MAKEMEAQUAL UNIVERSITY

Department of Mathematics

PhD Qualifying Examination

in

Mathematics

March 11, 2021

1 Question 1 (UGA 2014 #5)

Let $f, g \in L^1([0,1])$ and for all $x \in [0,1]$ define

$$F(x) := \int_0^x f(y)dy \quad \text{and} \quad G(x) := \int_0^x g(y)dy.$$

Prove that

$$\int_0^1 F(x)g(x)dx = F(1)G(1) - \int_0^1 f(x)G(x)dx$$

2 Question 2 (NUS 1970 #1fg)

Prove or disprove each of the following statements.

(f) If $E \subset \mathbb{R}$ and
$\mu(E) = \inf\{\sum_{I_i \in S} |I_i| : S = \{I_i\}_{i=1}^n$ such that $E \subset \bigcup_{i=1}^n I_i$ for some $n \in \mathbb{N}\}$
then μ coincides with the outer measure of E.

(g) If E is a Borel set and f is a measurable function, then $f^{-1}(E)$ is also measurable.

3 Question 3 (Emory 0 #0)

State and prove Fatou's Lemma on a general measurable space.

4 Question 4 (UGA 2016 #4)

Let $E \subset \mathbb{R}$ be measurable with $m(E) < \infty$. Define

$$f(x) = m(E \cap (E + x)).$$

Show that

1. $f \in L^1(\mathbb{R})$.
2. f is uniformly continuous.
3. $\lim_{|x| \to \infty} f(x) = 0$

Hint:

$$\chi_{E \cap (E+x)}(y) = \chi_E(y)\chi_E(y - x)$$

5 Question 5 (NUS 1970 #1)

If $\limsup_{n\to\infty} a_n \le l$, show that $\limsup_{n\to\infty} \sum_{i=1}^{n} a_i/n \le l$.

6 Question 6 (UGA 2015 #4)

Let $f : [1, \infty) \longrightarrow \mathbb{R}$ such that $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Show that the following limit exists and satisfies the equality

$$\lim_{x \to \infty} f(x) \leq 1 + \frac{\pi}{4}$$

7 Question 7 (UGA 2018 #5)

Let $f \geq 0$ be a measurable function on $\mathbb{R}$. Show that

$$\int_{\mathbb{R}} f = \int_0^\infty m(\{x : f(x) > t\})dt$$

8 Question 8 (UGA 2018 #2)

Let

$$f_n(x) := \frac{x}{1+x^n}, \quad x \geq 0.$$

a. Show that this sequence converges pointwise and find its limit. Is the convergence uniform on $[0, \infty)$?

b. Compute

$$\lim_{n\to\infty} \int_0^\infty f_n(x)dx$$

9 Question 9 (UGA 2015 #4)

Define

$$f(x,y) := \begin{cases} \frac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Carefully show that $f \in L^1(\mathbb{R}^2)$.

10 Question 10 (UGA 2018 #3)

Suppose $f(x)$ and $xf(x)$ are integrable on $\mathbb{R}$. Define F by

$$F(t) := \int_{-\infty}^{\infty} f(x)\cos(xt)dx$$

Show that

$$F'(t) = -\int_{-\infty}^{\infty} xf(x)\sin(xt)dx.$$