MAKEMEAQUAL UNIVERSITY

Department of Mathematics

PhD Qualifying Examination

in

Mathematics

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1 Algebra (140 Questions)

1.1 Question 1

Let G be a finite group with n distinct conjugacy classes. Let $g_1 \cdots g_n$ be representatives of the conjugacy classes of G.
Prove that if $g_i g_j = g_j g_i$ for all i, j then G is abelian.

1.2 Question 2

Let G be a group of order 105 and let P, Q, R be Sylow 3, 5, 7 subgroups respectively.

(a) Prove that at least one of Q and R is normal in G.

(b) Prove that G has a cyclic subgroup of order 35.

(c) Prove that both Q and R are normal in G.

(d) Prove that if P is normal in G then G is cyclic.

1.3 Question 3

Let R be a ring with the property that for every $a \in R, a^2 = a$.

(a) Prove that R has characteristic 2.

(b) Prove that R is commutative.

1.4 Question 4

Let F be a finite field with q elements.
Let n be a positive integer relatively prime to q and let ω be a primitive nth root of unity in an extension field of F.
Let $E = F[\omega]$ and let $k = [E : F]$.

(a) Prove that n divides $q^k - 1$.

(b) Let m be the order of q in $\mathbb{Z}/n\mathbb{Z}$. Prove that m divides k.

(c) Prove that $m = k$.

1.5 Question 5

Let R be a ring and M an R-module.

Recall that the set of torsion elements in M is defined by

$$\operatorname{Tor}(m) = \{m \in M \mid \exists r \in R,\ r \neq 0,\ rm = 0\}.$$

(a) Prove that if R is an integral domain, then $\mathrm{Tor}(M)$ is a submodule of M .

(b) Give an example where $\mathrm{Tor}(M)$ is not a submodule of M.

(c) If R has zero-divisors, prove that every non-zero R-module has non-zero torsion elements.

1.6 Question 6

Let R be a commutative ring with multiplicative identity. Assume Zorn's Lemma.

(a) Show that

$$N = \{r \in R \,\middle|\, r^n = 0 \text{ for some } n > 0\}$$

is an ideal which is contained in any prime ideal.

(b) Let r be an element of R not in N. Let S be the collection of all proper ideals of R not containing any positive power of r. Use Zorn's Lemma to prove that there is a prime ideal in S.

(c) Suppose that R has exactly one prime ideal P . Prove that every element r of R is either nilpotent or a unit.

1.7 Question 7

Let ζ_n denote a primitive nth root of $1 \in \mathbb{Q}$. You may assume the roots of the minimal polynomial $p_n(x)$ of ζ_n are exactly the primitive nth roots of 1.
Show that the field extension $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$ is Galois and prove its Galois group is $(\mathbb{Z}/n\mathbb{Z})^\times$.
How many subfields are there of $\mathbb{Q}(\zeta_{20})$?

1.8 Question 8

Let $\{e_1, \cdots, e_n\}$ be a basis of a real vector space V and let

$$\Lambda := \left\{\sum r_i e_i \,\middle|\, ri \in \mathbb{Z}\right\}$$

Let $\cdot$ be a non-degenerate ($v \cdot w = 0$ for all $w \in V \iff v = 0$) symmetric bilinear form on V such that the Gram matrix $M = (e_i \cdot e_j)$ has integer entries.
Define the dual of Λ to be

$$\Lambda^\vee := \{v \in V \,\middle|\, v \cdot x \in \mathbb{Z} \text{ for all } x \in \Lambda\}.$$

(a) Show that $\Lambda \subset \Lambda^\vee$.

(b) Prove that $\det M \neq 0$ and that the rows of M^{-1} span $\Lambda^\vee$.

(c) Prove that $\det M = |\Lambda^\vee/\Lambda|$.

1.9 Question 9

Let A be a square matrix over the complex numbers. Suppose that A is nonsingular and that A^{2019} is diagonalizable over $\mathbb{C}$.
Show that A is also diagonalizable over $\mathbb{C}$.

1.10 Question 10

Let $F = \mathbb{F}_p$, where p is a prime number.

(a) Show that if $\pi(x) \in F[x]$ is irreducible of degree d, then $\pi(x)$ divides $x^{p^d} - x$.

(b) Show that if $\pi(x) \in F[x]$ is an irreducible polynomial that divides $x^{p^n} - x$, then $\deg \pi(x)$ divides n.

1.11 Question 11

How many isomorphism classes are there of groups of order 45?
Describe a representative from each class.

1.12 Question 12

For a finite group G, let $c(G)$ denote the number of conjugacy classes of G.

(a) Prove that if two elements of G are chosen uniformly at random,then the probability they commute is precisely

$$\frac{c(G)}{|G|}.$$

(b) State the class equation for a finite group.

(c) Using the class equation (or otherwise) show that the probability in part (a) is at most

$$\frac{1}{2} + \frac{1}{2[G : Z(G)]}.$$

Here, as usual, $Z(G)$ denotes the center of G.

1.13 Question 13

Let R be an integral domain. Recall that if M is an R-module, the *rank* of M is defined to be the maximum number of R-linearly independent elements of M .

(a) Prove that for any R-module M, the rank of $\operatorname{Tor}(M)$ is 0.

(b) Prove that the rank of M is equal to the rank of of $M/\operatorname{Tor}(M)$.

(c) Suppose that M is a non-principal ideal of R.

(d) Prove that M is torsion-free of rank 1 but not free.

1.14 Question 14

Let R be a commutative ring with 1.

> Recall that $x \in R$ is nilpotent iff $xn = 0$ for some positive integer n.

(a) Show that every proper ideal of R is contained within a maximal ideal.

(b) Let $J(R)$ denote the intersection of all maximal ideals of R.
Show that $x \in J(R) \iff 1 + rx$ is a unit for all $r \in R$.

(c) Suppose now that R is finite. Show that in this case $J(R)$ consists precisely of the nilpotent elements in R.

1.15 Question 15

Let p be a prime number. Let A be a $p \times p$ matrix over a field F with 1 in all entries except 0 on the main diagonal.
Determine the Jordan canonical form (JCF) of A

(a) When $F = \mathbb{Q}$,

(b) When $F = \mathbb{F}_p$.

> Hint: In both cases, all eigenvalues lie in the ground field. In each case find a matrix P such that $P^{-1}AP$ is in JCF.

1.16 Question 16

Let $\zeta = e^{2\pi i/8}$.

(a) What is the degree of $\mathbb{Q}(\zeta)/\mathbb{Q}$?

(b) How many quadratic subfields of $\mathbb{Q}(\zeta)$ are there?

(c) What is the degree of $\mathbb{Q}(\zeta, \sqrt[4]{2})$ over $\mathbb{Q}$?

1.17 Question 17

Let G be a finite group whose order is divisible by a prime number p. Let P be a normal p-subgroup of G (so $|P| = p^c$ for some c).

(a) Show that P is contained in every Sylow p-subgroup of G.

(b) Let M be a maximal proper subgroup of G. Show that either $P \subseteq M$ or $|G/M| = p^b$ for some $b \leq c$.

1.18 Question 18

(a) Suppose the group G acts on the set X . Show that the stabilizers of elements in the same orbit are conjugate.

(b) Let G be a finite group and let H be a proper subgroup. Show that the union of the conjugates of H is strictly smaller than G, i.e.

$$\cup_{g \in G} gHg^{-1} \subsetneq G$$

(c) Suppose G is a finite group acting transitively on a set S with at least 2 elements. Show that there is an element of G with no fixed points in S.

1.19 Question 19

Let $F \subset K \subset L$ be finite degree field extensions. For each of the following assertions, give a proof or a counterexample.

(a) If L/F is Galois, then so is K/F.

(b) If L/F is Galois, then so is L/K.

(c) If K/F and L/K are both Galois, then so is L/F.

1.20 Question 20

Let V be a finite dimensional vector space over a field (the field is not necessarily algebraically closed).
Let $\phi : V \to V$ be a linear transformation. Prove that there exists a decomposition of V as $V = U \oplus W$, where U and W are ϕ-invariant subspaces of V , $\phi|_U$ is nilpotent, and $\phi|_W$ is nonsingular.

1.21 Question 21

Let A be an $n \times n$ matrix.

(a) Suppose that v is a column vector such that the set $\{v, Av, ..., A^{n-1}v\}$ is linearly independent. Show that any matrix B that commutes with A is a polynomial in A.

(b) Show that there exists a column vector v such that the set $\{v, Av, ..., A^{n-1}v\}$ is linearly independent $\iff$ the characteristic polynomial of A equals the minimal polynomial of A.

1.22 Question 22

Let R be a commutative ring, and let M be an R-module. An R-submodule N of M is maximal if there is no R-module P with $N \subsetneq P \subsetneq M$.

(a) Show that an R-submodule N of M is maximal $\iff$ M/N is a simple R-module: i.e., M/N is nonzero and has no proper, nonzero R-submodules.

(b) Let M be a $\mathbb{Z}$-module. Show that a $\mathbb{Z}$-submodule N of M is maximal $\iff$ $\#M/N$ is a prime number.

(c) Let M be the $\mathbb{Z}$-module of all roots of unity in $\mathbb{C}$ under multiplication. Show that there is no maximal $\mathbb{Z}$-submodule of M.

1.23 Question 23

Let R be a commutative ring.

(a) Let $r \in R$. Show that the map

$$\begin{aligned} r\bullet : R &\to R \\ x &\mapsto rx. \end{aligned}$$

is an R-module endomorphism of R.

(b) We say that r is a **zero-divisor** if r$\bullet$ is not injective. Show that if r is a zero-divisor and $r \neq 0$, then the kernel and image of R each consist of zero-divisors.

(c) Let $n \geq 2$ be an integer. Show: if R has exactly n zero-divisors, then $\#R \leq n^2$.

(d) Show that up to isomorphism there are exactly two commutative rings R with precisely 2 zero-divisors.

You may use without proof the following fact: every ring of order 4 is isomorphic to exactly one of the following:

$$\frac{\mathbb{Z}}{4\mathbb{Z}}, \quad \frac{\frac{\mathbb{Z}}{2\mathbb{Z}}[t]}{(t^2+t+1)}, \quad \frac{\frac{\mathbb{Z}}{2\mathbb{Z}}[t]}{(t^2-t)}, \quad \frac{\frac{\mathbb{Z}}{2\mathbb{Z}}[t]}{(t^2)}.$$

1.24 Question 24

(a) Use the Class Equation (equivalently, the conjugation action of a group on itself) to prove that any p-group (a group whose order is a positive power of a prime integer p) has a nontrivial center.

(b) Prove that any group of order p^2 (where p is prime) is abelian.

(c) Prove that any group of order $5^2 \cdot 7^2$ is abelian.

(d) Write down exactly one representative in each isomorphism class of groups of order $5^2 \cdot 7^2$.

1.25 Question 25

Let $f(x) = x^4 - 4x^2 + 2 \in \mathbb{Q}[x]$.

(a) Find the splitting field K of f, and compute $[K : \mathbb{Q}]$.

(b) Find the Galois group G of f, both as an explicit group of automorphisms, and as a familiar abstract group to which it is isomorphic.

(c) Exhibit explicitly the correspondence between subgroups of G and intermediate fields between $\mathbb{Q}$ and k.

1.26 Question 26

Let K be a Galois extension of $\mathbb{Q}$ with Galois group G, and let E_1, E_2 be intermediate fields of K which are the splitting fields of irreducible $f_i(x) \in \mathbb{Q}[x]$.
Let $E = E_1E_2 \subset K$.
Let $H_i = \mathrm{Gal}(K/E_i)$ and $H = \mathrm{Gal}(K/E)$.

(a) Show that $H = H_1 \cap H_2$.

(b) Show that H_1H_2 is a subgroup of G.

(c) Show that

$$\mathrm{Gal}(K/(E_1 \cap E_2)) = H_1H_2.$$

1.27 Question 27

Let

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 1 & -3 \\ 1 & 2 & -4 \end{bmatrix} \in M_3(\mathbb{C})$$

(a) Find the Jordan canonical form J of A.

(b) Find an invertible matrix P such that $P^{-1}AP = J$.

You should not need to compute P^{-1}.

1.28 Question 28

Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} x & u \\ -y & -v \end{pmatrix}$$

over a commutative ring R, where b and x are units of R. Prove that

$$MN = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \implies MN = 0.$$

1.29 Question 29

Let

$$M = \{(w,x,y,z) \in \mathbb{Z}^4 \;\Big|\; w+x+y+z \in 2\mathbb{Z}\},$$

and

$$N = \{(w,x,y,z) \in \mathbb{Z}^4 \;\Big|\; 4 \;\Big|\; (w-x),\ 4 \;\Big|\; (x-y),\ 4 \;\Big|\; (y-z)\}.$$

(a) Show that N is a $\mathbb{Z}$-submodule of M .

(b) Find vectors $u_1, u_2, u_3, u_4 \in \mathbb{Z}^4$ and integers d_1, d_2, d_3, d_4 such that

$$\{u_1, u_2, u_3, u_4\}$$

is a free basis for M, and

$$\{d_1u_1,\ d_2u_2,\ d_3u_3,\ d_4u_4\}$$

is a free basis for N .

(c) Use the previous part to describe M/N as a direct sum of cyclic $\mathbb{Z}$-modules.

1.30 Question 30

Let R be a PID and M be an R-module. Let p be a prime element of R. The module M is called $\langle p\rangle$ *-primary* if for every $m \in M$ there exists $k > 0$ such that $p^k m = 0$.

(a) Suppose M is $\langle p\rangle$ -primary. Show that if $m \in M$ and $t \in R,\ t \notin \langle p\rangle$, then there exists $a \in R$ such that $atm = m$.

(b) A submodule S of M is said to be *pure* if $S \cap rM = rS$ for all $r \in R$. Show that if M is $\langle p\rangle$ -primary, then S is pure if and only if $S \cap p^k M = p^k S$ for all $k \geq 0$.

1.31 Question 31

Let $R = C[0,1]$ be the ring of continuous real-valued functions on the interval $[0,1]$. Let I be an ideal of R.

(a) Show that if $f \in I, a \in [0,1]$ are such that $f(a) \neq 0$, then there exists $g \in I$ such that $g(x) \geq 0$ for all $x \in [0,1]$, and $g(x) > 0$ for all x in some open neighborhood of a.

(b) If $I \neq R$, show that the set $Z(I) = \{x \in [0,1] \;\Big|\; f(x) = 0 \text{ for all } f \in I\}$ is nonempty.

(c) Show that if I is maximal, then there exists $x_0 \in [0,1]$ such that $I = \{f \in R \;\Big|\; f(x_0) = 0\}$.

1.32 Question 32

Suppose the group G acts on the set A. Assume this action is faithful (recall that this means that the kernel of the homomorphism from G to $\mathrm{Sym}(A)$ which gives the action is trivial) and transitive (for all a, b in A, there exists g in G such that $g \cdot a = b$.)

(a) For $a \in A$, let G_a denote the stabilizer of a in G. Prove that for any $a \in A$,

$$\cap_{\sigma \in G} \sigma G_a \sigma^{-1} = \{1\} .$$

(b) Suppose that G is abelian. Prove that $|G| = |A|$. Deduce that every abelian transitive subgroup of S_n has order n.

1.33 Question 33

(a) Classify the abelian groups of order 36.

For the rest of the problem, assume that G is a non-abelian group of order 36.

You may assume that the only subgroup of order 12 in S_4 is A_4 and that A_4 has no subgroup of order 6.

(b) Prove that if the 2-Sylow subgroup of G is normal, G has a normal subgroup N such that G/N is isomorphic to A_4.

(c) Show that if G has a normal subgroup N such that G/N is isomorphic to A_4 and a subgroup H isomorphic to A_4 it must be the direct product of N and H.

(d) Show that the dihedral group of order 36 is a non-abelian group of order 36 whose Sylow-2 subgroup is not normal.

1.34 Question 34

Let F be a field. Let $f(x)$ be an irreducible polynomial in $F[x]$ of degree n and let $g(x)$ be any polynomial in $F[x]$. Let $p(x)$ be an irreducible factor (of degree m) of the polynomial $f(g(x))$.

Prove that n divides m. Use this to prove that if r is an integer which is not a perfect square, and n is a positive integer then every irreducible factor of $x^{2n} - r$ over $\mathbb{Q}[x]$ has even degree.

1.35 Question 35

(a) Let $f(x)$ be an irreducible polynomial of degree 4 in $\mathbb{Q}[x]$ whose splitting field K over $\mathbb{Q}$ has Galois group $G = S_4$.

Let θ be a root of $f(x)$. Prove that $\mathbb{Q}[\theta]$ is an extension of $\mathbb{Q}$ of degree 4 and that there are no intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}[\theta]$.

(b) Prove that if K is a Galois extension of $\mathbb{Q}$ of degree 4, then there is an intermediate subfield between K and $\mathbb{Q}$.

1.36 Question 36

A ring R is called *simple* if its only two-sided ideals are 0 and R.

(a) Suppose R is a commutative ring with 1. Prove R is simple if and only if R is a field.

(b) Let k be a field. Show the ring $M_n(k)$, $n \times n$ matrices with entries in k, is a simple ring.

1.37 Question 37

For a ring R, let $U(R)$ denote the multiplicative group of units in R. Recall that in an integral domain R, $r \in R$ is called *irreducible* if r is not a unit in R, and the only divisors of r have the form ru with u a unit in R.
We call a non-zero, non-unit $r \in R$ *prime* in R if $r \Bigm| ab \implies r \Bigm| a$ or $r \Bigm| b$. Consider the ring $R = \{a + b\sqrt{-5} \Bigm| a, b \in Z\}$.

(a) Prove R is an integral domain.

(b) Show $U(R) = \{\pm 1\}$.

(c) Show $3, 2 + \sqrt{-5}$, and $2 - \sqrt{-5}$ are irreducible in R.

(d) Show 3 is not prime in R.

(e) Conclude R is not a PID.

1.38 Question 38

Let F be a field and let V and W be vector spaces over F .
Make V and W into $F[x]$-modules via linear operators T on V and S on W by defining $X \cdot v = T(v)$ for all $v \in V$ and $X \cdot w = S(w)$ for all w ∈ W .
Denote the resulting $F[x]$-modules by V_T and W_S respectively.

(a) Show that an $F[x]$-module homomorphism from V_T to W_S consists of an F-linear transformation $R : V \to W$ such that $RT = SR$.

(b) Show that $VT \cong WS$ as $F[x]$-modules $\iff$ there is an F-linear isomorphism $P : V \to W$ such that $T = P^{-1}SP$.

(c) Recall that a module M is *simple* if $M \neq 0$ and any proper submodule of M must be zero. Suppose that V has dimension 2. Give an example of F, T with V_T simple.

(d) Assume F is algebraically closed. Prove that if V has dimension 2, then any V_T is not simple.

1.39 Question 39

Classify the groups of order $182 = 2 \cdot 7 \cdot 13$.

1.40 Question 40

Let G be a finite group of order $p^n m$ where p is a prime and m is not divisible by p. Prove that if H is a subgroup of G of order p^k for some $k < n$, then the normalizer of H in G properly contains H.

1.41 Question 41

Let H be a subgroup of S_n of index n. Prove:

1. There is an isomorphism $f : S_n \to S_n$ such that $f(H)$ is the subgroup of S_n stabilizing n. In particular, H is isomorphic to S_{n-1}.
2. The only subgroups of S_n containing H are S_n and H.

1.42 Question 42

- Prove that a group of order $351 = 3^3 \cdot 13$ cannot be simple.
- Prove that a group of order 33 must be cyclic.

1.43 Question 43

1. Let G be a group, and $Z(G)$ the center of G. Prove that if $G/Z(G)$ is cyclic, then G is abelian.
2. Prove that a group of order p^n, where p is a prime and $n \geq 1$, has non-trivial center.
3. Prove that a group of order p^2 must be abelian.

1.44 Question 44

Let G be a finite group.

1. Prove that if $H < G$ is a proper subgroup, then G is not the union of conjugates of H.
2. Suppose that G acts transitively on a set X with $|X| > 1$. Prove that there exists an element of G with no fixed points in X.

1.45 Question 45

Classify all groups of order 15 and of order 30.

1.46 Question 46

Count the number of p-Sylow subgroups of S_p.

1.47 Question 47

1. Let G be a group of order n. Suppose that for every divisor d of n, G contains at most one subgroup of order d. Show that G is clyclic.

2. Let F be a field. Show that every finite subgroup of the group of units $F^\times$ is cyclic.

1.48 Question 48

Let K and L be finite fields. Show that K is contained in L if and only if $\#K = p^r$ and $\#L = p^s$ for the same prime p, and $r \leq s$.

