# Real Analysis Qualifying Exam Questions

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## Contents

# 1 Undergraduate Analysis: Uniform Convergence

## 1.1 Fall 2018 # 1

Let $f(x) = \dfrac{1}{x}$. Show that $f$ is uniformly continuous on $(1, \infty)$ but not on $(0, \infty)$.

*Solution.*

## 1.2 1

Concepts used:

- Uniform continuity.

Show a stronger statement: $f(x) = \dfrac{1}{x}$ is uniformly continuous on any interval of the form $(c, \infty)$ where $c > 0$.

- Note that

$$|x|, |y| > c > 0 \implies |xy| = |x||y| > c^2 \implies \frac{1}{|xy|} < \frac{1}{c^2}.$$

- Letting $\varepsilon$ be arbitrary, choose $\delta < \varepsilon c^2$.
- Note that $\delta$ does not depend on $x, y$.

- Then

$$\begin{aligned} |f(x) - f(y)| &= \left|\frac{1}{x} - \frac{1}{y}\right| \\ &= \frac{|x-y|}{xy} \\ &\leq \frac{\delta}{xy} \\ &< \frac{\delta}{c^2} \\ &< \varepsilon, \end{aligned}$$

  which shows uniform continuity.

To see that $f$ is not uniformly continuous when $c = 0$:

> Note: negating uniform continuity says $\exists \varepsilon > 0$ such that $\forall \delta(\varepsilon)$ there exist $x, y$ such that $|x - y| < \delta$ *and* $|f(x) - f(y)| > \varepsilon$.

- Let $\varepsilon < 1$.
- Let $x_n = \dfrac{1}{n}$ for $n \geq 1$.
- Choose $n$ large enough such that $|x_n - x_{n+1}| = \dfrac{1}{n} - \dfrac{1}{n+1} < \delta$.
  - Why this can be done: by the archimedean property of $\mathbb{R}$, choose $n$ such that $\dfrac{1}{n} < \varepsilon$.
  - Then

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \leq \frac{1}{n} < \varepsilon \quad \text{since } n+1 > 1.$$

- Note $f(x_n) = n$ and thus

$$|f(x_n) - f(x_{n+1})| = n - (n+1) = 1 > \varepsilon.$$

## 1.3 Fall 2017 # 1

Let

$$f(x) = s \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which $f$ does and does not converge uniformly.

*Solution.*

Note that $f(x) = e^x$ is entire and thus equal to its power series. So $f(x) = \displaystyle\sum_{j=0}^{\infty} \frac{1}{j!} x^j$.

Letting $f_N(x) = \displaystyle\sum_{j=1}^{N} \frac{1}{j!} x^j$, we have $f_N(x) \longrightarrow f(x)$ pointwise on $(-\infty, \infty)$.

For any compact interval $[-M, M]$, we have

$$\begin{aligned}
\|f_N(x) - f(x)\|_\infty &= \sup_{-M \le x \le M} \left| \sum_{j=N+1}^{\infty} \frac{1}{j!} x^j \right| \\
&\le \sup_{-M \le x \le M} \sum_{j=N+1}^{\infty} \frac{1}{j!} |x|^j \\
&\le \sum_{j=N+1}^{\infty} \frac{1}{j!} M^j \\
&\le \sum_{j=0}^{\infty} \frac{1}{j!} M^j \\
&= e^M \\
&< \infty,
\end{aligned}$$

so $f_N \longrightarrow f$ uniformly on $[-M, M]$ by the M-test. Thus it converges on any bounded interval. It does not converge on $\mathbb{R}$, since $x^N$ is unbounded.

## 1.4 Fall 2014 # 1

Let $\{f_n\}$ be a sequence of continuous functions such that $\sum f_n$ converges uniformly.

Prove that $\sum f_n$ is also continuous.

## 1.5 Spring 2017 # 4

Let $f(x, y)$ on $[-1, 1]^2$ be defined by

$$f(x, y) = \begin{cases} \dfrac{xy}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Determine if $f$ is integrable.

*Solution.*
Switching to polar coordinates and integrating over a half-circle contained in $I^2$, we have

$$\int_{I^2} f \ge \int_0^\pi \int_0^1 \frac{\cos(\theta)\sin(\theta)}{r^2} \, dr \, d\theta = \infty,$$

so $f$ is not integrable.

## 1.6 Spring 2015 # 1

Let $(X, d)$ and $(Y, \rho)$ be metric spaces, $f : X \longrightarrow Y$, and $x_0 \in X$.

Prove that the following statements are equivalent:

1. For every $\varepsilon > 0 \quad \exists \delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$.
2. The sequence $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow f(x_0)$ for every sequence $\{x_n\} \longrightarrow x_0$ in $X$.

### 1.7 Fall 2014 # 2

Let $I$ be an index set and $\alpha : I \longrightarrow (0, \infty)$.

1. Show that

$$\sum_{i \in I} a(i) := \sup_{\substack{J \subset I \\ J \text{ finite}}} \sum_{i \in J} a(i) < \infty \implies I \text{ is countable.}$$

2. Suppose $I = \mathbb{Q}$ and $\sum_{q \in \mathbb{Q}} a(q) < \infty$. Define

$$f(x) := \sum_{\substack{q \in \mathbb{Q} \\ q \leq x}} a(q).$$

Show that $f$ is continuous at $x \iff x \notin \mathbb{Q}$.

### 1.8 Spring 2014 # 2

Let $\{a_n\}$ be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that $\sum a_n^2 < \infty$.

Note: Assume $a_n, b_n$ are all non-negative.

## 2 General Analysis

### 2.1 Spring 2020 # 1

Prove that if $f : [0, 1] \longrightarrow \mathbb{R}$ is continuous then

$$\lim_{k \longrightarrow \infty} \int_0^1 k x^{k-1} f(x)\, dx = f(1).$$

*Solution.*
Concepts used:

- DCT
- Weierstrass Approximation Theorem

**Solution**:

- Suppose $p$ is a polynomial, then

$$\begin{aligned}\lim_{k\longrightarrow\infty}\int_0^1 kx^{k-1}p(x)\,dx &= \lim_{k\longrightarrow\infty}\int_0^1 \left(\frac{\partial}{\partial x}x^k\right)p(x)\,dx \\ &= \lim_{k\longrightarrow\infty}\left[x^k p(x)\Big|_0^1 - \int_0^1 x^k\left(\frac{\partial}{\partial x}p(x)\right)dx\right] \quad \text{integrating by parts} \\ &= p(1) - \lim_{k\longrightarrow\infty}\int_0^1 x^k\left(\frac{\partial}{\partial x}p(x)\right)dx,\end{aligned}$$

- Thus it suffices to show that

$$\lim_{k\longrightarrow\infty}\int_0^1 x^k\left(\frac{\partial}{\partial x}p(x)\right)dx = 0.$$

- Integrating by parts a second time yields

$$\begin{aligned}\lim_{k\longrightarrow\infty}\int_0^1 x^k\left(\frac{\partial}{\partial x}p(x)\right)dx &= \lim_{k\longrightarrow\infty}\frac{x^{k+1}}{k+1}p'(x)\Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1}\left(\frac{\partial^2}{\partial x^2}p(x)\right)dx \\ &= -\lim_{k\longrightarrow\infty}\int_0^1 \frac{x^{k+1}}{k+1}\left(\frac{\partial^2}{\partial x^2}p(x)\right)dx \\ &= -\int_0^1 \lim_{k\longrightarrow\infty}\frac{x^{k+1}}{k+1}\left(\frac{\partial^2}{\partial x^2}p(x)\right)dx \quad \text{by DCT} \\ &= -\int_0^1 0\left(\frac{\partial^2}{\partial x^2}p(x)\right)dx \\ &= 0.\end{aligned}$$

  - The DCT can be applied here because $f''$ is continuous and $[0,1]$ is compact, so $f''$ is bounded on $[0,1]$ by a constant $M$ and

$$\int_0^1 \left|x^k f''(x)\right| \leq \int_0^1 1\cdot M = M < \infty.$$

- Now use the Weierstrass approximation theorem:
  - If $f:[a,b]\longrightarrow\mathbb{R}$ is continuous, then for every $\varepsilon>0$ there exists a polynomial $p_\varepsilon(x)$ such that $\|f-p_\varepsilon\|_\infty<\varepsilon$.
- Thus

$$\begin{aligned}\left|\int_0^1 kx^{k-1}p_\varepsilon(x)\,dx - \int_0^1 kx^{k-1}f(x)\,dx\right| &= \left|\int_0^1 kx^{k-1}(p_\varepsilon(x)-f(x))\,dx\right| \\ &\leq \left|\int_0^1 kx^{k-1}\|p_\varepsilon - f\|_\infty\,dx\right| \\ &= \|p_\varepsilon - f\|_\infty \cdot \left|\int_0^1 kx^{k-1}\,dx\right| \\ &= \|p_\varepsilon - f\|_\infty \cdot x^k\Big|_0^1 \\ &= \|p_\varepsilon - f\|_\infty \overset{\varepsilon\longrightarrow 0}{\Longrightarrow} 0\end{aligned}$$

and the integrals are equal.

- By the first argument,

$$\int_0^1 kx^{k-1} p_\varepsilon(x)\, dx = p_\varepsilon(1) \text{ for each } \varepsilon$$

- Since uniform convergence implies pointwise convergence, $p_\varepsilon(1) \overset{\varepsilon \longrightarrow 0}{\Longrightarrow} f(1)$.

## 2.2 Fall 2019 # 1.

Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers.

a. Prove that if $\lim_{n \longrightarrow \infty} a_n = 0$, then

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

b. Prove that if $\sum_{n=1}^\infty \frac{a_n}{n}$ converges, then

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

*Solution.*

Cesaro mean/summation. Break series apart into pieces that can be handled separately.

### 2.2.1 a

Prove a stronger result:

$$a_k \longrightarrow S \implies S_N := \frac{1}{N} \sum_{k=1}^N a_k \longrightarrow S.$$

Idea: once $N$ is large enough, $a_k \approx S$, and all smaller terms will die off as $N \longrightarrow \infty$. See this MSE answer.

- Use convergence $a_k \longrightarrow S$: choose $M$ large enough such that

$$k \geq M + 1 \implies |a_k - S| < \varepsilon.$$

Then

$$
\begin{aligned}
\left| \left( \frac{1}{N} \sum_{k=1}^{N} a_k \right) - S \right| &= \frac{1}{N} \left| \left( \sum_{k=1}^{N} a_k \right) - NS \right| \\
&= \frac{1}{N} \left| \left( \sum_{k=1}^{N} a_k \right) - \sum_{k=1}^{N} S \right| \\
&= \frac{1}{N} \left| \sum_{k=1}^{N} (a_k - S) \right| \\
&\leq \frac{1}{N} \sum_{k=1}^{N} |a_k - S| \\
&= \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} |a_k - S| \\
&\leq \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} \frac{\varepsilon}{2} \\
&= \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + (N - M)\frac{\varepsilon}{2} \\
&\overset{\varepsilon \longrightarrow 0}{\longrightarrow} \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + 0 \\
&\overset{N \longrightarrow \infty}{\longrightarrow} 0 + 0.
\end{aligned}
$$

Note: $M$ is fixed, so the last sum is some constant $c$, and $c/N \longrightarrow 0$ as $N \longrightarrow \infty$ for any constant. To be more careful, choose $M$ first to get $\varepsilon/2$ for the tail, then choose $N(M) > M$ for the remaining truncated part of the sum.

### 2.2.2 b

- Define

$$
\Gamma_n := \sum_{k=n}^{\infty} \frac{a_k}{k}.
$$

- $\Gamma_1 = \sum_{k=1}^{n} \frac{a_k}{k}$ is the original series and each $\Gamma_n$ is a tail of $\Gamma_1$, so by assumption $\Gamma_n \overset{n \longrightarrow \infty}{\longrightarrow} 0$.
- Compute

$$
\frac{1}{n} \sum_{k=1}^{n} a_k = \frac{1}{n}(\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \mathbf{\Gamma_{n+1}})
$$

.

- This comes from consider the following summation:

$$\begin{array}{l|llllll}
\Gamma_1: & a_1 & +\frac{a_2}{2} & +\frac{a_3}{3} & +\cdots & & \\
\Gamma_2: & & \frac{a_2}{2} & +\frac{a_3}{3} & +\cdots & & \\
\Gamma_3: & & & \frac{a_3}{3} & +\cdots & & \\
\hline
\sum_{i=1}^n \Gamma_i: & a_1 & +a_2 & +a_3 & +\cdots & a_n & +\frac{a_{n+1}}{n+1} & +\cdots
\end{array}$$

- Use part (a): since $\Gamma_n \overset{n\longrightarrow\infty}{\longrightarrow} 0$, we have $\frac{1}{n}\sum_{k=1}^n \Gamma_k \overset{n\longrightarrow\infty}{\longrightarrow} 0$.
- Also a minor check: $\Gamma_n \longrightarrow 0 \implies \frac{1}{n}\Gamma_n \longrightarrow 0$.
- Then

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^n a_k &= \frac{1}{n}(\Gamma_1+\Gamma_2+\cdots+\Gamma_n - \mathbf{\Gamma_{n+1}}) \\
&= \left(\frac{1}{n}\sum_{k=0}^n \Gamma_k\right) - \left(\frac{1}{n}\Gamma_{n+1}\right) \\
&\overset{n\longrightarrow\infty}{\longrightarrow} 0.
\end{aligned}$$

## 2.3 Fall 2018 # 4

Let $f \in L^1([0,1])$. Prove that

$$\lim_{n\longrightarrow\infty} \int_0^1 f(x)|\sin nx|\, dx = \frac{2}{\pi}\int_0^1 f(x)\, dx$$

Hint: Begin with the case that $f$ is the characteristic function of an interval.

*Solution.*
Case of characteristic function

- First suppose $f(x) = \chi_{[0,1]}(x)$.
- Note that $\sin(nx)$ has a period of $2\pi/n$, and thus $\left\lfloor \frac{n}{2\pi} \right\rfloor$ full periods in $[0,1]$.
- Taking the absolute value yields a new function with half the period, so a period of $\pi/n$ and $\lfloor \pi/n \rfloor$ full periods in $[0,1]$.
- We can compute the integral over one full period (which is independent of *which* period

is chosen), and since $\sin(x)$ is positive and agrees with $|\sin(nx)|$ on the first period, we have

$$\begin{aligned}\int_{\text{One Period}} |\sin(nx)|\,dx &= \int_0^{\pi/n} \sin(nx)\,dx \\ &= \frac{1}{n}\int_0^{\pi} \sin(u)\,du \quad u = nx \\ &= \frac{1}{n} - \cos(u)\,\Big|_0^{\pi} \\ &= \frac{2}{n}.\end{aligned}$$

- Then break the integral up into integrals over periods $P_1, P_2, \cdots, P_N$ where $N \coloneqq \lfloor n/\pi \rfloor$:

$$\begin{aligned}\int_0^1 |\sin(nx)|\,dx &= \left(\sum_{j=1}^{N} \int_{P_j} |\sin(nx)|\,dx\right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)|\,dx \\ &= \left(\sum_{j=1}^{N} \frac{2}{n}\right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)|\,dx \\ &= N\left(\frac{2}{n}\right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)|\,dx \\ &\coloneqq \left\lfloor \frac{n}{\pi} \right\rfloor \frac{2}{n} + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)|\,dx \\ &= \frac{2}{\pi} + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)|\,dx \\ &\coloneqq \frac{2}{\pi} + R(n)\end{aligned}$$

so it suffices to show that $R(n) \overset{n\longrightarrow\infty}{\longrightarrow} 0$.

Need to justify removing floor function and cancellation.

- Showing this: ????????????

No clue how to show this.

General case

Not sure. Approximate $f$ by simple functions...?

## 2.4 Fall 2017 # 4

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

1. Show that $f_n \longrightarrow 0$ pointwise but not uniformly on $[0,1]$.

Hint: Consider the maximum of $f_n$.

2.

$$\lim_{n\longrightarrow\infty} \int_0^1 n(1-x)^n \sin x\,dx = 0$$

*Solution.*

#### 2.4.1 a

Let $G(x) = \sum_{n=1}^{\infty} nx(1-x)^n$. Applying the ratio test, we have

$$\left|\frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n}\right| = \frac{n+1}{n}|1-x| \overset{n\longrightarrow\infty}{\longrightarrow} |1-x| < 1 \iff 0 \le x \le 2,$$

and in particular, this series converges on $[0,2]$. Thus its terms go to zero, and $nx(1-x)^n \longrightarrow 0$ on $[0,1] \subset [0,2]$.

To see that the convergence is not uniform, let $x_n = \frac{1}{n}$ and $\varepsilon > \frac{1}{e}$, then

$$\sup_{x\in[0,1]} |nx(1-x)^n - 0| \ge |nx_n(1-x_n)^n| = \left|\left(1-\frac{1}{n}\right)^n\right| \overset{n\longrightarrow\infty}{\longrightarrow} e^{-1} > \varepsilon.$$

#### 2.4.2 b

Note: could use the first part with $\sin(x) \le x$, but then integral ends up more complicated.

Noting that $\sin(x) \le 1$, we have We have

$$\begin{aligned}
\left|\int_0^1 n(1-x)^n \sin(x)\right| &\le \int_0^1 |n(1-x)^n \sin(x)| \\
&\le \int_0^1 |n(1-x)^n| \\
&= n\int_0^1 (1-x)^n \\
&= -\frac{n(1-x)^{n+1}}{n+1} \\
&\overset{n\longrightarrow\infty}{\longrightarrow} 0.
\end{aligned}$$

## 2.5 Spring 2017 # 3

Let

$$f_n(x) = ae^{-nax} - be^{-nbx} \qquad \text{where } 0 < a < b.$$

Show that

a. $\sum_{n=1}^{\infty} |f_n|$ is not in $L^1([0,\infty), m)$

Hint: $f_n(x)$ has a root $x_n$.

b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0,\infty), m) \quad \text{and} \quad \int_0^\infty \sum_{n=1}^{\infty} f_n(x)\, dm = \ln\frac{b}{a}$$

Not complete.

:::{.solution}

### 2.5.1 a

Letting $x_n := \frac{1}{n}$, we have

$$\sum_{k=1}^{\infty} |f_k(x)| \geq |f_n(x_n)| = \left| ae^{-ax} - be^{-bx} \right| := M.$$

In particular, $\sup_x |f_n(x)| \not\longrightarrow 0$, so the terms do not go to zero and the sum can not converge.

### 2.5.2 b

?

:::

## 2.6 Fall 2016 # 1

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that $f$ converges to a differentiable function on $(1, \infty)$ and that

$$f'(x) = \sum_{n=1}^{\infty} \left( \frac{1}{n^x} \right)'.$$

Hint:

$$\left( \frac{1}{n^x} \right)' = -\frac{1}{n^x} \ln n$$

*Solution.*

- Set $f_N(x) := \sum_{n=1}^{N} n^{-x}$, so $f(x) = \lim_{N \longrightarrow \infty} f_N(x)$.

