Complex Analysis Problems

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Contents

1 Integrals and Cauchy's Theorem

1.1 1

Suppose $f, g : [0,1] \longrightarrow \mathbb{R}$ where f is Riemann integrable and for $x, y \in [0,1]$,

$$|g(x) - g(y)| \leq |f(x) - f(y)|.$$

Prove that g is Riemann integrable.

1.2 2

State and prove Green's Theorem for rectangles.

Then use it to prove Cauchy's Theory for functions that are analytic in a rectangle.

1.3 3

Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of analytic functions on $\mathbb{D}^\circ \coloneqq \left\{z \in \mathbb{C} \;\middle|\; |z| < 1\right\}$.

Show that if $f_n \longrightarrow g$ for some $g : \mathbb{D}^\circ \longrightarrow \mathbb{C}$ uniformly on every compact $K \subset \mathbb{D}^\circ$, then g is analytic on $\mathbb{D}^\circ$.

1.4 4

Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of entire functions where

- $f_n \longrightarrow g$ pointwise for some $g : \mathbb{C} \longrightarrow \mathbb{C}$.
- On every line segment in $\mathbb{C}$, $f_n \longrightarrow g$ uniformly.

Show that

- g is entire, and
- $f_n \longrightarrow g$ uniformly on every compact subset of $\mathbb{C}$.

1.5 5

Prove that there is no sequence of polynomials that uniformly converge to $f(z) = \dfrac{1}{z}$ on S^1.

1.6 6

Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function that vanishes outside of some finite interval. For each $z \in \mathbb{C}$, define

$$g(z) = \int_{-\infty}^{\infty} f(t)e^{-izt}\,dt.$$

Show that g is entire.

1.7 7

Suppose $f : \mathbb{C} \longrightarrow \mathbb{C}$ is entire and

$$|f(z)| \leq |z|^{\frac{1}{2}} \qquad \text{when } |z| > 10.$$

Prove that f is constant.

1.8 8

Let γ be a smooth curve joining two distinct points $a, b \in \mathbb{C}$.

Prove that the function

$$f(z) := \int_\gamma \frac{g(w)}{w - z}\, dw$$

is analytic in $\mathbb{C} \setminus \gamma$.

1.9 9

Suppose that $f : \mathbb{C} \longrightarrow \mathbb{C}$ is continuous everywhere and analytic on $\mathbb{C} \setminus \mathbb{R}$ and prove that f is entire.

1.10 10

Prove Liouville's theorem: suppose $f : \mathbb{C} \longrightarrow \mathbb{C}$ is entire and bounded. Use Cauchy's formula to prove that $f' \equiv 0$ and hence f is constant.

2 Liouville's Theorem, Power Series

2.1 1

Suppose f is analytic on a region Ω such that $\mathbb{D} \subseteq \Omega \subseteq \mathbb{C}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series with radius of convergence exactly 1.

a. Give an example of such an f that converges at every point of S^1.

b. Given an example of such an f which is analytic at 1 but $\sum_{n=0}^{\infty} a_n$ diverges.

c. Prove that f can not be analytic at *every* point of S^1.

2.2 2

Suppose f is entire and has Taylor series $\sum a_n z^n$ about 0.

a. Express a_n as a contour integral along the circle $|z| = R$.

b. Apply (a) to show that the above Taylor series converges uniformly on every bounded subset of $\mathbb{C}$.

c. Determine those functions f for which the above Taylor series converges uniformly on all of $\mathbb{C}$.

2.3 3

Suppose D is a domain and f, g are analytic on D.

Prove that if $fg = 0$ on D, then either $f \equiv 0$ or $g \equiv 0$ on D.

2.4 4

Suppose f is analytic on $\mathbb{D}^\circ$. Determine with proof which of the following are possible:

a. $f\left(\dfrac{1}{n}\right) = (-1)^n$ for each $n > 1$.

b. $f\left(\dfrac{1}{n}\right) = e^{-n}$ for each even integer $n > 1$ while $f\left(\dfrac{1}{n}\right) = 0$ for each odd integer $n > 1$.

c. $f\left(\dfrac{1}{n^2}\right) = \dfrac{1}{n}$ for each integer $n > 1$.

d. $f\left(\dfrac{1}{n}\right) = \dfrac{n-2}{n-1}$ for each integer $n > 1$.

2.5 5

Prove the Fundamental Theorem of Algebra (using complex analysis).

2.6 6

Find all entire functions that satisfy

$$|f(z)| \geq |z| \quad \forall z \in \mathbb{C}.$$

Prove this list is complete.

2.7 7

Suppose $\sum_{n=0}^{\infty} a_n z^n$ converges for some $z_0 \neq 0$.

a. Prove that the series converges absolutely for each z with $|z| < |z|_0$.

b. Suppose $0 < r < |z_0|$ and show that the series converges uniformly on $|z| \leq r$.

2.8 8

Suppose f is entire and suppose that for some integer $n \geq 1$,

$$\lim_{z \longrightarrow \infty} \frac{f(z)}{z^n} = 0.$$

Prove that f is a polynomial of degree at most $n - 1$.

2.9 9

Find all entire functions satisfying

$$|f(z)| \leq |z|^{\frac{1}{2}} \quad \text{for } |z| > 10.$$

2.10 10

Prove that the following series converges uniformly on the set $\left\{z \;\middle|\; \Im(z) < \ln 2\right\}$:

$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{2^n}.$$

3 Spring 2020 Homework 1

3.1 1

Geometrically describe the following subsets of $\mathbb{C}$:

a. $|z-1| = 1$
b. $|z-1| = 2|z-2|$
c. $1/z = \bar{z}$
d. $\Re(z) = 3$
e. $\Im(z) = a$ with $a \in \mathbb{R}$.
f. $\Re(z) > a$ with $a \in \mathbb{R}$.
g. $|z-1| < 2|z-2|$

3.2 2

Prove the following inequality, and explain when equality holds:

$$|z+w| \geq ||z| - |w||.$$

3.3 3

Prove that the following polynomial has its roots outside of the unit circle:

$$p(z) = z^3 + 2z + 4.$$

> Hint: What is the maximum value of the modulus of the first two terms if $|z| \leq 1$?

3.4 4

a. Prove that if $c > 0$,

$$|w_1| = c|w_2| \implies \left|w_1 - c^2 w_2\right| = c|w_1 - w_2|.$$

b. Prove that if $c > 0$ and $c \neq 1$, with $z_1 \neq z_2$, then the following equation represents a circle:

$$\left|\frac{z - z_1}{z - z_2}\right| = c.$$

Find its center and radius.

> Hint: use part (a)

3.5 5

a. Let $z, w \in \mathbb{C}$ with $\bar{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1 \quad \text{if } |z| < 1,\ |w| < 1$$

with equality when $|z| = 1$ or $|w| = 1$.

b. Prove that for a fixed $w \in \mathbb{D}$, the mapping $F : z \mapsto \dfrac{w-z}{1-\overline{w}z}$ satisfies

- F maps $\mathbb{D}$ to itself and is holomorphic.
- $F(0) = w$ and $F(w) = 0$.
- $|z| = 1$ implies $|F(z)| = 1$.

3.6 6

Use nth roots of unity to show that

$$2^{n-1}\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\cdots\sin\left(\frac{(n-1)\pi}{n}\right) = n.$$

Hint:

$$\begin{aligned} 1-\cos(2\theta) &= 2\sin^2(\theta) \\ 2\sin(2\theta) &= 2\sin(\theta)\cos(\theta). \end{aligned}$$

3.7 7

3.8 8

3.9 9

3.10 10

3.11 11

4 Spring 2020 Homework 2

Note on notation: I sometimes use $f_x := \dfrac{\partial f}{\partial x}$ to denote partial derivatives, and $\partial_z^n f$ as $f^{(n)}(z)$.

