

Topology Qualifying Exam Solutions

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1 Problems to Revisit

- 4
- 6 (Without Heine-Borel)
- 8
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2 Definitions

- Hausdorff
- Second Countable: admits a countable basis.
- Closed (several characterizations)
- Closure in a subspace: $Y \subset X \implies \text{cl}_Y(A) := \text{cl}_X(A) \cap Y$.
- Bounded
- Compact: A topological space (X, τ) is **compact** if every open cover has a *finite* subcover. That is, if $\{U_j \mid j \in J\} \subset \tau$ is a collection of open sets such that $X \subseteq \bigcup_{j \in J} U_j$, then there exists a *finite* subset $J' \subset J$ such that $X \subseteq \bigcup_{j \in J'} U_j$.
- Locally compact For every $x \in X$, there exists a $K_x \ni x$ such that K_x is compact.
- Connected: There does not exist a disconnecting set $X = A \amalg B$ such that $\emptyset \neq A, B \subsetneq X$, i.e. X is the union of two proper disjoint nonempty sets. Equivalently, X contains no proper nonempty clopen sets.
 - Additional condition for a subspace $Y \subset X$: $\text{cl}_Y(A) \cap V = A \cap \text{cl}_Y(B) = \emptyset$.
- Locally connected: A space is locally connected at a point x iff $\forall N_x \ni x$, there exists a $U \subset N_x$ containing x that is connected.
- Retract: A subspace $A \subset X$ is a *retract* of X iff there exists a continuous map $f : X \rightarrow A$ such that $f|_A = \text{id}_A$. Equivalently it is a *left inverse* to the inclusion.
- Uniform Continuity: For $f : (X, d_X) \rightarrow (Y, d_Y)$ metric spaces,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$
- Lebesgue number: For (X, d) a compact metric space and $\{U_\alpha\} \Rightarrow X$, there exist $\delta_L > 0$ such that

$$A \subset X, \text{diam}(A) < \delta_L \implies A \subseteq U_\alpha \text{ for some } \alpha.$$

- Paracompact
- Components: Set $x \sim y$ iff there exists a connected set $U \ni x, y$ and take equivalence classes
- Path Components: Set $x \sim y$ iff there exists a path-connected set $U \ni x, y$ and take equivalence classes
- Separable: countable dense subset.
- Limit Point: For $A \subset X$, x is a limit point of A if every punctured neighborhood P_x of x satisfies $P_x \cap A \neq \emptyset$, i.e. every neighborhood of x intersects A in some point other than x itself. Equivalently, x is a limit point of A iff $x \in \text{cl}_X(A \setminus \{x\})$.

3 Theorems

- Closed subsets of Hausdorff spaces are compact? (check)
- Cantor's intersection theorem?
- Tube lemma
- Properties pushed forward through continuous maps:
 - Compactness?
 - Connectedness (when surjective)
 - Separability
 - Density **only when** f is surjective
 - **Not** openness
 - **Not** closedness
- Results that only work for metric spaces
 - ?
- A retract of a Hausdorff/connected/compact space is closed/connected/compact respectively.
- A continuous function on a compact set is uniformly continuous.
 - Proof: take $\{B_{\frac{\varepsilon}{2}}(y) \mid y \in Y\} \rightrightarrows Y$, pull back to an open cover of X , has Lebesgue number $\delta_L > 0$, then $x' \in B_{\delta_L}(x) \implies f(x), f(x') \in B_{\frac{\varepsilon}{2}}(y)$ for some y .
- Lipschitz continuity implies uniform continuity (take $\delta = \varepsilon/C$)
 - Counterexample to converse: $f(x) = \sqrt{x}$ on $[0, 1]$ has unbounded derivative.
- Extreme Value Theorem: for $f : X \rightarrow Y$ continuous with X compact and Y ordered in the order topology, there exist points $c, d \in X$ such that $f(x) \in [f(c), f(d)]$ for every x .

