# UGA Algebra Qualifying Exam Questions (Spring 2011 – Spring 2021)

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## **1** Group Theory: General

## 1.1 Spring 2020 #2

Let H be a normal subgroup of a finite group G where the order of H and the index of H in G are relatively prime. Prove that no other subgroup of G has the same order as H.

Work this problem

## 1.2 Spring 2019 #4 🔆

For a finite group G, let c(G) denote the number of conjugacy classes of G.

a. Prove that if two elements of G are chosen uniformly at random, then the probability they commute is precisely

$$\frac{c(G)}{|G|}.$$

- b. State the class equation for a finite group.
- c. Using the class equation (or otherwise) show that the probability in part (a) is at most

$$\frac{1}{2} + \frac{1}{2[G:Z(G)]}.$$

Here, as usual, Z(G) denotes the center of G.

Relevant concepts omitted.

Hint/strategy omitted.

Solution omitted.

## $\sim$ 1.3 Spring 2012 #2

Let G be a finite group and p a prime number such that there is a normal subgroup  $H \subseteq G$  with  $|H| = p^i > 1$ .

- a. Show that H is a subgroup of any Sylow p-subgroup of G.
- b. Show that G contains a nonzero abelian normal subgroup of order divisible by p.

## $\sim$ 1.4 Spring 2017 #1 $\stackrel{ extstyle }{\sim}$

Let G be a finite group and  $\pi: G \to \operatorname{Sym}(G)$  the Cayley representation.

(Recall that this means that for an element  $x \in G$ ,  $\pi(x)$  acts by left translation on G.)

Prove that  $\pi(x)$  is an odd permutation  $\iff$  the order  $|\pi(x)|$  of  $\pi(x)$  is even and  $|G|/|\pi(x)|$  is odd.

## $\sim$ 1.5 Fall 2016 #1 $\stackrel{ extstyle op}{\sim}$

Let G be a finite group and  $s, t \in G$  be two distinct elements of order 2. Show that subgroup of G generated by s and t is a dihedral group.

Recall that the dihedral groups of order 2m for  $m \geq 2$  are of the form

$$D_{2m} = \left\langle \sigma, \tau \mid \sigma^m = 1 = \tau^2, \tau \sigma = \sigma^{-1} \tau \right\rangle.$$

## $\sim$ 1.6 Fall 2015 #1 $\stackrel{ extstyle o}{\sim}$

Let G be a group containing a subgroup H not equal to G of finite index. Prove that G has a normal subgroup which is contained in every conjugate of H which is of finite index.



For a prime p, let G be a finite p-group and let N be a normal subgroup of G of order p. Prove that N is contained in the center of G.

1.4 Spring 2017 #1

### 1.8 Fall 2014 #6

Let G be a group and H, K < G be subgroups of finite index. Show that

$$[G:H\cap K]\leq [G:H]\ [G:K].$$

## $\sim$ 1.9 Spring 2013 #3 $\stackrel{\triangleright}{}$

Let P be a finite p-group. Prove that every nontrivial normal subgroup of P intersects the center of P nontrivially.

## $\sim$ 1.10 Fall 2019 Midterm #1 $\stackrel{ extstyle o}{\sim}$

Let G be a group of order  $p^2q$  for p,q prime. Show that G has a nontrivial normal subgroup.

## $\sim$ 1.11 Fall 2019 Midterm #4 $\stackrel{ extstyle op}{\sim}$

Let p be a prime. Show that  $S_p = \langle \tau, \sigma \rangle$  where  $\tau$  is a transposition and  $\sigma$  is a p-cycle.

## $\sim$ 1.12 Fall 2019 Midterm #5 $\sim$

Let G be a nonabelian group of order  $p^3$  for p prime. Show that Z(G) = [G, G].

## $\sim$ 1.13 Spring 2021 #2 $\sim$

Let  $H \leq G$  be a normal subgroup of a finite group G, where the order of H is the smallest prime p dividing |G|. Prove that H is contained in the center of G.

1.8 Fall 2014 #6

## 2 | Groups: Sylow Theory

## $\sim$ 2.1 Fall 2019 #1 $\rarrow$

Let G be a finite group with n distinct conjugacy classes. Let  $g_1 \cdots g_n$  be representatives of the conjugacy classes of G. Prove that if  $g_i g_j = g_j g_i$  for all i, j then G is abelian.

Relevant concepts omitted.

Solution omitted.



Let G be a finite group and let P be a sylow p-subgroup for p prime. Show that N(N(P)) = N(P) where N is the normalizer in G.



Let G be a group of order 30.

- a. Show that G has a subgroup of order 15.
- b. Show that every group of order 15 is cyclic.
- c. Show that G is isomorphic to some semidirect product  $\mathbb{Z}_{15} \rtimes \mathbb{Z}_2$ .
- d. Exhibit three nonisomorphic groups of order 30 and prove that they are not isomorphic. You are not required to use your answer to (c).



Let  $G \subset S_9$  be a Sylow-3 subgroup of the symmetric group on 9 letters.

a. Show that G contains a subgroup H isomorphic to  $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$  by exhibiting an appropriate set of cycles.

Groups: Sylow Theory 10

- b. Show that H is normal in G.
- c. Give generators and relations for G as an abstract group, such that all generators have order 3. Also exhibit elements of  $S_9$  in cycle notation corresponding to these generators.
- d. Without appealing to the previous parts of the problem, show that G contains an element of order 9.



Let G be a group of order 96.

- a. Show that G has either one or three 2-Sylow subgroups.
- b. Show that either G has a normal subgroup of order 32, or a normal subgroup of order 16.



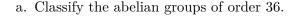
- a. State the three Sylow theorems.
- b. Prove that any group of order 1225 is abelian.
- c. Write down exactly one representative in each isomorphism class of abelian groups of order 1225.



- a. How many isomorphism classes of abelian groups of order 56 are there? Give a representative for one of each class.
- b. Prove that if G is a group of order 56, then either the Sylow-2 subgroup or the Sylow-7 subgroup is normal.
- c. Give two non-isomorphic groups of order 56 where the Sylow-7 subgroup is normal and the Sylow-2 subgroup is *not* normal. Justify that these two groups are not isomorphic.

2.5 Fall 2014 #2

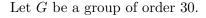
#### 2.8 Fall 2017 #2



For the rest of the problem, assume that G is a non-abelian group of order 36. You may assume that the only subgroup of order 12 in  $S_4$  is  $A_4$  and that  $A_4$  has no subgroup of order 6.

- b. Prove that if the 2-Sylow subgroup of G is normal, G has a normal subgroup N such that G/N is isomorphic to  $A_4$ .
- c. Show that if G has a normal subgroup N such that G/N is isomorphic to  $A_4$  and a subgroup H isomorphic to  $A_4$  it must be the direct product of N and H.
- d. Show that the dihedral group of order 36 is a non-abelian group of order 36 whose Sylow-2 subgroup is not normal.

### 2.9 Fall 2012 #2



- a. Show that G contains normal subgroups of orders 3, 5, and 15.
- b. Give all possible presentations and relations for G.
- c. Determine how many groups of order 30 there are up to isomorphism.

### 2.10 Fall 2018 #1 🦙

Let G be a finite group whose order is divisible by a prime number p. Let P be a normal p-subgroup of G (so  $|P| = p^c$  for some c).

- a. Show that P is contained in every Sylow p-subgroup of G.
- b. Let M be a maximal proper subgroup of G. Show that either  $P \subseteq M$  or  $|G/M| = p^b$  for some  $b \le c$ .

Relevant concepts omitted.

2.8 Fall 2017 #2

Solution omitted.

## 2.11 Fall 2019 #2 🦮

Let G be a group of order 105 and let P, Q, R be Sylow 3, 5, 7 subgroups respectively.

- a. Prove that at least one of Q and R is normal in G.
- b. Prove that G has a cyclic subgroup of order 35.
- c. Prove that both Q and R are normal in G.
- d. Prove that if P is normal in G then G is cyclic.

Relevant concepts omitted.

Solution omitted.

### 2.12 Spring 2021 #3

- a. Show that every group of order  $p^2$  with p prime is abelian.
- b. State the 3 Sylow theorems.
- c. Show that any group of order  $4225 = 5^2 \cdot 13^2$  is abelian.
- d. Write down one representative from each isomorphism class of abelian groups of order 4225.

## 2.13 Fall 2020 #1

- a. Using Sylow theory, show that every group of order 2p where p is prime is not simple.
- b. Classify all groups of order 2p and justify your answer. For the nonabelian group(s), give a presentation by generators and relations.

2.11 Fall 2019 #2 🔭

### 2.14 Fall 2020 #2

Let G be a group of order 60 whose Sylow 3-subgroup is normal.

- a. Prove that G is solvable.
- b. Prove that the Sylow 5-subgroup is also normal.

## **3** Groups: Group Actions

## 3.1 Fall 2012 #1

Let G be a finite group and X a set on which G acts.

