

# Real Analysis Qualifying Exam Questions

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## 1 Undergraduate Analysis: Uniform Convergence

### 1.1 Fall 2018 # 1

Let  $f(x) = \frac{1}{x}$ . Show that  $f$  is uniformly continuous on  $(1, \infty)$  but not on  $(0, \infty)$ .

*Solution.*

Concepts used:

- Uniform continuity.

**Solution:**

Show a stronger statement:  $f(x) = \frac{1}{x}$  is uniformly continuous on any interval of the form  $(c, \infty)$  where  $c > 0$ .

- Note that

$$|x|, |y| > c > 0 \implies |xy| = |x||y| > c^2 \implies \frac{1}{|xy|} < \frac{1}{c^2}.$$

- Letting  $\varepsilon$  be arbitrary, choose  $\delta < \varepsilon c^2$ .
- Note that  $\delta$  does not depend on  $x, y$ .

- Then

$$\begin{aligned}
 |f(x) - f(y)| &= \left| \frac{1}{x} - \frac{1}{y} \right| \\
 &= \frac{|x - y|}{xy} \\
 &\leq \frac{\delta}{xy} \\
 &< \frac{\delta}{c^2} \\
 &< \varepsilon,
 \end{aligned}$$

which shows uniform continuity.

To see that  $f$  is not uniformly continuous when  $c = 0$ :

Note: negating uniform continuity says  $\exists \varepsilon > 0$  such that  $\forall \delta(\varepsilon)$  there exist  $x, y$  such that  $|x - y| < \delta$  and  $|f(x) - f(y)| > \varepsilon$ .

- Let  $\varepsilon < 1$ .
- Let  $x_n = \frac{1}{n}$  for  $n \geq 1$ .
- Choose  $n$  large enough such that  $|x_n - x_{n+1}| = \frac{1}{n} - \frac{1}{n+1} < \delta$ .
  - Why this can be done: by the archimedean property of  $\mathbb{R}$ , choose  $n$  such that  $\frac{1}{n} < \varepsilon$ .
  - Then

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \leq \frac{1}{n} < \varepsilon \quad \text{since } n+1 > 1.$$

- Note  $f(x_n) = n$  and thus

$$|f(x_n) - f(x_{n+1})| = n - (n+1) = 1 > \varepsilon.$$

## 1.2 Fall 2017 # 1

Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which  $f$  does and does not converge uniformly.

Review and consolidate.

*Solution.*

Concepts used:

- ??

**Solution:**

Note that  $f(x) = e^x$  is entire and thus equal to its power series. So  $f(x) = \sum_{j=0}^{\infty} \frac{1}{j!} x^j$ .

Letting  $f_N(x) = \sum_{j=1}^N \frac{1}{j!} x^j$ , we have  $f_N(x) \rightarrow f(x)$  pointwise on  $(-\infty, \infty)$ .

For any compact interval  $[-M, M]$ , we have

$$\begin{aligned} \|f_N(x) - f(x)\|_\infty &= \sup_{-M \leq x \leq M} \left| \sum_{j=N+1}^{\infty} \frac{1}{j!} x^j \right| \\ &\leq \sup_{-M \leq x \leq M} \sum_{j=N+1}^{\infty} \frac{1}{j!} |x|^j \\ &\leq \sum_{j=N+1}^{\infty} \frac{1}{j!} M^j \\ &\leq \sum_{j=0}^{\infty} \frac{1}{j!} M^j \\ &= e^M \\ &< \infty, \end{aligned}$$

so  $f_N \rightarrow f$  uniformly on  $[-M, M]$  by the M-test. Thus it converges on any bounded interval. It does not converge on  $\mathbb{R}$ , since  $x^N$  is unbounded.

### 1.3 Fall 2014 # 1

Let  $\{f_n\}$  be a sequence of continuous functions such that  $\sum f_n$  converges uniformly.

Prove that  $\sum f_n$  is also continuous.

### 1.4 Spring 2017 # 4

Let  $f(x, y)$  on  $[-1, 1]^2$  be defined by

$$f(x, y) = \begin{cases} \frac{xy}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Determine if  $f$  is integrable.

Redo, may just be wrong.

*Solution.*

Concepts used:

- ??

**Solution:**

Switching to polar coordinates and integrating over a half-circle contained in  $I^2$ , we have

$$\int_{I^2} f \geq \int_0^\pi \int_0^1 \frac{\cos(\theta) \sin(\theta)}{r^2} dr d\theta = \infty,$$

so  $f$  is not integrable.

## 1.5 Spring 2015 # 1

Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces,  $f : X \rightarrow Y$ , and  $x_0 \in X$ .

Prove that the following statements are equivalent:

1. For every  $\varepsilon > 0$   $\exists \delta > 0$  such that  $\rho(f(x), f(x_0)) < \varepsilon$  whenever  $d(x, x_0) < \delta$ .
2. The sequence  $\{f(x_n)\}_{n=1}^{\infty} \rightarrow f(x_0)$  for every sequence  $\{x_n\} \rightarrow x_0$  in  $X$ .

## 1.6 Fall 2014 # 2

Let  $I$  be an index set and  $\alpha : I \rightarrow (0, \infty)$ .

1. Show that

$$\sum_{i \in I} a(i) := \sup_{\substack{J \subset I \\ J \text{ finite}}} \sum_{i \in J} a(i) < \infty \implies I \text{ is countable.}$$

2. Suppose  $I = \mathbb{Q}$  and  $\sum_{q \in \mathbb{Q}} a(q) < \infty$ . Define

$$f(x) := \sum_{\substack{q \in \mathbb{Q} \\ q \leq x}} a(q).$$

Show that  $f$  is continuous at  $x \iff x \notin \mathbb{Q}$ .

## 1.7 Spring 2014 # 2

Let  $\{a_n\}$  be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that  $\sum a_n^2 < \infty$ .

Note: Assume  $a_n, b_n$  are all non-negative.

## 2 General Analysis

### 2.1 Spring 2020 # 1

Prove that if  $f : [0, 1] \rightarrow \mathbb{R}$  is continuous then

$$\lim_{k \rightarrow \infty} \int_0^1 kx^{k-1} f(x) dx = f(1).$$

*Solution.*

Concepts used:

- DCT
- Weierstrass Approximation Theorem

**Solution:**

- Suppose  $p$  is a polynomial, then

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_0^1 kx^{k-1}p(x) dx &= \lim_{k \rightarrow \infty} \int_0^1 \left( \frac{\partial}{\partial x} x^k \right) p(x) dx \\ &= \lim_{k \rightarrow \infty} \left[ x^k p(x) \Big|_0^1 - \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx \right] \quad \text{integrating by parts} \\ &= p(1) - \lim_{k \rightarrow \infty} \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx, \end{aligned}$$

- Thus it suffices to show that

$$\lim_{k \rightarrow \infty} \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx = 0.$$

- Integrating by parts a second time yields

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx &= \lim_{k \rightarrow \infty} \left[ \frac{x^{k+1}}{k+1} p'(x) \Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1} \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx \right] \\ &= - \lim_{k \rightarrow \infty} \int_0^1 \frac{x^{k+1}}{k+1} \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx \\ &= - \int_0^1 \lim_{k \rightarrow \infty} \frac{x^{k+1}}{k+1} \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx \quad \text{by DCT} \\ &= - \int_0^1 0 \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx \\ &= 0. \end{aligned}$$

- The DCT can be applied here because  $f''$  is continuous and  $[0, 1]$  is compact, so  $f''$  is bounded on  $[0, 1]$  by a constant  $M$  and

$$\int_0^1 |x^k f''(x)| \leq \int_0^1 1 \cdot M = M < \infty.$$

- Now use the Weierstrass approximation theorem:
  - If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then for every  $\varepsilon > 0$  there exists a polynomial  $p_\varepsilon(x)$  such that  $\|f - p_\varepsilon\|_\infty < \varepsilon$ .
- Thus

$$\begin{aligned} \left| \int_0^1 kx^{k-1} p_\varepsilon(x) dx - \int_0^1 kx^{k-1} f(x) dx \right| &= \left| \int_0^1 kx^{k-1} (p_\varepsilon(x) - f(x)) dx \right| \\ &\leq \left| \int_0^1 kx^{k-1} \|p_\varepsilon - f\|_\infty dx \right| \\ &= \|p_\varepsilon - f\|_\infty \cdot \left| \int_0^1 kx^{k-1} dx \right| \\ &= \|p_\varepsilon - f\|_\infty \cdot x^k \Big|_0^1 \\ &= \|p_\varepsilon - f\|_\infty \xrightarrow{\varepsilon \rightarrow 0} 0 \end{aligned}$$



and the integrals are equal.

- By the first argument,

$$\int_0^1 kx^{k-1}p_\varepsilon(x) dx = p_\varepsilon(1) \text{ for each } \varepsilon$$

- Since uniform convergence implies pointwise convergence,  $p_\varepsilon(1) \xrightarrow{\varepsilon \rightarrow 0} f(1)$ .

## 2.2 Fall 2019 # 1.

Let  $\{a_n\}_{n=1}^\infty$  be a sequence of real numbers.

### 2.2.1 a

Prove that if  $\lim_{n \rightarrow \infty} a_n = 0$ , then

$$\lim_{n \rightarrow \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

### 2.2.2 b

Prove that if  $\sum_{n=1}^\infty \frac{a_n}{n}$  converges, then

$$\lim_{n \rightarrow \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

*Solution.*

Concepts used:

- Cesaro mean/summation.
- Break series apart into pieces that can be handled separately.

### 2.2.3 a

Prove a stronger result:

$$a_k \rightarrow S \implies S_N := \frac{1}{N} \sum_{k=1}^N a_k \rightarrow S.$$

Idea: once  $N$  is large enough,  $a_k \approx S$ , and all smaller terms will die off as  $N \rightarrow \infty$ .  
See this MSE answer.

- Use convergence  $a_k \rightarrow S$ : choose  $M$  large enough such that

$$k \geq M + 1 \implies |a_k - S| < \varepsilon.$$

Then

$$\begin{aligned}
 \left| \left( \frac{1}{N} \sum_{k=1}^N a_k \right) - S \right| &= \frac{1}{N} \left| \left( \sum_{k=1}^N a_k \right) - NS \right| \\
 &= \frac{1}{N} \left| \left( \sum_{k=1}^N a_k \right) - \sum_{k=1}^N S \right| \\
 &= \frac{1}{N} \left| \sum_{k=1}^N (a_k - S) \right| \\
 &\leq \frac{1}{N} \sum_{k=1}^N |a_k - S| \\
 &= \frac{1}{N} \sum_{k=1}^M |a_k - S| + \sum_{k=M+1}^N |a_k - S| \\
 &\leq \frac{1}{N} \sum_{k=1}^M |a_k - S| + \sum_{k=M+1}^N \frac{\varepsilon}{2} \\
 &= \frac{1}{N} \sum_{k=1}^M |a_k - S| + (N - M) \frac{\varepsilon}{2} \\
 &\xrightarrow{\varepsilon \rightarrow 0} \frac{1}{N} \sum_{k=1}^M |a_k - S| + 0 \\
 &\xrightarrow{N \rightarrow \infty} 0 + 0.
 \end{aligned}$$

Note:  $M$  is fixed, so the last sum is some constant  $c$ , and  $c/N \rightarrow 0$  as  $N \rightarrow \infty$  for any constant. To be more careful, choose  $M$  first to get  $\varepsilon/2$  for the tail, then choose  $N(M) > M$  for the remaining truncated part of the sum.

### 2.2.4 b

- Define

$$\Gamma_n := \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

- $\Gamma_1 = \sum_{k=1}^{\infty} \frac{a_k}{k}$  is the original series and each  $\Gamma_n$  is a tail of  $\Gamma_1$ , so by assumption  $\Gamma_n \xrightarrow{n \rightarrow \infty} 0$ .
- Compute

$$\frac{1}{n} \sum_{k=1}^n a_k = \frac{1}{n} (\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \Gamma_{n+1})$$

- This comes from consider the following summation:

$\Gamma_1 :$	$a_1$	$+\frac{a_2}{2}$	$+\frac{a_3}{3}$	$+\dots$	
$\Gamma_2 :$		$\frac{a_2}{2}$	$+\frac{a_3}{3}$	$+\dots$	
$\Gamma_3 :$			$\frac{a_3}{3}$	$+\dots$	
<hr/>					
$\sum_{i=1}^n \Gamma_i :$	$a_1$	$+a_2$	$+a_3$	$+\dots$	$a_n + \frac{a_{n+1}}{n+1} + \dots$

- Use part (a): since  $\Gamma_n \xrightarrow{n \rightarrow \infty} 0$ , we have  $\frac{1}{n} \sum_{k=1}^n \Gamma_k \xrightarrow{n \rightarrow \infty} 0$ .
- Also a minor check:  $\Gamma_n \rightarrow 0 \implies \frac{1}{n} \Gamma_n \rightarrow 0$ .
- Then

$$\begin{aligned}
 \frac{1}{n} \sum_{k=1}^n a_k &= \frac{1}{n} (\Gamma_1 + \Gamma_2 + \dots + \Gamma_n - \Gamma_{n+1}) \\
 &= \left( \frac{1}{n} \sum_{k=0}^n \Gamma_k \right) - \left( \frac{1}{n} \Gamma_{n+1} \right) \\
 &\xrightarrow{n \rightarrow \infty} 0.
 \end{aligned}$$

### 2.3 Fall 2018 # 4

Let  $f \in L^1([0, 1])$ . Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) |\sin nx| \, dx = \frac{2}{\pi} \int_0^1 f(x) \, dx$$

> Hint: Begin with the case that  $f$  is the characteristic function of an interval.

*Solution.*

Concepts used:

- ?

**Solution:**

Case of characteristic function

- First suppose  $f(x) = \chi_{[0,1]}(x)$ .
- Note that  $\sin(nx)$  has a period of  $2\pi/n$ , and thus  $\left\lfloor \frac{n}{2\pi} \right\rfloor$  full periods in  $[0, 1]$ .

- Taking the absolute value yields a new function with half the period, so a period of  $\pi/n$  and  $\lfloor \pi/n \rfloor$  full periods in  $[0, 1]$ .
- We can compute the integral over one full period (which is independent of *which* period is chosen), and since  $\sin(x)$  is positive and agrees with  $|\sin(nx)|$  on the first period, we have

$$\begin{aligned} \int_{\text{One Period}} |\sin(nx)| dx &= \int_0^{\pi/n} \sin(nx) dx \\ &= \frac{1}{n} \int_0^\pi \sin(u) du \quad u = nx \\ &= \frac{1}{n} - \cos(u) \Big|_0^\pi \\ &= \frac{2}{n}. \end{aligned}$$

- Then break the integral up into integrals over periods  $P_1, P_2, \dots, P_N$  where  $N := \lfloor n/\pi \rfloor$ :

$$\begin{aligned} \int_0^1 |\sin(nx)| dx &= \left( \sum_{j=1}^N \int_{P_j} |\sin(nx)| dx \right) + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)| dx \\ &= \left( \sum_{j=1}^N \frac{2}{n} \right) + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)| dx \\ &= N \left( \frac{2}{n} \right) + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)| dx \\ &:= \left\lfloor \frac{n}{\pi} \right\rfloor \frac{2}{n} + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)| dx \\ &= \frac{2}{\pi} + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)| dx \\ &:= \frac{2}{\pi} + R(n) \end{aligned}$$

so it suffices to show that  $R(n) \xrightarrow{n \rightarrow \infty} 0$ .