1.49 Question 49

Let K and L be finite fields with $K \subseteq L$. Prove that L is Galois over K and that $\mathrm{Gal}(L/K)$ is cyclic.

1.50 Question 50

Fix a field F, a separable polynomial $f \in F[x]$ of degree $n \geq 3$, and a splitting field L for f. Prove that if $[L : F] = n!$ then:

1. f is irreducible.

2. For each root r of f, r is the unique root of f in $F(r)$.

3. For every root r of f, there are no proper intermediate fields $F \subset L \subset F(r)$.

1.51 Question 51

1. Show that $\sqrt{2+\sqrt{2}}$ is a root of $p(x) = x^2 - 4x^2 + 2 \in \mathbb{Q}[x]$.

2. Prove that $\mathbb{Q}(\sqrt{2+\sqrt{2}})$ is a Galois extension of $\mathbb{Q}$ and find its Galois group. (Hint: note that $\sqrt{2-\sqrt{2}}$ is another root of $p(x)$).

3. Let $f(x) = x^3 - 5$. Determine the splitting field K of $f(x)$ over $\mathbb{Q}$ and the Galois group of $f(x)$. Give an example of a proper sub-extension $\mathbb{Q} \subset L \subset K$, such that $L/\mathbb{Q}$ is Galois.

1.52 Question 52

An integral domain R is said to be an *Euclidean domain* if there is a function $N : R \to \{n \in \mathbb{Z} \mid n \geq 0\}$ such that $N(0) = 0$ and for each $a, b \in R$ with $b \neq 0$, there exist elements $q, r \in R$ with

$$a = qb + r, \quad \text{and} \quad r = 0 \text{ or } N(r) < N(b).$$

Prove:

1. The ring $F[[x]]$ of power series over a field F is an Euclidean domain.
2. Every Euclidean domain is a PID.

1.53 Question 53

Let F be a field, and let R be the subring of $F[X]$ of polynomials with X coefficient equal to 0. Prove that R is not a UFD.

1.54 Question 54

R is a commutative ring with 1. Prove that if I is a maximal ideal in R, then R/I is a field. Prove that if R is a PID, then every nonzero prime ideal in R is maximal. Conclude that if R is a PID and $p \in R$ is prime, then $R/(p)$ is a field.

1.55 Question 55

Prove that any square matrix is conjugate to its transpose matrix. (You may prove it over $\mathbb{C}$).

1.56 Question 56

Determine the number of conjugacy classes of 16×16 matrices with entries in $\mathbb{Q}$ and minimal polynomial $(x^2+1)^2(x^3+2)^2$.

1.57 Question 57

Let V be a vector space over a field F. The evaluation map $e\colon V \to (V^\vee)^\vee$ is defined by $e(v)(f) := f(v)$ for $v \in V$ and $f \in V^\vee$.

1. Prove that e is an injection.
2. Prove that e is an isomorphism if and only if V is finite dimensional.

1.58 Question 58

Let R be a principal ideal domain that is not a field, and write F for its field of fractions. Prove that F is not a finitely generated R-module.

1.59 Question 59

Carefully state Zorn's lemma and use it to prove that every vector space has a basis.

1.60 Question 60

Show that no finite group is the union of conjugates of a proper subgroup.

1.61 Question 61

Classify all groups of order 18 up to isomorphism.

1.62 Question 62

Let α, β denote the unique positive real 5^{th} root of 7 and 4^{th} root of 5, respectively. Determine the degree of $\mathbb{Q}(\alpha, \beta)$ over $\mathbb{Q}$.

1.63 Question 63

Show that the field extension $\mathbb{Q} \subseteq \mathbb{Q}\left(\sqrt{2+\sqrt{2}}\right)$ is Galois and determine its Galois group.

1.64 Question 64

Let M be a square matrix over a field K. Use a suitable canonical form to show that M is similar to its transpose M^T.

1.65 Question 65

Let G be a finite group and π_0, π_1 be two irreducible representations of G. Prove or disprove the following assertion: π_0 and π_1 are equivalent if and only if $\det \pi_0(g) = \det \pi_1(g)$ for all $g \in G$.

1.66 Question 66

Let R be a Noetherian ring. Prove that $R[x]$ and $R[[x]]$ are both Noetherian. (The first part of the question is asking you to prove the Hilbert Basis Theorem, not to use it!)

1.67 Question 67

Classify (with proof) all fields with finitely many elements.

1.68 Question 68

Suppose A is a commutative ring and M is a finitely presented module. Given any surjection $\phi : A^n \to M$ from a finite free A-module, show that $\ker \phi$ is finitely generated.

1.69 Question 69

Classify all groups of order 57.

1.70 Question 70

Show that a finite simple group cannot have a 2-dimensional irreducible representation over $\mathbb{C}$.

> Hint: the determinant might prove useful.

1.71 Question 71

Let G be a finite simple group. Assume that every proper subgroup of G is abelian. Prove that then G is cyclic of prime order.

1.72 Question 72

Let $a \in \mathbb{N}$, $a > 0$. Compute the Galois group of the splitting field of the polynomial $x^5 - 5a^4x + a$ over $\mathbb{Q}$.

1.73 Question 73

Recall that an inner automorphism of a group is an automorphism given by conjugation by an element of the group. An outer automorphism is an automorphism that is not inner.

- Prove that S_5 has a subgroup of order 20.
- Use the subgroup from (a) to construct a degree 6 permutation representation of S_5 (i.e., an embedding $S_5 \hookrightarrow S_6$ as a transitive permutation group on 6 letters).
- Conclude that S_6 has an outer automorphism.

1.74 Question 74

Let A be a commutative ring and M a finitely generated A-module. Define

$$\text{Ann}(M) = \{a \in A : am = 0 \text{ for all } m \in M\}.$$

Show that for a prime ideal $\mathfrak{p} \subset A$, the following are equivalent:

- $\text{Ann}(M) \not\subset \mathfrak{p}$
- The localization of M at the prime ideal $\mathfrak{p}$ is 0.
- $M \otimes_A k(\mathfrak{p}) = 0$, where $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ is the residue field of A at $\mathfrak{p}$.

1.75 Question 75

Let $A = \mathbb{C}[x, y]/(y^2 - (x-1)^3 - (x-1)^2)$.

- Show that A is an integral domain and sketch the $\mathbb{R}$-points of Spec$A$.
- Find the integral closure of A. Recall that for an integral domain A with fraction field K, the integral closure of A in K is the set of all elements of K integral over A.

1.76 Question 76

Let $R = k[x, y]$ where k is a field, and let $I = (x, y)R$.

- Show that

$$0 \to R \xrightarrow{\phi} R \oplus R \xrightarrow{\psi} R \to k \to 0$$

  where $\phi(a) = (-ya, xa)$, $\psi((a, b)) = xa + yb$ for $a, b \in R$, is a projective resolution of the R-module $k \simeq R/I$.

- Show that I is not a flat R-module by computing $\mathrm{Tor}_i^R(I, k)$

1.77 Question 77

- Find an irreducible polynomial of degree 5 over the field $\mathbb{Z}/2$ of two elements and use it to construct a field of order 32 as a quotient of the polynomial ring $\mathbb{Z}/2[x]$.
- Using the polynomial found in part (a), find a 5×5 matrix M over $\mathbb{Z}/2$ of order 31, so that $M^{31} = I$ but $M \neq I$.

1.78 Question 78

Find the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$. Justify your answer.

1.79 Question 79

- Let R be a commutative ring with no nonzero nilpotent elements. Show that the only units in the polynomial ring $R[x]$ are the units of R, regarded as constant polynomials.
- Find all units in the polynomial ring $\mathbb{Z}_4[x]$.

1.80 Question 80

Let p, q be two distinct primes. Prove that there is at most one non-abelian group of order pq and describe the pairs (p, q) such that there is no non-abelian group of order pq.

1.81 Question 81

- Let L be a Galois extension of a field K of degree 4. What is the minimum number of subfields there could be strictly between K and L? What is the maximum number of such subfields? Give examples where these bounds are attained.
- How do these numbers change if we assume only that L is separable (but not necessarily Galois) over K?

1.82 Question 82

Let R be a commutative algebra over $\mathbb{C}$. A derivation of R is a $\mathbb{C}$-linear map $D : R \to R$ such that (i) $D(1) = 0$ and (ii) $D(ab) = D(a)b + aD(b)$ for all $a, b \in R$.

- Describe all derivations of the polynomial ring $\mathbb{C}[x]$.
- Let A be the subring (or $\mathbb{C}$-subalgebra) of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[x])$ generated by all derivations of $\mathbb{C}[x]$ and the left multiplications by x. Prove that $\mathbb{C}[x]$ is a simple left A-module. > Note that the inclusion $A \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[x])$ defines a natural left A-module structure on $\mathbb{C}[x]$.

1.83 Question 83

Let G be a non-abelian group of order p^3 with p a prime.

- Determine the order of the center Z of G.
- Determine the number of inequivalent complex 1-dimensional representations of G.
- Compute the dimensions of all the inequivalent irreducible representations of G and verify that the number of such representations equals the number of conjugacy classes of G.

1.84 Question 84

- Let G be a group (not necessarily finite) that contains a subgroup of index n. Show that G contains a *normal* subgroup N such that $n \leq [G : N] \leq n!$
- Use part (a) to show that there is no simple group of order 36.

1.85 Question 85

Let p be a prime, let $\mathbb{F}_p$ be the p-element field, and let $K = \mathbb{F}_p(t)$ be the field of rational functions in t with coefficients in $\mathbb{F}_p$. Consider the polynomial $f(x) = x^p - t \in K[x]$.

- Show that f does not have a root in K.
- Let E be the splitting field of f over K. Find the factorization of f over E.
- Conclude that f is irreducible over K.

1.86 Question 86

Recall that a ring A is called *graded* if it admits a direct sum decomposition $A = \oplus_{n=0}^{\infty} A_n$ as abelian groups, with the property that $A_i A_j \subseteq A_{i+j}$ for all $i, j \geq 0$. Prove that a graded commutative ring $A = \oplus_{n=0}^{\infty} A_n$ is Noetherian if and only if A_0 is Noetherian and A is finitely generated as an algebra over A_0.

1.87 Question 87

Let R be a ring with the property that $a^2 = a$ for all $a \in R$.

- Compute the Jacobson radical of R.
- What is the characteristic of R?
- Prove that R is commutative.
- Prove that if R is finite, then R is isomorphic (as a ring) to $(\mathbb{Z}/2\mathbb{Z})^d$ for some d.

1.88 Question 88

Let $\overline{\mathbb{F}_p}$ denote the algebraic closure of $\mathbb{F}_p$. Show that the Galois group $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ has no non-trivial finite subgroups.

1.89 Question 89

Let C_p denote the cyclic group of order p.

- Show that C_p has two irreducible representations over $\mathbb{Q}$ (up to isomorphism), one of dimension 1 and one of dimension $p - 1$.
- Let G be a finite group, and let $\rho : G \to \mathrm{GL}_n(\mathbb{Q})$ be a representation of G over $\mathbb{Q}$. Let $\rho_{\mathbb{C}} : G \to \mathrm{GL}_n(\mathbb{C})$ denote ρ followed by the inclusion $\mathrm{GL}_n(\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{C})$. Thus $\rho_{\mathbb{C}}$ is a representation of G over $\mathbb{C}$, called the *complexification* of ρ. We say that an irreducible representation ρ of G is *absolutely irreducible* if its complexification remains irreducible over $\mathbb{C}$.\ Now suppose G is abelian and that every representation of G over $\mathbb{Q}$ is absolutely irreducible. Show that $G \cong (C_2)^k$ for some k (i.e., is a product of cyclic groups of order 2).

1.90 Question 90

Let G be a finite group and $\mathbb{Z}[G]$ the internal group algebra. Let $\mathcal{Z}$ be the center of $\mathbb{Z}[G]$. For each conjugacy class $C \subseteq G$, let $P_C = \sum_{g \in C} g$.

- Show that the elements P_C form a $\mathbb{Z}$-basis for $\mathcal{Z}$. Hence $\mathcal{Z} \cong \mathbb{Z}^d$ as an abelian group, where d is the number of conjugacy classes in G.
- Show that if a ring R is isomorphic to $\mathbb{Z}^d$ as an abelian group, then every element in R satisfies a monic integral polynomial.

> **Hint:** Let $\{v_1, \ldots, v_d\}$ be a basis of R and for a fixed non-zero $r \in R$, write $rv_i = \sum_j a_{ij} v_j$. Use the Hamilton-Cayley theorem.

- Let $\pi : G \to \mathrm{GL}(V)$ be an irreducible representation of G (over $\mathbb{C}$). Show that $\pi(P_C)$ acts on V as multiplication by the scalar

$$\frac{|C|\chi_\pi(C)}{\dim V},$$

where $\chi_\pi(C)$ is the value of the character χ_π on any element of C.

- Conclude that $|C|\chi_\pi(C)/\dim V$ is an algebraic integer.

1.91 Question 91

- Suppose that G is a finitely generated group. Let n be a positive integer. Prove that G has only finitely many subgroups of index n
- Let p be a prime number. If G is any finitely-generated abelian group, let $t_p(G)$ denote the number of subgroups of G of index p. Determine the possible values of $t_p(G)$ as G varies over all finitely-generated abelian groups.

1.92 Question 92

Suppose that G is a finite group of order 2013. Prove that G has a normal subgroup N of index 3 and that N is a cyclic group. Furthermore, prove that the center of G has order divisible by 11. (You will need the factorization $2013 = 3 \cdot 11 \cdot 61$.)

1.93 Question 93

This question concerns an extension K of $\mathbb{Q}$ such that $[K : \mathbb{Q}] = 8$. Assume that $K/\mathbb{Q}$ is Galois and let $G = \mathrm{Gal}(K/\mathbb{Q})$. Furthermore, assume that G is non-abelian.

- Prove that K has a unique subfield F such that $F/\mathbb{Q}$ is Galois and $[F : \mathbb{Q}] = 4$.
- Prove that F has the form $F = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ where d_1, d_2 are non-zero integers.
- Suppose that G is the quaternionic group. Prove that d_1 and d_2 are positive integers.

1.94 Question 94

This question concerns the polynomial ring $R = \mathbb{Z}[x, y]$ and the ideal $I = (5, x^2 + 2)$ in R.

- Prove that I is a prime ideal of R and that R/I is a PID.
- Give an explicit example of a maximal ideal of R which contains I. (Give a set of generators for such an ideal.)
- Show that there are infinitely many distinct maximal ideals in R which contain I.

1.95 Question 95

Classify all groups of order 2012 up to isomorphism.

> Hint: 503 is prime.

1.96 Question 96

For any positive integer n, let G_n be the group generated by a and b subject to the following three relations:

$$a^2 = 1, \quad b^2 = 1, \quad \text{and} \quad (ab)^n = 1..$$

- Find the order of the group G_n

1.97 Question 97

Determine the Galois groups of the following polynomials over $\mathbb{Q}$.

- $f(x) = x^4 + 4x^2 + 1$
- $f(x) = x^4 + 4x^2 - 5$.

1.98 Question 98

Let R be a (commutative) principal ideal domain, let M and N be finitely generated free R-modules, and let $\varphi : M \to N$ be an R-module homomorphism.

- Let K be the kernel of φ. Prove that K is a direct summand of M.
- Let C be the image of φ. Show by example (specifying R, M, N, and φ) that C need not be a direct summand of N.

1.99 Question 99

In this problem, as you apply Sylow's Theorem, state precisely which portions you are using.

- Prove that there is no simple group of order 30.
- Suppose that G is a simple group of order 60. Determine the number of p-Sylow subgroups of G for each prime p dividing 60, then prove that G is isomorphic to the alternating group A_5.

  Note: in the second part, you needn't show that A_5 is simple. You need only show that if there is a simple group of order 60, then it must be isomorphic to A_5.

1.100 Question 100

Describe the Galois group and the intermediate fields of the cyclotomic extension $\mathbb{Q}(\zeta_{12})/\mathbb{Q}$.

1.101 Question 101

Let

$$R = \mathbb{Z}[x]/(x^2 + x + 1).$$

- Answer the following questions with suitable justification.
  - Is R a Noetherian ring?
  - Is R an Artinian ring?
- Prove that R is an integrally closed domain.

1.102 Question 102

Let R be a commutative ring. Recall that an element r of R is *nilpotent* if $r^n = 0$ for some positive integer n and that the *nilradical* of R is the set $N(R)$ of nilpotent elements.

- Prove that

  $$N(R) = \cap_{P \text{ prime}} P..$$

  Hint: given a non-nilpotent element r of R, you may wish to construct a prime ideal that does not contain r or its powers.

- Given a positive integer m, determine the nilradical of $\mathbb{Z}/(m)$.
- Determine the nilradical of $\mathbb{C}[x, y]/(y^2 - x^3)$.
- Let $p(x, y)$ be a polynomial in $\mathbb{C}[x, y]$ such that for any complex number a, $p(a, a^{3/2}) = 0$. Prove that $p(x, y)$ is divisible by $y^2 - x^3$.

1.103 Question 103

Given a finite group G, recall that its *regular representation* is the representation on the complex group algebra $\mathbb{C}[G]$ induced by left multiplication of G on itself and its *adjoint representation* is the representation on the complex group algebra $\mathbb{C}[G]$ induced by conjugation of G on itself.

- Let $G = \mathrm{GL}_2(\mathbb{F}_2)$. Describe the number and dimensions of the irreducible representations of G. Then describe the decomposition of its regular representation as a direct sum of irreducible representations.
- Let G be a group of order 12. Show that its adjoint representation is reducible; that is, there is an H-invariant subspace of $\mathbb{C}[H]$ besides 0 and $\mathbb{C}[H]$.

1.104 Question 104

Let R be a commutative integral domain. Show that the following are equivalent:

- R is a field;
- R is a semi-simple ring;
- Any R-module is projective.

1.105 Question 105

Let p be a positive prime number, $\mathbb{F}_p$ the field with p elements, and let $G = \mathrm{GL}_2(\mathbb{F}_p)$.

- Compute the order of G, $|G|$.
- Write down an explicit isomorphism from $\mathbb{Z}/p\mathbb{Z}$ to

$$U = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{F}_p \right\}.$$

- How many subgroups of order p does G have?

  Hint: compute gug^{-1} for $g \in G$ and $u \in U$; use this to find the size of the normalizer of U in G.

1.106 Question 106

- Give definitions of the following terms:

  (i) a finite length (left) module, (ii) a composition series for a module, and (iii) the length of a module,

- Let $l(M)$ denote the length of a module M. Prove that if

$$0 \to M_1 \to M_2 \to \cdots \to M_n \to 0.$$

  is an exact sequence of modules of finite length, then

$$\sum_{i=1}^{n} (-1)^k l(M_i) = 0..$$

1.107 Question 107

Let $\mathbb{F}$ be a field of characteristic p, and G a group of order p^n. Let $R = \mathbb{F}[G]$ be the group ring (group algebra) of G over $\mathbb{F}$, and let $u := \sum_{x \in G} x$ (so u is an element of R).

- Prove that u lies in the center of R.
- Verify that Ru is a 2-sided ideal of R.
- Show there exists a positive integer k such that $u^k = 0$. Conclude that for such a k, $(Ru)^k = 0$.
- Show that R is **not** a semi-simple ring.

> **Warning:** Please use the definition of a semi-simple ring: do **not** use the result that a finite length ring fails to be semisimple if and only if it has a non-zero nilpotent ideal.

1.108 Question 108

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$ (where $a_n \neq 0$) and let $R = \mathbb{Z}[x]/(f)$. Prove that R is a finitely generated module over $\mathbb{Z}$ if and only if $a_n = \pm 1$.

1.109 Question 109

Consider the ring

$$S = C[0,1] = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}.$$

with the usual operations of addition and multiplication of functions.

- What are the invertible elements of S?
- For $a \in [0,1]$, define $I_a = \{f \in S : f(a) = 0\}$. Show that I_a is a maximal ideal of S.
- Show that the elements of any proper ideal of S have a common zero, i.e., if I is a proper ideal of S, then there exists $a \in [0,1]$ such that $f(a) = 0$ for all $f \in I$. Conclude that every maximal ideal of S is of the form I_a for some $a \in [0,1]$.

> **Hint**: As $[0,1]$ is compact, every open cover of $[0,1]$ contains a finite sub-cover.

1.110 Question 110

Let F be a field of characteristic zero, and let K be an *algebraic* extension of F that possesses the following property: every polynomial $f \in F[x]$ has a root in K. Show that K is algebraically closed.\

> **Hint:** if $K(\theta)/K$ is algebraic, consider $F(\theta)/F$ and its normal closure; primitive elements might be of help.