- a. Let  $x \in X$  and  $G_x := \{g \in G \mid g \cdot x = x\}$ . Show that  $G_x$  is a subgroup of G.
- b. Let  $x \in X$  and  $G \cdot x := \{g \cdot x \mid g \in G\}$ . Prove that there is a bijection between elements in  $G \cdot x$  and the left cosets of  $G_x$  in G.

## 3.2 Fall 2015 #2

Let G be a finite group, H a p-subgroup, and P a sylow p-subgroup for p a prime. Let H act on the left cosets of P in G by left translation.

Prove that this is an orbit under this action of length 1.

Prove that xP is an orbit of length  $1 \iff H$  is contained in  $xPx^{-1}$ .

## $\sim$ 3.3 Spring 2016 #5 $\stackrel{ extstyle \sim}{ extstyle \sim}$

Let G be a finite group acting on a set X. For  $x \in X$ , let  $G_x$  be the stabilizer of x and  $G \cdot x$  be the orbit of x.

a. Prove that there is a bijection between the left cosets  $G/G_x$  and  $G \cdot x$ .

2.14 Fall 2020 #2

b. Prove that the center of every finite p-group G is nontrivial by considering that action of G on X = G by conjugation.

## 3.4 Fall 2017 #1

Suppose the group G acts on the set A. Assume this action is faithful (recall that this means that the kernel of the homomorphism from G to  $\operatorname{Sym}(A)$  which gives the action is trivial) and transitive (for all a, b in A, there exists g in G such that  $g \cdot a = b$ .)

a. For  $a \in A$ , let  $G_a$  denote the stabilizer of a in G. Prove that for any  $a \in A$ ,

$$\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = \{1\}.$$

b. Suppose that G is abelian. Prove that |G| = |A|. Deduce that every abelian transitive subgroup of  $S_n$  has order n.

#### 3.5 Fall 2018 #2 🥎

- a. Suppose the group G acts on the set X. Show that the stabilizers of elements in the same orbit are conjugate.
- b. Let G be a finite group and let H be a proper subgroup. Show that the union of the conjugates of H is strictly smaller than G, i.e.

$$\bigcup_{g \in G} gHg^{-1} \subsetneq G$$

c. Suppose G is a finite group acting transitively on a set S with at least 2 elements. Show that there is an element of G with no fixed points in S.

Relevant concepts omitted.

Solution omitted.

## 4 Groups: Classification

## 4.1 Spring 2020 #1

3.4 Fall 2017 #1

- a. Show that any group of order 2020 is solvable.
- b. Give (without proof) a classification of all abelian groups of order 2020.
- c. Describe one nonabelian group of order 2020.

Work this problem.

## $\sim$ 4.2 Spring 2019 #3 $^{"}$

How many isomorphism classes are there of groups of order 45?

Describe a representative from each class.

Relevant concepts omitted.

Solution omitted.

Revisit, seems short.



Let G be a group of order 70.

- a. Show that G is not simple.
- b. Exhibit 3 nonisomorphic groups of order 70 and prove that they are not isomorphic.



How many groups are there up to isomorphism of order pq where p < q are prime integers?

4.1 Spring 2020 #1

## 4.5 Spring 2018 #1 💝



- a. Use the Class Equation (equivalently, the conjugation action of a group on itself) to prove that any p-group (a group whose order is a positive power of a prime integer p) has a nontrivial center.
- b. Prove that any group of order  $p^2$  (where p is prime) is abelian.
- c. Prove that any group of order  $5^2 \cdot 7^2$  is abelian.
- d. Write down exactly one representative in each isomorphism class of groups of order  $5^2 \cdot 7^2$ .

Relevant concepts omitted.

Solution omitted.

## **5** Groups: Simple and Solvable



- a. Define what it means for a group G to be solvable.
- b. Show that every group G of order 36 is solvable.

Hint: you can use that  $S_4$  is solvable.



Let N be a positive integer, and let G be a finite group of order N.

a. Let  $\operatorname{Sym} G$  be the set of all bijections from  $G \to G$  viewed as a group under composition. Note that  $\operatorname{Sym} G \cong S_N$ . Prove that the Cayley map

$$C: G \to \operatorname{Sym} G$$
  
 $g \mapsto (x \mapsto gx)$ 

is an injective homomorphism.

- b. Let  $\Phi : \operatorname{Sym} G \to S_N$  be an isomorphism. For  $a \in G$  define  $\varepsilon(a) \in \{\pm 1\}$  to be the sign of the permutation  $\Phi(C(a))$ . Suppose that a has order d. Prove that  $\varepsilon(a) = -1 \iff d$  is even and N/d is odd.
- c. Suppose N > 2 and  $n \equiv 2 \pmod{4}$ . Prove that G is not simple.

Hint: use part (b).



Let p, n be integers such that p is prime and p does not divide n. Find a real number k = k(p, n) such that for every integer  $m \ge k$ , every group of order  $p^m n$  is not simple.



Let p, q be distinct primes.

- a. Let  $\bar{q} \in \mathbb{Z}_p$  be the class of  $q \pmod{p}$  and let k denote the order of  $\bar{q}$  as an element of  $\mathbb{Z}_p^{\times}$ . Prove that no group of order  $pq^k$  is simple.
- b. Let G be a group of order pq, and prove that G is not simple.



Define a simple group. Prove that a group of order 56 can not be simple.



Show that there exist no simple groups of order 148.

5.3 Spring 2014 #1

## **6** Commutative Algebra

## 6.1 Spring 2020 #5 🦙

Let R be a ring and  $f: M \to N$  and  $g: N \to M$  be R-module homomorphisms such that  $g \circ f = \mathrm{id}_M$ . Show that  $N \cong \mathrm{im} f \oplus \ker g$ .

## $\sim$ 6.2 Fall 2019 #3 $\stackrel{\triangleright}{}$

Let R be a ring with the property that for every  $a \in R, a^2 = a$ .

- a. Prove that R has characteristic 2.
- b. Prove that R is commutative.

Relevant concepts omitted.

Hint/strategy omitted.

Solution omitted.

$$\sim$$
 6.3 Fall 2019 #6  $^{\updownarrow}$ 

Let R be a commutative ring with multiplicative identity. Assume Zorn's Lemma.

a. Show that

$$N = \{ r \in R \mid r^n = 0 \text{ for some } n > 0 \}$$

is an ideal which is contained in any prime ideal.

- b. Let r be an element of R not in N. Let S be the collection of all proper ideals of R not containing any positive power of r. Use Zorn's Lemma to prove that there is a prime ideal in S.
- c. Suppose that R has exactly one prime ideal P. Prove that every element r of R is either nilpotent or a unit.

Commutative Algebra 19

Relevant concepts omitted.

Solution omitted.

## 6.4 Spring 2019 #6 💝

Let R be a commutative ring with 1.

Recall that  $x \in R$  is nilpotent iff xn = 0 for some positive integer n.

- a. Show that every proper ideal of R is contained within a maximal ideal.
- b. Let J(R) denote the intersection of all maximal ideals of R. Show that  $x \in J(R) \iff 1+rx$  is a unit for all  $r \in R$ .
- c. Suppose now that R is finite. Show that in this case J(R) consists precisely of the nilpotent elements in R.

Relevant concepts omitted.

Solution omitted.

## 6.5 Fall 2018 #7 🙀

Let R be a commutative ring.

a. Let  $r \in R$ . Show that the map

$$r \bullet : R \to R$$
  
 $x \mapsto rx.$ 

is an R-module endomorphism of R.

- b. We say that r is a **zero-divisor** if  $r \bullet$  is not injective. Show that if r is a zero-divisor and  $r \neq 0$ , then the kernel and image of R each consist of zero-divisors.
- c. Let  $n \geq 2$  be an integer. Show: if R has exactly n zero-divisors, then  $\#R \leq n^2$ .
- d. Show that up to isomorphism there are exactly two commutative rings R with precisely 2 zero-divisors.

You may use without proof the following fact: every ring of order 4 is isomorphic to exactly one of the following:

$$\frac{\mathbb{Z}}{4\mathbb{Z}}, \quad \frac{\frac{\mathbb{Z}}{2\mathbb{Z}}[t]}{(t^2+t+1)}, \quad \frac{\frac{\mathbb{Z}}{2\mathbb{Z}}[t]}{(t^2-t)}, \quad \frac{\frac{\mathbb{Z}}{2\mathbb{Z}}[t]}{(t^2)}.$$

Relevant concepts omitted.

Solution omitted.

## 6.6 Spring 2018 #5

Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} x & u \\ -y & -v \end{pmatrix}$$

over a commutative ring R, where b and x are units of R. Prove that

$$MN = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \implies MN = 0.$$

## 6.7 Spring 2018 #8

Let R = C[0,1] be the ring of continuous real-valued functions on the interval [0,1]. Let I be an ideal of R.