4 Sandbox of Spaces

- Finite discrete sets with the discrete topology
- Subspaces of \mathbb{R} : (a, b) , $(a, b]$, (a, ∞) , etc.
 - $\{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{Z}^{\geq 1} \right\}$
- \mathbb{Q}
- The topologist's sine curve
- One-point compactifications
- \mathbb{R}^ω

Alternative topologies to consider:

- Cofinite
- Discrete and Indiscrete
- Uniform

5 General Topology

5.1 2

- See definition section.
- Use Heine-Borel theorem: a set $U \subset \mathbb{R}^n$ is compact $\iff U$ is *closed* and *bounded*.

- X is closed in \mathbb{R} , since we can write its complement as an arbitrary union of open intervals:

$$X^c = (-\infty, 0) \cup \left(\bigcup_{n \in \mathbb{Z}^+} \left(\frac{1}{n}, \frac{1}{n+1} \right) \right) \cup (1, \infty)$$

- X is *bounded*, since we can pick $r = 1$, then $x, y \in X \implies d(x, y) \leq r = 1$.
- iii. Use Heine-Borel again: X is not closed because it does not contain all of its limit points, e.g. the sequence $\left\{ x_n := \frac{1}{n} \mid n \in \mathbb{Z}^{\geq 1} \right\} \subset X$ but $x_n \xrightarrow{n \rightarrow \infty} 0 \in X^c$. Thus X is **not** compact.

5.1.1 Alternate Proof of (ii)

See Munkres p.164

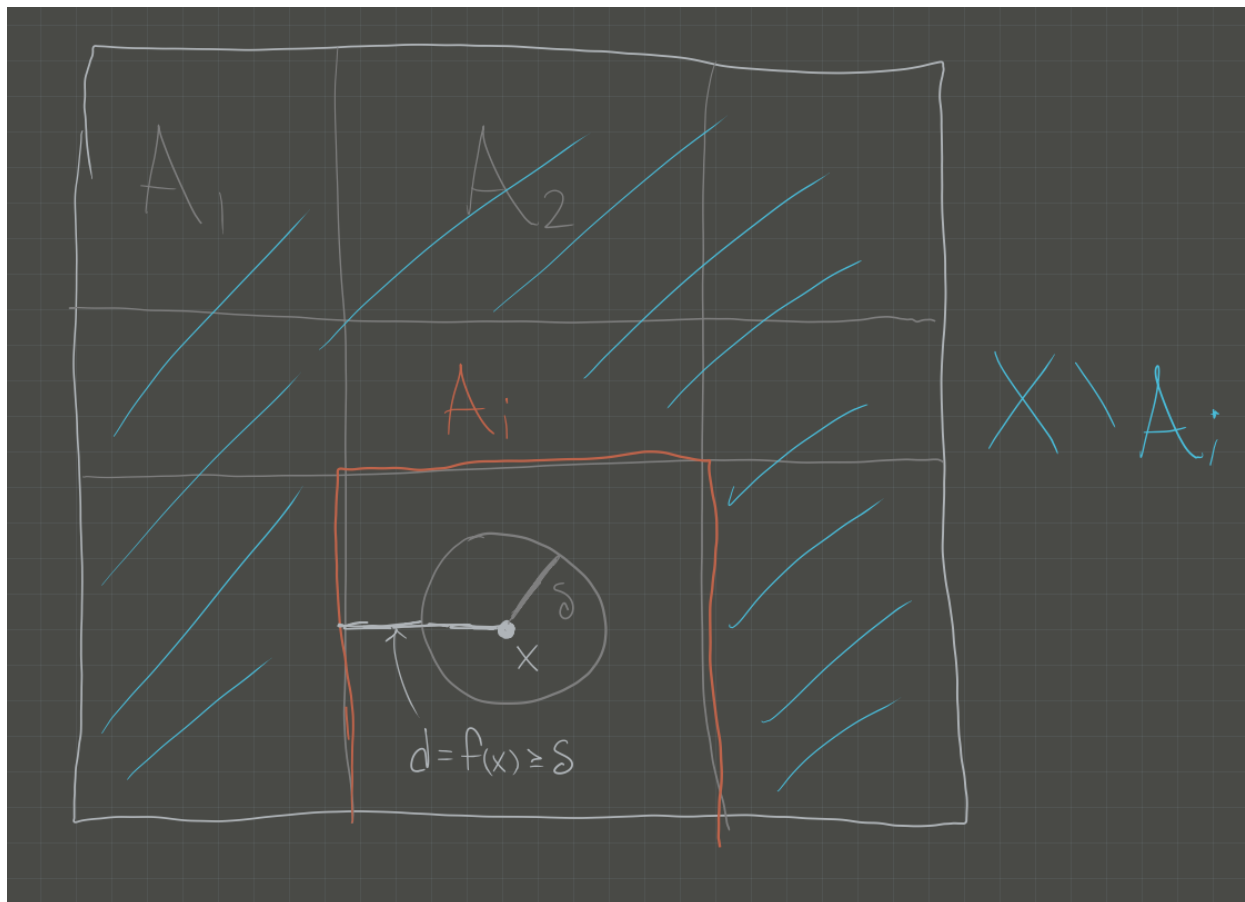
- Let $\{U_i \mid i \in J\} \Rightarrow X$; then $0 \in U_j$ for some $j \in J$.
- In the subspace topology, U_i is given by some $V \in \tau(\mathbb{R})$ such that $V \cap X = U_i$
 - A basis for the subspace topology on \mathbb{R} is open intervals, so write V as a union of open intervals $V = \bigcup_{k \in K} I_k$.
 - Since $0 \in U_j$, $0 \in I_k$ for some k .
- Since I_k is an interval, it contains infinitely many points of the form $x_n = \frac{1}{n} \in X$.
- Then $I_k \cap X \subset U_j$ contains infinitely many such points.
- So there are only *finitely* many points in $X \setminus U_j$, each of which is in $U_{j(n)}$ for some $j(n) \in J$ depending on n .
- So U_j and the *finitely* many $U_{j(n)}$ form a finite subcover of X . ■

