Topology Qualifying Exam Notes

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Contents

1 Definitions

- Topology: Closed under arbitrary unions and finite intersections.
- Basis: A subset $\{B_i\}$ is a basis iff
  - $x \in X \implies x \in B_i$ for some i.
  - $x \in B_i \bigcap B_j \implies x \in B_k \subset B_i \bigcap B_k$.
  - Topology generated by this basis: $x \in N_x \implies x \in B_i \subset N_x$ for some i.
- Dense: A subset $Q \subset X$ is dense iff $y \in N_y \subset X \implies N_y \bigcap Q \neq \emptyset$ iff $\overline{Q} = X$.
- Neighborhood: A neighborhood of a point x is any open set containing x.
- Hausdorff
- Second Countable: admits a countable basis.
- Closed (several characterizations)
- Closure in a subspace: $Y \subset X \implies \mathrm{cl}_Y(A) \coloneqq \mathrm{cl}_X(A) \bigcap Y$.
- Bounded
- Compact: A topological space (X, τ) is **compact** if every open cover has a *finite* subcover.

That is, if $\left\{U_j \;\middle|\; j \in J\right\} \subset \tau$ is a collection of open sets such that $X \subseteq \bigcup_{j\in J} U_j$, then there exists a *finite* subset $J' \subset J$ such that $X \subseteq \bigcup_{j\in J'} U_j$.

- Locally compact For every $x \in X$, there exists a $K_x \ni x$ such that K_x is compact.
- Connected: There does not exist a disconnecting set $X = A \coprod B$ such that $\emptyset \neq A, B \subsetneq$, i.e. X is the union of two proper disjoint nonempty sets.

  Equivalently, X contains no proper nonempty clopen sets.

  – Additional condition for a subspace $Y \subset X$: $\mathrm{cl}_Y(A) \bigcap V = A \bigcap \mathrm{cl}_Y(B) = \emptyset$.
- Locally connected: A space is locally connected at a point x iff $\forall N_x \ni x$, there exists a $U \subset N_x$ containing x that is connected.
- Retract: A subspace $A \subset X$ is a *retract* of X iff there exists a continuous map $f : X \longrightarrow A$ such that $f\Big|_A = \mathrm{id}_A$. Equivalently it is a *left* inverse to the inclusion.
- Uniform Continuity: For $f : (X, d_x) \longrightarrow (Y, d_Y)$ metric spaces,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \quad d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

- Lebesgue number: For (X, d) a compact metric space and $\{U_\alpha\} \rightrightarrows X$, there exist $\delta_L > 0$ such that

$$A \subset X,\ \mathrm{diam}(A) < \delta_L \implies A \subseteq U_\alpha \text{ for some } \alpha.$$

- Paracompact
- Components: Set $x \sim y$ iff there exists a connected set $U \ni x, y$ and take equivalence classes.
- Path Components: Set $x \sim y$ iff there exists a path-connected set $U \ni x, y$ and take equivalence classes.
- Separable: Contains a countable dense subset.
- Limit Point: For $A \subset X$, x is a limit point of A if every punctured neighborhood P_x of x satisfies $P_x \bigcap A \neq \emptyset$, i.e. every neighborhood of x intersects A in some point other than x itself.

  Equivalently, x is a limit point of A iff $x \in \mathrm{cl}_X(A \setminus \{x\})$.

2 Theorems

2.1 Point-Set

Theorem 2.1.
$U \subset X$ a Hausdorff spaces is closed $\iff$ it is compact.

Theorem 2.2 ***(Cantor's Intersection Theorem).***
A bounded collection of nested closed sets $C_1 \supset C_2 \supset \cdots$ in a metric space X is nonempty

$\iff X$ is complete.

- Tube lemma
- Properties pushed forward through continuous maps:
  - Compactness?
  - Connectedness (when surjective)
  - Separability
  - Density **only when** f is surjective
  - **Not** openness
  - **Not** closedness
- A retract of a Hausdorff/connected/compact space is closed/connected/compact respectively.

Proposition 2.3.
A continuous function on a compact set is uniformly continuous.

