# Topology Qualifying Exam Review

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# 1 Preface

References:

- Munkres [2]
- Hatcher [1]

Some fun resources:

- The Line with Two Origins

## 1.1 Notation

- $A \times B, \prod X_j$ are direct products.
- $A \oplus B, \bigoplus_j X_j$ are direct sums, the subset of $A \times B$ where only finitely many terms are nonzero.
  - $\mathbb{Z}^n$ denotes the direct sum of $n$ copies of the group $\mathbb{Z}$.
  - Note that $A \oplus B \hookrightarrow A \times B$.
- $A * B, *_j X_J$ are free products, $F_n \coloneqq \mathbb{Z}^{*n}$ is the free group on $n$ generators.
  - Note that the abelianization yields $(*_j X_j) = \bigoplus_j X_j$.

| Notation | Definition |
|---|---|
| $X \times Y, \prod_{j \in J} X_j, X^{\times n}$ | Products |
| $X \oplus Y, \bigoplus_{j \in J} X_j, X^{\oplus n}$ | Direct sums |
| $X * Y, *_{j \in J} X_j, X^{*n}$ | Free products |
| $X \otimes Y, \bigotimes_{j \in J} X_j, X^{\otimes n}$ | Tensor products |
| $\mathbb{Z}^n$ | The free abelian group of rank $n$ |
| $\mathbb{Z}^{*n}$ | The free group on $n$ generators |
| $\pi_0(X)$ | The *set* of path components of $X$ |
| $G = 1$ | The trivial abelian group |
| $G = 0$ | The trivial nonabelian group |

**Remark 1.1.1:** Both the product and direct sum have coordinate-wise operations. For finite index sets $|J| < \infty$, the direct sum and product coincide, but in general there is only an injection $\bigoplus_j X_i \hookrightarrow \prod_j X_j$. In the direct sum $\bigoplus_j X_j$ have only finitely many nonzero entries, while the product allows *infinitely* many nonzero entries. So in general, I always use the product notation.

**Remark 1.1.2:** The free group on $n$ generators is the free product of $n$ free abelian groups, but is not generally abelian! So we use multiplicative notation, and elements

$$x \in \mathbb{Z}^{*n} = \langle a_1, \dots, a_n \rangle$$

are finite words in the noncommuting symbols $a_i^k$ for $k \in \mathbb{Z}$. E.g. an element may look like

$$x = a_1^2 a_2^4 a_1 a_2^{-2}.$$

**Remark 1.1.3:** The free abelian group of rank $n$ is the abelianization of $\mathbb{Z}^{*n}$, and its elements are characterized by

$$x \in \mathbb{Z}^{*n} = \langle a_1, \cdots, a_n \rangle \implies x = \sum_n c_i a_i \text{ for some } c_i \in \mathbb{Z}$$

where the $a_i$ are some generating set of $n$ elements and we used additive notation since the group is abelian. E.g. such an element may look like

$$x = 2a_1 + 4a_2 + a_1 - a_2 = 3a_1 + 3a_2.$$

## 1.2 Conventions

- Spaces are assumed to be connected and path connected, so $\pi_0(X) = H_0(X) = \mathbb{Z}$.
- Graded objects like $\pi_*, H_*, H^*$ are sometimes represented as lists. In this case, all list entries start indexing at 1. Examples:

$$\pi_*(X) = [\pi_1(X), \pi_2(X), \pi_3(X), \cdots]$$
$$H_*(X) = [H_1(X), H_2(X), H_3(X), \cdots].$$

## 1.3 Some Prerequisite Algebra Facts

**Fact 1.3.1**
A group morphism $f : X \to Y$ can not be injective if $Y$ is trivial unless $X$ is also trivial.

**Proposition 1.3.2** ***(Morphisms between groups).***
There are no nontrivial homomorphisms from finite groups into free groups. In particular, any homomorphism $\mathbb{Z}_n \to \mathbb{Z}$ is trivial.

*Proof (?).*
Homomorphisms preserve torsion; the former has $n$-torsion while the latter does not. ■

**Remark 1.3.3** ***(How to use this fact):*** This is especially useful if you have some $f : A \to B$ and you look at the induced homomorphism $f_* : \pi_1(A) \to \pi_1(B)$. If the former is finite and the latter contains a copy of $\mathbb{Z}$, then $f_*$ has to be the trivial map $f_*([\alpha]) = e \in \pi_1(B)$ for every $[\alpha] \in \pi_1(A)$.

# 2 Summary and Topics: Point-Set Topology

- Connectedness
- Compactness
- Hausdorff Spaces
- Path-Connectedness

# 3 Definitions

## 3.1 Point-Set Topology

**Definition 3.1.1** (Bounded)
A set $S$ in a metric space $(X, d)$ is *bounded* iff there exists an $m \in \mathbb{R}$ such that $d(x, y) < m$ for every $x, y \in S$.

**Definition 3.1.2** (Connected)
There does not exist a disconnecting set $X = A \coprod B$ such that $\varnothing \neq A, B \subsetneq$, i.e. $X$ is the union of two proper disjoint nonempty sets. Additional condition for a subspace $Y \subset X$: $\mathrm{cl}_Y(A) \cap V = A \cap \mathrm{cl}_Y(B) = \varnothing$.
Equivalently, $X$ contains no proper nonempty clopen sets.

**Definition 3.1.3** (Connected Components)
Set $x \sim y$ iff there exists a connected set $U \ni x, y$ and take equivalence classes.

**Definition 3.1.4** (Closed Sets)

- A set is closed if and only if its complement is open.

- A set is closed iff it contains all of its limit points.
- A closet set in a subspace: $Y \subset X \implies \mathrm{cl}_Y(A) \coloneqq \mathrm{cl}_X(A) \cap Y$.

**Definition 3.1.5** (Closed Maps)
See Definition 3.1.22.

**Definition 3.1.6** (Compact)
A topological space $(X, \tau)$ is **compact** if every open cover has a *finite* subcover.
That is, if $\left\{U_j \,\middle|\, j \in J\right\} \subset \tau$ is a collection of open sets such that $X = \cup_{j\in J} U_j$, then there exists a *finite* subset $J' \subset J$ such that $X \subseteq \cup_{j\in J'} U_j$.

**Definition 3.1.7** (Continuous Map)
A map $f : X \to Y$ between topological spaces is **continuous** if and only if whenever $U \subseteq Y$ is open, $f^{-1}(U) \subseteq X$ is open.

**Definition 3.1.8** (Cover)
A collection of subsets $\{U_\alpha\}$ of $X$ is said to *cover* $X$ iff $X = \cup_\alpha U_\alpha$. If $A \subseteq X$ is a subspace, then this collection *covers* $A$ iff $A \subseteq \cup_\alpha U_\alpha$.

**Definition 3.1.9** (Dense)
A subset $Q \subset X$ is dense iff $y \in N_y \subset X \implies N_y \cap Q \neq \varnothing$ iff $\overline{Q} = X$.

**Definition 3.1.10** (First Countable)
A space is *first-countable* iff every point admits a countable neighborhood basis.

**Definition 3.1.11** (Hausdorff)
A topological space $X$ is *Hausdorff* iff for every $p \neq q \in X$ there exist disjoint open sets $U \ni p$ and $V \ni q$.

**Definition 3.1.12** (Injection)
A map $\iota$ with a **left** inverse $f$ satisfying $f \circ \iota = \mathrm{id}$

**Definition 3.1.13** (Lebesgue Number)
For $(X, d)$ a compact metric space and $\{U_\alpha\} \rightrightarrows X$, there exist $\delta_L > 0$ such that

$$A \subset X,\ \operatorname{diam}(A) < \delta_L \implies A \subseteq U_\alpha \text{ for some } \alpha.$$

**Definition 3.1.14** (Limit Point)
For $A \subset X$, $x$ is a limit point of $A$ if every punctured neighborhood $P_x$ of $x$ satisfies $P_x \cap A \neq \varnothing$, i.e. every neighborhood of $x$ intersects $A$ in some point other than $x$ itself.
Equivalently, $x$ is a limit point of $A$ iff $x \in \mathrm{cl}_X(A \setminus \{x\})$.

**Definition 3.1.15** (Locally Connected)
A space is *locally connected* at a point $x$ iff $\forall N_x \ni x$, there exists a $U \subset N_x$ containing $x$ that is connected.

**Definition 3.1.16** (Locally Compact)
A space $X$ is *locally compact* iff every $x \in X$ has a neighborhood contained in a compact subset of $X$.

**Definition 3.1.17** (Locally Finite)
A collection of subsets $\mathcal{S}$ of $X$ is *locally finite* iff each point of $M$ has a neighborhood that intersects at most finitely many elements of $\mathcal{S}$.

**Definition 3.1.18** (Locally Path-Connected)
A space is **locally path-connected** if it admits a basis of path-connected open subsets.

**Definition 3.1.19** (Neighborhood)
A **neighborhood** of a point $x$ is *any* open set containing $x$.

**Definition 3.1.20** (Normal)
A space is **normal** if any two disjoint closed subsets can be separated by neighborhoods.

**Definition 3.1.21** (Neighborhood Basis)
If $p \in X$, a **neighborhood basis** at $p$ is a collection $\mathcal{B}_p$ of neighborhoods of $p$ such that if $N_p$ is a neighborhood of $p$, then $N_p \supseteq B$ for at least one $B \in \mathcal{B}_p$.

**Definition 3.1.22** (Open and Closed Maps)
A map $f : X \to Y$ is an **open map** (respectively a **closed map**) if and only if whenever $U \subseteq X$ is open (resp. closed), $f(U)$ is again open (resp. closed)>

**Definition 3.1.23** (Paracompact)
A topological space $X$ is **paracompact** iff every open cover of $X$ admits an open locally finite refinement.

**Definition 3.1.24** (Quotient Map)
A map $q : X \to Y$ is a **quotient map** if and only if

1. $q$ is surjective, and
2. $U \subseteq Y$ is open if and only if $q^{-1}(U)$ is open.

**Definition 3.1.25** (Path Connected)
A space $X$ is **path connected** if and only if for every pair of points $x \neq y$ there exists a continuous map $f : I \to X$ such that $f(0) = x$ and $f(1) = y$.

**Definition 3.1.26** (Path Components)
Set $x \sim y$ iff there exists a path-connected set $U \ni x, y$ and take equivalence classes.

**Definition 3.1.27** (Precompact)
A subset $A \subseteq X$ is **precompact** iff $\mathrm{cl}_X(A)$ is compact.

**Definition 3.1.28** (The product topology)
For $(X, \tau_X)$ and $(Y, \tau_Y)$ topological spaces, defining

$$\tau_{X\times Y} \coloneqq \left\{ U \times V \;\middle|\; U \in \tau_X,\, V \in \tau_Y \right\}$$

yields the **product topology** on $X \times Y$.

**Definition 3.1.29** (Refinement)
A cover $\mathcal{V} \rightrightarrows X$ is a **refinement** of $\mathcal{U} \rightrightarrows X$ iff for each $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $V \subseteq U$.

**Definition 3.1.30** (Regular)
A space $X$ is **regular** if whenever $x \in X$ and $F \not\ni x$ is closed, $F$ and $x$ are separated by neighborhoods.

**Definition 3.1.31** (Retract)
A map $r$ in $A \underset{r}{\overset{\iota}{\rightleftarrows}} X$ satisfying

$$r \circ \iota = \mathrm{id}_A\,.$$

Equivalently $X \twoheadrightarrow_r A$ and $r|_A = \mathrm{id}_A$. If $X$ retracts onto $A$, then $i_*$ is injective.
Alt: Let $X$ be a topological space and $A \subset X$ be a subspace, then a **retraction** of $X$ onto $A$ is a map $r : X \to X$ such that the image of $X$ is $A$ and $r$ restricted to $A$ is the identity map on $A$.

**Definition 3.1.32** (Saturated)
A subset $U \subseteq X$ is **saturated** with respect to a surjective map $p : X \twoheadrightarrow Y$ if and only if whenever $U \cap p^{-1}(y) = V \neq \varnothing$, we have $V \subseteq U$, i.e. $U$ contains every set $p^{-1}(y)$ that it intersects. Equivalently, $U$ is the complete inverse image of a subset of $Y$.

**Definition 3.1.33** (Separable spaces)
A space $X$ is **separable** iff $X$ contains a countable dense subset.

**Definition 3.1.34** (Second Countable)
A space is *second-countable* iff it admits a countable basis.

**Definition 3.1.35** (The subspace topology)
For $(X, \tau)$ a topological space and $U \subseteq X$ an arbitrary subset, the space $(U, \tau_U)$ is a topological space with a **subspace topology** defined by

$$\tau_U \coloneqq \left\{ Y \cap U \;\middle|\; U \in \tau \right\}.$$

**Definition 3.1.36** (Surjection)
A map $\pi$ with a **right** inverse $f$ satisfying

$$\pi \circ f = \mathrm{id}$$

**Definition 3.1.37** ($T_n$ Spaces (Separation Axioms))

- $T_0$: For any 2 points $x_1 \neq x_2$, at least one $x_i$ (say $x_1$) admits a neighborhood not containing $x_2$.
- $T_1$: For any 2 points, *both* admit neighborhoods not containing the other.
- $T_2$: For any 2 points, both admit *disjoint* separating neighborhoods.
- $T_{2.5}$: For any 2 points, both admit *disjoint closed* separating neighborhoods.
- $T_3$: $T_0$ & *regular*. Given any point $x$ and any closed $F \not\ni x$, there are neighborhoods separating $F$ and $x$.
- $T_{3.5}$: $T_0$ & completely regular. Any point $x$ and closed $F \not\ni x$ can be separated by a continuous function.
- $T_4$: $T_1$ & normal. Any two disjoint closed subsets can be separated by neighborhoods.

**Definition 3.1.38** (Topology)
Closed under arbitrary unions and finite intersections.

**Definition 3.1.39** (Topology generated by a basis)
For $X$ an arbitrary set, a collection of subsets $\mathcal{B}$ is a *basis for* $X$ iff $\mathcal{B}$ is closed under intersections, and every intersection of basis elements contains another basis element. The set of unions of elements in $B$ is a topology and is denoted *the topology generated by* $\mathcal{B}$.

**Definition 3.1.40** (Topological Embedding)
A continuous map $f : X \to Y$ for which $X \cong f(X)$ are homeomorphic is called a **topological embedding**.

**Definition 3.1.41** (Uniform Continuity)
For $f : (X, d_x) \to (Y, d_Y)$ metric spaces,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \quad d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

## 3.2 Algebraic Topology

**Definition 3.2.1** (Acyclic)

Definitions

**Definition 3.2.2** (Alexander duality)

Definitions

**Definition 3.2.3** (Basis)
For an $R$-module $M$, a basis $B$ is a linearly independent generating set.

**Definition 3.2.4** (Boundary)

Definitions

**Definition 3.2.5** (Boundary of a manifold)
Points $x \in M^n$ defined by

$$\partial M = \{x \in M : H_n(M, M - \{x\}; \mathbb{Z}) = 0\}$$

**Definition 3.2.6** (Cap Product)
Denoting $\Delta^p \xrightarrow{\sigma} X \in C_p(X; G)$, a map that sends pairs ($p$-chains, $q$-cochains) to $(p-q)$-chains $\Delta^{p-q} \to X$ by

$$\begin{aligned} H_p(X; R) \times H^q(X; R) &\xrightarrow{\frown} H_{p-q}(X; R) \\ \sigma \frown \psi &= \psi(F_0^q(\sigma)) F_q^p(\sigma) \end{aligned}$$

where $F_i^j$ is the face operator, which acts on a simplicial map $\sigma$ by restriction to the face spanned by $[v_i \dots v_j]$, i.e. $F_i^j(\sigma) = \sigma|_{[v_i \dots v_j]}$.

**Definition 3.2.7** (Cellular Homology)

Definitions

**Definition 3.2.8** (Cellular Map)
A map $X \xrightarrow{f} Y$ is said to be cellular if $f(X^{(n)}) \subseteq Y^{(n)}$ where $X^{(n)}$ denotes the $n$- skeleton.

**Definition 3.2.9** (Chain)
An element $c \in C_p(X; R)$ can be represented as the singular $p$ simplex $\Delta^p \to X$.

**Definition 3.2.10** (Chain Homotopy)
Given two maps between chain complexes $(C_*, \partial_C) \xrightarrow{f,\ g} (D_*, \partial_D)$, a chain homotopy is a family $h_i : C_i \to B_{i+1}$ satisfying

$$f_i - g_i = \partial_{B,i-1} \circ h_n + h_{i+1} \circ \partial_{A,i}$$

.

**Definition 3.2.11** (Chain Map)
A map between chain complexes $(C_*, \partial_C) \xrightarrow{f} (D_*, \partial_D)$ is a chain map iff each component

$C_i \xrightarrow{f_i} D_i$ satisfies

$$f_{i-1} \circ \partial_{C,i} = \partial_{D,i} \circ f_i$$

(i.e this forms a commuting ladder)

**Definition 3.2.12** (Closed manifold)
A manifold that is compact, with or without boundary.

**Definition 3.2.13** (Coboundary)

Definitions

**Definition 3.2.14** (Cochain)
An cochain $c \in C^p(X;R)$ is a map $c \in \hom(C_p(X;R), R)$ on chains.

**Definition 3.2.15** (Cocycle)

Definitions

**Definition 3.2.16** (Constant Map)
A *constant map* $f : X \to Y$ iff $f(X) = y_0$ for some $y_0 \in Y$, i.e. for every $x \in X$ the output value $f(x) = y_0$ is the same.

**Definition 3.2.17** (Colimit)
For a directed system $(X_i, f_{ij}$, the **colimit** is an object $X$ with a sequence of projections $\pi_i : X \to X_i$ such that for any $Y$ mapping into the system, the following diagram commutes:

$$\begin{tikzcd}
 & & Y \arrow[d, dashed, "\exists!"] \arrow[ddl, "\psi_j"'] \arrow[ddr, "\psi_i"] & & \\
 & & X \arrow[dl, "\pi_j"] \arrow[dr, "\pi_i"'] & & \\
\cdots \arrow[r] & X_j \arrow[rr, "f_{ij}"] & & X_i \arrow[r] & \cdots
\end{tikzcd}$$

**Example 3.2.18** ***(of colimits):***

- Products
- Pullbacks
- Inverse / projective limits
- The $p$-adic integers $\mathbb{Z}_p$.

**Definition 3.2.19** (Cone)
For a space $X$, defined as

$$CX = \frac{X \times I}{X \times \{0\}}.$$

Example: The cone on the circle $CS^1$
Note that the cone embeds $X$ in a contractible space $CX$.

**Definition 3.2.20** (Contractible)
A space $X$ is **contractible** if $\mathrm{id}_X$ is nullhomotopic. i.e. the identity is homotopic to a constant map $c(x) = x_0$.
Equivalently, $X$ is contractible if $X \simeq \{x_0\}$ is homotopy equivalent to a point. This means that there exists a mutually inverse pair of maps $f : X \to \{x_0\}$ and $g : \{x_0\} \to X$ such that $f \circ g \simeq \mathrm{id}_{\{x_0\}}$ and $g \circ f \simeq \mathrm{id}_X$.[a]

[a] This is a useful property because it supplies you with a homotopy.

**Definition 3.2.21** (Coproduct)

Definitions

**Definition 3.2.22** (Covering Space)
A **covering space** of $X$ is the data $p : \tilde{X} \to X$ such that

1. Each $x \in X$ admits a neighborhood $U$ such that $p^{-1}(U)$ is a union of disjoint open sets in $\tilde{V}_i \subseteq X$ (the **sheets** of $\tilde{X}$ over $U$),
2. $p|_{V_i} : V_i \to U$ is a homeomorphism for each sheet.

An **isomorphism** of covering spaces $\tilde{X}_1 \cong \tilde{X}_2$ is a commutative diagram

$$\begin{tikzcd}
\tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\
& X &
\end{tikzcd}$$

*Link to diagram*

**Definition 3.2.23** (Cup Product)
A map taking pairs ($p$-cocycles, $q$-cocycles) to $(p+q)$-cocyles by

$$H^p(X;R) \times H^q(X;R) \xrightarrow{\smile} H^{p+q}(X;R)$$
$$(a \cup b)(\sigma) = a(\sigma \circ I_0^p)\; b(\sigma \circ I_p^{p+q})$$

where $\Delta^{p+q} \xrightarrow{\sigma} X$ is a singular $p+q$ simplex and

$$I_i^j : [i, \cdots, j] \hookrightarrow \Delta^{p+q}.$$

is an embedding of the $(j-i)$-simplex into a $(p+q)$-simplex.

**Example 3.2.24** ***(Applications of the cup product):*** On a manifold, the cup product is Poincaré dual to the intersection of submanifolds. Also used to show $T^2 \not\simeq S^2 \vee S^1 \vee S^1$.

**Definition 3.2.25** (CW Complex)

Definitions

**Definition 3.2.26** (CW Cell)
An $n$-cell of $X$, say $e^n$, is the image of a map $\Phi : B^n \to X$. That is, $e^n = \Phi(B^n)$. Attaching an $n$-cell to $X$ is equivalent to forming the space $B^n \coprod_f X$ where $f : \partial B^n \to X$.

- A 0-cell is a point.
- A 1-cell is an interval $[-1,1] = B^1 \subset \mathbb{R}^1$. Attaching requires a map from $S^0 = \{-1,+1\} \to X$
- A 2-cell is a solid disk $B^2 \subset \mathbb{R}^2$ in the plane. Attaching requires a map $S^1 \to X$.
- A 3-cell is a solid ball $B^3 \subset \mathbb{R}^3$. Attaching requires a map from the sphere $S^2 \to X$.

**Definition 3.2.27** (Cycle)

Definitions

**Definition 3.2.28** (Deck transformation)
For a covering space $\tilde{X} \xrightarrow{p} X$, self-isomorphisms $f : \tilde{X} \to \tilde{X}$ of covering spaces are referred to as **deck transformations**.

**Definition 3.2.29** (Deformation)

Definitions

**Definition 3.2.30** (Deformation Retract)
A map $r$ in $A \underset{r}{\overset{\iota}{\rightleftarrows}} X$ that is a retraction (so $r \circ \iota = \mathrm{id}_A$) *that also satisfies* $\iota \circ r \simeq \mathrm{id}_X$.

*Note that this is equality in one direction, but only homotopy equivalence in the other.*

Equivalently, a map $F : I \times X \to X$ such that

$$F_0(x) = \mathrm{id}_X \; F_t(x)\Big|_A \qquad = \mathrm{id}_A \; F_1(X) = A.$$

Alt:
A **deformation retract** is a homotopy $H : X \times I \to X$ from the identity on $X$ to the identity

on $A$ that fixes $A$ at all times:

$$\begin{aligned} H : X \times I &\to X \\ H(x,0) &= \mathrm{id}_X \\ H(x,1) &= \mathrm{id}_A \\ x \in A \implies H(x,t) &\in A \quad \forall t \end{aligned}$$

Equivalently, this requires that $H|_A = \mathrm{id}_A$

**Remark 3.2.31:** A deformation retract between a space and a subspace is a homotopy equivalence, and further $X \simeq Y$ iff there is a $Z$ such that both $X$ and $Y$ are deformation retracts of $Z$. Moreover, if $A$ and $B$ both have deformation retracts onto a common space $X$, then $A \simeq B$.

**Definition 3.2.32** (Degree of a Map of Spheres)
Given any $f : S^n \to S^n$, there are induced maps on homotopy and homology groups. Taking $f^* : H^n(S^n) \to H^n(S^n)$ and identifying $H^n(S^n) \cong \mathbb{Z}$, we have $f^* : \mathbb{Z} \to \mathbb{Z}$. But homomorphisms of free groups are entirely determined by their action on generators. So if $f^*(1) = n$, define $n$ to be the **degree** of $f$.

**Definition 3.2.33** (Derived Functor)
For a functor $T$ and an $R$-module $A$, a *left derived functor* $(L_{nT})$ is defined as $h_n(TP_A)$, where $P_A$ is a projective resolution of $A$.

**Definition 3.2.34** (Dimension of a manifold)
For $x \in M$, the only nonvanishing homology group $H_i(M, M - \{x\}; \mathbb{Z})$

**Definition 3.2.35** (Direct Limit)

Definitions

**Definition 3.2.36** (Direct Product)

Definitions

**Definition 3.2.37** (Direct Sum)

Definitions

**Definition 3.2.38** (Eilenberg-MacLane Space)

Definitions

**Definition 3.2.39** (Euler Characteristic)

Definitions

**Definition 3.2.40** (Exact Functor)
A functor $T$ is *right exact* if a short exact sequence

$$0 \to A \to B \to C \to 0$$

yields an exact sequence

$$\dots TA \to TB \to TC \to 0$$

and is *left exact* if it yields

$$0 \to TA \to TB \to TC \to \dots$$

Thus a functor is exact iff it is both left and right exact, yielding

$$0 \to TA \to TB \to TC \to 0.$$

**Example 3.2.41** ***(of an exact functor):*** $\cdot \otimes_R \cdot$ is a right exact bifunctor.

**Definition 3.2.42** (Exact Sequence)

Definitions

**Definition 3.2.43** (Excision)

Definitions

**Definition 3.2.44** (Ext Group)

Definitions

**Definition 3.2.45** (Flat)
An $R$-module is flat if $A \otimes_R \cdot$ is an exact functor.

**Definition 3.2.46** (Free and Properly Discontinuous)
An action $G \curvearrowright X$ is **properly discontinuous** if each $x \in X$ has a neighborhood $U$ such that all of the images $g(U)$ for $g \in G$ are disjoint, i.e. $g_1(U) \cap g_2(U) \neq \varnothing \implies g_1 = g_2$. The action is

**free** if there are no fixed points.
Sometimes a slightly weaker condition is used: every point $x \in X$ has a neighborhood $U$ such that $U \cap G(U) \neq \varnothing$ for only finitely many $G$.