5.2 4

Statement: show that the *Lebesgue number* is well-defined for compact metric spaces.

Note: this is a question about the *Lebesgue Number*. See Wikipedia for detailed proof.

- Write $U = \{U_i \mid i \in I\}$, then $X \subseteq \bigcup_{i \in I} U_i$. Need to construct a $\delta > 0$.
- By compactness of X , choose a finite subcover U_1, \dots, U_n .
- Define the distance between a point x and a set $Y \subset X$: $d(x, Y) = \inf_{y \in Y} d(x, y)$.
 - **Claim:** the function $d(\cdot, Y) : X \rightarrow \mathbb{R}$ is continuous for a fixed set.
 - Proof: Todo, not obvious.



- Define a function

$$f : X \longrightarrow \mathbb{R}$$

$$x \mapsto \frac{1}{n} \sum_{i=1}^n d(x, X \setminus U_i).$$

- Note this is a sum of continuous functions and thus continuous.

- **Claim:**

$$\delta := \inf_{x \in X} f(x) = \min_{x \in X} f(x) = f(x_{\min}) > 0$$

suffices.

- That the infimum is a minimum: f is a continuous function on a compact set, apply the extreme value theorem: it attains its minimum.
- That $\delta > 0$: otherwise, $\delta = 0 \implies \exists x_0$ such that $d(x_0, X \setminus U_i) = 0$ for all i .
 - * Forces $x_0 \in X \setminus U_i$ for all i , but $X \setminus \bigcup U_i = \emptyset$ since the U_i cover X .
- That it satisfies the Lebesgue condition:

$$\forall x \in X, \exists i \text{ such that } B_\delta(x) \subset U_i$$

- * Let $B_\delta(x) \ni x$; then by minimality $f(x) \geq \delta$.
- * Thus it can *not* be the case that $d(x, X \setminus U_i) < \delta$ for *every* i , otherwise

$$f(x) \leq \frac{1}{n}(\delta + \dots + \delta) = \frac{n\delta}{n} = \delta$$

- * So there is some particular i such that $d(x, X \setminus U_i) \geq \delta$.
- * But then $B_\delta \subseteq U_i$ as desired.

5.3 6

Statement: prove that $[0, 1] \subset \mathbb{R}$ is compact.

