Real Analysis Qualifying Exam Questions

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Tuesday 11$^{\text{th}}$ August, 2020

Contents

1 Undergraduate Analysis: Uniform Convergence

1.1 Fall 2018 # 1

Let $f(x) = \dfrac{1}{x}$. Show that f is uniformly continuous on $(1, \infty)$ but not on $(0, \infty)$.

1.2 Fall 2017 # 1

Let

$$f(x) = s \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which f does and does not converge uniformly.

1.3 Fall 2014 # 1

Let $\{f_n\}$ be a sequence of continuous functions such that $\sum f_n$ converges uniformly.

Prove that $\sum f_n$ is also continuous.

1.4 Spring 2017 # 4

Let $f(x, y)$ on $[-1, 1]^2$ be defined by

$$f(x, y) = \begin{cases} \dfrac{xy}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Determine if f is integrable.

1.5 Spring 2015 # 1

Let (X, d) and (Y, ρ) be metric spaces, $f : X \longrightarrow Y$, and $x_0 \in X$.

Prove that the following statements are equivalent:

1. For every $\varepsilon > 0$ $\exists \delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$.
2. The sequence $\{f(x_n)\}_{n=1}^\infty \longrightarrow f(x_0)$ for every sequence $\{x_n\} \longrightarrow x_0$ in X.

1.6 Fall 2014 # 2

Let I be an index set and $\alpha : I \longrightarrow (0, \infty)$.

1. Show that

$$\sum_{i \in I} a(i) := \sup_{\substack{J \subset I \\ J \text{ finite}}} \sum_{i \in J} a(i) < \infty \implies I \text{ is countable.}$$

2. Suppose $I = \mathbb{Q}$ and $\displaystyle\sum_{q \in \mathbb{Q}} a(q) < \infty$. Define

$$f(x) := \sum_{\substack{q \in \mathbb{Q} \\ q \leq x}} a(q).$$

   Show that f is continuous at $x \iff x \notin \mathbb{Q}$.

1.7 Spring 2014 # 2

Let $\{a_n\}$ be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that $\sum a_n^2 < \infty$.

Note: Assume a_n, b_n are all non-negative.

2 General Analysis

2.1 Spring 2020 # 1

Prove that if $f : [0, 1] \longrightarrow \mathbb{R}$ is continuous then

$$\lim_{k \to \infty} \int_0^1 kx^{k-1} f(x)\, dx = f(1).$$

Solution.
Concepts used:
- DCT
- Weierstrass Approximation Theorem

Solution:
- Suppose p is a polynomial, then

$$\lim_{k \to \infty} \int_0^1 kx^{k-1} p(x)\, dx = \lim_{k \to \infty} \int_0^1 \left(\frac{\partial}{\partial x} x^k \right) p(x)\, dx$$

$$= \lim_{k \to \infty} \left[x^k p(x) \Big|_0^1 - \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx \right] \quad \text{integrating by parts}$$

$$= p(1) - \lim_{k \to \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx,$$

- Thus it suffices to show that

$$\lim_{k \to \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx = 0.$$

- Integrating by parts a second time yields

$$\lim_{k \to \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx = \lim_{k \to \infty} \frac{x^{k+1}}{k+1} p'(x) \Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx$$

$$= - \lim_{k \to \infty} \int_0^1 \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx$$

$$= - \int_0^1 \lim_{k \to \infty} \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \quad \text{by DCT}$$

$$= - \int_0^1 0 \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx$$

$$= 0.$$

  - The DCT can be applied here because f'' is continuous and $[0, 1]$ is compact, so f'' is bounded on $[0, 1]$ by a constant M and

$$\int_0^1 \left| x^k f''(x) \right| \le \int_0^1 1 \cdot M = M < \infty.$$

- Now use the Weierstrass approximation theorem:
  - If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous, then for every $\varepsilon > 0$ there exists a polynomial $p_\varepsilon(x)$ such that $\| f - p_\varepsilon \|_\infty < \varepsilon$.

- Thus

$$\left| \int_0^1 kx^{k-1} p_\varepsilon(x)\, dx - \int_0^1 kx^{k-1} f(x)\, dx \right| = \left| \int_0^1 kx^{k-1}(p_\varepsilon(x) - f(x))\, dx \right|$$

$$\leq \left| \int_0^1 kx^{k-1} \|p_\varepsilon - f\|_\infty \, dx \right|$$

$$= \|p_\varepsilon - f\|_\infty \cdot \left| \int_0^1 kx^{k-1}\, dx \right|$$

$$= \|p_\varepsilon - f\|_\infty \cdot x^k \Big|_0^1$$

$$= \|p_\varepsilon - f\|_\infty \xrightarrow{\varepsilon \to 0} 0$$

  and the integrals are equal.
- By the first argument,

$$\int_0^1 kx^{k-1} p_\varepsilon(x)\, dx = p_\varepsilon(1) \text{ for each } \varepsilon$$

- Since uniform convergence implies pointwise convergence, $p_\varepsilon(1) \xrightarrow{\varepsilon \to 0} f(1)$.

2.2 Fall 2019 # 1.

Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers.

a. Prove that if $\lim\limits_{n \to \infty} a_n = 0$, then

$$\lim_{n \to \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

b. Prove that if $\sum\limits_{n=1}^\infty \frac{a_n}{n}$ converges, then

$$\lim_{n \to \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

2.3 Fall 2018 # 4

Let $f \in L^1([0,1])$. Prove that

$$\lim_{n \to \infty} \int_0^1 f(x)|\sin nx|\, dx = \frac{2}{\pi} \int_0^1 f(x)\, dx$$

Hint: Begin with the case that f is the characteristic function of an interval.

2.4 Fall 2017 # 4

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

1. Show that $f_n \longrightarrow 0$ pointwise but not uniformly on $[0, 1]$.

> Hint: Consider the maximum of f_n.