- If an interchange of limits is justified, we have

$$\begin{aligned}
\frac{\partial}{\partial x} \lim_{N \longrightarrow \infty} \sum_{n=1}^{N} n^{-x} &= \lim_{h \longrightarrow 0} \lim_{N \longrightarrow \infty} \frac{1}{h} \left[ \left( \sum_{n=1}^{N} n^{-x} \right) - \left( \sum_{n=1}^{N} n^{-(x+h)} \right) \right] \\
&\overset{?}{=} \lim_{N \longrightarrow \infty} \lim_{h \longrightarrow 0} \frac{1}{h} \left[ \left( \sum_{n=1}^{N} n^{-x} \right) - \left( \sum_{n=1}^{N} n^{-(x+h)} \right) \right] \\
&= \lim_{N \longrightarrow \infty} \lim_{h \longrightarrow 0} \frac{1}{h} \left[ \sum_{n=1}^{N} n^{-x} - n^{-(x+h)} \right] \quad (1) \\
&= \lim_{N \longrightarrow \infty} \sum_{n=1}^{N} \lim_{h \longrightarrow 0} \frac{1}{h} \left[ n^{-x} - n^{-(x+h)} \right] \quad \text{since this is a finite sum} \\
&:= \lim_{N \longrightarrow \infty} \sum_{n=1}^{N} \frac{\partial}{\partial x} \left( \frac{1}{n^x} \right) \\
&= \lim_{N \longrightarrow \infty} \sum_{n=1}^{N} - \frac{\ln(n)}{n^x},
\end{aligned}$$

  where the combining of sums in (1) is valid because $\sum n^{-x}$ is absolutely convergent for $x > 1$ by the $p$-test.
- Thus it suffices to justify the interchange of limits and show that the last sum converges on $(1, \infty)$.
- Claim: $\sum n^{-x} \ln(n)$ converges.
  - Use the fact that for any fixed $\varepsilon > 0$,

$$\lim_{n \longrightarrow \infty} \frac{\ln(n)}{n^{\varepsilon}} \overset{L.H.}{=} \lim_{n \longrightarrow \infty} \frac{1/n}{\varepsilon n^{\varepsilon - 1}} = \lim_{n \longrightarrow \infty} \frac{1}{\varepsilon n^{\varepsilon}} = 0,$$

  - This implies that for a fixed $\varepsilon > 0$ and for any constant $c > 0$ there exists an $N$ large enough such that $n \geq N$ implies $\ln(n)/n^{\varepsilon} < c$, i.e. $\ln(n) < cn^{\varepsilon}$.
  - Taking $c = 1$, we have $n \geq N \implies \ln(n) < n^{\varepsilon}$
  - We thus break up the sum:

$$\begin{aligned}
\sum_{n \in \mathbb{N}} \frac{\ln(n)}{n^x} &= \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{\ln(n)}{n^x} \\
&\leq \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{n^{\varepsilon}}{n^x} \\
&:= C_{\varepsilon} + \sum_{n=N}^{\infty} \frac{n^{\varepsilon}}{n^x} \quad \text{with } C_{\varepsilon} < \infty \text{ a constant} \\
&= C_{\varepsilon} + \sum_{n=N}^{\infty} \frac{1}{n^{x - \varepsilon}},
\end{aligned}$$

    where the last term converges by the $p$-test if $x - \varepsilon > 1$.
  - But $\varepsilon$ can depend on $x$, and if $x \in (1, \infty)$ is fixed we can choose $\varepsilon < |x - 1|$ to ensure this.
- Claim: the interchange of limits is justified.

?

### 2.7 Fall 2016 # 5

Let $\varphi \in L^\infty(\mathbb{R})$. Show that the following limit exists and satisfies the equality

$$\lim_{n \longrightarrow \infty} \left( \int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2}\, dx \right)^{\frac{1}{n}} = \|\varphi\|_\infty.$$

### 2.8 Fall 2016 # 6

Let $f, g \in L^2(\mathbb{R})$. Show that

$$\lim_{n \longrightarrow \infty} \int_{\mathbb{R}} f(x) g(x+n)\, dx = 0$$

### 2.9 Spring 2016 # 1

For $n \in \mathbb{N}$, define

$$]e_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad E_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

Show that $e_n < E_n$, and prove Bernoulli's inequality:

$$(1+x)^n \geq 1 + nx \text{ for } -1 < x < \infty \text{ and } n \in \mathbb{N}$$

Use this to show the following:

1. The sequence $e_n$ is increasing.
2. The sequence $E_n$ is decreasing.
3. $2 < e_n < E_n < 4$.
4. $\lim_{n \longrightarrow \infty} e_n = \lim_{n \longrightarrow \infty} E_n$.

### 2.10 Fall 2015 # 1

Define

$$f(x) = c_0 + c_1 x^1 + c_2 x^2 + \ldots + c_n x^n \text{ with } n \text{ even and } c_n > 0.$$

Show that there is a number $x_m$ such that $f(x_m) \leq f(x)$ for all $x \in \mathbb{R}$.

## 3 Measure Theory: Sets

### 3.1 Spring 2020 # 2

Let $m_*$ denote the Lebesgue outer measure on $\mathbb{R}$.

#### 3.1.1 a.

Prove that for every $E \subseteq \mathbb{R}$ there exists a Borel set $B$ containing $E$ such that

$$m_*(B) = m_*(E).$$

#### 3.1.2 b.

Prove that if $E \subseteq \mathbb{R}$ has the property that

$$m_*(A) = m_*(A \bigcap E) + m_*(A \bigcap E^c)$$

for every set $A \subseteq \mathbb{R}$, then there exists a Borel set $B \subseteq \mathbb{R}$ such that $E = B \setminus N$ with $m_*(N) = 0$.

Be sure to address the case when $m_*(E) = \infty$.

*Solution.*
Concepts used:

- Definition of outer measure: $m_*(E) = \inf_{\{Q_j\} \rightrightarrows E} \sum |Q_j|$ where $\{Q_j\}$ is a countable collection of closed cubes.
- Break $\mathbb{R}$ into $\coprod_{n \in \mathbb{Z}} [n, n+1)$, each with finite measure.
- Theorem: $m_*(Q) = |Q|$ for $Q$ a closed cube (i.e. the outer measure equals the volume).

**Proof (of Theorem)** Statement: if $Q$ is a closed cube, then $m_*(Q) = |Q|$, the usual volume.

- $m_*(Q) \leq |Q|$:
  - Since $Q \subseteq Q$, $Q \rightrightarrows Q$ and $m_*(Q) \leq |Q|$ since $m_*$ is an infimum over such coverings.
- $|Q| \leq m_*(Q)$:
  - Fix $\varepsilon > 0$.
  - Let $\{Q_i\}_{i=1}^\infty \rightrightarrows Q$ be arbitrary, it suffices to show that

$$|Q| \leq \left( \sum_{i=1}^\infty |Q_i| \right) + \varepsilon.$$

  - Pick open cubes $S_i$ such that $Q_i \subseteq S_i$ and $|Q_i| \leq |S_i| \leq (1+\varepsilon)|Q_i|$.
  - Then $\{S_i\} \rightrightarrows Q$, so by compactness of $Q$ pick a finite subcover with $N$ elements.
  - Note

$$Q \subseteq \bigcup_{i=1}^N S_i \implies |Q| \leq \sum_{i=1}^N |S_i| \leq \sum_{i=1}^N (1+\varepsilon)|Q_j| \leq (1+\varepsilon) \sum_{i=1}^\infty |Q_i|.$$

  - Taking an infimum over coverings on the RHS preserves the inequality, so

$$|Q| \leq (1+\varepsilon) m_*(Q)$$

  - Take $\varepsilon \longrightarrow 0$ to obtain final inequality.

#### 3.1.3 a

- If $m_*(E) = \infty$, then take $B = \mathbb{R}^n$ since $m(\mathbb{R}^n) = \infty$.
- Suppose $N \coloneqq m_*(E) < \infty$.
- Since $m_*(E)$ is an infimum, by definition, for every $\varepsilon > 0$ there exists a covering by closed cubes $\{Q_i(\varepsilon)\}_{i=1}^\infty \rightrightarrows E$ depending on $\varepsilon$ such that

$$\sum_{i=1}^\infty |Q_i(\varepsilon)| < N + \varepsilon.$$

- For each fixed $n$, set $\varepsilon_n = \frac{1}{n}$ to produce such a covering $\{Q_i(\varepsilon_n)\}_{i=1}^\infty$ and set $B_n \coloneqq \bigcup_{i=1}^\infty Q_i(\varepsilon_n)$.
- The outer measure of cubes is *equal* to the sum of their volumes, so

$$m_*(B_n) = \sum_{i=1}^\infty |Q_i(\varepsilon_n)| < N + \varepsilon_n = N + \frac{1}{n}.$$

- Now set $B \coloneqq \bigcap_{n=1}^\infty B_n$.
  - Since $E \subseteq B_n$ for every $n$, $E \subseteq B$
  - Since $B$ is a countable intersection of countable unions of closed sets, $B$ is Borel.
  - Since $B_n \subseteq B$ for every $n$, we can apply subadditivity to obtain the inequality

$$E \subseteq B \subseteq B_n \implies N \le m_*(B) \le m_*(B_n) < N + \frac{1}{n} \quad \text{for all} \quad n \in \mathbb{Z}^{\ge 1}.$$

- This forces $m_*(E) = m_*(B)$.

#### 3.1.4 b

Suppose $m_*(E) < \infty$.
- By (a), find a Borel set $B \supseteq E$ such that $m_*(B) = m_*(E)$
- Note that $E \subseteq B \implies B \bigcap E = E$ and $B \bigcap E^c = B \setminus E$.
- By assumption,

$$\begin{aligned} m_*(B) &= m_*(B \bigcap E) + m_*(B \bigcap E^c) \\ m_*(E) &= m_*(E) + m_*(B \setminus E) \\ m_*(E) - m_*(E) &= m_*(B \setminus E) && \text{since } m_*(E) < \infty \\ \implies m_*(B \setminus E) &= 0. \end{aligned}$$

- So take $N = B \setminus E$; this shows $m_*(N) = 0$ and $E = B \setminus (B \setminus E) = B \setminus N$.

If $m_*(E) = \infty$:
- Apply result to $E_R \coloneqq E \bigcap [R, R+1)^n \subset \mathbb{R}^n$ for $R \in \mathbb{Z}$, so $E = \coprod_R E_R$
- Obtain $B_R, N_R$ such that $E_R = B_R \setminus N_R$, $m_*(E_R) = m_*(B_R)$, and $m_*(N_R) = 0$.
- Note that
  - $B \coloneqq \bigcup_R B_R$ is a union of Borel sets and thus still Borel
  - $E = \bigcup_R E_R$
  - $N \coloneqq B \setminus E$
  - $N' \coloneqq \bigcup_R N_R$ is a union of null sets and thus still null
- Since $E_R \subset B_R$ for every $R$, we have $E \subset B$
- We can compute

$$N = B \setminus E = \left( \bigcup_R B_R \right) \setminus \left( \bigcup_R E_R \right) \subseteq \bigcup_R (B_R \setminus E_R) = \bigcup_R N_R \coloneqq N'$$

where $m_*(N') = 0$ since $N'$ is null, and thus subadditivity forces $m_*(N) = 0$.

## 3.2 Fall 2019 # 3.

Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) = 1$ and $\{B_n\}_{n=1}^\infty$ be a sequence of $\mathcal{B}$-measurable subsets of $X$, and

$$B := \left\{ x \in X \;\middle|\; x \in B_n \text{ for infinitely many } n \right\}.$$

a. Argue that $B$ is also a $\mathcal{B}$-measurable subset of $X$.

b. Prove that if $\displaystyle\sum_{n=1}^\infty \mu(B_n) < \infty$ then $\mu(B) = 0$.

c. Prove that if $\displaystyle\sum_{n=1}^\infty \mu(B_n) = \infty$ **and** the sequence of set complements $\{B_n^c\}_{n=1}^\infty$ satisfies

$$\mu\left(\bigcap_{n=k}^K B_n^c\right) = \prod_{n=k}^K (1 - \mu(B_n))$$

for all positive integers $k$ and $K$ with $k < K$, then $\mu(B) = 1$.

Hint: Use the fact that $1 - x \leq e^{-x}$ for all $x$.

*Solution.*
Concepts used:

- Borel-Cantelli: for a sequence of sets $X_n$,

$$\begin{aligned} \limsup_n X_n &= \left\{ x \;\middle|\; x \in X_n \text{ for infinitely many } n \right\} &&= \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} X_n \\ \liminf_n X_n &= \left\{ x \;\middle|\; x \in X_n \text{ for all but finitely many } n \right\} &&= \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} X_n. \end{aligned}$$

- Properties of logs and exponentials:

$$\prod_n e^{x_n} = e^{\Sigma_n x_n} \quad \text{and} \quad \sum_n \log(x_n) = \log\left(\prod_n x_n\right).$$

- Tails of convergent sums vanish.
- Continuity of measure: $B_n \searrow B$ and $\mu(B_0) < \infty$ implies $\lim_n \mu(B_n) = \mu(B)$, and $B_n \nearrow B \implies \lim_n \mu(B_n) = \mu(B)$.

### 3.2.1 a

- The Borel $\sigma$-algebra is closed under countable unions/intersections/complements,
- $B = \limsup_n B_n$ is an intersection of unions of measurable sets.

### 3.2.2 b

- Tails of convergent sums go to zero, so $\displaystyle\sum_{n \geq M} \mu(B_n) \xrightarrow{M \longrightarrow \infty} 0$,
- $\displaystyle B_M := \bigcap_{m=1}^M \bigcup_{n \geq m} B_n \searrow B.$

$$
\begin{aligned}
\mu(B_M) &= \mu\left(\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} B_n\right) \\
&\leq \mu\left(\bigcup_{n\geq m} B_n\right) \quad \text{for all } m \in \mathbb{N} \text{ by countable subadditivity} \\
&\longrightarrow 0,
\end{aligned}
$$

- The result follows by continuity of measure.

### 3.2.3 c

- To show $\mu(B) = 1$, we'll show $\mu(B^c) = 0$.
- Let $B_k = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{K} B_n$. Then

$$
\begin{aligned}
\mu(B_K^c) &= \mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{K} B_n^c\right) \\
&\leq \sum_{m=1}^{\infty} \mu\left(\bigcap_{n=m}^{K} B_n^c\right) \quad \text{by subadditivity} \\
&= \sum_{m=1}^{\infty} \prod_{n=m}^{K} (1 - \mu(B_n)) \quad \text{by assumption} \\
&\leq \sum_{m=1}^{\infty} \prod_{n=m}^{K} e^{-\mu(B_n^c)} \quad \text{by hint} \\
&= \sum_{m=1}^{\infty} \exp\left(-\sum_{n=m}^{K} \mu(B_n^c)\right) \\
&\overset{K\longrightarrow\infty}{\longrightarrow} 0
\end{aligned}
$$

since $\sum_{n=m}^{K} \mu(B_n^c) \overset{K\longrightarrow\infty}{\longrightarrow} \infty$ by assumption

- We can apply continuity of measure since $B_K^c \xrightarrow{K\longrightarrow\infty} B^c$.

Proving the hint: ?

## 3.3 Spring 2019 # 2

Let $\mathcal{B}$ denote the set of all Borel subsets of $\mathbb{R}$ and $\mu : \mathcal{B} \longrightarrow [0, \infty)$ denote a finite Borel measure on $\mathbb{R}$.

### 3.3.1 a

Prove that if $\{F_k\}$ is a sequence of Borel sets for which $F_k \supseteq F_{k+1}$ for all $k$, then

$$\lim_{k\to\infty} \mu\left(F_k\right) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right)$$

### 3.3.2 b

Suppose $\mu$ has the property that $\mu(E) = 0$ for every $E \in \mathcal{B}$ with Lebesgue measure $m(E) = 0$.

Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $E \in \mathcal{B}$ with $m(E) < \delta$, then $\mu(E) < \varepsilon$.

*Solution.*

### 3.3.3 a

See Folland p.26

- Lemma 1: $\mu(\coprod_{k=1}^{\infty} E_k) = \lim_{N\longrightarrow\infty} \sum_{k=1}^{N} \mu(E_k)$.
- Suppose $F_0 \supseteq F_1 \supseteq \cdots$.
- Let $A_k = F_k \setminus F_{k+1}$, since the $F_k$ are nested the $A_k$ are disjoint
- Set $A := \coprod_{k=1}^{\infty} A_k$ and $F := \bigcap_{k=1}^{\infty} F_k$.
- Note $X = X \setminus Y \coprod X \bigcap Y$ for any two sets (just write $X \setminus Y := X \bigcap Y^c$)
- Note that $A$ contains anything that was removed from $F_0$ when passing from any $F_j$ to $F_{j+1}$, while $F$ contains everything that is never removed at any stage, and these are disjoint possibilities.
- Thus $F_0 = F \coprod A$, so

$$\begin{aligned}
\mu(F_0) &= \mu(F) + \mu(A) \\
&= \mu(F) + \mu(\textstyle\coprod_{k=1}^{\infty} A_k) \\
&= \mu(F) + \lim_{n\longrightarrow\infty} \sum_{k=0}^{n} \mu(A_k) \quad \text{by countable additivity} \\
&= \mu(F) + \lim_{n\longrightarrow\infty} \sum_{k=0}^{n} \mu(F_k) - \mu(F_{k+1}) \\
&= \mu(F) + \lim_{n\longrightarrow\infty} \left(\mu(F_1) - \mu(F_n)\right) \quad \text{(Telescoping)} \\
&= \mu(F) + \mu(F_1) - \lim_{N\longrightarrow\infty} \mu(F_n),
\end{aligned}$$

- Since $\mu$ is a finite measure, $\mu(F_1) < \infty$ and can be subtracted, yielding

$$\begin{aligned}
\mu(F_1) &= \mu(F) + \mu(F_1) - \lim_{n\longrightarrow\infty} \mu(F_n) \\
\implies \mu(F) &= \lim_{n\longrightarrow\infty} \mu(F_n) \\
\implies \mu\left(\bigcap_{k=1}^{\infty} F_k\right) &= \lim_{n\longrightarrow\infty} \mu(F_n).
\end{aligned}$$

### 3.3.4 b

- Toward a contradiction, negate the implication: suppose there exists an $\varepsilon > 0$ such that for all $\delta$, we have $m(E) < \delta$ but $\mu(E) > \varepsilon$.
- The sequence $\left\{\delta_n := \frac{1}{2^n}\right\}_{n\in\mathbb{N}}$ and produce sets $A_n \in \mathcal{B}$ such $m(A_n) < \frac{1}{2^n}$ but $\mu(A_n) > \varepsilon$.
- Define

$$F_n := \bigcup_{j\geq n} A_j$$
$$C_m := \bigcap_{k=1}^{m} F_k$$
$$A := C_\infty := \bigcap_{k=1}^{\infty} F_k.$$

- Note that $F_1 \supseteq F_2 \supseteq \cdots$, since each increase in index unions fewer sets.
- By continuity for the Lebesgue measure,

$$m(A) = m\left(\bigcap_{k=1}^{\infty} F_k\right) = \lim_{k\longrightarrow\infty} m(F_k) = \lim_{k\longrightarrow\infty} m\left(\bigcup_{j\geq k} A_j\right) \leq \lim_{k\longrightarrow\infty} \sum_{j\geq k} m(A_j) = \lim_{k\longrightarrow\infty} \sum_{j\geq k} \frac{1}{2^n} = 0,$$

  which follows because this is the tail of a convergent sum
- Thus $m(A) = 0$ and by assumption, this implies $\mu(A) = 0$.
- However, by part (a),

$$\mu(A) = \lim_n \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \geq \lim_n \mu(A_n) = \lim_n \varepsilon = \varepsilon > 0.$$

All messed up!

## 3.4 Fall 2018 # 2

Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Show that there is a Borel set $B \subset E$ such that $m(E \setminus B) = 0$.

