Real Analysis Qualifying Exam Notes

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Contents

1 Integration

1.1 Useful Techniques

- Break integration domain up into disjoint annuli.
- Break real integrals up into $x < 1$ and $x > 1$.
- Calculus techniques: Taylor series, IVT, ...

1.2 Definitions

Definition 1.0.1 (L^+)**.**
$f \in L^+$ iff f is measurable and non-negative.

Definition 1.0.2 (Integrable)**.**
A measurable function is integrable iff $\|f\|_1 < \infty$.

Definition 1.0.3 (Essentially Bounded Functions)**.**
For $(X, \mathcal{M}, \mu)$ a measure space,

$$L^\infty(X) := \left\{ f \in \mathcal{M} \;\middle|\; f \text{ is essentially bounded} \right\},$$

where f is *essentially bounded* iff there exists a real number c such that $\mu(\{|f| > x\}) = 0$.
If $f \in L^\infty(X)$, then f is equal to some bounded function g almost everywhere.

Example:

- $f(x) = x\chi_{\mathbb{Q}}(x)$ is essentially bounded but not bounded.

1.3 Theorems

Useful facts about C_c functions:

- Bounded almost everywhere
- Uniformly continuous

Theorem ($p-$Test for Integrals) :

$$\int_0^1 x^{-p} < \infty \iff p < 1$$
$$\int_1^\infty x^{-p} < \infty \iff p > 1.$$

Yields a general technique: break integrals apart at $x = 1$.

1.4 Convergence Theorems

Theorem 1.1 ***(Monotone Convergence).***
If $f_n \in L^+$ and $f_n \nearrow f$ a.e., then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \quad \int f_n \longrightarrow \int f.$$

Needs to be positive and increasing.

Theorem 1.2 ***(Dominated Convergence).***

If $f_n \in L^1$ and $f_n \longrightarrow f$ a.e. with $|f_n| \leq g$ for some $g \in L^1$, then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \int f_n \longrightarrow \int f,$$

and more generally,

$$\int |f_n - f| \longrightarrow 0.$$

Positivity *not* needed.

Generalized DCT: can relax $|f_n| < g$ to $|f_n| < g_n \longrightarrow g \in L^1$.

Lemma 1.3.
If $f \in L^1$, then

$$\int |f_n - f| \longrightarrow 0 \iff \int |f_n| \longrightarrow |f|.$$

Proof .
Let $g_n = |f_n| - |f_n - f|$, then $g_n \longrightarrow |f|$ and

$$|g_n| = ||f_n| - |f_n - f|| \leq |f_n - (f_n - f)| = |f| \in L^1,$$

so the DCT applies to g_n and

$$\begin{aligned} \|f_n - f\|_1 = \int |f_n - f| + |f_n| - |f_n| &= \int |f_n| - g_n \\ &\longrightarrow_{DCT} \lim \int |f_n| - \int |f|. \end{aligned}$$

■

Fatou's Lemma If $f_n \in L^+$, then

$$\begin{aligned} \int \liminf_n f_n &\leq \liminf_n \int f_n \\ \limsup_n \int f_n &\leq \int \limsup_n f_n. \end{aligned}$$

Only need positivity.

Theorem 1.4 *(Tonelli).*
For $f(x,y)$ **non-negative and measurable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is a **measurable** function
- $F(x) = \int f(x,y)\, dy$ is a **measurable** function,
- For E measurable, the slices $E_x := \left\{ y \;\middle|\; (x,y) \in E \right\}$ are measurable.
- $\int f = \int \int F$, i.e. any iterated integral is equal to the original.

Theorem 1.5 ***(Fubini).***
For $f(x, y)$ **integrable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is an **integrable** function
- $F(x) := \displaystyle\int f(x, y)\, dy$ is an **integrable** function,
- For E measurable, the slices $E_x := \left\{ y \,\middle|\, (x, y) \in E \right\}$ are measurable.
- $\displaystyle\int f = \int \int f(x, y)$, i.e. any iterated integral is equal to the original

Theorem 1.6 ***(Fubini/Tonelli).***
If any iterated integral is **absolutely integrable**, i.e. $\displaystyle\int \int |f(x, y)| < \infty$, then f is integrable and $\displaystyle\int f$ equals any iterated integral.

Corollary 1.7 ***(Measurable Slices).***
Let E be a measurable subset of $\mathbb{R}^n$. Then

- For almost every $x \in \mathbb{R}^{n_1}$, the slice $E_x := \left\{ y \in \mathbb{R}^{n_2} \,\middle|\, (x, y) \in E \right\}$ is measurable in $\mathbb{R}^{n_2}$.
- The function

$$
\begin{aligned}
F : \mathbb{R}^{n_1} &\longrightarrow \mathbb{R} \\
x &\mapsto m(E_x) = \int_{\mathbb{R}^{n_2}} \chi_{E_x}\, dy
\end{aligned}
$$

is measurable and

$$
m(E) = \int_{\mathbb{R}^{n_1}} m(E_x)\, dx = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{E_x}\, dy\, dx
$$

Proof (Measurable Slices).

