Solutions to Mike's Compendium of Problems

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Contents

1 Preface

Note: usually (1/3) of the qual problems relate to point set (around 8 problems).

1.1 Problems to Revisit

- 4
- 6 (Without Heine-Borel)
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2 1: Point-Set

2.1 2

See Munkres p.164, especially for (ii).

i. See definitions in review doc.

ii. Direct proof:

- Let $\left\{U_i \;\middle|\; j \in J\right\} \rightrightarrows X$; then $0 \in U_j$ for some $j \in J$.
- In the subspace topology, U_i is given by some $V \in \tau(\mathbb{R})$ such that $V \bigcap X = U_i$
  - A basis for the subspace topology on $\mathbb{R}$ is open intervals, so write V as a union of open intervals $V = \displaystyle\bigcup_{k \in K} I_k$.
  - Since $0 \in U_j$, $0 \in I_k$ for some k.
- Since I_k is an interval, it contains infinitely many points of the form $x_n = \dfrac{1}{n} \in X$
- Then $I_k \bigcap X \subset U_j$ contains infinitely many such points.
- So there are only *finitely* many points in $X \setminus U_j$, each of which is in $U_{j(n)}$ for some $j(n) \in J$ depending on n.
- So U_j and the *finitely* many $U_{j(n)}$ form a finite subcover of X. ■

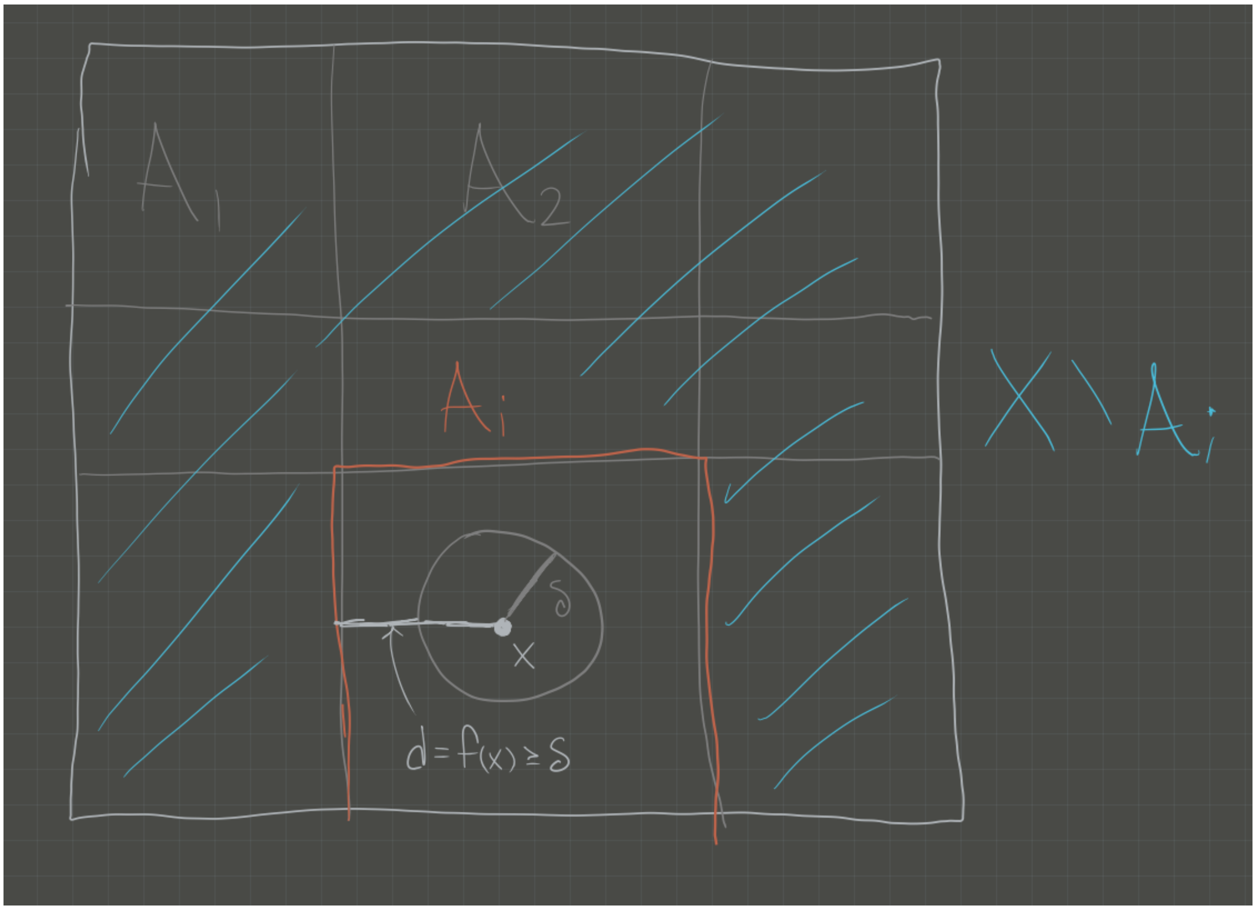
iii. Todo: Need direct proof.

2.2 4

Statement: show that the *Lebesgue number* is well-defined for compact metric spaces.

Note: this is a question about the *Lebesgue Number*. See Wikipedia for detailed proof.

- Write $U = \left\{U_i \;\middle|\; i \in I\right\}$, then $X \subseteq \displaystyle\bigcup_{i \in I} U_i$. Need to construct a $\delta > 0$.
- By compactness of X, choose a finite subcover $U_1, \cdots, U_n$.
- Define the distance between a point x and a set $Y \subset X$: $d(x, Y) = \displaystyle\inf_{y \in Y} d(x, y)$.
  - **Claim**: the function $d(\,\cdot\,, Y) : X \longrightarrow \mathbb{R}$ is continuous for a fixed set.
  - Proof: Todo, not obvious.



- Define a function

$$f : X \longrightarrow \mathbb{R}$$
$$x \mapsto \frac{1}{n}\sum_{i=1}^{n} d(x, X \setminus U_i).$$

  - Note this is a sum of continuous functions and thus continuous.

- **Claim**:

$$\delta := \inf_{x\in X} f(x) = \min_{x\in X} f(x) = f(x_{\min}) > 0$$

  suffices.
  - That the infimum is a minimum: f is a continuous function on a compact set, apply the extreme value theorem: it attains its minimum.
  - That $\delta > 0$: otherwise, $\delta = 0 \implies \exists x_0$ such that $d(x_0, X \setminus U_i) = 0$ for all i.
    * Forces $x_0 \in X \setminus U_i$ for all i, but $X \setminus \bigcup U_i = \emptyset$ since the U_i cover X.
  - That it satisfies the Lebesgue condition:

$$\forall x \in X, \exists i \quad \text{such that} \quad B_\delta(x) \subset U_i$$

    * Let $B_\delta(x) \ni x$; then by minimality $f(x) \geq \delta$.
    * Thus it can *not* be the case that $d(x, X \setminus U_i) < \delta$ for *every* i, otherwise

$$f(x) \leq \frac{1}{n}(\delta + \cdots + \delta) = \frac{n\delta}{n} = \delta$$

  - $*$ So there is some particular i such that $d(x, X \setminus U_i) \geq \delta$.
  - $*$ But then $B_\delta \subseteq U_i$ as desired.

2.3 6

Facts used:

- Cantor's Intersection Theorem
- Bases for standard topology on $\mathbb{R}$.
- Definition of compactness

- Toward a contradiction, let $\{U_\alpha\} \rightrightarrows [0,1]$ be an open cover with no finite subcover.
- Then either $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$ has no finite subcover; WLOG assume it is $[0, \frac{1}{2}]$.
- Then either $[0, \frac{1}{4}]$ or $[\frac{1}{4}, \frac{1}{2}]$ has no finite subcover
- Inductively defining $[a_n, b_n]$ this way yields a sequence of closed, bounded, nested intervals (each with no finite subcover) with $\operatorname{diam}([a_n, b_n]) \leq \frac{1}{2^n} \longrightarrow 0$, so Cantor's Nested Interval theorem applies and the intersection contains exactly one point $p \in [0,1]$.
- Since $p \in [0,1]$, $p \in U_\alpha$ for some α.
- Since a basis for $\tau(\mathbb{R})$ is given by open intervals, we can find an $\varepsilon > 0$ such that $(p-\varepsilon, p+\varepsilon) \subseteq U_\alpha$
- Then if $\frac{1}{2^N} < \varepsilon$, for $n \geq N$ we have

$$[a_n, b_n] \subseteq (p - \varepsilon, p + \varepsilon) \subseteq U_\alpha.$$

- But then $U_\alpha \rightrightarrows [a_n, b_n]$, yielding a finite subcover of $[a_n, b_n]$, a contradiction.

