Real Analysis Qualifying Exam Notes

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Contents

1 Practice Exam 2 (November 2014)

1.1 1: Fubini-Tonelli

1.1.1 a

Carefully state Tonelli's theorem for a nonnegative function $F(x,t)$ on $\mathbb{R}^n \times \mathbb{R}$.

1.1.2 b

Let $f : \mathbb{R}^n \longrightarrow [0, \infty]$ and define

$$\mathcal{A} \coloneqq \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; 0 \le t \le f(x) \right\}.$$

Prove the validity of the following two statements:

1. f is Lebesgue measurable on $\mathbb{R}^n$ $\iff$ $\mathcal{A}$ is a Lebesgue measurable subset of $\mathbb{R}^{n+1}$.
2. If f is Lebesgue measurable on $\mathbb{R}^n$ then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)dx = \int_0^\infty m\left(\left\{ x \in \mathbb{R}^n \;\middle|\; f(x) \ge t \right\} \right) dt.$$

Solution

- Define $A_y = \left\{ x \in \mathbb{R}^n \;\middle|\; (x,y) \in A \right\}$, and notice that $A_y = \left\{ x \in \mathbb{R}^n \;\middle|\; 0 \le y \le f(x) \right\}$.
- By the corollary, A_y is measurable and

$$m(\mathcal{A}) = \int m\left(\mathcal{A}_y\right) dy = \int_0^y m\left(\left\{ x \in \mathbb{R}^n \;\middle|\; f(x) \ge y \right\} \right) dy$$

1.2 2: Convolutions and the Fourier Transform

1.2.1 a

?

1.2.2 b

Facts:

- $\displaystyle\int \varphi = \int \varphi_t = 1$
- f bounded and uniformly continuous $\implies f * \varphi_t \rightrightarrows f$

Theorem (Norm Convergence of Approximate Identities):

$$\|f * \varphi_t - f\|_1 \xrightarrow{t \longrightarrow 0} 0.$$

Proof:

$$\begin{aligned}
\|f - f * \varphi_t\|_1 &= \int f(x) - \int f(x-y)\varphi_t(y)\ dydx \\
&= \int f(x) \int \varphi_t(y)\ dy - \int f(x-y)\varphi_t(y)\ dydx \\
&= \int \int \varphi_t(y)[f(x) - f(x-y)]\ dydx \\
&=_{FT} \int \int \varphi_t(y)[f(x) - f(x-y)]\ dxdy \\
&= \int \varphi_t(y) \int f(x) - f(x-y)\ dxdy \\
&= \int \varphi_t(y) \|f - \tau_y f\|_1 dy \\
&= \int_{y<\delta} \varphi_t(y) \|f - \tau_y f\|_1 dy + \int_{y \geq \delta} \varphi_t(y) \|f - \tau_y f\|_1 dy \\
&\leq \int_{y<\delta} \varphi_t(y)\varepsilon + \int_{y \geq \delta} \varphi_t(y) \left(\|f\|_1 + \|\tau_y f\|_1\right) dy \quad \text{by continuity in } L^1 \\
&\leq \varepsilon + 2\|f\|_1 \int_{y \geq \delta} \varphi_t(y) dy \\
&\leq \varepsilon + 2\|f\|_1 \varepsilon \quad \text{since } \varphi_t \text{ has small tails} \\
&\longrightarrow 0 \blacksquare.
\end{aligned}$$

Theorem (Convolutions Vanish at Infinity)

$$f, g \in L^1 \text{ and bounded} \implies \lim_{|x| \to \infty} (f * g)(x) = 0.$$

Proof:

- Choose $M \geq f, g$.
- By small tails, choose N such that $\displaystyle\int_{B_N^c} |f|, \int_{B_n^c} |g| < \varepsilon$
- Note

$$|f * g| \leq \int |f(x-y)|\ |g(y)|\ dy := I$$

- Use $|x| \leq |x-y| + |y|$, take $|x| \geq 2N$ so either

-
$$|x-y| \geq N \implies I \leq \int_{\{x-y \geq N\}} |f(x-y)| M \; dy \leq \varepsilon M \longrightarrow 0$$

-
$$|y| \geq N \implies I \leq \int_{\{y \geq N\}} M |g(y)| \; dy \leq M\varepsilon \longrightarrow 0$$

■

1.2.3 c

Definition (The Fourier Transform):

$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi} \; dx.$$

Facts:

- *Riemann-Lebesgue lemma:* $\widehat{f}$ vanishes at infinity
- $f \in L^1 \implies \widehat{f}$ is bounded and uniformly continuous
- $f, \widehat{f} \in L^1 \implies f$ is bounded, uniformly continuous, and vanishes at infinity
- $f \in L^1$ and $\widehat{f} = 0$ almost everywhere $\implies f = 0$ almost everywhere.

Theorem (Fourier Inversion):

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(x) e^{2\pi i x \cdot \xi} d\xi.$$

Proof: Idea: Fubini-Tonelli doesn't work directly, so introduce a convergence factor, take limits, and use uniqueness of limits.

Use the following facts:

- $f, g \in L^1 \implies \displaystyle\int \widehat{f} g = \int f \widehat{g}.$
- $g(x) \coloneqq e^{-\pi |t|^2} \implies \widehat{g}(\xi) = g(\xi)$
- $g_t(x) = t^{-n} g(x/t) = t^{-n} e^{-\pi |x|^2 / t^2}$
- $\widehat{g_t}(x) = g(tx) = e^{-\pi t^2 |x|^2}$
- $\varphi(\xi) \coloneqq e^{2\pi i x \cdot \xi} \; \widehat{g_t}(\xi)$
- $\widehat{\varphi}(\xi) = \mathcal{F}(\widehat{g_t}(\xi - x)) = g_t(x - \xi)$
- $\lim_{t \longrightarrow 0} \varphi(\xi) = e^{2\pi i x \cdot \xi}$

Take the modified integral:

$$\begin{aligned}
I_t(x) &= \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}\ e^{-\pi t^2 |\xi|^2} \\
&= \int \widehat{f}(\xi)\varphi(\xi) \\
&= \int f(\xi)\widehat{\varphi}(\xi) \\
&= \int f(\xi)\widehat{\widehat{g}}(\xi - x) \\
&= \int f(\xi) g_t(x-\xi)\ d\xi \\
&= \int f(y-x) g_t(y)\ dy \quad (\xi = y - x) \\
&= (f * g_t) \\
&\longrightarrow f \text{ in } L^1 \text{ as } t \longrightarrow 0,
\end{aligned}$$

but we also have

$$\begin{aligned}
\lim_{t\longrightarrow 0} I_t(x) &= \lim_{t\longrightarrow 0} \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}\ e^{-\pi t^2 |\xi|^2} \\
&= \lim_{t\longrightarrow 0} \int \widehat{f}(\xi)\varphi(\xi) \\
&=_{DCT} \int \widehat{f}(\xi) \lim_{t\longrightarrow 0} \varphi(\xi) \\
&= \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}
\end{aligned}$$

.