2.

$$\lim_{n \to \infty} \int_0^1 n(1-x)^n \sin x \, dx = 0$$

2.5 Spring 2017 # 3

Let

$$f_n(x) = ae^{-nax} - be^{-nbx} \quad \text{where } 0 < a < b.$$

Show that

a. $\sum_{n=1}^{\infty} |f_n|$ is not in $L^1([0, \infty), m)$

> Hint: $f_n(x)$ has a root x_n.

b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0, \infty), m) \quad \text{and} \quad \int_0^\infty \sum_{n=1}^{\infty} f_n(x) \, dm = \ln \frac{b}{a}$$

2.6 Fall 2016 # 1

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that f converges to a differentiable function on $(1, \infty)$ and that

$$f'(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n^x}\right)'.$$

> Hint:
>
> $$\left(\frac{1}{n^x}\right)' = -\frac{1}{n^x} \ln n$$

2.7 Fall 2016 # 5

Let $\varphi \in L^\infty(\mathbb{R})$. Show that the following limit exists and satisfies the equality

$$\lim_{n \to \infty} \left(\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1 + x^2} \, dx\right)^{\frac{1}{n}} = \|\varphi\|_\infty.$$

2.8 Fall 2016 # 6

Let $f, g \in L^2(\mathbb{R})$. Show that

$$\lim_{n \longrightarrow \infty} \int_{\mathbb{R}} f(x)g(x + n)\, dx = 0$$

2.9 Spring 2016 # 1

For $n \in \mathbb{N}$, define

$$]e_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad E_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

Show that $e_n < E_n$, and prove Bernoulli's inequality:

$$(1 + x)^n \geq 1 + nx \text{ for } -1 < x < \infty \text{ and } n \in \mathbb{N}$$

Use this to show the following:

1. The sequence e_n is increasing.
2. The sequence E_n is decreasing.
3. $2 < e_n < E_n < 4$.
4. $\lim_{n \longrightarrow \infty} e_n = \lim_{n \longrightarrow \infty} E_n$.

2.10 Fall 2015 # 1

Define

$$f(x) = c_0 + c_1 x^1 + c_2 x^2 + \ldots + c_n x^n \text{ with } n \text{ even and } c_n > 0.$$

Show that there is a number x_m such that $f(x_m) \leq f(x)$ for all $x \in \mathbb{R}$.

3 Measure Theory: Sets

3.1 Spring 2020 # 2

Let m_* denote the Lebesgue outer measure on $\mathbb{R}$.

3.1.1 a.

Prove that for every $E \subseteq \mathbb{R}$ there exists a Borel set B containing E such that

$$m_*(B) = m_*(E).$$

3.1.2 b.

Prove that if $E \subseteq \mathbb{R}$ has the property that

$$m_*(A) = m_*(A \bigcap E) + m_*(A \bigcap E^c)$$

for every set $A \subseteq \mathbb{R}$, then there exists a Borel set $B \subseteq \mathbb{R}$ such that $E = B \setminus N$ with $m_*(N) = 0$.

Be sure to address the case when $m_*(E) = \infty$.

Solution.
Concepts used:
- Definition of outer measure: $m_*(E) = \inf\limits_{\{Q_j\} \rightrightarrows E} \sum |Q_j|$ where $\{Q_j\}$ is a countable collection of closed cubes.
- Break $\mathbb{R}$ into $\coprod_{n \in \mathbb{Z}}[n, n+1)$, each with finite measure.
- Theorem: $m_*(Q) = |Q|$ for Q a closed cube (i.e. the outer measure equals the volume).

Proof (of Theorem) Statement: if Q is a closed cube, then $m_*(Q) = |Q|$, the usual volume.
- $m_*(Q) \leq |Q|$:
  - Since $Q \subseteq Q$, $Q \rightrightarrows Q$ and $m_*(Q) \leq |Q|$ since m_* is an infimum over such coverings.
- $|Q| \leq m_*(Q)$:
  - Fix $\varepsilon > 0$.
  - Let $\{Q_i\}_{i=1}^\infty \rightrightarrows Q$ be arbitrary, it suffices to show that

$$|Q| \leq \left(\sum_{i=1}^\infty |Q_i| \right) + \varepsilon.$$

  - Pick open cubes S_i such that $Q_i \subseteq S_i$ and $|Q_i| \leq |S_i| \leq (1+\varepsilon)|Q_i|$.
  - Then $\{S_i\} \rightrightarrows Q$, so by compactness of Q pick a finite subcover with N elements.
  - Note

$$Q \subseteq \bigcup_{i=1}^N S_i \implies |Q| \leq \sum_{i=1}^N |S_i| \leq \sum_{i=1}^N (1+\varepsilon)|Q_j| \leq (1+\varepsilon) \sum_{i=1}^\infty |Q_i|.$$

  - Taking an infimum over coverings on the RHS preserves the inequality, so

$$|Q| \leq (1+\varepsilon)m_*(Q)$$

  - Take $\varepsilon \longrightarrow 0$ to obtain final inequality.

3.1.3 a

- If $m_*(E) = \infty$, then take $B = \mathbb{R}^n$ since $m(\mathbb{R}^n) = \infty$.
- Suppose $N := m_*(E) < \infty$.
- Since $m_*(E)$ is an infimum, by definition, for every $\varepsilon > 0$ there exists a covering by closed cubes $\{Q_i(\varepsilon)\}_{i=1}^\infty \rightrightarrows E$ depending on ε such that

$$\sum_{i=1}^\infty |Q_i(\varepsilon)| < N + \varepsilon.$$

- For each fixed n, set $\varepsilon_n = \dfrac{1}{n}$ to produce such a covering $\{Q_i(\varepsilon_n)\}_{i=1}^\infty$ and set $B_n :=$
  $\displaystyle\bigcup_{i=1}^\infty Q_i(\varepsilon_n)$.
- The outer measure of cubes is *equal* to the sum of their volumes, so

$$m_*(B_n) = \sum_{i=1}^\infty |Q_i(\varepsilon_n)| < N + \varepsilon_n = N + \frac{1}{n}.$$

- Now set $B := \displaystyle\bigcap_{n=1}^\infty B_n$.
  - Since $E \subseteq B_n$ for every n, $E \subseteq B$
  - Since B is a countable intersection of countable unions of closed sets, B is Borel.
  - Since $B_n \subseteq B$ for every n, we can apply subadditivity to obtain the inequality

$$E \subseteq B \subseteq B_n \implies N \le m_*(B) \le m_*(B_n) < N + \frac{1}{n} \quad \text{for all} \quad n \in \mathbb{Z}^{\ge 1}.$$

- This forces $m_*(E) = m_*(B)$.

3.1.4 b

Suppose $m_*(E) < \infty$.
- By (a), find a Borel set $B \supseteq E$ such that $m_*(B) = m_*(E)$
- Note that $E \subseteq B \implies B \bigcap E = E$ and $B \bigcap E^c = B \setminus E$.
- By assumption,

$$m_*(B) = m_*(B \bigcap E) + m_*(B \bigcap E^c)$$
$$m_*(E) = m_*(E) + m_*(B \setminus E)$$
$$m_*(E) - m_*(E) = m_*(B \setminus E) \qquad \text{since } m_*(E) < \infty$$
$$\implies m_*(B \setminus E) = 0.$$

- So take $N = B \setminus E$; this shows $m_*(N) = 0$ and $E = B \setminus (B \setminus E) = B \setminus N$.