2.4 8

Topic: proof of the tube lemma.

Statement: show $X, Y \in \mathrm{Top}_{\text{compact}} \iff X \times Y \in \mathrm{Top}_{\text{compact}}$

2.4.1 Proof 1

$\Longleftarrow$:

- By universal properties, the product $X \times Y$ is equipped with continuous projections
- The continuous image of a compact set is compact, and $\pi_1(X \times Y) = X, p_2(X \times Y) = Y$
- So X, Y are compact.

$\Longrightarrow$:

- Let $\left\{U_j \;\middle|\; j \in J\right\} \rightrightarrows X \times Y$.
- Fix $x_0 \in X$, the slice $\{x_0\} \times Y$ is compact and can be covered by finitely many elements $\left\{U_j \;\middle|\; j \leq m\right\} \rightrightarrows \{x_0\} \times Y$.
  - Sum: write $N = \bigcup_{j=1}^{m} U_j$; then $\{x_0\} \times Y \subset N$.

  - Apply the tube lemma to N: produce $\{x_0\} \times Y \in W \times Y \subset N$; then $\left\{U_j \;\middle|\; j \le m\right\} \rightrightarrows W \times Y$.
- Now let $x \in X$ vary: for each $x \in X$, produce $W_x \times Y$ as above, then $\left\{W_x \times Y \;\middle|\; x \in X\right\} \rightrightarrows X$.
  - By above argument, every tube $W_x \times Y$ can be covered by *finitely* many U_j.
- Since $\left\{W_x \;\middle|\; x \in X\right\} \rightrightarrows X$ and X is compact, produce a finite subset $\left\{W_k \;\middle|\; k \le m'\right\} \rightrightarrows X$.
- Then $\left\{W_k \times Y \;\middle|\; k \le m'\right\} \rightrightarrows X \times Y$; the claim is that it is a finite cover.
  - Finitely many k
  - For each k, the tube $W_k \times Y$ is covered by finitely by U_j
  - And finite $\times$ finite $=$ finite. ■

Shorter mnemonic:

> ***19.U*** It is sufficient to consider a cover consisting of elementary sets. Since Y is compact, each fiber $x \times Y$ has a finite subcovering $\{U_i^x \times V_i^x\}$. Put $W^x = \cap U_i^x$. Since X is compact, the cover $\{W^x\}_{x \in X}$ has a finite subcovering W^{x_j}. Then $\{U_i^{x_j} \times V_i^{x_j}\}$ is the required finite subcovering.

2.4.2 Proof 2

Let π_X, π_Y denote the canonical projections, which we can note are continuous and preserve open sets.

$\implies$: Suppose $X \times Y$ is compact, and let $\{U_\alpha\}, \{V_\beta\}$ be open covers of X and Y respectively.

Let $T_{\alpha\beta} = U_\alpha \times V_\beta$; then $\{T_{\alpha\beta}\}$ is an open cover of $X \times Y$. So there is a finite subcover $\{T_{ij}\}$, $\{\pi_X(T_{ij})\}$ is an open cover of X, and similarly for Y. So both X, Y are compact.

$\impliedby$: Suppose X and Y are compact, and let $U_\alpha \rightrightarrows X \times Y$ be an open cover. Let $\pi_Y : X \times Y \longrightarrow Y$ be the canonical projection; then $\{\pi_Y(U_\alpha)\} \rightrightarrows Y$ and by compactness of Y there is a finite subcover of the form $\left\{\pi_Y(U_i) \;\middle|\; 1 \le i \le n\right\}$. Then $\{V_{x,i} := \{x\} \times U_i\}$ is an open cover of $\{x\} \times Y$ for any fixed x.

So if we fix an $x \in X$, we can let $V_{x,i} \rightrightarrows \{x\} \times Y$ be any finite subcollection covering this slice. By the Tube Lemma, there is an open set W_x such that $\{x\} \times Y \subset W_x \times Y \subset \bigcup V_{x,i} = \{x\} \times Y$.

Then $\{W_x\} \rightrightarrows X$ as x varies is an open cover of X, and by compactness of X, there are finitely many $x_j \in X$ such that $W_{x_j} \rightrightarrows X$. But then $X \times Y = \bigcup_j W_{x_j} \times Y = \bigcup_j \bigcup_i W_{x_j} \times V_{x_j,i} \subset \bigcup_\alpha U_\alpha$ is a finite cover.

2.4.3 Proof of Tube Lemma (Todo: Check)

Proof of Tube Lemma:

- Let $\left\{U_j \times V_j \;\middle|\; j \in J\right\} \rightrightarrows X \times Y$.
- Fix a point $x_0 \in X$, then $\{x_0\} \times Y \subset N$ for some open set N.
- By the tube lemma, there is a $U^x \subset X$ such that the tube $U^x \times Y \subset N$.

- Since $\{x_0\} \times Y \cong Y$ which is compact, there is a finite subcover $\left\{ U_j \times V_j \;\middle|\; j \leq n \right\} \rightrightarrows \{x_0\} \times Y$.
- "Integrate the X": write

$$W = \bigcap_{j=1}^{n} U_j,$$

  then $x_0 \in W$ and W is a finite intersection of open sets and thus open.
- Claim: $\left\{ U_j \times V_j \;\middle|\; j \leq n \right\} \rightrightarrows W \times Y$
  - Let $(x, y) \in W \times Y$; want to show $(x, y) \in U_j \times V_j$ for some $j \leq n$.
  - Then $(x_0, y) \in \{x_0\} \times Y$ is on the same horizontal line
  - $(x_0, y) \in U_j \times V_j$ for some j by construction
  - So $y \in V_j$ for this j
  - Since $x \in W$, $x \in U_j$ for *every* j, thus $x \in U_j$.
  - So $(x, y) \in U_j \times V_j$

2.5 9

2.6 10

2.6.1 Proof 1

X is connected:

- Write $X = L \coprod G$ where $L = \{0\} \times [-1, 1]$ and $G = \left\{ \Gamma(\sin(x)) \;\middle|\; x \in (0, 1] \right\}$ is the graph of $\sin(x)$.
- $L \cong [0, 1]$ which is connected
  - Claim: Every interval is connected (todo)
- Claim: G is connected (i.e. as the graph of a continuous function on a connected set)
  - The function

$$\begin{aligned} f : (0, 1] &\longrightarrow [-1, 1] \\ x &\mapsto \sin(x) \end{aligned}$$

    is continuous (how to prove?)
  - Products of continuous functions are continuous iff all of the components are continuous.
  - Claim: The diagonal map $\Delta : Y \longrightarrow Y \times Y$ where $\Delta(t) = (t, t)$ is continuous for any Y since $\Delta = (\mathrm{id}, \mathrm{id})$
    * Product of identity functions, which are continuous.
  - The composition of continuous function is continuous, therefore

$$\begin{aligned} F : (0, 1] \xrightarrow{\Delta} (0, 1]^2 &\xrightarrow{(\mathrm{id}, f)} (0, 1] \times [-1, 1] \\ t \mapsto (t, t) &\mapsto (t, f(t)) \end{aligned}$$

  - Then $G = F((0, 1])$ is the continuous image of a connected set and thus connected.
- Claim: X is connected
  - Suppose there is a disconnecting cover $X = A \coprod B$ such that $\overline{A} \bigcap B = A \bigcap \overline{B} = \emptyset$ and $A, B \neq \emptyset$.
  - WLOG let $(x, \sin(x)) \in B$ for $x > 0$ (otherwise just relabeling A, B)
  - Claim: $B = G$