Definitions

**Definition 3.2.47** (Free module)
A -module $M$ with a basis $S = \{s_i\}$ of generating elements. Every such module is the image of a unique map $\mathcal{F}(S) = R^S \twoheadrightarrow M$, and if $M = \left\langle S \mid \mathcal{R} \right\rangle$ for some set of relations $\mathcal{R}$, then $M \cong R^S / \mathcal{R}$.

**Definition 3.2.48** (Free Product)

Definitions

**Definition 3.2.49** (Free product with amalgamation)

Definitions

**Definition 3.2.50** (Fundamental Class)
For a connected, closed, orientable manifold, $[M]$ is a generator of $H_n(M; \mathbb{Z}) = \mathbb{Z}$.

**Definition 3.2.51** (Fundamental Group)

Definitions

**Definition 3.2.52** (Generating Set)
$S = \{s_i\}$ is a generating set for an $R$- module $M$ iff

$$x \in M \implies x = \sum r_i s_i$$

for some coefficients $r_i \in R$ (where this sum may be infinite).

**Definition 3.2.53** (Gluing Along a Map)

Definitions

**Definition 3.2.54** (Group Ring)

Definitions

**Definition 3.2.55** (Homologous)

Definitions

**Definition 3.2.56** (Homotopic)

Definitions

**Definition 3.2.57** (Homotopy)
Let $X, Y$ be topological spaces and $f, g : X \to Y$ continuous maps. Then a **homotopy** from $f$ to $g$ is a continuous function
$F : X \times I \to Y$
such that
$F(x, 0) = f(x)$ and $F(x, 1) = g(x)$
for all $x \in X$. If such a homotopy exists, we write $f \simeq g$. This is an equivalence relation on $\mathrm{Hom}(X, Y)$, and the set of such classes is denoted $[X, Y] \coloneqq \mathrm{hom}(X, Y)/\simeq$.

**Definition 3.2.58** (Homotopy Class)

Definitions

**Definition 3.2.59** (Homotopy Equivalence)
Let $f : X \to Y$ be a continuous map, then $f$ is said to be a *homotopy equivalence* if there exists a continuous map $g : X \to Y$ such that
$f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.
Such a map $g$ is called a homotopy inverse of $f$, the pair of maps is a homotopy equivalence. If such an $f$ exists, we write $X \simeq Y$ and say $X$ and $Y$ have the same homotopy type, or that they are homotopy equivalent.

*Note that homotopy equivalence is strictly weaker than homeomorphic equivalence, i.e., $X \cong Y$ implies $X \simeq Y$ but not necessarily the converse.*

**Definition 3.2.60** (Homotopy Extension Property)

Definitions

**Definition 3.2.61** (Homotopy Groups)

Definitions

**Definition 3.2.62** (Homotopy Lifting Property)

Definitions

**Definition 3.2.63** (Intersection Pairing)
For a manifold $M$, a map on homology defined by

$$\begin{aligned} H_{\widehat{i}}M \otimes H_{\widehat{j}}M &\to H_{\widehat{i+j}}X \\ \alpha \otimes \beta &\mapsto \langle \alpha, \beta \rangle \end{aligned}$$

obtained by conjugating the cup product with Poincaré Duality, i.e.

$$\langle \alpha, \beta \rangle = [M] \frown ([\alpha]^\vee \smile [\beta]^\vee).$$

Then, if $[A], [B]$ are transversely intersecting submanifolds representing $\alpha, \beta$, then

$$\langle \alpha, \beta \rangle = [A \cap B]$$

. If $\widehat{i} = j$ then $\langle \alpha, \beta \rangle \in H_0 M = \mathbb{Z}$ is the signed number of intersection points.
Alt: The pairing obtained from dualizing Poincare Duality to obtain

$$\mathrm{F}(H_i M) \otimes \mathrm{F}(H_{n-i} M) \to \mathbb{Z}$$

Computed as an oriented intersection number between two homology classes (perturbed to be transverse).

**Definition 3.2.64** (Inverse Limit)

Definitions

**Definition 3.2.65** (Intersection Form)
The nondegenerate bilinear form cohomology induced by the Kronecker Pairing:

$$I : H^k(M_n) \times H^{n-k}(M^n) \to \mathbb{Z}$$

where $n = 2k$.

- When $k$ is odd, $I$ is skew-symmetric and thus a *symplectic form.*
- When $k$ is even (and thus $n \equiv 0 \pmod 4$) this is a symmetric form.
- Satisfies $I(x, y) = (-1)^{k(n-k)} I(y, x)$

**Definition 3.2.66** (Kronecker Pairing)
A map pairing a chain with a cochain, given by

$$\begin{aligned} H^n(X; R) \times H_n(X; R) &\to R \\ ([\psi, \alpha]) &\mapsto \psi(\alpha) \end{aligned}$$

which is a nondegenerate bilinear form.

**Definition 3.2.67** (Kronecker Product)

Definitions

**Definition 3.2.68** (Lefschetz duality)

Definitions

**Definition 3.2.69** (Lefschetz Number)

Definitions

**Definition 3.2.70** (Lens Space)

Definitions

**Definition 3.2.71** (Local Degree)
At a point $x \in V \subset M$, a generator of $H_n(V, V - \{x\})$. The degree of a map $S^n \to S^n$ is the sum of its local degrees.

**Definition 3.2.72** (Local Orientation)

Definitions

**Definition 3.2.73** (Limit)

Definitions

**Definition 3.2.74** (Linear Independence)
A generating $S$ for a module $M$ is linearly independent if $\sum r_i s_i = 0_M \implies \forall i,\ r_i = 0$ where $s_i \in S, r_i \in R$.

**Definition 3.2.75** (Local homology)
$H_n(X, X - A; \mathbb{Z})$ is the local homology at $A$, also denoted $H_n(X \mid A)$

**Definition 3.2.76** (Local orientation of a manifold)
At a point $x \in M^n$, a choice of a generator $\mu_x$ of $H_n(M, M - \{x\}) = \mathbb{Z}$.

**Definition 3.2.77** (Long exact sequence)

Definitions

**Definition 3.2.78** (Loop Space)

Definitions

**Definition 3.2.79** (Manifold)
An $n$-manifold is a Hausdorff space in which each neighborhood has an open neighborhood homeomorphic to $\mathbb{R}^n$.

**Definition 3.2.80** (Manifold with boundary)
A manifold in which open neighborhoods may be isomorphic to either $\mathbb{R}^n$ or a half-space $\left\{\mathbf{x} \in \mathbb{R}^n \,\middle|\, x_i > 0\right\}$.

**Definition 3.2.81** (Mapping Cone)

Definitions

**Definition 3.2.82** (Mapping Cylinder)

Definitions

**Definition 3.2.83** (Mapping Path Space)

Definitions

**Definition 3.2.84** (Mayer-Vietoris Sequence)

Definitions

**Definition 3.2.85** (Monodromy)

Definitions

**Definition 3.2.86** (Moore Space)

Definitions

**Definition 3.2.87** (N-cell)

Definitions

**Definition 3.2.88** (N-connected)

Definitions

**Definition 3.2.89** (Normal covering space (a.k.a. 'regular'))
A covering space is **normal** if and only if for every $x \in X$ and every pair of lifts $\tilde{x}_1, \tilde{x}_2$, there is a deck transformation $f$ such that $f(\tilde{x}_1) = \tilde{x}_2$.

**Definition 3.2.90** (Nullhomotopic)
A map $X \xrightarrow{f} Y$ is *nullhomotopic* if it is homotopic to a constant map $X \xrightarrow{g} \{y_0\}$; that is, there exists a homotopy

$$\begin{aligned} &F : X \times I \to Y \\ &F\big|_{X \times \{0\}} = f \quad F(x,0) = f(x) \\ &F\big|_{X \times \{1\}} = g \quad F(x,1) = g(x) = y_0 \\ &. \end{aligned}$$

Alt:
If $f$ is homotopic to a constant map, say $f : x \mapsto y_0$ for some fixed $y_0 \in Y$, then $f$ is said to be *nullhomotopic*. In other words, if $f : X \to Y$ is nullhomotopic, then there exists a homotopy $H : X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = y_0$.
Note that constant maps (or anything homotopic) induce zero homomorphisms.

**Definition 3.2.91** (Orbit space)
For a group action $G \curvearrowright X$, the **orbit space** $X/G$ is defined as $X/\sim$ where $\forall x, y \in X, x \sim y \iff \exists g \in G \;\middle|\; g.x = y$.

**Definition 3.2.92** (Orientable manifold)
A manifold for which an orientation exists, see "Orientation of a Manifold".

**Definition 3.2.93** (Orientation cover)
For any manifold $M$, a two sheeted orientable covering space $\tilde{M}_o$. $M$ is orientable iff $\tilde{M}$ is disconnected. Constructed as

$$\tilde{M} = \coprod_{x \in M} \left\{ \mu_x \;\middle|\; \mu_x \text{ is a local orientation} \right\}.$$

**Definition 3.2.94** (Orientation of a manifold)
A family of $\{\mu_x\}_{x \in M}$ with local consistency: if $x, y \in U$ then $\mu_x, \mu_y$ are related via a propagation. Formally, a function

$$\begin{aligned} M^n \to \coprod_{x \in M} H\left(X \;\middle|\; \{x\}\right) \\ x \mapsto \mu_x \end{aligned}$$

such that $\forall x \exists N_x$ in which $\forall y \in N_x$, the preimage of each $\mu_y$ under the map $H_n(M \mid N_x) \twoheadrightarrow H_n(M \mid y)$ is a single generator $\mu_{N_x}$.
TFAE:

- $M$ is orientable.

- The map $W : (M, x) \to \mathbb{Z}_2$ is trivial.
- $\tilde{M}_o = M \coprod \mathbb{Z}_2$ (two sheets).
- $\tilde{M}_o$ is disconnected
- The projection $\tilde{M}_o \to M$ admits a section.

**Definition 3.2.95** (Oriented manifold)

Definitions

**Definition 3.2.96** (Path)

Definitions

**Definition 3.2.97** (Path Lifting Property)

Definitions

**Definition 3.2.98** (Perfect Pairing)
A pairing alone is an $R$-bilinear module map, or equivalently a map out of a tensor product since $p : M \otimes_R N \to L$ can be partially applied to yield $\varphi : M \to L^N = \hom_R(N, L)$. A pairing is **perfect** when $\varphi$ is an isomorphism.

**Definition 3.2.99** (Poincaré Duality)
For a closed, orientable $n$-manifold, following map $[M] \frown \cdot$ is an isomorphism:

$$D : H^k(M; R) \to H_{n-k}(M; R)$$
$$D(\alpha) = [M] \frown \alpha$$

**Definition 3.2.100** (Projective Resolution)

Definitions

**Definition 3.2.101** (Properly Discontinuous)

Definitions

**Definition 3.2.102** (Pullback)

Definitions

**Definition 3.2.103** (Pushout)

Definitions

**Definition 3.2.104** (Quasi-isomorphism)

Definitions

**Definition 3.2.105** (R-orientability)

Definitions

**Definition 3.2.106** (Relative boundaries)

Definitions

**Definition 3.2.107** (Relative cycles)

Definitions

**Definition 3.2.108** (Relative homotopy groups)

Definitions

**Definition 3.2.109** (Semilocally Simply Connected)
A space $X$ is **semilocally simply connected** if every $x \in X$ has a neighborhood $U$ such that $U \hookrightarrow X$ induces the trivial map $\pi_1(U; x) \to \pi_1(X, x)$.

**Definition 3.2.110** (Short exact sequence)

Definitions

**Definition 3.2.111** (Simplicial Complex)
Given a simplex $\sigma = [v_1 \cdots v_n]$, define the **face map**

$$\begin{aligned} \partial_i : \Delta^n &\to \Delta^{n-1} \\ \sigma &\mapsto [v_1 \cdots \widehat{v_i} \cdots v_n] \end{aligned}$$

A **simplicial complex** is a set $K$ satisfying

1. $\sigma \in K \implies \partial_i \sigma \in K$.

2. $\sigma, \tau \in K \implies \sigma \cap \tau = \varnothing,\ \partial_i \sigma,\ \text{or } \partial_i \tau$.

This amounts to saying that any collection of $(n-1)$-simplices uniquely determines an $n$-simplex (or its lack thereof), or that that map $\Delta^k \to X$ is a continuous injection from the standard simplex in $\mathbb{R}^n$.

3. $|K \cap B_\varepsilon(\sigma)| < \infty$ for every $\sigma \in K$, identifying $\sigma \subseteq \mathbb{R}^n$.

**Definition 3.2.112** (Simplicial Map)
For a map

$$K \xrightarrow{f} L$$

between simplicial complexes, $f$ is a simplicial map if for any set of vertices $\{v_i\}$ spanning a simplex in $K$, the set $\{f(v_i)\}$ are the vertices of a simplex in $L$.

**Definition 3.2.113** (Simply Connected)
A space $X$ is **simply connected** if and only if $X$ is path-connected and every loop $\gamma : S^1 \to X$ can be contracted to a point.
Equivalently, there exists a lift $\widehat{\gamma} : D^2 \to X$ such that $\widehat{\gamma}|_{\partial D^2} = \gamma$.
Equivalently, for any two paths $p_1, p_2 : I \to X$ such that $p_1(0) = p_2(0)$ and $p_1(1) = p_2(1)$, there exists a homotopy $F : I^2 \to X$ such that $F|_0 = p_1, \ F|_0 = p_2$.
Equivalently, $\pi_1 X = 1$ is trivial.

**Definition 3.2.114** (Singular Chain)

$$x \in C_n(x) \implies X = \sum_i n_i \sigma_i = \sum_i n_i (\Delta^n \xrightarrow{\sigma_i} X).$$

**Definition 3.2.115** (Singular Cochain)

$$x \in C^n(x) \implies X = \sum_i n_i \psi_i = \sum_i n_i (\sigma_i \xrightarrow{\psi_i} X).$$

**Definition 3.2.116** (Singular Homology)

Definitions

**Definition 3.2.117** (Smash Product)

Definitions

**Definition 3.2.118** (Suspension)
Compact represented as $\Sigma X = CX \coprod_{\mathrm{id}_X} CX$, two cones on $X$ glued along $X$. Explicitly given by

$$\Sigma X = \frac{X \times I}{(X \times \{0\}) \cup (X \times \{1\}) \cup (\{x_0\} \times I)}.$$

**Definition 3.2.119** (Tor Group)
For an $R$-module

$$\operatorname{Tor}_R^n(\,\cdot\,, B) = L_n(\,\cdot\, \otimes_R B),$$

where $L_n$ denotes the $n$th left derived functor.

**Definition 3.2.120** (Universal Cover)

Definitions

**Definition 3.2.121** (Weak Homotopy Equivalence)

Definitions

**Definition 3.2.122** (Weak Topology)

Definitions

**Definition 3.2.123** (Wedge Product)

Definitions

# 4 Examples

## 4.1 Point-Set

### 4.1.1 Common Spaces and Operations

**Example 4.1.1** ***(Nice spaces):*** The following are some standard "nice" spaces:

$$S^n, \mathbb{D}^n, T^n, \mathbb{RP}^n, \mathbb{CP}^n, \mathbb{M}, \mathbb{K}, \Sigma_g, \mathbb{RP}^\infty, \mathbb{CP}^\infty.$$

**Example 4.1.2** ***(A bank of counterexamples):*** The following are useful spaces to keep in mind to furnish counterexamples:

- Finite discrete sets with the discrete topology.
- Subspaces of $\mathbb{R}$: $(a, b), (a, b], (a, \infty)$, etc.

- Sets given by real sequences, such as $\{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{Z}^{\geq 1} \right\}$

- $\mathbb{Q}$
- The topologist's sine curve
- One-point compactifications
- $\mathbb{R}^{\omega}$ for $\omega$ the least uncountable ordinal (?)
- The Hawaiian earring
- The Cantor set

Examples of some more exotic spaces that show up less frequently:

- $\mathbb{HP}^n$, quaternionic projective space
- The Dunce Cap
- The Alexander Horned sphere

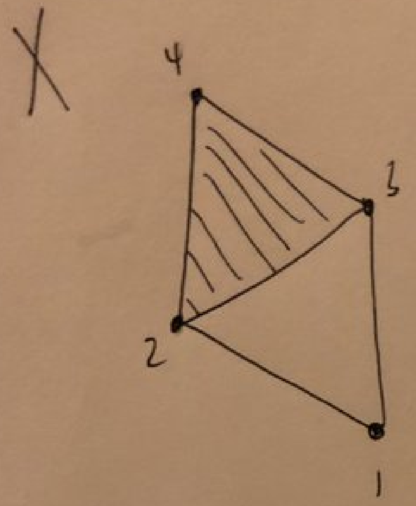
Break these into separate examples and explain properties.

**Example 4.1.3** ***(Non-Hausdorff spaces):*** The following spaces are non-Hausdorff:

- The cofinite topology on any infinite set.
- $\mathbb{R}/\mathbb{Q}$
- The line with two origins.
- Any variety $V(J) \subseteq \mathbb{A}^n_{/k}$ for $k$ a field and $J \trianglelefteq k[x_1, \cdots, x_n]$.

**Example 4.1.4** ***(Constructed spaces):*** The following are some examples of ways to construct specific spaces for examples or counterexamples:

- Knot complements in $S^3$
- Covering spaces (hyperbolic geometry)
- Lens spaces
- Matrix groups
- Prism spaces
- Pair of pants
- Seifert surfaces
- Surgery
- Simplicial Complexes
  - Nice minimal example:




$$
\begin{aligned}
X = \{ & \{1\}, \{2\}, \{3\}, \{4\}, && \text{0-simplices} \\
& \{1,2\}, \{1,3\}, \{2,3\}, \\
& \{2,4\}, \{3,4\}, && \text{1-simplices} \\
& \{2,3,4\} \} && \text{2-simplices}
\end{aligned}
$$

Operations

- Cartesian product $A \times B$

- Wedge product $A \vee B$
- Connect Sum $A \# B$
- Quotienting $A/B$
- Puncturing $A \setminus \{a_i\}$
- Smash product
- Join
- Cones
- Suspension
- Loop space
- Identifying a finite number of points

### 4.1.2 Alternative Topologies

**Example 4.1.5** ***(Nonstandard topologies):*** The following are some nice examples of topologies to put on familiar spaces to produce counterexamples:

- Discrete
- Cofinite
- Discrete and Indiscrete
- Uniform

**Example 4.1.6** ***(The cofinite topology):*** The cofinite topology on any space $X$ is always

- Non-Hausdorff
- Compact

**Proposition 4.1.7** ***(Topology is discrete if and only if points are open).***
A topology $(X, \tau)$ is the discrete topology iff points $x \in X$ are open.

*Proof (?).*
If $\{x\}_i$ is open for each $x_i \in X$, then

- Any set $U$ can be written as $U = \cup_{i \in I} x_I$ (for some $I$ depending on $U$), and
- Unions of open sets are open.

Thus $U$ is open.

■

**Example 4.1.8** ***(The discrete topology):*** Some facts about the discrete topology:

- Definition: every subset is open.
- Always Hausdorff
- Compact iff finite
- Totally disconnected

- If $X$ is discrete, every map $f : X \to Y$ for any $Y$ is continuous (obvious!)

**Example 4.1.9** ***(The indiscrete topology):*** Some facts about the indiscrete topology:

- Definition: the only open sets are $\varnothing, X$
- Never Hausdorff
- If $Y$ is indiscrete, every map $f : X \to Y$ is continuous (obvious!)
- Always compact

# 5 Theorems

**Proposition 5.0.1** ***(The continuous image of a...).***
The following properties are "pushed forward" through continuous maps, in the sense that if property $P$ holds for $X$ and $f : X \to Y$, then $f(X)$ also satisfies $P$:

- Compactness
- Separability
- If $f$ is surjective:
  - Connectedness
  - Density

The following are **not preserved**:

- Openness
- Closedness

*See more here.*

## 5.1 Metric Spaces and Analysis

**Theorem 5.1.1** ***(Cantor's Intersection Theorem).***
A bounded collection of nested closed sets $C_1 \supset C_2 \supset \cdots$ in a metric space $X$ is nonempty $\iff$ $X$ is complete.

**Theorem 5.1.2** ***(Cantor's Nested Intervals Theorem).***
If $\left\{ [a_n, b_n] \;\middle|\; n \in \mathbb{Z}^{\geq 0} \right\}$ is a nested sequence of **closed and bounded** intervals, then their intersection is nonempty. If $\operatorname{diam}([a_n, b_n]) \overset{n\to\infty}{\to} 0$, then the intersection contains exactly one point.

**Proposition 5.1.3** ***(Continuous on compact $\implies$ uniformly continuous).***
A continuous function on a compact set is uniformly continuous.

*Proof (?).*
Take $\left\{ B_{\frac{\varepsilon}{2}}(y) \;\middle|\; y \in Y \right\} \rightrightarrows Y$, pull back to an open cover of $X$, has Lebesgue number $\delta_L > 0$, then $x' \in B_{\delta_L}(x) \implies f(x), f(x') \in B_{\frac{\varepsilon}{2}}(y)$ for some $y$.

■

**Corollary 5.1.4** ***(Lipschitz implies uniformly continuous).***
Lipschitz continuity implies uniform continuity (take $\delta = \varepsilon/C$)

**Remark 5.1.5:** Counterexample to the converse: $f(x) = \sqrt{x}$ on $[0,1]$ has unbounded derivative.

**Theorem 5.1.6** ***(Extreme Value Theorem).***
For $f : X \to Y$ continuous with $X$ compact and $Y$ ordered in the order topology, there exist points $c, d \in X$ such that $f(x) \in [f(c), f(d)]$ for every $x$.

**Theorem 5.1.7** ***(Sequentially compact if and only if complete and totally bounded).***
A metric space $X$ is sequentially compact iff it is complete and totally bounded.

**Theorem 5.1.8** ***(Totally bounded if and only if Cauchy subsequences exist).***
A metric space is totally bounded iff every sequence has a Cauchy subsequence.

**Theorem 5.1.9** ***(Compact if and only if complete and totally bounded).***
A metric space is compact iff it is complete and totally bounded.

**Theorem 5.1.10** ***(Baire).***
If $X$ is a complete metric space, $X$ is a **Baire space**: the intersection of countably many dense open sets in $X$ is again dense in $X$.

## 5.2 Compactness

**Theorem 5.2.1** ***(Closed if and only if compact in Hausdorff spaces).***
$U \subset X$ a Hausdorff spaces is closed $\iff$ it is compact.

**Theorem 5.2.2** ***(Closed subset of compact is compact).***
A closed subset $A$ of a compact set $B$ is compact.

*Proof (?).*

- Let $\{A_i\} \rightrightarrows A$ be a covering of $A$ by sets open in $A$.
- Each $A_i = B_i \cap A$ for some $B_i$ open in $B$ (definition of subspace topology)

- Define $V = \{B_i\}$, then $V \rightrightarrows A$ is an open cover.
- Since $A$ is closed, $W \coloneqq B \setminus A$ is open
- Then $V \cup W$ is an open cover of $B$, and has a finite subcover $\{V_i\}$
- Then $\{V_i \cap A\}$ is a finite open cover of $A$.

■

**Theorem 5.2.3** ***(Continuous image of compact is compact).***
The continuous image of a compact set is compact.

**Theorem 5.2.4** ***(Closed in Hausdorff $\implies$ compact).***
A closed subset of a Hausdorff space is compact.

## 5.3 Separability

**Proposition 5.3.1** ***(Properties preserved under retracts).***
A retract of a Hausdorff/connected/compact space is closed/connected/compact respectively.

**Proposition 5.3.2** ***(?).***
Points are closed in $T_1$ spaces.

## 5.4 Maps and Homeomorphism

**Theorem 5.4.1** ***(Continuous bijections from compact to Hausdorff are homeomorphisms).***
A continuous bijection $f : X \to Y$ where $X$ is compact and $Y$ is Hausdorff is an open map and hence a homeomorphism.

**Remark 5.4.2** ***(On retractions):*** Every space has at least one retraction - for example, the constant map $r : X \to \{x_0\}$ for any $x\_0 \in X$.

