

Real Analysis Qualifying Exam Notes

D. Zack Garza

Sunday 5th July, 2020

Contents

1	Practice Exam 2 (November 2014)	2
1.1	1: Fubini-Tonelli	2
1.1.1	a	2
1.1.2	b	2
1.2	2: Convolutions and the Fourier Transform	2
1.2.1	a	2
1.2.2	b	2
1.2.3	c	3
1.3	3: Hilbert Spaces	3
1.3.1	a	3
1.3.2	b	4
1.3.3	c	4
1.4	4: L_p Spaces	5
1.4.1	a	5
1.4.2	c	7
1.5	5: Dual Spaces	8
1.5.1	b	9
1.5.2	c	9
2	Exam 2 (2018)	10
3	Exam 2 (2014)	10
4	Qual: Fall 2019	10
4.1	1	10
4.2	2	10
4.3	3	10
4.4	4	10
4.5	5	11
5	Basics	12
6	Uniform Convergence	14
7	Measure Theory	16

8	Integration	19
8.1	Convergence Theorems	20
8.2	L^1 Facts	23
8.3	L^p Spaces	25
9	Fourier Transform and Convolution	26
9.1	The Fourier Transform	26
9.2	Approximate Identities	27
10	Functional Analysis	29
11	Extra Problems	31
12	Inequalities and Equalities	33
12.1	Less Explicitly Used Inequalities	35

1 Practice Exam 2 (November 2014)

1.1 1: Fubini-Tonelli

1.1.1 a

Carefully state Tonelli's theorem for a nonnegative function $F(x, t)$ on $\mathbb{R}^n \times \mathbb{R}$.

1.1.2 b

Let $f : \mathbb{R}^n \rightarrow [0, \infty]$ and define

$$\mathcal{A} := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid 0 \leq t \leq f(x)\}.$$

Prove the validity of the following two statements:

1. f is Lebesgue measurable on $\mathbb{R}^n \iff \mathcal{A}$ is a Lebesgue measurable subset of \mathbb{R}^{n+1} .
2. If f is Lebesgue measurable on \mathbb{R}^n then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x) dx = \int_0^\infty m(\{x \in \mathbb{R}^n \mid f(x) \geq t\}) dt.$$

1.2 2: Convolutions and the Fourier Transform

1.2.1 a

Let $f, g \in L^1(\mathbb{R}^n)$ and give a definition of $f * g$.

1.2.2 b

Prove that if f, g are integrable and bounded, then

$$(f * g)(x) \xrightarrow{|x| \rightarrow \infty} 0.$$

1.2.3 c

1. Define the *Fourier transform* of an integrable function f on \mathbb{R}^n .
2. Give an outline of the proof of the Fourier inversion formula.
3. Give an example of a function $f \in L^1(\mathbb{R}^n)$ such that \hat{f} is not in $L^1(\mathbb{R}^n)$.

1.3 3: Hilbert Spaces**1.3.1 a**

Theorem (Bessel's Inequality):

$$\left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2$$

and thus

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

Proof: Let $S_N = \sum_{n=1}^N \langle x, u_n \rangle u_n$

$$\begin{aligned} \|x - S_N\|^2 &= \langle x - S_N, x - S_N \rangle \\ &= \|x\|^2 + \|S_N\|^2 - 2\Re \langle x, S_N \rangle \\ &= \|x\|^2 + \|S_N\|^2 - 2\Re \left\langle x, \sum_{n=1}^N \langle x, u_n \rangle u_n \right\rangle \\ &= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^N \langle x, \langle x, u_n \rangle u_n \rangle \\ &= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^N \overline{\langle x, u_n \rangle} \langle x, u_n \rangle \\ &= \|x\|^2 + \left\| \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|^2 - 2 \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\ &= \|x\|^2 + \sum_{n=1}^N |\langle x, u_n \rangle|^2 - 2 \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\ &= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2. \end{aligned}$$

And by continuity of the norm and inner product, we have

$$\begin{aligned}
\lim_{N \rightarrow \infty} \|x - S_N\|^2 &= \lim_{N \rightarrow \infty} \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\
\Rightarrow \left\| x - \lim_{N \rightarrow \infty} S_N \right\|^2 &= \|x\|^2 - \lim_{N \rightarrow \infty} \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\
\Rightarrow \left\| x - \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2.
\end{aligned}$$

Then noting that $0 \leq \|x - S_N\|^2$, we have

$$\begin{aligned}
0 &\leq \|x\|^2 - \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \\
\Rightarrow \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 &\leq \|x\|^2 \blacksquare.
\end{aligned}$$

1.3.2 b

- Take $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \Rightarrow$ the tails vanish.
- Define $x = \lim_{N \rightarrow \infty} S_N$ where $S_N = \sum_{k=1}^N a_k u_k$
- $\{S_N\}$ is Cauchy and H is complete, so $x \in H$.
- By construction, $\langle x, u_n \rangle = \left\langle \sum_k a_k u_k, u_n \right\rangle = \sum_k a_k \langle u_k, u_n \rangle = a_n$ since the u_k are all orthogonal.
- $\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$ by Pythagoras since the u_k are normal.

1.3.3 c

Let x and u_n be arbitrary. Then

$$\begin{aligned}
\left\langle x - \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k, u_n \right\rangle &= \langle x, u_n \rangle - \left\langle \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k, u_n \right\rangle \\
&= \langle x, u_n \rangle - \sum_{k=1}^{\infty} \langle \langle x, u_k \rangle u_k, u_n \rangle \\
&= \langle x, u_n \rangle - \sum_{k=1}^{\infty} \langle x, u_k \rangle \langle u_k, u_n \rangle \\
&= \langle x, u_n \rangle - \langle x, u_n \rangle = 0 \\
\implies x - \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k &= 0 \quad \text{by completeness.}
\end{aligned}$$

So

$$x = \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k \implies \|x\|^2 = \sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2. \blacksquare$$

1.4 4: Lp Spaces

p-test for integrals:

$$\begin{aligned}
\int_0^1 x^{-p} < \infty &\iff p < 1 \\
\int_1^{\infty} x^{-p} < \infty &\iff p > 1.
\end{aligned}$$

Yields a general technique: break integrals apart at $x = 1$.

Inclusions and relationships:

$$\begin{aligned}
m(X) < \infty &\implies L^\infty \subset L^2 \subset L^1 \\
\ell^2(\mathbb{Z}) &\subset \ell^1(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z}).
\end{aligned}$$

1.4.1 a

Theorem (Holder's Inequality):

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Proof:

It suffices to show this when $\|f\|_p = \|g\|_q = 1$, since

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \iff \int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \leq 1.$$

Using $AB \leq \frac{1}{p}A^p + \frac{1}{q}B^q$, we have

$$\int |f|g \leq \int \frac{|f|^p}{p} \frac{|g|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1 \blacksquare.$$

Theorem (Minkowski's Inequality):

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof:

We first note

$$|f + g|^p = |f + g| |f + g|^{p-1} \leq (|f| + |g|) |f + g|^{p-1}.$$

Note that if p, q are conjugate exponents then

$$\begin{aligned} \frac{1}{q} &= 1 - \frac{1}{p} = \frac{p-1}{p} \\ q &= \frac{p}{p-1}. \end{aligned}$$

