

Topology Qualifying Exam Notes

D. Zack Garza

Friday 29th May, 2020

Contents

0.1 Conventions	1
1 Table of Homotopy and Homology Structures	3
2 Useful Facts and Techniques	4
3 Other Interesting Things To Consider	6
4 Spheres	7
5 Table of Homotopy and Homology Structures	8
6 Euler Characteristics	11
7 Useful Facts and Techniques	12
8 Other Interesting Things To Consider	13
9 Spheres	14
10 Definitions	14
11 Examples	15
11.1 Common Spaces and Operations	15
11.2 Alternative Topologies	18
12 Theorems	18
12.1 Point-Set	18
12.2 Algebraic	20

0.1 Conventions

- $\pi_0(X)$ is the set of path components of X , and I write $\pi_0(X) = \mathbb{Z}$ if X is path-connected (although it is not a group). Similarly, $H_0(X)$ is a free abelian group on the set of path components of X .
- Lists start at entry 1, since all spaces are connected here and thus $\pi_0 = H_0 = \mathbb{Z}$. That is,

- $\pi_*(X) = [\pi_1(X), \pi_2(X), \pi_3(X), \dots]$
- $H_*(X) = [H_1(X), H_2(X), H_3(X), \dots]$

- For a finite index set I , it is the case that $\prod_I G = \bigoplus_I G$ in **Grp**, i.e. the finite direct product and finite direct sum coincide. Otherwise, if I is infinite, the direct sum requires cofinitely many zero entries (i.e. finitely many nonzero entries), so here we always use \prod .

In other words, there is an injective map

$$\bigoplus_I G \hookrightarrow \prod_I G$$

which is an isomorphism when $|I| < \infty$

- $\mathbb{Z}^n := \prod_{i=1}^n \mathbb{Z} = \mathbb{Z} \times \mathbb{Z} \times \dots \mathbb{Z}$ is the free abelian group of rank n .
 - $x \in \mathbb{Z}^n = \langle a_1, \dots, a_n \rangle \implies x = \sum_{i=1}^n c_i a_i$ for some $c_i \in \mathbb{Z}$, i.e. a_i form a basis.
 - Example: $x = 2a_1 + 4a_2 + a_1 - a_2 = 3a_1 + 3a_2$.
- $\mathbb{Z}^{*n} := \mathbb{Z} * \mathbb{Z} * \dots \mathbb{Z}$ is the free product of n free abelian groups, i.e. a free (nonabelian) group on n generators.
 - $x \in \mathbb{Z}^{*n} = \langle a_1, \dots, a_n \rangle$ implies that x is a finite word in the noncommuting symbols a_i^k for $k \in \mathbb{Z}$.
 - Example: $x = a_1^2 a_2^4 a_1 a_2^{-2}$
- $K(G, n)$ is an Eilenberg-MacLane space, the homotopy-unique space satisfying

$$\pi_k(K(G, n)) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

- $K(\mathbb{Z}, 1) = S^1$
- $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$
- $K(\mathbb{Z}_2, 1) = \mathbb{RP}^\infty$

- $M(G, n)$ is a Moore space, the homotopy-unique space satisfying

$$H_k(M(G, n); G) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

- $M(\mathbb{Z}, n) = S^n$
- $M(\mathbb{Z}_2, 1) = \mathbb{RP}^2$
- $M(\mathbb{Z}_p, n)$ is made by attaching e^{n+1} to S^n via a degree p map.

- $T^n = \prod_n S^1$ is the n -torus
- $D(k, X)$ is the space X with $k \in \mathbb{N}$ distinct points deleted, i.e. the punctured space $X - \{x_1, x_2, \dots, x_k\}$ where each $x_i \in X$.
- $\mathbb{RP}^n = S^n / S^0 = S^n / \mathbb{Z}_2$
- $\mathbb{CP}^n = S^{2n+1} / S^1$

- $B^n = \{\tilde{\lambda} \in \mathbb{R}^n \mid \|\tilde{\lambda}\| \leq 1\} \subset \mathbb{R}^n$
- $S^{n-1} = \partial B^n = \{\tilde{\lambda} \in \mathbb{R}^n \mid \|\tilde{\lambda}\| = 1\} \subset \mathbb{R}^n$

sphere ball correct

1 Table of Homotopy and Homology Structures

X	$\pi_*(X)$	$H_*(X)$	CW Structure	$H^*(X)$
\mathbb{R}^1	0	0	$\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot x$	0
\mathbb{R}^n	0	0	$(\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot x)^n$	0
$D(k, \mathbb{R}^n)$	$\pi_* \bigvee^k S^1$	$\bigoplus^k H_*M(\mathbb{Z}, 1)$	$1 + kx$	
B^n	$\pi_*(\mathbb{R}^n)$	$H_*(\mathbb{R}^n)$	$1 + x^n + x^{n+1}$	0
S^n	$[0 \dots, \mathbb{Z}, ? \dots]$	$H_*M(\mathbb{Z}, n)$	$1 + x^n$	$\mathbb{Z}[x]/(x^2)$
$D(k, S^n)$	$\pi_* \bigvee^{k-1} S^1$	$\bigoplus^{k-1} H_*M(\mathbb{Z}, 1)$	$1 + (k-1)x^1$	
T^2	$\pi_* S^1 \times \pi_* S^1$	$\bigoplus^{k-1} (H_*M(\mathbb{Z}, 1))^2 \times H_*M(\mathbb{Z}, 2)$	$1 + 2x + x^2$	$\Lambda(1x_1, 1x_2)$
T^n	$\prod^n \pi_* S^1$	$\prod_{i=1}^n (H_*M(\mathbb{Z}, i))^{\binom{n}{i}}$	$(1+x)^n$	$\Lambda(1x_1, 1x_2, \dots, 1x_n)$
$D(k, T^n)$	$[0, 0, 0, 0, \dots]?$	$[0, 0, 0, 0, \dots]?$	$1 + x$	
$S^1 \vee S^1$	$\pi_* S^1 * \pi_* S^1$	$(H_*M(\mathbb{Z}, 1))^2$	$1 + 2x$	
$\bigvee_n S^1$	$*^n \pi_* S^1$	$\prod H_*M(\mathbb{Z}, 1)$	$1 + x$	
\mathbb{RP}^1	$\pi_* S^1$	$H_*M(\mathbb{Z}, 1)$	$1 + x$	${}_0\mathbb{Z} \times {}_1\mathbb{Z}$
\mathbb{RP}^2	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^2$	$H_*M(\mathbb{Z}_2, 1)$	$1 + x + x^2$	${}_0\mathbb{Z} \times {}_2\mathbb{Z}_2$
\mathbb{RP}^3	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^3$	$H_*M(\mathbb{Z}_2, 1) \times H_*M(\mathbb{Z}, 3)$	$1 + x + x^2 + x^3$	${}_0\mathbb{Z} \times {}_2\mathbb{Z}_2 \times {}_3\mathbb{Z}$
\mathbb{RP}^4	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^4$	$H_*M(\mathbb{Z}_2, 1) \times H_*M(\mathbb{Z}_2, 3)$	$1 + x + x^2 + x^3 + x^4$	${}_0\mathbb{Z} \times ({}_2\mathbb{Z}_2)^2$
$\mathbb{RP}^n, n \geq 4$ even	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^n$	$\prod_{\text{odd } i < n} H_*M(\mathbb{Z}_2, i)$	$\sum_{i=1}^n x^i$	${}_0\mathbb{Z} \times \prod_{i=1}^{n/2} {}_2\mathbb{Z}_2$
$\mathbb{RP}^n, n \geq 4$ odd	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^n$	$\prod_{\text{odd } i \leq n-2} H_*M(\mathbb{Z}_2, i) \times H_*S^n$	$\sum_{i=1}^n x^i$	$H^*(\mathbb{RP}^{n-1}) \times {}_n\mathbb{Z}$