*Solution.*
Concepts used:

- Definition of measurability: there exists an open $O \supset E$ such that $m_*(O \setminus E) < \varepsilon$ for all $\varepsilon > 0$.
- Theorem: $E$ is Lebesgue measurable iff there exists a closed set $F \subseteq E$ such that $m_*(E \setminus F) < \varepsilon$ for all $\varepsilon > 0$.
- Every $F_\sigma, G_\delta$ is Borel.
- Claim: $E$ is measurable $\iff$ for every $\varepsilon$ there exist $F_\varepsilon \subset E \subset G_\varepsilon$ with $F_\varepsilon$ closed and $G_\varepsilon$ open and $m(G_\varepsilon \setminus E) < \varepsilon$ and $m(E \setminus F_\varepsilon) < \varepsilon$.
  - Proof: existence of $G_\varepsilon$ is the definition of measurability.
  - Existence of $F_\varepsilon$: ?

- Claim: $E$ is measurable $\implies$ there exists an open $O \supseteq E$ such that $m(O \setminus E) = 0$.
  - Since $E$ is measurable, for each $n \in \mathbb{N}$ choose $G_n \supseteq E$ such that $m_*(G_n \setminus E) < \frac{1}{n}$.
  - Set $O_N := \bigcap_{n=1}^{N} G_n$ and $O := \bigcap_{n=1}^{\infty} G_n$.
  - Suppose $E$ is bounded.
    * Note $O_N \searrow O$ and $m_*(O_1) < \infty$ if $E$ is bounded, since in this case

$$m_*(G_n \setminus E) = m_*(G_1) - m_*(E) < 1 \iff m_*(G_1) < m_*(E) + \frac{1}{n} < \infty.$$

    * Note $O_N \setminus E \searrow O \setminus E$ since $O_N \setminus E := O_N \bigcap E^c \supseteq O_{N+1} \bigcap E^c$ for all $N$, and again $m_*(O_1 \setminus E) < \infty$.
    * So it's valid to apply continuity of measure from above:

$$\begin{aligned} m_*(O \setminus E) &= \lim_{N \longrightarrow \infty} m_*(O_N \setminus E) \\ &\leq \lim_{N \longrightarrow \infty} m_*(G_N \setminus E) \\ &= \lim_{N \longrightarrow \infty} \frac{1}{N} = 0, \end{aligned}$$

    where the inequality uses subadditivity on $\bigcap_{n=1}^{N} G_n \subseteq G_N$
  - Suppose $E$ is unbounded.
    * Write $E^k = E \bigcap [k, k+1]^d \subset \mathbb{R}^d$ as the intersection of $E$ with an annulus, and note that $E = \coprod_{k \in \mathbb{N}} E_k$.
    * Each $E_k$ is bounded, so apply the previous case to obtain $O_k \supseteq E_k$ with $m(O_k \setminus E_k) = 0$.
    * So write $O_k = E_k \coprod N_k$ where $N_k := O_k \setminus E_k$ is a null set.
    * Define $O = \bigcup_{k \in \mathbb{N}} O_k$, note that $E \subseteq O$.
    * Now note

$$\begin{aligned} O \setminus E &= \left(\coprod_k O_k\right) \setminus \left(\coprod_K E_k\right) \\ &\subseteq \coprod_k (O_k \setminus E_k) \\ \implies m_*(O \setminus E) &\leq m_*\left(\coprod (O_k \setminus E_k)\right) = 0, \end{aligned}$$

    since any countable union of null sets is again null.
  - So $O \supseteq E$ with $m(O \setminus E) = 0$.
- Theorem: since $E$ is measurable, $E^c$ is measurable
  - Proof: It suffices to write $E^c$ as the union of two measurable sets, $E^c = S \bigcup (E^c - S)$, where $S$ is to be determined.
  - We'll produce an $S$ such that $m_*(E^c - S) = 0$ and use the fact that any subset of a null set is measurable.
  - Since $E$ is measurable, for every $\varepsilon > 0$ there exists an open $\mathcal{O}_\varepsilon \supseteq E$ such that $m_*(\mathcal{O}_\varepsilon \setminus E) < \varepsilon$.
  - Take the sequence $\left\{\varepsilon_n := \frac{1}{n}\right\}$ to produce a sequence of sets $\mathcal{O}_n$.

- Note that each $\mathcal{O}_n^c$ is closed and

$$\mathcal{O}_n \supseteq E \iff \mathcal{O}_n^c \subseteq E^c.$$

- Set $S := \bigcup_n \mathcal{O}_n^c$, which is a union of closed sets, thus an $F_\sigma$ set, thus Borel, thus measurable.
- Note that $S \subseteq E^c$ since each $\mathcal{O}_n \subseteq E^c$.
- Note that

$$\begin{aligned}
E^c \setminus S &:= E^c \setminus \left(\bigcup_{n=1}^{\infty} \mathcal{O}_n^c\right) \\
&:= E^c \bigcap \left(\bigcup_{n=1}^{\infty} \mathcal{O}_n^c\right)^c \quad \text{definition of set minus} \\
&= E^c \bigcap \left(\bigcap_{n=1}^{\infty} \mathcal{O}_n\right)^c \quad \text{De Morgan's law} \\
&= E^c \bigcup \left(\bigcap_{n=1}^{\infty} \mathcal{O}_n\right) \\
&:= \left(\bigcap_{n=1}^{\infty} \mathcal{O}_n\right) \setminus E \\
&\subseteq \mathcal{O}_N \setminus E \quad \text{for every } N \in \mathbb{N}.
\end{aligned}$$

- Then by subadditivity,

$$m_*(E^c \setminus S) \leq m_*(\mathcal{O}_N \setminus E) \leq \frac{1}{N} \quad \forall N \implies m_*(E^c \setminus S) = 0.$$

- Thus $E^c \setminus S$ is measurable.

### 3.4.1 Indirect Proof

- Since $E$ is measurable, $E^c$ is measurable.
- Since $E^c$ is measurable exists an open $O \supseteq E^c$ such that $m(O \setminus E^c) = 0$.
- Set $B := O^c$, then $O \supseteq E^c \iff \mathcal{O}^c \subseteq E \iff B \subseteq E$.
- Computing measures yields

$$E \setminus B := E \setminus \mathcal{O}^c := E \bigcap (\mathcal{O}^c)^c = E \bigcap \mathcal{O} = \mathcal{O} \bigcap (E^c)^c := \mathcal{O} \setminus E^c,$$

  thus $m(E \setminus B) = m(\mathcal{O} \setminus E^c) = 0$.
- Since $\mathcal{O}$ is open, $B$ is closed and thus Borel.

### 3.4.2 Direct Proof (Todo)

?

Try to construct the set.

## 3.5 Spring 2018 # 1

Define

$$E := \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N} \right\}.$$

Prove that $m(E) = 0$.

*Solution.*
Concepts used:

- Borel-Cantelli: If $\{E_k\}_{k \in \mathbb{Z}} \subset 2^{\mathbb{R}}$ is a countable collection of Lebesgue measurable sets with $\sum_{k \in \mathbb{Z}} m(E_k) < \infty$, then almost every $x \in \mathbb{R}$ is in *at most finitely* many $E_k$.
  - Equivalently (?), $m(\limsup_{k \longrightarrow \infty} E_k) = 0$, where $\limsup_{k \longrightarrow \infty} E_k = \bigcap_{k=1}^{\infty} \bigcup_{j \geq k} E_j$, the elements which are in $E_k$ for infinitely many $k$.

**Solution**:

- Strategy: Borel-Cantelli.
- We'll show that $m(E) \bigcap [n, n+1] = 0$ for all $n \in \mathbb{Z}$; then the result follows from

$$m(E) = m\left( \bigcup_{n \in \mathbb{Z}} E \bigcap [n, n+1] \right) \leq \sum_{n=1}^{\infty} m(E \bigcap [n, n+1]) = 0.$$

- By translation invariance of measure, it suffices to show $m(E \bigcap [0, 1]) = 0$.
  - So WLOG, replace $E$ with $E \bigcap [0, 1]$.
- Define

$$E_j := \left\{ x \in [0, 1] \;\middle|\; \exists p \in \mathbb{Z}^{\geq 0} \text{ s.t. } \left| x - \frac{p}{j} \right| < \frac{1}{j^3} \right\}.$$

  - Note that $E_j \subseteq \coprod_{p \in \mathbb{Z}^{\geq 0}} B_{j^{-3}}\left( \frac{p}{j} \right)$, i.e. a union over integers $p$ of intervals of radius $1/j^3$ around the points $p/j$. Since $1/j^3 < 1/j$, this union is in fact disjoint.
- Importantly, note that

$$\limsup_{j \longrightarrow \infty} E_j := \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j = E$$

  since

$$\begin{aligned} x \in \limsup_j E_j &\iff x \in E_j \text{ for infinitely many } j \\ &\iff \text{there are infinitely many } j \text{ for which there exist a } p \text{ such that } \left| x - \frac{p}{j} \right| < j^{-3} \\ &\iff \text{there are infinitely many such pairs } p, j \\ &\iff x \in E. \end{aligned}$$

- Intersecting with $[0,1]$, we can write $E_j$ as a union of intervals:

$$E_j = \left(0, j^{-3}\right) \quad \coprod \quad B_{j^{-3}}\left(\frac{1}{j}\right) \coprod B_{j^{-3}}\left(\frac{2}{j}\right) \coprod \cdots \coprod B_{j^{-3}}\left(\frac{j-1}{j}\right) \quad \coprod \quad (1-j^{-3}, 1),$$

  where we've separated out the "boundary" terms to emphasize that they are balls about 0 and 1 intersected with $[0,1]$.
- Since $E_j$ is a union of open sets, it is Borel and thus Lebesgue measurable.
- Computing the measure of $E_j$:
  - For a fixed $j$, there are exactly $j+1$ possible choices for a numerator $(0, 1, \cdots, j)$, thus there are exactly $j+1$ sets appearing in the above decomposition.
  - The first and last intervals are length $\frac{1}{j^3}$
  - The remaining $(j+1)-2 = j-1$ intervals are twice this length, $\frac{2}{j^3}$
  - Thus

$$m(E_j) = 2\left(\frac{1}{j^3}\right) + (j-1)\left(\frac{2}{j^3}\right) = \frac{2}{j^2}$$

- Note that

$$\sum_{j \in \mathbb{N}} m(E_j) = 2 \sum_{j \in \mathbb{N}} \frac{1}{j^2} < \infty,$$

  which converges by the $p$-test for sums.
- But then

$$\begin{aligned} m(E) &= m(\limsup_j E_j) \\ &= m\Big(\bigcap_{n \in \mathbb{N}} \bigcup_{j \geq n} E_j\Big) \\ &\leq m\Big(\bigcup_{j \geq N} E_j\Big) \quad \text{for every } N \\ &\leq \sum_{j \geq N} m(E_j) \\ &\overset{N \longrightarrow \infty}{\longrightarrow} 0 \quad . \end{aligned}$$

- Thus $E$ is measurable as a subset of a null set and $m(E) = 0$.

## 3.6 Fall 2017 # 2

Let $f(x) = x^2$ and $E \subset [0, \infty) := \mathbb{R}^+$.

1. Show that

$$m^*(E) = 0 \iff m^*(f(E)) = 0.$$

2. Deduce that the map

$$\varphi : \mathcal{L}(\mathbb{R}^+) \longrightarrow \mathcal{L}(\mathbb{R}^+)$$
$$E \mapsto f(E)$$

is a bijection from the class of Lebesgue measurable sets of $[0, \infty)$ to itself.

*Solution.*

### 3.6.1 a

It suffices to consider the bounded case, i.e. $E \subseteq B_M(0)$ for some $M$. Then write $E_n = B_n(0) \bigcap E$ and apply the theorem to $E_n$, and by subadditivity, $m^*(E) = m^*(\bigcup_n E_n) \leq \sum_n m^*(E_n) = 0$.

**Lemma:** $f(x) = x^2, f^{-1}(x) = \sqrt{x}$ are Lipschitz on any compact subset of $[0, \infty)$.
*Proof:* Let $g = f$ or $f^{-1}$. Then $g \in C^1([0, M])$ for any $M$, so $g$ is differentiable and $g'$ is continuous. Since $g'$ is continuous on a compact interval, it is bounded, so $|g'(x)| \leq L$ for all $x$. Applying the MVT,

$$|f(x) - f(y)| = f'(c)|x - y| \leq L|x - y|.$$

**Lemma:** If $g$ is Lipschitz on $\mathbb{R}^n$, then $m(E) = 0 \implies m(g(E)) = 0$.
*Proof:* If $g$ is Lipschitz, then

$$g(B_r(x)) \subseteq B_{Lr}(x),$$

which is a dilated ball/cube, and so

$$m^*(B_{Lr}(x)) \leq L^n \cdot m^*(B_r(x)).$$

Now choose $\{Q_j\} \rightrightarrows E$; then $\{g(Q_j)\} \rightrightarrows g(E)$.
By the above observation,

$$|g(Q_j)| \leq L^n |Q_j|,$$

and so

$$m^*(g(E)) \leq \sum_j |g(Q_j)| \leq \sum_j L^n |Q_j| = L^n \sum_j |Q_j| \longrightarrow 0.$$

Now just take $g(x) = x^2$ for one direction, and $g(x) = f^{-1}(x) = \sqrt{x}$ for the other. ■

### 3.6.2 b

Lemma: $E$ is measurable iff $E = K \coprod N$ for some $K$ compact, $N$ null.

Write $E = K \coprod N$ where $K$ is compact and $N$ is null.
Then $\varphi^{-1}(E) = \varphi^{-1}(K \coprod N) = \varphi^{-1}(K) \coprod \varphi^{-1}(N)$.
Since $\varphi^{-1}(N)$ is null by part (a) and $\varphi^{-1}(K)$ is the preimage of a compact set under a continuous map and thus compact, $\varphi^{-1}(E) = K' \coprod N'$ where $K'$ is compact and $N'$ is null, so $\varphi^{-1}(E)$ is measurable.
So $\varphi$ is a measurable function, and thus yields a well-defined map $\mathcal{L}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathbb{R})$ since it preserves measurable sets. Restricting to $[0, \infty)$, $f$ is bijection, and thus so is $\varphi$.

## 3.7 Spring 2017 # 2

### 3.7.1 a

Let $\mu$ be a measure on a measurable space $(X, \mathcal{M})$ and $f$ a positive measurable function.

Define a measure $\lambda$ by

$$\lambda(E) := \int_E f \, d\mu, \quad E \in \mathcal{M}$$

Show that for $g$ any positive measurable function,

$$\int_X g \, d\lambda = \int_X fg \, d\mu$$

### 3.7.2 b

Let $E \subset \mathbb{R}$ be a measurable set such that

$$\int_E x^2 \, dm = 0.$$

Show that $m(E) = 0$.

*Solution.*
Concepts used:

- Absolute continuity of measures: $\lambda \ll \mu \iff E \in \mathcal{M}, \mu(E) = 0 \implies \lambda(E) = 0$.
- Radon-Nikodym: if $\lambda \ll \mu$, then there exists a measurable function $\dfrac{\partial \lambda}{\partial \mu} := f$ where $\lambda(E) = \displaystyle\int_E f \, d\mu$.
- Chebyshev's inequality:

$$A_c := \left\{ x \in X \;\middle|\; |f(x)| \geq c \right\} \implies \mu(A_c) \leq c^{-p} \int_{A_c} |f|^p \, d\mu \quad \forall 0 < p < \infty.$$

### 3.7.3 a

- Strategy: use approximation by simple functions to show absolute continuity and apply Radon-Nikodym
- Claim: $\lambda \ll \mu$, i.e. $\mu(E) = 0 \implies \lambda(E) = 0$.
  - Note that if this holds, by Radon-Nikodym, $f = \dfrac{\partial \lambda}{\partial \mu} \implies d\lambda = f d\mu$, which would yield

$$\int g \, d\lambda = \int gf \, d\mu.$$

- So let $E$ be measurable and suppose $\mu(E) = 0$.
- Then

$$\lambda(E) := \int_E f \, d\mu = \lim_{n \longrightarrow \infty} \left\{ \int_E s_n \, d\mu \;\middle|\; s_n := \sum_{j=1}^{\infty} c_j \mu(E_j), \, s_n \nearrow f \right\}$$

where we take a sequence of simple functions increasing to $f$.

- But since each $E_j \subseteq E$, we must have $\mu(E_j) = 0$ for any such $E_j$, so every such $s_n$ must be zero and thus $\lambda(E) = 0$.

What is the final step in this approximation?

### 3.7.4 b

- Set $g(x) = x^2$, note that $g$ is positive and measurable.
- By part (a), there exists a positive $f$ such that for any $E \subseteq \mathbb{R}$,

$$\int_E g \, dm = \int_E gf \, d\mu$$

  - The LHS is zero by assumption and thus so is the RHS.
  - $m \ll \mu$ by construction.
  - Note that $gf$ is positive.
- Define $A_k = \left\{ x \in X \;\middle|\; gf \cdot \chi_E > \frac{1}{k} \right\}$, for $k \in \mathbb{Z}^{\geq 0}$
- Then by Chebyshev with $p = 1$, for every $k$ we have

$$\mu(A_k) \leq k \int_E gf \, d\mu = 0$$

- Then noting that $A_k \searrow A := \left\{ x \in X \;\middle|\; gf \cdot \chi_E(x) > 0 \right\}$, we have $\mu(A) = 0$.
- Since $gf$ is positive, we have

$$x \in E \iff gf\chi_E(x) > 0 \iff x \in A$$

  so $E = A$ and $\mu(E) = \mu(A)$.
- But $m \ll \mu$ and $\mu(E) = 0$, so we can conclude that $m(E) = 0$.

## 3.8 Fall 2016 # 4

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\{E_n\} \subset \mathcal{M}$ satisfies

$$\lim_{n \to \infty} \mu(X \backslash E_n) = 0.$$

Define

$$G := \left\{ x \in X \;\middle|\; x \in E_n \text{ for only finitely many } n \right\}.$$

Show that $G \in \mathcal{M}$ and $\mu(G) = 0$.

## 3.9 Spring 2016 # 3

Let $f$ be Lebesgue measurable on $\mathbb{R}$ and $E \subset \mathbb{R}$ be measurable such that

$$0 < A = \int_E f(x) dx < \infty.$$

Show that for every $0 < t < 1$, there exists a measurable set $E_t \subset E$ such that

$$\int_{E_t} f(x)dx = tA.$$

## 3.10 Spring 2016 # 5

Let $(X, \mathcal{M}, \mu)$ be a measure space. For $f \in L^1(\mu)$ and $\lambda > 0$, define

$$\varphi(\lambda) = \mu(\{x \in X | f(x) > \lambda\}) \quad \text{and} \quad \psi(\lambda) = \mu(\{x \in X | f(x) < -\lambda\})$$

Show that $\varphi, \psi$ are Borel measurable and

$$\int_X |f| \ d\mu = \int_0^\infty [\varphi(\lambda) + \psi(\lambda)] \ d\lambda$$

## 3.11 Fall 2015 # 2

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be Lebesgue measurable.

1. Show that there is a sequence of simple functions $s_n(x)$ such that $s_n(x) \longrightarrow f(x)$ for all $x \in \mathbb{R}$.
2. Show that there is a Borel measurable function $g$ such that $g = f$ almost everywhere.

## 3.12 Spring 2015 # 3

Let $\mu$ be a finite Borel measure on $\mathbb{R}$ and $E \subset \mathbb{R}$ Borel. Prove that the following statements are equivalent:

1. $\forall \varepsilon > 0$ there exists $G$ open and $F$ closed such that

$$F \subseteq E \subseteq G \quad \text{and} \quad \mu(G \setminus F) < \varepsilon.$$

2. There exists a $V \in G_\delta$ and $H \in F_\sigma$ such that

$$H \subseteq E \subseteq V \quad \text{and} \quad \mu(V \setminus H) = 0$$

## 3.13 Spring 2014 # 3

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and suppose

$$\forall x \in \mathbb{R}, \quad f(x) \geq \limsup_{y \to x} f(y)$$

Prove that $f$ is Borel measurable.