Then taking integrals yields

$$\begin{aligned} \|f + g\|_p^p &= \int |f + g|^p \\ &\leq \int (|f| + |g|) |f + g|^{p-1} \\ &= \int |f| |f + g|^{p-1} + \int |g| |f + g|^{p-1} \\ &= \|f(f + g)^{p-1}\|_1 + \|g(f + g)^{p-1}\|_1 \\ &\leq \|f\|_p \|(f + g)^{p-1}\|_q + \|g\|_p \|(f + g)^{p-1}\|_q \\ &= (\|f\|_p + \|g\|_p) \|(f + g)^{p-1}\|_q \\ &= (\|f\|_p + \|g\|_p) \left(\int |f + g|^{(p-1)q} \right)^{\frac{1}{q}} \\ &= (\|f\|_p + \|g\|_p) \left(\int |f + g|^p \right)^{1 - \frac{1}{p}} \\ &= (\|f\|_p + \|g\|_p) \frac{\int |f + g|^p}{\left(\int |f + g|^p \right)^{\frac{1}{p}}} \\ &= (\|f\|_p + \|g\|_p) \frac{\|f + g\|_p^p}{\|f + g\|_p} \end{aligned}$$

and canceling common terms yields

$$\begin{aligned} 1 &\leq (\|f\|_p + \|g\|_p) \frac{1}{\|f + g\|_p} \\ \implies \|f + g\|_p &\leq \|f\|_p + \|g\|_p \blacksquare. \end{aligned}$$

1.4.2 c**Definition (Infinity Norm):**

$$L^\infty(X) = \{f : X \longrightarrow \mathbb{C} \mid \|f\|_\infty < \infty\}$$

where

$$\|f\|_\infty = \inf_{\alpha \geq 0} \left\{ \alpha \mid m\{|f| \geq \alpha\} = 0 \right\}.$$

Theorem:

$$m(X) < \infty \implies \lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty.$$

Proof: Let $M = \|f\|_\infty$. For any $L < M$, let $S = \{|f| \geq L\}$. Then $m(S) > 0$ and

$$\begin{aligned} \|f\|_p &= \left(\int_X |f|^p \right)^{\frac{1}{p}} \\ &\geq \left(\int_S |f|^p \right)^{\frac{1}{p}} \\ &\geq L m(S)^{\frac{1}{p}} \xrightarrow{p \rightarrow \infty} L \\ &\implies \liminf_p \|f\|_p \geq M. \end{aligned}$$

We also have

$$\begin{aligned} \|f\|_p &= \left(\int_X |f|^p \right)^{\frac{1}{p}} \\ &\leq \left(\int_X M^p \right)^{\frac{1}{p}} \\ &= M m(X)^{\frac{1}{p}} \xrightarrow{p \rightarrow \infty} M \\ &\implies \limsup_p \|f\|_p \leq M \blacksquare. \end{aligned}$$

Note: this doesn't help with this problem at all.

Solution:

$$\begin{aligned} \int f^p &= \int_{x \leq 1} f^p + \int_{x=1} f^p + \int_{x \geq 1} f^p \\ &= \int_{x \leq 1} f^p + \int_{x=1} 1 + \int_{x \geq 1} f^p \\ &= \int_{x \leq 1} f^p + m(\{f = 1\}) + \int_{x \geq 1} f^p \\ &\xrightarrow{p \rightarrow \infty} 0 + m(\{f = 1\}) + \begin{cases} 0 & m(\{x \geq 1\}) = 0 \\ \infty & m(\{x \geq 1\}) > 0. \end{cases} \end{aligned}$$

1.5 5: Dual Spaces

Definition: A map $L : X \rightarrow \mathbb{C}$ is a linear functional iff

$$L(\alpha \mathbf{x} + \mathbf{y}) = \alpha L(\mathbf{x}) + L(\mathbf{y}).$$

Theorem (Riesz Representation for Hilbert Spaces): If Λ is a continuous linear functional on a Hilbert space H , then there exists a unique $y \in H$ such that

$$\forall x \in H, \quad \Lambda(x) = \langle x, y \rangle.$$

Proof:

- Define $M := \ker \Lambda$.
- Then M is a closed subspace and so $H = M \oplus M^\perp$.
- There is some $z \in M^\perp$ such that $\|z\| = 1$.
- Set $u := \Lambda(x)z - \Lambda(z)x$
- Check

$$\Lambda(u) = \Lambda(\Lambda(x)z - \Lambda(z)x) = \Lambda(x)\Lambda(z) - \Lambda(z)\Lambda(x) = 0 \implies u \in M$$

- Compute

$$\begin{aligned} 0 &= \langle u, z \rangle \\ &= \langle \Lambda(x)z - \Lambda(z)x, z \rangle \\ &= \langle \Lambda(x)z, z \rangle - \langle \Lambda(z)x, z \rangle \\ &= \Lambda(x)\langle z, z \rangle - \Lambda(z)\langle x, z \rangle \\ &= \Lambda(x)\|z\|^2 - \Lambda(z)\langle x, z \rangle \\ &= \Lambda(x) - \Lambda(z)\langle x, z \rangle \\ &= \Lambda(x) - \langle x, \overline{\Lambda(z)}z \rangle, \end{aligned}$$

- Choose $y := \overline{\Lambda(z)}z$.
- Check uniqueness:

$$\begin{aligned} &\langle x, y \rangle = \langle x, y' \rangle \quad \forall x \\ \implies &\langle x, y - y' \rangle = 0 \quad \forall x \\ \implies &\langle y - y', y - y' \rangle = 0 \\ \implies &\|y - y'\| = 0 \\ \implies &y - y' = \mathbf{0} \implies y = y'. \end{aligned}$$

1.5.1 b

Theorem (Continuous iff Bounded): Let $L : X \rightarrow \mathbb{C}$ be a linear functional, then the following are equivalent:

1. L is continuous
2. L is continuous at zero
3. L is bounded, i.e. $\exists c \geq 0 \mid |L(x)| \leq c\|x\|$ for all $x \in H$

2 \implies 3: Choose $\delta < 1$ such that

$$\|x\| \leq \delta \implies |L(x)| < 1.$$

Then

$$\begin{aligned} |L(x)| &= \left| L\left(\frac{\|x\|}{\delta} \frac{\delta}{\|x\|} x\right) \right| \\ &= \frac{\|x\|}{\delta} \left| L\left(\delta \frac{x}{\|x\|}\right) \right| \\ &\leq \frac{\|x\|}{\delta} 1, \end{aligned}$$

so we can take $c = \frac{1}{\delta}$. ■

3 \implies 1:

We have $|L(x - y)| \leq c\|x - y\|$, so given $\varepsilon \geq 0$ simply choose $\delta = \frac{\varepsilon}{c}$.

1.5.2 c

Definition (Dual Space):

$$X^\vee := \left\{ L : X \rightarrow \mathbb{C} \mid L \text{ is continuous} \right\}$$

Definition (Operator Norm):

$$\|L\|_{X^\vee} := \sup_{\substack{x \in X \\ \|x\|=1}} |L(x)|$$

Theorem: (Operator Norm is a Norm)

Proof: The only nontrivial property is the triangle inequality, but

$$\|L_1 + L_2\| = \sup |L_1(x) + L_2(x)| \leq \sup L_1(x) + \sup L_2(x) = \|L_1\| + \|L_2\|.$$

Theorem (Completeness in Operator Norm):

X^\vee equipped with the operator norm is a Banach space.

Proof:

- Let $\{L_n\}$ be Cauchy in X^\vee .

-
- Then for all $x \in C$, $\{L_n(x)\} \subset \mathbb{C}$ is Cauchy and converges to something denoted $L(x)$.
 - Need to show L is continuous and $\|L_n - L\| \rightarrow 0$.
 - Since $\{L_n\}$ is Cauchy in X^\vee , choose N large enough so that

$$n, m \geq N \implies \|L_n - L_m\| < \varepsilon \implies |L_m(x) - L_n(x)| < \varepsilon \quad \forall x \mid \|x\| = 1.$$

- Take $n \rightarrow \infty$ to obtain

$$\begin{aligned} m \geq N &\implies |L_m(x) - L(x)| < \varepsilon \quad \forall x \mid \|x\| = 1 \\ &\implies \|L_m - L\| < \varepsilon \rightarrow 0. \end{aligned}$$

- Continuity:

$$\begin{aligned} |L(x)| &= |L(x) - L_n(x) + L_n(x)| \\ &\leq |L(x) - L_n(x)| + |L_n(x)| \\ &\leq \varepsilon\|x\| + c\|x\| \\ &= (\varepsilon + c)\|x\| \blacksquare. \end{aligned}$$

2 Exam 2 (2018)

[Link to PDF File](#)

3 Exam 2 (2014)

[Link to PDF File](#)

4 Qual: Fall 2019

4.1 1

See phone photo?