5.3.1 Proof 1: Direct (DZG)

Reference

- Let $I = [0, 1] = A \cup B$ be a disconnection, so
 - $A, B \neq \emptyset$
 - $A \cap B = \emptyset$
 - $\text{cl}_I(A) \cap B = A \cap \text{cl}_I(B) = \emptyset$.
- Let $a \in A$ and $b \in B$ where WLOG $a < b$
 - (since either $a < b$ or $b < a$, and $a \neq b$ since A, B are disjoint)
- Let $K = [a, b]$ and define $A_K := A \cap K$ and $B_K := B \cap K$.
- Now A_K, B_K is a disconnection of K .
- Let $s = \sup(A_K)$, which exists since \mathbb{R} is complete and has the LUB property
- Claim: $s \in \text{cl}_I(A_K)$. Proof:
 - If $s \in A_K$ there's nothing to show since $A_K \subset \text{cl}_I(A_K)$, so assume $s \in I \setminus A_K$.
 - Now let N_s be an arbitrary neighborhood of s , then using ??? we can find an $\varepsilon > 0$ such that $B_\varepsilon(s) \subset N_s$
 - Since s is a supremum, there exists an $a \in A_K$ such that $s - \varepsilon < a$.
 - But then $a \in B_\varepsilon(s)$ and $a \in N_s$ with $a \neq s$.
 - Since N_s was arbitrary, every N_s contains a point of A_K not equal to s , so s is a limit point by definition.
- Since $s \in \text{cl}_I(A_K)$ and $\text{cl}_I(A_K) \cap B_K = \emptyset$, we have $s \notin B_K$.
- Then the subinterval $(x, b] \cap A_K = \emptyset$ for every $x > c$ since $c := \sup A_K$.
- But since $A_K \cup B_K = K$, we must have $(x, b] \subset B_K$, and thus $s \in \text{cl}_I(B_K)$.
- Since A_K, B_K were assumed disconnecting, $s \notin A_K$
- But then $s \in K$ but $s \notin A_K \cup B_K = K$, a contradiction.

■

A potentially shorter proof

5.4 8

Topic: proof of the tube lemma.

Statement: show $X, Y \in \text{Top}_{\text{compact}} \iff X \times Y \in \text{Top}_{\text{compact}}$

5.4.1 Proof 1 (DZG)

\Leftarrow :

- By universal properties, the product $X \times Y$ is equipped with continuous projections

- The continuous image of a compact set is compact, and $\pi_1(X \times Y) = X, \pi_2(X \times Y) = Y$
- So X, Y are compact.

\implies :

Proof of Tube Lemma:

- Let $\{U_j \times V_j \mid j \in J\} \rightrightarrows X \times Y$.
- Fix a point $x_0 \in X$, then $\{x_0\} \times Y \subset N$ for some open set N .
- By the tube lemma, there is a $U^x \subset X$ such that the tube $U^x \times Y \subset N$.
- Since $\{x_0\} \times Y \cong Y$ which is compact, there is a finite subcover $\{U_j \times V_j \mid j \leq n\} \rightrightarrows \{x_0\} \times Y$.
- “Integrate the X ”: write

$$W = \bigcap_{j=1}^n U_j,$$

then $x_0 \in W$ and W is a finite intersection of open sets and thus open.

- Claim: $\{U_j \times V_j \mid j \leq n\} \rightrightarrows W \times Y$
 - Let $(x, y) \in W \times Y$; want to show $(x, y) \in U_j \times V_j$ for some $j \leq n$.
 - Then $(x_0, y) \in \{x_0\} \times Y$ is on the same horizontal line
 - $(x_0, y) \in U_j \times V_j$ for some j by construction
 - So $y \in V_j$ for this j
 - Since $x \in W$, $x \in U_j$ for *every* j , thus $x \in U_j$.
 - So $(x, y) \in U_j \times V_j$

Actual Proof:

- Let $\{U_j \mid j \in J\} \rightrightarrows X \times Y$.
- Fix $x_0 \in X$, the slice $\{x_0\} \times Y$ is compact and can be covered by finitely many elements $\{U_j \mid j \leq m\} \rightrightarrows \{x_0\} \times Y$.
 - Sum: write $N = \bigcup_{j=1}^m U_j$; then $\{x_0\} \times Y \subset N$.
 - Apply the tube lemma to N : produce $\{x_0\} \times Y \in W \times Y \subset N$; then $\{U_j \mid j \leq m\} \rightrightarrows W \times Y$.
- Now let $x \in X$ vary: for each $x \in X$, produce $W_x \times Y$ as above, then $\{W_x \times Y \mid x \in X\} \rightrightarrows X$.
 - By above argument, every tube $W_x \times Y$ can be covered by *finitely* many U_j .
- Since $\{W_x \mid x \in X\} \rightrightarrows X$ and X is compact, produce a finite subset $\{W_k \mid k \leq m'\} \rightrightarrows X$.
- Then $\{W_k \times Y \mid k \leq m'\} \rightrightarrows X \times Y$; the claim is that it is a finite cover.
 - Finitely many k
 - For each k , the tube $W_k \times Y$ is covered by finitely by U_j
 - And finite \times finite = finite. ■

Shorter mnemonic:

19.U It is sufficient to consider a cover consisting of elementary sets. Since Y is compact, each fiber $x \times Y$ has a finite subcovering $\{U_i^x \times V_i^x\}$. Put $W^x = \cap U_i^x$. Since X is compact, the cover $\{W^x\}_{x \in X}$ has a finite subcovering W^{x_j} . Then $\{U_i^{x_j} \times V_i^{x_j}\}$ is the required finite subcovering.

5.5 10

X is connected:

- Write $X = L \coprod G$ where $L = \{0\} \times [-1, 1]$ and $G = \{\Gamma(\sin(x)) \mid x \in (0, 1]\}$ is the graph of $\sin(x)$.
- $L \cong [0, 1]$ which is connected
 - Claim: Every interval is connected (todo)
- Claim: G is connected
 - The function

$$\begin{aligned} f : (0, 1] &\longrightarrow [-1, 1] \\ x &\mapsto \sin(x) \end{aligned}$$

is continuous (how to prove?)

- Claim: The diagonal map $\Delta : Y \longrightarrow Y \times Y$ where $\Delta(t) = (t, t)$ is continuous for any Y since $\Delta = (\text{id}, \text{id})$
- The composition of continuous function is continuous
- So the composition is continuous:

$$\begin{aligned} F : (0, 1] &\xrightarrow{\Delta} (0, 1]^2 \xrightarrow{(\text{id}, f)} (0, 1] \times [-1, 1] \\ t &\mapsto (t, t) \mapsto (t, f(t)) \end{aligned}$$

- Then $G = F((0, 1])$ is the continuous image of a connected set and thus connected.
- Claim: X is connected
 - Suppose there is a disconnecting cover $X = A \coprod B$ such that $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ and $A, B \neq \emptyset$.
 - WLOG suppose $(x, \sin(x)) \in B$ for $x > 0$.
 - Claim: $B = G$
 - * It can't be the case that A intersects G : otherwise $X = A \coprod B \implies G = (A \cap G) \coprod (B \cap G)$ disconnects G . So $A \cap G = \emptyset$, forcing $A \subseteq L$
 - * Similarly L can not be disconnected, so $B \cap L = \emptyset$ forcing $B \subset G$
 - * So $A \subset L$ and $B \subset G$, and since $X = A \coprod B$, this forces $A = L$ and $B = G$.
 - But any open set U in the subspace topology $L \subset \mathbb{R}^2$ (generated by open balls) containing $(0, 0) \in L$ is the restriction of a ball $V \subset \mathbb{R}^2$ of positive radius $r > 0$, i.e. $U = V \cap X$.
 - * But any such ball contains points of G : namely take n large enough such that $\frac{1}{n\pi} < r$.
 - * So $U \cap L \cap G \neq \emptyset$, contradicting $L \cap G = \emptyset$.