$\implies$:

- Let f be measurable on $\mathbb{R}^n$.
- Then the cylinders $F(x, y) = f(x)$ and $G(x, y) = f(y)$ are both measurable on $\mathbb{R}^{n+1}$.
- Write $\mathcal{A} = \{G \leq F\} \bigcap \{G \geq 0\}$; both are measurable.

$\impliedby$:

- Let A be measurable in $\mathbb{R}^{n+1}$.
- Define $A_x = \left\{ y \in \mathbb{R} \,\middle|\, (x, y) \in \mathcal{A} \right\}$, then $m(A_x) = f(x)$.
- By the corollary, A_x is measurable set, $x \mapsto A_x$ is a measurable function, and $m(A) = \displaystyle\int f(x)\, dx$.
- Then explicitly, $f(x) = \chi_A$, which makes f a measurable function.

■

Proposition 1.8 ***(Differentiating Under an Integral).***

If $\left|\frac{\partial}{\partial t} f(x,t)\right| \leq g(x) \in L^1$, then letting $F(t) = \int f(x,t)\, dt$,

$$\frac{\partial}{\partial t} F(t) := \lim_{h \to 0} \int \frac{f(x,t+h) - f(x,t)}{h} dx$$
$$\overset{\text{DCT}}{=} \int \frac{\partial}{\partial t} f(x,t)\, dx.$$

To justify passing the limit, let $h_k \longrightarrow 0$ be any sequence and define

$$f_k(x,t) = \frac{f(x,t+h_k) - f(x,t)}{h_k},$$

so $f_k \overset{\text{pointwise}}{\longrightarrow} \frac{\partial}{\partial t} f$.
Apply the MVT to f_k to get $f_k(x,t) = f_k(\xi, t)$ for some $\xi \in [0, h_k]$, and show that $f_k(\xi, t) \in L_1$.

Proposition 1.9 ***(Swapping Sum and Integral).***
If f_n are non-negative and $\sum \int |f|_n < \infty$, then $\sum \int f_n = \int \sum f_n$.

Proof .

MCT. Let $F_N = \sum^N f_n$ be a finite partial sum; then there are simple functions $\varphi_n \nearrow f_n$ and so $\sum^N \varphi_n \nearrow F_N$, so apply MCT.

■

Lemma 1.10.
If $f_k \in L^1$ and $\sum \|f_k\|_1 < \infty$ then $\sum f_k$ converges almost everywhere and in L^1.

Proof .

Define $F_N = \sum^N f_k$ and $F = \lim_N F_N$, then $\|F_N\|_1 \leq \sum^N \|f_k\| < \infty$ so $F \in L^1$ and $\|F_N - F\|_1 \longrightarrow 0$ so the sum converges in L^1. Almost everywhere convergence: ?

■

1.5 L^1 Facts

Lemma 1.11 ***(Translation Invariance).***
The Lebesgue integral is translation invariant, i.e. $\int f(x)\, dx = \int f(x+h)\, dx$ for any h.

Proof .

- For characteristic functions, $\int_E f(x+h) = \int_{E+h} f(x) = m(E+h) = m(E) = \int_E f$ by translation invariance of measure.

- So this also holds for simple functions by linearity
- For $f \in L^+$, choose $\varphi_n \nearrow f$ so $\int \varphi_n \longrightarrow \int f$.
- Similarly, $\tau_h \varphi_n \nearrow \tau_h f$ so $\int \tau_h f \longrightarrow \int f$
- Finally $\left\{ \int \tau_h \varphi \right\} = \left\{ \int \varphi \right\}$ by step 1, and the suprema are equal by uniqueness of limits.

■

Lemma 1.12 ***(Integrals Distribute Over Disjoint Sets).***
If $X \subseteq A \bigcup B$, then $\int_X f \leq \int_A f + \int_{A^c} f$ with equality iff $X = A \coprod B$.

Lemma 1.13 ***(Unif Cts L1 Functions Decay Rapidly).***
If $f \in L^1$ and f is uniformly continuous, then $f(x) \overset{|x| \longrightarrow \infty}{\longrightarrow} 0$.

Doesn't hold for general L^1 functions, take any train of triangles with height 1 and summable areas.

Lemma 1.14 ***(L1 Functions Have Small Tails).***
If $f \in L^1$, then for every ε there exists a radius R such that if $A = B_R(0)^c$, then $\int_A |f| < \varepsilon$.

