Topology Qual Problems

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Friday 29th May, 2020

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1 Problems

1.1 Homotopy

1. Show that any non-surjective map $f : X \to S^n$ is homotopic to the constant map.
2. Let $f, g \to S^n$ be such that $\forall x \in X, f(x) \neq -g(x)$. Show that $f \simeq g$.
3. Let $\alpha : S^n \longrightarrow S^n,\ \alpha(p) = -p$ be the antipodal map on S^n. Show that n odd $\implies f \simeq \mathrm{id}$.
4. Show that X is homotopy-equivalent to a point $\iff \mathrm{id}_X \simeq g$ for some constant map g.
5. Show that $S^1 \times I \simeq M$, the Mobius strip.
6. Show that $\mathbb{R}^3 - S^1 \simeq S^1 \vee S^2$.
7. Classify the letters of the alphabet up to homeomorphism, and up to homotopy.
8. **REVISIT** Let $f, g : S^1 \to X$, $P = X \cup_f B^2 \cong X \coprod B^2 / \sim$, where $x \sim f(x)$, $Q = X \cup_g B^2$. Show that $f \simeq g \implies P \simeq Q$.

1.2 Fundamental Group

1. Show that $x, y \in X$ path & simply-connected $\implies$ all paths from x to y are homotopic rel $\{0, 1\}$.
2. Show that for X path connected, $\pi_1(X) = \not\Vdash \iff \forall$cts. $f : S^1 \to X$ f, extends to a continuous map $F : B^2 \to X$.
3. Show $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

4. Show $\pi_1(S^n) = 1$ for $n \geq 2$.
5. Show that $S^2 - \{p_0, p_1\} \simeq S^1$.
6. Show that $S^3 - \{p_0, p_1\} \simeq S^2$
7. Show that $S^2 \not\simeq S^3$.
8. For each of the following $f : S^1 \to S^1$, identify the corresponding $f_* : \mathbb{Z} \longrightarrow \mathbb{Z}$:
   1. $z \mapsto z^n$
   2. $\bar{x} \mapsto -\bar{x}$
   3. $e^{i\theta} \mapsto e^{2\pi i \sin\theta}$
9. Determine the winding number of the following map: $f : S^1 \longrightarrow \mathbb{C} - \{0\}, z \mapsto 8z^4 + 4z^3 + 2z^2 + z^{-1}$
10. Identify $\pi_1(M, [(1, \frac{1}{2})])$, and identify the class of ∂M.
11. Let $X = S^1 \times S^1$ and γ a loop based at x_0. What is the induced map $\gamma_\sharp$?

1.3 Group Actions

1. Show that octagon pasting is homeomorphic to the $T = \mathbb{R}^2/\mathbb{Z}^2$.
2. Let x_0 be the image of 0, show that there is an order 6 homeomorphism $f : T \longrightarrow T$ fixing x_0. Find a representation of f_* as a matrix, and find its determinant.
3. Show that $\pi_1(K)$, the Klein bottle, is given by pairs (m, n) where $(m, n) \star (p, q) = (m + (-1)^n p, n + q)$
   1. Show this is torsion-free
   2. Show that T is a double cover of K.
4. For each of these actions of $\mathbb{Z}_2$ on S^n, compute $\pi_1(S^n/\mathbb{Z}_2)$
   1. $S^1, z \mapsto -z$
   2. $S^2, (x, y, z) \mapsto (-x, -y, z)$
   3. $S^3, (z, w) \mapsto (-z, -w)$

1.4 Applications

1. Let $i : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^3$, induced by $S^2 \hookrightarrow S^3$ as the equator. Show that $i \not\simeq$ const.
2. Show that there is no map $f : S^2 \longrightarrow S^1$ that commutes with the antipodal map.
3. Prove that for any $f : S^2 \longrightarrow \mathbb{R}^2$, there exists $x \in S^2$ such that $f(x) = f(-x)$.
4. Prove the Ham Sandwich theorem.
5. Show that K can not be a topological group.