4.2 2

DCT?

4.3 3

“Follow your nose.”

4.4 4

See Problem Set 8.

Bessel's Inequality: For any orthonormal set in a Hilbert space (not necessarily a basis), we have

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

Proof:

$$0 \leq \left\| x - \sum_{k=1}^n \langle x, e_k \rangle e_k \right\|^2$$

Corollary (Parseval's Identity): If $\text{span}\{u_n\}$ is dense in \mathcal{H} , so it is a basis, then this is an equality.

Riesz-Fischer: Let $U = \{u_n\}_{n=1}^{\infty}$ be an orthonormal set (not necessarily a basis), then

1. There is an isometric surjection

$$\begin{aligned} \mathcal{H} &\longrightarrow \ell^2(\mathbb{N}) \\ \mathbf{x} &\mapsto \{\langle \mathbf{x}, \mathbf{u}_n \rangle\}_{n=1}^{\infty} \end{aligned}$$

i.e. if $\{a_n\} \in \ell^2(\mathbb{N})$, so $\sum |a_n|^2 < \infty$, then there exists a $\mathbf{x} \in \mathcal{H}$ such that

$$a_n = \langle \mathbf{x}, \mathbf{u}_n \rangle \quad \forall n.$$

2. \mathbf{x} can be chosen such that

$$\|\mathbf{x}\|^2 = \sum |a_n|^2$$

Note: the choice of \mathbf{x} is unique $\iff \{u_n\}$ is **complete**, i.e. $\langle \mathbf{x}, \mathbf{u}_n \rangle = 0$ for all n implies $\mathbf{x} = \mathbf{0}$.

Proof:

- Given $\{a_n\}$, define $S_N = \sum_{n=1}^N a_n \mathbf{u}_n$.
- S_N is Cauchy in \mathcal{H} and so $S_N \longrightarrow \mathbf{x}$ for some $\mathbf{x} \in \mathcal{H}$.
- $\langle x, u_n \rangle = \langle x - S_N, u_n \rangle + \langle S_N, u_n \rangle \longrightarrow a_n$
- By construction, $\|x - S_N\|^2 = \|x\|^2 - \sum_{n=1}^N |a_n|^2 \longrightarrow 0$, so $\|x\|^2 = \sum_{n=1}^{\infty} |a_n|^2$.

4.5 5

See Problem Set 5.

Heine-Cantor theorem: Every continuous function on a compact set is uniformly continuous.

Uniform continuity:

$$\begin{aligned} & \forall \varepsilon \quad \exists \delta(\varepsilon) \mid \quad \forall x, y, \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \\ \iff & \forall \varepsilon \quad \exists \delta(\varepsilon) \mid \quad \forall x, y, \quad |y| < \delta \implies |f(x - y) - f(y)| < \varepsilon \end{aligned}$$

Fubini-Tonelli interchange of integrals, where the change of bounds becomes very important.

Continuity in L^1 :

$$\lim_{y \rightarrow 0} \|\tau_y f - f\|_1 = 0.$$

5 Basics

Useful Technique: $\lim f_n = \limsup f_n = \liminf f_n$ iff the limit exists, so $\limsup f_n \leq g \leq \liminf f_n$ implies that $g = \lim f$. Similarly, a limit does not exist iff $\liminf f_n > \limsup f_n$.

Lemma 5.1 (Convergent Sums Have Small Tails).

$$\sum a_n < \infty \implies a_n \rightarrow 0 \quad \text{and} \quad \sum_{k=N}^{\infty} a_k \xrightarrow{N \rightarrow \infty} 0$$

Theorem 5.2 (Heine-Borel).

$X \subseteq \mathbb{R}^n$ is compact $\iff X$ is closed and bounded.

Lemma 5.3 (Geometric Series).

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \iff |x| < 1.$$

Corollary: $\sum_{k=0}^{\infty} \frac{1}{2^k} = 1.$

Definition 5.3.1.

A set S is **nowhere dense** iff the closure of S has empty interior iff every interval contains a subinterval that does not intersect S .

Definition 5.3.2.

A set is **meager** if it is a *countable* union of nowhere dense sets.

Note that a *finite* union of nowhere dense is still nowhere dense.

Lemma 5.4.

The Cantor set is closed with empty interior.

Proof .

Its complement is a union of open intervals, and can't contain an interval since intervals have positive measure and $m(C_n)$ tends to zero.

Corollary: The Cantor set is nowhere dense. ■

Definition 5.4.1.

An F_σ set is a union of closed sets, and a G_δ set is an intersection of opens.

Mnemonic: “F” stands for ferme, which is “closed” in French, and σ corresponds to a “sum”, i.e. a union.

Lemma 5.5.

Singleton sets in \mathbb{R} are closed, and thus \mathbb{Q} is an F_σ set.

Theorem 5.6 (Baire).

\mathbb{R} is a **Baire space** (countable intersections of open, dense sets are still dense). Thus \mathbb{R} can not be written as a countable union of nowhere dense sets.

Lemma 5.7.

There is a function discontinuous precisely on \mathbb{Q} .

Proof .

$f(x) = \frac{1}{n}$ if $x = r_n \in \mathbb{Q}$ is an enumeration of the rationals, and zero otherwise. The limit at every point is 0. ■

Lemma 5.8.

There *do not* exist functions that are discontinuous precisely on $\mathbb{R} \setminus \mathbb{Q}$.

Proof .

D_f is always an F_σ set, which follows by considering the oscillation ω_f . $\omega_f(x) = 0 \iff f$ is continuous at x , and $D_f = \bigcup_n A_{\frac{1}{n}}$ where $A_\varepsilon = \{\omega_f \geq \varepsilon\}$ is closed. ■

Lemma 5.9.

Any nonempty set which is bounded from above (resp. below) has a well-defined supremum (resp. infimum).

6 Uniform Convergence

Theorem 6.1 (Egorov).

Let $E \subseteq \mathbb{R}^n$ be measurable with $m(E) > 0$ and $\{f_k : E \rightarrow \mathbb{R}\}$ be measurable functions such that

$$f(x) := \lim_{k \rightarrow \infty} f_k(x) < \infty$$

exists almost everywhere.

Then $f_k \rightarrow f$ *almost uniformly*, i.e.

$$\forall \varepsilon > 0, \exists F \subseteq E \text{ closed such that } m(E \setminus F) < \varepsilon \text{ and } f_k \xrightarrow{u} f \text{ on } F.$$

Proposition 6.2.

The space $X = C([0, 1])$, continuous functions $f : [0, 1] \rightarrow \mathbb{R}$, equipped with the norm $\|f\| = \sup_{x \in [0, 1]} |f(x)|$, is a **complete** metric space.

Proof .

1. Let $\{f_k\}$ be Cauchy in X .
2. Define a candidate limit using pointwise convergence:
Fix an x ; since

$$|f_k(x) - f_j(x)| \leq \|f_k - f_j\| \rightarrow 0$$

the sequence $\{f_k(x)\}$ is Cauchy in \mathbb{R} . So define $f(x) := \lim_k f_k(x)$.

3. Show that $\|f_k - f\| \rightarrow 0$:

$$|f_k(x) - f_j(x)| < \varepsilon \quad \forall x \implies \lim_j |f_k(x) - f_j(x)| < \varepsilon \quad \forall x$$

Alternatively, $\|f_k - f\| \leq \|f_k - f_N\| + \|f_N - f_j\|$, where N, j can be chosen large enough to bound each term by $\varepsilon/2$.

