Topology Qualifying Exam Notes

D. Zack Garza

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Preface

Some fun resources:

- The Line with Two Origins

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- Connectedness
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- Path-Connectedness

2 Definitions: Point-Set Topology

2.1 Basics

Definition (Topology) Closed under arbitrary unions and finite intersections.

Definition (Neighborhood) A neighborhood of a point x is *any* open set containing x.

Definition (Limit Point) For $A \subset X$, x is a limit point of A if every punctured neighborhood P_x of x satisfies $P_x \bigcap A \neq \emptyset$, i.e. every neighborhood of x intersects A in some point other than x itself.

Equivalently, x is a limit point of A iff $x \in \text{cl}_X(A \setminus \{x\})$.

Definition (Closed) There are several characterizations of a closed set:

- Closure of a subspace: $Y \subset X \implies \text{cl}_Y(A) \coloneqq \text{cl}_X(A) \bigcap Y$.
- A set is closed iff it contains all of its limit points.

Definition (Basis) For X a space and $\mathcal{B}$ a collection of subsets, $\mathcal{B}$ is a *basis* for (X, τ_X) iff every open $U \in \tau_X$ is a union of elements in $\mathcal{B}$.

Definition (Topology Generated by a Basis) For X an arbitrary set, a collection of subsets $\mathcal{B}$ is a *basis for* X iff $\mathcal{B}$ is closed under intersections, and every intersection of basis elements contains another basis element. The set of unions of elements in B is a topology and is denoted *the topology generated by* $\mathcal{B}$.

Definition (Neighborhood Basis) If $p \in X$, a *neighborhood basis* at p is a collection $\mathcal{B}_p$ of neighborhoods of p such that if N_p is a neighborhood of p, then $N_p \supseteq B$ for at least one $B \in \mathcal{B}_p$.

Definition (Cover) A collection of subsets $\{U_\alpha\}$ of X is said to *cover* X iff $X = \bigcup_\alpha U_\alpha$. If $A \subseteq X$ is a subspace, then this collection *covers* A iff $A \subseteq \bigcup_\alpha U_\alpha$

Definition (Refinement) A cover $\mathcal{V} \rightrightarrows X$ is a *refinement* of $\mathcal{U} \rightrightarrows X$ iff for each $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $V \subseteq U$.

2.2 Analysis

Definition (Dense) A subset $Q \subset X$ is dense iff $y \in N_y \subset X \implies N_y \bigcap Q \neq \emptyset$ iff $\overline{Q} = X$.

Definition (Bounded) A set S in a metric space (X, d) is *bounded* iff there exists an $m \in \mathbb{R}$ such that $d(x, y) < m$ for every $x, y \in S$.

Definition (Uniform Continuity) For $f : (X, d_x) \longrightarrow (Y, d_Y)$ metric spaces,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \quad d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

Definition (Lebesgue number) For (X, d) a compact metric space and $\{U_\alpha\} \rightrightarrows X$, there exist $\delta_L > 0$ such that

$$A \subset X,\ \operatorname{diam}(A) < \delta_L \implies A \subseteq U_\alpha \text{ for some } \alpha.$$

2.3 Connectedness

Definition (Connected) There does not exist a disconnecting set $X = A \coprod B$ such that $\emptyset \neq A, B \subsetneq$, i.e. X is the union of two proper disjoint nonempty sets.

Equivalently, X contains no proper nonempty clopen sets.

> Additional condition for a subspace $Y \subset X$: $\operatorname{cl}_Y(A) \bigcap V = A \bigcap \operatorname{cl}_Y(B) = \emptyset$.

Definition (Locally Connected) A space is *locally connected* at a point x iff $\forall N_x \ni x$, there exists a $U \subset N_x$ containing x that is connected.

Definition (Locally Path-Connected) A space is *locally path-connected* if it admits a basis of path-connected open subsets.

Definition (Components) Set $x \sim y$ iff there exists a connected set $U \ni x, y$ and take equivalence classes.

Definition (Path Components) Set $x \sim y$ iff there exists a path-connected set $U \ni x, y$ and take equivalence classes.

2.4 Compactness

Definition (Compact) A topological space (X, τ) is **compact** if every open cover has a *finite* subcover.

That is, if $\left\{U_j \;\middle|\; j \in J\right\} \subset \tau$ is a collection of open sets such that $X = \bigcup_{j \in J} U_j$, then there exists a *finite* subset $J' \subset J$ such that $X \subseteq \bigcup_{j \in J'} U_j$.

Definition (Locally Compact) A space X is *locally compact* iff every $x \in X$ has a neighborhood contained in a compact subset of X.

Definition (Paracompact) A topological space X is *paracompact* iff every open cover of X admits an open locally finite refinement.

Definition (Precompact) A subset $A \subseteq X$ is *precompact* iff $\text{cl}_X(A)$ is compact.

2.5 Separability

Definition (Locally Finite) A collection of subsets $\mathcal{S}$ of X is *locally finite* iff each point of M has a neighborhood that intersects at most finitely many elements of $\mathcal{S}$.

Definition (Separable) A space X is *separable* iff X contains a countable dense subset.

Definition (Hausdorff) A topo.ogical space X is *Hausdorff* iff for every $p \neq q \in X$ there exist disjoint open sets $U \ni p$ and $V \ni q$.

Definition (First Countable) A space is *first-countable* iff every point admits a countable neighborhood basis.

Definition (Second Countable) A space is *second-countable* iff it admits a countable basis.

Definition (Regular)

Todo

Definition (Normal)

Todo

2.6 Misc

Definition (Normal)

Todo

2.7 Todo

- Saturated
- Quotient Map
- The subspace topology
- The quotient topology
- The product topology

- Topological Embedding
- Continuous map
- Open and Closed maps

3 Examples

3.1 Common Spaces and Operations

Point-Set:

- Finite discrete sets with the discrete topology
- Subspaces of $\mathbb{R}$: $(a, b), (a, b], (a, \infty)$, etc.
  - $\{0\} \bigcup \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{Z}^{\geq 1} \right\}$
- $\mathbb{Q}$
- The topologist's sine curve
- One-point compactifications
- $\mathbb{R}^\omega$
- Hawaiian earring
- Cantor set

Non-Hausdorff spaces:

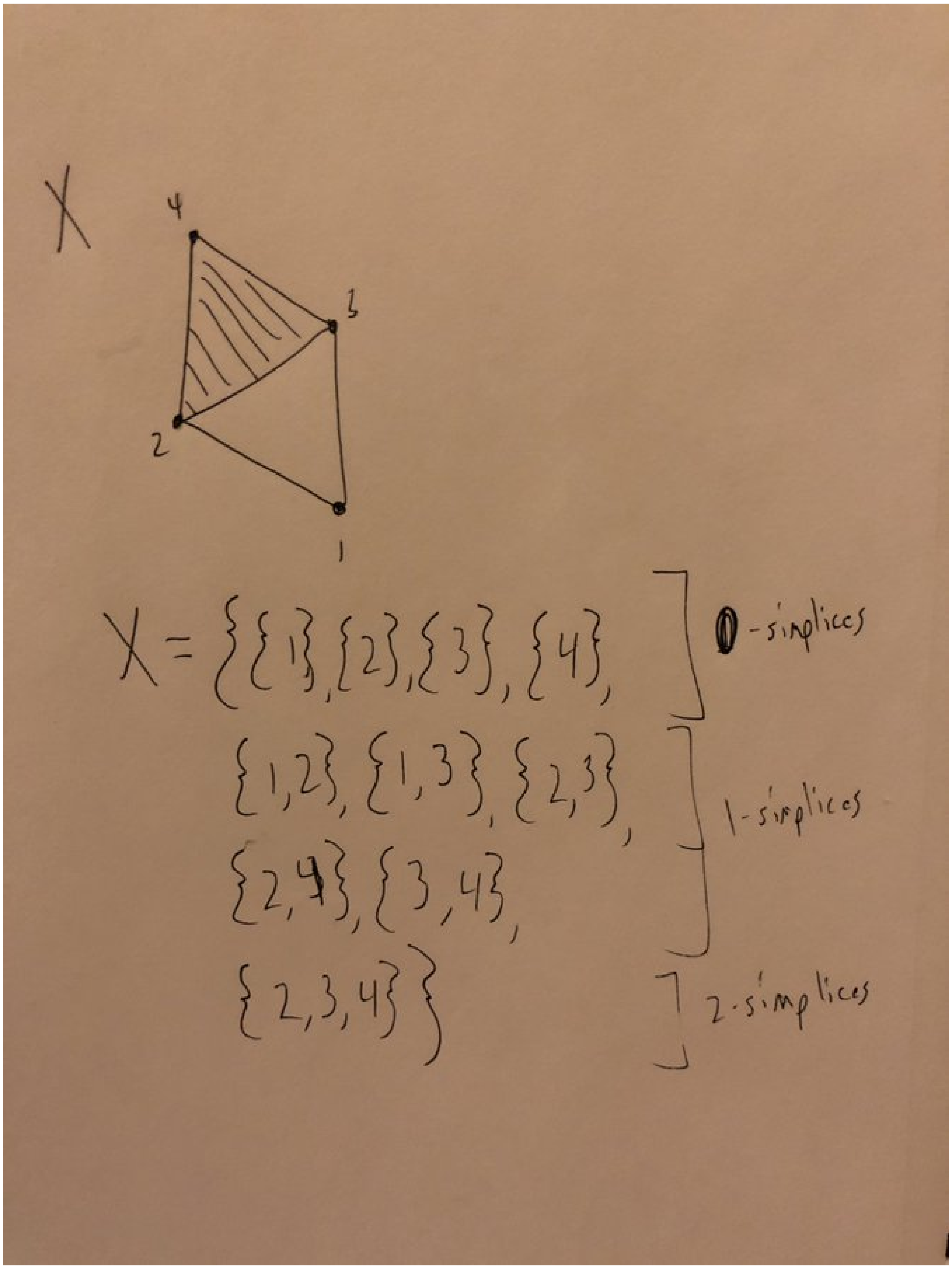
- The cofinite topology on any infinite set.
- $\mathbb{R}/\mathbb{Q}$
- The line with two origins.

General Spaces:

$$S^n, \mathbb{D}^n, T^n, \mathbb{RP}^n, \mathbb{CP}^n, \mathbb{M}, \mathbb{K}, \Sigma_g, \mathbb{RP}^\infty, \mathbb{CP}^\infty.$$

"Constructed" Spaces

- Knot complements in S^3
- Covering spaces (hyperbolic geometry)
- Lens spaces
- Matrix groups
- Prism spaces
- Pair of pants
- Seifert surfaces
- Surgery
- Simplicial Complexes
  - Nice minimal example:



Exotic/Pathological Spaces

- $\mathbb{HP}^n$
- Dunce Cap

- Horned sphere

Operations

- Cartesian product $A \times B$
- Wedge product $A \vee B$
- Connect Sum $A \# B$
- Quotienting A/B
- Puncturing $A \setminus \{a_i\}$
- Smash product
- Join
- Cones
- Suspension
- Loop space
- Identifying a finite number of points

3.2 Alternative Topologies

- Discrete
- Cofinite
- Discrete and Indiscrete
- Uniform

The cofinite topology:

- Non-Hausdorff
- Compact

The discrete topology:

- Discrete iff points are open
- Always Hausdorff
- Compact iff finite
- Totally disconnected
- If the domain, every map is continuous

The indiscrete topology:

- Only open sets are $\emptyset, X$
- Non-Hausdorff
- If the codomain, every map is continuous
- Compact

4 Theorems

Properties preserved and not preserved by continuous functions: Link

- Properties pushed forward through continuous maps:
  - Compactness?
  - Connectedness (when surjective)
  - Separability

- Density **only when** f is surjective
- **Not** openness
- **Not** closedness

4.1 Metric Spaces and Analysis

Theorem (Cantor's Intersection Theorem) A bounded collection of nested closed sets $C_1 \supset C_2 \supset \cdots$ in a metric space X is nonempty $\iff$ X is complete.

Theorem (Cantor's Nested Intervals Theorem) If $\left\{ [a_n, b_n] \,\middle|\, n \in \mathbb{Z}^{\geq 0} \right\}$ is a nested sequence of **closed and bounded** intervals, then their intersection is nonempty. If $\operatorname{diam}([a_n, b_n]) \overset{n \longrightarrow \infty}{0}$, then the intersection contains exactly one point.

Proposition A continuous function on a compact set is uniformly continuous.

Proof Take $\left\{ B_{\frac{\varepsilon}{2}}(y) \,\middle|\, y \in Y \right\} \rightrightarrows Y$, pull back to an open cover of X, has Lebesgue number $\delta_L > 0$, then $x' \in B_{\delta_L}(x) \implies f(x), f(x') \in B_{\frac{\varepsilon}{2}}(y)$ for some y.

Corollary Lipschitz continuity implies uniform continuity (take $\delta = \varepsilon / C$)

Counterexample to the converse: $f(x) = \sqrt{x}$ on $[0, 1]$ has unbounded derivative.

Theorem (Extreme Value Theorem) For $f : X \longrightarrow Y$ continuous with X compact and Y ordered in the order topology, there exist points $c, d \in X$ such that $f(x) \in [f(c), f(d)]$ for every x.

Theorem A metric space X is sequentially compact iff it is complete and totally bounded.

Theorem A metric space is totally bounded iff every sequence has a Cauchy subsequence.

Theorem A metric space is compact iff it is complete and totally bounded.

Theorem (Baire) If X is a complete metric space, then the intersection of countably many dense open sets is dense in X.

4.2 Connectedness

Theorem (Tube Lemma)

Todo

4.3 Compactness

Theorem $U \subset X$ a Hausdorff spaces is closed $\iff$ it is compact.

Theorem A closed subset A of a compact set B is compact.

Proof

- Let $\{A_i\} \rightrightarrows A$ be a covering of A by sets open in A.
- Each $A_i = B_i \bigcap A$ for some B_i open in B (definition of subspace topology)
- Define $V = \{B_i\}$, then $V \rightrightarrows A$ is an open cover.
- Since A is closed, $W := B \setminus A$ is open
- Then $V \bigcup W$ is an open cover of B, and has a finite subcover $\{V_i\}$
- Then $\left\{ V_i \bigcap A \right\}$ is a finite open cover of A.

Theorem The continuous image of a compact set is compact.

Theorem A closed subset of a Hausdorff space is compact.

Theorem A continuous bijection $f : X \longrightarrow Y$ where X is compact and Y is Hausdorff is an open map and hence a homeomorphism.

4.4 Separability

Proposition A retract of a Hausdorff/connected/compact space is closed/connected/compact respectively.

Theorem Points are closed in T_1 spaces.

4.5 Maps and Homeomorphism

Theorem A continuous bijective open map is a homeomorphism.

Theorem (Munkres 18.1) For $f : X \longrightarrow Y$, TFAE:

- f is continuous
- $A \subset X \implies f(\mathrm{cl}_X(A)) \subset \mathrm{cl}_X(f(A))$
- B closed in $Y \implies f^{-1}(B)$ closed in X.
- For each $x \in X$ and each neighborhood $V \ni f(x)$, there is a neighborhood $U \ni x$ such that $f(U) \subset V$.

Proof

Todo, see Munkres page 104

Theorem (Lee A.52) If $f : X \longrightarrow Y$ is continuous where X is compact and Y is Hausdorff, then

- f is a closed map.
- If f is surjective, f is a quotient map.
- If f is injective, f is a topological embedding.
- If f is bijective, it is a homeomorphism.