Proof .
Take $\left\{B_{\frac{\varepsilon}{2}}(y) \;\middle|\; y \in Y\right\} \rightrightarrows Y$, pull back to an open cover of X, has Lebesgue number $\delta_L > 0$, then $x' \in B_{\delta_L}(x) \implies f(x), f(x') \in B_{\frac{\varepsilon}{2}}(y)$ for some y.
■

Corollary 2.4.
Lipschitz continuity implies uniform continuity (take $\delta = \varepsilon/C$)

Counterexample to converse: $f(x) = \sqrt{x}$ on $[0,1]$ has unbounded derivative.

Theorem 2.5 ***(Extreme Value Theorem).***
For $f : X \longrightarrow Y$ continuous with X compact and Y ordered in the order topology, there exist points $c, d \in X$ such that $f(x) \in [f(c), f(d)]$ for every x.

Theorem 2.6.
Points are closed in T_1 spaces.

Theorem 2.7.
A metric space X is sequentially compact iff it is complete and totally bounded.

Theorem 2.8.
A metric space is totally bounded iff every sequence has a Cauchy subsequence.

Theorem 2.9.
A metric space is compact iff it is complete and totally bounded.

Theorem 2.10 ***(Baire).***
If X is a complete metric space, then the intersection of countably many dense open sets is dense in X.

Theorem 2.11.
A continuous bijective open map is a homeomorphism.

Theorem 2.12.
A closed subset A of a compact set B is compact.

Proof .

- Let $\{A_i\} \rightrightarrows A$ be a covering of A by sets open in A.
- Each $A_i = B_i \bigcap A$ for some B_i open in B (definition of subspace topology)
- Define $V = \{B_i\}$, then $V \rightrightarrows A$ is an open cover.
- Since A is closed, $W := B \setminus A$ is open
- Then $V \bigcup W$ is an open cover of B, and has a finite subcover $\{V_i\}$
- Then $\left\{V_i \bigcap A\right\}$ is a finite open cover of A.

∎

Theorem 2.13.
The continuous image of a compact set is compact.

Theorem 2.14.
A closed subset of a Hausdorff space is compact.

2.2 Algebraic

Todo: Merge the two van Kampen theorems.

Theorem 2.15 ***(Van Kampen).***
The pushout is the northwest colimit of the following diagram

$$\begin{tikzcd}
A \coprod_Z B & A \arrow[l] \\
B \arrow[u] & Z \arrow[l, hook', "\iota_B"] \arrow[u, hook, "\iota_A"]
\end{tikzcd}$$

For groups, the pushout is given by the amalgamated free product: if $A = \left\langle G_A \;\middle|\; R_A \right\rangle, B = \left\langle G_B \;\middle|\; R_B \right\rangle$, then

$$A *_Z B = \left\langle G_A, G_B \;\middle|\; R_A, R_B, T \right\rangle$$

where T is a set of relations given by

$$T = \left\{ \iota_A(z)\iota_B(z)^{-1} \;\middle|\; z \in Z \right\}.$$

Suppose $X = U_1 \bigcup U_2$ such that $U_1 \bigcap U_2 \neq \emptyset$ is path connected. Then taking $x_0 \in U := U_1 \bigcap U_2$ yields a pushout of fundamental groups

$$\pi_1(X;x_0) = \pi_1(U_1;x_0) *_{\pi_1(U;x_0)} \pi_1(U_2;x_0).$$

Example 2.1.
$A = \mathbb{Z}/4\mathbb{Z} = \left\langle x \;\middle|\; x^4 \right\rangle, B = \mathbb{Z}/6\mathbb{Z} = \left\langle y \;\middle|\; x^6 \right\rangle, Z = \mathbb{Z}/2\mathbb{Z} = \left\langle z \;\middle|\; z^2 \right\rangle$. Then we can identify Z as a subgroup of A, B using $\iota_A(z) = x^2$ and $\iota_B(z) = y^3$. So

$$A *_Z B = \left\langle x, y \;\middle|\; x^4, y^6, x^2y^{-3} \right\rangle$$

.