Proof .
Approximate with compactly supported functions. Take $g \overset{L_1}{\rightarrow} f$ with $g \in C_c$, then choose N large enough so that $g = 0$ on $E := B_N(0)^c$, then $\int_E |f| \leq \int_E |f - g| + \int_E |g|$.

■

Lemma 1.15 ***(L1 Functions Have Absolutely Continuity).***
$m(E) \longrightarrow 0 \implies \int_E f \longrightarrow 0$.

Proof .
Approximate with compactly supported functions. Take $g \overset{L_1}{\rightarrow} f$, then $g \leq M$ so $\int_E f \leq \int_E f - g + \int_E g \longrightarrow 0 + M \cdot m(E) \longrightarrow 0$.

■

Lemma 1.16 ***(L1 Functions Are Finite a.e.).***
If $f \in L^1$, then $m(\{f(x) = \infty\}) = 0$.

Proof .
Idea: Split up domain Let $A = \{f(x) = \infty\}$, then $\infty > \int f = \int_A f + \int_{A^c} f = \infty \cdot m(A) + \int_{A^c} f \implies m(X) = 0$.

■

Proposition 1.17 ***(Continuity in L1).***

$$\|\tau_h f - f\|_1 \overset{h \to 0}{\longrightarrow} 0$$

Proof .
Approximate with compactly supported functions. Take $g \xrightarrow{L_1} f$ with $g \in C_c$.

$$\begin{aligned}\int f(x+h) - f(x) \leq \\ \int f(x+h) - g(x+h) + \int g(x+h) - g(x) + \int g(x) - f(x) \\ \longrightarrow 2\varepsilon + \int g(x+h) - g(x) \\ = \int_K g(x+h) - g(x) + \int_{K^c} g(x+h) - g(x) \longrightarrow 0,\end{aligned}$$

which follows because we can enlarge the support of g to K where the integrand is zero on K^c, then apply uniform continuity on K.

■

Proposition 1.18 ***(Integration by Parts, Special Case).***

$$F(x) := \int_0^x f(y)dy \quad \text{and} \quad G(x) := \int_0^x g(y)dy$$
$$\implies \int_0^1 F(x)g(x)dx = F(1)G(1) - \int_0^1 f(x)G(x)dx.$$

Proof .
Fubini-Tonelli, and sketch region to change integration bounds.

■

Theorem 1.19 ***(Lebesgue Density).***

$$A_h(f)(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(y)dy \implies \|A_h(f) - f\| \overset{h \to 0}{\longrightarrow} 0.$$

Proof .
Fubini-Tonelli, and sketch region to change integration bounds, and continuity in L^1.

■

1.6 L^p Spaces

Lemma 1.20.
The following are dense subspaces of $L^2([0,1])$:
- Simple functions
- Step functions
- $C_0([0,1])$
- Smoothly differentiable functions $C_0^\infty([0,1])$
- Smooth compactly supported functions C_c^∞

Theorem 1.21 ***(Dual Lp Spaces).***
For $p \neq \infty$, $(L^p)^\vee \cong L^q$.

Proof (p=1).
?

■

Proof (p=2).
Use Riesz Representation for Hilbert spaces.

■

Proof (p =).
$L^1 \subset (L^\infty)^\vee$, since the isometric mapping is always injective, but *never* surjective. So this containment is always proper (requires Hahn-Banach Theorem).

■

2 Fourier Transform and Convolution

2.1 The Fourier Transform

Definition 2.0.1 (Convolution)**.**

$$f * g(x) = \int f(x-y)g(y)dy.$$

Definition 2.0.2 (The Fourier Transform)**.**

$$\widehat{f}(\xi) = \int f(x)\ e^{2\pi i x\cdot\xi}\ dx.$$

Lemma 2.1.
If $\widehat{f} = \widehat{g}$ then $f = g$ almost everywhere.

Lemma 2.2 ***(Riemann-Lebesgue: Fourier transforms have small tails).***

$$f \in L^1 \implies \widehat{f}(\xi) \to 0 \text{ as } |\xi| \to \infty.$$

Lemma 2.3.
If $f \in L^1$, then $\widehat{f}$ is continuous and bounded.

Proof .

- Boundedness:

$$\left|\widehat{f}(\xi)\right| \leq \int |f| \cdot \left|e^{2\pi i x\cdot\xi}\right| = \|f\|_1.$$

- Continuity:
  - Apply DCT to show $\left|\widehat{f}(\xi_n) - \widehat{f}(\xi)\right| \overset{n\longrightarrow\infty}{\longrightarrow} 0.$

■

Theorem (Fourier Inversion) :

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(x) e^{2\pi i x\cdot\xi} d\xi.$$

Proof .
Idea: Fubini-Tonelli doesn't work directly, so introduce a convergence factor, take limits, and use uniqueness of limits.