Need to justify removing floor function and cancellation.

- Showing this: ????????????

No clue how to show this.

General case

Not sure. Approximate  $f$  by simple functions...?

## 2.4 Fall 2017 # 4

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

1. Show that  $f_n \rightarrow 0$  pointwise but not uniformly on  $[0, 1]$ .

Hint: Consider the maximum of  $f_n$ .

2.

$$\lim_{n \rightarrow \infty} \int_0^1 n(1-x)^n \sin x \, dx = 0$$

*Solution.*

Concepts used:

- ?

**2.4.1 a**

Let  $G(x) = \sum_{n=1}^{\infty} nx(1-x)^n$ . Applying the ratio test, we have

$$\left| \frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n} \right| = \frac{n+1}{n} |1-x| \xrightarrow{n \rightarrow \infty} |1-x| < 1 \iff 0 \leq x \leq 2,$$

and in particular, this series converges on  $[0, 2]$ . Thus its terms go to zero, and  $nx(1-x)^n \rightarrow 0$  on  $[0, 1] \subset [0, 2]$ .

To see that the convergence is not uniform, let  $x_n = \frac{1}{n}$  and  $\varepsilon > \frac{1}{e}$ , then

$$\sup_{x \in [0,1]} |nx(1-x)^n - 0| \geq |nx_n(1-x_n)^n| = \left| \left(1 - \frac{1}{n}\right)^n \right| \xrightarrow{n \rightarrow \infty} e^{-1} > \varepsilon.$$

**2.4.2 b**

Note: could use the first part with  $\sin(x) \leq x$ , but then integral ends up more complicated.

Noting that  $\sin(x) \leq 1$ , we have We have

$$\begin{aligned} \left| \int_0^1 n(1-x)^n \sin(x) \, dx \right| &\leq \int_0^1 |n(1-x)^n \sin(x)| \, dx \\ &\leq \int_0^1 |n(1-x)^n| \, dx \\ &= n \int_0^1 (1-x)^n \, dx \\ &= -\frac{n(1-x)^{n+1}}{n+1} \Big|_0^1 \\ &\xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

**2.5 Spring 2017 # 3**

Let

$$f_n(x) = ae^{-nax} - be^{-nbx} \quad \text{where } 0 < a < b.$$

Show that

a.  $\sum_{n=1}^{\infty} |f_n|$  is not in  $L^1([0, \infty), m)$

Hint:  $f_n(x)$  has a root  $x_n$ .

b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0, \infty), m) \quad \text{and} \quad \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x) dm = \ln \frac{b}{a}$$

Not complete.

:::{.solution} Concepts used:

- ?

### 2.5.1 a

Letting  $x_n := \frac{1}{n}$ , we have

$$\sum_{k=1}^{\infty} |f_k(x)| \geq |f_n(x_n)| = \left| ae^{-ax} - be^{-bx} \right| := M.$$

In particular,  $\sup_x |f_n(x)| \not\rightarrow 0$ , so the terms do not go to zero and the sum can not converge.

### 2.5.2 b

?

:::

## 2.6 Fall 2016 # 1

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that  $f$  converges to a differentiable function on  $(1, \infty)$  and that

$$f'(x) = \sum_{n=1}^{\infty} \left( \frac{1}{n^x} \right)'.$$

Hint:

$$\left( \frac{1}{n^x} \right)' = -\frac{1}{n^x} \ln n$$

*Solution.*

Concepts used:

• ?

**Solution:**

- Set  $f_N(x) := \sum_{n=1}^N n^{-x}$ , so  $f(x) = \lim_{N \rightarrow \infty} f_N(x)$ .
- If an interchange of limits is justified, we have

$$\begin{aligned}
 \frac{\partial}{\partial x} \lim_{N \rightarrow \infty} \sum_{n=1}^N n^{-x} &= \lim_{h \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{h} \left[ \left( \sum_{n=1}^N n^{-x} \right) - \left( \sum_{n=1}^N n^{-(x+h)} \right) \right] \\
 &\stackrel{?}{=} \lim_{N \rightarrow \infty} \lim_{h \rightarrow 0} \frac{1}{h} \left[ \left( \sum_{n=1}^N n^{-x} \right) - \left( \sum_{n=1}^N n^{-(x+h)} \right) \right] \\
 &= \lim_{N \rightarrow \infty} \lim_{h \rightarrow 0} \frac{1}{h} \left[ \sum_{n=1}^N n^{-x} - n^{-(x+h)} \right] \quad (1) \\
 &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \lim_{h \rightarrow 0} \frac{1}{h} [n^{-x} - n^{-(x+h)}] \quad \text{since this is a finite sum} \\
 &:= \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{\partial}{\partial x} \left( \frac{1}{n^x} \right) \\
 &= \lim_{N \rightarrow \infty} \sum_{n=1}^N -\frac{\ln(n)}{n^x},
 \end{aligned}$$

where the combining of sums in (1) is valid because  $\sum n^{-x}$  is absolutely convergent for  $x > 1$  by the  $p$ -test.

- Thus it suffices to justify the interchange of limits and show that the last sum converges on  $(1, \infty)$ .
- Claim:  $\sum n^{-x} \ln(n)$  converges.
  - Use the fact that for any fixed  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \frac{\ln(n)}{n^\varepsilon} \stackrel{L.H.}{=} \lim_{n \rightarrow \infty} \frac{1/n}{\varepsilon n^{\varepsilon-1}} = \lim_{n \rightarrow \infty} \frac{1}{\varepsilon n^\varepsilon} = 0,$$

- This implies that for a fixed  $\varepsilon > 0$  and for any constant  $c > 0$  there exists an  $N$  large enough such that  $n \geq N$  implies  $\ln(n)/n^\varepsilon < c$ , i.e.  $\ln(n) < cn^\varepsilon$ .
- Taking  $c = 1$ , we have  $n \geq N \implies \ln(n) < n^\varepsilon$
- We thus break up the sum:

$$\begin{aligned}
 \sum_{n \in \mathbb{N}} \frac{\ln(n)}{n^x} &= \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{\ln(n)}{n^x} \\
 &\leq \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{n^\varepsilon}{n^x} \\
 &:= C_\varepsilon + \sum_{n=N}^{\infty} \frac{n^\varepsilon}{n^x} \quad \text{with } C_\varepsilon < \infty \text{ a constant} \\
 &= C_\varepsilon + \sum_{n=N}^{\infty} \frac{1}{n^{x-\varepsilon}},
 \end{aligned}$$

where the last term converges by the  $p$ -test if  $x - \varepsilon > 1$ .

- But  $\varepsilon$  can depend on  $x$ , and if  $x \in (1, \infty)$  is fixed we can choose  $\varepsilon < |x - 1|$  to ensure this.
- Claim: the interchange of limits is justified.

?

**2.7 Fall 2016 # 5**

Let  $\varphi \in L^\infty(\mathbb{R})$ . Show that the following limit exists and satisfies the equality

$$\lim_{n \rightarrow \infty} \left( \int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} dx \right)^{\frac{1}{n}} = \|\varphi\|_\infty.$$

Review and clean up.

:::{.solution} Concepts used:

- ??

**Solution:**

Let  $L$  be the LHS and  $R$  be the RHS.

Claim:  $L \leq R$ . - Since  $|\varphi| \leq \|\varphi\|_\infty$  a.e., we can write

$$\begin{aligned} L^{\frac{1}{n}} &:= \int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} \\ &\leq \int_{\mathbb{R}} \frac{\|\varphi\|_\infty^n}{1+x^2} \\ &= \|\varphi\|_\infty^n \int_{\mathbb{R}} \frac{1}{1+x^2} \\ &= \|\varphi\|_\infty^n \arctan(x) \Big|_{-\infty}^{\infty} \\ &= \|\varphi\|_\infty^n \left( \frac{\pi}{2} - \frac{-\pi}{2} \right) \\ &= \pi \|\varphi\|_\infty^n \\ \implies L^{\frac{1}{n}} &\leq \sqrt[n]{\pi \|\varphi\|_\infty^n} \\ \implies L &\leq \pi^{\frac{1}{n}} \|\varphi\|_\infty \\ &\xrightarrow{n \rightarrow \infty} \|\varphi\|_\infty, \end{aligned}$$

where we've used the fact that  $c^{\frac{1}{n}} \xrightarrow{n \rightarrow \infty} 1$  for any constant  $c$ .

Actually true? Need conditions?

Claim:  $R \leq L$ .

- We will show that  $R \leq L + \varepsilon$  for every  $\varepsilon > 0$ .



- Set

$$S_\varepsilon := \left\{ x \in \mathbb{R}^n \mid |\varphi(x)| \geq \|\varphi\|_\infty - \varepsilon \right\}.$$

- Then we have

$$\begin{aligned} \int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} dx &\geq \int_{S_\varepsilon} \frac{|\varphi(x)|^n}{1+x^2} dx \quad S_\varepsilon \subset \mathbb{R} \\ &\geq \int_{S_\varepsilon} \frac{(\|\varphi\|_\infty - \varepsilon)^n}{1+x^2} dx \quad \text{by definition of } S_\varepsilon \\ &= (\|\varphi\|_\infty - \varepsilon)^n \int_{S_\varepsilon} \frac{1}{1+x^2} dx \\ &= (\|\varphi\|_\infty - \varepsilon)^n C_\varepsilon \quad \text{where } C_\varepsilon \text{ is some constant} \end{aligned}$$

$$\begin{aligned} \Rightarrow \left( \int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} dx \right)^{\frac{1}{n}} &\geq (\|\varphi\|_\infty - \varepsilon) C_\varepsilon^{\frac{1}{n}} \\ &\xrightarrow{n \rightarrow \infty} (\|\varphi\|_\infty - \varepsilon) \cdot 1 \\ &\xrightarrow{\varepsilon \rightarrow 0} \|\varphi\|_\infty, \end{aligned}$$

where we've again used the fact that  $c^{\frac{1}{n}} \rightarrow 1$  for any constant.

...

## 2.8 Fall 2016 # 6

Let  $f, g \in L^2(\mathbb{R})$ . Show that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x)g(x+n) dx = 0$$

*Solution.*

Concepts used:

- ??

**Solution:**

- Use the fact that  $L^p$  has small tails: if  $h \in L^2(\mathbb{R})$ , then for any  $\varepsilon > 0$ ,

$$\forall \varepsilon, \exists N \in \mathbb{N} \quad \text{such that} \quad \int_{|x| \geq N} |h(x)|^2 dx < \varepsilon.$$

How to prove small tails in  $L^p$ ?

- So choose  $n$  large enough so the tails of both  $f$  and  $g$  are smaller than  $\varepsilon$ .
- Apply Cauchy-Schwarz:

$$\begin{aligned} \left| \int_{\mathbb{R}} f(x)g(x+n) dx \right| &\leq \int_{\mathbb{R}} |f(x)g(x+n)| dx \\ &\leq \int_{\mathbb{R}} \dots \end{aligned}$$

**2.9 Spring 2016 # 1**

For  $n \in \mathbb{N}$ , define

$$e_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad E_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

Show that  $e_n < E_n$ , and prove Bernoulli's inequality:

$$(1+x)^n \geq 1+nx \quad \text{for } -1 < x < \infty \text{ and } n \in \mathbb{N}$$

Use this to show the following:

1. The sequence  $e_n$  is increasing.
2. The sequence  $E_n$  is decreasing.
3.  $2 < e_n < E_n < 4$ .
4.  $\lim_{n \rightarrow \infty} e_n = \lim_{n \rightarrow \infty} E_n$ .

**2.10 Fall 2015 # 1**

Define

$$f(x) = c_0 + c_1x^1 + c_2x^2 + \dots + c_nx^n \quad \text{with } n \text{ even and } c_n > 0.$$

Show that there is a number  $x_m$  such that  $f(x_m) \leq f(x)$  for all  $x \in \mathbb{R}$ .

**3 Measure Theory: Sets****3.1 Spring 2020 # 2**

Let  $m_*$  denote the Lebesgue outer measure on  $\mathbb{R}$ .

**3.1.1 a.**

Prove that for every  $E \subseteq \mathbb{R}$  there exists a Borel set  $B$  containing  $E$  such that

$$m_*(B) = m_*(E).$$

**3.1.2 b.**

Prove that if  $E \subseteq \mathbb{R}$  has the property that

$$m_*(A) = m_*(A \cap E) + m_*(A \cap E^c)$$

for every set  $A \subseteq \mathbb{R}$ , then there exists a Borel set  $B \subseteq \mathbb{R}$  such that  $E = B \setminus N$  with  $m_*(N) = 0$ .

Be sure to address the case when  $m_*(E) = \infty$ .

*Solution.*

Concepts used:

- Definition of outer measure:  $m_*(E) = \inf_{\{Q_j\} \Rightarrow E} \sum |Q_j|$  where  $\{Q_j\}$  is a countable collection of closed cubes.
- Break  $\mathbb{R}$  into  $\prod_{n \in \mathbb{Z}} [n, n+1)$ , each with finite measure.
- Theorem:  $m_*(Q) = |Q|$  for  $Q$  a closed cube (i.e. the outer measure equals the volume).

*Proof.*

- $m_*(Q) \leq |Q|$ :
- Since  $Q \subseteq Q$ ,  $Q \Rightarrow Q$  and  $m_*(Q) \leq |Q|$  since  $m_*$  is an infimum over such coverings.
- $|Q| \leq m_*(Q)$ :
- Fix  $\varepsilon > 0$ .
- Let  $\{Q_i\}_{i=1}^\infty \Rightarrow Q$  be arbitrary, it suffices to show that

$$|Q| \leq \left( \sum_{i=1}^\infty |Q_i| \right) + \varepsilon.$$

- Pick open cubes  $S_i$  such that  $Q_i \subseteq S_i$  and  $|Q_i| \leq |S_i| \leq (1 + \varepsilon)|Q_i|$ .
- Then  $\{S_i\} \Rightarrow Q$ , so by compactness of  $Q$  pick a finite subcover with  $N$  elements.
- Note

$$Q \subseteq \bigcup_{i=1}^N S_i \implies |Q| \leq \sum_{i=1}^N |S_i| \leq \sum_{i=1}^N (1 + \varepsilon)|Q_i| \leq (1 + \varepsilon) \sum_{i=1}^\infty |Q_i|.$$

- Taking an infimum over coverings on the RHS preserves the inequality, so

$$|Q| \leq (1 + \varepsilon)m_*(Q)$$

- Take  $\varepsilon \rightarrow 0$  to obtain final inequality. ■

**Solution:**

### 3.1.3 a

- If  $m_*(E) = \infty$ , then take  $B = \mathbb{R}^n$  since  $m(\mathbb{R}^n) = \infty$ .
- Suppose  $N := m_*(E) < \infty$ .
- Since  $m_*(E)$  is an infimum, by definition, for every  $\varepsilon > 0$  there exists a covering by closed cubes  $\{Q_i(\varepsilon)\}_{i=1}^\infty \Rightarrow E$  depending on  $\varepsilon$  such that

$$\sum_{i=1}^\infty |Q_i(\varepsilon)| < N + \varepsilon.$$

- For each fixed  $n$ , set  $\varepsilon_n = \frac{1}{n}$  to produce such a covering  $\{Q_i(\varepsilon_n)\}_{i=1}^\infty$  and set  $B_n := \bigcup_{i=1}^\infty Q_i(\varepsilon_n)$ .

