Midterm and Final Exam Questions, Fall 2019

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Contents

1 Midterm

1. Let G be a group of order p^2q for p, q prime. Show that G has a nontrivial normal subgroup.
2. Let G be a finite group and let P be a sylow p-subgroup for p prime. Show that $N(N(P)) = N(P)$ where N is the normalizer in G.
3. Show that there exist no simple groups of order 148.
4. Let p be a prime. Show that $S_p = \langle \tau, \sigma \rangle$ where τ is a transposition and σ is a p-cycle.
5. Let G be a nonabelian group of order p^3 for p prime. Show that $Z(G) = [G, G]$
6. Compute the Galois group of $f(x) = x^3 - 3x - 3 \in \mathbb{Q}[x]/\mathbb{Q}$.
7. Show that a field k of characteristic $p \neq 0$ is perfect $\iff$ for every $x \in k$ there exists a $y \in k$ such that $y^p = x$.
8. Let k be a field of characteristic $p \neq 0$ and $f \in k[x]$ irreducible. Show that $f(x) = g(x^{p^d})$ where $g(x) \in k[x]$ is irreducible and separable. Concluded that every root of f has the same multiplicity p^d in the splitting field of f over k.
9. Let $n \geq 3$ and ζ_n be a primitive nth root of unity. Show that $[\mathbb{Q}(\zeta_n + \zeta_n^{-1}) : \mathbb{Q}] = \varphi(n)/2$ for φ the totient function.
10. Let L/K be a finite normal extension

- Show that if L/K is cyclic and E/K is normal with $L/E/K$ then L/E and E/K are cyclic.
- Show that if L/K is cyclic then there exists exactly one extension E/K of degree n with $L/E/K$ for each divisor n of $[L : K]$.

2 Final

1. Let A be an abelian group, and show A is a $\mathbb{Z}$-module in a unique way.
2. Consider the $\mathbb{Z}$-submodule N of $\mathbb{Z}^3$ spanned by $f_1 = [-1, 0, 1]$, $f_2 = [2, -3, 1]$, $f_3 = [0, 3, 1]$, $f_4 = [3, 1, 5]$. Find a basis for N and describe $\mathbb{Z}^3/N$.
3. Let $R = k[x]$ for k a field and let M be the R-module given by

$$M = \frac{k[x]}{(x-1)^3} \oplus \frac{k[x]}{(x^2+1)^2} \oplus \frac{k[x]}{(x-1)(x^2+1)^4} \oplus \frac{k[x]}{(x+2)(x^2+1)^2}.$$

   Describe the elementary divisors and invariant factors of M.
4. Let $I = (2, x)$ be an ideal in $R = \mathbb{Z}[x]$, and show that I is not a direct sum of nontrivial cyclic R-modules.
5. Let R be a PID.

- Classify irreducible R-modules up to isomorphism.
- Classify indecomposable R-modules up to isomorphism.

6. Let V be a finite-dimensional k-vector space and $T : V \longrightarrow V$ a non-invertible k-linear map. Show that there exists a k-linear map $S : V \longrightarrow V$ with $T \circ S = 0$ but $S \circ T \neq 0$.
7. Let $A \in M_n(\mathbb{C})$ with $A^2 = A$. Show that A is similar to a diagonal matrix, and exhibit an explicit diagonal matrix similar to A.
8. Exhibit the rational canonical form for

- $A \in M_6(\mathbb{Q})$ with minimal polynomial $(x-1)(x^2+1)^2$.
- $A \in M_{10}(\mathbb{Q})$ with minimal polynomial $(x^2+1)^2(x^3+1)$.

9. Exhibit the rational and Jordan canonical forms for the following matrix $A \in M_4(\mathbb{C})$:

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{pmatrix}.$$

10. Show that the eigenvalues of a Hermitian matrix A are real and that $A = PDP^{-1}$ where P is an invertible matrix with orthogonal columns.