## 3.14 Spring 2014 # 4

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $f$ is a measurable function on $X$. Show that

$$\lim_{n \to \infty} \int_X f^n \ d\mu = \begin{cases} \infty & \text{or} \\ \mu(f^{-1}(1)), \end{cases}$$

and characterize the collection of functions of each type.

## 3.15 Spring 2017 # 1

Let $K$ be the set of numbers in $[0,1]$ whose decimal expansions do not use the digit 4.

> We use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 with $399\cdots$. For example, $0.8754 = 0.8753999\cdots$.

Show that $K$ is a compact, nowhere dense set without isolated points, and find the Lebesgue measure $m(K)$.

*Solution.*
Concepts used:

- Definition: $A$ is *nowhere dense* $\iff$ every interval $I$ contains a subinterval $S \subseteq A^c$.
  - Equivalently, the interior of the closure is empty, $\left(\overline{K}\right)^\circ = \emptyset$.

**Solution**
Claim: $K$ **is compact**.

- It suffices to show that $K^c := [0,1] \setminus K$ is open; Then $K$ will be a closed and bounded subset of $\mathbb{R}$ and thus compact by Heine-Borel.
- Strategy: write $K^c$ as the union of open balls (since these form a basis for the Euclidean topology on $\mathbb{R}$).
  - Do this by showing every point $x \in K^c$ is an interior point, i.e. $x$ admits a neighborhood $N_x$ such that $N_x \subseteq K^c$.
- Identify $K^c$ as the set of real numbers in $[0,1]$ whose decimal expansion **does** contain a 4.
  - We will show that there exists a neighborhood small enough such that all points in it contain a 4 in their decimal expansions.
- Let $x \in K^c$, suppose a 4 occurs as the $k$th digit, and write

$$x = 0.d_1 d_2 \cdots d_{k-1}\ 4\ d_{k+1} \cdots = \left(\sum_{j=1}^{k} d_j 10^{-j}\right) + \left(4 \cdot 10^{-k}\right) + \left(\sum_{j=k+1}^{\infty} d_j 10^{-j}\right).$$

- Set $r_x < 10^{-k}$ and let $y \in [0,1] \bigcap B_{r_x}(x)$ be arbitrary and write

$$y = \sum_{j=1}^{\infty} c_j 10^{-j}.$$

- Thus $|x-y| < r_x < 10^{-k}$, and the first $k$ digits of $x$ and $y$ must agree:
  - We first compute the difference:

$$x - y = \sum_{i=1}^{\infty} d_j 10^{-j} - \sum_{i=1}^{\infty} c_j 10^{-j} = \sum_{i=1}^{\infty} (d_j - c_j) 10^{-j}$$

  - Thus (claim)

$$|x-y| \le \sum_{j=1}^{\infty} |d_j - c_j| 10^{j} < 10^{-k} \iff |d_j - c_j| = 0 \quad \forall j \le k.$$

- Otherwise we can note that any term $|d_j - c_j| \geq 1$ and there is a contribution to $|x - y|$ of at least $1 \cdot 10^{-j}$ for some $j < k$, whereas

$$j < k \iff 10^{-j} > 10^{-k},$$

a contradiction.

- This means that for all $j \leq k$ we have $d_j = c_j$, and in particular $d_k = 4 = c_k$, so $y$ has a 4 in its decimal expansion.
- But then $K^c = \bigcup_x B_{r_x}(x)$ is a union of open sets and thus open.

Claim: $K$ **is nowhere dense and** $m(K) = 0$**:**

- Strategy: Show $\left(\overline{K}\right)^\circ = \emptyset$.
- Since $K$ is closed, $\overline{K} = K$, so it suffices to show that $K$ does not properly contain any interval.
- It suffices to show $m(K^c) = 1$, since this implies $m(K) = 0$ and since any interval has strictly positive measure, this will mean $K$ can not contain an interval.
- As in the construction of the Cantor set, let
  - $K_0$ denote $[0, 1]$ with 1 interval $\left(\frac{4}{10}, \frac{5}{10}\right)$ of length $\frac{1}{10}$ deleted, so

$$m(K_0^c) = \frac{1}{10}.$$

  - $K_1$ denote $K_0$ with 9 intervals $\left(\frac{1}{100}, \frac{5}{100}\right), \left(\frac{14}{100}, \frac{15}{100}\right), \cdots \left(\frac{94}{100}, \frac{95}{100}\right)$ of length $\frac{1}{100}$ deleted, so

$$m(K_1^c) = \frac{1}{10} + \frac{9}{100}.$$

  - $K_n$ denote $K_{n-1}$ with $9^n$ such intervals of length $\frac{1}{10^{n+1}}$ deleted, so

$$m(K_n^c) = \frac{1}{10} + \frac{9}{100} + \cdots + \frac{9^n}{10^{n+1}}.$$

- Then compute

$$m(K^c) = \sum_{j=0}^{\infty} \frac{9^n}{10^{n+1}} = \frac{1}{10} \sum_{j=0}^{\infty} \left(\frac{9}{10}\right)^n = \frac{1}{10}\left(\frac{1}{1 - \frac{9}{10}}\right) = 1.$$

Claim: $K$ **has no isolated points**:

- A point $x \in K$ is isolated iff there there is an open ball $B_r(x)$ containing $x$ such that $B_r(x) \subsetneq K^c$.
  - So every point in this ball **should** have a 4 in its decimal expansion.
- Strategy: show that if $x \in K$, every neighborhood of $x$ intersects $K$.
- Note that $m(K_n) = \left(\frac{9}{10}\right)^n \overset{n \longrightarrow \infty}{\longrightarrow} 0$
- Also note that we deleted open intervals, and the endpoints of these intervals are never deleted.
  - Thus endpoints of deleted intervals are elements of $K$.

- Fix $x$. Then for every $\varepsilon$, by the Archimedean property of $\mathbb{R}$, choose $n$ such that $\left(\frac{9}{10}\right)^n < \varepsilon$.
- Then there is an endpoint $x_n$ of some deleted interval $I_n$ satisfying

$$|x - x_n| \leq \left(\frac{9}{10}\right)^n < \varepsilon.$$

- So every ball containing $x$ contains some endpoint of a removed interval, and thus an element of $K$.

### 3.16 Spring 2016 # 2

Let $0 < \lambda < 1$ and construct a Cantor set $C_\lambda$ by successively removing middle intervals of length $\lambda$.

Prove that $m(C_\lambda) = 0$.

## 4 Measure Theory: Functions

### 4.1 Fall 2016 # 2

Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be measurable with

$$\int_a^b f(x)\ dx = \int_a^b g(x)\ dx.$$

Show that either

1. $f(x) = g(x)$ almost everywhere, or
2. There exists a measurable set $E \subset [a, b]$ such that

$$]\int_E f(x)\, dx > \int_E g(x)\, dx$$

*Solution.* • Suppose it is *not* the case that $f = g$ almost everywhere; then letting $A \coloneqq \left\{x \in [a, b] \;\middle|\; f(x) \neq g(x)\right\}$, we have $m(A) > 0$.

- Write

$$A = A_1 \coprod A_2 \coloneqq \{f > g\} \coprod \{f < g\},$$

then $m(A_1) > 0$ or $m(A_2) > 0$ (or both).
- Wlog (by relabeling $f, g$ if necessary), suppose $m(A_1) > 0$, and take $E \coloneqq A_1$.
- Then on $E$, we have $f(x) > g(x)$ pointwise. This is preserved by monotonicity of the integral, thus

$$f(x) > g(x) \text{ on } E \implies \int_E f(x)\, dx > \int_E g(x)\, dx.$$

### 4.2 Spring 2016 # 4

Let $E \subset \mathbb{R}$ be measurable with $m(E) < \infty$. Define

$$f(x) = m(E \cap (E + x)).$$

Show that

1. $f \in L^1(\mathbb{R})$.
2. $f$ is uniformly continuous.
3. $\lim_{|x| \longrightarrow \infty} f(x) = 0$.

Hint:

$$\chi_{E \cap (E+x)}(y) = \chi_E(y)\chi_E(y - x)$$

## 5 Integrals: Convergence

### 5.1 Fall 2019 # 2.

Prove that

$$\left| \frac{d^n}{dx^n} \frac{\sin x}{x} \right| \leq \frac{1}{n}$$

for all $x \neq 0$ and positive integers $n$.

Hint: Consider $\displaystyle\int_0^1 \cos(tx)\,dt$

*Solution.*
Concepts used:

- DCT
- Bounding in the right place. Don't evaluate the actual integral!

**Solution**:

- By induction on the number of limits we can pass through the integral.
- For $n = 1$ we first pass one derivative into the integral: let $x_n \longrightarrow x$ be any sequence

converging to $x$, then

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\sin(x)}{x} &= \frac{\partial}{\partial x}\int_0^1 \cos(tx)\,dt \\
&= \lim_{x_n\longrightarrow x}\frac{1}{x_n - x}\left(\int_0^1 \cos(tx_n)\,dt - \int_0^1 \cos(tx)\,dt\right) \\
&= \lim_{x_n\longrightarrow x}\left(\int_0^1 \frac{\cos(tx_n)-\cos(tx)}{x_n - x}\,dt\right) \\
&= \lim_{x_n\longrightarrow x}\left(\int_0^1 \left(t\sin(tx)\,\Big|_{x=\xi_n}\right)dt\right) \quad \text{where} \quad \xi_n \in [x_n, x] \text{ by MVT}, \xi_n \longrightarrow x \\
&= \lim_{\xi_n\longrightarrow x}\left(\int_0^1 t\sin(t\xi_n)\,dt\right) \\
&=_{\text{DCT}} \int_0^1 \lim_{\xi_n\longrightarrow x} t\sin(t\xi_n)\,dt \\
&= \int_0^1 t\sin(tx)\,dt
\end{aligned}$$

.

- Taking absolute values we obtain an upper bound

$$\begin{aligned}
\left|\frac{\partial}{\partial x}\frac{\sin(x)}{x}\right| &= \left|\int_0^1 t\sin(tx)\,dt\right| \\
&\leq \int_0^1 |t\sin(tx)|\,dt \\
&\leq \int_0^1 1\,dt = 1,
\end{aligned}$$

since $t \in [0,1] \implies |t| < 1$, and $|\sin(xt)| \leq 1$ for any $x$ and $t$.

- Note that this bound also justifies the DCT, since the functions $f_n(t) = t\sin(t\xi_n)$ are uniformly dominated by $g(t) = 1$ on $L^1([0,1])$.

Note: integrating by parts here yields the actual formula:

$$\begin{aligned}
\int_0^1 t\sin(tx)\,dt &=_{\text{IBP}} \left(\frac{-t\cos(tx)}{x}\right)\Big|_{t=0}^{t=1} - \int_0^1 \frac{\cos(tx)}{x}\,dt \\
&= \frac{-\cos(x)}{x} - \frac{\sin(x)}{x^2} \\
&= \frac{x\cos(x) - \sin(x)}{x^2}.
\end{aligned}$$

- For the inductive step, we assume that we can pass $n-1$ limits through the integral and show we can pass the $n$th through as well.

$$\begin{aligned}
\frac{\partial^n}{\partial x^n}\frac{\sin(x)}{x} &= \frac{\partial^n}{\partial x^n}\int_0^1 \cos(tx)\,dt \\
&= \frac{\partial}{\partial x}\int_0^1 \frac{\partial^{n-1}}{\partial x^{n-1}}\cos(tx)\,dt \\
&= \frac{\partial}{\partial x}\int_0^1 t^{n-1} f_{n-1}(x,t)\,dt
\end{aligned}$$

- Note that $f_n(x,t) = \pm\sin(tx)$ when $n$ is odd and $f_n(x,t) = \pm\cos(tx)$ when $n$ is even, and a constant factor of $t$ is multiplied when each derivative is taken.
- We continue as in the base case:

$$
\begin{aligned}
\frac{\partial}{\partial x}\int_0^1 t^{n-1} f_{n-1}(x,t)\,dt &= \lim_{x_k \longrightarrow x}\int_0^1 t^{n-1}\left(\frac{f_{n-1}(x_n,t) - f_{n-1}(x,t)}{x_n - x}\right)dt \\
&=_{\text{IVT}} \lim_{x_k \longrightarrow x}\int_0^1 t^{n-1}\frac{\partial f_{n-1}}{\partial x}(\xi_k,t)\,dt \quad \text{where } \xi_k \in [x_k, x],\ \xi_k \longrightarrow x \\
&=_{\text{DCT}} \int_0^1 \lim_{x_k \longrightarrow x} t^{n-1}\frac{\partial f_{n-1}}{\partial x}(\xi_k,t)\,dt \\
&:= \int_0^1 \lim_{x_k \longrightarrow x} t^n f_n(\xi_k,t)\,dt \\
&:= \int_0^1 t^n f_n(x,t)\,dt.
\end{aligned}
$$

  - We've used the fact that $f_0(x) = \cos(tx)$ is smooth as a function of $x$, and in particular continuous
  - The DCT is justified because the functions $h_{n,k}(x,t) = t^n f_n(\xi_k,t)$ are again uniformly (in $k$) bounded by 1 since $t \le 1 \implies t^n \le 1$ and each $f_n$ is a sin or cosine.
- Now take absolute values

$$
\begin{aligned}
\left|\frac{\partial^n}{\partial x^n}\frac{\sin(x)}{x}\right| &= \left|\int_0^1 -t^n f_n(x,t)\ dt\right| \\
&\le \int_0^1 |t^n f_n(x,t)|\ dt \\
&\le \int_0^1 |t^n||f_n(x,t)|\ dt \\
&\le \int_0^1 |t^n|\cdot 1\ dt \\
&\le \int_0^1 t^n\ dt \quad \text{since } t \text{ is positive} \\
&= \frac{1}{n+1} \\
&< \frac{1}{n}.
\end{aligned}
$$

  - We've again used the fact that $f_n(x,t)$ is of the form $\pm\cos(tx)$ or $\pm\sin(tx)$, both of which are bounded by 1.

## 5.2 Spring 2020 # 5

Compute the following limit and justify your calculations:

$$
\lim_{n \longrightarrow \infty}\int_0^n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx.
$$

Not finished, flesh out.

*Solution.*
Concepts used:

- DCT
- Passing limits through products and quotients

Note that

$$\begin{aligned}
\lim_n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} &= \frac{1}{\lim_n \left(1 + \frac{x^2}{n}\right)^1 \left(1 + \frac{x^2}{n}\right)^n} \\
&= \frac{1}{1 \cdot e^{x^2}} \\
&= e^{-x^2}.
\end{aligned}$$

If passing the limit through the integral is justified, we will have

$$\begin{aligned}
\lim_{n \longrightarrow \infty} \int_0^n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx &= \lim_{n \longrightarrow \infty} \int_{\mathbb{R}} \chi_{[0,n]} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \\
&= \int_{\mathbb{R}} \lim_{n \longrightarrow \infty} \chi_{[0,n]} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \quad \text{by the DCT} \\
&= \int_{\mathbb{R}} \lim_{n \longrightarrow \infty} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \\
&= \int_0^\infty e^{-x^2} \\
&= \frac{\sqrt{\pi}}{2}.
\end{aligned}$$

Computing the last integral:

$$\begin{aligned}
\left(\int_{\mathbb{R}} e^{-x^2}\, dx\right)^2 &= \left(\int_{\mathbb{R}} e^{-x^2}\, dx\right) \left(\int_{\mathbb{R}} e^{-y^2}\, dx\right) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(x+y)^2}\, dx \\
&= \int_0^{2\pi} \int_0^\infty e^{-r^2} r\, dr\, d\theta \qquad u = r^2 \\
&= \frac{1}{2} \int_0^{2\pi} \int_0^\infty e^{-u}\, du\, d\theta \\
&= \frac{1}{2} \int_0^{2\pi} 1 \\
&= \pi,
\end{aligned}$$

and now use the fact that the function is even so $\displaystyle\int_0^\infty f = \frac{1}{2} \int_{\mathbb{R}} f$.
Justifying the DCT:

- Apply Bernoulli's inequality:

$$1+\frac{x^2}{n}^{n+1} \geq 1+\frac{x^2}{n}\left(1+x^2\right) \geq 1+x^2,$$

where the last inequality follows from the fact that $1+\dfrac{x^2}{n} \geq 1$

## 5.3 Spring 2019 # 3

Let $\{f_k\}$ be any sequence of functions in $L^2([0,1])$ satisfying $\|f_k\|_2 \leq M$ for all $k \in \mathbb{N}$.

Prove that if $f_k \longrightarrow f$ almost everywhere, then $f \in L^2([0,1])$ with $\|f\|_2 \leq M$ and

$$\lim_{k\to\infty} \int_0^1 f_k(x)dx = \int_0^1 f(x)dx$$

Hint: Try using Fatou's Lemma to show that $\|f\|_2 \leq M$ and then try applying Egorov's Theorem.

*Solution.*
Concepts used:

- Definition of $L^+$: space of measurable function $X \longrightarrow [0,\infty]$.
- Fatou: For any sequence of $L^+$ functions, $\displaystyle\int \liminf f_n \leq \liminf \int f_n$.
- Egorov's Theorem: If $E \subseteq \mathbb{R}^n$ is measurable, $m(E) > 0$, $f_k : E \longrightarrow \mathbb{R}$ a sequence of measurable functions where $\displaystyle\lim_{n\longrightarrow\infty} f_n(x)$ exists and is finite a.e., then $f_n \longrightarrow f$ *almost uniformly*: for every $\varepsilon > 0$ there exists a closed subset $F_\varepsilon \subseteq E$ with $m(E \setminus F) < \varepsilon$ and $f_n \longrightarrow f$ uniformly on $F$.

$L^2$ bound:

- Since $f_k \longrightarrow f$ almost everywhere, $\liminf_n f_n(x) = f(x)$ a.e.
- $\|f_n\|_2 < \infty$ implies each $f_n$ is measurable and thus $|f_n|^2 \in L^+$, so we can apply Fatou:

$$\begin{aligned}
\|f\|_2^2 &= \int |f(x)|^2 \\
&= \int \liminf_n |f_n(x)|^2 \\
&\underset{\text{Fatou}}{\leq} \liminf_n \int |f_n(x)|^2 \\
&\leq \liminf_n M \\
&= M.
\end{aligned}$$

- Thus $\|f\|_2 \leq \sqrt{M} < \infty$ implying $f \in L^2$.

What is the "right" proof here that uses the first part?

Equality of Integrals:

- Take the sequence $\varepsilon_n = \dfrac{1}{n}$

- Apply Egorov's theorem: obtain a set $F_\varepsilon$ such that $f_n \longrightarrow f$ uniformly on $F_\varepsilon$ and $m(I \setminus F_\varepsilon) < \varepsilon$.

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \left| \int_0^1 f_n - f \right| &\leq \lim_{n\longrightarrow\infty} \int_0^1 |f_n - f| \\
&= \lim_{n\longrightarrow\infty} \left( \int_{F_\varepsilon} |f_n - f| + \int_{I\setminus F_\varepsilon} |f_n - f| \right) \\
&= \int_{F_\varepsilon} \lim_{n\longrightarrow\infty} |f_n - f| + \lim_{n\longrightarrow\infty} \int_{I\setminus F_\varepsilon} |f_n - f| \quad \text{by uniform convergence} \\
&= 0 + \lim_{n\longrightarrow\infty} \int_{I\setminus F_\varepsilon} |f_n - f|,
\end{aligned}$$

so it suffices to show $\displaystyle\int_{I\setminus F_\varepsilon} |f_n - f| \overset{n\longrightarrow\infty}{\longrightarrow} 0$.

