

Real Analysis Qualifying Exam Questions

D. Zack Garza

Tuesday 11th August, 2020

Contents

1 Undergraduate Analysis: Uniform Convergence	3
1.1 Fall 2018 # 1	3
1.2 Fall 2017 # 1	3
1.3 Fall 2014 # 1	3
1.4 Spring 2017 # 4	4
1.5 Spring 2015 # 1	4
1.6 Fall 2014 # 2	4
1.7 Spring 2014 # 2	4
2 General Analysis	5
2.1 Spring 2020 # 1	5
2.2 Fall 2019 # 1.	6
2.2.1 a	6
2.2.2 b	7
2.3 Fall 2018 # 4	8
2.4 Fall 2017 # 4	8
2.5 Spring 2017 # 3	9
2.6 Fall 2016 # 1	9
2.7 Fall 2016 # 5	9
2.8 Fall 2016 # 6	10
2.9 Spring 2016 # 1	10
2.10 Fall 2015 # 1	10
3 Measure Theory: Sets	10
3.1 Spring 2020 # 2	10
3.1.1 a.	10
3.1.2 b.	11
3.1.3 a	11
3.1.4 b	12
3.2 Fall 2019 # 3.	13
3.3 Spring 2019 # 2	13
3.4 Fall 2018 # 2	13
3.5 Spring 2018 # 1	13
3.6 Fall 2017 # 2	14

3.7	Spring 2017 # 2	14
3.8	Fall 2016 # 4	14
3.9	Spring 2016 # 3	15
3.10	Spring 2016 # 5	15
3.11	Fall 2015 # 2	15
3.12	Spring 2015 # 3	15
3.13	Spring 2014 # 3	15
3.14	Spring 2014 # 4	16
3.15	Spring 2017 # 1	16
3.16	Spring 2016 # 2	16
4	Measure Theory: Functions	16
4.1	Fall 2016 # 2	16
4.2	Spring 2016 # 4	16
5	Integrals: Convergence	17
5.1	Fall 2019 # 2	17
5.2	Spring 2020 # 5	19
5.3	Spring 2019 # 3	20
5.4	Fall 2018 # 6	21
5.5	Fall 2018 # 3	21
5.6	Spring 2018 # 5	21
5.7	Spring 2018 # 2	21
5.8	Fall 2016 # 3	21
5.9	Fall 2015 # 3	22
5.10	Fall 2015 # 4	22
6	Integrals: Approximation	22
6.1	Spring 2018 # 3	22
6.2	Spring 2018 # 4	22
6.3	Spring 2015 # 2	22
6.4	Fall 2014 # 4	23
7	L^1	23
7.1	Spring 2020 # 3	23
7.1.1	a	23
7.1.2	b	24
7.1.3	c	27
7.2	Fall 2019 # 5	27
7.3	Fall 2017 # 3	27
7.4	Spring 2015 # 4	27
7.5	Fall 2014 # 3	28
7.6	Spring 2014 # 1	28
8	Fubini-Tonelli	28
8.1	Spring 2020 # 4	28
8.2	Spring 2019 # 4	29
8.3	Fall 2018 # 5	30

8.4	Fall 2015 # 5	30
8.5	Spring 2014 # 5	30
9	L^2 and Fourier Analysis	30
9.1	Spring 2020 # 6	30
9.1.1	a	30
9.1.2	b	31
9.1.3	a	31
9.2	Fall 2017 # 5	32
9.3	Spring 2017 # 5	33
9.4	Spring 2015 # 6	33
9.5	Fall 2014 # 5	33
10	Functional Analysis: General	33
10.1	Fall 2019 # 4.	33
10.2	Spring 2019 # 5	34
10.2.1	a	34
10.2.2	b	34
10.3	Spring 2016 # 6	34
10.4	Spring 2015 # 5	34
10.5	Fall 2015 # 6	35
10.6	Fall 2014 # 6	35
11	Functional Analysis: Banach Spaces	35
11.1	Spring 2019 # 1	35
11.2	Spring 2017 # 5	35
11.3	Fall 2017 # 6	35

1 Undergraduate Analysis: Uniform Convergence

1.1 Fall 2018 # 1

Let $f(x) = \frac{1}{x}$. Show that f is uniformly continuous on $(1, \infty)$ but not on $(0, \infty)$.

1.2 Fall 2017 # 1

Let

$$f(x) = s \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which f does and does not converge uniformly.

1.3 Fall 2014 # 1

Let $\{f_n\}$ be a sequence of continuous functions such that $\sum f_n$ converges uniformly.

Prove that $\sum f_n$ is also continuous.

1.4 Spring 2017 # 4

Let $f(x, y)$ on $[-1, 1]^2$ be defined by

$$f(x, y) = \begin{cases} \frac{xy}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Determine if f is integrable.

1.5 Spring 2015 # 1

Let (X, d) and (Y, ρ) be metric spaces, $f : X \rightarrow Y$, and $x_0 \in X$.

Prove that the following statements are equivalent:

1. For every $\varepsilon > 0$ $\exists \delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$.
2. The sequence $\{f(x_n)\}_{n=1}^\infty \rightarrow f(x_0)$ for every sequence $\{x_n\} \rightarrow x_0$ in X .

1.6 Fall 2014 # 2

Let I be an index set and $\alpha : I \rightarrow (0, \infty)$.

1. Show that

$$\sum_{i \in I} a(i) := \sup_{\substack{J \subset I \\ J \text{ finite}}} \sum_{i \in J} a(i) < \infty \implies I \text{ is countable.}$$

2. Suppose $I = \mathbb{Q}$ and $\sum_{q \in \mathbb{Q}} a(q) < \infty$. Define

$$f(x) := \sum_{\substack{q \in \mathbb{Q} \\ q \leq x}} a(q).$$

Show that f is continuous at $x \iff x \notin \mathbb{Q}$.

1.7 Spring 2014 # 2

Let $\{a_n\}$ be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that $\sum a_n^2 < \infty$.

Note: Assume a_n, b_n are all non-negative.

2 General Analysis

2.1 Spring 2020 # 1

Prove that if $f : [0, 1] \rightarrow \mathbb{R}$ is continuous then

$$\lim_{k \rightarrow \infty} \int_0^1 kx^{k-1} f(x) dx = f(1).$$

Solution.

Concepts used:

- DCT
- Weierstrass Approximation Theorem

Solution:

- Suppose p is a polynomial, then

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_0^1 kx^{k-1} p(x) dx &= \lim_{k \rightarrow \infty} \int_0^1 \left(\frac{\partial}{\partial x} x^k \right) p(x) dx \\ &= \lim_{k \rightarrow \infty} \left[x^k p(x) \Big|_0^1 - \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx \right] \quad \text{integrating by parts} \\ &= p(1) - \lim_{k \rightarrow \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx, \end{aligned}$$

- Thus it suffices to show that

$$\lim_{k \rightarrow \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx = 0.$$

- Integrating by parts a second time yields

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx &= \lim_{k \rightarrow \infty} \frac{x^{k+1}}{k+1} p'(x) \Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \\ &= - \lim_{k \rightarrow \infty} \int_0^1 \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \\ &= - \int_0^1 \lim_{k \rightarrow \infty} \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \quad \text{by DCT} \\ &= - \int_0^1 0 \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \\ &= 0. \end{aligned}$$

- The DCT can be applied here because f'' is continuous and $[0, 1]$ is compact, so f'' is bounded on $[0, 1]$ by a constant M and

$$\int_0^1 |x^k f''(x)| \leq \int_0^1 1 \cdot M = M < \infty.$$

- Now use the Weierstrass approximation theorem:
 - If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then for every $\varepsilon > 0$ there exists a polynomial $p_\varepsilon(x)$ such that $\|f - p_\varepsilon\|_\infty < \varepsilon$.

- Thus

$$\begin{aligned}
 \left| \int_0^1 kx^{k-1} p_\varepsilon(x) dx - \int_0^1 kx^{k-1} f(x) dx \right| &= \left| \int_0^1 kx^{k-1} (p_\varepsilon(x) - f(x)) dx \right| \\
 &\leq \left| \int_0^1 kx^{k-1} \|p_\varepsilon - f\|_\infty dx \right| \\
 &= \|p_\varepsilon - f\|_\infty \cdot \left| \int_0^1 kx^{k-1} dx \right| \\
 &= \|p_\varepsilon - f\|_\infty \cdot x^k \Big|_0^1 \\
 &= \|p_\varepsilon - f\|_\infty \xrightarrow{\varepsilon \rightarrow 0} 0
 \end{aligned}$$

and the integrals are equal.