1.111 Question 111

Let G be the unique non-abelian group of order 21.

- Describe all 1-dimensional complex representations of G.
- How many (non-isomorphic) irreducible complex representations does G have and what are their dimensions?
- Determine the character table of G.

1.112 Question 112

- Classify all groups of order $2009 = 7^2 \times 41$.
- Suppose that G is a group of order 2009. How many intermediate groups are there—that is, how many groups H are there with $1 \subsetneq H \subsetneq G$, where both inclusions are proper? (There may be several cases to consider.)

1.113 Question 113

Let K be a field. A discrete valuation on K is a function $\nu : K \setminus \{0\} \to \mathbb{Z}$ such that

- $\nu(ab) = \nu(a) + \nu(b)$
- ν is surjective
- $\nu(a + b) \geq \min\{(\nu(a), \nu(b)\}$ for $a, b \in K \setminus \{0\}$ with $a + b \neq 0$.

Let $R := \{x \in K \setminus \{0\} : \nu(x) \geq 0\} \cup \{0\}$. Then R is called the valuation ring of ν.
Prove the following:

- R is a subring of K containing the 1 in K.
- for all $x \in K \setminus \{0\}$, either x or x^{-1} is in R.
- x is a unit of R if and only if $\nu(x) = 0$.
- Let p be a prime number, $K = \mathbb{Q}$, and $\nu_p : \mathbb{Q} \setminus \{0\} \to \mathbb{Z}$ be the function defined by $\nu_p(\frac{a}{b}) = n$ where $\frac{a}{b} = p^n \frac{c}{d}$ and p does not divide c and d. Prove that the corresponding valuation ring R is the ring of all rational numbers whose denominators are relatively prime to p.

1.114 Question 114

Let F be a field of characteristic not equal to 2.

- Prove that any extension K of F of degree 2 is of the form $F(\sqrt{D})$ where $D \in F$ is not a square in F and, conversely, that each such extension has degree 2 over F.
- Let $D_1, D_2 \in F$ neither of which is a square in F. Prove that $[F(\sqrt{D_1}, \sqrt{D_2}) : F] = 4$ if $D_1 D_2$ is not a square in F and is of degree 2 otherwise.

1.115 Question 115

Let F be a field and $p(x) \in F[x]$ an irreducible polynomial.

- Prove that there exists a field extension K of F in which $p(x)$ has a root.
- Determine the dimension of K as a vector space over F and exhibit a vector space basis for K.
- If $\theta \in K$ denotes a root of $p(x)$, express θ^{-1} in terms of the basis found in part (b).
- Suppose $p(x) = x^3 + 9x + 6$. Show $p(x)$ is irreducible over $\mathbb{Q}$. If θ is a root of $p(x)$, compute the inverse of $(1 + \theta)$ in $\mathbb{Q}(\theta)$.

1.116 Question 116

Fix a ring R, an R-module M, and an R-module homomorphism $f : M \to M$.

- If M satisfies the descending chain condition on submodules, show that if f is injective, then f is surjective.

  Hint: note that if f is injective, so are $f \circ f$, $f \circ f \circ f$, etc.

- Give an example of a ring R, an R-module M, and an injective R-module homomorphism $f : M \to M$ which is not surjective.
- If M satisfies the ascending chain condition on submodules, show that if f is surjective, then f is injective.
- Give an exampe of a ring R, and R-module M, and a surjective R-module homomorphism $f : M \to M$ which is not injective.

1.117 Question 117

Let G be a finite group, k an algebraically closed field, and V an irreducible k-linear representation of G.

- Show that $\hom_{kG}(V, V)$ is a division algebra with k in its center.
- Show that V is finite-dimensional over k, and conclude that $\hom_{kG}(V, V)$ is also finite dimensional.
- Show the inclusion $k \hookrightarrow \hom_{kG}(V, V)$ found in (a) is an isomorphism. (For $f \in \hom_{kG}(V, V)$, view f as a linear transformation and consider $f - \alpha I$, where α is an eigenvalue of f).

1.118 Question 118

Let $f(x)$ be an irreducible polynomial of degree 5 over the field $\mathbb{Q}$ of rational numbers with exactly 3 real roots.

- Show that $f(x)$ is not solvable by radicals.
- Let E be the splitting field of f over $\mathbb{Q}$. Construct a Galois extension K of degree 2 over $\mathbb{Q}$ lying in E such that *no* field F strictly between K and E is Galois over $\mathbb{Q}$.

1.119 Question 119

Let F be a finite field. Show for any positive integer n that there are irreducible polynomials of degree n in $F[x]$.

1.120 Question 120

Show that the order of the group $\mathrm{GL}_n(\mathbb{F}_q)$ of invertible $n \times n$ matrices over the field $\mathbb{F}_q$ of q elements is given by $(q^n - 1)(q^n - q) \ldots (q^n - q^{n-1})$.

1.121 Question 121

- Let R be a commutative principal ideal domain. Show that any R-module M generated by two elements takes the form $R/(a) \oplus R/(b)$ for some $a, b \in R$. What more can you say about a and b?
- Give a necessary and sufficient condition for two direct sums as in part (a) to be isomorphic as R-modules.

1.122 Question 122

Let G be the subgroup of $\mathrm{GL}_3(\mathbb{C})$ generated by the three matrices

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $i^2 = -1$. Here $\mathbb{C}$ denotes the complex field.

- Compute the order of G.
- Find a matrix in G of largest possible order (as an element of G) and compute this order.
- Compute the number of elements in G with this largest order.

1.123 Question 123

- Let G be a group of (finite) order n. Show that any irreducible left module over the group algebra $\mathbb{C}G$ has complex dimension at least $\sqrt{n}$.
- Give an example of a group G of order $n \geq 5$ and an irreducible left module over $\mathbb{C}G$ of complex dimension $\lfloor \sqrt{n} \rfloor$, the greatest integer to $\sqrt{n}$.

1.124 Question 124

Use the rational canonical form to show that any square matrix M over a field k is similar to its transpose M^t, recalling that $p(M) = 0$ for some $p \in k[t]$ if and only if $p(M^t) = 0$.

1.125 Question 125

Let K be a field of characteristic zero and L a Galois extension of K. Let f be an irreducible polynomial in $K[x]$ of degree 7 and suppose f has no zeroes in L. Show that f is irreducible in $L[x]$.

1.126 Question 126

Let K be a field of characteristic zero and $f \in K[x]$ an irreducible polynomial of degree n. Let L be a splitting field for f. Let G be the group of automorphisms of L which act trivially on K.

- Show that G embeds in the symmetric group S_n.
- For each n, give an example of a field K and polynomial f such that $G = S_n$.
- What are the possible groups G when $n = 3$. Justify your answer.

1.127 Question 127

Show there are exactly two groups of order 21 up to isomorphism.

1.128 Question 128

Let K be the field $\mathbb{Q}(z)$ of rational functions in a variable z with coeffiecients in the rational field $\mathbb{Q}$. Let n be a positive integer. Consider the polynomial $x^n - z \in K[x]$.

- Show that the polynomial $x^n - z$ is irreducible over K.
- Describe the splitting field of $x^n - z$ over K.
- Determine the Galois group of the splitting field of $x^5 - z$ over the field K.

1.129 Question 129

- Let $p < q < r$ be prime integers. Show that a group of order pqr cannot be simple.
- Consider groups of orders $2^2 \cdot 3 \cdot p$ where p has the values 5, 7, and 11. For each of those values of p, either display a simple group of order $2^2 \cdot 3 \cdot p$, or show that there cannot be a simple group of that order.

1.130 Question 130

Let K/F be a finite Galois extension and let $n = [K : F]$. There is a theorem (often referred to as the "normal basis theorem") which states that there exists an irreducible polynomial $f(x) \in F[x]$ whose roots form a basis for K as a vector space over F. You may assume that theorem in this problem.

- Let $G = \mathrm{Gal}(K/F)$. The action of G on K makes K into a finite-dimensional representation space for G over F. Prove that K is isomorphic to the regular representation for G over F.

  > The regular representation is defined by letting G act on the group algebra $F[G]$ by multiplication on the left.

- Suppose that the Galois group G is cyclic and that F contains a primitive n^{th} root of unity. Show that there exists an injective homomorphism $\chi : G \to F^\times$.
- Show that K contains a non-zero element a with the following property:

  $$g(a) = \chi(g) \cdot a.$$

  for all $g \in G$.
- If a has the property stated in (c), show that $K = F(a)$ and that $a^n \in F^\times$.

1.131 Question 131

Let G be the group of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

with entries in the finite field $\mathbb{F}_p$ of p element, where p is a prime.

- Prove that G is non-abelian.
- Suppose p is odd. Prove that $g^p = I_3$ for all $g \in G$.
- Suppose that $p = 2$. It is known that there are exactly two non-abelian groups of order 8, up to isomorphism: the dihedral group D_8 and the quaternionic group. Assuming this fact without proof, determine which of these groups G is isomorphic to.

1.132 Question 132

There are five nonisomorphic groups of order 8. For each of those groups G, find the smallest positive integer n such that there is an injective homomorphism $\varphi : G \to S_n$.

1.133 Question 133

For any group G we define $\Omega(G)$ to be the image of the group homomorphism $\rho : G \to \mathrm{Aut}(G)$ where ρ maps $g \in G$ to the conjugation automorphism $x \mapsto gxg^{-1}$. Starting with a group G_0, we define $G_1 = \Omega(G_0)$ and $G_{i+1} = \Omega(G_i)$ for all $i \geq 0$. If G_0 is of order p^e for a prime p and integer $e \geq 2$, prove that G_{e-1} is the trivial group.

1.134 Question 134

Let $\mathbb{F}_2$ be the field with two elements.

- What is the order of $\mathrm{GL}_3(\mathbb{F}_2)$?
- Use the fact that $\mathrm{GL}_3(\mathbb{F}_2)$ is a simple group (which you should not prove) to find the number of elements of order 7 in $\mathrm{GL}_3(\mathbb{F}_2)$.

1.135 Question 135

Let G be a finite abelian group. Let $f : \mathbb{Z}^m \to G$ be a surjection of abelian groups. We may think of f as a homomorphism of $\mathbb{Z}$-modules. Let K be the kernel of f.

- Prove that K is isomorphic to $\mathbb{Z}^m$.
- We can therefore write the inclusion map $K \to \mathbb{Z}^m$ as $\mathbb{Z}^m \to \mathbb{Z}^m$ and represent it by an $m \times m$ integer matrix A. Prove that $|\det A| = |G|$.

1.136 Question 136

Let $R = C([0,1])$ be the ring of all continuous real-valued functions on the closed interval $[0,1]$, and for each $c \in [0,1]$, denote by M_c the set of all functions $f \in R$ such that $f(c) = 0$.

- Prove that $g \in R$ is a unit if and only if $g(c) \neq 0$ for all $c \in [0,1]$.
- Prove that for each $c \in [0,1]$, M_c is a maximal ideal of R.
- Prove that if M is a maximal ideal of T, then $M = M_c$ for some $c \in [0,1]$.

  Hint: compactness of $[0,1]$ may be relevant.

1.137 Question 137

Let R and S be commutative rings, and $f : R \to S$ a ring homomorphism.

- Show that if I is a prime ideal of S, then

  $$f^{-1}(I) = \{r \in R : f(r) \in I\}$$

  is a prime ideal of R.

- Let N be the set of nilpotent elements of R:

  $$N = \{r \in R : r^m = 0 \text{ for some } m \geq 1\}..$$

  N is called the *nilradical* of R. Prove that it is an ideal which is contained in every prime ideal.

- Part (a) lets us define a function

  $$f^* : \{\text{prime ideals of } S\} \to \{\text{prime ideals of } R\}.I \qquad \mapsto f^{-1}(I)..$$

  Let N be the nilradical of R. Show that if $S = R/N$ and $f : R \to R/N$ is the quotient map, then f^* is a bijection

1.138 Question 138

Consider the polynomial $f(x) = x^{10} + x^5 + 1 \in \mathbb{Q}[x]$ with splitting field K over $\mathbb{Q}$.

- Determine whether $f(x)$ is irreducible over $\mathbb{Q}$ and find $[K : \mathbb{Q}]$.
- Determine the structure of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

1.139 Question 139

For each prime number p and each positive integer n, how many elements α are there in $\mathbb{F}_{p^n}$ such that $F_p(\alpha) = F_{p^6}$?

1.140 Question 140

Assume that K is a cyclic group, H is an arbitrary group, and φ_1 and φ_2 are homomorphisms from K into $\mathrm{Aut}(H)$ such that $\varphi_1(K)$ and $\varphi_2(K)$ are conjugate subgroups of $\mathrm{Aut}(H)$. Prove by constructing an explicit isomorphism that $H \rtimes_{\varphi_1} K \cong H \rtimes_{\varphi_2} K$.

> Suppose $\sigma_{\varphi_1}(K)\sigma^{-1} = \varphi_2(K)$ so that for some $a \in \mathbb{Z}$ we have $\sigma\varphi_1(k)\sigma^{-1} = \varphi_2(k)^a$ for all $k \in K$. Show that the map $\psi : H \rtimes_{\varphi_1} K \to H \rtimes_{\varphi_2} K$ defined by $\psi((h, k)) = (\sigma(h), k^a)$ is a homomorphism. Show ψ is bijective by construcing a 2-sided inverse.

2 Real Analysis (85 Questions)

2.1 Question 1

Let $C([0,1])$ denote the space of all continuous real-valued functions on $[0,1]$.

a. Prove that $C([0,1])$ is complete under the uniform norm $\|f\|_u := \sup_{x\in[0,1]} |f(x)|$.

b. Prove that $C([0,1])$ is not complete under the L^1-norm $\|f\|_1 = \int_0^1 |f(x)|\, dx$.

2.2 Question 2

Let $\mathcal{B}$ denote the set of all Borel subsets of $\mathbb{R}$ and $\mu : \mathcal{B} \to [0,\infty)$ denote a finite Borel measure on $\mathbb{R}$.

a. Prove that if $\{F_k\}$ is a sequence of Borel sets for which $F_k \supseteq F_{k+1}$ for all k, then

$$\lim_{k\to\infty} \mu\left(F_k\right) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right)$$

b. Suppose mu has the property that $mu(E) = 0$ for every $E \in \mathcal{B}$ with Lebesgue measure $m(E) = 0$. Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $E \in \mathcal{B}$ with $m(E) < \delta$, then $mu(E) < \varepsilon$.

2.3 Question 3

Let $\{f_k\}$ be any sequence of functions in $L^2([0,1])$ satisfying $\|f_k\|_2 \leq M$ for all $k \in \mathbb{N}$. Prove that if $f_k \to f$ almost everywhere, then $f \in L^2([0,1])$ with $\|f\|_2 \leq M$ and

$$\lim_{k\to\infty} \int_0^1 f_k(x)dx = \int_0^1 f(x)dx$$

Hint: Try using Fatou's Lemma to show that $\|f\|_2 \leq M$ and then try applying Egorov's Theorem.

2.4 Question 4

Let f be a non-negative function on $\mathbb{R}^n$ and $\mathcal{A} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq f(x)\}$. Prove the validity of the following two statements:

a. f is a Lebesgue measurable function on $\mathbb{R}^n$ $\iff$ $\mathcal{A}$ is a Lebesgue measurable subset of $\mathbb{R}^{n+1}$

b. If f is a Lebesgue measurable function on $\mathbb{R}^n$, then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)dx = \int_0^{\infty} m\left(\{x \in \mathbb{R}^n : f(x) \geq t\}\right) dt$$

2.5 Question 5

a. Show that $L^2([0,1]) \subseteq L^1([0,1])$ and argue that $L^2([0,1])$ in fact forms a dense subset of $L^1([0,1])$.

b. Let Λ be a continuous linear functional on $L^1([0,1])$.

   Prove the Riesz Representation Theorem for $L^1([0,1])$ by following the steps below:

   i. Establish the existence of a function $g \in L^2([0,1])$ which represents Λ in the sense that
   $$\Lambda(f) = f(x)g(x)dx \text{ for all } f \in L^2([0,1]).$$
   Hint: You may use, without proof, the Riesz Representation Theorem for $L^2([0,1])$.

   ii. Argue that the g obtained above must in fact belong to $L^\infty([0,1])$ and represent Λ in the sense that
   $$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}dx \quad \text{for all } f \in L^1([0,1])$$
   with
   $$\|g\|_{L^\infty([0,1])} = \|\Lambda\|_{L^1([0,1])^\vee}$$

2.6 Question 6

Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers.

a. Prove that if $\lim\limits_{n\to\infty} a_n = 0$, then $\lim\limits_{n\to\infty} a_1 + \cdots + a_n = 0$.
$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

b. Prove that if $\sum\limits_{n=1}^{\infty} \frac{a_n}{n}$ converges, then
$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

2.7 Question 7

Prove that
$$\left| \frac{d^n}{dx^n} \frac{\sin x}{x} \right| \leq \frac{1}{n}$$
for all $x \neq 0$ and positive integers n.

Hint: Consider $\displaystyle\int_0^1 \cos(tx)dt$

2.8 Question 8

Let $(X, \mathcal{B}, mu)$ be a measure space with $mu(X) = 1$ and $\{B_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{B}$-measurable subsets of X, and

$$B := \left\{ x \in X \;\middle|\; x \in B_n \text{ for infinitely many } n \right\}.$$

a. Argue that B is also a $\mathcal{B}$-measurable subset of X.

b. Prove that if $\sum_{n=1}^{\infty} \mu(B_n) < \infty$ then $\mu(B) = 0$.

c. Prove that if $\sum_{n=1}^{\infty} \mu(B_n) = \infty$ **and** the sequence of set complements $\{B_n^c\}_{n=1}^{\infty}$ satisfies

$$\mu\left(\bigcap_{n=k}^{K} B_n^c\right) = \prod_{n=k}^{K} \left(1 - \mu\left(B_n\right)\right)$$

for all positive integers k and K with $k < K$, then $mu(B) = 1$.

Hint: Use the fact that $1 - x \leq e^{-x}$ for all x.

2.9 Question 9

Let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space $\mathcal{H}$.

a. Prove that for every $x \in \mathcal{H}$ one has

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

b. Prove that for any sequence $\{a_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$ there exists an element $x \in \mathcal{H}$ such that

$$a_n = \langle x,\ u_n \rangle \text{ for all } n \in \mathbb{N}$$

and

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$$

2.10 Question 10

a. Show that if f is continuous with compact support on $\mathbb{R}$, then

$$\lim_{y \to 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dx = 0$$

b. Let $f \in L^1(\mathbb{R})$ and for each $h > 0$ let

$$\mathcal{A}_h f(x) := \frac{1}{2h} \int_{|y| \leq h} f(x-y) dy$$

c. Prove that $\|\mathcal{A}_h f\|_1 \leq \|f\|_1$ for all $h > 0$.

ii. Prove that $\mathcal{A}_h f \to f$ in $L^1(\mathbb{R})$ as $h \to 0^+$.

2.11 Question 11

Define

$$E := \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N} \right\}.$$

Prove that $m(E) = 0$.

2.12 Question 12

Let

$$f_n(x) := \frac{x}{1 + x^n}, \quad x \geq 0.$$

a. Show that this sequence converges pointwise and find its limit. Is the convergence uniform on $[0, \infty)$?

b. Compute

$$\lim_{n \to \infty} \int_0^\infty f_n(x) dx$$

2.13 Question 13

Let f be a non-negative measurable function on $[0, 1]$.
Show that

$$\lim_{p \to \infty} \left(\int_{[0,1]} f(x)^p dx \right)^{\frac{1}{p}} = \|f\|_\infty.$$

2.14 Question 14

Let $f \in L^2([0, 1])$ and suppose

$$\int_{[0,1]} f(x) x^n dx = 0 \text{ for all integers } n \geq 0.$$

Show that $f = 0$ almost everywhere.

2.15 Question 15

Suppose that

- $f_n, f \in L^1$,
- $f_n \to f$ almost everywhere, and
- $\int |f_n| \to \int |f|$.

Show that $\int f_n \to \int f$

2.16 Question 16

Let $f(x) = \frac{1}{x}$. Show that f is uniformly continuous on $(1, \infty)$ but not on $(0, \infty)$.

2.17 Question 17

Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Show that there is a Borel set $B \subset E$ such that $m(E \setminus B) = 0$.

2.18 Question 18

Suppose $f(x)$ and $xf(x)$ are integrable on $\mathbb{R}$. Define F by

$$F(t) := \int_{-\infty}^{\infty} f(x) \cos(xt) dx$$

Show that

$$F'(t) = -\int_{-\infty}^{\infty} xf(x) \sin(xt) dx.$$

2.19 Question 19

Let $f \in L^1([0,1])$. Prove that

$$\lim_{n \to \infty} \int_0^1 f(x) |\sin nx| \; dx = \frac{2}{\pi} \int_0^1 f(x) \; dx$$

Hint: Begin with the case that f is the characteristic function of an interval.