If $m_*(E) = \infty$:
- Apply result to $E_R := E \bigcap [R, R+1)^n \subset \mathbb{R}^n$ for $R \in \mathbb{Z}$, so $E = \coprod_R E_R$
- Obtain B_R, N_R such that $E_R = B_R \setminus N_R$, $m_*(E_R) = m_*(B_R)$, and $m_*(N_R) = 0$.
- Note that
  - $B := \bigcup_R B_R$ is a union of Borel sets and thus still Borel
  - $E = \bigcup_R E_R$
  - $N := B \setminus E$
  - $N' := \bigcup_R N_R$ is a union of null sets and thus still null
- Since $E_R \subset B_R$ for every R, we have $E \subset B$
- We can compute

$$N = B \setminus E = \left(\bigcup_R B_R\right) \setminus \left(\bigcup_R E_R\right) \subseteq \bigcup_R (B_R \setminus E_R) = \bigcup_R N_R := N'$$

where $m_*(N') = 0$ since N' is null, and thus subadditivity forces $m_*(N) = 0$.

3.2 Fall 2019 # 3.

Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) = 1$ and $\{B_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{B}$-measurable subsets of X, and

$$B := \left\{ x \in X \mid x \in B_n \text{ for infinitely many } n \right\}.$$

a. Argue that B is also a $\mathcal{B}$-measurable subset of X.

b. Prove that if $\displaystyle\sum_{n=1}^{\infty} \mu(B_n) < \infty$ then $\mu(B) = 0$.

c. Prove that if $\displaystyle\sum_{n=1}^{\infty} \mu(B_n) = \infty$ **and** the sequence of set complements $\{B_n^c\}_{n=1}^{\infty}$ satisfies

$$\mu\left(\bigcap_{n=k}^{K} B_n^c \right) = \prod_{n=k}^{K} \left(1 - \mu\left(B_n\right)\right)$$

for all positive integers k and K with $k < K$, then $\mu(B) = 1$.

> Hint: Use the fact that $1 - x \le e^{-x}$ for all x.

3.3 Spring 2019 # 2

Let $\mathcal{B}$ denote the set of all Borel subsets of $\mathbb{R}$ and $\mu : \mathcal{B} \longrightarrow [0, \infty)$ denote a finite Borel measure on $\mathbb{R}$.

a. Prove that if $\{F_k\}$ is a sequence of Borel sets for which $F_k \supseteq F_{k+1}$ for all k, then

$$\lim_{k \to \infty} \mu\left(F_k\right) = \mu\left(\bigcap_{k=1}^{\infty} F_k \right)$$

b. Suppose μ has the property that $\mu(E) = 0$ for every $E \in \mathcal{B}$ with Lebesgue measure $m(E) = 0$. Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $E \in \mathcal{B}$ with $m(E) < \delta$, then $\mu(E) < \varepsilon$.

3.4 Fall 2018 # 2

Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Show that there is a Borel set $B \subset E$ such that $m(E \setminus B) = 0$.

3.5 Spring 2018 # 1

Define

$$E := \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N} \right\}.$$

Prove that $m(E) = 0$.

3.6 Fall 2017 # 2

Let $f(x) = x^2$ and $E \subset [0, \infty) := \mathbb{R}^+$.

1. Show that

$$m^*(E) = 0 \iff m^*(f(E)) = 0.$$

2. Deduce that the map

$$\varphi : \mathcal{L}(\mathbb{R}^+) \longrightarrow \mathcal{L}(\mathbb{R}^+)$$
$$E \mapsto f(E)$$

is a bijection from the class of Lebesgue measurable sets of $[0, \infty)$ to itself.

3.7 Spring 2017 # 2

a. Let μ be a measure on a measurable space $(X, \mathcal{M})$ and f a positive measurable function.

Define a measure λ by

$$\lambda(E) := \int_E f \ d\mu, \quad E \in \mathcal{M}$$

Show that for g any positive measurable function,

$$\int_X g \ d\lambda = \int_X fg \ d\mu$$

b. Let $E \subset \mathbb{R}$ be a measurable set such that

$$\int_E x^2 \ dm = 0.$$

Show that $m(E) = 0$.

3.8 Fall 2016 # 4

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\{E_n\} \subset \mathcal{M}$ satisfies

$$\lim_{n \to \infty} \mu(X \backslash E_n) = 0.$$

Define

$$G := \left\{ x \in X \mid x \in E_n \text{ for only finitely many } n \right\}.$$

Show that $G \in \mathcal{M}$ and $\mu(G) = 0$.

3.9 Spring 2016 # 3

Let f be Lebesgue measurable on $\mathbb{R}$ and $E \subset \mathbb{R}$ be measurable such that

$$0 < A = \int_E f(x)dx < \infty.$$

Show that for every $0 < t < 1$, there exists a measurable set $E_t \subset E$ such that

$$\int_{E_t} f(x)dx = tA.$$

3.10 Spring 2016 # 5

Let $(X, \mathcal{M}, \mu)$ be a measure space. For $f \in L^1(\mu)$ and $\lambda > 0$, define

$$\varphi(\lambda) = \mu(\{x \in X | f(x) > \lambda\}) \quad \text{and} \quad \psi(\lambda) = \mu(\{x \in X | f(x) < -\lambda\})$$

Show that φ, ψ are Borel measurable and

$$\int_X |f| \, d\mu = \int_0^\infty [\varphi(\lambda) + \psi(\lambda)] \, d\lambda$$

3.11 Fall 2015 # 2

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be Lebesgue measurable.

1. Show that there is a sequence of simple functions $s_n(x)$ such that $s_n(x) \longrightarrow f(x)$ for all $x \in \mathbb{R}$.
2. Show that there is a Borel measurable function g such that $g = f$ almost everywhere.

3.12 Spring 2015 # 3

Let μ be a finite Borel measure on $\mathbb{R}$ and $E \subset \mathbb{R}$ Borel. Prove that the following statements are equivalent:

1. $\forall \varepsilon > 0$ there exists G open and F closed such that

$$F \subseteq E \subseteq G \quad \text{and} \quad \mu(G \setminus F) < \varepsilon.$$

2. There exists a $V \in G_\delta$ and $H \in F_\sigma$ such that

$$H \subseteq E \subseteq V \quad \text{and} \quad \mu(V \setminus H) = 0$$

3.13 Spring 2014 # 3

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and suppose

$$\forall x \in \mathbb{R}, \quad f(x) \geq \limsup_{y \to x} f(y)$$

Prove that f is Borel measurable.

3.14 Spring 2014 # 4

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose f is a measurable function on X. Show that

$$\lim_{n \to \infty} \int_X f^n \, d\mu = \begin{cases} \infty & \\ \mu(f^{-1}(1)), \end{cases} \quad \text{or}$$

and characterize the collection of functions of each type.

3.15 Spring 2017 # 1

Let K be the set of numbers in $[0, 1]$ whose decimal expansions do not use the digit 4.

> We use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 with $399\cdots$. For example, $0.8754 = 0.8753999\cdots$.

Show that K is a compact, nowhere dense set without isolated points, and find the Lebesgue measure $m(K)$.

3.16 Spring 2016 # 2

Let $0 < \lambda < 1$ and construct a Cantor set C_λ by successively removing middle intervals of length λ.