■

- Take the modified integral:

$$\begin{aligned}
I_t(x) &= \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}\ e^{-\pi t^2|\xi|^2} \\
&= \int \widehat{f}(\xi)\varphi(\xi) \\
&= \int f(\xi)\widehat{\varphi}(\xi) \\
&= \int f(\xi)\widehat{\widehat{g}}(\xi - x) \\
&= \int f(\xi) g_t(x-\xi)\ d\xi \\
&= \int f(y-x) g_t(y)\ dy \quad (\xi = y - x) \\
&= (f * g_t) \\
&\longrightarrow f \text{ in } L^1 \text{ as } t \longrightarrow 0.
\end{aligned}$$

- We also have

$$\begin{aligned}\lim_{t\longrightarrow 0} I_t(x) &= \lim_{t\longrightarrow 0}\int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}\ e^{-\pi t^2|\xi|^2} \\ &= \lim_{t\longrightarrow 0}\int \widehat{f}(\xi)\varphi(\xi) \\ &=_{DCT} \int \widehat{f}(\xi) \lim_{t\longrightarrow 0}\varphi(\xi) \\ &= \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}\end{aligned}$$

.

- So

$$I_t(x) \longrightarrow \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi} \text{ pointwise and } \|I_t(x) - f(x)\|_1 \longrightarrow 0.$$

- So there is a subsequence I_{t_n} such that $I_{t_n}(x) \longrightarrow f(x)$ almost everywhere
- Thus $f(x) = \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}$ almost everywhere by uniqueness of limits.

Proposition (Eigenfunction of the Fourier Transform) :

$$\begin{aligned} g(x) := e^{-\pi|t|^2} \implies \widehat{g}(\xi) = g(\xi) \quad \text{and} \\ \widehat{g}_t(x) = g(tx) = e^{-\pi t^2|x|^2}, .\end{aligned}$$

Proposition 2.4*(Properties of the Fourier Transform).*

?????.

2.2 Approximate Identities

Definition 2.4.1 (Dilation)**.**

$$\varphi_t(x) = t^{-n}\varphi\left(t^{-1}x\right).$$

Definition 2.4.2 (Approximation to the Identity)**.**
For $\varphi \in L^1$, the dilations satisfy $\int \varphi_t = \int \varphi$, and if $\int \varphi = 1$ then φ is an *approximate identity.*
Example: $\varphi(x) = e^{-\pi x^2}$

Theorem 2.5*(Convolution Against Approximate Identities Converge in L^1).*

$$\|f * \varphi_t - f\|_1 \overset{t\longrightarrow 0}{\longrightarrow} 0.$$

Proof .

$$
\begin{aligned}
\|f - f * \varphi_t\|_1 &= \int f(x) - \int f(x-y)\varphi_t(y)\ dydx \\
&= \int f(x) \int \varphi_t(y)\ dy - \int f(x-y)\varphi_t(y)\ dydx \\
&= \int \int \varphi_t(y)[f(x) - f(x-y)]\ dydx \\
&=_{FT} \int \int \varphi_t(y)[f(x) - f(x-y)]\ dxdy \\
&= \int \varphi_t(y) \int f(x) - f(x-y)\ dxdy \\
&= \int \varphi_t(y) \|f - \tau_y f\|_1 dy \\
&= \int_{y<\delta} \varphi_t(y) \|f - \tau_y f\|_1 dy + \int_{y \geq \delta} \varphi_t(y) \|f - \tau_y f\|_1 dy \\
&\leq \int_{y<\delta} \varphi_t(y)\varepsilon + \int_{y\geq\delta} \varphi_t(y) \left(\|f\|_1 + \|\tau_y f\|_1\right) dy \quad \text{by continuity in } L^1 \\
&\leq \varepsilon + 2\|f\|_1 \int_{y\geq\delta} \varphi_t(y) dy \\
&\leq \varepsilon + 2\|f\|_1 \cdot \varepsilon \quad \text{since } \varphi_t \text{ has small tails} \\
&\overset{\varepsilon \longrightarrow 0}{\Longrightarrow} 0.
\end{aligned}
$$

■

Theorem 2.6 ***(Convolutions Vanish at Infinity).***

$$
f, g \in L^1 \text{ and bounded } \implies \lim_{|x| \to \infty} (f * g)(x) = 0.
$$

Proof .