- a. Show that if  $f \in I$ ,  $a \in [0,1]$  are such that  $f(a) \neq 0$ , then there exists  $g \in I$  such that  $g(x) \geq 0$  for all  $x \in [0,1]$ , and g(x) > 0 for all x in some open neighborhood of a.
- b. If  $I \neq R$ , show that the set  $Z(I) = \{x \in [0,1] \mid f(x) = 0 \text{ for all } f \in I\}$  is nonempty.
- c. Show that if I is maximal, then there exists  $x_0 \in [0,1]$  such that  $I = \{f \in R \mid f(x_0) = 0\}$ .

## 6.8 Fall 2017 #5

A ring R is called *simple* if its only two-sided ideals are 0 and R.

- a. Suppose R is a commutative ring with 1. Prove R is simple if and only if R is a field.
- b. Let k be a field. Show the ring  $M_n(k)$ ,  $n \times n$  matrices with entries in k, is a simple ring.

### 6.9 Fall 2017 #6

For a ring R, let U(R) denote the multiplicative group of units in R. Recall that in an integral domain R,  $r \in R$  is called *irreducible* if r is not a unit in R, and the only divisors of r have the form ru with u a unit in R.

We call a non-zero, non-unit  $r \in R$  prime in R if  $r \mid ab \implies r \mid a$  or  $r \mid b$ . Consider the ring  $R = \{a + b\sqrt{-5} \mid a, b \in Z\}$ .

- a. Prove R is an integral domain.
- b. Show  $U(R) = \{\pm 1\}.$
- c. Show  $3, 2 + \sqrt{-5}$ , and  $2 \sqrt{-5}$  are irreducible in R.
- d. Show 3 is not prime in R.
- e. Conclude R is not a PID.

## 6.10 Spring 2017 #3

Let R be a commutative ring with 1. Suppose that M is a free R-module with a finite basis X.

- a. Let  $I \subseteq R$  be a proper ideal. Prove that M/IM is a free R/I-module with basis X', where X' is the image of X under the canonical map  $M \to M/IM$ .
- b. Prove that any two bases of M have the same number of elements. You may assume that the result is true when R is a field.

## 6.11 Spring 2017 #4

a. Let R be an integral domain with quotient field F. Suppose that p(x), a(x), b(x) are monic polynomials in F[x] with p(x) = a(x)b(x) and with  $p(x) \in R[x]$ , a(x) not in R[x], and both a(x), b(x) not constant.

6.9 Fall 2017 #6

Prove that R is not a UFD.

(You may assume Gauss' lemma)

b. Prove that  $\mathbb{Z}[2\sqrt{2}]$  is not a UFD.

*Hint:* let  $p(x) = x^2 - 2$ .

### 6.12 Spring 2016 #8

Let R be a simple rng (a nonzero ring which is not assume to have a 1, whose only two-sided ideals are (0) and R) satisfying the following two conditions:

- i. R has no zero divisors, and
- ii. If  $x \in R$  with  $x \neq 0$  then  $2x \neq 0$ , where 2x := x + x.

Prove the following:

- a. For each  $x \in R$  there is one and only one element  $y \in R$  such that x = 2y.
- b. Suppose  $x, y \in R$  such that  $x \neq 0$  and 2(xy) = x, then yz = zy for all  $z \in R$ .

You can get partial credit for (b) by showing it in the case R has a 1.

## 6.13 Fall 2015 #3

Let R be a rng (a ring without 1) which contains an element u such that for all  $y \in R$ , there exists an  $x \in R$  such that xu = y.

Prove that R contains a maximal left ideal.

Hint: imitate the proof (using Zorn's lemma) in the case where R does have a 1.

## 6.14 Fall 2015 #4

6.12 Spring 2016 #8

Let R be a PID and  $(a_1) < (a_2) < \cdots$  be an ascending chain of ideals in R. Prove that for some n, we have  $(a_j) = (a_n)$  for all  $j \ge n$ .



Let R be a commutative ring, and  $S \subset R$  be a nonempty subset that does not contain 0 such that for all  $x, y \in S$  we have  $xy \in S$ . Let  $\mathcal{I}$  be the set of all ideals  $I \subseteq R$  such that  $I \cap S = \emptyset$ .

Show that for every ideal  $I \in \mathcal{I}$ , there is an ideal  $J \in \mathcal{I}$  such that  $I \subset J$  and J is not properly contained in any other ideal in  $\mathcal{I}$ .

Prove that every such ideal J is prime.



Give a careful proof that  $\mathbb{C}[x,y]$  is not a PID.



Let R be a nonzero commutative ring without unit such that R does not contain a proper maximal ideal. Prove that for all  $x \in R$ , the ideal xR is proper.

You may assume the axiom of choice.



Let R be a commutative ring and  $a \in R$ . Prove that a is not nilpotent  $\iff$  there exists a commutative ring S and a ring homomorphism  $\varphi : R \to S$  such that  $\varphi(a)$  is a unit.

Note: by definition, a is nilpotent  $\iff$  there is a natural number n such that  $a^n = 0$ .

6.14 Fall 2015 #4

### 6.19 Spring 2014 #6

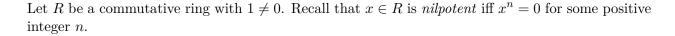
R be a commutative ring with identity and let n be a positive integer.

- a. Prove that every surjective R-linear endomorphism  $T: \mathbb{R}^n \to \mathbb{R}^n$  is injective.
- b. Show that an injective R-linear endomorphism of  $R^n$  need not be surjective.

## 6.20 Fall 2013 #3

- a. Define *prime ideal*, give an example of a nontrivial ideal in the ring  $\mathbb{Z}$  that is not prime, and prove that it is not prime.
- b. Define  $maximal\ ideal$ , give an example of a nontrivial maximal ideal in  $\mathbb{Z}$  and prove that it is maximal.

### 6.21 Fall 2013 #4



- a. Show that the collection of nilpotent elements in R forms an ideal.
- b. Show that if x is nilpotent, then x is contained in every prime ideal of R.
- c. Suppose  $x \in R$  is not nilpotent and let  $S = \{x^n \mid n \in \mathbb{N}\}$ . There is at least on ideal of R disjoint from S, namely (0).

By Zorn's lemma the set of ideals disjoint from S has a maximal element with respect to inclusion, say I. In other words, I is disjoint from S and if J is any ideal disjoint from S with  $I \subseteq J \subseteq R$  then J = I or J = R.

Show that I is a prime ideal.

d. Deduce from (a) and (b) that the set of nilpotent elements of R is the intersection of all prime ideals of R.

25

6.19 Spring 2014 #6

#### 6.22 Spring 2013 #1

Let R be a commutative ring.

- a. Define a  $maximal\ ideal$  and prove that R has a maximal ideal.
- b. Show than an element  $r \in R$  is not invertible  $\iff r$  is contained in a maximal ideal.
- c. Let M be an R-module, and recall that for  $0 \neq \mu \in M$ , the annihilator of  $\mu$  is the set

$$\operatorname{Ann}(\mu) = \left\{ r \in R \mid r\mu = 0 \right\}.$$

Suppose that I is an ideal in R which is maximal with respect to the property that there exists an element  $\mu \in M$  such that  $I = \operatorname{Ann}(\mu)$  for some  $\mu \in M$ . In other words,  $I = \operatorname{Ann}(\mu)$  but there does not exist  $\nu \in M$  with  $J = \operatorname{Ann}(\nu) \subsetneq R$  such that  $I \subsetneq J$ .

Prove that I is a prime ideal.

#### 6.23 Spring 2013 #2

- a. Define a Euclidean domain.
- b. Define a unique factorization domain.
- c. Is a Euclidean domain an UFD? Give either a proof or a counterexample with justification.
- d. Is a UFD a Euclidean domain? Give either a proof or a counterexample with justification.

### 6.24 Spring 2021 #5

Suppose that  $f(x) \in (\mathbb{Z}/n\mathbb{Z})[x]$  is a zero divisor. Show that there is a nonzero  $a \in \mathbb{Z}/n\mathbb{Z}$  with af(x) = 0.

#### 6.25 Spring 2021 #6

a. Carefully state the definition of **Noetherian** for a commutative ring R.

b. Let R be a subset of  $\mathbb{Z}[x]$  consisting of all polynomials

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

such that  $a_k$  is even for  $1 \leq k \leq n$ . Show that R is a subring of  $\mathbb{Z}[x]$ .

c. Show that R is not Noetherian.

Hint: consider the ideal generated by 
$$\left\{2x^k \mid 1 \leq k \in \mathbb{Z}\right\}$$
.

## **7** | Fields and Galois Theory



How many monic irreducible polynomials over  $\mathbb{F}_p$  of prime degree  $\ell$  are there? Justify your answer.

$$\sim$$
 7.2  $\star$  Fall 2013 #7  $\stackrel{ extstyle }{ extstyle }\sim$ 

Let  $F = \mathbb{F}_2$  and let  $\overline{F}$  denote its algebraic closure.

- a. Show that  $\overline{F}$  is not a finite extension of F.
- b. Suppose that  $\alpha \in \overline{F}$  satisfies  $\alpha^{17} = 1$  and  $\alpha \neq 1$ . Show that  $F(\alpha)/F$  has degree 8.

$$\sim$$
 7.3 Fall 2019 #4  $\rat{}$   $\sim$ 

Let F be a finite field with q elements. Let n be a positive integer relatively prime to q and let  $\omega$  be a primitive nth root of unity in an extension field of F. Let  $E = F[\omega]$  and let k = [E : F].

