Title

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Contents

1 Modules

1.1 General Questions

1.1.1 Fall 2019 Final #2

Consider the $\mathbb{Z}$-submodule N of $\mathbb{Z}^3$ spanned by $f_1 = [-1,0,1], f_2 = [2,-3,1], f_3 = [0,3,1], f_4 = [3,1,5]$. Find a basis for N and describe $\mathbb{Z}^3/N$.

1.1.2 Spring 2018 #6.

Let

$$M = \{(w,x,y,z) \in \mathbb{Z}^4 \;\middle|\; w+x+y+z \in 2\mathbb{Z}\},$$

and

$$N = \{(w,x,y,z) \in \mathbb{Z}^4 \;\middle|\; 4 \;\middle|\; (w-x),\; 4 \;\middle|\; (x-y),\; 4 \;\middle|\; (y-z)\}.$$

a. Show that N is a $\mathbb{Z}$-submodule of M .

b. Find vectors $u_1, u_2, u_3, u_4 \in \mathbb{Z}^4$ and integers d_1, d_2, d_3, d_4 such that

$$\{u_1, u_2, u_3, u_4\}$$

is a free basis for M, and

$$\{d_1 u_1,\ d_2 u_2,\ d_3 u_3,\ d_4 u_4\}$$

is a free basis for N .

c. Use the previous part to describe M/N as a direct sum of cyclic $\mathbb{Z}$-modules.

1.1.3 Fall 2018 #6 $\bowtie$

Let R be a commutative ring, and let M be an R-module. An R-submodule N of M is maximal if there is no R-module P with $N \subsetneq P \subsetneq M$.

a. Show that an R-submodule N of M is maximal $\iff$ M/N is a simple R-module: i.e., M/N is nonzero and has no proper, nonzero R-submodules.

b. Let M be a $\mathbb{Z}$-module. Show that a $\mathbb{Z}$-submodule N of M is maximal $\iff$ $\#M/N$ is a prime number.

c. Let M be the $\mathbb{Z}$-module of all roots of unity in $\mathbb{C}$ under multiplication. Show that there is no maximal $\mathbb{Z}$-submodule of M.

Solution.
a
By the correspondence theorem, submodules of M/N biject with submodules A of M containing N.
So

- M is maximal:
- $\iff$ no such (proper, nontrivial) submodule A exists
- $\iff$ there are no (proper, nontrivial) submodules of M/N
- $\iff$ M/N is simple.

b
Identify $\mathbb{Z}$-modules with abelian groups, then by (a), N is maximal $\iff$ M/N is simple $\iff$ M/N has no nontrivial proper subgroups.

By Cauchy's theorem, if $|M/N| = ab$ is a composite number, then $a \mathrel{\Big|} ab \implies$ there is an element (and thus a subgroup) of order a. In this case, M/N contains a nontrivial proper cyclic subgroup, so M/N is not simple. So $|M/N|$ can not be composite, and therefore must be prime.

c
Let $G = \left\{ x \in \mathbb{C} \mathrel{\Big|} x^n = 1 \text{ for some } n \in \mathbb{N} \right\}$, and suppose $H < G$ is a proper subgroup.
Then there must be a prime p such that the $\zeta_{p^k} \notin H$ for all k greater than some constant m – otherwise, we can use the fact that if $\zeta_{p^k} \in H$ then $\zeta_{p^\ell} \in H$ for all $\ell \leq k$, and if $\zeta_{p^k} \in H$ for all p and all k then $H = G$.
But this means there are infinitely many elements in $G \setminus H$, and so $\infty = [G : H] = |G/H|$ is not a prime. Thus by (b), H can not be maximal, a contradiction.

1.1.4 Spring 2018 #7.

Let R be a PID and M be an R-module. Let p be a prime element of R. The module M is called $\langle p \rangle$ *-primary* if for every $m \in M$ there exists $k > 0$ such that $p^k m = 0$.

a. Suppose M is $\langle p \rangle$-primary. Show that if $m \in M$ and $t \in R,\ t \notin \langle p \rangle$, then there exists $a \in R$ such that $atm = m$.

b. A submodule S of M is said to be *pure* if $S \cap rM = rS$ for all $r \in R$. Show that if M is $\langle p \rangle$-primary, then S is pure if and only if $S \cap p^k M = p^k S$ for all $k \geq 0$.

1.1.5 Fall 2016 #6

Let R be a ring and $f : M \longrightarrow N$ and $g : N \longrightarrow M$ be R-module homomorphisms such that $g \circ f = \mathrm{id}_M$. Show that $N \cong \operatorname{im} f \oplus \ker g$.