Prove that $m(C_\lambda) = 0$.

4 Measure Theory: Functions

4.1 Fall 2016 # 2

Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be measurable with

$$\int_a^b f(x) \, dx = \int_a^b g(x) \, dx.$$

Show that either

1. $f(x) = g(x)$ almost everywhere, or
2. There exists a measurable set $E \subset [a, b]$ such that

$$\int_E f(x) \, dx > \int_E g(x) \, dx$$

4.2 Spring 2016 # 4

Let $E \subset \mathbb{R}$ be measurable with $m(E) < \infty$. Define

$$f(x) = m(E \cap (E + x)).$$

Show that

1. $f \in L^1(\mathbb{R})$.

2. f is uniformly continuous.
3. $\lim\limits_{|x|\longrightarrow\infty} f(x) = 0$.

> Hint:
> $$\chi_{E\cap(E+x)}(y) = \chi_E(y)\chi_E(y-x)$$

5 Integrals: Convergence

5.1 Fall 2019 # 2.

Prove that

$$\left| \frac{d^n}{dx^n} \frac{\sin x}{x} \right| \leq \frac{1}{n}$$

for all $x \neq 0$ and positive integers n.

> Hint: Consider $\displaystyle\int_0^1 \cos(tx)\,dt$

5.2 Spring 2020 # 5

Compute the following limit and justify your calculations:

$$\lim_{n\longrightarrow\infty} \int_0^n \left(1 + \frac{x^2}{n}\right)^{-(n+1)}\,dx.$$

> Not finished, flesh out.

> *Solution.*
> Concepts used:
> - DCT
> - Passing limits through products and quotients
>
> Note that
> $$\lim_n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} = \frac{1}{\lim_n \left(1 + \frac{x^2}{n}\right)^1 \left(1 + \frac{x^2}{n}\right)^n}$$
> $$= \frac{1}{1 \cdot e^{x^2}}$$
> $$= e^{-x^2}.$$

If passing the limit through the integral is justified, we will have

$$\lim_{n \to \infty} \int_0^n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx = \lim_{n \to \infty} \int_{\mathbb{R}} \chi_{[0,n]} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx$$

$$= \int_{\mathbb{R}} \lim_{n \to \infty} \chi_{[0,n]} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \quad \text{by the DCT}$$

$$= \int_{\mathbb{R}} \lim_{n \to \infty} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx$$

$$= \int_0^\infty e^{-x^2}$$

$$= \frac{\sqrt{\pi}}{2}.$$

Computing the last integral:

$$\left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2 = \left(\int_{\mathbb{R}} e^{-x^2} dx\right)\left(\int_{\mathbb{R}} e^{-y^2} dx\right)$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(x+y)^2} dx$$

$$= \int_0^{2\pi} \int_0^\infty e^{-r^2} r \, dr \, d\theta \qquad u = r^2$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^\infty e^{-u} \, du \, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} 1$$

$$= \pi,$$

and now use the fact that the function is even so $\int_0^\infty f = \frac{1}{2} \int_{\mathbb{R}} f$.

Justifying the DCT:

- Apply Bernoulli's inequality:

$$1 + \frac{x^2}{n}^{n+1} \geq 1 + \frac{x^2}{n}\left(1 + x^2\right) \geq 1 + x^2,$$

where the last inequality follows from the fact that $1 + \frac{x^2}{n} \geq 1$

5.3 Spring 2019 # 3

Let $\{f_k\}$ be any sequence of functions in $L^2([0,1])$ satisfying $\|f_k\|_2 \leq M$ for all $k \in \mathbb{N}$.

Prove that if $f_k \longrightarrow f$ almost everywhere, then $f \in L^2([0,1])$ with $\|f\|_2 \leq M$ and

$$\lim_{k \to \infty} \int_0^1 f_k(x) dx = \int_0^1 f(x) dx$$

> Hint: Try using Fatou's Lemma to show that $\|f\|_2 \le M$ and then try applying Egorov's Theorem.

5.4 Fall 2018 # 6

Compute the following limit and justify your calculations:

$$\lim_{n \to \infty} \int_1^n \frac{dx}{\left(1 + \frac{x}{n}\right)^n \sqrt[n]{x}}$$

5.5 Fall 2018 # 3

Suppose $f(x)$ and $xf(x)$ are integrable on $\mathbb{R}$. Define F by

$$F(t) := \int_{-\infty}^{\infty} f(x)\cos(xt)dx$$

Show that

$$F'(t) = -\int_{-\infty}^{\infty} xf(x)\sin(xt)dx.$$

5.6 Spring 2018 # 5

Suppose that

- $f_n, f \in L^1$,
- $f_n \longrightarrow f$ almost everywhere, and
- $\int |f_n| \to \int |f|$.

Show that $\int f_n \to \int f$

5.7 Spring 2018 # 2

Let

$$f_n(x) := \frac{x}{1 + x^n}, \quad x \ge 0.$$

a. Show that this sequence converges pointwise and find its limit. Is the convergence uniform on $[0, \infty)$?

b. Compute

$$\lim_{n \to \infty} \int_0^{\infty} f_n(x)dx$$

5.8 Fall 2016 # 3

Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{x \longrightarrow 0} \int_{\mathbb{R}} |f(y - x) - f(y)|\, dy = 0$$

5.9 Fall 2015 # 3

Compute the following limit:

$$\lim_{n\to\infty} \int_1^n \frac{ne^{-x}}{1+nx^2} \sin\left(\frac{x}{n}\right) dx$$

5.10 Fall 2015 # 4

Let $f : [1, \infty) \longrightarrow \mathbb{R}$ such that $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Show that the following limit exists and satisfies the equality

$$\lim_{x\to\infty} f(x) \le 1 + \frac{\pi}{4}$$

6 Integrals: Approximation

6.1 Spring 2018 # 3

Let f be a non-negative measurable function on $[0, 1]$.

Show that

$$\lim_{p\to\infty} \left(\int_{[0,1]} f(x)^p dx \right)^{\frac{1}{p}} = \|f\|_\infty.$$

6.2 Spring 2018 # 4

Let $f \in L^2([0,1])$ and suppose

$$\int_{[0,1]} f(x)x^n dx = 0 \text{ for all integers } n \ge 0.$$

Show that $f = 0$ almost everywhere.

6.3 Spring 2015 # 2

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be continuous with period 1. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N f(n\alpha) = \int_0^1 f(t)dt \quad \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

Hint: show this first for the functions $f(t) = e^{2\pi ikt}$ for $k \in \mathbb{Z}$.