Theorem 2.16 ***(Van Kampen).***
If $X = U \bigcup V$ where $U, V, U \bigcap V$ are all path-connected then

$$\pi_1(X) = \pi_1 U *_{\pi_1(U \bigcap V)} \pi_1 V,$$

where the amalgamated product can be computed as follows: If we have presentations

$$\begin{aligned}
\pi_1(U, w) &= \left\langle u_1, \cdots, u_k \;\middle|\; \alpha_1, \cdots, \alpha_l \right\rangle \\
\pi_1(V, w) &= \left\langle v_1, \cdots, v_m \;\middle|\; \beta_1, \cdots, \beta_n \right\rangle \\
\pi_1(U \cap V, w) &= \left\langle w_1, \cdots, w_p \;\middle|\; \gamma_1, \cdots, \gamma_q \right\rangle
\end{aligned}$$

then

$$\begin{aligned}
\pi_1(X, w) = {} & \langle u_1, \cdots, u_k, v_1, \cdots, v_m \rangle \\
& \quad \bmod \left\langle \alpha_1, \cdots, \alpha_l, \beta_1, \cdots, \beta_n, I\left(w_1\right) J\left(w_1\right)^{-1}, \cdots, I\left(w_p\right) J\left(w_p\right)^{-1} \right\rangle \\
= {} & \frac{\pi_1(U) * \pi_1(B)}{\left\langle \left\{ I(w_i) J(w_i)^{-1} \;\middle|\; 1 \le i \le p \right\} \right\rangle}
\end{aligned}$$

where

$$\begin{aligned}
I &: \pi_1(U \cap V, w) \to \pi_1(U, w) \\
J &: \pi_1(U \cap V, w) \to \pi_1(V, w).
\end{aligned}$$

3 Examples

3.1 Common Spaces and Operations

Point-Set:

- Finite discrete sets with the discrete topology
- Subspaces of $\mathbb{R}$: $(a, b), (a, b], (a, \infty)$, etc.
  - $\{0\} \bigcup \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{Z}^{\geq 1} \right\}$
- $\mathbb{Q}$
- The topologist's sine curve
- One-point compactifications
- $\mathbb{R}^\omega$
- Hawaiian earring
- Cantor set

Non-Hausdorff spaces:

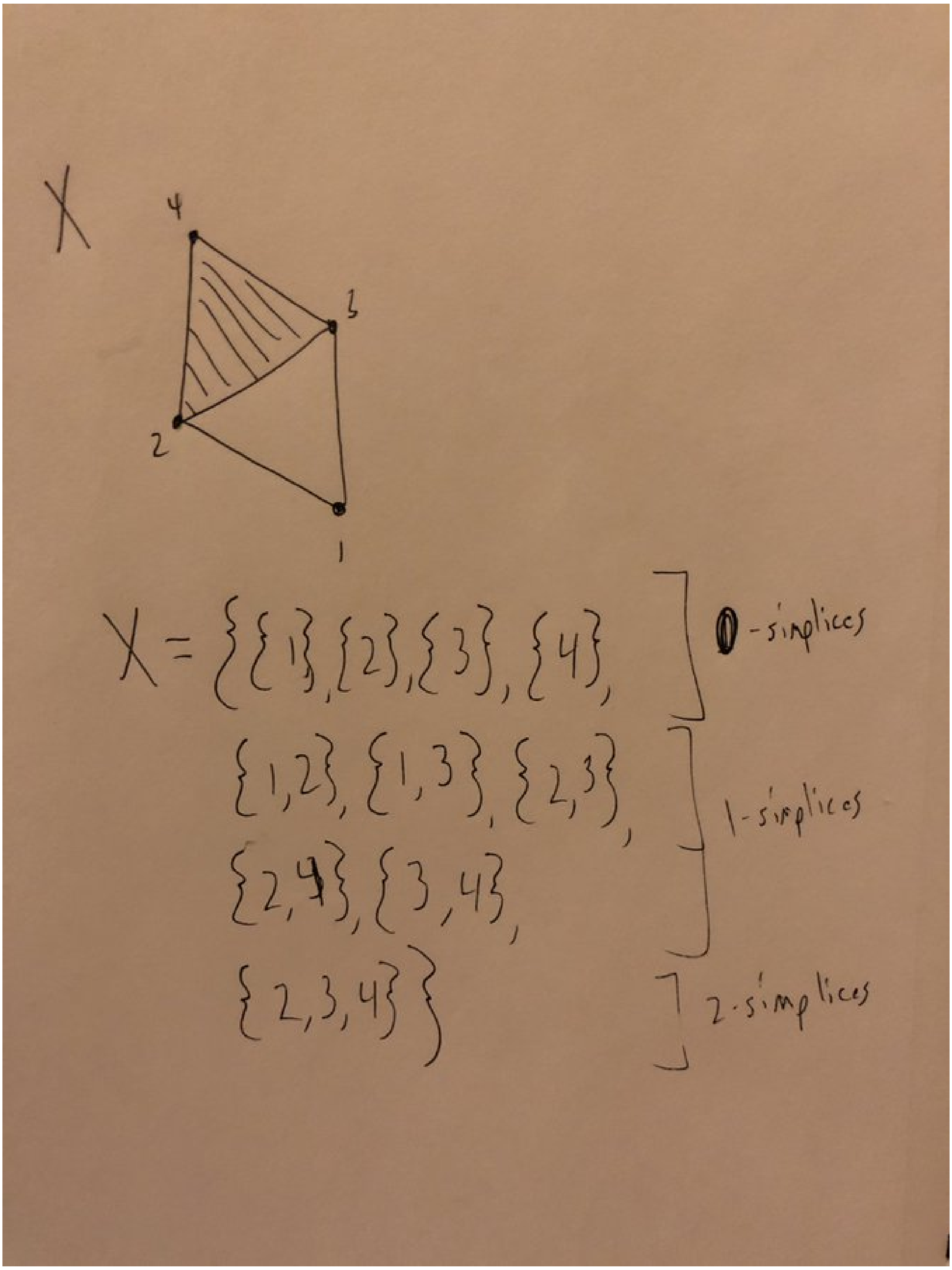
- The cofinite topology on any infinite set.
- $\mathbb{R}/\mathbb{Q}$
- The line with two origins.