- a. Prove that n divides  $q^k 1$ .
- b. Let m be the order of q in  $\mathbb{Z}/n\mathbb{Z}^{\times}$ . Prove that m divides k.
- c. Prove that m = k.

#### Revisit, tricky!

Relevant concepts omitted.

Solution omitted.

## 7.4 Fall 2019 #7 🦮

Let  $\zeta_n$  denote a primitive nth root of  $1 \in \mathbb{Q}$ . You may assume the roots of the minimal polynomial  $p_n(x)$  of  $\zeta_n$  are exactly the primitive nth roots of 1.

Show that the field extension  $\mathbb{Q}(\zeta_n)$  over  $\mathbb{Q}$  is Galois and prove its Galois group is  $(\mathbb{Z}/n\mathbb{Z})^{\times}$ .

How many subfields are there of  $\mathbb{Q}(\zeta_{20})$ ?

Relevant concepts omitted.

Solution omitted.

## 7.5 Spring 2019 #2 🦙

Let  $F = \mathbb{F}_p$  , where p is a prime number.

- a. Show that if  $\pi(x) \in F[x]$  is irreducible of degree d, then  $\pi(x)$  divides  $x^{p^d} x$ .
- b. Show that if  $\pi(x) \in F[x]$  is an irreducible polynomial that divides  $x^{p^n} x$ , then  $\deg \pi(x)$  divides n.

Relevant concepts omitted.

Solution omitted.

## 7.6 Spring 2019 #8 😽

Let  $\zeta = e^{2\pi i/8}$ .

a. What is the degree of  $\mathbb{Q}(\zeta)/\mathbb{Q}$ ?

- b. How many quadratic subfields of  $\mathbb{Q}(\zeta)$  are there?
- c. What is the degree of  $\mathbb{Q}(\zeta, \sqrt[4]{2})$  over  $\mathbb{Q}$ ?

Relevant concepts omitted.

Solution omitted.

#### 7.7 Fall 2018 #3 🦮

Let  $F \subset K \subset L$  be finite degree field extensions. For each of the following assertions, give a proof or a counterexample.

- a. If L/F is Galois, then so is K/F.
- b. If L/F is Galois, then so is L/K.
- c. If K/F and L/K are both Galois, then so is L/F.

Relevant concepts omitted.

Solution omitted.

## 7.8 Spring 2018 #2 🦙

Let  $f(x) = x^4 - 4x^2 + 2 \in \mathbb{Q}[x]$ .

- a. Find the splitting field K of f, and compute  $[K : \mathbb{Q}]$ .
- b. Find the Galois group G of f, both as an explicit group of automorphisms, and as a familiar abstract group to which it is isomorphic.
- c. Exhibit explicitly the correspondence between subgroups of G and intermediate fields between  $\mathbb{Q}$  and k.

Not the nicest proof! Would be better to replace the ad-hoc computations at the end.

Relevant concepts omitted.

Solution omitted.

## 7.9 Spring 2018 #3 🦙

Let K be a Galois extension of  $\mathbb{Q}$  with Galois group G, and let  $E_1, E_2$  be intermediate fields of K which are the splitting fields of irreducible  $f_i(x) \in \mathbb{Q}[x]$ .

Let  $E = E_1 E_2 \subset K$ .

Let  $H_i = \operatorname{Gal}(K/E_i)$  and  $H = \operatorname{Gal}(K/E)$ .

- a. Show that  $H = H_1 \cap H_2$ .
- b. Show that  $H_1H_2$  is a subgroup of G.
- c. Show that

$$Gal(K/(E_1 \cap E_2)) = H_1H_2.$$

Relevant concepts omitted.

Solution omitted.

## 7.10 Spring 2020 #4

Let  $f(x) = x^4 - 2 \in \mathbb{Q}[x]$ .

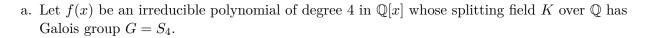
- a. Define what it means for a finite extension field E of a field F to be a Galois extension.
- b. Determine the Galois group  $\operatorname{Gal}(E/\mathbb{Q})$  for the polynomial f(x), and justify your answer carefully.
- c. Exhibit a subfield K in (b) such that  $\mathbb{Q} \leq K \leq E$  with K not a Galois extension over  $\mathbb{Q}$ . Explain.

### 7.11 Spring 2020 #3

Let E be an extension field of F and  $\alpha \in E$  be algebraic of odd degree over F.

- a. Show that  $F(\alpha) = F(\alpha^2)$ .
- b. Prove that  $\alpha^{2020}$  is algebraic of odd degree over F.

#### 7.12 Fall 2017 #4



Let  $\theta$  be a root of f(x). Prove that  $\mathbb{Q}[\theta]$  is an extension of  $\mathbb{Q}$  of degree 4 and that there are no intermediate fields between  $\mathbb{Q}$  and  $\mathbb{Q}[\theta]$ .

b. Prove that if K is a Galois extension of  $\mathbb{Q}$  of degree 4, then there is an intermediate subfield between K and  $\mathbb{Q}$ .

## 7.13 Fall 2017 #3

Let F be a field. Let f(x) be an irreducible polynomial in F[x] of degree n and let g(x) be any polynomial in F[x]. Let p(x) be an irreducible factor (of degree m) of the polynomial f(g(x)).

Prove that n divides m. Use this to prove that if r is an integer which is not a perfect square, and n is a positive integer then every irreducible factor of  $x^{2n} - r$  over  $\mathbb{Q}[x]$  has even degree.

## $\sim$ 7.14 Spring 2017 #7 $\stackrel{ extstyle }{\sim}$

Let F be a field and let  $f(x) \in F[x]$ .

- a. Define what a splitting field of f(x) over F is.
- b. Let F now be a finite field with q elements. Let E/F be a finite extension of degree n > 0. Exhibit an explicit polynomial  $g(x) \in F[x]$  such that E/F is a splitting field of g(x) over F. Fully justify your answer.
- c. Show that the extension E/F in (b) is a Galois extension.

## $\sim$ 7.15 Spring 2017 #8 $\stackrel{ extstyle extstyl$

7.12 Fall 2017 #4 31

a. Let K denote the splitting field of  $x^5-2$  over  $\mathbb Q$ . Show that the Galois group of  $K/\mathbb Q$  is isomorphic to the group of invertible matrices

$$\left(\begin{array}{cc} a & b \\ 0 & 1 \end{array}\right)$$
 where  $a \in \mathbb{F}_5^{\times}$  and  $b \in \mathbb{F}_5$ .

b. Determine all intermediate fields between K and  $\mathbb{Q}$  which are Galois over  $\mathbb{Q}$ .

## 7.16 Fall 2016 #4

Set 
$$f(x) = x^3 - 5 \in \mathbb{Q}[x]$$
.

- a. Find the splitting field K of f(x) over  $\mathbb{Q}$ .
- b. Find the Galois group G of K over  $\mathbb{Q}$ .
- c. Exhibit explicitly the correspondence between subgroups of G and intermediate fields between  $\mathbb{Q}$  and K.

## 7.17 Spring 2016 #2 $\stackrel{\triangleright}{}$

Let  $K = \mathbb{Q}[\sqrt{2} + \sqrt{5}].$ 

- a. Find  $[K:\mathbb{Q}]$ .
- b. Show that  $K/\mathbb{Q}$  is Galois, and find the Galois group G of  $K/\mathbb{Q}$ .
- c. Exhibit explicitly the correspondence between subgroups of G and intermediate fields between  $\mathbb{Q}$  and K.

## 7.18 Spring 2016 #6 $\stackrel{ extstyle o}{\sim}$

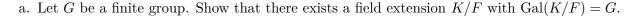
Let K be a Galois extension of a field F with [K : F] = 2015. Prove that K is an extension by radicals of the field F.

#### 7.19 Fall 2015 #5

Let  $u = \sqrt{2 + \sqrt{2}}$ ,  $v = \sqrt{2 - \sqrt{2}}$ , and  $E = \mathbb{Q}(u)$ .

- a. Find (with justification) the minimal polynomial f(x) of u over  $\mathbb{Q}$ .
- b. Show  $v \in E$ , and show that E is a splitting field of f(x) over  $\mathbb{Q}$ .
- c. Determine the Galois group of E over  $\mathbb Q$  and determine all of the intermediate fields F such that  $\mathbb Q \subset F \subset E$ .

### 7.20 Fall 2015 #6



You may assume that for any natural number n there is a field extension with Galois group  $S_n$ .

- b. Let K be a Galois extension of F with |Gal(K/F)| = 12. Prove that there exists an intermediate field E of K/F with [E:F] = 3.
- c. With K/F as in (b), does an intermediate field L necessarily exist satisfying [L:F]=2? Give a proof or counterexample.

## 7.21 Spring 2015 #2

Let  $\mathbb{F}$  be a finite field.

