Title

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1 Spring 2017

1.1 1

Concepts used:

- Definition: A is *nowhere dense* $\iff$ every interval I contains a subinterval $S \subseteq A^c$.
  - Equivalently, the interior of the closure is empty, $\left(\overline{K}\right)^\circ = \emptyset$.

Solution

Claim: K **is compact**.

- It suffices to show that $K^c := [0,1] \setminus K$ is open; Then K will be a closed and bounded subset of $\mathbb{R}$ and thus compact by Heine-Borel.
- Strategy: write K^c as the union of open balls (since these form a basis for the Euclidean topology on $\mathbb{R}$).
  - Do this by showing every point $x \in K^c$ is an interior point, i.e. x admits a neighborhood N_x such that $N_x \subseteq K^c$.
- Identify K^c as the set of real numbers in $[0,1]$ whose decimal expansion **does** contain a 4.
  - We will show that there exists a neighborhood small enough such that all points in it contain a 4 in their decimal expansions.

- Let $x \in K^c$, suppose a 4 occurs as the kth digit, and write

$$x = 0.d_1 d_2 \cdots d_{k-1} \; 4 \; d_{k+1} \cdots = \left(\sum_{j=1}^{k} d_j 10^{-j} \right) + \left(4 \cdot 10^{-k} \right) + \left(\sum_{j=k+1}^{\infty} d_j 10^{-j} \right).$$

- Set $r_x < 10^{-k}$ and let $y \in [0,1] \bigcap B_{r_x}(x)$ be arbitrary and write

$$y = \sum_{j=1}^{\infty} c_j 10^{-j}.$$

- Thus $|x-y| < r_x < 10^{-k}$, and the first k digits of x and y must agree:
  - We first compute the difference:

$$x - y = \sum_{i=1}^{\infty} d_j 10^{-j} - \sum_{i=1}^{\infty} c_j 10^{-j} = \sum_{i=1}^{\infty} (d_j - c_j) 10^{-j}$$

  - Thus (claim)

$$|x-y| \leq \sum_{j=1}^{\infty} |d_j - c_j| 10^{j} < 10^{-k} \iff |d_j - c_j| = 0 \quad \forall j \leq k.$$

  - Otherwise we can note that any term $|d_j - c_j| \geq 1$ and there is a contribution to $|x-y|$ of at least $1 \cdot 10^{-j}$ for some $j < k$, whereas

$$j < k \iff 10^{-j} > 10^{-k},$$

    a contradiction.
- This means that for all $j \leq k$ we have $d_j = c_j$, and in particular $d_k = 4 = c_k$, so y has a 4 in its decimal expansion.
- But then $K^c = \bigcup_x B_{r_x}(x)$ is a union of open sets and thus open.

Claim: K **is nowhere dense and** $m(K) = 0$**:**

- Strategy: Show $\left(\overline{K}\right)^\circ = \emptyset$.
- Since K is closed, $\overline{K} = K$, so it suffices to show that K does not properly contain any interval.
- It suffices to show $m(K^c) = 1$, since this implies $m(K) = 0$ and since any interval has strictly positive measure, this will mean K can not contain an interval.
- As in the construction of the Cantor set, let
  - K_0 denote $[0,1]$ with 1 interval $\left(\frac{4}{10}, \frac{5}{10}\right)$ of length $\frac{1}{10}$ deleted, so

$$m(K_0^c) = \frac{1}{10}.$$

- K_1 denote K_0 with 9 intervals $\left(\frac{1}{100}, \frac{5}{100}\right)$, $\left(\frac{14}{100}, \frac{15}{100}\right), \cdots \left(\frac{94}{100}, \frac{95}{100}\right)$ of length $\frac{1}{100}$ deleted, so

$$m(K_1^c) = \frac{1}{10} + \frac{9}{100}.$$

- K_n denote K_{n-1} with 9^n such intervals of length $\frac{1}{10^{n+1}}$ deleted, so

$$m(K_n^c) = \frac{1}{10} + \frac{9}{100} + \cdots + \frac{9^n}{10^{n+1}}.$$

- Then compute

$$m(K^c) = \sum_{j=0}^{\infty} \frac{9^n}{10^{n+1}} = \frac{1}{10} \sum_{j=0}^{\infty} \left(\frac{9}{10}\right)^n = \frac{1}{10}\left(\frac{1}{1 - \frac{9}{10}}\right) = 1.$$

Claim: K **has no isolated points**:

- A point $x \in K$ is isolated iff there there is an open ball $B_r(x)$ containing x such that $B_r(x) \subsetneq K^c$.
  - So every point in this ball **should** have a 4 in its decimal expansion.
- Strategy: show that if $x \in K$, every neighborhood of x intersects K.
- Note that $m(K_n) = \left(\frac{9}{10}\right)^n \overset{n \longrightarrow \infty}{\Longrightarrow} 0$
- Also note that we deleted open intervals, and the endpoints of these intervals are never deleted.
  - Thus endpoints of deleted intervals are elements of K.
- Fix x. Then for every ε, by the Archimedean property of $\mathbb{R}$, choose n such that $\left(\frac{9}{10}\right)^n < \varepsilon$.
- Then there is an endpoint x_n of some deleted interval I_n satisfying

$$|x - x_n| \leq \left(\frac{9}{10}\right)^n < \varepsilon.$$

- So every ball containing x contains some endpoint of a removed interval, and thus an element of K.