- The outer measure of cubes is *equal* to the sum of their volumes, so

$$m_*(B_n) = \sum_{i=1}^{\infty} |Q_i(\varepsilon_n)| < N + \varepsilon_n = N + \frac{1}{n}.$$

- Now set  $B := \bigcap_{n=1}^{\infty} B_n$ .
  - Since  $E \subseteq B_n$  for every  $n$ ,  $E \subseteq B$
  - Since  $B$  is a countable intersection of countable unions of closed sets,  $B$  is Borel.
  - Since  $B_n \subseteq B$  for every  $n$ , we can apply subadditivity to obtain the inequality

$$E \subseteq B \subseteq B_n \implies N \leq m_*(B) \leq m_*(B_n) < N + \frac{1}{n} \quad \text{for all } n \in \mathbb{Z}^{\geq 1}.$$

- This forces  $m_*(E) = m_*(B)$ .

### 3.1.4 b

Suppose  $m_*(E) < \infty$ .

- By (a), find a Borel set  $B \supseteq E$  such that  $m_*(B) = m_*(E)$
- Note that  $E \subseteq B \implies B \cap E = E$  and  $B \cap E^c = B \setminus E$ .
- By assumption,

$$\begin{aligned} m_*(B) &= m_*(B \cap E) + m_*(B \cap E^c) \\ m_*(E) &= m_*(E) + m_*(B \setminus E) \\ m_*(E) - m_*(E) &= m_*(B \setminus E) \quad \text{since } m_*(E) < \infty \\ \implies m_*(B \setminus E) &= 0. \end{aligned}$$

- So take  $N = B \setminus E$ ; this shows  $m_*(N) = 0$  and  $E = B \setminus (B \setminus E) = B \setminus N$ .

If  $m_*(E) = \infty$ :

- Apply result to  $E_R := E \cap [R, R+1)^n \subset \mathbb{R}^n$  for  $R \in \mathbb{Z}$ , so  $E = \bigsqcup_R E_R$
- Obtain  $B_R, N_R$  such that  $E_R = B_R \setminus N_R$ ,  $m_*(E_R) = m_*(B_R)$ , and  $m_*(N_R) = 0$ .
- Note that
  - $B := \bigcup_R B_R$  is a union of Borel sets and thus still Borel
  - $E = \bigcup_R E_R$
  - $N := B \setminus E$
  - $N' := \bigcup_R N_R$  is a union of null sets and thus still null
- Since  $E_R \subset B_R$  for every  $R$ , we have  $E \subset B$
- We can compute

$$N = B \setminus E = \left( \bigcup_R B_R \right) \setminus \left( \bigcup_R E_R \right) \subseteq \bigcup_R (B_R \setminus E_R) = \bigcup_R N_R := N'$$

where  $m_*(N') = 0$  since  $N'$  is null, and thus subadditivity forces  $m_*(N) = 0$ .

**3.2 Fall 2019 # 3.**

Let  $(X, \mathcal{B}, \mu)$  be a measure space with  $\mu(X) = 1$  and  $\{B_n\}_{n=1}^\infty$  be a sequence of  $\mathcal{B}$ -measurable subsets of  $X$ , and

$$B := \left\{ x \in X \mid x \in B_n \text{ for infinitely many } n \right\}.$$

- Argue that  $B$  is also a  $\mathcal{B}$ -measurable subset of  $X$ .
- Prove that if  $\sum_{n=1}^\infty \mu(B_n) < \infty$  then  $\mu(B) = 0$ .
- Prove that if  $\sum_{n=1}^\infty \mu(B_n) = \infty$  **and** the sequence of set complements  $\{B_n^c\}_{n=1}^\infty$  satisfies

$$\mu\left(\bigcap_{n=k}^K B_n^c\right) = \prod_{n=k}^K (1 - \mu(B_n))$$

for all positive integers  $k$  and  $K$  with  $k < K$ , then  $\mu(B) = 1$ .

Hint: Use the fact that  $1 - x \leq e^{-x}$  for all  $x$ .

*Solution.*

Concepts used:

- Borel-Cantelli: for a sequence of sets  $X_n$ ,

$$\begin{aligned} \limsup_n X_n &= \left\{ x \mid x \in X_n \text{ for infinitely many } n \right\} &= \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} X_n \\ \liminf_n X_n &= \left\{ x \mid x \in X_n \text{ for all but finitely many } n \right\} &= \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} X_n. \end{aligned}$$

- Properties of logs and exponentials:

$$\prod_n e^{x_n} = e^{\sum_n x_n} \quad \text{and} \quad \sum_n \log(x_n) = \log\left(\prod_n x_n\right).$$

- Tails of convergent sums vanish.
- Continuity of measure:  $B_n \searrow B$  and  $\mu(B_0) < \infty$  implies  $\lim_n \mu(B_n) = \mu(B)$ , and  $B_n \nearrow B \implies \lim_n \mu(B_n) = \mu(B)$ .

**Solution:**

**3.2.1 a**

- The Borel  $\sigma$ -algebra is closed under countable unions/intersections/complements,
- $B = \limsup_n B_n$  is an intersection of unions of measurable sets.

**3.2.2 b**

- Tails of convergent sums go to zero, so  $\sum_{n \geq M} \mu(B_n) \xrightarrow{M \rightarrow \infty} 0$ ,

- $B_M := \bigcap_{m=1}^M \bigcup_{n \geq m} B_n \searrow B.$

$$\begin{aligned} \mu(B_M) &= \mu \left( \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} B_n \right) \\ &\leq \mu \left( \bigcup_{n \geq m} B_n \right) \quad \text{for all } m \in \mathbb{N} \text{ by countable subadditivity} \\ &\longrightarrow 0, \end{aligned}$$

- The result follows by continuity of measure.

### 3.2.3 c

- To show  $\mu(B) = 1$ , we'll show  $\mu(B^c) = 0$ .
- Let  $B_K = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^K B_n$ . Then

$$\begin{aligned} \mu(B_K^c) &= \mu \left( \bigcup_{m=1}^{\infty} \bigcap_{n=m}^K B_n^c \right) \\ &\leq \sum_{m=1}^{\infty} \mu \left( \bigcap_{n=m}^K B_n^c \right) \quad \text{by subadditivity} \\ &= \sum_{m=1}^{\infty} \prod_{n=m}^K (1 - \mu(B_n)) \quad \text{by assumption} \\ &\leq \sum_{m=1}^{\infty} \prod_{n=m}^K e^{-\mu(B_n^c)} \quad \text{by hint} \\ &= \sum_{m=1}^{\infty} \exp \left( - \sum_{n=m}^K \mu(B_n^c) \right) \\ &\xrightarrow{K \rightarrow \infty} 0 \end{aligned}$$

since  $\sum_{n=m}^K \mu(B_n^c) \xrightarrow{K \rightarrow \infty} \infty$  by assumption

- We can apply continuity of measure since  $B_K^c \xrightarrow{K \rightarrow \infty} B^c$ .

Proving the hint: ?

## 3.3 Spring 2019 # 2

Let  $\mathcal{B}$  denote the set of all Borel subsets of  $\mathbb{R}$  and  $\mu : \mathcal{B} \rightarrow [0, \infty)$  denote a finite Borel measure on  $\mathbb{R}$ .

**3.3.1 a**

Prove that if  $\{F_k\}$  is a sequence of Borel sets for which  $F_k \supseteq F_{k+1}$  for all  $k$ , then

$$\lim_{k \rightarrow \infty} \mu(F_k) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right)$$

**3.3.2 b**

Suppose  $\mu$  has the property that  $\mu(E) = 0$  for every  $E \in \mathcal{B}$  with Lebesgue measure  $m(E) = 0$ .

Prove that for every  $\varepsilon > 0$  there exists  $\delta > 0$  so that if  $E \in \mathcal{B}$  with  $m(E) < \delta$ , then  $\mu(E) < \varepsilon$ .

*Solution.*

Concepts used:

- ??

**Solution:**

**3.3.3 a**

See Folland p.26

- Lemma 1:  $\mu(\coprod_{k=1}^{\infty} E_k) = \lim_{N \rightarrow \infty} \sum_{k=1}^N \mu(E_k)$ .
- Suppose  $F_0 \supseteq F_1 \supseteq \dots$ .
- Let  $A_k = F_k \setminus F_{k+1}$ , since the  $F_k$  are nested the  $A_k$  are disjoint
- Set  $A := \coprod_{k=1}^{\infty} A_k$  and  $F := \bigcap_{k=1}^{\infty} F_k$ .
- Note  $X = X \setminus Y \coprod X \cap Y$  for any two sets (just write  $X \setminus Y := X \cap Y^c$ )
- Note that  $A$  contains anything that was removed from  $F_0$  when passing from any  $F_j$  to  $F_{j+1}$ , while  $F$  contains everything that is never removed at any stage, and these are disjoint possibilities.
- Thus  $F_0 = F \coprod A$ , so

$$\begin{aligned} \mu(F_0) &= \mu(F) + \mu(A) \\ &= \mu(F) + \mu(\coprod_{k=1}^{\infty} A_k) \\ &= \mu(F) + \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu(A_k) \quad \text{by countable additivity} \\ &= \mu(F) + \lim_{n \rightarrow \infty} \sum_{k=1}^n (\mu(F_k) - \mu(F_{k+1})) \\ &= \mu(F) + \lim_{n \rightarrow \infty} (\mu(F_1) - \mu(F_n)) \quad (\text{Telescoping}) \\ &= \mu(F) + \mu(F_1) - \lim_{N \rightarrow \infty} \mu(F_N), \end{aligned}$$

- Since  $\mu$  is a finite measure,  $\mu(F_1) < \infty$  and can be subtracted, yielding

$$\begin{aligned}\mu(F_1) &= \mu(F) + \mu(F_1) - \lim_{n \rightarrow \infty} \mu(F_n) \\ \implies \mu(F) &= \lim_{n \rightarrow \infty} \mu(F_n) \\ \implies \mu\left(\bigcap_{k=1}^{\infty} F_k\right) &= \lim_{n \rightarrow \infty} \mu(F_n).\end{aligned}$$

**3.3.4 b**

- Toward a contradiction, negate the implication: suppose there exists an  $\varepsilon > 0$  such that for all  $\delta$ , we have  $m(E) < \delta$  but  $\mu(E) > \varepsilon$ .
- The sequence  $\left\{\delta_n := \frac{1}{2^n}\right\}_{n \in \mathbb{N}}$  and produce sets  $A_n \in \mathcal{B}$  such  $m(A_n) < \frac{1}{2^n}$  but  $\mu(A_n) > \varepsilon$ .
- Define

$$\begin{aligned}F_n &:= \bigcup_{j \geq n} A_j \\ C_m &:= \bigcap_{k=1}^m F_k \\ A &:= C_{\infty} := \bigcap_{k=1}^{\infty} F_k.\end{aligned}$$

- Note that  $F_1 \supseteq F_2 \supseteq \dots$ , since each increase in index unions fewer sets.
- By continuity for the Lebesgue measure,

$$m(A) = m\left(\bigcap_{k=1}^{\infty} F_k\right) = \lim_{k \rightarrow \infty} m(F_k) = \lim_{k \rightarrow \infty} m\left(\bigcup_{j \geq k} A_j\right) \leq \lim_{k \rightarrow \infty} \sum_{j \geq k} m(A_j) = \lim_{k \rightarrow \infty} \sum_{j \geq k} \frac{1}{2^j} = 0,$$

which follows because this is the tail of a convergent sum

- Thus  $m(A) = 0$  and by assumption, this implies  $\mu(A) = 0$ .
- However, by part (a),

$$\mu(A) = \lim_n \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \geq \lim_n \mu(A_n) = \lim_n \varepsilon = \varepsilon > 0.$$

All messed up!

**3.4 Fall 2018 # 2**

Let  $E \subset \mathbb{R}$  be a Lebesgue measurable set. Show that there is a Borel set  $B \subset E$  such that  $m(E \setminus B) = 0$ .

Move this to review notes to clean things up.



*Solution.*

Concepts used:

- Definition of measurability: there exists an open  $O \supset E$  such that  $m_*(O \setminus E) < \varepsilon$  for all  $\varepsilon > 0$ .
- Theorem:  $E$  is Lebesgue measurable iff there exists a closed set  $F \subseteq E$  such that  $m_*(E \setminus F) < \varepsilon$  for all  $\varepsilon > 0$ .
- Every  $F_\sigma, G_\delta$  is Borel.
- Claim:  $E$  is measurable  $\iff$  for every  $\varepsilon$  there exist  $F_\varepsilon \subset E \subset G_\varepsilon$  with  $F_\varepsilon$  closed and  $G_\varepsilon$  open and  $m(G_\varepsilon \setminus E) < \varepsilon$  and  $m(E \setminus F_\varepsilon) < \varepsilon$ .
  - Proof: existence of  $G_\varepsilon$  is the definition of measurability.
  - Existence of  $F_\varepsilon$ : ?
- Claim:  $E$  is measurable  $\implies$  there exists an open  $O \supseteq E$  such that  $m(O \setminus E) = 0$ .
  - Since  $E$  is measurable, for each  $n \in \mathbb{N}$  choose  $G_n \supseteq E$  such that  $m_*(G_n \setminus E) < \frac{1}{n}$ .
  - Set  $O_N := \bigcap_{n=1}^N G_n$  and  $O := \bigcap_{n=1}^\infty G_n$ .
  - Suppose  $E$  is bounded.

\* Note  $O_N \searrow O$  and  $m_*(O_1) < \infty$  if  $E$  is bounded, since in this case

$$m_*(G_n \setminus E) = m_*(G_1) - m_*(E) < 1 \iff m_*(G_1) < m_*(E) + \frac{1}{n} < \infty.$$

\* Note  $O_N \setminus E \searrow O \setminus E$  since  $O_N \setminus E := O_N \cap E^c \supseteq O_{N+1} \cap E^c$  for all  $N$ , and again  $m_*(O_1 \setminus E) < \infty$ .

\* So it's valid to apply continuity of measure from above:

$$\begin{aligned} m_*(O \setminus E) &= \lim_{N \rightarrow \infty} m_*(O_N \setminus E) \\ &\leq \lim_{N \rightarrow \infty} m_*(G_N \setminus E) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} = 0, \end{aligned}$$

where the inequality uses subadditivity on  $\bigcap_{n=1}^N G_n \subseteq G_N$

- Suppose  $E$  is unbounded.
  - \* Write  $E^k = E \cap [k, k+1]^d \subset \mathbb{R}^d$  as the intersection of  $E$  with an annulus, and note that  $E = \coprod_{k \in \mathbb{N}} E_k$ .
  - \* Each  $E_k$  is bounded, so apply the previous case to obtain  $O_k \supseteq E_k$  with  $m(O_k \setminus E_k) = 0$ .
  - \* So write  $O_k = E_k \coprod N_k$  where  $N_k := O_k \setminus E_k$  is a null set.
  - \* Define  $O = \bigcup_{k \in \mathbb{N}} O_k$ , note that  $E \subseteq O$ .
  - \* Now note

$$\begin{aligned} O \setminus E &= \left( \coprod_k O_k \right) \setminus \left( \coprod_k E_k \right) \\ &\subseteq \coprod_k (O_k \setminus E_k) \\ \implies m_*(O \setminus E) &\leq m_*\left( \coprod_k (O_k \setminus E_k) \right) = 0, \end{aligned}$$

since any countable union of null sets is again null.