- Choose $M \geq f, g$.
- By small tails, choose N such that $\int_{B_N^c} |f|, \int_{B_n^c} |g| < \varepsilon$
- Note

$$
|f * g| \leq \int |f(x-y)|\ |g(y)|\ dy := I.
$$

- Use $|x| \leq |x-y| + |y|$, take $|x| \geq 2N$ so either

$$
|x-y| \geq N \implies I \leq \int_{\{x-y \geq N\}} |f(x-y)| M\ dy \leq \varepsilon M \longrightarrow 0
$$

  then

$$
|y| \geq N \implies I \leq \int_{\{y \geq N\}} M |g(y)|\ dy \leq M\varepsilon \longrightarrow 0.
$$

■

Proposition (Young's Inequality?) :

$$\frac{1}{r} \coloneqq \frac{1}{p} + \frac{1}{q} - 1 \implies \|f * g\|_r \leq \|f\|_p \|g\| q.$$

Corollary 2.7.
Take $q = 1$ to obtain

$$\|f * g\|_p \leq \|f\| p \|g\| 1.$$

Corollary 2.8.
If $f, g \in L^1$ then $f * g \in L^1$.

3 Functional Analysis

3.1 Definitions

Notation: H denotes a Hilbert space.

Definition 3.0.1 (Orthonormal Sequence)**.**
?

Definition 3.0.2 (Basis)**.**
?

Definition 3.0.3 (Complete)**.**
A collection of vectors $\{u_n\} \subset H$ is *complete* iff $\langle x,\ u_n \rangle = 0$ for all $n \iff x = 0$ in H.

Definition 3.0.4 (Dual Space)**.**

$$X^\vee \coloneqq \left\{ L : X \longrightarrow \mathbb{C} \;\middle|\; L \text{ is continuous} \right\}.$$

Definition 3.0.5.
A map $L : X \longrightarrow \mathbb{C}$ is a linear functional iff

$$L(\alpha \mathbf{x} + \mathbf{y}) = \alpha L(\mathbf{x}) + L(\mathbf{y})..$$

Definition 3.0.6 (Operator Norm)**.**

$$\|L\|_{X^\vee} \coloneqq \sup_{\substack{x \in X \\ \|x\| = 1}} |L(x)|.$$

Definition 3.0.7 (Banach Space)**.**
A complete normed vector space.

Definition 3.0.8 (Hilbert Space)**.**
An inner product space which is a Banach space under the induced norm.

3.2 Theorems

Theorem (Bessel's Inequality) :

$$\left\| x - \sum_{n=1}^{N} \langle x, u_n \rangle \, u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n \rangle|^2$$

and thus

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2.$$

Proof .

- Let $S_N = \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n$

$$\begin{aligned}
\|x - S_N\|^2 &= \langle x - S_n,\ x - S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\langle x,\ S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\left\langle x,\ \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n \right\rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^{N} \langle x,\ \langle x,\ u_n \rangle u_n \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^{N} \overline{\langle x,\ u_n \rangle} \langle x,\ u_n \rangle \\
&= \|x\|^2 + \left\| \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n \right\|^2 - 2 \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 - 2 \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
&= \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2.
\end{aligned}$$

- By continuity of the norm and inner product, we have

$$\begin{aligned}
\lim_{N\longrightarrow\infty} \|x - S_N\|^2 &= \lim_{N\longrightarrow\infty} \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n\rangle|^2 \\
\implies \left\| x - \lim_{N\longrightarrow\infty} S_N \right\|^2 &= \|x\|^2 - \lim_{N\longrightarrow\infty} \sum_{n=1}^{N} |\langle x,\ u_n\rangle|^2 \\
\implies \left\| x - \sum_{n=1}^{\infty} \langle x,\ u_n\rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^{\infty} |\langle x,\ u_n\rangle|^2.
\end{aligned}$$

- Then noting that $0 \le \|x - S_N\|^2$,

$$\begin{aligned}
0 &\le \|x\|^2 - \sum_{n=1}^{\infty} |\langle x,\ u_n\rangle|^2 \\
\implies \sum_{n=1}^{\infty} |\langle x,\ u_n\rangle|^2 &\le \|x\|^2 \blacksquare.
\end{aligned}$$

■

Theorem 3.1 ***(Riesz Representation for Hilbert Spaces).***
If Λ is a continuous linear functional on a Hilbert space H, then there exists a unique $y \in H$ such that

$$\forall x \in H, \quad \Lambda(x) = \langle x,\ y\rangle..$$

Proof .

- Define $M := \ker \Lambda$.
- Then M is a closed subspace and so $H = M \oplus M^\perp$
- There is some $z \in M^\perp$ such that $\|z\| = 1$.
- Set $u := \Lambda(x)z - \Lambda(z)x$
- Check

$$\Lambda(u) = \Lambda(\Lambda(x)z - \Lambda(z)x) = \Lambda(x)\Lambda(z) - \Lambda(z)\Lambda(x) = 0 \implies u \in M$$