X	$\pi_*(X)$	$H_*(X)$	CW Structure	$H^*(X)$
\mathbb{CP}^{1^a}	$\pi_*K(\mathbb{Z}, 2) \times \pi_*S^3$	H_*S^2	$x^0 + x^2$	$\mathbb{Z}[2x]/(2x^2)$
$\overline{\mathbb{CP}^1} \simeq S^2$.				
\mathbb{CP}^2	$\pi_*K(\mathbb{Z}, 2) \times \pi_*S^5$	$H_*S^2 \times H_*S^4$	$x^0 + x^2 + x^4$	$\mathbb{Z}[2x]/(2x^3)$
$\mathbb{CP}^n, n \geq 2$	$\pi_*K(\mathbb{Z}, 2) \times \pi_*S^{2n+1}$	$\prod_{i=1}^n H_*S^{2i}$	$\sum_{i=1}^n x^{2i}$	$\mathbb{Z}[2x]/(2x^{n+1})$
Mobius Band ^a	π_*S^1	H_*S^1	$1 + x$	
^a Uses the fact that $M \simeq S^1$ by deformation-retracting onto the center circle.				
Klein Bottle	$K(\mathbb{Z} \rtimes_{x \mapsto -x} \mathbb{Z}, 1)$	$H_*S^1 \times H_*\mathbb{RP}^\infty$	$1 + 2x + x^2$	

- \mathbb{R}^n is a contractible space, and so $[S^m, \mathbb{R}^n] = 0$ for all n, m which makes its homotopy groups all zero.
- All calculations follow from the fact that $D(k, \mathbb{R}^n) = \mathbb{R}^n - \{x_1 \dots x_k\} \simeq \bigvee_{i=1}^k S^1$ by a deformation retract.
- This uses the fact that $S^n \cong B^n / \partial B^n$ and employs an attaching map

$$\begin{aligned} \phi : (D^n, \partial D^n) &\longrightarrow S^n \\ (D^n, \partial D^n) &\mapsto (e^n, e^0) \end{aligned}$$

- $B^n \simeq \mathbb{R}^n$ by normalizing vectors.
- Use the inclusion $S^n = B^{n+1}$ as the attaching map.
- $\mathbb{CP}^1 \simeq S^2$.
- $\mathbb{RP}^1 \simeq S^1$.
- Use $\pi_1 \prod = \prod \pi_1$ and the universal cover $\mathbb{R}^1 \rightarrow S^1$ to yield the cover $\mathbb{R}^n \rightarrow T^n$.

2 Useful Facts and Techniques

- Fundamental group:
 - Van Kampen
- Homotopy Groups

-
- Hurewicz map
 - Homology
 - Mayer-Vietoris
 - * $(X = A \bigcup B) \mapsto (\bigcap, \oplus, \bigcup)$ in homology
 - LES of a pair
 - * $(A \hookrightarrow X) \mapsto (A, X, X/A)$
 - Excision
 - $\pi_{i \geq 2}(X)$ is always abelian.
 - The ranks of π_0 and H_0 are the number of path components, and $\pi_0(X) = \mathbb{Z}$ iff X is simply connected.
 - X simply connected implies $\pi_k(X) \cong H_k(X)$ up to and including the first nonvanishing H_k
 - $H_1(X) = \pi_1 X / [\pi_1 X, \pi_1 X]$, the abelianization.
 - General mantra: homotopy plays nicely with products, homology with wedge products.¹

In general, homotopy groups behave nicely under homotopy pull-backs (e.g., fibrations and products), but not homotopy push-outs (e.g., cofibrations and wedges). Homology is the opposite.

- $\pi_k \prod X = \prod \pi_k X$ by LES.²
- $H_k \prod X \neq \prod H_k X$ due to torsion.
 - Nice case: $H_k(A \times B) = \prod_{i+j=k} H_i A \otimes H_j B$ by Kunneth when all groups are torsion-free.³
- $H_k \bigvee X = \prod H_k X$ by Mayer-Vietoris.⁴
- $\pi_k \bigvee X \neq \prod \pi_k X$ (counterexample: $S^1 \vee S^2$)
 - Nice case: $\pi_1 \bigvee X = * \pi_1 X$ by Van Kampen.
- $\pi_i(\widehat{X}) \cong \pi_i(X)$ for $i \geq 2$ whenever $\widehat{X} \rightarrow X$ is a universal cover.
- Groups and Group Actions

¹More generally, in **Top**, we can look at $A \leftarrow \{\text{pt}\} \rightarrow B$ – then $A \times B$ is the pullback and $A \vee B$ is the pushout. In this case, homology $h : \mathbf{Top} \rightarrow \mathbf{Grp}$ takes pushouts to pullbacks but doesn't behave well with pullbacks. Similarly, while π takes pullbacks to pullbacks, it doesn't behave nicely with pushouts.

