Extra Problems

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Contents

1 Problems

1.1 Point Set

1.1.1 Compactness

- Show that $\mathbb{R}$ with the cofinite topology is compact.
- Show that $[0, 1]$ is compact without using the Heine-Borel theorem.
- Let X be a compact space and let A be a closed subspace. Show that A is compact. Solution

  Suggested by Ernest

- Let $f : X \longrightarrow Y$ be a continuous function, with X compact. Show that $f(X)$ is compact. Solution

  Suggested by Ernest

1.1.2 Connectedness

- Show that $[0, 1]$ is connected. Solution

1.1.3 Hausdorff Spaces

- Let A be a compact subspace of a Hausdorff space X. Show that A is closed. Solution

  Suggested by Ernest

- Show that a closed subset of a Hausdorff space need not be compact.
- Show that in a *compact* Hausdorff space, A is closed iff A is compact.
- Show that a local homeomorphism between compact Hausdorff spaces is a covering space.
- Show that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism. Solution

  Suggested by Ernest

1.2 Algebraic Topology

1.2.1 Fundamental Group

- Compute $\pi_1(X)$ where $X \coloneqq S^2/\sim$, where $x \sim -x$ only for x on the equator $S^1 \hookrightarrow S^2$.
  - Hint: try cellular homology. Should yield $[\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}, 0, \cdots]$.
- Show that if $X = S^2 \coprod_{\text{id}} S^2$ is a pushout along the equators, then $H_n(X) = [\mathbb{Z}, 0, \mathbb{Z}^3, 0, \cdots]$.

1.2.2 Covering Spaces

- Describe all connected covering spaces of $\mathbb{RP}^2 \vee \mathbb{RP}^2$.

1.2.3 Homology

- Compute the homology of the Klein bottle using the Mayer-Vietoris sequence and a decomposition $K = M \coprod_f M$
- Use the Kunneth formula to compute $H^*(S^2 \times S^2; \mathbb{Z})$.
  - Known to be $[\mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z}, 0, 0, \cdots]$.
- Compute $H^*(S^2 \vee S^2 \vee S^4)$
  - Known to be $[\mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z}, 0, 0, \cdots]$.
- Show that $\chi(\Sigma_g + \Sigma_h) = \chi(\Sigma_g) + \chi(\Sigma_h) - 2$.

2 Solutions

2.1 Point Set

2.1.1 Connectedness

1. Problem Statement

Reference
A potentially shorter proof

- Let $I = [0, 1] = A \bigcup B$ be a disconnection, so
  - $A, B \neq \emptyset$

- $A\coprod B = I$
- $\text{cl}_I(A)\bigcap B = A\bigcap \text{cl}_I(B) = \emptyset$.

- Let $a \in A$ and $b \in B$ where WLOG $a < b$
  - (since either $a < b$ or $b < a$, and $a \neq b$ since A, B are disjoint)
- Let $K = [a, b]$ and define $A_K := A\bigcap K$ and $B_K := B\bigcap K$.
- Now A_K, B_K is a disconnection of K.
- Let $s = \sup(A_K)$, which exists since $\mathbb{R}$ is complete and has the LUB property
- Claim: $s \in \text{cl}_I(A_K)$. Proof:
  - If $s \in A_K$ there's nothing to show since $A_K \subset \text{cl}_I(A_K)$, so assume $s \in I \setminus A_K$.
  - Now let N_s be an arbitrary neighborhood of s, then using ??? we can find an $\varepsilon > 0$ such that $B_\varepsilon(s) \subset N_s$
  - Since s is a supremum, there exists an $a \in A_K$ such that $s - \varepsilon < a$.
  - But then $a \in B_\varepsilon(s)$ and $a \in N_s$ with $a \neq s$.
  - Since N_s was arbitrary, every N_s contains a point of A_K not equal to s, so s is a limit point by definition.
- Since $s \in \text{cl}_I(A_K)$ and $\text{cl}_I(A_K)\bigcap B_K = \emptyset$, we have $s \notin B_K$.
- Then the subinterval $(x, b]\bigcap A_K = \emptyset$ for every $x > c$ since $c := \sup A_K$.
- But since $A_K\coprod B_K = K$, we must have $(x, b] \subset B_K$, and thus $s \in \text{cl}_I(B_K)$.
- Since A_K, B_K were assumed disconnecting, $s \notin A_K$
- But then $s \in K$ but $s \notin A_K\coprod B_K = K$, a contradiction.