- We can obtain a bound using Holder's inequality with $p = q = 2$:

$$\begin{aligned}
\int_{I\setminus F_\varepsilon} |f_n - f| &\leq \left( \int_{I\setminus F_\varepsilon} |f_n - f|^2 \right) \left( \int_{I\setminus F_\varepsilon} |1|^2 \right) \\
&= \left( \int_{I\setminus F_\varepsilon} |f_n - f|^2 \right) \mu(F_\varepsilon) \\
&\leq \|f_n - f\|_2 \mu(F_\varepsilon) \\
&\leq (\|f_n\|_2 + \|f\|_2) \mu(F_\varepsilon) \\
&\leq 2M \cdot \mu(F_\varepsilon)
\end{aligned}$$

where $M$ is now a constant not depending on $\varepsilon$ or $n$.

- Now take a nested sequence of sets $F_\varepsilon$ with $\mu(F_\varepsilon) \longrightarrow 0$ and applying continuity of measure yields the desired statement.

## 5.4 Fall 2018 # 6

Compute the following limit and justify your calculations:

$$\lim_{n\to\infty} \int_1^n \frac{dx}{\left(1 + \frac{x}{n}\right)^n \sqrt[n]{x}}$$

*Solution.*

- Note that $x^{\frac{1}{n}} \overset{n\longrightarrow\infty}{\longrightarrow} 1$ for any $0 < x < \infty$.
- Thus the integrand converges to $\dfrac{1}{e^x}$, which is integrable on $(0, \infty)$ and integrates to 1.
- Break the integrand up:

$$\int_0^\infty \frac{1}{(1 + \frac{x}{n})^n x^{\frac{1}{n}}} \, dx = \int_0^1 \frac{1}{(1 + \frac{x}{n})^n x^{\frac{1}{n}}} \, dx = \int_1^\infty \frac{1}{(1 + \frac{x}{n})^n x^{\frac{1}{n}}} \, dx.$$

## 5.5 Fall 2018 # 3

Suppose $f(x)$ and $xf(x)$ are integrable on $\mathbb{R}$. Define $F$ by

$$F(t) := \int_{-\infty}^{\infty} f(x)\cos(xt)dx$$

Show that

$$F'(t) = -\int_{-\infty}^{\infty} xf(x)\sin(xt)dx.$$

*Solution.*
Concepts used:

- Mean Value Theorem
- DCT

$$\begin{aligned}
\frac{\partial}{\partial t} F(t) &= \frac{\partial}{\partial t} \int_{\mathbb{R}} f(x)\cos(xt)\ dx \\
&\overset{DCT}{=} \int_{\mathbb{R}} f(x) \frac{\partial}{\partial t} \cos(xt)\ dx \\
&= \int_{\mathbb{R}} xf(x)\cos(xt)\ dx,
\end{aligned}$$

so it only remains to justify the DCT.

- Fix $t$, then let $t_n \longrightarrow t$ be arbitrary.
- Define

$$h_n(x,t) = f(x)\left(\frac{\cos(tx)-\cos(t_n x)}{t_n - t}\right) \overset{n\longrightarrow\infty}{\longrightarrow} \frac{\partial}{\partial t}(f(x)\cos(xt))$$

  since $\cos(tx)$ is differentiable in $t$ and this is the limit definition of differentiability.
- Note that

$$\begin{aligned}
\frac{\partial}{\partial t}\cos(tx) &:= \lim_{t_n \longrightarrow t} \frac{\cos(tx)-\cos(t_n x)}{t_n - t} \\
&\overset{MVT}{=} \frac{\partial}{\partial t}\cos(tx)\ \Big|_{t=\xi_n} && \text{for some } \xi_n \in [t, t_n] \text{ or } [t_n, t] \\
&= x\sin(\xi_n x)
\end{aligned}$$

  where $\xi_n \overset{n\longrightarrow\infty}{\longrightarrow} t$ since wlog $t_n \leq \xi_n \leq t$ and $t_n \nearrow t$.
- We then have

$$|h_n(x)| = |f(x) x \sin(\xi_n x)| \leq |xf(x)| \quad \text{since } |\sin(\xi_n x)| \leq 1$$

  for every $x$ and every $n$.
- Since $xf(x) \in L^1(\mathbb{R})$ by assumption, the DCT applies.

## 5.6 Spring 2018 # 5

Suppose that

- $f_n, f \in L^1$,
- $f_n \longrightarrow f$ almost everywhere, and
- $\int |f_n| \to \int |f|$.

Show that $\int f_n \to \int f$.

*Solution.*
Concepts used:

- $\int |f_n - f| \longrightarrow \iff \int f_n = \int f$.
- Fatou:

$$\int \liminf f_n \le \liminf \int f_n$$
$$\int \limsup f_n \ge \limsup \int f_n.$$

**Solution**:

- Since $\int |f_n| \overset{n\longrightarrow\infty}{\longrightarrow} \int |f|$, define

$$h_n = |f_n - f| \qquad \overset{n\longrightarrow\infty}{\longrightarrow} 0 \;\; a.e.$$
$$g_n = |f_n| + |f| \qquad \overset{n\longrightarrow\infty}{\longrightarrow} 2|f| \;\; a.e.$$

  - Note that $g_n - h_n \overset{n\longrightarrow\infty}{\longrightarrow} 2|f| - 0 = 2|f|$.
- Then

$$\begin{aligned}
\int 2|f| &= \int \liminf_n (g_n - h_n) \\
&= \int \liminf_n (g_n) + \int \liminf_n (-h_n) \\
&= \int \liminf_n (g_n) - \int \limsup_n (h_n) \\
&= \int 2|f| - \int \limsup_n (h_n) \\
&\le \int 2|f| - \limsup_n \int h_n \quad \text{by Fatou,}
\end{aligned}$$

- Since $f \in L^1$, $\int 2|f| = 2\|f\|_1 < \infty$ and it makes sense to subtract it from both sides, thus

$$\begin{aligned}
0 &\le -\limsup_n \int h_n \\
&\coloneqq -\limsup_n \int |f_n - f|.
\end{aligned}$$

  which forces $\limsup_n \int |f_n - f| = 0$, since
  - The integral of a nonnegative function is nonnegative, so $\int |f_n - f| \ge 0$.

- So $\left(-\int |f_n - f|\right) \leq 0$.
- But the above inequality shows $\left(-\int |f_n - f|\right) \geq 0$ as well.

- Since $\liminf_n \int h_n \leq \limsup_n \int h_n = 0$, $\lim_n \int h_n$ exists and is equal to zero.
- But then

$$\left|\int f_n - \int f\right| = \left|\int f_n - f\right| \leq \int |f_n - f|,$$

and taking $\lim_{n\longrightarrow\infty}$ on both sides yields

$$\lim_{n\longrightarrow\infty} \left|\int f_n - \int f\right| \leq \lim_{n\longrightarrow\infty} \int |f_n - f| = 0,$$

so $\lim_{n\longrightarrow\infty} \int f_n = \int f$.

## 5.7 Spring 2018 # 2

Let

$$f_n(x) := \frac{x}{1+x^n}, \quad x \geq 0.$$

a. Show that this sequence converges pointwise and find its limit. Is the convergence uniform on $[0, \infty)$?

b. Compute

$$\lim_{n\to\infty} \int_0^\infty f_n(x)dx$$

*Solution.*

### 5.7.1 a

Claim: $f_n$ does not converge uniformly to its limit.

- Note each $f_n(x)$ is clearly continuous on $(0, \infty)$, since it is a quotient of continuous functions where the denominator is never zero.
- Note

$$x < 1 \implies x^n \overset{n\longrightarrow\infty}{\longrightarrow} 0 \quad \text{and} \quad x > 1 \implies x^n \overset{n\longrightarrow\infty}{\longrightarrow} \infty.$$

- Thus

$$f_n(x) = \frac{x}{1+x^n} \overset{n\longrightarrow\infty}{\longrightarrow} f(x) := \begin{cases} x, & 0 \leq x < 1 \\ \frac{1}{2}, & x = 1 \\ 0, & x > 1 \end{cases}.$$

- If $f_n \longrightarrow f$ uniformly on $[0, \infty)$, it would converge uniformly on every subset and thus uniformly on $(0, \infty)$.

  - Then $f$ would be a uniform limit of continuous functions on $(0, \infty)$ and thus continuous on $(0, \infty)$.
  - By uniqueness of limits, $f_n$ would converge to the pointwise limit $f$ above, which is not continuous at $x = 1$, a contradiction.

### 5.7.2 b

- If the DCT applies, interchange the limit and integral:

$$\begin{aligned}
\lim_{n \longrightarrow \infty} \int_0^\infty f_n(x)\,dx &= \int_0^\infty \lim_{n \longrightarrow \infty} f_n(x)\,dx \quad \text{DCT} \\
&= \int_0^\infty f(x)\,dx \\
&= \int_0^1 x\,dx + \int_1^\infty 0\,dx \\
&= \frac{1}{2} x^2 \Big|_0^1 \\
&= \frac{1}{2}.
\end{aligned}$$

- To justify the DCT, write

$$\int_0^\infty f_n(x) = \int_0^1 f_n(x) + \int_1^\infty f_n(x).$$

- $f_n$ restricted to $(0, 1)$ is uniformly bounded by $g_0(x) = 1$ in the first integral, since

$$x \in [0, 1] \implies \frac{x}{1 + x^n} < \frac{1}{1 + x^n} < 1 := g(x)$$

  so

$$\int_0^1 f_n(x)\,dx \leq \int_0^1 1\,dx = 1 < \infty.$$

  Also note that $g_0 \cdot \chi_{(0,1)} \in L^1((0, \infty))$.
- The $f_n$ restricted to $(1, \infty)$ are uniformly bounded by $g_1(x) = \dfrac{1}{x^2}$ on $[1, \infty)$, since

$$x \in (1, \infty) \implies \frac{x}{1 + x^n} \leq \frac{x}{x^n} = \frac{1}{x^{n-1}} \leq \frac{1}{x^2} \in L^1([1, \infty) \text{ when } n \geq 3,$$

  by the $p$-test for integrals.
- So set

$$g := g_0 \cdot \chi_{(0,1)} + g_1 \cdot \chi_{[1,\infty)},$$

  then by the above arguments $g \in L^1((0, \infty))$ and $f_n \leq g$ everywhere, so the DCT applies.

## 5.8 Fall 2016 # 3

Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{x \longrightarrow 0} \int_{\mathbb{R}} |f(y-x) - f(y)| \, dy = 0$$

*Solution.*
Concepts used:

- $C_c^\infty \hookrightarrow L^p$ is dense.
- If $f
- Fixing notation, set $\tau_x f(y) := f(y-x)$; we then want to show

$$\|\tau_x f - f\|_{L^1} \overset{x \longrightarrow 0}{\longrightarrow} 0.$$

- Claim: by an $\varepsilon/3$ argument, it suffices to show this for compactly supported functions:
  - Since $f \in L^1$, choose $g_n \subset C_c^\infty(\mathbb{R}^1)$ smooth and compactly supported so that

$$\|f - g\|_{L^1} < \varepsilon.$$

  - Claim: $\|\tau_x f - \tau_x g\| < \varepsilon$.
    * Proof 1: translation invariance of the integral.
    * Proof 2: Apply a change of variables:

$$\begin{aligned}
\|\tau_x f - \tau_x g\|_1 &:= \int_{\mathbb{R}} |\tau_x f(y) - \tau_x g(y)| \, dy \\
&= \int_{\mathbb{R}} |f(y-x) - g(y-x)| \, dy \\
&= \int_{\mathbb{R}} |f(u) - g(u)| \, du \qquad (u = y - x,\, du = dy) \\
&= \|f - g\|_1 \\
&< \varepsilon.
\end{aligned}$$

  - Then

$$\begin{aligned}
\|\tau_x f - f\|_1 &= \|\tau_x f - \tau_x g + \tau_x g - g + g - f\|_1 \\
&\leq \|\tau_x f - \tau_x g\|_1 + \|\tau_x g - g\|_1 + \|g - f\|_1 \\
&\leq 2\varepsilon + \|\tau_x g - g\|_1.
\end{aligned}$$

- To show this for compactly supported functions:
  - Let $g \in C_c^\infty(\mathbb{R}^1)$, let $E = \operatorname{supp}(g)$, and write

$$\begin{aligned}
\|\tau_x g - g\|_1 &= \int_{\mathbb{R}} |g(y-x) - g(y)| \, dy \\
&= \int_E |g(y-x) - g(y)| \, dy + \int_{E^c} |g(y-x) - g(y)| \, dy \\
&= \int_E |g(y-x) - g(y)| \, dy.
\end{aligned}$$

- But $g$ is smooth and compactly supported on $E$, and thus uniformly continuous on $E$, so

$$\begin{aligned}\lim_{x \longrightarrow 0} \int_E |g(y-x) - g(y)|\, dy &= \int_E \lim_{x \longrightarrow 0} |g(y-x) - g(y)|\, dy \\ &= \int_E 0\, dy \\ &= 0.\end{aligned}$$

### 5.9 Fall 2015 # 3

Compute the following limit:

$$\lim_{n \to \infty} \int_1^n \frac{ne^{-x}}{1+nx^2} \sin\left(\frac{x}{n}\right) dx$$

### 5.10 Fall 2015 # 4

Let $f : [1, \infty) \longrightarrow \mathbb{R}$ such that $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Show that the following limit exists and satisfies the equality

$$\lim_{x \to \infty} f(x) \leq 1 + \frac{\pi}{4}$$

## 6 Integrals: Approximation

### 6.1 Spring 2018 # 3

Let $f$ be a non-negative measurable function on $[0, 1]$.

Show that

$$\lim_{p \to \infty} \left( \int_{[0,1]} f(x)^p dx \right)^{\frac{1}{p}} = \|f\|_\infty.$$

*Solution.*
Concepts used:

- $\|f\|_\infty := \inf_t \left\{ t \;\middle|\; m\left(\left\{ x \in \mathbb{R}^n \;\middle|\; f(x) > t \right\}\right) = 0 \right\}$, i.e. this is the lowest upper bound that holds almost everywhere.

**Solution**:

- $\|f\|_p \leq \|f\|_\infty$:
  - Note $|f(x)| \leq \|f\|_\infty$ almost everywhere and taking $p$th powers preserves this inequality.

- Thus

$$
\begin{aligned}
|f(x)| &\leq \|f\|_\infty \quad \text{a.e. by definition} \\
\implies |f(x)|^p &\leq \|f\|_\infty^p \quad \text{for } p \geq 0 \\
\implies \|f\|_p^p &= \int_X |f(x)|^p \, dx \\
&\leq \int_X \|f\|_\infty^p \, dx \\
&= \|f\|_\infty^p \int_X 1 \, dx \\
&= \|f\|_\infty^p \cdot m(X) \quad \text{since the norm doesn't depend on } x \\
&= \|f\|_\infty^p \qquad \text{since } m(X) = 1.
\end{aligned}
$$

  * Thus $\|f\|_p \leq \|f\|_\infty$ for all $p$ and taking $\lim_{p \longrightarrow \infty}$ preserves this inequality.

- $\|f\|_p \geq \|f\|_\infty$:
  - Fix $\varepsilon > 0$.
  - Define

$$
S_\varepsilon := \left\{ x \in \mathbb{R}^n \;\middle|\; |f(x)| \geq \|f\|_\infty - \varepsilon \right\}.
$$

    * Note that $m(S_\varepsilon) > 0$; otherwise if $m(S_\varepsilon) = 0$, then $t := \|f\|_\infty - \varepsilon < \|f\|_\varepsilon$. But this produces a *smaller* upper bound almost everywhere than $\|f\|_\varepsilon$, contradicting the definition of $\|f\|_\varepsilon$ as an infimum over such bounds.
  - Then

$$
\begin{aligned}
\|f\|_p^p &= \int_X |f(x)|^p \, dx \\
&\geq \int_{S_\varepsilon} |f(x)|^p \, dx \quad \text{since } S_\varepsilon \subseteq X \\
&\geq \int_{S_\varepsilon} |\|f\|_\infty - \varepsilon|^p \, dx \quad \text{since on } S_\varepsilon, |f| \geq \|f\|_\infty - \varepsilon \\
&= |\|f\|_\infty - \varepsilon|^p \cdot m(S_\varepsilon) \quad \text{since the integrand is independent of } x \\
&\geq 0 \quad \text{since } m(S_\varepsilon) > 0
\end{aligned}
$$

  - Taking $p$th roots for $p \geq 1$ preserves the inequality, so

$$
\implies \|f\|_p \geq |\|f\|_\infty - \varepsilon| \cdot m(S_\varepsilon)^{\frac{1}{p}} \overset{p \longrightarrow \infty}{\longrightarrow} |\|f\|_\infty - \varepsilon| \overset{\varepsilon \longrightarrow 0}{\longrightarrow} \|f\|_\infty
$$

    where we've used the fact that above arguments work
  - Thus $\|f\|_p \geq \|f\|_\infty$.

## 6.2 Spring 2018 # 4

Let $f \in L^2([0,1])$ and suppose

$$
\int_{[0,1]} f(x) x^n dx = 0 \text{ for all integers } n \geq 0.
$$

Show that $f = 0$ almost everywhere.

## 6.3 Spring 2015 # 2

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be continuous with period 1. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(t)dt \quad \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

Hint: show this first for the functions $f(t) = e^{2\pi ikt}$ for $k \in \mathbb{Z}$.

*Solution.*

### 6.3.1 Proof 1: Using Fourier Transforms

Concepts used:

- Weierstrass Approximation: A uniformly continuous function on a compact set can be uniformly approximated by polynomials.

**Solution**:

- Fix $k \in \mathbb{Z}$.
- Since $e^{2\pi ikx}$ is continuous on the compact interval $[0, 1]$, it is uniformly continuous.
- Thus there is a sequence of polynomials $P_\ell$ such that

$$P_{\ell,k} \overset{\ell\longrightarrow\infty}{\longrightarrow} e^{2\pi ikx} \text{ uniformly on } [0, 1].$$

- Note applying linearity to the assumption $\int f(x)\, x^n$, we have

$$\int f(x)x^n\, dx = 0\ \forall n \implies \int f(x)p(x)\, dx = 0$$

  for any polynomial $p(x)$, and in particular for $P_{\ell,k}(x)$ for every $\ell$ and every $k$.
- But then

$$\begin{aligned}
\langle f,\ e_k\rangle &= \int_0^1 f(x)e^{-2\pi ikx}\, dx \\
&= \int_0^1 f(x) \lim_{\ell\longrightarrow\infty} P_\ell(x) \\
&= \lim_{\ell\longrightarrow\infty} \int_0^1 f(x)P_\ell(x) && \text{by uniform convergence on a compact interval} \\
&= \lim_{\ell\longrightarrow\infty} 0 \quad \text{by assumption} \\
&= 0 \quad \forall k \in \mathbb{Z},
\end{aligned}$$

  so $f$ is orthogonal to every $e_k$.
- Thus $f \in S^\perp := \operatorname{span}_{\mathbb{C}} \{e_k\}_{k\in\mathbb{Z}}^\perp \subseteq L^2([0, 1])$, but since this is a basis, $S$ is dense and thus $S^\perp = \{0\}$ in $L^2([0, 1])$.
- Thus $f \equiv 0$ in $L^2([0, 1])$, which implies that $f$ is zero almost everywhere.

■

### 6.3.2 Alternative Proof

Concepts used

- $C^1([0,1])$ is dense in $L^2([0,1])$
- Polynomials are dense in $L^p(X, \mathcal{M}, \mu)$ for any $X \subseteq \mathbb{R}^n$ compact and $\mu$ a finite measure, for all $1 \leq p < \infty$.
  - Use Weierstrass Approximation, then uniform convergence implies $L^p(\mu)$ convergence by DCT.