5 Topics

- Algebraic topology topics:
  - Classification of compact surfaces
  - Euler characteristic
  - Connect sum
  - Homology and cohomology groups
  - Fundamental group
  - Singular/cellular/simplicial homology
  - Mayer-Vietoris long exact sequences for homology and cohomology
  - Diagram chasing
  - Degree of maps from $S^n \longrightarrow S^n$
  - Orientability, compactness
  - Top-level homology and cohomology
  - Reduced homology and cohomology
  - Relative homology

- Homotopy and homotopy invariance
  - Deformation retract
  - Retract
  - Excision
  - Kunneth formula
  - Factoring maps
  - Fundamental theorem of algebra
- Algebraic topology theorems:
  - Brouwer fixed point theorem
  - Poincare lemma
  - Poincare duality
  - de Rham theorem
  - Seifert-van Kampen theorem
- Covering space theory topics:
  - Covering maps
  - Free actions
  - Properly discontinuous action
  - Universal cover
  - Correspondence between covering spaces and subgroups of the fundamental group of the base.
  - Lifting paths
  - Homotopy lifting property
  - Deck transformations
  - The action of the fundamental group
  - Normal/regular cover

6 AT Summary

6.1 Different Types of Product/Sum Structures

- Cartesian Product $X \times Y, \prod_i X_i$
- Direct Sum $X \oplus Y, \bigoplus_i X_i$
- Direct Product $X * Y, *_i X_i$
  - Element-wise multiplication, allows infinitely many entries
  - $*_i X_i = \oplus_i X_i$ for $i < \infty$
- Tensor Product $X \otimes Y, \bigotimes_i X_i, X^{\otimes i}$
- For a finite index set I, $\prod_I G = \bigoplus_I G$ in **Grp**, i.e. the finite direct product and finite direct sum coincide.

  Otherwise, if I is infinite, the direct sum requires cofinitely many zero entries (i.e. finitely many nonzero entries), so here we always use $\prod$.

In other words, there is an injective map

$$\bigoplus_I G \hookrightarrow \prod_I G$$

which is an isomorphism when $|I| < \infty$

- The free abelian group of rank n:

$$\mathbb{Z}^n \coloneqq \prod_{i=1}^{n} \mathbb{Z} = \mathbb{Z} \times \mathbb{Z} \times \dots \mathbb{Z}.$$

  - $x \in \mathbb{Z}^n = \langle a_1, \cdots, a_n \rangle \implies x = \sum_n c_i a_i$ for some $c_i \in \mathbb{Z}$, i.e. a_i form a basis.
  - Example: $x = 2a_1 + 4a_2 + a_1 - a_2 = 3a_1 + 3a_2$.

- The **free product** of n free abelian groups:

$$\mathbb{Z}^{*n} \coloneqq \mathop{\ast}_{i=1}^{n} \mathbb{Z} = \mathbb{Z} * \mathbb{Z} * \dots \mathbb{Z}$$

  This is a free *nonabelian* group on n generators.

  - $x \in \mathbb{Z}^{*n} = \langle a_1, \dots, a_n \rangle$ implies that x is a finite word in the noncommuting symbols a_i^k for $k \in \mathbb{Z}$.
  - Example: $x = a_1^2 a_2^4 a_1 a_2^{-2}$

Proposition There are no nontrivial homomorphisms from finite groups into free groups.

In particular, any homomorphism $\mathbb{Z}_n \longrightarrow \mathbb{Z}$ is trivial.

Proof Homomorphisms preserve torsion; the former has n-torsion while the latter does not.

This is especially useful if you have some $f : A \longrightarrow B$ and you look at the induced homomorphism $f_* : \pi_1(A) \longrightarrow \pi_1(B)$. If the former is finite and the latter contains a copy of $\mathbb{Z}$, then f_* has to be the trivial map $f_*([\alpha]) = e \in \pi_1(B)$ for every $[\alpha] \in \pi_1(A)$.

6.2 Conventions

- Generally assume spaces are connected.
- $\pi_0(X)$ is the set of path components of X, and I write $\pi_0(X) = \mathbb{Z}$ if X is path-connected (although it is not a group). Similarly, $H_0(X)$ is a free abelian group on the set of path components of X.
- Lists start at entry 1, since all spaces are connected here and thus $\pi_0 = H_0 = \mathbb{Z}$. That is,
  - $\pi_*(X) = [\pi_1(X), \pi_2(X), \pi_3(X), \cdots]$
  - $H_*(X) = [H_1(X), H_2(X), H_3(X), \cdots]$

7 Definitions: Algebraic Topology

- Acyclic
- Alexander duality

- Basis
  - For an R-module M, a basis B is a linearly independent generating set.
- Boundary
- Boundary of a manifold
  - Points $x \in M^n$ defined by

$$\partial M = \{x \in M : H_n(M, M - \{x\}\,; \mathbb{Z}) = 0\}$$

- Cap Product
  - Denoting $\Delta^p \xrightarrow{\sigma} X \in C_p(X; G)$, a map that sends pairs (p-chains, q-cochains) to $(p - q)$-chains $\Delta^{p-q} \longrightarrow X$ by

$$\begin{aligned} H_p(X; R) \times H^q(X; R) &\xrightarrow{\frown} H_{p-q}(X; R) \\ \sigma \frown \psi &= \psi(F_0^q(\sigma))F_q^p(\sigma) \end{aligned}$$

    where F_i^j is the face operator, which acts on a simplicial map σ by restriction to the face spanned by $[v_i \dots v_j]$, i.e. $F_i^j(\sigma) = \sigma|_{[v_i \dots v_j]}$.
- Cellular Homology
- CW Cell
  - An n-cell of X, say e^n, is the image of a map $\Phi : B^n \longrightarrow X$. That is, $e^n = \Phi(B^n)$. Attaching an n-cell to X is equivalent to forming the space $B^n \coprod_f X$ where $f : \partial B^n \longrightarrow X$.
    * A 0-cell is a point.
    * A 1-cell is an interval $[-1, 1] = B^1 \subset \mathbb{R}^1$. Attaching requires a map from $S^0 = \{-1, +1\} \longrightarrow X$
    * A 2-cell is a solid disk $B^2 \subset \mathbb{R}^2$ in the plane. Attaching requires a map $S^1 \longrightarrow X$.
    * A 3-cell is a solid ball $B^3 \subset \mathbb{R}^3$. Attaching requires a map from the sphere $S^2 \longrightarrow X$.
- Cellular Map
  - A map $X \xrightarrow{f} Y$ is said to be cellular if $f(X^{(n)}) \subseteq Y^{(n)}$ where $X^{(n)}$ denotes the n- skeleton.
- Chain
  - An element $c \in C_p(X; R)$ can be represented as the singular p simplex $\Delta^p \longrightarrow X$.
- Chain Homotopy
  - Given two maps between chain complexes $(C_*, \partial_C) \xrightarrow{f,\ g} (D_*, \partial_D)$, a chain homotopy is a family $h_i : C_i \longrightarrow B_{i+1}$ satisfying

$$f_i - g_i = \partial_{B,i-1} \circ h_n + h_{i+1} \circ \partial_{A,i}$$

    .

$$
\begin{tikzcd}
\cdots \arrow[dr, "h_{n-2}"] & A_{n-1} \arrow[l, "d_{A,n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dr, "h_{n-1}"] & A_n \arrow[l, "d_{A,n}"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dr, "h_n"] & A_{n+1} \arrow[l, "d_{A,n+1}"'] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] \arrow[dr, "h_{n+1}"] & \cdots \arrow[l, "d_{A,n+2}"'] \\
\cdots & B_{n-1} \arrow[l, "d_{B,n-1}"'] & B_n \arrow[l, "d_{B,n}"'] & B_{n+1} \arrow[l, "d_{B,n+1}"'] & \cdots \arrow[l, "d_{B,n+2}"']
\end{tikzcd}
$$

- Chain Map
  - A map between chain complexes $(C_*, \partial_C) \xrightarrow{f} (D_*, \partial_D)$ is a chain map iff each component $C_i \xrightarrow{f_i} D_i$ satisfies

$$f_{i-1} \circ \partial_{C,i} = \partial_{D,i} \circ f_i$$

    (i.e this forms a commuting ladder)

$$
\begin{tikzcd}
\cdots & A_{n-1} \arrow[l, "d_{A,n-1}"'] \arrow[d, "f_{n-1}"'] & A_n \arrow[l, "d_{A,n}"'] \arrow[d, "f_n"'] & A_{n+1} \arrow[l, "d_{A,n+1}"'] \arrow[d, "f_{n+1}"'] & \cdots \arrow[l, "d_{A,n+2}"'] \\
\cdots & B_{n-1} \arrow[l, "d_{B,n-1}"'] & B_n \arrow[l, "d_{B,n}"'] & B_{n+1} \arrow[l, "d_{B,n+1}"'] & \cdots \arrow[l, "d_{B,n+2}"']
\end{tikzcd}
$$

- Closed manifold
  - A manifold that is compact, with or without boundary.
- Coboundary
- Cochain
  - An cochain $c \in C^p(X; R)$ is a map $c \in \hom(C_p(X; R), R)$ on chains.
- Cocycle

Definition 7.0.1 (Constant Map)**.**
A *constant map* $f : X \longrightarrow Y$ iff $f(X) = y_0$ for some $y_0 \in Y$, i.e. for every $x \in X$ the output value $f(x) = y_0$ is the same.

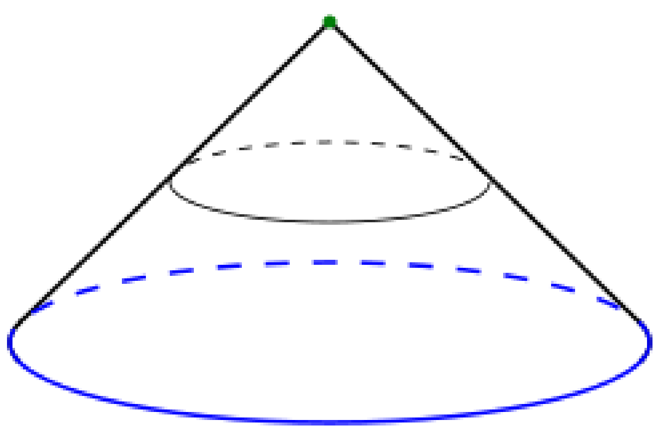
- Colimit :::{.definition title="Colimit"} For a directed system $(X_i, f_{ij}$, the *colimit* is an object X with a sequence of projections $\pi_i : X \longrightarrow X_i$ such that for any Y mapping into the system, the following diagram commutes:

$$\begin{tikzcd}
 & & Y \arrow[dl, phantom] \arrow[d, dashed, "\exists!"] \arrow[ddl, "\psi_j"'] \arrow[ddr, "\psi_i"] & & \\
 & & X \arrow[dl, "\pi_j"] \arrow[dr, "\pi_i"'] & & \\
\cdots \arrow[r] & X_j \arrow[rr, "f_{ij}"] & & X_i \arrow[r] & \cdots
\end{tikzcd}$$

Examples:

- Products
- Pullbacks
- Inverse/Projective limits
- The p-adic integers $\mathbb{Z}_p$. :::
- Compact
  - A space X is compact iff every open cover of X has a finite subcover.
- Cone
  - For a space X, defined as

$$CX = \frac{X \times I}{X \times \{0\}}.$$



Example: The cone on the circle CS^1
Note that the cone embeds X in a contractible space CX.

- Contractible
  - A space is contractible if its identity map is nullhomotopic.
- Contractible
- Coproduct
- Covering Space
- Cup Product

- A map taking pairs (p-cocycles, q-cocycles) to $(p+q)$-cocyles by

$$\begin{aligned}H^p(X;R) \times H^q(X;R) &\xrightarrow{\smile} H^{p+q}(X;R)\\ (a \cup b)(\sigma) &= a(\sigma \circ I_0^p)\ b(\sigma \circ I_p^{p+q})\end{aligned}$$

where $\Delta^{p+q} \xrightarrow{\sigma} X$ is a singular $p+q$ simplex and

$$I_i^j : [i, \cdots, j] \hookrightarrow \Delta^{p+q}$$

is an embedding of the $(j-i)$-simplex into a $(p+q)$-simplex. On a manifold, the cup product is Poincare dual to the intersection of submanifolds.
- Applications
  * $T^2 \not\simeq S^2 \vee S^1 \vee S^1$.

Proof

- CW Complex
- Cycle
- Deck Transformation
- Deformation
- Deformation Retract
  - A map r in $A \underset{r}{\overset{\iota}{\leftrightarrows}} X$ that is a retraction (so $r \circ \iota = \mathrm{id}_A$) **that also satisfies** $\iota \circ r \simeq \mathrm{id}_X$.

Note that this is equality in one direction, but only homotopy equivalence in the other.

  - Equivalently, a map $F : I \times X \longrightarrow X$ such that
    * $F_0(x) = \mathrm{id}_X$
    * $F_t(x)\Big|_A = \mathrm{id}_A$
    * $F_1(X) = A$
- Degree of a Map
- Derived Functor
  - For a functor T and an R-module A, a *left derived functor* (L_nT) is defined as $h_n(TP_A)$, where P_A is a projective resolution of A.
- Dimension of a manifold
  - For $x \in M$, the only nonvanishing homology group $H_i(M, M - \{x\}; \mathbb{Z})$
- Direct Limit
- Direct Product
- Direct Sum
- Eilenberg-MacLane Space
- Euler Characteristic
- Exact Functor

- A functor T is *right exact* if a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

yields an exact sequence

$$\ldots TA \longrightarrow TB \longrightarrow TC \longrightarrow 0,$$

and is *left exact* if it yields

$$0 \longrightarrow TA \longrightarrow TB \longrightarrow TC \longrightarrow \ldots$$

Thus a functor is exact iff it is both left and right exact, yielding

$$0 \longrightarrow TA \longrightarrow TB \longrightarrow TC \longrightarrow 0$$

- Examples:
  * $\cdot \otimes_R \cdot$ is a right exact bifunctor.

- Exact Sequence
- Excision
- Ext Group
- Flat
  - An R-module is flat if $A \otimes_R \cdot$ is an exact functor.
- Free and Properly Discontinuous
- Free module
  - A -module M with a basis $S = \{s_i\}$ of generating elements. Every such module is the image of a unique map $\mathcal{F}(S) = R^S \twoheadrightarrow M$, and if $M = \left\langle S \mid \mathcal{R} \right\rangle$ for some set of relations $\mathcal{R}$, then $M \cong R^S/\mathcal{R}$.
- Free Product
- Free product with amalgamation
- Fundamental Class
  - For a connected, closed, orientable manifold, $[M]$ is a generator of $H_n(M;\mathbb{Z}) = \mathbb{Z}$.
- Fundamental classes
- Fundamental Group
- Generating Set
  - $S = \{s_i\}$ is a generating set for an R- module M iff

$$x \in M \implies x = \sum r_i s_i$$

for some coefficients $r_i \in R$ (where this sum may be infinite).

- Gluing Along a Map

- Group Ring
- Homologous
- Homotopic
- Homotopy
- Homotopy Class
- Homotopy Equivalence
- Homotopy Extension Property
- Homotopy Groups
- Homotopy Lifting Property
- Injection
  - A map ι with a **left** inverse f satisfying $f \circ \iota = \mathrm{id}$
- Intersection Pairing For a manifold M, a map on homology defined by

  $$\begin{aligned} H_{\widehat{i}}M \otimes H_{\widehat{j}}M &\longrightarrow H_{\widehat{i+j}}X \\ \alpha \otimes \beta &\mapsto \langle \alpha, \beta \rangle \end{aligned}$$

  obtained by conjugating the cup product with Poincare Duality, i.e.

  $$\langle \alpha, \beta \rangle = [M] \frown ([\alpha]^\vee \smile [\beta]^\vee)$$

  Then, if $[A], [B]$ are transversely intersecting submanifolds representing α, β, then

  $$\langle \alpha, \beta \rangle = [A \bigcap B]$$

  .

  If $\widehat{i} = j$ then $\langle \alpha, \beta \rangle \in H_0 M = \mathbb{Z}$ is the signed number of intersection points.
- Inverse Limit
- Intersection Pairing
  - The pairing obtained from dualizing Poincare Duality to obtain

    $$\mathrm{F}(H_i M) \otimes \mathrm{F}(H_{n-i} M) \longrightarrow \mathbb{Z}$$

    Computed as an oriented intersection number between two homology classes (perturbed to be transverse).
- Intersection Form
  - The nondegenerate bilinear form cohomology induced by the Kronecker Pairing:

    $$I : H^k(M_n) \times H^{n-k}(M^n) \longrightarrow \mathbb{Z}$$

    where $n = 2k$.
    * When k is odd, I is skew-symmetric and thus a *symplectic form.*

    * When k is even (and thus $n \equiv 0 \mod 4$) this is a symmetric form.
    * Satisfies $I(x,y) = (-1)^{k(n-k)} I(y,x)$

- Kronecker Pairing
  - A map pairing a chain with a cochain, given by

$$\begin{aligned} H^n(X;R) \times H_n(X;R) &\longrightarrow R \\ ([\psi, \alpha]) &\mapsto \psi(\alpha) \end{aligned}$$

    which is a nondegenerate bilinear form.