    - It can't be the case that A intersects G: otherwise

$$X = A\coprod B \implies G = (A\bigcap G)\coprod(B\bigcap V)$$

      disconnects G. So $A\bigcap G = \emptyset$, forcing $A \subseteq L$
    - Similarly L can not be disconnected, so $B\bigcap L = \emptyset$ forcing $B \subset G$
    - So $A \subset L$ and $B \subset G$, and since $X = A\coprod B$, this forces $A = L$ and $B = G$.
  - But any open set U in the subspace topology $L \subset \mathbb{R}^2$ (generated by open balls) containing $(0,0) \in L$ is the restriction of a ball $V \subset \mathbb{R}^2$ of radius $r > 0$, i.e. $U = V\bigcap X$.
    - But any such ball contains points of G:

$$n \gg 0 \implies \frac{1}{n\pi} < r \implies \exists g \in G \text{ s.t. } g \in U.$$

    - So $U\bigcap L\bigcap G \neq \emptyset$, contradicting $L\bigcap G = \emptyset$.
- Claim: X is *not* path-connected.
  - Todo: "can't get from L to G in finite time".
  - Toward a contradiction, choose a continuous function $f : I \longrightarrow X$ with $f(0) \in G$ and $f(1) \in L$.
    - Since $L \cong [0,1]$, use path-connectedness to create a path $f(1) \longrightarrow (0,1)$
    - Concatenate paths and reparameterize to obtain $f(1) = (0,1) \in L \subset \mathbb{R}^2$.
  - Let $\varepsilon = \frac{1}{2}$; by continuity there exists a $\delta \in I$ such that

$$t \in B_\delta(1) \subset I \implies f(t) \in B_\varepsilon(\mathbf{0}) \in X$$

  - Using the fact that $[1-\delta, 1]$ is connected, $f([1-\delta,1]) \subset X$ is connected.
  - Let $f(1-\delta) = \mathbf{x}_0 = (x_0, y_0) \subset X \subset \mathbb{R}^2$.
  - Define a composite map

$$F : [0,1] \longrightarrow \mathbb{R}F \qquad := \mathfrak{p}_{x\text{-axis}} \circ f.$$

    - F is continuous as a composition of continuous functions.
  - Then $F([1-\delta,1]) \subset \mathbb{R}$ is connected and thus must be an interval (a,b)
  - Since $f(1) = \mathbf{0}$ which has x-component zero, $[0,b] \subset (a,b)$.
  - Since $f(1-\delta) = \mathbf{x}$, $F(\mathbf{x}) = x_0$ and this $[0, x_0] \subset (a,b)$.
  - Thus for all $x \in (0, x_0]$ there exists a $t \in [1-\delta, 1]$ such that $f(t) = (x, \sin\left(\frac{1}{x}\right))$.
  - Now toward the contradiction, choose $x = \dfrac{1}{2n\pi - \pi/2} \in \mathbb{R}$ with n large enough such that $x \in (0, x_0)$.
    - Note that $\sin\left(\frac{1}{x}\right) = -1$ by construction.
    - Apply the previous statement: there exists a t such that $f(t) = (x, \sin\left(\frac{1}{x}\right)) = (x, -1)$.
    - But then

$$\|f(t) - f(x)\| = \|(x,-1) - (0,1)\| = \|(x,2)\| > \frac{1}{2},$$

      contradicting continuity of f.

2.6.2 Proof 2?

Let $X = A \bigcup B$ with $A = \left\{(0, y) \;\middle|\; y \in [-1, 1]\right\}$ and $B = \left\{(x, \sin(1/x)) \;\middle|\; x \in (0, 1]\right\}$. Since B is the graph of a continuous function, which is always connected. Moreover, $X = \overline{A}$, and the closure of a connected set is still connected.

> Alternative direct argument: the subspace $X' = B \bigcup \{\mathbf{0}\}$ is not connected. If it were, write $X' = U \coprod V$, where wlog $\mathbf{0} \in U$. Then there is an open such that $\mathbf{0} \in N_r(\mathbf{0}) \subset U$. But any neighborhood about zero intersects B, so we must have $V \subset B$ as a strict inclusion. But then $U \bigcap B$ and V disconnects B, a connected set, which is a contradiction.

To see that X is not path-connected, suppose toward a contradiction that there is a continuous function $f : I \longrightarrow X \subset \mathbb{R}^2$. In particular, f is continuous at $\mathbf{0}$, and so

$$\forall \varepsilon \quad \exists \delta \;\middle|\; \|\mathbf{x}\| < \delta \implies \|f(\mathbf{x})\| < \varepsilon.$$

where the norm is the standard Euclidean norm.

However, we can pick $\varepsilon < 1$, say, and consider points of the form $\mathbf{x}_n = (\frac{1}{2n\pi}, 0)$. In particular, we can pick n large enough such that $\|\mathbf{x}_n\|$ is as small as we like, whereas $\|f(\mathbf{x}_n)\| = 1 > \varepsilon$ for all n, a contradiction.

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Consider the (continuous) projection $\pi : \mathbb{R}^2 \longrightarrow \mathbb{RP}^1$ given by $(x, y) \mapsto [y/x, 1]$ in homogeneous coordinates. (I.e. this sends points to lines through the origin with rational slope).

Note that the image of π is $\mathbb{RP}^1 \setminus \{\infty\}$, which is homeomorphic to $\mathbb{R}$.

If we now define $f = \pi|_X$, we have $f(X) \twoheadrightarrow \mathbb{Q} \subset \mathbb{R}$. If X were connected, then $f(X)$ would also be connected, but $\mathbb{Q} \subset \mathbb{R}$ is disconnected, a contradiction.

2.8 12 (Todo: Not Finished)

- Using the fact that $[0, \infty) \subset \mathbb{R}$ is Hausdorff, any retract must be closed, so any closed interval $[\varepsilon, N]$ for $0 \leq \varepsilon \leq N \leq \infty$.
  - Note that $\varepsilon = N$ yields all one point sets $\{x_0\}$ for $x_0 \geq 0$.
- No finite discrete sets occur, since the retract of a connected set is connected.
- ?

2.9 14

2.9.1 Proof 1

- Take two connected sets X, Y; then there exists $p \in X \bigcap Y$.
- Toward a contradiction: write $X \bigcup Y = A \coprod B$ with both $A, B \subset A \coprod B$ open.
- Since $p \in X \bigcup Y = A \coprod B$, WLOG $p \in A$. We will show B must be empty.

- Claim: $A \bigcap X$ is clopen in X.
  - $A \bigcap X$ is open in X: ?
  - $A \bigcap X$ is closed in X: ?
- The only clopen sets of a connected set are empty or the entire thing, and since $p \in A$, we must have $A \bigcap X = X$.
- By the same argument, $A \bigcap Y = Y$.
- So $A \bigcap \left(X \bigcup Y \right) = \left(A \bigcap X \right) \bigcup \left(A \bigcap Y \right) = X \bigcup Y$
- Since $A \subset X \bigcup Y$, $A \bigcap \left(X \bigcup Y \right) = A$
- Thus $A = X \bigcup Y$, forcing $B = \emptyset$.

2.9.2 Proof 2?

Let $X := \bigcup_\alpha X_\alpha$, and let $p \in \bigcap X_\alpha$. Suppose toward a contradiction that $X = A \coprod B$ with A, B nonempty, disjoint, and relatively open as subspaces of X. Wlog, suppose $p \in A$, so let $q \in B$ be arbitrary.

Then $q \in X_\alpha$ for some α, so $q \in B \bigcap X_\alpha$. We also have $p \in A \bigcap X_\alpha$.

But then these two sets disconnect X_α, which was assumed to be connected – a contradiction.

2.10 16

2.10.1 Proof 1

Topic: closure and connectedness in the subspace topology. See Munkres p.148

- $S \subset X$ is **not** connected if S with the subspace topology is not connected.
  - I.e. there exist $A, B \subset S$ such that
    - $A, B \neq \emptyset$,
    - $A \bigcap B = \emptyset$,
    - $A \coprod B = S$.
- Or equivalently, there exists a nontrivial $A \subset S$ that is clopen in S.