1.5 Van Kampen's Theorem

1. Compute a presentation of $\pi_1(T)$ and prove it is isomorphic to $\mathbb{Z}_2$.
2. (Images)
3. Show that $T - D^1 := X \simeq S^1 \vee S^1$.
   1. Show there does not exist a retraction $r : X \longrightarrow \partial X$.
4. Images
5. IMages
6. Images
7. Calculate a presentation of $\pi_1(S^3 - K)$
8. Show that all 3 presentations of $\pi_1(K)$ are isomorphic
   1. Square with sides glued

   2. Two mobius strips glues along boundary
   3. Multiplication rule
9. Given a group $G =< A : R >$, show how to construct a CW-complex X such that $\pi_1(X) = G$.
10. Write down the fundamental group of the following spaces:
11. $\mathbb{R}^2 - \{0, 1\}$
12. $\mathbb{R}^2 - I$
13. The symbol $\oplus \in \mathbb{R}^2$
14. $S^2 - \{p_i\}_{i=1}^4$
15. $T - \{p_0\}$
16. $S^2/\mathbb{Z}_2$ via the antipodal map
17. $S^2/\mathbb{Z}_3$ via a $2\pi/3$ rotation about the z-axis.
18. $S_2 \cup \{(0, 0, z) \,\Big|\, -1 \le z \le 1\}$
19. $\mathbb{R}^3 - \{(x, y, 0) \,\Big|\, x^2 + y^2 = 1\}$
20. $\mathbb{R}^2 - H$, the Hopf link
21. Prove that the homophony group is trivial.

1.6 Mayer Vietoris (Sheet 7)

1. Compute the homology of:
   1. $\mathbb{RP}^2 = M \bigcup_{\partial} D^2$
   2. $T^2 = S^1 \times S^1 = (S^1 \times I) \bigcup_f (S^1 \times I)$ where $(x, 0) \sim (x, 1) \sim (\bar{x}, 0) \in \mathbb{C}$
   3. $S^1 \bigcup_f B^2$ attached along ∂B^2 using $z \mapsto z^n$
2. Show $\tilde{H}_i(\Sigma X) \cong \tilde{H}_{i-1}(X)$
   1. Show $\Sigma S^n \cong S^{n+1}$
3. For $f : S^n \circlearrowleft$, show $\deg f = \deg \Sigma f$
   1. Conclude $\pi_n(S^n) = \mathbb{Z}$
4. Let $\{A_i\}^n \in \mathbf{Ab}$ be finitely generated, show $\exists X \,\Big|\, H_i(X) \cong A_i$ for $i \le n$ and 0 otherwise.
5. Suppose $X = \bigcup_i^n A_i$ such that for any $1 \le k \le n$, $\bigcap_i^k A_i$ is either empty or contractible, show $i \ge n - 1 \implies \tilde{H}_i(X) = 0$ and that this bound is sharp.
6. Compute $H_*(X \times S^n)$ in terms of $H_*(X)$
   1. Compute $H_*(T^n)$
7. Let $M = (S^1 \times B^2) \bigcup_{\mathrm{id}_\partial} (S^1 \times B^2)$ and compute $H_*(M; \mathbb{Z})$
8. Let $X = S^n \times I$ with its ends glued together by a map $S^n \circlearrowleft$ of degree d, calculate $H_*(X)$.
9. Compute $H_*(X)$ for $X = S^3 - N$, with N a knotted solid torus and $\partial N = T$ its boundary torus