**Theorem 5.4.3** ***(When open maps are homeomorphisms).***
A continuous bijective open map is a homeomorphism.

**Theorem 5.4.4** ***(Characterizations of continuous maps, Munkres 18.1).***
For $f : X \to Y$, TFAE:

- $f$ is continuous
- $A \subset X \implies f(\mathrm{cl}_X(A)) \subset \mathrm{cl}_X(f(A))$
- $B$ closed in $Y \implies f^{-1}(B)$ closed in $X$.

- For each $x \in X$ and each neighborhood $V \ni f(x)$, there is a neighborhood $U \ni x$ such that $f(U) \subset V$.

*Proof (?).*
See Munkres page 104. ∎

**Theorem 5.4.5** ***(Maps from compact to Hausdorff spaces, Lee A.52).***
If $f : X \to Y$ is continuous where $X$ is compact and $Y$ is Hausdorff, then

- $f$ is a closed map.
- If $f$ is surjective, $f$ is a quotient map.
- If $f$ is injective, $f$ is a topological embedding.
- If $f$ is bijective, it is a homeomorphism.

## 5.5 The Tube Lemma

**Theorem 5.5.1** ***(The Tube Lemma).***
Let $X, Y$ be spaces with $Y$ compact. For each $U \subseteq X \times Y$ and each slice $\{x\} \times Y \subseteq U$, there is an open $O \subseteq X$ such that

$$\{x\} \times Y \subseteq O \times Y \subseteq U.$$

*Proof (Sketch).*

- For each $y \in Y$ choose neighborhoods $A_y, B_y \subseteq Y$ such that

$$(x, y) \in A_y \times B_y \subseteq U.$$

- By compactness of $Y$, reduce this to finitely many $B_y \rightrightarrows Y$ so $Y = \bigcup_{j=1}^{n} B_{y_j}$
- Set $O \coloneqq \cap_{j=1}^{n} B_{y_j}$; this works.

∎

# 6 Summary of Standard Topics

- Algebraic topology topics:
  - Classification of compact surfaces
  - Euler characteristic
  - Connect sum

  - Homology and cohomology groups
  - Fundamental group
  - Singular/cellular/simplicial homology
  - Mayer-Vietoris long exact sequences for homology and cohomology
  - Diagram chasing
  - Degree of maps from $S^n \to S^n$
  - Orientability, compactness
  - Top-level homology and cohomology
  - Reduced homology and cohomology
  - Relative homology
  - Homotopy and homotopy invariance
  - Deformation retract
  - Retract
  - Excision
  - Kunneth formula
  - Factoring maps
  - Fundamental theorem of algebra

- Algebraic topology theorems:
  - Brouwer fixed point theorem
  - Poincaré lemma
  - Poincaré duality
  - de Rham theorem
  - Seifert-van Kampen theorem

- Covering space theory topics:
  - Covering maps
  - Free actions
  - Properly discontinuous action
  - Universal cover
  - Correspondence between covering spaces and subgroups of the fundamental group of the base.
  - Lifting paths
  - Homotopy lifting property
  - Deck transformations
  - The action of the fundamental group
  - Normal/regular cover

# 7 Examples: Algebraic Topology

## 7.1 Standard Spaces and Modifications

**Example 7.1.1** ***(Spheres and Balls):***

$$\mathbb{D}^n = \mathbb{B}^n \coloneqq \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \|\mathbf{x}\| \leq 1 \right\} \mathbb{S}^n \qquad \coloneqq \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; \|\mathbf{x}\| = 1 \right\} = \partial \mathbb{D}^n$$

.

> *Note: I'll immediately drop the blackboard notation, this is just to emphasize that they're "canonical" objects.*

The sphere can be constructed in several equivalent ways:

- $S^n \cong D^n/\partial D^n$: collapsing the boundary of a disc is homeomorphic to a sphere.
- $S^n \cong D^n \coprod_{\partial D^n} D^n$: gluing two discs along their boundary.

Note the subtle differences in dimension: $S^n$ is a manifold of dimension $n$ embedded in a space of dimension $n+1$.

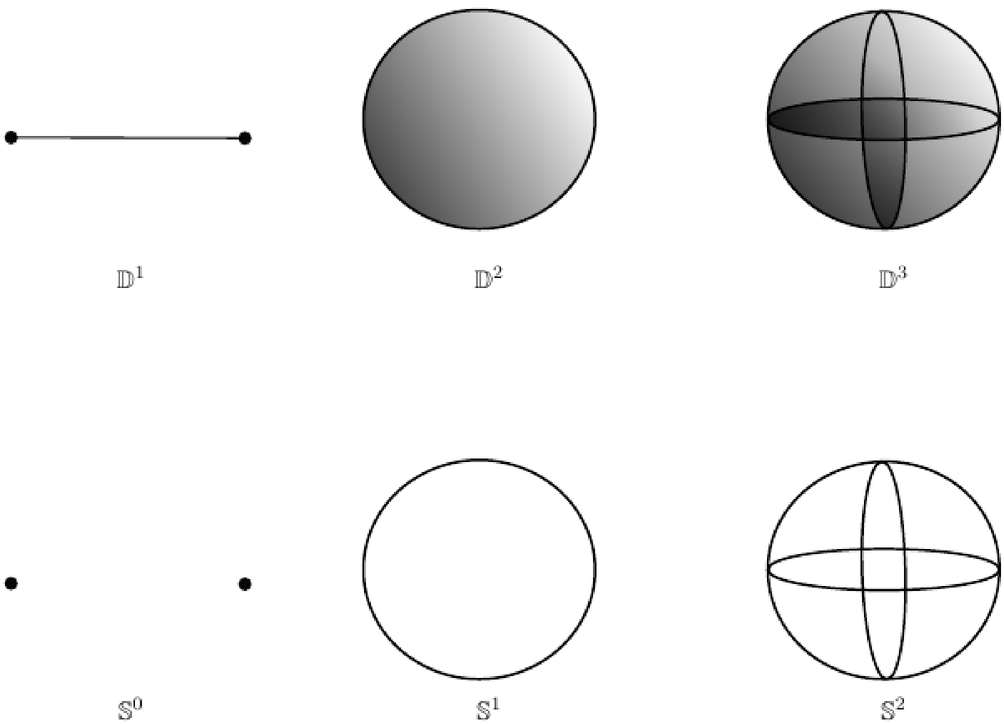



Figure 1: Low Dimensional Discs/Balls vs Spheres

**Example 7.1.2** ***(Real Projective Space):*** Constructed in one of several equivalent ways:

- $S^n/\sim$ where $\mathbf{x} \sim -\mathbf{x}$, i.e. antipodal points are identified.
- The space of lines in $\mathbb{R}^{n+1}$.

One can also define $\mathbb{RP}^\infty \coloneqq \varinjlim_n \mathbb{RP}^n$. Fits into a fiber bundle of the form

$$\begin{tikzcd} S^0 \arrow[r] & S^n \arrow[d] \\ & \mathbb{RP}^n \end{tikzcd}$$

**Example 7.1.3** ***(Complex Projective Space):*** Defined in a similar ways,

- Taking the unit sphere in $\mathbb{C}^n$ and identifying $\mathbf{z} \sim -\mathbf{z}$.
- The space of lines in $\mathbb{C}^{n+1}$

Can similarly define $\mathbb{CP}^\infty \coloneqq \varinjlim_n \mathbb{CP}^n$. Fits into a fiber bundle of the form

$$\begin{tikzcd} S^1 \arrow[r] & S^{2n+1} \arrow[d] \\ & \mathbb{CP}^n \end{tikzcd}$$

**Example 7.1.4** ***(Torii):*** The $n$-torus, defined as

$$T^n \coloneqq \prod_{j=1}^n S^1 = S^1 \times S^1 \times \cdots.$$

**Example 7.1.5** ***(Grassmannians):*** The real Grassmannian, $\mathrm{Gr}(n,k)_{/\mathbb{R}}$, i.e. the set of $k$ dimensional subspaces of $\mathbb{R}^n$. One can similar define $\mathrm{Gr}(n,k)_{\mathbb{C}}$ for complex subspaces. Note that $\mathbb{RP}^n = \mathrm{Gr}(n,1)_{\mathbb{R}}$ and $\mathbb{CP}^n = \mathrm{Gr}(n,1)_{/\mathbb{C}}$.

**Example 7.1.6** ***(Stiefel Manifolds):*** The Stiefel manifold $V_n(k)_{\mathbb{R}}$, the space of orthonormal $k$-frames in $\mathbb{R}^n$?

**Example 7.1.7** ***(Lie Groups):*** Lie Groups:

- The general linear group, $\mathrm{GL}_n(\mathbb{R})$
  - The special linear group $SL_n(\mathbb{R})$
- The orthogonal group, $O_n(\mathbb{R})$
  - The special orthogonal group, $SO_n(\mathbb{R})$
- The real unitary group, $U_n(\mathbb{C})$
  - The special unitary group, $SU_n(\mathbb{R})$
- The symplectic group $Sp(2n)$

**Example 7.1.8** ***(More random geometric examples):*** Some other spaces that show up, but don't usually have great algebraic topological properties:

- Affine $n$-space over a field $\mathbb{A}^n(k) = k^n \rtimes GL_n(k)$
- The projective space $\mathbb{P}^n(k)$
- The projective linear group over a ring $R$, $PGL_n(R)$
- The projective special linear group over a ring $R$, $PSL_n(R)$
- The modular groups $PSL_n(\mathbb{Z})$
  - Specifically $PSL_2(\mathbb{Z})$

**Example 7.1.9** ***(Eilenberg-MacLane Spaces):*** $K(G,n)$ is an Eilenberg-MacLane space, the homotopy-unique space satisfying

$$\pi_k(K(G,n)) = \begin{cases} G & k = n, \\ 0 & \text{else} \end{cases}$$

Some known examples:

- $K(\mathbb{Z},1) = S^1$
- $K(\mathbb{Z},2) = \mathbb{CP}^\infty$
- $K(\mathbb{Z}/2\mathbb{Z},1) = \mathbb{RP}^\infty$

**Example 7.1.10** ***(Moore Spaces):*** $M(G,n)$ is a Moore space, the homotopy-unique space satisfying

$$H_k(M(G,n);G) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

Some known examples:

- $M(\mathbb{Z},n) = S^n$
- $M(\mathbb{Z}/2\mathbb{Z},1) = \mathbb{RP}^2$

- $M(\mathbb{Z}/p\mathbb{Z}, n)$ is made by attaching $e^{n+1}$ to $S^n$ via a degree $p$ map.

**Fact 7.1.11** (about standard low-dimensional spaces)

- $\mathcal{M} \simeq S^1$ where $\mathcal{M}$ is the Mobius band.
- $\mathbb{CP}^n = \mathbb{C}^n \coprod \mathbb{CP}^{n-1} = \coprod_{i=0}^{n} \mathbb{C}^i$
- $\mathbb{CP}^n = S^{2n+1}/S^n$
- $S^n/S^k \simeq S^n \vee \Sigma S^k$.

**Remark 7.1.12** ***(Accidental isomorphisms):*** In low dimensions, there are some "accidental" homeomorphisms:

- $\mathbb{RP}^1 \cong S^1$
- $\mathbb{CP}^1 \cong S^2$
- $\mathrm{SO}(3) \cong \mathbb{RP}^2$?

## 7.2 Modifying Known Spaces

**Example 7.2.1** ***(Deleting points):*** Write $D(k, X)$ for the space $X$ with $k \in \mathbb{N}$ distinct points deleted, i.e. the punctured space $X - \{x_1, x_2, \ldots x_k\}$ where each $x_i \in X$.

**Example 7.2.2** ***(Bouquets of Spheres):*** The "generalized uniform bouquet"? $\mathcal{B}^n(m) = \bigvee_{i=1}^{n} S^m$. There's no standard name for this, but it's an interesting enough object to consider!

**Example 7.2.3** ***(Other ways to modify a known space):*** Possible modifications to a space $X$:

- Remove a line segment
- Remove an entire line/axis
- Remove a hole
- Quotient by a group action (e.g. antipodal map, or rotation)
- Remove a knot
- Take complement in ambient space

# 8 Low Dimensional Homology Examples

**Fact 8.0.1** (Table of low-dimensional homology)

$$
\begin{array}{rcllllllll}
S^1 & = & [ & \mathbb{Z}, & \mathbb{Z}, & 0, & 0, & 0, & 0 \to & ] \\
\mathcal{M} & = & [ & \mathbb{Z}, & \mathbb{Z}, & 0, & 0, & 0, & 0 \to & ] \\
\mathbb{RP}^1 & = & [ & \mathbb{Z}, & \mathbb{Z}, & 0, & 0, & 0, & 0 \to & ] \\
\mathbb{RP}^2 & = & [ & \mathbb{Z}, & \mathbb{Z}_2, & 0, & 0, & 0, & 0 \to & ] \\
\mathbb{RP}^3 & = & [ & \mathbb{Z}, & \mathbb{Z}_2, & 0, & \mathbb{Z}, & 0, & 0 \to & ] \\
\mathbb{RP}^4 & = & [ & \mathbb{Z}, & \mathbb{Z}_2, & 0, & \mathbb{Z}_2, & 0, & 0 \to & ] \\
S^2 & = & [ & \mathbb{Z}, & 0, & \mathbb{Z}, & 0, & 0, & 0 \to & ] \\
\mathbb{T}^2 & = & [ & \mathbb{Z}, & \mathbb{Z}^2, & \mathbb{Z}, & 0, & 0, & 0 \to & ] \\
\mathbb{K} & = & [ & \mathbb{Z}, & \mathbb{Z} \oplus \mathbb{Z}_2, & 0, & 0, & 0, & 0 \to & ] \\
\mathbb{CP}^1 & = & [ & \mathbb{Z}, & 0, & \mathbb{Z}, & 0, & 0, & 0 \to & ] \\
\mathbb{CP}^2 & = & [ & \mathbb{Z}, & 0, & \mathbb{Z}, & 0, & \mathbb{Z}, & 0 \to & ]
\end{array}.
$$

# 9 Table of Homotopy and Homology Structures

**Remark 9.0.1:** The following is a giant list of known homology/homotopy.

| $X$ | $\pi_*(X)$ | $H_*(X)$ | CW Structure | $H^*(X)$ |
|---|---|---|---|---|
| $\mathbb{R}^1$ | $0$ | $0$ | $\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot x$ | $0$ |
| $\mathbb{R}^n$ | $0$ | $0$ | $(\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot x)^n$ | $0$ |
| $D(k, \mathbb{R}^n)$ | $\pi_* \bigvee^k S^1$ | $\bigoplus_k H_* M(\mathbb{Z}, 1)$ | $1 + kx$ | ? |
| $B^n$ | $\pi_*(\mathbb{R}^n)$ | $H_*(\mathbb{R}^n)$ | $1 + x^n + x^{n+1}$ | $0$ |
| $S^n$ | $[0 \ldots, \mathbb{Z}, ? \ldots]$ | $H_* M(\mathbb{Z}, n)$ | $1 + x^n$ or $\sum_{i=0}^n 2x^i$ | $\mathbb{Z}[_n x]/(x^2)$ |
| $D(k, S^n)$ | $\pi_* \bigvee^{k-1} S^1$ | $\bigoplus_{k-1} H_* M(\mathbb{Z}, 1)$ | $1 + (k-1)x^1$ | ? |
| $T^2$ | $\pi_* S^1 \times \pi_* S^1$ | $(H_* M(\mathbb{Z}, 1))^2 \times H_* M(\mathbb{Z}, 2)$ | $1 + 2x + x^2$ | $\Lambda(_1x_1, {}_1x_2)$ |
| $T^n$ | $\prod^n \pi_* S^1$ | $\prod_{i=1}^n (H_* M(\mathbb{Z}, i))^{\binom{n}{i}}$ | $(1+x)^n$ | $\Lambda(_1x_1, {}_1x_2, \ldots {}_1x_n)$ |
| $D(k, T^n)$ | $[0, 0, 0, 0, \ldots]$? | $[0, 0, 0, 0, \ldots]$? | $1 + x$ | ? |
| $S^1 \vee S^1$ | $\pi_* S^1 * \pi_* S^1$ | $(H_* M(\mathbb{Z}, 1))^2$ | $1 + 2x$ | ? |
| $\bigvee^n S^1$ | $*^n \pi_* S^1$ | $\prod H_* M(\mathbb{Z}, 1)$ | $1 + x$ | ? |
| $\mathbb{RP}^1$ | $\pi_* S^1$ | $H_* M(\mathbb{Z}, 1)$ | $1 + x$ | $_0\mathbb{Z} \times {}_1\mathbb{Z}$ |
| $\mathbb{RP}^2$ | $\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) + \pi_* S^2$ | $H_* M(\mathbb{Z}/2\mathbb{Z}, 1)$ | $1 + x + x^2$ | $_0\mathbb{Z} \times {}_2\mathbb{Z}/2\mathbb{Z}$ |

| $X$ | $\pi_*(X)$ | $H_*(X)$ | CW Structure | $H^*(X)$ |
|---|---|---|---|---|
| $\mathbb{RP}^3$ | $\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) + \pi_* S^3$ | $H_* M(\mathbb{Z}/2\mathbb{Z}, 1) + H_* M(\mathbb{Z}, 3)$ | $1 + x + x^2 + x^3$ | ${}_0\mathbb{Z} \times {}_2\mathbb{Z}/2\mathbb{Z} \times {}_3\mathbb{Z}$ |
| $\mathbb{RP}^4$ | $\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) + \pi_* S^4$ | $H_* M(\mathbb{Z}/2\mathbb{Z}, 1) + H_* M(\mathbb{Z}/2\mathbb{Z}, 3)$ | $1 + x + x^2 + x^3 + x^4$ | ${}_0\mathbb{Z} \times ({}_2\mathbb{Z}/2\mathbb{Z})^2$ |
| $\mathbb{RP}^n, n \geq 4$ even | $\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) + \pi_* S^n$ | $\prod_{\text{odd } i<n} H_* M(\mathbb{Z}/2\mathbb{Z}, i)$ | $\sum_{i=1}^n x^i$ | ${}_0\mathbb{Z} \times \prod_{i=1}^{n/2} {}_2\mathbb{Z}/2\mathbb{Z}$ |
| $\mathbb{RP}^n, n \geq 4$ odd | $\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) + \pi_* S^n$ | $\prod_{\text{odd } i\leq n-2} H_* M(\mathbb{Z}/2\mathbb{Z}, i) \times H_* S^n$ | $\sum_{i=1}^n x^i$ | $H^*(\mathbb{RP}^{n-1}) \times {}_n\mathbb{Z}$ |
| $\mathbb{CP}^1$ | $\pi_* K(\mathbb{Z}, 2) + \pi_* S^3$ | $H_* S^2$ | $x^0 + x^2$ | $\mathbb{Z}[{}_2x]/({}_2x^2)$ |
| $\mathbb{CP}^2$ | $\pi_* K(\mathbb{Z}, 2) + \pi_* S^5$ | $H_* S^2 \times H_* S^4$ | $x^0 + x^2 + x^4$ | $\mathbb{Z}[{}_2x]/({}_2x^3)$ |
| $\mathbb{CP}^n, n \geq 2$ | $\pi_* K(\mathbb{Z}, 2) + \pi_* S^{2n+1}$ | $\prod_{i=1}^n H_* S^{2i}$ | $\sum_{i=1}^n x^{2i}$ | $\mathbb{Z}[{}_2x]/({}_2x^{n+1})$ |
| Mobius Band | $\pi_* S^1$ | $H_* S^1$ | $1 + x$ | ? |
| Klein Bottle | $K(\mathbb{Z} \rtimes_{-1} \mathbb{Z}, 1)$ | $H_* S^1 \times H_* \mathbb{RP}^\infty$ | $1 + 2x + x^2$ | ? |

**Fact 9.0.2** (used to fill out the above table)

- $\mathbb{R}^n$ is a contractible space, and so $[S^m, \mathbb{R}^n] = 0$ for all $n, m$ which makes its homotopy groups all zero.
- $D(k, \mathbb{R}^n) = \mathbb{R}^n - \{x_1 \ldots x_k\} \simeq \bigvee_{i=1}^k S^1$ by a deformation retract.
- $S^n \cong B^n/\partial B^n$ and employs an attaching map

$$\begin{aligned} \varphi : (D^n, \partial D^n) &\to S^n \\ (D^n, \partial D^n) &\mapsto (e^n, e^0). \end{aligned}$$

- $B^n \simeq \mathbb{R}^n$ by normalizing vectors.
- Use the inclusion $S^n \hookrightarrow B^{n+1}$ as the attaching map.
- $\mathbb{CP}^1 \cong S^2$.
- $\mathbb{RP}^1 \cong S^1$.
- Use $[\pi_1, \prod] = 0$ and the universal cover $\mathbb{R}^1 \twoheadrightarrow S^1$ to yield the cover $\mathbb{R}^n \twoheadrightarrow T^n$.
- Take the universal double cover $S^n \twoheadrightarrow^{\times 2} \mathbb{RP}^n$ to get equality in $\pi_{i\geq 2}$.

- Use $\mathbb{CP}^n = S^{2n+1}/S^1$
- Alternatively, the fundamental group is $\mathbb{Z} * \mathbb{Z}/bab^{-1}a$. Use the fact the $\tilde{K} = \mathbb{R}^2$.
- $M \simeq S^1$ by deformation-retracting onto the center circle.
- $D(1, S^n) \cong \mathbb{R}^n$ and thus $D(k, S^n) \cong D(k-1, \mathbb{R}^n) \cong \bigvee^{k-1} S^1$

# 10 Theorems: Algebraic Topology

## 10.1 General Homotopies

**Fact 10.1.1** (Contracting Spaces in Products)

$$X \times \mathbb{R}^n \simeq X \times \{\mathrm{pt}\} \cong X.$$

**Fact 10.1.2** ($\pi_0, H_0$ detect path components)
The ranks of $\pi_0$ and $H_0$ are the number of path components.

**Theorem 10.1.3** ***(Convex sets admit homotopies).***
Any two continuous functions into a convex set are homotopic.

*Proof (?).*
The linear homotopy. Supposing $X$ is convex, for any two points $x, y \in X$, the line $tx + (1-t)y$ is contained in $X$ for every $t \in [0,1]$. So let $f, g : Z \to X$ be any continuous functions into $X$. Then define $H : Z \times I \to X$ by $H(z,t) = tf(z) + (1-t)g(z)$, the linear homotopy between $f, g$. By convexity, the image is contained in $X$ for every $t, z$, so this is a homotopy between $f, g$. ■

## 10.2 Fundamental Group

### 10.2.1 Definition

**Definition 10.2.1** (The Fundamental Group)
Given a pointed space $(X, x_0)$, we define the fundamental group $\pi_1(X)$ as follows:

- Take the set

$$L \coloneqq \left\{ \alpha : S^1 \to X \,\middle|\, \alpha(0) = \alpha(1) = x_0 \right\}.$$

- Define an equivalence relation $\alpha \sim \beta$ iff $\alpha \simeq \beta$ in $X$, so there exists a homotopy

$$\begin{aligned} H : & S^1 \times I \to X \\ & \begin{cases} H(s,0) = \alpha(s) \\ H(s,1) = \beta(s), \end{cases} \end{aligned}$$

- Check that this relation is

- Symmetric: Follows from considering $H(s, 1-t)$.
- Reflexive: Take $H(s,t) = \alpha(s)$ for all $t$.
- Transitive: Follows from reparameterizing.
- Define $L/\sim$, which contains elements like $[\alpha]$ and $[\mathrm{id}_{x_0}]$, the equivalence classes of loops after quotienting by this relation.
- Define a product structure: for $[\alpha], [\beta] \in L/\sim$, define $[\alpha][\beta] = [\alpha \cdot \beta]$, where we just need to define a product structure on actual loops. Do this by reparameterizing:

$$(\alpha \cdot \beta)(s) \coloneqq \begin{cases} \alpha(2s) & s \in [0, 1/2] \\ \beta(2s-1) & s \in [1/2, 1]. \end{cases}$$

- Check that this map is:
  - Continuous: by the pasting lemma and assumed continuity of $f, g$.
  - Well-defined: ?
- Check that this is actually a group
  - Identity element: The constant loop $\mathrm{id}_{x_0} : I \to X$ where $\mathrm{id}_{x_0}(t) = x_0$ for all $t$.
  - Inverses: The reverse loop $\bar{\alpha}(t) \coloneqq \alpha(1-t)$.
  - Closure: Follows from the fact that start/end points match after composing loops, and reparameterizing.
  - Associativity: Follows from reparameterizing.

**Remark 10.2.2** ***(a summary):*** Elements of the fundamental group are *homotopy classes of loops*, and every continuous map between spaces induces a homomorphism on fundamental groups.

### 10.2.2 Conjugacy in $\pi_1$:

- See Hatcher 1.19, p.28
- See Hatcher's proof that $\pi_1$ is a group
- See change of basepoint map

### 10.2.3 Calculating $\pi_1$

**Proposition 10.2.3** ***(Using universal covers).***
If $\tilde{X} \to X$ the universal cover of $X$ and $G \curvearrowright \tilde{X}$ with $\tilde{X}/G = X$ then $\pi_1(X) = G$.