²This follows because $X \times Y \rightarrow X$ is a fiber bundle, so use LES in homotopy and the fact that $\pi_{i \geq 2} \in \mathbf{Ab}$.

³The generalization of Kunneth is as follows: write $\mathcal{P}(n, k)$ be the set of partitions of n into k parts, i.e. $\curvearrowright \in \mathcal{P}(n, k) \implies \curvearrowright = (x_1, x_2, \dots, x_k)$ where $\sum x_i = n$. Then

$$H_n\left(\prod_{j=1}^k X_j\right) = \bigoplus_{\curvearrowright \in \mathcal{P}(n, k)} \bigotimes_{i=1}^k H_{x_i}(X_i).$$

⁴ \bigvee is the coproduct in the category **Top**₀ of pointed topological spaces, and alternatively, $X \vee Y$ is the pushout in **Top** of $X \leftarrow \{\text{pt}\} \rightarrow Y$

-
- $\pi_0(G) = G$ for G a discrete topological group.
 - $\pi_k(G/H) = \pi_k(G)$ if $\pi_k(H) = \pi_{k-1}(H) = 0$.
 - $\pi_1(X/G) = \pi_0(G)$ when G acts freely/transitively on X .

- Manifolds

- $H^n(M^n) = \mathbb{Z}$ if M^n is orientable and zero if M^n is nonorientable.
- Poincare Duality: $H_i M^n \cong H^{n-i} M^n$ iff M^n is closed and orientable.

3 Other Interesting Things To Consider

- The “generalized uniform bouquet”? $B^n(m) = \bigvee_{i=1}^n S^m$
- Lie Groups
 - The real general linear group, $GL_n(\mathbb{R})$
 - * The real special linear group $SL_n(\mathbb{R})$
 - * The real orthogonal group, $O_n(\mathbb{R})$
 - The real special orthogonal group, $SO_n(\mathbb{R})$
 - * The real unitary group, $U_n(\mathbb{R})$
 - The real special unitary group, $SU_n(\mathbb{R})$
 - * The real symplectic group $Sp(n)$
- “Geometric” Stuff
 - Affine n -space over a field $\mathbb{A}^n(k) = k^n \rtimes GL_n(k)$
 - The projective space $\mathbb{P}^n(k)$
 - * The projective linear group over a ring R , $PGL_n(R)$
 - * The projective special linear group over a ring R , $PSL_n(R)$
 - * The modular groups $PSL_n(\mathbb{Z})$
 - Specifically $PSL_2(\mathbb{Z})$
- The real Grassmannian, $Gr(n, k, \mathbb{R})$, i.e. the set of k dimensional subspaces of \mathbb{R}^n
- The Stiefel manifold $V_n(k)$
- Possible modifications to a space X :
 - Remove k points by taking $D(k, X)$
 - Remove a line segment
 - Remove an entire line/axis
 - Remove a hole
 - Quotient by a group action (e.g. antipodal map, or rotation)
 - Remove a knot
 - Take complement in ambient space
- Assorted info about other Lie Groups:
- $O_n, U_n, SO_n, SU_n, Sp_n$
- $\pi_k(U_n) = \mathbb{Z} \cdot 1$ [k odd]
 - $\pi_1(U_n) = 1$

-
- $\pi_k(SU_n) = \mathbb{Z} \cdot \mathbb{1} [k \text{ odd}]$
 - $\pi_1(SU_n) = 0$
 - $\pi_k(U_n) = \mathbb{Z}_2 \cdot \mathbb{1} [k = 0, 1 \pmod{8}] + \mathbb{Z} \cdot \mathbb{1} [k = 3, 7 \pmod{8}]$
 - $\pi_k(SP_n) = \mathbb{Z}_2 \cdot \mathbb{1} [k = 4, 5 \pmod{8}] + \mathbb{Z} \cdot \mathbb{1} [k = 3, 7 \pmod{8}]$

4 Spheres

- $\pi_i(S^n) = 0$ for $i < n$, $\pi_n(S^n) = \mathbb{Z}$
 - Not necessarily true that $\pi_i(S^n) = 0$ when $i > n$!!!
 - * E.g. $\pi_3(S^2) = \mathbb{Z}$ by Hopf fibration
- $H_i(S^n) = \mathbb{1} [i \in \{0, n\}]$
- $H_n(\bigvee_i X_i) \cong \prod_i H_n(X_i)$ for “good pairs”
 - Corollary: $H_n(\bigvee_k S^n) = \mathbb{Z}^k$
- $S^n/S^k \simeq S^n \vee \Sigma S^k$
 - $\Sigma S^n = S^{n+1}$
- S^n has the CW complex structure of 2 k -cells for each $0 \leq k \leq n$.
- $\pi_0(X)$ is the set of path components of X , and I write $\pi_0(X) = \mathbb{Z}$ if X is path-connected (although it is not a group). Similarly, $H_0(X)$ is a free abelian group on the set of path components of X .
- Lists start at entry 1, since all spaces are connected here and thus $\pi_0 = H_0 = \mathbb{Z}$. That is,
 - $\pi_*(X) = [\pi_1(X), \pi_2(X), \pi_3(X), \dots]$
 - $H_*(X) = [H_1(X), H_2(X), H_3(X), \dots]$
- For a finite index set I , it is the case that $\prod_I G = \bigoplus_I G$ in **Grp**, i.e. the finite direct product and finite direct sum coincide. Otherwise, if I is infinite, the direct sum requires cofinitely many zero entries (i.e. finitely many nonzero entries), so here we always use \prod .

In other words, there is an injective map

$$\bigoplus_I G \hookrightarrow \prod_I G$$

which is an isomorphism when $|I| < \infty$

- $\mathbb{Z}^n := \prod_{i=1}^n \mathbb{Z} = \mathbb{Z} \times \mathbb{Z} \times \dots \mathbb{Z}$ is the free abelian group of rank n .
 - $x \in \mathbb{Z}^n = \langle a_1, \dots, a_n \rangle \implies x = \sum_{i=1}^n c_i a_i$ for some $c_i \in \mathbb{Z}$, i.e. a_i form a basis.
 - Example: $x = 2a_1 + 4a_2 + a_1 - a_2 = 3a_1 + 3a_2$.