6.4 Fall 2014 # 4

Let $g \in L^\infty([0,1])$ Prove that

$$\int_{[0,1]} f(x)g(x)\,dx = 0 \quad \text{for all continuous } f : [0,1] \longrightarrow \mathbb{R} \implies g(x) = 0 \text{ almost everywhere.}$$

7 L^1

7.1 Spring 2020 # 3

a. Prove that if $g \in L^1(\mathbb{R})$ then

$$\lim_{N \longrightarrow \infty} \int_{|x| \geq N} |f(x)|\,dx = 0,$$

and demonstrate that it is not necessarily the case that $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.

b. Prove that if $f \in L^1([1,\infty])$ and is decreasing, then $\lim\limits_{x \longrightarrow \infty} f(x) = 0$ and in fact $\lim\limits_{x \longrightarrow \infty} xf(x) = 0$.

c. If $f : [1,\infty) \longrightarrow [0,\infty)$ is decreasing with $\lim\limits_{x \longrightarrow \infty} xf(x) = 0$, does this ensure that $f \in L^1([1,\infty))$?

Solution.

Concepts used:

- Limits
- Cauchy Criterion for Integrals: $\int_a^\infty f(x)\,dx$ converges iff for every $\varepsilon > 0$ there exists an M_0 such that $A, B \geq M_0$ implies $\left| \int_A^B f \right| < \varepsilon$, i.e. $\left| \int_A^B f \right| \overset{A \longrightarrow \infty}{\Longrightarrow} 0$.
- Integrals of L^1 functions have vanishing tails: $\int_N^\infty |f| \overset{N \longrightarrow \infty}{\Longrightarrow} 0$.
- Mean Value Theorem for Integrals: $\int_a^b f(t)\,dt = (b-a)f(c)$ for some $c \in [a,b]$.

7.1.1 a

Stated integral equality:

- Let $\varepsilon > 0$
- $C_c(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$ is dense so choose $\{f_n\} \longrightarrow f$ with $\|f_n - f\|_1 \longrightarrow 0$.
- Since $\{f_n\}$ are compactly supported, choose $N_0 \gg 1$ such that f_n is zero outside of $B_{N_0}(\mathbf{0})$.

- Then

$$N \geq N_0 \implies \int_{|x|>N} |f| = \int_{|x|>N} |f - f_n + f_n|$$

$$\leq \int_{|x|>N} |f - f_n| + \int_{|x|>N} |f_n|$$

$$= \int_{|x|>N} |f - f_n|$$

$$\leq \int_{|x|>N} \|f - f_n\|_1$$

$$= \|f_n - f\|_1 \left(\int_{|x|>N} 1 \right)$$

$$\xrightarrow{n \to \infty} 0 \left(\int_{|x|>N} 1 \right)$$

$$= 0$$

$$\xrightarrow{N \to \infty} 0.$$

To see that this doesn't force $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$:
- Take $f(x)$ to be a train of rectangles of height 1 and area $1/2^j$ centered on even integers.
- Then

$$\int_{|x|>N} |f| = \sum_{j=N}^{\infty} 1/2^j \xrightarrow{N \to \infty} 0$$

as the tail of a convergent sum.
- However $f(x) = 1$ for infinitely many even integers $x > N$, so $f(x) \not\longrightarrow 0$ as $|x| \longrightarrow \infty$.

7.1.2 b

Solution 1 ("Trick")
- Since f is decreasing on $[1, \infty)$, for any $t \in [x - n, x]$ we have

$$x - n \leq t \leq x \implies f(x) \leq f(t) \leq f(x - n).$$

- Integrate over $[x, 2x]$, using monotonicity of the integral:

$$\int_x^{2x} f(x) \, dt \leq \int_x^{2x} f(t) \, dt \leq \int_x^{2x} f(x - n) \, dt$$

$$\implies f(x) \int_x^{2x} dt \leq \int_x^{2x} f(t) \, dt \leq f(x - n) \int_x^{2x} dt$$

$$\implies x f(x) \leq \int_x^{2x} f(t) \, dt \leq x f(x - n).$$

- By the Cauchy Criterion for integrals, $\displaystyle \lim_{x \to \infty} \int_x^{2x} f(t) \, dt = 0.$
- So the LHS term $x f(x) \xrightarrow{x \to \infty} 0.$
- Since $x > 1$, $|f(x)| \leq |x f(x)|$
- Thus $f(x) \xrightarrow{x \to \infty} 0$ as well.

Solution 2 (Variation on the Trick)

- Use mean value theorem for integrals:

$$\int_x^{2x} f(t)\, dt = x f(c_x) \quad \text{for some } c_x \in [x, 2x] \text{ depending on } x.$$

- Since f is decreasing,

$$
\begin{aligned}
x \le c_x \le 2x \implies\ & f(2x) \le f(c_x) \le f(x) \\
\implies\ & 2x f(2x) \le 2x f(c_x) \le 2x f(x) \\
\implies\ & 2x f(2x) \le 2x \int_x^{2x} f(t)\, dt \le 2x f(x)
\end{aligned}
$$

.

- By Cauchy Criterion, $\displaystyle\int_x^{2x} f \longrightarrow 0$.
- So $2x f(2x) \longrightarrow 0$, which by a change of variables gives $u f(u) \longrightarrow 0$.
- Since $u \ge 1$, $f(u) \le u f(u)$ so $f(u) \longrightarrow 0$ as well.

Solution 3 (Contradiction)

Just showing $f(x) \overset{x \to \infty}{\longrightarrow} 0$:

- Toward a contradiction, suppose not.
- Since f is decreasing, it can not diverge to $+\infty$
- If $f(x) \longrightarrow -\infty$, then $f \notin L^1(\mathbb{R})$: choose $x_0 \gg 1$ so that $t \ge x_0 \implies f(t) < -1$, then
- Then $t \ge x_0 \implies |f(t)| \ge 1$, so

$$\int_1^\infty |f| \ge \int_{x_0}^\infty |f(t)|\, dt \ge \int_{x_0}^\infty 1 = \infty.$$

- Otherwise $f(x) \longrightarrow L \ne 0$, some finite limit.
- If $L > 0$:
  - Fix $\varepsilon > 0$, choose $x_0 \gg 1$ such that $t \ge x_0 \implies L - \varepsilon \le f(t) \le L$
  - Then

$$\int_1^\infty f \ge \int_{x_0}^\infty f \ge \int_{x_0}^\infty (L - \varepsilon)\, dt = \infty$$

- If $L < 0$:
  - Fix $\varepsilon > 0$, choose $x_0 \gg 1$ such that $t \ge x_0 \implies L \le f(t) \le L + \varepsilon$.
  - Then

$$\int_1^\infty f \ge \int_{x_0}^\infty f \ge \int_{x_0}^\infty (L)\, dt = \infty$$

Showing $x f(x) \overset{x \to \infty}{\longrightarrow} 0$.