- By the first argument,

$$\int_0^1 kx^{k-1} p_\varepsilon(x) dx = p_\varepsilon(1) \text{ for each } \varepsilon$$

- Since uniform convergence implies pointwise convergence, $p_\varepsilon(1) \xrightarrow{\varepsilon \rightarrow 0} f(1)$.

2.2 Fall 2019 # 1.

Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers.

- a. Prove that if $\lim_{n \rightarrow \infty} a_n = 0$, then

$$\lim_{n \rightarrow \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

- b. Prove that if $\sum_{n=1}^\infty \frac{a_n}{n}$ converges, then

$$\lim_{n \rightarrow \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

Solution.

Cesaro mean/summation. Break series apart into pieces that can be handled separately.

2.2.1 a

Prove a stronger result:

$$a_k \rightarrow S \implies S_N := \frac{1}{N} \sum_{k=1}^N a_k \rightarrow S.$$

Idea: once N is large enough, $a_k \approx S$, and all smaller terms will die off as $N \rightarrow \infty$.
See this MSE answer.

- Use convergence $a_k \rightarrow S$: choose M large enough such that

$$k \geq M + 1 \implies |a_k - S| < \varepsilon.$$

Then

$$\begin{aligned} \left| \left(\frac{1}{N} \sum_{k=1}^N a_k \right) - S \right| &= \frac{1}{N} \left| \left(\sum_{k=1}^N a_k \right) - NS \right| \\ &= \frac{1}{N} \left| \left(\sum_{k=1}^N a_k \right) - \sum_{k=1}^N S \right| \\ &= \frac{1}{N} \left| \sum_{k=1}^N (a_k - S) \right| \\ &\leq \frac{1}{N} \sum_{k=1}^N |a_k - S| \\ &= \frac{1}{N} \sum_{k=1}^M |a_k - S| + \sum_{k=M+1}^N |a_k - S| \\ &\leq \frac{1}{N} \sum_{k=1}^M |a_k - S| + \sum_{k=M+1}^N \frac{\varepsilon}{2} \\ &= \frac{1}{N} \sum_{k=1}^M |a_k - S| + (N - M) \frac{\varepsilon}{2} \\ &\xrightarrow{\varepsilon \rightarrow 0} \frac{1}{N} \sum_{k=1}^M |a_k - S| + 0 \\ &\xrightarrow{N \rightarrow \infty} 0 + 0. \end{aligned}$$

Note: M is fixed, so the last sum is some constant c , and $c/N \rightarrow 0$ as $N \rightarrow \infty$ for any constant. To be more careful, choose M first to get $\varepsilon/2$ for the tail, then choose $N(M) > M$ for the remaining truncated part of the sum.

2.2.2 b

- Define

$$\Gamma_n := \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

- $\Gamma_1 = \sum_{k=1}^{\infty} \frac{a_k}{k}$ is the original series and each Γ_n is a tail of Γ_1 , so by assumption $\Gamma_n \xrightarrow{n \rightarrow \infty} 0$.
- Compute

$$\frac{1}{n} \sum_{k=1}^n a_k = \frac{1}{n} (\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \Gamma_{n+1})$$

- This comes from consider the following summation:

$\Gamma_1 :$	a_1	$+\frac{a_2}{2}$	$+\frac{a_3}{3}$	$+\cdots$
$\Gamma_2 :$		$\frac{a_2}{2}$	$+\frac{a_3}{3}$	$+\cdots$
$\Gamma_3 :$			$\frac{a_3}{3}$	$+\cdots$
<hr/>				
$\sum_{i=1}^n \Gamma_i :$	a_1	$+a_2$	$+a_3$	$+\cdots$
				a_n
				$+\frac{a_{n+1}}{n+1}$
				$+\cdots$

- Use part (a): since $\Gamma_n \xrightarrow{n \rightarrow \infty} 0$, we have $\frac{1}{n} \sum_{k=1}^n \Gamma_k \xrightarrow{n \rightarrow \infty} 0$.
- Also a minor check: $\Gamma_n \rightarrow 0 \implies \frac{1}{n} \Gamma_n \rightarrow 0$.
- Then

$$\begin{aligned}
 \frac{1}{n} \sum_{k=1}^n a_k &= \frac{1}{n} (\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \Gamma_{n+1}) \\
 &= \left(\frac{1}{n} \sum_{k=0}^n \Gamma_k \right) - \left(\frac{1}{n} \Gamma_{n+1} \right) \\
 &\xrightarrow{n \rightarrow \infty} 0.
 \end{aligned}$$

2.3 Fall 2018 # 4

Let $f \in L^1([0, 1])$. Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) |\sin nx| \, dx = \frac{2}{\pi} \int_0^1 f(x) \, dx$$

Hint: Begin with the case that f is the characteristic function of an interval.

2.4 Fall 2017 # 4

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

1. Show that $f_n \rightarrow 0$ pointwise but not uniformly on $[0, 1]$.

Hint: Consider the maximum of f_n .

2.

$$\lim_{n \rightarrow \infty} \int_0^1 n(1-x)^n \sin x \, dx = 0$$

2.5 Spring 2017 # 3

Let

$$f_n(x) = ae^{-nax} - be^{-nbx} \quad \text{where } 0 < a < b.$$

Show that

a. $\sum_{n=1}^{\infty} |f_n|$ is not in $L^1([0, \infty), m)$

Hint: $f_n(x)$ has a root x_n .

b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0, \infty), m) \quad \text{and} \quad \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x) \, dm = \ln \frac{b}{a}$$

2.6 Fall 2016 # 1

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that f converges to a differentiable function on $(1, \infty)$ and that

$$f'(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n^x} \right)'.$$

Hint:

$$\left(\frac{1}{n^x} \right)' = -\frac{1}{n^x} \ln n$$

2.7 Fall 2016 # 5

Let $\varphi \in L^\infty(\mathbb{R})$. Show that the following limit exists and satisfies the equality

$$\lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} \, dx \right)^{\frac{1}{n}} = \|\varphi\|_\infty.$$

2.8 Fall 2016 # 6

Let $f, g \in L^2(\mathbb{R})$. Show that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x)g(x+n) dx = 0$$

2.9 Spring 2016 # 1

For $n \in \mathbb{N}$, define

$$e_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad E_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

Show that $e_n < E_n$, and prove Bernoulli's inequality:

$$(1+x)^n \geq 1+nx \text{ for } -1 < x < \infty \text{ and } n \in \mathbb{N}$$

Use this to show the following:

1. The sequence e_n is increasing.
2. The sequence E_n is decreasing.
3. $2 < e_n < E_n < 4$.
4. $\lim_{n \rightarrow \infty} e_n = \lim_{n \rightarrow \infty} E_n$.

2.10 Fall 2015 # 1

Define

$$f(x) = c_0 + c_1x^1 + c_2x^2 + \dots + c_nx^n \text{ with } n \text{ even and } c_n > 0.$$

Show that there is a number x_m such that $f(x_m) \leq f(x)$ for all $x \in \mathbb{R}$.

3 Measure Theory: Sets**3.1 Spring 2020 # 2**

Let m_* denote the Lebesgue outer measure on \mathbb{R} .

3.1.1 a.

Prove that for every $E \subseteq \mathbb{R}$ there exists a Borel set B containing E such that

$$m_*(B) = m_*(E).$$

3.1.2 b.

Prove that if $E \subseteq \mathbb{R}$ has the property that

$$m_*(A) = m_*(A \cap E) + m_*(A \cap E^c)$$

for every set $A \subseteq \mathbb{R}$, then there exists a Borel set $B \subseteq \mathbb{R}$ such that $E = B \setminus N$ with $m_*(N) = 0$.

Be sure to address the case when $m_*(E) = \infty$.

Solution.

Concepts used:

- Definition of outer measure: $m_*(E) = \inf_{\{Q_j\} \Rightarrow E} \sum |Q_j|$ where $\{Q_j\}$ is a countable collection of closed cubes.
- Break \mathbb{R} into $\prod_{n \in \mathbb{Z}} [n, n+1)$, each with finite measure.
- Theorem: $m_*(Q) = |Q|$ for Q a closed cube (i.e. the outer measure equals the volume).