-
- $\mathbb{Z}^{*n} := \mathbb{Z} * \mathbb{Z} * \dots * \mathbb{Z}$ is the free product of n free abelian groups, i.e. a free (nonabelian) group on n generators.
 - $x \in \mathbb{Z}^{*n} = \langle a_1, \dots, a_n \rangle$ implies that x is a finite word in the noncommuting symbols a_i^k for $k \in \mathbb{Z}$.
 - Example: $x = a_1^2 a_2^4 a_1 a_2^{-2}$
 - $K(G, n)$ is an Eilenberg-MacLane space, the homotopy-unique space satisfying

$$\pi_k(K(G, n)) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

- $K(\mathbb{Z}, 1) = S^1$
- $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$
- $K(\mathbb{Z}_2, 1) = \mathbb{RP}^\infty$

- $M(G, n)$ is a Moore space, the homotopy-unique space satisfying

$$H_k(M(G, n); G) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

- $M(\mathbb{Z}, n) = S^n$
- $M(\mathbb{Z}_2, 1) = \mathbb{RP}^2$
- $M(\mathbb{Z}_p, n)$ is made by attaching e^{n+1} to S^n via a degree p map.

- $T^n = \prod_n S^1$ is the n -torus
- $D(k, X)$ is the space X with $k \in \mathbb{N}$ distinct points deleted, i.e. the punctured space $X - \{x_1, x_2, \dots, x_k\}$ where each $x_i \in X$.
- $\mathbb{RP}^n = S^n / S^0 = S^n / \mathbb{Z}_2$
- $\mathbb{CP}^n = S^{2n+1} / S^1$
- $B^n = \{ \tilde{\gamma} \in \mathbb{R}^n \mid \|\tilde{\gamma}\| \leq 1 \} \subset \mathbb{R}^n$
- $S^{n-1} = \partial B^n = \{ \tilde{\gamma} \in \mathbb{R}^n \mid \|\tilde{\gamma}\| = 1 \} \subset \mathbb{R}^n$

sphere ball correct

5 Table of Homotopy and Homology Structures

X	$\pi_*(X)$	$H_*(X)$	CW	$H^*(X)$
\mathbb{R}^n ^a	0	0	$(\bigcup_{\mathbb{Z}} e^0 + \bigcup_{\mathbb{Z}} e^1)^n$	0
^a \mathbb{R}^n is a contractible space, and so $[S^m, \mathbb{R}^n] = 0$ for all n, m which makes its homotopy groups all zero.				
$D(k, \mathbb{R}^n)$ ^a	$\pi_* \bigvee_k S^1$	$\bigoplus_k H_*M(\mathbb{Z}, 1)$	$e^0 + ke^1$	
^a All calculations follow from the fact that $D(k, \mathbb{R}^n) = \mathbb{R}^n - \{x_1 \dots x_k\} \simeq \bigvee_{i=1}^k S^1$ by a deformation retract.				
B^n ^a	$\pi_*(\mathbb{R}^n)$	$H_*(\mathbb{R}^n)$	$e^0 + e^n + e^{n+1}$ ^a	0
^a $B^n \simeq \mathbb{R}^n$ by normalizing vectors.			^a Use the inclusion $S^n = B^{n+1}$ as the attaching map.	
S^n	$[0 \dots, \mathbb{Z}, ? \dots]$	$H_*M(\mathbb{Z}, n)$	$e^0 + e^n$ ^a	$\mathbb{Z}[x]/(x^2)$
			^a This uses the fact that $S^n \cong B^n/\partial B^n$ and employs an attaching map	
			$\phi : (D^n, \partial D^n) \longrightarrow S^n$ $(D^n, \partial D^n) \mapsto (e^n, e^0)$	

X	$\pi_*(X)$	$H_*(X)$	CW	$H^*(X)$
$D(k, S^n)^a$	$\pi_* \bigvee_{k-1}^{k-1} S^1$	$\bigoplus_{k-1} H_*M(\mathbb{Z}, 1)$	$e^0 + (k-1)e^1$	
$\overline{\text{Use the fact that } D(1, S^n) \cong \mathbb{R}^n \text{ and thus } D(k, S^n) \cong D(k-1, \mathbb{R}^n) \cong \bigvee_{k-1}^{k-1} S^1}$ Conventions				
T^2	$\pi_* S^1 \times \pi_* S^1$	$(H_*M(\mathbb{Z}, 1))^2 \times H_*M(\mathbb{Z}, 2)$	$e^0 + 2e^1 + e^2$	$\Lambda(1x_1, 1x_2)$
T^n	$\prod_{i=1}^n \pi_* S^{1a}$			$\Lambda(1x_1, 1x_2, \dots, 1x_n)$
$\overline{\text{Use } \pi_1 \prod_{i=1}^n \pi_* S^{1a} = \prod_{i=1}^n \pi_1 \text{ and the universal cover } \mathbb{R}^1 \twoheadrightarrow S^1 \text{ to yield the cover } \mathbb{R}^n \twoheadrightarrow T^n.}$				
$D(k, T^n)$	$[0, 0, 0, 0, \dots]?$	$[0, 0, 0, 0, \dots]?$	$e^0 + e^1$	
$S^1 \vee S^1$	$\pi_* S^1 * \pi_* S^1$	$(H_*M(\mathbb{Z}, 1))^2$	$e^0 + 2e^1$	
$\bigvee_n^n S^{1a}$	$*^n \pi_* S^1$	$\prod H_*M(\mathbb{Z}, 1)$	$e^0 + e^1$	
$\overline{a?}$				
\mathbb{RP}^{1a}	$\pi_* S^1$	$H_*M(\mathbb{Z}, 1)$	$e^0 + e^1$	${}_0\mathbb{Z} \times {}_1\mathbb{Z}$
$\overline{a\mathbb{RP}^1 \simeq S^1.}$				
\mathbb{RP}^2	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^2$	$H_*M(\mathbb{Z}_2, 1)$	$e^0 + e^1 + e^2$	${}_0\mathbb{Z} \times {}_2\mathbb{Z}_2$
\mathbb{RP}^3	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^3$	$H_*M(\mathbb{Z}_2, 1) \times H_*M(\mathbb{Z}, 3)$	$e^0 + e^1 + e^2 + e^3$	${}_0\mathbb{Z} \times {}_2\mathbb{Z}_2 \times {}_3\mathbb{Z}$
\mathbb{RP}^4	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^4$	$H_*M(\mathbb{Z}_2, 1) \times H_*M(\mathbb{Z}_2, 3)$	$e^0 + e^1 + e^2 + e^3 + e^4$	${}_0\mathbb{Z} \times ({}_2\mathbb{Z}_2)^2$
$\mathbb{RP}^n, n \geq 4$ even	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^{na}$	$\prod_{\text{odd } i < n} H_*M(\mathbb{Z}_2, i)$	$\sum_{i=1}^n e^i$	${}_0\mathbb{Z} \times \prod_{i=1}^{n/2} {}_2\mathbb{Z}_2$
$\overline{\text{Take the universal double cover } S^n \twoheadrightarrow^{\times 2} \mathbb{RP}^n \text{ to get equality in } \pi_{i \geq 2}.}$				