- Toward a contradiction, suppose not.
- (How to show that $x f(x) \not\longrightarrow +\infty$?)
- If $x f(x) \longrightarrow -\infty$
  - Choose a sequence $\Gamma = \{\widehat{x}_i\}$ such that $x_i \longrightarrow \infty$ and $x_i f(x_i) \longrightarrow -\infty$.
  - Choose a subsequence $\Gamma' = \{x_i\}$ such that $x_i f(x_i) \le -1$ for all i and $x_i \le x_{i+1}$.

- – Choose a further subsequence $S = \left\{ x_i \in \Gamma' \mid 2x_i < x_{i+1} \right\}$.
- – Then since f is always decreasing, for $t \geq x_0$, $|f|$ is increasing, and $|f(x_i)| \leq |f(2x_i)|$, so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f| \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(t)|\, dt \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(x_i)| = \sum_{x_i \in S} x_i f(x_i) \longrightarrow \infty.$$

- If $xf(x) \longrightarrow L \neq 0$ for $0 < L < \infty$:
  - – Fix $\varepsilon > 0$, choose an infinite sequence $\{x_i\}$ such that $L - \varepsilon \leq x_i f(x_i) \leq L$ for all i.

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)|\, dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i)\, dt = \sum_S x_i f(x_i) \geq \sum_S (L - \varepsilon) \longrightarrow \infty.$$

- If $xf(x) \longrightarrow L \neq 0$ for $-\infty < L < 0$:
  - – Fix $\varepsilon > 0$, choose an infinite sequence $\{x_i\}$ such that $L \leq x_i f(x_i) \leq L + \varepsilon$ for all i.

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)|\, dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i)\, dt = \sum_S x_i f(x_i) \geq \sum_S (L) \longrightarrow \infty.$$

Solution 4 (Akos's Suggestion)   For $x \geq 1$,

$$|xf(x)| = \left| \int_x^{2x} f(x)\, dt \right| \leq \int_x^{2x} |f(x)|\, dt \leq \int_x^{2x} |f(t)|\, dt \leq \int_x^\infty |f(t)|\, dt \overset{x \to \infty}{\longrightarrow} 0$$

where we've used

- Since f is decreasing and $\lim_{x \to \infty} f(x) = 0$ from part (a), f is non-negative.
- Since f is positive and decreasing, for every $t \in [a, b]$ we have $|f(a)| \leq |f(t)|$.
- By part (a), the last integral goes to zero.

Solution 5 (Peter's)

- Toward a contradiction, produce a sequence $x_i \longrightarrow \infty$ with $x_i f(x_i) \longrightarrow \infty$ and $x_i f(x_i) > \varepsilon > 0$, then

$$\int f(x)\, dx \geq \sum_{i=1}^\infty \int_{x_i}^{x_{i+1}} f(x)\, dx$$

$$\geq \sum_{i=1}^\infty \int_{x_i}^{x_{i+1}} f(x_{i+1})\, dx$$

$$= \sum_{i=1}^\infty f(x_{i+1}) \int_{x_i}^{x_{i+1}} dx$$

$$\geq \sum_{i=1}^\infty (x_{i+1} - x_i) f(x_{i+1})$$

$$\geq \sum_{i=1}^\infty (x_{i+1} - x_i) \frac{\varepsilon}{x_{i+1}}$$

$$= \varepsilon \sum_{i=1}^\infty \left(1 - \frac{x_{i-1}}{x_i} \right) \longrightarrow \infty$$

which can be ensured by passing to a subsequence where $\sum \frac{x_{i-1}}{x_i} < \infty$.

7.1.3 c

- No: take $f(x) = \dfrac{1}{x \ln x}$
- Then by a u-substitution,

$$\int_0^x f = \ln(\ln(x)) \xrightarrow{x \to \infty} \infty$$

  is unbounded, so $f \notin L^1([1, \infty))$.
- But

$$xf(x) = \frac{1}{\ln(x)} \xrightarrow{x \to \infty} 0.$$

7.2 Fall 2019 # 5.

a. Show that if f is continuous with compact support on $\mathbb{R}$, then

$$\lim_{y \to 0} \int_{\mathbb{R}} |f(x - y) - f(x)| dx = 0$$

b. Let $f \in L^1(\mathbb{R})$ and for each $h > 0$ let

$$\mathcal{A}_h f(x) := \frac{1}{2h} \int_{|y| \leq h} f(x - y) dy$$

c. Prove that $\|\mathcal{A}_h f\|_1 \leq \|f\|_1$ for all $h > 0$.

ii. Prove that $\mathcal{A}_h f \longrightarrow f$ in $L^1(\mathbb{R})$ as $h \longrightarrow 0^+$.

7.3 Fall 2017 # 3

Let

$$S = \operatorname{span}_{\mathbb{C}} \left\{ \chi_{(a,b)} \ \middle| \ a, b \in \mathbb{R} \right\},$$

the complex linear span of characteristic functions of intervals of the form (a, b).

Show that for every $f \in L^1(\mathbb{R})$, there exists a sequence of functions $\{f_n\} \subset S$ such that

$$\lim_{n \to \infty} \|f_n - f\|_1 = 0$$

7.4 Spring 2015 # 4

Define

$$f(x, y) := \begin{cases} \dfrac{x^{1/3}}{(1 + xy)^{3/2}} & \text{if } 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Carefully show that $f \in L^1(\mathbb{R}^2)$.

7.5 Fall 2014 # 3

Let $f \in L^1(\mathbb{R})$. Show that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that} \qquad m(E) < \delta \implies \int_E |f(x)|\, dx < \varepsilon$$

7.6 Spring 2014 # 1

1. Give an example of a continuous $f \in L^1(\mathbb{R})$ such that $f(x) \not\to 0$ as $|x| \longrightarrow \infty$.

2. Show that if f is *uniformly* continuous, then

$$\lim_{|x| \longrightarrow \infty} f(x) = 0.$$

8 Fubini-Tonelli

8.1 Spring 2020 # 4

Let $f, g \in L^1(\mathbb{R})$. Argue that $H(x,y) := f(y)g(x-y)$ defines a function in $L^1(\mathbb{R}^2)$ and deduce from this fact that

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x-y)\, dy$$

defines a function in $L^1(\mathbb{R})$ that satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