Proof (of Theorem) Statement: if Q is a closed cube, then $m_*(Q) = |Q|$, the usual volume.

- $m_*(Q) \leq |Q|$:
 - Since $Q \subseteq Q$, $Q \Rightarrow Q$ and $m_*(Q) \leq |Q|$ since m_* is an infimum over such coverings.
- $|Q| \leq m_*(Q)$:
 - Fix $\varepsilon > 0$.
 - Let $\{Q_i\}_{i=1}^\infty \Rightarrow Q$ be arbitrary, it suffices to show that

$$|Q| \leq \left(\sum_{i=1}^\infty |Q_i| \right) + \varepsilon.$$

- Pick open cubes S_i such that $Q_i \subseteq S_i$ and $|Q_i| \leq |S_i| \leq (1 + \varepsilon)|Q_i|$.
- Then $\{S_i\} \Rightarrow Q$, so by compactness of Q pick a finite subcover with N elements.
- Note

$$Q \subseteq \bigcup_{i=1}^N S_i \implies |Q| \leq \sum_{i=1}^N |S_i| \leq \sum_{i=1}^N (1 + \varepsilon)|Q_i| \leq (1 + \varepsilon) \sum_{i=1}^\infty |Q_i|.$$

- Taking an infimum over coverings on the RHS preserves the inequality, so

$$|Q| \leq (1 + \varepsilon)m_*(Q)$$

- Take $\varepsilon \rightarrow 0$ to obtain final inequality.

3.1.3 a

- If $m_*(E) = \infty$, then take $B = \mathbb{R}^n$ since $m(\mathbb{R}^n) = \infty$.
- Suppose $N := m_*(E) < \infty$.
- Since $m_*(E)$ is an infimum, by definition, for every $\varepsilon > 0$ there exists a covering by closed cubes $\{Q_i(\varepsilon)\}_{i=1}^\infty \Rightarrow E$ depending on ε such that

$$\sum_{i=1}^\infty |Q_i(\varepsilon)| < N + \varepsilon.$$

- For each fixed n , set $\varepsilon_n = \frac{1}{n}$ to produce such a covering $\{Q_i(\varepsilon_n)\}_{i=1}^\infty$ and set $B_n := \bigcup_{i=1}^\infty Q_i(\varepsilon_n)$.
- The outer measure of cubes is *equal* to the sum of their volumes, so

$$m_*(B_n) = \sum_{i=1}^\infty |Q_i(\varepsilon_n)| < N + \varepsilon_n = N + \frac{1}{n}.$$

- Now set $B := \bigcap_{n=1}^\infty B_n$.
 - Since $E \subseteq B_n$ for every n , $E \subseteq B$
 - Since B is a countable intersection of countable unions of closed sets, B is Borel.
 - Since $B_n \subseteq B$ for every n , we can apply subadditivity to obtain the inequality

$$E \subseteq B \subseteq B_n \implies N \leq m_*(B) \leq m_*(B_n) < N + \frac{1}{n} \quad \text{for all } n \in \mathbb{Z}^{\geq 1}.$$

- This forces $m_*(E) = m_*(B)$.

3.1.4 b

Suppose $m_*(E) < \infty$.

- By (a), find a Borel set $B \supseteq E$ such that $m_*(B) = m_*(E)$
- Note that $E \subseteq B \implies B \cap E = E$ and $B \cap E^c = B \setminus E$.
- By assumption,

$$\begin{aligned} m_*(B) &= m_*(B \cap E) + m_*(B \cap E^c) \\ m_*(E) &= m_*(E) + m_*(B \setminus E) \\ m_*(E) - m_*(E) &= m_*(B \setminus E) \quad \text{since } m_*(E) < \infty \\ \implies m_*(B \setminus E) &= 0. \end{aligned}$$

- So take $N = B \setminus E$; this shows $m_*(N) = 0$ and $E = B \setminus (B \setminus E) = B \setminus N$.

If $m_*(E) = \infty$:

- Apply result to $E_R := E \cap [R, R+1)^n \subset \mathbb{R}^n$ for $R \in \mathbb{Z}$, so $E = \bigsqcup_R E_R$
- Obtain B_R, N_R such that $E_R = B_R \setminus N_R$, $m_*(E_R) = m_*(B_R)$, and $m_*(N_R) = 0$.
- Note that
 - $B := \bigcup_R B_R$ is a union of Borel sets and thus still Borel
 - $E = \bigcup_R E_R$
 - $N := B \setminus E$
 - $N' := \bigcup_R N_R$ is a union of null sets and thus still null
- Since $E_R \subset B_R$ for every R , we have $E \subset B$
- We can compute

$$N = B \setminus E = \left(\bigcup_R B_R \right) \setminus \left(\bigcup_R E_R \right) \subseteq \bigcup_R (B_R \setminus E_R) = \bigcup_R N_R := N'$$

where $m_*(N') = 0$ since N' is null, and thus subadditivity forces $m_*(N) = 0$.

3.2 Fall 2019 # 3.

Let (X, \mathcal{B}, μ) be a measure space with $\mu(X) = 1$ and $\{B_n\}_{n=1}^\infty$ be a sequence of \mathcal{B} -measurable subsets of X , and

$$B := \left\{x \in X \mid x \in B_n \text{ for infinitely many } n\right\}.$$

- Argue that B is also a \mathcal{B} -measurable subset of X .
- Prove that if $\sum_{n=1}^\infty \mu(B_n) < \infty$ then $\mu(B) = 0$.
- Prove that if $\sum_{n=1}^\infty \mu(B_n) = \infty$ **and** the sequence of set complements $\{B_n^c\}_{n=1}^\infty$ satisfies

$$\mu\left(\bigcap_{n=k}^K B_n^c\right) = \prod_{n=k}^K (1 - \mu(B_n))$$

for all positive integers k and K with $k < K$, then $\mu(B) = 1$.

Hint: Use the fact that $1 - x \leq e^{-x}$ for all x .

3.3 Spring 2019 # 2

Let \mathcal{B} denote the set of all Borel subsets of \mathbb{R} and $\mu : \mathcal{B} \rightarrow [0, \infty)$ denote a finite Borel measure on \mathbb{R} .

- Prove that if $\{F_k\}$ is a sequence of Borel sets for which $F_k \supseteq F_{k+1}$ for all k , then

$$\lim_{k \rightarrow \infty} \mu(F_k) = \mu\left(\bigcap_{k=1}^\infty F_k\right)$$

- Suppose μ has the property that $\mu(E) = 0$ for every $E \in \mathcal{B}$ with Lebesgue measure $m(E) = 0$. Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $E \in \mathcal{B}$ with $m(E) < \delta$, then $\mu(E) < \varepsilon$.

3.4 Fall 2018 # 2

Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Show that there is a Borel set $B \subset E$ such that $m(E \setminus B) = 0$.

3.5 Spring 2018 # 1

Define

$$E := \left\{x \in \mathbb{R} : \left|x - \frac{p}{q}\right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N}\right\}.$$

Prove that $m(E) = 0$.

3.6 Fall 2017 # 2

Let $f(x) = x^2$ and $E \subset [0, \infty) := \mathbb{R}^+$.

1. Show that

$$m^*(E) = 0 \iff m^*(f(E)) = 0.$$

2. Deduce that the map

$$\begin{aligned} \varphi : \mathcal{L}(\mathbb{R}^+) &\longrightarrow \mathcal{L}(\mathbb{R}^+) \\ E &\mapsto f(E) \end{aligned}$$

is a bijection from the class of Lebesgue measurable sets of $[0, \infty)$ to itself.

3.7 Spring 2017 # 2

- a. Let μ be a measure on a measurable space (X, \mathcal{M}) and f a positive measurable function.

Define a measure λ by

$$\lambda(E) := \int_E f \, d\mu, \quad E \in \mathcal{M}$$

Show that for g any positive measurable function,

$$\int_X g \, d\lambda = \int_X fg \, d\mu$$

- b. Let $E \subset \mathbb{R}$ be a measurable set such that

$$\int_E x^2 \, dm = 0.$$

Show that $m(E) = 0$.

3.8 Fall 2016 # 4

Let (X, \mathcal{M}, μ) be a measure space and suppose $\{E_n\} \subset \mathcal{M}$ satisfies

$$\lim_{n \rightarrow \infty} \mu(X \setminus E_n) = 0.$$

Define

$$G := \left\{ x \in X \mid x \in E_n \text{ for only finitely many } n \right\}.$$

Show that $G \in \mathcal{M}$ and $\mu(G) = 0$.