2.20 Question 20

Let $f \geq 0$ be a measurable function on $\mathbb{R}$. Show that

$$\int_{\mathbb{R}} f = \int_0^{\infty} m(\{x : f(x) > t\}) dt$$

2.21 Question 21

Compute the following limit and justify your calculations:

$$\lim_{n \to \infty} \int_1^n \frac{dx}{\left(1 + \frac{x}{n}\right)^n \sqrt[n]{x}}$$

2.22 Question 22

Let K be the set of numbers in $[0,1]$ whose decimal expansions do not use the digit 4.

> We use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 with $399 \cdots$. For example, $0.8754 = 0.8753999 \cdots$.

Show that K is a compact, nowhere dense set without isolated points, and find the Lebesgue measure $m(K)$.

2.23 Question 23

a. Let μ be a measure on a measurable space $(X, \mathcal{M})$ and f a positive measurable function.

Define a measure λ by

$$\lambda(E) := \int_E f \, d\mu, \quad E \in \mathcal{M}$$

Show that for g any positive measurable function,

$$\int_X g \, d\lambda = \int_X fg \, d\mu$$

b. Let $E \subset \mathbb{R}$ be a measurable set such that

$$\int_E x^2 \, dm = 0.$$

Show that $m(E) = 0$.

2.24 Question 24

Let

$$f_n(x) = ae^{-nax} - be^{-nbx} \qquad \text{where } 0 < a < b.$$

Show that

a. $\sum_{n=1}^{\infty} |f_n|$ is not in $L^1([0, \infty), m)$

Hint: $f_n(x)$ has a root x_n.

b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0, \infty), m) \quad \text{and} \quad \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x) \, dm = \ln \frac{b}{a}$$

2.25 Question 25

Let $f(x, y)$ on $[-1, 1]^2$ be defined by

$$f(x, y) = \begin{cases} \frac{xy}{(x^2+y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Determine if f is integrable.

2.26 Question 26

Let $f, g \in L^2(\mathbb{R})$. Prove that the formula

$$h(x) := \int_{-\infty}^{\infty} f(t)g(x - t)dt$$

defines a uniformly continuous function h on $\mathbb{R}$.

2.27 Question 27

Show that the space $C^1([a,b])$ is a Banach space when equipped with the norm

$$\|f\| := \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |f'(x)| .$$

2.28 Question 28

Let

$$f(x) = s\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which f does and does not converge uniformly.

2.29 Question 29

Let $f(x) = x^2$ and $E \subset [0,\infty) := \mathbb{R}^+$.

1. Show that

$$m^*(E) = 0 \iff m^*(f(E)) = 0.$$

2. Deduce that the map

$$\begin{aligned} \phi : \mathcal{L}(\mathbb{R}^+) &\to \mathcal{L}(\mathbb{R}^+) \\ E &\mapsto f(E) \end{aligned}$$

is a bijection from the class of Lebesgue measurable sets of $[0,\infty)$ to itself.

2.30 Question 30

Let

$$S = \operatorname{span}_{\mathbb{C}} \left\{ \chi_{(a,b)} \;\middle|\; a,b \in \mathbb{R} \right\},$$

the complex linear span of characteristic functions of intervals of the form (a,b).
Show that for every $f \in L^1(\mathbb{R})$, there exists a sequence of functions $\{f_n\} \subset S$ such that

$$\lim_{n\to\infty} \|f_n - f\|_1 = 0$$

2.31 Question 31

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

1. Show that $f_n \to 0$ pointwise but not uniformly on $[0,1]$.

    Hint: Consider the maximum of f_n.

2.

$$\lim_{n\to\infty} \int_0^1 n(1-x)^n \sin x dx = 0$$

2.32 Question 32

Let ϕ be a compactly supported smooth function that vanishes outside of an interval $[-N, N]$ such that $\int_{\mathrm{R}} \phi(x)dx = 1$.
For $f \in L^1(\mathbb{R})$, define

$$K_j(x) := j\phi(jx), \qquad f * K_j(x) := \int_{\mathbb{R}} f(x-y)K_j(y)\ dy$$

and prove the following:

1. Each $f * K_j$ is smooth and compactly supported.
2. $$\lim_{j\to\infty} \|f * K_j - f\|_1 = 0$$

   Hint:
   $$\lim_{y\to 0} \int_{\mathbb{R}} |f(x-y) - f(x)|dy = 0$$

2.33 Question 33

Let X be a complete metric space and define a norm

$$\|f\| := \max\{|f(x)| : x \in X\}.$$

Show that $(C^0(\mathbb{R}), \|\cdot\|)$ (the space of continuous functions $f : X \to \mathbb{R}$) is complete.

2.34 Question 34

For $n \in \mathbb{N}$, define

$$e_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad E_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

Show that $e_n < E_n$, and prove Bernoulli's inequality:

$$(1+x)^n \geq 1 + nx \text{ for } -1 < x < \infty \text{ and } n \in \mathbb{N}$$

Use this to show the following:

1. The sequence e_n is increasing.
2. The sequence E_n is decreasing.
3. $2 < e_n < E_n < 4$.
4. $\lim_{n\to\infty} e_n = \lim_{n\to\infty} E_n$.

2.35 Question 35

Let $0 < \lambda < 1$ and construct a Cantor set C_λ by successively removing middle intervals of length λ.
Prove that $m(C_\lambda) = 0$.

2.36 Question 36

Let f be Lebesgue measurable on $\mathbb{R}$ and $E \subset \mathbb{R}$ be measurable such that

$$0 < A = \int_E f(x)dx < \infty.$$

Show that for every $0 < t < 1$, there exists a measurable set $E_t \subset E$ such that

$$\int_{E_t} f(x)dx = tA.$$

2.37 Question 37

Let $E \subset \mathbb{R}$ be measurable with $m(E) < \infty$. Define

$$f(x) = m(E \cap (E + x)).$$

Show that

1. $f \in L^1(\mathbb{R})$.
2. f is uniformly continuous.
3. $\lim_{|x|\to\infty} f(x) = 0$

   Hint:

$$\chi_{E\cap(E+x)}(y) = \chi_E(y)\chi_E(y - x)$$

2.38 Question 38

Let $(X, \mathcal{M}, \mu)$ be a measure space. For $f \in L^1(\mu)$ and $\lambda > 0$, define

$$\phi(\lambda) = \mu(\{x \in X | f(x) > \lambda\}) \quad \text{and} \quad \psi(\lambda) = \mu(\{x \in X | f(x) < -\lambda\})$$

Show that ϕ, ψ are Borel measurable and

$$\int_X |f| \; d\mu = \int_0^\infty [\phi(\lambda) + \psi(\lambda)] \; d\lambda$$

2.39 Question 39

Without using the Riesz Representation Theorem, compute

$$\sup\left\{\left|\int_0^1 f(x)e^x dx\right| \;\middle|\; f \in L^2([0, 1], m), \; \|f\|_2 \le 1\right\}$$

2.40 Question 40

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that f converges to a differentiable function on $(1, \infty)$ and that

$$f'(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n^x}\right)'.$$

Hint:

$$\left(\frac{1}{n^x}\right)' = -\frac{1}{n^x} \ln n$$

2.41 Question 41

Let $f, g : [a, b] \to \mathbb{R}$ be measurable with

$$\int_a^b f(x)\ dx = \int_a^b g(x)\ dx.$$

Show that either

1. $f(x) = g(x)$ almost everywhere, or
2. There exists a measurable set $E \subset [a, b]$ such that

$$\int_E f(x)\ dx > \int_E g(x)\ dx$$

2.42 Question 42

Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{x \to 0} \int_{\mathbb{R}} |f(y - x) - f(y)| dy = 0$$

2.43 Question 43

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\{E_n\} \subset \mathcal{M}$ satisfies

$$\lim_{n \to \infty} \mu\left(X \backslash E_n\right) = 0.$$

Define

$$G \coloneqq \left\{ x \in X \;\middle|\; x \in E_n \text{ for only finitely many } n \right\}.$$

Show that $G \in \mathcal{M}$ and $\mu(G) = 0$.

2.44 Question 44

Let $\phi \in L^\infty(\mathbb{R})$. Show that the following limit exists and satisfies the equality

$$\lim_{n\to\infty} \left(\int_{\mathbb{R}} \frac{|\phi(x)|^n}{1+x^2} dx \right)^{\frac{1}{n}} = \|\phi\|_\infty.$$

2.45 Question 45

Let $f, g \in L^2(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)g(x+n)dx = 0$$

2.46 Question 46

Let (X, d) and (Y, ρ) be metric spaces, $f : X \to Y$, and $x_0 \in X$.
Prove that the following statements are equivalent:

1. For every $\varepsilon > 0 \quad \exists \delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$.
2. The sequence $\{f(x_n)\}_{n=1}^\infty \to f(x_0)$ for every sequence $\{x_n\} \to x_0$ in X.

2.47 Question 47

Let $f : \mathbb{R} \to \mathbb{C}$ be continuous with period 1. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(t)dt \quad \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

Hint: show this first for the functions $f(t) = e^{2\pi ikt}$ for $k \in \mathbb{Z}$.

2.48 Question 48

Let μ be a finite Borel measure on $\mathbb{R}$ and $E \subset \mathbb{R}$ Borel. Prove that the following statements are equivalent:

1. $\forall \varepsilon > 0$ there exists G open and F closed such that
$$F \subseteq E \subseteq G \quad \text{and} \quad \mu(G \setminus F) < \varepsilon.$$
2. There exists a $V \in G_\delta$ and $H \in F_\sigma$ such that
$$H \subseteq E \subseteq V \quad \text{and} \quad \mu(V \setminus H) = 0$$

2.49 Question 49

Define

$$f(x, y) := \begin{cases} \frac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0 \le x \le y \\ 0 & \text{otherwise} \end{cases}$$

Carefully show that $f \in L^1(\mathbb{R}^2)$.

2.50 Question 50

Let $\mathcal{H}$ be a Hilbert space.

1. Let $x \in \mathcal{H}$ and $\{u_n\}_{n=1}^N$ be an orthonormal set. Prove that the best approximation to x in $\mathcal{H}$ by an element in $\text{span}_{\mathbb{C}}\{u_n\}$ is given by

$$\widehat{x} := \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n.$$

2. Conclude that finite dimensional subspaces of $\mathcal{H}$ are always closed.

2.51 Question 51

Let $f \in L^1(\mathbb{R})$ and g be a bounded measurable function on $\mathbb{R}$.

1. Show that the convolution $f * g$ is well-defined, bounded, and uniformly continuous on $\mathbb{R}$.
2. Prove that one further assumes that $g \in C^1(\mathbb{R})$ with bounded derivative, then $f * g \in C^1(\mathbb{R})$ and

$$\frac{d}{dx}(f * g) = f * \left(\frac{d}{dx} g\right)$$

2.52 Question 52

Define

$$f(x) = c_0 + c_1 x^1 + c_2 x^2 + \ldots + c_n x^n \text{ with } n \text{ even and } c_n > 0.$$

Show that there is a number x_m such that $f(x_m) \leq f(x)$ for all $x \in \mathbb{R}$.

2.53 Question 53

Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable.

1. Show that there is a sequence of simple functions $s_n(x)$ such that $s_n(x) \to f(x)$ for all $x \in \mathbb{R}$.
2. Show that there is a Borel measurable function g such that $g = f$ almost everywhere.

2.54 Question 54

Compute the following limit:

$$\lim_{n\to\infty} \int_1^n \frac{ne^{-x}}{1 + nx^2} \sin\left(\frac{x}{n}\right) \, dx$$

2.55 Question 55

Let $f : [1, \infty) \to \mathbb{R}$ such that $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Show that the following limit exists and satisfies the equality

$$\lim_{x \to \infty} f(x) \leq 1 + \frac{\pi}{4}$$

2.56 Question 56

Let $f, g \in L^1(\mathbb{R})$ be Borel measurable.

1. Show that

- The function
$$F(x, y) := f(x - y)g(y)$$
is Borel measurable on $\mathbb{R}^2$, and
- For almost every $y \in \mathbb{R}$,
$$F_y(x) := f(x - y)g(y)$$
is integrable with respect to y.

2. Show that $f * g \in L^1(\mathbb{R})$ and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

2.57 Question 57

Let $f : [0, 1] \to \mathbb{R}$ be continuous. Show that

$$\sup \left\{ \|fg\|_1 \;\middle|\; g \in L^1[0, 1],\ \|g\|_1 \leq 1 \right\} = \|f\|_\infty$$

2.58 Question 58

1. Give an example of a continuous $f \in L^1(\mathbb{R})$ such that $f(x) \not\to 0$ as$|x| \to \infty$.

2. Show that if f is *uniformly* continuous, then

$$\lim_{|x| \to \infty} f(x) = 0.$$

2.59 Question 59

Let $\{a_n\}$ be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that $\sum a_n^2 < \infty$.

Note: Assume a_n, b_n are all non-negative.

2.60 Question 60

Let $f : \mathbb{R} \to \mathbb{R}$ and suppose

$$\forall x \in \mathbb{R}, \quad f(x) \geq \limsup_{y \to x} f(y)$$

Prove that f is Borel measurable.

2.61 Question 61

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose f is a measurable function on X. Show that

$$\lim_{n\to\infty} \int_X f^n \, d\mu = \begin{cases} \infty & \textit{or} \\ \mu(f^{-1}(1)), \end{cases}$$

and characterize the collection of functions of each type.

2.62 Question 62

Let $f, g \in L^1([0,1])$ and for all $x \in [0,1]$ define

$$F(x) := \int_0^x f(y)dy \quad \text{and} \quad G(x) := \int_0^x g(y)dy.$$

Prove that

$$\int_0^1 F(x)g(x)dx = F(1)G(1) - \int_0^1 f(x)G(x)dx$$

2.63 Question 63

Let $\{f_n\}$ be a sequence of continuous functions such that $\sum f_n$ converges uniformly. Prove that $\sum f_n$ is also continuous.

2.64 Question 64

Let I be an index set and $\alpha : I \to (0, \infty)$.

1. Show that

$$\sum_{i\in I} a(i) := \sup_{\substack{J\subset I \\ J \text{ finite}}} \sum_{i\in J} a(i) < \infty \implies I \text{ is countable.}$$

2. Suppose $I = \mathbb{Q}$ and $\sum_{q\in\mathbb{Q}} a(q) < \infty$. Define

$$f(x) := \sum_{\substack{q\in\mathbb{Q} \\ q\leq x}} a(q).$$

Show that f is continuous at $x \iff x \notin \mathbb{Q}$.

2.65 Question 65

Let $f \in L^1(\mathbb{R})$. Show that

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } m(E) < \delta \implies \int_E |f(x)|dx < \varepsilon$$

2.66 Question 66

Let $g \in L^\infty([0,1])$ Prove that

$$\int_{[0,1]} f(x)g(x)dx = 0 \quad \text{for all continuous } f : [0,1] \to \mathbb{R} \implies g(x) = 0 \text{ almost everywhere.}$$

2.67 Question 67

1. Let $f \in C_c^0(\mathbb{R}^n)$, and show

$$\lim_{t \to 0} \int_{\mathbb{R}^n} |f(x+t) - f(x)|dx = 0.$$

2. Extend the above result to $f \in L^1(\mathbb{R}^n)$ and show that

$$f \in L^1(\mathbb{R}^n),\ g \in L^\infty(\mathbb{R}^n) \implies f * g \text{ is bounded and uniformly continuous.}$$

2.68 Question 68

Let $1 \leq p, q \leq \infty$ be conjugate exponents, and show that

$$f \in L^p(\mathbb{R}^n) \implies \|f\|_p = \sup_{\|g\|_q = 1} \left| \int f(x)g(x)dx \right|$$

2.69 Question 69

Prove or disprove each of the following statements.

(a) If f is of bounded variation on $[0,1]$, then it is continuous on $[0,1]$.

(b) If $f : [0,1] \to [0,1]$ is a continuous function, then there exists $x_0 \in [0,1]$ such that $f(x_0) = x_0$.

(c) Let $\{f_n\}$ be a sequence of uniformly continuous functions on an interval I. If $\{f_n\}$ converges uniformly to a function f on I, then f is also uniformly continuous on I.

(d) If f is differentiable on a connected set $E \subset \mathbb{R}^n$, then for any $x, y \in E$, there exists $z \in E$ such that $f(x) - f(y) = \nabla f(z)(x - y)$.

2.70 Question 70

Prove or disprove each of the following statements.

(d) If $\lim_{n\to\infty} |a_n + 1/a_n|$ exists, then $\lim_{n\to\infty} |a_n|^{1/n}$ exists and the two limits are equal.

(e) If $\sum_{n=1}^{\infty} a_n x^n$ converges for all $x \in [0,1]$, then $\lim_{x\to 1^-} \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_n$

2.71 Question 71

Prove or disprove each of the following statements.

(f) If $E \subset \mathbb{R}$ and
$\mu(E) = \inf\{\sum_{I_i \in S} |I_i| : S = \{I_i\}_{i=1}^n \text{ such that } E \subset \cup_{i=1}^n I_i \text{ for some } n \in \mathbb{N}\}$
then μ coincides with the outer measure of E.

(g) If E is a Borel set and f is a measurable function, then $f^{-1}(E)$ is also measurable.

2.72 Question 72

If f is a finite real valued measurable function on a measurable set $E \subset \mathbb{R}$, show that the set $\{(x, f(x)) : x \in E\}$ is measurable.

2.73 Question 73

Let g : $[0,1] \times [0,1] \to [0,1]$ be a continuous function and let $\{f_n\}$ be a sequence of functions such that

$$f_n(x) = \begin{cases} 0, 0 \le x \le 1/n, \\ \int_0^{x-\frac{1}{n}} g(t, f_n(t))dt, 1/n \le x \le 1. \end{cases}$$

With the help of the Arzela-Ascoli theorem or otherwise, show that there exists a continuous function $f : [0,1] \to \mathbb{R}$ such that
$f(x) = \int_0^x g(t, f(t))dt$
for all $x \in [0,1]$.

Hint: first show that $|f_n(x_1) - f_n(x_2)| \le |x_1 - x_2|$.

2.74 Question 74

If $\limsup_{n\to\infty} a_n \le l$, show that $\limsup_{n\to\infty} \sum_{i=1}^n a_i/n \le l$.

2.75 Question 75

If f is a nonnegative measurable function on $\mathbb{R}$ and $p > 0$, show that

$$\int f^p \, dx = \int_0^\infty pt^{p-1} |\{x : f(x) > t\}| \, dt$$

where $|\{x : f(x) > t\}|$ is the Lebesgue measure of the set $\{x : f(x) > t\}$.

2.76 Question 76

If f is a nonnegative measurable function on $[0, \pi]$ and $\int_0^\pi f(x)^3 \, dx < \infty$, show that

$$\lim_{\alpha\to\infty} \int_{\{x: f(x)>\alpha\}} f(x)^2 \, dx = 0.$$

2.77 Question 77

Prove or disprove each of the following statements.

(a) If $f : [0, 1] \to \mathbb{R}$ is a measurable function, then given any $\varepsilon > 0$, there exists a compact set $K \subset [0, 1]$ such that f is continuous on K relative to K.

(b) If f is Borel measurable on $\mathbb{R} \times \mathbb{R}$, then for any $x \in \mathbb{R}$, the function $g(y) = f(x, y)$ is also Borel measurable on $\mathbb{R}$.

(c) If $E \subset \mathbb{R}$, then E is measurable if and only if given any $\varepsilon > 0$, there exist a closed set F and an open set G such that $F \subset E \subset G$ and the measure of $G - F$ is less than ε.

2.78 Question 78

Prove or disprove each of the following statements.

(b) If f_n is a sequence of measurable functions that converges uniformly to f on $\mathbb{R}$, then $\int f = \lim_{k\to\infty} \int f_k$

(c) If $\{f_k\}$ is a sequence of function in $L_p[0, \infty)$ that converges to a function $f \in L_p[0, \infty)$, then $\{f_k\}$ has a subsequence that converges to f almost everywhere.

2.79 Question 79

Prove or disprove each of the following statements.

(f) If f is Riemann integrable on $[\varepsilon, 1]$ for all $0 < \varepsilon < 1$, then f is Lebesgue integrable on $[0, 1]$ if f is nonnegative and the following limit exists $\lim_{\varepsilon\to 0^+} \int_\varepsilon^1 f dx$.

(g) If f is integrable on $[0, 1]$, then $\lim_{n\to\infty} \int_0^1 f(x) \sin(n\pi x) dx = 0$.