Show stronger statement: this is an iff.

$\implies$:

- Suppose S is not connected; we then have sets $A \bigcup B = S$ from above and it suffices to show $\mathrm{cl}_Y(A) \bigcap B = A \bigcap \mathrm{cl}_X(B) = \emptyset$.
- A is open by assumption and $Y \setminus A = B$ is closed in Y, so A is clopen.
- Write $\mathrm{cl}_Y(A) := \mathrm{cl}_X(A) \bigcap Y$.
- Since A is closed in Y, $A = \mathrm{cl}_Y(A)$ by definition, so $A = \mathrm{cl}_Y(A) = \mathrm{cl}_X(A) \bigcap Y$.
- Since $A \bigcap B = \emptyset$, we then have $\mathrm{cl}_Y(A) \bigcap B = \emptyset$.
- The same argument applies to B, so $\mathrm{cl}_Y(B) \bigcap A = \emptyset$.

$\impliedby$:

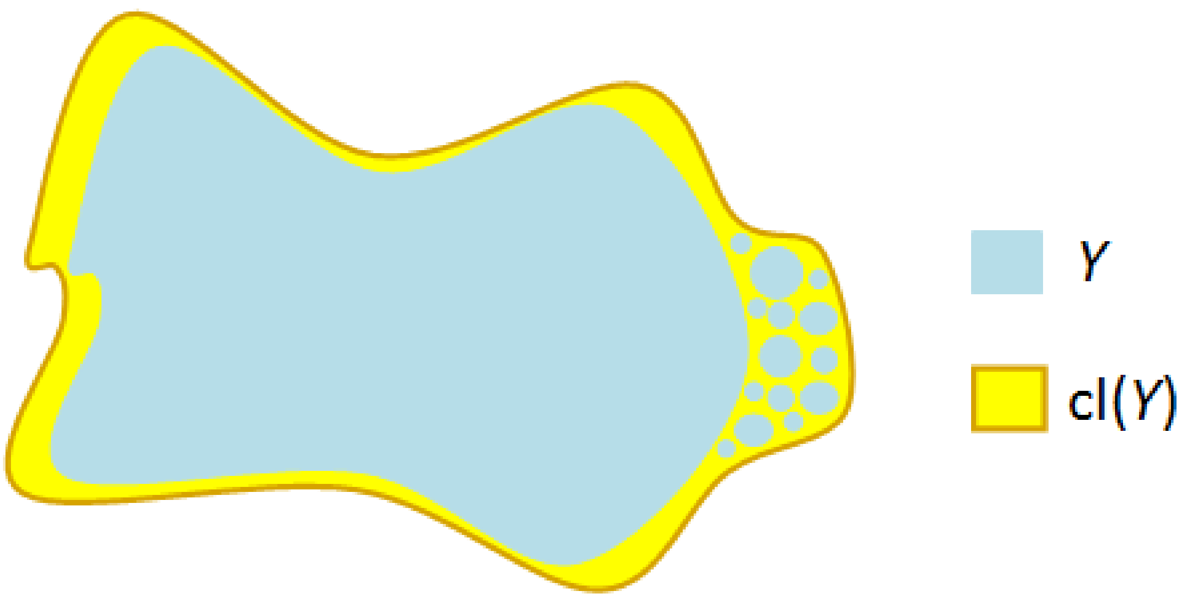
- Suppose displayed condition holds; given such A, B we will show they are clopen in Y.

- Since $\mathrm{cl}_Y(A) \bigcap B = \emptyset$, (claim) we have $\mathrm{cl}_Y(A) = A$ and thus A is closed in Y.
  - Why?

$$\begin{aligned}
\mathrm{cl}_Y(A) &\coloneqq \mathrm{cl}_X(A) \bigcap Y \\
&= \mathrm{cl}_X(A) \bigcap \left(A \coprod B\right) \\
&= \left(\mathrm{cl}_X(A) \bigcap A\right) \coprod \left(\mathrm{cl}_X(A) \bigcap B\right) \\
&= A \coprod \left(\mathrm{cl}_X(A) \bigcap B\right) \quad \text{since } A \subset \mathrm{cl}_Y(A) \\
&= A \coprod \left(\mathrm{cl}_Y(A) \bigcap B\right) \quad \text{since } B \subset Y \\
&= A \coprod \emptyset \quad \text{using the assumption} \\
&= A.
\end{aligned}$$

- But $A = Y \setminus B$ where B is closed, so A is open and thus a nontrivial clopen subset.

■



2.10.2 Proof 2

Lemma: X is connected iff the only subsets of X that are closed and open are $\emptyset, X$.

If $S \subset X$ is not connected, then there exists a subset $A \subset S$ that is both open and closed in the subspace topology, where $A \neq \emptyset, S$.

Suppose S is not connected, then choose A as above. Then $B = S \setminus A$ yields a pair A, B that disconnects S. Since A is closed in S, $\overline{A} = A$ and thus $\overline{A} \cap B = A \cap B = \emptyset$. Similarly, since A is open, B is closed, and $\overline{B} = B \implies \overline{B} \cap A = B \cap A = \emptyset$.

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- Define a new function

$$\begin{aligned} g : X &\longrightarrow \mathbb{R} \\ x &\mapsto d_X(x, f(x)). \end{aligned}$$

- Attempt to minimize. Claim: g is a continuous function.
- Given claim, a continuous function on a compact space attains its infimum, so set

$$m \coloneqq \inf_{x \in X} g(x)$$

  and produce $x_0 \in X$ such that $g(x) = m$.
- Then

$$m > 0 \iff d(x_0, f(x_0)) > 0 \iff x_0 \neq f(x_0).$$

- Now apply f and use the assumption that f is a contraction to contradict minimality of m:

$$\begin{aligned} d(f(f(x_0)), f(x_0)) &\leq C \cdot d(f(x_0), x_0) \\ &< d(f(x_0), x_0) \quad \text{since } C < 1 \\ &\leq m \end{aligned}$$

- Proof that g is continuous: use the definition of g, the triangle inequality, and that f is a contraction:

$$\begin{aligned} d(x, f(x)) &\leq d(x, y) + d(y, f(y)) + d(f(x), f(y)) \\ \implies d(x, f(x)) - d(y, f(y)) &\leq d(x, y) + d(f(x), f(y)) \\ \implies g(x) - g(y) &\leq d(x, y) + C \cdot d(x, y) = (C + 1) \cdot d(x, y) \end{aligned}$$

  - This shows that g is Lipschitz continuous with constant $C + 1$ (implies uniformly continuous, but not used).

2.12 19

Statement: prove that the product of two connected spaces is connected.

Solution:

Use the fact that a union of spaces containing a common point is still connected. Fix a point $(a, b) \in X \times Y$. Since the horizontal slice $X_b \coloneqq X \times \{b\}$ is homeomorphic to X which is connected, as are all of the vertical slices $Y_x \coloneqq \{x\} \times Y \cong Y$ (for any x), the "T-shaped" space $T_x \coloneqq X_b \bigcup Y_x$ is connected for each x.

Note that $(a, b) \in T_x$ for every x, so $\bigcup_{x \in X} T_x = X \times Y$ is connected.

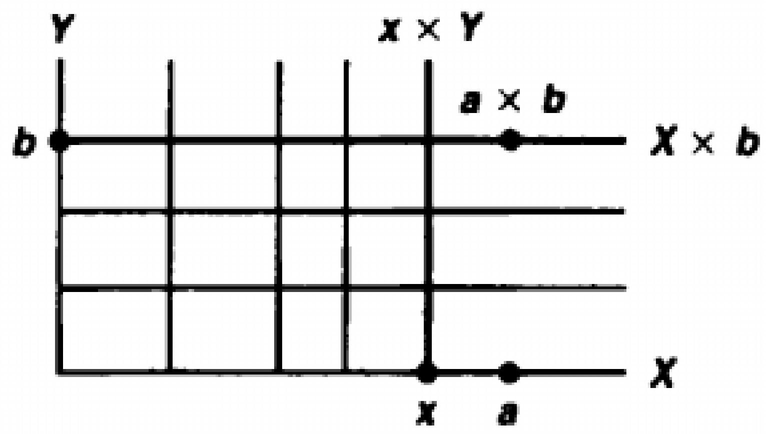


Figure 23.2

2.13 20

a. See definitions in intro.

b. Claim: the Topologist's sine curve X suffices.