- Kronecker Product
- Lefschetz duality
- Lefschetz Number
- Lens Space
- Local Degree
  - At a point $x \in V \subset M$, a generator of $H_n(V, V - \{x\})$. The degree of a map $S^n \longrightarrow S^n$ is the sum of its local degrees.
- Local Orientation
- Limit
- Linear Independence
  - A generating S for a module M is linearly independent if $\sum r_i s_i = 0_M \implies \forall i,\ r_i = 0$ where $s_i \in S, r_i \in R$.
- Local homology
  - $H_n(X, X - A; \mathbb{Z})$ is the local homology at A, also denoted $H_n(X \mid A)$
- Local Homology
- Local orientation of a manifold
  - At a point $x \in M^n$, a choice of a generator μ_x of $H_n(M, M - \{x\}) = \mathbb{Z}$.
- Long exact sequence
- Loop Space
- Manifold
  - An n-manifold is a Hausdorff space in which each neighborhood has an open neighborhood homeomorphic to $\mathbb{R}^n$.
- Manifold with boundary
  - A manifold in which open neighborhoods may be isomorphic to either $\mathbb{R}^n$ or a half-space $\left\{\mathbf{x} \in \mathbb{R}^n \;\middle|\; x_i > 0\right\}$.
- Mapping Cone

- Mapping Cylinder
- Mapping Path Space
- Mayer-Vietoris Sequence
- Monodromy
- Moore Space
- N-cell
- N-connected

Definition 7.0.2 (Nullhomotopic)**.**
A map $X \xrightarrow{f} Y$ is *nullhomotopic* if it is homotopic to a constant map $X \xrightarrow{g} \{y_0\}$; that is, there exists a homotopy

$$\begin{aligned} F : X \times I &\longrightarrow Y \\ F|_{X\times\{0\}} &= f \quad F(x,0) = f(x) \\ F|_{X\times\{1\}} &= g \quad F(x,1) = g(x) = y_0 \end{aligned}$$

.

- Orientable manifold
  - A manifold for which an orientation exists, see "Orientation of a Manifold".
- Orientation Cover
  - For any manifold M, a two sheeted orientable covering space $\tilde{M}_o$. M is orientable iff $\tilde{M}$ is disconnected. Constructed as

$$\tilde{M} = \coprod_{x\in M} \left\{ \mu_x \;\middle|\; \mu_x \text{ is a local orientation} \right\}$$

- Orientation of a manifold
  - A family of $\{\mu_x\}_{x\in M}$ with local consistency: if $x, y \in U$ then μ_x, μ_y are related via a propagation.
    - Formally, a function

$$\begin{aligned} M^n &\longrightarrow \coprod_{x\in M} H(X \;\middle|\; \{x\}) \\ x &\mapsto \mu_x \end{aligned}$$

      such that $\forall x \exists N_x$ in which $\forall y \in N_x$, the preimage of each μ_y under the map $H_n(M \mid N_x) \twoheadrightarrow H_n(M \mid y)$ is a single generator μ_{N_x}.
  - TFAE:
    - M is orientable.
    - The map $W : (M, x) \longrightarrow \mathbb{Z}_2$ is trivial.
    - $\tilde{M}_o = M \coprod \mathbb{Z}_2$ (two sheets).
    - $\tilde{M}_o$ is disconnected

    * The projection $\tilde{M}_o \longrightarrow M$ admits a section.

- Oriented manifold
- Path
- Path Lifting Property
- Perfect Pairing
  - A pairing alone is an R-bilinear module map, or equivalently a map out of a tensor product since $p : M \otimes_R N \longrightarrow L$ can be partially applied to yield $\varphi : M \longrightarrow L^N = \hom_R(N, L)$. A pairing is **perfect** when φ is an isomorphism.
    * Example: $\underset{M}{\det} : k^2 \times k^2 \longrightarrow k$
- Poincare Duality
  - For a closed, orientable n-manifold, following map $[M] \frown \cdot$ is an isomorphism:

$$\begin{aligned} D : H^k(M; R) &\longrightarrow H_{n-k}(M; R) \\ D(\alpha) &= [M] \frown \alpha \end{aligned}$$

- Projective Resolution
- Properly Discontinuous
- Pullback
- Pushout
- Quasi-isomorphism
- R-orientability
- Relative boundaries
- Relative cycles
- Relative homotopy groups
- Retraction
  - A map r in $A \underset{r}{\overset{\iota}{\leftrightarrows}} X$ satisfying

$$r \circ \iota = \mathrm{id}_A.$$

  Equivalently $X \twoheadrightarrow_r A$ and $r|_A = \mathrm{id}_A$. If X retracts onto A, then i_* is injective.

- Short exact sequence
- Simplicial Complex
- Simplicial Map
  - For a map

$$K \xrightarrow{f} L$$

  between simplicial complexes, f is a simplicial map if for any set of vertices $\{v_i\}$ spanning a simplex in K, the set $\{f(v_i)\}$ are the vertices of a simplex in L.

- Simply Connected
- Singular Chain

$$x \in C_n(x) \implies X = \sum_i n_i \sigma_i = \sum_i n_i (\Delta^n \xrightarrow{\sigma_i} X)$$

- Singular Cochain

$$x \in C^n(x) \implies X = \sum_i n_i \psi_i = \sum_i n_i (\sigma_i \xrightarrow{\psi_i} X)$$

- Singular Homology
- Smash Product
- Surjection
  - A map π with a **right** inverse f satisfying

$$\pi \circ f = \mathrm{id}$$

- Suspension Compact represented as $\Sigma X = CX \coprod_{\mathrm{id}_X} CX$, two cones on X glued along X. Explicitly given by

$$\Sigma X = \frac{X \times I}{(X \times \{0\}) \bigcup (X \times \{1\}) \bigcup (\{x_0\} \times I)}$$

- Tor Group
  - For an R-module

$$\mathrm{Tor}_R^n(\,\cdot\,, B) = L_n(\,\cdot\, \otimes_R B)$$

    where L_n denotes the nth left derived functor.
- Universal Cover
- Universal Coefficient Theorem for Cohomology
- Universal Coefficient Theorem for Change of Coefficient Ring
- Weak Homotopy Equivalence
- Weak Topology
- Wedge Product

8 Examples: Algebraic Topology

8.1 Standard Spaces and Modifications

- $K(G, n)$ is an Eilenberg-MacLane space, the homotopy-unique space satisfying

$$\pi_k(K(G,n)) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

  - $K(\mathbb{Z}, 1) = S^1$
  - $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$
  - $K(\mathbb{Z}/2\mathbb{Z}, 1) = \mathbb{RP}^\infty$

- $M(G, n)$ is a Moore space, the homotopy-unique space satisfying

$$H_k(M(G, n); G) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

  - $M(\mathbb{Z}, n) = S^n$
  - $M(\mathbb{Z}/2\mathbb{Z}, 1) = \mathbb{RP}^2$
  - $M(\mathbb{Z}/p\mathbb{Z}, n)$ is made by attaching e^{n+1} to S^n via a degree p map.

- $\mathbb{RP}^n = S^n/S^0 = S^n/\mathbb{Z}/2\mathbb{Z}$
- $\mathbb{CP}^n = S^{2n+1}/S^1$
- $T^n = \prod_n S^1$ is the n-torus
- $D(k, X)$ is the space X with $k \in \mathbb{N}$ distinct points deleted, i.e. the punctured space $X - \{x_1, x_2, \ldots x_k\}$ where each $x_i \in X$.
- $B^n = \left\{ \mathbf{v} \in \mathbb{R}^n \;\middle|\; \|\mathbf{v}\| \leq 1 \right\} \subset \mathbb{R}^n$
- $S^{n-1} = \partial B^n = \left\{ \mathbf{v} \in \mathbb{R}^n \;\middle|\; \|\mathbf{v}\| = 1 \right\} \subset \mathbb{R}^n$

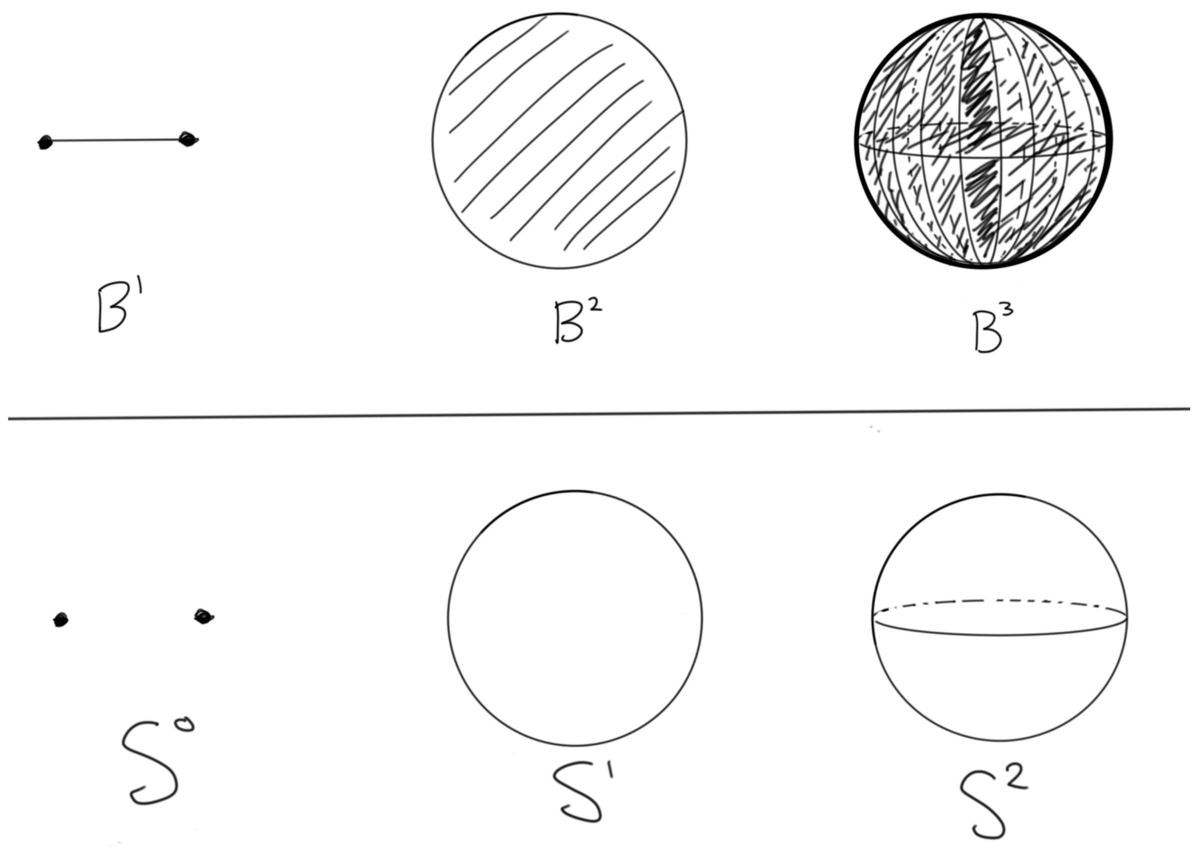


Figure 1: Low-Dimensional Spheres/Discs/Balls

- The "generalized uniform bouquet"? $\mathcal{B}^n(m) = \bigvee_{i=1}^{n} S^m$
- The real Grassmannian, $Gr(n, k, \mathbb{R})$, i.e. the set of k dimensional subspaces of $\mathbb{R}^n$
- The Stiefel manifold $V_n(k)$
- Possible modifications to a space X:
  - Remove k points by taking $D(k, X)$
  - Remove a line segment
  - Remove an entire line/axis
  - Remove a hole
  - Quotient by a group action (e.g. antipodal map, or rotation)
  - Remove a knot
  - Take complement in ambient space
- Lie Groups
  - The real general linear group, $\mathrm{GL}_n(\mathbb{R})$
    * The real special linear group $SL_n(\mathbb{R})$

    * The real orthogonal group, $O_n(\mathbb{R})$
        · The real special orthogonal group, $SO_n(\mathbb{R})$
    * The real unitary group, $U_n(\mathbb{R})$
        · The real special unitary group, $SU_n(\mathbb{R})$
    * The real symplectic group $Sp(n)$

- "Geometric" Stuff
  - Affine n-space over a field $\mathbb{A}^n(k) = k^n \rtimes GL_n(k)$
  - The projective space $\mathbb{P}^n(k)$
    * The projective linear group over a ring R, $PGL_n(R)$
    * The projective special linear group over a ring R, $PSL_n(R)$
    * The modular groups $PSL_n(\mathbb{Z})$
      · Specifically $PSL_2(\mathbb{Z})$

8.2 Facts About Low Dimensional and/or Standard Spaces

- $\mathbb{RP}^1 \cong S^1$
- $\mathbb{CP}^1 \cong S^2$
- $\mathcal{M} \simeq S^1$
- $\mathbb{CP}^n = \mathbb{C}^n \coprod \mathbb{CP}^{n-1} = \coprod_{i=0}^{n} \mathbb{C}^i$
- $S^{2n+1} \subset \mathbb{C}^{n+1}$
- $\mathbb{CP}^n = S^{2n+1}/S^n$

8.3 Table of Homotopy and Homology Structures

9 Low Dimensional Homology Examples

$$
\begin{array}{rcllllllll}
S^1 & = & [& \mathbb{Z}, & \mathbb{Z}, & 0, & 0, & 0, & 0 \to &] \\
\mathcal{M} & = & [& \mathbb{Z}, & \mathbb{Z}, & 0, & 0, & 0, & 0 \to &] \\
\mathbb{RP}^1 & = & [& \mathbb{Z}, & \mathbb{Z}, & 0, & 0, & 0, & 0 \to &] \\
\mathbb{RP}^2 & = & [& \mathbb{Z}, & \mathbb{Z}_2, & 0, & 0, & 0, & 0 \to &] \\
\mathbb{RP}^3 & = & [& \mathbb{Z}, & \mathbb{Z}_2, & 0, & \mathbb{Z}, & 0, & 0 \to &] \\
\mathbb{RP}^4 & = & [& \mathbb{Z}, & \mathbb{Z}_2, & 0, & \mathbb{Z}_2, & 0, & 0 \to &] \\
S^2 & = & [& \mathbb{Z}, & 0, & \mathbb{Z}, & 0, & 0, & 0 \to &] \\
\mathbb{T}^2 & = & [& \mathbb{Z}, & \mathbb{Z}^2, & \mathbb{Z}, & 0, & 0, & 0 \to &] \\
\mathbb{K} & = & [& \mathbb{Z}, & \mathbb{Z} \oplus \mathbb{Z}_2, & 0, & 0, & 0, & 0 \to &] \\
\mathbb{CP}^1 & = & [& \mathbb{Z}, & 0, & \mathbb{Z}, & 0, & 0, & 0 \to &] \\
\mathbb{CP}^2 & = & [& \mathbb{Z}, & 0, & \mathbb{Z}, & 0, & \mathbb{Z}, & 0 \to &]
\end{array}.
$$

X	$\pi_*(X)$	$H_*(X)$	CW Structure	$H^*(X)$
$\mathbb{R}^1$	0	0	$\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot x$	0
$\mathbb{R}^n$	0	0	$(\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot x)^n$	0