So

$$I_t(x) \longrightarrow \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi} \text{ pointwise and } \|I_t(x) - f(x)\|_1 \longrightarrow 0.$$

So there is a subsequence I_{t_n} such that $I_{t_n}(x) \longrightarrow f(x)$ almost everywhere, so $f(x) = \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}$ almost everywhere by uniqueness of limits. ■

1.3 3: Hilbert Spaces

1.3.1 a

Theorem (Bessel's Inequality):

$$\left\| x - \sum_{n=1}^{N} \langle x, u_n\rangle\, u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n\rangle|^2$$

and thus

$$\sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 \leq \|x\|^2$$

Proof: Let $S_N = \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n$

$$\begin{aligned}
\|x - S_N\|^2 &= \langle x - S_n,\ x - S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\langle x,\ S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\left\langle x,\ \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n \right\rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^{N} \langle x,\ \langle x,\ u_n \rangle u_n \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^{N} \overline{\langle x,\ u_n \rangle} \langle x,\ u_n \rangle \\
&= \|x\|^2 + \left\| \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n \right\|^2 - 2 \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 - 2 \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
&= \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2.
\end{aligned}$$

And by continuity of the norm and inner product, we have

$$\begin{aligned}
\lim_{N \longrightarrow \infty} \|x - S_N\|^2 &= \lim_{N \longrightarrow \infty} \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
\implies \left\| x - \lim_{N \longrightarrow \infty} S_N \right\|^2 &= \|x\|^2 - \lim_{N \longrightarrow \infty} \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
\implies \left\| x - \sum_{n=1}^{\infty} \langle x,\ u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2.
\end{aligned}$$

Then noting that $0 \leq \|x - S_N\|^2$, we have

$$\begin{aligned}
0 &\leq \|x\|^2 - \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2 \\
\implies \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2 &\leq \|x\|^2 \blacksquare.
\end{aligned}$$

1.3.2 b

- Take $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
- Define $x = \lim_{N \longrightarrow \infty} S_N$ where $S_N = \sum_{k=1}^{N} a_k u_k$

- $\{S_N\}$ is Cauchy and H is complete, so $x \in H$.
- By construction, $\langle x,\ u_n \rangle = \left\langle \sum_k a_k u_k,\ u_n \right\rangle = \sum_k a_k \langle u_k,\ u_n \rangle = a_n$ since the u_k are all orthogonal.
- $\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$ by Pythagoras since the u_k are normal.

1.3.3 c

Let x and u_n be arbitrary. Then

$$\begin{aligned}
\left\langle x - \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k,\ u_n \right\rangle &= \langle x,\ u_n \rangle - \left\langle \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k,\ u_n \right\rangle \\
&= \langle x,\ u_n \rangle - \sum_{k=1}^{\infty} \langle \langle x,\ u_k \rangle u_k,\ u_n \rangle \\
&= \langle x,\ u_n \rangle - \sum_{k=1}^{\infty} \langle x,\ u_k \rangle \langle u_k,\ u_n \rangle \\
&= \langle x,\ u_n \rangle - \langle x,\ u_n \rangle = 0 \\
\implies x - \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k = 0 \quad & \text{by completeness.}
\end{aligned}$$

So

$$x = \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k \implies \|x\|^2 = \sum_{k=1}^{\infty} |\langle x,\ u_k \rangle|^2. \blacksquare$$

1.4 4: Lp Spaces

p-test for integrals:

$$\begin{aligned}
\int_0^1 x^{-p} < \infty &\iff p < 1 \\
\int_1^{\infty} x^{-p} < \infty &\iff p > 1.
\end{aligned}$$

Yields a general technique: break integrals apart at $x = 1$.

Inclusions and relationships:

$$\begin{aligned}
m(X) < \infty &\implies L^{\infty} \subset L^2 \subset L^1 \\
& \ell^2(\mathbb{Z}) \subset \ell^1(\mathbb{Z}) \subset \ell^{\infty}(\mathbb{Z}).
\end{aligned}$$

1.4.1 a

Theorem (Holder's Inequality):

$$\|fg\|_1 \le \|f\|_p \|g\|_q.$$

Proof:

It suffices to show this when $\|f\|_p = \|g\|_q = 1$, since

$$\|fg\|_1 \le \|f\|_p \|f\|_q \iff \int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \le 1.$$

Using $AB \le \frac{1}{p}A^p + \frac{1}{q}B^q$, we have

$$\int |f||g| \le \int \frac{|f|^p}{p} \frac{|g|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1 \blacksquare.$$

Theorem (Minkowski's Inequality):

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

Proof:

We first note

$$|f+g|^p = |f+g||f+g|^{p-1} \le (|f| + |g|) |f+g|^{p-1}.$$

Note that if p, q are conjugate exponents then

$$\begin{aligned} \frac{1}{q} &= 1 - \frac{1}{p} = \frac{p-1}{p} \\ q &= \frac{p}{p-1}. \end{aligned}$$

Then taking integrals yields

$$\begin{aligned}
\|f+g\|_p^p &= \int |f+g|^p \\
&\leq \int (|f|+|g|)\,|f+g|^{p-1} \\
&= \int |f||f+g|^{p-1} + \int |g||f+g|^{p-1} \\
&= \left\| f(f+g)^{p-1} \right\|_1 + \left\| g(f+g)^{p-1} \right\|_1 \\
&\leq \|f\|_p \left\| (f+g)^{p-1}) \right\|_q + \|g\|_p \left\| (f+g)^{p-1}) \right\|_q \\
&= \left(\|f\|_p + \|g\|_p \right) \left\| (f+g)^{p-1}) \right\|_q \\
&= \left(\|f\|_p + \|g\|_p \right) \left(\int |f+g|^{(p-1)q} \right)^{\frac{1}{q}} \\
&= \left(\|f\|_p + \|g\|_p \right) \left(\int |f+g|^{p} \right)^{1-\frac{1}{p}} \\
&= \left(\|f\|_p + \|g\|_p \right) \frac{\int |f+g|^p}{\left(\int |f+g|^p\right)^{\frac{1}{p}}} \\
&= \left(\|f\|_p + \|g\|_p \right) \frac{\|f+g\|_p^p}{\|f+g\|_p}
\end{aligned}$$

and canceling common terms yields

$$\begin{aligned}
1 &\leq \left(\|f\|_p + \|g\|_p \right) \frac{1}{\|f+g\|_p} \\
&\implies \|f+g\|_p \leq \|f\|_p + \|g\|_p \blacksquare.
\end{aligned}$$

1.4.2 c

Definition (Infinity Norm):

$$\begin{aligned}
L^\infty(X) = \left\{ f : X \longrightarrow \mathbb{C} \;\middle|\; \|f\|_\infty < \infty \right\} \\
\text{where} \\
\|f\|_\infty = \inf_{\alpha \geq 0} \left\{ \alpha \;\middle|\; m\left\{ |f| \geq \alpha \right\} = 0 \right\}.
\end{aligned}$$

Theorem:

$$m(X) < \infty \implies \lim_{p \longrightarrow \infty} \|f\|_p = \|f\|_\infty.$$

Proof: Let $M = \|f\|_\infty$. For any $L < M$, let $S = \{|f| \geq L\}$. Then $m(S) > 0$ and

$$\begin{aligned}
\|f\|_p &= \left(\int_X |f|^p\right)^{\frac{1}{p}} \\
&\geq \left(\int_S |f|^p\right)^{\frac{1}{p}} \\
&\geq L\ m(S)^{\frac{1}{p}} \xrightarrow{p \longrightarrow \infty} L \\
&\implies \liminf_p \|f\|_p \geq M.
\end{aligned}$$

We also have

$$\begin{aligned}
\|f\|_p &= \left(\int_X |f|^p\right)^{\frac{1}{p}} \\
&\leq \left(\int_X M^p\right)^{\frac{1}{p}} \\
&= M\ m(X)^{\frac{1}{p}} \xrightarrow{p \longrightarrow \infty} M \\
&\implies \limsup_p \|f\|_p \leq M\blacksquare.
\end{aligned}$$

Note: this doesn't help with this problem at all.