Proof:

- Claim 1: X is connected.
  - Intervals and graphs of cts functions are connected, so the only problem point is 0.
- Claim 2: X is **not** locally connected.
  - Take any $B_\varepsilon(0) \in \mathbb{R}^2$; then projecting onto the subspace $\pi_X(B_\varepsilon(0))$ yields infinitely many arcs, each intersecting the graph at two points on $\partial B_\varepsilon(0)$.
  - These are homeomorphic to a collection of disjoint embedded open intervals, and any disjoint union of intervals is clearly not connected. ■

Space	Connected	Locally Connected
$\mathbb{R}$	✓	✓
$[0,1]\bigcup[2,3]$		✓
Sine Curve	✓	
$\mathbb{Q}$		

Todo: what's the picture?

2.14 23

Note: this is precisely the cofinite topology.

1. $\mathbb{R} \in \tau$ since $\mathbb{R} \setminus \mathbb{R} = \emptyset$ is trivially a finite set, and $\emptyset \in \tau$ by definition.
2. If $U_i \in \tau$ then $(\bigcup_i U_i)^c = \bigcap U_i^c$ is an intersection of finite sets and thus finite, so $\bigcup_i U_i \in \tau$.
3. If $U_i \in \tau$, then $(\bigcap_{i=1}^n U_i)^c = \bigcup_{i=1}^n U_i^c$ is a finite union of finite sets and thus finite, so $\bigcap U_i \in \tau$.

So τ forms a topology.

To see that $(\mathbb{R}, \tau)$ is compact, let $\{U_i\} \rightrightarrows \mathbb{R}$ be an open cover by elements in τ.

Fix any U_α, then $U_\alpha^c = \{p_1, \cdots, p_n\}$ is finite, say of size n. So pick $U_1 \ni p_1, \cdots, U_n \ni p_n$; then $\mathbb{R} \subset U_\alpha \bigcup_{i=1}^n U_i$ is a finite cover.

2.15 27

Notes: use diagonal trick to construct the Cauchy sequence.

2.15.1 a

$\Longrightarrow$:

If X is totally bounded, let $\varepsilon = \frac{1}{n}$ for each n, and let $\{x_i\}$ be an arbitrary sequence. For $n = 1$, pick a finite open cover $\{U_i\}_n$ such that $\mathrm{diam} U_i < \frac{1}{n}$ for every i.

Choose V_1 such that there are infinitely many $x_i \in V_1$. (Why?) Note that $\mathrm{diam} V_i < 1$. Now choose $x_i \in V_1$ arbitrarily and define it to be y_1.

Then since V_1 is totally bounded, repeat this process to obtain $V_2 \subseteq V_1$ with $\mathrm{diam}(V_2) < \frac{1}{2}$, and choose $x_i \in V_2$ arbitrarily and define it to be y_2.

This yields a nested family of sets $V_1 \supseteq V_2 \supseteq \cdots$ and a sequence $\{y_i\}$ such that $d(y_i, y_j) < \max(\frac{1}{i}, \frac{1}{j}) \longrightarrow 0$, so $\{y_i\}$ is a Cauchy subsequence.

$\Longleftarrow$:

Then fix $\varepsilon > 0$ and pick x_1 arbitrarily and define $S_1 = B(\varepsilon, x_1)$. Then pick $x_2 \in S_1^c$ and define $S_2 = S_1 \bigcup B(\varepsilon, x_2)$, and so on. Continue by picking $x_{n+1} \in S_n^c$ (Since X is not totally bounded, this can always be done) and defining $S_{n+1} = S_n \bigcup B(\varepsilon, x_{n+1})$.

Then $\{x_n\}$ is not Cauchy, because $d(x_i, x_j) > \varepsilon$ for every $i \neq j$.

2.15.2 b

Take $X = C^0([0, 1])$ with the sup-norm, then $f_n(x) = x^n$ are all bounded by 1, but $\|f_i - f_j\| = 1$ for every i, j, so no subsequence can be Cauchy, so X can not be totally bounded.

Moreover, $\{f_n\}$ is closed. (Why?)

2.16 30

Let $A \subset X$ be compact, and pick a fixed $x \in X \setminus A$. Since X is Hausdorff, for arbitrary $a \in A$, there exists opens $U_a \ni a$ and $U_{x,a} \ni x$ such that $V_a \bigcap U_{x,a} = \emptyset$. Then $\left\{ U_a \;\middle|\; a \in A \right\} \rightrightarrows A$, so by compactness there is a finite subcover $\{U_{a_i}\} \rightrightarrows A$.

Now take $U = \bigcup_i U_{a_i}$ and $V_x = \bigcap_i V_{a_i,x}$, so $U \bigcap V = \emptyset$. Note that both U and V_x are open.

But then defining $V \coloneqq \bigcup_{x \in X \setminus A} V_x$, we have $X \setminus A \subset V$ and $V \bigcap A = \emptyset$, so $V = X \setminus A$, which is open and thus A is closed.

2.17 31

2.17.1 a

Theorems used:

- Continuous bijection + open map (or closed map) $\implies$ homeomorphism.
- **Closed** subsets of compact sets are compact.
- The continuous image of a compact set is compact.
- Closed subsets of Hausdorff spaces are compact.

So we'll show that f is a closed map.

Let $U \in X$ be closed.

- Since X is compact, U is compact
- Since f is continuous, $f(U)$ is compact
- Since Y is Hausdorff, $f(U)$ is closed.

2.17.2 b

Note that any finite space is clearly compact.

Take $f : ([2], \tau_1) \longrightarrow ([2], \tau_2)$ to be the identity map, where τ_1 is the discrete topology and τ_2 is the indiscrete topology. Any map into an indiscrete topology is continuous, and f is clearly a bijection.

Let g be the inverse map; then note that $1 \in \tau_1$ but $g^{-1}(1) = 1$ is not in τ_2, so g is not continuous.

$\blacksquare$

2.18 32

$\implies$:

- Let $p \in X^2 \setminus \Delta$.
- Then p is of the form (x, y) where $x \neq y$ and $x, y \in X$.
- Since X is Hausdorff, pick N_x, N_y in X such that $N_x \bigcap N_y = \emptyset$.

- Then $N_p := N_x \times N_y$ is an open set in X^2 containing p.
- Claim: $N_p \bigcap \Delta = \emptyset$.
  - If $q \in N_p \bigcap \Delta$, then $q = (z,z)$ where $z \in X$, and $q \in N_p \implies q \in N_x \bigcap N_y = \emptyset$.
- Then $X^2 \setminus \Delta = \bigcup_p N_p$ is open.

$\Longleftarrow$:

- Let $x \neq y \in X$.
- Consider $(x,y) \in \Delta^c \subset X^2$, which is open.
- Thus $(x,y) \in B$ for some box in the product topology.
- $B = U \times V$ where $U \ni x, V \ni y$ are open in X, and $B \subset X^2 \setminus \Delta$.
- Claim: $U \bigcap V = \emptyset$.
  - Otherwise, $z \in U \bigcap V \implies (z,z) \in B \bigcap \Delta$, but $B \subset X^2 \setminus \Delta \implies B \bigcap \Delta = \emptyset$.

2.19 38

$\mathbb{R}$ is clearly Hausdorff, and $\mathbb{R}/\mathbb{Q}$ has the indiscrete topology, and is thus non-Hausdorff. So take the quotient map $\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Q}$.

Direct proof that $\mathbb{R}/\mathbb{Q}$ isn't Hausdorff:

- Pick $[x] \subset U \neq [y] \subset V \in \mathbb{R}/\mathbb{Q}$ and suppose $U \cap V = \emptyset$.
- Pull back $U \longrightarrow A, V \longrightarrow B$ open disjoint sets in $\mathbb{R}$
- Both A, B contain intervals, so they contain rationals $p \in A, q \in B$
- Then $[p] = [q] \in U \bigcap V$.