X	$\pi_*(X)$	$H_*(X)$	CW Structure	$H^*(X)$
$D(k,\mathbb{R}^n)$	$\pi_* \bigvee^k S^1$	$\bigoplus_k H_*M(\mathbb{Z},1)$	$1+kx$	?
B^n	$\pi_*(\mathbb{R}^n)$	$H_*(\mathbb{R}^n)$	$1+x^n+x^{n+1}$	0
S^n	$[0\ldots,\mathbb{Z},?\ldots]$	$H_*M(\mathbb{Z},n)$	$1+x^n$ or $\sum_{i=0}^n 2x^i$	$\mathbb{Z}[_nx]/(x^2)$
$D(k,S^n)$	$\pi_* \bigvee^{k-1} S^1$	$\bigoplus_{k-1} H_*M(\mathbb{Z},1)$	$1+(k-1)x^1$	?
T^2	$\pi_*S^1\times\pi_*S^1$	$(H_*M(\mathbb{Z},1))^2\times H_*M(\mathbb{Z},2)$	$1+2x+x^2$	$\Lambda(_1x_1,{}_1x_2)$
T^n	$\prod^n \pi_*S^1$	$\prod_{i=1}^n (H_*M(\mathbb{Z},i))^{\binom{n}{i}}$	$(1+x)^n$	$\Lambda(_1x_1,{}_1x_2,\ldots{}_1x_n)$
$D(k,T^n)$	$[0,0,0,0,\ldots]$?	$[0,0,0,0,\ldots]$?	$1+x$	?
$S^1\vee S^1$	$\pi_*S^1 * \pi_*S^1$	$(H_*M(\mathbb{Z},1))^2$	$1+2x$	?
$\bigvee^n S^1$	$*^n\pi_*S^1$	$\prod H_*M(\mathbb{Z},1)$	$1+x$	?
$\mathbb{RP}^1$	π_*S^1	$H_*M(\mathbb{Z},1)$	$1+x$	$_0\mathbb{Z}\times{}_1\mathbb{Z}$
$\mathbb{RP}^2$	$\pi_*K(\mathbb{Z}/2\mathbb{Z},1)+\pi_*S^2$	$H_*M(\mathbb{Z}/2\mathbb{Z},1)$	$1+x+x^2$	$_0\mathbb{Z}\times{}_2\mathbb{Z}/2\mathbb{Z}$
$\mathbb{RP}^3$	$\pi_*K(\mathbb{Z}/2\mathbb{Z},1)+\pi_*S^3$	$H_*M(\mathbb{Z}/2\mathbb{Z},1)+H_*M(\mathbb{Z},3)$	$1+x+x^2+x^3$	$_0\mathbb{Z}\times{}_2\mathbb{Z}/2\mathbb{Z}\times{}_3\mathbb{Z}$
$\mathbb{RP}^4$	$\pi_*K(\mathbb{Z}/2\mathbb{Z},1)+\pi_*S^4$	$H_*M(\mathbb{Z}/2\mathbb{Z},1)+H_*M(\mathbb{Z}/2\mathbb{Z},3)$	$1+x+x^2+x^3+x^4$	$_0\mathbb{Z}\times({}_2\mathbb{Z}/2\mathbb{Z})^2$
$\mathbb{RP}^n, n\geq 4$ even	$\pi_*K(\mathbb{Z}/2\mathbb{Z},1)+\pi_*S^n$	$\prod_{\text{odd } i<n} H_*M(\mathbb{Z}/2\mathbb{Z},i)$	$\sum_{i=1}^n x^i$	$_0\mathbb{Z}\times\prod_{i=1}^{n/2}{}_2\mathbb{Z}/2\mathbb{Z}$
$\mathbb{RP}^n, n\geq 4$ odd	$\pi_*K(\mathbb{Z}/2\mathbb{Z},1)+\pi_*S^n$	$\prod_{\text{odd } i\leq n-2} H_*M(\mathbb{Z}/2\mathbb{Z},i)\times H_*S^n$	$\sum_{i=1}^n x^i$	$H^*(\mathbb{RP}^{n-1})\times{}_n\mathbb{Z}$
$\mathbb{CP}^1$	$\pi_*K(\mathbb{Z},2)+\pi_*S^3$	H_*S^2	x^0+x^2	$\mathbb{Z}[_2x]/(_2x^2)$
$\mathbb{CP}^2$	$\pi_*K(\mathbb{Z},2)+\pi_*S^5$	$H_*S^2\times H_*S^4$	$x^0+x^2+x^4$	$\mathbb{Z}[_2x]/(_2x^3)$
$\mathbb{CP}^n, n\geq 2$	$\pi_*K(\mathbb{Z},2)+\pi_*S^{2n+1}$	$\prod_{i=1}^n H_*S^{2i}$	$\sum_{i=1}^n x^{2i}$	$\mathbb{Z}[_2x]/(_2x^{n+1})$
Mobius Band	π_*S^1	H_*S^1	$1+x$	?
Klein Bottle	$K(\mathbb{Z}\rtimes_{-1}\mathbb{Z},1)$	$H_*S^1\times H_*\mathbb{RP}^\infty$	$1+2x+x^2$	?

Facts used to compute the above table:

- $\mathbb{R}^n$ is a contractible space, and so $[S^m,\mathbb{R}^n]=0$ for all n,m which makes its homotopy groups all zero.
- $D(k,\mathbb{R}^n)=\mathbb{R}^n-\{x_1\ldots x_k\}\simeq\bigvee_{i=1}^k S^1$ by a deformation retract.
- $S^n\cong B^n/\partial B^n$ and employs an attaching map

$$\begin{aligned}\varphi:(D^n,\partial D^n)&\longrightarrow S^n\\ (D^n,\partial D^n)&\mapsto(e^n,e^0).\end{aligned}$$

- $B^n\simeq\mathbb{R}^n$ by normalizing vectors.
- Use the inclusion $S^n\hookrightarrow B^{n+1}$ as the attaching map.
- $\mathbb{CP}^1\cong S^2$.

- $\mathbb{RP}^1 \cong S^1$.
- Use $\left[\pi_1, \prod\right] = 0$ and the universal cover $\mathbb{R}^1 \twoheadrightarrow S^1$ to yield the cover $\mathbb{R}^n \twoheadrightarrow T^n$.
- Take the universal double cover $S^n \twoheadrightarrow^{\times 2} \mathbb{RP}^n$ to get equality in $\pi_{i\geq 2}$.
- Use $\mathbb{CP}^n = S^{2n+1}/S^1$
- Alternatively, the fundamental group is $\mathbb{Z} * \mathbb{Z}/bab^{-1}a$. Use the fact the $\tilde{K} = \mathbb{R}^2$.
- $M \simeq S^1$ by deformation-retracting onto the center circle.
- $D(1, S^n) \cong \mathbb{R}^n$ and thus $D(k, S^n) \cong D(k-1, \mathbb{R}^n) \cong \bigvee^{k-1} S^1$

10 Theorems: Algebraic Topology

10.1 Fundamental Group

Conjugacy in π_1:

- See Hatcher 1.19, p.28
- See Hatcher's proof that π_1 is a group
- See change of basepoint map
- For a graph G, we always have $\pi_1(G) \cong \mathbb{Z}^n$ where $n = |E(G-T)|$, the complement of the set of edges in any maximal tree. Equivalently, $n = 1 - \chi(G)$. Moreover, $X \simeq \bigvee^n S^1$ in this case.

To calculate $\pi_1(X)$: construct a universal cover $\tilde{X}$, then find a group $G \curvearrowright \tilde{X}$ such that $\tilde{X}/G = X$; then $\pi_1(X) = G$ by uniqueness of universal covers.

10.2 Homotopy

Merge Van Kampen theorems.

Theorem (Van Kampen) The pushout is the northwest colimit of the following diagram

$$\begin{tikzcd}
A\coprod_Z B & A \arrow[l] \\
B \arrow[u] & Z \arrow[l, hook', "\iota_B"'] \arrow[u, hook, "\iota_A"']
\end{tikzcd}$$

For groups, the pushout is given by the amalgamated free product: if $A = \left\langle G_A \;\middle|\; R_A \right\rangle$, $B = \left\langle G_B \;\middle|\; R_B \right\rangle$, then

$$A *_Z B = \left\langle G_A, G_B \;\middle|\; R_A, R_B, T \right\rangle$$

where T is a set of relations given by

$$T = \left\{ \iota_A(z)\iota_B(z)^{-1} \;\middle|\; z \in Z \right\}.$$

Suppose $X = U_1 \bigcup U_2$ such that $U_1 \bigcap U_2 \neq \emptyset$ is **path connected** (necessary condition). Then taking $x_0 \in U \coloneqq U_1 \bigcap U_2$ yields a pushout of fundamental groups

$$\pi_1(X; x_0) = \pi_1(U_1; x_0) *_{\pi_1(U; x_0)} \pi_1(U_2; x_0).$$

Theorem (Van Kampen) If $X = U \bigcup V$ where $U, V, U \bigcap V$ are all path-connected then

$$\pi_1(X) = \pi_1 U *_{\pi_1(U \bigcap V)} \pi_1 V,$$

where the amalgamated product can be computed as follows: If we have presentations

$$\begin{aligned}\pi_1(U, w) &= \left\langle u_1, \cdots, u_k \;\middle|\; \alpha_1, \cdots, \alpha_l \right\rangle \\ \pi_1(V, w) &= \left\langle v_1, \cdots, v_m \;\middle|\; \beta_1, \cdots, \beta_n \right\rangle \\ \pi_1(U \cap V, w) &= \left\langle w_1, \cdots, w_p \;\middle|\; \gamma_1, \cdots, \gamma_q \right\rangle\end{aligned}$$

then

$$\begin{aligned}\pi_1(X, w) = &\langle u_1, \cdots, u_k, v_1, \cdots, v_m \rangle \\ &\quad \bmod \left\langle \alpha_1, \cdots, \alpha_l, \beta_1, \cdots, \beta_n, I(w_1) J(w_1)^{-1}, \cdots, I(w_p) J(w_p)^{-1} \right\rangle \\ = &\frac{\pi_1(U) * \pi_1(B)}{\left\langle \left\{ I(w_i) J(w_i)^{-1} \;\middle|\; 1 \leq i \leq p \right\} \right\rangle}\end{aligned}$$

where

$$\begin{aligned}I &: \pi_1(U \cap V, w) \to \pi_1(U, w) \\ J &: \pi_1(U \cap V, w) \to \pi_1(V, w).\end{aligned}$$

Theorem (Seifert-van Kampen Theorem) Suppose $X = U_1 \bigcup U_2$ where $U \coloneqq U_1 \bigcap U_2 \neq \emptyset$ is path-connected, and let $\{\mathrm{pt}\} \in U$. Then the maps $i_1 : U_1 \longrightarrow X$ and $i_2 : U_2 \longrightarrow X$ induce the following group homomorphisms:

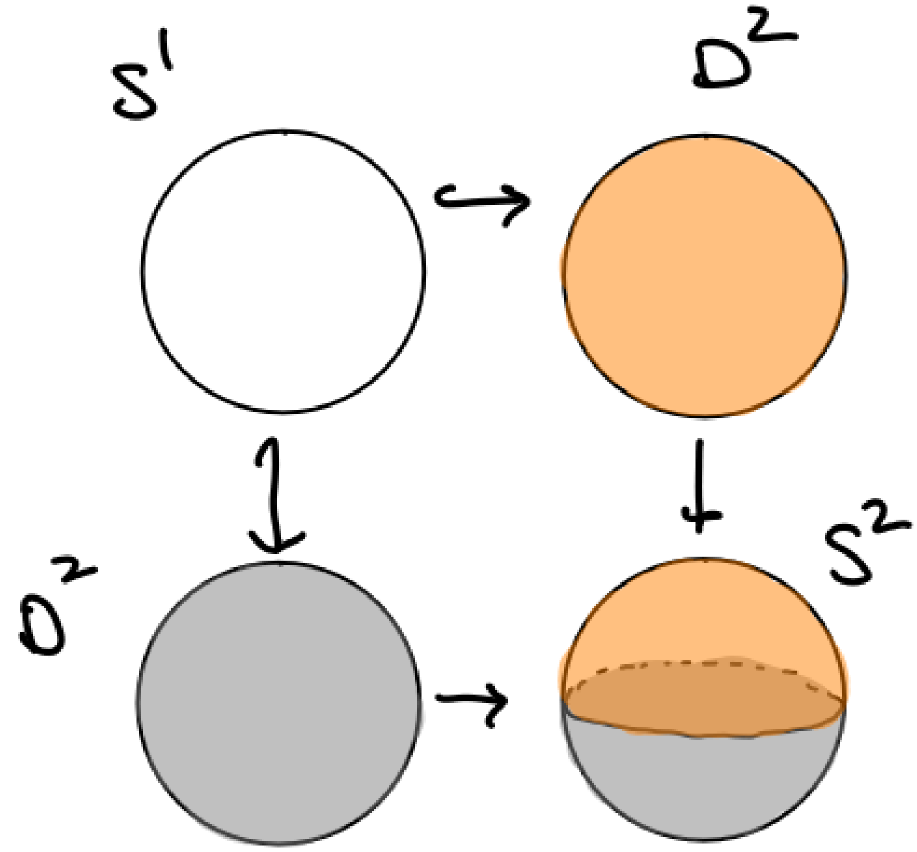
$$\begin{aligned}i_1^* &: \pi_1(U_1, \{\mathrm{pt}\}) \longrightarrow \pi_1(X, \{\mathrm{pt}\}) \\ i_2^* &: \pi_1(U_2, \{\mathrm{pt}\}) \longrightarrow \pi_1(X, \{\mathrm{pt}\})\end{aligned}$$

and letting $P = \pi_1(U), \{\mathrm{pt}\}$, there is a natural isomorphism

$$\pi_1(X, \{\text{pt}\}) \cong \pi_1(U_1, \{\text{pt}\}) *_P \pi_1(U_2, \{\text{pt}\})$$

where $*_P$ is the amalgamated free product over P.

Formulate in terms of pushouts.



Note that the hypothesis that $U \cap V$ is path-connected is necessary: take S^1 with U, V neighborhoods of the poles, whose intersection is two disjoint components.

Example (of pushing out with Van Kampen) $A = \mathbb{Z}/4\mathbb{Z} = \left\langle x \;\middle|\; x^4 \right\rangle, B = \mathbb{Z}/6\mathbb{Z} = \left\langle y \;\middle|\; x^6 \right\rangle, Z = \mathbb{Z}/2\mathbb{Z} = \left\langle z \;\middle|\; z^2 \right\rangle$.

Then we can identify Z as a subgroup of A, B using $\iota_A(z) = x^2$ and $\iota_B(z) = y^3$.

So

$$A *_Z B = \left\langle x, y \;\middle|\; x^4, y^6, x^2 y^{-3} \right\rangle$$

.

Theorem (Whitehead) A map $X \xrightarrow{f} Y$ on CW complexes that is a weak homotopy equivalence (inducing isomorphisms in homotopy) is in fact a homotopy equivalence.

Warning Individual maps may not work: take $S^2 \times \mathbb{RP}^3$ and $S^3 \times \mathbb{RP}^2$ which have isomorphic homotopy but not homology.

Theorem (Hurewicz) The Hurewicz map on an $n-1$-connected space X is an isomorphism $\pi_{k \leq n} X \longrightarrow H_{k \leq n} X$.

> I.e. for the minimal $i \geq 2$ for which $\pi_i X \neq 0$ but $\pi_{\leq i-1} X = 0$, $\pi_i X \cong H_i X$.

Theorem (Cellular Approximation) Any continuous map between CW complexes is homotopy equivalent to a cellular map.