5.6 12

- Using the fact that $[0, \infty) \subset \mathbb{R}$ is Hausdorff, any retract must be closed, so any closed interval $[\varepsilon, N]$ for $0 \leq \varepsilon \leq N \leq \infty$.
 - Note that $\varepsilon = N$ yields all one point sets $\{x_0\}$ for $x_0 \geq 0$.
- No finite discrete sets occur, since the retract of a connected set is connected.
- ?

5.7 14

- Take two connected sets X, Y ; then there exists $p \in X \cap Y$.
- Write $X \cup Y = A \coprod B$ with both $A, B \subset A \coprod B$ open.
- Since $p \in X \cup Y = A \coprod B$, WLOG $p \in A$. We will show B must be empty.
- Claim: $A \cap X$ is clopen in X .
 - $A \cap X$ is open in X : ?
 - $A \cap X$ is closed in X : ?
- The only clopen sets of a connected set are empty or the entire thing, and since $p \in A$, we must have $A \cap X = X$.
- By the same argument, $A \cap Y = Y$.
- So $A \cap (X \cup Y) = (A \cap X) \cup (A \cap Y) = X \cup Y$
- Since $A \subset X \cup Y$, $A \cap (X \cup Y) = A$
- Thus $A = X \cup Y$, forcing $B = \emptyset$.

5.8 16

Topic: closure and connectedness in the subspace topology. See Munkres p.148

- $S \subset X$ is **not** connected if S with the subspace topology is not connected.
 - I.e. there exist $A, B \subset S$ such that
 - * $A, B \neq \emptyset$,
 - * $A \cap B = \emptyset$,
 - * $A \coprod B = S$.
- Or equivalently, there exists a nontrivial $A \subset S$ that is clopen in S .

Show stronger statement: this is an iff.

\implies :

- Suppose S is not connected; we then have sets $A \cup B = S$ from above and it suffices to show $\text{cl}_Y(A) \cap B = A \cap \text{cl}_X(B) = \emptyset$.
- A is open by assumption and $Y \setminus A = B$ is closed in Y , so A is clopen.
- Write $\text{cl}_Y(A) := \text{cl}_X(A) \cap Y$.
- Since A is closed in Y , $A = \text{cl}_Y(A)$ by definition, so $A = \text{cl}_Y(A) = \text{cl}_X(A) \cap Y$.
- Since $A \cap B = \emptyset$, we then have $\text{cl}_Y(A) \cap B = \emptyset$.
- The same argument applies to B , so $\text{cl}_Y(B) \cap A = \emptyset$.