Solution:

$$\begin{aligned}
\int f^p &= \int_{x\leq 1} f^p + \int_{x=1} f^p + \int_{x\geq 1} f^p \\
&= \int_{x\leq 1} f^p + \int_{x=1} 1 + \int_{x\geq 1} f^p \\
&= \int_{x\leq 1} f^p + m(\{f=1\}) + \int_{x\geq 1} f^p \\
&\xrightarrow{p \longrightarrow \infty} 0 + m(\{f=1\}) + \begin{cases} 0 & m(\{x \geq 1\}) = 0 \\ \infty & m(\{x \geq 1\}) > 0. \end{cases}
\end{aligned}$$

1.5 5: Dual Spaces

Definition: A map $L : X \longrightarrow \mathbb{C}$ is a linear functional iff

$$L(\alpha\mathbf{x} + \mathbf{y}) = \alpha L(\mathbf{x}) + L(\mathbf{y}).$$

Theorem (Riesz Representation for Hilbert Spaces): If Λ is a continuous linear functional on a Hilbert space H, then there exists a unique $y \in H$ such that

$$\forall x \in H, \quad \Lambda(x) = \langle x,\ y\rangle.$$

Proof:

- Define $M \coloneqq \ker \Lambda$.
- Then M is a closed subspace and so $H = M \oplus M^\perp$
- There is some $z \in M^\perp$ such that $\|z\| = 1$.
- Set $u \coloneqq \Lambda(x)z - \Lambda(z)x$
- Check

$$\Lambda(u) = \Lambda(\Lambda(x)z - \Lambda(z)x) = \Lambda(x)\Lambda(z) - \Lambda(z)\Lambda(x) = 0 \implies u \in M$$

- Compute

$$\begin{aligned}
0 &= \langle u,\ z\rangle \\
&= \langle \Lambda(x)z - \Lambda(z)x,\ z\rangle \\
&= \langle \Lambda(x)z,\ z\rangle - \langle \Lambda(z)x,\ z\rangle \\
&= \Lambda(x)\langle z,\ z\rangle - \Lambda(z)\langle x,\ z\rangle \\
&= \Lambda(x)\|z\|^2 - \Lambda(z)\langle x,\ z\rangle \\
&= \Lambda(x) - \Lambda(z)\langle x,\ z\rangle \\
&= \Lambda(x) - \left\langle x,\ \overline{\Lambda(z)}z\right\rangle,
\end{aligned}$$

- Choose $y \coloneqq \overline{\Lambda(z)}z$.
- Check uniqueness:

$$\begin{aligned}
\langle x,\ y\rangle = \langle x,\ y'\rangle \quad \forall x \\
\implies \langle x,\ y - y'\rangle = 0 \quad \forall x \\
\implies \langle y - y',\ y - y'\rangle = 0 \\
\implies \|y - y'\| = 0 \\
\implies y - y' = \mathbf{0} \implies y = y'.
\end{aligned}$$

1.5.1 b

Theorem (Continuous iff Bounded): Let $L : X \longrightarrow \mathbb{C}$ be a linear functional, then the following are equivalent:

1. L is continuous
2. L is continuous at zero
3. L is bounded, i.e. $\exists c \geq 0 \mathrel{\Big|} |L(x)| \leq c\|x\|$ for all $x \in H$

$2 \implies 3$: Choose $\delta < 1$ such that

$$\|x\| \leq \delta \implies |L(x)| < 1.$$

Then

$$\begin{aligned}
|L(x)| &= \left| L\left(\frac{\|x\|}{\delta} \frac{\delta}{\|x\|} x \right) \right| \\
&= \frac{\|x\|}{\delta} \left| L\left(\delta \frac{x}{\|x\|} \right) \right| \\
&\leq \frac{\|x\|}{\delta} 1,
\end{aligned}$$

so we can take $c = \dfrac{1}{\delta}$. ■

$3 \implies 1$:

We have $|L(x-y)| \leq c\|x-y\|$, so given $\varepsilon \geq 0$ simply choose $\delta = \dfrac{\varepsilon}{c}$.

1.5.2 c

Definition (Dual Space):

$$X^\vee := \left\{ L : X \longrightarrow \mathbb{C} \;\middle|\; L \text{ is continuous } \right\}$$

Definition (Operator Norm):

$$\|L\|_{X^\vee} := \sup_{\substack{x \in X \\ \|x\|=1}} |L(x)|$$

Theorem: (Operator Norm is a Norm)

Proof: The only nontrivial property is the triangle inequality, but

$$\|L_1 + L_2\| = \sup |L_1(x) + L_2(x)| \leq \sup L_1(x) + \sup L_2(x) = \|L_1\| + \|L_2\|.$$

Theorem (Completeness in Operator Norm):

$X^\vee$ equipped with the operator norm is a Banach space.

Proof:

- Let $\{L_n\}$ be Cauchy in $X^\vee$.
- Then for all $x \in C$, $\{L_n(x)\} \subset \mathbb{C}$ is Cauchy and converges to something denoted $L(x)$.
- Need to show L is continuous and $\|L_n - L\| \longrightarrow 0$.
- Since $\{L_n\}$ is Cauchy in $X^\vee$, choose N large enough so that

$$n, m \geq N \implies \|L_n - L_m\| < \varepsilon \implies |L_m(x) - L_n(x)| < \varepsilon \quad \forall x \;\middle|\; \|x\| = 1.$$

- Take $n \longrightarrow \infty$ to obtain

$$\begin{aligned}
m \geq N &\implies |L_m(x) - L(x)| < \varepsilon \quad \forall x \;\middle|\; \|x\| = 1 \\
&\implies \|L_m - L\| < \varepsilon \longrightarrow 0.
\end{aligned}$$

- Continuity:

$$\begin{aligned}|L(x)| &= |L(x) - L_n(x) + L_n(x)| \\ &\leq |L(x) - L_n(x)| + |L_n(x)| \\ &\leq \varepsilon\|x\| + c\|x\| \\ &= (\varepsilon + c)\|x\| \blacksquare.\end{aligned}$$

2 Exam 2 (2018)

Link to PDF File

3 Exam 2 (2014)

Link to PDF File

4 Qual: Fall 2019

4.1 1

See phone photo?

4.2 2

DCT?

4.3 3

“Follow your nose.”

4.4 4

See Problem Set 8.

Bessel's Inequality: For any orthonormal set in a Hilbert space (not necessarily a basis), we have

$$\sum_{n=1}^{\infty} |\langle x,\ u_n\rangle|^2 \leq \|x\|^2$$

Proof:

$$0 \leq \left\| x - \sum_{k=1}^{n} \langle x, e_k \rangle \, e_k \right\|^2$$

Corollary (Parseval's Identity): If span $\{u_n\}$ is dense in $\mathcal{H}$, so it is a basis, then this is an equality.

Riesz-Fischer: Let $U = \{u_n\}_{n=1}^{\infty}$ be an orthonormal set (not necessarily a basis), then

1. There is an isometric surjection

$$\begin{aligned}\mathcal{H} &\longrightarrow \ell^2(\mathbb{N}) \\ \mathbf{x} &\mapsto \{\langle \mathbf{x},\ \mathbf{u}_n \rangle\}_{n=1}^{\infty}\end{aligned}$$

i.e. if $\{a_n\} \in \ell^2(\mathbb{N})$, so $\sum |a_n|^2 < \infty$, then there exists a $\mathbf{x} \in \mathcal{H}$ such that

$$a_n = \langle \mathbf{x},\ \mathbf{u}_n \rangle \quad \forall n.$$

2. $\mathbf{x}$ can be chosen such that

$$\|\mathbf{x}\|^2 = \sum |a_n|^2$$

Note: the choice of $\mathbf{x}$ is unique $\iff$ $\{u_n\}$ is **complete**, i.e. $\langle \mathbf{x},\ \mathbf{u}_n \rangle = 0$ for all n implies $\mathbf{x} = \mathbf{0}$.