**Solution**:

- By density of polynomials, for $f \in L^2([0,1])$ choose $p_\varepsilon(x)$ such that $\|f - p_\varepsilon\| < \varepsilon$ by Weierstrass approximation.
- Then on one hand,

$$\begin{aligned} \|f(f - p_\varepsilon)\|_1 &= \left\|f^2\right\|_1 - \|f \cdot p_\varepsilon\|_1 \\ &= \left\|f^2\right\|_1 - 0 \quad \text{by assumption} \\ &= \|f\|_2^2. \end{aligned}$$

  - Where we've used that $\left\|f^2\right\|_1 = \int \left|f^2\right| = \int |f|^2 = \|f\|_2^2$.
- On the other hand

$$\begin{aligned} \|f(f - p_\varepsilon)\| &\leq \|f\|_1 \|f - p_\varepsilon\|_\infty \quad \text{by Holder} \\ &\leq \varepsilon \|f\|_1 \\ &\leq \varepsilon \|f\|_2 \sqrt{m(X)} \\ &= \varepsilon \|f\|_2 \quad \text{since } m(X) = 1. \end{aligned}$$

  - Where we've used that $\|fg\|_1 = \int |fg| = \int |f||g| \leq \int \|f\|_\infty |g| = \|f\|_\infty \|g\|_1$.
- Combining these,

$$\|f\|_2^2 \leq \|f\|_2 \varepsilon \implies \|f\|_2 < \varepsilon \longrightarrow 0, .$$

  so $\|f\|_2 = 0$, which implies $f = 0$ almost everywhere.

## 6.4 Fall 2014 # 4

Let $g \in L^\infty([0,1])$ Prove that

$$\int_{[0,1]} f(x) g(x) \, dx = 0 \quad \text{for all continuous } f : [0,1] \longrightarrow \mathbb{R} \implies g(x) = 0 \text{ almost everywhere.}$$

# 7 $L^1$

## 7.1 Spring 2020 # 3

a. Prove that if $g \in L^1(\mathbb{R})$ then

$$\lim_{N \longrightarrow \infty} \int_{|x| \geq N} |f(x)| \, dx = 0,$$

and demonstrate that it is not necessarily the case that $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.

b. Prove that if $f \in L^1([1, \infty])$ and is decreasing, then $\lim_{x \longrightarrow \infty} f(x) = 0$ and in fact $\lim_{x \longrightarrow \infty} x f(x) = 0$.

c. If $f : [1, \infty) \longrightarrow [0, \infty)$ is decreasing with $\lim_{x \longrightarrow \infty} x f(x) = 0$, does this ensure that $f \in L^1([1, \infty))$?

*Solution.*
Concepts used:

- Limits
- Cauchy Criterion for Integrals: $\int_a^\infty f(x) \, dx$ converges iff for every $\varepsilon > 0$ there exists an $M_0$ such that $A, B \geq M_0$ implies $\left| \int_A^B f \right| < \varepsilon$, i.e. $\left| \int_A^B f \right| \overset{A \longrightarrow \infty}{\longrightarrow} 0$.
- Integrals of $L^1$ functions have vanishing tails: $\int_N^\infty |f| \overset{N \longrightarrow \infty}{\longrightarrow} 0$.
- Mean Value Theorem for Integrals: $\int_a^b f(t) \, dt = (b - a) f(c)$ for some $c \in [a, b]$.

#### 7.1.1 a

Stated integral equality:

- Let $\varepsilon > 0$
- $C_c(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$ is dense so choose $\{f_n\} \longrightarrow f$ with $\|f_n - f\|_1 \longrightarrow 0$.
- Since $\{f_n\}$ are compactly supported, choose $N_0 \gg 1$ such that $f_n$ is zero outside of $B_{N_0}(\mathbf{0})$.

- Then

$$\begin{aligned}
N \geq N_0 \implies \int_{|x|>N} |f| &= \int_{|x|>N} |f - f_n + f_n| \\
&\leq \int_{|x|>N} |f - f_n| + \int_{|x|>N} |f_n| \\
&= \int_{|x|>N} |f - f_n| \\
&\leq \int_{|x|>N} \|f - f_n\|_1 \\
&= \|f_n - f\|_1 \left( \int_{|x|>N} 1 \right) \\
&\overset{n\longrightarrow\infty}{\longrightarrow} 0 \left( \int_{|x|>N} 1 \right) \\
&= 0 \\
&\overset{N\longrightarrow\infty}{\longrightarrow} 0.
\end{aligned}$$

To see that this doesn't force $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$:

- Take $f(x)$ to be a train of rectangles of height 1 and area $1/2^j$ centered on even integers.
- Then

$$\int_{|x|>N} |f| = \sum_{j=N}^{\infty} 1/2^j \overset{N\longrightarrow\infty}{\longrightarrow} 0$$

  as the tail of a convergent sum.
- However $f(x) = 1$ for infinitely many even integers $x > N$, so $f(x) \not\longrightarrow 0$ as $|x| \longrightarrow \infty$.

#### 7.1.2 b

**Solution 1 ("Trick")**

- Since $f$ is decreasing on $[1, \infty)$, for any $t \in [x - n, x]$ we have

$$x - n \leq t \leq x \implies f(x) \leq f(t) \leq f(x - n).$$

- Integrate over $[x, 2x]$, using monotonicity of the integral:

$$\begin{aligned}
\int_x^{2x} f(x)\, dt \leq \int_x^{2x} f(t)\, dt &\leq \int_x^{2x} f(x-n)\, dt \\
\implies f(x) \int_x^{2x} dt \leq \int_x^{2x} f(t)\, dt &\leq f(x-n) \int_x^{2x} dt \\
\implies x f(x) \leq \int_x^{2x} f(t)\, dt &\leq x f(x-n).
\end{aligned}$$

- By the Cauchy Criterion for integrals, $\lim_{x\longrightarrow\infty} \int_x^{2x} f(t)\ dt = 0$.
- So the LHS term $x f(x) \overset{x\longrightarrow\infty}{\longrightarrow} 0$.
- Since $x > 1$, $|f(x)| \leq |x f(x)|$
- Thus $f(x) \overset{x\longrightarrow\infty}{\longrightarrow} 0$ as well.

**Solution 2 (Variation on the Trick)**

- Use mean value theorem for integrals:

$$\int_x^{2x} f(t)\,dt = xf(c_x) \quad \text{for some } c_x \in [x, 2x] \text{ depending on } x.$$

- Since $f$ is decreasing,

$$\begin{aligned} x \leq c_x \leq 2x &\implies f(2x) \leq f(c_x) \leq f(x) \\ &\implies 2xf(2x) \leq 2xf(c_x) \leq 2xf(x) \\ &\implies 2xf(2x) \leq 2x \int_x^{2x} f(t)\,dt \leq 2xf(x) \end{aligned}$$

.

- By Cauchy Criterion, $\int_x^{2x} f \longrightarrow 0$.
- So $2xf(2x) \longrightarrow 0$, which by a change of variables gives $uf(u) \longrightarrow 0$.
- Since $u \geq 1$, $f(u) \leq uf(u)$ so $f(u) \longrightarrow 0$ as well.

**Solution 3 (Contradiction)**

Just showing $f(x) \overset{x\longrightarrow\infty}{\longrightarrow} 0$:

- Toward a contradiction, suppose not.
- Since $f$ is decreasing, it can not diverge to $+\infty$
- If $f(x) \longrightarrow -\infty$, then $f \notin L^1(\mathbb{R})$: choose $x_0 \gg 1$ so that $t \geq x_0 \implies f(t) < -1$, then
- Then $t \geq x_0 \implies |f(t)| \geq 1$, so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f(t)|\,dt \geq \int_{x_0}^\infty 1 = \infty.$$

- Otherwise $f(x) \longrightarrow L \neq 0$, some finite limit.
- If $L > 0$:
  - Fix $\varepsilon > 0$, choose $x_0 \gg 1$ such that $t \geq x_0 \implies L - \varepsilon \leq f(t) \leq L$
  - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L-\varepsilon)\,dt = \infty$$

- If $L < 0$:
  - Fix $\varepsilon > 0$, choose $x_0 \gg 1$ such that $t \geq x_0 \implies L \leq f(t) \leq L + \varepsilon$.
  - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L)\,dt = \infty$$

Showing $xf(x) \overset{x\longrightarrow\infty}{\longrightarrow} 0$.

- Toward a contradiction, suppose not.
- (How to show that $xf(x) \not\longrightarrow +\infty$?)
- If $xf(x) \longrightarrow -\infty$
  - Choose a sequence $\Gamma = \{\widehat{x}_i\}$ such that $x_i \longrightarrow \infty$ and $x_i f(x_i) \longrightarrow -\infty$.
  - Choose a subsequence $\Gamma' = \{x_i\}$ such that $x_i f(x_i) \leq -1$ for all $i$ and $x_i \leq x_{i+1}$.

– Choose a further subsequence $S = \left\{ x_i \in \Gamma' \;\middle|\; 2x_i < x_{i+1} \right\}$.
– Then since $f$ is always decreasing, for $t \geq x_0$, $|f|$ is increasing, and $|f(x_i)| \leq |f(2x_i)|$, so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f| \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(t)|\,dt \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(x_i)| = \sum_{x_i \in S} x_i f(x_i) \longrightarrow \infty.$$

- If $xf(x) \longrightarrow L \neq 0$ for $0 < L < \infty$:
  – Fix $\varepsilon > 0$, choose an infinite sequence $\{x_i\}$ such that $L - \varepsilon \leq x_i f(x_i) \leq L$ for all $i$.

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)|\,dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i)\,dt = \sum_S x_i f(x_i) \geq \sum_S (L - \varepsilon) \longrightarrow \infty.$$

- If $xf(x) \longrightarrow L \neq 0$ for $-\infty < L < 0$:
  – Fix $\varepsilon > 0$, choose an infinite sequence $\{x_i\}$ such that $L \leq x_i f(x_i) \leq L + \varepsilon$ for all $i$.

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)|\,dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i)\,dt = \sum_S x_i f(x_i) \geq \sum_S (L) \longrightarrow \infty.$$

**Solution 4 (Akos's Suggestion)** For $x \geq 1$,

$$|xf(x)| = \left| \int_x^{2x} f(x)\,dt \right| \leq \int_x^{2x} |f(x)|\,dt \leq \int_x^{2x} |f(t)|\,dt \leq \int_x^\infty |f(t)|\,dt \overset{x \longrightarrow \infty}{\longrightarrow} 0$$

where we've used

- Since $f$ is decreasing and $\lim_{x \longrightarrow \infty} f(x) = 0$ from part (a), $f$ is non-negative.
- Since $f$ is positive and decreasing, for every $t \in [a, b]$ we have $|f(a)| \leq |f(t)|$.
- By part (a), the last integral goes to zero.

**Solution 5 (Peter's)**

- Toward a contradiction, produce a sequence $x_i \longrightarrow \infty$ with $x_i f(x_i) \longrightarrow \infty$ and $x_i f(x_i) > \varepsilon > 0$, then

$$\begin{aligned}
\int f(x)\,dx &\geq \sum_{i=1}^\infty \int_{x_i}^{x_{i+1}} f(x)\,dx \\
&\geq \sum_{i=1}^\infty \int_{x_i}^{x_{i+1}} f(x_{i+1})\,dx \\
&= \sum_{i=1}^\infty f(x_{i+1}) \int_{x_i}^{x_{i+1}} dx \\
&\geq \sum_{i=1}^\infty (x_{i+1} - x_i) f(x_{i+1}) \\
&\geq \sum_{i=1}^\infty (x_{i+1} - x_i) \frac{\varepsilon}{x_{i+1}} \\
&= \varepsilon \sum_{i=1}^\infty \left( 1 - \frac{x_{i-1}}{x_i} \right) \longrightarrow \infty
\end{aligned}$$

which can be ensured by passing to a subsequence where $\sum \frac{x_{i-1}}{x_i} < \infty$.

#### 7.1.3 c

- No: take $f(x) = \dfrac{1}{x \ln x}$
- Then by a $u$-substitution,

$$\int_0^x f = \ln(\ln(x)) \overset{x \longrightarrow \infty}{\longrightarrow} \infty$$

  is unbounded, so $f \notin L^1([1, \infty))$.
- But

$$x f(x) = \frac{1}{\ln(x)} \overset{x \longrightarrow \infty}{\longrightarrow} 0.$$

## 7.2 Fall 2019 # 5.

### 7.2.1 a

Show that if $f$ is continuous with compact support on $\mathbb{R}$, then

$$\lim_{y \to 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dx = 0$$

### 7.2.2 b

Let $f \in L^1(\mathbb{R})$ and for each $h > 0$ let

$$\mathcal{A}_h f(x) := \frac{1}{2h} \int_{|y| \leq h} f(x-y) dy$$

i. Prove that $\|\mathcal{A}_h f\|_1 \leq \|f\|_1$ for all $h > 0$.

ii. Prove that $\mathcal{A}_h f \longrightarrow f$ in $L^1(\mathbb{R})$ as $h \longrightarrow 0^+$.

*Solution.*

Continuity in $L^1$ (recall that DCT won't work! Notes 19.4, prove it for a dense subset first).
Lebesgue differentiation in 1-dimensional case. See HW 5.6.

### 7.3 a

Choose $g \in C_c^0$ such that $\|f - g\|_1 \longrightarrow 0$.
By translation invariance, $\|\tau_h f - \tau_h g\|_1 \longrightarrow 0$.
Write

$$\begin{aligned}
\|\tau f - f\|_1 &= \|\tau_h f - g + g - \tau_h g + \tau_h g - f\|_1 \\
&\leq \|\tau_h f - \tau_h g\| + \|g - f\| + \|\tau_h g - g\| \\
&\longrightarrow \|\tau_h g - g\|,
\end{aligned}$$

so it suffices to show that $\|\tau_h g - g\| \longrightarrow 0$ for $g \in C_c^0$.

Fix $\varepsilon > 0$. Enlarge the support of $g$ to $K$ such that

$$|h| \leq 1 \text{ and } x \in K^c \implies |g(x-h) - g(x)| = 0.$$

By uniform continuity of $g$, pick $\delta \leq 1$ small enough such that

$$x \in K,\ |h| \leq \delta \implies |g(x-h) - g(x)| < \varepsilon,$$

then

$$\int_K |g(x-h) - g(x)| \leq \int_K \varepsilon = \varepsilon \cdot m(K) \longrightarrow 0.$$

### 7.4 b

We have

$$\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x)|\, dx &= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y)\, dy \right| dx \\
&\leq \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)|\, dy\, dx \\
&=_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)|\, \mathbf{dx}\, \mathbf{dy} \\
&= \int_{\mathbb{R}} |f(y)|\, dy \\
&= \|f\|_1.
\end{aligned}$$

and (rough sketch)

$$\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x) - f(x)|\, dx &= \int_{\mathbb{R}} \left| \left( \frac{1}{2h} \int_{B(h,x)} f(y)\, dy \right) - f(x) \right| dx \\
&= \int_{\mathbb{R}} \left| \left( \frac{1}{2h} \int_{B(h,x)} f(y)\, dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x)\, dy \right| dx \\
&\leq_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{B(h,x)} |f(y-x) - f(x)|\, \mathbf{dx}\, \mathbf{dy} \\
&\leq \frac{1}{2h} \int_{\mathbb{R}} \|\tau_x f - f\|_1\, dy \\
&\longrightarrow 0 \quad \text{by (a)}.
\end{aligned}$$

## 7.5 Fall 2017 # 3

Let

$$S = \operatorname{span}_{\mathbb{C}} \left\{ \chi_{(a,b)} \;\middle|\; a, b \in \mathbb{R} \right\},$$

the complex linear span of characteristic functions of intervals of the form $(a, b)$.

Show that for every $f \in L^1(\mathbb{R})$, there exists a sequence of functions $\{f_n\} \subset S$ such that

$$\lim_{n\to\infty} \|f_n - f\|_1 = 0$$

*Solution.*

From homework: $E$ is Lebesgue measurable iff there exists a finite union of closed cubes $A$ such that $m(E\Delta A) < \varepsilon$.

It suffices to show that $S$ is dense in simple functions, and since simple functions are *finite* linear combinations of characteristic functions, it suffices to show this for $\chi_A$ for $A$ a measurable set.

Let $s = \chi_A$. By regularity of the Lebesgue measure, choose an open set $O \supseteq A$ such that $m(O \setminus A) < \varepsilon$.

$O$ is an open subset of $\mathbb{R}$, and thus $O = \coprod_{j\in\mathbb{N}} I_j$ is a disjoint union of countably many open intervals.

Now choose $N$ large enough such that $m(O\Delta I_{N,n}) < \varepsilon = \dfrac{1}{n}$ where we define $I_{N,n} \coloneqq \coprod_{j=1}^{N} I_j$.

Now define $f_n = \chi_{I_{N,n}}$, then

$$\|s - f_n\|_1 = \int \left|\chi_A - \chi_{I_{N,n}}\right| = m(A\Delta I_{N,n}) \overset{n\to\infty}{\longrightarrow} 0.$$

Since any simple function is a finite linear combination of $\chi_{A_i}$, we can do this for each $i$ to extend this result to all simple functions. But simple functions are dense in $L^1$, so $S$ is dense in $L^1$.

## 7.6 Spring 2015 # 4

Define

$$f(x,y) := \begin{cases} \dfrac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0 \le x \le y \\ 0 & \text{otherwise} \end{cases}$$

Carefully show that $f \in L^1(\mathbb{R}^2)$.

## 7.7 Fall 2014 # 3

Let $f \in L^1(\mathbb{R})$. Show that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that} \qquad m(E) < \delta \implies \int_E |f(x)|\, dx < \varepsilon$$

## 7.8 Spring 2014 # 1

1. Give an example of a continuous $f \in L^1(\mathbb{R})$ such that $f(x) \not\longrightarrow 0$ as$|x| \longrightarrow \infty$.
2. Show that if $f$ is *uniformly* continuous, then

$$\lim_{|x|\longrightarrow\infty} f(x) = 0.$$

# 8 Fubini-Tonelli

## 8.1 Spring 2020 # 4

Let $f, g \in L^1(\mathbb{R})$. Argue that $H(x,y) \coloneqq f(y)g(x-y)$ defines a function in $L^1(\mathbb{R}^2)$ and deduce from this fact that

$$(f * g)(x) \coloneqq \int_{\mathbb{R}} f(y)g(x-y)\,dy$$

defines a function in $L^1(\mathbb{R})$ that satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

*Solution.*
Relevant concepts:

- Tonelli: non-negative and measurable yields measurability of slices and equality of iterated integrals
- Fubini: $f(x,y) \in L^1$ yields *integrable* slices and equality of iterated integrals
- F/T: apply Tonelli to $|f|$; if finite, $f \in L^1$ and apply Fubini to $f$

$$\begin{aligned}
\|H(x)\|_1 &= \int_{\mathbb{R}} |H(x,y)|\,dx \\
&= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y)g(x-y)\,dy \right| dx \\
&\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)g(x-y)|\,dy \right) dx \\
&= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)g(x-y)|\,dx \right) dy \quad \text{by Tonelli} \\
&= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)g(t)|\,dt \right) dy \quad \text{setting } t = x-y,\, dt = -dx \\
&= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)| \cdot |g(t)|\,dt \right) dy \\
&= \int_{\mathbb{R}} |f(y)| \cdot \left( \int_{\mathbb{R}} |g(t)|\,dt \right) dy \\
&\coloneqq \int_{\mathbb{R}} |f(y)| \cdot \|g\|_1\,dy \\
&= \|g\|_1 \int_{\mathbb{R}} |f(y)|\,dy \\
&\coloneqq \|g\|_1 \|f\|_1 \\
&< \infty \quad \text{by assumption} \quad .
\end{aligned}$$

- $H$ is measurable on $\mathbb{R}^2$:
  - If we can show $\tilde{f}(x,y) \coloneqq f(y)$ and $\tilde{g}(x,y) \coloneqq g(x-y)$ are both measurable on $\mathbb{R}^2$, then $H = \tilde{f} \cdot \tilde{g}$ is a product of measurable functions and thus measurable.
  - $f \in L^1$, and $L^1$ functions are measurable by definition.
  - The function $(x,y) \mapsto g(x-y)$ is measurable on $\mathbb{R}^2$:
    * Let $g$ be measurable on $\mathbb{R}$, then the cylinder function $G(x,y) = g(x)$ on $\mathbb{R}^2$ is always measurable

* Define a linear transformation $T := [1, -1; 0, 1]$ which sends $(x, y) \longrightarrow (x - y, y)$, then $T \in \mathrm{GL}(2, \mathbb{R})$ is linear and thus measurable.
* Then $(G \circ T)(x, y) = G(x - y, y) = \tilde{g}(x - y)$, so $\tilde{g}$ is a composition of measurable functions and thus measurable.