**Proposition 10.2.4** ***(Killing homotopy).***
$\pi_1 X$ for $X$ a CW-complex only depends on the 2-skeleton $X^2$, and in general $\pi_k(X)$ only depends on the $k+2$-skeleton. Thus attaching $k+2$ or higher cells does not change $\pi_k$.

**Theorem 10.2.5** ***(Seifert-van Kampen).***
Suppose $X = U_1 \cup U_2$ where $U_1, U_2$, and $U \coloneqq U_1 \cap U_2 \neq \emptyset$ are open and path-connected [a]
, and let $x_0 \in U$.
Then the inclusion maps $i_1 : U_1 \hookrightarrow X$ and $i_2 : U_2 \hookrightarrow X$ induce the following group homomorphisms:

$$\begin{aligned} i_1^* &: \pi_1(U_1, x_0) \to \pi_1(X, x_0) \\ i_2^* &: \pi_1(U_2, x_0) \to \pi_1(X, x_0) \end{aligned}$$

There is a natural isomorphism

$$\pi_1(X) \cong \pi_1 U *_{\pi_1(U \cap V)} \pi_1 V,$$

where the amalgamated product can be computed as follows: A **pushout** is the colimit of the following diagram

$$\begin{tikzcd} A \coprod_Z B & A \arrow[l] \\ B \arrow[u] & Z \arrow[l, "\iota_B"] \arrow[u, "\iota_A"'] \end{tikzcd}$$

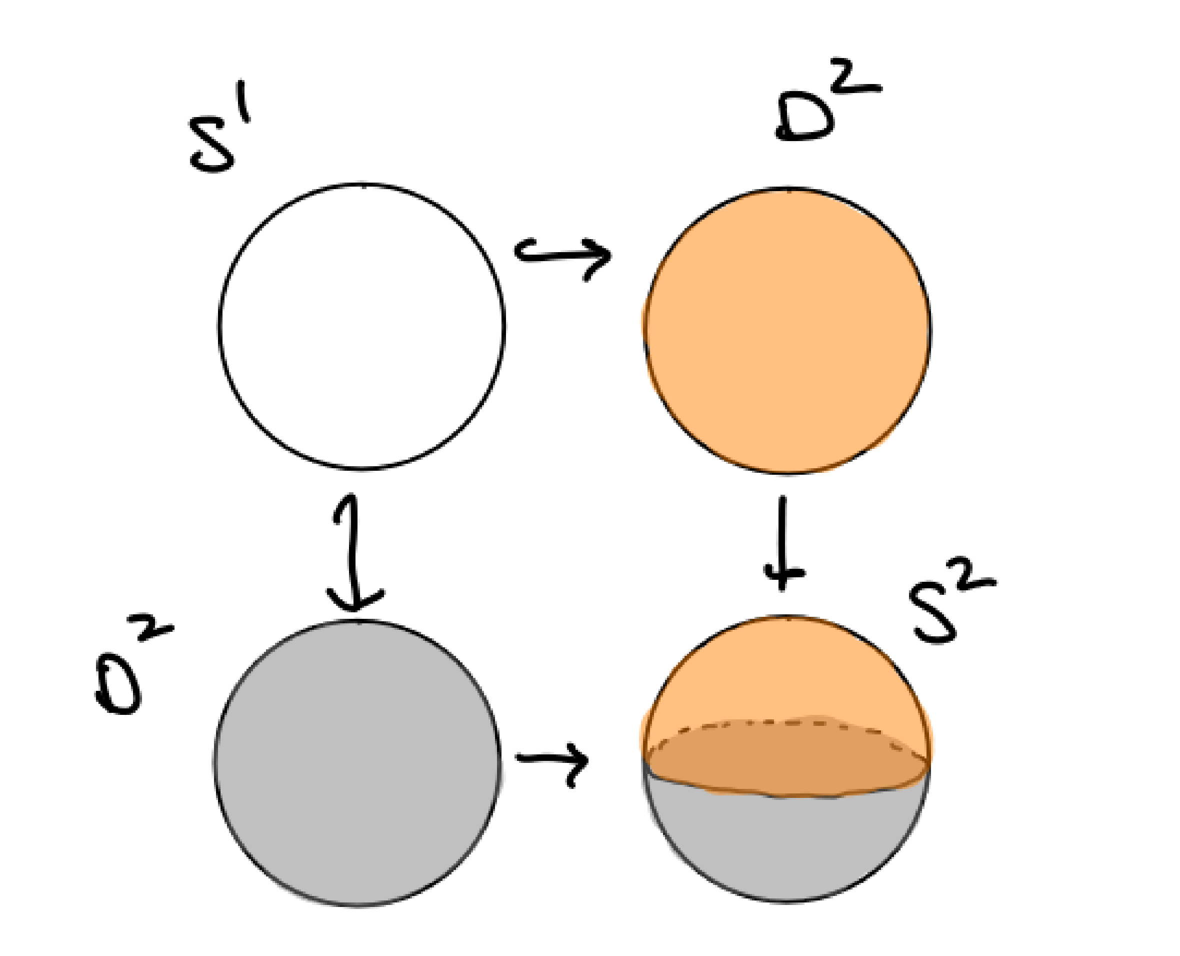



Figure 2: Example of a pushout of spaces

For groups, the pushout is realized by the amalgamated free product: if

$$\begin{cases} \pi_1 U_1 = A = \left\langle G_A \;\middle|\; R_A \right\rangle \\ \pi_1 U_2 = B = \left\langle G_B \;\middle|\; R_B \right\rangle \end{cases} \implies A *_Z B \coloneqq \left\langle G_A, G_B \;\middle|\; R_A, R_B, T \right\rangle$$

where $T$ is a set of relations given by

$$T = \left\{ \iota_1^*(z)\iota_2^*(z)^{-1} \;\middle|\; z \in \pi_1(U_1 \cap U_2) \right\},$$

where $\iota_2^*(z)^{-1}$ denotes the inverse group element. If we have presentations

$$\pi_1(U, x_0) = \left\langle u_1, \cdots, u_k \;\middle|\; \alpha_1, \cdots, \alpha_l \right\rangle$$
$$\pi_1(V, w) = \left\langle v_1, \cdots, v_m \;\middle|\; \beta_1, \cdots, \beta_n \right\rangle$$
$$\pi_1(U \cap V, x_0) = \left\langle w_1, \cdots, w_p \;\middle|\; \gamma_1, \cdots, \gamma_q \right\rangle$$

then

$$\pi_1(X,w) = \left\langle u_1, \cdots, u_k, v_1, \cdots, v_m \middle| \begin{cases} \alpha_1, \cdots, \alpha_l \\ \beta_1, \cdots, \beta_n \\ I(w_1) J(w_1)^{-1}, \cdots, I(w_p) J(w_p)^{-1} \end{cases} \right\rangle$$

$$= \frac{\pi_1(U_1) * \pi_1(U_2)}{\left\langle \left\{ \iota_1^*(w_i)\iota_2^*(w_i)^{-1} \;\middle|\; 1 \le i \le p \right\} \right\rangle}$$

[a] Note that the hypothesis that $U_1 \cap U_2$ is path-connected is necessary: take $S^1$ with $U, V$ neighborhoods of the poles, whose intersection is two disjoint components.

*Proof (Sketch).*

- Construct a map going backwards
- Show it is surjective
  - "There and back" paths
- Show it is injective
  - Divide $I \times I$ into a grid

■

**Example 10.2.6** ***(Pushing out with van Kampen):*** $A = \mathbb{Z}/4\mathbb{Z} = \left\langle x \mid x^4 \right\rangle, B = \mathbb{Z}/6\mathbb{Z} = \left\langle y \mid x^6 \right\rangle, Z = \mathbb{Z}/2\mathbb{Z} = \left\langle z \mid z^2 \right\rangle$. Then we can identify $Z$ as a subgroup of $A, B$ using $\iota_A(z) = x^2$ and $\iota_B(z) = y^3$. So

$$A *_Z B = \left\langle x, y \mid x^4, y^6, x^2 y^{-3} \right\rangle$$

.

**Proposition 10.2.7** ***($\pi_1$ of a wedge).***

$$\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y).$$

*Proof (?).*
By van Kampen, this is equivalent to the amalgamated product over $\pi_1(x_0) = 1$, which is just a free product.

■

### 10.2.4 Facts

**Fact 10.2.8**
$H_1$ is the abelianization of $\pi_1$.

**Proposition 10.2.9** ***($\pi_1$ of a product, Hatcher 1.12).***
If $X, Y$ are path-connected, then

$$\pi_1(X \times Y) = \pi_1(X) \times \pi_2(Y).$$

*Proof (sketch).*

- A loop in $X \times Y$ is a continuous map $\gamma : I \to X \times Y$ given by $\gamma(t) = (f(t), g(t)$ in components.
- $\gamma$ being continuous in the product topology is equivalent to $f, g$ being continuous maps to $X, Y$ respectively.
- Similarly a homotopy $F : I^2 \to X \times Y$ is equivalent to a pair of homotopies $f_t, g_t$ of the corresponding loops.
- So the map $[\gamma] \mapsto ([f], [g])$ is the desired bijection.

$\blacksquare$

**Proposition 10.2.10** ***($\pi_1$ detects simply-connectedness).***
$\pi_1(X) = 1$ iff $X$ is simply connected.

*Proof (?).*
$\Rightarrow$: Suppose $X$ is simply connected. Then every loop in $X$ contracts to a point, so if $\alpha$ is a loop in $X$, $[\alpha] = [\mathrm{id}_{x_0}]$, the identity element of $\pi_1(X)$. But then there is only one element in in this group.
$\Leftarrow$: Suppose $\pi_1(X) = 0$. Then there is just one element in the fundamental group, the identity element, so if $\alpha$ is a loop in $X$ then $[\alpha] = [\mathrm{id}_{x_0}]$. So there is a homotopy taking $\alpha$ to the constant map, which is a contraction of $\alpha$ to a point.

$\blacksquare$

:::{.fact "Unsorted facts"}

- For a graph $G$, we always have $\pi_1(G) \cong \mathbb{Z}^n$ where $n = |E(G - T)|$, the complement of the set of edges in any maximal tree. Equivalently, $n = 1 - \chi(G)$. Moreover, $X \simeq \bigvee^n S^1$ in this case.

:::

## 10.3 General Homotopy Theory

**Theorem 10.3.1** ***(Whitehead's Theorem).***
A map $X \xrightarrow{f} Y$ on CW complexes that is a weak homotopy equivalence (inducing isomorphisms in homotopy) is in fact a homotopy equivalence.

**Warning 10.3.2**
Individual maps may not work: take $S^2 \times \mathbb{RP}^3$ and $S^3 \times \mathbb{RP}^2$ which have isomorphic homotopy but

not homology.

**Theorem 10.3.3** ***(Hurewicz).***
The Hurewicz map on an $n-1$-connected space $X$ is an isomorphism $\pi_{k\leq n}X \to H_{k\leq n}X$.

*I.e. for the minimal $i \geq 2$ for which $\pi_{iX} \neq 0$ but $\pi_{\leq i-1}X = 0$, $\pi_{iX} \cong H_{iX}$.*

**Theorem 10.3.4** ***(Cellular Approximation).***
Any continuous map between CW complexes is homotopy equivalent to a cellular map.

**Example 10.3.5** ***(Applications of cellular approximation):***

- $\pi_{k\leq n}S^n = 0$
- $\pi_n(X) \cong \pi_n(X^{(n)})$

**Theorem 10.3.6** ***(Freudenthal Suspension).***

Theorem

:::{.fact title="Unsorted facts about higher homotopy groups}

- $\pi_{i\geq 2}(X)$ is always abelian.

  – $X$ simply connected $\implies \pi_k(X) \cong H_k(X)$ up to and including the first nonvanishing $H_k$

- $\pi_k \bigvee X \neq \prod \pi_k X$ (counterexample: $S^1 \vee S^2$)

  – Nice case: $\pi_1 \bigvee X = *\pi_1 X$ by Van Kampen.

- $\pi_i(\widehat{X}) \cong \pi_i(X)$ for $i \geq 2$ whenever $\widehat{X} \twoheadrightarrow X$ is a universal cover.

- $\pi_i(S^n) = 0$ for $i < n$, $\pi_n(S^n) = \mathbb{Z}$

  – Not necessarily true that $\pi_i(S^n) = 0$ when $i > n$!!!
    - E.g. $\pi_3(S^2) = \mathbb{Z}$ by Hopf fibration

- $S^n/S^k \simeq S^n \vee \Sigma S^k$

  – $\Sigma S^n = S^{n+1}$

- General mantra: homotopy plays nicely with products, homology with wedge products.[1]

[1] More generally, in **Top**, we can look at $A \leftarrow \{pt\} \to B$ – then $A \times B$ is the pullback and $A \vee B$ is the pushout. In this case, homology $h : \mathbf{Top} \to \mathbf{Grp}$ takes pushouts to pullbacks but doesn't behave well with pullbacks. Similarly, while $\pi$ takes pullbacks to pullbacks, it doesn't behave nicely with pushouts.

- $\pi_k \prod X = \prod \pi_k X$ by LES.[2]
- In general, homotopy groups behave nicely under homotopy pull-backs (e.g., fibrations and products), but not homotopy push-outs (e.g., cofibrations and wedges). Homology is the opposite.
- Constructing a $K(\pi, 1)$: since $\pi = \left\langle S \mid R \right\rangle = F(S)/R$, take $\bigvee^{|S|} S^1 \cup_{|R|} e^2$. In English, wedge a circle for each generator and attach spheres for relations.

:::

# 11 | Covering Spaces

Some pictures to keep in mind when it comes to covers and path lifting:

[2]This follows because $X \times Y \twoheadrightarrow X$ is a fiber bundle, so use LES in homotopy and the fact that $\pi_{i \geq 2} \in \mathbf{Ab}$.

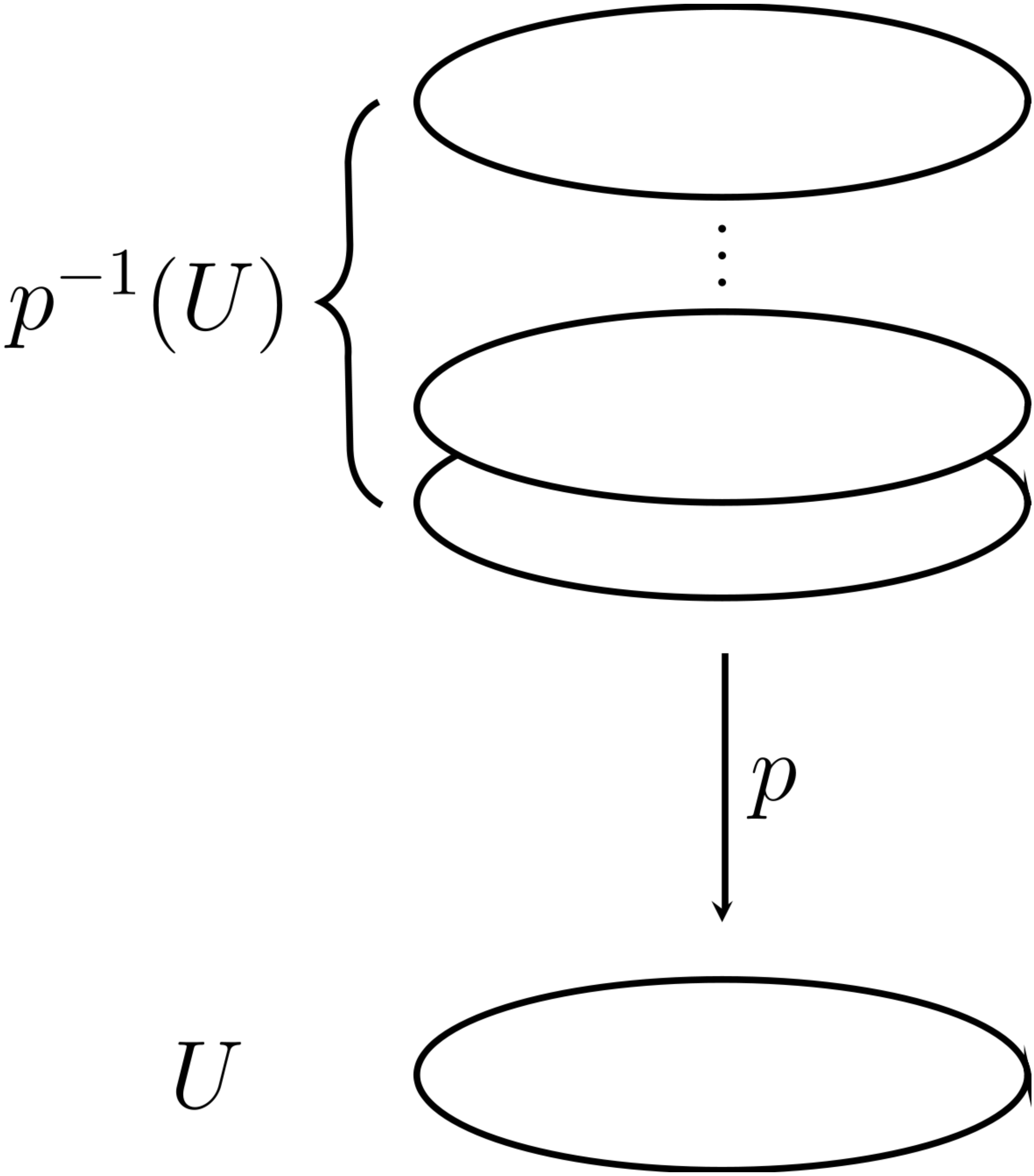



Figure 3: Picture to keep in mind

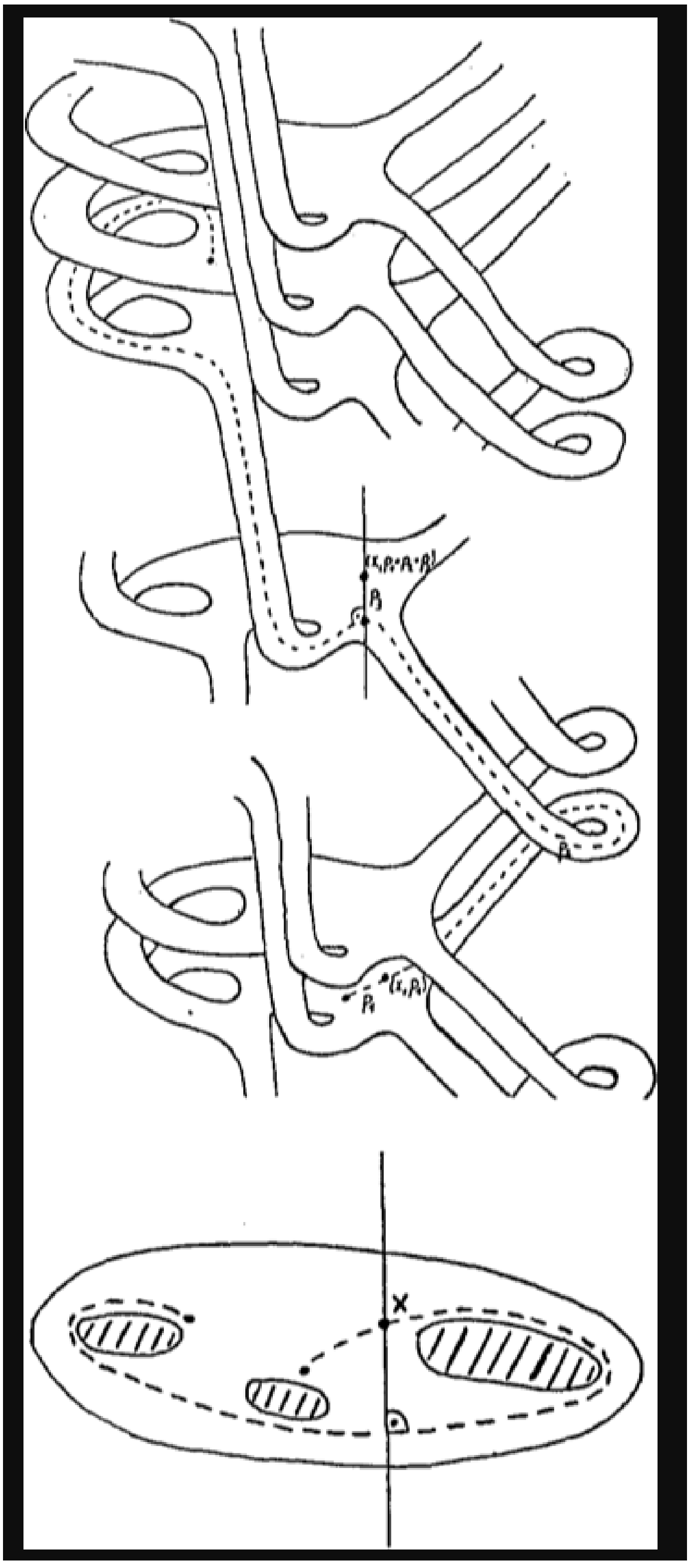



Figure 4: A more complicated situation

## 11.1 Useful Facts

**Remark 11.1.1:** When covering spaces are involved in any way, try computing Euler characteristics - this sometimes yields nice numerical constraints.

**Fact 11.1.2** (Euler characteristics are multiplicative on covering spaces)
For $p : A \to B$ an $n$-fold cover,

$$\chi(A) = n\,\chi(B).$$

**Fact 11.1.3**
Covering spaces of orientable manifolds are orientable.

**Fact 11.1.4**
The preimage of a boundary point under a covering map must also be a boundary point

**Fact 11.1.5**
Normal subgroups correspond to *normal/regular* coverings, where automorphisms act freely/transitively. These are "maximally symmetric".

## 11.2 Universal Covers

**Proposition 11.2.1** ***(Existence of universal covers).***
If $X$ is

- Connected,
- Locally path-connected, and
- Semilocally simply connected,

then $X$ admits a universal cover: if $C \xrightarrow{q} X$ is a covering map with $C$ connected, then there exists a covering map $\tilde{p} : \tilde{X} \to C$ making the following diagram commute:

$$\begin{tikzcd}
C \arrow[d, two heads, "q"] & \tilde{X} \arrow[l, dashed, two heads, "\tilde{p}"'] \arrow[dl, two heads, "p"] \\
X &
\end{tikzcd}$$

*Link to diagram*

That is, any other cover $C$ of $X$ is itself covered by $\tilde{X}$. Note that by this universal property, $\tilde{X}$ is unique up to homeomorphism when it exists.

**Theorem 11.2.2** ***(Homotopy lifting property for covers, Hatcher 1.30).***
Let $p : \tilde{X} \to X$ be any covering space, $F : Y \times I \to X$ be any homotopy, and $\tilde{F}_0 : Y \to \tilde{X}$ be any lift of $F_0$. Then there exists a unique homotopy $\tilde{F} : Y \to \tilde{X}$ of $\tilde{F}_0$ that lifts $F$:

$$\begin{tikzcd}
Y \arrow[r, "\tilde{F}_0"] \arrow[d, hook, "y\mapsto(y,0)"'] & \tilde{X} \arrow[d, "p"] \\
Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists\tilde{F}"] & X
\end{tikzcd}$$

Link to diagram

**Theorem 11.2.3** ***(Lifting criterion for covers, Hatcher 1.33).***
If $f : Y \to X$ with $Y$ path-connected and locally path-connected, then there exists a unique lift $\tilde{f} : Y \to \tilde{X}$ if and only if $f_*(\pi_1(Y)) \subset \pi_*(\pi_1(\tilde{X}))$:

$$\begin{tikzcd}
 & \tilde{X} \arrow[d, "p"] \\
Y \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & X
\end{tikzcd}$$

Link to diagram

Moreover, lifts are *unique* if they agree at a single point.

**Remark 11.2.4** ***(Automatic lifts):*** Note that if $Y$ is simply connected, then $\pi_1(Y) = 0$ and this holds automatically!

**Proposition 11.2.5** ***(Covering spaces induce injections on $\pi_1$, Hatcher 1.31).***
Given a covering space $\tilde{X} \xrightarrow{p} X$, the induced map $p^* : \pi_1(\tilde{X}) \to \pi_1(X)$ is injective. The image consists of classes $[\gamma]$ whose lifts to $\tilde{X}$ are again loops.

**Theorem 11.2.6** ***(Fundamental theorem of covering spaces, Hatcher 1.39).***
For $\tilde{X} \xrightarrow{p} X$ a covering space with

- $\tilde{X}$ path-connected,
- $X$ path-connected and locally path-connected,

letting $H$ be the image of $\pi_1(\tilde{X})$ in $\pi_1(X)$, we have

1. $\tilde{X}$ is normal if and only if $H \trianglelefteq \pi_1(X)$,
2. $G(\tilde{X}) \cong \mathrm{Aut}_{\mathrm{Cov}(\tilde{X})} N_{\pi_1(X)}(H)$, the normalizer of $H$ in $\pi_1(X)$.

In particular, if $\tilde{X}$ is normal, $\mathrm{Aut}(\tilde{X}) \cong \pi_1(X)/H$, and if $\tilde{X}$ is the universal cover, $\mathrm{Aut}(\tilde{X}) = \pi_1(X)$.