- Compute

$$\begin{aligned}
0 &= \langle u,\ z\rangle \\
&= \langle \Lambda(x)z - \Lambda(z)x,\ z\rangle \\
&= \langle \Lambda(x)z,\ z\rangle - \langle \Lambda(z)x,\ z\rangle \\
&= \Lambda(x)\langle z,\ z\rangle - \Lambda(z)\langle x,\ z\rangle \\
&= \Lambda(x)\|z\|^2 - \Lambda(z)\langle x,\ z\rangle \\
&= \Lambda(x) - \Lambda(z)\langle x,\ z\rangle \\
&= \Lambda(x) - \left\langle x,\ \overline{\Lambda(z)}z \right\rangle,
\end{aligned}$$

- Choose $y := \overline{\Lambda(z)}z$.
- Check uniqueness:

$$\begin{aligned}
&\langle x,\ y\rangle = \langle x,\ y'\rangle \quad \forall x \\
\implies &\langle x,\ y - y'\rangle = 0 \quad \forall x \\
\implies &\langle y - y',\ y - y'\rangle = 0 \\
\implies &\|y - y'\| = 0 \\
\implies &y - y' = \mathbf{0} \implies y = y'.
\end{aligned}$$

■

Theorem 3.2 ***(Continuous iff Bounded).***
Let $L : X \longrightarrow \mathbb{C}$ be a linear functional, then the following are equivalent:

1. L is continuous
2. L is continuous at zero
3. L is bounded, i.e. $\exists c \geq 0 \;\middle|\; |L(x)| \leq c\|x\|$ for all $x \in H$

Proof .

$2 \implies 3$: Choose $\delta < 1$ such that

$$\|x\| \leq \delta \implies |L(x)| < 1.$$

Then

$$\begin{aligned}
|L(x)| &= \left| L\left(\frac{\|x\|}{\delta}\frac{\delta}{\|x\|}x\right)\right| \\
&= \frac{\|x\|}{\delta}\left| L\left(\delta\frac{x}{\|x\|}\right)\right| \\
&\leq \frac{\|x\|}{\delta}1,
\end{aligned}$$

so we can take $c = \dfrac{1}{\delta}$. ■

$3 \implies 1$:
We have $|L(x - y)| \leq c\|x - y\|$, so given $\varepsilon \geq 0$ simply choose $\delta = \dfrac{\varepsilon}{c}$.

■

Theorem: (Operator Norm is a Norm) If H is a Hilbert space, then $(H^\vee, \|\cdot\|_{\text{op}})$ is a normed space.

Proof .
The only nontrivial property is the triangle inequality, but

$$\|L_1 + L_2\| = \sup |L_1(x) + L_2(x)| \leq \sup L_1(x) + \sup L_2(x) = \|L_1\| + \|L_2\|.$$

■

Theorem 3.3 ***(Completeness in Operator Norm).***
If X is a normed vector space, then $(X^\vee, \|\cdot\|_{\text{op}}$ is a Banach space.

Proof .

- Let $\{L_n\}$ be Cauchy in $X^\vee$.
- Then for all $x \in C$, $\{L_n(x)\} \subset \mathbb{C}$ is Cauchy and converges to something denoted $L(x)$.
- Need to show L is continuous and $\|L_n - L\| \longrightarrow 0$.
- Since $\{L_n\}$ is Cauchy in $X^\vee$, choose N large enough so that

$$n, m \geq N \implies \|L_n - L_m\| < \varepsilon \implies |L_m(x) - L_n(x)| < \varepsilon \quad \forall x \mid \|x\| = 1.$$

- Take $n \longrightarrow \infty$ to obtain

$$\begin{aligned} m \geq N \implies& |L_m(x) - L(x)| < \varepsilon \quad \forall x \mid \|x\| = 1 \\ \implies& \|L_m - L\| < \varepsilon \longrightarrow 0. \end{aligned}$$

- Continuity:

$$\begin{aligned} |L(x)| &= |L(x) - L_n(x) + L_n(x)| \\ &\leq |L(x) - L_n(x)| + |L_n(x)| \\ &\leq \varepsilon\|x\| + c\|x\| \\ &= (\varepsilon + c)\|x\| \blacksquare. \end{aligned}$$

■

4 Extra Problems

Integration

- Show that if $f \in C^1(\mathbb{R})$ and $\lim_{x \longrightarrow \infty} f(x), f'(x)$ exist, then $\lim f'(x) = 0$.

Basics

- If f is continuous, is it necessarily the case that f' is continuous?
- If $f_n \longrightarrow f$, is it necessarily the case that f_n' converges to f' (or at all)?
- Is it true that the sum of differentiable functions is differentiable?
- Is it true that the limit of integrals equals the integral of the limit?
- Is it true that a limit of continuous functions is continuous?
- Prove that uniform convergence implies pointwise convergence implies a.e. convergence, but none of the implications may be reversed.
- Show that if K is compact and F is closed with K, F disjoint then $\text{dist}(K, F) > 0$.
- Show that if $f_n \longrightarrow f$ uniformly with each f_n continuous then f is continuous.
- Show that a subset of a metric space is closed iff it is complete.
- Show that if a subset of a metric space is complete and totally bounded, then it is compact.
- Show that every compact set is closed and bounded.
- Show that a uniform limit of bounded functions is bounded.