X	$\pi_*(X)$	$H_*(X)$	CW	$H^*(X)$
$\mathbb{RP}^n, n \geq 4$ odd	$\pi_*K(\mathbb{Z}_2, 1) \times \pi_*S^{n-1}$ $\xrightarrow{\text{Take the universal double cover } S^n \twoheadrightarrow \mathbb{RP}^n \text{ to get equality in } \pi_{i \geq 2} \cdot}$	$\prod_{\text{odd } i \leq n-2} H_*M(\mathbb{Z}_2, i) \times H_*S^n$	$\sum_{i=1}^n e^i$	$H^*(\mathbb{RP}^{n-1}) \times {}_n\mathbb{Z}$
\mathbb{CP}^{1a}	$\pi_*K(\mathbb{Z}, 2) \times \pi_*S^3$	H_*S^2	$e^0 + e^2$	$\mathbb{Z}[2x]/(2x^2)$
\mathbb{CP}^2	$\pi_*K(\mathbb{Z}, 2) \times \pi_*S^5$	$H_*S^2 \times H_*S^4$	$e^0 + e^2 + e^4$	$\mathbb{Z}[2x]/(2x^3)$
$\mathbb{CP}^n, n \geq 2$	$\pi_*K(\mathbb{Z}, 2) \times \pi_*S^{2n+1}$ $\xrightarrow{\text{Use } \mathbb{CP}^n = S^{2n+1}/S^1}$	$\prod_{i=1}^n H_*S^{2i}$	$\sum_{i=1}^n e^{2i}$	$\mathbb{Z}[2x]/(2x^{n+1})$
Mobius Band ^a	π_*S^1	H_*S^1	$e^0 + e^1$	
Klein Bottle	$K(\mathbb{Z} \rtimes_{x \mapsto -x} \mathbb{Z}, 1)^a$ $\xrightarrow{\text{Alternatively, the fundamental group is } \mathbb{Z} * \mathbb{Z}/bab^{-1}a. \text{ Use the fact the } \tilde{K} = \mathbb{R}^2.}$	$H_*S^1 \times H_*\mathbb{RP}^\infty$	$e^0 + 2e^1 + e^2$	

6 Euler Characteristics

- Only surfaces with positive χ :
 - $\chi S^2 = 2$
 - $\chi \mathbb{RP}^2 = 1$
 - $\chi B^2 = 1$
- Manifolds with zero χ
 - $T^2, K, M, S^1 \times I$
- Manifolds with negative χ
 - $\Sigma_{g \geq 2}$ by $\chi(X) = 2 - 2g$.

7 Useful Facts and Techniques

- Fundamental group:
 - Van Kampen
- Homotopy Groups
 - Hurewicz map
- Homology
 - Mayer-Vietoris
 - * $(X = A \cup B) \mapsto (\bigcap, \oplus, \bigcup)$ in homology
 - LES of a pair
 - * $(A \hookrightarrow X) \mapsto (A, X, X/A)$
 - Excision
- $\pi_{i \geq 2}(X)$ is always abelian.
- The ranks of π_0 and H_0 are the number of path components, and $\pi_0(X) = \mathbb{Z}$ iff X is simply connected.
 - X simply connected implies $\pi_k(X) \cong H_k(X)$ up to and including the first nonvanishing H_k
 - $H_1(X) = \pi_1 X / [\pi_1 X, \pi_1 X]$, the abelianization.
- General mantra: homotopy plays nicely with products, homology with wedge products.⁵

In general, homotopy groups behave nicely under homotopy pull-backs (e.g., fibrations and products), but not homotopy push-outs (e.g., cofibrations and wedges). Homology is the opposite.

- $\pi_k \prod X = \prod \pi_k X$ by LES.⁶
- $H_k \prod X \neq \prod H_k X$ due to torsion.
 - Nice case: $H_k(A \times B) = \prod_{i+j=k} H_i A \otimes H_j B$ by Kunneth when all groups are torsion-free.⁷
- $H_k \bigvee X = \prod H_k X$ by Mayer-Vietoris.⁸
- $\pi_k \bigvee X \neq \prod \pi_k X$ (counterexample: $S^1 \vee S^2$)

⁵More generally, in **Top**, we can look at $A \leftarrow \{\text{pt}\} \rightarrow B$ – then $A \times B$ is the pullback and $A \vee B$ is the pushout. In this case, homology $h : \mathbf{Top} \rightarrow \mathbf{Grp}$ takes pushouts to pullbacks but doesn't behave well with pullbacks. Similarly, while π takes pullbacks to pullbacks, it doesn't behave nicely with pushouts.

⁶This follows because $X \times Y \rightarrow X$ is a fiber bundle, so use LES in homotopy and the fact that $\pi_{i \geq 2} \in \mathbf{Ab}$.