1.1.6 Spring 2016 #4

Let R be a ring with the following commutative diagram of R-modules, where each row represents a short exact sequence of R-modules:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

Prove that if α and γ are isomorphisms then β is an isomorphism.

1.1.7 Spring 2015 #8

Let R be a PID and M a finitely generated R-module.

a. Prove that there are R-submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that for all $0 \leq i \leq n-1$, the module M_{i+1}/M_i is cyclic.

b. Is the integer n in part (a) uniquely determined by M? Prove your answer.

1.1.8 Fall 2012 #6

Let R be a ring and M an R-module. Recall that M is *Noetherian* iff any strictly increasing chain of submodule $M_1 \subsetneq M_2 \subsetneq \cdots$ is finite. Call a proper submodule $M' \subsetneq M$ *intersection-decomposable* if it can not be written as the intersection of two proper submodules $M' = M_1 \bigcap M_2$ with $M_i \subsetneq M$.

Prove that for every Noetherian module M, any proper submodule $N \subsetneq M$ can be written as a finite intersection $N = N_1 \bigcap \cdots \bigcap N_k$ of intersection-indecomposable modules.

1.1.9 Fall 2019 Final #1

Let A be an abelian group, and show A is a $\mathbb{Z}$-module in a unique way.

1.2 Torsion and the Structure Theorem

1.2.1 ⋆ Fall 2019 #5 ⋈

Let R be a ring and M an R-module.

> Recall that the set of torsion elements in M is defined by
>
> $$\operatorname{Tor}(m) = \{m \in M \mid \exists r \in R,\ r \neq 0,\ rm = 0\}.$$

a. Prove that if R is an integral domain, then $\operatorname{Tor}(M)$ is a submodule of M .

b. Give an example where $\operatorname{Tor}(M)$ is not a submodule of M.

c. If R has zero-divisors, prove that every non-zero R-module has non-zero torsion elements.

Solution.

> One-step submodule test.

a It suffices to show that

$$r \in R,\ t_1, t_2 \in \operatorname{Tor}(M) \implies rt_1 + t_2 \in \operatorname{Tor}(M).$$

We have

$$\begin{aligned} t_1 \in \operatorname{Tor}(M) &\implies \exists s_1 \neq 0 \text{ such that } s_1 t_1 = 0 \\ t_2 \in \operatorname{Tor}(M) &\implies \exists s_2 \neq 0 \text{ such that } s_2 t_2 = 0. \end{aligned}$$

Since R is an integral domain, $s_1 s_2 \neq 0$. Then

$$\begin{aligned} s_1 s_2 (rt_1 + t_2) &= s_1 s_2 r t_1 + s_1 s_2 t_2 \\ &= s_2 r (s_1 t_1) + s_1 (s_2 t_2) \quad \text{since } R \text{ is commutative} \\ &= s_2 r(0) + s_1(0) \\ &= 0. \end{aligned}$$

b Let $R = \mathbb{Z}/6\mathbb{Z}$ as a $\mathbb{Z}/6\mathbb{Z}$-module, which is not an integral domain as a ring.
Then $[3]_6 \curvearrowright [2]_6 = [0]_6$ and $[2]_6 \curvearrowright [3]_6 = [0]_6$, but $[2]_6 + [3]_6 = [5]_6$, where 5 is coprime to 6, and thus $[n]_6 \curvearrowright [5]_6 = [0] \implies [n]_6 = [0]_6$. So $[5]_6$ is *not* a torsion element.
So the set of torsion elements are not closed under addition, and thus not a submodule.

c Suppose R has zero divisors $a, b \neq 0$ where $ab = 0$. Then for any $m \in M$, we have $b \curvearrowright m := bm \in M$ as well, but then

$$a \curvearrowright bm = (ab) \curvearrowright m = 0 \curvearrowright m = 0_M,$$

so m is a torsion element for any m.

1.2.2 $\star$ Spring 2019 #5 $\bowtie$

Let R be an integral domain. Recall that if M is an R-module, the *rank* of M is defined to be the maximum number of R-linearly independent elements of M .

a. Prove that for any R-module M, the rank of $\text{Tor}(M)$ is 0.

b. Prove that the rank of M is equal to the rank of of $M/\text{Tor}(M)$.

c. Suppose that M is a non-principal ideal of R.

Prove that M is torsion-free of rank 1 but not free.