Applications:

- $\pi_{k \leq n} S^n = 0$
- $\pi_n(X) \cong \pi_n(X^{(n)})$

Theorem (Freudenthal Suspension)

Todo

- $\pi_{i \geq 2}(X)$ is always abelian.
- The ranks of π_0 and H_0 are the number of path components, and $\pi_0(X) = \mathbb{Z}$ iff X is simply connected.
  - X simply connected $\implies \pi_k(X) \cong H_k(X)$ up to and including the first nonvanishing H_k
  - $H_1(X) = \mathrm{Ab}(\pi_1 X)$, the abelianization.
- $\pi_k \bigvee X \neq \prod \pi_k X$ (counterexample: $S^1 \vee S^2$)
  - Nice case: $\pi_1 \bigvee X = * \pi_1 X$ by Van Kampen.
- $\pi_i(\widehat{X}) \cong \pi_i(X)$ for $i \geq 2$ whenever $\widehat{X} \twoheadrightarrow X$ is a universal cover.
- $\pi_i(S^n) = 0$ for $i < n$, $\pi_n(S^n) = \mathbb{Z}$
  - Not necessarily true that $\pi_i(S^n) = 0$ when $i > n$!!!
    - E.g. $\pi_3(S^2) = \mathbb{Z}$ by Hopf fibration
- $S^n / S^k \simeq S^n \vee \Sigma S^k$
  - $\Sigma S^n = S^{n+1}$
- General mantra: homotopy plays nicely with products, homology with wedge products.[1]
- $\pi_k \prod X = \prod \pi_k X$ by LES.[2]

In general, homotopy groups behave nicely under homotopy pull-backs (e.g., fibrations and products), but not homotopy push-outs (e.g., cofibrations and wedges). Homology is the opposite.

Constructing a $K(\pi, 1)$: since $\pi = \left\langle S \,\middle|\, R \right\rangle = F(S)/R$, take $\bigvee^{|S|} S^1 \bigcup_{|R|} e^2$. In English, wedge a circle for each generator and attach spheres for relations.

[1]More generally, in **Top**, we can look at $A \leftarrow \{\mathrm{pt}\} \longrightarrow B$ – then $A \times B$ is the pullback and $A \vee B$ is the pushout. In this case, homology $h : \mathbf{Top} \longrightarrow \mathbf{Grp}$ takes pushouts to pullbacks but doesn't behave well with pullbacks. Similarly, while π takes pullbacks to pullbacks, it doesn't behave nicely with pushouts.

[2]This follows because $X \times Y \twoheadrightarrow X$ is a fiber bundle, so use LES in homotopy and the fact that $\pi_{i \geq 2} \in \mathbf{Ab}$.

Proposition (Contracting Spaces in Products)

$$X \times \mathbb{R}^n \simeq X \times \{\text{pt}\} \cong X.$$

---

11 Covering Spaces

When covering spaces are involved in any way, try computing Euler characteristics - this sometimes yields nice numerical constraints.

11.1 Useful Covering Spaces

- $\mathbb{R} \xrightarrow{\pi} S^1 \leftarrow \mathbb{Z}$
- $\mathbb{R}^n \xrightarrow{\pi} T^n \leftarrow \mathbb{Z}^n$
- $\mathbb{RP}^n \xrightarrow{\pi} S^n \leftarrow \mathbb{Z}_2$
- $\vee_n S^1 \xrightarrow{\pi} C^n \leftarrow \mathbb{Z}^{*n}$ where C^n is the n-valent Cayley Graph
- $M \xrightarrow{\pi} \tilde{M} \leftarrow \mathbb{Z}_2$, the orientation double cover
- $T^2 \xrightarrow{\times 2} \mathbb{K}$
- $L_{p/q} \xrightarrow{\pi} S^3 \leftarrow \mathbb{Z}_q$
- $\mathbb{C}^* \xrightarrow{z^n} \mathbb{C} \leftarrow \mathbb{Z}_n$
- For $A \xrightarrow{\pi(\times d)} B$, we have $\chi(A) = d\chi(B)$
- Covering spaces of orientable manifolds are orientable.

11.2 Theorems

Theorem 11.1 ***(Lifts to Universal Cover (H).***
If $f : Y \longrightarrow X$ with Y path-connected and locally path-connected, then there is a unique lift $\widehat{f} : Y \longrightarrow \widehat{X} \iff f_*(\pi_1 Y) \subset \pi_*(\pi_1 \widehat{X})$.

12 Homology

12.1 Unsorted

- $H_n(X/A) \cong \tilde{H}_n(X, A)$ when $A \subset X$ has a neighborhood that deformation retracts onto it.
- $H_n(\bigvee_\alpha X_\alpha) = \bigoplus_\alpha H_n X_\alpha$
- Useful fact: since $\mathbb{Z}$ is free, any exact sequence of the form $0 \longrightarrow \mathbb{Z}^n \longrightarrow A \longrightarrow \mathbb{Z}^m \longrightarrow 0$ splits and $A \cong \mathbb{Z}^n \times \mathbb{Z}^m$.

- Useful fact: $\tilde{H}_*(A \vee B) \cong H_*(A) \times H_*(B)$.
- $H_n(\bigvee_\alpha X_\alpha) = \bigoplus_\alpha H_n X_\alpha$
- $H_n(X, A) \cong H_n(X/A)$
- $H_n(X) = 0 \iff X$ has no n-cells.
- $C^0 X = \{\text{pt}\} \implies d_1 : C^1 \longrightarrow C^0$ is the zero map.
- $H^*(X; \mathbb{F}) = \hom(H_*(X; \mathbb{F}), \mathbb{F})$ for a field.
- Useful tools:
  - Mayer-Vietoris
    * $(X = A \bigcup B) \mapsto (\bigcap, \oplus, \bigcup)$ in homology
  - LES of a pair
    * $(A \hookrightarrow X) \mapsto (A, X, X/A)$
  - Excision
- $H_k \prod X \neq \prod H_k X$ due to torsion.
  - Nice case: $H_k(A \times B) = \prod_{i+j=k} H_i A \otimes H_j B$ by Kunneth when all groups are torsion-free.[3]
- $H_k \bigvee X = \prod H_k X$ by Mayer-Vietoris.[4]
- $H_i(S^n) = \mathbb{1}[i \in \{0, n\}]$
- $H_n(\bigvee_i X_i) \cong \prod_i H_n(X_i)$ for "good pairs"
  - Corollary: $H_n(\bigvee_k S^n) = \mathbb{Z}^k$

$$X = A \cup B \implies A \cap B \longrightarrow A \oplus B \longrightarrow A \cup B \xrightarrow{\delta} \cdots (X, A) \implies \quad A \longrightarrow X \longrightarrow X, A \xrightarrow{\delta} \cdots$$
$$A \longrightarrow B \longrightarrow C \implies \operatorname{Tor}(A, G) \longrightarrow \operatorname{Tor}(B, G) \longrightarrow \operatorname{Tor}(C, G) \xrightarrow{\delta_\downarrow} \cdots$$
$$A \longrightarrow B \longrightarrow C \implies \operatorname{Ext}(A, G) \longrightarrow \operatorname{Ext}(B, G) \longrightarrow \operatorname{Ext}(C, G) \xrightarrow{\delta_\uparrow} \cdots$$

12.2 Constructing a CW Complex with Prescribed Homology

- Given $G = \bigoplus G_i$, and want a space such that $H_i X = G$? Construct $X = \bigvee X_i$ and then $H_i(\bigvee X_i) = \bigoplus H_i X_i$. Reduces problem to: given a group H, find a space Y such that

[3] The generalization of Kunneth is as follows: write $\mathcal{P}(n, k)$ be the set of partitions of n into k parts, i.e. $\mathbf{x} \in \mathcal{P}(n, k) \implies \mathbf{x} = (x_1, x_2, \ldots, x_k)$ where $\sum x_i = n$. Then

$$H_n\left(\prod_{j=1}^k X_j\right) = \bigoplus_{\mathbf{x} \in \mathcal{P}(n,k)} \bigotimes_{i=1}^k H_{x_i}(X_i).$$

[4] $\bigvee$ is the coproduct in the category $\mathbf{Top}_0$ of pointed topological spaces, and alternatively, $X \vee Y$ is the pushout in $\mathbf{Top}$ of $X \leftarrow \{\text{pt}\} \longrightarrow Y$

$H_n(Y) = G$.
  – Attach an e^n to a point to get $H_n = \mathbb{Z}$
  – Then attach an e^{n+1} with attaching map of degree d to get $H_n = \mathbb{Z}_d$

12.3 Mayer-Vietoris

Theorem (Mayer Vietoris) Let $X = A^\circ \bigcup B^\circ$; then there is a SES of chain complexes

$$0 \longrightarrow C_n(A \bigcap B) \xrightarrow{x \mapsto (x,-x)} C_n(A) \oplus C_n(B) \xrightarrow{(x,y) \mapsto x+y} C_n(A+B) \longrightarrow 0$$

where $C_n(A+B)$ denotes the chains that are sums of chains in A and chains in B. This yields a LES in homology:

$$\cdots \longrightarrow H_n(A \bigcap B) \xrightarrow{x \mapsto (x,-x)} H_n(A) \oplus H_n(B) \xrightarrow{(x,y) \mapsto x+y} H_n(X) \longrightarrow \cdots.$$

Given $A, B \subset X$ such that $A^\circ \cup B^\circ = X$, there is a long exact sequence in homology:

$$\begin{tikzcd}
 & & & \cdots \arrow[dlll, rounded corners, "\delta_3"'] \\
H_2(A \cap B) \arrow[r, "(i^*,-j^*)_2"] & H_2A \oplus H_2B \arrow[r, "(l^*-r^*)_2"] & H_2(A \cup B) \arrow[dll, rounded corners, "\delta_2"'] & \\
H_1(A \cap B) \arrow[r, "(i^*,-j^*)_1"] & H_1A \oplus H_1B \arrow[r, "(l^*-r^*)_1"] & H_1(A \cup B) \arrow[dll, rounded corners, "\delta_1"'] & \\
H_0(A \cap B) \arrow[r, "(i^*,-j^*)_0"] & H_0A \oplus H_0B \arrow[r, "(l^*-r^*)_0"] & H_0(A \cup B) \arrow[dll, rounded corners, "\delta_0"'] & \\
0 & & &
\end{tikzcd}$$

This is sometimes written in the following compact form:

$$\cdots H_n(A \cap B) \xrightarrow{(i^*,\, j^*)} H_n(A) \oplus H_n(B) \xrightarrow{l^* - r^*} H_n(X) \xrightarrow{\delta} H_{n-1}(A \cap B) \cdots$$

Where

- $i : A \cap B \hookrightarrow A$ induces $i^* : H_*(A \cap B) \longrightarrow H_*(A)$
- $j : A \cap B \hookrightarrow B$ induces $j^* : H_*(A \cap B) \longrightarrow H_*(B)$
- $l : A \hookrightarrow A \cup B$ induces $l^* : H_*(A) \longrightarrow H_*(X)$
- $r : B \hookrightarrow A \cup B$ induces $r^* : H_*(B) \longrightarrow H_*(X)$

The connecting homomorphisms $\delta_n : H_n(X) \longrightarrow H_{n-1}(X)$ are defined by taking a class $[\alpha] \in H_n(X)$, writing it as an n-cycle z, then decomposing $z = \sum c_i$ where each c_i is an $x + y$ chain. Then $\partial(c_i) = \partial(x + y) = 0$, since the boundary of a cycle is zero, so $\partial(x) = -\partial(y)$. So then just define $\delta([\alpha]) = [\partial x] = [-\partial y]$.

Handy mnemonic diagram:

$$\begin{tikzcd} & A \bigcap B \arrow[dl, dash] \arrow[dr, dash] & \\ A \bigcup B & & A \oplus B \arrow[ll] \end{tikzcd}.$$

12.3.1 Application: Isomorphisms in the homology of spheres.

Claim: $H^i(S^n) \cong H^{i-1}(S^{n-1})$.

Write $X = A \cup B$, the northern and southern hemispheres, so that $A \cap B = S^{n-1}$, the equator. In the LES, we have:

$$H^{i+1}(S^n) \to H^i(S^{n-1}) \to H^i A \oplus H^i B \to H^i S^n \to H^{i-1}(S^{n-1}) \to H^{i-1} A \oplus H^{i-1} B.$$

But A, B are contractible, so $H^i A = H^i B = 0$, so we have

$$H^{i+1}(S^n) \to H^i(S^{n-1}) \to 0 \oplus 0 \to H^i(S^n) \to H^{i-1}(S^{n-1}) \to 0.$$

And in particular, we have the shape $0 \longrightarrow A \longrightarrow B \longrightarrow 0$ in an exact sequence, which is always an isomorphism.

Theorem (Eilenberg-Zilber) Given two spaces X, Y, there are chain maps

$$\begin{aligned} F : C_*(X \times Y; R) &\longrightarrow C_*(X; R) \otimes_R C_*(Y; R) \\ G : C_*(X; R) \otimes_R C_*(Y; R) &\longrightarrow C_*(X \times Y; R) \end{aligned}$$

such that $FG = \mathrm{id}$ and $GF \simeq \mathrm{id}$. In particular,

$$H_*(X \times Y; R) \cong H_*(X; R) \otimes_R H_*(Y; R).$$

Theorem (Kunneth) There exists a short exact sequence

$$0 \longrightarrow \bigoplus_{i+j=k} H_j(X; R) \otimes_R H_i(Y; R) \longrightarrow H_k(X \times Y; R) \longrightarrow \bigoplus_{i+j=k-1} \mathrm{Tor}^1_R(H_i(X; R), H_j(Y; R))$$

It has a non-canonical splitting given by

$$H_k(X \times Y) = \left(\bigoplus_{i+j=k} H_i X \oplus H_j Y \right) \oplus \bigoplus_{i+j=k-1} \mathrm{Tor}(H_i X, H_j Y)$$

Theorem (Universal Coefficients for Change of Group) For changing coefficients from $\mathbb{Z}$ to G an arbitrary group, there are short exact sequences

$$0 \longrightarrow H_iX \otimes G \longrightarrow H_i(X;G) \longrightarrow \mathrm{Tor}(H_{i-1}X,G) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Ext}(H_{i-1}X,G) \longrightarrow H^i(X;G) \longrightarrow \hom(H_iX,G) \longrightarrow 0$$

which split unnaturally:

$$H_i(X;G) = (H_iX \otimes G) \oplus \mathrm{Tor}(H_{i-1}X;G)$$

$$H^i(X;G) = \hom(H_iX,G) \oplus \mathrm{Ext}(H_{i-1}X;G)$$

When H_iX are all finitely generated, write $H_i(X;\mathbb{Z}) = \mathbb{Z}^{\beta_i} \oplus T_i$. Then

$$H^i(X;\mathbb{Z}) = \mathbb{Z}^{\beta_i} \oplus T_{i-1}.$$

12.3.2 Useful long exact sequences

$$\cdots \longrightarrow H^i(X) \longrightarrow H^i(U) \oplus H^i(V) \longrightarrow H^i(U \cap V) \xrightarrow{\delta} H^{i+1}(X) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_i(A) \longrightarrow H_i(X) \longrightarrow H_i(X,A) \xrightarrow{\delta} H_{i-1}(A) \longrightarrow \cdots$$

12.3.3 Useful Short Exact Sequences

Note that $\mathrm{Ext}^0_R = \hom_R$ and $\mathrm{Tor}^0_R = \otimes_R$

Homology to cohomology:

$$0 \longrightarrow \mathrm{Tor}^0_{\mathbb{Z}}(H_i(X;\mathbb{Z}),A) \longrightarrow H_i(X;A) \longrightarrow \mathrm{Tor}^1_{\mathbb{Z}}(H_{i-1}(X;\mathbb{Z}),A) \longrightarrow 0.$$

Cohomology to dual space:

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}(X;\mathbb{Z}),A) \longrightarrow H^i(X;A) \longrightarrow \mathrm{Ext}^0_{\mathbb{Z}}(H_i(X;\mathbb{Z}),A) \longrightarrow 0.$$

Product of spaces to tensor product of homology:

$$0 \longrightarrow \bigoplus_{i+j=k} H_i(X;R) \otimes_R H_j(Y;R) \longrightarrow H_k(X \times Y;R) \longrightarrow \bigoplus_{i+j=k-1} \mathrm{Tor}^R_1(H_i(X;R),H_j(Y;R)) \longrightarrow 0$$

12.3.4 Useful Shortcuts

- Cohomology: If A is a field, then

$$H^i(X;A) \cong \hom(H_i(X;A), A)$$

- Kunneth: If R is a freely generated free R-module (or a PID or a field), then

$$H_k(X \times Y) \cong \bigoplus_{i+j=k} H_i(X) \otimes H_j(Y) \bigoplus_{i+j=k-1} \mathrm{Tor}(H_i(X), H_j(X))$$

- Universal Coefficients Theorem: If X is a finite CW complex then

$$H^i(X;\mathbb{Z}) = F(H_i(X;\mathbb{Z})) \times T(H_{i-1}(X;\mathbb{Z}))$$
$$H_i(X;\mathbb{Z}) = F(H^i(X;\mathbb{Z})) \times T(H^{i+1}(X;\mathbb{Z}))$$

12.4 Cellular Homology

- S^n has the CW complex structure of 2 k-cells for each $0 \leq k \leq n$.