- Apply **Tonelli** to $|H|$
  - $H$ measurable implies $|H|$ is measurable
  - $|H|$ is non-negative
  - So the iterated integrals are equal in the extended sense
  - The calculation shows the iterated integral is finite, to $\int |H|$ is finite and $H$ is thus integrable on $\mathbb{R}^2$.

Note: Fubini is not needed, since we're not calculating the actual integral, just showing $H$ is integrable.

## 8.2 Spring 2019 # 4

Let $f$ be a non-negative function on $\mathbb{R}^n$ and $\mathcal{A} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq f(x)\}$.

Prove the validity of the following two statements:

a. $f$ is a Lebesgue measurable function on $\mathbb{R}^n \iff \mathcal{A}$ is a Lebesgue measurable subset of $\mathbb{R}^{n+1}$

b. If $f$ is a Lebesgue measurable function on $\mathbb{R}^n$, then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x) dx = \int_0^\infty m(\{x \in \mathbb{R}^n : f(x) \geq t\}) \, dt$$

*Solution.*

See S&S p.82.

### 8.2.1 a

$\implies$:

- Suppose $f$ is a measurable function.
- Note that $\mathcal{A} = \{f(x) - t \geq 0\} \bigcap \{t \geq 0\}$.
- Define $F(x, t) = f(x)$, $G(x, t) = t$, which are cylinders on measurable functions and thus measurable.
- Define $H(x, y) = F(x, t) - G(x, t)$, which are linear combinations of measurable functions and thus measurable.
- Then $\mathcal{A} = \{H \geq 0\} \bigcap \{G \geq 0\}$ as a countable intersection of measurable sets, which is again measurable.

$\impliedby$:

- Suppose $\mathcal{A}$ is a measurable set.
- Then FT on $\chi_{\mathcal{A}}$ implies that for almost every $x \in \mathbb{R}^n$, the $x$-slices $\mathcal{A}_x$ are measurable and $

$$\mathcal{A}_x := \left\{ t \in \mathbb{R} \;\middle|\; (x, t) \in \mathcal{A} \right\} = [0, f(x)] \implies m(\mathcal{A}_x) = f(x) - 0 = f(x)$$

- But $x \mapsto m(\mathcal{A}_x)$ is a measurable function, and is exactly the function $x \mapsto f(x)$, so $f$ is measurable.

### 8.2.2 b

- Note

$$\mathcal{A} = \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; 0 \leq t \leq f(x) \right\}$$
$$\mathcal{A}_t = \left\{ x \in \mathbb{R}^n \;\middle|\; t \leq f(x) \right\}.$$

- Then

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)\, dx &= \int_{\mathbb{R}^n} \int_0^{f(x)} 1\, dt\, dx \\ &= \int_{\mathbb{R}^n} \int_0^{\infty} \chi_{\mathcal{A}}\, dt\, dx \\ &\overset{F.T.}{=} \int_0^{\infty} \int_{\mathbb{R}^n} \chi_{\mathcal{A}}\, dx\, dt \\ &= \int_0^{\infty} m(\mathcal{A}_t)\, dt,\end{aligned}$$

  where we just use that $\int \int \chi_{\mathcal{A}} = m(\mathcal{A})$

- By F.T., all of these integrals are equal.

  Why is FT justified.

## 8.3 Fall 2018 # 5

Let $f \geq 0$ be a measurable function on $\mathbb{R}$. Show that

$$\int_{\mathbb{R}} f = \int_0^{\infty} m(\{x : f(x) > t\}) dt$$

*Solution.*

Concepts used:

- Claim: If $E \subseteq \mathbb{R}^a \times \mathbb{R}^b$ is a measurable set, then for almost every $y \in \mathbb{R}^b$, the slice $E^y$ is measurable and

$$m(E) = \int_{\mathbb{R}^b} m(E^y)\, dy.$$

  - Set $g = \chi_E$, which is non-negative and measurable, so apply Tonelli.
  - Conclude that $g^y = \chi_{E^y}$ is measurable, the function $y \mapsto \int g^y(x)\, dx$ is measurable, and $\int \int g^y(x)\, dx\, dy = \int g.$
  - But $\int g = m(E)$ and $\int \int g^y(x)\, dx\, dy = \int m(E^y)\, dy.$

**Solution**

Note: $f$ is a function $\mathbb{R} \longrightarrow \mathbb{R}$ in the original problem, but here I've assumed $f : \mathbb{R}^n \longrightarrow \mathbb{R}$.

- Since $f \geq 0$, set

$$E := \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; f(x) > t \right\} = \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; 0 \leq t < f(x) \right\}.$$

- Claim: since $f$ is measurable, $E$ is measurable and thus $m(E)$ makes sense.
  - Since $f$ is measurable, $F(x,t) \coloneqq t - f(x)$ is measurable on $\mathbb{R}^n \times \mathbb{R}$.
  - Then write $E = \{F < 0\} \bigcap \{t \geq 0\}$ as an intersection of measurable sets.
- We have slices

$$E^t \coloneqq \left\{ x \in \mathbb{R}^n \;\middle|\; (x,t) \in E \right\} = \left\{ x \in \mathbb{R}^n \;\middle|\; 0 \leq t < f(x) \right\}$$
$$E^x \coloneqq \left\{ t \in \mathbb{R} \;\middle|\; (x,t) \in E \right\} = \left\{ t \in \mathbb{R} \;\middle|\; 0 \leq t \leq f(x) \right\} = [0, f(x)].$$

  - $E_t$ is precisely the set that appears in the original RHS integrand.
  - $m(E^x) = f(x)$.
- Claim: $\chi_E$ satisfies the conditions of Tonelli, and thus $m(E) = \displaystyle\int \chi_E$ is equal to any iterated integral.
  - Non-negative: clear since $0 \leq \chi_E \leq 1$
  - Measurable: characteristic functions of measurable sets are measurable.
- Conclude:
  1. For almost every $x$, $E^x$ is a measurable set, $x \mapsto m(E^x)$ is a measurable function, and $m(E) = \displaystyle\int_{\mathbb{R}^n} m(E^x)\, dx$
  2. For almost every $t$, $E^t$ is a measurable set, $t \mapsto m(E^t)$ is a measurable function, and $m(E) = \displaystyle\int_{\mathbb{R}} m(E^t)\, dt$
- On one hand,

$$\begin{aligned} m(E) &= \int_{\mathbb{R}^{n+1}} \chi_E(x,t) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_E(x,t)\, dt\, dx \quad \text{by Tonelli} \\ &= \int_{\mathbb{R}^n} m(E^x)\, dx \quad \text{first conclusion} \\ &= \int_{\mathbb{R}^n} f(x)\, dx. \end{aligned}$$

- On the other hand,

$$\begin{aligned} m(E) &= \int_{\mathbb{R}^{n+1}} \chi_E(x,t) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_E(x,t)\, dx\, dt \quad \text{by Tonelli} \\ &= \int_{\mathbb{R}} m(E^t)\, dt \quad \text{second conclusion.} \end{aligned}$$

- Thus

$$\int_{\mathbb{R}^n} f\, dx = m(E) = \int_{\mathbb{R}} m(E^t)\, dt = \int_{\mathbb{R}} m\left( \left\{ x \;\middle|\; f(x) > t \right\} \right).$$

## 8.4 Fall 2015 # 5

Let $f, g \in L^1(\mathbb{R})$ be Borel measurable.

1. Show that

- The function

$$F(x,y) \coloneqq f(x-y)g(y)$$

  is Borel measurable on $\mathbb{R}^2$, and
- For almost every $y \in \mathbb{R}$,

$$F_y(x) \coloneqq f(x-y)g(y)$$

  is integrable with respect to $y$.

2. Show that $f * g \in L^1(\mathbb{R})$ and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

### 8.5 Spring 2014 # 5

Let $f, g \in L^1([0,1])$ and for all $x \in [0,1]$ define

$$F(x) \coloneqq \int_0^x f(y)\,dy \quad \text{and} \quad G(x) \coloneqq \int_0^x g(y)\,dy.$$

Prove that

$$\int_0^1 F(x)g(x)\,dx = F(1)G(1) - \int_0^1 f(x)G(x)\,dx$$

## 9 $L^2$ and Fourier Analysis

### 9.1 Spring 2020 # 6

#### 9.1.1 a

Show that

$$L^2([0,1]) \subseteq L^1([0,1]) \quad \text{and} \quad \ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z}).$$

#### 9.1.2 b

For $f \in L^1([0,1])$ define

$$\widehat{f}(n) \coloneqq \int_0^1 f(x) e^{-2\pi i n x}\,dx.$$

Prove that if $f \in L^1([0,1])$ and $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$ then

$$S_N f(x) \coloneqq \sum_{|n| \leq N} \widehat{f}(n) e^{2\pi i n x}.$$

converges uniformly on $[0,1]$ to a continuous function $g$ such that $g = f$ almost everywhere.

Hint: One approach is to argue that if $f \in L^1([0,1])$ with $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$ then $f \in L^2([0,1])$.

*Solution.*
Concepts used:

- For $e_n(x) := e^{2\pi i n x}$, the set $\{e_n\}$ is an orthonormal basis for $L^2([0,1])$.
- For any orthonormal sequence in a Hilbert space, we have Bessel's inequality:

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

- When $\{e_n\}$ is a basis, the above is an *equality* (Parseval)
- Arguing uniform convergence: since $\left\{ \widehat{f}(n) \right\} \in \ell^1(\mathbb{Z})$, we should be able to apply the $M$ test.

### 9.1.3 a

Claim: $\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$.

- Set $\mathbf{c} = \left\{ c_k \;\middle|\; k \in \mathbb{Z} \right\} \in \ell^1(\mathbb{Z})$.
- It suffices to show that if $\displaystyle\sum_{k \in \mathbb{Z}} |c_k| < \infty$ then $\displaystyle\sum_{k \in \mathbb{Z}} |c_k|^2 < \infty$.
- Let $S = \left\{ c_k \;\middle|\; |c_k| \leq 1 \right\}$, then $c_k \in S \implies |c_k|^2 \leq |c_k|$
- Claim: $S^c$ can only contain finitely many elements, all of which are finite.
  - If not, either $S^c := \{c_j\}_{j=1}^{\infty}$ is infinite with every $|c_j| > 1$, which forces

$$\sum_{c_k \in S^c} |c_k| = \sum_{j=1}^{\infty} |c_j| > \sum_{j=1}^{\infty} 1 = \infty.$$

  - If any $c_j = \infty$, then $\displaystyle\sum_{k \in \mathbb{Z}} |c_k| \geq c_j = \infty$.
- So $S^c$ is a finite set of finite integers, let $N = \max \left\{ |c_j|^2 \;\middle|\; c_j \in S^c \right\} < \infty$.
- Rewrite the sum

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} |c_k|^2 &= \sum_{c_k \in S} |c_k|^2 + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \sum_{c_k \in S} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \sum_{k \in \mathbb{Z}} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \quad \text{since the } |c_k| \text{ are all positive} \\
&= \|\mathbf{c}\|_{\ell^1} + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \|\mathbf{c}\|_{\ell^1} + |S^c| \cdot N \\
&< \infty.
\end{aligned}$$

Claim: $L^2([0,1]) \subseteq L^1([0,1])$.

- It suffices to show that $\displaystyle\int |f|^2 < \infty \implies \int |f| < \infty$.
- Define $S = \left\{ x \in [0,1] \;\middle|\; |f(x)| \leq 1 \right\}$, then $x \in S^c \implies |f(x)|^2 \geq |f(x)|$.

- Break up the integral:

$$\begin{aligned}
\int_{\mathbb{R}} |f| &= \int_S |f| + \int_{S^c} |f| \\
&\leq \int_S |f| + \int_{S^c} |f|^2 \\
&\leq \int_S |f| + \|f\|_2 \\
&\leq \sup_{x \in S} \{|f(x)|\} \cdot \mu(S) + \|f\|_2 \\
&= 1 \cdot \mu(S) + \|f\|_2 \quad \text{by definition of } S \\
&\leq 1 \cdot \mu([0,1]) + \|f\|_2 \quad \text{since } S \subseteq [0,1] \\
&= 1 + \|f\|_2 \\
&< \infty.
\end{aligned}$$

Note: this proof shows $L^2(X) \subseteq L^1(X)$ whenever $\mu(X) < \infty$.

## 9.2 Fall 2017 # 5

Let $\varphi$ be a compactly supported smooth function that vanishes outside of an interval $[-N, N]$ such that $\int_{\mathbb{R}} \varphi(x)\, dx = 1$.

For $f \in L^1(\mathbb{R})$, define

$$K_j(x) \coloneqq j\varphi(jx), \qquad f * K_j(x) \coloneqq \int_{\mathbb{R}} f(x-y) K_j(y)\, dy$$

and prove the following:

1. Each $f * K_j$ is smooth and compactly supported.
2. $$\lim_{j \longrightarrow \infty} \|f * K_j - f\|_1 = 0$$

Hint:

$$\lim_{y \longrightarrow 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dy = 0$$

*Solution.*

### 9.2.1 a

**Lemma:** If $\varphi \in C_c^1$, then $(f * \varphi)' = f * \varphi'$ almost everywhere.
*Silly Proof:*

$$\begin{aligned}\mathcal{F}((f*\varphi)') &= 2\pi i\xi\ \mathcal{F}(f*\varphi) \\ &= 2\pi i\xi\ \mathcal{F}(f)\ \mathcal{F}(\varphi) \\ &= \mathcal{F}(f)\cdot(2\pi i\xi\ \mathcal{F}(\varphi)) \\ &= \mathcal{F}(f)\cdot\mathcal{F}(\varphi') \\ &= \mathcal{F}(f*\varphi').\end{aligned}$$

*Actual proof*:

$$\begin{aligned}(f*\varphi)'(x) &= (\varphi*f)'(x) \\ &= \lim_{h\longrightarrow 0}\frac{(\varphi*f)'(x+h)-(\varphi*f)'(x)}{h} \\ &= \lim_{h\longrightarrow 0}\int\frac{\varphi(x+h-y)-\varphi(x-y)}{h}f(y) \\ &\overset{DCT}{=}\int\lim_{h\longrightarrow 0}\frac{\varphi(x+h-y)-\varphi(x-y)}{h}f(y) \\ &= \int\varphi'(x-y)f(y) \\ &= (\varphi'*f)(x) \\ &= (f*\varphi')(x).\end{aligned}$$

To see that the DCT is justified, we can apply the MVT on the interval $[0,h]$ to $f$ to obtain

$$\frac{\varphi(x+h-y)-\varphi(x-y)}{h}=\varphi'(c)\quad c\in[0,h],$$

and since $\varphi'$ is continuous and compactly supported, $\varphi'$ is bounded by some $M<\infty$ by the extreme value theorem and thus

$$\begin{aligned}\int\left|\frac{\varphi(x+h-y)-\varphi(x-y)}{h}f(y)\right| &= \int|\varphi'(c)f(y)| \\ &\leq\int|M||f| \\ &= |M|\int|f|<\infty,\end{aligned}$$

since $f\in L^1$ by assumption, so we can take $g:=|M||f|$ as the dominating function.
Applying this theorem infinitely many times shows that $f*\varphi$ is smooth.
To see that $f*\varphi$ is compactly supported, approximate $f$ by a *continuous* compactly supported function $h$, so $\|h-f\|_1\xrightarrow{L^1}0$.
Now let $g_x(y)=\varphi(x-y)$, and note that $\operatorname{supp}(g)=x-\operatorname{supp}(\varphi)$ which is still compact.
But since $\operatorname{supp}(h)$ is bounded, there is some $N$ such that

$$|x|>N\implies A_x:=\operatorname{supp}(h)\bigcap\operatorname{supp}(g_x)=\emptyset$$

and thus

$$\begin{aligned}
(h * \varphi)(x) &= \int_{\mathbb{R}} \varphi(x-y)h(y)\, dy \\
&= \int_{A_x} g_x(y)h(y) \\
&= 0,
\end{aligned}$$

so $\left\{ x \;\middle|\; f * g(x) = 0 \right\}$ is open, and its complement is closed and bounded and thus compact.

### 9.2.2 b

$$\begin{aligned}
\|f * K_j - f\|_1 &= \int \left| \int f(x-y)K_j(y)\, dy - f(x) \right| dx \\
&= \int \left| \int f(x-y)K_j(y)\, dy - \int f(x)K_j(y)\, dy \right| dx \\
&= \int \left| \int (f(x-y) - f(x))K_j(y)\, dy \right| dx \\
&\leq \int \int |(f(x-y) - f(x))| \cdot |K_j(y)|\, dy\, dx \\
&\overset{FT}{=} \int \int |(f(x-y) - f(x))| \cdot |K_j(y)|\, \mathbf{dx\, dy} \\
&= \int |K_j(y)| \left( \int |(f(x-y) - f(x))|\, dx \right) dy \\
&= \int |K_j(y)| \cdot \|f - \tau_y f\|_1\, dy.
\end{aligned}$$

We now split the integral up into pieces.

1. Chose $\delta$ small enough such that $|y| < \delta \implies \|f - \tau_y f\|_1 < \varepsilon$ by continuity of translation in $L^1$, and
2. Since $\varphi$ is compactly supported, choose $J$ large enough such that

$$j > J \implies \int_{|y| \geq \delta} |K_j(y)|\, dy = \int_{|y| \geq \delta} |j\varphi(jy)| = 0$$

Then

$$\begin{aligned}
\|f * K_j - f\|_1 &\leq \int |K_j(y)| \cdot \|f - \tau_y f\|_1\, dy \\
&= \int_{|y| < \delta} |K_j(y)| \cdot \|f - \tau_y f\|_1\, dy + \int_{|y| \geq \delta} |K_j(y)| \cdot \|f - \tau_y f\|_1\, dy \\
&= \varepsilon \int_{|y| \geq \delta} |K_j(y)| + 0 \\
&\leq \varepsilon(1) \longrightarrow 0.
\end{aligned}$$

### 9.3 Spring 2017 # 5

Let $f, g \in L^2(\mathbb{R})$. Prove that the formula

$$h(x) := \int_{-\infty}^{\infty} f(t)g(x-t)\,dt$$

defines a uniformly continuous function $h$ on $\mathbb{R}$.

### 9.4 Spring 2015 # 6

Let $f \in L^1(\mathbb{R})$ and $g$ be a bounded measurable function on $\mathbb{R}$.