- a. Give (with proof) the decomposition of the additive group  $(\mathbb{F}, +)$  into a direct sum of cyclic groups.
- b. The *exponent* of a finite group is the least common multiple of the orders of its elements. Prove that a finite abelian group has an element of order equal to its exponent.
- c. Prove that the multiplicative group  $(\mathbb{F}^{\times}, \cdot)$  is cyclic.

## 7.22 Spring 2015 #5

7.19 Fall 2015 #5

Let  $f(x) = x^4 - 5 \in \mathbb{Q}[x]$ .

- a. Compute the Galois group of f over  $\mathbb{Q}$ .
- b. Compute the Galois group of f over  $\mathbb{Q}(\sqrt{5})$ .

#### 7.23 Fall 2014 #1

Let  $f \in \mathbb{Q}[x]$  be an irreducible polynomial and L a finite Galois extension of  $\mathbb{Q}$ . Let  $f(x) = g_1(x)g_2(x)\cdots g_r(x)$  be a factorization of f into irreducibles in L[x].

- a. Prove that each of the factors  $g_i(x)$  has the same degree.
- b. Give an example showing that if L is not Galois over  $\mathbb{Q}$ , the conclusion of part (a) need not hold.

#### 7.24 Fall 2014 #3

Consider the polynomial  $f(x) = x^4 - 7 \in \mathbb{Q}[x]$  and let  $E/\mathbb{Q}$  be the splitting field of f.

- a. What is the structure of the Galois group of  $E/\mathbb{Q}$ ?
- b. Give an explicit description of all of the intermediate subfields  $\mathbb{Q} \subset K \subset E$  in the form  $K = \mathbb{Q}(\alpha), \mathbb{Q}(\alpha, \beta), \cdots$  where  $\alpha, \beta$ , etc are complex numbers. Describe the corresponding subgroups of the Galois group.

## 7.25 Spring 2014 #3

Let  $F \subset C$  be a field extension with C algebraically closed.

- a. Prove that the intermediate field  $C_{\text{alg}} \subset C$  consisting of elements algebraic over F is algebraically closed.
- b. Prove that if  $F \to E$  is an algebraic extension, there exists a homomorphism  $E \to C$  that is the identity on F.

### 7.26 Spring 2014 #4

Let  $E \subset \mathbb{C}$  denote the splitting field over  $\mathbb{Q}$  of the polynomial  $x^3 - 11$ .

a. Prove that if n is a squarefree positive integer, then  $\sqrt{n} \notin E$ .

Hint: you can describe all quadratic extensions of  $\mathbb{Q}$  contained in E.

- b. Find the Galois group of  $(x^3 11)(x^2 2)$  over  $\mathbb{Q}$ .
- c. Prove that the minimal polynomial of  $11^{1/3} + 2^{1/2}$  over  $\mathbb Q$  has degree 6.

#### 7.27 Fall 2013 #5

Let L/K be a finite extension of fields.

- a. Define what it means for L/K to be separable.
- b. Show that if K is a finite field, then L/K is always separable.
- c. Give an example of a finite extension L/K that is not separable.

## 7.28 Fall 2013 #6

Let K be the splitting field of  $x^4 - 2$  over  $\mathbb{Q}$  and set  $G = \operatorname{Gal}(K/\mathbb{Q})$ .

- a. Show that  $K/\mathbb{Q}$  contains both  $\mathbb{Q}(i)$  and  $\mathbb{Q}(\sqrt[4]{2})$  and has degree 8 over  $\mathbb{Q}/$
- b. Let  $N = \operatorname{Gal}(K/\mathbb{Q}(i))$  and  $H = \operatorname{Gal}(K/\mathbb{Q}(\sqrt[4]{2}))$ . Show that N is normal in G and NH = G.

Hint: what field is fixed by NH?

c. Show that  $\operatorname{Gal}(K/\mathbb{Q})$  is generated by elements  $\sigma, \tau$ , of orders 4 and 2 respectively, with  $\tau \sigma \tau^{-1} = \sigma^{-1}$ .

Equivalently, show it is the dihedral group of order 8.

d. How many distinct quartic subfields of K are there? Justify your answer.

7.26 Spring 2014 #4 35

### 7.29 Spring 2013 #7

Let  $f(x) = g(x)h(x) \in \mathbb{Q}[x]$  and  $E, B, C/\mathbb{Q}$  be the splitting fields of f, g, h respectively.

- a. Prove that Gal(E/B) and Gal(E/C) are normal subgroups of  $Gal(E/\mathbb{Q})$ .
- b. Prove that  $Gal(E/B) \cap Gal(E/C) = \{1\}.$
- c. If  $B \cap C = \mathbb{Q}$ , show that  $Gal(E/B) Gal(E/C) = Gal(E/\mathbb{Q})$ .
- d. Under the hypothesis of (c), show that  $Gal(E/\mathbb{Q}) \cong Gal(E/B) \times Gal(E/C)$ .
- e. Use (d) to describe  $Gal(\mathbb{Q}[\alpha]/\mathbb{Q})$  where  $\alpha = \sqrt{2} + \sqrt{3}$ .

### 7.30 Spring 2013 #8

Let F be the field with 2 elements and K a splitting field of  $f(x) = x^6 + x^3 + 1$  over F. You may assume that f is irreducible over F.

- a. Show that if r is a root of f in K, then  $r^9 = 1$  but  $r^3 \neq 1$ .
- b. Find  $\operatorname{Gal}(K/F)$  and express each intermediate field between F and K as  $F(\beta)$  for an appropriate  $\beta \in K$ .

### 7.31 Fall 2012 #3

Let  $f(x) \in \mathbb{Q}[x]$  be an irreducible polynomial of degree 5. Assume that f has all but two roots in  $\mathbb{R}$ . Compute the Galois group of f(x) over  $\mathbb{Q}$  and justify your answer.

## 7.32 Fall 2012 #4

Let  $f(x) \in \mathbb{Q}[x]$  be a polynomial and K be a splitting field of f over  $\mathbb{Q}$ . Assume that  $[K : \mathbb{Q}] = 1225$  and show that f(x) is solvable by radicals.

### 7.33 Spring 2012 #1

Suppose that  $F \subset E$  are fields such that E/F is Galois and |Gal(E/F)| = 14.

- a. Show that there exists a unique intermediate field K with  $F \subset K \subset E$  such that [K : F] = 2.
- b. Assume that there are at least two distinct intermediate subfields  $F \subset L_1, L_2 \subset E$  with  $[L_i : F] = 7$ . Prove that Gal(E/F) is nonabelian.

## 

Let  $f(x) = x^7 - 3 \in \mathbb{Q}[x]$  and  $E/\mathbb{Q}$  be a splitting field of f with  $\alpha \in E$  a root of f.

- a. Show that E contains a primitive 7th root of unity.
- b. Show that  $E \neq \mathbb{Q}(\alpha)$ .

# $\sim$ 7.35 Fall 2019 Midterm #6 $\stackrel{ wo}{\sim}$

Compute the Galois group of  $f(x) = x^3 - 3x - 3 \in \mathbb{Q}[x]/\mathbb{Q}$ .

## $\sim$ 7.36 Fall 2019 Midterm #7 $\stackrel{ extstyle o}{}$

Show that a field k of characteristic  $p \neq 0$  is perfect  $\iff$  for every  $x \in k$  there exists a  $y \in k$  such that  $y^p = x$ .

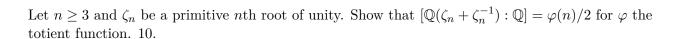
## 

Let k be a field of characteristic  $p \neq 0$  and  $f \in k[x]$  irreducible. Show that  $f(x) = g(x^{p^d})$  where  $g(x) \in k[x]$  is irreducible and separable.

Conclude that every root of f has the same multiplicity  $p^d$  in the splitting field of f over k.

7.33 Spring 2012 #1

### 7.38 Fall 2019 Midterm #9



Let L/K be a finite normal extension.

- a. Show that if L/K is cyclic and E/K is normal with L/E/K then L/E and E/K are cyclic.
- b. Show that if L/K is cyclic then there exists exactly one extension E/K of degree n with L/E/K for each divisor n of [L:K].

### 7.39 Spring 2021 #4

Define

$$f(x) := x^4 + 4x^2 + 64 \in \mathbb{Q}[x].$$

- a. Find the splitting field K of f over  $\mathbb{Q}$ .
- b. Find the Galois group G of f.
- c. Exhibit explicitly the correspondence between subgroups of G and intermediate fields between  $\mathbb{Q}$  and K.

## 7.40 Spring 2021 #7 🦙

Let p be a prime number and let F be a field of characteristic p. Show that if  $a \in F$  is not a pth power in F, then  $x^p - a \in F[x]$  is irreducible.

Hint/strategy omitted.

Relevant concepts omitted.

Solution omitted.