10. Let CA be the cone on A, show that $\tilde{H}_*(X \bigcup CA) \cong \tilde{H}_*(X, A)$.

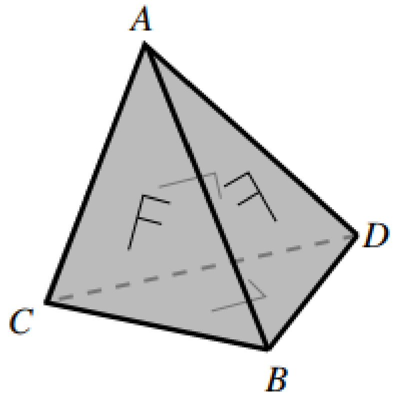
11. Show that the Mayer-Vietoris sequence is natural, i.e. If $X \xrightarrow{f} Y$ where $f(A) \subset C$ and $f(B) \subset D$, then this commutes:

$$\begin{array}{ccccccc}
H_n(X) & \longrightarrow & H_n(A \bigcap B) & \longrightarrow & H_n(A) \oplus H_n(B) & \longrightarrow & H_{n-1}(X) \\
\downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\
H_n(Y) & \longrightarrow & H_n(C \bigcap D) & \longrightarrow & H_n(C) \oplus H_n(D) & \longrightarrow & H_{n-1}(Y)
\end{array}$$

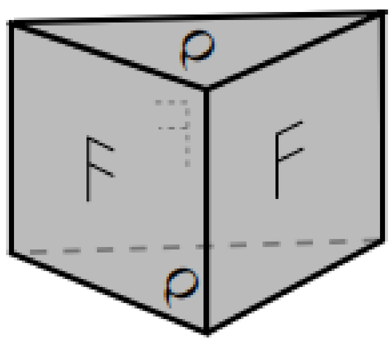
1.7 Cellular Homology (Sheet 8)

Compute the homology of these spaces

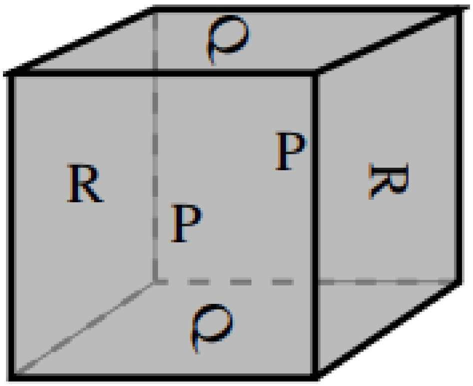
1. $S_m \vee S_n$
2. $S^m \times S^n$
3. A hexagon with the identifications $a + b + c - a - b - c$
4. Orientable surface of genus g
    1. $g = 2$ is given by $a + b - a - b + c + d - c - d$
5. Nonorientable surface of genus g Obtain by removing g discs from S^2 and attaching g mobius strips
6. $S_1 \vee S_1$ with two discs attached via $(ab)^3$ and $(ab)^6$



7. This identification space:



8. This identification space: 262



9. This identification space: a natural number) is defined by the
10. Describe a CW complex structure for the lens space $L(p,1)$ and compute π_1, H_* for it.

1.8 Degree

1. Let $p(x) = \sum_i^n a_i x^i$, view $p : \mathbb{C}\bigcup\infty\circlearrowleft$ and determine its topological degree
2. Let $p(z) = \frac{\prod_i^n z - a_i}{\prod_j^m z - b_j}$ with all a_i, b_j distinct. What is its topological degree?
3. Show that if $f : S^m \longrightarrow S^n$ and $\exists U \subset S^m$ such that $f|_U \cong f(U)$, then $m = n$ and f is surjective.