- So  $O \supseteq E$  with  $m(O \setminus E) = 0$ .
- Theorem: since  $E$  is measurable,  $E^c$  is measurable
  - Proof: It suffices to write  $E^c$  as the union of two measurable sets,  $E^c = S \cup (E^c - S)$ , where  $S$  is to be determined.
  - We'll produce an  $S$  such that  $m_*(E^c - S) = 0$  and use the fact that any subset of a null set is measurable.
  - Since  $E$  is measurable, for every  $\varepsilon > 0$  there exists an open  $\mathcal{O}_\varepsilon \supseteq E$  such that  $m_*(\mathcal{O}_\varepsilon \setminus E) < \varepsilon$ .
  - Take the sequence  $\left\{ \varepsilon_n := \frac{1}{n} \right\}$  to produce a sequence of sets  $\mathcal{O}_n$ .
  - Note that each  $\mathcal{O}_n^c$  is closed and

$$\mathcal{O}_n \supseteq E \iff \mathcal{O}_n^c \subseteq E^c.$$

- Set  $S := \bigcup_n \mathcal{O}_n^c$ , which is a union of closed sets, thus an  $F_\sigma$  set, thus Borel, thus measurable.
- Note that  $S \subseteq E^c$  since each  $\mathcal{O}_n \subseteq E^c$ .
- Note that

$$\begin{aligned}
 E^c \setminus S &:= E^c \setminus \left( \bigcup_{n=1}^{\infty} \mathcal{O}_n^c \right) \\
 &:= E^c \cap \left( \bigcup_{n=1}^{\infty} \mathcal{O}_n^c \right)^c \quad \text{definition of set minus} \\
 &= E^c \cap \left( \bigcap_{n=1}^{\infty} \mathcal{O}_n \right)^c \quad \text{De Morgan's law} \\
 &= E^c \cup \left( \bigcap_{n=1}^{\infty} \mathcal{O}_n \right) \\
 &:= \left( \bigcap_{n=1}^{\infty} \mathcal{O}_n \right) \setminus E \\
 &\subseteq \mathcal{O}_N \setminus E \quad \text{for every } N \in \mathbb{N}.
 \end{aligned}$$

- Then by subadditivity,

$$m_*(E^c \setminus S) \leq m_*(\mathcal{O}_N \setminus E) \leq \frac{1}{N} \quad \forall N \implies m_*(E^c \setminus S) = 0.$$

- Thus  $E^c \setminus S$  is measurable.

### Solution

#### 3.4.1 Indirect Proof

- Since  $E$  is measurable,  $E^c$  is measurable.
- Since  $E^c$  is measurable exists an open  $O \supseteq E^c$  such that  $m(O \setminus E^c) = 0$ .
- Set  $B := O^c$ , then  $O \supseteq E^c \iff \mathcal{O}^c \subseteq E \iff B \subseteq E$ .
- Computing measures yields

$$E \setminus B := E \setminus \mathcal{O}^c := E \cap (\mathcal{O}^c)^c = E \cap \mathcal{O} = \mathcal{O} \cap (E^c)^c := \mathcal{O} \setminus E^c,$$

thus  $m(E \setminus B) = m(\mathcal{O} \setminus E^c) = 0$ .

- Since  $\mathcal{O}$  is open,  $B$  is closed and thus Borel.

## 3.4.2 Direct Proof (Todo)

?

Try to construct the set.

## 3.5 Spring 2018 # 1

Define

$$E := \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N} \right\}.$$

Prove that  $m(E) = 0$ .*Solution.*

Concepts used:

- Borel-Cantelli: If  $\{E_k\}_{k \in \mathbb{Z}} \subset 2^{\mathbb{R}}$  is a countable collection of Lebesgue measurable sets with  $\sum_{k \in \mathbb{Z}} m(E_k) < \infty$ , then almost every  $x \in \mathbb{R}$  is in *at most finitely* many  $E_k$ .
  - Equivalently (?),  $m(\limsup_{k \rightarrow \infty} E_k) = 0$ , where  $\limsup_{k \rightarrow \infty} E_k = \bigcap_{k=1}^{\infty} \bigcup_{j \geq k} E_j$ , the elements which are in  $E_k$  for infinitely many  $k$ .

**Solution:**

- Strategy: Borel-Cantelli.
- We'll show that  $m(E) \cap [n, n+1] = 0$  for all  $n \in \mathbb{Z}$ ; then the result follows from

$$m(E) = m\left(\bigcup_{n \in \mathbb{Z}} E \cap [n, n+1]\right) \leq \sum_{n=1}^{\infty} m(E \cap [n, n+1]) = 0.$$

- By translation invariance of measure, it suffices to show  $m(E \cap [0, 1]) = 0$ .
  - So WLOG, replace  $E$  with  $E \cap [0, 1]$ .
- Define

$$E_j := \left\{ x \in [0, 1] \mid \exists p \in \mathbb{Z}^{\geq 0} \text{ s.t. } \left| x - \frac{p}{j} \right| < \frac{1}{j^3} \right\}.$$

- Note that  $E_j \subseteq \coprod_{p \in \mathbb{Z}^{\geq 0}} B_{j^{-3}}\left(\frac{p}{j}\right)$ , i.e. a union over integers  $p$  of intervals of radius  $1/j^3$  around the points  $p/j$ . Since  $1/j^3 < 1/j$ , this union is in fact disjoint.
- Importantly, note that

$$\limsup_{j \rightarrow \infty} E_j := \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j = E$$

since

$$x \in \limsup_j E_j \iff x \in E_j \text{ for infinitely many } j$$

$$\iff \text{there are infinitely many } j \text{ for which there exist a } p \text{ such that } \left| x - \frac{p}{j} \right| < j^{-3}$$

$$\iff \text{there are infinitely many such pairs } p, j$$

$$\iff x \in E.$$

- Intersecting with  $[0, 1]$ , we can write  $E_j$  as a union of intervals:

$$E_j = (0, j^{-3}) \amalg B_{j^{-3}}\left(\frac{1}{j}\right) \amalg B_{j^{-3}}\left(\frac{2}{j}\right) \amalg \cdots \amalg B_{j^{-3}}\left(\frac{j-1}{j}\right) \amalg (1 - j^{-3}, 1),$$

where we've separated out the "boundary" terms to emphasize that they are balls about 0 and 1 intersected with  $[0, 1]$ .

- Since  $E_j$  is a union of open sets, it is Borel and thus Lebesgue measurable.
- Computing the measure of  $E_j$ :
  - For a fixed  $j$ , there are exactly  $j + 1$  possible choices for a numerator  $(0, 1, \dots, j)$ , thus there are exactly  $j + 1$  sets appearing in the above decomposition.
  - The first and last intervals are length  $\frac{1}{j^3}$
  - The remaining  $(j + 1) - 2 = j - 1$  intervals are twice this length,  $\frac{2}{j^3}$
  - Thus

$$m(E_j) = 2\left(\frac{1}{j^3}\right) + (j - 1)\left(\frac{2}{j^3}\right) = \frac{2}{j^2}$$

- Note that

$$\sum_{j \in \mathbb{N}} m(E_j) = 2 \sum_{j \in \mathbb{N}} \frac{1}{j^2} < \infty,$$

which converges by the  $p$ -test for sums.

- But then

$$\begin{aligned} m(E) &= m(\limsup_j E_j) \\ &= m\left(\bigcap_{n \in \mathbb{N}} \bigcup_{j \geq n} E_j\right) \\ &\leq m\left(\bigcup_{j \geq N} E_j\right) \quad \text{for every } N \\ &\leq \sum_{j \geq N} m(E_j) \\ &\xrightarrow{N \rightarrow \infty} 0. \end{aligned}$$

- Thus  $E$  is measurable as a subset of a null set and  $m(E) = 0$ .

## 3.6 Fall 2017 # 2

Let  $f(x) = x^2$  and  $E \subset [0, \infty) := \mathbb{R}^+$ .

1. Show that

$$m^*(E) = 0 \iff m^*(f(E)) = 0.$$

2. Deduce that the map

$$\begin{aligned} \varphi : \mathcal{L}(\mathbb{R}^+) &\longrightarrow \mathcal{L}(\mathbb{R}^+) \\ E &\mapsto f(E) \end{aligned}$$

is a bijection from the class of Lebesgue measurable sets of  $[0, \infty)$  to itself.

*Solution.*

Concepts used:

- ??

**Solution:**

## 3.6.1 a

It suffices to consider the bounded case, i.e.  $E \subseteq B_M(0)$  for some  $M$ . Then write  $E_n = B_n(0) \cap E$  and apply the theorem to  $E_n$ , and by subadditivity,  $m^*(E) = m^*\left(\bigcup_n E_n\right) \leq$

$$\sum_n m^*(E_n) = 0.$$

**Lemma:**  $f(x) = x^2, f^{-1}(x) = \sqrt{x}$  are Lipschitz on any compact subset of  $[0, \infty)$ .

*Proof:* Let  $g = f$  or  $f^{-1}$ . Then  $g \in C^1([0, M])$  for any  $M$ , so  $g$  is differentiable and  $g'$  is continuous. Since  $g'$  is continuous on a compact interval, it is bounded, so  $|g'(x)| \leq L$  for all  $x$ . Applying the MVT,

$$|f(x) - f(y)| = |f'(c)| |x - y| \leq L |x - y|.$$

**Lemma:** If  $g$  is Lipschitz on  $\mathbb{R}^n$ , then  $m(E) = 0 \implies m(g(E)) = 0$ .

*Proof:* If  $g$  is Lipschitz, then

$$g(B_r(x)) \subseteq B_{Lr}(x),$$

which is a dilated ball/cube, and so

$$m^*(B_{Lr}(x)) \leq L^n \cdot m^*(B_r(x)).$$

Now choose  $\{Q_j\} \rightrightarrows E$ ; then  $\{g(Q_j)\} \rightrightarrows g(E)$ .

By the above observation,

$$|g(Q_j)| \leq L^n |Q_j|,$$

and so

$$m^*(g(E)) \leq \sum_j |g(Q_j)| \leq \sum_j L^n |Q_j| = L^n \sum_j |Q_j| \longrightarrow 0.$$

Now just take  $g(x) = x^2$  for one direction, and  $g(x) = f^{-1}(x) = \sqrt{x}$  for the other. ■

**3.6.2 b**

Lemma:  $E$  is measurable iff  $E = K \coprod N$  for some  $K$  compact,  $N$  null.

Write  $E = K \coprod N$  where  $K$  is compact and  $N$  is null.

Then  $\varphi^{-1}(E) = \varphi^{-1}(K \coprod N) = \varphi^{-1}(K) \coprod \varphi^{-1}(N)$ .

Since  $\varphi^{-1}(N)$  is null by part (a) and  $\varphi^{-1}(K)$  is the preimage of a compact set under a continuous map and thus compact,  $\varphi^{-1}(E) = K' \coprod N'$  where  $K'$  is compact and  $N'$  is null, so  $\varphi^{-1}(E)$  is measurable.

So  $\varphi$  is a measurable function, and thus yields a well-defined map  $\mathcal{L}(\mathbb{R}) \rightarrow \mathcal{L}(\mathbb{R})$  since it preserves measurable sets. Restricting to  $[0, \infty)$ ,  $f$  is bijection, and thus so is  $\varphi$ .

**3.7 Spring 2017 # 2****3.7.1 a**

Let  $\mu$  be a measure on a measurable space  $(X, \mathcal{M})$  and  $f$  a positive measurable function.

Define a measure  $\lambda$  by

$$\lambda(E) := \int_E f \, d\mu, \quad E \in \mathcal{M}$$

Show that for  $g$  any positive measurable function,

$$\int_X g \, d\lambda = \int_X fg \, d\mu$$

**3.7.2 b**

Let  $E \subset \mathbb{R}$  be a measurable set such that

$$\int_E x^2 \, dm = 0.$$

Show that  $m(E) = 0$ .

*Solution.*

Concepts used:

- Absolute continuity of measures:  $\lambda \ll \mu \iff E \in \mathcal{M}, \mu(E) = 0 \implies \lambda(E) = 0$ .
- Radon-Nikodym: if  $\lambda \ll \mu$ , then there exists a measurable function  $\frac{\partial \lambda}{\partial \mu} := f$  where

$$\lambda(E) = \int_E f \, d\mu.$$

- Chebyshev's inequality:

$$A_c := \{x \in X \mid |f(x)| \geq c\} \implies \mu(A_c) \leq c^{-p} \int_{A_c} |f|^p \, d\mu \quad \forall 0 < p < \infty.$$

**Solutions**

**3.7.3 a**

- Strategy: use approximation by simple functions to show absolute continuity and apply Radon-Nikodym
- Claim:  $\lambda \ll \mu$ , i.e.  $\mu(E) = 0 \implies \lambda(E) = 0$ .
  - Note that if this holds, by Radon-Nikodym,  $f = \frac{\partial \lambda}{\partial \mu} \implies d\lambda = f d\mu$ , which would yield

$$\int g d\lambda = \int gf d\mu.$$

- So let  $E$  be measurable and suppose  $\mu(E) = 0$ .
- Then

$$\lambda(E) := \int_E f d\mu = \lim_{n \rightarrow \infty} \left\{ \int_E s_n d\mu \mid s_n := \sum_{j=1}^{\infty} c_j \mu(E_j), s_n \nearrow f \right\}$$

where we take a sequence of simple functions increasing to  $f$ .

- But since each  $E_j \subseteq E$ , we must have  $\mu(E_j) = 0$  for any such  $E_j$ , so every such  $s_n$  must be zero and thus  $\lambda(E) = 0$ .

What is the final step in this approximation?

**3.7.4 b**

- Set  $g(x) = x^2$ , note that  $g$  is positive and measurable.
- By part (a), there exists a positive  $f$  such that for any  $E \subseteq \mathbb{R}$ ,

$$\int_E g dm = \int_E gf d\mu$$

- The LHS is zero by assumption and thus so is the RHS.
- $m \ll \mu$  by construction.
- Note that  $gf$  is positive.
- Define  $A_k = \left\{ x \in X \mid gf \cdot \chi_E > \frac{1}{k} \right\}$ , for  $k \in \mathbb{Z}^{\geq 0}$
- Then by Chebyshev with  $p = 1$ , for every  $k$  we have

$$\mu(A_k) \leq k \int_E gf d\mu = 0$$

- Then noting that  $A_k \searrow A := \left\{ x \in X \mid gf \cdot \chi_E(x) > 0 \right\}$ , we have  $\mu(A) = 0$ .
- Since  $gf$  is positive, we have

$$x \in E \iff gf \chi_E(x) > 0 \iff x \in A$$

so  $E = A$  and  $\mu(E) = \mu(A)$ .

- But  $m \ll \mu$  and  $\mu(E) = 0$ , so we can conclude that  $m(E) = 0$ .

**3.8 Fall 2016 # 4**

Let  $(X, \mathcal{M}, \mu)$  be a measure space and suppose  $\{E_n\} \subset \mathcal{M}$  satisfies

$$\lim_{n \rightarrow \infty} \mu(X \setminus E_n) = 0.$$

Define

$$G := \left\{ x \in X \mid x \in E_n \text{ for only finitely many } n \right\}.$$

Show that  $G \in \mathcal{M}$  and  $\mu(G) = 0$ .

*Solution.*

Concepts used:

- ??

**Solution:**

- Claim:  $G \in \mathcal{M}$ .  
– Claim:

$$G = \left( \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n \right)^c = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n^c.$$

\* This follows because  $x$  is in the RHS  $\iff x \in E_n^c$  for all but finitely many  $n$   
 $\iff x \in E_n$  for at most finitely many  $n$ .