3.9 Spring 2016 # 3

Let f be Lebesgue measurable on \mathbb{R} and $E \subset \mathbb{R}$ be measurable such that

$$0 < A = \int_E f(x) dx < \infty.$$

Show that for every $0 < t < 1$, there exists a measurable set $E_t \subset E$ such that

$$\int_{E_t} f(x) dx = tA.$$

3.10 Spring 2016 # 5

Let (X, \mathcal{M}, μ) be a measure space. For $f \in L^1(\mu)$ and $\lambda > 0$, define

$$\varphi(\lambda) = \mu(\{x \in X | f(x) > \lambda\}) \quad \text{and} \quad \psi(\lambda) = \mu(\{x \in X | f(x) < -\lambda\})$$

Show that φ, ψ are Borel measurable and

$$\int_X |f| d\mu = \int_0^\infty [\varphi(\lambda) + \psi(\lambda)] d\lambda$$

3.11 Fall 2015 # 2

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be Lebesgue measurable.

1. Show that there is a sequence of simple functions $s_n(x)$ such that $s_n(x) \rightarrow f(x)$ for all $x \in \mathbb{R}$.
2. Show that there is a Borel measurable function g such that $g = f$ almost everywhere.

3.12 Spring 2015 # 3

Let μ be a finite Borel measure on \mathbb{R} and $E \subset \mathbb{R}$ Borel. Prove that the following statements are equivalent:

1. $\forall \varepsilon > 0$ there exists G open and F closed such that

$$F \subseteq E \subseteq G \quad \text{and} \quad \mu(G \setminus F) < \varepsilon.$$

2. There exists a $V \in G_\delta$ and $H \in F_\sigma$ such that

$$H \subseteq E \subseteq V \quad \text{and} \quad \mu(V \setminus H) = 0$$

3.13 Spring 2014 # 3

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and suppose

$$\forall x \in \mathbb{R}, \quad f(x) \geq \limsup_{y \rightarrow x} f(y)$$

Prove that f is Borel measurable.

3.14 Spring 2014 # 4

Let (X, \mathcal{M}, μ) be a measure space and suppose f is a measurable function on X . Show that

$$\lim_{n \rightarrow \infty} \int_X f^n d\mu = \begin{cases} \infty \\ \mu(f^{-1}(1)), \end{cases} \quad \text{or}$$

and characterize the collection of functions of each type.

3.15 Spring 2017 # 1

Let K be the set of numbers in $[0, 1]$ whose decimal expansions do not use the digit 4.

We use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 with $399\cdots$. For example, $0.8754 = 0.8753999\cdots$.

Show that K is a compact, nowhere dense set without isolated points, and find the Lebesgue measure $m(K)$.

3.16 Spring 2016 # 2

Let $0 < \lambda < 1$ and construct a Cantor set C_λ by successively removing middle intervals of length λ .

Prove that $m(C_\lambda) = 0$.

4 Measure Theory: Functions**4.1 Fall 2016 # 2**

Let $f, g : [a, b] \rightarrow \mathbb{R}$ be measurable with

$$\int_a^b f(x) dx = \int_a^b g(x) dx.$$

Show that either

1. $f(x) = g(x)$ almost everywhere, or
2. There exists a measurable set $E \subset [a, b]$ such that

$$\int_E f(x) dx > \int_E g(x) dx$$

4.2 Spring 2016 # 4

Let $E \subset \mathbb{R}$ be measurable with $m(E) < \infty$. Define

$$f(x) = m(E \cap (E + x)).$$

Show that

1. $f \in L^1(\mathbb{R})$.

2. f is uniformly continuous.
3. $\lim_{|x| \rightarrow \infty} f(x) = 0$.

Hint:

$$\chi_{E \cap (E+x)}(y) = \chi_E(y) \chi_E(y-x)$$

5 Integrals: Convergence

5.1 Fall 2019 # 2.

Prove that

$$\left| \frac{d^n \sin x}{dx^n} \right| \leq \frac{1}{n}$$

for all $x \neq 0$ and positive integers n .

Hint: Consider $\int_0^1 \cos(tx) dt$

Solution.

Concepts used:

- DCT
- Bounding in the right place. Don't evaluate the actual integral!

Solution:

- By induction on the number of limits we can pass through the integral.
- For $n = 1$ we first pass one derivative into the integral: let $x_n \rightarrow x$ be any sequence converging to x , then

$$\begin{aligned} \frac{\partial}{\partial x} \frac{\sin(x)}{x} &= \frac{\partial}{\partial x} \int_0^1 \cos(tx) dt \\ &= \lim_{x_n \rightarrow x} \frac{1}{x_n - x} \left(\int_0^1 \cos(tx_n) dt - \int_0^1 \cos(tx) dt \right) \\ &= \lim_{x_n \rightarrow x} \left(\int_0^1 \frac{\cos(tx_n) - \cos(tx)}{x_n - x} dt \right) \\ &= \lim_{x_n \rightarrow x} \left(\int_0^1 \left(t \sin(tx) \Big|_{x=\xi_n} \right) dt \right) \quad \text{where } \xi_n \in [x_n, x] \text{ by MVT, } \xi_n \rightarrow x \\ &= \lim_{\xi_n \rightarrow x} \left(\int_0^1 t \sin(t\xi_n) dt \right) \\ &=_{\text{DCT}} \int_0^1 \lim_{\xi_n \rightarrow x} t \sin(t\xi_n) dt \\ &= \int_0^1 t \sin(tx) dt \end{aligned}$$

- Taking absolute values we obtain an upper bound

$$\begin{aligned} \left| \frac{\partial}{\partial x} \frac{\sin(x)}{x} \right| &= \left| \int_0^1 t \sin(tx) dt \right| \\ &\leq \int_0^1 |t \sin(tx)| dt \\ &\leq \int_0^1 1 dt = 1, \end{aligned}$$

since $t \in [0, 1] \implies |t| < 1$, and $|\sin(xt)| \leq 1$ for any x and t .

- Note that this bound also justifies the DCT, since the functions $f_n(t) = t \sin(t\xi_n)$ are uniformly dominated by $g(t) = 1$ on $L^1([0, 1])$.

Note: integrating by parts here yields the actual formula:

$$\begin{aligned} \int_0^1 t \sin(tx) dt &=_{\text{IBP}} \left(\frac{-t \cos(tx)}{x} \right) \Big|_{t=0}^{t=1} - \int_0^1 \frac{\cos(tx)}{x} dt \\ &= \frac{-\cos(x)}{x} - \frac{\sin(x)}{x^2} \\ &= \frac{x \cos(x) - \sin(x)}{x^2}. \end{aligned}$$

- For the inductive step, we assume that we can pass $n - 1$ limits through the integral and show we can pass the n th through as well.

$$\begin{aligned} \frac{\partial^n}{\partial x^n} \frac{\sin(x)}{x} &= \frac{\partial^n}{\partial x^n} \int_0^1 \cos(tx) dt \\ &= \frac{\partial}{\partial x} \int_0^1 \frac{\partial^{n-1}}{\partial x^{n-1}} \cos(tx) dt \\ &= \frac{\partial}{\partial x} \int_0^1 t^{n-1} f_{n-1}(x, t) dt \end{aligned}$$

- Note that $f_n(x, t) = \pm \sin(tx)$ when n is odd and $f_n(x, t) = \pm \cos(tx)$ when n is even, and a constant factor of t is multiplied when each derivative is taken.

- We continue as in the base case:

$$\begin{aligned} \frac{\partial}{\partial x} \int_0^1 t^{n-1} f_{n-1}(x, t) dt &= \lim_{x_k \rightarrow x} \int_0^1 t^{n-1} \left(\frac{f_{n-1}(x_k, t) - f_{n-1}(x, t)}{x_k - x} \right) dt \\ &=_{\text{IVT}} \lim_{x_k \rightarrow x} \int_0^1 t^{n-1} \frac{\partial f_{n-1}}{\partial x}(\xi_k, t) dt \quad \text{where } \xi_k \in [x_k, x], \xi_k \rightarrow x \\ &=_{\text{DCT}} \int_0^1 \lim_{x_k \rightarrow x} t^{n-1} \frac{\partial f_{n-1}}{\partial x}(\xi_k, t) dt \\ &:= \int_0^1 \lim_{x_k \rightarrow x} t^n f_n(\xi_k, t) dt \\ &:= \int_0^1 t^n f_n(x, t) dt. \end{aligned}$$

- We've used the fact that $f_0(x) = \cos(tx)$ is smooth as a function of x , and in particular continuous
- The DCT is justified because the functions $h_{n,k}(x, t) = t^n f_n(\xi_k, t)$ are again uniformly (in k) bounded by 1 since $t \leq 1 \implies t^n \leq 1$ and each f_n is a sin or cosine.