General Spaces:

$$S^n, \mathbb{D}^n, T^n, \mathbb{RP}^n, \mathbb{CP}^n, \mathbb{M}, \mathbb{K}, \Sigma_g, \mathbb{RP}^\infty, \mathbb{CP}^\infty.$$

"Constructed" Spaces

- Knot complements in S^3
- Covering spaces (hyperbolic geometry)
- Lens spaces
- Matrix groups
- Prism spaces
- Pair of pants
- Seifert surfaces
- Surgery
- Simplicial Complexes
  - Nice minimal example:



Exotic/Pathological Spaces

- $\mathbb{HP}^n$
- Dunce Cap

- Horned sphere

Operations

- Cartesian product $A \times B$
- Wedge product $A \vee B$
- Connect Sum $A \# B$
- Quotienting A/B
- Puncturing $A \setminus \{a_i\}$
- Smash product
- Join
- Cones
- Suspension
- Loop space
- Identifying a finite number of points

3.2 Alternative Topologies

- Discrete
- Cofinite
- Discrete and Indiscrete
- Uniform

The cofinite topology:

- Non-Hausdorff
- Compact

The discrete topology:

- Discrete iff points are open
- Always Hausdorff
- Compact iff finite
- Totally disconnected
- If the domain, every map is continuous

The indiscrete topology:

- Only open sets are $\emptyset, X$
- Non-Hausdorff
- If the codomain, every map is continuous
- Compact

- Acyclic

- Alexander duality

- Basis

  - For an R-module M, a basis B is a linearly independent generating set.

- Boundary

- Boundary of a manifold

  - Points $x \in M^n$ defined by

$$\partial M = \{x \in M : H_n(M, M - \{x\}\,; \mathbb{Z}) = 0\}$$

- Cap Product
  - Denoting $\Delta^p \xrightarrow{\sigma} X \in C_p(X; G)$, a map that sends pairs (p-chains, q-cochains) to $(p-q)$-chains $\Delta^{p-q} \longrightarrow X$ by

$$\begin{aligned} H_p(X; R) \times H^q(X; R) &\xrightarrow{\frown} H_{p-q}(X; R) \\ \sigma \frown \psi &= \psi(F_0^q(\sigma))F_q^p(\sigma) \end{aligned}$$

    where F_i^j is the face operator, which acts on a simplicial map σ by restriction to the face spanned by $[v_i \dots v_j]$, i.e. $F_i^j(\sigma) = \sigma|_{[v_i \dots v_j]}$.