- But  $\mathcal{M}$  is a  $\sigma$ -algebra, and this shows  $G$  is obtained by countable unions/intersections/complements of measurable sets, so  $G \in \mathcal{M}$ .
- Claim:  $\mu(G) = 0$ .  
– We have

$$\begin{aligned} \mu(G) &= \mu \left( \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n^c \right) \\ &\leq \sum_{N=1}^{\infty} \mu \left( \bigcap_{n=N}^{\infty} E_n^c \right) \\ &\leq \sum_{N=1}^{\infty} \mu(E_N^c) \\ &:= \sum_{N=1}^{\infty} \mu(X \setminus E_N) \\ &\xrightarrow{N \rightarrow \infty} 0. \end{aligned}$$

Last step seems wrong!

**3.9 Spring 2016 # 3**

Let  $f$  be Lebesgue measurable on  $\mathbb{R}$  and  $E \subset \mathbb{R}$  be measurable such that

$$0 < A = \int_E f(x) dx < \infty.$$



Show that for every  $0 < t < 1$ , there exists a measurable set  $E_t \subset E$  such that

$$\int_{E_t} f(x) dx = tA.$$

### 3.10 Spring 2016 # 5

Let  $(X, \mathcal{M}, \mu)$  be a measure space. For  $f \in L^1(\mu)$  and  $\lambda > 0$ , define

$$\varphi(\lambda) = \mu(\{x \in X | f(x) > \lambda\}) \quad \text{and} \quad \psi(\lambda) = \mu(\{x \in X | f(x) < -\lambda\})$$

Show that  $\varphi, \psi$  are Borel measurable and

$$\int_X |f| d\mu = \int_0^\infty [\varphi(\lambda) + \psi(\lambda)] d\lambda$$

### 3.11 Fall 2015 # 2

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be Lebesgue measurable.

1. Show that there is a sequence of simple functions  $s_n(x)$  such that  $s_n(x) \rightarrow f(x)$  for all  $x \in \mathbb{R}$ .
2. Show that there is a Borel measurable function  $g$  such that  $g = f$  almost everywhere.

### 3.12 Spring 2015 # 3

Let  $\mu$  be a finite Borel measure on  $\mathbb{R}$  and  $E \subset \mathbb{R}$  Borel. Prove that the following statements are equivalent:

1.  $\forall \varepsilon > 0$  there exists  $G$  open and  $F$  closed such that

$$F \subseteq E \subseteq G \quad \text{and} \quad \mu(G \setminus F) < \varepsilon.$$

2. There exists a  $V \in G_\delta$  and  $H \in F_\sigma$  such that

$$H \subseteq E \subseteq V \quad \text{and} \quad \mu(V \setminus H) = 0$$

### 3.13 Spring 2014 # 3

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and suppose

$$\forall x \in \mathbb{R}, \quad f(x) \geq \limsup_{y \rightarrow x} f(y)$$

Prove that  $f$  is Borel measurable.

### 3.14 Spring 2014 # 4

Let  $(X, \mathcal{M}, \mu)$  be a measure space and suppose  $f$  is a measurable function on  $X$ . Show that

$$\lim_{n \rightarrow \infty} \int_X f^n d\mu = \begin{cases} \infty \\ \mu(f^{-1}(1)) \end{cases} \quad \text{or}$$

and characterize the collection of functions of each type.

## 3.15 Spring 2017 # 1

Let  $K$  be the set of numbers in  $[0, 1]$  whose decimal expansions do not use the digit 4.

We use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 with  $399\ldots$ . For example,  $0.8754 = 0.8753999\ldots$ .

Show that  $K$  is a compact, nowhere dense set without isolated points, and find the Lebesgue measure  $m(K)$ .

*Solution.*

Concepts used:

- Definition:  $A$  is *nowhere dense*  $\iff$  every interval  $I$  contains a subinterval  $S \subseteq A^c$ .  
– Equivalently, the interior of the closure is empty,  $(\overline{K})^\circ = \emptyset$ .

**Solution:**

Claim:  $K$  is compact.

- It suffices to show that  $K^c := [0, 1] \setminus K$  is open; Then  $K$  will be a closed and bounded subset of  $\mathbb{R}$  and thus compact by Heine-Borel.
- Strategy: write  $K^c$  as the union of open balls (since these form a basis for the Euclidean topology on  $\mathbb{R}$ ).  
– Do this by showing every point  $x \in K^c$  is an interior point, i.e.  $x$  admits a neighborhood  $N_x$  such that  $N_x \subseteq K^c$ .
- Identify  $K^c$  as the set of real numbers in  $[0, 1]$  whose decimal expansion **does** contain a 4.  
– We will show that there exists a neighborhood small enough such that all points in it contain a 4 in their decimal expansions.
- Let  $x \in K^c$ , suppose a 4 occurs as the  $k$ th digit, and write

$$x = 0.d_1d_2\cdots d_{k-1} 4 d_{k+1}\cdots = \left(\sum_{j=1}^k d_j 10^{-j}\right) + (4 \cdot 10^{-k}) + \left(\sum_{j=k+1}^{\infty} d_j 10^{-j}\right).$$

- Set  $r_x < 10^{-k}$  and let  $y \in [0, 1] \cap B_{r_x}(x)$  be arbitrary and write

$$y = \sum_{j=1}^{\infty} c_j 10^{-j}.$$

- Thus  $|x - y| < r_x < 10^{-k}$ , and the first  $k$  digits of  $x$  and  $y$  must agree:  
– We first compute the difference:

$$x - y = \sum_{i=1}^{\infty} d_i 10^{-i} - \sum_{i=1}^{\infty} c_i 10^{-i} = \sum_{i=1}^{\infty} (d_i - c_i) 10^{-i}$$

- Thus (claim)

$$|x - y| \leq \sum_{j=1}^{\infty} |d_j - c_j| 10^j < 10^{-k} \iff |d_j - c_j| = 0 \quad \forall j \leq k.$$

- Otherwise we can note that any term  $|d_j - c_j| \geq 1$  and there is a contribution to  $|x - y|$  of at least  $1 \cdot 10^{-j}$  for some  $j < k$ , whereas

$$j < k \iff 10^{-j} > 10^{-k},$$

a contradiction.

- This means that for all  $j \leq k$  we have  $d_j = c_j$ , and in particular  $d_k = 4 = c_k$ , so  $y$  has a 4 in its decimal expansion.
- But then  $K^c = \bigcup_x B_{r_x}(x)$  is a union of open sets and thus open.

Claim:  $K$  is nowhere dense and  $m(K) = 0$ :

- Strategy: Show  $(\overline{K})^\circ = \emptyset$ .
- Since  $K$  is closed,  $\overline{K} = K$ , so it suffices to show that  $K$  does not properly contain any interval.
- It suffices to show  $m(K^c) = 1$ , since this implies  $m(K) = 0$  and since any interval has strictly positive measure, this will mean  $K$  can not contain an interval.
- As in the construction of the Cantor set, let
  - $K_0$  denote  $[0, 1]$  with 1 interval  $\left(\frac{4}{10}, \frac{5}{10}\right)$  of length  $\frac{1}{10}$  deleted, so

$$m(K_0^c) = \frac{1}{10}.$$

- $K_1$  denote  $K_0$  with 9 intervals  $\left(\frac{1}{100}, \frac{5}{100}\right), \left(\frac{14}{100}, \frac{15}{100}\right), \dots, \left(\frac{94}{100}, \frac{95}{100}\right)$  of length  $\frac{1}{100}$  deleted, so

$$m(K_1^c) = \frac{1}{10} + \frac{9}{100}.$$

- $K_n$  denote  $K_{n-1}$  with  $9^n$  such intervals of length  $\frac{1}{10^{n+1}}$  deleted, so

$$m(K_n^c) = \frac{1}{10} + \frac{9}{100} + \dots + \frac{9^n}{10^{n+1}}.$$

- Then compute

$$m(K^c) = \sum_{j=0}^{\infty} \frac{9^j}{10^{j+1}} = \frac{1}{10} \sum_{j=0}^{\infty} \left(\frac{9}{10}\right)^j = \frac{1}{10} \left(\frac{1}{1 - \frac{9}{10}}\right) = 1.$$

Claim:  $K$  has no isolated points:

- A point  $x \in K$  is isolated iff there is an open ball  $B_r(x)$  containing  $x$  such that  $B_r(x) \subsetneq K^c$ .
  - So every point in this ball **should** have a 4 in its decimal expansion.
- Strategy: show that if  $x \in K$ , every neighborhood of  $x$  intersects  $K$ .
- Note that  $m(K_n) = \left(\frac{9}{10}\right)^n \xrightarrow{n \rightarrow \infty} 0$
- Also note that we deleted open intervals, and the endpoints of these intervals are never deleted.
  - Thus endpoints of deleted intervals are elements of  $K$ .

- Fix  $x$ . Then for every  $\varepsilon$ , by the Archimedean property of  $\mathbb{R}$ , choose  $n$  such that  $\left(\frac{9}{10}\right)^n < \varepsilon$ .
- Then there is an endpoint  $x_n$  of some deleted interval  $I_n$  satisfying

$$|x - x_n| \leq \left(\frac{9}{10}\right)^n < \varepsilon.$$

- So every ball containing  $x$  contains some endpoint of a removed interval, and thus an element of  $K$ .

### 3.16 Spring 2016 # 2

Let  $0 < \lambda < 1$  and construct a Cantor set  $C_\lambda$  by successively removing middle intervals of length  $\lambda$ .

Prove that  $m(C_\lambda) = 0$ .

## 4 Measure Theory: Functions

### 4.1 Fall 2016 # 2

Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be measurable with

$$\int_a^b f(x) \, dx = \int_a^b g(x) \, dx.$$

Show that either

1.  $f(x) = g(x)$  almost everywhere, or
2. There exists a measurable set  $E \subset [a, b]$  such that

$$\int_E f(x) \, dx > \int_E g(x) \, dx$$

*Solution.*

Concepts used:

- ??

**Solution:**

- Suppose it is *not* the case that  $f = g$  almost everywhere; then letting  $A := \{x \in [a, b] \mid f(x) \neq g(x)\}$ , we have  $m(A) > 0$ .
- Write

$$A = A_1 \amalg A_2 := \{f > g\} \amalg \{f < g\},$$

then  $m(A_1) > 0$  or  $m(A_2) > 0$  (or both).

- Wlog (by relabeling  $f, g$  if necessary), suppose  $m(A_1) > 0$ , and take  $E := A_1$ .
- Then on  $E$ , we have  $f(x) > g(x)$  pointwise. This is preserved by monotonicity of the integral, thus

$$f(x) > g(x) \text{ on } E \implies \int_E f(x) \, dx > \int_E g(x) \, dx.$$

**4.2 Spring 2016 # 4**

Let  $E \subset \mathbb{R}$  be measurable with  $m(E) < \infty$ . Define

$$f(x) = m(E \cap (E + x)).$$

Show that

1.  $f \in L^1(\mathbb{R})$ .
2.  $f$  is uniformly continuous.
3.  $\lim_{|x| \rightarrow \infty} f(x) = 0$ .

Hint:

$$\chi_{E \cap (E+x)}(y) = \chi_E(y) \chi_E(y-x)$$

**5 Integrals: Convergence****5.1 Fall 2019 # 2.**

Prove that

$$\left| \frac{d^n}{dx^n} \frac{\sin x}{x} \right| \leq \frac{1}{n}$$

for all  $x \neq 0$  and positive integers  $n$ .

Hint: Consider  $\int_0^1 \cos(tx) dt$

*Solution.*

Concepts used:

- DCT
- Bounding in the right place. Don't evaluate the actual integral!

**Solution:**

- By induction on the number of limits we can pass through the integral.
- For  $n = 1$  we first pass one derivative into the integral: let  $x_n \rightarrow x$  be any sequence

converging to  $x$ , then

$$\begin{aligned}
 \frac{\partial}{\partial x} \frac{\sin(x)}{x} &= \frac{\partial}{\partial x} \int_0^1 \cos(tx) dt \\
 &= \lim_{x_n \rightarrow x} \frac{1}{x_n - x} \left( \int_0^1 \cos(tx_n) dt - \int_0^1 \cos(tx) dt \right) \\
 &= \lim_{x_n \rightarrow x} \left( \int_0^1 \frac{\cos(tx_n) - \cos(tx)}{x_n - x} dt \right) \\
 &= \lim_{x_n \rightarrow x} \left( \int_0^1 \left( t \sin(tx) \Big|_{x=\xi_n} \right) dt \right) \quad \text{where } \xi_n \in [x_n, x] \text{ by MVT, } \xi_n \rightarrow x \\
 &= \lim_{\xi_n \rightarrow x} \left( \int_0^1 t \sin(t\xi_n) dt \right) \\
 &=_{\text{DCT}} \int_0^1 \lim_{\xi_n \rightarrow x} t \sin(t\xi_n) dt \\
 &= \int_0^1 t \sin(tx) dt
 \end{aligned}$$

- Taking absolute values we obtain an upper bound

$$\begin{aligned}
 \left| \frac{\partial}{\partial x} \frac{\sin(x)}{x} \right| &= \left| \int_0^1 t \sin(tx) dt \right| \\
 &\leq \int_0^1 |t \sin(tx)| dt \\
 &\leq \int_0^1 1 dt = 1,
 \end{aligned}$$

since  $t \in [0, 1] \implies |t| < 1$ , and  $|\sin(xt)| \leq 1$  for any  $x$  and  $t$ .

- Note that this bound also justifies the DCT, since the functions  $f_n(t) = t \sin(t\xi_n)$  are uniformly dominated by  $g(t) = 1$  on  $L^1([0, 1])$ .

Note: integrating by parts here yields the actual formula:

$$\begin{aligned}
 \int_0^1 t \sin(tx) dt &=_{\text{IBP}} \left( \frac{-t \cos(tx)}{x} \right) \Big|_{t=0}^{t=1} - \int_0^1 \frac{\cos(tx)}{x} dt \\
 &= \frac{-\cos(x)}{x} - \frac{\sin(x)}{x^2} \\
 &= \frac{x \cos(x) - \sin(x)}{x^2}.
 \end{aligned}$$

- For the inductive step, we assume that we can pass  $n - 1$  limits through the integral and show we can pass the  $n$ th through as well.

$$\begin{aligned}
 \frac{\partial^n}{\partial x^n} \frac{\sin(x)}{x} &= \frac{\partial^n}{\partial x^n} \int_0^1 \cos(tx) dt \\
 &= \frac{\partial}{\partial x} \int_0^1 \frac{\partial^{n-1}}{\partial x^{n-1}} \cos(tx) dt \\
 &= \frac{\partial}{\partial x} \int_0^1 t^{n-1} f_{n-1}(x, t) dt
 \end{aligned}$$

- Note that  $f_n(x, t) = \pm \sin(tx)$  when  $n$  is odd and  $f_n(x, t) = \pm \cos(tx)$  when  $n$  is even, and a constant factor of  $t$  is multiplied when each derivative is taken.
- We continue as in the base case:

$$\begin{aligned}
 \frac{\partial}{\partial x} \int_0^1 t^{n-1} f_{n-1}(x, t) dt &= \lim_{x_k \rightarrow x} \int_0^1 t^{n-1} \left( \frac{f_{n-1}(x_k, t) - f_{n-1}(x, t)}{x_k - x} \right) dt \\
 &=_{\text{IVT}} \lim_{x_k \rightarrow x} \int_0^1 t^{n-1} \frac{\partial f_{n-1}}{\partial x}(\xi_k, t) dt \quad \text{where } \xi_k \in [x_k, x], \xi_k \rightarrow x \\
 &=_{\text{DCT}} \int_0^1 \lim_{x_k \rightarrow x} t^{n-1} \frac{\partial f_{n-1}}{\partial x}(\xi_k, t) dt \\
 &:= \int_0^1 \lim_{x_k \rightarrow x} t^n f_n(\xi_k, t) dt \\
 &:= \int_0^1 t^n f_n(x, t) dt.
 \end{aligned}$$

- We've used the fact that  $f_0(x) = \cos(tx)$  is smooth as a function of  $x$ , and in particular continuous
- The DCT is justified because the functions  $h_{n,k}(x, t) = t^n f_n(\xi_k, t)$  are again uniformly (in  $k$ ) bounded by 1 since  $t \leq 1 \implies t^n \leq 1$  and each  $f_n$  is a sin or cosine.
- Now take absolute values

$$\begin{aligned}
 \left| \frac{\partial^n}{\partial x^n} \frac{\sin(x)}{x} \right| &= \left| \int_0^1 -t^n f_n(x, t) dt \right| \\
 &\leq \int_0^1 |t^n f_n(x, t)| dt \\
 &\leq \int_0^1 |t^n| |f_n(x, t)| dt \\
 &\leq \int_0^1 |t^n| \cdot 1 dt \\
 &\leq \int_0^1 t^n dt \quad \text{since } t \text{ is positive} \\
 &= \frac{1}{n+1} \\
 &< \frac{1}{n}.
 \end{aligned}$$

- We've again used the fact that  $f_n(x, t)$  is of the form  $\pm \cos(tx)$  or  $\pm \sin(tx)$ , both of which are bounded by 1.