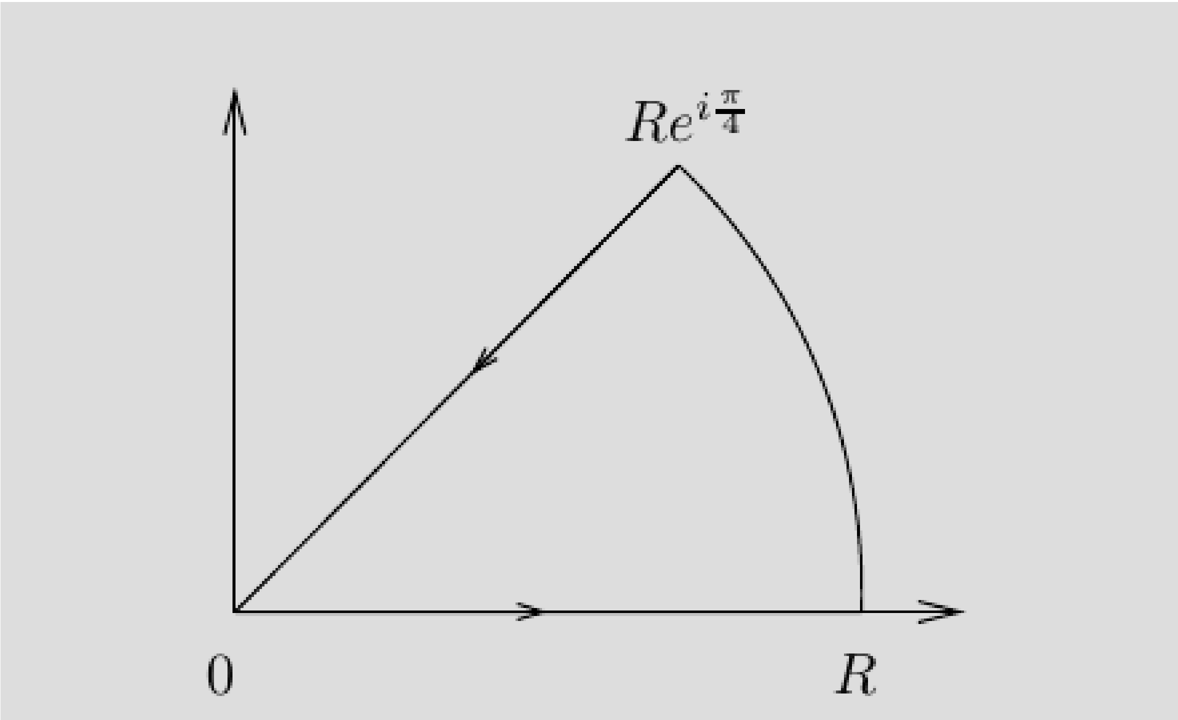
4.1 Stein And Shakarchi

4.1.1 2.6.1

Show that

$$\int_0^\infty \sin\left(x^2\right)dx = \int_0^\infty \cos\left(x^2\right)dx = \frac{\sqrt{2\pi}}{4}.$$

Hint: integrate e^{-x^2} over the following contour, using the fact that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$:



4.1.2 2.6.2

Show that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Hint: use the fact that this integral equals $\frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{ix}-1}{x} dx$, and integrate around an indented semicircle.

4.1.3 2.6.5

Suppose $f \in C^1_{\mathbb{C}}(\Omega)$ and $T \subset \Omega$ is a triangle with $T^\circ \subset \Omega$. Apply Green's theorem to show that $\int_T f(z)\, dz = 0$.

Assume that f' is continuous and prove Goursat's theorem.

Hint: Green's theorem states

$$\int_T F dx + G dy = \int_{T^\circ} \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right) dxdy.$$

4.1.4 2.6.6

Suppose that f is holomorphic on a punctured open set $\Omega \setminus \{w_0\}$ and let $T \subset \Omega$ be a triangle containing w_0. Prove that if f is bounded near w_0, then $\int_T f(z)\, dz = 0$.

4.1.5 2.6.7

Suppose $f : \mathbb{D} \longrightarrow \mathbb{C}$ is holomorphic and let $d \coloneqq \sup_{z,w \in \mathbb{D}} |f(z) - f(w)|$ be the diameter of the image of f. Show that $2\left|f'(0)\right| \leq d$, and that equality holds iff f is linear, so $f(z) = a_1 z + a_2$.

> Hint: $2f'(0) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi) - f(-\xi)}{\xi^2}\, d\xi$ whenever $0 < r < 1$.

4.1.6 2.6.8

Suppose that f is holomorphic on the strip $S = \left\{ x + iy \;\middle|\; x \in \mathbb{R},\ -1 < y < 1 \right\}$ with $|f(z)| \leq A(1 + |z|)^\nu$ for ν some fixed real number. Show that for all $z \in S$, for each integer $n \geq 0$ there exists an $A_n \geq 0$ such that $\left| f^{(n)}(x) \right| \leq A_n (1 + |x|)^\nu$ for all $x \in \mathbb{R}$.

> Hint: Use the Cauchy inequalities.

4.1.7 2.6.9

Let $\Omega \subset \mathbb{C}$ be open and bounded and $\varphi : \Omega \longrightarrow \Omega$ holomorphic. Prove that if there exists a point $z_0 \in \Omega$ such that $\varphi(z_0) = z_0$ and $\varphi'(z_0) = 1$, then φ is linear.

> Hint: assume $z_0 = 0$ (explain why this can be done) and write $\varphi(z) = z + a_n z^n + O(z^{n+1})$ near 0. Let $\varphi_k = \varphi \circ \varphi \circ \cdots \circ \varphi$ and prove that $\varphi_k(z) = z + k a_n z^n + O(z^{n+1})$. Apply Cauchy's inequalities and let $k \longrightarrow \infty$ to conclude.

4.1.8 2.6.10

Can every continuous function on $\overline{\mathbb{D}}$ be uniformly approximated by polynomials in the variable z?

> Hint: compare to Weierstrass for the real interval.

4.1.9 2.6.13

Suppose f is analytic, defined on all of $\mathbb{C}$, and for each $z_0 \in \mathbb{C}$ there is at least one coefficient in the expansion $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ is zero. Prove that f is a polynomial.

> Hint: use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.

4.1.10 2.6.14

Suppose that f is holomorphic in an open set containing $\mathbb{D}$ except for a pole $z_0 \in \partial\mathbb{D}$. Let $\sum_{n=0}^{\infty} a_n z^n$ be the power series expansion of f in $\mathbb{D}$, and show that $\lim \frac{a_n}{a_{n+1}} = z_0$.

4.1.11 2.6.15

Suppose f is continuous and nonvanishing on $\overline{\mathbb{D}}$, and holomorphic in $\mathbb{D}$. Prove that if $|z| = 1 \implies |f(z)| = 1$, then f is constant.

> Hint: Extend f to all of $\mathbb{C}$ by $f(z) = 1/\overline{f(1/\bar{z})}$ for any $|z| > 1$, and argue as in the Schwarz reflection principle.

4.2 Additional Problems

4.2.1 1

Let $a_n \neq 0$ and show that

$$\lim_{n \longrightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L \implies \lim_{n \longrightarrow \infty} |a_n|^{\frac{1}{n}} = L.$$

In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

4.2.2 2

Let f be a power series centered at the origin. Prove that f has a power series expansion about any point in its disc of convergence.

4.2.3 3

Prove the following:

a. $\sum_n nz^n$ does not converge at any point of S^1

b. $\sum_n \frac{z^n}{n^2}$ converges at every point of S^1.

c. $\sum_n \frac{z^n}{n}$ converges at every point of S^1 except $z = 1$.

4.2.4 4

Without using Cauchy's integral formula, show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{dz}{(z-\alpha)(z-\beta)} = \frac{2\pi i}{\alpha - \beta}$$

where γ denotes the circle centered at the origin of radius r with positive orientation.

4.2.5 5

Assume f is continuous in the region $\left\{x+iy \,\middle|\, x \geq x_0,\ 0 \leq y \leq b\right\}$, and the following limit exists independent of y:

$$\lim_{x \longrightarrow +\infty} f(x+iy) = A.$$

Show that if $\gamma_x \coloneqq \left\{z = x+it \,\middle|\, 0 \leq t \leq b\right\}$, then

$$\lim_{x \longrightarrow +\infty} \int_{\gamma_x} f(z)\, dz = iAb.$$

4.2.6 6

Show by example that there exists a function $f(z)$ that is holomorphic on $\left\{z \in \mathbb{C} \,\middle|\, 0 < |z| < 1\right\}$ and for all $r < 1$,

$$\int_{|z|=r} f(z)\, dz = 0,$$

but f is not holomorphic at $z = 0$.

4.2.7 7

Let f be analytic on a region R and suppose $f'(z_0) \neq 0$ for some $z_0 \in R$. Show that if C is a circle of sufficiently small radius centered at z_0, then

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{dz}{f(z) - f(z_0)}.$$

Hint: use the inverse function theorem.

4.2.8 8

Assume two functions $u, b : \mathbb{R}^2 \longrightarrow \mathbb{R}$ have continuous partial derivatives at (x_0, y_0). Show that $f \coloneqq u + iv$ has derivative $f'(z_0)$ at $z_0 == x_0 + iy_0$ if and only if

$$\lim_{r \to 0} \frac{1}{\pi r^2} \int_{|z-z_0|=r} f(z) dz = 0.$$

4.2.9 9 (Cauchy's Formula for Exterior Regions)

Let γ be a piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2. Assume f' exists in an open set containing γ and Ω_2 with $\lim_{z \longrightarrow \infty} f(z) = A$. Show that

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1 \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}.$$

4.2.10 10

Let $f(z)$ be bounded and analytic in $\mathbb{C}$. Let $a \neq b$ be any fixed complex numbers. Show that the following limit exists:

$$\lim_{R \longrightarrow \infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}\, dz.$$

Use this to show that $f(z)$ must be constant.

4.2.11 11

Suppose $f(z)$ is entire and

$$\lim_{z \longrightarrow \infty} \frac{f(z)}{z} = 0.$$

Show that $f(z)$ is a constant.

4.2.12 12

Let f be analytic in a domain D and γ be a closed curve in D. For any $z_0 \in D$ not on γ, show that

$$\int_\gamma \frac{f'(z)}{(z-z_0)} dz = \int_\gamma \frac{f(z)}{(z-z_0)^2} dz.$$

Give a generalization of this result.

4.2.13 13

Compute

$$\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z}$$

and use it to show that

$$\in_0^{2\pi} \cos^{2n}(\theta)\, d\theta = 2\pi \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right).$$

5 Spring 2020 Homework 3

5.1 Stein and Shakarchi

5.1.1 3.8.1

Use the following formula to show that the complex zeros of $\sin(\pi z)$ are exactly the integers, and they are each of order 1:

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}.$$

Calculate the residue of $\dfrac{1}{\sin(\pi z)}$ at $z = n \in \mathbb{Z}$.