⁷The generalization of Kunneth is as follows: write $\mathcal{P}(n, k)$ be the set of partitions of n into k parts, i.e. $\curvearrowright \in \mathcal{P}(n, k) \implies \curvearrowright = (x_1, x_2, \dots, x_k)$ where $\sum x_i = n$. Then

$$H_n(\prod_{j=1}^k X_j) = \bigoplus_{\curvearrowright \in \mathcal{P}(n, k)} \bigotimes_{i=1}^k H_{x_i}(X_i).$$

⁸ \bigvee is the coproduct in the category **Top**₀ of pointed topological spaces, and alternatively, $X \vee Y$ is the pushout in **Top** of $X \leftarrow \{\text{pt}\} \rightarrow Y$

-
- Nice case: $\pi_1 \bigvee X = * \pi_1 X$ by Van Kampen.
 - $\pi_i(\widehat{X}) \cong \pi_i(X)$ for $i \geq 2$ whenever $\widehat{X} \twoheadrightarrow X$ is a universal cover.
 - Groups and Group Actions
 - $\pi_0(G) = G$ for G a discrete topological group.
 - $\pi_k(G/H) = \pi_k(G)$ if $\pi_k(H) = \pi_{k-1}(H) = 0$.
 - $\pi_1(X/G) = \pi_0(G)$ when G acts freely/transitively on X .
 - Manifolds
 - $H^n(M^n) = \mathbb{Z}$ if M^n is orientable and zero if M^n is nonorientable.
 - Poincare Duality: $H_i M^n \cong H^{n-i} M^n$ iff M^n is closed and orientable.

8 Other Interesting Things To Consider

- The “generalized uniform bouquet”? $\mathcal{B}^n(m) = \bigvee_{i=1}^m S^n$
- Lie Groups
 - The real general linear group, $GL_n(\mathbb{R})$
 - * The real special linear group $SL_n(\mathbb{R})$
 - * The real orthogonal group, $O_n(\mathbb{R})$
 - The real special orthogonal group, $SO_n(\mathbb{R})$
 - * The real unitary group, $U_n(\mathbb{R})$
 - The real special unitary group, $SU_n(\mathbb{R})$
 - * The real symplectic group $Sp(n)$
- “Geometric” Stuff
 - Affine n -space over a field $\mathbb{A}^n(k) = k^n \rtimes GL_n(k)$
 - The projective space $\mathbb{P}^n(k)$
 - * The projective linear group over a ring R , $PGL_n(R)$
 - * The projective special linear group over a ring R , $PSL_n(R)$
 - * The modular groups $PSL_n(\mathbb{Z})$
 - Specifically $PSL_2(\mathbb{Z})$
- The real Grassmannian, $Gr(n, k, \mathbb{R})$, i.e. the set of k dimensional subspaces of \mathbb{R}^n
- The Stiefel manifold $V_n(k)$
- Possible modifications to a space X :
 - Remove k points by taking $D(k, X)$
 - Remove a line segment
 - Remove an entire line/axis
 - Remove a hole
 - Quotient by a group action (e.g. antipodal map, or rotation)
 - Remove a knot
 - Take complement in ambient space

-
- Assorted info about other Lie Groups:
 - $O_n, U_n, SO_n, SU_n, Sp_n$
 - $\pi_k(U_n) = \mathbb{Z} \cdot \mathbb{1} [k \text{ odd}]$
 - $\pi_1(U_n) = 1$
 - $\pi_k(SU_n) = \mathbb{Z} \cdot \mathbb{1} [k \text{ odd}]$
 - $\pi_1(SU_n) = 0$
 - $\pi_k(U_n) = \mathbb{Z}_2 \cdot \mathbb{1} [k = 0, 1 \pmod 8] + \mathbb{Z} \cdot \mathbb{1} [k = 3, 7 \pmod 8]$
 - $\pi_k(Sp_n) = \mathbb{Z}_2 \cdot \mathbb{1} [k = 4, 5 \pmod 8] + \mathbb{Z} \cdot \mathbb{1} [k = 3, 7 \pmod 8]$

9 Spheres

- $\pi_i(S^n) = 0$ for $i < n$, $\pi_n(S^n) = \mathbb{Z}$
 - Not necessarily true that $\pi_i(S^n) = 0$ when $i > n!!!$
 - * E.g. $\pi_3(S^2) = \mathbb{Z}$ by Hopf fibration
- $H_i(S^n) = \mathbb{1} [i \in \{0, n\}]$
- $H_n(\bigvee_i X_i) \cong \prod_i H_n(X_i)$ for “good pairs”
 - Corollary: $H_n(\bigvee_k S^n) = \mathbb{Z}^k$
- $S^n/S^k \simeq S^n \vee \Sigma S^k$
 - $\Sigma S^n = S^{n+1}$
- S^n has the CW complex structure of 2 k -cells for each $0 \leq k \leq n$.

10 Definitions

- Topology: Closed under arbitrary unions and finite intersections.
- Basis: A subset $\{B_i\}$ is a basis iff
 - $x \in X \implies x \in B_i$ for some i .
 - $x \in B_i \cap B_j \implies x \in B_k \subset B_i \cap B_j$.
 - Topology generated by this basis: $x \in N_x \implies x \in B_i \subset N_x$ for some i .
- Dense: A subset $Q \subset X$ is dense iff $y \in N_y \subset X \implies N_y \cap Q \neq \emptyset$ iff $\overline{Q} = X$.
- Neighborhood: A neighborhood of a point x is any open set containing x .
- Hausdorff
- Second Countable: admits a countable basis.
- Closed (several characterizations)

-
- Closure in a subspace: $Y \subset X \implies \text{cl}_Y(A) := \text{cl}_X(A) \cap Y$.
 - Bounded
 - Compact: A topological space (X, τ) is **compact** if every open cover has a *finite* subcover.
That is, if $\{U_j \mid j \in J\} \subset \tau$ is a collection of open sets such that $X \subseteq \bigcup_{j \in J} U_j$, then there exists a *finite* subset $J' \subset J$ such that $X \subseteq \bigcup_{j \in J'} U_j$.
 - Locally compact For every $x \in X$, there exists a $K_x \ni x$ such that K_x is compact.
 - Connected: There does not exist a disconnecting set $X = A \amalg B$ such that $\emptyset \neq A, B \subsetneq X$, i.e. X is the union of two proper disjoint nonempty sets.
Equivalently, X contains no proper nonempty clopen sets.
 - Additional condition for a subspace $Y \subset X$: $\text{cl}_Y(A) \cap V = A \cap \text{cl}_Y(B) = \emptyset$.
 - Locally connected: A space is locally connected at a point x iff $\forall N_x \ni x$, there exists a $U \subset N_x$ containing x that is connected.
 - Retract: A subspace $A \subset X$ is a *retract* of X iff there exists a continuous map $f : X \longrightarrow A$ such that $f|_A = \text{id}_A$. Equivalently it is a *left inverse* to the inclusion.
 - Uniform Continuity: For $f : (X, d_x) \longrightarrow (Y, d_Y)$ metric spaces,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$
 - Lebesgue number: For (X, d) a compact metric space and $\{U_\alpha\} \rightrightarrows X$, there exist $\delta_L > 0$ such that