How to compute:

1. Write cellular complex

$$0 \longrightarrow C^n \longrightarrow C^{n-1} \longrightarrow \cdots C^2 \longrightarrow C^1 \longrightarrow C^0 \longrightarrow 0$$

2. Compute differentials $\partial_i : C^i \longrightarrow C^{i-1}$
   3. *Note: if C^0 is a point, ∂_1 is the zero map.*
   4. *Note: $H_n X = 0 \iff C^n = \emptyset$.*
   5. Compute degrees: Use $\partial_n(e_i^n) = \sum_i d_i e_i^{n-1}$ where

$$d_i = \deg(\text{Attach } e_i^n \longrightarrow \text{Collapse } X^{n-1}\text{-skeleton}),$$

   which is a map $S^{n-1} \longrightarrow S^{n-1}$.
      1. Alternatively, choose orientations for both spheres. Then pick a point in the target, and look at points in the fiber. Sum them up with a weight of +1 if the orientations match and -1 otherwise.
   6. Note that $\mathbb{Z}^m \xrightarrow{f} \mathbb{Z}^n$ has an $n \times m$ matrix
   7. Row reduce, image is span of rows with pivots. Kernel can be easily found by taking RREF, padding with zeros so matrix is square and has all diagonals, then reading down diagonal - if a zero is encountered on nth element, take that column vector as a basis element with -1 substituted in for the nth entry. e.g.

$$\begin{array}{cccc} \mathbf{1} & 2 & 0 & 2 \\ 0 & 0 & \mathbf{1} & -1 \\ 0 & 0 & 0 & \mathbf{0} \end{array} \longrightarrow \begin{array}{cccccccc} \mathbf{1} & 2 & 0 & 2 & \mathbf{1} & 2 & 0 & 2 \\ 0 & \mathbf{0} & 0 & 0 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & \mathbf{1} & -1 & 0 & 0 & \mathbf{1} & -1 \\ 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & \mathbf{0} \end{array}$$

$$\ker = \begin{array}{cc} 2 & 3 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{array}$$

$$\mathrm{im} \ = \langle a + 2b + 2d, c - d \rangle \,.$$

6. Or look at elementary divisors, say n_i, then the image is isomorphic to $\bigoplus n_i \mathbb{Z}$

13 Fixed Points and Degree Theory

Theorem (Lefschetz Fixed Point) If $\Lambda_f \neq 0$ then f has a fixed point, where $X \circlearrowleft_f$ and $\Lambda_f = \sum_{k \geq 0} (-1)^k \operatorname{Tr}(H_k(X; \mathbb{Q}) \circlearrowleft_{f_*})$.

Theorem: (Brouwer Fixed Point) Every $B^n \circlearrowleft_f$ has a fixed point.

Theorem (Hairy Ball) There is no non-vanishing tangent vector field on even dimensional spheres.

Theorem (Borsuk-Ulam) For every $S^n \xrightarrow{f} \mathbb{R}^n \exists x \in S^n$ such that $f(x) = f(-x)$.

Theorem (Ham Sandwich)

Todo

Review and collect notes from Hatcher.

14 Surfaces and Manifolds

14.1 Classification of Surfaces

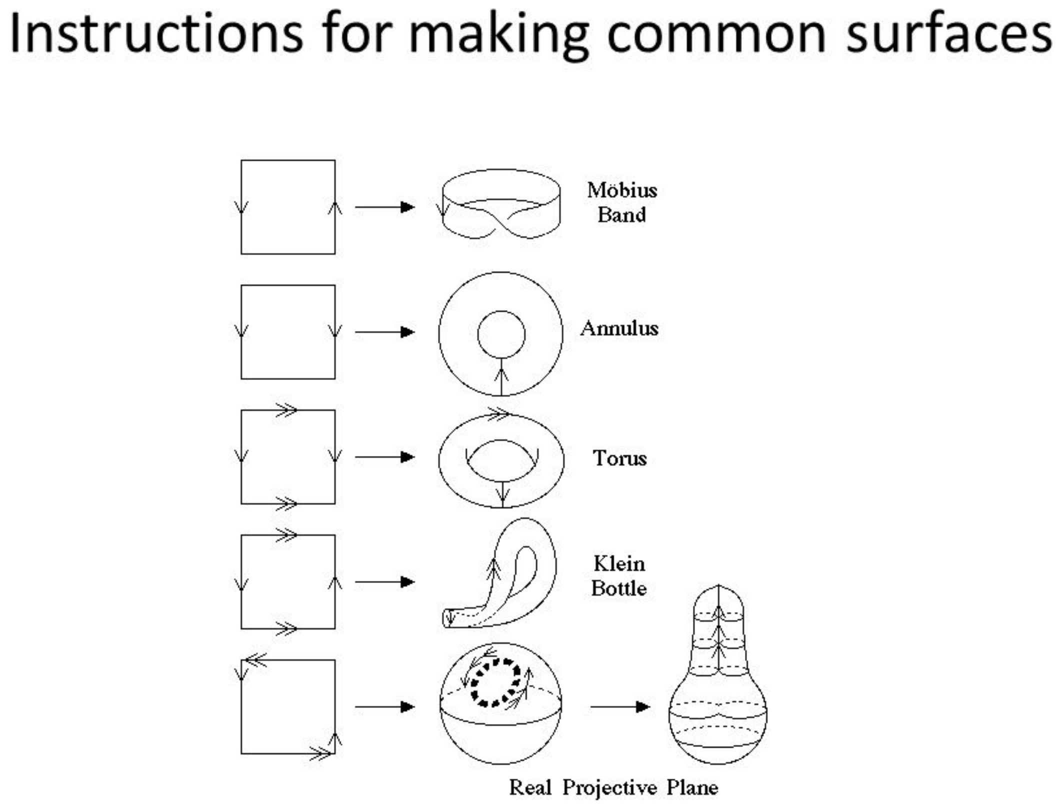


Figure 2: Pasting Diagrams for Surfaces

The most common spaces appearing in this theory:

- $\mathbb{M}$ the Möbius Strip
- $\mathbb{S}^2$,
- $\mathbb{T}^2 \coloneqq S^1 \times S^1$,
- $\mathbb{RP}^2$
- $\mathbb{K}$ the Klein bottle
- $\Sigma_n \coloneqq \mathop{\#}\limits_{i=1}^{n} \mathbb{T}^2$.

Theorem 14.1 ***(Classification of Surfaces).***
The set of surfaces under connect sum forms a monoid with the presentation

$$\left\langle \mathbb{S}^2, \mathbb{RP}^2, \mathbb{T} \;\middle|\; \mathbb{S}^2 = 0, 3\mathbb{RP}^2 = \mathbb{RP}^2 + \mathbb{T}^2 \right\rangle.$$

Surfaces are classified up to homeomorphism by orientability and χ, or equivalently "genus"

- In orientable case, actual genus, g equals the number of copies of $\mathbb{T}^2$.
- In nonorientable case, k equals the number of copies of $\mathbb{RP}^2$.

In each case, there is a formula

$$\chi(X) = \begin{cases} 2 - 2g - b & \text{orientable} \\ 2 - k & \text{non-orientable.} \end{cases}$$

Orientable?	-4	-3	-2	-1	0	1	2
Yes	Σ_3	$\emptyset$	Σ_2	$\emptyset$	$\mathbb{T}^2, S^1 \times I$	$\mathbb{D}^2$	$\mathbb{S}^2$
No	?	?	?	?	$\mathbb{K}, \mathbb{M}$	$\mathbb{RP}^2$	$\emptyset$

Proposition 14.2 ***(Inclusion-Exclusion).***

$$X = U \cup V \implies \chi(X) = \chi(U) + \chi(V) - \chi(U \cap V).$$

Proof .
Todo ∎

Proof.

Corollary 14.3 ***(Euler for Connect Sums).***

$$\chi(A \# B) = \chi(A) + \chi(B) - 2.$$

Proof .
Set $U = A, B = V$, then by definition of the connect sum, $A \cap B = \mathbb{S}^2$ where $\chi(\mathbb{S}^2) = 2$ ∎

Proposition 14.4 ***(Decomposing $\mathbb{RP}^2$).***

$$\mathbb{RP}^2 = \mathbb{M} \coprod_{\mathrm{id}_{\partial \mathbb{M}}} \mathbb{M}.$$

Proposition 14.5 ***(Decomposing a Klein Bottle).***

$$\mathbb{K} \cong \mathbb{RP}^2 \# \mathbb{RP}^2.$$

Proof .
Todo ∎

Proof.

Proposition 14.6 ***(Rewriting a Klein Bottle).***

$$\mathbb{RP}^2 \# \mathbb{K} \cong \mathbb{RP}^2 \# \mathbb{T}^2.$$

Proof .
Todo

■

Proof.

14.2 Manifolds

To show something is not a manifold, try looking at local homology. Can use point-set style techniques like removing points, i.e. $H_1(X, X - \{\mathrm{pt}\})$; this should essentially always yield $\mathbb{Z}$ by excision arguments.

- M^n closed/connected $\implies H_n = \mathbb{Z}$ and $\operatorname{Tor}(H_{n-1}) = 0$
- 3-manifolds:
  - Orientable: $H_* = (\mathbb{Z}, \mathbb{Z}^r, \mathbb{Z}^r, \mathbb{Z})$
  - Nonorientable: $H_* = (\mathbb{Z}, \mathbb{Z}^r, \mathbb{Z}^{r-1} \oplus \mathbb{Z}_2, \mathbb{Z})$
- $H^n(M^n) = \mathbb{Z}$ if M^n is orientable and zero if M^n is nonorientable.
- Poincaré Duality: $H_i M^n = \cong H^{n-i} M^n$ iff M^n is closed and orientable.

On the complements of spaces in $\mathbb{R}^3$:

My personal crutch is to just think about complements in S^3, which are usually easier since knot complements in S^3 are always $K(\pi, 1)$s. Now if K is a knot and X is its complement in S^3, then you can prove that its complement in $\mathbb{R}^3$ is homotopy equivalent to $S^2 \vee X$

If M is a closed 3-manifold and K is a nullhomologous knot in M, then $H_1(X - n(K)) \cong H_1(X) \times \mathbb{Z}$ where $n(K)$ is a tubular neighborhood.

Proposition 14.7 ***(Homology of Sphere minus a knot).***
For $M = S^3 \setminus K$, $H_*(M) = [\mathbb{Z}, \mathbb{Z}, 0, 0, \cdots]$.

Proof .
Apply Mayer-Vietoris, taking $S^3 = n(K) \cup (S^3 - K)$, where $n(K) \simeq S^1$ and $S^3 - K \cap n(K) \simeq T^2$. Use the fact that $S^3 - K$ is a connected, open 3-manifold, so $H^3(S^3 - K) = 0$.

■

- Every $\mathbb{C}$-manifold is canonically orientable.
- If M^n is **closed and connected**, then $H_{\geq n}(X) = 0$ and M^n is orientable iff $H_n(X) = \mathbb{Z}$.
- If M^n is a **closed orientable manifold without boundary**, then $H^k(M^n; F) \cong H_{n-k}(M^n; F)$ for a field F.
- This is a strict implication, so failure of the RHS implies missing conditions on the LHS.
- The intersection pairing is nondegenerate modulo torsion.
- If M^n is a **closed orientable manifold with boundary** then $H_k(M^n; \mathbb{Z}) \cong H^{n-k}(M^n, \partial M^n; \mathbb{Z})$
- M^n closed, connected, and orientable $\implies H_n = \mathbb{Z}$ and $\operatorname{Tor}(H_{n-1}) = 0$
- M^n closed and n odd implies $\chi(M^n) = 0$.
- Any map $X \longrightarrow Y$ with X factors through the orientation cover $\tilde{Y}_o$.

- If Y is non-orientable, this is a double cover.
- If n is odd, $\chi(M^n) = 0$ by Poincaré Duality.

Theorem 14.8 ***(Poincare Duality).***
Todo

Theorem 14.9 ***(Lefschetz Duality).***
Todo

15 Extra Problems: Algebraic Topology

15.1 Homotopy 101

- Show that if $X \xrightarrow{f} X^n$ is not surjective, then f is nullhomotopic.

15.2 π_1

- Compute $\pi_1(S^1 \vee S^1)$
- Compute $\pi_1(S^1 \times S^1)$

15.3 Surfaces

- Show that if $M^{\text{orientable}} \xrightarrow{\pi_k} M^{\text{non-orientable}}$ is a k-fold cover, then k is even or ∞.
- Show that M is orientable if $\pi_1(M)$ has no subgroup of index 2.

16 Fall 2014

16.1 1

Let $X = \mathbb{R}^3 - \Delta^{(1)}$, the complement of the skeleton of regular tetrahedron, and compute $\pi_1(X)$ and $H_*(X)$.

Lay the graph out flat in the plane, then take a maximal tree - these leaves 3 edges, and so $\pi_1(X) = \mathbb{Z}^{*3}$.

Moreover $X \simeq S^1 \vee S^1 \vee S^1$ which has only a 1-skeleton, thus $H_*(X) = [\mathbb{Z}, \mathbb{Z}^3, 0 \to]$.

16.2 2

Let $X = S^1 \times B^2 - L$ where L is two linked solid torii inside a larger solid torus. Compute $H_*(X)$. ?

16.3 3

Let L be a 3-manifold with homology $[\mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z}, \ldots]$ and let $X = L \times \Sigma L$. Compute $H_*(X), H^*(X)$.

Useful facts:

- $H_k(X \times Y) \cong \bigoplus_{i+j=k} H_i(X) \otimes H_j(Y) \bigoplus_{i+j=k-1} \mathrm{Tor}(H_i(X), H_j(Y))$
- $\tilde{H}_i(\Sigma X) = \tilde{H}_{i-1}(X)$

We will use the fact that $H_*(\Sigma L) = [\mathbb{Z}, \mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z}]$.

Represent $H_*(L)$ by $p(x, y) = 1 + yx + x^3$ and $H_*(\Sigma L)$ by $q(x, y) = 1 + x + yx^2 + x^4$, we can extract the free part of $H_*(X)$ by multiplying

$$p(x, y)q(x, y) = 1 + (1 + y)x + 2yx^2 + (y^2 + 1)x^3 + 2x^4 + 2yx^5 + x^7$$

where multiplication corresponds to the tensor product, addition to the direct sum/product.

So the free portion is

$$\begin{aligned} H_*(X) &= [\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}_3 \otimes \mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z}_3 \otimes \mathbb{Z}_3, \mathbb{Z}^2, \mathbb{Z}_3^2, 0, \mathbb{Z}] \\ &= [\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}^2, \mathbb{Z}_3^2, 0, \mathbb{Z}] \end{aligned}$$

We can add in the correction from torsion by noting that only terms of the form $\mathrm{Tor}(\mathbb{Z}_3, \mathbb{Z}_3) = \mathbb{Z}_3$ survive. These come from the terms $i = 1, j = 2$, so $i + j = k - 1 \implies k = 1 + 2 + 1 = 4$ and there is thus an additional torsion term appearing in dimension 4. So we have

$$\begin{aligned} H_*(X) &= [\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z} \times \mathbb{Z}_3, \mathbb{Z}^2 \times \mathbb{Z}_3, \mathbb{Z}_3^2, 0, \mathbb{Z}] \\ &= [\mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}^2, 0, 0, \mathbb{Z}] \times [0, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3^2, 0, 0] \end{aligned}$$

and

$$\begin{aligned} H^*(X) &= [\mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}^2, 0, 0, \mathbb{Z}] \times [0, 0, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3^2, 0] \\ &= [\mathbb{Z}, \mathbb{Z}, \mathbb{Z}_3, \mathbb{Z} \times \mathbb{Z}_3, \mathbb{Z}^2 \times \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3^2, \mathbb{Z}]. \end{aligned}$$

■

16.4 4

Let M be a closed, connected, oriented 4-manifold such that $H_2(M; \mathbb{Z})$ has rank 1. Show that there is not a free $\mathbb{Z}_2$ action on M.