> *Solution.*
> Relevant concepts:
> - Tonelli: non-negative and measurable yields measurability of slices and equality of iterated integrals
> - Fubini: $f(x,y) \in L^1$ yields *integrable* slices and equality of iterated integrals
> - F/T: apply Tonelli to $|f|$; if finite, $f \in L^1$ and apply Fubini to f

$$\|H(x)\|_1 = \int_{\mathbb{R}} |H(x,y)| \, dx$$

$$= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y)g(x-y) \, dy \right| dx$$

$$\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(x-y)| \, dy \right) dx$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(x-y)| \, dx \right) dy \quad \text{by Tonelli}$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(t)| \, dt \right) dy \quad \text{setting } t = x - y, \, dt = -dx$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)| \cdot |g(t)| \, dt \right) dy$$

$$= \int_{\mathbb{R}} |f(y)| \cdot \left(\int_{\mathbb{R}} |g(t)| \, dt \right) dy$$

$$:= \int_{\mathbb{R}} |f(y)| \cdot \|g\|_1 \, dy$$

$$= \|g\|_1 \int_{\mathbb{R}} |f(y)| \, dy$$

$$:= \|g\|_1 \|f\|_1$$

$$< \infty \quad \text{by assumption} \quad .$$

- H is measurable on $\mathbb{R}^2$:
  - If we can show $\tilde{f}(x,y) := f(y)$ and $\tilde{g}(x,y) := g(x-y)$ are both measurable on $\mathbb{R}^2$, then $H = \tilde{f} \cdot \tilde{g}$ is a product of measurable functions and thus measurable.
  - $f \in L^1$, and L^1 functions are measurable by definition.
  - The function $(x,y) \mapsto g(x-y)$ is measurable on $\mathbb{R}^2$:
    * Let g be measurable on $\mathbb{R}$, then the cylinder function $G(x,y) = g(x)$ on $\mathbb{R}^2$ is always measurable
    * Define a linear transformation $T := [1, -1; 0, 1]$ which sends $(x,y) \longrightarrow (x-y, y)$, then $T \in \mathrm{GL}(2, \mathbb{R})$ is linear and thus measurable.
    * Then $(G \circ T)(x,y) = G(x-y, y) = \tilde{g}(x-y)$, so $\tilde{g}$ is a composition of measurable functions and thus measurable.
- Apply **Tonelli** to $|H|$
  - H measurable implies $|H|$ is measurable
  - $|H|$ is non-negative
  - So the iterated integrals are equal in the extended sense
  - The calculation shows the iterated integral is finite, to $\int |H|$ is finite and H is thus integrable on $\mathbb{R}^2$.

Note: Fubini is not needed, since we're not calculating the actual integral, just showing H is integrable.

8.2 Spring 2019 # 4

Let f be a non-negative function on $\mathbb{R}^n$ and $\mathcal{A} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq f(x)\}$.

Prove the validity of the following two statements:

   a. f is a Lebesgue measurable function on $\mathbb{R}^n$ $\iff$ $\mathcal{A}$ is a Lebesgue measurable subset of $\mathbb{R}^{n+1}$

   b. If f is a Lebesgue measurable function on $\mathbb{R}^n$, then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)dx = \int_0^\infty m\left(\{x \in \mathbb{R}^n : f(x) \geq t\}\right) dt$$

8.3 Fall 2018 # 5

Let $f \geq 0$ be a measurable function on $\mathbb{R}$. Show that

$$\int_{\mathbb{R}} f = \int_0^\infty m(\{x : f(x) > t\})dt$$

8.4 Fall 2015 # 5

Let $f, g \in L^1(\mathbb{R})$ be Borel measurable.

   1. Show that

    • The function

$$F(x, y) := f(x - y)g(y)$$

    is Borel measurable on $\mathbb{R}^2$, and

    • For almost every $y \in \mathbb{R}$,

$$F_y(x) := f(x - y)g(y)$$

    is integrable with respect to y.

   2. Show that $f * g \in L^1(\mathbb{R})$ and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

8.5 Spring 2014 # 5

Let $f, g \in L^1([0, 1])$ and for all $x \in [0, 1]$ define

$$F(x) := \int_0^x f(y)\, dy \quad \text{and} \quad G(x) := \int_0^x g(y)\, dy.$$

Prove that

$$\int_0^1 F(x)g(x)\, dx = F(1)G(1) - \int_0^1 f(x)G(x)\, dx$$

9 L^2 and Fourier Analysis

9.1 Spring 2020 # 6

9.1.1 a

Show that

$$L^2([0, 1]) \subseteq L^1([0, 1]) \quad \text{and} \quad \ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z}).$$

9.1.2 b

For $f \in L^1([0,1])$ define

$$\widehat{f}(n) := \int_0^1 f(x)e^{-2\pi i n x}\, dx.$$

Prove that if $f \in L^1([0,1])$ and $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$ then

$$S_N f(x) := \sum_{|n| \leq N} \widehat{f}(n)e^{2\pi i n x}.$$

converges uniformly on $[0,1]$ to a continuous function g such that $g = f$ almost everywhere.

> Hint: One approach is to argue that if $f \in L^1([0,1])$ with $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$ then $f \in L^2([0,1])$.

Solution.
Concepts used:
- For $e_n(x) := e^{2\pi i n x}$, the set $\{e_n\}$ is an orthonormal basis for $L^2([0,1])$.
- For any orthonormal sequence in a Hilbert space, we have Bessel's inequality:

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

- When $\{e_n\}$ is a basis, the above is an *equality* (Parseval)
- Arguing uniform convergence: since $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$, we should be able to apply the M test.

9.1.3 a

Claim: $\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$.
- Set $\mathbf{c} = \left\{ c_k \mid k \in \mathbb{Z} \right\} \in \ell^1(\mathbb{Z})$.
- It suffices to show that if $\sum_{k \in \mathbb{Z}} |c_k| < \infty$ then $\sum_{k \in \mathbb{Z}} |c_k|^2 < \infty$.
- Let $S = \left\{ c_k \mid |c_k| \leq 1 \right\}$, then $c_k \in S \implies |c_k|^2 \leq |c_k|$
- Claim: S^c can only contain finitely many elements, all of which are finite.
  - If not, either $S^c := \{c_j\}_{j=1}^{\infty}$ is infinite with every $|c_j| > 1$, which forces

$$\sum_{c_k \in S^c} |c_k| = \sum_{j=1}^{\infty} |c_j| > \sum_{j=1}^{\infty} 1 = \infty.$$

  - If any $c_j = \infty$, then $\sum_{k \in \mathbb{Z}} |c_k| \geq c_j = \infty$.
- So S^c is a finite set of finite integers, let $N = \max \left\{ |c_j|^2 \mid c_j \in S^c \right\} < \infty$.

- Rewrite the sum

$$
\begin{aligned}
\sum_{k \in \mathbb{Z}} |c_k|^2 &= \sum_{c_k \in S} |c_k|^2 + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \sum_{c_k \in S} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \sum_{k \in \mathbb{Z}} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \quad \text{since the } |c_k| \text{ are all positive} \\
&= \|\mathbf{c}\|_{\ell^1} + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \|\mathbf{c}\|_{\ell^1} + |S^c| \cdot N \\
&< \infty.
\end{aligned}
$$

Claim: $L^2([0,1]) \subseteq L^1([0,1])$.