$$A \subset X, \text{diam}(A) < \delta_L \implies A \subseteq U_\alpha \text{ for some } \alpha.$$
 - Paracompact
 - Components: Set $x \sim y$ iff there exists a connected set $U \ni x, y$ and take equivalence classes.
 - Path Components: Set $x \sim y$ iff there exists a path-connected set $U \ni x, y$ and take equivalence classes.
 - Separable: Contains a countable dense subset.
 - Limit Point: For $A \subset X$, x is a limit point of A if every punctured neighborhood P_x of x satisfies $P_x \cap A \neq \emptyset$, i.e. every neighborhood of x intersects A in some point other than x itself.
Equivalently, x is a limit point of A iff $x \in \text{cl}_X(A \setminus \{x\})$.

11 Examples

11.1 Common Spaces and Operations

Point-Set:

- Finite discrete sets with the discrete topology
- Subspaces of \mathbb{R} : (a, b) , $(a, b]$, (a, ∞) , etc.
 - $\{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{Z}^{\geq 1} \right\}$
- \mathbb{Q}
- The topologist's sine curve
- One-point compactifications
- \mathbb{R}^ω
- Hawaiian earring
- Cantor set

Non-Hausdorff spaces:

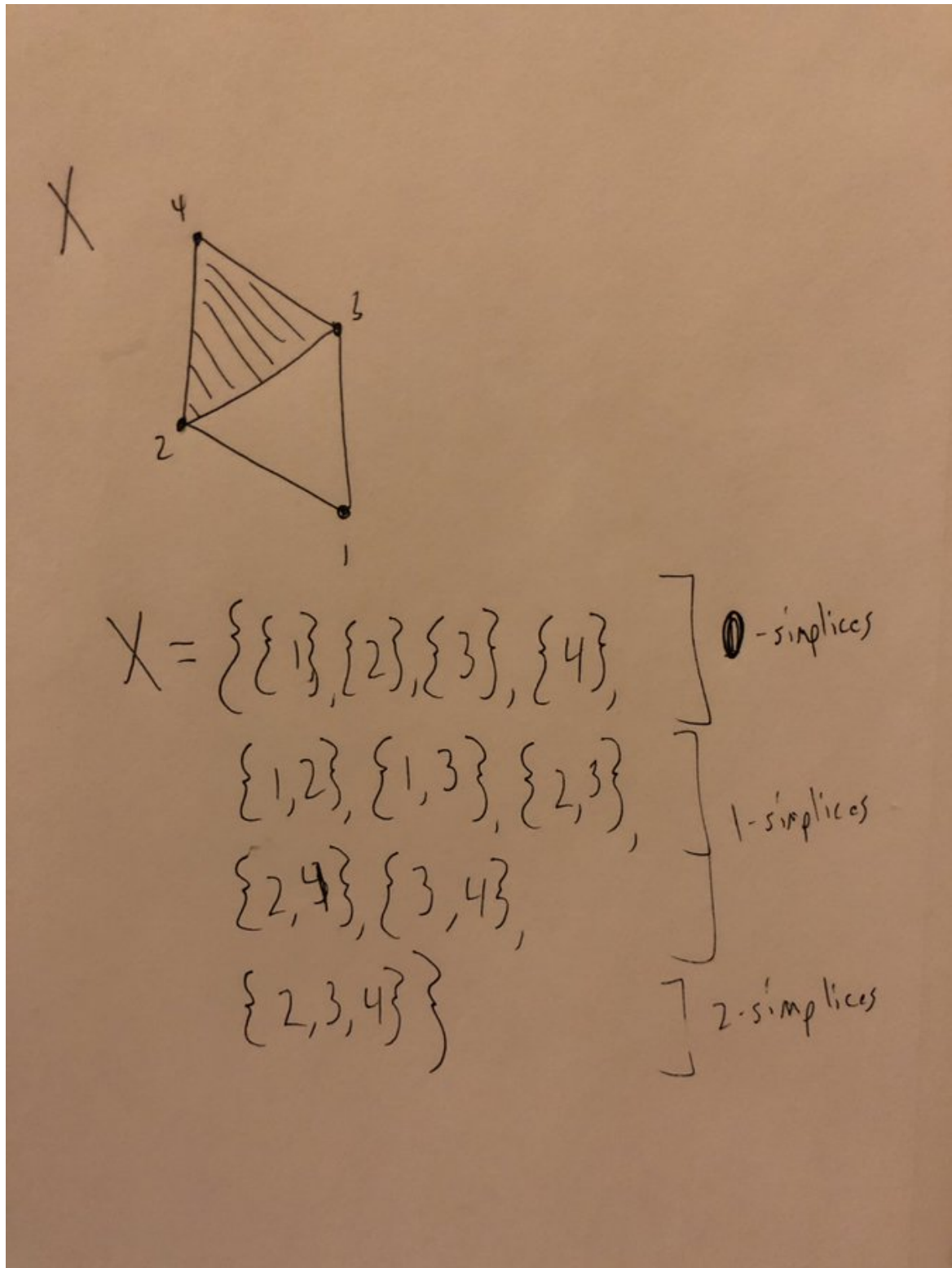
- The cofinite topology on any infinite set.
- \mathbb{R}/\mathbb{Q}
- The line with two origins.