- Show that a uniform limit of continuous function is continuous.
- Show that if $f_n \longrightarrow f$ pointwise, $f_n' \longrightarrow g$ uniformly for some f, g, then f is differentiable and $g = f'$.

Measure Theory

- $\star$: Show that for $E \subseteq \mathbb{R}^n$, TFAE:
  1. E is measurable
  2. $E = H \bigcup Z$ here H is F_σ and Z is null
  3. $E = V \setminus Z'$ where $V \in G_\delta$ and Z' is null.
- Show that continuity of measure from above/below holds for outer measures.
- $\star$: Show that if $E \subseteq \mathbb{R}^n$ is measurable then $m(E) = \sup\limits_{K \subset E \text{ compact}} m(K)$ iff for all $\varepsilon > 0$ there exists a compact $K \subseteq E$ such that $m(K) \geq m(E) - \varepsilon$.
- Show that a countable union of null sets is null.

Continuity

- Show that a continuous function on a compact set is uniformly continuous.

Measurability

- Show that $f = 0$ a.e. iff $\displaystyle\int_E f = 0$ for every measurable set E.
- $\star$: Show that cylinder functions are measurable, i.e. if f is measurable on $\mathbb{R}^s$, then $F(x, y) := f(x)$ is measurable on $\mathbb{R}^s \times \mathbb{R}^t$ for any t.
- Show that if f is a measurable function, then $f = 0$ a.e. iff $\displaystyle\int f = 0$.

Integrability

- $\star$: Prove that the Lebesgue integral is translation invariant, i.e. if $\tau_h(x) = x + h$ then $\displaystyle\int \tau_h f = \int f$.
- $\star$: Prove that the Lebesgue integral is dilation invariant, i.e. if $f_\delta(x) = \dfrac{f(\frac{x}{\delta})}{\delta^n}$ then $\displaystyle\int f_\delta = \int f$.
- $\star$: Prove continuity in L^1, i.e.

$$f \in L^1 \implies \lim_{h \to 0} \int |f(x+h) - f(x)| = 0.$$

- Show that a bounded function is Lebesgue integrable iff it is measurable.
- Show that simple functions are dense in L^1.
- Show that step functions are dense in L^1.
- Show that smooth compactly supposed functions are dense in L^1.

Convergence

- Prove Fatou's lemma using the Monotone Convergence Theorem.
- Show that if $\{f_n\}$ is in L^1 and $\sum \int |f_n| < \infty$ then $\sum f_n$ convergence to an L^1 function and $\displaystyle\int \sum f_n = \sum \int f_n$.

Convolution

- Show that $f, g \in L^1 \implies f * g \in L^1$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.
- Show that $f \in L^1, g \leq M \implies f * g \leq M'$ and is uniformly continuous.
- Show that if $f, g \in L^1$ with $f \leq M, g \leq M'$, then $f * g \overset{x \longrightarrow \infty}{\longrightarrow} 0$.
- Show that if $f \in L^1$ and g' exists with $\dfrac{\partial g}{\partial x_i}$ all bounded, then $\dfrac{\partial}{\partial x_i}(f * g) = f * \dfrac{\partial g}{\partial x_i}$
- Show that if f, g are smooth and compactly supported then $f * g$ is smooth and $f * g \overset{x \longrightarrow \infty}{\longrightarrow} 0$.
- $\star$: show that if $f, g \in L^1$, then $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.
- Is it the case that $f, g \in C_c$ implies that $f * g \in C_c$?
- Show that if $f \in L^1$ and $g \in C_c^\infty$ then $f * g$ is smooth and $f * g$ vanishes at infinity.
- Show that if $f, g \in L^1$ and g is bounded, then $\lim_{|x| \longrightarrow \infty} (f * g)(x) = 0$.

Fourier Analysis

- Show that if $f \in L^1$ then $\widehat{f}$ is bounded and uniformly continuous.
- Is it the case that $f \in L^1$ implies $\widehat{f} \in L^1$?
- Show that if $f, \widehat{f} \in L^1$ then f is bounded, uniformly continuous, and vanishes at infinity.
  - Show that this is not true for arbitrary L^1 functions.
- Show that if $f \in L^1$ and $\widehat{f} = 0$ almost everywhere then $f = 0$ almost everywhere.
  - Prove that $\widehat{f} = \widehat{g}$ implies that $f = g$ a.e.
- Show that if $f, g \in L^1$ then $\displaystyle\int \widehat{f} g = \int f \widehat{g}$.
  - Give an example showing that this fails if g is not bounded.
- Show that if $f \in C^1$ then f is equal to its Fourier *series*.