5.1.2 3.8.2

Evaluate the integral

$$\int_{\mathbb{R}} \frac{dx}{1+x^4}.$$

What are the poles of $\dfrac{1}{1+z^4}$?

5.1.3 3.8.4

Show that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2} dx = \pi e^{-a}, \qquad \text{for all } a > 0.$$

5.1.4 3.8.5

Show that if $\xi \in \mathbb{R}$, then

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{(1+x^2)^2} dx = \frac{\pi}{2}(1+2\pi|\xi|)e^{-2\pi|\xi|}.$$

5.1.5 3.8.6

Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \pi.$$

5.1.6 3.8.7

Show that

$$\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{3/2}}, \qquad \text{whenever } a > 1.$$

5.1.7 3.8.8

Show that if $a, b \in \mathbb{R}$ with $a > |b|$, then

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}.$$

5.1.8 3.8.9

Show that

$$\int_0^1 \log(\sin \pi x) dx = -\log 2.$$

Hint: use the following contour.

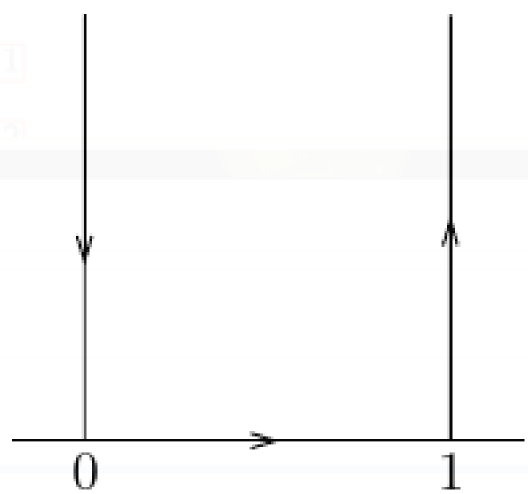


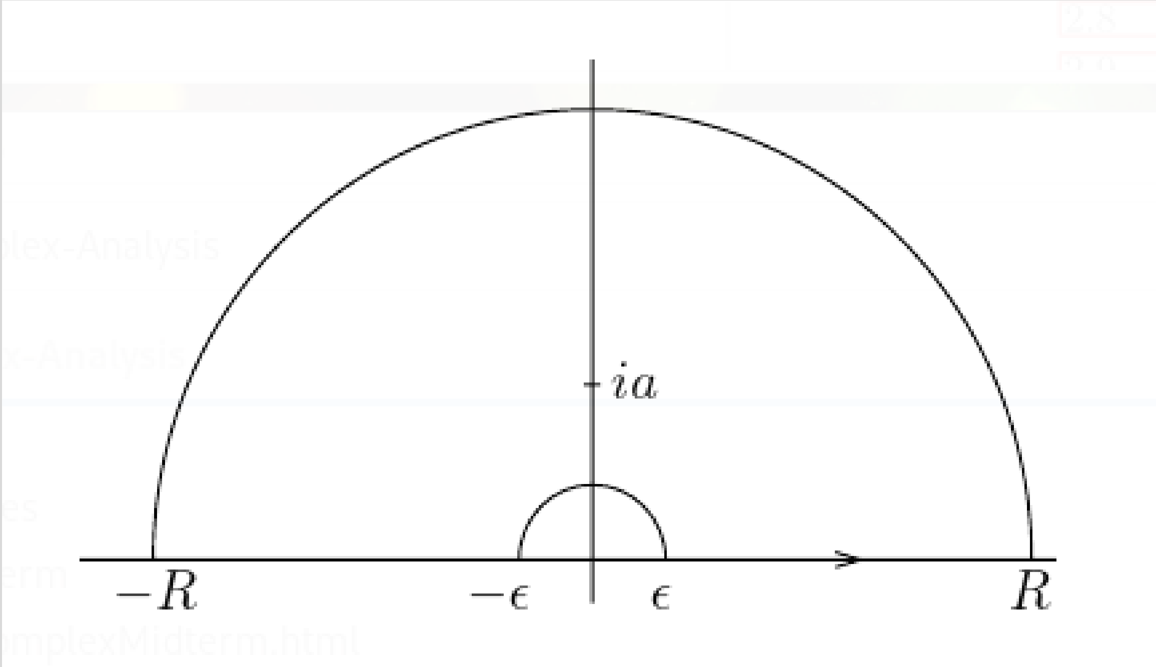
Figure 9. Contour in Exercise 9

5.1.9 3.8.10

Show that if $a > 0$, then

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx = \frac{\pi}{2a} \log a.$$

Hint: use the following contour.



5.1.10 3.8.14

Prove that all entire functions that are injective are of the form $f(z) = az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$.

Hint: Apply the Casorati-Weierstrass theorem to $f(1/z)$.

5.1.11 3.8.15

Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

a. Prove that if f is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \leq AR^k + B$$

for all $R > 0$, some integer $k \geq 0$, and some constants $A, B > 0$, then f is a polynomial of degree $\leq k$.

b. Show that if f is holomorphic in the unit disc, is bounded, and converges uniformly to zero in the sector $\theta < \arg(z) < \varphi$ as $|z| \longrightarrow 0$, then $f \equiv 0$.

c. Let $w_1, \cdots w_n$ be points on $S^1 \subset \mathbb{C}$. Prove that there exists a point $z \in S^1$ such that the product of the distances from z to the points w_j is at least 1.

Conclude that there exists a point $w \in S^1$ such that the product of the above distances is *exactly* 1.

d. Show that if the real part of an entire function is bounded, then f is constant.

5.1.12 3.8.17

Let f be non-constant and holomorphic in an open set containing the closed unit disc.

a. Show that if $|f(z)| = 1$ whenever $|z| = 1$, then the image of f contains the unit disc.

> Hint: Show that $f(z) = w_0$ has a root for every $w_0 \in \mathbb{D}$, for which it suffices to show that $f(z) = 0$ has a root. Conclude using the maximum modulus principle.

b. If $|f(z)| \geq 1$ whenever $|z| = 1$ and there exists a $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$, then the image of f contains the unit disc.

5.1.13 3.8.19

Prove that maximum principle for harmonic functions, i.e.

a. If u is a non-constant real-valued harmonic function in a region Ω, then u can not attain a maximum or a minimum in Ω.

b. Suppose Ω is a region with compact closure $\overline{\Omega}$. If u is harmonic in Ω and continuous in $\overline{\Omega}$, then

$$\sup_{z \in \Omega} |u(z)| \leq \sup_{z \in \overline{\Omega} - \Omega} |u(z)|.$$

> Hint: to prove (a), assume u attains a local maximum at z_0. Let f be holomorphic near z_0 with $\Re(f) = u$, and show that f is not an open map. Then (a) implies (b).

5.2 Problems From Tie

5.2.1 1

Prove that if f has two Laurent series expansions,

$$f(z) = \sum c_n (z - a)^n \quad \text{and} \quad f(z) = \sum c_n' (z - a)^n$$

then $c_n = c_n'$.

5.2.2 2

Find Laurent series expansions of

$$\frac{1}{1 - z^2} + \frac{1}{3 - z}$$

How many such expansions are there? In what domains are each valid?

5.2.3 3

Let P, Q be polynomials with no common zeros. Assume a is a root of Q. Find the principal part of P/Q at $z = a$ in terms of P and Q if a is (1) a simple root, and (2) a double root.

5.2.4 4

Let f be non-constant, analytic in $|z| > 0$, where $f(z_n) = 0$ for infinitely many points z_n with $\lim_{n \longrightarrow \infty} z_n = 0$.
Show that $z = 0$ is an essential singularity for f.

Example: $f(z) = \sin(1/z)$.

5.2.5 5

Show that if f is entire and $\lim_{z \longrightarrow \infty} f(z) = \infty$, then f is a polynomial.

5.2.6 6

a. Show (without using 3.8.9 in the S&S) that

$$\int_0^{2\pi} \log \left|1 - e^{i\theta}\right| \, d\theta = 0$$

b. Show that this identity is equivalent to S&S 3.8.9:

$$\int_0^1 \log(\sin(\pi x)) \, dx = -\log 2.$$

5.2.7 7

Let $0 < a < 4$ and evaluate

$$\int_0^\infty \frac{x^{\alpha-1}}{1 + x^3} \, dx$$

5.2.8 8

Prove the fundamental theorem of Algebra using
a. Rouche's Theorem.
b. The maximum modulus principle.

5.2.9 9

Let f be analytic in a region D and γ a rectifiable curve in D with interior in D. Prove that if $f(z)$ is real for all $z \in \gamma$, then f is constant.

5.2.10 10

For $a > 0$, evaluate

$$\int_0^{\pi/2} \frac{d\theta}{a + \sin^2 \theta}$$

5.2.11 11

Find the number of roots of $p(z) = 4z^4 - 6z + 3$ in $|z| < 1$ and $1 < |z| < 2$ respectively.

5.2.12 12

Prove that $z^4 + 2z^3 - 2z + 10$ has exactly one root in each open quadrant.

5.2.13 13

Prove that for $a > 0$, $z \tan z - a$ has only real roots.

5.2.14 14

Let f be nonzero, analytic on a bounded region Ω and continuous on its closure $\overline{\Omega}$. Show that if $|f(z)| \equiv M$ is constant for $z \in \partial\Omega$, then $f(z) \equiv Me^{i\theta}$ for some real constant θ.