**Fact 11.2.7**
There is a contravariant bijective correspondence

$$\left\{ \substack{\text{Connected covering spaces} \\ p:\tilde{X} \to X} \right\}_{/\sim} \rightleftharpoons \left\{ \substack{\text{Conjugacy classes of subgroups} \\ \text{of } \pi_1(X)} \right\}.$$

If one fixes $\tilde{x}_0$ as a basepoint for $\pi_1(\tilde{X})$, this yields

$$\left\{ \substack{\text{Connected covering spaces} \\ p:\tilde{X} \to X} \right\}_{/\sim} \rightleftharpoons \left\{ \text{Subgroups of } \pi_1(X) \right\}.$$

**Proposition 11.2.8** ***(Number of sheets in a covering space, Hatcher 1.32).***
For $X, \tilde{X}$ both path-connected, the number of sheets of a covering space is equal to the index

$$[p^*(\pi_1(\tilde{X})) : \pi_1(X)].$$

Note that the number of sheets is always equal to the cardinality of $p^{-1}(x_0)$.

### 11.2.1 Examples

**Example 11.2.9** ***(The circle $S^1$):*** Identify $S^1 \subset \mathbb{C}$, then every map $p_n : S^1 \to S^1$ given by $z \mapsto z^n$ a yields a covering space $\tilde{X}_n$. The induced map can be described on generators as

$$\begin{aligned} p_n^* : \pi_1(S^1) &\to \pi_1(S^1) \\ [\omega_1] &\mapsto [\omega_n] = n[\omega_1] \end{aligned}$$

and so the image is isomorphic to $n\mathbb{Z}$ and thus

$$p_n^*(\pi_1(S^1)) = \mathrm{Aut}_{\mathrm{Cov}}(\tilde{X}_n) = \mathbb{Z}/n\mathbb{Z}.$$

where the deck transformations are rotations of the circle by $2\pi/n$. The universal cover of $S^1$ is $\mathbb{R}$; this is an infinitely sheeted cover, and the fiber above $x_0$ has cardinality $|\mathbb{Z}|$.

**Example 11.2.10** ***(Projective $n$-space $\mathbb{RP}^n$):*** The universal cover of $\mathbb{RP}^n$ is $S^n$; this is a two-sheeted cover. The fiber above $x_0$ contains the two antipodal points.

**Example 11.2.11** ***(The torus):*** The universal cover of $T = S^1 \times S^1$ is $\tilde{X} = \mathbb{R} \times \mathbb{R}$. The fiber above the base point contains every point on the integer lattice $\mathbb{Z} \times \mathbb{Z} = \pi_1(T) = \mathrm{Aut}(\tilde{X})$

**Fact 11.2.12**
For a wedge product $X = \bigvee_i^n \tilde{X}_i$, the covering space $\tilde{X}$ is constructed as a infinite tree with $n$-colored vertices:

- Each vertex corresponds to one of the universal covers $\tilde{X}_i$,
- The color corresponds to which summand $\tilde{X}_i$ appears,
- T The neighborhood of each colored vertex has edges corresponding (not bijectively) to generators of $\pi_1(X_i)$.

**Example 11.2.13*(Covering spaces of wedges of spheres):*** The fundamental group of $S^1 \vee S^1$ is $\mathbb{Z} * \mathbb{Z}$, and the universal cover is the following 4-valent Cayley graph:

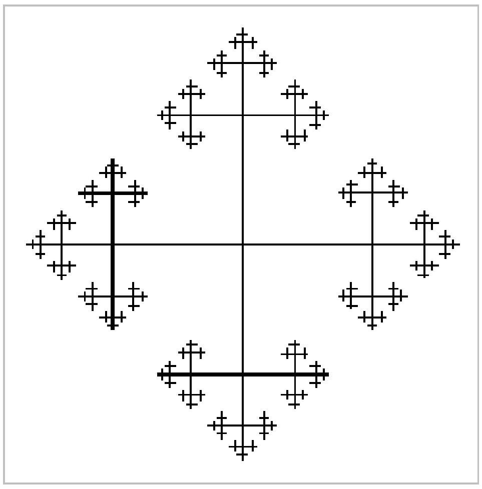


Figure 5: The universal cover of $\S^1 \vee S^1$

See Hatcher p.58 for other covers.

**Corollary 11.2.14*(Every subgroup of a free group is free).***

Idea for a particular case: use the fact that $\pi_1(\bigvee^k S^1) = \mathbb{Z}^{*k}$, so if $G \leq \mathbb{Z}^{*k}$ then there is a covering space $X \twoheadrightarrow \bigvee^k S^1$ such that $\pi_1(X) = G$. Since $X$ can be explicitly constructed as a graph, i.e. a CW complex with only a 1-skeleton, $\pi_1(X)$ is free on its maximal tree. ■

**Example 11.2.15*(of a universal covering space):*** The fundamental group of $\mathbb{RP}^2 \vee \mathbb{RP}^2$ is $\mathbb{Z}_2 * \mathbb{Z}_2$, corresponding to an infinite string of copies of 2-valent $S^2$s:

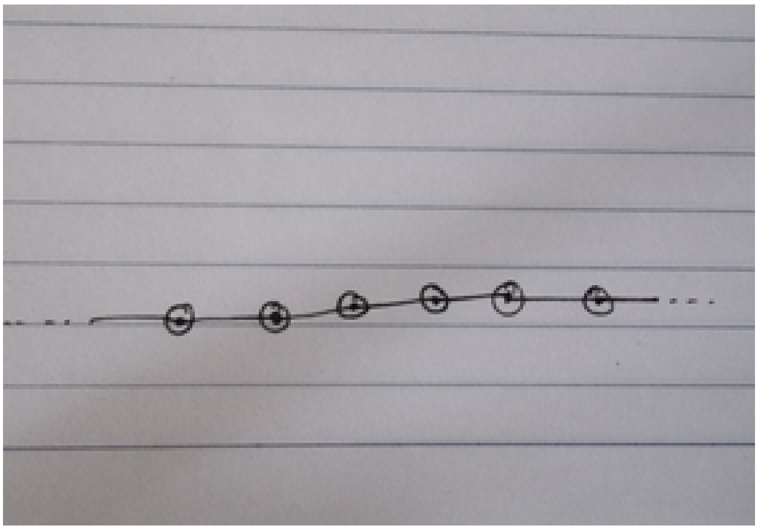


Figure 6: Another universal cover.

**Example 11.2.16** ***(of a universal covering space):*** The fundamental group of $\mathbb{RP}^2 \vee T^2$ is $\mathbb{Z}_2 * \mathbb{Z}$, and the universal cover is shown in the following image. Each red vertex corresponds to a copy of $S^2$ covering $\mathbb{RP}^2$ (having exactly 2 neighbors each), and each blue vertex corresponds to $\mathbb{R}^2$ cover $\mathbb{T}^2$, with $\left|\mathbb{Z}^2\right|$ many vertices as neighbors.

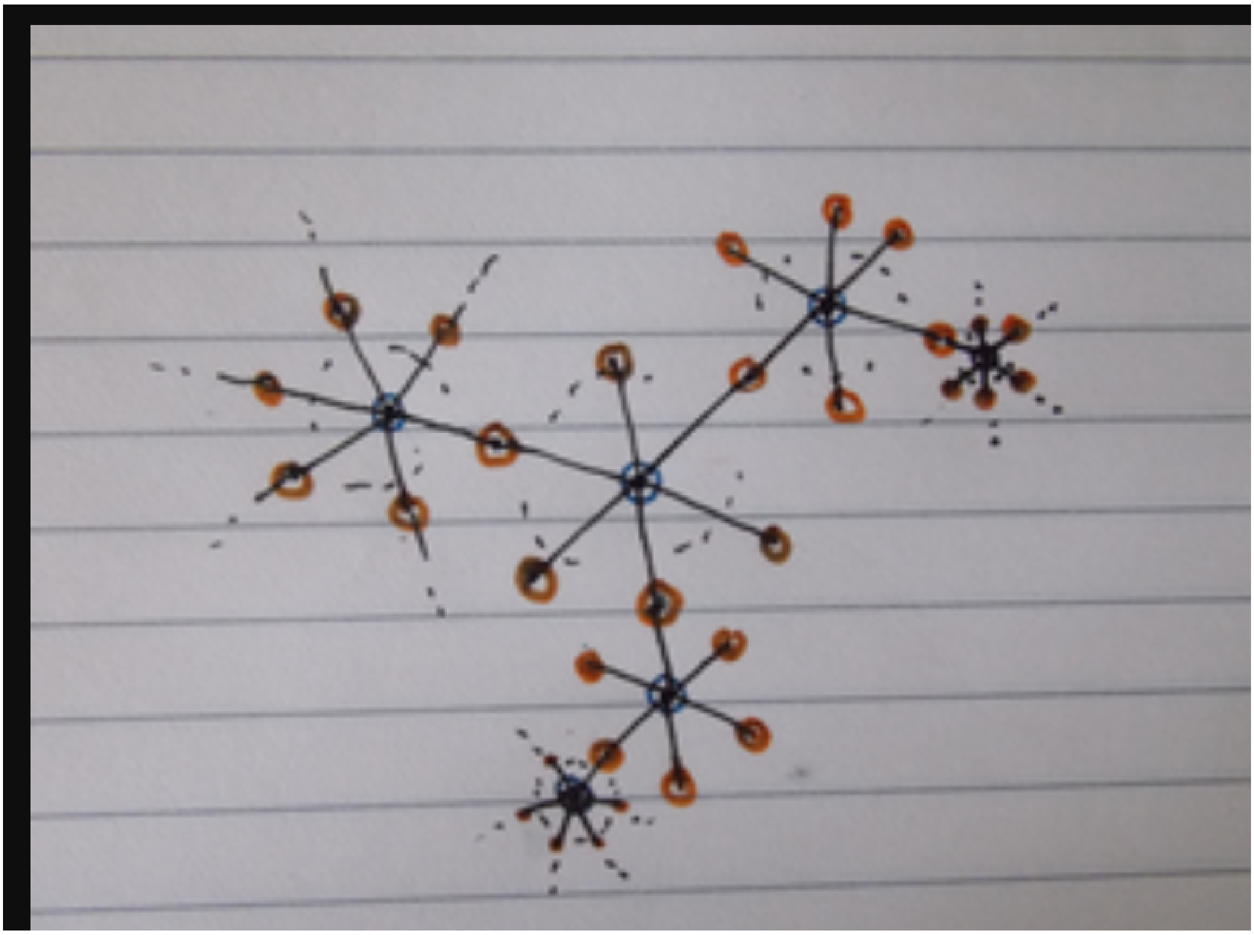


Figure 7: Universal cover of $\mathbb{T}^2 \vee \mathbb{RP}^2$

### 11.2.2 Applications

**Theorem 11.2.17** ***(Maps into contractible spaces are always nullhomotopic).***
If $X$ is contractible, every map $f : Y \to X$ is nullhomotopic.

*Proof (?).*
If $X$ is contractible, there is a homotopy $H : X \times I \to X$ between $\mathrm{id}_X$ and a constant map $c : x \mapsto x_0$. So construct

$$H' : Y \times I \to X$$

$$H'(y,t) := \begin{cases} H(f(y),0) = (\mathrm{id}_X \circ f)(y) = f(y) & t = 0 \\ H(f(y),1) = (c \circ f)(y) = c(y) = x_0 & t = 1 \\ H(f(y),t) & \text{else.} \end{cases}$$

Then $H'$ is a homotopy between $f$ and a constant map, and $f$ is nullhomotopic.

■

**Corollary 11.2.18** ***(Factoring through a contractible space implies nullhomotopic).***
Any map $f : X \to Y$ that factors through a contractible space $Z$ is nullhomotopic.

*Proof (?).*
We have the following situation where $f = p \circ \tilde{f}$:

$$\begin{tikzcd} & Z \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

*Link to diagram*

Since every map into a contractible space is nullhomotopic, there is a homotopy $\tilde{H} : Y \times I \to Z$ from $\tilde{f}$ to a constant map $c : Y \to Z$, say $c(y) = z_0$ for all $y$. But then $p \circ \tilde{H} : X \times I \to Y$ is also a homotopy from $f$ to the map $p \circ c$, which satisfies $(p \circ c)(y) = p(z_0) = x_0$ for some $x_0 \in X$, and is in particular a constant map.

■

**Proposition 11.2.19** ***(Application: showing one space can not cover another).***
There is no covering map $p : \mathbb{RP}^2 \to \mathbb{T}^2$.

*Proof (?).*

- Use the fact that $\pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2$ and $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$ are known.
- The universal cover of $\mathbb{T}^2$ is $\mathbb{R}^2$, which is contractible.
- Using the following two facts, $p_*$ is the trivial map:
  - By the previous results, $p$ is thus nullhomotopic.
  - Since $p$ is a covering map, $p_* : \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}^2$ is injective.
- Since $p$ was supposed a cover, this can be used to imply that $\mathrm{id}_{\mathbb{T}^2}$ is nullhomotopic.
- Covering maps induce injections on $\pi_1$, and the only way the trivial map can be injective is if $\pi_1(T^2) = 0$, a contradiction.

■

**Theorem 11.2.20** ***(When actions yield covering maps onto their quotients, Hatcher 1.40).***
If $G \curvearrowright X$ is a free and properly discontinuous action, then

1. The quotient map $p : X \to X/G$ given by $p(y) = Gy$ is a normal covering space,
2. If $X$ is path-connected, then $G = \mathrm{Aut}_{\mathrm{Cov}}(X)$ is the group of deck transformations for the cover $p$,

3. If $X$ is path-connected and locally path-connected, then $G \cong \pi_1(X/G)/p_*(\pi_1(X))$.

**Fact 11.2.21** (Some common covering spaces)

$$\begin{tikzcd} \mathbb{Z} \arrow[r] & \mathbb{R} \arrow[d] \\ & S^1 \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z}^n \arrow[r] & \mathbb{R}^n \arrow[d] \\ & \mathbb{T}^n \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z}/2\mathbb{Z} \arrow[r] & S^n \arrow[d] \\ & \mathbb{RP}^n \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z}^{*n} \arrow[r] & \mathrm{Cayley}(n) \arrow[d] \\ & \bigvee_n S^1 \end{tikzcd}$$

given by the $n$-valent Cayley graph covering a wedge of circles.

- $T^2 \xrightarrow{\times 2} \mathbb{K}$
- $\mathbb{Z}/q\mathbb{Z} \to L_{p/q} \xrightarrow{\pi} S^3$
- $\mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^* \xrightarrow{z^n} \mathbb{C}$

# 12 | CW and Simplicial Complexes

Missing a lot on CW complexes

## 12.1 Degrees

**Fact 12.1.1** (Useful properties of the degree of a map between spheres)

- deg $\mathrm{id}_{S^n} = 1$
- $\deg(f \circ g) = \deg f \cdot \deg g$
- deg $r = -1$ where $r$ is any rotation about a hyperplane, i.e. $r([x_1 \cdots x_i \cdots x_n]) = [x_1 \cdots - x_i \cdots x_n]$.
- The antipodal map on $S^n \subset \mathbb{R}^{n+1}$ is the composition of $n+1$ reflections, so deg $\alpha = (-1)^{n+1}$.

## 12.2 Examples of CW Complexes/Structures

**Example 12.2.1** ***(Spheres):*** $S^n = e^0 \cup e^n$: a point and an $n$-cell.

**Example 12.2.2** ***(Real Projective Space):*** $\mathbb{RP}^n = e^1 \cup e^2 \cup \cdots \cup e^n$: one cell in each dimension.

**Example 12.2.3** ***(Complex Projective Space):*** $\mathbb{CP}^n = e^2 \cup e^4 \cup \cdots e^{2n}$

**Examples 4.17.** The common surfaces $\mathbb{S}^2$, $\mathbb{T}^2$, $K$ and $\mathbb{P}^2$ all have presentations:

(1) The sphere: $\langle a \mid aa^{-1} \rangle$ or $\langle a, b \mid abb^{-1}a^{-1} \rangle$
(2) The torus: $\langle a, b \mid aba^{-1}b^{-1} \rangle$
(3) The projective plane: $\langle a \mid aa \rangle$ or $\langle a, b \mid abab \rangle$
(4) The Klein Bottle: $\langle a, b \mid abab^{-1} \rangle$

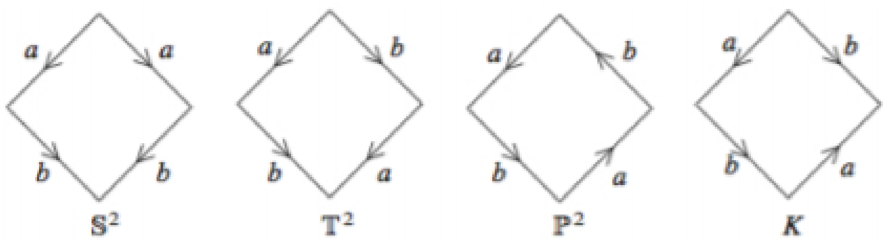



FIGURE 14. Polygonal presentation of $\mathbb{S}^2$, $\mathbb{T}^2$, $\mathbb{P}^2$, and $K$.

Figure 8: Fundamental domains

**Example 12.2.4** ***(Surfaces):***

## 12.3 Examples of Simplicial Complexes

**Remark 12.3.1:** To write down a simplicial complex, label the vertices with increasing integers. Then each $n$-cell will correspond to a set of $n+1$ of these integers - throw them in a list.

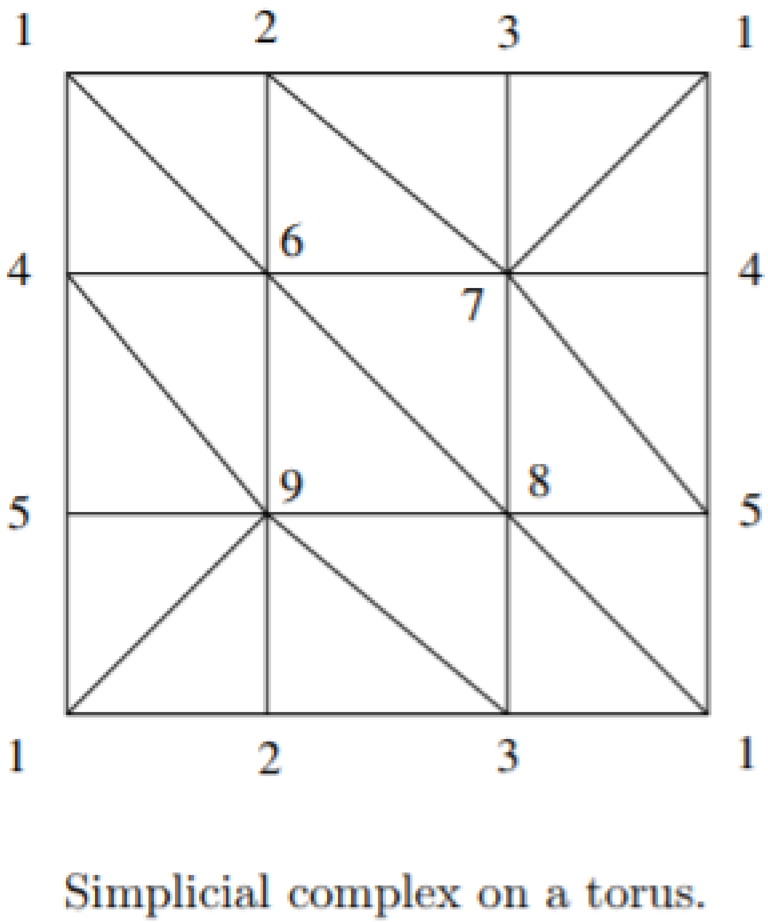



Figure 9: Torus

**Example 12.3.2** ***(Torus):***

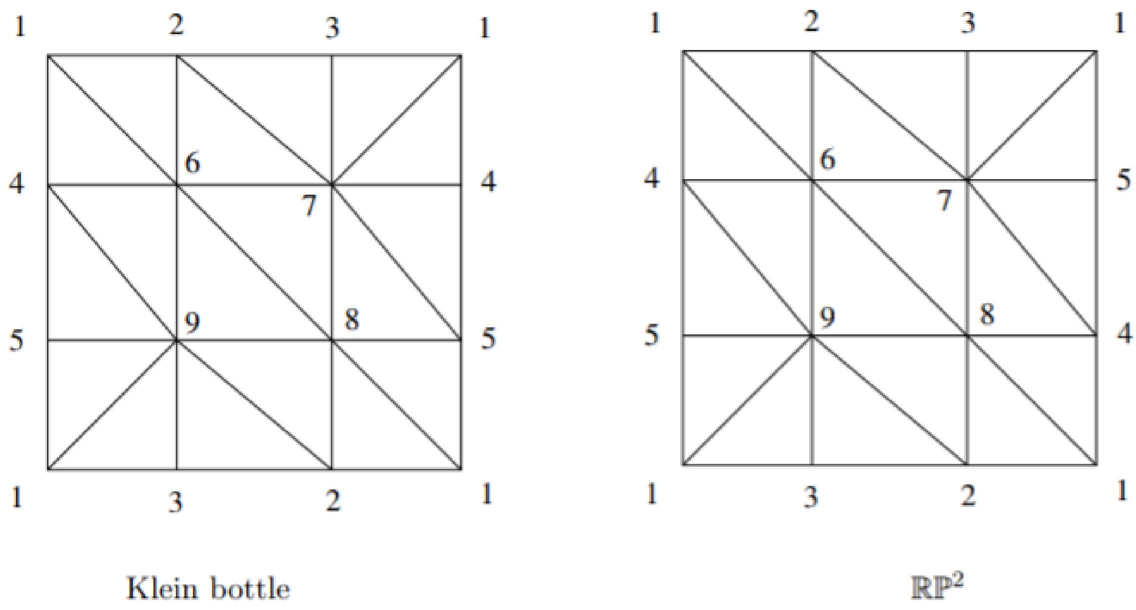



Figure 10: Klein Bottle and $\mathbb{RP}^2$

**Example 12.3.3** ***(Klein Bottle and*** $\mathbb{RP}^2$***):***

**Example 12.3.4** ***(Non-example):*** For counterexamples, note that this fails to be a triangulation of $T$:

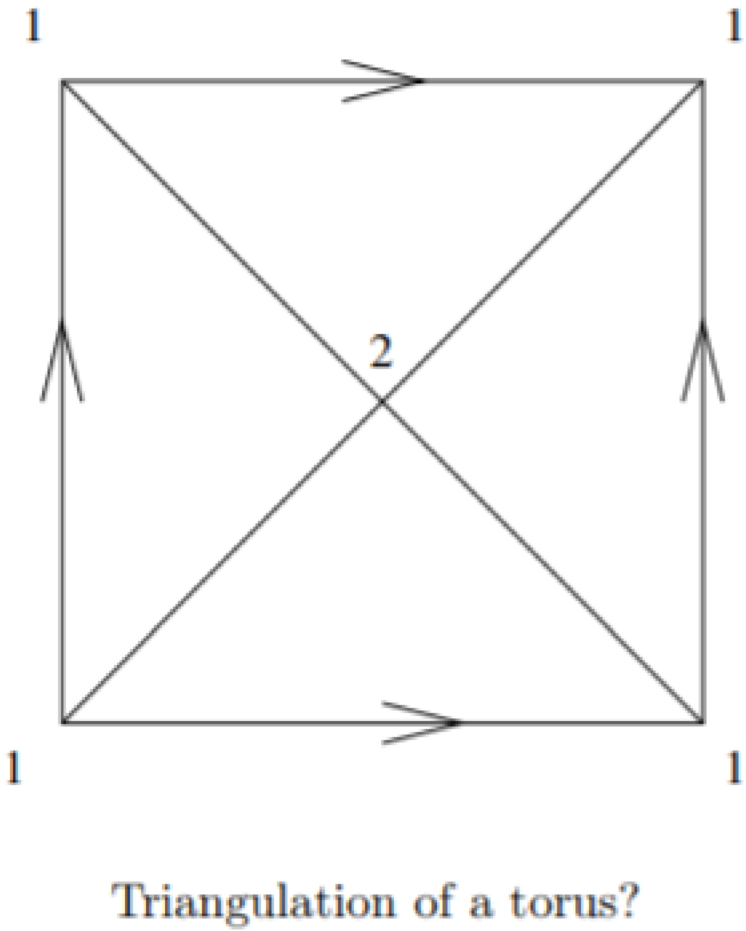



Figure 11: Not a Torus

This fails - for example, the specification of a simplex $[1,2,1]$ does not uniquely determine a triangle in the this picture.

## 12.4 Cellular Homology

- $S^n$ has the CW complex structure of 2 $k$-cells for each $0 \le k \le n$.

How to compute:

1. Write cellular complex

$$0 \to C^n \to C^{n-1} \to \cdots C^2 \to C^1 \to C^0 \to 0$$

2. Compute differentials $\partial_i : C^i \to C^{i-1}$

3. *Note: if $C^0$ is a point, $\partial_1$ is the zero map.*

4. *Note:* $H_n X = 0 \iff C^n = \varnothing$.