(h) If f is continuous on $[0, 1]$, then it is of bounded variation on [0, 1]$.

2.80 Question 80

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $f'(-1) < 2$ and $f'(1) > 2$, show that there exists $x_0 \in (i1, 1)$ such that $f'(x_0) = 2$.

Hint: consider the function $f(x)-2x$ and recall the proof of Rolle's theorem.)

(b) Let $f : (-1, 1) \to \mathbb{R}$ be a differentiable function on $(-1, 0)\cup(0, 1)$ such that $\lim_{x\to 0} f'(x) = L$. If f is continuous on $(-1, 1)$, show that f is indeed differentiable at 0 and $f'(0) = L$.

2.81 Question 81

Describe the process that extends a measure on an algebra $\mathcal{A}$ of subsets of X, to a complete measure defined on a σ-algebra $\mathcal{B}$ containing $\mathcal{A}$. State the corresponding definitions and results (without proofs).

2.82 Question 82

State and prove Fatou's Lemma on a general measurable space.

2.83 Question 83

1. State the Dominated Convergence Theorem for Lebesgue integrals.
2. Let $\{f_n\}$ be a sequence of measurable functions on a Lebesgue measurable set E which converges *in measure* to a function f on E. Suppose that for every n, $|f_n| \leq g$ with g integrable on E. Using the above theorem show that

$$\int_E |f_n - f| \longrightarrow 0\,.$$

2.84 Question 84

Let $f \in L^1([0,1])$. Show that

1. The limit $\lim_{p\to 0^+} \|f\|_p$ exists.
2. If $m\{x : f(x) = 0\} > 0$, then the above limit is zero.

2.85 Question 85

Let f be a continuous function on $[0,1]$. Show that the following statements are equivalent.

1. f is absolutely continuous.
2. For any $\epsilon > 0$ there exists $\delta > 0$ such that $m(f(E)) < \epsilon$ for any set $E \subseteq [0,1]$ with $m(E) < \delta$.
3. $m(f(E)) = 0$ for any set $E \subseteq [0,1]$ with $m(E) = 0$.

3 Complex Analysis (125 Questions)

3.1 Question 1

(1) Assume $f(z) = \displaystyle\sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}\,.$$

(2) Deduce Liouville's theorem from (1).

3.2 Question 2

Let f be a continuous function in the region

$$D = \{z \;\Big|\; |z| > R, 0 \le \arg z \le \theta\} \quad \text{where} \quad 1 \le \theta \le 2\pi.$$

If there exists k such that $\lim_{z\to\infty} zf(z) = k$ for z in the region D. Show that

$$\lim_{R'\to\infty} \int_L f(z)dz = i\theta k,$$

where L is the part of the circle $|z| = R'$ which lies in the region D.

3.3 Question 3

Suppose that f is an analytic function in the region D which contains the point a. Let

$$F(z) = z - a - qf(z), \quad \text{where } q \text{ is a complex parameter.}$$

(1) Let $K \subset D$ be a circle with the center at point a and also we assume that $f(z) \neq 0$ for $z \in K$. Prove that the function F has one and only one zero $z = w$ on the closed disc $\overline{K}$ whose boundary is the circle K if $|q| < \min_{z\in K} \dfrac{|z-a|}{|f(z)|}$.

(2) Let $G(z)$ be an analytic function on the disk $\overline{K}$. Apply the residue theorem to prove that $\dfrac{G(w)}{F'(w)} = \dfrac{1}{2\pi i}\displaystyle\int_K \dfrac{G(z)}{F(z)}dz$, where w is the zero from (1).

(3) If $z \in K$, prove that the function $\dfrac{1}{F(z)}$ can be represented as a convergent series with respect to q: $\dfrac{1}{F(z)} = \displaystyle\sum_{n=0}^{\infty} \dfrac{(qf(z))^n}{(z-a)^{n+1}}$.

3.4 Question 4

Evaluate

$$\int_0^\infty \frac{x\sin x}{x^2+a^2}\,dx.$$

3.5 Question 5

Let $f = u + iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

3.6 Question 6

Show that $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^n}dx = \frac{\pi}{n\sin\frac{a\pi}{n}}$ using complex analysis, $0 < a < n$. Here n is a positive integer.

3.7 Question 7

For $s > 0$, the **gamma function** is defined by $\Gamma(s) = \displaystyle\int_0^\infty e^{-t}t^{s-1}dt$.

1. Show that the gamma function is analytic in the half-plane $\Re(s) > 0$, and is still given there by the integral formula above.

2. Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}.$$

Hint: You may need $\Gamma(1-s) = t\displaystyle\int_0^\infty e^{-vt}(vt)^{-s}dv$ for $t > 0$.

3.8 Question 8

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree n, then it has n zeros in $\mathbb{C}$.

3.9 Question 9

Suppose f is entire and there exist $A, R > 0$ and natural number N such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that

(i) f is a polynomial and

(ii) the degree of f is at least N.

3.10 Question 10

Let $f : \mathbb{C} \to \mathbb{C}$ be an injective analytic (also called *univalent*) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

3.11 Question 11

Let g be analytic for $|z| \leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

1. Show that g has a unique fixed point in $|z| < 1$.
2. What happens if we replace $|g(z)| < 1$ with $|g(z)| \leq 1$ for $|z| = 1$? Give an example if (a) is not true or give an proof if (a) is still true.
3. What happens if we simply assume that f is analytic for $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z) \not\equiv z$. Can f have more than one fixed point in $|z| < 1$?

   Hint: The map $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ may be useful.

3.12 Question 12

Find a conformal map from $D = \{z : \ |z| < 1, \ |z - 1/2| > 1/2\}$ to the unit disk $\Delta = \{z : |z| < 1\}$.

3.13 Question 13

Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω. Show without using Picard's theorems that $f(z)$ is a constant.

3.14 Question 14

(1) Assume $f(z) = \displaystyle\sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \ .$$

(2) Deduce Liouville's theorem from (1).

3.15 Question 15

Let $f(z)$ be entire and assume that $f(z) \leq M|z|^2$ outside some disk for some constant M. Show that $f(z)$ is a polynomial in z of degree ≤ 2.

3.16 Question 16

Let $a_n(z)$ be an analytic sequence in a domain D such that $\displaystyle\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D. Show that $\displaystyle\sum_{n=0}^{\infty} |a_n'(z)|$ converges uniformly on bounded and closed sub-regions of D.

3.17 Question 17

Let $f(z)$ be analytic in an open set Ω except possibly at a point z_0 inside Ω. Show that if $f(z)$ is bounded in near z_0, then $\int_\Delta f(z)dz = 0$ for all triangles Δ in Ω.

3.18 Question 18

Assume f is continuous in the region: $0 < |z-a| \leq R,\ 0 \leq \arg(z-a) \leq \beta_0\ (0 < \beta_0 \leq 2\pi)$ and the limit $\lim_{z\to a}(z-a)f(z) = A$ exists. Show that

$$\lim_{r\to 0}\int_{\gamma_r} f(z)dz = iA\beta_0\ ,$$

where

$$\gamma_r := \{z \mid z = a + re^{it},\ 0 \leq t \leq \beta_0\}.$$

3.19 Question 19

Show that $f(z) = z^2$ is uniformly continuous in any open disk $|z| < R$, where $R > 0$ is fixed, but it is not uniformly continuous on $\mathbb{C}$.

3.20 Question 20

(1) Show that the function $u = u(x,y)$ given by

$$u(x,y) = \frac{e^{ny} - e^{-ny}}{2n^2}\sin nx \quad \text{for } n \in \mathbf{N}$$

is the solution on $D = \{(x,y)\ |x^2 + y^2 < 1\}$ of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,0) = \frac{\sin nx}{n}.$$

(2) Show that there exist points $(x,y) \in D$ such that $\limsup_{n\to\infty}|u(x,y)| = \infty$.

3.21 Question 21

(1) Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}\ .$$

(2) Deduce Liouville's theorem from (1).

3.22 Question 22

Let f be a continuous function in the region

$$D = \{z \,||z| > R, 0 \leq \arg Z \leq \theta\} \quad \text{where} \quad 0 \leq \theta \leq 2\pi.$$

If there exists k such that $\lim\limits_{z\to\infty} zf(z) = k$ for z in the region D. Show that

$$\lim_{R'\to\infty} \int_L f(z)dz = i\theta k,$$

where L is the part of the circle $|z| = R'$ which lies in the region D.

3.23 Question 23

Evaluate $\displaystyle\int_0^\infty \frac{x \sin x}{x^2 + a^2}\, dx$.

3.24 Question 24

Let $f = u + iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

3.25 Question 25

Show that $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^n}dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$ using complex analysis, $0 < a < n$. Here n is a positive integer.

3.26 Question 26

For $s > 0$, the **gamma function** is defined by $\Gamma(s) = \displaystyle\int_0^\infty e^{-t}t^{s-1}dt$.

1. Show that the gamma function is analytic in the half-plane $\Re(s) > 0$, and is still given there by the integral formula above.

2. Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need $\Gamma(1-s) = t\displaystyle\int_0^\infty e^{-vt}(vt)^{-s}dv$ for $t > 0$.

3.27 Question 27

Suppose f is entire and there exist $A, R > 0$ and natural number N such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that

(i) f is a polynomial and

(ii) the degree of f is at least N.

3.28 Question 28

Let $f : \mathbb{C} \to \mathbb{C}$ be an injective analytic (also called univalent) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

3.29 Question 29

Let g be analytic for $|z| \leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

- Show that g has a unique fixed point in $|z| < 1$.
- What happens if we replace $|g(z)| < 1$ with $|g(z)| \leq 1$ for $|z| = 1$? Give an example if (a) is not true or give an proof if (a) is still true.
- What happens if we simply assume that f is analytic for $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z) \not\equiv z$. Can f have more than one fixed point in $|z| < 1$?

  Hint: The map $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ may be useful.

3.30 Question 30

Find a conformal map from $D = \{z : \ |z| < 1, \ |z - 1/2| > 1/2\}$ to the unit disk $\Delta = \{z : |z| < 1\}$.

3.31 Question 31

Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω. Show without using Picard's theorems that $f(z)$ is a constant.

3.32 Question 32

(1) Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} .$$

(2) Deduce Liouville's theorem from (1).

3.33 Question 33

Let $f(z)$ be entire and assume that $f(z) \leq M|z|^2$ outside some disk for some constant M. Show that $f(z)$ is a polynomial in z of degree ≤ 2.

3.34 Question 34

Let $a_n(z)$ be an analytic sequence in a domain D such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D. Show that $\sum_{n=0}^{\infty} |a_n'(z)|$ converges uniformly on bounded and closed sub-regions of D.

3.35 Question 35

Let $f(z)$ be analytic in an open set Ω except possibly at a point z_0 inside Ω. Show that if $f(z)$ is bounded in near z_0, then $\int_{\Delta} f(z)dz = 0$ for all triangles Δ in Ω.

3.36 Question 36

Assume f is continuous in the region: $0 < |z-a| \leq R,\ 0 \leq \arg(z-a) \leq \beta_0$ $(0 < \beta_0 \leq 2\pi)$ and the limit $\lim_{z\to a}(z-a)f(z) = A$ exists. Show that

$$\lim_{r\to 0} \int_{\gamma_r} f(z)dz = iA\beta_0 \ ,$$

where

$$\gamma_r := \{z \mid z = a + re^{it},\ 0 \leq t \leq \beta_0\}.$$

3.37 Question 37

Show that $f(z) = z^2$ is uniformly continuous in any open disk $|z| < R$, where $R > 0$ is fixed, but it is not uniformly continuous on $\mathbb{C}$.

(1) Show that the function $u = u(x, y)$ given by

$$u(x,y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx \quad \text{for } n \in \mathbf{N}$$

is the solution on $D = \{(x, y)\ |x^2 + y^2 < 1\}$ of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,0) = \frac{\sin nx}{n}.$$

3.38 Question 38

This question provides some insight into Cauchy's theorem. Solve the problem without using Cauchy's theorem.

1. Evaluate the integral $\int_\gamma z^n dz$ for all integers n. Here γ is any circle centered at the origin with the positive (counterclockwise) orientation.
2. Same question as (a), but with γ any circle not containing the origin.
3. Show that if $|a| < r < |b|$, then $\int_\gamma \frac{dz}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$. Here γ denotes the circle centered at the origin, of radius r, with the positive orientation.

3.39 Question 39

(1) Assume the infinite series $\sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$ and let $f(z)$ be the limit. Show that for $r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} .$$

(2) Deduce Liouville's theorem from (1).

Liouville's theorem: If $f(z)$ is entire and bounded, then f is constant.

3.40 Question 40

Let f be a continuous function in the region

$$D = \{z \, ||z| > R, 0 \leq \arg Z \leq \theta\} \quad \text{where} \quad 0 \leq \theta \leq 2\pi.$$

If there exists k such that $\lim_{z\to\infty} zf(z) = k$ for z in the region D. Show that

$$\lim_{R'\to\infty} \int_L f(z)dz = i\theta k,$$

where L is the part of the circle $|z| = R'$ which lies in the region D.

3.41 Question 41

Evaluate $\int_0^\infty \frac{x \sin x}{x^2 + a^2} \, dx.$

3.42 Question 42

Let $f = u + iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

3.43 Question 43

Show that $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^n}dx = \frac{\pi}{n\sin\frac{a\pi}{n}}$ using complex analysis, $0 < a < n$. Here n is a positive integer.

3.44 Question 44

For $s > 0$, the **gamma function** is defined by $\Gamma(s) = \displaystyle\int_0^\infty e^{-t}t^{s-1}dt$.

- Show that the gamma function is analytic in the half-plane $\Re(s) > 0$, and is still given there by the integral formula above.
- Apply the formula in the previous question to show that

  $$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

  Hint: You may need $\Gamma(1-s) = t\displaystyle\int_0^\infty e^{-vt}(vt)^{-s}dv$ for $t > 0$.

3.45 Question 45

Suppose f is entire and there exist $A, R > 0$ and natural number N such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that

(i) f is a polynomial and

(ii) the degree of f is at least N.

3.46 Question 46

Let $f : \mathbb{C} \to \mathbb{C}$ be an injective analytic (also called univalent) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

3.47 Question 47

Let g be analytic for $|z| \leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

- Show that g has a unique fixed point in $|z| < 1$.
- What happens if we replace $|g(z)| < 1$ with $|g(z)| \leq 1$ for $|z| = 1$? Give an example if (a) is not true or give an proof if (a) is still true.
- What happens if we simply assume that f is analytic for $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z) \not\equiv z$. Can f have more than one fixed point in $|z| < 1$?

  Hint: The map $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ may be useful.

3.48 Question 48

Find a conformal map from $D = \{z : \ |z| < 1, \ |z - 1/2| > 1/2\}$ to the unit disk $\Delta = \{z : |z| < 1\}$.

3.49 Question 49

Let $a_n \neq 0$ and assume that $\lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = L$. Show that $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L$. In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

3.50 Question 50

(a) Let z, w be complex numbers, such that $\bar{z}w \neq 1$. Prove that

$$\left|\frac{w - z}{1 - \overline{w}z}\right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left|\frac{w - z}{1 - \overline{w}z}\right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

(b) Prove that for fixed w in the unit disk $\mathbb{D}$, the mapping

$$F : z \mapsto \frac{w - z}{1 - \overline{w}z}$$

satisfies the following conditions:

(c) F maps $\mathbb{D}$ to itself and is holomorphic.

(ii) F interchanges 0 and w, namely, $F(0) = w$ and $F(w) = 0$.

(iii) $|F(z)| = 1$ if $|z| = 1$.

(iv) $F : \mathbb{D} \mapsto \mathbb{D}$ is bijective.

Hint: Calculate $F \circ F$.

3.51 Question 51

Use n-th roots of unity (i.e. solutions of $z^n - 1 = 0$) to show that

$$2^{n-1} \sin\frac{\pi}{n} \sin\frac{2\pi}{n} \cdots \sin\frac{(n-1)\pi}{n} = n\ .$$

Hint: $1 - \cos 2\theta = 2\sin^2\theta,\ \sin 2\theta = 2\sin\theta\cos\theta$.

3.52 Question 52

(a) Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

(b) Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \ \text{ where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

is a holomorphic function in the region $r > 0,\ -\pi < \theta < \pi$. Also show that $\log z$ defined above is not continuous in $r > 0$.

3.53 Question 53

Assume f is continuous in the region: $x \geq x_0,\ 0 \leq y \leq b$ and the limit

$$\lim_{x\to+\infty} f(x+iy) = A$$

exists uniformly with respect to y (independent of y).
Show that

$$\lim_{x\to+\infty} \int_{\gamma_x} f(z)dz = iAb\ ,$$

where $\gamma_x := \{z \mid z = x + it,\ 0 \leq t \leq b\}$.

3.54 Question 54

(Cauchy's formula for "exterior" region) Let γ be piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2. Assume $f'(z)$ exists in an open set containing γ and Ω_2 and $\lim_{z\to\infty} f(z) = A$. Show that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

3.55 Question 55

Let $f(z)$ be bounded and analytic in $\mathbb{C}$. Let $a \neq b$ be any fixed complex numbers. Show that the following limit exists

$$\lim_{R\to\infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

Use this to show that $f(z)$ must be a constant (Liouville's theorem).

3.56 Question 56

Prove by *justifying all steps* that for all $\xi \in \mathbb{C}$ we have $e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\xi} dx$.

> Hint: You may use that fact in Example 1 on p. 42 of the textbook without proof, i.e., you may assume the above is true for real values of ξ.

3.57 Question 57

Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Let $f(z) = \sum_{n=1}^{\infty} c_n z^n$ denote the power series in the open disc. Show that

(1) $c_n \neq 0$ for all large enough n's, and

(2) $\lim_{n\to\infty} \dfrac{c_n}{c_{n+1}} = z_0$.

3.58 Question 58

Let $f(z)$ be a non-constant analytic function in $|z| > 0$ such that $f(z_n) = 0$ for infinite many points z_n with $\lim_{n\to\infty} z_n = 0$. Show that $z = 0$ is an essential singularity for $f(z)$. (An example of such a function is $f(z) = \sin(1/z)$.)

3.59 Question 59

Let f be entire and suppose that $\lim_{z\to\infty} f(z) = \infty$. Show that f is a polynomial.

3.60 Question 60

Expand the following functions into Laurent series in the indicated regions:

(a) $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$, $2 < |z| < 3$, $3 < |z| < +\infty$.

(b) $f(z) = \sin \dfrac{z}{1-z}$, $0 < |z-1| < +\infty$

3.61 Question 61

Assume $f(z)$ is analytic in region D and Γ is a rectifiable curve in D with interior in D. Prove that if $f(z)$ is real for all $z \in \Gamma$, then $f(z)$ is a constant.

3.62 Question 62

Find the number of roots of $z^4 - 6z + 3 = 0$ in $|z| < 1$ and $1 < |z| < 2$ respectively.

3.63 Question 63

Prove that $z^4 + 2z^3 - 2z + 10 = 0$ has exactly one root in each open quadrant.

3.64 Question 64

(1) Let $f(z) \in H(\mathbb{D})$, $\mathrm{Re}(f(z)) > 0$, $f(0) = a > 0$. Show that

$$\left|\frac{f(z) - a}{f(z) + a}\right| \leq |z|, \quad |f'(0)| \leq 2a.$$

(2) Show that the above is still true if $\mathrm{Re}(f(z)) > 0$ is replaced with $\mathrm{Re}(f(z)) \geq 0$.

3.65 Question 65

Assume $f(z)$ is analytic in $\mathbb{D}$ and $f(0) = 0$ and is not a rotation (i.e. $f(z) \neq e^{i\theta}z$). Show that $\displaystyle\sum_{n=1}^{\infty} f^n(z)$ converges uniformly to an analytic function on compact subsets of $\mathbb{D}$, where $f^{n+1}(z) = f(f^n(z))$.

3.66 Question 66

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic and one-to-one in $|z| < 1$. For $0 < r < 1$, let D_r be the disk $|z| < r$. Show that the area of $f(D_r)$ is finite and is given by

$$S = \pi \sum_{n=1}^{\infty} n|c_n|^2 r^{2n}.$$

(Note that in general the area of $f(D_1)$ is infinite.)

3.67 Question 67

Let $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ be analytic and one-to-one in $r_0 < |z| < R_0$. For $r_0 < r < R < R_0$, let $D(r, R)$ be the annulus $r < |z| < R$. Show that the area of $f(D(r, R))$ is finite and is given by

$$S = \pi \sum_{n=-\infty}^{\infty} n|c_n|^2 (R^{2n} - r^{2n}).$$

3.68 Question 68

Let $a_n(z)$ be an analytic sequence in a domain D such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D. Show that $\sum_{n=0}^{\infty} |a_n'(z)|$ converges uniformly on bounded and closed sub-regions of D.