- Now take absolute values

$$\begin{aligned}
 \left| \frac{\partial^n}{\partial x^n} \frac{\sin(x)}{x} \right| &= \left| \int_0^1 -t^n f_n(x, t) \, dt \right| \\
 &\leq \int_0^1 |t^n f_n(x, t)| \, dt \\
 &\leq \int_0^1 |t^n| |f_n(x, t)| \, dt \\
 &\leq \int_0^1 |t^n| \cdot 1 \, dt \\
 &\leq \int_0^1 t^n \, dt \quad \text{since } t \text{ is positive} \\
 &= \frac{1}{n+1} \\
 &< \frac{1}{n}.
 \end{aligned}$$

- We’ve again used the fact that $f_n(x, t)$ is of the form $\pm \cos(tx)$ or $\pm \sin(tx)$, both of which are bounded by 1.

5.2 Spring 2020 # 5

Compute the following limit and justify your calculations:

$$\lim_{n \rightarrow \infty} \int_0^n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx.$$

Not finished, flesh out.

Solution.

Concepts used:

- DCT
- Passing limits through products and quotients

Note that

$$\begin{aligned}
 \lim_n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} &= \frac{1}{\lim_n \left(1 + \frac{x^2}{n}\right)^1 \left(1 + \frac{x^2}{n}\right)^n} \\
 &= \frac{1}{1 \cdot e^{x^2}} \\
 &= e^{-x^2}.
 \end{aligned}$$

If passing the limit through the integral is justified, we will have

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \int_0^n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \chi_{[0,n]} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \\
 &= \int_{\mathbb{R}} \lim_{n \rightarrow \infty} \chi_{[0,n]} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \quad \text{by the DCT} \\
 &= \int_{\mathbb{R}} \lim_{n \rightarrow \infty} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \\
 &= \int_0^\infty e^{-x^2} dx \\
 &= \frac{\sqrt{\pi}}{2}.
 \end{aligned}$$

Computing the last integral:

$$\begin{aligned}
 \left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2 &= \left(\int_{\mathbb{R}} e^{-x^2} dx\right) \left(\int_{\mathbb{R}} e^{-y^2} dx\right) \\
 &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(x+y)^2} dx \\
 &= \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta \quad u = r^2 \\
 &= \frac{1}{2} \int_0^{2\pi} \int_0^\infty e^{-u} du d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} 1 d\theta \\
 &= \pi,
 \end{aligned}$$

and now use the fact that the function is even so $\int_0^\infty f = \frac{1}{2} \int_{\mathbb{R}} f$.

Justifying the DCT:

- Apply Bernoulli's inequality:

$$1 + \frac{x^{2n+1}}{n} \geq 1 + \frac{x^2}{n} (1 + x^2) \geq 1 + x^2,$$

where the last inequality follows from the fact that $1 + \frac{x^2}{n} \geq 1$

5.3 Spring 2019 # 3

Let $\{f_k\}$ be any sequence of functions in $L^2([0, 1])$ satisfying $\|f_k\|_2 \leq M$ for all $k \in \mathbb{N}$.

Prove that if $f_k \rightarrow f$ almost everywhere, then $f \in L^2([0, 1])$ with $\|f\|_2 \leq M$ and

$$\lim_{k \rightarrow \infty} \int_0^1 f_k(x) dx = \int_0^1 f(x) dx$$

Hint: Try using Fatou's Lemma to show that $\|f\|_2 \leq M$ and then try applying Egorov's Theorem.

5.4 Fall 2018 # 6

Compute the following limit and justify your calculations:

$$\lim_{n \rightarrow \infty} \int_1^n \frac{dx}{\left(1 + \frac{x}{n}\right)^n \sqrt[n]{x}}$$

5.5 Fall 2018 # 3

Suppose $f(x)$ and $xf(x)$ are integrable on \mathbb{R} . Define F by

$$F(t) := \int_{-\infty}^{\infty} f(x) \cos(xt) dx$$

Show that

$$F'(t) = - \int_{-\infty}^{\infty} xf(x) \sin(xt) dx.$$

5.6 Spring 2018 # 5

Suppose that

- $f_n, f \in L^1$,
- $f_n \rightarrow f$ almost everywhere, and
- $\int |f_n| \rightarrow \int |f|$.

Show that $\int f_n \rightarrow \int f$

5.7 Spring 2018 # 2

Let

$$f_n(x) := \frac{x}{1 + x^n}, \quad x \geq 0.$$

- a. Show that this sequence converges pointwise and find its limit. Is the convergence uniform on $[0, \infty)$?
- b. Compute

$$\lim_{n \rightarrow \infty} \int_0^{\infty} f_n(x) dx$$

5.8 Fall 2016 # 3

Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{x \rightarrow 0} \int_{\mathbb{R}} |f(y-x) - f(y)| dy = 0$$

5.9 Fall 2015 # 3

Compute the following limit:

$$\lim_{n \rightarrow \infty} \int_1^n \frac{ne^{-x}}{1+nx^2} \sin\left(\frac{x}{n}\right) dx$$

5.10 Fall 2015 # 4

Let $f : [1, \infty) \rightarrow \mathbb{R}$ such that $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Show that the following limit exists and satisfies the equality

$$\lim_{x \rightarrow \infty} f(x) \leq 1 + \frac{\pi}{4}$$

6 Integrals: Approximation**6.1 Spring 2018 # 3**

Let f be a non-negative measurable function on $[0, 1]$.

Show that

$$\lim_{p \rightarrow \infty} \left(\int_{[0,1]} f(x)^p dx \right)^{\frac{1}{p}} = \|f\|_{\infty}.$$

6.2 Spring 2018 # 4

Let $f \in L^2([0, 1])$ and suppose

$$\int_{[0,1]} f(x)x^n dx = 0 \text{ for all integers } n \geq 0.$$

Show that $f = 0$ almost everywhere.

6.3 Spring 2015 # 2

Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be continuous with period 1. Prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(n\alpha) = \int_0^1 f(t) dt \quad \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

Hint: show this first for the functions $f(t) = e^{2\pi i k t}$ for $k \in \mathbb{Z}$.

6.4 Fall 2014 # 4

Let $g \in L^\infty([0, 1])$ Prove that

$$\int_{[0,1]} f(x)g(x) dx = 0 \quad \text{for all continuous } f : [0, 1] \longrightarrow \mathbb{R} \implies g(x) = 0 \text{ almost everywhere.}$$

7 L^1 **7.1 Spring 2020 # 3**

a. Prove that if $g \in L^1(\mathbb{R})$ then

$$\lim_{N \rightarrow \infty} \int_{|x| \geq N} |f(x)| dx = 0,$$

and demonstrate that it is not necessarily the case that $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

b. Prove that if $f \in L^1([1, \infty))$ and is decreasing, then $\lim_{x \rightarrow \infty} f(x) = 0$ and in fact $\lim_{x \rightarrow \infty} xf(x) = 0$.

c. If $f : [1, \infty) \rightarrow [0, \infty)$ is decreasing with $\lim_{x \rightarrow \infty} xf(x) = 0$, does this ensure that $f \in L^1([1, \infty))$?

Solution.

Concepts used:

- Limits
- Cauchy Criterion for Integrals: $\int_a^\infty f(x) dx$ converges iff for every $\varepsilon > 0$ there exists an M_0 such that $A, B \geq M_0$ implies $\left| \int_A^B f \right| < \varepsilon$, i.e. $\left| \int_A^B f \right| \xrightarrow{A \rightarrow \infty} 0$.
- Integrals of L^1 functions have vanishing tails: $\int_N^\infty |f| \xrightarrow{N \rightarrow \infty} 0$.
- Mean Value Theorem for Integrals: $\int_a^b f(t) dt = (b - a)f(c)$ for some $c \in [a, b]$.