### 7.41 Fall 2020 #3

- a. Define what it means for a finite extension of fields E over F to be a Galois extension.
- b. Determine the Galois group of  $f(x) = x^3 7$  over  $\mathbb{Q}$ , and justify your answer carefully.
- c. Find all subfields of the splitting field of f(x) over  $\mathbb{Q}$ .

### 7.42 Fall 2020 #4

Let K be a Galois extension of F, and let  $F \subset E \subset K$  be inclusions of fields. Let  $G := \operatorname{Gal}(K/F)$  and  $H := \operatorname{Gal}(K/E)$ , and suppose H contains  $N_G(P)$ , where P is a Sylow p-subgroup of G for p a prime. Prove that  $[E:F] \equiv 1 \pmod{p}$ .

# 8 | Modules

#### 8.1 General Questions



### 8.1.1 Fall 2018 #6 💝

Let R be a commutative ring, and let M be an R-module. An R-submodule N of M is maximal if there is no R-module P with  $N \subsetneq P \subsetneq M$ .

- a. Show that an R-submodule N of M is maximal  $\iff M/N$  is a simple R-module: i.e., M/N is nonzero and has no proper, nonzero R-submodules.
- b. Let M be a  $\mathbb{Z}$ -module. Show that a  $\mathbb{Z}$ -submodule N of M is maximal  $\iff \#M/N$  is a prime number.
- c. Let M be the  $\mathbb{Z}$ -module of all roots of unity in  $\mathbb{C}$  under multiplication. Show that there is no maximal  $\mathbb{Z}$ -submodule of M.

Relevant concepts omitted.

Solution omitted.

7.41 Fall 2020 #3

### 8.1.2 Fall 2019 Final #2

Consider the  $\mathbb{Z}$ -submodule N of  $\mathbb{Z}^3$  spanned by

$$f_1 = [-1, 0, 1],$$
  

$$f_2 = [2, -3, 1],$$
  

$$f_3 = [0, 3, 1],$$
  

$$f_4 = [3, 1, 5].$$

Find a basis for N and describe  $\mathbb{Z}^3/N$ .

### 8.1.3 Spring 2018 #6

Let

$$\begin{split} M &= \{ (w, x, y, z) \in \mathbb{Z}^4 \mid w + x + y + z \in 2\mathbb{Z} \} \\ N &= \left\{ (w, x, y, z) \in \mathbb{Z}^4 \mid 4 \mid (w - x), \ 4 \mid (x - y), \ 4 \mid (y - z) \right\}. \end{split}$$

- a. Show that N is a  $\mathbb{Z}$ -submodule of M .
- b. Find vectors  $u_1, u_2, u_3, u_4 \in \mathbb{Z}^4$  and integers  $d_1, d_2, d_3, d_4$  such that

$$\{u_1,u_2,u_3,u_4\} \qquad \qquad \text{is a free basis for } M$$
 
$$\{d_1u_1,\ d_2u_2,\ d_3u_3,\ d_4u_4\} \qquad \qquad \text{is a free basis for } N$$

c. Use the previous part to describe M/N as a direct sum of cyclic  $\mathbb{Z}$ -modules.

### 8.1.4 Spring 2018 #7

Let R be a PID and M be an R-module. Let p be a prime element of R. The module M is called  $\langle p \rangle$  -primary if for every  $m \in M$  there exists k > 0 such that  $p^k m = 0$ .

- a. Suppose M is  $\langle p \rangle$ -primary. Show that if  $m \in M$  and  $t \in R$ ,  $t \notin \langle p \rangle$ , then there exists  $a \in R$  such that atm = m.
- b. A submodule S of M is said to be *pure* if  $S \cap rM = rS$  for all  $r \in R$ . Show that if M is  $\langle p \rangle$ -primary, then S is pure if and only if  $S \cap p^k M = p^k S$  for all  $k \geq 0$ .

### 8.1.5 Fall 2016 #6

Let R be a ring and  $f: M \to N$  and  $g: N \to M$  be R-module homomorphisms such that  $g \circ f = \mathrm{id}_M$ . Show that  $N \cong \mathrm{im} f \oplus \ker g$ .

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### 8.1.6 Spring 2016 #4

Let R be a ring with the following commutative diagram of R-modules, where each row represents a short exact sequence of R-modules:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

$$\downarrow^{\alpha} \qquad \downarrow^{\beta} \qquad \downarrow^{\gamma}$$

$$0 \longrightarrow A' \xrightarrow{f'} B' \xrightarrow{g'} C' \longrightarrow 0$$

Prove that if  $\alpha$  and  $\gamma$  are isomorphisms then  $\beta$  is an isomorphism.

### 8.1.7 Spring 2015 #8

Let R be a PID and M a finitely generated R-module.

a. Prove that there are R-submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that for all  $0 \le i \le n-1$ , the module  $M_{i+1}/M_i$  is cyclic.

b. Is the integer n in part (a) uniquely determined by M? Prove your answer.

### 8.1.8 Fall 2012 #6

Let R be a ring and M an R-module. Recall that M is *Noetherian* iff any strictly increasing chain of submodule  $M_1 \subsetneq M_2 \subsetneq \cdots$  is finite. Call a proper submodule  $M' \subsetneq M$  intersection-decomposable if it can not be written as the intersection of two proper submodules  $M' = M_1 \cap M_2$  with  $M_i \subsetneq M$ .

Prove that for every Noetherian module M, any proper submodule  $N \subseteq M$  can be written as a finite intersection  $N = N_1 \cap \cdots \cap N_k$  of intersection-indecomposable modules.

### 8.1.9 Fall 2019 Final #1

Let A be an abelian group, and show A is a  $\mathbb{Z}$ -module in a unique way.

8.1 General Questions 41

### 8.1.10 Fall 2020 #6

Let R be a ring with 1 and let M be a left R-module. If I is a left ideal of R, define

$$IM := \left\{ \sum_{i=1}^{N < \infty} a_i m_i \mid a_i \in I, m_i \in M, n \in \mathbb{N} \right\},\,$$

i.e. the set of finite sums of elements of the form am where  $a \in I, m \in M$ .

- a. Prove that  $IM \leq M$  is a submodule.
- b. Let M,N be left R-modules, I a nilpotent left ideal of R, and  $f:M\to N$  an R-module morphism. Prove that if the induced morphism  $\overline{f}:M/IM\to N/IN$  is surjective, then f is surjective.

#### 8.2 Torsion and the Structure Theorem



### 8.2.1 \* Fall 2019 #5 \*

Let R be a ring and M an R-module.

Recall that the set of torsion elements in M is defined by

$$\operatorname{Tor}(M) = \{ m \in M \mid \exists r \in R, \ r \neq 0, \ rm = 0 \}.$$

- a. Prove that if R is an integral domain, then Tor(M) is a submodule of M.
- b. Give an example where Tor(M) is not a submodule of M.
- c. If R has zero-divisors, prove that every non-zero R-module has non-zero torsion elements.

Relevant concepts omitted.

Solution omitted.

### 8.2.2 \* Spring 2019 #5 \*

Let R be an integral domain. Recall that if M is an R-module, the rank of M is defined to be the maximum number of R-linearly independent elements of M.

a. Prove that for any R-module M, the rank of Tor(M) is 0.

b. Prove that the rank of M is equal to the rank of  $M/\operatorname{Tor}(M)$ .

c. Suppose that M is a non-principal ideal of R.

Prove that M is torsion-free of rank 1 but not free.

Relevant concepts omitted.

:::{.solution}

Proof omitted.

Proof omitted.

Proof omitted.

Proof omitted.

Proof omitted.

#### 8.2.3 ★ Spring 2020 #6 💝

Let R be a ring with unity.

- a. Give a definition for a free module over R.
- b. Define what it means for an R-module to be torsion free.
- c. Prove that if F is a free module, then any short exact sequence of R-modules of the following form splits:

$$0 \to N \to M \to F \to 0.$$

d. Let R be a PID. Show that any finitely generated R-module M can be expressed as a direct sum of a torsion module and a free module.

You may assume that a finitely generated torsionfree module over a PID is free.

Solution omitted.

### 8.2.4 Spring 2012 #5

Let M be a finitely generated module over a PID R.

a.  $M_t$  be the set of torsion elements of M, and show that  $M_t$  is a submodule of M.

- b. Show that  $M/M_t$  is torsion free.
- c. Prove that  $M \cong M_t \oplus F$  where F is a free module.

### 8.2.5 Spring 2017 #5

Let R be an integral domain and let M be a nonzero torsion R-module.

- a. Prove that if M is finitely generated then the annihilator in R of M is nonzero.
- b. Give an example of a non-finitely generated torsion R-module whose annihilator is (0), and justify your answer.

### 8.2.6 Fall 2019 Final #3

Let R = k[x] for k a field and let M be the R-module given by

$$M = \frac{k[x]}{(x-1)^3} \oplus \frac{k[x]}{(x^2+1)^2} \oplus \frac{k[x]}{(x-1)(x^2+1)^4} \oplus \frac{k[x]}{(x+2)(x^2+1)^2}.$$

Describe the elementary divisors and invariant factors of M.