Approximate Identities

- Show that if φ is an approximate identity, then $\|f * \varphi_t - f\|_1 \overset{t \longrightarrow 0}{\longrightarrow} 0$.
  - Show that if additionally $|\varphi(x)| \leq c(1 + |x|)^{-n-\varepsilon}$ for some $c, \varepsilon > 0$, then this converges is almost everywhere.
- Show that is f is bounded and uniformly continuous and φ_t is an approximation to the identity, then $f * \varphi_t$ uniformly converges to f.

L^p Spaces

- Show that if $E \subseteq \mathbb{R}^n$ is measurable with $\mu(E) < \infty$ and $f \in L^p(X)$ then $\|f\|_{L^p(X)} \overset{p \longrightarrow \infty}{\longrightarrow} \|f\|_\infty$.
- Is it true that the converse to the DCT holds? I.e. if $\displaystyle\int f_n \longrightarrow \int f$, is there a $g \in L^p$ such that $f_n < g$ a.e. for every n?
- Prove continuity in L^p: If f is uniformly continuous then $\|\tau_h f - f\|_p \longrightarrow 0$ as $h \longrightarrow 0$ for all p.
- Prove the following inclusions of L^p spaces for $m(X) < \infty$:

$$
\begin{aligned}
&L^\infty(X) \subset L^2(X) \subset L^1(X) \\
&\ell^2(\mathbb{Z}) \subset \ell^1(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z}).
\end{aligned}
$$

5 Inequalities and Equalities

Proposition 5.1 ***(Reverse Triangle Inequality).***

$$\big|\|x\| - \|y\|\big| \leq \|x - y\|.$$

Proposition 5.2 ***(Chebyshev's Inequality).***

$$\mu(\{x : |f(x)| > \alpha\}) \leq \left(\frac{\|f\|_p}{\alpha}\right)^p.$$

Proposition 5.3 ***(Holder's Inequality (when surjective).***

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Application: For finite measure spaces,

$$1 \leq p < q \leq \infty \implies L^q \subset L^p \quad (\text{ and } \ell^p \subset \ell^q).$$

Proof (Holder's Inequality).
Fix p, q, let $r = \frac{q}{p}$ and $s = \frac{r}{r-1}$ so $r^{-1} + s^{-1} = 1$. Then let $h = |f|^p$:

■

$$\|f\|_p^p = \|h \cdot 1\|_1 \leq \|1\|_s \|h\|_r = \mu(X)^{\frac{1}{s}} \|f\|_q^{\frac{q}{r}} \implies \|f\|_p \leq \mu(X)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q.$$

Note: doesn't work for ℓ_p spaces, but just note that $\sum |x_n| < \infty \implies x_n < 1$ for large enough n, and thus $p < q \implies |x_n|^q \leq |x_n|^q$.

Proposition 5.4 ***(Cauchy-Schwarz Inequality).***

$$|\langle f,\ g\rangle| = \|fg\|_1 \leq \|f\|_2 \|g\|_2 \quad \text{with equality} \quad \iff f = \lambda g.$$

Note: Relates inner product to norm, and only happens to relate norms in L^1.

Proof .
?

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Proposition 5.5 ***(Minkowski's Inequality:).***

$$1 \le p < \infty \implies \|f+g\|_p \le \|f\|_p + \|g\|_p.$$

Note: does not handle $p = \infty$ case. Use to prove L^p is a normed space.

Proposition 5.6 ***(Young's Inequality*).***

[

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \implies \|f * g\|_r \le \|f\|_p \|g\|_q.$$

]

Application: Some useful specific cases:

$$\begin{aligned}
\|f * g\|_1 &\le \|f\|_1 \|g\|_1 \\
\|f * g\|_p &\le \|f\|_1 \|g\|p, \\
\|f * g\|_\infty &\le \|f\|_2 \|g\|_2 \\
\|f * g\|_\infty &\le \|f\|_p \|g\|_q.
\end{aligned}$$

Proposition 5.7 ***(? Inequality).***

$$(a+b)^p \le 2^p (a^p + b^p).$$

Proposition 5.8 ***(Bezel's Inequality:).***

For $x \in H$ a Hilbert space and $\{e_k\}$ an orthonormal sequence,

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \le \|x\|^2.$$

Note: this does not need to be a basis.

Proposition 5.9 ***(Parseval's Identity:).***
Equality in Bessel's inequality, attained when $\{e_k\}$ is a *basis*, i.e. it is complete, i.e. the span of its closure is all of H.

5.1 Less Explicitly Used Inequalities

Proposition 5.10 ***(AM-GM Inequality).***

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Proposition 5.11 ***(Jensen's Inequality).***

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$