7.1.1 a

Stated integral equality:

- Let $\varepsilon > 0$
- $C_c(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$ is dense so choose $\{f_n\} \rightarrow f$ with $\|f_n - f\|_1 \rightarrow 0$.
- Since $\{f_n\}$ are compactly supported, choose $N_0 \gg 1$ such that f_n is zero outside of $B_{N_0}(\mathbf{0})$.

- Then

$$\begin{aligned}
N \geq N_0 &\implies \int_{|x|>N} |f| = \int_{|x|>N} |f - f_n + f_n| \\
&\leq \int_{|x|>N} |f - f_n| + \int_{|x|>N} |f_n| \\
&= \int_{|x|>N} |f - f_n| \\
&\leq \int_{|x|>N} \|f - f_n\|_1 \\
&= \|f_n - f\|_1 \left(\int_{|x|>N} 1 \right) \\
&\stackrel{n \rightarrow \infty}{\longrightarrow} 0 \left(\int_{|x|>N} 1 \right) \\
&= 0 \\
&\stackrel{N \rightarrow \infty}{\longrightarrow} 0.
\end{aligned}$$

To see that this doesn't force $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$:

- Take $f(x)$ to be a train of rectangles of height 1 and area $1/2^j$ centered on even integers.
- Then

$$\int_{|x|>N} |f| = \sum_{j=N}^{\infty} 1/2^j \stackrel{N \rightarrow \infty}{\longrightarrow} 0$$

as the tail of a convergent sum.

- However $f(x) = 1$ for infinitely many even integers $x > N$, so $f(x) \not\rightarrow 0$ as $|x| \rightarrow \infty$.

7.1.2 b

Solution 1 (“Trick”)

- Since f is decreasing on $[1, \infty)$, for any $t \in [x - n, x]$ we have

$$x - n \leq t \leq x \implies f(x) \leq f(t) \leq f(x - n).$$

- Integrate over $[x, 2x]$, using monotonicity of the integral:

$$\begin{aligned}
&\int_x^{2x} f(x) dt \leq \int_x^{2x} f(t) dt \leq \int_x^{2x} f(x - n) dt \\
\implies f(x) \int_x^{2x} dt &\leq \int_x^{2x} f(t) dt \leq f(x - n) \int_x^{2x} dt \\
&\implies xf(x) \leq \int_x^{2x} f(t) dt \leq xf(x - n).
\end{aligned}$$

- By the Cauchy Criterion for integrals, $\lim_{x \rightarrow \infty} \int_x^{2x} f(t) dt = 0$.
- So the LHS term $xf(x) \stackrel{x \rightarrow \infty}{\longrightarrow} 0$.
- Since $x > 1$, $|f(x)| \leq |xf(x)|$
- Thus $f(x) \stackrel{x \rightarrow \infty}{\longrightarrow} 0$ as well.

Solution 2 (Variation on the Trick)

- Use mean value theorem for integrals:

$$\int_x^{2x} f(t) dt = xf(c_x) \quad \text{for some } c_x \in [x, 2x] \text{ depending on } x.$$

- Since f is decreasing,

$$\begin{aligned} x \leq c_x \leq 2x &\implies f(2x) \leq f(c_x) \leq f(x) \\ &\implies 2xf(2x) \leq 2xf(c_x) \leq 2xf(x) \\ &\implies 2xf(2x) \leq 2x \int_x^{2x} f(t) dt \leq 2xf(x) \end{aligned}$$

- By Cauchy Criterion, $\int_x^{2x} f \rightarrow 0$.
- So $2xf(2x) \rightarrow 0$, which by a change of variables gives $uf(u) \rightarrow 0$.
- Since $u \geq 1$, $f(u) \leq uf(u)$ so $f(u) \rightarrow 0$ as well.

Solution 3 (Contradiction)

Just showing $f(x) \xrightarrow{x \rightarrow \infty} 0$:

- Toward a contradiction, suppose not.
- Since f is decreasing, it can not diverge to $+\infty$
- If $f(x) \rightarrow -\infty$, then $f \notin L^1(\mathbb{R})$: choose $x_0 \gg 1$ so that $t \geq x_0 \implies f(t) < -1$, then
- Then $t \geq x_0 \implies |f(t)| \geq 1$, so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f(t)| dt \geq \int_{x_0}^\infty 1 = \infty.$$

- Otherwise $f(x) \rightarrow L \neq 0$, some finite limit.
- If $L > 0$:
 - Fix $\varepsilon > 0$, choose $x_0 \gg 1$ such that $t \geq x_0 \implies L - \varepsilon \leq f(t) \leq L$
 - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L - \varepsilon) dt = \infty$$

- If $L < 0$:
 - Fix $\varepsilon > 0$, choose $x_0 \gg 1$ such that $t \geq x_0 \implies L \leq f(t) \leq L + \varepsilon$.
 - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L) dt = \infty$$

Showing $xf(x) \xrightarrow{x \rightarrow \infty} 0$.

- Toward a contradiction, suppose not.
- (How to show that $xf(x) \not\rightarrow +\infty$?)
- If $xf(x) \rightarrow -\infty$
 - Choose a sequence $\Gamma = \{\hat{x}_i\}$ such that $x_i \rightarrow \infty$ and $x_i f(x_i) \rightarrow -\infty$.
 - Choose a subsequence $\Gamma' = \{x_i\}$ such that $x_i f(x_i) \leq -1$ for all i and $x_i \leq x_{i+1}$.

- Choose a further subsequence $S = \{x_i \in \Gamma' \mid 2x_i < x_{i+1}\}$.
- Then since f is always decreasing, for $t \geq x_0$, $|f|$ is increasing, and $|f(x_i)| \leq |f(2x_i)|$, so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f| \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(t)| dt \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(x_i)| = \sum_{x_i \in S} x_i f(x_i) \longrightarrow \infty.$$

- If $xf(x) \longrightarrow L \neq 0$ for $0 < L < \infty$:
 - Fix $\varepsilon > 0$, choose an infinite sequence $\{x_i\}$ such that $L - \varepsilon \leq x_i f(x_i) \leq L$ for all i .

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)| dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i) dt = \sum_S x_i f(x_i) \geq \sum_S (L - \varepsilon) \longrightarrow \infty.$$

- If $xf(x) \longrightarrow L \neq 0$ for $-\infty < L < 0$:
 - Fix $\varepsilon > 0$, choose an infinite sequence $\{x_i\}$ such that $L \leq x_i f(x_i) \leq L + \varepsilon$ for all i .

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)| dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i) dt = \sum_S x_i f(x_i) \geq \sum_S (L) \longrightarrow \infty.$$

Solution 4 (Akos's Suggestion) For $x \geq 1$,

$$|xf(x)| = \left| \int_x^{2x} f(x) dt \right| \leq \int_x^{2x} |f(x)| dt \leq \int_x^{2x} |f(t)| dt \leq \int_x^\infty |f(t)| dt \xrightarrow{x \rightarrow \infty} 0$$

where we've used

- Since f is decreasing and $\lim_{x \rightarrow \infty} f(x) = 0$ from part (a), f is non-negative.
- Since f is positive and decreasing, for every $t \in [a, b]$ we have $|f(a)| \leq |f(t)|$.
- By part (a), the last integral goes to zero.

Solution 5 (Peter's)

- Toward a contradiction, produce a sequence $x_i \longrightarrow \infty$ with $x_i f(x_i) \longrightarrow \infty$ and $x_i f(x_i) > \varepsilon > 0$, then

$$\begin{aligned} \int f(x) dx &\geq \sum_{i=1}^{\infty} \int_{x_i}^{x_{i+1}} f(x) dx \\ &\geq \sum_{i=1}^{\infty} \int_{x_i}^{x_{i+1}} f(x_{i+1}) dx \\ &= \sum_{i=1}^{\infty} f(x_{i+1}) \int_{x_i}^{x_{i+1}} dx \\ &\geq \sum_{i=1}^{\infty} (x_{i+1} - x_i) f(x_{i+1}) \\ &\geq \sum_{i=1}^{\infty} (x_{i+1} - x_i) \frac{\varepsilon}{x_{i+1}} \\ &= \varepsilon \sum_{i=1}^{\infty} \left(1 - \frac{x_{i-1}}{x_i}\right) \longrightarrow \infty \end{aligned}$$

which can be ensured by passing to a subsequence where $\sum \frac{x_{i-1}}{x_i} < \infty$.