2.20 42

Proof that $\mathbb{R}/\mathbb{Q}$ has the indiscrete topology:

- Let $U \subset \mathbb{R}/\mathbb{Q}$ be open and nonempty, show $U = \mathbb{R}/\mathbb{Q}$.
- Let $[x] \in U$, then $x \in \pi^{-1}(U) := V \subset \mathbb{R}$ is open.
- Then V contains an interval (a,b)
- Every $y \in V$ satisfies $y + q \in V$ for all $q \in \mathbb{Q}$, since $y + q - y \in \mathbb{Q} \implies [y+q] = [y]$.
- So $(a - q, b + q) \in V$ for all $q \in \mathbb{Q}$.
- So $\bigcup_{q \in \mathbb{Q}} (a - q, b + q) \in V \implies \mathbb{R} \subset V$.
- So $\pi(V) = \mathbb{R}/\mathbb{Q} = U$, and thus the only open sets are the entire space and the empty set.

2.21 44

2.21.1 a

- Suppose X has a countable basis $B = \{B_i\}$.
- Choose an arbitrary $x_i \in B_i$ for each i. Define $Q = \{x_i\}$.
- Let $y \in N_y \subset X$.
- By definition of a basis, there exists some B_i such that $y \in B_i \subset N_y$.
- Since $x_i \in B_i$, $Q \bigcap N_y \neq \emptyset$.
- Thus Q is dense in X.

2.21.2 b

- Let $\{q_i\}$ be a countable dense subset.
- Define $B_{i,j} = B_{\frac{1}{i}}(q_j)$, which is still countable.
- Property 1: Every $x \in B_{i,j}$
  - Take $x \in N_{\frac{1}{2}}(x) \ni q_j$ by density.
  - Then $x \in B_{\frac{1}{2},j}$.
- Property 2: $x \in B_{i_1,j_1} \bigcap B_{i_2,j_2} \implies x \in B_{i_3,j_3} \subset B_{i_1,j_1} \cap B_{i_2,j_2}$:
  - Take $i < \min(i_1, i_2)$, then $N_i(x) \ni q_j$. for some j.
  - Thus $x \in B_{i,j}$.

3 2: Fundamental Group

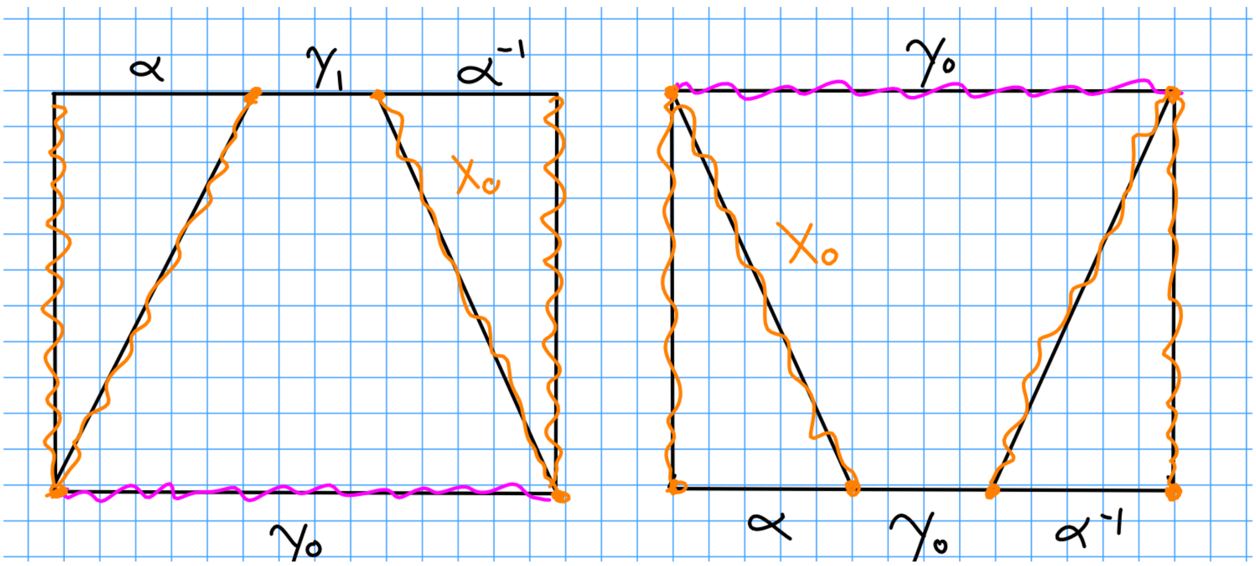
3.1 1

Proposition: $\gamma_1 \simeq \gamma_2 \iff \gamma_1, \gamma_2$ are conjugate in $\pi_1(X, x_0)$, i.e. $\exists[\alpha] \in \pi_1$ such that $[\gamma_1] = [\alpha][\gamma_2][\alpha]^{-1}$.

Proof:

$\implies$: Clear, since $\gamma_1 \sim \gamma_2 \implies [\gamma_1] = [\gamma_2] \in \pi_1(X)$, so take $\alpha(t) = x_0$ the constant loop for all t.

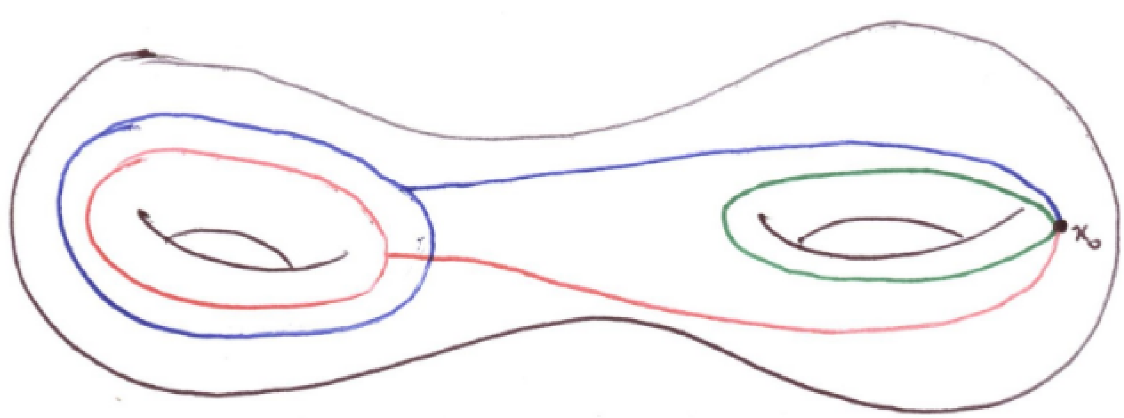
$\impliedby$: ? Forgot how these arguments go.



∎

Counterexample where homotopic loops are not equal in π_1, but just conjugate:

It's not a great picture, but the blue and red loops below are freely homotopic, but not homotopic relative to the basepoint x_0. In $\pi_1(X, x_0)$, they are conjugate via the green loop.



4 Covering Spaces

4.1 1b

Homotopy lifting property:

$$\begin{tikzcd}
 & \tilde{X} \arrow[d, "\pi"] \\
Y \times I \arrow[ur, dashed, "\exists \tilde{H}"] \arrow[r, "H"] & X
\end{tikzcd}$$

π clearly induces a map p_* on π_1 by functoriality, so we'll show that $\ker p_*$ is trivial. Let $\gamma : S^1 \longrightarrow \tilde{X} \in \pi_1(\tilde{X})$ and suppose $\alpha := p_*(\gamma) = [e] \in \pi_1(X)$. We'll show $\gamma \simeq [e]$ in $\pi_1(\tilde{X})$.

Since $\alpha = [e]$, $\alpha \simeq$ const. and thus there is a homotopy $H : I \times S^1 \longrightarrow X$ such that $H_0 = \text{const.}(x_0)$ and $H_1 = \gamma$. By the HLP, this lifts to $\tilde{H} : I \times S^1 \longrightarrow \tilde{X}$. Noting that $\pi^{-1}(\text{const.}(x_0))$ is still a constant loop, this says that γ is homotopic to a constant loop and thus nullhomotopic.