4. Show that $f \in X$:

The uniform limit of continuous functions is continuous. (Note: in other cases, you may need to show the limit is bounded, or has bounded derivative, or whatever other conditions define X .)

■

Lemma 6.3.

Metric spaces are compact iff they are sequentially compact, (i.e. every sequence has a convergent subsequence).

Proposition 6.4.

The unit ball in $C([0, 1])$ with the sup norm is not compact.

Proof .

Take $f_k(x) = x^n$, which converges to a dirac delta at 1. The limit is not continuous, so no subsequence can converge. ■

Lemma 6.5.

A uniform limit of continuous functions is continuous.

Lemma 6.6 (Testing Uniform Convergence).

$f_n \rightarrow f$ uniformly iff there exists an M_n such that $\|f_n - f\|_\infty \leq M_n \rightarrow 0$.

Negating: find an x which depends on n for which the norm is bounded below.

Useful Technique: If f_n has a global maximum (computed using f'_n and the first derivative test) $M_n \rightarrow 0$, then $f_n \rightarrow 0$ uniformly.

Lemma 6.7 (Baby Commuting Limits with Integrals).

If $f_n \rightarrow f$ uniformly, then $\int f_n = \int f$.

Lemma 6.8 (Uniform Convergence and Derivatives).

If $f'_n \rightarrow g$ uniformly for some g and $f_n \rightarrow f$ pointwise (or at least at one point), then $g = f'$.

Lemma 6.9 (Uniform Convergence of Series).

If $f_n(x) \leq M_n$ for a fixed x where $\sum M_n < \infty$, then the series $f(x) = \sum f_n(x)$ converges pointwise.

Lemma 6.10.

If $\sum f_n$ converges then $f_n \rightarrow 0$ uniformly.

Useful Technique: For a fixed x , if $f = \sum f_n$ converges *uniformly* on some $B_r(x)$ and each f_n is continuous at x , then f is also continuous at x .

Lemma 6.11 (M-test for Series).

If $|f_n(x)| \leq M_n$ which does not depend on x , then $\sum f_n$ converges uniformly.

Lemma 6.12 (p-tests).

Let n be a fixed dimension and set $B = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.

$$\begin{aligned}\sum \frac{1}{n^p} < \infty &\iff p > 1 \\ \int_{\varepsilon}^{\infty} \frac{1}{x^p} < \infty &\iff p > 1 \\ \int_0^1 \frac{1}{x^p} < \infty &\iff p < 1 \\ \int_B \frac{1}{|x|^p} < \infty &\iff p < n \\ \int_{B^c} \frac{1}{|x|^p} < \infty &\iff p > n\end{aligned}$$

Proposition 6.13.

A function $f : (a, b) \rightarrow \mathbb{R}$ is Lipschitz $\iff f$ is differentiable and f' is bounded. In this case, $|f'(x)| \leq C$, the Lipschitz constant.

Proposition 6.14.

There exist smooth compactly supported functions, e.g. take

$$f(x) = e^{\frac{-1}{x^2}} \chi_{(0, \infty)}(x).$$

7 Measure Theory

Useful Technique: $s = \inf \{x \in X\} \implies$ for every ε there is an $x \in X$ such that $x \leq s + \varepsilon$.

Useful Techniques: Always consider bounded sets, and if E is unbounded write $E = \bigcup_n B_n(0) \cap E$ and use countable subadditivity or continuity of measure.

Lemma 7.1.

Every open subset of \mathbb{R} (resp \mathbb{R}^n) can be written as a unique countable union of disjoint (resp. almost disjoint) intervals (resp. cubes).

Definition 7.1.1.

The outer measure of a set is given by

$$m_*(E) = \inf_{\substack{\{Q_i\} \rightrightarrows E \\ \text{closed cubes}}} \sum |Q_i|.$$

Lemma 7.2 (Properties of Outer Measure).

- Monotonicity: $E \subseteq F \implies m_*(E) \leq m_*(F)$.

- Countable Subadditivity: $m_*(\bigcup E_i) \leq \sum m_*(E_i)$.
- Approximation: For all E there exists a $G \supseteq E$ such that $m_*(G) \leq m_*(E) + \varepsilon$.
- Disjoint^a Additivity: $m_*(A \amalg B) = m_*(A) + m_*(B)$.

^aThis holds for outer measure iff $\text{dist}(A, B) > 0$.

Lemma 7.3 (Subtraction of Measure).

$$m(A) = m(B) + m(C) \quad \text{and} \quad m(C) < \infty \implies m(A) - m(C) = m(B).$$

Lemma 7.4 (Continuity of Measure).

$$E_i \nearrow E \implies m(E_i) \longrightarrow m(E)$$

$$m(E_1) < \infty \text{ and } E_i \searrow E \implies m(E_i) \longrightarrow m(E).$$

Proof .

1. Break into disjoint annuli $A_2 = E_2 \setminus E_1$, etc then apply countable disjoint additivity to $E = \amalg A_i$.
2. Use $E_1 = (\amalg E_j \setminus E_{j+1}) \amalg (\bigcap E_j)$, taking measures yields a telescoping sum, and use countable disjoint additivity.

■

Theorem 7.5.

Suppose E is measurable; then for every $\varepsilon > 0$,

1. There exists an open $O \supset E$ with $m(O \setminus E) < \varepsilon$
2. There exists a closed $F \subset E$ with $m(E \setminus F) < \varepsilon$
3. There exists a compact $K \subset E$ with $m(E \setminus K) < \varepsilon$.

Proof .

- (1): Take $\{Q_i\} \rightrightarrows E$ and set $O = \bigcup Q_i$.
- (2): Since E^c is measurable, produce $O \supset E^c$ with $m(O \setminus E^c) < \varepsilon$.
 - Set $F = O^c$, so F is closed.
 - Then $F \subset E$ by taking complements of $O \supset E^c$
 - $E \setminus F = O \setminus E^c$ and taking measures yields $m(E \setminus F) < \varepsilon$
- (3): Pick $F \subset E$ with $m(E \setminus F) < \varepsilon/2$.
 - Set $K_n = F \cap \mathbb{D}_n$, a ball of radius n about 0.
 - Then $E \setminus K_n \searrow E \setminus F$
 - Since $m(E) < \infty$, there is an N such that $n \geq N \implies m(E \setminus K_n) < \varepsilon$.

■

Lemma 7.6.

Lebesgue measure is translation and dilation invariant.

Proof .

Obvious for cubes; if $Q_i \rightrightarrows E$ then $Q_i + k \rightrightarrows E + k$, etc. ■

Theorem 7.7 (Non-Measurable Sets).

There is a non-measurable set.

Proof .

- Use AOC to choose one representative from every coset of \mathbb{R}/\mathbb{Q} on $[0, 1)$, which is countable, and assemble them into a set N
- Enumerate the rationals in $[0, 1]$ as q_j , and define $N_j = N + q_j$. These intersect trivially.
- Define $M := \coprod N_j$, then $[0, 1) \subseteq M \subseteq [-1, 2)$, so the measure must be between 1 and 3. By translation invariance, $m(N_j) = m(N)$, and disjoint additivity forces $m(M) = 0$, a contradiction. ■

Proposition 7.8 (Borel Characterization of Measurable Sets).

If E is Lebesgue measurable, then $E = H \coprod N$ where $H \in F_\sigma$ and N is null.

Useful technique: F_σ sets are Borel, so establish something for Borel sets and use this to extend it to Lebesgue.

Proof .