7.1.3 c

- No: take $f(x) = \frac{1}{x \ln x}$
- Then by a u -substitution,

$$\int_0^x f = \ln(\ln(x)) \xrightarrow{x \rightarrow \infty} \infty$$

is unbounded, so $f \notin L^1([1, \infty))$.

- But

$$xf(x) = \frac{1}{\ln(x)} \xrightarrow{x \rightarrow \infty} 0.$$

7.2 Fall 2019 # 5.

- a. Show that if f is continuous with compact support on \mathbb{R} , then

$$\lim_{y \rightarrow 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dx = 0$$

- b. Let $f \in L^1(\mathbb{R})$ and for each $h > 0$ let

$$\mathcal{A}_h f(x) := \frac{1}{2h} \int_{|y| \leq h} f(x-y) dy$$

- c. Prove that $\|\mathcal{A}_h f\|_1 \leq \|f\|_1$ for all $h > 0$.
 ii. Prove that $\mathcal{A}_h f \rightarrow f$ in $L^1(\mathbb{R})$ as $h \rightarrow 0^+$.

7.3 Fall 2017 # 3

Let

$$S = \text{span}_{\mathbb{C}} \left\{ \chi_{(a,b)} \mid a, b \in \mathbb{R} \right\},$$

the complex linear span of characteristic functions of intervals of the form (a, b) .

Show that for every $f \in L^1(\mathbb{R})$, there exists a sequence of functions $\{f_n\} \subset S$ such that

$$\lim_{n \rightarrow \infty} \|f_n - f\|_1 = 0$$

7.4 Spring 2015 # 4

Define

$$f(x, y) := \begin{cases} \frac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Carefully show that $f \in L^1(\mathbb{R}^2)$.

7.5 Fall 2014 # 3

Let $f \in L^1(\mathbb{R})$. Show that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } m(E) < \delta \implies \int_E |f(x)| dx < \varepsilon$$

7.6 Spring 2014 # 1

1. Give an example of a continuous $f \in L^1(\mathbb{R})$ such that $f(x) \not\rightarrow 0$ as $|x| \rightarrow \infty$.
2. Show that if f is *uniformly* continuous, then

$$\lim_{|x| \rightarrow \infty} f(x) = 0.$$

8 Fubini-Tonelli**8.1 Spring 2020 # 4**

Let $f, g \in L^1(\mathbb{R})$. Argue that $H(x, y) := f(y)g(x - y)$ defines a function in $L^1(\mathbb{R}^2)$ and deduce from this fact that

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x - y) dy$$

defines a function in $L^1(\mathbb{R})$ that satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

Solution.

Relevant concepts:

- Tonelli: non-negative and measurable yields measurability of slices and equality of iterated integrals
- Fubini: $f(x, y) \in L^1$ yields *integrable* slices and equality of iterated integrals
- F/T: apply Tonelli to $|f|$; if finite, $f \in L^1$ and apply Fubini to f

$$\begin{aligned}
\|H(x)\|_1 &= \int_{\mathbb{R}} |H(x, y)| dx \\
&= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y)g(x-y) dy \right| dx \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(x-y)| dy \right) dx \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(x-y)| dx \right) dy \quad \text{by Tonelli} \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(t)| dt \right) dy \quad \text{setting } t = x - y, dt = -dx \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)| \cdot |g(t)| dt \right) dy \\
&= \int_{\mathbb{R}} |f(y)| \cdot \left(\int_{\mathbb{R}} |g(t)| dt \right) dy \\
&:= \int_{\mathbb{R}} |f(y)| \cdot \|g\|_1 dy \\
&= \|g\|_1 \int_{\mathbb{R}} |f(y)| dy \\
&:= \|g\|_1 \|f\|_1 \\
&< \infty \quad \text{by assumption} .
\end{aligned}$$

- H is measurable on \mathbb{R}^2 :
 - If we can show $\tilde{f}(x, y) := f(y)$ and $\tilde{g}(x, y) := g(x - y)$ are both measurable on \mathbb{R}^2 , then $H = \tilde{f} \cdot \tilde{g}$ is a product of measurable functions and thus measurable.
 - $f \in L^1$, and L^1 functions are measurable by definition.
 - The function $(x, y) \mapsto g(x - y)$ is measurable on \mathbb{R}^2 :
 - * Let g be measurable on \mathbb{R} , then the cylinder function $G(x, y) = g(x)$ on \mathbb{R}^2 is always measurable
 - * Define a linear transformation $T := [1, -1; 0, 1]$ which sends $(x, y) \rightarrow (x - y, y)$, then $T \in \text{GL}(2, \mathbb{R})$ is linear and thus measurable.
 - * Then $(G \circ T)(x, y) = G(x - y, y) = \tilde{g}(x - y)$, so \tilde{g} is a composition of measurable functions and thus measurable.
- Apply **Tonelli** to $|H|$
 - H measurable implies $|H|$ is measurable
 - $|H|$ is non-negative
 - So the iterated integrals are equal in the extended sense
 - The calculation shows the iterated integral is finite, to $\int |H|$ is finite and H is thus integrable on \mathbb{R}^2 .

Note: Fubini is not needed, since we're not calculating the actual integral, just showing H is integrable.

8.2 Spring 2019 # 4

Let f be a non-negative function on \mathbb{R}^n and $\mathcal{A} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq f(x)\}$.

Prove the validity of the following two statements:

- a. f is a Lebesgue measurable function on $\mathbb{R}^n \iff \mathcal{A}$ is a Lebesgue measurable subset of \mathbb{R}^{n+1}
- b. If f is a Lebesgue measurable function on \mathbb{R}^n , then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x) dx = \int_0^\infty m(\{x \in \mathbb{R}^n : f(x) \geq t\}) dt$$

8.3 Fall 2018 # 5

Let $f \geq 0$ be a measurable function on \mathbb{R} . Show that

$$\int_{\mathbb{R}} f = \int_0^\infty m(\{x : f(x) > t\}) dt$$

8.4 Fall 2015 # 5

Let $f, g \in L^1(\mathbb{R})$ be Borel measurable.

1. Show that
 - The function

$$F(x, y) := f(x - y)g(y)$$

is Borel measurable on \mathbb{R}^2 , and

- For almost every $y \in \mathbb{R}$,

$$F_y(x) := f(x - y)g(y)$$

is integrable with respect to y .

2. Show that $f * g \in L^1(\mathbb{R})$ and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

8.5 Spring 2014 # 5

Let $f, g \in L^1([0, 1])$ and for all $x \in [0, 1]$ define

$$F(x) := \int_0^x f(y) dy \quad \text{and} \quad G(x) := \int_0^x g(y) dy.$$

Prove that

$$\int_0^1 F(x)g(x) dx = F(1)G(1) - \int_0^1 f(x)G(x) dx$$

9 L^2 and Fourier Analysis

9.1 Spring 2020 # 6

9.1.1 a

Show that

$$L^2([0, 1]) \subseteq L^1([0, 1]) \quad \text{and} \quad \ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z}).$$

9.1.2 b

For $f \in L^1([0, 1])$ define

$$\hat{f}(n) := \int_0^1 f(x) e^{-2\pi i n x} dx.$$

Prove that if $f \in L^1([0, 1])$ and $\{\hat{f}(n)\} \in \ell^1(\mathbb{Z})$ then

$$S_N f(x) := \sum_{|n| \leq N} \hat{f}(n) e^{2\pi i n x}.$$

converges uniformly on $[0, 1]$ to a continuous function g such that $g = f$ almost everywhere.

Hint: One approach is to argue that if $f \in L^1([0, 1])$ with $\{\hat{f}(n)\} \in \ell^1(\mathbb{Z})$ then $f \in L^2([0, 1])$.

Solution.

Concepts used:

- For $e_n(x) := e^{2\pi i n x}$, the set $\{e_n\}$ is an orthonormal basis for $L^2([0, 1])$.
- For any orthonormal sequence in a Hilbert space, we have Bessel's inequality:

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

- When $\{e_n\}$ is a basis, the above is an *equality* (Parseval)
- Arguing uniform convergence: since $\{\hat{f}(n)\} \in \ell^1(\mathbb{Z})$, we should be able to apply the M test.

9.1.3 a

Claim: $\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$.

