R}$ be a differentiable function on $(-1,0)\bigcup(0,1)$ such that $\lim_{x\longrightarrow 0} f'(x)=L$. If f is continuous on $(-1,1)$, show that f is indeed differentiable at 0 and $f'(0)=L$.

3 Question 3 (UGA 2019 #2)

Let $\mathcal{B}$ denote the set of all Borel subsets of $\mathbb{R}$ and $\mu:\mathcal{B}\longrightarrow[0,\infty)$ denote a finite Borel measure on $\mathbb{R}$.

a. Prove that if $\{F_k\}$ is a sequence of Borel sets for which $F_k\supseteq F_{k+1}$ for all k, then

$$\lim_{k\to\infty}\mu(F_k)=\mu\left(\bigcap_{k=1}^{\infty}F_k\right)$$

b. Suppose μ has the property that $\mu(E)=0$ for every $E\in\mathcal{B}$ with Lebesgue measure $m(E)=0$. Prove that for every $\varepsilon>0$ there exists $\delta>0$ so that if $E\in\mathcal{B}$ with $m(E)<\delta$, then $\mu(E)<\varepsilon$.

4 Question 4 (NUS 1970 #3)

Let g : $[0,1]\times[0,1]\longrightarrow[0,1]$ be a continuous function and let $\{f_n\}$ be a sequence of functions such that

$$f_n(x)=\begin{cases}0, 0\le x\le 1/n,\\ \displaystyle\int_0^{x-\frac{1}{n}} g(t,f_n(t))dt, 1/n\le x\le 1.\end{cases}$$

With the help of the Arzela-Ascoli theorem or otherwise, show that there exists a continuous function $f:[0,1]\longrightarrow\mathbb{R}$ such that

$$f(x)=\int_0^x g(t,f(t))dt$$

for all $x\in[0,1]$.

Hint: first show that $|f_n(x_1)-f_n(x_2)|\le|x_1-x_2|$.

5 Question 5 (UGA 2015 #4)

Define

$$f(x,y):=\begin{cases}\dfrac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0\le x\le y\\ 0 & \text{otherwise}\end{cases}$$

Carefully show that $f\in L^1(\mathbb{R}^2)$.

6 Question 6 (Emory 0 #0)

Describe the process that extends a measure on an algebra $\mathcal{A}$ of subsets of X, to a complete measure defined on a σ-algebra $\mathcal{B}$ containing $\mathcal{A}$. State the corresponding definitions and results (without proofs).

7 Question 7 (UGA 2018 #2)

Let

$$f_n(x) := \frac{x}{1+x^n}, \quad x \geq 0.$$

a. Show that this sequence converges pointwise and find its limit. Is the convergence uniform on $[0, \infty)$?

b. Compute

$$\lim_{n\to\infty} \int_0^\infty f_n(x)dx$$

8 Question 8 (UGA 2016 #1)

Define

$$f(x) = \sum_{n=1}^\infty \frac{1}{n^x}.$$

Show that f converges to a differentiable function on $(1, \infty)$ and that

$$f'(x) = \sum_{n=1}^\infty \left(\frac{1}{n^x}\right)'.$$

Hint:

$$\left(\frac{1}{n^x}\right)' = -\frac{1}{n^x}\ln n$$

9 Question 9 (UGA 2017 #2)

a. Let μ be a measure on a measurable space $(X, \mathcal{M})$ and f a positive measurable function.

Define a measure λ by

$$\lambda(E) := \int_E f \, d\mu, \quad E \in \mathcal{M}$$

Show that for g any positive measurable function,

$$\int_X g \, d\lambda = \int_X fg \, d\mu$$

b. Let $E \subset \mathbb{R}$ be a measurable set such that

$$\int_E x^2 \, dm = 0.$$

Show that $m(E) = 0$.

10 Question 10 (UGA 2016 #4)

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\{E_n\} \subset \mathcal{M}$ satisfies

$$\lim_{n \to \infty} \mu\left(X \backslash E_n\right) = 0.$$

Define

$$G \coloneqq \left\{ x \in X \;\middle|\; x \in E_n \text{ for only finitely many } n \right\}.$$

Show that $G \in \mathcal{M}$ and $\mu(G) = 0$.