4.2 1c

Since both spaces are path-connected, the degree o the covering map π is precisely the index of the included fundamental group. This forces π to be a degree 1 covering and hence a homeomorphism.

4.3 6

Note $\pi_1 \mathbb{RP}^2 = \mathbb{Z}/2\mathbb{Z}$, so $\pi_1 X = (\mathbb{Z}/2\mathbb{Z})^2$.

The pullback of any neighborhood of the basepoint needs to be locally homeomorphic to one of

- $S^2 \vee S^2$
- $\mathbb{RP}^2 \vee S^2$

And so *all* possibilities for regular covering spaces are given by

- $\bigvee^{2k} S^2$ "beads" wrapped into a necklace for any $k \geq 1$
- $\mathbb{RP}^2 \vee (\bigvee^{k} S^2) \vee \mathbb{RP}^2$
- $\vee^\infty S^2$, the universal cover

To get a threefold cover, we want the basepoint to lift to three preimages, so we can take

- $S^2 \vee S^2 \vee S^2$ wrapped
- $\mathbb{RP}^2 \vee S^2 \vee \mathbb{RP}^2$.

4.4 7

- $\mathbb{RP}_3 \vee S^2 \vee \mathbb{RP}^3$, which has $\pi_2 = 0 * \mathbb{Z} * 0 = \mathbb{Z}$ since $\pi_{i \geq 1} X = \pi_{i \geq 1} \tilde{X}$ and $\tilde{\mathbb{RP}}^3 = S^3$.
- $\mathbb{RP}^2 \vee S^3 \vee \mathbb{RP}^2$, which was $\pi_2 = \mathbb{Z} * 0 * \mathbb{Z} = \mathbb{Z} * \mathbb{Z} \neq \mathbb{Z}$

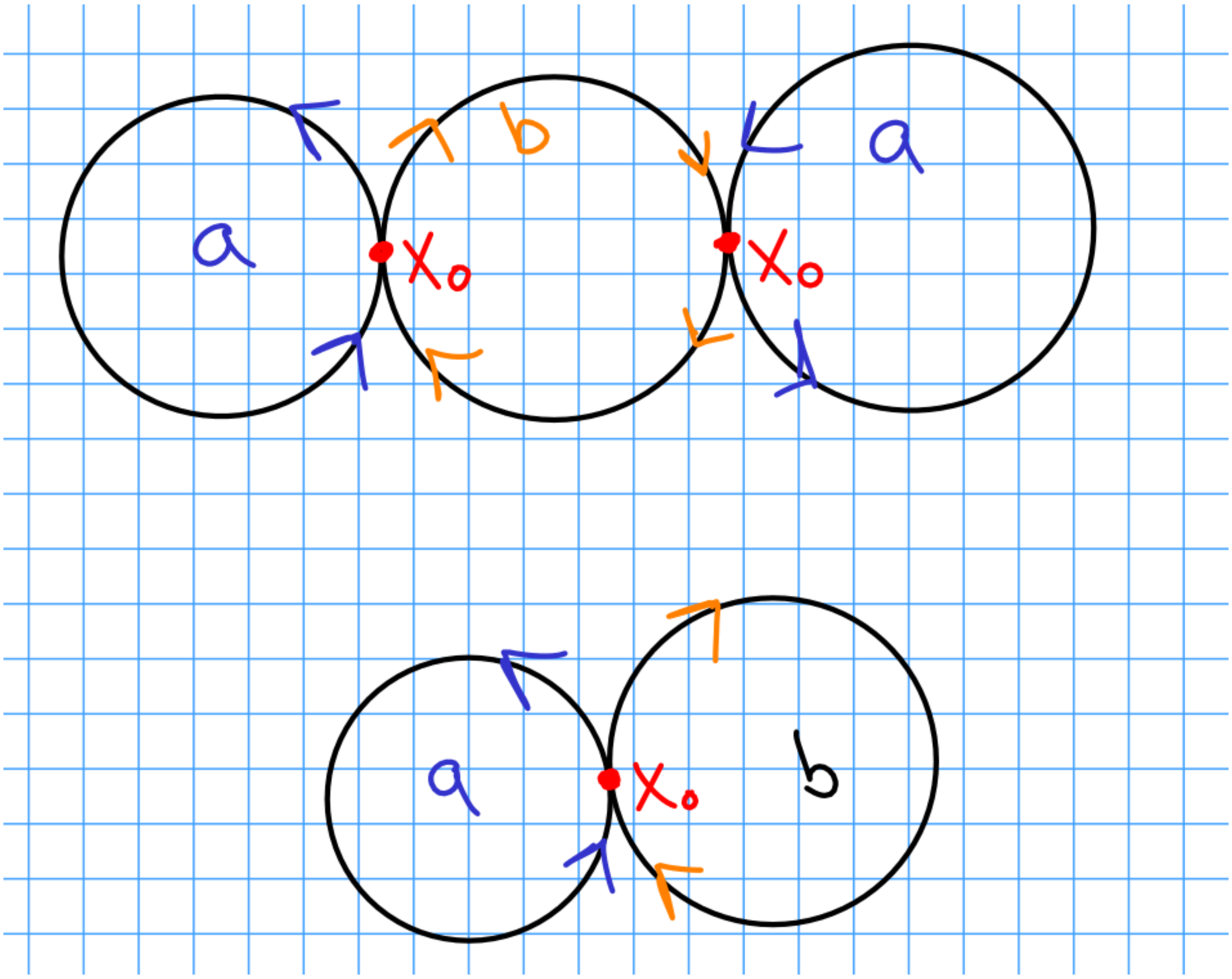


Figure 1: Image

4.5 8

Yes,

5 Spring 2019

5.1 Problem 1

Complete and **totally** bounded $\implies$ compact. - Definition: A space X is *totally bounded* if for every $\varepsilon > 0$, there is a finite cover $X \subseteq \bigcup_\alpha B_\alpha(\varepsilon)$ such that the radius of each ball is less than ε.

- Definition: A subset of a space $S \subset X$ is *bounded* if there exists a $B(r)$ such that $r < \infty$ and $S \subseteq B(r)$ - Totally bounded $\implies$ bounded - Counterexample to converse: $\mathbb{N}$ with the discrete metric. - Equivalent for Euclidean metric - Compact $\implies$ totally bounded.

Counterexample for problem: the unit ball in any Hilbert (or Banach) space of infinite dimenion is closed, bounded, and not compact.

> Proof: Inductively, let $\mathbf{x}_1 \in B(1, \mathbf{0})$ and $A_1 = \operatorname{span}(\mathbf{x}_1)$, then choose $s = \mathbf{x} + A_1 \in B(1, 0)/A_1$ such that $\|s\| = \frac{1}{2}$ and then a representative $\mathbf{x}_2$ such that $\|\mathbf{x}_2\| \leq 1$. Then $\|\mathbf{x}_2 - \mathbf{x}_1\| \geq \frac{1}{2}$
> Then, let $A_2 = \operatorname{span}(\mathbf{x}_1, \mathbf{x}_2)$, (which is closed) and repeat this for $s = \mathbf{x} + A_2 \in B(1, \mathbf{0})/A_2$ to get an $\mathbf{x}_3$ such that $\|\mathbf{x}_3 - \mathbf{x}_{\leq 2}\| \geq \frac{1}{2}$.
> This produces a non-convergent sequence in the closed ball, so it can not be compact.

Second counterexample: $(\mathbb{R}, (x, y) \mapsto \frac{|x-y|}{1+|x-y|})$.

Best counterexample: $X = \left(\mathbb{Z},\ \rho(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}\right)$. This metric makes X complete for any X, then take $\mathbb{N} \subset X$. All sets are closed, and bounded, so we have a complete, closed, bounded set that is not compact – take that cover $U_i = B(1, i)$.