6 Extra Questions from Jingzhi Tie

6.1 Fall 2009

6.1.1 ?

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  (1) Assume $\displaystyle f(z) = \sum_{n=0}^\infty c_n z^n$
converges in $|z| < R$. Show that for $r <R$,
$$\frac{1}{2 \pi} \int_0^{2 \pi} |f(r e^{i \theta})|^2 d \theta =
\sum_{n=0}^\infty |c_n|^2 r^{2n} \; .$$

(2) Deduce Liouville’s theorem from (1).
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6.1.2 ?

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  Let $f$ be a continuous function in the region
$$D=\{z \suchthat  \abs{z}>R, 0\leq \arg z\leq \theta\}\quad\text{where}\quad
1\leq \theta \leq 2\pi.$$ If there exists $k$ such that
$\displaystyle{\lim_{z\to\infty} zf(z)=k}$ for $z$ in the region
$D$.
Show that $$\lim_{R’\to\infty} \int_{L} f(z) dz=i\theta k,$$
where $L$ is the part of the circle $|z|=R’$ which lies in the
region $D$.
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6.1.3 ?

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  Suppose that $f$ is an analytic function in the region $D$ which
contains the point $a$. Let
$$F(z)= z-a-qf(z),\quad \text{where}~ q \ \text{is a complex
parameter}.$$

(1) Let $K\subset D$ be a circle with the center at
point $a$ and also we assume that $f(z)\not =0$ for $z\in K$. Prove
that the function $F$ has one and only one zero $z=w$ on the closed
disc $\bar{K}$ whose boundary is the circle $K$ if $\displaystyle{
|q|<\min_{z\in K} \frac{|z-a|}{|f(z)|}.}$\

(2) Let $G(z)$ be an analytic function on the disk $\bar{K}$. Apply
the residue theorem to prove that
$\displaystyle{ \frac{G(w)}{F’(w)}=\frac{1}{2\pi
i}\int_K \frac{G(z)}{F(z)} dz,}$ where $w$ is the zero from (1).\