## 5.2 Spring 2020 # 5

Compute the following limit and justify your calculations:

$$\lim_{n \rightarrow \infty} \int_0^n \left( 1 + \frac{x^2}{n} \right)^{-(n+1)} dx.$$

Not finished, flesh out.

*Solution.*

Concepts used:

- DCT
- Passing limits through products and quotients

**Solution:**

Note that

$$\begin{aligned}\lim_n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} &= \frac{1}{\lim_n \left(1 + \frac{x^2}{n}\right)^1 \left(1 + \frac{x^2}{n}\right)^n} \\ &= \frac{1}{1 \cdot e^{x^2}} \\ &= e^{-x^2}.\end{aligned}$$

If passing the limit through the integral is justified, we will have

$$\begin{aligned}\lim_{n \rightarrow \infty} \int_0^n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \chi_{[0,n]} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \\ &= \int_{\mathbb{R}} \lim_{n \rightarrow \infty} \chi_{[0,n]} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \quad \text{by the DCT} \\ &= \int_{\mathbb{R}} \lim_{n \rightarrow \infty} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \\ &= \int_0^\infty e^{-x^2} dx \\ &= \frac{\sqrt{\pi}}{2}.\end{aligned}$$

Computing the last integral:

$$\begin{aligned}\left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2 &= \left(\int_{\mathbb{R}} e^{-x^2} dx\right) \left(\int_{\mathbb{R}} e^{-y^2} dy\right) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(x+y)^2} dx dy \\ &= \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta \quad u = r^2 \\ &= \frac{1}{2} \int_0^{2\pi} \int_0^\infty e^{-u} du d\theta \\ &= \frac{1}{2} \int_0^{2\pi} 1 d\theta \\ &= \pi,\end{aligned}$$

and now use the fact that the function is even so  $\int_0^\infty f = \frac{1}{2} \int_{\mathbb{R}} f$ .

Justifying the DCT:



- Apply Bernoulli's inequality:

$$1 + \frac{x^{2n+1}}{n} \geq 1 + \frac{x^2}{n} (1 + x^2) \geq 1 + x^2,$$

where the last inequality follows from the fact that  $1 + \frac{x^2}{n} \geq 1$

### 5.3 Spring 2019 # 3

Let  $\{f_k\}$  be any sequence of functions in  $L^2([0, 1])$  satisfying  $\|f_k\|_2 \leq M$  for all  $k \in \mathbb{N}$ .

Prove that if  $f_k \rightarrow f$  almost everywhere, then  $f \in L^2([0, 1])$  with  $\|f\|_2 \leq M$  and

$$\lim_{k \rightarrow \infty} \int_0^1 f_k(x) dx = \int_0^1 f(x) dx$$

Hint: Try using Fatou's Lemma to show that  $\|f\|_2 \leq M$  and then try applying Egorov's Theorem.

*Solution.*

Concepts used:

- Definition of  $L^+$ : space of measurable function  $X \rightarrow [0, \infty]$ .
- Fatou: For any sequence of  $L^+$  functions,  $\int \liminf f_n \leq \liminf \int f_n$ .
- Egorov's Theorem: If  $E \subseteq \mathbb{R}^n$  is measurable,  $m(E) > 0$ ,  $f_k : E \rightarrow \mathbb{R}$  a sequence of measurable functions where  $\lim_{n \rightarrow \infty} f_n(x)$  exists and is finite a.e., then  $f_n \rightarrow f$  *almost uniformly*: for every  $\varepsilon > 0$  there exists a closed subset  $F_\varepsilon \subseteq E$  with  $m(E \setminus F_\varepsilon) < \varepsilon$  and  $f_n \rightarrow f$  uniformly on  $F_\varepsilon$ .

**Solution:**

$L^2$  bound:

- Since  $f_k \rightarrow f$  almost everywhere,  $\liminf_n f_n(x) = f(x)$  a.e.
- $\|f_n\|_2 < \infty$  implies each  $f_n$  is measurable and thus  $|f_n|^2 \in L^+$ , so we can apply Fatou:

$$\begin{aligned} \|f\|_2^2 &= \int |f(x)|^2 \\ &= \int \liminf_n |f_n(x)|^2 \\ &\leq \liminf_n \int |f_n(x)|^2 \\ &\leq \liminf_n M \\ &= M. \end{aligned}$$

- Thus  $\|f\|_2 \leq \sqrt{M} < \infty$  implying  $f \in L^2$ .

What is the "right" proof here that uses the first part?

Equality of Integrals:

- Take the sequence  $\varepsilon_n = \frac{1}{n}$

- Apply Egorov's theorem: obtain a set  $F_\varepsilon$  such that  $f_n \rightarrow f$  uniformly on  $F_\varepsilon$  and  $m(I \setminus F_\varepsilon) < \varepsilon$ .

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left| \int_0^1 f_n - f \right| &\leq \lim_{n \rightarrow \infty} \int_0^1 |f_n - f| \\
 &= \lim_{n \rightarrow \infty} \left( \int_{F_\varepsilon} |f_n - f| + \int_{I \setminus F_\varepsilon} |f_n - f| \right) \\
 &= \int_{F_\varepsilon} \lim_{n \rightarrow \infty} |f_n - f| + \lim_{n \rightarrow \infty} \int_{I \setminus F_\varepsilon} |f_n - f| \quad \text{by uniform convergence} \\
 &= 0 + \lim_{n \rightarrow \infty} \int_{I \setminus F_\varepsilon} |f_n - f|,
 \end{aligned}$$

so it suffices to show  $\int_{I \setminus F_\varepsilon} |f_n - f| \xrightarrow{n \rightarrow \infty} 0$ .

- We can obtain a bound using Holder's inequality with  $p = q = 2$ :

$$\begin{aligned}
 \int_{I \setminus F_\varepsilon} |f_n - f| &\leq \left( \int_{I \setminus F_\varepsilon} |f_n - f|^2 \right)^{1/2} \left( \int_{I \setminus F_\varepsilon} |1|^2 \right)^{1/2} \\
 &= \left( \int_{I \setminus F_\varepsilon} |f_n - f|^2 \right)^{1/2} \mu(F_\varepsilon)^{1/2} \\
 &\leq \|f_n - f\|_2 \mu(F_\varepsilon)^{1/2} \\
 &\leq (\|f_n\|_2 + \|f\|_2) \mu(F_\varepsilon)^{1/2} \\
 &\leq 2M \cdot \mu(F_\varepsilon)^{1/2}
 \end{aligned}$$

where  $M$  is now a constant not depending on  $\varepsilon$  or  $n$ .

- Now take a nested sequence of sets  $F_\varepsilon$  with  $\mu(F_\varepsilon) \rightarrow 0$  and applying continuity of measure yields the desired statement.

## 5.4 Fall 2018 # 6

Compute the following limit and justify your calculations:

$$\lim_{n \rightarrow \infty} \int_1^n \frac{dx}{\left(1 + \frac{x}{n}\right)^n \sqrt[n]{x}}$$

*Solution.*

Concepts used:

- ??

**Solution:**

- Note that  $x^{\frac{1}{n}} \xrightarrow{n \rightarrow \infty} 1$  for any  $0 < x < \infty$ .
- Thus the integrand converges to  $\frac{1}{e^x}$ , which is integrable on  $(0, \infty)$  and integrates to 1.
- Break the integrand up:

$$\int_0^\infty \frac{1}{\left(1 + \frac{x}{n}\right)^n x^{\frac{1}{n}}} dx = \int_0^1 \frac{1}{\left(1 + \frac{x}{n}\right)^n x^{\frac{1}{n}}} dx = \int_1^\infty \frac{1}{\left(1 + \frac{x}{n}\right)^n x^{\frac{1}{n}}} dx.$$

**5.5 Fall 2018 # 3**

Suppose  $f(x)$  and  $xf(x)$  are integrable on  $\mathbb{R}$ . Define  $F$  by

$$F(t) := \int_{-\infty}^{\infty} f(x) \cos(xt) dx$$

Show that

$$F'(t) = - \int_{-\infty}^{\infty} xf(x) \sin(xt) dx.$$

*Solution.*

Concepts used:

- Mean Value Theorem
- DCT

**Solution:**

$$\begin{aligned} \frac{\partial}{\partial t} F(t) &= \frac{\partial}{\partial t} \int_{\mathbb{R}} f(x) \cos(xt) dx \\ &\stackrel{DCT}{=} \int_{\mathbb{R}} f(x) \frac{\partial}{\partial t} \cos(xt) dx \\ &= \int_{\mathbb{R}} xf(x) \cos(xt) dx, \end{aligned}$$

so it only remains to justify the DCT.

- Fix  $t$ , then let  $t_n \rightarrow t$  be arbitrary.
- Define

$$h_n(x, t) = f(x) \left( \frac{\cos(tx) - \cos(t_n x)}{t_n - t} \right) \xrightarrow{n \rightarrow \infty} \frac{\partial}{\partial t} (f(x) \cos(xt))$$

since  $\cos(tx)$  is differentiable in  $t$  and this is the limit definition of differentiability.

- Note that

$$\begin{aligned} \frac{\partial}{\partial t} \cos(tx) &:= \lim_{t_n \rightarrow t} \frac{\cos(tx) - \cos(t_n x)}{t_n - t} \\ &\stackrel{MVT}{=} \frac{\partial}{\partial t} \cos(tx) \Big|_{t=\xi_n} \quad \text{for some } \xi_n \in [t, t_n] \text{ or } [t_n, t] \\ &= x \sin(\xi_n x) \end{aligned}$$

where  $\xi_n \xrightarrow{n \rightarrow \infty} t$  since wlog  $t_n \leq \xi_n \leq t$  and  $t_n \nearrow t$ .

- We then have

$$|h_n(x)| = |f(x)x \sin(\xi_n x)| \leq |xf(x)| \quad \text{since } |\sin(\xi_n x)| \leq 1$$

for every  $x$  and every  $n$ .

- Since  $xf(x) \in L^1(\mathbb{R})$  by assumption, the DCT applies.

## 5.6 Spring 2018 # 5

Suppose that

- $f_n, f \in L^1$ ,
- $f_n \rightarrow f$  almost everywhere, and
- $\int |f_n| \rightarrow \int |f|$ .

Show that  $\int f_n \rightarrow \int f$ .

*Solution.*

Concepts used:

- $\int |f_n - f| \rightarrow 0 \iff \int f_n = \int f$ .
- Fatou:

$$\begin{aligned} \int \liminf f_n &\leq \liminf \int f_n \\ \int \limsup f_n &\geq \limsup \int f_n. \end{aligned}$$

**Solution:**

- Since  $\int |f_n| \xrightarrow{n \rightarrow \infty} \int |f|$ , define

$$\begin{aligned} h_n &= |f_n - f| && \xrightarrow{n \rightarrow \infty} 0 \text{ a.e.} \\ g_n &= |f_n| + |f| && \xrightarrow{n \rightarrow \infty} 2|f| \text{ a.e.} \end{aligned}$$

– Note that  $g_n - h_n \xrightarrow{n \rightarrow \infty} 2|f| - 0 = 2|f|$ .

- Then

$$\begin{aligned} \int 2|f| &= \int \liminf_n (g_n - h_n) \\ &= \int \liminf_n (g_n) + \int \liminf_n (-h_n) \\ &= \int \liminf_n (g_n) - \int \limsup_n (h_n) \\ &= \int 2|f| - \int \limsup_n (h_n) \\ &\leq \int 2|f| - \limsup_n \int h_n \quad \text{by Fatou,} \end{aligned}$$

- Since  $f \in L^1$ ,  $\int 2|f| = 2\|f\|_1 < \infty$  and it makes sense to subtract it from both sides, thus

$$\begin{aligned} 0 &\leq -\limsup_n \int h_n \\ &:= -\limsup_n \int |f_n - f|. \end{aligned}$$

which forces  $\limsup_n \int |f_n - f| = 0$ , since

- The integral of a nonnegative function is nonnegative, so  $\int |f_n - f| \geq 0$ .
- So  $\left(-\int |f_n - f|\right) \leq 0$ .
- But the above inequality shows  $\left(-\int |f_n - f|\right) \geq 0$  as well.
- Since  $\liminf_n \int h_n \leq \limsup_n \int h_n = 0$ ,  $\lim_n \int h_n$  exists and is equal to zero.
- But then

$$\left|\int f_n - \int f\right| = \left|\int f_n - f\right| \leq \int |f_n - f|,$$

and taking  $\lim_{n \rightarrow \infty}$  on both sides yields

$$\lim_{n \rightarrow \infty} \left|\int f_n - \int f\right| \leq \lim_{n \rightarrow \infty} \int |f_n - f| = 0,$$

$$\text{so } \lim_{n \rightarrow \infty} \int f_n = \int f.$$

## 5.7 Spring 2018 # 2

Let

$$f_n(x) := \frac{x}{1+x^n}, \quad x \geq 0.$$

- a. Show that this sequence converges pointwise and find its limit. Is the convergence uniform on  $[0, \infty)$ ?
- b. Compute

$$\lim_{n \rightarrow \infty} \int_0^\infty f_n(x) dx$$

*Solution.*

Concepts used:

- ??

**Solution:**

### 5.7.1 a

Claim:  $f_n$  does not converge uniformly to its limit.

- Note each  $f_n(x)$  is clearly continuous on  $(0, \infty)$ , since it is a quotient of continuous functions where the denominator is never zero.
- Note

$$x < 1 \implies x^n \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad x > 1 \implies x^n \xrightarrow{n \rightarrow \infty} \infty.$$

- Thus

$$f_n(x) = \frac{x}{1+x^n} \xrightarrow{n \rightarrow \infty} f(x) := \begin{cases} x, & 0 \leq x < 1 \\ \frac{1}{2}, & x = 1 \\ 0, & x > 1 \end{cases}.$$

- If  $f_n \rightarrow f$  uniformly on  $[0, \infty)$ , it would converge uniformly on every subset and thus uniformly on  $(0, \infty)$ .
  - Then  $f$  would be a uniform limit of continuous functions on  $(0, \infty)$  and thus continuous on  $(0, \infty)$ .
  - By uniqueness of limits,  $f_n$  would converge to the pointwise limit  $f$  above, which is not continuous at  $x = 1$ , a contradiction.