Proof:

- Given $\{a_n\}$, define $S_N = \sum^N a_n \mathbf{u}_n$.
- S_N is Cauchy in $\mathcal{H}$ and so $S_N \longrightarrow \mathbf{x}$ for some $\mathbf{x} \in \mathcal{H}$.
- $\langle x,\ u_n \rangle = \langle x - S_N,\ u_n \rangle + \langle S_N,\ u_n \rangle \longrightarrow a_n$
- By construction, $\|x - S_N\|^2 = \|x\|^2 - \sum^N |a_n|^2 \longrightarrow 0$, so $\|x\|^2 = \sum^\infty |a_n|^2$.

4.5 5

See Problem Set 5.

Heine-Cantor theorem: Every continuous function on a compact set is uniformly continuous.

Uniform continuity:

$$\begin{aligned} &\forall \varepsilon \quad \exists \delta(\varepsilon) \Big| \quad \forall x, y, \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \\ \iff &\forall \varepsilon \quad \exists \delta(\varepsilon) \Big| \quad \forall x, y, \quad |y| < \delta \implies |f(x - y) - f(y)| < \varepsilon \end{aligned}$$

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Fubini-Tonelli interchange of integrals, where the change of bounds becomes very important.

Continuity in L^1:

$$\lim_{y \longrightarrow 0} \|\tau_y f - f\|_1 = 0.$$

5 Basics

Useful Technique: $\lim f_n = \limsup f_n = \liminf f_n$ iff the limit exists, so $\limsup f_n \leq g \leq \liminf f_n$ implies that $g = \lim f$. Similarly, a limit does not exist iff $\liminf f_n > \limsup f_n$.

Lemma 5.1 ***(Convergent Sums Have Small Tails).***

$$\sum a_n < \infty \implies a_n \longrightarrow 0 \quad \text{and} \quad \sum_{k=N}^{\infty} \overset{N\longrightarrow\infty}{\longrightarrow} 0$$

Theorem 5.2 ***(Heine-Borel).***
$X \subseteq \mathbb{R}^n$ is compact $\iff$ X is closed and bounded.

Lemma 5.3 ***(Geometric Series).***

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \iff |x| < 1.$$

Corollary: $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k} = 1.$

Definition 5.3.1.
A set S is **nowhere dense** iff the closure of S has empty interior iff every interval contains a subinterval that does not intersect S.

Definition 5.3.2.
A set is **meager** if it is a *countable* union of nowhere dense sets.

Note that a *finite* union of nowhere dense is still nowhere dense.

Lemma 5.4.
The Cantor set is closed with empty interior.

Proof .
Its complement is a union of open intervals, and can't contain an interval since intervals have positive measure and $m(C_n)$ tends to zero.
Corollary: The Cantor set is nowhere dense.

■

Definition 5.4.1.
An F_σ set is a union of closed sets, and a G_δ set is an intersection of opens.
Mnemonic: "F" stands for ferme*, which is "closed" in French, and σ corresponds to a "sum", i.e. a union.*

Lemma 5.5.
Singleton sets in $\mathbb{R}$ are closed, and thus $\mathbb{Q}$ is an F_σ set.

Theorem 5.6 ***(Baire).***
$\mathbb{R}$ is a **Baire space** (countable intersections of open, dense sets are still dense). Thus $\mathbb{R}$ can not be written as a countable union of nowhere dense sets.

Lemma 5.7.
There is a function discontinuous precisely on $\mathbb{Q}$.

Proof .
$f(x) = \frac{1}{n}$ if $x = r_n \in \mathbb{Q}$ is an enumeration of the rationals, and zero otherwise. The limit at every point is 0.

■

Lemma 5.8.
There *do not* exist functions that are discontinuous precisely on $\mathbb{R} \setminus \mathbb{Q}$.

Proof .
D_f is always an F_σ set, which follows by considering the oscillation ω_f. $\omega_f(x) = 0 \iff f$ is continuous at x, and $D_f = \bigcup_n A_{\frac{1}{n}}$ where $A_\varepsilon = \{\omega_f \geq \varepsilon\}$ is closed.

■

Lemma 5.9.
Any nonempty set which is bounded from above (resp. below) has a well-defined supremum (resp. infimum).

6 Uniform Convergence

Theorem 6.1 ***(Egorov).***
Let $E \subseteq \mathbb{R}^n$ be measurable with $m(E) > 0$ and $\{f_k : E \longrightarrow \mathbb{R}\}$ be measurable functions such that

$$f(x) \coloneqq \lim_{k \longrightarrow \infty} f_k(x) < \infty$$

exists almost everywhere.
Then $f_k \longrightarrow f$ *almost uniformly*, i.e.

$$\forall \varepsilon > 0,\ \exists F \subseteq E \text{ closed such that } m(E \setminus F) < \varepsilon \quad \text{and} \quad f_k \xrightarrow{u} f \text{ on } F.$$

Proposition 6.2.
The space $X = C([0,1])$, continuous functions $f : [0,1] \longrightarrow \mathbb{R}$, equipped with the norm $\|f\| = \sup_{x \in [0,1]} |f(x)|$, is a **complete** metric space.

Proof .

1. Let $\{f_k\}$ be Cauchy in X.
2. Define a candidate limit using pointwise convergence:
   Fix an x; since

   $$|f_k(x) - f_j(x)| \leq \|f_k - f_k\| \longrightarrow 0$$

   the sequence $\{f_k(x)\}$ is Cauchy in $\mathbb{R}$. So define $f(x) := \lim_k f_k(x)$.
3. Show that $\|f_k - f\| \longrightarrow 0$:

   $$|f_k(x) - f_j(x)| < \varepsilon \ \forall x \implies \lim_j |f_k(x) - f_j(x)| < \varepsilon \ \forall x$$

   Alternatively, $\|f_k - f\| \leq \|f_k - f_N\| + \|f_N - f_j\|$, where N, j can be chosen large enough to bound each term by $\varepsilon/2$.
4. Show that $f \in X$:
   The uniform limit of continuous functions is continuous. (Note: in other cases, you may need to show the limit is bounded, or has bounded derivative, or whatever other conditions define X.)

■

Lemma 6.3.
Metric spaces are compact iff they are sequentially compact, (i.e. every sequence has a convergent subsequence).

Proposition 6.4.
The unit ball in $C([0,1])$ with the sup norm is not compact.

Proof .
Take $f_k(x) = x^n$, which converges to a dirac delta at 1. The limit is not continuous, so no subsequence can converge.

■

Lemma 6.5.
A uniform limit of continuous functions is continuous.

Lemma 6.6 ***(Testing Uniform Convergence).***
$f_n \longrightarrow f$ uniformly iff there exists an M_n such that $\|f_n - f\|_\infty \leq M_n \longrightarrow 0$.
Negating: find an x which depends on n for which the norm is bounded below.

Useful Technique: If f_n has a global maximum (computed using f_n' and the first derivative test) $M_n \longrightarrow 0$, then $f_n \longrightarrow 0$ uniformly.

Lemma 6.7 ***(Baby Commuting Limits with Integrals).***
If $f_n \longrightarrow f$ uniformly, then $\displaystyle\int f_n = \int f$.

Lemma 6.8 ***(Uniform Convergence and Derivatives).***
If $f_n' \longrightarrow g$ uniformly for some g and $f_n \longrightarrow f$ pointwise (or at least at one point), then $g = f'$.

Lemma 6.9 ***(Uniform Convergence of Series).***
If $f_n(x) \leq M_n$ **for a fixed** x where $\sum M_n < \infty$, then the series $f(x) = \sum f_n(x)$ converges pointwise.

Lemma 6.10.
If $\sum f_n$ converges then $f_n \longrightarrow 0$ uniformly.

Useful Technique: For a fixed x, if $f = \sum f_n$ converges *uniformly* on some $B_r(x)$ and each f_n is continuous at x, then f is also continuous at x .