General Spaces:

$$S^n, \mathbb{D}^n, T^n, \mathbb{R}P^n, \mathbb{C}P^n, \mathbb{M}, \mathbb{K}, \Sigma_g, \mathbb{R}P^\infty, \mathbb{C}P^\infty.$$

“Constructed” Spaces

- Knot complements in S^3
- Covering spaces (hyperbolic geometry)
- Lens spaces
- Matrix groups
- Prism spaces
- Pair of pants
- Seifert surfaces
- Surgery
- Simplicial Complexes
 - Nice minimal example:



Exotic/Pathological Spaces

- $\mathbb{H}\mathbb{P}^n$
- Dunce Cap

- Horned sphere

Operations

- Cartesian product $A \times B$
- Wedge product $A \vee B$
- Connect Sum $A \# B$
- Quotienting A/B
- Puncturing $A \setminus \{a_i\}$
- Smash product
- Join
- Cones
- Suspension
- Loop space
- Identifying a finite number of points

11.2 Alternative Topologies

- Discrete
- Cofinite
- Discrete and Indiscrete
- Uniform

The cofinite topology:

- Non-Hausdorff
- Compact

The discrete topology:

- Discrete iff points are open
- Always Hausdorff
- Compact iff finite
- Totally disconnected
- If the domain, every map is continuous

The indiscrete topology:

- Only open sets are \emptyset, X
- Non-Hausdorff
- If the codomain, every map is continuous
- Compact

12 Theorems

12.1 Point-Set

- Closed subsets of Hausdorff spaces are compact? (check)
- Cantor's intersection theorem?
- Tube lemma

- Properties pushed forward through continuous maps:
 - Compactness?
 - Connectedness (when surjective)
 - Separability
 - Density **only when** f is surjective
 - **Not** openness
 - **Not** closedness
- A retract of a Hausdorff/connected/compact space is closed/connected/compact respectively.

Proposition 12.1.

A continuous function on a compact set is uniformly continuous.

Proof.

Take $\{B_{\frac{\varepsilon}{2}}(y) \mid y \in Y\} \Rightarrow Y$, pull back to an open cover of X , has Lebesgue number $\delta_L > 0$, then $x' \in B_{\delta_L}(x) \Rightarrow f(x), f(x') \in B_{\frac{\varepsilon}{2}}(y)$ for some y . ■

- Lipschitz continuity implies uniform continuity (take $\delta = \varepsilon/C$)
 - Counterexample to converse: $f(x) = \sqrt{x}$ on $[0, 1]$ has unbounded derivative.
- Extreme Value Theorem: for $f : X \rightarrow Y$ continuous with X compact and Y ordered in the order topology, there exist points $c, d \in X$ such that $f(x) \in [f(c), f(d)]$ for every x .

Theorem 12.2.

Points are closed in T_1 spaces.

Theorem 12.3.

A metric space X is sequentially compact iff it is complete and totally bounded.

Theorem 12.4.

A metric space is totally bounded iff every sequence has a Cauchy subsequence.

Theorem 12.5.

A metric space is compact iff it is complete and totally bounded.

Theorem 12.6 (Baire).

If X is a complete metric space, then the intersection of countably many dense open sets is dense in X .

Theorem 12.7.

A continuous bijective open map is a homeomorphism.

Theorem 12.8.

A closed subset A of a compact set B is compact.

Proof .

- Let $\{A_i\} \rightrightarrows A$ be a covering of A by sets open in A .
- Each $A_i = B_i \cap A$ for some B_i open in B (definition of subspace topology)
- Define $V = \{B_i\}$, then $V \rightrightarrows A$ is an open cover.
- Since A is closed, $W := B \setminus A$ is open
- Then $V \cup W$ is an open cover of B , and has a finite subcover $\{V_i\}$
- Then $\{V_i \cap A\}$ is a finite open cover of A .

■

Theorem 12.9.

The continuous image of a compact set is compact.

Theorem 12.10.

A closed subset of a Hausdorff space is compact.

12.2 Algebraic

Todo: Merge the two van Kampen theorems.

Theorem 12.11 (Van Kampen).

The pushout is the northwest colimit of the following diagram

$$\begin{array}{ccc} A \amalg_Z B & \longleftarrow & A \\ & & \uparrow \iota_A \\ B & \xleftarrow{\iota_B} & Z \end{array}$$

For groups, the pushout is given by the amalgamated free product: if $A = \langle G_A \mid R_A \rangle$, $B = \langle G_B \mid R_B \rangle$, then $A *_Z B = \langle G_A, G_B \mid R_A, R_B, T \rangle$ where T is a set of relations given by $T = \{ \iota_A(z) \iota_B(z)^{-1} \mid z \in Z \}$.

Example: $A = \mathbb{Z}/4\mathbb{Z} = \langle x \mid x^4 \rangle$, $B = \mathbb{Z}/6\mathbb{Z} = \langle y \mid y^6 \rangle$, $Z = \mathbb{Z}/2\mathbb{Z} = \langle z \mid z^2 \rangle$. Then we can identify Z as a subgroup of A, B using $\iota_A(z) = x^2$ and $\iota_B(z) = y^3$. So

$$A *_Z B = \langle x, y \mid x^4, y^6, x^2 y^{-3} \rangle$$

Suppose $X = U_1 \cup U_2$ such that $U_1 \cap U_2 \neq \emptyset$ is path connected. Then taking $x_0 \in U := U_1 \cap U_2$ yields a pushout of fundamental groups

$$\pi_1(X; x_0) = \pi_1(U_1; x_0) *_{\pi_1(U; x_0)} \pi_1(U_2; x_0).$$

Theorem 12.12 (Van Kampen).

If $X = U \bigcup V$ where $U, V, U \cap V$ are all path-connected then

$$\pi_1(X) = \pi_1 U *_{\pi_1(U \cap V)} \pi_1 V,$$

where the amalgamated product can be computed as follows: If we have presentations

$$\begin{aligned}\pi_1(U, w) &= \langle u_1, \dots, u_k \mid \alpha_1, \dots, \alpha_l \rangle \\ \pi_1(V, w) &= \langle v_1, \dots, v_m \mid \beta_1, \dots, \beta_n \rangle \\ \pi_1(U \cap V, w) &= \langle w_1, \dots, w_p \mid \gamma_1, \dots, \gamma_q \rangle\end{aligned}$$

then

$$\begin{aligned}\pi_1(X, w) &= \langle u_1, \dots, u_k, v_1, \dots, v_m \rangle \\ &\quad \text{mod } \langle \alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_n, I(w_1)J(w_1)^{-1}, \dots, I(w_p)J(w_p)^{-1} \rangle \\ &= \frac{\pi_1(U) * \pi_1(V)}{\langle \{I(w_i)J(w_i)^{-1} \mid 1 \leq i \leq p\} \rangle}\end{aligned}$$

where

$$\begin{aligned}I &: \pi_1(U \cap V, w) \rightarrow \pi_1(U, w) \\ J &: \pi_1(U \cap V, w) \rightarrow \pi_1(V, w).\end{aligned}$$