- Set $\mathbf{c} = \{c_k \mid k \in \mathbb{Z}\} \in \ell^1(\mathbb{Z})$.
- It suffices to show that if $\sum_{k \in \mathbb{Z}} |c_k| < \infty$ then $\sum_{k \in \mathbb{Z}} |c_k|^2 < \infty$.
- Let $S = \{c_k \mid |c_k| \leq 1\}$, then $c_k \in S \implies |c_k|^2 \leq |c_k|$
- Claim: S^c can only contain finitely many elements, all of which are finite.
 - If not, either $S^c := \{c_j\}_{j=1}^{\infty}$ is infinite with every $|c_j| > 1$, which forces

$$\sum_{c_k \in S^c} |c_k| = \sum_{j=1}^{\infty} |c_j| > \sum_{j=1}^{\infty} 1 = \infty.$$

- If any $c_j = \infty$, then $\sum_{k \in \mathbb{Z}} |c_k| \geq c_j = \infty$.

- So S^c is a finite set of finite integers, let $N = \max \{|c_j|^2 \mid c_j \in S^c\} < \infty$.

- Rewrite the sum

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} |c_k|^2 &= \sum_{c_k \in S} |c_k|^2 + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \sum_{c_k \in S} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \sum_{k \in \mathbb{Z}} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \quad \text{since the } |c_k| \text{ are all positive} \\
&= \|\mathbf{c}\|_{\ell^1} + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \|\mathbf{c}\|_{\ell^1} + |S^c| \cdot N \\
&< \infty.
\end{aligned}$$

Claim: $L^2([0, 1]) \subseteq L^1([0, 1])$.

- It suffices to show that $\int |f|^2 < \infty \implies \int |f| < \infty$.
- Define $S = \{x \in [0, 1] \mid |f(x)| \leq 1\}$, then $x \in S^c \implies |f(x)|^2 \geq |f(x)|$.
- Break up the integral:

$$\begin{aligned}
\int_{\mathbb{R}} |f| &= \int_S |f| + \int_{S^c} |f| \\
&\leq \int_S |f| + \int_{S^c} |f|^2 \\
&\leq \int_S |f| + \|f\|_2 \\
&\leq \sup_{x \in S} \{|f(x)|\} \cdot \mu(S) + \|f\|_2 \\
&= 1 \cdot \mu(S) + \|f\|_2 \quad \text{by definition of } S \\
&\leq 1 \cdot \mu([0, 1]) + \|f\|_2 \quad \text{since } S \subseteq [0, 1] \\
&= 1 + \|f\|_2 \\
&< \infty.
\end{aligned}$$

Note: this proof shows $L^2(X) \subseteq L^1(X)$ whenever $\mu(X) < \infty$.

9.2 Fall 2017 # 5

Let φ be a compactly supported smooth function that vanishes outside of an interval $[-N, N]$ such that $\int_{\mathbb{R}} \varphi(x) dx = 1$.

For $f \in L^1(\mathbb{R})$, define

$$K_j(x) := j\varphi(jx), \quad f * K_j(x) := \int_{\mathbb{R}} f(x-y)K_j(y) dy$$

and prove the following:

1. Each $f * K_j$ is smooth and compactly supported.

2.

$$\lim_{j \rightarrow \infty} \|f * K_j - f\|_1 = 0$$

Hint:

$$\lim_{y \rightarrow 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dy = 0$$

9.3 Spring 2017 # 5

Let $f, g \in L^2(\mathbb{R})$. Prove that the formula

$$h(x) := \int_{-\infty}^{\infty} f(t)g(x-t) dt$$

defines a uniformly continuous function h on \mathbb{R} .

9.4 Spring 2015 # 6

Let $f \in L^1(\mathbb{R})$ and g be a bounded measurable function on \mathbb{R} .

1. Show that the convolution $f * g$ is well-defined, bounded, and uniformly continuous on \mathbb{R} .
2. Prove that one further assumes that $g \in C^1(\mathbb{R})$ with bounded derivative, then $f * g \in C^1(\mathbb{R})$ and

$$\frac{d}{dx}(f * g) = f * \left(\frac{d}{dx}g\right)$$

9.5 Fall 2014 # 5

1. Let $f \in C_c^0(\mathbb{R}^n)$, and show

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^n} |f(x+t) - f(x)| dx = 0.$$

2. Extend the above result to $f \in L^1(\mathbb{R}^n)$ and show that

$$f \in L^1(\mathbb{R}^n), \quad g \in L^\infty(\mathbb{R}^n) \quad \implies \quad f * g \text{ is bounded and uniformly continuous.}$$

10 Functional Analysis: General**10.1 Fall 2019 # 4.**

Let $\{u_n\}_{n=1}^\infty$ be an orthonormal sequence in a Hilbert space \mathcal{H} .

- a. Prove that for every $x \in \mathcal{H}$ one has

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

- b. Prove that for any sequence $\{a_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$ there exists an element $x \in \mathcal{H}$ such that

$$a_n = \langle x, u_n \rangle \text{ for all } n \in \mathbb{N}$$

and

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$$

10.2 Spring 2019 # 5

10.2.1 a

Show that $L^2([0, 1]) \subseteq L^1([0, 1])$ and argue that $L^2([0, 1])$ in fact forms a dense subset of $L^1([0, 1])$.

10.2.2 b

Let Λ be a continuous linear functional on $L^1([0, 1])$.

Prove the Riesz Representation Theorem for $L^1([0, 1])$ by following the steps below:

- i. Establish the existence of a function $g \in L^2([0, 1])$ which represents Λ in the sense that

$$\Lambda(f) = \int_0^1 f(x)g(x)dx \text{ for all } f \in L^2([0, 1]).$$

Hint: You may use, without proof, the Riesz Representation Theorem for $L^2([0, 1])$.

- ii. Argue that the g obtained above must in fact belong to $L^\infty([0, 1])$ and represent Λ in the sense that

$$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}dx \quad \text{for all } f \in L^1([0, 1])$$

with

$$\|g\|_{L^\infty([0,1])} = \|\Lambda\|_{L^1([0,1])^\vee}$$

10.3 Spring 2016 # 6

Without using the Riesz Representation Theorem, compute

$$\sup \left\{ \left| \int_0^1 f(x)e^x dx \right| \mid f \in L^2([0, 1], m), \|f\|_2 \leq 1 \right\}$$

10.4 Spring 2015 # 5

Let \mathcal{H} be a Hilbert space.

1. Let $x \in \mathcal{H}$ and $\{u_n\}_{n=1}^N$ be an orthonormal set. Prove that the best approximation to x in \mathcal{H} by an element in $\text{span}_{\mathbb{C}} \{u_n\}$ is given by

$$\hat{x} := \sum_{n=1}^N \langle x, u_n \rangle u_n.$$

2. Conclude that finite dimensional subspaces of \mathcal{H} are always closed.

10.5 Fall 2015 # 6

Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous. Show that

$$\sup \left\{ \|fg\|_1 \mid g \in L^1[0, 1], \|g\|_1 \leq 1 \right\} = \|f\|_\infty$$

10.6 Fall 2014 # 6

Let $1 \leq p, q \leq \infty$ be conjugate exponents, and show that

$$f \in L^p(\mathbb{R}^n) \implies \|f\|_p = \sup_{\|g\|_q=1} \left| \int f(x)g(x)dx \right|$$

11 Functional Analysis: Banach Spaces**11.1 Spring 2019 # 1**

Let $C([0, 1])$ denote the space of all continuous real-valued functions on $[0, 1]$.

- Prove that $C([0, 1])$ is complete under the uniform norm $\|f\|_\infty := \sup_{x \in [0, 1]} |f(x)|$.
- Prove that $C([0, 1])$ is not complete under the L^1 -norm $\|f\|_1 = \int_0^1 |f(x)| dx$.

11.2 Spring 2017 # 5

Show that the space $C^1([a, b])$ is a Banach space when equipped with the norm

$$\|f\| := \sup_{x \in [a, b]} |f(x)| + \sup_{x \in [a, b]} |f'(x)|.$$

11.3 Fall 2017 # 6

Let X be a complete metric space and define a norm

$$\|f\| := \max\{|f(x)| : x \in X\}.$$

Show that $(C^0(\mathbb{R}), \|\cdot\|)$ (the space of continuous functions $f : X \rightarrow \mathbb{R}$) is complete.