1.9 Universal Coefficient Theorem (Sheet 10)

1. Identify the following groups up to isomorphism
   1. $\mathbb{Z}_m \otimes \mathbb{Z}_n$
   2. $\mathbb{Z}_{60}^4 \otimes (\mathbb{Z}_{24}^3 \oplus \mathbb{Z}_8^4 \oplus \mathbb{Z}_{120})$
   3. $\mathbb{Z}_n \otimes \mathbb{Q}$
   4. $(\mathbb{Z} \oplus \mathbb{Z}_n) \otimes (\mathbb{Q}/\mathbb{Z})$
2. Compute:
   1. $\mathrm{Tor}(\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8, \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4)$
   2. $\mathrm{Ext}(\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5)$
3. Compute the following directly from chain complexes and check using UCT:
   1. $H_*(\mathbb{RP}^n; \mathbb{Z}_2)$
   2. $H_*(\mathbb{RP}^n, \mathbb{Z}_3)$
   3. $H^*(\mathbb{RP}^n, \mathbb{Z}_6)$
4. For any space X, show that $H^1(X)$ is free abelian
5. Show that $H_*(X; \mathbb{Q}) = H_*(X; \mathbb{Z}) \otimes \mathbb{Q}$ $H^*(X; \mathbb{Z}) = \hom(H_*(X; \mathbb{Z}), \mathbb{Q})$
6. Construct a space X such that $H_*(X; \mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}_6, \mathbb{Z}_{12}, \mathbb{Z} \oplus \mathbb{Z}_4, 0 \cdots)$ Compute $H^*(X; \mathbb{Z})$
7. Compute $H_*(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}_2)$
8. Compute $H_*(\Sigma\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z})$
9. Compute $H_*(\mathbb{RP}^2 \times \mathbb{RP}^3; \mathbb{Z})$
10. Let G be a topological group. Show that $H_*(G)$ is an algebra. Show that $G \curvearrowright H_*(G)$, which factors through the homomorphism $G \longrightarrow \pi_0(G)$ yielding a trivial action if G is path-connected.

1.10 Homological Algebra (Sheet 11)

1. Show that $\ker A \longrightarrow A \otimes \mathbb{Q}$ given by $a \mapsto a \otimes 1$ is the torsion subgroup of A.
2. Show that $A \hookrightarrow B \implies A \otimes \mathbb{Q} \hookrightarrow B \otimes \mathbb{Q}$
3. Find a free resolution of $\mathbb{Q}$ as a $\mathbb{Z}$-module.
4. Compute $\mathrm{Tor}(\mathbb{Q}, A)$
    1. Compute $\mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, A)$
5.
6. Let $R = \mathbb{Z}[x, y]$, and $M = R/(x - y), N = R/(x, y)$. Construct free resolutions of M, N to compute:
    - $\mathrm{Ext}^*_R(M, M)$
    - $\mathrm{Ext}^*_R(M, N)$
    - $\mathrm{Ext}^*_R(N, M)$
    - $\mathrm{Ext}^*_R(N, N)$
7. Let Λ_* be the exterior algebra generated by the symbols $\{dx_i\}^n$ over a field k. Show that letting $d = \cdot \vee dx_1$ yields a chain complex $0 \longrightarrow \Lambda^0 \longrightarrow \Lambda^1 \longrightarrow \cdots \longrightarrow \Lambda^n \longrightarrow 0$ with trivial homology. Compute what happens when dx_1 is replaced with an arbitrary non-zero element in Λ^1.
8. Define M as the group ring $R = \mathbb{Z}[\mathbb{Z}_2]$ with the action $(\cdot) \times -1$. Construct a free resolution of M and compute $\mathrm{Tor}^*_R(M, M)$.
9. Show $\mathrm{Tor}^*_R(\cdot, \cdot)$ is symmetric in the following way: Given M, N, take free resolutions, view $M_* \longrightarrow M$ as a chain map and tensor with N_* to get a chain map$\psi : M_* \otimes_R N_* \longrightarrow M \otimes_R N_*$. Show that ψ is a quasi-isomorphism using the exact sequence $0 \longrightarrow (Z_n, 0) \longrightarrow (N_n, 0) \longrightarrow (B_{n-1}, 0) \longrightarrow 0$, then switch the roles of M, N.
10. Prove that for a SES $0 \longrightarrow A \longrightarrow B \longrightarrow C$, the group $\mathrm{Ext}(C, A)$ classifies extensions of C by A up to isomorphism.

1.11 Cohomology Ring (Sheet 12)

Todo