1.2 2

Concepts used:

- Absolute continuity of measures: $\lambda \ll \mu \iff E \in \mathcal{M}, \mu(E) = 0 \implies \lambda(E) = 0$.
- Radon-Nikodym: if $\lambda \ll \mu$, then there exists a measurable function $\frac{\partial \lambda}{\partial \mu} := f$ where $\lambda(E) = \int_E f \, d\mu$.
- Chebyshev's inequality:

$$A_c := \left\{ x \in X \;\middle|\; |f(x)| \geq c \right\} \implies \mu(A_c) \leq c^{-p} \int_{A_c} |f|^p \, d\mu \quad \forall 0 < p < \infty.$$

1.2.1 a

- Strategy: use approximation by simple functions to show absolute continuity and apply Radon-Nikodym
- Claim: $\lambda \ll \mu$, i.e. $\mu(E) = 0 \implies \lambda(E) = 0$.
  - Note that if this holds, by Radon-Nikodym, $f = \dfrac{\partial \lambda}{\partial \mu} \implies d\lambda = f d\mu$, which would yield

$$\int g \, d\lambda = \int gf \, d\mu.$$

- So let E be measurable and suppose $\mu(E) = 0$.
- Then

$$\lambda(E) \coloneqq \int_E f \, d\mu = \lim_{n \longrightarrow \infty} \left\{ \int_E s_n \, d\mu \;\middle|\; s_n \coloneqq \sum_{j=1}^{\infty} c_j \mu(E_j),\, s_n \nearrow f \right\}$$

  where we take a sequence of simple functions increasing to f.
- But since each $E_j \subseteq E$, we must have $\mu(E_j) = 0$ for any such E_j, so every such s_n must be zero and thus $\lambda(E) = 0$.

What is the final step in this approximation?

1.2.2 b

- Set $g(x) = x^2$, note that g is positive and measurable.
- By part (a), there exists a positive f such that for any $E \subseteq \mathbb{R}$,

$$\int_E g \, dm = \int_E gf \, d\mu$$

  - The LHS is zero by assumption and thus so is the RHS.
  - $m \ll \mu$ by construction.
  - Note that gf is positive.
- Define $A_k = \left\{ x \in X \;\middle|\; gf \cdot \chi_E > \dfrac{1}{k} \right\}$, for $k \in \mathbb{Z}^{\geq 0}$
- Then by Chebyshev with $p = 1$, for every k we have

$$\mu(A_k) \leq k \int_E gf \, d\mu = 0$$

- Then noting that $A_k \searrow A \coloneqq \left\{ x \in E \;\middle|\; x^2 > 0 \right\} = \{g > 0\}$, we have $\mu(A) = 0$.
- Since gf is positive, we have

$$x \in E \iff gf\chi_E(x) > 0 \iff x \in A$$

  so $E = A$ and $\mu(E) = \mu(A)$.
- But $m \ll \mu$ and $\mu(E) = 0$, so we can conclude that $m(E) = 0$.

1.3 3

1.3.1 a

Letting $x_n := \frac{1}{n}$, we have

$$\sum_{k=1}^{\infty} |f_k(x)| \geq |f_n(x_n)| = \left| ae^{-ax} - be^{-bx} \right| := M.$$

In particular, $\sup_x |f_n(x)| \not\longrightarrow 0$, so the terms do not go to zero and the sum can not converge.

1.3.2 b

?

1.4 4

Switching to polar coordinates and integrating over a half-circle contained in I^2, we have

$$\int_{I^2} f \geq \int_0^{\pi} \int_0^1 \frac{\cos(\theta)\sin(\theta)}{r^2} \; dr \; d\theta = \infty,$$

so f is not integrable.

1.5 5

> See https://math.stackexchange.com/questions/507263/prove-that-c1a-b-with-the-c1-norm-is-a-banach-space

This is clearly a norm, which we'll write $\|\cdot\|_u$

Let f_n be a Cauchy sequence and define a candidate limit $f(x) = \lim_n f_n(x)$.

Then noting that $\|f_n\|_\infty, \|f_n'\|\infty \leq \|f_n\|_u < \infty$, both f_n, f_n are Cauchy sequences in $C^0([a,b], \|\cdot\|_\infty)$, which is a Banach space.

So $f_n \longrightarrow f$ uniformly, and $f_n' \longrightarrow g$ uniformly for some g, and moreover $f, g \in C^0([a,b])$.

We thus have

$$\begin{aligned} f_n(x) - f_n(a) &\xrightarrow{u} f(x) - f(a) \\ \int_a^x f_n' &\xrightarrow{u} \int_a^x g, \end{aligned}$$

and by the FTC, the left-hand sides are equal, and by uniqueness of limits so are the right-hand sides, so $f' = g$.

Since $f, f' \in C^0([a,b])$, they are bounded, and so $\|f\|_u < \infty$. This means that $\|f_n - f\|_u \longrightarrow 0$, so f_n converges to f, which is in the same space.

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