- Cellular Homology
- CW Cell
  - An n-cell of X, say e^n, is the image of a map $\Phi : B^n \longrightarrow X$. That is, $e^n = \Phi(B^n)$. Attaching an n-cell to X is equivalent to forming the space $B^n \coprod_f X$ where $f : \partial B^n \longrightarrow X$.
    * A 0-cell is a point.
    * A 1-cell is an interval $[-1, 1] = B^1 \subset \mathbb{R}^1$. Attaching requires a map from $S^0 = \{-1, +1\} \longrightarrow X$
    * A 2-cell is a solid disk $B^2 \subset \mathbb{R}^2$ in the plane. Attaching requires a map $S^1 \longrightarrow X$.
    * A 3-cell is a solid ball $B^3 \subset \mathbb{R}^3$. Attaching requires a map from the sphere $S^2 \longrightarrow X$.
- Cellular Map
  - A map $X \xrightarrow{f} Y$ is said to be cellular if $f(X^{(n)}) \subseteq Y^{(n)}$ where $X^{(n)}$ denotes the n- skeleton.
- Chain
  - An element $c \in C_p(X; R)$ can be represented as the singular p simplex $\Delta^p \longrightarrow X$.
- Chain Homotopy
  - Given two maps between chain complexes $(C_*, \partial_C) \xrightarrow{f,\, g} (D_*, \partial_D)$, a chain homotopy is a family $h_i : C_i \longrightarrow B_{i+1}$ satisfying

$$f_i - g_i = \partial_{B,i-1} \circ h_n + h_{i+1} \circ \partial_{A,i}$$

.

$$\begin{tikzcd}
\cdots \arrow[dr, "h_{n-2}"] & A_{n-1} \arrow[l, "d_{A,n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dr, "h_{n-1}"] & A_n \arrow[l, "d_{A,n}"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dr, "h_n"] & A_{n+1} \arrow[l, "d_{A,n+1}"'] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] \arrow[dr, "h_{n+1}"] & \cdots \arrow[l, "d_{A,n+2}"'] \\
\cdots & B_{n-1} \arrow[l, "d_{B,n-1}"'] & B_n \arrow[l, "d_{B,n}"'] & B_{n+1} \arrow[l, "d_{B,n+1}"'] & \cdots \arrow[l, "d_{B,n+2}"']
\end{tikzcd}$$

- Chain Map
  - A map between chain complexes $(C_*, \partial_C) \xrightarrow{f} (D_*, \partial_D)$ is a chain map iff each component $C_i \xrightarrow{f_i} D_i$ satisfies

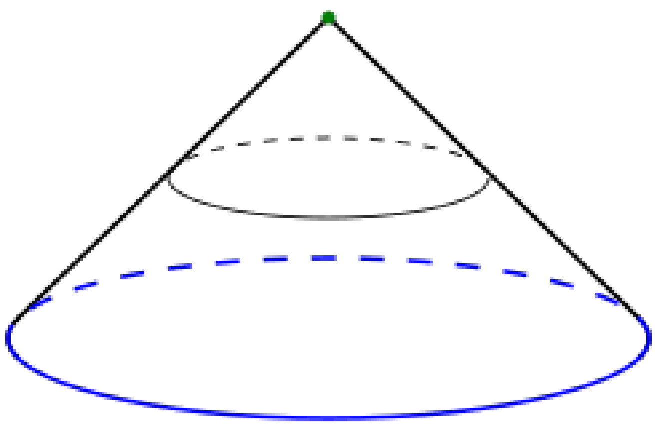
$$f_{i-1} \circ \partial_{C,i} = \partial_{D,i} \circ f_i$$

(i.e this forms a commuting ladder)

$$\begin{tikzcd}
\cdots & A_{n-1} \arrow[l, "d_{A,n-1}"'] \arrow[d, "f_{n-1}"'] & A_n \arrow[l, "d_{A,n}"'] \arrow[d, "f_n"'] & A_{n+1} \arrow[l, "d_{A,n+1}"'] \arrow[d, "f_{n+1}"'] & \cdots \arrow[l, "d_{A,n+2}"'] \\
\cdots & B_{n-1} \arrow[l, "d_{B,n-1}"'] & B_n \arrow[l, "d_{B,n}"'] & B_{n+1} \arrow[l, "d_{B,n+1}"'] & \cdots \arrow[l, "d_{B,n+2}"']
\end{tikzcd}$$