3.69 Question 69

Let f_n, f be analytic functions on the unit disk $\mathbb{D}$. Show that the following are equivalent.

(i) $f_n(z)$ converges to $f(z)$ uniformly on compact subsets in $\mathbb{D}$.

(ii) $\int_{|z|=r} |f_n(z) - f(z)|\,|dz|$ converges to 0 if $0 < r < 1$.

3.70 Question 70

Let f and g be non-zero analytic functions on a region Ω. Assume $|f(z)| = |g(z)|$ for all z in Ω. Show that $f(z) = e^{i\theta} g(z)$ in Ω for some $0 \leq \theta < 2\pi$.

3.71 Question 71

Suppose f is analytic in an open set containing the unit disc $\mathbb{D}$ and $|f(z)| = 1$ when $|z|$=1. Show that either $f(z) = e^{i\theta}$ for some $\theta \in \mathbb{R}$ or there are finite number of $z_k \in \mathbb{D}$, $k \leq n$ and $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} \prod_{k=1}^{n} \frac{z - z_k}{1 - \bar{z}_k z}$.

Also cf. Stein et al, 1.4.7, 3.8.17

3.72 Question 72

(1) Let $p(z)$ be a polynomial, $R > 0$ any positive number, and $m \geq 1$ an integer. Let

$$M_R = \sup\{|z^m p(z) - 1| : |z| = R\}.$$

Show that $M_R > 1$.

(2) Let $m \geq 1$ be an integer and $K = \{z \in \mathbb{C} : r \leq |z| \leq R\}$ where $r < R$. Show (i) using (1) as well as, (ii) without using (1) that there exists a positive number $\varepsilon_0 > 0$ such that for each polynomial $p(z)$,

$$\sup\{|p(z) - z^{-m}| : z \in K\} \geq \varepsilon_0\,.$$

3.73 Question 73

Let $f(z) = \dfrac{1}{z} + \dfrac{1}{z^2 - 1}$. Find all the Laurent series of f and describe the largest annuli in which these series are valid.

3.74 Question 74

Suppose f is entire and there exist $A, R > 0$ and natural number N such that $|f(z)| \leq A|z|^N$ for $|z| \geq R$. Show that

(i) f is a polynomial and

(ii) the degree of f is at most N.

3.75 Question 75

(1) Explicitly write down an example of a non-zero analytic function in $|z| < 1$ which has infinitely zeros in $|z| < 1$.

(2) Why does not the phenomenon in (1) contradict the uniqueness theorem?

3.76 Question 76

(1) Assume u is harmonic on open set O and z_n is a sequence in O such that $u(z_n) = 0$ and $\lim z_n \in O$. Prove or disprove that u is identically zero. What if O is a region?

(2) Assume u is harmonic on open set O and $u(z) = 0$ on a disc in O. Prove or disprove that u is identically zero. What if O is a region?

(3) Formulate and prove a Schwarz reflection principle for harmonic functions

    cf. Theorem 5.6 on p.60 of Stein et al.

    Hint: Verify the mean value property for your new function obtained by Schwarz reflection principle.

3.77 Question 77

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s < r$, there exists a constant $c > 0$ such that

$$||f||_{(\infty,s)} \leq c||f||_{(1,r)},$$

where $|f||_{(\infty,s)} = \sup_{z \in D_s(z_0)} |f(z)|$ and $||f||_{(1,r)} = \displaystyle\int_{D_r(z_0)} |f(z)| dxdy$.

> Note: Exercise 3.8.20 on p.107 in Stein et al is a straightforward consequence of this stronger result using the integral form of the Cauchy-Schwarz inequality in real analysis.

3.78 Question 78

(1) Let f be analytic in $\Omega : 0 < |z-a| < r$ except at a sequence of poles $a_n \in \Omega$ with $\lim_{n\to\infty} a_n = a$. Show that for any $w \in \mathbb{C}$, there exists a sequence $z_n \in \Omega$ such that $\lim_{n\to\infty} f(z_n) = w$.

(2) Explain the similarity and difference between the above assertion and the Weierstrass-Casorati theorem.

3.79 Question 79

Compute the following integrals.
$i \displaystyle\int_0^\infty \frac{1}{(1+x^n)^2}\,dx,\ n \geq 1$ (ii) $\displaystyle\int_0^\infty \frac{\cos x}{(x^2+a^2)^2}\,dx,\ a \in \mathbb{R}$ (iii) $\displaystyle\int_0^\pi \frac{1}{a+\sin\theta}\,d\theta,\ a > 1$
$iv \displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{a+\sin^2\theta},\ a > 0.$ (v) $\displaystyle\int_{|z|=2} \frac{1}{(z^5-1)(z-3)}\,dz$ (v) $\displaystyle\int_{-\infty}^\infty \frac{\sin \pi a}{\cosh \pi x + \cos \pi a} e^{-ix\xi}\,dx,\ 0 < a < 1,\ \xi \in \mathbb{R}$ (vi) $\displaystyle\int_{|z|=1} \cot^2 z\,dz.$

3.80 Question 80

Compute the following integrals.
$i \displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ (ii) $\displaystyle\int_0^\infty (\frac{\sin x}{x})^2\,dx$ (iii) $\displaystyle\int_0^\infty \frac{x^{a-1}}{(1+x)^2}\,dx,\ 0 < a < 2$
$i \displaystyle\int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx,\ a, b > 0$ (ii) $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^n}\,dx,\ 0 < a < n$
$iii \displaystyle\int_0^\infty \frac{\log x}{1+x^n}\,dx,\ n \geq 2$ (iv) $\displaystyle\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$ (v) $\displaystyle\int_0^\pi \log|1 - a\sin\theta| d\theta,\ a \in \mathbb{C}$

3.81 Question 81

Let $0 < r < 1$. Show that polynomials $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ have no zeros in $|z| < r$ for all sufficiently large n's.

3.82 Question 82

Let f be an analytic function on a region Ω. Show that f is a constant if there is a simple closed curve γ in Ω such that its image $f(\gamma)$ is contained in the real axis.

3.83 Question 83

(1) Show that $\dfrac{\pi^2}{\sin^2 \pi z}$ and $g(z) = \displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ have the same principal part at each integer point.

(2) Show that $h(z) = \dfrac{\pi^2}{\sin^2 \pi z} - g(z)$ is bounded on $\mathbb{C}$ and conclude that $\dfrac{\pi^2}{\sin^2 \pi z} = \displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$.

3.84 Question 84

Let $f(z)$ be an analytic function on $\mathbb{C}\backslash\{z_0\}$, where z_0 is a fixed point. Assume that $f(z)$ is bijective from $\mathbb{C}\backslash\{z_0\}$ onto its image, and that $f(z)$ is bounded outside $D_r(z_0)$, where r is some fixed positive number. Show that there exist $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$, $c \neq 0$ such that $f(z) = \dfrac{az+b}{cz+d}$.

3.85 Question 85

Assume $f(z)$ is analytic in $\mathbb{D} : |z| < 1$ and $f(0) = 0$ and is not a rotation (i.e. $f(z) \neq e^{i\theta} z$). Show that $\displaystyle\sum_{n=1}^{\infty} f^n(z)$ converges uniformly to an analytic function on compact subsets of $\mathbb{D}$, where $f^{n+1}(z) = f(f^n(z))$.

3.86 Question 86

Let f be a non-constant analytic function on $\mathbb{D}$ with $f(\mathbb{D}) \subseteq \mathbb{D}$. Use $\psi_a(f(z))$ (where $a = f(0)$, $\psi_a(z) = \dfrac{a-z}{1-\bar{a}z}$) to prove that

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}.$$

3.87 Question 87

Find a conformal map

1. From $\{z : |z - 1/2| > 1/2, \mathrm{Re}(z) > 0\}$ to $\mathbb{H}$
2. From $\{z : |z - 1/2| > 1/2, |z| < 1\}$ to $\mathbb{D}$
3. From the intersection of the disk $|z + i| < \sqrt{2}$ with $\mathbb{H}$ to $\mathbb{D}$.
4. From $\mathbb{D}\backslash[a, 1)$ to $\mathbb{D}\backslash[0, 1)$ $(0 < a < 1)$.

   Hint: Short solution possible using Blaschke factor

5. From $\{z : |z| < 1, \mathrm{Re}(z) > 0\}\backslash(0, 1/2]$ to $\mathbb{H}$.

3.88 Question 88

Let C and C' be two circles and let $z_1 \in C$, $z_2 \notin C$, $z_1' \in C'$, $z_2' \notin C'$. Show that there is a unique fractional linear transformation f with $f(C) = C'$ and $f(z_1) = z_1'$, $f(z_2) = z_2'$.

3.89 Question 89

Assume $f_n \in H(\Omega)$ is a sequence of holomorphic functions on the region Ω that are uniformly bounded on compact subsets and $f \in H(\Omega)$ is such that the set $\{z \in \Omega : \lim_{n\to\infty} f_n(z) = f(z)\}$ has a limit point in Ω. Show that f_n converges to f uniformly on compact subsets of Ω.

3.90 Question 90

Let $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ with $|\alpha| < 1$ and $\mathbb{D} = \{z : \ |z| < 1\}$. Prove that

- $\dfrac{1}{\pi} \displaystyle\iint_{\mathbb{D}} |\psi_\alpha'|^2 dxdy = 1.$
- $\dfrac{1}{\pi} \displaystyle\iint_{\mathbb{D}} |\psi_\alpha'| dxdy = \dfrac{1 - |\alpha|^2}{|\alpha|^2} \log \dfrac{1}{1 - |\alpha|^2}.$

3.91 Question 91

Prove that $f(z) = -\dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$ is a conformal map from half disc $\{z = x + iy : \ |z| < 1, \ y > 0\}$ to upper half plane $\mathbb{H} = \{z = x + iy : \ y > 0\}$.

3.92 Question 92

Let Ω be a simply connected open set and let γ be a simple closed contour in Ω and enclosing a bounded region U anticlockwise. Let $f : \ \Omega \to \mathbb{C}$ be a holomorphic function and $|f(z)| \leq M$ for all $z \in \gamma$. Prove that $|f(z)| \leq M$ for all $z \in U$.

3.93 Question 93

Compute the following integrals.

(i) $\displaystyle\int_0^\infty \frac{x^{a-1}}{1 + x^n}\, dx$, $0 < a < n$

(ii) $\displaystyle\int_0^\infty \frac{\log x}{(1 + x^2)^2}\, dx$

3.94 Question 94

Let $0 < r < 1$. Show that the polynomials $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ have no zeros in $|z| < r$ for all sufficiently large n's.

3.95 Question 95

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s < r$, there exists a constant $c > 0$ such that

$$\|f\|_{(\infty,s)} \leq c\|f\|_{(1,r)},$$

where $\|f\|_{(\infty,s)} = \sup_{z \in D_s(z_0)} |f(z)|$ and $\|f\|_{(1,r)} = \displaystyle\int_{D_r(z_0)} |f(z)| dxdy$.

3.96 Question 96

Let $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ with $|\alpha| < 1$ and $\mathbb{D} = \{z : \ |z| < 1\}$. Prove that

- $\displaystyle\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_\alpha|^2 dxdy = 1.$
- $\displaystyle\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_\alpha| dxdy = \frac{1 - |\alpha|^2}{|\alpha|^2} \log \frac{1}{1 - |\alpha|^2}.$

3.97 Question 97

Let Ω be a simply connected open set and let γ be a simple closed contour in Ω and enclosing a bounded region U anticlockwise. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function and $|f(z)| \leq M$ for all $z \in \gamma$. Prove that $|f(z)| \leq M$ for all $z \in U$.

3.98 Question 98

Compute the following integrals.

(i) $\displaystyle\int_0^\infty \frac{x^{a-1}}{1 + x^n} dx$, $0 < a < n$

(ii) $\displaystyle\int_0^\infty \frac{\log x}{(1 + x^2)^2} dx$

3.99 Question 99

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s < r$, there exists a constant $c > 0$ such that

$$\|f\|_{(\infty,s)} \leq c\|f\|_{(1,r)},$$

where $\|f\|_{(\infty,s)} = \sup_{z \in D_s(z_0)} |f(z)|$ and $\|f\|_{(1,r)} = \displaystyle\int_{D_r(z_0)} |f(z)| dxdy$.

3.100 Question 100

Let $u(x, y)$ be harmonic and have continuous partial derivatives of order three in an open disc of radius $R > 0$.

(a) Let two points $(a, b), (x, y)$ in this disk be given. Show that the following integral is independent of the path in this disk joining these points:

$$v(x, y) = \int_{a,b}^{x,y} (-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy).$$

(b)

(i) Prove that $u(x, y) + iv(x, y)$ is an analytic function in this disc.

(ii) Prove that $v(x, y)$ is harmonic in this disc.

3.101 Question 101

(a) $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain $D \subset \mathbb{C}$. Let $z_0 = (x_0, y_0)$ be a point in D which is in the intersection of the curves $u(x, y) = c_1$ and $v(x, y) = c_2$, where c_1 and c_2 are constants. Suppose that $f'(z_0) \neq 0$. Prove that the lines tangent to these curves at z_0 are perpendicular.

(b) Let $f(z) = z^2$ be defined in $\mathbb{C}$.

(c) Describe the level curves of $\text{Re}(f)$ and of $\text{Im}(f)$.

(ii) What are the angles of intersections between the level curves $\text{Re}(f) = 0$ and $\text{Im}(f)$? Is your answer in agreement with part a) of this question?

3.102 Question 102

(a) Let $f : D \to \mathbb{C}$ be a continuous function, where $D \subset \mathbb{C}$ is a domain. Let $\alpha : [a, b] \to D$ be a smooth curve. Give a precise definition of the *complex line integral*

$$\int_\alpha f.$$

(b) Assume that there exists a constant M such that $|f(\tau)| \leq M$ for all $\tau \in \text{Image}(\alpha)$. Prove that

$$|\int_\alpha f| \leq M \times \text{length}(\alpha).$$

(c) Let C_R be the circle $|z| = R$, described in the counterclockwise direction, where $R > 1$. Provide an upper bound for $\left| \int_{C_R} \frac{\log(z)}{z^2} \right|$ which depends *only* on R and other constants.

3.103 Question 103

(a) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Assume the existence of a non-negative integer m, and of positive constants L and R, such that for all z with $|z| > R$ the inequality

$$|f(z)| \leq L|z|^m$$

holds. Prove that f is a polynomial of degree $\leq m$.

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose that there exists a real number M such that for all $z \in \mathbb{C}$

$$\mathrm{Re}(f) \leq M.$$

Prove that f must be a constant.

3.104 Question 104

Prove that all the roots of the complex polynomial

$$z^7 - 5z^3 + 12 = 0$$

lie between the circles $|z| = 1$ and $|z| = 2$.

3.105 Question 105

Let F be an analytic function inside and on a simple closed curve C, except for a pole of order $m \geq 1$ at $z = a$ inside C. Prove that

$$\frac{1}{2\pi i} \oint_C F(\tau) d\tau = \lim_{\tau \to a} \frac{d^{m-1}}{d\tau^{m-1}} \big((\tau - a)^m F(\tau)\big).$$

3.106 Question 106

Find the conformal map that takes the upper half-plane comformally onto the half-strip $\{w = x + iy : \ -\pi/2 < x < \pi/2 \ y > 0\}$.

3.107 Question 107

Compute the integral $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{\cosh \pi x} dx$ where $\cosh z = \dfrac{e^z + e^{-z}}{2}$.

3.108 Question 108

Find the number of zeroes, counting multiplicities, of the polynomial
$f(z) = 2z^5 - 6z^2 - z + 1 = 0$
in the annulus $1 \leq |z| \leq 2$.

3.109 Question 109

Find an analytic isomorphism from the open region between $|z| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$ to the upper half plane $\Im z > 0$. (You may leave your result as a composition of functions).

3.110 Question 110

Use Green theorem or otherwise to prove the Cauchy theorem.

3.111 Question 111

State and prove the divergence theorem on any rectangle in $\mathbb{R}^2$.

3.112 Question 112

Find an analytic isomorphism from the open region between $x = 1$ and $x = 3$ to the upper half unit disk $\{|z| < 1, \Im z > 0\}$. (You may leave your result as a composition of functions)

3.113 Question 113

Use Cauchy's theorem to prove the argument principle.

3.114 Question 114

Evaluate the following by the method of residues: $\int_0^{\pi/2} \frac{1}{3+\sin^2 x} dx$

3.115 Question 115

Evaluate the improper integral
$\int_0^{\infty} \frac{x^2\ dx}{(x^2+1)(x^2+4)}$

3.116 Question 116

Use residues to compute the integral

$$\int_0^{\infty} \frac{\cos x}{(x^2+1)^2} \mathrm{d}x$$

3.117 Question 117

State and prove the Cauchy integral formula for holomorphic functions.

3.118 Question 118

Let f be an entire function and suppose that $|f(z)| \leq A|z|^2$ for all z and some constant A. Show that f is a polynomial of degree ≤ 2.

3.119 Question 119

1. State the Schwarz lemma for analytic functions in the unit disc.

2. Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic map from the unit disc $\mathbb{D}$ into itself. Use the Schwarz lemma to show that for each $a \in \mathbb{D}$ we have

$$\frac{|f'(a)|}{1-|f(a)|^2} \leq \frac{1}{1-|a|^2}$$

3.120 Question 120

State the Riemann mapping theorem and prove the uniqueness part.

3.121 Question 121

Compute the integrals

$$\int_{|z-2|=1} \frac{e^z}{z(z-1)^2}\,\mathrm{d}z, \quad \int_0^\infty \frac{\cos 2x}{x^2+2}\,\mathrm{d}x$$

3.122 Question 122

Let (f_n) be a sequence of holomorphic functions in a domain D. Suppose that $f_n \to f$ uniformly on each compact subset of D. Show that

- f is holomorphic on D.
- $f_n' \to f'$ uniformly on each compact subset of D.

3.123 Question 123

If f is a non-constant entire function, then $f(\mathbb{C})$ is dense in the plane.

3.124 Question 124

1. State Rouche's theorem.

2. Let f be analytic in a neighborhood of 0, and satisfying $f'(0) \neq 0$. Use Rouche's theorem to show that there exists a neighborhood U of 0 such that f is a bijection in U.

3.125 Question 125

Let f be a meromorphic function in the plane such that

$$\lim_{|z|\to\infty} |f(z)| = \infty$$

1. Show that f has only finitely many poles.

2. Show that f is a rational function.

4 Topology (158 Questions)

4.1 Question 1

Suppose (X, d) is a metric space. State criteria for continuity of a function $f : X \to X$ in terms of:

i. open sets;

ii. ε's and δ's; and

iii. convergent sequences.

Then prove that (iii) implies (i).

4.2 Question 2

Let X be a topological space.

i. State what it means for X to be compact.

ii. Let $X = \{0\} \cup \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{Z}^+ \right\}$. Is X compact?

iii. Let $X = (0, 1]$. Is X compact?

4.3 Question 3

Let (X, d) be a compact metric space, and let $f : X \to X$ be an isometry:

$$\forall\, x, y \in X, \qquad d(f(x), f(y)) = d(x, y).$$

Prove that f is a bijection.

4.4 Question 4

Suppose (X, d) is a compact metric space and U is an open covering of X.
Prove that there is a number $\delta > 0$ such that for every $x \in X$, the ball of radius δ centered at x is contained in some element of U.

4.5 Question 5

Let X be a topological space, and $B \subset A \subset X$. Equip A with the subspace topology, and write $\mathrm{cl}_X(B)$ or $\mathrm{cl}_A(B)$ for the closure of B as a subset of, respectively, X or A.
Determine, with proof, the general relationship between $\mathrm{cl}_X(B) \cap A$ and $\mathrm{cl}_A(B)$

> I.e., are they always equal? Is one always contained in the other but not conversely? Neither?

4.6 Question 6

Prove that the unit interval I is compact. Be sure to explicitly state any properties of $\mathbb{R}$ that you use.

4.7 Question 7

A topological space is **sequentially compact** if every infinite sequence in X has a convergent subsequence.
Prove that every compact metric space is sequentially compact.

4.8 Question 8

Show that for any two topological spaces X and Y , $X \times Y$ is compact if and only if both X and Y are compact.

4.9 Question 9

Recall that a topological space is said to be **connected** if there does not exist a pair U, V of disjoint nonempty subsets whose union is X.

i. Prove that X is connected if and only if the only subsets of X that are both open and closed are X and the empty set.

ii. Suppose that X is connected and let $f : X \to \mathbb{R}$ be a continuous map. If a and b are two points of X and r is a point of $\mathbb{R}$ lying between $f(a)$ and $f(b)$ show that there exists a point c of X such that $f(c) = r$.