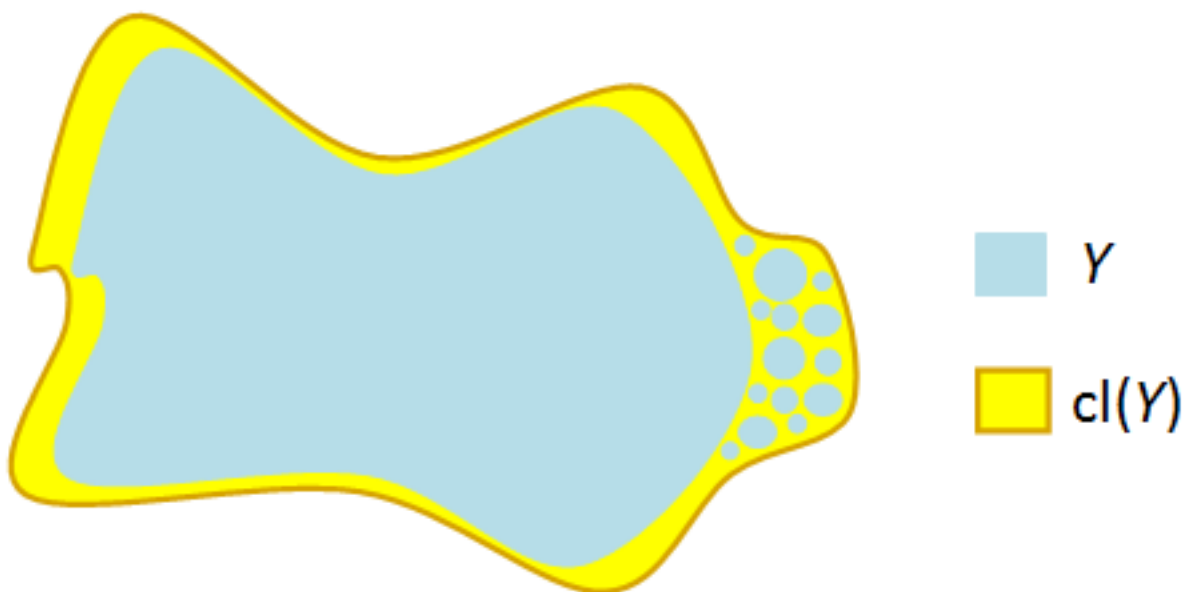
$\Leftarrow :$

- Suppose displayed condition holds; given such A, B we will show they are clopen in Y .
- Since $\text{cl}_Y(A) \cap B = \emptyset$, (claim) we have $\text{cl}_Y(A) = A$ and thus A is closed in Y .
 - Why?

$$\begin{aligned}
 \text{cl}_Y(A) &:= \text{cl}_X(A) \cap Y \\
 &= \text{cl}_X(A) \cap (A \amalg B) \\
 &= (\text{cl}_X(A) \cap A) \amalg (\text{cl}_X(A) \cap B) \\
 &= A \amalg (\text{cl}_X(A) \cap B) \quad \text{since } A \subset \text{cl}_Y(A) \\
 &= A \amalg (\text{cl}_Y(A) \cap B) \quad \text{since } B \subset Y \\
 &= A \amalg \emptyset \quad \text{using the assumption} \\
 &= A.
 \end{aligned}$$

- But $A = Y \setminus B$ where B is closed, so A is open and thus a nontrivial clopen subset.

■



5.9 18

- Define a new function

$$\begin{aligned}
 g : X &\longrightarrow \mathbb{R} \\
 x &\mapsto d_X(x, f(x)).
 \end{aligned}$$

- Attempt to minimize. Claim: g is a continuous function.

- Given claim, a continuous function on a compact space attains its infimum, so set

$$m := \inf_{x \in X} g(x)$$

and produce $x_0 \in X$ such that $g(x) = m$.

- Then

$$m > 0 \iff d(x_0, f(x_0)) > 0 \iff x_0 \neq f(x_0).$$

- Now apply f and use the assumption that f is a contraction to contradict minimality of m :

$$\begin{aligned} d(f(f(x_0)), f(x_0)) &\leq C \cdot d(f(x_0), x_0) \\ &< d(f(x_0), x_0) \quad \text{since } C < 1 \\ &\leq m \end{aligned}$$

- Proof that g is continuous: use the definition of g , the triangle inequality, and that f is a contraction:

$$\begin{aligned} d(x, f(x)) &\leq d(x, y) + d(y, f(y)) + d(f(x), f(y)) \\ \implies d(x, f(x)) - d(y, f(y)) &\leq d(x, y) + d(f(x), f(y)) \\ \implies g(x) - g(y) &\leq d(x, y) + C \cdot d(x, y) = (C + 1) \cdot d(x, y) \end{aligned}$$

- This shows that g is Lipschitz continuous with constant $C + 1$ (implies uniformly continuous, but not used).

5.10 20

Space	Connected	Locally Connected
\mathbb{R}	✓	✓
$[0, 1] \cup [2, 3]$		✓
Sine Curve	✓	
\mathbb{Q}		

- See definitions in intro.
- Claim: the Topologist's sine curve X suffices.

Proof:

- Claim 1: X is connected.
 - Intervals and graphs of cts functions are connected, so the only problem point is 0.
- Claim 2: X is **not** locally path connected.
 - Take any $B_\varepsilon(0) \in \mathbb{R}^2$; then $\pi_X B_\varepsilon(0)$ yields infinitely many arcs, each intersecting the graph at two points on $\partial B_\varepsilon(0)$.

- These are homeomorphic to a collection of disjoint embedded open intervals, and any disjoint union of intervals is clearly not connected. ■

Todo: what's the picture?