1. Show that the convolution $f * g$ is well-defined, bounded, and uniformly continuous on $\mathbb{R}$.
2. Prove that one further assumes that $g \in C^1(\mathbb{R})$ with bounded derivative, then $f * g \in C^1(\mathbb{R})$ and

$$\frac{d}{dx}(f * g) = f * \left(\frac{d}{dx} g\right)$$

### 9.5 Fall 2014 # 5

1. Let $f \in C_c^0(\mathbb{R}^n)$, and show

$$\lim_{t \longrightarrow 0} \int_{\mathbb{R}^n} |f(x+t) - f(x)|\,dx = 0.$$

2. Extend the above result to $f \in L^1(\mathbb{R}^n)$ and show that

$$f \in L^1(\mathbb{R}^n), \quad g \in L^\infty(\mathbb{R}^n) \quad \implies f * g \text{ is bounded and uniformly continuous.}$$

## 10 Functional Analysis: General

### 10.1 Fall 2019 # 4.

Let $\{u_n\}_{n=1}^\infty$ be an orthonormal sequence in a Hilbert space $\mathcal{H}$.

#### 10.1.1 a

Prove that for every $x \in \mathcal{H}$ one has

$$\sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 \le \|x\|^2$$

#### 10.1.2 b

Prove that for any sequence $\{a_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$ there exists an element $x \in \mathcal{H}$ such that

$$a_n = \langle x,\ u_n\rangle \text{ for all } n \in \mathbb{N}$$

and

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$$

*Solution.*
Concepts used:

- Bessel's Inequality
- Pythagoras
- Surjectivity of the Riesz map
- Parseval's Identity
- Trick – remember to write out finite sum $S_N$, and consider $\|x - S_N\|$.

### 10.1.3 a

**Claim:**

$$0 \leq \left\| x - \sum_{n=1}^{N} \langle x, u_n \rangle u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n \rangle|^2$$
$$\implies \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2.$$

*Proof:* Let $S_N = \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n$. Then

$$\begin{aligned} 0 &\leq \|x - S_N\|^2 \\ &= \langle x - S_n,\ x - S_N \rangle \\ &= \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\ &\xrightarrow{N \longrightarrow \infty} \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2. \end{aligned}$$

### 10.1.4 b

1. Fix $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
2. Define

$$x := \lim_{N \longrightarrow \infty} S_N = \lim_{N \longrightarrow \infty} \sum_{k=1}^{N} a_k u_k$$

3. $\{S_N\}$ Cauchy (by 1) and $H$ complete $\implies x \in H$.
4.

$$\langle x,\ u_n \rangle = \left\langle \sum_k a_k u_k,\ u_n \right\rangle = \sum_k a_k \langle u_k,\ u_n \rangle = a_n \quad \forall n \in \mathbb{N}$$

since the $u_k$ are all orthogonal.

5.

$$\|x\|^2 = \left\|\sum_k a_k u_k\right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$$

by Pythagoras since the $u_k$ are normal.

Bonus: We didn't use completeness here, so the Fourier series may not actually converge to $x$. If $\{u_n\}$ is **complete** (so $x = 0 \iff \langle x,\ u_n \rangle = 0\ \forall n$) then the Fourier series *does* converge to $x$ and $\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 = \|x\|^2$ for all $x \in H$.

## 10.2 Spring 2019 # 5

### 10.2.1 a

Show that $L^2([0,1]) \subseteq L^1([0,1])$ and argue that $L^2([0,1])$ in fact forms a dense subset of $L^1([0,1])$.

### 10.2.2 b

Let $\Lambda$ be a continuous linear functional on $L^1([0,1])$.

Prove the Riesz Representation Theorem for $L^1([0,1])$ by following the steps below:

i. Establish the existence of a function $g \in L^2([0,1])$ which represents $\Lambda$ in the sense that

$$\Lambda(f) = f(x)g(x)dx \text{ for all } f \in L^2([0,1]).$$

Hint: You may use, without proof, the Riesz Representation Theorem for $L^2([0,1])$.

ii. Argue that the $g$ obtained above must in fact belong to $L^\infty([0,1])$ and represent $\Lambda$ in the sense that

$$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}dx \quad \text{for all } f \in L^1([0,1])$$

with

$$\|g\|_{L^\infty([0,1])} = \|\Lambda\|_{L^1([0,1])^\vee}$$

*Solution.*
Concepts used:

- Holders' inequality: $\|fg\|_1 \leq \|f\|_p \|f\|_q$
- Riesz Representation for $L^2$: If $\Lambda \in (L^2)^\vee$ then there exists a unique $g \in L^2$ such that $\Lambda(f) = \int fg$.
- $\|f\|_{L^\infty(X)} \coloneqq \inf \left\{ t \geq 0 \;\middle|\; |f(x)| \leq t \text{ almost everywhere} \right\}$.
- **Lemma**: $m(X) < \infty \implies L^p(X) \subset L^2(X)$.

*Proof:* Write Holder's inequality as $\|fg\|_1 \leq \|f\|_a \|g\|_b$ where $\dfrac{1}{a} + \dfrac{1}{b} = 1$, then

$$\|f\|_p^p = \||f|^p\|_1 \leq \||f|^p\|_a \, \|1\|_b.$$

Now take $a = \dfrac{2}{p}$ and this reduces to

$$\begin{aligned} \|f\|_p^p &\leq \|f\|_2^p \, m(X)^{\frac{1}{b}} \\ \implies \|f\|_p &\leq \|f\|_2 \cdot O(m(X)) < \infty. \end{aligned}$$

### 10.2.3 a

- Note $X = [0,1] \implies m(X) = 1$.
- By Holder's inequality with $p = q = 2$,

$$\|f\|_1 = \|f \cdot 1\|_1 \leq \|f\|_2 \cdot \|1\|_2 = \|f\|_2 \cdot m(X)^{\frac{1}{2}} = \|f\|_2,$$

- Thus $L^2(X) \subseteq L^1(X)$
- Since they share a common dense subset (simple functions) $L^2$ is dense in $L^1$

What theorem is this using?

### 10.2.4 b

Let $\Lambda \in L^1(X)^\vee$ be arbitrary.

**(i): Existence of $g$ Representing $\Lambda$.**

- Let $f \in L^2 \subseteq L^1$ be arbitrary
- Claim: $\Lambda \in L^1(X)^\vee \implies \Lambda \in L^2(X)^\vee$.
  - Suffices to show that $\|\Gamma\|_{L^2(X)^\vee} := \sup\limits_{\|f\|_2 = 1} |\Gamma(f)| < \infty$, since bounded implies continuous.
  - By the lemma, $\|f\|_1 \leq C\|f\|_2$ for some constant $C \approx m(X)$.
  - Note

$$\|\Lambda\|_{L^1(X)^\vee} := \sup_{\|f\|_1 = 1} |\Lambda(f)|$$

  - Define $\widehat{f} = \dfrac{f}{\|f\|_1}$ so $\left\|\widehat{f}\right\|_1 = 1$
  - Since $\|\Lambda\|_{1^\vee}$ is a supremum over *all* $f \in L^1(X)$ with $\|f\|_1 = 1$,

$$\left|\Lambda(\widehat{f})\right| \leq \|\Lambda\|_{(L^1(X))^\vee},$$

  - Then

$$\begin{aligned} \frac{|\Lambda(f)|}{\|f\|_1} &= \left|\Lambda(\widehat{f})\right| \leq \|\Lambda\|_{L^1(X)^\vee} \\ \implies |\Lambda(f)| &\leq \|\Lambda\|_{1^\vee} \cdot \|f\|_1 \\ &\leq \|\Lambda\|_{1^\vee} \cdot C\|f\|_2 < \infty \quad \text{by assumption,} \end{aligned}$$

- So $\Lambda \in (L^2)^\vee$.

- Now apply Riesz Representation for $L^2$: there is a $g \in L^2$ such that

$$f \in L^2 \implies \Lambda(f) = \langle f,\ g \rangle := \int_0^1 f(x)\overline{g(x)}\,dx.$$

**(ii): $g$ is in $L^\infty$**

- It suffices to show $\|g\|_{L^\infty(X)} < \infty$.
- Since we're assuming $\|\Gamma\|_{L^1(X)^\vee} < \infty$, it suffices to show the stated equality.

Is this assumed..? Or did we show it..?

- Claim: $\|\Lambda\|_{L^1(X)^\vee} = \|g\|_{L^\infty(X)}$
  - The result follows because $\Lambda$ was assumed to be in $L^1(X)^\vee$, so $\|\Lambda\|_{L^1(X)^\vee} < \infty$.
  - $\leq$:

$$\begin{aligned}
\|\Lambda\|_{L^1(X)^\vee} &= \sup_{\|f\|_1 = 1} |\Lambda(f)| \\
&= \sup_{\|f\|_1 = 1} \left| \int_X f\bar{g} \right| \quad \text{by (i)} \\
&= \sup_{\|f\|_1 = 1} \int_X |f\bar{g}| \\
&:= \sup_{\|f\|_1 = 1} \|fg\|_1 \\
&\leq \sup_{\|f\|_1 = 1} \|f\|_1 \|g\|_\infty \quad \text{by Holder with } p = 1, q = \infty \\
&= \|g\|_\infty,
\end{aligned}$$

  - $\geq$:
    * Suppose toward a contradiction that $\|g\|_\infty > \|\Lambda\|_{1^\vee}$.
    * Then there exists some $E \subseteq X$ with $m(E) > 0$ such that

$$x \in E \implies |g(x)| > \|\Lambda\|_{L^1(X)^\vee}.$$

    * Define

$$h = \frac{1}{m(E)} \frac{\overline{g}}{|g|} \chi_E.$$

    * Note $\|h\|_{L^1(X)} = 1$.
    * Then

$$\begin{aligned}
\Lambda(h) &= \int_X hg \\
&:= \int_X \frac{1}{m(E)} \frac{g\overline{g}}{|g|} \chi_E \\
&= \frac{1}{m(E)} \int_E |g| \\
&\geq \frac{1}{m(E)} \|g\|_\infty m(E) \\
&= \|g\|_\infty \\
&> \|\Lambda\|_{L^1(X)^\vee},
\end{aligned}$$

a contradiction since $\|\Lambda\|_{L^1(X)^\vee}$ is the supremum over all $h_\alpha$ with $\|h_\alpha\|_{L^1(X)} = 1$.

### 10.3 Spring 2016 # 6

Without using the Riesz Representation Theorem, compute

$$\sup \left\{ \left| \int_0^1 f(x) e^x dx \right| \;\middle|\; f \in L^2([0,1], m),\ \|f\|_2 \leq 1 \right\}$$

### 10.4 Spring 2015 # 5

Let $\mathcal{H}$ be a Hilbert space.

1. Let $x \in \mathcal{H}$ and $\{u_n\}_{n=1}^N$ be an orthonormal set. Prove that the best approximation to $x$ in $\mathcal{H}$ by an element in $\operatorname{span}_{\mathbb{C}} \{u_n\}$ is given by

$$\widehat{x} := \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n.$$

2. Conclude that finite dimensional subspaces of $\mathcal{H}$ are always closed.

### 10.5 Fall 2015 # 6

Let $f : [0,1] \longrightarrow \mathbb{R}$ be continuous. Show that

$$\sup \left\{ \|fg\|_1 \;\middle|\; g \in L^1[0,1],\ \|g\|_1 \leq 1 \right\} = \|f\|_\infty$$

### 10.6 Fall 2014 # 6

Let $1 \leq p, q \leq \infty$ be conjugate exponents, and show that

$$f \in L^p(\mathbb{R}^n) \implies \|f\|_p = \sup_{\|g\|_q = 1} \left| \int f(x) g(x) dx \right|$$

## 11 Functional Analysis: Banach Spaces

### 11.1 Spring 2019 # 1

Let $C([0,1])$ denote the space of all continuous real-valued functions on $[0,1]$.

a. Prove that $C([0,1])$ is complete under the uniform norm $\|f\|_u := \sup_{x \in [0,1]} |f(x)|$.

b. Prove that $C([0,1])$ is not complete under the $L^1$-norm $\|f\|_1 = \int_0^1 |f(x)|\, dx$.

*Solution.*

### 11.1.1 a

- Let $\{f_n\}$ be a Cauchy sequence in $C(I, \|\cdot\|_\infty)$, so $\lim_n \lim_m \|f_m - f_n\|_\infty = 0$, we will show it converges to some $f$ in this space.
- For each fixed $x_0 \in [0,1]$, the sequence of real numbers $\{f_n(x_0)\}$ is Cauchy in $\mathbb{R}$ since

$$x_0 \in I \implies |f_m(x_0) - f_n(x_0)| \leq \sup_{x \in I} |f_m(x) - f_n(x)| := \|f_m - f_n\|_\infty \overset{m>n\longrightarrow\infty}{\longrightarrow} 0,$$

- Since $\mathbb{R}$ is complete, this sequence converges and we can define $f(x) := \lim_{k \longrightarrow \infty} f_n(x)$.
- Thus $f_n \longrightarrow f$ pointwise by construction
- Claim: $\|f - f_n\| \overset{n\longrightarrow\infty}{\longrightarrow} 0$, so $f_n$ converges to $f$ in $C([0,1], \|\cdot\|_\infty)$.
  - Proof:
    - Fix $\varepsilon > 0$; we will show there exists an $N$ such that $n \geq N \implies \|f_n - f\| < \varepsilon$
    - Fix an $x_0 \in I$. Since $f_n \longrightarrow f$ pointwise, choose $N_1$ large enough so that

$$n \geq N_1 \implies |f_n(x_0) - f(x_0)| < \varepsilon/2.$$

    - Since $\|f_n - f_m\|_\infty \longrightarrow 0$, choose and $N_2$ large enough so that

$$n, m \geq N_2 \implies \|f_n - f_m\|_\infty < \varepsilon/2.$$

    - Then for $n, m \geq \max(N_1, N_2)$, we have

$$\begin{aligned}
|f_n(x_0) - f(x_0)| &= |f_n(x_0) - f(x_0) + f_m(x_0) - f_m(x_0)| \\
&= |f_n(x_0) - f_m(x_0) + f_m(x_0) - f(x_0)| \\
&\leq |f_n(x_0) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \\
&< |f_n(x_0) - f_m(x_0)| + \frac{\varepsilon}{2} \\
&\leq \sup_{x \in I} |f_n(x) - f_m(x)| + \frac{\varepsilon}{2} \\
&< \|f_n - f_m\|_\infty + \frac{\varepsilon}{2} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
\implies |f_n(x_0) - f(x_0)| &< \varepsilon \\
\implies \sup_{x \in I} |f_n(x_0) - f(x_0)| &\leq \sup_{x \in I} \varepsilon \quad \text{by order limit laws} \\
\implies \|f_n - f\| &\leq \varepsilon
\end{aligned}$$

.

- $f$ is the uniform limit of continuous functions and thus continuous, so $f \in C([0,1])$.

### 11.1.2 b

- It suffices to produce a Cauchy sequence that does not converge to a continuous function.
- Take the following sequence of functions:
  - $f_1$ increases linearly from 0 to 1 on $[0, 1/2]$ and is 1 on $[1/2, 1]$

- $f_2$ is 0 on $[0, 1/4]$ increases linearly from 0 to 1 on $[1/4, 1/2]$ and is 1 on $[1/2, 1]$
- $f_3$ is 0 on $[0, 3/8]$ increases linearly from 0 to 1 on $[3/8, 1/2]$ and is 1 on $[1/2, 1]$
- $f_3$ is 0 on $[0, (1/2 - 3/8)/2]$ increases linearly from 0 to 1 on $[(1/2 - 3/8)/2, 1/2]$ and is 1 on $[1/2, 1]$

Idea: take sequence starting points for the triangles: $0, 0 + \frac{1}{4}, 0 + \frac{1}{4} + \frac{1}{8}, \cdots$ which converges to $1/2$ since $\sum_{k=1}^{\infty} \frac{1}{2^k} = -\frac{1}{2} + \sum_{k=0}^{\infty} \frac{1}{2^k}$.

- Then each $f_n$ is clearly integrable, since its graph is contained in the unit square.
- $\{f_n\}$ is Cauchy: geometrically subtracting areas yields a single triangle whose area tends to 0.
- But $f_n$ converges to $\chi_{[\frac{1}{2},1]}$ which is discontinuous.

show that $\int_0^1 |f_n(x) - f_m(x)|\, dx \longrightarrow 0$ rigorously, show that no $g \in L^1([0,1])$ can converge to this indicator function.

## 11.2 Spring 2017 # 6

Show that the space $C^1([a, b])$ is a Banach space when equipped with the norm

$$\|f\| := \sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |f'(x)|.$$

*Solution.*

See https://math.stackexchange.com/questions/507263/prove-that-c1a-b-with-the-c1-norm-is-a-banach-space

- Denote this norm $\|\cdot\|_u$
- Let $f_n$ be a Cauchy sequence in this space, so $\|f_n\|_u < \infty$ for every $n$ and $\|f_j - f_k\|_u \overset{j,k \longrightarrow \infty}{\longrightarrow} 0$.

and define a candidate limit: for each $x \in I$, set

$$f(x) := \lim_{n \longrightarrow \infty} f_n(x).$$

- Note that

$$\begin{aligned} \|f_n\|_\infty &\leq \|f_n\|_u < \infty \\ \|f_n'\|_\infty &\leq \|f_n\|_u < \infty. \end{aligned}$$

  - Thus both $f_n, f_n'$ are Cauchy sequences in $C^0([a, b], \|\cdot\|_\infty)$, which is a Banach space, so they converge.
- So
  - $f_n \longrightarrow f$ uniformly (by uniqueness of limits),
  - $f_n' \longrightarrow g$ uniformly for some $g$, and
  - $f, g \in C^0([a, b])$.
- Claim: $g = f'$

- For any fixed $a \in I$, we have

$$\begin{aligned} f_n(x) - f_n(a) &\xrightarrow{u} f(x) - f(a) \\ \int_a^x f_n' &\xrightarrow{u} \int_a^x g. \end{aligned}$$

- By the FTC, the left-hand sides are equal.
- By uniqueness of limits so are the right-hand sides, so $f' = g$.

- Claim: the limit $f$ is an element in this space.
  - Since $f, f' \in C^0([a,b])$, they are bounded, and so $\|f\|_u < \infty$.
- Claim: $\|f_n - f\|_u \overset{n \longrightarrow \infty}{\longrightarrow} 0$
- Thus the Cauchy sequence $\{f_n\}$ converges to a function $f$ in the $u$-norm where $f$ is an element of this space, making it complete.

## 11.3 Fall 2017 # 6

Let $X$ be a complete metric space and define a norm

$$\|f\| := \max\{|f(x)| : x \in X\}.$$

Show that $(C^0(\mathbb{R}), \|\cdot\|)$ (the space of continuous functions $f : X \longrightarrow \mathbb{R}$) is complete.

*Solution.*

Should be supremum maybe..?

Let $\{f_k\}$ be a Cauchy sequence, so $\|f_k\| < \infty$ for all $k$. Then for a fixed $x$, the sequence $f_k(x)$ is Cauchy in $\mathbb{R}$ and thus converges to some $f(x)$, so define $f$ by $f(x) := \lim_{k \longrightarrow \infty} f_k(x)$.

Then $\|f_k - f\| = \max_{x \in X} |f_k(x) - f(x)| \overset{k \longrightarrow \infty}{\longrightarrow} 0$, and thus $f_k \longrightarrow f$ uniformly and thus $f$ is continuous. It just remains to show that $f$ has bounded norm.

Choose $N$ large enough so that $\|f - f_N\| < \varepsilon$, and write $\|f_N\| := M < \infty$

$$\|f\| \leq \|f - f_N\| + \|f_N\| < \varepsilon + M < \infty.$$