Useful facts:

- $X \twoheadrightarrow_{\times p} Y$ induces $\chi(X) = p\chi(Y)$
- Moral: always try a simple Euler characteristic argument first!

We know that $H_*(M) = [\mathbb{Z}, A, \mathbb{Z} \times G, A, \mathbb{Z}]$ for some group A and some torsion group G. Letting $n = \mathrm{rank}(A)$ and taking the Euler characteristic, we have $\chi(M) = (1)1 + (-1)n + (1)1 + (-1)n + (1)1 = 3 - 2n$. Note that this is odd for any n.

However, a free action of $\mathbb{Z}_2 \curvearrowright M$ would produce a double covering $M \twoheadrightarrow_{\times 2} M/\mathbb{Z}_2$, and multiplicativity of Euler characteristics would force $\chi(M) = 2\chi(M/\mathbb{Z}_2)$ and thus $3 - 2n = 2k$ for some integer k. This would require $3 - 2n$ to be even, so we have a contradiction. ■

16.5 5

Let X be T^2 with a 2-cell attached to the interior along a longitude. Compute $\pi_2(X)$.

Useful facts:

- $T^2 = e^0 + e^1_1 + e^1_2 + e^2$ as a CW complex.
- $S^2/(x_0 \sim x_1) \simeq S^2 \wedge S^1$ when x_0, x_1 are two distinct points. (Picture: sphere with a string handle connecting north/south poles.)
- $\pi_{\geq 2}(\tilde{X}) \cong \pi_{\geq 2}(X)$ for $\tilde{X} \twoheadrightarrow X$ the universal cover.

Write $T^2 = e^0 + e^1_1 + e^1_2 + e^2$, where the first and second 1-cells denote the longitude and meridian respectively. By symmetry, we could have equivalently attached a disk to the meridian instead of the longitude, filling the center hole in the torus. Contract this disk to a point, then pull it vertically in both directions to obtain S^2 with two points identified, which is homotopy-equivalent to $S^2 \vee S_1$.

Take the universal cover, which is $\mathbb{R}^1 \bigcup_{\mathbb{Z}} S^2$ and has the same π_2. This is homotopy-equivalent to $\bigvee_{i \in \mathbb{Z}} S^2$ and so $\pi_2(X) = \prod_{i \in \mathbb{Z}} \mathbb{Z}$ generated by each distinct copy of S^2. (Alternatively written as $\mathbb{Z}[t, t^{-1}]$).

17 Summer 2003

17.1 1

Describe all possible covering maps between S^2, T^2, K

Useful facts:

1. $\tilde{X} \twoheadrightarrow X$ induces $\pi_1(\tilde{X}) \hookrightarrow \pi_1(X)$
2. $\chi(\tilde{X}) = n\chi(X)$
3. $\pi_n(X) = [S^n, X]$
4. $Y \longrightarrow X$ with $\pi_1(Y) = 0$ and $\tilde{X} \simeq \{\text{pt}\} \implies$ every $Y \xrightarrow{f} X$ is nullhomotopic.
5. $\pi_*(T^2) = [\mathbb{Z} * \mathbb{Z}, 0 \rightarrow]$
6. $\pi_*(K) = [\mathbb{Z} \rtimes_{\mathbb{Z}_2} \mathbb{Z}, 0 \rightarrow]$
7. Universal covers are homeomorphic.
8. $\pi_{\geq 2}(\tilde{X}) \cong \pi_{\geq 2}(X)$

Spaces

- $S^2 \twoheadrightarrow T^2$
- $S^2 \twoheadrightarrow K$
- $K \twoheadrightarrow S^2$
- $T^2 \twoheadrightarrow S^2$
  - All covered by the fact that
  $$\mathbb{Z} = \pi_2(S^2) \neq \pi_2(X) = 0$$
  for $X = T^2, K$.
- $K \twoheadrightarrow T^2$
  - Doesn't cover, would induce $\pi_1(K) \hookrightarrow \pi_1(T^2) \implies \mathbb{Z} \rtimes \mathbb{Z} \hookrightarrow \mathbb{Z}^2$ but this would be a non-abelian subgroup of an abelian group.

- $T^2 \twoheadrightarrow K$
  - ?

$\blacksquare$

17.2 2

Show that $\mathbb{Z}^{*2}$ has subgroups isomorphic to $\mathbb{Z}^{*n}$ for every n.

Facts Used 1. $\pi_1(\bigvee^k S^1) = \mathbb{Z}^{*k}$ 2. $\tilde{X} \twoheadrightarrow X \implies \pi_1(\tilde{X}) \hookrightarrow \pi_1(X)$ 3. Every subgroup $G \leq \pi_1(X)$ corresponds to a covering space $X_G \twoheadrightarrow X$ 4. $A \subseteq B \implies F(A) \leq F(B)$ for free groups.

It is easier to prove the stronger claim that $\mathbb{Z}^{\mathbb{N}} \leq \mathbb{Z}^{*2}$ (i.e. the free group on countably many generators) and use fact 4 above.

Just take the covering space $\tilde{X} \twoheadrightarrow S^1 \vee S^1$ defined via the gluing map $\mathbb{R} \bigcup_{\mathbb{Z}} S^1$ which attaches a circle to each integer point, taking 0 as the base point. Then let a denote a translation and b denote traversing a circle, so we have $\pi_1(\tilde{X}) = \left\langle \bigcup_{n\in\mathbb{Z}} a^n b a^{-n} \right\rangle$ which is a free group on countably many generators. Since $\tilde{X}$ is a covering space, $\pi_1(\tilde{X}) \hookrightarrow \pi_1(S^1 \vee S^1) = \mathbb{Z}^{*2}$. By 4, we can restrict this to n generators for any n to get a subgroup, and $A \leq B \leq C \implies A \leq C$ as groups.

$\blacksquare$

17.3 3

Construct a space having $H_*(X) = [\mathbb{Z}, 0, 0, 0, 0, \mathbb{Z}_4, 0 \to]$.

Facts used: - Construction of Moore Spaces - $\tilde{H}_n(\Sigma X) = \tilde{H}_{n-1}(X)$, using $\Sigma X = C_X \bigcup_X C_X$ and Mayer-Vietoris.

Take $X = e^0 \bigcup_{\Phi_1} e^5 \bigcup_{\Phi_2} e^6$, where

$$\Phi_1 : \partial B^5 = S^4 \xrightarrow{z \mapsto z^0} e^0$$

$$\Phi_2 : \partial B^6 = S^5 \xrightarrow{z \mapsto z^4} e^5.$$

where $\deg \Phi_2 = 4$. $\blacksquare$

17.4 4

Compute the complement of a knotted solid torus in S^3.

Facts used:

- $H_*(T^2) = [\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}, 0 \to]$
- $N^{(1)} \simeq S^1$, so $H_{\geq 2}(N) = 0$.
- A SES $0 \longrightarrow A \longrightarrow B \longrightarrow F \longrightarrow 0$ with F free splits.
- $0 \longrightarrow A \longrightarrow B \xrightarrow{\cong} C \longrightarrow D \longrightarrow 0$ implies $A = D = 0$.

Let N be the knotted solid torus, so that $\partial N = T^2$, and let $X = S^3 - N$. Then

- $S^3 = N \bigcup_{T^2} X$
- $N \cap X = T^2$

and we apply Mayer-Vietoris to S^3:

$$\begin{array}{ll}
4 & H_4(T^2) \longrightarrow H_4(N) \times H_4(X) \longrightarrow H_4(S^3) \\
3 & H_3(T^2) \longrightarrow H_3(N) \times H_3(X) \longrightarrow H_3(S^3) \\
2 & H_2(T^2) \longrightarrow H_2(N) \times H_2(X) \longrightarrow H_2(S^3) \\
1 & H_1(T^2) \longrightarrow H_1(N) \times H_1(X) \longrightarrow H_1(S^3) \\
0 & H_0(T^2) \longrightarrow H_0(N) \times H_0(X) \longrightarrow H_0(S^3)
\end{array}.$$

where we can plug in known information and deduce some maps:

$$\begin{array}{llrl}
4 & 0 \longrightarrow & 0 & \longrightarrow 0 \xrightarrow{\partial_4} \\
3 & 0 \longrightarrow & H_3(X) & \longrightarrow \mathbb{Z} \xrightarrow{\partial_3} \\
2 & \mathbb{Z} \longrightarrow & H_2(X) & \longrightarrow 0 \xrightarrow{\partial_2} \\
1 & \mathbb{Z}^2 \cong & \mathbb{Z} \times H_1(X) & \longrightarrow 0 \xrightarrow{\partial_1} \\
0 & \mathbb{Z} \longrightarrow & \mathbb{Z} \times H_0(X) & \longrightarrow \mathbb{Z} \longrightarrow 0
\end{array}$$

(1)
(2)
(3)
(4)
(5)
(6)

We then deduce: - $H_0(X) = \mathbb{Z}$ by the splitting of the line 0 SES

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \times H_0(X) \longrightarrow \mathbb{Z} \longrightarrow 0$$

yielding $Z \times H_0(X) \cong \mathbb{Z} \times \mathbb{Z}$. - $H_1(X) = \mathbb{Z}$ by the line 1 SES

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \times H_1(X) \longrightarrow 0$$

which yields an isomorphism. - $H_2(X) = H_3(X) = 0$ by examining the SES spanning lines 3 and 2:

$$0 \hookrightarrow H_3(X) \hookrightarrow \mathbb{Z} \xrightarrow{\cong \partial_3} \mathbb{Z} \twoheadrightarrow H_2(X) \twoheadrightarrow 0$$

Since ∂_3 must be an isomorphism, this forces the edge terms to be zero.

■

17.5 5

Compute the homology and cohomology of a closed, connected, oriented 3-manifold M with $\pi_1(M) = \mathbb{Z}^{*2}$.

Facts used: - M closed, connected, oriented $\implies H_i(M) \cong H^{n-i}(M)$ - $H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$
- For orientable manifolds $H_n(M^n) = \mathbb{Z}$

Homology

- Since M is connected, $H_0 = \mathbb{Z}$
- Since $\pi_1(M) = \mathbb{Z}^{*2}$, H_1 is the abelianization and $H_1(X) = \mathbb{Z}^2$
- Since M is closed/connected/oriented, Poincare Duality holds and $H_2 = H^{3-2} = H^1 = \mathbf{F}H_1 + \mathbf{T}H_0$ by UCT. Since $H_0 = \mathbb{Z}$ is torsion-free, we have $H_2(M) = H_1(M) = \mathbb{Z}^2$.
- Since M is an orientable manifold, $H_3(M) = \mathbb{Z}$
- So $H_*(M) = [\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^2, \mathbb{Z}, 0 \to]$

Cohomology

- By Poincare Duality, $H^*(M) = \widehat{H_*(M)} = [\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^2, \mathbb{Z}, 0 \to]$. (Where the hat denotes reversing the list.)

■

17.6 6

Compute $\operatorname{Ext}(\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_5)$

Facts Used:

1. $\operatorname{Ext}(\mathbb{Z}, \mathbb{Z}_m) = \mathbb{Z}_m$
2. $\operatorname{Ext}(\mathbb{Z}_m, \mathbb{Z}) = 0$
3. $\operatorname{Ext}(\prod_i A_i, \prod_j B_j) = \prod_i \prod_j \operatorname{Ext}(A_i, B_j)$

Break it up into a bigraded complex, take Ext of the pieces, and sum over the complex: $\operatorname{Ext}(\downarrow, \to)$ | $\mathbb{Z}$ | $\mathbb{Z}_4$ | $\mathbb{Z}_5$ ————————————|———|———|——— $\mathbb{Z}$ | 0 | 0 | 0 $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 $\mathbb{Z}_3$ | $\mathbb{Z}_3$ | 0 | 0

So the answer is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 = \mathbb{Z}_{12}$.

■

17.7 7

Show there is no homeomorphism $\mathbb{CP}^2 \circlearrowleft_f$ such that $f(\mathbb{CP}^1)$ is disjoint from $\mathbb{CP}_1 \subset \mathbb{CP}_2$.

Facts used:

1. Every homeomorphism induces isomorphisms on homotopy/homology/cohomology.
2. $H^*(\mathbb{CP}^2) = \mathbb{Z}[\alpha]/(\alpha^2)$ where $\deg \alpha = 2$.
3. $[f(X)] = f_*([X])$
4. $a\cdot b = 0 \implies a = 0$ or $b = 0$ (nondegeneracy).

Supposing such a homeomorphism exists, we would have $[\mathbb{CP}^1]\cdot[f(\mathbb{CP}^1)] = 0$ by the definition of these submanifolds being disjoint.

But $[\mathbb{CP}^1]\cdot[f(\mathbb{CP}^1)] = [\mathbb{CP}^1]\cdot f_*([\mathbb{CP}^1])$, where

$$f_* : H^*(\mathbb{CP}^2) \longrightarrow H^*(\mathbb{CP}^2)$$

is the induced map on cohomology.

Since the intersection pairing is nondegenerate, either $[\mathbb{CP}^1] = 0$ or $f_*([\mathbb{CP}^1]) = 0$.

We know that $H^*(\mathbb{CP}^2) = \mathbb{Z}[\alpha]/\alpha^2$ where $\alpha = [\mathbb{CP}^1]$, however, so this forces $f_*([\mathbb{CP}^1]) = 0$. But since this was a generator of H^*, we have $f_*(H^*(\mathbb{CP}^2)) = 0$, so f is not an isomorphism on cohomology. ■

17.8 8

Describe the universal cover of $X = (S^1 \times S^1) \vee S^2$ and compute $\pi_2(X)$.

Facts used: - $\pi_{\geq 2}(\tilde{X}) \cong \pi_{\geq 2}(X)$ - Structure of the universal cover of a wedge product - $\mathbb{R}^2 \twoheadrightarrow_p T^2 = S^1 \times S^1$

$\tilde{X} = \mathbb{R}^2 \bigcup_{\mathbb{Z}^2} S^2$, so $\pi_2(X) \cong \pi_2(\tilde{X}) = \prod_{i,j \in \mathbb{Z}^2} \mathbb{Z} = \mathbb{Z}^{\mathbb{Z}^2} = \mathbb{Z}^{\aleph_0}$. ■

17.9 9

Let $S^3 \longrightarrow E \longrightarrow S^5$ be a fiber bundle and compute $H_3(E)$.

Facts used: - Homotopy LES - Hurewicz - $0 \longrightarrow A \longrightarrow B \longrightarrow 0$ exact iff $A \cong B$

From the LES in homotopy we have

$$
\begin{aligned}
&4 \quad \pi_4(S^3) \longrightarrow \pi_4(E) \longrightarrow \pi_4(S^5) && (7)\\
&3 \quad \pi_3(S^3) \longrightarrow \pi_3(E) \longrightarrow \pi_3(S^5) && (8)\\
&2 \quad \pi_2(S^3) \longrightarrow \pi_2(E) \longrightarrow \pi_2(S^5) && (9)\\
&1 \quad \pi_1(S^3) \longrightarrow \pi_1(E) \longrightarrow \pi_1(S^5) && (10)\\
&0 \quad \pi_0(S^3) \longrightarrow \pi_0(E) \longrightarrow \pi_0(S^5) && (11)\\
& && (12)
\end{aligned}
$$

and plugging in known information yields

$$
\begin{aligned}
&4 \quad \pi_4(S^3) \longrightarrow && \pi_4(E) && \longrightarrow 0 && (13)\\
&3 \quad \mathbb{Z} \longrightarrow && \pi_3(E) && \longrightarrow 0 && (14)\\
&2 \quad 0 \longrightarrow && \pi_2(E) && \longrightarrow 0 && (15)\\
&1 \quad 0 \longrightarrow && \pi_1(E) && \longrightarrow 0 && (16)\\
&0 \quad \mathbb{Z} \longrightarrow && \pi_0(E) && \longrightarrow \mathbb{Z} && (17)\\
& && && && (18)
\end{aligned}
$$

where rows 3 and 4 force $\pi_3(E) \cong \mathbb{Z}$, rows 0 and 1 force $\pi_0(E) = \mathbb{Z}$, and the remaining rows force $\pi_1(E) = \pi_2(E) = 0$.