(3) If $z\in K$, prove that the function
$\displaystyle{\frac{1}{F(z)}}$ can be represented as a convergent
series with respect to $q$: $\displaystyle{
\frac{1}{F(z)}=\sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z-a)^{n+1}}.}$
```

6.1.4 ?

Evaluate $$\displaystyle{ \int_{0}^{\infty}\frac{x\sin x}{x^2+a^2} \,
dx }.$$

6.1.5 ?

Let $f=u+iv$ be differentiable (i.e. $f'(z)$ exists) with continuous
partial derivatives at a point $z=re^{i\theta}$, $r\not= 0$. Show
that
$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\quad
\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

6.1.6 ?

Show that $\displaystyle \int_0^\infty \frac{x^{a-1}}{1+x^n}
dx=\frac{\pi}{n\sin \frac{a\pi}{n}}$ using complex analysis, $0< a <
n$. Here n is a positive integer.

6.1.7 ?

For $s>0$, the **gamma function** is defined by
$\displaystyle{\Gamma(s)=\int_0^{\infty} e^{-t}t^{s-1} dt}$.

1. Show that the gamma function is analytic in the half-plane
   $\Re (s)>0$, and is still given there by the integral formula
   above.

2. Apply the formula in the previous question to show that
   $$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin \pi s}.$$

> Hint: You may need $\displaystyle{\Gamma(1-s)=t \int_0^{\infty}e^{-vt}(vt)^{-s} dv}$ for $t>0$.

6.1.8 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If
$$P_n(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n\quad (a_n \neq 0)$$
is a polynomial of degree n, then it has n zeros in $\mathbb C$.

6.1.9 ?

Suppose f is entire and there exist $A, R >0$ and natural number
N such that $$|f(z)| \geq A |z|^N\ \text{for}\ |z| \geq R.$$ Show
that
(i) f is a polynomial and
(ii) the degree of f is at least N.

6.1.10 ?

Let $f: {\mathbb C} \rightarrow {\mathbb C}$ be an injective
analytic (also called *univalent*) function. Show that there exist
complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

6.1.11 ?

Let g be analytic for $|z|\leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

1. Show that g has a unique fixed point in $|z| < 1$.

2. What happens if we replace $|g(z)| < 1$ with $|g(z)|\leq 1$ for
$|z|=1$? Give an example if (a) is not true or give an proof
if (a) is still true.

3. What happens if we simply assume that f is analytic for
$|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z)
\not\equiv z$. Can f have more than one fixed point in
$|z| < 1$?

> Hint: The map $\displaystyle{\psi_{\alpha}(z)=\frac{\alpha-z}{1-\bar{\alpha}z}}$ may be useful.

6.1.12 ?

Find a conformal map from $D = \{z :\ |z| < 1,\ |z - 1/2| > 1/2\}$
to the unit disk $\Delta=\{z: \ |z|<1\}$.

6.1.13 ?

Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded*
open set Ω. Show without using Picard's theorems that $f(z)$ is a constant.

6.1.14 ?

(1) Assume $\displaystyle f(z) = \sum_{n=0}^\infty c_n z^n$ converges
in $|z| < R$. Show that for $r <R$,
$$\frac{1}{2 \pi} \int_0^{2 \pi} |f(r e^{i \theta})|^2 d \theta
= \sum_{n=0}^\infty |c_n|^2 r^{2n} \; .$$

(2) Deduce Liouville's theorem from (1).

6.1.15 ?

Let $f(z)$ be entire and assume that $f(z) \leq M |z|^2$ outside some
disk for some constant M. Show that $f(z)$ is a polynomial in z of degree ≤ 2.

6.1.16 ?

Let $a_n(z)$ be an analytic sequence in a domain D such that

$\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D. Show that $\sum_{n=0}^{\infty} |a_n'(z)|$ converges uniformly on bounded and closed sub-regions of D.

6.1.17 ?

Let $f(z)$ be analytic in an open set Ω except possibly at a
point z_0 inside Ω. Show that if $f(z)$ is bounded in near
z_0, then $\displaystyle \int_\Delta f(z) dz = 0$ for all triangles
Δ in Ω.

6.1.18 ?

Assume f is continuous in the region:

$0 < |z-a| \leq R,\ 0 \leq \arg(z-a) \leq \beta_0\ (0 < \beta_0 \leq 2\pi)$ and the limit $\lim_{z\to a}(z-a)f(z) = A$ exists. Show that

$$\lim_{r\to 0}\int_{\gamma_r} f(z)dz = iA\beta_0\ ,$$

where $\gamma_r := \{z \mid z = a + re^{it},\ 0 \leq t \leq \beta_0\}$.

6.1.19 ?

Show that $f(z) = z^2$ is uniformly continuous in any open disk

$|z| < R$, where $R > 0$ is fixed, but it is not uniformly continuous on $\mathbb{C}$.

6.1.20 ?

(1) Show that the function $u=u(x,y)$ given by
$$u(x,y)=\frac{e^{ny}-e^{-ny}}{2n^2}\sin nx\quad \text{for}\ n\in {\mathbf N}$$
is the solution on $D=\{(x,y)\ | x^2+y^2<1\}$ of the Cauchy problem for
the Laplace equation
$$\frac{\partial ^2u}{\partial x^2}+\frac{\partial ^2u}{\partial y^2}=0,\quad
u(x,0)=0,\quad \frac{\partial u}{\partial y}(x,0)=\frac{\sin nx}{n}.$$
(2) Show that there exist points $(x,y)\in D$ such that
$\displaystyle{\limsup_{n\to\infty} |u(x,y)|=\infty}$.

6.2 Fall 2011

6.2.1 ?

(1) Assume $\displaystyle f(z) = \sum_{n=0}^\infty c_n z^n$
converges in $|z| < R$. Show that for $r <R$,

$$\frac{1}{2 \pi} \int_0^{2 \pi} |f(r e^{i \theta})|^2 d \theta =
\sum_{n=0}^\infty |c_n|^2 r^{2n} \; .$$

(2) Deduce Liouville's theorem from (1).

6.2.2 ?

Let f be a continuous function in the region
$$D=\{z\ | |z|>R, 0\leq \arg Z\leq \theta\}\quad\text{where}\quad
0\leq \theta \leq 2\pi.$$ If there exists k such that
$\displaystyle{\lim_{z\to\infty} zf(z)=k}$ for z in the region
D. Show that $$\lim_{R'\to\infty} \int_{L} f(z) dz=i\theta k,$$
where L is the part of the circle $|z|=R'$ which lies in the
region D.

6.2.3 ?

Suppose that f is an analytic function in the region D which
contains the point a. Let
$$F(z)= z-a-qf(z),\quad \text{where}\quad q \ \text{is a complex
parameter}.$$

(1) Let $K\subset D$ be a circle with the center at
point a and also we assume that $f(z)\not =0$ for $z\in K$. Prove
that the function F has one and only one zero $z=w$ on the closed
disc $\bar{K}$ whose boundary is the circle K if $\displaystyle{
|q|<\min_{z\in K} \frac{|z-a|}{|f(z)|}.}$\

(2) Let $G(z)$ be an analytic function on the disk $\bar{K}$. Apply
the residue theorem to prove that
$\displaystyle{ \frac{G(w)}{F'(w)}=\frac{1}{2\pi
i}\int_K \frac{G(z)}{F(z)} dz,}$ where w is the zero from (1).\

(3) If $z\in K$, prove that the function
$\displaystyle{\frac{1}{F(z)}}$ can be represented as a convergent
series with respect to q: $\displaystyle{
\frac{1}{F(z)}=\sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z-a)^{n+1}}.}$

6.2.4 ?

Evaluate $\displaystyle{ \int_{0}^{\infty}\frac{x\sin x}{x^2+a^2} \,
dx }$.

6.2.5 ?

Let $f=u+iv$ be differentiable (i.e. $f'(z)$ exists) with continuous
partial derivatives at a point $z=re^{i\theta}$, $r\not= 0$. Show

that
$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\quad
\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

6.2.6 ?

Show that $\displaystyle \int_0^\infty \frac{x^{a-1}}{1+x^n}
dx=\frac{\pi}{n\sin \frac{a\pi}{n}}$ using complex analysis, $0< a <
n$. Here n is a positive integer.

6.2.7 ?

For $s>0$, the **gamma function** is defined by
$\displaystyle{\Gamma(s)=\int_0^{\infty} e^{-t}t^{s-1} dt}$.

1. Show that the gamma function is analytic in the half-plane
   $\Re (s)>0$, and is still given there by the integral formula
   above.

2. Apply the formula in the previous question to show that
   $$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin \pi s}.$$

> Hint: You may need $\displaystyle{\Gamma(1-s)=t \int_0^{\infty}e^{-vt}(vt)^{-s} dv}$ for $t>0$.

6.2.8 ?

Apply Rouché’s Theorem to prove the Fundamental Theorem of Algebra: If
$$P_n(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n\quad (a_n \neq 0)$$
is a polynomial of degree n, then it has n zeros in $\mathbb C$.

6.2.9 ?

Suppose f is entire and there exist $A, R >0$ and natural number
N such that $$|f(z)| \geq A |z|^N\ \text{for}\ |z| \geq R.$$ Show
that (i) f is a polynomial and (ii) the degree of f is at least
N.

6.2.10 ?

Let $f: {\mathbb C} \rightarrow {\mathbb C}$ be an injective
analytic (also called univalent) function. Show that there exist
complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

6.2.11 ?

Let g be analytic for $|z|\leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

- Show that g has a unique fixed point in $|z| < 1$.

- What happens if we replace $|g(z)| < 1$ with $|g(z)|\leq 1$ for
  $|z|=1$? Give an example if (a) is not true or give an proof
  if (a) is still true.

- What happens if we simply assume that f is analytic for
  $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z)
  \not\equiv z$. Can f have more than one fixed point in
  $|z| < 1$?

> Hint: The map
$\displaystyle{\psi_{\alpha}(z)=\frac{\alpha-z}{1-\bar{\alpha}z}}$
> may be useful.

6.2.12 ?

Find a conformal map from $D = \{z :\ |z| < 1,\ |z - 1/2| > 1/2\}$
to the unit disk $\Delta=\{z: \ |z|<1\}$.

6.2.13 ?

Let $f(z)$ be entire and assume values of $f(z)$ lie outside a
bounded open set Ω. Show without using Picard's theorems
that $f(z)$ is a constant.