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2.1.2 Suggested by Ernest

1. Problem Statement

- Let X be compact, $A \subset X$ closed, and $\{U_\alpha\} \rightrightarrows A$ be an open cover.
- By definition of the subspace topology, each $U_\alpha = V_\alpha\bigcap A$ for some open $V_\alpha \subset X$, and $A \subset \bigcup_\alpha V_\alpha$.
- Since A is closed in X, $X \setminus A$ is open.
- Then $\{V_\alpha\}\bigcup\{X \setminus A\} \rightrightarrows X$ is an open cover, since every point is either in A or $X \setminus A$.
- By compactness of X, there is a finite subcover $\left\{U_j \;\middle|\; j \leq N\right\}\bigcup\{X \setminus A\}$
- Then $\left(\{U_j\}\bigcup\{X \setminus A\}\right)\bigcap A := \{V_j\}$ is a finite cover of A.

2. Problem Statement

- Let $f : X \longrightarrow Y$ be continuous with X compact, and $\{U_\alpha\} \rightrightarrows f(X)$ be an open cover.
- Then $\left\{f^{-1}(U_\alpha)\right\} \rightrightarrows X$ is an open cover of X, since $x \in X \implies f(x) \in f(X) \implies f(x) \in U_\alpha$ for some α, so $x \in f^{-1}(U_\alpha)$ by definition.
- By compactness of X there is a finite subcover $\left\{f^{-1}(U_j) \;\middle|\; j \leq N\right\} \rightrightarrows X$.
- Then the finite subcover $\left\{U_j \;\middle|\; j \leq N\right\} \rightrightarrows f(X)$, since if $y \in f(X)$, $y \in U_\alpha$ for some α and thus $f^{-1}(y) \in f^{-1}(U_j)$ for some j since $\{U_j\}$ is a cover of X.

3. Problem Statement

Note, alternative definition of "open":

- Let A be a compact subset of X a Hausdorff space, we will show $X \setminus A$ is open
- Fix $x \in X \setminus A$.
- Since X is Hausdorff, for every $y \in A$ we can find $U_y \ni y$ and $V_x(y) \ni x$ depending on y such that $U_x(y) \bigcap U_y = \emptyset$.
- Then $\left\{U_y \;\middle|\; y \in A\right\} \rightrightarrows A$, and by compactness of A there is a finite subcover corresponding to a finite collection $\{y_1, \cdots, y_n\}$.
- Set $U = \bigcup U_{y_i}$ and $V = \bigcap V_x(y_i)$;
  - Note $A \subset U$ and $x \in V$
  - Note $U \bigcap V = \emptyset$.
- Done: for every $x \in X \setminus A$, we have found an open set $V \ni x$ such that $V \bigcap A = \emptyset$, so x is an interior point and a set is open iff every point is an interior point.

4. Problem Statement

- Since $f : X \longrightarrow Y$ is a bijection, set $g \coloneqq f^{-1} : Y \longrightarrow X$ (to distinguish images from preimages), we will show g is continuous by showing that $U \in X$ closed implies $g^{-1}(U) \in X$ closed.
- Let $U \in X$ be closed; since X is closed, U is compact (since closed subsets of compact spaces are compact)
- Since f is continuous, $f(U)$ is compact (since the continuous image of a compact set is compact)
- Since Y is Hausdorff and $f(U)$ is compact, $f(U)$ is closed (since compact subsets of Hausdorff spaces are closed)
- Since $f \coloneqq g^{-1}$, $f(U) = g^{-1}(U)$ is thus closed.