4.10 Question 10

Let

$$X = \left\{(0, y) \;\middle|\; -1 \leq y \leq 1\right\} \cup \left\{\left(x, s = \sin\left(\frac{1}{x}\right)\right) \;\middle|\; 0 < x \leq 1\right\}.$$

Prove that X is connected but not path connected.

4.11 Question 11

Let

$$X = \left\{(x, y) \in \mathbb{R}^2 | x > 0, y \geq 0, \text{ and } \frac{y}{x} \text{ is rational } \right\}$$

and equip X with the subspace topology induced by the usual topology on $\mathbb{R}^2$.
Prove or disprove that X is connected.

4.12 Question 12

Write Y for the interval $[0, \infty)$, equipped with the usual topology.
Find, with proof, all subspaces Z of Y which are retracts of Y.

4.13 Question 13

a. Prove that if the space X is connected and locally path connected then X is path connected.

b. Is the converse true? Prove or give a counterexample.

4.14 Question 14

Let $\left\{ X_\alpha \;\middle|\; \alpha \in A \right\}$ be a family of connected subspaces of a space X such that there is a point $p \in X$ which is in each of the X_α.
Show that the union of the X_α is connected.

4.15 Question 15

Let X be a topological space.

a. Prove that X is connected if and only if there is no continuous nonconstant map to the discrete two-point space $\{0, 1\}$.

b. Suppose in addition that X is compact and Y is a connected Hausdorff space. Suppose further that there is a continuous map $f : X \to Y$ such that every preimage $f^{-1}(y)$ for $y \in Y$, is a connected subset of X.

   Show that X is connected.

c. Give an example showing that the conclusion of (b) may be false if X is not compact.

4.16 Question 16

If X is a topological space and $S \subset X$, define in terms of open subsets of X what it means for S **not** to be connected.
Show that if S is not connected there are nonempty subsets $A, B \subset X$ such that

$$A \cup B = S \quad \text{and} \quad A \cap \overline{B} = \overline{A} \cap B = \emptyset$$

Here $\overline{A}$ and $\overline{B}$ denote closure with respect to the topology on the ambient space X.

4.17 Question 17

A topological space is **totally disconnected** if its only connected subsets are one-point sets.
Is it true that if X has the discrete topology, it is totally disconnected?
Is the converse true? Justify your answers.

4.18 Question 18

Prove that if (X, d) is a compact metric space, $f : X \to X$ is a continuous map, and C is a constant with $0 < C < 1$ such that

$$d(f(x), f(y)) \leq C \cdot d(x, y) \quad \forall x, y,$$

then f has a fixed point.

4.19 Question 19

Prove that the product of two connected topological spaces is connected.

4.20 Question 20

a. Define what it means for a topological space to be:

   i. **Connected**

   ii. **Locally connected**

b. Give, with proof, an example of a space that is connected but not locally connected.

4.21 Question 21

Let X and Y be topological spaces and let $f : X \to Y$ be a function.
Suppose that $X = A \cup B$ where A and B are closed subsets, and that the restrictions $f \Big|_A$ and $f \Big|_B$ are continuous (where A and B have the subspace topology).
Prove that f is continuous.

4.22 Question 22

Let X be a compact space and let $f : X \times R \to R$ be a continuous function such that $f(x, 0) > 0$ for all $x \in X$.
Prove that there is $\varepsilon > 0$ such that $f(x, t) > 0$ whenever $|t| < \varepsilon$.
Moreover give an example showing that this conclusion may not hold if X is not assumed compact.

4.23 Question 23

Define a family $\mathcal{T}$ of subsets of $\mathbb{R}$ by saying that $A \in T$ is $\iff$ $A = \emptyset$ or $\mathbb{R} \setminus A$ is a finite set.
Prove that $\mathcal{T}$ is a topology on $\mathbb{R}$, and that $\mathbb{R}$ is compact with respect to this topology.

4.24 Question 24

In each part of this problem X is a compact topological space.
Give a proof or a counterexample for each statement.

a. If $\{F_n\}_{n=1}^{\infty}$ is a sequence of nonempty *closed* subsets of X such that $F_{n+1} \subset F_n$ for all n then

$$\cap_{n=1}^{\infty} F_n \neq \emptyset.$$

b. If $\{O_n\}_{n=1}^{\infty}$ is a sequence of nonempty *open* subsets of X such that $O_{n+1} \subset O_n$ for all n then

$$\cap_{n=1}^{\infty} O_n \neq \emptyset.$$

4.25 Question 25

Let $\mathcal{S}, \mathcal{T}$ be topologies on a set X. Show that $\mathcal{S} \cap \mathcal{T}$ is a topology on X.
Give an example to show that $\mathcal{S} \cup \mathcal{T}$ need not be a topology.

4.26 Question 26

Let $f : X \to Y$ be a continuous function between topological spaces.
Let A be a subset of X and let $f(A)$ be its image in Y .
One of the following statements is true and one is false. Decide which is which, prove the true statement, and provide a counterexample to the false statement:

1. If A is closed then $f(A)$ is closed.

2. If A is compact then $f(A)$ is compact.

4.27 Question 27

A metric space is said to be **totally bounded** if for every $\varepsilon > 0$ there exists a finite cover of X by open balls of radius ε.

a. Show: a metric space X is totally bounded iff every sequence in X has a Cauchy subsequence.

b. Exhibit a complete metric space X and a closed subset A of X that is bounded but not totally bounded.

   You are not required to prove that your example has the stated properties.

4.28 Question 28

Suppose that X is a Hausdorff topological space and that $A \subset X$.
Prove that if A is compact in the subspace topology then A is closed as a subset of X.

4.29 Question 29

a. Show that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

b. Give an example that shows that the "Hausdorff" hypothesis in part (a) is necessary.

4.30 Question 30

Let X be a topological space and let

$$\Delta = \left\{ (x, y) \in X \times X \;\middle|\; x = y \right\}.$$

Show that X is a Hausdorff space if and only if Δ is closed in $X \times X$.

4.31 Question 31

If f is a function from X to Y , consider the graph

$$G = \left\{ (x, y) \in X \times Y \;\middle|\; f(x) = y \right\}.$$

a. Prove that if f is continuous and Y is Hausdorff, then G is a closed subset of $X \times Y$.

b. Prove that if G is closed and Y is compact, then f is continuous.

4.32 Question 32

Let X be a noncompact locally compact Hausdorff space, with topology $\mathcal{T}$. Let $\tilde{X} = X \cup \{\infty\}$ (X with one point adjoined), and consider the family $\mathcal{B}$ of subsets of $\tilde{X}$ defined by

$$\mathcal{B} = T \cup \left\{ S \cup \{\infty\} \;\middle|\; S \subset X,\ X \backslash S \text{ is compact} \right\}.$$

a. Prove that $\mathcal{B}$ is a topology on $\tilde{X}$, that the resulting space is compact, and that X is dense in $\tilde{X}$.

b. Prove that if $Y \supset X$ is a compact space such that X is dense in Y and $Y \backslash X$ is a singleton, then Y is homeomorphic to $\tilde{X}$.

   The space $\tilde{X}$ is called the **one-point compactification** of X.

c. Find familiar spaces that are homeomorphic to the one point compactifications of

   i. $X = (0, 1)$ and

4.33 Question 33

Prove that a metric space X is **normal**, i.e. if $A, B \subset X$ are closed and disjoint then there exist open sets $A \subset U \subset X,\ B \subset V \subset X$ such that $U \cap V = \emptyset$.

4.34 Question 34

Prove that every compact, Hausdorff topological space is normal.

4.35 Question 35

Show that a connected, normal topological space with more than a single point is uncountable.

4.36 Question 36

Give an example of a quotient map in which the domain is Hausdorff, but the quotient is not.

4.37 Question 37

Let X be a compact Hausdorff space and suppose $R \subset X \times X$ is a closed equivalence relation. Show that the quotient space X/R is Hausdorff.

4.38 Question 38

Let $U \subset \mathbb{R}^n$ be an open set which is bounded in the standard Euclidean metric.
Prove that the quotient space $\mathbb{R}^n/U$ is not Hausdorff.

4.39 Question 39

Let A be a closed subset of a normal topological space X.
Show that both A and the quotient X/A are normal.

4.40 Question 40

Define an equivalence relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if $x - y \in Q$. Let X be the set of equivalence classes, endowed with the quotient topology induced by the canonical projection $\pi : \mathbb{R} \to X$.
Describe, with proof, all open subsets of X with respect to this topology.

4.41 Question 41

Let A denote a subset of points of S^2 that looks exactly like the capital letter A. Let Q be the quotient of S^2 given by identifying all points of A to a single point.
Show that Q is homeomorphic to a familiar topological space and identify that space.

4.42 Question 42

a. Prove that a topological space that has a countable base for its topology also contains a countable dense subset.

b. Prove that the converse to (a) holds if the space is a metric space.

4.43 Question 43

Recall that a topological space is **regular** if for every point $p \in X$ and for every closed subset $F \subset X$ not containing p, there exist disjoint open sets $U, V \subset X$ with $p \in U$ and $F \subset V$.
Let X be a regular space that has a countable basis for its topology, and let U be an open subset of X.

a. Show that U is a countable union of closed subsets of X.

b. Show that there is a continuous function $f : X \to [0, 1]$ such that $f(x) > 0$ for $x \in U$ and $f(x) = 0$ for $x \in U$.

4.44 Question 44

Let S^1 denote the unit circle in C, X be any topological space, $x_0 \in X$, and

$$\gamma_0, \gamma_1 : S^1 \to X$$

be two continuous maps such that $\gamma_0(1) = \gamma_1(1) = x_0$.
Prove that γ_0 is homotopic to γ_1 if and only if the elements represented by γ_0 and γ_1 in $\pi_1(X, x_0)$ are conjugate.

4.45 Question 45

a. State van Kampen's theorem.

b. Calculate the fundamental group of the space obtained by taking two copies of the torus $T = S^1 \times S^1$ and gluing them along a circle $S^1 \times p$ where p is a point in S^1.

c. Calculate the fundamental group of the Klein bottle.

d. Calculate the fundamental group of the one-point union of $S^1 \times S^1$ and S^1.

e. Calculate the fundamental group of the one-point union of $S^1 \times S^1$ and $\mathbb{RP}^2$.

Note: multiple appearances!!

4.46 Question 46

Prove the following portion of van Kampen's theorem. If $X = A \cup B$ and A, B, and $A \cap B$ are nonempty and path connected with $\{\text{pt}\} \in A \cap B$, then there is a surjection

$$\pi_1(A, \{\text{pt}\}) * \pi_1(B, \{\text{pt}\}) \to \pi_1(X, \{\text{pt}\}).$$

4.47 Question 47

Let X denote the quotient space formed from the sphere S^2 by identifying two distinct points.
Compute the fundamental group and the homology groups of X.

4.48 Question 48

Start with the unit disk $\mathbb{D}^2$ and identify points on the boundary if their angles, thought of in polar coordinates, differ a multiple of $\pi/2$.
Let X be the resulting space. Use van Kampen's theorem to compute $\pi_1(X, *)$.

4.49 Question 49

Let L be the union of the z-axis and the unit circle in the xy-plane. Compute $\pi_1(\mathbb{R}^3 \backslash L, *)$.

4.50 Question 50

Let A be the union of the unit sphere in $\mathbb{R}^3$ and the interval $\{(t, 0, 0) : -1 \leq t \leq 1\} \subset \mathbb{R}^3$.
Compute $\pi_1(A)$ and give an explicit description of the universal cover of X.

4.51 Question 51

a. Let S_1 and S_2 be disjoint surfaces. Give the definition of their connected sum $S^1 \# S^2$.

b. Compute the fundamental group of the connected sum of the projective plane and the two-torus.

4.52 Question 52

Compute the fundamental group, using any technique you like, of $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$.

4.53 Question 53

Let

$$V = \mathbb{D}^2 \times S^1 = \left\{ (z, e^{it}) \;\middle|\; \|z\| \leq 1,\ 0 \leq t < 2\pi \right\}$$

be the "solid torus" with boundary given by the torus $T = S^1 \times S^1$.
For $n \in Z$ define

$$\begin{aligned} \phi_n &: T \to T \\ (e^{is}, e^{it}) &\mapsto (e^{is}, e^{i(ns+t)}). \end{aligned}$$

Find the fundamental group of the identification space

$$V_n = \frac{V \coprod V}{\sim n}$$

where the equivalence relation $\sim_n$ identifies a point x on the boundary T of the first copy of V with the point $\phi_n(x)$ on the boundary of the second copy of V.

4.54 Question 54

Let S_k be the space obtained by removing k disjoint open disks from the sphere S^2. Form X_k by gluing k Möbius bands onto S_k , one for each circle boundary component of S_k (by identifying the boundary circle of a Möbius band homeomorphically with a given boundary component circle).
Use van Kampen's theorem to calculate $\pi_1(X_k)$ for each $k > 0$ and identify X_k in terms of the classification of surfaces.

4.55 Question 55

i. Let A be a subspace of a topological space X. Define what it means for A to be a **deformation retract** of X.

ii. Consider X_1 the "planar figure eight" and

$$X_2 = S^1 \cup (0 \times [-1, 1])$$

(the "theta space"). Show that X_1 and X_2 have isomorphic fundamental groups.

iii. Prove that the fundamental group of X_2 is a free group on two generators.

4.56 Question 56

a. Give the definition of a **covering space** $\widehat{X}$ (and **covering map** $p : \widehat{X} \to X$) for a topological space X.

b. State the homotopy lifting property of covering spaces. Use it to show that a covering map $p : \widehat{X} \to X$ induces an injection

$$p^* : \pi_1(\widehat{X}, \widehat{x}) \to \pi_1(X, p(\widehat{x}))$$

on fundamental groups.

c. Let $p : \widehat{X} \to X$ be a covering map with Y and X path-connected. Suppose that the induced map p^* on π_1 is an isomorphism. Prove that p is a homeomorphism.

4.57 Question 57

a. Give the definitions of **covering space** and **deck transformation** (or covering transformation).

b. Describe the universal cover of the Klein bottle and its group of deck transformations.

c. Explicitly give a collection of deck transformations on

$$\Big\{(x,y)\ \Big|\ -1 \le x \le 1, -\infty < y < \infty\Big\}$$

such that the quotient is a Möbius band.

d. Find the universal cover of $\mathbb{RP}^2 \times S^1$ and explicitly describe its group of deck transformations.

4.58 Question 58

a. What is the definition of a **regular** (or Galois) covering space?

b. State, without proof, a criterion in terms of the fundamental group for a covering map $p : \tilde{X} \to X$ to be regular.

c. Let Θ be the topological space formed as the union of a circle and its diameter (so this space looks exactly like the letter Θ). Give an example of a covering space of Θ that is not regular.

4.59 Question 59

Let S be the closed orientable surface of genus 2 and let C be the commutator subgroup of $\pi_1(S, *)$. Let $\tilde{S}$ be the cover corresponding to C. Is the covering map $\tilde{S} \to S$ regular?

The term "normal" is sometimes used as a synonym for regular in this context.

What is the group of deck transformations?
Give an example of a nontrivial element of $\pi_1(S, *)$ which lifts to a trivial deck transformation.

4.60 Question 60

Describe the 3-fold connected covering spaces of $S^1 \vee S^1$.

4.61 Question 61

Find all three-fold covers of the wedge of two copies of $\mathbb{RP}^2$. Justify your answer.

4.62 Question 62

Describe, as explicitly as you can, two different (non-homeomorphic) connected two-sheeted covering spaces of $\mathbb{RP}^2 \vee \mathbb{RP}^3$, and prove that they are not homeomorphic.

4.63 Question 63

Is there a covering map from

$$X_3 = \big\{x^2 + y^2 = 1\big\} \cup \big\{(x-2)^2 + y^2 = 1\big\} \cup \big\{(x+2)^2 + y^2 = 1\big\} \subset \mathbb{R}^2$$

to the wedge of two S^1's? If there is, give an example; if not, give a proof.

4.64 Question 64

a. Suppose Y is an n-fold connected covering space of the torus $S^1 \times S^1$. Up to homeomorphism, what is Y? Justify your answer.

b. Let X be the topological space obtained by deleting a disk from a torus. Suppose Y is a 3-fold covering space of X.

   What surfaces could Y be? Justify your answer, but you need not exhibit the covering maps explicitly.

4.65 Question 65

Let S be a connected surface, and let U be a connected open subset of S. Let $p : \tilde{S} \to S$ be the universal cover of S. Show that $p^{-1}(U)$ is connected if and only if the homeomorphism $i_* : \pi_1(U) \to \pi_1(S)$ induced by the inclusion $i : U \to S$ is onto.

4.66 Question 66

Suppose that X has universal cover $p : \tilde{X} \to X$ and let $A \subset X$ be a subspace with $p(\tilde{a}) = a \in A$. Show that there is a group isomorphism

$$\ker(\pi_1(A, a) \to \pi_1(X, a)) \cong \pi_1(p^{-1}A, \bar{a}).$$

4.67 Question 67

Prove that every continuous map $f : \mathbb{RP}^2 \to S^1$ is homotopic to a constant.

Hint: think about covering spaces.

4.68 Question 68

Prove that the free group on two generators contains a subgroup isomorphic to the free group on five generators by constructing an appropriate covering space of $S^1 \vee S^1$.

4.69 Question 69

Use covering space theory to show that $\mathbb{Z}_2 * \mathbb{Z}$ (that is, the free product of $\mathbb{Z}_2$ and $\mathbb{Z}$) has two subgroups of index 2 which are not isomorphic to each other.

4.70 Question 70

a. Show that any finite index subgroup of a finitely generated free group is free. State clearly any facts you use about the fundamental groups of graphs.

b. Prove that if N is a nontrivial normal subgroup of infinite index in a finitely generated free group F , then N is not finitely generated.

4.71 Question 71

Let $p : X \to Y$ be a covering space, where X is compact, path-connected, and locally path-connected.
Prove that for each $x \in X$ the set $p^{-1}(\{p(x)\})$ is finite, and has cardinality equal to the index of $p_*(\pi_1(X, x))$ in $\pi_1(Y, p(x))$.

4.72 Question 72

Compute the homology of the one-point union of $S^1 \times S^1$ and S^1.

4.73 Question 73

a. State the **Mayer-Vietoris theorem**.

b. Use it to compute the homology of the space X obtained by gluing two solid tori along their boundary as follows. Let $\mathbb{D}^2$ be the unit disk and let S^1 be the unit circle in the complex plane $\mathbb{C}$. Let $A = S^1 \times \mathbb{D}^2$ and $B = \mathbb{D}^2 \times S^1$.

   Then X is the quotient space of the disjoint union $A \coprod B$ obtained by identifying $(z, w) \in A$ with $(zw^3, w) \in B$ for all $(z, w) \in S^1 \times S^1$.

4.74 Question 74

Let A and B be circles bounding disjoint disks in the plane $z = 0$ in $\mathbb{R}^3$. Let X be the subset of the upper half-space of $\mathbb{R}^3$ that is the union of the plane $z = 0$ and a (topological) cylinder that intersects the plane in $\partial C = A \cup B$.
Compute $H_*(X)$ using the Mayer–Vietoris sequence.

4.75 Question 75

Compute the integral homology groups of the space $X = Y \cup Z$ which is the union of the sphere

$$Y = \left\{x^2 + y^2 + z^2 = 1\right\}$$

and the ellipsoid

$$Z = \left\{x^2 + y^2 + \frac{z^2}{4} = 1\right\}.$$

4.76 Question 76

Let X consist of two copies of the solid torus $\mathbb{D}^2 \times S^1$, glued together by the identity map along the boundary torus $S^1 \times S^1$. Compute the homology groups of X.

4.77 Question 77

Use the circle along which the connected sum is performed and the Mayer-Vietoris long exact sequence to compute the homology of $\mathbb{RP}^2 \# \mathbb{RP}^2$.

4.78 Question 78

Express a Klein bottle as the union of two annuli.
Use the Mayer Vietoris sequence and this decomposition to compute its homology.

4.79 Question 79

Let X be the topological space obtained by identifying three distinct points on S^2. Calculate $H_*(X; Z)$.

4.80 Question 80

Compute H_0 and H_1 of the complete graph K_5 formed by taking five points and joining each pair with an edge.

4.81 Question 81

Compute the homology of the subset $X \subset \mathbb{R}^3$ formed as the union of the unit sphere, the z-axis, and the xy-plane.

4.82 Question 82

Let X be the topological space formed by filling in two circles $S^1 \times \{p_1\}$ and $S^1 \times \{p_2\}$ in the torus $S^1 \times S^1$ with disks.
Calculate the fundamental group and the homology groups of X.