6.2.14 ?

Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded*
open set Ω. Show without using Picard's theorems that $f(z)$ is a
constant.

6.2.15 ?

(1) Assume $\displaystyle f(z) = \sum_{n=0}^\infty c_n z^n$ converges
in $|z| < R$. Show that for $r <R$,
$$\frac{1}{2 \pi} \int_0^{2 \pi} |f(r e^{i \theta})|^2 d \theta
= \sum_{n=0}^\infty |c_n|^2 r^{2n} \; .$$

(2) Deduce Liouville's theorem from (1).

6.2.16 ?

Let $f(z)$ be entire and assume that $f(z) \leq M |z|^2$ outside some
disk for some constant M. Show that $f(z)$ is a polynomial in z of
degree ≤ 2.

6.2.17 ?

Let $a_n(z)$ be an analytic sequence in a domain D such that
$\displaystyle \sum_{n=0}^\infty |a_n(z)|$ converges uniformly on
bounded and closed sub-regions of D. Show that
$\displaystyle \sum_{n=0}^\infty |a'_n(z)|$ converges uniformly on
bounded and closed sub-regions of D.

6.2.18 ?

Let $f(z)$ be analytic in an open set Ω except possibly at a
point z_0 inside Ω. Show that if $f(z)$ is bounded in near
z_0, then $\displaystyle \int_\Delta f(z) dz = 0$ for all triangles
Δ in Ω.

6.2.19 ?

Assume f is continuous in the region:
$0< |z-a| \leq R, \; 0 \leq \arg(z-a) \leq \beta_0$
($0 < \beta_0 \leq 2 \pi$) and the limit
$\displaystyle \lim_{z \rightarrow a} (z-a) f(z) = A$ exists. Show that
$$\lim_{r \rightarrow 0} \int_{\gamma_r} f(z) dz = i A \beta_0 \; , \; \;$$
where
$\gamma_r : = \{ z \; | \; z = a + r e^{it}, \; 0 \leq t \leq \beta_0 \}.$

6.2.20 ?

Show that $f(z) = z^2$ is uniformly continuous in any open disk
$|z| < R$, where $R>0$ is fixed, but it is not uniformly continuous on
$\mathbb C$.

(1) Show that the function $u=u(x,y)$ given by
$$u(x,y)=\frac{e^{ny}-e^{-ny}}{2n^2}\sin nx\quad \text{for}\ n\in {\mathbf N}$$
is the solution on $D=\{(x,y)\ | x^2+y^2<1\}$ of the Cauchy problem for
the Laplace equation
$$\frac{\partial ^2u}{\partial x^2}+\frac{\partial ^2u}{\partial y^2}=0,\quad
u(x,0)=0,\quad \frac{\partial u}{\partial y}(x,0)=\frac{\sin nx}{n}.$$
(2) Show that there exist points $(x,y)\in D$ such that
$\displaystyle{\limsup_{n\to\infty} |u(x,y)|=\infty}$.

6.3 Spring 2014

6.3.1 ?

The question provides some insight into Cauchy's theorem. Solve the
problem without using the Cauchy theorem.

1. Evaluate the integral $\displaystyle{\int_{\gamma} z^n dz}$ for

all integers n. Here γ is any circle centered at the origin with the positive (counterclockwise) orientation.

2. Same question as (a), but with γ any circle not containing the origin.

3. Show that if $|a|<r<|b|$, then $\displaystyle{\int_{\gamma}\frac{dz}{(z-a)(z-b)} dz=\frac{2\pi i}{a-b}}$. Here γ denotes the circle centered at the origin, of radius r, with the positive orientation.

6.3.2 ?

(1) Assume the infinite series $\displaystyle \sum_{n=0}^\infty c_n z^n$ converges in $|z| < R$ and let $f(z)$ be the limit. Show that for $r <R$, $$\frac{1}{2 \pi} \int_0^{2 \pi} |f(r e^{i \theta})|^2 d \theta = \sum_{n=0}^\infty |c_n|^2 r^{2n} \; .$$

(2) Deduce Liouville's theorem from (1). Liouville's theorem: If $f(z)$ is entire and bounded, then f is constant.

6.3.3 ?

Let f be a continuous function in the region $$D=\{z\ | |z|>R, 0\leq \arg Z\leq \theta\}\quad\text{where}\quad 0\leq \theta \leq 2\pi.$$ If there exists k such that $\displaystyle{\lim_{z\to\infty} zf(z)=k}$ for z in the region D. Show that $$\lim_{R'\to\infty} \int_{L} f(z) dz=i\theta k,$$ where L is the part of the circle $|z|=R'$ which lies in the region D.

6.3.4 ?

Evaluate $\displaystyle{ \int_{0}^{\infty}\frac{x\sin x}{x^2+a^2} \, dx }$.

6.3.5 ?

Let $f=u+iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z=re^{i\theta}$, $r\not= 0$. Show that $$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\quad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

6.3.6 ?

Show that $\displaystyle \int_0^\infty \frac{x^{a-1}}{1+x^n}
dx=\frac{\pi}{n\sin \frac{a\pi}{n}}$ using complex analysis, $0< a <
n$. Here n is a positive integer.

6.3.7 ?

For $s>0$, the **gamma function** is defined by
$\displaystyle{\Gamma(s)=\int_0^{\infty} e^{-t}t^{s-1} dt}$.

- Show that the gamma function is analytic in the half-plane
  $\Re (s)>0$, and is still given there by the integral formula
  above.

- Apply the formula in the previous question to show that
  $$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin \pi s}.$$

> Hint: You may need $\displaystyle{\Gamma(1-s)=t \int_0^{\infty}e^{-vt}(vt)^{-s} dv}$ for $t>0$.

6.3.8 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If
$$P_n(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n\quad (a_n \neq 0)$$
is a polynomial of degree n, then it has n zeros in $\mathbf C$.

6.3.9 ?

Suppose f is entire and there exist $A, R >0$ and natural number
N such that $$|f(z)| \geq A |z|^N\ \text{for}\ |z| \geq R.$$ Show
that (i) f is a polynomial and (ii) the degree of f is at least
N.

6.3.10 ?

Let $f: {\mathbb C} \rightarrow {\mathbb C}$ be an injective
analytic (also called univalent) function. Show that there exist
complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

6.3.11 ?

Let g be analytic for $|z|\leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

- Show that g has a unique fixed point in $|z| < 1$.

- What happens if we replace $|g(z)| < 1$ with $|g(z)|\leq 1$ for
  $|z|=1$? Give an example if (a) is not true or give an proof

if (a) is still true.

- What happens if we simply assume that f is analytic for
$|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z)
\not\equiv z$. Can f have more than one fixed point in
$|z| < 1$?

> Hint: The map
$\displaystyle{\psi_{\alpha}(z)=\frac{\alpha-z}{1-\bar{\alpha}z}}$
> may be useful.

6.3.12 ?

Find a conformal map from $D = \{z :\ |z| < 1,\ |z - 1/2| > 1/2\}$
to the unit disk $\Delta=\{z: \ |z|<1\}$.

6.4 Fall 2015

6.4.1 ?

Let $a_n \neq 0$ and assume that $\displaystyle
\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L$. Show that
$\displaystyle
\lim_{n \rightarrow \infty}
\sqrt[n]{|a_n|} = L.
%p_n^{\frac{1}{n}} = L.$ In particular, this shows that when
applicable, the ratio test can be used to calculate the radius of
convergence of a power series.

6.4.2 ?

(a) Let z, w be complex numbers, such that $\bar{z} w \neq 1$.
Prove that
$$\abs{\frac{w - z}{1 - \bar{w} z}} < 1 \; \; \; \mbox{if} \; |z| < 1 \; \mbox{and}\; |w| < 1,$$
and also that
$$\abs{\frac{w - z}{1 - \bar{w} z}} = 1 \; \; \; \mbox{if} \; |z| = 1 \; \mbox{or}\; |w| = 1.$$

(b) Prove that for fixed w in the unit disk $\mathbb D$, the
mapping $$F: z \mapsto \frac{w - z}{1 - \bar{w} z}$$ satisfies the
following conditions:

(i) F maps $\mathbb D$ to itself and is holomorphic.

(ii) F interchanges 0 and w, namely, $F(0) = w$ and
$F(w) = 0$.

(iii) $|F(z)| = 1$ if $|z| = 1$.

(iv) $F: {\mathbb D} \mapsto {\mathbb D}$ is bijective.

> Hint: Calculate $F \circ F$.

6.4.3 ?

Use n-th roots of unity (i.e. solutions of $z^n - 1 =0$) to show
that
$$2^{n-1} \sin\frac{\pi}{n} \sin\frac{2\pi}{n} \cdots \sin\frac{(n-1)\pi}{n}
= n
\; .$$

> Hint: $1 - \cos 2 \theta = 2 \sin^2 \theta,\; \sin 2 \theta = 2 \sin \theta \cos \theta$.

(a) Show that in polar coordinates, the Cauchy-Riemann
equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}
\; \; \; \text{and} \; \; \;
\frac{\partial v}{\partial r} = - \frac{1}{r} \frac{\partial u}{\partial \theta}$$

(b) Use these equations to show that the logarithm function
defined by $$\log z = \log r + i \theta \; \;
\mbox{where} \; z = r e^{i \theta } \; \mbox{with} \; - \pi < \theta < \pi$$
is a holomorphic function in the region
$r>0, \; - \pi < \theta < \pi$. Also show that $\log z$ defined
above is not continuous in $r>0$.

6.4.4 ?

Assume f is continuous in the region:
$x \geq x_0, \; 0 \leq y \leq b$ and the limit
$$\displaystyle \lim_{x \rightarrow + \infty} f(x + iy) = A$$ exists
uniformly with respect to y (independent of y). Show that
$$\lim_{x \rightarrow + \infty} \int_{\gamma_x} f(z) dz = iA b \; , \; \;$$
where $\gamma_x : = \{ z \; | \; z = x + it, \; 0 \leq t \leq b\}.$

6.4.5 ?

(Cauchy's formula for "exterior" region) Let γ be piecewise
smooth simple closed curve with interior Ω_1 and exterior
Ω_2. Assume $f'(z)$ exists in an open set containing γ
and Ω_2 and $\lim_{z \rightarrow \infty } f(z) = A$. Show
that
$$\frac{1}{2 \pi i} \int_\gamma \frac{f(\xi)}{\xi - z} \, d \xi =
\begin{cases}
A,   &   \text{if\ $z \in \Omega_1$}, \\

-f (z) + A, & \text{if\ $z \in \Omega_2$}
\end{cases}$$

6.4.6 ?

Let $f(z)$ be bounded and analytic in $\mathbb C$. Let $a \neq b$ be
any fixed complex numbers. Show that the following limit exists
$$\lim_{R \rightarrow \infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$
Use this to show that $f(z)$ must be a constant (Liouville's