- Closed manifold
  - A manifold that is compact, with or without boundary.
- Coboundary
- Cochain
  - An cochain $c \in C^p(X; R)$ is a map $c \in \hom(C_p(X; R), R)$ on chains.
- Cocycle
- Colimit
- Compact
  - A space X is compact iff every open cover of X has a finite subcover.
- Cone
  - For a space X, defined as

$$CX = \frac{X \times I}{X \times \{0\}}$$



Example: The cone on the circle CS^1

Note that the cone embeds X in a contractible space CX.

- Contractible
  - A space is contractible if its identity map is nullhomotopic.
- Contractible
- Coproduct
- Covering Space
- Cup Product
  - A map taking pairs (p-cocycles, q-cocycles) to $(p+q)$-cocyles by

$$\begin{aligned} H^p(X;R) \times H^q(X;R) &\xrightarrow{\smile} H^{p+q}(X;R) \\ (a \cup b)(\sigma) &= a(\sigma \circ I_0^p)\ b(\sigma \circ I_p^{p+q}) \end{aligned}$$

    where $\Delta^{p+q} \xrightarrow{\sigma} X$ is a singular $p+q$ simplex and

$$I_i^j : [i, \cdots, j] \hookrightarrow \Delta^{p+q}$$

    is an embedding of the $(j-i)$-simplex into a $(p+q)$-simplex. On a manifold, the cup product is Poincare dual to the intersection of submanifolds.
  - Applications
    * $T^2 \not\simeq S^2 \vee S^1 \vee S^1$.

Proof: todo

- CW Complex
- Cycle
- Deck Transformation
- Deformation
- Deformation Retract
  - A map r in $A \overset{\iota}{\underset{r}{\rightleftarrows}} X$ that is a retraction (so $r \circ \iota = \mathrm{id}_A$) **that also satisfies** $\iota \circ r \simeq \mathrm{id}_X$.
  - Note that this is equality in one direction, but only homotopy equivalence in the other.
- Degree of a Map
- Derived Functor
  - For a functor T and an R-module A, a *left derived functor* $(L_n T)$ is defined as $h_n(TP_A)$, where P_A is a projective resolution of A.
- Dimension of a manifold
  - For $x \in M$, the only nonvanishing homology group $H_i(M, M - \{x\}; \mathbb{Z})$
- Direct Limit
- Direct Product
- Direct Sum

- Eilenberg-MacLane Space
- Euler Characteristic
- Exact Functor
  - A functor T is *right exact* if a short exact sequence

    $$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

    yields an exact sequence

    $$\dots TA \longrightarrow TB \longrightarrow TC \longrightarrow 0,$$

    and is *left exact* if it yields

    $$0 \longrightarrow TA \longrightarrow TB \longrightarrow TC \longrightarrow \dots$$

    Thus a functor is exact iff it is both left and right exact, yielding

    $$0 \longrightarrow TA \longrightarrow TB \longrightarrow TC \longrightarrow 0$$

  - Examples:
    * $\cdot \otimes_R \cdot$ is a right exact bifunctor.
- Exact Sequence
- Excision
- Ext Group
- Flat
  - An R-module is flat if $A \otimes_R \cdot$ is an exact functor.
- Free and Properly Discontinuous
- Free module
  - A -module M with a basis $S = \{s_i\}$ of generating elements. Every such module is the image of a unique map $\mathcal{F}(S) = R^S \twoheadrightarrow M$, and if $M = \left\langle S \mathrel{}\middle|\mathrel{} \mathcal{R} \right\rangle$ for some set of relations $\mathcal{R}$, then $M \cong R^S/\mathcal{R}$.
- Free Product
- Free product with amalgamation
- Fundamental Class
  - For a connected, closed, orientable manifold, $[M]$ is a generator of $H_n(M;\mathbb{Z}) = \mathbb{Z}$.
- Fundamental classes
- Fundamental Group
- Generating Set

  - $S = \{s_i\}$ is a generating set for an R- module M iff

$$x \in M \implies x = \sum r_i s_i$$

    for some coefficients $r_i \in R$ (where this sum may be infinite).