### 5.7.2 b

- If the DCT applies, interchange the limit and integral:

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\infty f_n(x) dx &= \int_0^\infty \lim_{n \rightarrow \infty} f_n(x) dx \quad \text{DCT} \\ &= \int_0^\infty f(x) dx \\ &= \int_0^1 x dx + \int_1^\infty 0 dx \\ &= \frac{1}{2} x^2 \Big|_0^1 \\ &= \frac{1}{2}. \end{aligned}$$

- To justify the DCT, write

$$\int_0^\infty f_n(x) dx = \int_0^1 f_n(x) dx + \int_1^\infty f_n(x) dx.$$

- $f_n$  restricted to  $(0, 1)$  is uniformly bounded by  $g_0(x) = 1$  in the first integral, since

$$x \in [0, 1] \implies \frac{x}{1+x^n} < \frac{1}{1+x^n} < 1 := g(x)$$

so

$$\int_0^1 f_n(x) dx \leq \int_0^1 1 dx = 1 < \infty.$$

Also note that  $g_0 \cdot \chi_{(0,1)} \in L^1((0, \infty))$ .

- The  $f_n$  restricted to  $(1, \infty)$  are uniformly bounded by  $g_1(x) = \frac{1}{x^2}$  on  $[1, \infty)$ , since

$$x \in (1, \infty) \implies \frac{x}{1+x^n} \leq \frac{x}{x^n} = \frac{1}{x^{n-1}} \leq \frac{1}{x^2} \in L^1([1, \infty)) \text{ when } n \geq 3,$$

by the  $p$ -test for integrals.

- So set

$$g := g_0 \cdot \chi_{(0,1)} + g_1 \cdot \chi_{[1,\infty)},$$

then by the above arguments  $g \in L^1((0, \infty))$  and  $f_n \leq g$  everywhere, so the DCT applies.

**5.8 Fall 2016 # 3**

Let  $f \in L^1(\mathbb{R})$ . Show that

$$\lim_{x \rightarrow 0} \int_{\mathbb{R}} |f(y-x) - f(y)| dy = 0$$

Missing some stuff.

:::{.solution}

Concepts used:

- $C_c^\infty \hookrightarrow L^p$  is dense.
- If  $f \dots$ ?

**Solution:**

- Fixing notation, set  $\tau_x f(y) := f(y-x)$ ; we then want to show

$$\|\tau_x f - f\|_{L^1} \xrightarrow{x \rightarrow 0} 0.$$

- Claim: by an  $\varepsilon/3$  argument, it suffices to show this for compactly supported functions:
  - Since  $f \in L^1$ , choose  $g_n \in C_c^\infty(\mathbb{R}^1)$  smooth and compactly supported so that

$$\|f - g\|_{L^1} < \varepsilon.$$

- Claim:  $\|\tau_x f - \tau_x g\| < \varepsilon$ .
  - \* Proof 1: translation invariance of the integral.
  - \* Proof 2: Apply a change of variables:

$$\begin{aligned} \|\tau_x f - \tau_x g\|_1 &:= \int_{\mathbb{R}} |\tau_x f(y) - \tau_x g(y)| dy \\ &= \int_{\mathbb{R}} |f(y-x) - g(y-x)| dy \\ &= \int_{\mathbb{R}} |f(u) - g(u)| du \quad (u = y-x, du = dy) \\ &= \|f - g\|_1 \\ &< \varepsilon. \end{aligned}$$

- Then

$$\begin{aligned} \|\tau_x f - f\|_1 &= \|\tau_x f - \tau_x g + \tau_x g - g + g - f\|_1 \\ &\leq \|\tau_x f - \tau_x g\|_1 + \|\tau_x g - g\|_1 + \|g - f\|_1 \\ &\leq 2\varepsilon + \|\tau_x g - g\|_1. \end{aligned}$$

- To show this for compactly supported functions:
  - Let  $g \in C_c^\infty(\mathbb{R}^1)$ , let  $E = \text{supp}(g)$ , and write

$$\begin{aligned} \|\tau_x g - g\|_1 &= \int_{\mathbb{R}} |g(y-x) - g(y)| dy \\ &= \int_E |g(y-x) - g(y)| dy + \int_{E^c} |g(y-x) - g(y)| dy \\ &= \int_E |g(y-x) - g(y)| dy. \end{aligned}$$

– But  $g$  is smooth and compactly supported on  $E$ , and thus uniformly continuous on  $E$ , so

$$\begin{aligned}\lim_{x \rightarrow 0} \int_E |g(y-x) - g(y)| dy &= \int_E \lim_{x \rightarrow 0} |g(y-x) - g(y)| dy \\ &= \int_E 0 dy \\ &= 0.\end{aligned}$$

...

### 5.9 Fall 2015 # 3

Compute the following limit:

$$\lim_{n \rightarrow \infty} \int_1^n \frac{ne^{-x}}{1+nx^2} \sin\left(\frac{x}{n}\right) dx$$

### 5.10 Fall 2015 # 4

Let  $f : [1, \infty) \rightarrow \mathbb{R}$  such that  $f(1) = 1$  and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Show that the following limit exists and satisfies the equality

$$\lim_{x \rightarrow \infty} f(x) \leq 1 + \frac{\pi}{4}$$

## 6 Integrals: Approximation

### 6.1 Spring 2018 # 3

Let  $f$  be a non-negative measurable function on  $[0, 1]$ .

Show that

$$\lim_{p \rightarrow \infty} \left( \int_{[0,1]} f(x)^p dx \right)^{\frac{1}{p}} = \|f\|_{\infty}.$$

*Solution.*

Concepts used:

- $\|f\|_{\infty} := \inf_t \left\{ t \mid m\left(\left\{x \in \mathbb{R}^n \mid f(x) > t\right\}\right) = 0 \right\}$ , i.e. this is the lowest upper bound that holds almost everywhere.

**Solution:**

- $\|f\|_p \leq \|f\|_{\infty}$ :  
– Note  $|f(x)| \leq \|f\|_{\infty}$  almost everywhere and taking  $p$ th powers preserves this inequality.



– Thus

$$\begin{aligned}
 |f(x)| &\leq \|f\|_\infty \quad \text{a.e. by definition} \\
 \implies |f(x)|^p &\leq \|f\|_\infty^p \quad \text{for } p \geq 0 \\
 \implies \|f\|_p^p &= \int_X |f(x)|^p dx \\
 &\leq \int_X \|f\|_\infty^p dx \\
 &= \|f\|_\infty^p \int_X 1 dx \\
 &= \|f\|_\infty^p \cdot m(X) \quad \text{since the norm doesn't depend on } x \\
 &= \|f\|_\infty^p \quad \text{since } m(X) = 1.
 \end{aligned}$$

\* Thus  $\|f\|_p \leq \|f\|_\infty$  for all  $p$  and taking  $\lim_{p \rightarrow \infty}$  preserves this inequality.

- $\|f\|_p \geq \|f\|_\infty$ :
  - Fix  $\varepsilon > 0$ .
  - Define

$$S_\varepsilon := \{x \in \mathbb{R}^n \mid |f(x)| \geq \|f\|_\infty - \varepsilon\}.$$

\* Note that  $m(S_\varepsilon) > 0$ ; otherwise if  $m(S_\varepsilon) = 0$ , then  $t := \|f\|_\infty - \varepsilon < \|f\|_\varepsilon$ . But this produces a *smaller* upper bound almost everywhere than  $\|f\|_\varepsilon$ , contradicting the definition of  $\|f\|_\varepsilon$  as an infimum over such bounds.

– Then

$$\begin{aligned}
 \|f\|_p^p &= \int_X |f(x)|^p dx \\
 &\geq \int_{S_\varepsilon} |f(x)|^p dx \quad \text{since } S_\varepsilon \subseteq X \\
 &\geq \int_{S_\varepsilon} (\|f\|_\infty - \varepsilon)^p dx \quad \text{since on } S_\varepsilon, |f| \geq \|f\|_\infty - \varepsilon \\
 &= (\|f\|_\infty - \varepsilon)^p \cdot m(S_\varepsilon) \quad \text{since the integrand is independent of } x \\
 &\geq 0 \quad \text{since } m(S_\varepsilon) > 0
 \end{aligned}$$

– Taking  $p$ th roots for  $p \geq 1$  preserves the inequality, so

$$\implies \|f\|_p \geq \|f\|_\infty - \varepsilon \cdot m(S_\varepsilon)^{\frac{1}{p}} \xrightarrow{p \rightarrow \infty} \|f\|_\infty - \varepsilon \xrightarrow{\varepsilon \rightarrow 0} \|f\|_\infty$$

where we've used the fact that above arguments work

– Thus  $\|f\|_p \geq \|f\|_\infty$ .

## 6.2 Spring 2018 # 4

Let  $f \in L^2([0, 1])$  and suppose

$$\int_{[0,1]} f(x) x^n dx = 0 \text{ for all integers } n \geq 0.$$

Show that  $f = 0$  almost everywhere.

### 6.3 Spring 2015 # 2

Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be continuous with period 1. Prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(n\alpha) = \int_0^1 f(t) dt \quad \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

Hint: show this first for the functions  $f(t) = e^{2\pi i k t}$  for  $k \in \mathbb{Z}$ .

*Solution.*

#### 6.3.1 Proof 1: Using Fourier Transforms

Concepts used:

- Weierstrass Approximation: A uniformly continuous function on a compact set can be uniformly approximated by polynomials.

**Solution:**

- Fix  $k \in \mathbb{Z}$ .
- Since  $e^{2\pi i k x}$  is continuous on the compact interval  $[0, 1]$ , it is uniformly continuous.
- Thus there is a sequence of polynomials  $P_\ell$  such that

$$P_{\ell,k} \xrightarrow{\ell \rightarrow \infty} e^{2\pi i k x} \text{ uniformly on } [0, 1].$$

- Note applying linearity to the assumption  $\int f(x) x^n$ , we have

$$\int f(x) x^n dx = 0 \quad \forall n \implies \int f(x) p(x) dx = 0$$

for any polynomial  $p(x)$ , and in particular for  $P_{\ell,k}(x)$  for every  $\ell$  and every  $k$ .

- But then

$$\begin{aligned} \langle f, e_k \rangle &= \int_0^1 f(x) e^{-2\pi i k x} dx \\ &= \int_0^1 f(x) \lim_{\ell \rightarrow \infty} P_\ell(x) \\ &= \lim_{\ell \rightarrow \infty} \int_0^1 f(x) P_\ell(x) \quad \text{by uniform convergence on a compact interval} \\ &= \lim_{\ell \rightarrow \infty} 0 \quad \text{by assumption} \\ &= 0 \quad \forall k \in \mathbb{Z}, \end{aligned}$$

so  $f$  is orthogonal to every  $e_k$ .

- Thus  $f \in S^\perp := \text{span}_{\mathbb{C}} \{e_k\}_{k \in \mathbb{Z}}^\perp \subseteq L^2([0, 1])$ , but since this is a basis,  $S$  is dense and thus  $S^\perp = \{0\}$  in  $L^2([0, 1])$ .
- Thus  $f \equiv 0$  in  $L^2([0, 1])$ , which implies that  $f$  is zero almost everywhere. ■

### 6.3.2 Alternative Proof

Concepts used

- $C^1([0, 1])$  is dense in  $L^2([0, 1])$
- Polynomials are dense in  $L^p(X, \mathcal{M}, \mu)$  for any  $X \subseteq \mathbb{R}^n$  compact and  $\mu$  a finite measure, for all  $1 \leq p < \infty$ .
  - Use Weierstrass Approximation, then uniform convergence implies  $L^p(\mu)$  convergence by DCT.

**Solution:**

- By density of polynomials, for  $f \in L^2([0, 1])$  choose  $p_\varepsilon(x)$  such that  $\|f - p_\varepsilon\| < \varepsilon$  by Weierstrass approximation.
- Then on one hand,

$$\begin{aligned} \|f(f - p_\varepsilon)\|_1 &= \|f^2\|_1 - \|f \cdot p_\varepsilon\|_1 \\ &= \|f^2\|_1 - 0 \quad \text{by assumption} \\ &= \|f\|_2^2. \end{aligned}$$

- Where we've used that  $\|f^2\|_1 = \int |f^2| = \int |f|^2 = \|f\|_2^2$ .
- On the other hand

$$\begin{aligned} \|f(f - p_\varepsilon)\| &\leq \|f\|_1 \|f - p_\varepsilon\|_\infty \quad \text{by Holder} \\ &\leq \varepsilon \|f\|_1 \\ &\leq \varepsilon \|f\|_2 \sqrt{m(X)} \\ &= \varepsilon \|f\|_2 \quad \text{since } m(X) = 1. \end{aligned}$$

- Where we've used that  $\|fg\|_1 = \int |fg| = \int |f||g| \leq \int \|f\|_\infty |g| = \|f\|_\infty \|g\|_1$ .
- Combining these,

$$\|f\|_2^2 \leq \|f\|_2 \varepsilon \implies \|f\|_2 < \varepsilon \longrightarrow 0, .$$

so  $\|f\|_2 = 0$ , which implies  $f = 0$  almost everywhere.

## 6.4 Fall 2014 # 4

Let  $g \in L^\infty([0, 1])$  Prove that

$$\int_{[0,1]} f(x)g(x) dx = 0 \quad \text{for all continuous } f : [0, 1] \longrightarrow \mathbb{R} \implies g(x) = 0 \text{ almost everywhere.}$$

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## 7 $L^1$

### 7.1 Spring 2020 # 3

a. Prove that if  $g \in L^1(\mathbb{R})$  then

$$\lim_{N \rightarrow \infty} \int_{|x| \geq N} |f(x)| dx = 0,$$

and demonstrate that it is not necessarily the case that  $f(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ .

b. Prove that if  $f \in L^1([1, \infty))$  and is decreasing, then  $\lim_{x \rightarrow \infty} f(x) = 0$  and in fact  $\lim_{x \rightarrow \infty} xf(x) = 0$ .

c. If  $f : [1, \infty) \rightarrow [0, \infty)$  is decreasing with  $\lim_{x \rightarrow \infty} xf(x) = 0$ , does this ensure that  $f \in L^1([1, \infty))$ ?

*Solution.*

Concepts used:

- Limits
- Cauchy Criterion for Integrals:  $\int_a^\infty f(x) dx$  converges iff for every  $\varepsilon > 0$  there exists an  $M_0$  such that  $A, B \geq M_0$  implies  $\left| \int_A^B f \right| < \varepsilon$ , i.e.  $\left| \int_A^B f \right| \xrightarrow{A \rightarrow \infty} 0$ .
- Integrals of  $L^1$  functions have vanishing tails:  $\int_N^\infty |f| \xrightarrow{N \rightarrow \infty} 0$ .
- Mean Value Theorem for Integrals:  $\int_a^b f(t) dt = (b - a)f(c)$  for some  $c \in [a, b]$ .