Lemma 6.11 ***(M-test for Series).***
If $|f_n(x)| \leq M_n$ which does not depend on x, then $\sum f_n$ converges uniformly.

Lemma 6.12 ***(p-tests).***
Let n be a fixed dimension and set $B = \left\{ x \in \mathbb{R}^n \;\middle|\; \|x\| \leq 1 \right\}$.

$$
\begin{aligned}
\sum \frac{1}{n^p} < \infty &\iff p > 1 \\
\int_{\varepsilon}^{\infty} \frac{1}{x^p} < \infty &\iff p > 1 \\
\int_{0}^{1} \frac{1}{x^p} < \infty &\iff p < 1 \\
\int_{B} \frac{1}{|x|^p} < \infty &\iff p < n \\
\int_{B^c} \frac{1}{|x|^p} < \infty &\iff p > n
\end{aligned}
$$

.

Proposition 6.13.
A function $f : (a, b) \longrightarrow \mathbb{R}$ is Lipschitz $\iff$ f is differentiable and f' is bounded. In this case, $|f'(x)| \leq C$, the Lipschitz constant.

Proposition 6.14.
There exist smooth compactly supported functions, e.g. take

$$f(x) = e^{\frac{-1}{x^2}} \chi_{(0,\infty)}(x).$$

7 Measure Theory

Useful Technique: $s = \inf \{x \in X\} \implies$ for every ε there is an $x \in X$ such that $x \leq s + \varepsilon$.

Useful Techniques: Always consider bounded sets, and if E is unbounded write $E = \bigcup_n B_n(0) \bigcap E$ and use countable subadditivity or continuity of measure.

Lemma 7.1.
Every open subset of $\mathbb{R}$ (resp $\mathbb{R}^n$) can be written as a unique countable union of disjoint (resp. almost disjoint) intervals (resp. cubes).

Definition 7.1.1.
The outer measure of a set is given by

$$m_*(E) = \inf_{\substack{\{Q_i\} \rightrightarrows E \\ \text{closed cubes}}} \sum |Q_i|.$$

Lemma 7.2 ***(Properties of Outer Measure).***

- Montonicity: $E \subseteq F \implies m_*(E) \leq m_*(F)$.
- Countable Subadditivity: $m_*(\bigcup E_i) \leq \sum m_*(E_i)$.
- Approximation: For all E there exists a $G \supseteq E$ such that $m_*(G) \leq m_*(E) + \varepsilon$.
- Disjoint[a] Additivity: $m_*(A \coprod B) = m_*(A) + m_*(B)$.

[a] This holds for outer measure **iff** $\operatorname{dist}(A, B) > 0$.

Lemma 7.3 ***(Subtraction of Measure).***

$$m(A) = m(B) + m(C) \quad \text{and} \quad m(C) < \infty \implies m(A) - m(C) = m(B).$$

Lemma 7.4 ***(Continuity of Measure).***

$$E_i \nearrow E \implies m(E_i) \longrightarrow m(E)$$
$$m(E_1) < \infty \text{ and } E_i \searrow E \implies m(E_i) \longrightarrow m(E).$$

Proof .
1. Break into disjoint annuli $A_2 = E_2 \setminus E_1$, etc then apply countable disjoint additivity to $E = \coprod A_i$.
2. Use $E_1 = (\coprod E_j \setminus E_{j+1}) \coprod (\bigcap E_j)$, taking measures yields a telescoping sum,and use countable disjoint additivity.

■

Theorem 7.5.
Suppose E is measurable; then for every $\varepsilon > 0$,
1. There exists an open $O \supset E$ with $m(O \setminus E) < \varepsilon$
2. There exists a closed $F \subset E$ with $m(E \setminus F) < \varepsilon$

3. There exists a compact $K \subset E$ with $m(E \setminus K) < \varepsilon$.

Proof .

- (1): Take $\{Q_i\} \rightrightarrows E$ and set $O = \bigcup Q_i$.
- (2): Since E^c is measurable, produce $O \supset E^c$ with $m(O \setminus E^c) < \varepsilon$.
  - Set $F = O^c$, so F is closed.
  - Then $F \subset E$ by taking complements of $O \supset E^c$
  - $E \setminus F = O \setminus E^c$ and taking measures yields $m(E \setminus F) < \varepsilon$
- (3): Pick $F \subset E$ with $m(E \setminus F) < \varepsilon/2$.
  - Set $K_n = F \bigcap \mathbb{D}_n$, a ball of radius n about 0.
  - Then $E \setminus K_n \searrow E \setminus F$
  - Since $m(E) < \infty$, there is an N such that $n \geq N \implies m(E \setminus K_n) < \varepsilon$.

■

Lemma 7.6.
Lebesgue measure is translation and dilation invariant.

Proof .
Obvious for cubes; if $Q_i \rightrightarrows E$ then $Q_i + k \rightrightarrows E + k$, etc.

■

Theorem 7.7 ***(Non-Measurable Sets).***
There is a non-measurable set.

Proof .

- Use AOC to choose one representative from every coset of $\mathbb{R}/\mathbb{Q}$ on $[0,1)$, which is countable, and assemble them into a set N
- Enumerate the rationals in $[0,1]$ as q_j, and define $N_j = N + q_j$. These intersect trivially.
- Define $M := \coprod N_j$, then $[0,1) \subseteq M \subseteq [-1,2)$, so the measure must be between 1 and 3. By translation invariance, $m(N_j) = m(N)$, and disjoint additivity forces $m(M) = 0$, a contradiction.

■

Proposition 7.8 ***(Borel Characterization of Measurable Sets).***
If E is Lebesgue measurable, then $E = H \coprod N$ where $H \in F_\sigma$ and N is null.

Useful technique: F_σ sets are Borel, so establish something for Borel sets and use this to extend it to Lebesgue.

Proof .
For every $\frac{1}{n}$ there exists a closed set $K_n \subset E$ such that $m(E \setminus K_n) \leq \frac{1}{n}$. Take $K = \bigcup K_n$, wlog $K_n \nearrow K$ so $m(K) = \lim m(K_n) = m(E)$. Take $N := E \setminus K$, then $m(N) = 0$.

■

Definition 7.8.1.

$$\limsup_n A_n \coloneqq \bigcap_n \bigcup_{j\geq n} A_j = \left\{x \;\middle|\; x \in A_n \text{ for inf. many } n\right\}$$
$$\liminf_n A_n \coloneqq \bigcup_n \bigcap_{j\geq n} A_j = \left\{x \;\middle|\; x \in A_n \text{ for all except fin. many } n\right\}$$

.

Lemma 7.9.
If A_n are all measurable, $\limsup A_n$ and $\liminf A_n$ are measurable.

Proof .
Measurable sets form a sigma algebra, and these are expressed as countable unions/intersections of measurable sets. ■

Theorem 7.10 ***(Borel-Cantelli).***
Let $\{E_k\}$ be a countable collection of measurable sets. Then

$$\sum_k m(E_k) < \infty \implies \text{almost every } x \in \mathbb{R} \text{ is in at most finitely many } E_k.$$

Proof .

- If $E = \limsup_j E_j$ with $\sum m(E_j) < \infty$ then $m(E) = 0$.
- If E_j are measurable, then $\limsup_j E_j$ is measurable.
- If $\sum_j m(E_j) < \infty$, then $\sum_{j=N}^{\infty} m(E_j) \overset{N\to\infty}{\longrightarrow} 0$ as the tail of a convergent sequence.
- $E = \limsup_j E_j = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j \implies E \subseteq \bigcup_{j=k}^{\infty}$ for all k
- $E \subset \bigcup_{j=k}^{\infty} \implies m(E) \leq \sum_{j=k}^{\infty} m(E_j) \overset{k\to\infty}{\longrightarrow} 0.$

■

Lemma 7.11.