Useful tool: $(X, d) \cong_{\text{Top}} (X, \min\left(d(x, y), 1\right)$ where the RHS is now a bounded space. This preserves all topological properties (e.g. compactness).

5.2 Problem 2

Definition: (X, τ) where $\tau \subseteq \mathcal{P}(X)$ is a *topological space* iff

- $\emptyset, X \in \tau$
- $\{U_i\}_{i \in I} \subseteq \tau \implies \bigcup_{i \in I} U_i \in \tau$
- $\{U_i\}_{i \in \mathbb{N}} \subseteq \tau \implies \bigcap_{i \in \mathbb{N}} U_i \in \tau$

We can write $\overline{(X, \tau)} = (X \coprod \{\text{pt}\}, \tau \bigcup \tau')$ where $\tau' = \left\{ U \coprod \{\text{pt}\} \;\middle|\; X - U \text{ is compact} \right\}$. We need to show that $T \coloneqq \tau \bigcup \tau'$ forms a topology.

- We have $\emptyset, X \in \tau \implies \emptyset, X \in \tau \bigcup \tau'$.
- We just need to check that τ' is closed under arbitrary unions. Let $\{U_i\} \subset \tau'$, so $X - U_i = K_i$ a compact set for each i. Then $\bigcup_i U_i = \bigcup_i X - (X - U_i) = \bigcup_i X - K_i = X - \bigcup_i K_i$

5.3 7

Let $f : S^1 \xrightarrow{\times k} S^1$.

Claim: The inclusion $S^1 \longrightarrow C_\varphi$ induces an isomorphism $\pi_1(C_\varphi) \cong \pi_1(S^1)/H$ where $H = N_{\pi_1(S^1)}(\langle f^* \rangle)$ is the normal subgroup generated by the induced map $f^*\pi_1(S^1) \longrightarrow \pi_1(S^1)$.

Since f is a k-fold cover, the induced map is multiplication by k on the generator $\alpha \in \pi_1(S^1)$, i.e. $\alpha \mapsto \alpha^k$. But then $\pi_1(S^1) \cong \mathbb{Z}$ and $H \cong k\mathbb{Z}$, so $\pi_1(C_\varphi) \cong \mathbb{Z}/m\mathbb{Z}$.

$\blacksquare$

6 Extra Problem Solutions

6.1 Point Set

6.1.1 Connectedness

1. Problem Statement

Reference
A potentially shorter proof

- Let $I = [0,1] = A \bigcup B$ be a disconnection, so
  - $A, B \neq \emptyset$
  - $A \coprod B = I$
  - $\mathrm{cl}_I(A) \bigcap B = A \bigcap \mathrm{cl}_I(B) = \emptyset$.
- Let $a \in A$ and $b \in B$ where WLOG $a < b$
  - (since either $a < b$ or $b < a$, and $a \neq b$ since A, B are disjoint)
- Let $K = [a, b]$ and define $A_K \coloneqq A \bigcap K$ and $B_K \coloneqq B \bigcap K$.
- Now A_K, B_K is a disconnection of K.
- Let $s = \sup(A_K)$, which exists since $\mathbb{R}$ is complete and has the LUB property
- Claim: $s \in \mathrm{cl}_I(A_K)$. Proof:
  - If $s \in A_K$ there's nothing to show since $A_K \subset \mathrm{cl}_I(A_K)$, so assume $s \in I \setminus A_K$.
  - Now let N_s be an arbitrary neighborhood of s, then using ??? we can find an $\varepsilon > 0$ such that $B_\varepsilon(s) \subset N_s$
  - Since s is a supremum, there exists an $a \in A_K$ such that $s - \varepsilon < a$.
  - But then $a \in B_\varepsilon(s)$ and $a \in N_s$ with $a \neq s$.
  - Since N_s was arbitrary, every N_s contains a point of A_K not equal to s, so s is a limit point by definition.
- Since $s \in \mathrm{cl}_I(A_K)$ and $\mathrm{cl}_I(A_K) \bigcap B_K = \emptyset$, we have $s \notin B_K$.
- Then the subinterval $(x, b] \bigcap A_K = \emptyset$ for every $x > c$ since $c \coloneqq \sup A_K$.
- But since $A_K \coprod B_K = K$, we must have $(x, b] \subset B_K$, and thus $s \in \mathrm{cl}_I(B_K)$.
- Since A_K, B_K were assumed disconnecting, $s \notin A_K$
- But then $s \in K$ but $s \notin A_K \coprod B_K = K$, a contradiction.

■

6.2 From Problem Sessions

1. Problem Statement

- Let X be compact, $A \subset X$ closed, and $\{U_\alpha\} \rightrightarrows A$ be an open cover.
- By definition of the subspace topology, each $U_\alpha = V_\alpha \bigcap A$ for some open $V_\alpha \subset X$, and $A \subset \bigcup_\alpha V_\alpha$.
- Since A is closed in X, $X \setminus A$ is open.
- Then $\{V_\alpha\} \bigcup \{X \setminus A\} \rightrightarrows X$ is an open cover, since every point is either in A or $X \setminus A$.
- By compactness of X, there is a finite subcover $\left\{U_j \;\middle|\; j \leq N\right\} \bigcup \{X \setminus A\}$
- Then $\left(\{U_j\} \bigcup \{X \setminus A\}\right) \bigcap A \coloneqq \{V_j\}$ is a finite cover of A.

2. Problem Statement

- Let $f : X \longrightarrow Y$ be continuous with X compact, and $\{U_\alpha\} \rightrightarrows f(X)$ be an open cover.
- Then $\left\{f^{-1}(U_\alpha)\right\} \rightrightarrows X$ is an open cover of X, since $x \in X \implies f(x) \in f(X) \implies f(x) \in U_\alpha$ for some α, so $x \in f^{-1}(U_\alpha)$ by definition.
- By compactness of X there is a finite subcover $\left\{f^{-1}(U_j) \;\middle|\; j \leq N\right\} \rightrightarrows X$.
- Then the finite subcover $\left\{U_j \;\middle|\; j \leq N\right\} \rightrightarrows f(X)$, since if $y \in f(X)$, $y \in U_\alpha$ for some α and thus $f^{-1}(y) \in f^{-1}(U_j)$ for some j since $\{U_j\}$ is a cover of X.

3. Problem Statement

Note, alternative definition of "open":

- Let A be a compact subset of X a Hausdorff space, we will show $X \setminus A$ is open
- Fix $x \in X \setminus A$.
- Since X is Hausdorff, for every $y \in A$ we can find $U_y \ni y$ and $V_x(y) \ni x$ depending on y such that $U_x(y) \bigcap U_y = \emptyset$.
- Then $\left\{U_y \;\middle|\; y \in A\right\} \rightrightarrows A$, and by compactness of A there is a finite subcover corresponding to a finite collection $\{y_1, \cdots, y_n\}$.
- **Magic Step**: set $U = \bigcup U_{y_i}$ and $V = \bigcap V_x(y_i)$;
  - Note $A \subset U$ and $x \in V$
  - Note $U \bigcap V = \emptyset$.
- Done: for every $x \in X \setminus A$, we have found an open set $V \ni x$ such that $V \bigcap A = \emptyset$, so x is an interior point and a set is open iff every point is an interior point.

4. Problem Statement

- It suffices to show that f is a closed map, i.e. if $U \subseteq X$ is closed then $f(U) \subseteq Y$ is again closed.
- Let $U \in X$ be closed; since X is closed, U is compact
  - Since closed subsets of compact spaces are compact.
- Since f is continuous, $f(U)$ is compact
  - Since the continuous image of a compact set is compact.
- Since Y is Hausdorff and $f(U)$ is compact, $f(U)$ is closed
  - Since compact subsets of Hausdorff spaces are closed.

5. Todo

6. Problem Statement

- Take $[0, 1] \subset [0, 1] \subset \mathbb{R}$. Then $[0, 1]$ is tautologically open in $[0, 1]$ as it is the entire space, but $[0, 1]$ is not open in $\mathbb{R}$ since (e.g.) $\{1\}$ is not an interior point (every neighborhood intersects the complement $\mathbb{R} \setminus [0, 1]$).

7. Todo