For every $\frac{1}{n}$ there exists a closed set $K_n \subset E$ such that $m(E \setminus K_n) \leq \frac{1}{n}$. Take $K = \bigcup_n K_n$, wlog $K_n \nearrow K$ so $m(K) = \lim_n m(K_n) = m(E)$. Take $N := E \setminus K$, then $m(N) = 0$. ■

Definition 7.8.1.

$$\limsup_n A_n := \bigcap_n \bigcup_{j \geq n} A_j = \left\{ x \mid x \in A_n \text{ for inf. many } n \right\}$$

$$\liminf_n A_n := \bigcup_n \bigcap_{j \geq n} A_j = \left\{ x \mid x \in A_n \text{ for all except fin. many } n \right\}$$

Lemma 7.9.

If A_n are all measurable, $\limsup A_n$ and $\liminf A_n$ are measurable.

Proof .

Measurable sets form a sigma algebra, and these are expressed as countable unions/intersections of measurable sets. ■

Theorem 7.10 (Borel-Cantelli).

Let $\{E_k\}$ be a countable collection of measurable sets. Then

$$\sum_k m(E_k) < \infty \implies \text{almost every } x \in \mathbb{R} \text{ is in at most finitely many } E_k.$$

Proof .

- If $E = \limsup_j E_j$ with $\sum m(E_j) < \infty$ then $m(E) = 0$.
 - If E_j are measurable, then $\limsup_j E_j$ is measurable.
 - If $\sum_j m(E_j) < \infty$, then $\sum_{j=N}^{\infty} m(E_j) \xrightarrow{N \rightarrow \infty} 0$ as the tail of a convergent sequence.
 - $E = \limsup_j E_j = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j \implies E \subseteq \bigcup_{j=k}^{\infty} E_j$ for all k
 - $E \subseteq \bigcup_{j=k}^{\infty} E_j \implies m(E) \leq \sum_{j=k}^{\infty} m(E_j) \xrightarrow{k \rightarrow \infty} 0$.
-

Lemma 7.11.

- Characteristic functions are measurable
- If f_n are measurable, so are $|f_n|$, $\limsup f_n$, $\liminf f_n$, $\lim f_n$,
- Sums and differences of measurable functions are measurable,
- Cones $F(x, y) = f(x)$ are measurable,
- Compositions $f \circ T$ for T a linear transformation are measurable,
- “Convolution-ish” transformations $(x, y) \mapsto f(x - y)$ are measurable

Proof (Convolution).

Take the cone on f to get $F(x, y) = f(x)$, then compose F with the linear transformation $T = [1, -1; 1, 0]$. ■

8 Integration

Definition 8.0.1.

$f \in L^+$ iff f is measurable and non-negative.

Useful techniques:

- Break integration domain up into disjoint annuli.
- Break real integrals up into $x < 1$ and $x > 1$.

Definition 8.0.2.

A measurable function is integrable iff $\|f\|_1 < \infty$.

Useful facts about C_c functions:

- Bounded almost everywhere
- Uniformly continuous

8.1 Convergence Theorems

Theorem 8.1 (Monotone Convergence).

If $f_n \in L^+$ and $f_n \nearrow f$ a.e., then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \quad \int f_n \longrightarrow \int f.$$

Needs to be positive and increasing.

Theorem 8.2 (Dominated Convergence).

If $f_n \in L^1$ and $f_n \longrightarrow f$ a.e. with $|f_n| \leq g$ for some $g \in L^1$, then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \quad \int f_n \longrightarrow \int f,$$

and more generally,

$$\int |f_n - f| \longrightarrow 0.$$

Positivity *not* needed.

Generalized DCT: can relax $|f_n| < g$ to $|f_n| < g_n \longrightarrow g \in L^1$.

Lemma 8.3.

If $f \in L^1$, then

$$\int |f_n - f| \longrightarrow 0 \iff \int |f_n| \longrightarrow \int |f|.$$

Proof.

Let $g_n = |f_n| - |f_n - f|$, then $g_n \longrightarrow |f|$ and

$$|g_n| = ||f_n| - |f_n - f|| \leq |f_n - (f_n - f)| = |f| \in L^1,$$

so the DCT applies to g_n and

$$\begin{aligned}\|f_n - f\|_1 &= \int |f_n - f| + |f_n| - |f| = \int |f_n| - g_n \\ &\rightarrow_{DCT} \lim \int |f_n| - \int |f|.\end{aligned}$$

■

Fatou's Lemma If $f_n \in L^+$, then

$$\begin{aligned}\int \liminf_n f_n &\leq \liminf_n \int f_n \\ \limsup_n \int f_n &\leq \int \limsup_n f_n.\end{aligned}$$

Only need positivity.

Theorem 8.4 (Tonelli).

For $f(x, y)$ **non-negative and measurable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is a **measurable** function
- $F(x) = \int f(x, y) dy$ is a **measurable** function,
- For E measurable, the slices $E_x := \{y \mid (x, y) \in E\}$ are measurable.
- $\int f = \int \int F$, i.e. any iterated integral is equal to the original.

Theorem 8.5 (Fubini).

For $f(x, y)$ **integrable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is an **integrable** function
- $F(x) := \int f(x, y) dy$ is an **integrable** function,
- For E measurable, the slices $E_x := \{y \mid (x, y) \in E\}$ are measurable.
- $\int f = \int \int f(x, y)$, i.e. any iterated integral is equal to the original

Theorem 8.6 (Fubini/Tonelli).

If any iterated integral is **absolutely integrable**, i.e. $\int \int |f(x, y)| < \infty$, then f is integrable and $\int f$ equals any iterated integral.

Corollary 8.7 (Measurable Slices).

Let E be a measurable subset of \mathbb{R}^n . Then

- For almost every $x \in \mathbb{R}^{n_1}$, the slice $E_x := \{y \in \mathbb{R}^{n_2} \mid (x, y) \in E\}$ is measurable in \mathbb{R}^{n_2} .
- The function

$$F : \mathbb{R}^{n_1} \longrightarrow \mathbb{R}$$

$$x \mapsto m(E_x) = \int_{\mathbb{R}^{n_2}} \chi_{E_x} dy$$

is measurable and

$$m(E) = \int_{\mathbb{R}^{n_1}} m(E_x) dx = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{E_x} dy dx$$

Proof (Measurable Slices).

$\implies :$

- Let f be measurable on \mathbb{R}^n .
- Then the cylinders $F(x, y) = f(x)$ and $G(x, y) = f(y)$ are both measurable on \mathbb{R}^{n+1} .
- Write $\mathcal{A} = \{G \leq F\} \cap \{G \geq 0\}$; both are measurable.

$\impliedby :$

- Let A be measurable in \mathbb{R}^{n+1} .
- Define $A_x = \{y \in \mathbb{R} \mid (x, y) \in A\}$, then $m(A_x) = f(x)$.
- By the corollary, A_x is measurable set, $x \mapsto A_x$ is a measurable function, and $m(A) = \int f(x) dx$.
- Then explicitly, $f(x) = \chi_A$, which makes f a measurable function. ■

Proposition 8.8 (Differentiating Under an Integral).

If $\left| \frac{\partial}{\partial t} f(x, t) \right| \leq g(x) \in L^1$, then letting $F(t) = \int f(x, t) dx$,

$$\begin{aligned} \frac{\partial}{\partial t} F(t) &:= \lim_{h \rightarrow 0} \int \frac{f(x, t+h) - f(x, t)}{h} dx \\ &\stackrel{\text{DCT}}{=} \int \frac{\partial}{\partial t} f(x, t) dx. \end{aligned}$$

To justify passing the limit, let $h_k \rightarrow 0$ be any sequence and define

$$f_k(x, t) = \frac{f(x, t+h_k) - f(x, t)}{h_k},$$

so $f_k \xrightarrow{\text{pointwise}} \frac{\partial}{\partial t} f$.

Apply the MVT to f_k to get $f_k(x, t) = f_k(\xi, t)$ for some $\xi \in [0, h_k]$, and show that $f_k(\xi, t) \in L_1$.