### 8.2.7 Fall 2019 Final #4

Let I = (2, x) be an ideal in  $R = \mathbb{Z}[x]$ , and show that I is not a direct sum of nontrivial cyclic R-modules.

### 8.2.8 Fall 2019 Final #5

Let R be a PID.

- a. Classify irreducible R-modules up to isomorphism.
- b. Classify indecomposable R-modules up to isomorphism.

### 8.2.9 Fall 2019 Final #6

Let V be a finite-dimensional k-vector space and  $T: V \to V$  a non-invertible k-linear map. Show that there exists a k-linear map  $S: V \to V$  with  $T \circ S = 0$  but  $S \circ T \neq 0$ .

### 8.2.10 Fall 2019 Final #7

Let  $A \in M_n(\mathbb{C})$  with  $A^2 = A$ . Show that A is similar to a diagonal matrix, and exhibit an explicit diagonal matrix similar to A.

### 8.2.11 Fall 2019 Final #10

Show that the eigenvalues of a Hermitian matrix A are real and that  $A = PDP^{-1}$  where P is an invertible matrix with orthogonal columns.

### 8.2.12 Fall 2020 #7

Let  $A \in \operatorname{Mat}(n \times n, \mathbb{R})$  be arbitrary. Make  $\mathbb{R}^n$  into an  $\mathbb{R}[x]$ -module by letting  $f(x).\mathbf{v} := f(A)(\mathbf{v})$  for  $f(\mathbf{v}) \in \mathbb{R}[x]$  and  $\mathbf{v} \in \mathbb{R}^n$ . Suppose that this induces the following direct sum decomposition:

$$\mathbb{R}^n \cong \frac{\mathbb{R}[x]}{\langle (x-1)^3 \rangle} \oplus \frac{\mathbb{R}[x]}{\langle (x^2+1)^2 \rangle} \oplus \frac{\mathbb{R}[x]}{\langle (x-1)(x^2-1)(x^2+1)^4 \rangle} \oplus \frac{\mathbb{R}[x]}{\langle (x+2)(x^2+1)^2 \rangle}.$$

- a. Determine the elementary divisors and invariant factors of A.
- b. Determine the minimal polynomial of A.
- c. Determine the characteristic polynomial of A.

# **9** Linear Algebra: Diagonalizability

# 9.1 Fall 2017 #7

Let F be a field and let V and W be vector spaces over F .

Make V and W into F[x]-modules via linear operators T on V and S on W by defining  $X \cdot v = T(v)$  for all  $v \in V$  and  $X \cdot w = S(w)$  for all  $w \in W$ .

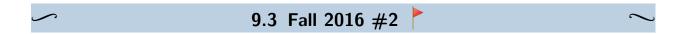
Denote the resulting F[x]-modules by  $V_T$  and  $W_S$  respectively.

- a. Show that an F[x]-module homomorphism from  $V_T$  to  $W_S$  consists of an F-linear transformation  $R: V \to W$  such that RT = SR.
- b. Show that  $VT \cong WS$  as F[x]-modules  $\iff$  there is an F-linear isomorphism  $P: V \to W$  such that  $T = P^{-1}SP$ .
- c. Recall that a module M is *simple* if  $M \neq 0$  and any proper submodule of M must be zero. Suppose that V has dimension 2. Give an example of F, T with  $V_T$  simple.
- d. Assume F is algebraically closed. Prove that if V has dimension 2, then any  $V_T$  is not simple.



Let F be a field and V a finite dimensional F-vector space, and let  $A, B : V \to V$  be commuting F-linear maps. Suppose there is a basis  $\mathcal{B}_1$  with respect to which A is diagonalizable and a basis  $\mathcal{B}_2$  with respect to which B is diagonalizable.

Prove that there is a basis  $\mathcal{B}_3$  with respect to which A and B are both diagonalizable.



Let A, B be two  $n \times n$  matrices with the property that AB = BA. Suppose that A and B are diagonalizable. Prove that A and B are simultaneously diagonalizable.



Let A be a square matrix over the complex numbers. Suppose that A is nonsingular and that  $A^{2019}$  is diagonalizable over  $\mathbb{C}$ .

Show that A is also diagonalizable over  $\mathbb{C}$ .

Relevant concepts omitted.

Solution omitted.

9.2 Spring 2015 #3

# $oldsymbol{10}$ Linear Algebra: Misc

### 10.1 ★ Spring 2012 #6

Let k be a field and let the group  $G = GL(m, k) \times GL(n, k)$  acts on the set of  $m \times n$  matrices  $M_{m,n}(k)$  as follows:

$$(A, B) \cdot X = AXB^{-1}$$

where  $(A, B) \in G$  and  $X \in M_{m,n}(k)$ .

- a. State what it means for a group to act on a set. Prove that the above definition yields a group action.
- b. Exhibit with justification a subset S of  $M_{m,n}(k)$  which contains precisely one element of each orbit under this action.

# 10.2 ★ Spring 2014 #7

Let  $G = \mathrm{GL}(3,\mathbb{Q}[x])$  be the group of invertible  $3 \times 3$  matrices over  $\mathbb{Q}[x]$ . For each  $f \in \mathbb{Q}[x]$ , let  $S_f$  be the set of  $3 \times 3$  matrices A over  $\mathbb{Q}[x]$  such that  $\det(A) = cf(x)$  for some nonzero constant  $c \in \mathbb{Q}$ .

a. Show that for  $(P,Q) \in G \times G$  and  $A \in S_f$ , the formula

$$(P,Q) \cdot A := PAQ^{-1}$$

gives a well defined map  $G \times G \times S_f \to S_f$  and show that this map gives a group action of  $G \times G$  on  $S_f$ .

b. For  $f(x) = x^3(x^2 + 1)^2$ , give one representative from each orbit of the group action in (a), and justify your assertion.

### 10.3 Fall 2012 #7

Let k be a field of characteristic zero and  $A, B \in M_n(k)$  be two square  $n \times n$  matrices over k such that AB - BA = A. Prove that det A = 0.

Moreover, when the characteristic of k is 2, find a counterexample to this statement.

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### 10.4 Fall 2012 #8

Prove that any nondegenerate matrix  $X \in M_n(\mathbb{R})$  can be written as X = UT where U is orthogonal and T is upper triangular.

### 10.5 Fall 2012 #5

Let U be an infinite-dimensional vector space over a field k,  $f:U\to U$  a linear map, and  $\{u_1,\cdots,u_m\}\subset U$  vectors such that U is generated by  $\{u_1,\cdots,u_m,f^d(u_1),\cdots,f^d(u_m)\}$  for some  $d\in\mathbb{N}$ 

Prove that U can be written as a direct sum  $U \cong V \oplus W$  such that

- 1. V has a basis consisting of some vector  $v_1, \dots, v_n, f^d(v_1), \dots, f^d(v_n)$  for some  $d \in \mathbb{N}$ , and
- $2. \ W$  is finite-dimensional.

Moreover, prove that for any other decomposition  $U \cong V' \oplus W'$ , one has  $W' \cong W$ .

### 10.6 Fall 2015 #7

- a. Show that two  $3 \times 3$  matrices over  $\mathbb{C}$  are similar  $\iff$  their characteristic polynomials are equal and their minimal polynomials are equal.
- b. Does the conclusion in (a) hold for  $4 \times 4$  matrices? Justify your answer with a proof or counterexample.

### 10.7 Fall 2014 #4

Let F be a field and T an  $n \times n$  matrix with entries in F. Let I be the ideal consisting of all polynomials  $f \in F[x]$  such that f(T) = 0.

Show that the following statements are equivalent about a polynomial  $g \in I$ :

- a. g is irreducible.
- b. If  $k \in F[x]$  is nonzero and of degree strictly less than g, then k[T] is an invertible matrix.

10.4 Fall 2012 #8

### 10.8 Fall 2015 #8

Let V be a vector space over a field F and V its dual. A symmetric bilinear form (-,-) on V is a map  $V \times V \to F$  satisfying

$$(av_1 + bv_2, w) = a(v_1, w) + b(v_2, w)$$
 and  $(v_1, v_2) = (v_2, v_1)$ 

for all  $a, b \in F$  and  $v_1, v_2 \in V$ . The form is nondegenerate if the only element  $w \in V$  satisfying (v, w) = 0 for all  $v \in V$  is w = 0.

Suppose (-,-) is a nondegenerate symmetric bilinear form on V. If W is a subspace of V, define

$$W^{\perp} := \left\{ v \in V \mid (v, w) = 0 \text{ for all } w \in W \right\}.$$

- a. Show that if X, Y are subspaces of V with  $Y \subset X$ , then  $X^{\perp} \subseteq Y^{\perp}$ .
- b. Define an injective linear map

$$\psi: Y^{\perp}/X^{\perp} \hookrightarrow (X/Y)^{\tilde{}}$$

which is an isomorphism if V is finite dimensional.

## 10.9 Fall 2018 #4 🐪

Let V be a finite dimensional vector space over a field (the field is not necessarily algebraically closed).