5. Compute degrees: Use $\partial_n(e_i^n) = \sum_i d_i e_i^{n-1}$ where

$$d_i = \deg(\text{Attach } e_i^n \to \text{Collapse } X^{n-1}\text{-skeleton}),$$

   which is a map $S^{n-1} \to S^{n-1}$.

Alternatively, choose orientations for both spheres. Then pick a point in the target, and look at points in the fiber. Sum them up with a weight of +1 if the orientations match and -1 otherwise.

6. Note that $\mathbb{Z}^m \xrightarrow{f} \mathbb{Z}^n$ has an $n \times m$ matrix

7. Row reduce, image is span of rows with pivots. Kernel can be easily found by taking RREF, padding with zeros so matrix is square and has all diagonals, then reading down diagonal - if a zero is encountered on $n$th element, take that column vector as a basis element with $-1$ substituted in for the $n$th entry.

For example:

$$\begin{array}{cccc} \mathbf{1} & 2 & 0 & 2 \\ 0 & 0 & \mathbf{1} & -1 \\ 0 & 0 & 0 & \mathbf{0} \end{array} \to \begin{array}{cccccccc} \mathbf{1} & 2 & 0 & 2 & \mathbf{1} & 2 & 0 & 2 \\ 0 & \mathbf{0} & 0 & 0 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & \mathbf{1} & -1 & 0 & 0 & \mathbf{1} & -1 \\ 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & \mathbf{0} \end{array}$$

$$\ker = \begin{array}{cc} 2 & 3 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{array}$$

$$\text{im} = \langle a + 2b + 2d, c - d \rangle .$$

6. Or look at elementary divisors, say $n_i$, then the image is isomorphic to $\bigoplus n_i \mathbb{Z}$

## 12.5 Constructing a CW Complex with Prescribed Homology

Given $G = \bigoplus G_i$, and want a space such that $H_i X = G$? Construct $X = \bigvee X_i$ and then $H_i(\bigvee X_i) = \bigoplus H_i X_i$. Reduces problem to: given a group $H$, find a space $Y$ such that $H_n(Y) = G$. By the structure theorem of finitely generated abelian groups, it suffices to know how to do this for $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$, since their powers are just obtained by wedging (previous remark). Recipe:

1. Attach an $e^n$ to a point to get $H_n = \mathbb{Z}$

2. Attach an $e^{n+1}$ with attaching map of degree $d$ to get $H_n = \mathbb{Z}_d$

# 13 | Homology

## 13.1 Useful Facts

**Fact 13.1.1**
$H_0(X)$ is a free abelian group on the set of path components of $X$. Thus if $X$ is path connected, $H_0(X) \cong \mathbb{Z}$. In general, $H_0(X) \cong \mathbb{Z}^{|\pi_0(X)|}$, where $|\pi_0(X)|$ is the number of path components of $X$.

**Proposition 13.1.2** ***(Homology commutes with wedge products).***

$$\tilde{H}_*(A \vee B) \cong H_*(A) \times H_*(B)$$
$$H_n\left(\bigvee_\alpha X_\alpha\right) \cong \prod_\alpha H_n X_\alpha$$

See footnote for categorical interpretation.[a]

[a] $\bigvee$ is the coproduct in the category $\mathbf{Top}_0$ of pointed topological spaces, and alternatively, $X \vee Y$ is the pushout in $\mathbf{Top}$ of $X \leftarrow \{\mathrm{pt}\} \rightarrow Y$

May need some good pair condition?

**Example 13.1.3** ***(Application):***

$$H_n(\bigvee_k S^n) = \mathbb{Z}^k.$$

*Proof (?).*
Mayer-Vietoris. ∎

**Warning 13.1.4**
$H_k\left(\prod_\alpha X_\alpha\right)$ is **not** generally equal to $\prod_\alpha (H_k X_\alpha)$. The obstruction is due to torsion – if all groups are torsionfree, then the Kunneth theorem[3] yields

$$H_k(A \times B) = \prod_{i+j=k} H_i A \otimes H_j B$$

[3] The generalization of Kunneth is as follows: write $\mathcal{P}(n,k)$ be the set of partitions of $n$ into $k$ parts, i.e. $\mathbf{x} \in \mathcal{P}(n,k) \implies \mathbf{x} = (x_1, x_2, \dots, x_k)$ where $\sum x_i = n$. Then

$$H_n\left(\prod_{j=1}^k X_j\right) = \bigoplus_{\mathbf{x} \in \mathcal{P}(n,k)} \bigotimes_{i=1}^k H_{x_i}(X_i).$$

**Theorem 13.1.5** ***(Excision).***
Todo

Excision.

:::{.fact title="Assorted facts}

- $H_n(X) = 0 \iff X$ has no $n$-cells.
- $C^0 X = \{\mathrm{pt}\} \implies d_1 : C^1 \to C^0$ is the zero map. :::

## 13.2 Known Homology

**Example 13.2.1** ***(Spheres):***

$$H_i(S^n) = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{else.} \end{cases}$$

**Example 13.2.2** ***(Real Projective Spaces):***
**Example 13.2.3** ***(Complex Projective Spaces):***
**Example 13.2.4** ***(Surfaces):***

Homology examples.

## 13.3 Mayer-Vietoris

**Fact 13.3.1** (Useful algebra fact)
Since $\mathbb{Z}$ is free and thus projective, any exact sequence of the form $0 \to \mathbb{Z}^n \to A \to \mathbb{Z}^m \to 0$ splits and $A \cong \mathbb{Z}^n \times \mathbb{Z}^m$.

**Theorem 13.3.2** ***(Mayer-Vietoris).***

*Mnemonic:* $X = A \cup B \rightsquigarrow (\cap, \oplus, \cup)$

Let $X = A^\circ \cup B^\circ$; then there is a SES of chain complexes

$$0 \to C_n(A \cap B) \xrightarrow{x \mapsto (x, -x)} C_n(A) \oplus C_n(B) \xrightarrow{(x,y) \mapsto x+y} C_n(A+B) \to 0$$

where $C_n(A+B)$ denotes the chains that are sums of chains in $A$ and chains in $B$. This yields a LES in homology:

$$\cdots H_n(A \cap B) \xrightarrow{(i^*,\ j^*)} H_n(A) \oplus H_n(B) \xrightarrow{l^* - r^*} H_n(X) \xrightarrow{\delta} H_{n-1}(A \cap B) \cdots$$

where

- $i : A \cap B \hookrightarrow A$ induces $i^* : H_*(A \cap B) \to H_*(A)$
- $j : A \cap B \hookrightarrow B$ induces $j^* : H_*(A \cap B) \to H_*(B)$
- $l : A \hookrightarrow A \cup B$ induces $l^* : H_*(A) \to H_*(X)$
- $r : B \hookrightarrow A \cup B$ induces $r^* : H_*(B) \to H_*(X)$

More explicitly,

$$
\begin{tikzcd}
 & & \cdots \arrow[dll, "\delta_3"'] \\
H_2(A \cap B) \arrow[r, "(i^*{,}-j^*)_2"] & H_2A \oplus H_2B \arrow[r, "(l^*-r^*)_2"] & H_2(A \cup B) \arrow[dll, "\delta_2"'] \\
H_1(A \cap B) \arrow[r, "(i^*{,}-j^*)_1"] & H_1A \oplus H_1B \arrow[r, "(l^*-r^*)_1"] & H_1(A \cup B) \arrow[dll, "\delta_1"'] \\
H_0(A \cap B) \arrow[r, "(i^*{,}-j^*)_0"] & H_0A \oplus H_0B \arrow[r, "(l^*-r^*)_0"] & H_0(A \cup B) \arrow[dll, "\delta_0"'] \\
0 & &
\end{tikzcd}
$$

The connecting homomorphisms $\delta_n : H_n(X) \to H_{n-1}(X)$ are defined by taking a class $[\alpha] \in H_n(X)$, writing it as an $n$-cycle $z$, then decomposing $z = \sum c_i$ where each $c_i$ is an $x + y$ chain. Then $\partial(c_i) = \partial(x + y) = 0$, since the boundary of a cycle is zero, so $\partial(x) = -\partial(y)$. So then just define $\delta([\alpha]) = [\partial x] = [-\partial y]$.
Handy mnemonic diagram:

$$
\begin{array}{ccccc}
 & & A \cap B & & \\
 & \swarrow & & \searrow & \\
A \cup B & & \longleftarrow & & A \oplus B
\end{array}.
$$

**Example 13.3.3** ***(Application: computing the homology of a connect sum):*** $H_*(A \# B)$: Use the fact that $A \# B = A \cup_{S^n} B$ to apply Mayer-Vietoris.

**Proposition 13.3.4** ***(Application: isomorphisms in the homology of spheres).***

$$H^i(S^n) \cong H^{i-1}(S^{n-1}).$$

*Proof .*
Write $X = A \cup B$, the northern and southern hemispheres, so that $A \cap B = S^{n-1}$, the equator. In the LES, we have:

$$H^{i+1}(S^n) \to H^i(S^{n-1}) \to H^iA \oplus H^iB \to H^iS^n \to H^{i-1}(S^{n-1}) \to H^{i-1}A \oplus H^{i-1}B.$$

But $A, B$ are contractible, so $H^iA = H^iB = 0$, so we have

$$H^{i+1}(S^n) \to H^i(S^{n-1}) \to 0 \oplus 0 \to H^i(S^n) \to H^{i-1}(S^{n-1}) \to 0.$$

In particular, we have the shape $0 \to A \to B \to 0$ in an exact sequence, which is always an isomorphism.

■

## 13.4 More Exact Sequences

**Theorem 13.4.1** ***(Kunneth).***
There exists a short exact sequence

$$0 \to \prod_{i+j=k} H_j(X;R) \otimes_R H_i(Y;R) \to H_k(X \times Y;R) \to \prod_{i+j=k-1} \operatorname{Tor}_R^1(H_i(X;R), H_j(Y;R)).$$

If $R$ is a free $R$-module, a PID, or a field, then there is a (non-canonical) splitting given by

$$H_k(X \times Y) \cong \left( \prod_{i+j=k} H_iX \oplus H_jY \right) \times \prod_{i+j=k-1} \operatorname{Tor}(H_iX, H_jY)$$

**Theorem 13.4.2** ***(UCT for Change of Group).***
For changing coefficients from $\mathbb{Z}$ to $G$ an arbitrary group, there are short exact sequences

$$0 \to \operatorname{Tor}_{\mathbb{Z}}^0(H_i(X;\mathbb{Z}), A) \to H_i(X;A) \to \operatorname{Tor}_{\mathbb{Z}}^1(H_{i-1}(X;\mathbb{Z}), A) \to 0$$
$$\Downarrow$$
$$0 \to H_iX \otimes G \to H_i(X;G) \to \operatorname{Tor}_{\mathbb{Z}}^1(H_{i-1}X, G) \to 0$$

and

$$0 \to \operatorname{Ext}_{\mathbb{Z}}^1(H_{i-1}(X;\mathbb{Z}), A) \to H^i(X;A) \to \operatorname{Ext}_{\mathbb{Z}}^0(H_i(X;\mathbb{Z}), A) \to 0$$
$$\Downarrow$$
$$0 \to \operatorname{Ext}(H_{i-1}X, G) \to H^i(X;G) \to \hom(H_iX, G) \to 0.$$

These split unnaturally:

$$\begin{aligned} H_i(X;G) &= (H_{iX} \otimes G) \oplus \operatorname{Tor}(H_{i-1}X;G) \\ H^i(X;G) &= \hom(H_iX, G) \oplus \operatorname{Ext}(H_{i-1}X;G) \end{aligned}$$

When all of the $H_iX$ are all finitely generated (e.g. if $G$ is a field), writing $H_i(X;\mathbb{Z}) = \mathbb{Z}^{\beta_i} \oplus T_i$ as the sum of a free and a torsionfree module, we have

$$\begin{aligned} H^i(X;\mathbb{Z}) &\cong \mathbb{Z}^{\beta_i} \times T_{i-1} \\ H^i(X;A) &\cong (H_i(X;G))^\vee \coloneqq \hom_{\mathbb{Z}}(H_i(X;G), G). \end{aligned}$$

In other words, letting $F(\cdot)$ be the free part and $T(\cdot)$ be the torsion part, we have

$$\begin{aligned} H^i(X;\mathbb{Z}) &= F(H_i(X;\mathbb{Z})) \times T(H_{i-1}(X;\mathbb{Z})) \\ H_i(X;\mathbb{Z}) &= F(H^i(X;\mathbb{Z})) \times T(H^{i+1}(X;\mathbb{Z})) \end{aligned}$$

Might need assumptions: finite CW complex?

## 13.5 Relative Homology

**Fact 13.5.1** (Some assorted facts)

- $H_n(X/A) \cong \tilde{H}_n(X,A)$ when $A \subset X$ has a neighborhood that deformation retracts onto it.
- LES of a pair
  - $(A \hookrightarrow X) \mapsto (A, X, X/A)$
- For CW complexes $X = \left\{X^{(i)}\right\}$, we have

$$H_n(X^{(k)}, X^{(k-1)}) \cong \begin{cases} \mathbb{Z}[\{e^n\}] & k = n, \\ 0 & \text{otherwise} \end{cases} \qquad \text{since } X^k/X^{k-1} \cong \bigvee S^k$$

- $H_n(X,A) \cong_? H_n(X/A, \{\mathrm{pt}\})$

# 14 Fixed Points and Degree Theory

**Theorem 14.0.1** ***(Lefschetz Fixed Point).***
For $f : X \to X$, define the **trace** of $f$ to be

$$\Lambda_f := \sum_{k \geq 0} (-1)^k \operatorname{Tr}(f_* \Big| H_k(X; \mathbb{Q}))$$

where $f_* : H_k(X; \mathbb{Q}) \to H_k(X; \mathbb{Q})$ is the induced map on homology. If $\Lambda_f \neq 0$ then $f$ has a fixed point.

**Theorem 14.0.2** ***(?).***
Every $f : B^n \to B^n$ has a fixed point.

*Proof (?).* ∎

Proof

**Theorem 14.0.3** ***(Hairy Ball).***
There is no non-vanishing tangent vector field on even dimensional spheres $S^{2n}$.

**Theorem 14.0.4** ***(Borsuk-Ulam).***
For every $S^n \xrightarrow{f} \mathbb{R}^n \exists x \in S^n$ such that $f(x) = f(-x)$.

# 15 Surfaces and Manifolds

**Remark 15.0.1:** The most common spaces appearing in this theory:

- $\mathbb{S}^2$,
- $\mathbb{T}^2 := S^1 \times S^1$,
- $\mathbb{RP}^2$
- $\mathbb{K}$ the Klein bottle
- $\mathbb{M}$ the Möbius Strip
- $\Sigma_n := \#_{i=1}^n \mathbb{T}^2$.

The first 4 can be obtained from the following pasting diagrams:

# Instructions for making common surfaces

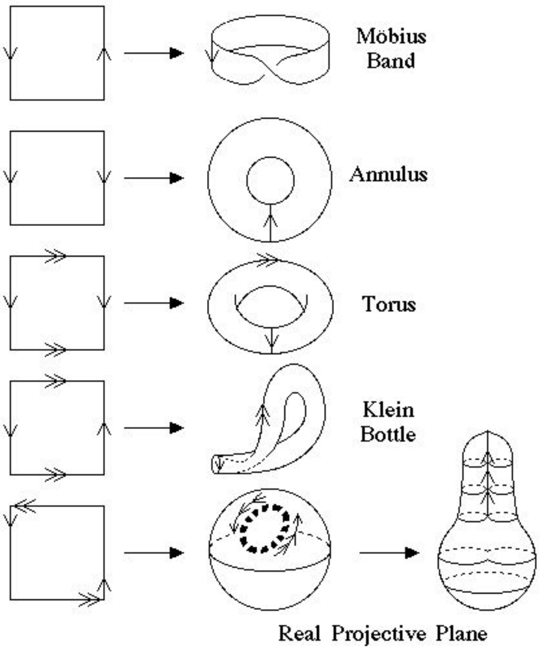



Figure 12: Pasting Diagrams for Surfaces

## 15.1 Classification of Surfaces

**Theorem 15.1.1** ***(Classification of Surfaces).***
The set of surfaces under connect sum forms a monoid with the presentation

$$\left\langle \mathbb{S}^2, \mathbb{RP}^2, \mathbb{T} \;\middle|\; \mathbb{S}^2 = 0, 3\mathbb{RP}^2 = \mathbb{RP}^2 + \mathbb{T}^2 \right\rangle.$$

Surfaces are classified up to homeomorphism by orientability and $\chi$, or equivalently "genus"

- In orientable case, actual genus, $g$ equals the number of copies of $\mathbb{T}^2$.
- In nonorientable case, $k$ equals the number of copies of $\mathbb{RP}^2$.

In each case, there is a formula

$$\chi(X) = \begin{cases} 2 - 2g - b & \text{orientable} \\ 2 - k & \text{non-orientable.} \end{cases}$$

| Orientable? | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|---|---|
| Yes | $\Sigma_3$ | $\varnothing$ | $\Sigma_2$ | $\varnothing$ | $\mathbb{T}^2, S^1 \times I$ | $\mathbb{D}^2$ | $\mathbb{S}^2$ |
| No | ? | ? | ? | ? | $\mathbb{K}, \mathbb{M}$ | $\mathbb{RP}^2$ | $\varnothing$ |

**Fact 15.1.2**

**Proposition 15.1.3** ***(Inclusion-Exclusion).***

$$X = U \cup V \implies \chi(X) = \chi(U) + \chi(V) - \chi(U \cap V).$$

*Proof .*
Todo ■

Proof.

**Corollary 15.1.4** ***(Euler for Connect Sums).***

$$\chi(A \# B) = \chi(A) + \chi(B) - 2.$$

*Proof .*
Set $U = A, B = V$, then by definition of the connect sum, $A \cap B = \mathbb{S}^2$ where $\chi(\mathbb{S}^2) = 2$ ■

**Proposition 15.1.5** ***(Decomposing $\mathbb{RP}^2$).***

$$\mathbb{RP}^2 = \mathbb{M} \coprod_{\mathrm{id}_{\partial \mathbb{M}}} \mathbb{M}.$$

**Proposition 15.1.6** ***(Decomposing a Klein Bottle).***

$$\mathbb{K} \cong \mathbb{RP}^2 \# \mathbb{RP}^2.$$

*Proof .*
Todo ■

Proof.

**Proposition 15.1.7** ***(Rewriting a Klein Bottle).***

$$\mathbb{RP}^2 \# \mathbb{K} \cong \mathbb{RP}^2 \# \mathbb{T}^2.$$

*Proof .*
Todo ■

Proof.

## 15.2 Manifolds

**Remark 15.2.1:** To show something is not a manifold, try looking at local homology. Can use point-set style techniques like removing points, i.e. $H_1(X, X - \{\mathrm{pt}\})$; this should essentially always yield $\mathbb{Z}$ by excision arguments.

**Proposition 15.2.2** ***(Dimension vanishing for homology of manifolds).***
If $M^n$ is a closed and connected $n$-manifold, then $H^{\geq n} X = 0$.

**Proposition 15.2.3** ***(Top homology for manifolds).***
If $M^n$ is a closed connected manifold, then $H_n = \mathbb{Z}$ and $\mathrm{Tor}(H_{n-1}) = 0$. More generally,

$$\begin{cases} \mathbb{Z} & M^n \text{ is orientable} \\ 0 & \text{else.} \end{cases}$$

**Proposition 15.2.4** ***(Poincaré Duality for manifolds).***
For $M^n$ a closed orientable manifold without boundary and $\mathbb{F}$ a field,

$$H_k(M^n; \mathbb{F}) \cong H^{n-k}(M^n; \mathbb{F}) \iff M^n \text{ is closed and orientable.}$$

**Proposition 15.2.5** ***(Relative Poincaré Duality for manifolds).***
If $M^n$ is a closed orientable manifold with boundary then

$$H_k(M^n; \mathbb{Z}) \cong H^{n-k}(M^n, \partial M^n; \mathbb{Z}).$$

**Proposition 15.2.6** ***(Known Euler characteristics).***
If $M^n$ is closed and $n$ is odd, then $\chi(M^n) = 0$.

*Proof (?).*
Todo. Uses Poincaré duality? ■

Proof!

**Proposition 15.2.7** ***(Nondegenerate intersection pairings).***
For $M^n$ closed and orientable, the intersection pairing is nondegenerate modulo torsion.

**Proposition 15.2.8** ***(Orientation covers).***
For any manifold $X$ there exists a covering space $p : \tilde{X}_o \to X$, the **orientation cover**, where any map $Y \to X$ factors through $\tilde{X}_o$. If $X$ is nonorientable, then $p$ is a double cover.

**Theorem 15.2.9** ***(Lefschetz Duality).***
Todo

Statement of Lefschetz duality.

### 15.2.1 3-Manifolds, and Knot Complements

**Fact 15.2.10**
Every $\mathbb{C}$-manifold is canonically orientable.

**Proposition 15.2.11** ***(Homology of 3-manifolds).***
Let $M^3$ be a 3-manifold, then its homology is given by the following (by cases):

- Orientable: $H_* = (\mathbb{Z}, \mathbb{Z}^r, \mathbb{Z}^r, \mathbb{Z})$
- Nonorientable: $H_* = (\mathbb{Z}, \mathbb{Z}^r, \mathbb{Z}^{r-1} \oplus \mathbb{Z}_2, \mathbb{Z})$

**Proposition 15.2.12** ***(Homotopy type of knot complements).***
For $K$ a knot, $S^3 \setminus K$ is a $K(\pi, 1)$, and $\mathbb{R}^3 \setminus K \simeq S^2 \vee \left(S^3 \setminus K\right)$. Moreover, if $K$ is nullhomologous and $X$ is any 3-manifold,

$$H_1\left(X \setminus \nu(K)\right) \cong H_1 X \times \mathbb{Z}$$

where $\nu(K)$ is a tubular neighborhood of $K$.

*Proof (?).*
Todo

■

todo

**Proposition 15.2.13** *(Homology of knot complements in $S^3$).*
For $K$ a knot,

$$H_*(S^3 \setminus K) = [\mathbb{Z}, \mathbb{Z}, 0, 0, \cdots].$$

*Proof .*
Apply Mayer-Vietoris, taking $S^3 = n(K) \cup (S^3 - K)$, where $n(K) \simeq S^1$ and $S^3 - K \cap n(K) \simeq T^2$. Use the fact that $S^3 - K$ is a connected, open 3-manifold, so $H^3(S^3 - K) = 0$.

■

# 16 Extra Problems: Algebraic Topology

## 16.1 Homotopy 101

- Show that if $X \xrightarrow{f} X^n$ is not surjective, then $f$ is nullhomotopic.

## 16.2 $\pi_1$

- Compute $\pi_1(S^1 \vee S^1)$
- Compute $\pi_1(S^1 \times S^1)$

## 16.3 Surfaces

- Show that if $M^{\text{orientable}} \xrightarrow{\pi_k} M^{\text{non-orientable}}$ is a $k$-fold cover, then $k$ is even or $\infty$.
- Show that $M$ is orientable if $\pi_1(M)$ has no subgroup of index 2.

# 17 Fall 2014

## 17.1 1

Let $X = \mathbb{R}^3 - \Delta^{(1)}$, the complement of the skeleton of regular tetrahedron, and compute $\pi_1(X)$ and $H_*(X)$.

**Solution:**
Lay the graph out flat in the plane, then take a maximal tree - these leaves 3 edges, and so $\pi_1(X) = \mathbb{Z}^{*3}$.
Moreover $X \simeq S^1 \vee S^1 \vee S^1$ which has only a 1-skeleton, thus $H_*(X) = [\mathbb{Z}, \mathbb{Z}^3, 0 \to]$.

## 17.2 2

Let $X = S^1 \times B^2 - L$ where $L$ is two linked solid torii inside a larger solid torus. Compute $H_*(X)$.

Solution

## 17.3 3

Let $L$ be a 3-manifold with homology $[\mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z}, \dots]$ and let $X = L \times \Sigma L$. Compute $H_*(X), H^*(X)$.