- Gluing Along a Map
- Group Ring
- Homologous
- Homotopic
- Homotopy
- Homotopy Class
- Homotopy Equivalence
- Homotopy Extension Property
- Homotopy Groups
- Homotopy Lifting Property
- Injection
  - A map ι with a **left** inverse f satisfying $f \circ \iota = \mathrm{id}$
- Intersection Pairing For a manifold M, a map on homology defined by

$$\begin{aligned} H_{\widehat{i}} M \otimes H_{\widehat{j}} M &\longrightarrow H_{\widehat{i+j}} X \\ \alpha \otimes \beta &\mapsto \langle \alpha, \beta \rangle \end{aligned}$$

  obtained by conjugating the cup product with Poincare Duality, i.e.

$$\langle \alpha, \beta \rangle = [M] \frown ([\alpha]^\vee \smile [\beta]^\vee)$$

  Then, if $[A], [B]$ are transversely intersecting submanifolds representing α, β, then

$$\langle \alpha, \beta \rangle = [A \bigcap B]$$

  .

  If $\widehat{i} = j$ then $\langle \alpha, \beta \rangle \in H_0 M = \mathbb{Z}$ is the signed number of intersection points.

- Inverse Limit
- Intersection Pairing
  - The pairing obtained from dualizing Poincare Duality to obtain

$$\mathrm{F}(H_i M) \otimes \mathrm{F}(H_{n-i} M) \longrightarrow \mathbb{Z}$$

    Computed as an oriented intersection number between two homology classes (perturbed to be transverse).

- Intersection Form
  - The nondegenerate bilinear form cohomology induced by the Kronecker Pairing:
$$I : H^k(M_n) \times H^{n-k}(M^n) \longrightarrow \mathbb{Z}$$
    where $n = 2k$.
    * When k is odd, I is skew-symmetric and thus a *symplectic form.*
    * When k is even (and thus $n \equiv 0 \mod 4$) this is a symmetric form.
    * Satisfies $I(x, y) = (-1)^{k(n-k)} I(y, x)$
- Kronecker Pairing
  - A map pairing a chain with a cochain, given by
$$\begin{aligned} H^n(X; R) \times H_n(X; R) &\longrightarrow R \\ ([\psi, \alpha]) &\mapsto \psi(\alpha) \end{aligned}$$
    which is a nondegenerate bilinear form.
- Kronecker Product
- Lefschetz duality
- Lefschetz Number
- Lens Space
- Local Degree
  - At a point $x \in V \subset M$, a generator of $H_n(V, V - \{x\})$. The degree of a map $S^n \longrightarrow S^n$ is the sum of its local degrees.
- Local Orientation
- Limit
- Linear Independence
  - A generating S for a module M is linearly independent if $\sum r_i s_i = 0_M \implies \forall i,\ r_i = 0$ where $s_i \in S, r_i \in R$.
- Local homology
  - $H_n(X, X - A; \mathbb{Z})$ is the local homology at A, also denoted $H_n(X \mid A)$
- Local Homology
- Local orientation of a manifold
  - At a point $x \in M^n$, a choice of a generator μ_x of $H_n(M, M - \{x\}) = \mathbb{Z}$.
- Long exact sequence
- Loop Space
- Manifold
  - An n-manifold is a Hausdorff space in which each neighborhood has an open neighborhood homeomorphic to $\mathbb{R}^n$.

- Manifold with boundary
  - A manifold in which open neighborhoods may be isomorphic to either $\mathbb{R}^n$ or a half-space $\left\{\mathbf{x} \in \mathbb{R}^n \;\middle|\; x_i > 0\right\}$.
- Mapping Cone
- Mapping Cylinder
- Mapping Path Space
- Mayer-Vietoris Sequence
- Monodromy
- Moore Space
- N-cell
- N-connected
- Nullhomotopic
  - A map $X \xrightarrow{f} Y$ is nullhomotopic if it is homotopic to a constant map $X \xrightarrow{c} \{y_0\}$; that is, there exists a homotopy
- Orientable manifold
  - A manifold for which an orientation exists, see "Orientation of a Manifold".
- Orientation Cover
  - For any manifold M, a two sheeted orientable covering space $\tilde{M}_o$. M is orientable iff $\tilde{M}$ is disconnected. Constructed as

$$\tilde{M} = \coprod_{x \in M} \left\{\mu_x \;\middle|\; \mu_x \text{ is a local orientation}\right\}$$