- Characteristic functions are measurable
- If f_n are measurable, so are $|f_n|, \limsup f_n, \liminf f_n, \lim f_n,$
- Sums and differences of measurable functions are measurable,
- Cones $F(x, y) = f(x)$ are measurable,
- Compositions $f \circ T$ for T a linear transformation are measurable,
- "Convolution-ish" transformations $(x, y) \mapsto f(x - y)$ are measurable

Proof (Convolution).
Take the cone on f to get $F(x,y) = f(x)$, then compose F with the linear transformation $T = [1,-1;1,0]$.

■

8 Integration

Definition 8.0.1.
$f \in L^+$ iff f is measurable and non-negative.

Useful techniques:

- Break integration domain up into disjoint annuli.
- Break real integrals up into $x < 1$ and $x > 1$.

Definition 8.0.2.
A measurable function is integrable iff $\|f\|_1 < \infty$.

Useful facts about C_c functions:

- Bounded almost everywhere
- Uniformly continuous

8.1 Convergence Theorems

Theorem 8.1 ***(Monotone Convergence).***
If $f_n \in L^+$ and $f_n \nearrow f$ a.e., then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \int f_n \longrightarrow \int f.$$

Needs to be positive and increasing.

Theorem 8.2 ***(Dominated Convergence).***
If $f_n \in L^1$ and $f_n \longrightarrow f$ a.e. with $|f_n| \leq g$ for some $g \in L^1$, then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \int f_n \longrightarrow \int f,$$

and more generally,

$$\int |f_n - f| \longrightarrow 0.$$

Positivity *not* needed.

Generalized DCT: can relax $|f_n| < g$ to $|f_n| < g_n \longrightarrow g \in L^1$.

Lemma 8.3.
If $f \in L^1$, then

$$\int |f_n - f| \longrightarrow 0 \iff \int |f_n| \longrightarrow |f|.$$

Proof .
Let $g_n = |f_n| - |f_n - f|$, then $g_n \longrightarrow |f|$ and

$$|g_n| = ||f_n| - |f_n - f|| \le |f_n - (f_n - f)| = |f| \in L^1,$$

so the DCT applies to g_n and

$$\begin{aligned} \|f_n - f\|_1 = \int |f_n - f| + |f_n| - |f_n| &= \int |f_n| - g_n \\ &\longrightarrow_{DCT} \lim \int |f_n| - \int |f|. \end{aligned}$$

■

Fatou's Lemma If $f_n \in L^+$, then

$$\begin{aligned} \int \liminf_n f_n &\le \liminf_n \int f_n \\ \limsup_n \int f_n &\le \int \limsup_n f_n. \end{aligned}$$

Only need positivity.

Theorem 8.4 ***(Tonelli).***
For $f(x,y)$ **non-negative and measurable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is a **measurable** function
- $F(x) = \displaystyle\int f(x,y)\, dy$ is a **measurable** function,
- For E measurable, the slices $E_x := \left\{ y \;\middle|\; (x,y) \in E \right\}$ are measurable.
- $\displaystyle\int f = \int\int F$, i.e. any iterated integral is equal to the original.

Theorem 8.5 ***(Fubini).***
For $f(x,y)$ **integrable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is an **integrable** function
- $F(x) := \displaystyle\int f(x,y)\, dy$ is an **integrable** function,
- For E measurable, the slices $E_x := \left\{ y \;\middle|\; (x,y) \in E \right\}$ are measurable.
- $\displaystyle\int f = \int\int f(x,y)$, i.e. any iterated integral is equal to the original

Theorem 8.6 ***(Fubini/Tonelli).***

If any iterated integral is **absolutely integrable**, i.e. $\int\int |f(x,y)| < \infty$, then f is integrable and $\int f$ equals any iterated integral.

Corollary 8.7 ***(Measurable Slices).***
Let E be a measurable subset of $\mathbb{R}^n$. Then

- For almost every $x \in \mathbb{R}^{n_1}$, the slice $E_x := \left\{ y \in \mathbb{R}^{n_2} \;\middle|\; (x,y) \in E \right\}$ is measurable in $\mathbb{R}^{n_2}$.
- The function

$$\begin{aligned} F : \mathbb{R}^{n_1} &\longrightarrow \mathbb{R} \\ x &\mapsto m(E_x) = \int_{\mathbb{R}^{n_2}} \chi_{E_x} \, dy \end{aligned}$$

is measurable and

$$m(E) = \int_{\mathbb{R}^{n_1}} m(E_x) \, dx = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{E_x} \, dy \, dx$$

Proof (Measurable Slices).

$\Longrightarrow$:

- Let f be measurable on $\mathbb{R}^n$.
- Then the cylinders $F(x,y) = f(x)$ and $G(x,y) = f(y)$ are both measurable on $\mathbb{R}^{n+1}$.
- Write $\mathcal{A} = \{G \le F\} \bigcap \{G \ge 0\}$; both are measurable.

$\Longleftarrow$:

- Let A be measurable in $\mathbb{R}^{n+1}$.
- Define $A_x = \left\{ y \in \mathbb{R} \;\middle|\; (x,y) \in \mathcal{A} \right\}$, then $m(A_x) = f(x)$.
- By the corollary, A_x is measurable set, $x \mapsto A_x$ is a measurable function, and $m(A) = \int f(x) \, dx$.
- Then explicitly, $f(x) = \chi_A$, which makes f a measurable function.

∎

Proposition 8.8 ***(Differentiating Under an Integral).***
If $\left| \frac{\partial}{\partial t} f(x,t) \right| \le g(x) \in L^1$, then letting $F(t) = \int f(x,t) \, dt$,

$$\begin{aligned} \frac{\partial}{\partial t} F(t) &:= \lim_{h \to 0} \int \frac{f(x,t+h) - f(x,t)}{h} dx \\ &\overset{\text{DCT}}{=} \int \frac{\partial}{\partial t} f(x,t) \, dx. \end{aligned}$$

To justify passing the limit, let $h_k \longrightarrow 0$ be any sequence and define

$$f_k(x,t) = \frac{f(x,t+h_k) - f(x,t)}{h_k},$$

so $f_k \overset{\text{pointwise}}{\longrightarrow} \frac{\partial}{\partial t} f$.

Apply the MVT to f_k to get $f_k(x,t) = f_k(\xi, t)$ for some $\xi \in [0, h_k]$, and show that $f_k(\xi, t) \in L_1$.

Proposition 8.9 ***(Swapping Sum and Integral).***
If f_n are non-negative and $\sum \int |f|_n < \infty$, then $\sum \int f_n = \int \sum f_n$.

Proof .

MCT. Let $F_N = \sum^N f_n$ be a finite partial sum; then there are simple functions $\varphi_n \nearrow f_n$ and so $\sum^N \varphi_n \nearrow F_N$, so apply MCT. ■

Lemma 8.10.
If $f_k \in L^1$ and $\sum \|f_k\|_1 < \infty$ then $\sum f_k$ converges almost everywhere and in L^1.

Proof .

Define $F_N = \sum^N f_k$ and $F = \lim_N F_N$, then $\|F_N\|_1 \leq \sum^N \|f_k\| < \infty$ so $F \in L^1$ and $\|F_N - F\|_1 \longrightarrow 0$ so the sum converges in L^1. Almost everywhere convergence: ? ■

8.2 L^1 Facts

Lemma 8.11 ***(Translation Invariance).***
The Lebesgue integral is translation invariant, i.e. $\int f(x)\, dx = \int f(x+h)\, dx$ for any h.

Proof .