4.83 Question 83

a. Consider the quotient space

$$T^2 = \mathbb{R}^2/\sim \quad \text{where} \quad (x, y) \sim (x + m, y + n) \text{ for } m, n \in \mathbb{Z},$$

and let A be any 2×2 matrix whose entries are integers such that $\det A = 1$.

Prove that the action of A on $\mathbb{R}^2$ descends via the quotient $\mathbb{R}^2 \to T^2$ to induce a homeomorphism $T^2 \to T^2$.

b. Using this homeomorphism of T^2, we define a new quotient space

$$T^3_A := \frac{T^2 \times \mathbb{R}}{\sim} \quad \text{where} \quad ((x, y), t) \sim (A(x, y), t + 1)$$

Compute $H_1(T^3_A)$ if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

4.84 Question 84

Give a self-contained proof that the zeroth homology $H_0(X)$ is isomorphic to $\mathbb{Z}$ for every path-connected space X.

4.85 Question 85

Give a self-contained proof that the zeroth homology $H_0(X)$ is isomorphic to $\mathbb{Z}$ for every path-connected space X.

4.86 Question 86

It is a fact that if X is a single point then $H_1(X) = \{0\}$.
One of the following is the correct justification of this fact in terms of the singular chain complex.
Which one is correct and why is it correct?

a. $C_1(X) = \{0\}$.

b. $C_1(X) \neq \{0\}$ but $\ker \partial_1 = 0$ with $\partial_1 : C_1(X) \to C_0(X)$.

c. $\ker \partial_1 \neq 0$ but $\ker \partial_1 = \operatorname{im} \partial_2$ with $\partial_2 : C_2(X) \to C_1(X)$.

4.87 Question 87

Compute the homology groups of $S^2 \times S^2$.

4.88 Question 88

Let Σ be a closed orientable surface of genus g. Compute $H_i(S^1 \times \Sigma; Z)$ for $i = 0, 1, 2, 3$.

4.89 Question 89

Prove that if A is a retract of the topological space X, then for all nonnegative integers n there is a group G_n such that $H_n(X) \cong H_n(A) \oplus G_n$.

Here H_n denotes the nth singular homology group with integer coefficients.

4.90 Question 90

Does there exist a map of degree 2013 from $S^2 \to S^2$.

4.91 Question 91

For each $n \in \mathbb{Z}$ give an example of a map $f_n : S^2 \to S^2$.
For which n must any such map have a fixed point?

4.92 Question 92

a. What is the degree of the antipodal map on the n-sphere? (No justification required)

b. Define a CW complex homeomorphic to the real projective n-space $\mathbb{RP}^n$.

c. Let $\pi : \mathbb{RP}^n \to X$ be a covering map. Show that if n is even, π is a homeomorphism.

4.93 Question 93

Let $A \subset X$. Prove that the relative homology group $H_0(X, A)$ is trivial if and only if A intersects every path component of X.

4.94 Question 94

Let $\mathbb{D}$ be a closed disk embedded in the torus $T = S^1 \times S^1$ and let X be the result of removing the interior of $\mathbb{D}$ from T . Let B be the boundary of X, i.e. the circle boundary of the original closed disk $\mathbb{D}$.

4.95 Question 95

Let $\mathbb{D}$ be a closed disk embedded in the torus $T = S^1 \times S^1$ and let X be the result of removing the interior of $\mathbb{D}$ from T . Let B be the boundary of X, i.e. the circle boundary of the original closed disk $\mathbb{D}$.
Compute $H_i(T, B)$ for all i.

4.96 Question 96

For any $n \geq 1$ let $S^n = \left\{(x_0, \cdots, x_n) \,\middle|\, \sum x_i^2 = 1\right\}$ denote the n dimensional unit sphere and let

$$E = \left\{(x_0, ..., x_n) \,\middle|\, x_n = 0\right\}$$

denote the "equator".
Find, for all k, the relative homology $H_k(S^n, E)$.

4.97 Question 97

Suppose that U and V are open subsets of a space X, with $X = U \cup V$. Find, with proof, a general formula relating the Euler characteristics of X, U, V, and $U \cap V$.

> You may assume that the homologies of $U, V, U \cap V, X$ are finite-dimensional so that their Euler characteristics are well defined.

4.98 Question 98

Describe a cell complex structure on the torus $T = S^1 \times S^1$ and use this to compute the homology groups of T.

> To justify your answer you will need to consider the attaching maps in detail.

4.99 Question 99

Let X be the space formed by identifying the boundary of a Möbius band with a meridian of the torus T^2.
Compute $\pi_1(X)$ and $H_*(X)$.

4.100 Question 100

Compute the homology of the space X obtained by attaching a Möbius band to $\mathbb{RP}^2$ via a homeomorphism of its boundary circle to the standard $\mathbb{RP}^1$ in $\mathbb{RP}^2$.

4.101 Question 101

Let X be a space obtained by attaching two 2-cells to the torus $S^1 \times S^1$, one along a simple closed curve $\{x\} \times S^1$ and the other along $\{y\} \times S^1$ for two points $x \neq y$ in S^1 .

a. Draw an embedding of X in $\mathbb{R}^3$ and calculate its fundamental group.

b. Calculate the homology groups of X.

4.102 Question 102

Let X be the space obtained as the quotient of a disjoint union of a 2-sphere S^2 and a torus $T = S^1 \times S^1$ by identifying the equator in S^2 with a circle $S^1 \times \{p\}$ in T.
Compute the homology groups of X.

4.103 Question 103

Let $X = S^2 / \{p_1 = \cdots = p_k\}$ be the topological space obtained from the 2-sphere by identifying k distinct points on it ($k \geq 2$).
Find:

a. The fundamental group of X.

b. The Euler characteristic of X.

c. The homology groups of X.

4.104 Question 104

Let X be the topological space obtained as the quotient of the sphere $S^2 = \left\{ \mathbf{x} \in \mathbb{R}^3 \;\middle|\; \|\mathbf{x}\| = 1 \right\}$ under the equivalence relation $\mathbf{x} \sim -\mathbf{x}$ for $\mathbf{x}$ in the equatorial circle, i.e. for $\mathbf{x} = (x_1, x_2, 0)$.
Calculate $H_*(X; \mathbb{Z})$ from a CW complex description of X.

4.105 Question 105

Compute, by any means available, the fundamental group and all the homology groups of the space obtained by gluing one copy A of S^2 to another copy B of S^2 via a two-sheeted covering space map from the equator of A onto the equator of B.

4.106 Question 106

Use cellular homology to calculate the homology groups of $S^n \times S^m$.

4.107 Question 107

Denote the points of $S^1 \times I$ by (z, t) where z is a unit complex number and $0 \leq t \leq 1$. Let X denote the quotient of $S^1 \times I$ given by identifying $(z, 1)$ and $(z_2, 0)$ for all $z \in S^1$.
Give a cell structure, with attaching maps, for X, and use it to compute $\pi_1(X, *)$ and $H_1(X)$.

4.108 Question 108

Let $X = S_1 \cup S_2 \subset \mathbb{R}^3$ be the union of two spheres of radius 2, one about $(1, 0, 0)$ and the other about $(-1, 0, 0)$, i.e.

$$S_1 = \left\{(x, y, z) \,\middle|\, (x-1)^2 + y^2 + z^2 = 4\right\}$$
$$S_2 = \left\{(x, y, z) \,\middle|\, (x+1)^2 + y^2 + z^2 = 4\right\}.$$

a. Give a description of X as a CW complex.

b. Write out the cellular chain complex of X.

c. Calculate $H_*(X; Z)$.

4.109 Question 109

Let M and N be finite CW complexes.

a. Describe a cellular structure of $M \times N$ in terms of the cellular structures of M and N.

b. Show that the Euler characteristic of $M \times N$ is the product of the Euler characteristics of M and N.

4.110 Question 110

Suppose the space X is obtained by attaching a 2-cell to the torus $S^1 \times S^1$.
In other words, X is the quotient space of the disjoint union of the closed disc $\mathbb{D}^2$ and the torus $S^1 \times S^1$ by the identification $x \sim f(x)$ where S^1 is the boundary of the unit disc and $f : S^1 \to S^1 \times S^1$ is a continuous map.
What are the possible homology groups of X? Justify your answer.

4.111 Question 111

Let X be the topological space constructed by attaching a closed 2-disk $\mathbb{D}^2$ to the circle S^1 by a continuous map $\partial\mathbb{D}^2 \to S^1$ of degree $d > 0$ on the boundary circle.

a. Show that every continuous map $X \to X$ has a fixed point.

b. Explain how to obtain all the connected covering spaces of X.

4.112 Question 112

Let X be a topological space obtained by attaching a 2-cell to $\mathbb{RP}^2$ via some map $f : S^1 \to \mathbb{RP}^2$.
What are the possibilities for the homology $H_*(X; Z)$?

4.113 Question 113

For any integer $n \geq 2$ let X_n denote the space formed by attaching a 2-cell to the circle S^1 via the attaching map

$$\begin{aligned} a_n : S^1 &\to S^1 \\ e^{i\theta} &\mapsto e^{in\theta}. \end{aligned}$$

a. Compute the fundamental group and the homology of X_n.

b. Exactly one of the X_n (for $n \geq 2$) is homeomorphic to a surface. Identify, with proof, both this value of n and the surface that X_n is homeomorphic to (including a description of the homeomorphism).

4.114 Question 114

Let X be a CW complex and let $\pi : Y \to X$ be a covering space.

a. Show that Y is compact iff X is compact and π has finite degree.

b. Assume that π has finite degree d. Show show that $\chi(Y) = d\chi(X)$.

c. Let $\pi : \mathbb{RP}^N \to X$ be a covering map. Show that if N is even, π is a homeomorphism.

4.115 Question 115

For topological spaces X, Y the **mapping cone** $C(f)$ of a map $f : X \to Y$ is defined to be the quotient space

$$\begin{aligned} (X \times [0,1])\coprod Y/\sim \quad &\text{where} \\ &(x,0) \sim (x',0) \quad \text{forall} \quad x, x' \in X \text{ and} \\ &(x,1) \sim f(x) \quad \text{forall} \quad x \in X. \end{aligned}$$

Let $\phi_k : S^n \to S^n$ be a degree k map for some integer k.
Find $H_i(C(\phi_k))$ for all i.

4.116 Question 116

Prove that a finite CW complex must be Hausdorff.

4.117 Question 117

State the classification theorem for surfaces (compact, without boundary, but not necessarily orientable). For each surface in the classification, indicate the structure of the first homology group and the value of the Euler characteristic.
Also, explain briefly how the 2-holed torus and the connected sum $\mathbb{RP}^2\#\mathbb{RP}^2$ fit into the classification.

4.118 Question 118

Give a list without repetitions of all compact surfaces (orientable or non-orientable and with or without boundary) that have Euler characteristic negative one.
Explain why there are no repetitions on your list.

4.119 Question 119

Describe the topological classification of all compact connected surfaces M without boundary having Euler characteristic $\chi(M) \geq -2$.
No proof is required.

4.120 Question 120

How many surfaces are there, up to homeomorphism, which are:

- Connected,
- Compact,
- Possibly with boundary,
- Possibly nonorientable, and
- With Euler characteristic -3?

Describe one representative from each class.

4.121 Question 121

Prove that the Euler characteristic of a compact surface with boundary which has k boundary components is less than or equal to $2 - k$.

4.122 Question 122

Let X be the topological space obtained as the quotient space of a regular $2n$-gon ($n \geq 2$) in $\mathbb{R}^2$ by identifying opposite edges via translations in the plane.
First show that X is a compact, orientable surface without boundary, and then identify its genus as a function of n.

4.123 Question 123

a. Show that any compact connected surface with nonempty boundary is homotopy equivalent to a wedge of circles

   Hint: you may assume that any compact connected surface without boundary is given by identifying edges of a polygon in pairs.

b. For each surface appearing in the classification of compact surfaces with nonempty boundary, say how many circles are needed in the wedge from part (a).

   Hint: you should be able to do this even if you have not done part (a).

4.124 Question 124

Let M_g^2 be the compact oriented surface of genus g.
Show that there exists a continuous map $f : M_g^2 \to S^2$ which is not homotopic to a constant map.

4.125 Question 125

Show that $\mathbb{RP}^2 \vee S^1$ is *not* homotopy equivalent to a compact surface (possibly with boundary).

4.126 Question 126

Identify (with proof, but of course you can appeal to the classification of surfaces) all of the compact surfaces without boundary that have a cell decomposition having exactly one 0-cell and exactly two 1-cells (with no restriction on the number of cells of dimension larger than 1).

4.127 Question 127

For any natural number g let Σ_g denote the (compact, orientable) surface of genus g.
Determine, with proof, all valued of g with the property that there exists a covering space $\pi : \Sigma_5 \to \Sigma_g$.

Hint: How does the Euler characteristic behave for covering spaces?

4.128 Question 128

Find *all* surfaces, orientable and non-orientable, which can be covered by a closed surface (i.e. compact with empty boundary) of genus 2. Prove that your answer is correct.

4.129 Question 129

a. Write down (without proof) a presentation for $\pi_1(\Sigma_2, p)$ where Σ_2 is a closed, connected, orientable genus 2 surface and p is any point on Σ_2 .

b. Show that $\pi_1(\Sigma_2, p)$ is not abelian by showing that it surjects onto a free group of rank 2.

c. Show that there is no covering space map from Σ_2 to $S^1 \times S^1$. You may use the fact that $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ together with the result in part (b) above.

4.130 Question 130

Give an example, with explanation, of a closed curve in a surfaces which is not nullhomotopic but is nullhomologous.

4.131 Question 131

Let M be a compact orientable surface of genus 2 without boundary.
Give an example of a pair of loops

$$\gamma_0, \gamma_1 : S^1 \to M$$

with $\gamma_0(1) = \gamma_1(1)$ such that there is a continuous map $\Gamma : [0,1] \times S^1 \to M$ such that

$$\Gamma(0,t) = \gamma_0(t), \quad \Gamma(1,t) = \gamma_1(t) \quad \text{forall} \quad t \in S^1,$$

but such that there is no such map Γ with the additional property that $\Gamma_s(1) = \gamma_0(1)$ for all $s \in [0,1]$.
(You are not required to prove that your example satisfies the stated property.)

4.132 Question 132

Let C be cylinder. Let I and J be disjoint closed intervals contained in ∂C.
What is the Euler characteristic of the surface S obtained by identifying I and J?
Can all surface with nonempty boundary and with this Euler characteristic be obtained from this construction?

4.133 Question 133

Let Σ be a compact connected surface and let $p_1, \cdots, p_k \in \Sigma$.
Prove that $H_2\left(\Sigma \setminus \cup_{i=1}^k p_i\right) = 0$.

4.134 Question 134

Prove or disprove:
Every continuous map from S^2 to S^2 has a fixed point.

4.135 Question 135

a. State the **Lefschetz Fixed Point Theorem** for a finite simplicial complex X.

b. Use degree theory to prove this theorem in case $X = S^n$.

4.136 Question 136

a. Prove that for every continuous map $f : S^2 \to S^2$ there is some x such that either $f(x) = x$ or $f(x) = -x$.

   Hint: Where $A : S^2 \to S^2$ is the antipodal map, you are being asked to prove that either f or $A \circ f$ has a fixed point.

b. Exhibit a continuous map $f : S^3 \to S^3$ such that for every $x \in S^3$, $f(x)$ is equal to neither x nor $-x$.

   Hint: It might help to first think about how you could do this for a map from S^1 to S^1.

4.137 Question 137

Show that a map $S^n \to S^n$ has a fixed point unless its degree is equal to the degree of the antipodal map $a : x \to -x$.

4.138 Question 138

Give an example of a homotopy class of maps of $S^1 \vee S^1$ each member of which must have a fixed point, and also an example of a map of $S^1 \vee S^1$ which doesn't have a fixed point.

4.139 Question 139

Prove or disprove:
Every map from $\mathbb{RP}^2 \vee \mathbb{RP}^2$ to itself has a fixed point.

4.140 Question 140

Find all homotopy classes of maps from $S^1 \times \mathbb{D}^2$ to itself such that every element of the homotopy class has a fixed point.

4.141 Question 141

Let X and Y be finite connected simplicial complexes and let $f : X \to Y$ and $g : Y \to X$ be basepoint-preserving maps.
Show that no matter how you homotope $f \vee g : X \vee Y \to X \vee Y$, there will always be a fixed point.

4.142 Question 142

Let $f = \mathrm{id}_{\mathbb{RP}^2} \vee *$ and $g = * \vee id_{S^1}$ be two maps of $\mathbb{RP}^2 \vee S^1$ to itself where $*$ denotes the constant map of a space to its basepoint.
Show that one map is homotopic to a map with no fixed points, while the other is not.

4.143 Question 143

View the torus T as the quotient space $\mathbb{R}^2/\mathbb{Z}^2$.
Let A be a 2×2 matrix with $\mathbb{Z}$ coefficients.

a. Show that the linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ descends to a continuous map $\mathcal{A} : T \to T$.

b. Show that, with respect to a suitable basis for $H_1(T; \mathbb{Z})$, the matrix A represents the map induced on H_1 by $\mathcal{A}$.

c. Find a necessary and sufficient condition on A for $\mathcal{A}$ to be homotopic to the identity.

d. Find a necessary and sufficient condition on A for $\mathcal{A}$ to be homotopic to a map with no fixed points.

4.144 Question 144

a. Use the Lefschetz fixed point theorem to show that any degree-one map $f : S^2 \to S^2$ has at least one fixed point.

b. Give an example of a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ having no fixed points.

c. Give an example of a degree-one map $f : S^2 \to S^2$ having exactly one fixed point.

4.145 Question 145

For which compact connected surfaces Σ (with or without boundary) does there exist a continuous map $f : \Sigma \to \Sigma$ that is homotopic to the identity and has no fixed point?
Explain your answer fully.

4.146 Question 146

Use the Brouwer fixed point theorem to show that an $n \times n$ matrix with nonnegative entries has a real eigenvalue.

4.147 Question 147

Prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n > 2$.

4.148 Question 148

Prove that any finite tree is contractible, where a **tree** is a connected graph that contains no closed edge paths.

4.149 Question 149

Show that any continuous map $f : \mathbb{RP}^2 \to S^1 \times S^1$ is necessarily null-homotopic.

4.150 Question 150

Prove that, for $n \geq 2$, every continuous map $f : \mathbb{RP}^n \to S^1$ is null-homotopic.

4.151 Question 151

Let $S^2 \to \mathbb{RP}^2$ be the universal covering map.
Is this map null-homotopic? Give a proof of your answer.

4.152 Question 152

Suppose that a map $f : S^3 \times S^3 \to \mathbb{RP}^3$ is not surjective.
Prove that f is homotopic to a constant function.

4.153 Question 153

Prove that there does not exist a continuous map $f : S^2 \to S^2$ from the unit sphere in $\mathbb{R}^3$ to itself such that $f(\mathbf{x}) \perp \mathbf{x}$ (as vectors in $\mathbb{R}^3$ for all $\mathbf{x} \in S^2$).

4.154 Question 154

Let f be the map of $S^1 \times [0,1]$ to itself defined by

$$f(e^{i\theta}, s) = (e^{i(\theta+2\pi s)}, s),$$

so that f restricts to the identity on the two boundary circles of $S^1 \times [0,1]$.
Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy that is stationary on both boundary circles.

> Hint: Consider what f does to the path $s \mapsto (e^{i\theta_0}, s)$ for fixed $e^{i\theta_0} \in S^1$.

4.155 Question 155

Show that $S^1 \times S^1$ is not the union of two disks (where there is no assumption that the disks intersect along their boundaries).

4.156 Question 156

Suppose that $X \subset Y$ and X is a deformation retract of Y.
Show that if X is a path connected space, then Y is path connected.

4.157 Question 157

Do one of the following:

a. Give (with justification) a contractible subset $X \subset \mathbb{R}^2$ which is not a retract of $\mathbb{R}^2$.

b. Give (with justification) two topological spaces that have the same homology groups but that are not homotopy equivalent.

4.158 Question 158

Recall that the **suspension** of a topological space, denoted SX, is the quotient space formed from $X \times [-1, 1]$ by identifying $(x, 1)$ with $(y, 1)$ for all $x, y \in X$, and also identifying $(x, -1)$ with $(y, -1)$ for all $x, y \in X$.

a. Show that SX is the union of two contractible subspaces.

b. Prove that if X is path-connected then $\pi_1(SX) = \{0\}$.

c. For all $n \geq 1$, prove that $H_n(X) \cong H_{n+1}(SX)$.