theorem).

6.4.7 ?

Prove by *justifying all steps* that for all $\xi \in {\mathbb C}$
we have $\displaystyle
e^{- \pi \xi^2} = \int_{- \infty}^\infty e^{- \pi x^2} e^{2 \pi i x \xi} dx \; .$

> Hint: You may use that fact in Example 1 on p. 42 of the textbook
without proof, i.e., you may assume the above is true for real
values of ξ.

6.4.8 ?

Suppose that f is holomorphic in an open set containing the closed
unit disc, except for a pole at z_0 on the unit circle. Let
$\displaystyle
%f(z) = \sum_{n = 1}^\infty a_n z^n
f(z) = \sum_{n = 1}^\infty c_n z^n$ denote the the power series in
the open disc. Show that (1) $c_n \neq 0$ for all large enough
n's, and (2)
$\displaystyle \lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}}= z_0$.

6.4.9 ?

Let $f(z)$ be a non-constant analytic function in $|z|>0$ such that
$f(z_n) = 0$ for infinite many points z_n with
$\lim_{n \rightarrow \infty} z_n =0$. Show that $z=0$ is an
essential singularity for $f(z)$. (An example of such a function is
$f(z) = \sin (1/z)$.)

6.4.10 ?

Let f be entire and suppose that
$\lim_{z \rightarrow \infty} f(z) = \infty$. Show that f is a
polynomial.

6.4.11 ?

Expand the following functions into Laurent series in the indicated regions:

(a)
$\displaystyle f(z) = \frac{z^2 - 1}{ (z+2)(z+3)}, \; \; 2 < |z| < 3$, $3 < |z| < + \infty$.

(b)
$\displaystyle f(z) = \sin \frac{z}{1-z}, \; \; 0 < |z-1| < + \infty$

6.4.12 ?

Assume $f(z)$ is analytic in region D and Γ is a rectifiable curve in D with interior in D. Prove that if $f(z)$ is real for all $z \in \Gamma$, then $f(z)$ is a constant.

6.4.13 ?

Find the number of roots of $z^4 - 6z + 3 =0$ in $|z|<1$ and $1 < |z| < 2$ respectively.

6.4.14 ?

Prove that $z^4 + 2 z^3 - 2z + 10 =0$ has exactly one root in each open quadrant.

6.4.15 ?

(1) Let $f(z) \in H({\mathbb D})$, $\text{Re}(f(z)) >0$, $f(0)= a>0$. Show that $$|\frac{f(z)-a}{f(z)+a}| \leq |z|, \; \; \; |f'(0)| \leq 2a.$$

(2) Show that the above is still true if $\text{Re}(f(z)) >0$ is replaced with $\text{Re}(f(z)) \geq 0$.

6.4.16 ?

Assume $f(z)$ is analytic in ${\mathbb D}$ and $f(0)=0$ and is not a rotation (i.e. $f(z) \neq e^{i \theta} z$). Show that $\displaystyle \sum_{n=1}^\infty f^{n}(z)$ converges uniformly to an analytic function on compact subsets of ${\mathbb D}$, where $f^{n+1}(z) = f(f^{n}(z))$.

6.4.17 ?

Let $f(z) = \sum_{n=0}^\infty c_n z^n$ be analytic and one-to-one in
$|z| < 1$. For $0<r<1$, let D_r be the disk $|z|<r$. Show that the
area of $f(D_r)$ is finite and is given by
$$S = \pi \sum_{n=1}^\infty n |c_n|^2 r^{2n}.$$ (Note that in
general the area of $f(D_1)$ is infinite.)

6.4.18 ?

Let $f(z) = \sum_{n= -\infty}^\infty c_n z^n$ be analytic and
one-to-one in $r_0< |z| < R_0$. For $r_0<r<R<R_0$, let $D(r,R)$ be
the annulus $r<|z|<R$. Show that the area of $f(D(r,R))$ is finite
and is given by
$$S = \pi \sum_{n=- \infty}^\infty n |c_n|^2 (R^{2n} - r^{2n}).$$

6.5 Spring 2015

6.5.1 ?

Let $a_n(z)$ be an analytic sequence in a domain D such that
$\displaystyle \sum_{n=0}^\infty |a_n(z)|$ converges uniformly on
bounded and closed sub-regions of D. Show that
$\displaystyle \sum_{n=0}^\infty |a'_n(z)|$ converges uniformly on
bounded and closed sub-regions of D.

6.5.2 ?

Let f_n, f be analytic functions on the unit disk ${\mathbb D}$.
Show that the following are equivalent.

(i) $f_n(z)$ converges to $f(z)$ uniformly on compact subsets in
$\mathbb D$.

(ii) $\int_{|z|= r} |f_n(z) - f(z)| \, |dz|$ converges to 0 if
$0< r<1$.

6.5.3 ?

Let f and g be non-zero analytic functions on a region Ω.
Assume $|f(z)| = |g(z)|$ for all z in Ω. Show that
$f(z) = e^{i \theta} g(z)$ in Ω for some
$0 \leq \theta < 2 \pi$.

6.5.4 ?

Suppose f is analytic in an open set containing the unit disc
$\mathbb D$ and $|f(z)| =1$ when $|z|$=1. Show that either

$f(z) = e^{i \theta}$ for some $\theta \in \mathbb R$ or there are
finite number of $z_k \in \mathbb D$, $k \leq n$ and
$\theta \in \mathbb R$ such that
$\displaystyle f(z) = e^{i\theta} \prod_{k=1}^n \frac{z-z_k}{1 - \bar{z}_k z } \, .$

> Also cf. Stein et al, 1.4.7, 3.8.17

6.5.5 ?

(1) Let $p(z)$ be a polynomial, $R>0$ any positive number, and
$m \geq 1$ an integer. Let
$M_R = \sup \{ |z^{m} p(z) - 1|: |z| = R \}$. Show that $M_R>1$.

(2) Let $m \geq 1$ be an integer and
$K = \{z \in {\mathbb C}: r \leq |z| \leq R \}$ where $r<R$.
Show (i) using (1) as well as, (ii) without using (1) that there
exists a positive number $\varepsilon_0>0$ such that for each
polynomial $p(z)$,
$$\sup \{|p(z) - z^{-m}|: z \in K \} \geq \varepsilon_0 \, .$$

6.5.6 ?

Let $\displaystyle f(z) = \frac{1}{z} + \frac{1}{z^2 -1}$. Find all
the Laurent series of f and describe the largest annuli in which
these series are valid.

6.5.7 ?

Suppose f is entire and there exist $A, R >0$ and natural number
N such that $|f(z)| \leq A |z|^N$ for $|z| \geq R$. Show that (i)
f is a polynomial and (ii) the degree of f is at most N.

6.5.8 ?

Suppose f is entire and there exist $A, R >0$ and natural number
N such that $|f(z)| \geq A |z|^N$ for $|z| \geq R$. Show that (i)
f is a polynomial and (ii) the degree of f is at least N.

6.5.9 ?

(1) Explicitly write down an example of a non-zero analytic
function in $|z|<1$ which has infinitely zeros in $|z|<1$.

(2) Why does not the phenomenon in (1) contradict the uniqueness
theorem?

6.5.10 ?

(1) Assume u is harmonic on open set O and z_n is a sequence
in O such that $u(z_n) = 0$ and $\lim z_n \in O$. Prove or
disprove that u is identically zero. What if O is a region?

(2) Assume u is harmonic on open set O and $u(z) = 0$ on a
disc in O. Prove or disprove that u is identically zero. What if
O is a region?

(3) Formulate and prove a Schwarz reflection principle for
harmonic functions

> cf. Theorem 5.6 on p.60 of Stein et al.

> Hint: Verify the mean value property for your new function obtained by
Schwarz reflection principle.

6.5.11 ?

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that
for any $s<r$, there exists a constant $c>0$ such that
$$||f||_{(\infty, s)} \leq c ||f||_{(1, r)},$$ where
$\displaystyle |f||_{(\infty, s)} = \text{sup}_{z \in D_s(z_0)}|f(z)|$
and $\displaystyle ||f||_{(1, r)} = \int_{D_r(z_0)} |f(z)|dx dy$.

> Note: Exercise 3.8.20 on p.107 in Stein et al is a
straightforward consequence of this stronger result using the
integral form of the Cauchy-Schwarz inequality in real analysis.

6.5.12 ?

(1) Let f be analytic in $\Omega: 0<|z-a|<r$ except at a
sequence of poles $a_n \in \Omega$ with
$\lim_{n \rightarrow \infty} a_n = a$. Show that for any
$w \in \mathbb C$, there exists a sequence $z_n \in \Omega$ such
that $\lim_{n \rightarrow \infty} f(z_n) = w$.

(2) Explain the similarity and difference between the above
assertion and the Weierstrass-Casorati theorem.

6.5.13 ?

Compute the following integrals.

\(i\) $\displaystyle \int_0^\infty \frac{1}{(1 + x^n)^2} \, dx$,
$n \geq 1$ (ii)
$\displaystyle \int_0^\infty \frac{\cos x}{(x^2 + a^2)^2} \, dx$,

$a \in \mathbb R$ (iii)
$\displaystyle \int_0^\pi \frac{1}{a + \sin \theta} \, d \theta$,
$a>1$

\(iv\) $\displaystyle \int_0^{\frac{\pi}{2}}
\frac{d \theta}{a+ \sin ^2 \theta},$ $a >0$. (v)
$\displaystyle \int_{|z|=2} \frac{1}{(z^{5} -1) (z-3)} \, dz$ (v)
$\displaystyle \int_{- \infty}^{\infty} \frac{\sin \pi a}{\cosh \pi x + \cos \pi a} e^{- i x \xi} \,
$0< a <1$, $\xi \in \mathbb R$ (vi)
$\displaystyle \int_{|z| = 1} \cot^2 z \, dz$.

6.5.14 ?

Compute the following integrals.

\(i\) $\displaystyle \int_0^\infty \frac{\sin x}{x} \, dx$ (ii)
$\displaystyle \int_0^\infty (\frac{\sin x}{x})^2 \, dx$ (iii)
$\displaystyle \int_0^\infty \frac{x^{a-1}}{(1 + x)^2} \, dx$,
$0< a < 2$

\(i\)
$\displaystyle \int_0^\infty \frac{\cos a x - \cos bx}{x^2} dx$,
$a, b >0$ (ii)
$\displaystyle \int_0^\infty \frac{x^{a-1}}{1 + x^n} \, dx$,
$0< a < n$

\(iii\) $\displaystyle \int_0^\infty \frac{\log x}{1 + x^n} \, dx$,
$n \geq 2$ (iv)
$\displaystyle \int_0^\infty \frac{\log x}{(1 + x^2)^2} dx$ (v)
$\displaystyle \int_0^{\pi} \log|1 - a \sin \theta| d \theta$,
$a \in \mathbb C$

6.5.15 ?

Let $0<r<1$. Show that polynomials
$P_n(z) = 1 + 2z + 3 z^2 + \cdots + n z^{n-1}$ have no zeros in
$|z|<r$ for all sufficiently large n's.

6.5.16 ?

Let f be an analytic function on a region Ω. Show that f
is a constant if there is a simple closed curve γ in Ω
such that its image $f(\gamma)$ is contained in the real axis.

6.5.17 ?

(1) Show that $\displaystyle \frac{\pi^2}{\sin^2 \pi z}$ and

$\displaystyle g(z) = \sum_{n = - \infty}^{ \infty} \frac{1}{(z-n)^2}$
have the same principal part at each integer point.

(2) Show that
$\displaystyle h(z) = \frac{\pi^2}{\sin^2 \pi z} - g(z)$ is bounded