**Solution:**
Useful facts:

- $H_k(X \times Y) \cong \bigoplus_{i+j=k} H_i(X) \otimes H_j(Y) \bigoplus_{i+j=k-1} \mathrm{Tor}(H_i(X), H_j(Y))$
- $\tilde{H}_i(\Sigma X) = \tilde{H}_{i-1}(X)$

We will use the fact that $H_*(\Sigma L) = [\mathbb{Z}, \mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z}]$.
Represent $H_*(L)$ by $p(x, y) = 1 + yx + x^3$ and $H_*(\Sigma L)$ by $q(x, y) = 1 + x + yx^2 + x^4$, we can extract the free part of $H_*(X)$ by multiplying

$$p(x, y)q(x, y) = 1 + (1 + y)x + 2yx^2 + (y^2 + 1)x^3 + 2x^4 + 2yx^5 + x^7$$

where multiplication corresponds to the tensor product, addition to the direct sum/product. So the free portion is

$$\begin{aligned} H_*(X) &= [\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}_3 \otimes \mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z}_3 \otimes \mathbb{Z}_3, \mathbb{Z}^2, \mathbb{Z}_3^2, 0, \mathbb{Z}] \\ &= [\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}^2, \mathbb{Z}_3^2, 0, \mathbb{Z}] \end{aligned}$$

We can add in the correction from torsion by noting that only terms of the form $\mathrm{Tor}(\mathbb{Z}_3, \mathbb{Z}_3) = \mathbb{Z}_3$ survive. These come from the terms $i = 1, j = 2$, so $i + j = k - 1 \implies k = 1 + 2 + 1 = 4$ and there is thus an additional torsion term appearing in dimension 4. So we have

$$\begin{aligned}H_*(X) &= [\mathbb{Z}, \mathbb{Z}\times\mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}\times\mathbb{Z}_3, \mathbb{Z}^2\times\mathbb{Z}_3, \mathbb{Z}_3^2, 0, \mathbb{Z}]\\ &= [\mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}^2, 0, 0, \mathbb{Z}]\times[0, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3^2, 0, 0]\end{aligned}$$

and

$$\begin{aligned}H^*(X) &= [\mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}^2, 0, 0, \mathbb{Z}]\times[0, 0, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3^2, 0]\\ &= [\mathbb{Z}, \mathbb{Z}, \mathbb{Z}_3, \mathbb{Z}\times\mathbb{Z}_3, \mathbb{Z}^2\times\mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3^2, \mathbb{Z}].\end{aligned}$$

## 17.4 4

Let $M$ be a closed, connected, oriented 4-manifold such that $H_2(M;\mathbb{Z})$ has rank 1. Show that there is not a free $\mathbb{Z}_2$ action on $M$.

**Solution:**
Useful facts:

- $X \twoheadrightarrow_{\times p} Y$ induces $\chi(X) = p\chi(Y)$
- Moral: always try a simple Euler characteristic argument first!

We know that $H_*(M) = [\mathbb{Z}, A, \mathbb{Z}\times G, A, \mathbb{Z}]$ for some group $A$ and some torsion group $G$. Letting $n = \operatorname{rank}(A)$ and taking the Euler characteristic, we have $\chi(M) = (1)1 + (-1)n + (1)1 + (-1)n + (1)1 = 3 - 2n$. Note that this is odd for any $n$.
However, a free action of $\mathbb{Z}_2 \curvearrowright M$ would produce a double covering $M \twoheadrightarrow_{\times 2} M/\mathbb{Z}_2$, and multiplicativity of Euler characteristics would force $\chi(M) = 2\chi(M/\mathbb{Z}_2)$ and thus $3 - 2n = 2k$ for some integer $k$. This would require $3 - 2n$ to be even, so we have a contradiction.

## 17.5 5

Let $X$ be $T^2$ with a 2-cell attached to the interior along a longitude. Compute $\pi_2(X)$.

**Solution:**
Useful facts:

- $T^2 = e^0 + e^1_1 + e^1_2 + e^2$ as a CW complex.
- $S^2/(x_0 \sim x_1) \simeq S^2 \wedge S^1$ when $x_0, x_1$ are two distinct points. (Picture: sphere with a string handle connecting north/south poles.)
- $\pi_{\geq 2}(\tilde{X}) \cong \pi_{\geq 2}(X)$ for $\tilde{X} \twoheadrightarrow X$ the universal cover.

Write $T^2 = e^0 + e^1_1 + e^1_2 + e^2$, where the first and second 1-cells denote the longitude and meridian respectively. By symmetry, we could have equivalently attached a disk to the meridian instead

of the longitude, filling the center hole in the torus. Contract this disk to a point, then pull it vertically in both directions to obtain $S^2$ with two points identified, which is homotopy-equivalent to $S^2 \vee S_1$.
Take the universal cover, which is $\mathbb{R}^1 \cup_{\mathbb{Z}} S^2$ and has the same $\pi_2$. This is homotopy-equivalent to $\bigvee_{i \in \mathbb{Z}} S^2$ and so $\pi_2(X) = \prod_{i \in \mathbb{Z}} \mathbb{Z}$ generated by each distinct copy of $S^2$. (Alternatively written as $\mathbb{Z}[t, t^{-1}]$).

# 18 | Summer 2003

## 18.1 1

Describe all possible covering maps between $S^2, T^2, K$

**Solution:**

**Concepts Used:**

1. $\tilde{X} \twoheadrightarrow X$ induces $\pi_1(\tilde{X}) \hookrightarrow \pi_1(X)$
2. $\chi(\tilde{X}) = n\chi(X)$
3. $\pi_n(X) = [S^n, X]$
4. $Y \to X$ with $\pi_1(Y) = 0$ and $\tilde{X} \simeq \{\mathrm{pt}\} \implies$ every $Y \xrightarrow{f} X$ is nullhomotopic.
5. $\pi_*(T^2) = [\mathbb{Z} * \mathbb{Z}, 0 \to]$
6. $\pi_*(K) = [\mathbb{Z} \rtimes_{\mathbb{Z}_2} \mathbb{Z}, 0 \to]$
7. Universal covers are homeomorphic.
8. $\pi_{\geq 2}(\tilde{X}) \cong \pi_{\geq 2}(X)$

Spaces

- $S^2 \twoheadrightarrow T^2$
- $S^2 \twoheadrightarrow K$
- $K \twoheadrightarrow S^2$
- $T^2 \twoheadrightarrow S^2$
  - All covered by the fact that
  $$\mathbb{Z} = \pi_2(S^2) \neq \pi_2(X) = 0$$
  for $X = T^2, K$.
- $K \twoheadrightarrow T^2$
  - Doesn't cover, would induce $\pi_1(K) \hookrightarrow \pi_1(T^2) \implies \mathbb{Z} \rtimes \mathbb{Z} \hookrightarrow \mathbb{Z}^2$ but this would be a non-abelian subgroup of an abelian group.

- $T^2 \twoheadrightarrow K$
  - ?

Not complete!

## 18.2 2

Show that $\mathbb{Z}^{*2}$ has subgroups isomorphic to $\mathbb{Z}^{*n}$ for every $n$.

**Solution:**

**Concepts Used:**

1. $\pi_1(\bigvee^k S^1) = \mathbb{Z}^{*k}$
2. $\tilde{X} \twoheadrightarrow X \implies \pi_1(\tilde{X}) \hookrightarrow \pi_1(X)$
3. Every subgroup $G \leq \pi_1(X)$ corresponds to a covering space $X_G \twoheadrightarrow X$
4. $A \subseteq B \implies F(A) \leq F(B)$ for free groups.

It is easier to prove the stronger claim that $\mathbb{Z}^{\mathbb{N}} \leq \mathbb{Z}^{*2}$ (i.e. the free group on countably many generators) and use fact 4 above. Just take the covering space $\tilde{X} \twoheadrightarrow S^1 \vee S^1$ defined via the gluing map $\mathbb{R} \cup_{\mathbb{Z}} S^1$ which attaches a circle to each integer point, taking 0 as the base point. Then let $a$ denote a translation and $b$ denote traversing a circle, so we have $\pi_1(\tilde{X}) = \langle \cup_{n \in \mathbb{Z}} a^n b a^{-n} \rangle$ which is a free group on countably many generators. Since $\tilde{X}$ is a covering space, $\pi_1(\tilde{X}) \hookrightarrow \pi_1(S^1 \vee S^1) = \mathbb{Z}^{*2}$. By 4, we can restrict this to $n$ generators for any $n$ to get a subgroup, and $A \leq B \leq C \implies A \leq C$ as groups.

## 18.3 3

Construct a space having $H_*(X) = [\mathbb{Z}, 0, 0, 0, 0, \mathbb{Z}_4, 0, \cdots]$.

**Solution:**

**Concepts Used:**

- Construction of Moore Spaces
- $\tilde{H}_n(\Sigma X) = \tilde{H}_{n-1}(X)$, using $\Sigma X = C_X \cup_X C_X$ and Mayer-Vietoris.

Take $X = e^0 \cup_{\Phi_1} e^5 \cup_{\Phi_2} e^6$, where

$$\Phi_1 : \partial B^5 = S^4 \xrightarrow{z \mapsto z^0} e^0$$
$$\Phi_2 : \partial B^6 = S^5 \xrightarrow{z \mapsto z^4} e^5.$$

where $\deg \Phi_2 = 4$.

## 18.4 4

Compute the complement of a knotted solid torus in $S^3$.

**Solution:**

**Concepts Used:**

- $H_*(T^2) = [\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}, 0 \to]$
- $N^{(1)} \simeq S^1$, so $H_{\geq 2}(N) = 0$.
- A SES $0 \to A \to B \to F \to 0$ with $F$ free splits.
- $0 \to A \to B \xrightarrow{\cong} C \to D \to 0$ implies $A = D = 0$.

Let $N$ be the knotted solid torus, so that $\partial N = T^2$, and let $X = S^3 - N$. Then

- $S^3 = N \cup_{T^2} X$
- $N \cap X = T^2$

and we apply Mayer-Vietoris to the reduced homology of $S^3$:

$$\begin{tikzcd}
H_4(T^2) \arrow[r] & H_4(N) \oplus H_4(X) \arrow[r] & H_4(S^3) \arrow[dll, out=0, in=180] \\
H_3(T^2) \arrow[r] & H_3(N) \oplus H_3(X) \arrow[r] & H_3(S^3) \arrow[dll, out=0, in=180] \\
H_2(T^2) \arrow[r] & H_2(N) \oplus H_2(X) \arrow[r] & H_2(S^3) \arrow[dll, out=0, in=180] \\
H_1(S^3) \arrow[r] & H_1(N) \oplus H_1(X) \arrow[r] & H_1(S^3)
\end{tikzcd}$$

We can plug in known information and deduce some maps:

$$
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
0 \arrow[r] & H_3(X) \arrow[r] & \mathbb{Z} \arrow[dll] \\
\mathbb{Z} \arrow[r] & H_2(X) \arrow[r] & 0 \arrow[dll] \\
\mathbb{Z}^{\oplus 2} \arrow[r, hook, two heads, "\sim"] & \mathbb{Z} \oplus H_1(X) \arrow[r] & 0
\end{tikzcd}
$$

We then deduce:

- $H_0(X) = \mathbb{Z}$: ? (Appeal to some path-connectedness argument?)
- $H_1(X) = \mathbb{Z}$ using the SES appearing on the first row:

$$0 \to \mathbb{Z}^{\oplus 2} \to \mathbb{Z} \oplus H_1(X) \to 0$$

  which is thus an isomorphism.

- $H_2(X) = H_3(X) = 0$ by examining the SES spanning lines 3 and 2:

$$0 \hookrightarrow H_3(X) \hookrightarrow \mathbb{Z} \xrightarrow{\cong \partial_3} \mathbb{Z} \twoheadrightarrow H_2(X) \twoheadrightarrow 0$$

  Claim: $\partial_3$ must be an isomorphism. If this is true, $H_3(X) \cong \ker \partial_3 = 0$ and $H_2(X) \cong \operatorname{coker}(\partial_3) \coloneqq \mathbb{Z}/\operatorname{im}(\partial_3) \cong \mathbb{Z}/\mathbb{Z} = 0$.

  Why is this true?

## 18.5 5

Compute the homology and cohomology of a closed, connected, oriented 3-manifold $M$ with $\pi_1(M) = \mathbb{Z}^{*2}$.

**Solution:**
Facts used:

- $M$ closed, connected, oriented $\implies H_i(M) \cong H^{n-i}(M)$
- $H_1(X) = \operatorname{Ab}(\pi_1(X))$.
- For orientable manifolds $H_n(M^n) = \mathbb{Z}$

**Homology**

- Since $M$ is connected, $H_0 = \mathbb{Z}$
- Since $\pi_1(M) = \mathbb{Z}^{*2}$, $H_1$ is the abelianization and $H_1(X) = \mathbb{Z}^2$
- Since $M$ is closed/connected/oriented, Poincaré Duality holds and $H_2 = H^{3-2} = H^1 = \mathbf{F}H_1 + \mathbf{T}H_0$ by UCT. Since $H_0 = \mathbb{Z}$ is torsion-free, we have $H_2(M) = H_1(M) = \mathbb{Z}^2$.
- Since $M$ is an orientable manifold, $H_3(M) = \mathbb{Z}$
- So $H_*(M) = [\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^2, \mathbb{Z}, 0 \to]$

**Cohomology**

- By Poincaré Duality, $H^*(M) = \widehat{H_*(M)} = [\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^2, \mathbb{Z}, 0, \cdots]$. (Where the hat denotes reversing the list.)

## 18.6 6

Compute $\mathrm{Ext}(\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3, \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/5)$.

**Solution:**

**Concepts Used:**

Facts Used:[a]

- Since $\mathbb{Z}$ is a free $\mathbb{Z}$-module,

$$\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}/m) = 0$$

- Using the usual projective resolution $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 0$,

$$\mathrm{Ext}(\mathbb{Z}/n, \mathbb{Z}) = \mathbb{Z}/n.$$

- $$\mathrm{Ext}(\mathbb{Z}/n, \mathbb{Z}/m) = (\mathbb{Z}/m)/(n \cdot \mathbb{Z}/m) \cong (\mathbb{Z}/m)/(d \cdot \mathbb{Z}/m) \quad \text{where } d \coloneqq \gcd(m, n).$$

  General principle: $\mathrm{Ext}(\mathbb{Z}/n, G) = G/nG$

  By applying $\mathrm{Hom}_{\mathbb{Z}}(\,\cdot\,, G)$ to the above resolution:

$$\begin{tikzcd}
 & & 0 & \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, G) \arrow[l] \\
\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, G) \arrow[urrr, two heads] & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, G) \arrow[l, "\cdot n"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, G) \arrow[l, hook'] & 0 \arrow[l]
\end{tikzcd}$$

*Link to Diagram*

which can be identified with:

$$\begin{tikzcd}
 & & 0 & G/nG \arrow[l] \\
G \arrow[urrr, two heads] & G \arrow[l, "\cdot n"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, G) \arrow[l, hook'] & 0 \arrow[l]
\end{tikzcd}$$

*Link to Diagram*

3. Contravariant Hom takes coproducts to products:

$$\mathrm{Ext}(\bigoplus_{i \in I} A_i, \prod_{k \in K} B_k) = \prod_{i \in I} \prod_{k \in K} \mathrm{Ext}(A_i, B_k).$$

[a] Thanks to Oskar Henriksson for some fixes/clarifications and further explanations here!

Write

$$\begin{aligned} A_\cdot &\coloneqq A_1 \oplus A_2 \oplus A_3 \coloneqq \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3 \\ B_\cdot &\coloneqq B_1 \oplus B_2 \oplus B_3 \coloneqq \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/5. \end{aligned}$$

We can then define the bicomplex

$$C_{\cdot,\cdot} \coloneqq \mathrm{Ext}(A_\cdot, B_\cdot) = \bigoplus_{0 \le i,k \le 3} \mathrm{Ext}(A_i, B_k),$$

i.e. $C_{i,k} \coloneqq \mathrm{Ext}(A_i, B_k)$, which can be organized into the following diagram where we take the Ext at each position and sum them all together:

$$\begin{tikzcd}
\mathrm{Ext}(A_1, B_1) & \mathrm{Ext}(A_1, B_2) & \mathrm{Ext}(A_1, B_3) \\
\mathrm{Ext}(A_2, B_1) & \mathrm{Ext}(A_2, B_2) & \mathrm{Ext}(A_2, B_3) \\
\mathrm{Ext}(A_3, B_1) & \mathrm{Ext}(A_3, B_2) & \mathrm{Ext}(A_3, B_3)
\end{tikzcd}$$

*Link to Diagram*

This equals the following:

$$\begin{tikzcd}
\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}) & \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}/4) & \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}/5) \\
\mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z}) & \mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z}/4) & \mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z}/5) \\
\mathrm{Ext}(\mathbb{Z}/3, \mathbb{Z}) & \mathrm{Ext}(\mathbb{Z}/3, \mathbb{Z}/4) & \mathrm{Ext}(\mathbb{Z}/3, \mathbb{Z}/5)
\end{tikzcd}$$

*Link to Diagram*

Which simplifies to:

$$\begin{tikzcd}
0 & 0 & 0 \\
\mathbb{Z}/2 & \mathbb{Z}/2 & 0 \\
\mathbb{Z}/3 & 0 & 0
\end{tikzcd}$$

Link to Diagram

So the answer is $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/6$.

## 18.7 7

Show there is no homeomorphism $\mathbb{CP}^2 \xrightarrow{f} \mathbb{CP}^2$ such that $f(\mathbb{CP}^1)$ is disjoint from $\mathbb{CP}_1 \subset \mathbb{CP}_2$.

**Solution:**

**Concepts Used:**

1. Every homeomorphism induces isomorphisms on homotopy/homology/cohomology.
2. $H^*(\mathbb{CP}^2) = \mathbb{Z}[\alpha]/(\alpha^2)$ where $\deg \alpha = 2$.
3. $[f(X)] = f_*([X])$
4. $a \frown b = 0 \implies a = 0$ or $b = 0$ (nondegeneracy).

Supposing such a homeomorphism exists, we would have $[\mathbb{CP}^1] \frown [f(\mathbb{CP}^1)] = 0$ by the definition of these submanifolds being disjoint. But $[\mathbb{CP}^1] \frown [f(\mathbb{CP}^1)] = [\mathbb{CP}^1] \frown f_*([\mathbb{CP}^1])$, where

$$f_* : H^*(\mathbb{CP}^2) \to H^*(\mathbb{CP}^2)$$

is the induced map on cohomology. Since the intersection pairing is nondegenerate, either $[\mathbb{CP}^1] = 0$ or $f_*([\mathbb{CP}^1]) = 0$. We know that $H^*(\mathbb{CP}^2) = \mathbb{Z}[\alpha]/\alpha^2$ where $\alpha = [\mathbb{CP}^1]$, however, so this forces $f_*([\mathbb{CP}^1]) = 0$. But since this was a generator of $H^*$, we have $f_*(H^*(\mathbb{CP}^2)) = 0$, so $f$ is not an isomorphism on cohomology.

## 18.8 8

Describe the universal cover of $X = (S^1 \times S^1) \vee S^2$ and compute $\pi_2(X)$.

**Solution:**

**Concepts Used:**

- $\pi_{\geq 2}(\overline{X}) \cong \pi_{\geq 2}(X)$ for $\overline{X}$ the universal cover of $X$
- Structure of the universal cover of a wedges
- $\overline{T^2} = \mathbb{R}^2$ and $\overline{S^2} = S^2$
- By Mayer-Vietoris, $H_n(\bigvee X_i) = \bigoplus H_n(X_i)$.

The universal cover can be identified as

$$\overline{X} = \mathbb{R}^2 \bigvee_{i,j\in\mathbb{Z}^2} S^2,$$

i.e. the plane with a sphere wedged onto every integer lattice point. We can then check

$$\begin{aligned}
\pi_1(X) &\cong \pi_1(\overline{X}) \\
&= \pi_1(\mathbb{R}^2 \bigvee_{i,j\in\mathbb{Z}^2} S^2) \\
&= \pi_1(\mathbb{R}^2 \bigvee_{i,j\in\mathbb{Z}^2} S^2) \\
&= \prod_{i,j\in\mathbb{Z}^2} \pi_1(\mathbb{R}^2) \times \pi_1(S^2) \\
&= 0,
\end{aligned}$$

using that $\pi_1(S^2) = 0$. Then by Hurewicz, $\pi_2(X) \cong H_2(X)$, so we can compute

$$\begin{aligned}
H_2(X) &= H_2(\mathbb{R}^2 \bigvee_{i,j\in\mathbb{Z}^2} S^2) \\
&= \bigoplus_{i,j\in\mathbb{Z}^2} H_2(\mathbb{R}^2) \oplus H_2(S^2) \\
&= \bigoplus_{i,j\in\mathbb{Z}^2} \mathbb{Z}.
\end{aligned}$$

## 18.9 9

Let $S^3 \to E \to S^5$ be a fiber bundle and compute $H_3(E)$.

**Solution** ***(Using the LES in Homotopy):***

**Concepts Used:**

- Homotopy LES: $F \to E \to B \rightsquigarrow \pi_* F() \to \pi_*(E) \to \pi_*(B)$.
- Hurewicz: $\pi_{\leq n}(X) = 0, \pi_n(X) \neq 0 \implies \pi_n(X) \cong H_n(X)$.
- $0 \to A \to B \to 0$ exact iff $A \cong B$

From the LES in homotopy we have

$$
\begin{tikzcd}
\pi_4(S^3) \arrow[r] & \pi_4(E) \arrow[r] & \pi_4(S^5) \arrow[dll, out=0, in=180] \\
\pi_3(S^3) \arrow[r] & \pi_3(E) \arrow[r] & \pi_3(S^5) \arrow[dll, out=0, in=180] \\
\pi_2(S^3) \arrow[r] & \pi_2(E) \arrow[r] & \pi_2(S^5) \arrow[dll, out=0, in=180] \\
\pi_1(S^3) \arrow[r] & \pi_1(E) \arrow[r] & \pi_1(S^5) \arrow[dll, out=0, in=180] \\
\pi_0(S^3) \arrow[r] & \pi_0(E) \arrow[r] & \pi_0(S^5)
\end{tikzcd}
$$

*Link to Diagram*

and plugging in known information yields

$$
\begin{tikzcd}
\pi_4(S^3) \arrow[r] & \pi_4(E) \arrow[r] & 0 \arrow[dll, out=0, in=180] \\
\mathbb{Z} \arrow[r, hook, two heads, "\sim"] & \pi_3(E) \arrow[r] & 0 \arrow[dll, out=0, in=180] \\
0 \arrow[r] & \pi_2(E) = 0 \arrow[r] & 0 \arrow[dll, out=0, in=180] \\
0 \arrow[r] & \pi_1(E) = 0 \arrow[r] & 0 \arrow[dll, out=0, in=180] \\
\mathbb{Z} \arrow[r, hook] & \pi_0(E) \arrow[r, two heads] & \mathbb{Z}
\end{tikzcd}
$$

*Link to Diagram*

where

- Rows 3 and 4 force $\pi_3(E) \cong \mathbb{Z}$,
- Rows 0 and 1 force $\pi_0(E) = \mathbb{Z}$ (todo: not clear if this is true... is it even needed here?)

- The remaining rows force $\pi_1(E) = \pi_2(E) = 0$.

By Hurewicz, we thus have $H_3(E) = \pi_3(E) = \mathbb{Z}$.

**Solution** *(Using the Serre spectral sequence):*

Four-corner spectral sequences, only homology in degrees 1,3,5,8. No differentials hit anything!

# 19 Fall 2017 Final

## 19.1 1

Let $X$ be the subspace of the unit cube $I^3$ consisting of the union of the 6 faces and the 4 internal diagonals. Compute $\pi_1(X)$.

## 19.2 2

Let $X$ be an arbitrary topological space, and compute $\pi_1(\Sigma X)$.

**Solution:**
Write $\Sigma X = U \cup V$ where $U = \Sigma X - (X \times [0, 1/2])$ and $U = \Sigma X - X \times [1/2, 1])$. Then $U \cap V = X \times \{1/2\} \cong X$, so $\pi_1(U \cap V) = \pi_1(X)$.
But both $U$ and $V$ can be identified by the cone on $X$, given by $CX = \dfrac{X \times I}{X \times 1}$, by just rescaling the interval with the maps:
$i_U : U \to CX$ where $(x, s) \mapsto (x, 2s - 1)$ (The second component just maps $[1/2, 1] \to [0, 1]$. )
$i_V : V \to CX$ where $(x, s) \mapsto (x, 2s)$. (The second component just maps $[0, 1/2] \to [0, 1]$)
But $CX$ is contractible by the homotopy $H : CX \times I \to CX$ where $H((c, s), t) = (c, s(1 - t))$.
So $\pi_1(U) = \pi_1(V) = 0$.
By Van Kampen, we have $\pi_1(X) = 0 *_{\pi_1(X)} 0 = 0$.

## 19.3 3

Let $X = S^1 \times S^1$ and $A \subset X$ be a subspace with $A \cong S^1 \vee S^1$. Show that there is no retraction from $X$ to $A$.

**Solution:**
We have $\pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1)$ since $S^1$ is path-connected (by a lemma from the problem sets), and this equals $\mathbb{Z} \times \mathbb{Z}$.
We also have $\pi_1(S^1 \vee S^1) = \pi_1(S^1) *_{\{pt\}} \pi_1(S^1)$, which by Van-Kampen is $\mathbb{Z} * \mathbb{Z}$.
Suppose $X$ retracts onto $A$, we can then look at the inclusion $\iota : A \hookrightarrow X$. The induced homomorphism $\iota_* : \pi_1(A) \hookrightarrow \pi_1(X)$ is then also injective, so we've produced an injection from $f : \mathbb{Z} * \mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Z}$.
This is a contradiction, because no such injection can exists. In particular, the commutator $[a, b]$ is nontrivial in the source. But $f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1}$ since $f$ is a homomorphism, but since the target is a commutative group, this has to equal $f(a)f(a)^{-1}f(b)f(b)^{-1} = e$. So there is a non-trivial element in the kernel of $f$, and $f$ can not be injective - a contradiction.