- For characteristic functions, $\int_E f(x+h) = \int_{E+h} f(x) = m(E+h) = m(E) = \int_E f$ by translation invariance of measure.
- So this also holds for simple functions by linearity
- For $f \in L^+$, choose $\varphi_n \nearrow f$ so $\int \varphi_n \longrightarrow \int f$.
- Similarly, $\tau_h \varphi_n \nearrow \tau_h f$ so $\int \tau_h f \longrightarrow \int f$
- Finally $\left\{ \int \tau_h \varphi \right\} = \left\{ \int \varphi \right\}$ by step 1, and the suprema are equal by uniqueness of limits.

■

Lemma 8.12 ***(Integrals Distribute Over Disjoint Sets).***
If $X \subseteq A \bigcup B$, then $\int_X f \leq \int_A f + \int_{A^c} f$ with equality iff $X = A \coprod B$.

Lemma 8.13 ***(Unif Cts L1 Functions Decay Rapidly).***
If $f \in L^1$ and f is uniformly continuous, then $f(x) \overset{|x| \longrightarrow \infty}{\longrightarrow} 0$.

Doesn't hold for general L^1 functions, take any train of triangles with height 1 and summable areas.

Lemma 8.14 ***(L1 Functions Have Small Tails).***
If $f \in L^1$, then for every ε there exists a radius R such that if $A = B_R(0)^c$, then $\int_A |f| < \varepsilon$.

Proof .
Approximate with compactly supported functions. Take $g \xrightarrow{L_1} f$ with $g \in C_c$, then choose N large enough so that $g = 0$ on $E := B_N(0)^c$, then $\int_E |f| \leq \int_E |f - g| + \int_E |g|$.

∎

Lemma 8.15 ***(L1 Functions Have Absolutely Continuity).***
$m(E) \longrightarrow 0 \implies \int_E f \longrightarrow 0$.

Proof .
Approximate with compactly supported functions. Take $g \xrightarrow{L_1} f$, then $g \leq M$ so $\int_E f \leq \int_E f - g + \int_E g \longrightarrow 0 + M \cdot m(E) \longrightarrow 0$.

∎

Lemma 8.16 ***(L1 Functions Are Finite a.e.).***
If $f \in L^1$, then $m(\{f(x) = \infty\}) = 0$.

Proof .
Idea: Split up domain Let $A = \{f(x) = \infty\}$, then $\infty > \int f = \int_A f + \int_{A^c} f = \infty \cdot m(A) + \int_{A^c} f \implies m(X) = 0$.

∎

Proposition 8.17 ***(Continuity in L1).***

$$\|\tau_h f - f\|_1 \overset{h \longrightarrow 0}{\longrightarrow} 0$$

Proof .

Approximate with compactly supported functions. Take $g \xrightarrow{L_1} f$ with $g \in C_c$.

$$\begin{aligned}\int f(x+h) - f(x) \leq \\ \int f(x+h) - g(x+h) + \int g(x+h) - g(x) + \int g(x) - f(x) \\ \longrightarrow 2\varepsilon + \int g(x+h) - g(x) \\ = \int_K g(x+h) - g(x) + \int_{K^c} g(x+h) - g(x) \longrightarrow 0,\end{aligned}$$

which follows because we can enlarge the support of g to K where the integrand is zero on K^c, then apply uniform continuity on K.

■

Proposition 8.18 ***(Integration by Parts, Special Case).***

$$\begin{gathered}F(x) := \int_0^x f(y)dy \quad \text{and} \quad G(x) := \int_0^x g(y)dy \\ \implies \int_0^1 F(x)g(x)dx = F(1)G(1) - \int_0^1 f(x)G(x)dx.\end{gathered}$$

Proof .
Fubini-Tonelli, and sketch region to change integration bounds.

■

Theorem 8.19 ***(Lebesgue Density).***

$$A_h(f)(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(y)dy \implies \|A_h(f) - f\| \xrightarrow{h \longrightarrow 0} 0.$$

Proof .
Fubini-Tonelli, and sketch region to change integration bounds, and continuity in L^1.

■

8.3 L^p Spaces

Lemma 8.20.
The following are dense subspaces of $L^2([0,1])$:

- Simple functions
- Step functions
- $C_0([0,1])$
- Smoothly differentiable functions $C_0^\infty([0,1])$
- Smooth compactly supported functions C_c^∞

Theorem 8.21 ***(Dual Lp Spaces).***
For $p \neq \infty$, $(L^p)^\vee \cong L^q$.

Proof (p=1).
?

■

Proof (p=2) > Use Riesz Representation for Hilbert spaces.

For the $p = \infty$ case: $L^1 \subset (L^\infty)^\vee$, since the isometric mapping is always injective, but *never* surjective. So this containment is always proper (requires Hahn-Banach Theorem).

9 Fourier Series and Convolution

Definition 9.0.1 (Convolution)**.**

$$f * g(x) = \int f(x-y) g(y) dy.$$

Definition 9.0.2 (Dilation)**.**

$$\varphi_t(x) = t^{-n} \varphi\left(t^{-1} x\right).$$

Definition 9.0.3 (The Fourier Transform)**.**

$$\widehat{f}(\xi) = \int f(x)\ e^{2\pi i x \cdot \xi}\ dx.$$

Lemma 9.1.
$\widehat{f} = \widehat{g} \implies f = g$ almost everywhere.

Lemma 9.2 ***(Riemann-Lebesgue: Fourier transforms have small tails.).***

$$f \in L^1 \implies \widehat{f}(\xi) \to 0 \text{ as } |\xi| \to \infty.$$

Lemma 9.3.
If $f \in L^1$, then $\widehat{f}$ is continuous and bounded.

Proof .
$\left|\widehat{f}\right| \leq \int |f| \cdot |e^{\cdots}| \leq \|f\|_1$, and the DCT shows that $\left|\widehat{f}(\xi_n) - \widehat{f}(\xi)\right| \longrightarrow 0$.

■

Lemma 9.4.
Young's Inequality?

$$\frac{1}{r} := \frac{1}{p} + \frac{1}{q} - 1 \implies \|f * g\|_r \leq \|f\|_p \|g\|_q.$$

- Useful variant - take $q = 1$ to get $\|f * g\|_p \leq \|f\|_p \|g\|_1$
- Take $p = 1$ to show L_1 is closed under $*$.

Definition 9.4.1 (Approximation to the Identity)**.**
?

$$\varphi(x) = e^{-\pi x^2}$$
$$\varphi_t(x) = t^{-n} \varphi(\frac{x}{t}).$$

10 Extra Problems

Integration

- Show that if $f \in C^1(\mathbb{R})$ and $\lim_{x \longrightarrow \infty} f(x), f'(x)$ exist, then $\lim f'(x) = 0$.

Basics

- If f is continuous, is it necessarily the case that f' is continuous?
- If $f_n \longrightarrow f$, is it necessarily the case that f_n' converges to f' (or at all)?
- Is it true that the sum of differentiable functions is differentiable?
- Is it true that the limit of integrals equals the integral of the limit?
- Is it true that a limit of continuous functions is continuous?
- Prove that uniform convergence implies pointwise convergence implies a.e. convergence, but none of the implications may be reversed.
- Show that if K is compact and F is closed with K, F disjoint then $\text{dist}(K, F) > 0$.
- Show that if $f_n \longrightarrow f$ uniformly with each f_n continuous then f is continuous.
- Show that a subset of a metric space is closed iff it is complete.
- Show that if a subset of a metric space is complete and totally bounded, then it is compact.
- Show that every compact set is closed and bounded.
- Show that a uniform limit of bounded functions is bounded.
- Show that a uniform limit of continuous function is continuous.
- Show that if $f_n \longrightarrow f$ pointwise, $f_n' \longrightarrow g$ uniformly for some f, g, then f is differentiable and $g = f'$.

Measure Theory

- $\star$: Show that for $E \subseteq \mathbb{R}^n$, TFAE:
  1. E is measurable
  2. $E = H \bigcup Z$ here H is F_σ and Z is null
  3. $E = V \setminus Z'$ where $V \in G_\delta$ and Z' is null.
- Show that continuity of measure from above/below holds for outer measures.