By Hurewicz, we thus have $H_3(E) = \pi_3(E) = \mathbb{Z}$.

$\blacksquare$

18 Fall 2017 Final

18.1 1

Let X be the subspace of the unit cube I^3 consisting of the union of the 6 faces and the 4 internal diagonals. Compute $\pi_1(X)$.

Solution:

18.2 2

Let X be an arbitrary topological space, and compute $\pi_1(\Sigma X)$.

Solution:

Write $\Sigma X = U \bigcup V$ where $U = \Sigma X - (X \times [0, 1/2])$ and $U = \Sigma X - X \times [1/2, 1])$. Then $U \bigcap V = X \times \{1/2\} \cong X$, so $\pi_1(U \bigcap V) = \pi_1(X)$.

But both U and V can be identified by the cone on X, given by $CX = \dfrac{X \times I}{X \times 1}$, by just rescaling the interval with the maps:

$i_U : U \longrightarrow CX$ where $(x, s) \mapsto (x, 2s - 1)$ (The second component just maps $[1/2, 1] \longrightarrow [0, 1]$.)

$i_V : V \longrightarrow CX$ where $(x, s) \mapsto (x, 2s)$. (The second component just maps $[0, 1/2] \longrightarrow [0, 1]$)

But CX is contractible by the homotopy $H : CX \times I \longrightarrow CX$ where $H((c, s), t) = (c, s(1 - t))$.

So $\pi_1(U) = \pi_1(V) = 0$.

By Van Kampen, we have $\pi_1(X) = 0 *_{\pi_1(X)} 0 = 0$.

18.3 3

Let $X = S^1 \times S^1$ and $A \subset X$ be a subspace with $A \cong S^1 \vee S^1$. Show that there is no retraction from X to A.

Solution:

We have $\pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1)$ since S^1 is path-connected (by a lemma from the problem sets), and this equals $\mathbb{Z} \times \mathbb{Z}$.

We also have $\pi_1(S^1 \vee S^1) = \pi_1(S^1) *_{\{pt\}} \pi_1(S^1)$, which by Van-Kampen is $\mathbb{Z} * \mathbb{Z}$.

Suppose X retracts onto A, we can then look at the inclusion $\iota : A \hookrightarrow X$. The induced homomorphism $\iota_* : \pi_1(A) \hookrightarrow \pi_1(X)$ is then also injective, so we've produced an injection from $f : \mathbb{Z} * \mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Z}$.

This is a contradiction, because no such injection can exists. In particular, the commutator $[a, b]$ is nontrivial in the source. But $f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1}$ since f is a homomorphism, but since the target is a commutative group, this has to equal $f(a)f(a)^{-1}f(b)f(b)^{-1} = e$. So there is a non-trivial element in the kernel of f, and f can not be injective - a contradiction.

18.4 4

Show that for every map $f : S^2 \longrightarrow S^1$, there is a point $x \in S^2$ such that $f(x) = f(-x)$.

Solution:

Suppose towards a contradiction that f does not possess this property, so there is no $x \in S^2$ such that $f(x) = f(-x)$.

Then define $g : S^2 \longrightarrow S^1$ by $g(x) = f(x) - f(-x)$; by assumption, this is a nontrivial map, i.e. $g(x) \neq 0$ for *any* $x \in S^2$.

In particular, $-g(-x) = -(f(-x) - f(x)) = f(x) - f(-x) = g(x)$, so $-g(x) = g(-x)$ and thus g commutes with the antipodal map $\alpha : S^2 \longrightarrow S^2$.

This means g is constant on the fibers of the quotient map $p : S^2 \longrightarrow \mathbb{RP}2$, and thus descends to a well defined map $\tilde{g} : \mathbb{RP}2 \longrightarrow S^1$, and since $S^1 \cong \mathbb{RP}1$, we can identify this with a map $\tilde{g} : \mathbb{RP}2 \longrightarrow \mathbb{RP}1$ which thus induces a homomorphism $\tilde{g}_* : \pi_1(\mathbb{RP}2) \longrightarrow \pi_1(\mathbb{RP}1)$.

Since g was nontrivial, $\tilde{g}$ is nontrivial, and by functoriality of π_1, $\tilde{g}_*$ is nontrivial.

But $\pi_1(\mathbb{RP}2) = \mathbb{Z}_2$ and $\pi_1(\mathbb{RP}1) = \mathbb{Z}$, and $\tilde{g}_* : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ can only be the trivial homomorphism - a contradiction.

Alternate Solution

Use covering space $\mathbb{R} \twoheadrightarrow S^1$?

18.5 5

How many path-connected 2-fold covering spaces does $S^1 \vee \mathbb{RP}2$ have? What are the total spaces?

Solution:

First note that $\pi_1(X) = \pi_1(S^1) *_{\{\text{pt}\}} \pi_1(\mathbb{RP}2)$ by Van-Kampen, and this is equal to $\mathbb{Z} * \mathbb{Z}_2$.

18.6 6

Let $G =< a, b >$ and $H \leq G$ where $H =< aba^{-1}b^{-1},\ a^2ba^{-2}b^{-1},\ a^{-1}bab^{-1},\ aba^{-2}b^{-1}a >$. To what well-known group is H isomorphic?

Solution:

19 Appendix: Homological Algebra

19.1 Free Resolutions

The canonical example:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \xrightarrow{\text{mod } m} \mathbb{Z}_m \longrightarrow 0$$

Or more generally for a finitely generated group $G = \langle g_1, g_2, \cdots, g_n \rangle$,

$$\cdots \longrightarrow \ker(f) \longrightarrow F[g_1, g_2, \cdots, g_n] \xrightarrow{f} G \longrightarrow 0$$

where F denotes taking the free group.

Every abelian groups has a resolution of this form and length 2.

19.2 Computing Tor

$$\mathrm{Tor}(A,B) = h[\cdots \longrightarrow A_n \otimes B \longrightarrow A_{n-1} \otimes B \longrightarrow \cdots A_1 \otimes B \longrightarrow 0]$$

where A_* is any free resolution of A.

Shorthand/mnemonic:

$$\mathrm{Tor} : \mathcal{F}(A) \longrightarrow (\,\cdot \otimes B) \longrightarrow H_*$$

19.3 Computing Ext

$$\mathrm{Ext}(A,B) = h[\cdots \hom(A,B_n) \longrightarrow \hom(A,B_{n-1}) \longrightarrow \cdots \longrightarrow \hom(A,B_1) \longrightarrow 0]$$

where B_* is a any free resolution of B.

Shorthand/mnemonic:

$$\mathrm{Ext} : \mathcal{F}(B) \longrightarrow \hom(A,\,\cdot\,) \longrightarrow H_*$$

19.4 Properties of Tensor Products

- $A \otimes B \cong B \otimes A$
- $(\,\cdot\,) \otimes_R R^n = \mathrm{id}$
- $\bigoplus_i A_i \otimes \bigoplus_j B_j = \bigoplus_i \bigoplus_j (A_i \otimes B_j)$
- $\mathbb{Z}_m \otimes \mathbb{Z}_n = \mathbb{Z}_d$
- $\mathbb{Z}_n \otimes A = A/nA$

19.5 Properties of Hom

- $\hom_R(\bigoplus_i A_i, \prod B_j) = \bigoplus_i \prod_j \hom(A_i, B_j)$
- Contravariant in first slot, covariant in second
- Exact over vector spaces

19.6 Properties of Tor

- $\mathrm{Tor}_R^0(A,B) = A \otimes_R B$
- $\mathrm{Tor}(\bigoplus_i A_i, \bigoplus_j B) = \bigoplus_i \bigoplus_j \mathrm{Tor}(\mathbf{T}A_i, \mathbf{T}B_j)$ where $\mathbf{T}G$ is the torsion component of G.
- $\mathrm{Tor}(\mathbb{Z}_n, G) = \ker(g \mapsto ng) = \left\{ g \in G \;\middle|\; ng = 0 \right\}$
- $\mathrm{Tor}(A,B) = \mathrm{Tor}(B,A)$

19.7 Properties of Ext

- $\operatorname{Ext}_R^0(A,B) = \hom_R(A,B)$
- $\operatorname{Ext}(\bigoplus_i A_i, \prod_j B_j) = \bigoplus_i \prod_j \operatorname{Ext}(\mathbf{T}A_i, B_j)$
- $\operatorname{Ext}(F,G) = 0$ if F is free
- $\operatorname{Ext}(\mathbb{Z}_n, G) \cong G/nG$

19.8 Hom/Ext/Tor Tables

hom	$\mathbb{Z}_m$	$\mathbb{Z}$	$\mathbb{Q}$
$\mathbb{Z}_n$	$\mathbb{Z}_d$	0	0
$\mathbb{Z}$	$\mathbb{Z}_m$	$\mathbb{Z}$	$\mathbb{Q}$
$\mathbb{Q}$	0	0	$\mathbb{Q}$

Tor	$\mathbb{Z}_m$	$\mathbb{Z}$	$\mathbb{Q}$
$\mathbb{Z}_n$	$\mathbb{Z}_d$	0	0
$\mathbb{Z}$	0	0	0
$\mathbb{Q}$	0	0	0

Ext	$\mathbb{Z}_m$	$\mathbb{Z}$	$\mathbb{Q}$
$\mathbb{Z}_n$	$\mathbb{Z}_d$	$\mathbb{Z}_n$	0
$\mathbb{Z}$	0	0	0
$\mathbb{Q}$	0	$\mathcal{A}_{\surd}/\mathbb{Q}$	0

Where $d = \gcd(m,n)$ and $\mathbb{Z}_0 \coloneqq 0$.

Things that behave like "the zero functor":

- $\operatorname{Ext}(\mathbb{Z}, \cdot\,)$
- $\operatorname{Tor}(\,\cdot\,, \mathbb{Z}), \operatorname{Tor}(\mathbb{Z}, \cdot\,)$
- $\operatorname{Tor}(\,\cdot\,, \mathbb{Q}), \operatorname{Tor}(\mathbb{Q}, \cdot\,)$

Thins that behave like "the identity functor":

- $\hom(\mathbb{Z}, \cdot\,)$
- $\cdot \otimes_{\mathbb{Z}} \mathbb{Z}$ and $\mathbb{Z} \otimes_{\mathbb{Z}} \cdot$

For description of $\mathcal{A}_{\surd}$, see here. This is a certain ring of adeles.

20 Appendix: ?

- Assorted info about other Lie Groups:
- $O_n, U_n, SO_n, SU_n, Sp_n$

- $\pi_k(U_n) = \mathbb{Z} \cdot \mathbb{1}\,[k \text{ odd}]$
  - $\pi_1(U_n) = 1$
- $\pi_k(SU_n) = \mathbb{Z} \cdot \mathbb{1}\,[k \text{ odd}]$
  - $\pi_1(SU_n) = 0$
- $\pi_k(U_n) = \mathbb{Z}/2\mathbb{Z} \cdot \mathbb{1}\,[k = 0, 1 \mod 8] + \mathbb{Z} \cdot \mathbb{1}\,[k = 3, 7 \mod 8]$
- $\pi_k(SP_n) = \mathbb{Z}/2\mathbb{Z} \cdot \mathbb{1}\,[k = 4, 5 \mod 8] + \mathbb{Z} \cdot \mathbb{1}\,[k = 3, 7 \mod 8]$
- Groups and Group Actions
  - $\pi_0(G) = G$ for G a discrete topological group.
  - $\pi_k(G/H) = \pi_k(G)$ if $\pi_k(H) = \pi_{k-1}(H) = 0$.
  - $\pi_1(X/G) = \pi_0(G)$ when G acts freely/transitively on X.

20.1 Cap and Cup Products

$$\cup : H^p \times H^q \longrightarrow H^{p+q}; (a^p \cup b^q)(\sigma) = a^p(\sigma \circ F_p) b^q(\sigma \circ B_q)$$

where F_p, B_q is embedding into a $p + q$ simplex.

For f continuous, $f^*(a \cup b) = f^* a \cup f^* b$

It satisfies the Leibniz rule

$$\partial(a^p \cup b^q) = \partial a^p \cup b^q + (-1)^p (a^p \cup \partial b^q)$$

$$\cap : H_p \times H^q \longrightarrow H_{p-q}; \sigma \cap \psi = \psi(F \circ \sigma)(B \circ \sigma)$$

where F, B are the front/back face maps.

Given $\psi \in C^q, \varphi \in C^p, \sigma : \Delta^{p+q} \longrightarrow X$, we have

$$\begin{gathered} \psi(\sigma \cap \varphi) = (\varphi \cup \psi)(\sigma) \\ \langle \varphi \cup \psi,\ \sigma \rangle = \langle \psi,\ \sigma \cap \varphi \rangle \end{gathered}$$

Let M^n be a closed oriented smooth manifold, and $A^{\widehat{i}}, B^{\widehat{j}} \subseteq X$ be submanifolds of codimension i and j respectively that intersect transversely (so $\forall p \in A \bigcap B$, the inclusion-induced map $T_p A \times T_p B \longrightarrow T_p X$ is surjective.)

Then $A \cap B$ is a submanifold of codimension $i + j$ and there is a short exact sequence

$$0 \longrightarrow T_p(A \bigcap B) \longrightarrow T_p A \times T_p B \longrightarrow T_p X \longrightarrow 0$$

which determines an orientation on $A \bigcap B$.

Then the images under inclusion define homology classes

- $[A] \in H_{\widehat{i}}X$
- $[B] \in H_{\widehat{j}}X$
- $[A \bigcap B] \in H_{\widehat{i+j}}X$.

Denoting their Poincare duals by

- $[A]^\vee \in H^i X$
- $[B]^\vee \in H^j X$
- $[A \bigcap B]^\vee \in H^{i+j} X$

We then have

$$[A]^\vee \smile [B]^\vee = [A \bigcap B]^\vee \in H^{i+j} X$$

Example: in $\mathbb{CP}^n$, each even-dimensional cohomology $H^{2i}\mathbb{CP}^n$ has a generator α_i with is Poincare dual to an $\widehat{i}$ plane. A generic $\widehat{i}$ plane intersects a $\widehat{j}$ plane in a $\widehat{i+j}$ plane, yielding $\alpha_i \smile \alpha_j = \alpha_{i+j}$ for $i + j \leq n$.

Example: For T^2, we have - $H_1 T^2 = \mathbb{Z}^2$ generated by $[A], [B]$, the longitudinal and meridian circles. - $H_0 T^2 = \mathbb{Z}$ generated by $[p]$, the class of a point.

Then $A \cap B = \pm[p]$, and so

$$\begin{aligned} [A]^\vee \smile [B]^\vee &= [p]^\vee \\ [B]^\vee \smile [A]^\vee &= -[p]^\vee \end{aligned}$$

20.2 The Long Exact Sequence of a Pair

LES of pair $(A, B) \implies \cdots H_n(B) \longrightarrow H_n(A) \longrightarrow H_n(A, B) \longrightarrow H_{n-1}(B) \cdots$

$$\begin{array}{ccccc} & & B & & \\ & \diagup & & \diagdown & \\ (A, B) & & \longleftarrow & & A \end{array}.$$

3.1.3 Example. The cases $n = 1, 2$ and part of the case $n = 3$ are shown in the figure below.

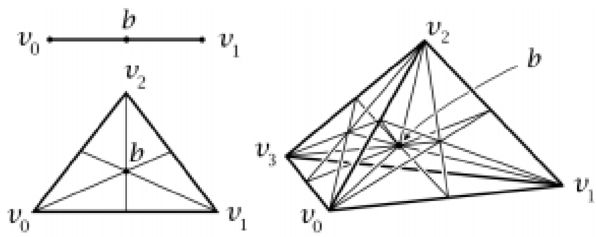


Figure 3.1: Barycentric subdivision [10].

Figure 3: Barycentric Subdivision