Let  $\varphi:V\to V$  be a linear transformation. Prove that there exists a decomposition of V as  $V=U\oplus W$ , where U and W are  $\varphi$ -invariant subspaces of V,  $\varphi|_U$  is nilpotent, and  $\varphi|_W$  is nonsingular.

Revisit

Solution omitted.

## 10.10 Fall 2018 #5 🦙

Let A be an  $n \times n$  matrix.

a. Suppose that v is a column vector such that the set  $\{v, Av, ..., A^{n-1}v\}$  is linearly independent. Show that any matrix B that commutes with A is a polynomial in A.

10.8 Fall 2015 #8

b. Show that there exists a column vector v such that the set  $\{v, Av, ..., A^{n-1}v\}$  is linearly independent  $\iff$  the characteristic polynomial of A equals the minimal polynomial of A.

Relevant concepts omitted.

Hint/strategy omitted.

Solution omitted.

### 10.11 Fall 2019 #8

Let  $\{e_1, \dots, e_n\}$  be a basis of a real vector space V and let

$$\Lambda := \left\{ \sum r_i e_i \mid r_i \in \mathbb{Z} \right\}$$

Let  $\cdot$  be a non-degenerate  $(v \cdot w = 0 \text{ for all } w \in V \iff v = 0)$  symmetric bilinear form on V such that the Gram matrix  $M = (e_i \cdot e_j)$  has integer entries.

Define the dual of  $\Lambda$  to be

$$\check{\Lambda} \coloneqq \{ v \in V \mid v \cdot x \in \mathbb{Z} \text{ for all } x \in \Lambda \}.$$

- a. Show that  $\Lambda \subset \mathring{\Lambda}$ .
- b. Prove that  $\det M \neq 0$  and that the rows of  $M^{-1}$  span  $\Lambda$ .
- c. Prove that  $\det M = |\Lambda/\Lambda|$ .

Todo, missing part (c).

Solution omitted.

Solution omitted.

### 10.12 Spring 2013 #6

Let V be a finite dimensional vector space over a field F and let  $T: V \to V$  be a linear operator with characteristic polynomial  $f(x) \in F[x]$ .

a. Show that f(x) is irreducible in  $F[x] \iff$  there are no proper nonzero subspaces W < V with  $T(W) \subseteq W$ .

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b. If f(x) is irreducible in F[x] and the characteristic of F is 0, show that T is diagonalizable when we extend the field to its algebraic closure.

Is there a proof without matrices? What if V is infinite dimensional?

How to extend basis?

Relevant concepts omitted.

Solution omitted.

## 10.13 Fall 2020 #8

Let  $A \in \operatorname{Mat}(n \times n, \mathbb{C})$  such that the group generated by A under multiplication is finite. Show that

$$\operatorname{Tr}(A^{-1}) = \overline{\operatorname{Tr}(A)},$$

where  $\overline{(-)}$  denotes taking the complex conjugate and Tr(-) is the trace.

# 11 | Linear Algebra: Canonical Forms

# $\sim$ 11.1 $_{\star}$ Spring 2012 #8 $\stackrel{ extstyle }{ extstyle }$

Let V be a finite-dimensional vector space over a field k and  $T: V \to V$  a linear transformation.

- a. Provide a definition for the minimal polynomial in k[x] for T.
- b. Define the *characteristic polynomial* for T.
- c. Prove the Cayley-Hamilton theorem: the linear transformation T satisfies its characteristic polynomial.



Let  $T:V\to V$  be a linear transformation where V is a finite-dimensional vector space over  $\mathbb{C}$ . Prove the Cayley-Hamilton theorem: if p(x) is the characteristic polynomial of T, then p(T)=0.

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You may use canonical forms.

## 11.3 \* Spring 2012 #7

Consider the following matrix as a linear transformation from  $V \coloneqq \mathbb{C}^5$  to itself:

$$A = \left(\begin{array}{ccccc} -1 & 1 & 0 & 0 & 0 \\ -4 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right).$$

- a. Find the invariant factors of A.
- b. Express V in terms of a direct sum of indecomposable  $\mathbb{C}[x]$ -modules.
- c. Find the Jordan canonical form of A.

### 11.4 Fall 2019 Final #8

Exhibit the rational canonical form for

- A ∈ M<sub>6</sub>(Q) with minimal polynomial (x 1)(x² + 1)².
  A ∈ M<sub>10</sub>(Q) with minimal polynomial (x² + 1)²(x³ + 1).

## 11.5 Fall 2019 Final #9

Exhibit the rational and Jordan canonical forms for the following matrix  $A \in M_4(\mathbb{C})$ :

$$A = \left(\begin{array}{cccc} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{array}\right).$$

### 11.6 Spring 2016 #7

Let  $D = \mathbb{Q}[x]$  and let M be a  $\mathbb{Q}[x]$ -module such that

$$M \cong \frac{\mathbb{Q}[x]}{(x-1)^3} \oplus \frac{\mathbb{Q}[x]}{(x^2+1)^3} \oplus \frac{\mathbb{Q}[x]}{(x-1)(x^2+1)^5} \oplus \frac{\mathbb{Q}[x]}{(x+2)(x^2+1)^2}.$$

Determine the elementary divisors and invariant factors of M.

### 11.7 Spring 2020 #7

Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 6 & 1 \\ -16 & -16 & -2 \end{bmatrix} \in M_3(\mathbf{C}).$$

- a. Find the Jordan canonical form J of A.
- b. Find an invertible matrix P such that  $P^{-1}AP = J$ .

You should not need to compute  $P^{-1}$ .

c. Write down the minimal polynomial of A.

## 11.8 Spring 2019 #7 🦙

Let p be a prime number. Let A be a  $p \times p$  matrix over a field F with 1 in all entries except 0 on the main diagonal.

Determine the Jordan canonical form (JCF) of A

- a. When  $F = \mathbb{Q}$ ,
- b. When  $F = \mathbb{F}_p$ .

Hint: In both cases, all eigenvalues lie in the ground field. In each case find a matrix P such that  $P^{-1}AP$  is in JCF.

Hint/strategy omitted.

Relevant concepts omitted.

Solution omitted.

### 11.9 Spring 2018 #4

Let

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 1 & -3 \\ 1 & 2 & -4 \end{bmatrix} \in M_3(\mathbb{C})$$

- a. Find the Jordan canonical form J of A.
- b. Find an invertible matrix P such that  $P^{-1}AP = J$ .

You should not need to compute  $P^{-1}$ .

## 11.10 Spring 2017 #6

Let A be an  $n \times n$  matrix with all entries equal to 0 except for the n-1 entries just above the diagonal being equal to 2.

- a. What is the Jordan canonical form of A, viewed as a matrix in  $M_n(\mathbb{C})$ ?
- b. Find a nonzero matrix  $P \in M_n(\mathbb{C})$  such that  $P^{-1}AP$  is in Jordan canonical form.

### 11.11 Spring 2016 #1

Let

$$A = \begin{pmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{pmatrix} \in M_3(\mathbf{C}).$$

- a. Find the Jordan canonical form J of A.
- b. Find an invertible matrix P such that  $P^{-1}AP = J$ . You do not need to compute  $P^{-1}$ .

### 11.12 Spring 2015 #6



$$A = \left[ \begin{array}{ccc} 1 & \dots & 1 \\ & \ddots & \\ 1 & \dots & 1 \end{array} \right] \in M_n(F).$$

Show that A has a Jordan normal form over F and find it.

Hint: treat the cases  $n \cdot 1 \neq 0$  in F and  $n \cdot 1 = 0$  in F separately.

### 11.13 Fall 2014 #5

Let T be a  $5 \times 5$  complex matrix with characteristic polynomial  $\chi(x) = (x-3)^5$  and minimal polynomial  $m(x) = (x-3)^2$ . Determine all possible Jordan forms of T.

### 11.14 Spring 2013 #5

Let  $T: V \to V$  be a linear map from a 5-dimensional  $\mathbb{C}$ -vector space to itself and suppose f(T) = 0 where  $f(x) = x^2 + 2x + 1$ .

- a. Show that there does not exist any vector  $v \in V$  such that Tv = v, but there does exist a vector  $w \in V$  such that  $T^2w = w$ .
- b. Give all of the possible Jordan canonical forms of T.

## 11.15 Spring 2021 #1

Let m

$$A \coloneqq \begin{bmatrix} r & 1 & -1 \\ -6 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \in \operatorname{Mat}(3 \times 3, \mathbb{C}).$$

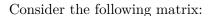
a. Find the Jordan canonical form J of A.

b. Find an invertible matrix P such that  $J = P^{-1}AP$ .

You should not need to compute  $P^{-1}$ 

c. Write down the minimal polynomial of A.

## 11.16 Fall 2020 #5



$$B \coloneqq \begin{bmatrix} 1 & 3 & 3 \\ 2 & 2 & 3 \\ -1 & -2 & -2 \end{bmatrix}.$$

- a. Find the minimal polynomial of B.
- b. Find a  $3 \times 3$  matrix J in Jordan canonical form such that  $B = JPJ^{-1}$  where P is an invertible matrix.