Proposition 8.9 (Swapping Sum and Integral).

If f_n are non-negative and $\sum \int |f_n| < \infty$, then $\sum \int f_n = \int \sum f_n$.

Proof .

MCT. Let $F_N = \sum_{n=1}^N f_n$ be a finite partial sum; then there are simple functions $\varphi_n \nearrow f_n$ and so $\sum_{n=1}^N \varphi_n \nearrow F_N$, so apply MCT. ■

Lemma 8.10.

If $f_k \in L^1$ and $\sum \|f_k\|_1 < \infty$ then $\sum f_k$ converges almost everywhere and in L^1 .

Proof .

Define $F_N = \sum_{k=1}^N f_k$ and $F = \lim_N F_N$, then $\|F_N\|_1 \leq \sum_{k=1}^N \|f_k\|_1 < \infty$ so $F \in L^1$ and $\|F_N - F\|_1 \rightarrow 0$ so the sum converges in L^1 . Almost everywhere convergence: ? ■

8.2 L^1 Facts

Lemma 8.11 (Translation Invariance).

The Lebesgue integral is translation invariant, i.e. $\int f(x) dx = \int f(x+h) dx$ for any h .

Proof .

- For characteristic functions, $\int_E f(x+h) = \int_{E+h} f(x) = m(E+h) = m(E) = \int_E f$ by translation invariance of measure.
- So this also holds for simple functions by linearity
- For $f \in L^+$, choose $\varphi_n \nearrow f$ so $\int \varphi_n \rightarrow \int f$.
- Similarly, $\tau_h \varphi_n \nearrow \tau_h f$ so $\int \tau_h f \rightarrow \int f$
- Finally $\left\{ \int \tau_h \varphi \right\} = \left\{ \int \varphi \right\}$ by step 1, and the suprema are equal by uniqueness of limits. ■

Lemma 8.12 (Integrals Distribute Over Disjoint Sets).

If $X \subseteq A \sqcup B$, then $\int_X f \leq \int_A f + \int_{A^c} f$ with equality iff $X = A \sqcup B$.

Lemma 8.13 (Unif Cts L^1 Functions Decay Rapidly).

If $f \in L^1$ and f is uniformly continuous, then $f(x) \xrightarrow{|x| \rightarrow \infty} 0$.

Doesn't hold for general L^1 functions, take any train of triangles with height 1 and summable areas.

Lemma 8.14 (L^1 Functions Have Small Tails).

If $f \in L^1$, then for every ε there exists a radius R such that if $A = B_R(0)^c$, then $\int_A |f| < \varepsilon$.

Proof .

Approximate with compactly supported functions. Take $g \xrightarrow{L^1} f$ with $g \in C_c$, then choose N large enough so that $g = 0$ on $E := B_N(0)^c$, then $\int_E |f| \leq \int_E |f - g| + \int_E |g|$. ■

Lemma 8.15 (L^1 Functions Have Absolutely Continuity).

$m(E) \rightarrow 0 \implies \int_E f \rightarrow 0$.

Proof .

Approximate with compactly supported functions. Take $g \xrightarrow{L^1} f$, then $g \leq M$ so $\int_E f \leq \int_E f - g + \int_E g \rightarrow 0 + M \cdot m(E) \rightarrow 0$. ■

Lemma 8.16 (L^1 Functions Are Finite a.e.).

If $f \in L^1$, then $m(\{f(x) = \infty\}) = 0$.

Proof .

Idea: Split up domain Let $A = \{f(x) = \infty\}$, then $\infty > \int f = \int_A f + \int_{A^c} f = \infty \cdot m(A) + \int_{A^c} f \implies m(A) = 0$. ■

Proposition 8.17 (Continuity in L^1).

$$\|\tau_h f - f\|_1 \xrightarrow{h \rightarrow 0} 0$$

Proof .

Approximate with compactly supported functions. Take $g \xrightarrow{L^1} f$ with $g \in C_c$.

$$\begin{aligned} \int f(x+h) - f(x) &\leq \\ \int f(x+h) - g(x+h) + \int g(x+h) - g(x) + \int g(x) - f(x) & \\ \longrightarrow 2\varepsilon + \int g(x+h) - g(x) & \\ = \int_K g(x+h) - g(x) + \int_{K^c} g(x+h) - g(x) &\longrightarrow 0, \end{aligned}$$

which follows because we can enlarge the support of g to K where the integrand is zero on K^c ,

then apply uniform continuity on K . ■

Proposition 8.18 (*Integration by Parts, Special Case*).

$$F(x) := \int_0^x f(y)dy \quad \text{and} \quad G(x) := \int_0^x g(y)dy \\ \implies \int_0^1 F(x)g(x)dx = F(1)G(1) - \int_0^1 f(x)G(x)dx.$$

Proof.

Fubini-Tonelli, and sketch region to change integration bounds. ■

Theorem 8.19 (*Lebesgue Density*).

$$A_h(f)(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(y)dy \implies \|A_h(f) - f\| \xrightarrow{h \rightarrow 0} 0.$$

Proof.

Fubini-Tonelli, and sketch region to change integration bounds, and continuity in L^1 . ■

8.3 L^p Spaces

Lemma 8.20.

The following are dense subspaces of $L^2([0, 1])$:

- Simple functions
- Step functions
- $C_0([0, 1])$
- Smoothly differentiable functions $C_0^\infty([0, 1])$
- Smooth compactly supported functions C_c^∞

Theorem 8.21 (*Dual L^p Spaces*).

For $p \neq \infty$, $(L^p)^\vee \cong L^q$.

Proof ($p=1$).

?

■

Proof ($p=2$) > Use Riesz Representation for Hilbert spaces.

For the $p = \infty$ case: $L^1 \subset (L^\infty)^\vee$, since the isometric mapping is always injective, but *never* surjective. So this containment is always proper (requires Hahn-Banach Theorem).

9 Fourier Transform and Convolution

9.1 The Fourier Transform

Definition 9.0.1 (Convolution).

$$f * g(x) = \int f(x-y)g(y)dy.$$

Definition 9.0.2 (The Fourier Transform).

$$\hat{f}(\xi) = \int f(x) e^{2\pi i x \cdot \xi} dx.$$

Lemma 9.1.

If $\hat{f} = \hat{g}$ then $f = g$ almost everywhere.

Lemma 9.2 (*Riemann-Lebesgue: Fourier transforms have small tails*).

$$f \in L^1 \implies \hat{f}(\xi) \rightarrow 0 \text{ as } |\xi| \rightarrow \infty.$$

Lemma 9.3.

If $f \in L^1$, then \hat{f} is continuous and bounded.

Proof .

- Boundedness:

$$|\hat{f}(\xi)| \leq \int |f| \cdot |e^{2\pi i x \cdot \xi}| = \|f\|_1.$$

- Continuity:

– Apply DCT to show $|\hat{f}(\xi_n) - \hat{f}(\xi)| \xrightarrow{n \rightarrow \infty} 0$.

■

Theorem (Fourier Inversion) :

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

Proof .

Idea: Fubini-Tonelli doesn't work directly, so introduce a convergence factor, take limits, and use uniqueness of limits.