on $\mathbb C$ and conclude that
$\displaystyle \frac{\pi^2}{\sin^2 \pi z} = \sum_{n = - \infty}^{ \infty} \frac{1}{(z-n)^2} \, .$

6.5.18 ?

Let $f(z)$ be an analytic function on
${\mathbb C} \backslash \{ z_0 \}$, where z_0 is a fixed point.
Assume that $f(z)$ is bijective from
${\mathbb C} \backslash \{ z_0 \}$ onto its image, and that $f(z)$
is bounded outside $D_r(z_0)$, where r is some fixed positive
number. Show that there exist $a, b, c, d \in \mathbb C$ with
$ad-bc \neq 0$, $c \neq 0$ such that
$\displaystyle f(z) = \frac{az + b}{cz + d}$.

6.5.19 ?

Assume $f(z)$ is analytic in ${\mathbb D}: |z|<1$ and $f(0)=0$ and
is not a rotation (i.e. $f(z) \neq e^{i \theta} z$). Show that
$\displaystyle \sum_{n=1}^\infty f^{n}(z)$ converges uniformly to an
analytic function on compact subsets of ${\mathbb D}$, where
$f^{n+1}(z) = f(f^{n}(z))$.

6.5.20 ?

Let f be a non-constant analytic function on $\mathbb D$ with
$f(\mathbb D) \subseteq \mathbb D$. Use $\psi_{a} (f(z))$ (where
$a=f(0)$, $\displaystyle \psi_a(z) = \frac{a - z}{1 - \bar{a}z}$) to
prove that $\displaystyle
\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq
\frac{|f(0)| + |z|}{1 - |f(0)||z|}$.

6.5.21 ?

Find a conformal map

1. from $\{ z: |z - 1/2| > 1/2, \text{Re}(z)>0 \}$ to $\mathbb H$

2. from $\{ z: |z - 1/2| > 1/2, |z| <1 \}$ to $\mathbb D$

3. from the intersection of the disk $|z + i| < \sqrt{2}$ with
   ${\mathbb H}$ to ${\mathbb D}$.

4. from ${\mathbb D} \backslash [a, 1)$ to
   ${\mathbb D} \backslash [0, 1)$ ($0<a<1)$. \[Short solution
   possible using Blaschke factor\]

5. from $\{ z: |z| < 1, \text{Re}(z) > 0 \} \backslash (0, 1/2]$ to
   $\mathbb H$.

6.5.22 ?

Let C and C' be two circles and let $z_1 \in C$, $z_2 \notin C$,
$z'_1 \in C'$, $z'_2 \notin C'$. Show that there is a unique
fractional linear transformation f with $f(C) = C'$ and
$f(z_1) = z'_1$, $f(z_2) = z'_2$.

6.5.23 ?

Assume $f_n \in H(\Omega)$ is a sequence of holomorphic functions on
the region Ω that are uniformly bounded on compact subsets
and $f \in H(\Omega)$ is such that the set
$\displaystyle \{z \in \Omega: \lim_{n \rightarrow \infty} f_n(z) = f(z) \}$
has a limit point in Ω. Show that f_n converges to f
uniformly on compact subsets of Ω.

6.5.24 ?

Let
$\displaystyle{\psi_{\alpha}(z)=\frac{\alpha-z}{1-\bar{\alpha}z}}$
with $|\alpha|<1$ and ${\mathbb D}=\{z:\ |z|<1\}$. Prove that

- $\displaystyle{\frac{1}{\pi}\iint_{{\mathbb D}} |\psi'_{\alpha}|^2 dx dy =1}$.

- $\displaystyle{\frac{1}{\pi}\iint_{{\mathbb D}} |\psi'_{\alpha}| dx dy =\frac{1-|\alpha|^2}{|\alph
  \log \frac{1}{1-|\alpha|^2}}$.

6.5.25 ?

Prove that
$\displaystyle{f(z)=-\frac{1}{2}\left(z+\frac{1}{z}\right)}$ is a
conformal map from half disc $\{z=x+iy:\ |z|<1,\ y>0\}$ to upper
half plane ${\mathbb H}=\{z=x+iy:\ y>0\}$.

6.5.26 ?

Let Ω be a simply connected open set and let γ be a
simple closed contour in Ω and enclosing a bounded region U
anticlockwise. Let $f: \ \Omega \to {\mathbb C}$ be a holomorphic
function and $|f(z)|\leq M$ for all $z\in \gamma$. Prove that

$|f(z)|\leq M$ for all $z\in U$.

6.5.27 ?

Compute the following integrals. (i)
$\displaystyle \int_0^\infty \frac{x^{a-1}}{1 + x^n} \, dx$,
$0< a < n$ (ii)
$\displaystyle \int_0^\infty \frac{\log x}{(1 + x^2)^2}\, dx$

6.5.28 ?

Let $0<r<1$. Show that polynomials
$P_n(z) = 1 + 2z + 3 z^2 + \cdots + n z^{n-1}$ have no zeros in
$|z|<r$ for all sufficiently large n's.

6.5.29 ?

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that
for any $s<r$, there exists a constant $c>0$ such that
$$\|f\|_{(\infty, s)} \leq c \|f\|_{(1, r)},$$ where
$\displaystyle \|f\|_{(\infty, s)} = \text{sup}_{z \in D_s(z_0)}|f(z)|$
and $\displaystyle \|f\|_{(1, r)} = \int_{D_r(z_0)} |f(z)|dx dy$.

6.5.30 ?

Let $\displaystyle{\psi_{\alpha}(z)=\frac{\alpha-z}{1-\bar{\alpha}z}}$
with $|\alpha|<1$ and ${\mathbb D}=\{z:\ |z|<1\}$. Prove that

- $\displaystyle{\frac{1}{\pi}\iint_{{\mathbb D}} |\psi'_{\alpha}|^2 dx dy =1}$.

- $\displaystyle{\frac{1}{\pi}\iint_{{\mathbb D}} |\psi'_{\alpha}| dx dy =\frac{1-|\alpha|^2}{|\alph
  \log \frac{1}{1-|\alpha|^2}}$.

Prove that $\displaystyle{f(z)=-\frac{1}{2}\left(z+\frac{1}{z}\right)}$
is a conformal map from half disc $\{z=x+iy:\ |z|<1,\ y>0\}$ to upper
half plane $\mathbb H=\{z=x+iy:\ y>0\}$.

6.5.31 ?

Let Ω be a simply connected open set and let γ be a simple
closed contour in Ω and enclosing a bounded region U
anticlockwise. Let $f: \ \Omega \to {\mathbb C}$ be a holomorphic
function and $|f(z)|\leq M$ for all $z\in \gamma$. Prove that
$|f(z)|\leq M$ for all $z\in U$.

6.5.32 ?

Compute the following integrals. (i)
$\displaystyle \int_0^\infty \frac{x^{a-1}}{1 + x^n} \, dx$, $0< a < n$
(ii) $\displaystyle \int_0^\infty \frac{\log x}{(1 + x^2)^2}\, dx$

6.5.33 ?

Let $0<r<1$. Show that polynomials
$P_n(z) = 1 + 2z + 3 z^2 + \cdots + n z^{n-1}$ have no zeros in $|z|<r$
for all sufficiently large n's.

6.5.34 ?

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for
any $s<r$, there exists a constant $c>0$ such that
$$\|f\|_{(\infty, s)} \leq c \|f\|_{(1, r)},$$ where
$\displaystyle \|f\|_{(\infty, s)} = \text{sup}_{z \in D_s(z_0)}|f(z)|$
and $\displaystyle \|f\|_{(1, r)} = \int_{D_r(z_0)} |f(z)|dx dy$.

6.6 Fall 2016

6.6.1 ?

Let $u(x,y)$ be harmonic and have continuous partial derivatives of
order three in an open disc of radius $R>0$.

(a) Let two points $(a,b), (x,y)$ in this disk be given. Show that
the following integral is independent of the path in this disk
joining these points:
$$v(x,y) = \int_{a,b}^{x,y} (-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy).$$

(b) \hfill

(i) Prove that $u(x,y)+i v(x,y)$ is an analytic function in this
disc.

(ii) Prove that $v(x,y)$ is harmonic in this disc.

6.6.2 ?

(a) $f(z)= u(x,y) +i v(x,y)$ be analytic in a domain
$D\subset {\mathbb C}$. Let $z_0=(x_0,y_0)$ be a point in D which
is in the intersection of the curves $u(x,y)= c_1$ and $v(x,y)=c_2$,
where c_1 and c_2 are constants. Suppose that $f'(z_0)\neq 0$.
Prove that the lines tangent to these curves at z_0 are
perpendicular.

(b) Let $f(z)=z^2$ be defined in ${\mathbb C}$.
  (i) Describe the
  level curves of $\mbox{\textrm Re}{(f)}$ and of $\mbox{Im}{(f)}$.

  (ii) What are the angles of intersections between the level curves
  $\mbox{\textrm Re}{(f)}=0$ and $\mbox{\textrm Im}{(f)}$? Is your answer in
  agreement with part a) of this question?

6.6.3 ?

(a) $f: D\rightarrow {\mathbb C}$ be a continuous function, where
$D\subset {\mathbb C}$ is a domain.Let $\alpha:[a,b]\rightarrow D$
be a smooth curve. Give a precise definition of the *complex line
integral* $$\int_{\alpha} f.$$

(b) Assume that there exists a constant M such that
$|f(\tau)|\leq M$ for all $\tau\in \mbox{\textrm Image}(\alpha$). Prove
that
$$\big | \int_{\alpha} f \big |\leq M \times \mbox{\textrm length}(\alpha).$$

(c) Let C_R be the circle $|z|=R$, described in the
counterclockwise direction, where $R>1$. Provide an upper bound for
$\big | \int_{C_R} \dfrac{\log{(z)} }{z^2} \big |,$ which depends
[only]{.underline} on R and other constants.

6.6.4 ?

(a) Let Let $f:{\mathbb C}\rightarrow {\mathbb C}$ be an entire
function. Assume the existence of a non-negative integer m, and of
positive constants L and R, such that for all z with $|z|>R$
the inequality $$|f(z)| \leq L |z|^m$$ holds. Prove that f is a
polynomial of degree $\leq m$.

(b) Let $f:{\mathbb C}\rightarrow {\mathbb C}$ be an entire
function. Suppose that there exists a real number M such that for
all $z\in {\mathbb C}$ $$\mbox{\textrm Re} (f) \leq M.$$ Prove that f
must be a constant.

6.6.5 ?

Prove that all the roots of the complex polynomial
$$z^7 - 5 z^3 +12 =0$$ lie between the circles $|z|=1$ and $|z|=2$.

6.6.6 ?

(a) Let F be an analytic function inside and on a simple closed
curve C, except for a pole of order $m\geq 1$ at $z=a$ inside C.

```
Prove that

$$\frac{1}{2 \pi i}\oint_{C} F(\tau) d\tau =
\lim_{\tau\rightarrow a} \frac{d^{m-1}}{d\tau^{m-1}}\big((\tau-a)^m F(\tau))\big).$$

(b) Evaluate $$\oint_{C}\frac{e^{\tau}}{(\tau^2+\pi^2)^2}d\tau$$
where $C$ is the circle $|z|=4$.
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6.6.7 ?

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  Find the conformal map that takes the upper half-plane comformally
onto the half-strip $\{
w=x+iy:\ -\pi/2<x<\pi/2\ y>0\}$.
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6.6.8 ?

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  Compute the integral
$\displaystyle{\int_{-\infty}^{\infty} \frac{e^{-2\pi ix\xi}}{\cosh\pi x}dx}$
where $\displaystyle{\cosh z=\frac{e^{z}+e^{-z}}{2}}$.
```