- Orientation of a manifold
  - A family of $\{\mu_x\}_{x \in M}$ with local consistency: if $x, y \in U$ then μ_x, μ_y are related via a propagation.
    * Formally, a function

$$\begin{aligned} M^n &\longrightarrow \coprod_{x \in M} H(X \mid \{x\}) \\ x &\mapsto \mu_x \end{aligned}$$

      such that $\forall x \exists N_x$ in which $\forall y \in N_x$, the preimage of each μ_y under the map $H_n(M \mid N_x) \twoheadrightarrow H_n(M \mid y)$ is a single generator μ_{N_x}.
  - TFAE:
    * M is orientable.
    * The map $W : (M, x) \longrightarrow \mathbb{Z}_2$ is trivial.
    * $\tilde{M}_o = M \coprod \mathbb{Z}_2$ (two sheets).
    * $\tilde{M}_o$ is disconnected
    * The projection $\tilde{M}_o \longrightarrow M$ admits a section.

- Oriented manifold
- Path
- Path Lifting Property
- Perfect Pairing
  - A pairing alone is an R-bilinear module map, or equivalently a map out of a tensor product since $p : M \otimes_R N \longrightarrow L$ can be partially applied to yield $\varphi : M \longrightarrow L^N = \hom_R(N, L)$. A pairing is **perfect** when φ is an isomorphism.
    * Example: $\det_M : k^2 \times k^2 \longrightarrow k$
- Poincare Duality
  - For a closed, orientable n-manifold, following map $[M] \frown \cdot$ is an isomorphism:

$$\begin{aligned} D : H^k(M; R) &\longrightarrow H_{n-k}(M; R) \\ D(\alpha) &= [M] \frown \alpha \end{aligned}$$

- Projective Resolution
- Properly Discontinuous
- Pullback
- Pushout
- Quasi-isomorphism
- R-orientability
- Relative boundaries
- Relative cycles
- Relative homotopy groups
- Retraction
  - A map r in $A \underset{r}{\overset{\iota}{\rightleftarrows}} X$ satisfying

$$r \circ \iota = \mathrm{id}_A.$$

  Equivalently $X \twoheadrightarrow_r A$ and $r|_A = \mathrm{id}_A$. If X retracts onto A, then i_* is injective.
- Short exact sequence
- Simplicial Complex
- Simplicial Map
  - For a map

$$K \xrightarrow{f} L$$

  between simplicial complexes, f is a simplicial map if for any set of vertices $\{v_i\}$ spanning a simplex in K, the set $\{f(v_i)\}$ are the vertices of a simplex in L.

- Simply Connected
- Singular Chain

$$x \in C_n(x) \implies X = \sum_i n_i \sigma_i = \sum_i n_i(\Delta^n \xrightarrow{\sigma_i} X)$$

- Singular Cochain

$$x \in C^n(x) \implies X = \sum_i n_i \psi_i = \sum_i n_i(\sigma_i \xrightarrow{\psi_i} X)$$

- Singular Homology
- Smash Product
- Surjection
  - A map π with a **right** inverse f satisfying

$$\pi \circ f = \mathrm{id}$$

- Suspension Compact represented as $\Sigma X = CX \coprod_{\mathrm{id}_X} CX$, two cones on X glued along X. Explicitly given by

$$\Sigma X = \frac{X \times I}{(X \times \{0\}) \bigcup (X \times \{1\}) \bigcup (\{x_0\} \times I)}$$

- Tor Group
  - For an R-module

$$\mathrm{Tor}_R^n(\,\cdot\,, B) = L_n(\,\cdot\, \otimes_R B)$$

    where L_n denotes the nth left derived functor.

- Universal Cover
- Universal Coefficient Theorem for Cohomology
- Universal Coefficient Theorem for Change of Coefficient Ring
- Weak Homotopy Equivalence
- Weak Topology
- Wedge Product

4 Notation

- C_X
- $\Sigma(X)$
- Σ_g
- ι, π
- $\widehat{i+j}$: for an n-dimensional manifold, the "dual" dimension $\widehat{i+j} \coloneqq n - (i+j)$.