- It suffices to show that $\displaystyle\int |f|^2 < \infty \implies \int |f| < \infty$.
- Define $S = \left\{ x \in [0,1] \,\middle|\, |f(x)| \leq 1 \right\}$, then $x \in S^c \implies |f(x)|^2 \geq |f(x)|$.
- Break up the integral:

$$
\begin{aligned}
\int_{\mathbb{R}} |f| &= \int_S |f| + \int_{S^c} |f| \\
&\leq \int_S |f| + \int_{S^c} |f|^2 \\
&\leq \int_S |f| + \|f\|_2 \\
&\leq \sup_{x \in S} \{|f(x)|\} \cdot \mu(S) + \|f\|_2 \\
&= 1 \cdot \mu(S) + \|f\|_2 \quad \text{by definition of } S \\
&\leq 1 \cdot \mu([0,1]) + \|f\|_2 \quad \text{since } S \subseteq [0,1] \\
&= 1 + \|f\|_2 \\
&< \infty.
\end{aligned}
$$

> Note: this proof shows $L^2(X) \subseteq L^1(X)$ whenever $\mu(X) < \infty$.

9.2 Fall 2017 # 5

Let φ be a compactly supported smooth function that vanishes outside of an interval $[-N, N]$ such that $\displaystyle\int_{\mathbb{R}} \varphi(x)\, dx = 1$.

For $f \in L^1(\mathbb{R})$, define

$$
K_j(x) := j\varphi(jx), \qquad f * K_j(x) := \int_{\mathbb{R}} f(x-y)K_j(y)\, dy
$$

and prove the following:

1. Each $f * K_j$ is smooth and compactly supported.

2.

$$\lim_{j \longrightarrow \infty} \|f * K_j - f\|_1 = 0$$

Hint:

$$\lim_{y \longrightarrow 0} \int_{\mathbb{R}} |f(x - y) - f(x)| dy = 0$$

9.3 Spring 2017 # 5

Let $f, g \in L^2(\mathbb{R})$. Prove that the formula

$$h(x) := \int_{-\infty}^{\infty} f(t)g(x - t)\, dt$$

defines a uniformly continuous function h on $\mathbb{R}$.

9.4 Spring 2015 # 6

Let $f \in L^1(\mathbb{R})$ and g be a bounded measurable function on $\mathbb{R}$.

1. Show that the convolution $f * g$ is well-defined, bounded, and uniformly continuous on $\mathbb{R}$.
2. Prove that one further assumes that $g \in C^1(\mathbb{R})$ with bounded derivative, then $f * g \in C^1(\mathbb{R})$ and

$$\frac{d}{dx}(f * g) = f * \left(\frac{d}{dx} g \right)$$

9.5 Fall 2014 # 5

1. Let $f \in C_c^0(\mathbb{R}^n)$, and show

$$\lim_{t \longrightarrow 0} \int_{\mathbb{R}^n} |f(x + t) - f(x)|\, dx = 0.$$

2. Extend the above result to $f \in L^1(\mathbb{R}^n)$ and show that

$$f \in L^1(\mathbb{R}^n), \quad g \in L^\infty(\mathbb{R}^n) \quad \implies \quad f * g \text{ is bounded and uniformly continuous.}$$

10 Functional Analysis: General

10.1 Fall 2019 # 4.

Let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space $\mathcal{H}$.

a. Prove that for every $x \in \mathcal{H}$ one has

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

b. Prove that for any sequence $\{a_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$ there exists an element $x \in \mathcal{H}$ such that

$$a_n = \langle x,\ u_n \rangle \text{ for all } n \in \mathbb{N}$$

and

$$\|x\|^2 = \sum_{n=1}^\infty |\langle x, u_n \rangle|^2$$

10.2 Spring 2019 # 5

10.2.1 a

Show that $L^2([0,1]) \subseteq L^1([0,1])$ and argue that $L^2([0,1])$ in fact forms a dense subset of $L^1([0,1])$.

10.2.2 b

Let Λ be a continuous linear functional on $L^1([0,1])$.

Prove the Riesz Representation Theorem for $L^1([0,1])$ by following the steps below:

i. Establish the existence of a function $g \in L^2([0,1])$ which represents Λ in the sense that

$$\Lambda(f) = f(x)g(x)dx \text{ for all } f \in L^2([0,1]).$$

Hint: You may use, without proof, the Riesz Representation Theorem for $L^2([0,1])$.

ii. Argue that the g obtained above must in fact belong to $L^\infty([0,1])$ and represent Λ in the sense that

$$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}dx \quad \text{for all } f \in L^1([0,1])$$

with

$$\|g\|_{L^\infty([0,1])} = \|\Lambda\|_{L^1([0,1])^\vee}$$

10.3 Spring 2016 # 6

Without using the Riesz Representation Theorem, compute

$$\sup\left\{ \left| \int_0^1 f(x)e^x dx \right| \ \Big| \ f \in L^2([0,1], m),\ \|f\|_2 \le 1 \right\}$$

10.4 Spring 2015 # 5

Let $\mathcal{H}$ be a Hilbert space.

1. Let $x \in \mathcal{H}$ and $\{u_n\}_{n=1}^N$ be an orthonormal set. Prove that the best approximation to x in $\mathcal{H}$ by an element in $\mathrm{span}_\mathbb{C}\{u_n\}$ is given by

$$\widehat{x} := \sum_{n=1}^N \langle x,\ u_n \rangle u_n.$$

2. Conclude that finite dimensional subspaces of $\mathcal{H}$ are always closed.

10.5 Fall 2015 # 6

Let $f : [0, 1] \longrightarrow \mathbb{R}$ be continuous. Show that

$$\sup\left\{ \|fg\|_1 \ \middle|\ g \in L^1[0, 1], \ \|g\|_1 \leq 1 \right\} = \|f\|_\infty$$

10.6 Fall 2014 # 6

Let $1 \leq p, q \leq \infty$ be conjugate exponents, and show that

$$f \in L^p(\mathbb{R}^n) \implies \|f\|_p = \sup_{\|g\|_q = 1} \left| \int f(x)g(x)dx \right|$$

11 Functional Analysis: Banach Spaces

11.1 Spring 2019 # 1

Let $C([0, 1])$ denote the space of all continuous real-valued functions on $[0, 1]$.

  a. Prove that $C([0, 1])$ is complete under the uniform norm $\|f\|_u := \sup_{x \in [0,1]} |f(x)|$.

  b. Prove that $C([0, 1])$ is not complete under the L^1-norm $\|f\|_1 = \int_0^1 |f(x)| \ dx$.

11.2 Spring 2017 # 5

Show that the space $C^1([a, b])$ is a Banach space when equipped with the norm

$$\|f\| := \sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |f'(x)| \,.$$

11.3 Fall 2017 # 6

Let X be a complete metric space and define a norm

$$\|f\| := \max\{|f(x)| : x \in X\}.$$

Show that $(C^0(\mathbb{R}), \|\cdot\|)$ (the space of continuous functions $f : X \longrightarrow \mathbb{R}$) is complete.