■

- Take the modified integral:

$$\begin{aligned}
 I_t(x) &= \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi} e^{-\pi t^2 |\xi|^2} \\
 &= \int \widehat{f}(\xi) \varphi(\xi) \\
 &= \int f(\xi) \widehat{\varphi}(\xi) \\
 &= \int f(\xi) \widehat{g}_t(\xi - x) \\
 &= \int f(\xi) g_t(x - \xi) d\xi \\
 &= \int f(y - x) g_t(y) dy \quad (\xi = y - x) \\
 &= (f * g_t) \\
 &\longrightarrow f \text{ in } L^1 \text{ as } t \longrightarrow 0.
 \end{aligned}$$

- We also have

$$\begin{aligned}
 \lim_{t \rightarrow 0} I_t(x) &= \lim_{t \rightarrow 0} \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi} e^{-\pi t^2 |\xi|^2} \\
 &= \lim_{t \rightarrow 0} \int \widehat{f}(\xi) \varphi(\xi) \\
 &=_{DCT} \int \widehat{f}(\xi) \lim_{t \rightarrow 0} \varphi(\xi) \\
 &= \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi}
 \end{aligned}$$

- So

$$I_t(x) \longrightarrow \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi} \text{ pointwise and } \|I_t(x) - f(x)\|_1 \longrightarrow 0.$$

- So there is a subsequence I_{t_n} such that $I_{t_n}(x) \longrightarrow f(x)$ almost everywhere
- Thus $f(x) = \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi}$ almost everywhere by uniqueness of limits.

Proposition (Eigenfunction of the Fourier Transform) :

$$\begin{aligned}
 g(x) &:= e^{-\pi |x|^2} \implies \widehat{g}(\xi) = g(\xi) \quad \text{and} \\
 \widehat{g}_t(x) &= g(tx) = e^{-\pi t^2 |x|^2}, .
 \end{aligned}$$

Proposition 9.4 (Properties of the Fourier Transform).

?????.

9.2 Approximate Identities

Definition 9.4.1 (Dilation).

$$\varphi_t(x) = t^{-n} \varphi(t^{-1}x).$$

Definition 9.4.2 (Approximation to the Identity).

For $\varphi \in L^1$, the dilations satisfy $\int \varphi_t = \int \varphi$, and if $\int \varphi = 1$ then φ is an *approximate identity*.

Example: $\varphi(x) = e^{-\pi x^2}$

Theorem 9.5 (*Convolution Against Approximate Identities Converge in L^1*).

$$\|f * \varphi_t - f\|_1 \xrightarrow{t \rightarrow 0} 0.$$

Proof .

$$\begin{aligned} \|f - f * \varphi_t\|_1 &= \int f(x) - \int f(x-y)\varphi_t(y) dy dx \\ &= \int f(x) \int \varphi_t(y) dy - \int f(x-y)\varphi_t(y) dy dx \\ &= \int \int \varphi_t(y)[f(x) - f(x-y)] dy dx \\ &=_{FT} \int \int \varphi_t(y)[f(x) - f(x-y)] dx dy \\ &= \int \varphi_t(y) \int f(x) - f(x-y) dx dy \\ &= \int \varphi_t(y) \|f - \tau_y f\|_1 dy \\ &= \int_{y < \delta} \varphi_t(y) \|f - \tau_y f\|_1 dy + \int_{y \geq \delta} \varphi_t(y) \|f - \tau_y f\|_1 dy \\ &\leq \int_{y < \delta} \varphi_t(y) \varepsilon + \int_{y \geq \delta} \varphi_t(y) (\|f\|_1 + \|\tau_y f\|_1) dy \quad \text{by continuity in } L^1 \\ &\leq \varepsilon + 2\|f\|_1 \int_{y \geq \delta} \varphi_t(y) dy \\ &\leq \varepsilon + 2\|f\|_1 \cdot \varepsilon \quad \text{since } \varphi_t \text{ has small tails} \\ &\xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

■

Theorem 9.6 (*Convolutions Vanish at Infinity*).

$$f, g \in L^1 \text{ and bounded} \implies \lim_{|x| \rightarrow \infty} (f * g)(x) = 0.$$

Proof .

- Choose $M \geq f, g$.
- By small tails, choose N such that $\int_{B_N^c} |f|, \int_{B_N^c} |g| < \varepsilon$
- Note

$$|f * g| \leq \int |f(x-y)| |g(y)| dy := I.$$

- Use $|x| \leq |x-y| + |y|$, take $|x| \geq 2N$ so either

$$|x-y| \geq N \implies I \leq \int_{\{|x-y| \geq N\}} |f(x-y)| M dy \leq \varepsilon M \longrightarrow 0$$

then

$$|y| \geq N \implies I \leq \int_{\{|y| \geq N\}} M |g(y)| dy \leq M \varepsilon \longrightarrow 0.$$

■

Proposition (Young's Inequality?) :

$$\frac{1}{r} := \frac{1}{p} + \frac{1}{q} - 1 \implies \|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Corollary 9.7.

Take $q = 1$ to obtain

$$\|f * g\|_p \leq \|f\|_p \|g\|_1.$$

Corollary 9.8.

If $f, g \in L^1$ then $f * g \in L^1$.

10 Functional Analysis

Theorem (Bessel's Inequality) :

$$\left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2$$

and thus

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2.$$

Proof .

- Let $S_N = \sum_{n=1}^N \langle x, u_n \rangle u_n$

$$\begin{aligned}
\|x - S_N\|^2 &= \langle x - S_N, x - S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \langle x, S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \left\langle x, \sum_{n=1}^N \langle x, u_n \rangle u_n \right\rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^N \langle x, \langle x, u_n \rangle u_n \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^N \overline{\langle x, u_n \rangle} \langle x, u_n \rangle \\
&= \|x\|^2 + \left\| \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|^2 - 2 \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\
&= \|x\|^2 + \sum_{n=1}^N |\langle x, u_n \rangle|^2 - 2 \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\
&= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2.
\end{aligned}$$

- By continuity of the norm and inner product, we have

$$\begin{aligned}
\lim_{N \rightarrow \infty} \|x - S_N\|^2 &= \lim_{N \rightarrow \infty} \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\
\Rightarrow \left\| x - \lim_{N \rightarrow \infty} S_N \right\|^2 &= \|x\|^2 - \lim_{N \rightarrow \infty} \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\
\Rightarrow \left\| x - \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2.
\end{aligned}$$

- Then noting that $0 \leq \|x - S_N\|^2$,

$$\begin{aligned}
0 &\leq \|x\|^2 - \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \\
\Rightarrow \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 &\leq \|x\|^2 \blacksquare.
\end{aligned}$$

■

11 Extra Problems

Integration

- Show that if $f \in C^1(\mathbb{R})$ and $\lim_{x \rightarrow \infty} f(x), f'(x)$ exist, then $\lim_{x \rightarrow \infty} f'(x) = 0$.

Basics

- If f is continuous, is it necessarily the case that f' is continuous?
- If $f_n \rightarrow f$, is it necessarily the case that f'_n converges to f' (or at all)?
- Is it true that the sum of differentiable functions is differentiable?
- Is it true that the limit of integrals equals the integral of the limit?
- Is it true that a limit of continuous functions is continuous?
- Prove that uniform convergence implies pointwise convergence implies a.e. convergence, but none of the implications may be reversed.
- Show that if K is compact and F is closed with K, F disjoint then $\text{dist}(K, F) > 0$.
- Show that if $f_n \rightarrow f$ uniformly with each f_n continuous then f is continuous.
- Show that a subset of a metric space is closed iff it is complete.
- Show that if a subset of a metric space is complete and totally bounded, then it is compact.
- Show that every compact set is closed and bounded.
- Show that a uniform limit of bounded functions is bounded.
- Show that a uniform limit of continuous function is continuous.
- Show that if $f_n \rightarrow f$ pointwise, $f'_n \rightarrow g$ uniformly for some f, g , then f is differentiable and $g = f'$.

Measure Theory

- \star : Show that for $E \subseteq \mathbb{R}^n$, TFAE:
 1. E is measurable
 2. $E = H \bigcup Z$ here H is F_σ and Z is null
 3. $E = V \setminus Z'$ where $V \in G_\delta$ and Z' is null.
- Show that continuity of measure from above/below holds for outer measures.
- \star : Show that if $E \subseteq \mathbb{R}^n$ is measurable then $m(E) = \sup_{K \subseteq E \text{ compact}} m(K)$ iff for all $\varepsilon > 0$ there exists a compact $K \subseteq E$ such that $m(E) - m(K) < \varepsilon$.
- Show that a countable union of null sets is null.