- $\star$: Show that if $E \subseteq \mathbb{R}^n$ is measurable then $m(E) = \sup_{K \subset E \text{ compact}} m(K)$ iff for all $\varepsilon > 0$ there exists a compact $K \subseteq E$ such that $m(K) \geq m(E) - \varepsilon$.
- Show that a countable union of null sets is null.

Continuity

- Show that a continuous function on a compact set is uniformly continuous.

Measurability

- Show that $f = 0$ a.e. iff $\int_E f = 0$ for every measurable set E.
- $\star$: Show that cylinder functions are measurable, i.e. if f is measurable on $\mathbb{R}^s$, then $F(x, y) := f(x)$ is measurable on $\mathbb{R}^s \times \mathbb{R}^t$ for any t.
- Show that if f is a measurable function, then $f = 0$ a.e. iff $\int f = 0$.

Integrability

- $\star$: Prove that the Lebesgue integral is translation invariant, i.e. if $\tau_h(x) = x + h$ then $\int \tau_h f = \int f$.
- $\star$: Prove that the Lebesgue integral is dilation invariant, i.e. if $f_\delta(x) = \frac{f(\frac{x}{\delta})}{\delta^n}$ then $\int f_\delta = \int f$.
- $\star$: Prove continuity in L^1, i.e.

$$f \in L^1 \implies \lim_{h \to 0} \int |f(x+h) - f(x)| = 0.$$

- Show that a bounded function is Lebesgue integrable iff it is measurable.
- Show that simple functions are dense in L^1.
- Show that step functions are dense in L^1.
- Show that smooth compactly supposed functions are dense in L^1.

Convergence

- Prove Fatou's lemma using the Monotone Convergence Theorem.
- Show that if $\{f_n\}$ is in L^1 and $\sum \int |f_n| < \infty$ then $\sum f_n$ convergence to an L^1 function and $\int \sum f_n = \sum \int f_n$.

Fourier Analysis

- Show that if $f \in L^1$ then $\widehat{f}$ is bounded and uniformly continuous.
- Is it the case that $f \in L^1$ implies $\widehat{f} \in L^1$?
- Show that if $f, \widehat{f} \in L^1$ then f is bounded, continuous, and vanishes at infinity.
  - Show that this is not true for arbitrary L^1 functions.
- Show that if $f \in L^1$ and $\widehat{f} = 0$ a.e. then $f = 0$ a.e.
  - Prove that $\widehat{f} = \widehat{g}$ implies that $f = g$ a.e.
- Show that if $f, g \in L^1$ then $\int \widehat{f} g = \int f \widehat{g}$.
  - Give an example showing that this fails if g is not bounded.
- Show that if $f \in C^1$ then f is equal to its Fourier series.

Convolution

- Show that $f, g \in L^1 \implies f * g \in L^1$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.
- Show that $f \in L^1, g \leq M \implies f * g \leq M'$ and is uniformly continuous.
- Show that if $f, g \in L^1$ with $f \leq M, g \leq M'$, then $f * g \overset{x \longrightarrow \infty}{\longrightarrow} 0$.
- Show that if $f \in L^1$ and g' exists with $\dfrac{\partial g}{\partial x_i}$ all bounded, then $\dfrac{\partial}{\partial x_i}(f * g) = f * \dfrac{\partial g}{\partial x_i}$
- Show that if f, g are smooth and compactly supported then $f * g$ is smooth and $f * g \overset{x \longrightarrow \infty}{\longrightarrow} 0$.
- $\star$: show that if $f, g \in L^1$, then $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.
- Is it the case that $f, g \in C_c$ implies that $f * g \in C_c$?
- Show that if $f \in L^1$ and $g \in C_c^\infty$ then $f * g$ is smooth and $f * g$ vanishes at infinity.
- Show that if $f, g \in L^1$ and g is bounded, then $\lim_{|x| \longrightarrow \infty} (f * g)(x) = 0$.

Lp

- Show that if $E \subseteq \mathbb{R}^n$ is measurable with $\mu(E) < \infty$ and $f \in L^p(X)$ then $\|f\|_{L^p(X)} \overset{p \longrightarrow \infty}{\longrightarrow} \|f\|_\infty$.
- Is it true that the converse to the DCT holds? I.e. if $\int f_n \longrightarrow \int f$, is there a $g \in L^p$ such that $f_n < g$ a.e. for every n?
- Prove continuity in L^p: If f is uniformly continuous then $\|\tau_h f - f\|_p \longrightarrow 0$ as $h \longrightarrow 0$ for all p.

11 Inequalities and Equalities

Proposition 11.1 ***(Reverse Triangle Inequality).***

$$\|\|x\| - \|y\|\| \leq \|x - y\|.$$

Proposition 11.2 ***(Chebyshev's Inequality).***

$$\mu(\{x : |f(x)| > \alpha\}) \leq \left(\frac{\|f\|_p}{\alpha}\right)^p.$$

Proposition 11.3 ***(Holder's Inequality (when surjective).***

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Application: For finite measure spaces,

$$1 \leq p < q \leq \infty \implies L^q \subset L^p \quad (\text{ and } \ell^p \subset \ell^q).$$

Proof (Holder's Inequality).
Fix p, q, let $r = \dfrac{q}{p}$ and $s = \dfrac{r}{r-1}$ so $r^{-1} + s^{-1} = 1$. Then let $h = |f|^p$:

■

$$\|f\|_p^p = \|h \cdot 1\|_1 \leq \|1\|_s \|h\|_r = \mu(X)^{\frac{1}{s}} \|f\|_q^{\frac{q}{r}} \implies \|f\|_p \leq \mu(X)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q.$$

Note: doesn't work for ℓ_p spaces, but just note that $\sum |x_n| < \infty \implies x_n < 1$ for large enough n, and thus $p < q \implies |x_n|^q \leq |x_n|^q$.

Proposition 11.4 ***(Cauchy-Schwarz Inequality).***

$$|\langle f, \ g \rangle| = \|fg\|_1 \leq \|f\|_2 \|g\|_2 \quad \text{with equality} \quad \iff f = \lambda g.$$

Note: Relates inner product to norm, and only happens to relate norms in L^1.

Proof .
?

■

Proposition 11.5 ***(Minkowski's Inequality:).***

$$1 \leq p < \infty \implies \|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Note: does not handle $p = \infty$ case. Use to prove L^p is a normed space.

Proposition 11.6 ***(Young's Inequality*).***

[

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \implies \|f * g\|_r \leq \|f\|_p \|g\|_q.$$

]

Application: Some useful specific cases:

$$\begin{aligned}
\|f * g\|_1 &\leq \|f\|_1 \|g\|_1 \\
\|f * g\|_p &\leq \|f\|_1 \|g\|_p, \\
\|f * g\|_\infty &\leq \|f\|_2 \|g\|_2 \\
\|f * g\|_\infty &\leq \|f\|_p \|g\|_q.
\end{aligned}$$

Proposition 11.7 ***(? Inequality).***

$$(a+b)^p \leq 2^p(a^p+b^p).$$

Proposition 11.8 ***(Bezel's Inequality:).***

For $x \in H$ a Hilbert space and $\{e_k\}$ an orthonormal sequence,

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2.$$

Note: this does not need to be a basis.

Proposition 11.9 ***(Parseval's Identity:).***
Equality in Bessel's inequality, attained when $\{e_k\}$ is a *basis*, i.e. it is complete, i.e. the span of its closure is all of H.

11.1 Less Explicitly Used Inequalities

Proposition 11.10 ***(AM-GM Inequality).***

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Proposition 11.11 ***(Jensen's Inequality).***

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y).$$