## 19.4 4

Show that for every map $f : S^2 \to S^1$, there is a point $x \in S^2$ such that $f(x) = f(-x)$.

**Solution:**
Suppose towards a contradiction that $f$ does not possess this property, so there is no $x \in S^2$ such that $f(x) = f(-x)$.
Then define $g : S^2 \to S^1$ by $g(x) = f(x) - f(-x)$; by assumption, this is a nontrivial map, i.e. $g(x) \neq 0$ for *any* $x \in S^2$.
In particular, $-g(-x) = -(f(-x) - f(x)) = f(x) - f(-x) = g(x)$, so $-g(x) = g(-x)$ and thus $g$ commutes with the antipodal map $\alpha : S^2 \to S^2$.
This means $g$ is constant on the fibers of the quotient map $p : S^2 \to \mathbb{RP}2$, and thus descends to a well defined map $\tilde{g} : \mathbb{RP}2 \to S^1$, and since $S^1 \cong \mathbb{RP}1$, we can identify this with a map $\tilde{g} : \mathbb{RP}2 \to \mathbb{RP}1$ which thus induces a homomorphism $\tilde{g}_* : \pi_1(\mathbb{RP}2) \to \pi_1(\mathbb{RP}1)$.
Since $g$ was nontrivial, $\tilde{g}$ is nontrivial, and by functoriality of $\pi_1$, $\tilde{g}_*$ is nontrivial.
But $\pi_1(\mathbb{RP}2) = \mathbb{Z}_2$ and $\pi_1(\mathbb{RP}1) = \mathbb{Z}$, and $\tilde{g}_* : \mathbb{Z}^2 \to \mathbb{Z}$ can only be the trivial homomorphism - a contradiction.

**Remark 19.4.1: Alternate Solution** Use covering space $\mathbb{R} \twoheadrightarrow S^1$?

## 19.5 5

How many path-connected 2-fold covering spaces does $S^1 \vee \mathbb{RP}2$ have? What are the total spaces?

**Solution:**
First note that $\pi_1(X) = \pi_1(S^1) *_{\{\mathrm{pt}\}} \pi_1(\mathbb{RP}2)$ by Van-Kampen, and this is equal to $\mathbb{Z} * \mathbb{Z}_2$.

## 19.6 6

Let $G = < a, b >$ and $H \leq G$ where $H = < aba^{-1}b^{-1},\ a^2ba^{-2}b^{-1},\ a^{-1}bab^{-1},\ aba^{-2}b^{-1}a >$. To what well-known group is $H$ isomorphic?

# 20 Appendix: Homological Algebra

## 20.1 Exact Sequences

**Proposition 20.1.1** *(?).*
The sequence $A \xrightarrow{f_1} B \xrightarrow{f_2} C$ is exact if and only if $\operatorname{im} f_i = \ker f_{i+1}$ and thus $f_2 \circ f_1 = 0$.

**Fact 20.1.2**
Some useful results:

- $0 \to A \hookrightarrow_f B$ is exact iff $f$ is **injective**
- $B \twoheadrightarrow_f C \to 0$ is exact iff $f$ is **surjective**
- $0 \to A \to B \to 0$ is exact iff $A \cong B$.
- $A \hookrightarrow B \to C \to D \twoheadrightarrow E$ iff $C = 0$
- $0 \to A \to B \xrightarrow{\cong} C \to D \to 0$ iff $A = D = 0$.
  - Todo: Proof
- $0 \to A \to B \to C \to 0$ splits iff $C$ is free.
- Can think of $C \cong \dfrac{B}{\operatorname{im} f_1}$.

**Definition 20.1.3** (Splitting an exact sequence)
The sequences *splits* when a morphism $f_2^{-1} : C \to B$ exists. In **Ab**, this means $B \cong A \oplus C$, in **Grp** it's $B \cong A \rtimes_\varphi C$.

**Example 20.1.4** *(of exact sequences):*

- $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \dfrac{\mathbb{Z}}{2\mathbb{Z}} \to 0$
- $1 \to N \xrightarrow{\iota} G \xrightarrow{p} \dfrac{G}{N} \to 1$
  - Groups and normal subgroups
- $1 \to \dfrac{\mathbb{Z}}{n\mathbb{Z}} \xrightarrow{\iota} D_{2n} \xrightarrow{?} \dfrac{\mathbb{Z}}{2\mathbb{Z}} \to 1$

- Dihedral group and cyclic groups

- $0 \to I \cap J \xrightarrow{\Delta : x \mapsto (x,x)} I \oplus J \xrightarrow{f:(x,y) \mapsto x-y} I + J \to 0$
  - $R$-Modules
- $0 \to \dfrac{R}{I \cap J} \xrightarrow{\Delta : x \mapsto (x,x)} \dfrac{R}{I} \oplus \dfrac{R}{J} \xrightarrow{f:(x,y) \mapsto x-y} \dfrac{R}{I+J} \to 0$
- $0 \to \mathbb{H}_1 \xrightarrow{\nabla} \mathbb{H}_{\text{curl}} \xrightarrow{\nabla \times} \mathbb{H}_{\text{div}} \xrightarrow{\nabla \cdot} \mathbb{L}_2 \to 0$
  - Since $\nabla \times \nabla F = \nabla \cdot \nabla \times \bar{v} = 0$ in Hilbert spaces

**Remark 20.1.5:** Is $f_1 \circ f_2 = 0$ equivalent to exactness..? Answer: yes, every exact sequence is a chain complex with trivial homology. Therefore homology measures the failure of exactness.

*Alternatively stated: Exact sequences are chain complexes with no cycles.*

**Remark 20.1.6:** Any LES $A_1 \to \cdots \to A_6$ decomposes into a twisted collection of SES's; define $C_k = \ker(A_k \to A_{k+1}) \cong \operatorname{im}(A_{k-1} \to A_k)) \cong \operatorname{coker}(A_{k-2} \to A_{k-1})$, then all diagonals here are exact:

## 20.2 Five Lemma

**Theorem 20.2.1** *(?).*
If $m, p$ are isomorphisms, $l$ is an **surjection**, and $q$ is an **injection**, then $n$ is an **isomorphism**.
Proof: diagram chase two "four lemmas", one on each side. Full proof here.

## 20.3 Free Resolutions

**Example 20.3.1** ***(?):*** The canonical example:

$$0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \xrightarrow{(\text{mod } m)} \mathbb{Z}_m \to 0$$

Or more generally for a finitely generated group $G = \langle g_1, g_2, \cdots, g_n \rangle$,

$$\cdots \to \ker(f) \to F[g_1, g_2, \cdots, g_n] \xrightarrow{f} G \to 0$$

where $F$ denotes taking the free group.

Every abelian groups has a resolution of this form and length 2.

## 20.4 Properties of Tensor Products

- $A \otimes B \cong B \otimes A$
- $(\,\cdot\,) \otimes_R R^n = \mathrm{id}$
- $\bigoplus_i A_i \otimes \bigoplus_j B_j = \bigoplus_i \bigoplus_j (A_i \otimes B_j)$
- $\mathbb{Z}_m \otimes \mathbb{Z}_n = \mathbb{Z}_d$
- $\mathbb{Z}_n \otimes A = A/nA$

## 20.5 Properties of Hom

- $\hom_R(\bigoplus_i A_i, \prod B_j) = \bigoplus_i \prod_j \hom(A_i, B_j)$
- Contravariant in first slot, covariant in second
- Exact over vector spaces

## 20.6 Properties of Tor

- $\mathrm{Tor}_R^0(A, B) = A \otimes_R B$
- $\mathrm{Tor}(\bigoplus_i A_i, \bigoplus_j B) = \bigoplus_i \bigoplus_j \mathrm{Tor}(\mathbf{T}A_i, \mathbf{T}B_j)$ where $\mathbf{T}G$ is the torsion component of $G$.
- $\mathrm{Tor}(\mathbb{Z}_n, G) = \ker(g \mapsto ng) = \left\{ g \in G \;\middle|\; ng = 0 \right\}$
- $\mathrm{Tor}(A, B) = \mathrm{Tor}(B, A)$

## 20.7 Properties of Ext

- $\mathrm{Ext}_R^0(A, B) = \hom_R(A, B)$
- $\mathrm{Ext}(\bigoplus_i A_i, \prod_j B_j) = \bigoplus_i \prod_j \mathrm{Ext}(\mathbf{T}A_i, B_j)$
- $\mathrm{Ext}(F, G) = 0$ if $F$ is free
- $\mathrm{Ext}(\mathbb{Z}_n, G) \cong G/nG$

## 20.8 Computing Tor

$$\mathrm{Tor}(A, B) = h[\cdots \to A_n \otimes B \to A_{n-1} \otimes B \to \cdots A_1 \otimes B \to 0]$$

where $A_*$ is any free resolution of $A$.

Shorthand/mnemonic:

$$\mathrm{Tor} : \mathcal{F}(A) \to (\,\cdot \otimes B) \to H_*$$

## 20.9 Computing Ext

$$\mathrm{Ext}(A,B) = h[\cdots \hom(A,B_n) \to \hom(A,B_{n-1}) \to \cdots \to \hom(A,B_1) \to 0]$$

where $B_*$ is a any free resolution of $B$.

Shorthand/mnemonic:

$$\mathrm{Ext} : \mathcal{F}(B) \to \hom(A,\,\cdot\,) \to H_*$$

## 20.10 Hom/Ext/Tor Tables

| hom | $\mathbb{Z}_m$ | $\mathbb{Z}$ | $\mathbb{Q}$ |
|---|---|---|---|
| $\mathbb{Z}_n$ | $\mathbb{Z}_d$ | 0 | 0 |
| $\mathbb{Z}$ | $\mathbb{Z}_m$ | $\mathbb{Z}$ | $\mathbb{Q}$ |
| $\mathbb{Q}$ | 0 | 0 | $\mathbb{Q}$ |

| Tor | $\mathbb{Z}_m$ | $\mathbb{Z}$ | $\mathbb{Q}$ |
|---|---|---|---|
| $\mathbb{Z}_n$ | $\mathbb{Z}_d$ | 0 | 0 |
| $\mathbb{Z}$ | 0 | 0 | 0 |
| $\mathbb{Q}$ | 0 | 0 | 0 |

| Ext | $\mathbb{Z}_m$ | $\mathbb{Z}$ | $\mathbb{Q}$ |
|---|---|---|---|
| $\mathbb{Z}_n$ | $\mathbb{Z}_d$ | $\mathbb{Z}_n$ | 0 |
| $\mathbb{Z}$ | 0 | 0 | 0 |
| $\mathbb{Q}$ | 0 | $\mathcal{A}/\mathbb{Q}$ | 0 |

Where $d = \gcd(m, n)$ and $\mathbb{Z}_0 \coloneqq 0$.

Things that behave like "the zero functor":

- $\operatorname{Ext}(\mathbb{Z}, \cdot)$
- $\operatorname{Tor}(\cdot, \mathbb{Z}), \operatorname{Tor}(\mathbb{Z}, \cdot)$
- $\operatorname{Tor}(\cdot, \mathbb{Q}), \operatorname{Tor}(\mathbb{Q}, \cdot)$

Thins that behave like "the identity functor":

- $\hom(\mathbb{Z}, \cdot)$
- $\cdot \otimes_{\mathbb{Z}} \mathbb{Z}$ and $\mathbb{Z} \otimes_{\mathbb{Z}} \cdot$

For description of $\mathcal{A}$, see here. This is a certain ring of adeles.

# 21 | Appendix: Unsorted Stuff

- Assorted info about other Lie Groups:
- $O_n, U_n, SO_n, SU_n, Sp_n$
- $\pi_k(U_n) = \mathbb{Z} \cdot \mathbb{1}[k \text{ odd}]$
  - $\pi_1(U_n) = 1$
- $\pi_k(SU_n) = \mathbb{Z} \cdot \mathbb{1}[k \text{ odd}]$
  - $\pi_1(SU_n) = 0$
- $\pi_k(U_n) = \mathbb{Z}/2\mathbb{Z} \cdot \mathbb{1}[k = 0, 1 \pmod 8] + \mathbb{Z} \cdot \mathbb{1}[k = 3, 7 \pmod 8]$
- $\pi_k(SP_n) = \mathbb{Z}/2\mathbb{Z} \cdot \mathbb{1}[k = 4, 5 \pmod 8] + \mathbb{Z} \cdot \mathbb{1}[k = 3, 7 \pmod 8]$
- Groups and Group Actions
  - $\pi_0(G) = G$ for $G$ a discrete topological group.
  - $\pi_k(G/H) = \pi_k(G)$ if $\pi_k(H) = \pi_{k-1}(H) = 0$.
  - $\pi_1(X/G) = \pi_0(G)$ when $G$ acts freely/transitively on $X$.

## 21.1 Cap and Cup Products

$$\cup : H^p \times H^q \to H^{p+q}; (a^p \cup b^q)(\sigma) = a^p(\sigma \circ F_p) b^q(\sigma \circ B_q)$$

where $F_p, B_q$ is embedding into a $p+q$ simplex.

For $f$ continuous, $f^*(a \cup b) = f^* a \cup f^* b$

It satisfies the Leibniz rule

$$\partial(a^p \cup b^q) = \partial a^p \cup b^q + (-1)^p (a^p \cup \partial b^q)$$

$$\cap : H_p \times H^q \to H_{p-q}; \sigma \cap \psi = \psi(F \circ \sigma)(B \circ \sigma)$$

where $F, B$ are the front/back face maps.

Given $\psi \in C^q, \varphi \in C^p, \sigma : \Delta^{p+q} \to X$, we have

$$\psi(\sigma \cap \varphi) = (\varphi \cup \psi)(\sigma)$$
$$\langle \varphi \cup \psi,\ \sigma \rangle = \langle \psi,\ \sigma \cap \varphi \rangle$$

Let $M^n$ be a closed oriented smooth manifold, and $A^{\widehat{i}}, B^{\widehat{j}} \subseteq X$ be submanifolds of codimension $i$ and $j$ respectively that intersect transversely (so $\forall p \in A \cap B$, the inclusion-induced map $T_p A \times T_p B \to T_p X$ is surjective.)

Then $A \cap B$ is a submanifold of codimension $i + j$ and there is a short exact sequence

$$0 \to T_p(A \cap B) \to T_p A \times T_p B \to T_p X \to 0$$

which determines an orientation on $A \cap B$.

Then the images under inclusion define homology classes

- $[A] \in H_{\widehat{i}} X$
- $[B] \in H_{\widehat{j}} X$
- $[A \cap B] \in H_{\widehat{i+j}} X$.

Denoting their Poincare duals by

- $[A]^\vee \in H^i X$
- $[B]^\vee \in H^j X$
- $[A \cap B]^\vee \in H^{i+j} X$

We then have

$$[A]^\vee \smile [B]^\vee = [A \cap B]^\vee \in H^{i+j} X$$

Example: in $\mathbb{CP}^n$, each even-dimensional cohomology $H^{2i}\mathbb{CP}^n$ has a generator $\alpha_i$ with is Poincare dual to an $\widehat{i}$ plane. A generic $\widehat{i}$ plane intersects a $\widehat{j}$ plane in a $\widehat{i+j}$ plane, yielding $\alpha_i \smile \alpha_j = \alpha_{i+j}$ for $i + j \leq n$.

Example: For $T^2$, we have - $H_1 T^2 = \mathbb{Z}^2$ generated by $[A], [B]$, the longitudinal and meridian circles. - $H_0 T^2 = \mathbb{Z}$ generated by $[p]$, the class of a point.

Then $A \cap B = \pm[p]$, and so

$$\begin{aligned} [A]^\vee \smile [B]^\vee &= [p]^\vee \\ [B]^\vee \smile [A]^\vee &= -[p]^\vee \end{aligned}$$

## 21.2 The Long Exact Sequence of a Pair

LES of pair $(A, B) \implies \cdots H_n(B) \to H_n(A) \to H_n(A, B) \to H_{n-1}(B) \cdots$

$$\begin{tikzcd} & B \arrow[dl, dash] \arrow[dr, dash] & \\ (A, B) & & A \arrow[ll] \end{tikzcd} .$$

**3.1.3 Example.** The cases $n = 1, 2$ and part of the case $n = 3$ are shown in the figure below.

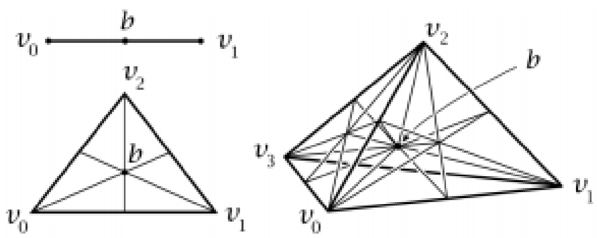



Figure 3.1: Barycentric subdivision [10].

Figure 13: Barycentric Subdivision

## 21.3 Tables

**Homotopy groups of real projective spaces**

| | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ | $\pi_7$ | $\pi_8$ | $\pi_9$ | $\pi_{10}$ | $\pi_{11}$ | $\pi_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $RP^1$ | $Z$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $RP^2$ | $Z_2$ | $Z$ | $Z$ | $Z_2$ | $Z_2$ | $Z_{12}$ | $Z_2$ | $Z_2$ | $Z_3$ | $Z_{15}$ | $Z_2$ | $Z_2 \times Z_2$ |
| $RP^3$ | $Z_2$ | 0 | $Z$ | $Z_2$ | $Z_2$ | $Z_{12}$ | $Z_2$ | $Z_2$ | $Z_3$ | $Z_{15}$ | $Z_2$ | $Z_2 \times Z_2$ |
| $RP^4$ | $Z_2$ | 0 | 0 | $Z$ | $Z_2$ | $Z_2$ | $Z \times Z_{12}$ | $Z_2 \times Z_2$ | $Z_2 \times Z_2$ | $Z_{24} \times Z_3$ | $Z_{15}$ | $Z_2$ |

Figure 14: Higher homotopy groups of $\mathbb{RP}^n$

**Homotopy groups of complex projective spaces**

| | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ | $\pi_7$ | $\pi_8$ | $\pi_9$ | $\pi_{10}$ | $\pi_{11}$ | $\pi_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $CP^1$ | 0 | $Z$ | $Z$ | $Z_2$ | $Z_2$ | $Z_{12}$ | $Z_2$ | $Z_2$ | $Z_3$ | $Z_{15}$ | $Z_2$ | $Z_2 \times Z_2$ |
| $CP^2$ | 0 | $Z$ | 0 | 0 | $Z$ | $Z_2$ | $Z_2$ | $Z_{24}$ | $Z_2$ | $Z_2$ | $Z_2$ | $Z_{30}$ |
| $CP^3$ | 0 | $Z$ | 0 | 0 | 0 | 0 | $Z$ | $Z_2$ | $Z_2$ | $Z_{24}$ | 0 | 0 |
| $CP^4$ | 0 | $Z$ | 0 | 0 | 0 | 0 | 0 | 0 | $Z$ | $Z_2$ | $Z_2$ | $Z_{24}$ |

Figure 15: Higher homotopy groups of $\mathbb{CP}^n$

**Homotopy groups of spheres**

| | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ | $\pi_7$ | $\pi_8$ | $\pi_9$ | $\pi_{10}$ | $\pi_{11}$ | $\pi_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $S^1$ | $Z$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $S^2$ | 0 | $\boxed{Z}$ | $Z$ | $Z_2$ | $Z_2$ | $Z_{12}$ | $Z_2$ | $Z_2$ | $Z_3$ | $Z_{15}$ | $Z_2$ | $Z_2 \times Z_2$ |
| $S^3$ | 0 | 0 | $Z$ | $\boxed{Z_2}$ | $Z_2$ | $Z_{12}$ | $Z_2$ | $Z_2$ | $Z_3$ | $Z_{15}$ | $Z_2$ | $Z_2 \times Z_2$ |
| $S^4$ | 0 | 0 | 0 | $Z$ | $Z_2$ | $\boxed{Z_2}$ | $Z \times Z_{12}$ | $Z_2 \times Z_2$ | $Z_2 \times Z_2$ | $Z_{24} \times Z_3$ | $Z_{15}$ | $Z_2$ |
| $S^5$ | 0 | 0 | 0 | 0 | $Z$ | $Z_2$ | $Z_2$ | $\boxed{Z_{24}}$ | $Z_2$ | $Z_2$ | $Z_2$ | $Z_{30}$ |
| $S^6$ | 0 | 0 | 0 | 0 | 0 | $Z$ | $Z_2$ | $Z_2$ | $Z_{24}$ | $\boxed{0}$ | $Z$ | $Z_2$ |
| $S^7$ | 0 | 0 | 0 | 0 | 0 | 0 | $Z$ | $Z_2$ | $Z_2$ | $Z_{24}$ | 0 | $\boxed{0}$ |
| $S^8$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $Z$ | $Z_2$ | $Z_2$ | $Z_{24}$ | 0 |

Figure 16: Homotopy groups of spheres.

*A1.1.3.4 Exceptional groups*

**Homotopy groups of exceptional groups**

| | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ | $\pi_7$ | $\pi_8$ | $\pi_9$ | $\pi_{10}$ | $\pi_{11}$ | $\pi_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $G_2$ | 0 | 0 | $Z$ | 0 | 0 | $Z_3$ | 0 | $Z_2$ | $Z_6$ | 0 | $Z \times Z_2$ | 0 |
| $F_4$ | 0 | 0 | $Z$ | 0 | 0 | 0 | 0 | $Z_2$ | $Z_2$ | 0 | $Z \times Z_2$ | 0 |
| $E_6$ | 0 | 0 | $Z$ | 0 | 0 | 0 | 0 | 0 | $Z$ | 0 | $Z$ | $Z_{12}$ |
| $E_7$ | 0 | 0 | $Z$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $Z$ | $Z_2$ |
| $E_8$ | 0 | 0 | $Z$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Figure 17: Homotopy groups of exceptional groups

## 21.4 Homotopy Groups of Lie Groups

- $O(n)$: $\pi_k O_n =?$

- $U(n) : \pi_k U_n$ is $\mathbb{Z}$ in odd degrees and $\pi_1 U_n = 1$

Check

- $SU(n) : \pi_k U_n$ is $\mathbb{Z}$ in odd degrees and $\pi_1 U_n = 0$.
- $U_n : \pi_k(U_n)$ is $\mathbb{Z}/2\mathbb{Z}$ in degrees?

## 21.5 Higher Homotopy

- $n \geq 2 \implies \pi_n(X) \in \mathbf{Ab}$
- $\Sigma S^n = S^{n+1}$
- $[\Sigma^n X, Y] \cong [X, \Omega^n Y]$
- $\pi * n(\Omega X) = \pi * n + 1(X)$
  - $\pi_n(X) \cong \pi_0(\Omega^n X)$
- $n \geq 2 \implies \pi_n(S^1) = 0$
- $k < n \implies \pi_k(S^n) = 0$
- $\pi_n(X)$ is the obstruction to $f : S^n \to X$ being lifted to $\widehat{f} : D^{n+1} \to X$
- $\pi_n(X) \cong H_n(X)$ for the first $n$ such that $\pi_n(X) \neq 0$; $\forall k < n, \; H_k(X) = 0$.
- $k + 2 \leq 2n \implies \pi_k(S^n) \cong \pi_{k+1}(S^{n+1})$
- $\pi_k(S^n) = \pi_{k+1} S^{n+1} = \cdots = \pi_{k+i} S^{n+i}$
- $F \to E \to B$ a fibration yields $\cdots \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi * n - 1(F) \cdots$
- Freundenthal suspension, stable homotopy groups

## 21.6 Higher Homotopy Groups of the Sphere

- $\pi_n(S^n) = \mathbb{Z}$
- $\pi_{n+1} S^n = \mathbb{Z}_2$ for $n \geq 4$
- $\pi_{n+2}(S^n) \cong \mathbb{Z}_2$
- $\pi_{n+3} S^n = \mathbb{Z}_8$ for $n \geq 5$
- $\pi_5 S^2 = \mathbb{Z}_2$
- $\pi_6 S^3 = \mathbb{Z}_4$

- $\pi_7 S^4 = \mathbb{Z} \oplus \mathbb{Z}_4$
- $\pi_k S^2 \cong \pi_k S^3$
- $\pi_3 S^2 \cong \mathbb{Z}$
- $\pi_4 S^2 \cong \mathbb{Z}_2$

## 21.7 Misc

- $\Omega(\cdot)$ is an exact functor.

## 21.8 Building a Moore Space

- To build a Moore space $M(n, \mathbb{Z}_p)$, take $X = S^n$ and attach $e^{n+1}$ via a map $\Phi : S^n = \partial B^{n+1} \to X^{(n)} = S^n$ of degree $p$.
  - To obtain $M(n, \prod G_i)$ take the corresponding $\bigvee X_i$
  - Can also use Mayer Vietoris to conclude $H_{n+1}(\Sigma X) = H_n(X)$, and just suspend spaces with known homology.

# Bibliography

[1] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[2] James Raymond Munkres. *Topology*. Pearson, 2018.