Continuity

- Show that a continuous function on a compact set is uniformly continuous.

Measurability

- Show that $f = 0$ a.e. iff $\int_E f = 0$ for every measurable set E .
- \star : Show that cylinder functions are measurable, i.e. if f is measurable on \mathbb{R}^s , then $F(x, y) := f(x)$ is measurable on $\mathbb{R}^s \times \mathbb{R}^t$ for any t .
- Show that if f is a measurable function, then $f = 0$ a.e. iff $\int f = 0$.

Integrability

-
- ★: Prove that the Lebesgue integral is translation invariant, i.e. if $\tau_h(x) = x + h$ then $\int \tau_h f = \int f$.
 - ★: Prove that the Lebesgue integral is dilation invariant, i.e. if $f_\delta(x) = \frac{f(\frac{x}{\delta})}{\delta^n}$ then $\int f_\delta = \int f$.
 - ★: Prove continuity in L^1 , i.e.

$$f \in L^1 \implies \lim_{h \rightarrow 0} \int |f(x+h) - f(x)| = 0.$$

- Show that a bounded function is Lebesgue integrable iff it is measurable.
- Show that simple functions are dense in L^1 .
- Show that step functions are dense in L^1 .
- Show that smooth compactly supported functions are dense in L^1 .

Convergence

- Prove Fatou's lemma using the Monotone Convergence Theorem.
- Show that if $\{f_n\}$ is in L^1 and $\sum \int |f_n| < \infty$ then $\sum f_n$ convergence to an L^1 function and $\int \sum f_n = \sum \int f_n$.

Convolution

- Show that $f, g \in L^1 \implies f * g \in L^1$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.
- Show that $f \in L^1, g \leq M \implies f * g \leq M'$ and is uniformly continuous.
- Show that if $f, g \in L^1$ with $f \leq M, g \leq M'$, then $f * g \xrightarrow{x \rightarrow \infty} 0$.
- Show that if $f \in L^1$ and g' exists with $\frac{\partial g}{\partial x_i}$ all bounded, then $\frac{\partial}{\partial x_i} (f * g) = f * \frac{\partial g}{\partial x_i}$.
- Show that if f, g are smooth and compactly supported then $f * g$ is smooth and $f * g \xrightarrow{x \rightarrow \infty} 0$.
- ★: show that if $f, g \in L^1$, then $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.
- Is it the case that $f, g \in C_c$ implies that $f * g \in C_c$?
- Show that if $f \in L^1$ and $g \in C_c^\infty$ then $f * g$ is smooth and $f * g$ vanishes at infinity.
- Show that if $f, g \in L^1$ and g is bounded, then $\lim_{|x| \rightarrow \infty} (f * g)(x) = 0$.

Fourier Analysis

- Show that if $f \in L^1$ then \hat{f} is bounded and uniformly continuous.
- Is it the case that $f \in L^1$ implies $\hat{f} \in L^1$?
- Show that if $f, \hat{f} \in L^1$ then f is bounded, uniformly continuous, and vanishes at infinity.
 - Show that this is not true for arbitrary L^1 functions.
- Show that if $f \in L^1$ and $\hat{f} = 0$ almost everywhere then $f = 0$ almost everywhere.
 - Prove that $\hat{f} = \hat{g}$ implies that $f = g$ a.e.
- Show that if $f, g \in L^1$ then $\int \hat{f}g = \int f\hat{g}$.
 - Give an example showing that this fails if g is not bounded.
- Show that if $f \in C^1$ then f is equal to its Fourier series.

Approximate Identities

- Show that if φ is an approximate identity, then $\|f * \varphi_t - f\|_1 \xrightarrow{t \rightarrow 0} 0$.
 - Show that if additionally $|\varphi(x)| \leq c(1 + |x|)^{-n-\varepsilon}$ for some $c, \varepsilon > 0$, then this converges is almost everywhere.

- Show that if f is bounded and uniformly continuous and φ_t is an approximation to the identity, then $f * \varphi_t$ uniformly converges to f .

L_p

- Show that if $E \subseteq \mathbb{R}^n$ is measurable with $\mu(E) < \infty$ and $f \in L^p(X)$ then $\|f\|_{L^p(X)} \xrightarrow{p \rightarrow \infty} \|f\|_\infty$.
- Is it true that the converse to the DCT holds? I.e. if $\int f_n \rightarrow \int f$, is there a $g \in L^p$ such that $f_n < g$ a.e. for every n ?
- Prove continuity in L^p : If f is uniformly continuous then $\|\tau_h f - f\|_p \rightarrow 0$ as $h \rightarrow 0$ for all p .

12 Inequalities and Equalities

Proposition 12.1 (Reverse Triangle Inequality).

$$|\|x\| - \|y\|| \leq \|x - y\|.$$

Proposition 12.2 (Chebyshev's Inequality).

$$\mu(\{x : |f(x)| > \alpha\}) \leq \left(\frac{\|f\|_p}{\alpha} \right)^p.$$

Proposition 12.3 (Holder's Inequality (when surjective)).

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Application: For finite measure spaces,

$$1 \leq p < q \leq \infty \implies L^q \subset L^p \quad (\text{and } \ell^p \subset \ell^q).$$

Proof (Holder's Inequality).

Fix p, q , let $r = \frac{q}{p}$ and $s = \frac{r}{r-1}$ so $r^{-1} + s^{-1} = 1$. Then let $h = |f|^p$:

■

$$\|f\|_p^p = \|h \cdot 1\|_1 \leq \|1\|_s \|h\|_r = \mu(X)^{\frac{1}{s}} \|f\|_q^{\frac{q}{r}} \implies \|f\|_p \leq \mu(X)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q.$$

Note: doesn't work for ℓ_p spaces, but just note that $\sum |x_n| < \infty \implies x_n < 1$ for large enough n , and thus $p < q \implies |x_n|^q \leq |x_n|^p$.

Proposition 12.4 (Cauchy-Schwarz Inequality).

$$|\langle f, g \rangle| = \|fg\|_1 \leq \|f\|_2 \|g\|_2 \quad \text{with equality} \iff f = \lambda g.$$

Note: Relates inner product to norm, and only happens to relate norms in L^1 .

Proof .
?



Proposition 12.5 (Minkowski's Inequality:).

$$1 \leq p < \infty \implies \|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Note: does not handle $p = \infty$ case. Use to prove L^p is a normed space.

Proposition 12.6 (Young's Inequality*).

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \implies \|f * g\|_r \leq \|f\|_p \|g\|_q.$$

]

Application: Some useful specific cases:

$$\begin{aligned} \|f * g\|_1 &\leq \|f\|_1 \|g\|_1 \\ \|f * g\|_p &\leq \|f\|_1 \|g\|_p, \\ \|f * g\|_\infty &\leq \|f\|_2 \|g\|_2 \\ \|f * g\|_\infty &\leq \|f\|_p \|g\|_q. \end{aligned}$$

Proposition 12.7 (? Inequality).

$$(a + b)^p \leq 2^p(a^p + b^p).$$

Proposition 12.8 (Bezel's Inequality:).

For $x \in H$ a Hilbert space and $\{e_k\}$ an orthonormal sequence,

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

Note: this does not need to be a basis.

Proposition 12.9 (*Parseval's Identity*).

Equality in Bessel's inequality, attained when $\{e_k\}$ is a *basis*, i.e. it is complete, i.e. the span of its closure is all of H .

12.1 Less Explicitly Used Inequalities

Proposition 12.10 (*AM-GM Inequality*).

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Proposition 12.11 (*Jensen's Inequality*).

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$