**Solution:**

#### 7.1.1 a

Stated integral equality:

- Let  $\varepsilon > 0$
- $C_c(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$  is dense so choose  $\{f_n\} \rightarrow f$  with  $\|f_n - f\|_1 \rightarrow 0$ .
- Since  $\{f_n\}$  are compactly supported, choose  $N_0 \gg 1$  such that  $f_n$  is zero outside of  $B_{N_0}(\mathbf{0})$ .

- Then

$$\begin{aligned}
 N \geq N_0 &\implies \int_{|x|>N} |f| = \int_{|x|>N} |f - f_n + f_n| \\
 &\leq \int_{|x|>N} |f - f_n| + \int_{|x|>N} |f_n| \\
 &= \int_{|x|>N} |f - f_n| \\
 &\leq \int_{|x|>N} \|f - f_n\|_1 \\
 &= \|f_n - f\|_1 \left( \int_{|x|>N} 1 \right) \\
 &\stackrel{n \rightarrow \infty}{\longrightarrow} 0 \left( \int_{|x|>N} 1 \right) \\
 &= 0 \\
 &\stackrel{N \rightarrow \infty}{\longrightarrow} 0.
 \end{aligned}$$

To see that this doesn't force  $f(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ :

- Take  $f(x)$  to be a train of rectangles of height 1 and area  $1/2^j$  centered on even integers.
- Then

$$\int_{|x|>N} |f| = \sum_{j=N}^{\infty} 1/2^j \stackrel{N \rightarrow \infty}{\longrightarrow} 0$$

as the tail of a convergent sum.

- However  $f(x) = 1$  for infinitely many even integers  $x > N$ , so  $f(x) \not\rightarrow 0$  as  $|x| \rightarrow \infty$ .

### 7.1.2 b

#### Solution 1 (“Trick”)

- Since  $f$  is decreasing on  $[1, \infty)$ , for any  $t \in [x - n, x]$  we have

$$x - n \leq t \leq x \implies f(x) \leq f(t) \leq f(x - n).$$

- Integrate over  $[x, 2x]$ , using monotonicity of the integral:

$$\begin{aligned}
 \int_x^{2x} f(x) dt &\leq \int_x^{2x} f(t) dt \leq \int_x^{2x} f(x - n) dt \\
 \implies f(x) \int_x^{2x} dt &\leq \int_x^{2x} f(t) dt \leq f(x - n) \int_x^{2x} dt \\
 \implies xf(x) &\leq \int_x^{2x} f(t) dt \leq xf(x - n).
 \end{aligned}$$

- By the Cauchy Criterion for integrals,  $\lim_{x \rightarrow \infty} \int_x^{2x} f(t) dt = 0$ .
- So the LHS term  $xf(x) \stackrel{x \rightarrow \infty}{\longrightarrow} 0$ .
- Since  $x > 1$ ,  $|f(x)| \leq |xf(x)|$
- Thus  $f(x) \stackrel{x \rightarrow \infty}{\longrightarrow} 0$  as well.

**Solution 2 (Variation on the Trick)**

- Use mean value theorem for integrals:

$$\int_x^{2x} f(t) dt = xf(c_x) \quad \text{for some } c_x \in [x, 2x] \text{ depending on } x.$$

- Since  $f$  is decreasing,

$$\begin{aligned} x \leq c_x \leq 2x &\implies f(2x) \leq f(c_x) \leq f(x) \\ &\implies 2xf(2x) \leq 2xf(c_x) \leq 2xf(x) \\ &\implies 2xf(2x) \leq 2x \int_x^{2x} f(t) dt \leq 2xf(x) \end{aligned}$$

- By Cauchy Criterion,  $\int_x^{2x} f \rightarrow 0$ .
- So  $2xf(2x) \rightarrow 0$ , which by a change of variables gives  $uf(u) \rightarrow 0$ .
- Since  $u \geq 1$ ,  $f(u) \leq uf(u)$  so  $f(u) \rightarrow 0$  as well.

**Solution 3 (Contradiction)**

Just showing  $f(x) \xrightarrow{x \rightarrow \infty} 0$ :

- Toward a contradiction, suppose not.
- Since  $f$  is decreasing, it can not diverge to  $+\infty$
- If  $f(x) \rightarrow -\infty$ , then  $f \notin L^1(\mathbb{R})$ : choose  $x_0 \gg 1$  so that  $t \geq x_0 \implies f(t) < -1$ , then
- Then  $t \geq x_0 \implies |f(t)| \geq 1$ , so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f(t)| dt \geq \int_{x_0}^\infty 1 = \infty.$$

- Otherwise  $f(x) \rightarrow L \neq 0$ , some finite limit.
- If  $L > 0$ :
  - Fix  $\varepsilon > 0$ , choose  $x_0 \gg 1$  such that  $t \geq x_0 \implies L - \varepsilon \leq f(t) \leq L$
  - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L - \varepsilon) dt = \infty$$

- If  $L < 0$ :
  - Fix  $\varepsilon > 0$ , choose  $x_0 \gg 1$  such that  $t \geq x_0 \implies L \leq f(t) \leq L + \varepsilon$ .
  - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L) dt = \infty$$

Showing  $xf(x) \xrightarrow{x \rightarrow \infty} 0$ .

- Toward a contradiction, suppose not.
- (How to show that  $xf(x) \not\rightarrow +\infty$ ?)
- If  $xf(x) \rightarrow -\infty$ 
  - Choose a sequence  $\Gamma = \{\hat{x}_i\}$  such that  $x_i \rightarrow \infty$  and  $x_i f(x_i) \rightarrow -\infty$ .
  - Choose a subsequence  $\Gamma' = \{x_i\}$  such that  $x_i f(x_i) \leq -1$  for all  $i$  and  $x_i \leq x_{i+1}$ .

- Choose a further subsequence  $S = \{x_i \in \Gamma' \mid 2x_i < x_{i+1}\}$ .
- Then since  $f$  is always decreasing, for  $t \geq x_0$ ,  $|f|$  is increasing, and  $|f(x_i)| \leq |f(2x_i)|$ , so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f| \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(t)| dt \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(x_i)| = \sum_{x_i \in S} x_i f(x_i) \longrightarrow \infty.$$

- If  $xf(x) \longrightarrow L \neq 0$  for  $0 < L < \infty$ :
  - Fix  $\varepsilon > 0$ , choose an infinite sequence  $\{x_i\}$  such that  $L - \varepsilon \leq x_i f(x_i) \leq L$  for all  $i$ .

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)| dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i) dt = \sum_S x_i f(x_i) \geq \sum_S (L - \varepsilon) \longrightarrow \infty.$$

- If  $xf(x) \longrightarrow L \neq 0$  for  $-\infty < L < 0$ :
  - Fix  $\varepsilon > 0$ , choose an infinite sequence  $\{x_i\}$  such that  $L \leq x_i f(x_i) \leq L + \varepsilon$  for all  $i$ .

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)| dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i) dt = \sum_S x_i f(x_i) \geq \sum_S (L) \longrightarrow \infty.$$

**Solution 4 (Akos's Suggestion)** For  $x \geq 1$ ,

$$|xf(x)| = \left| \int_x^{2x} f(x) dt \right| \leq \int_x^{2x} |f(x)| dt \leq \int_x^{2x} |f(t)| dt \leq \int_x^\infty |f(t)| dt \xrightarrow{x \rightarrow \infty} 0$$

where we've used

- Since  $f$  is decreasing and  $\lim_{x \rightarrow \infty} f(x) = 0$  from part (a),  $f$  is non-negative.
- Since  $f$  is positive and decreasing, for every  $t \in [a, b]$  we have  $|f(a)| \leq |f(t)|$ .
- By part (a), the last integral goes to zero.

**Solution 5 (Peter's)**

- Toward a contradiction, produce a sequence  $x_i \longrightarrow \infty$  with  $x_i f(x_i) \longrightarrow \infty$  and  $x_i f(x_i) > \varepsilon > 0$ , then

$$\begin{aligned} \int f(x) dx &\geq \sum_{i=1}^{\infty} \int_{x_i}^{x_{i+1}} f(x) dx \\ &\geq \sum_{i=1}^{\infty} \int_{x_i}^{x_{i+1}} f(x_{i+1}) dx \\ &= \sum_{i=1}^{\infty} f(x_{i+1}) \int_{x_i}^{x_{i+1}} dx \\ &\geq \sum_{i=1}^{\infty} (x_{i+1} - x_i) f(x_{i+1}) \\ &\geq \sum_{i=1}^{\infty} (x_{i+1} - x_i) \frac{\varepsilon}{x_{i+1}} \\ &= \varepsilon \sum_{i=1}^{\infty} \left(1 - \frac{x_{i-1}}{x_i}\right) \longrightarrow \infty \end{aligned}$$

which can be ensured by passing to a subsequence where  $\sum \frac{x_{i-1}}{x_i} < \infty$ .

**7.1.3 c**

- No: take  $f(x) = \frac{1}{x \ln x}$
- Then by a  $u$ -substitution,

$$\int_0^x f = \ln(\ln(x)) \xrightarrow{x \rightarrow \infty} \infty$$

is unbounded, so  $f \notin L^1([1, \infty))$ .

- But

$$xf(x) = \frac{1}{\ln(x)} \xrightarrow{x \rightarrow \infty} 0.$$

**7.2 Fall 2019 # 5.****7.2.1 a**

Show that if  $f$  is continuous with compact support on  $\mathbb{R}$ , then

$$\lim_{y \rightarrow 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dx = 0$$

**7.2.2 b**

Let  $f \in L^1(\mathbb{R})$  and for each  $h > 0$  let

$$\mathcal{A}_h f(x) := \frac{1}{2h} \int_{|y| \leq h} f(x-y) dy$$

- Prove that  $\|\mathcal{A}_h f\|_1 \leq \|f\|_1$  for all  $h > 0$ .
- Prove that  $\mathcal{A}_h f \rightarrow f$  in  $L^1(\mathbb{R})$  as  $h \rightarrow 0^+$ .

Fix up.

*Solution.*

Concepts used:

- Continuity in  $L^1$  (recall that DCT won't work! Notes 19.4, prove it for a dense subset first).
- Lebesgue differentiation in 1-dimensional case. See HW 5.6.

**Solution:**

**7.3 a**

Choose  $g \in C_c^0$  such that  $\|f - g\|_1 \rightarrow 0$ .

By translation invariance,  $\|\tau_h f - \tau_h g\|_1 \rightarrow 0$ .



Write

$$\begin{aligned}\|\tau f - f\|_1 &= \|\tau_h f - g + g - \tau_h g + \tau_h g - f\|_1 \\ &\leq \|\tau_h f - \tau_h g\| + \|g - f\| + \|\tau_h g - g\| \\ &\longrightarrow \|\tau_h g - g\|,\end{aligned}$$

so it suffices to show that  $\|\tau_h g - g\| \longrightarrow 0$  for  $g \in C_c^0$ .

Fix  $\varepsilon > 0$ . Enlarge the support of  $g$  to  $K$  such that

$$|h| \leq 1 \text{ and } x \in K^c \implies |g(x-h) - g(x)| = 0.$$

By uniform continuity of  $g$ , pick  $\delta \leq 1$  small enough such that

$$x \in K, |h| \leq \delta \implies |g(x-h) - g(x)| < \varepsilon,$$

then

$$\int_K |g(x-h) - g(x)| \leq \int_K \varepsilon = \varepsilon \cdot m(K) \longrightarrow 0.$$

#### 7.4 b

We have

$$\begin{aligned}\int_{\mathbb{R}} |A_h(f)(x)| \, dx &= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y) \, dy \right| \, dx \\ &\leq \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)| \, dy \, dx \\ &=_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)| \, \mathbf{d}\mathbf{x} \, \mathbf{d}y \\ &= \int_{\mathbb{R}} |f(y)| \, dy \\ &= \|f\|_1.\end{aligned}$$

and (rough sketch)

$$\begin{aligned}\int_{\mathbb{R}} |A_h(f)(x) - f(x)| \, dx &= \int_{\mathbb{R}} \left| \left( \frac{1}{2h} \int_{B(h,x)} f(y) \, dy \right) - f(x) \right| \, dx \\ &= \int_{\mathbb{R}} \left| \left( \frac{1}{2h} \int_{B(h,x)} f(y) \, dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x) \, dy \right| \, dx \\ &\leq_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{B(h,x)} |f(y-x) - f(x)| \, \mathbf{d}\mathbf{x} \, \mathbf{d}y \\ &\leq \frac{1}{2h} \int_{\mathbb{R}} \|\tau_x f - f\|_1 \, dy \\ &\longrightarrow 0 \quad \text{by (a).}\end{aligned}$$

## 7.5 Fall 2017 # 3

Let

$$S = \text{span}_{\mathbb{C}} \left\{ \chi_{(a,b)} \mid a, b \in \mathbb{R} \right\},$$

the complex linear span of characteristic functions of intervals of the form  $(a, b)$ .

Show that for every  $f \in L^1(\mathbb{R})$ , there exists a sequence of functions  $\{f_n\} \subset S$  such that

$$\lim_{n \rightarrow \infty} \|f_n - f\|_1 = 0$$

*Solution.*

Concepts used:

- From homework:  $E$  is Lebesgue measurable iff there exists a finite union of closed cubes  $A$  such that  $m(E \Delta A) < \varepsilon$ .

**Solution:**

It suffices to show that  $S$  is dense in simple functions, and since simple functions are *finite* linear combinations of characteristic functions, it suffices to show this for  $\chi_A$  for  $A$  a measurable set.

Let  $s = \chi_A$ . By regularity of the Lebesgue measure, choose an open set  $O \supseteq A$  such that  $m(O \setminus A) < \varepsilon$ .

$O$  is an open subset of  $\mathbb{R}$ , and thus  $O = \coprod_{j \in \mathbb{N}} I_j$  is a disjoint union of countably many open intervals.

Now choose  $N$  large enough such that  $m(O \Delta I_{N,n}) < \varepsilon = \frac{1}{n}$  where we define  $I_{N,n} := \coprod_{j=1}^N I_j$ .

Now define  $f_n = \chi_{I_{N,n}}$ , then

$$\|s - f_n\|_1 = \int |\chi_A - \chi_{I_{N,n}}| = m(A \Delta I_{N,n}) \xrightarrow{n \rightarrow \infty} 0.$$

Since any simple function is a finite linear combination of  $\chi_{A_i}$ , we can do this for each  $i$  to extend this result to all simple functions. But simple functions are dense in  $L^1$ , so  $S$  is dense in  $L^1$ .

## 7.6 Spring 2015 # 4

Define

$$f(x, y) := \begin{cases} \frac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Carefully show that  $f \in L^1(\mathbb{R}^2)$ .

## 7.7 Fall 2014 # 3

Let  $f \in L^1(\mathbb{R})$ . Show that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } m(E) < \delta \implies \int_E |f(x)| dx < \varepsilon$$

**7.8 Spring 2014 # 1**

1. Give an example of a continuous  $f \in L^1(\mathbb{R})$  such that  $f(x) \not\rightarrow 0$  as  $|x| \rightarrow \infty$ .
2. Show that if  $f$  is *uniformly* continuous, then

$$\lim_{|x| \rightarrow \infty} f(x) = 0.$$

**8 Fubini-Tonelli****8.1 Spring 2020 # 4**

Let  $f, g \in L^1(\mathbb{R})$ . Argue that  $H(x, y) := f(y)g(x - y)$  defines a function in  $L^1(\mathbb{R}^2)$  and deduce from this fact that

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x - y) dy$$

defines a function in  $L^1(\mathbb{R})$  that satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

*Solution.*

Relevant concepts:

- Tonelli: non-negative and measurable yields measurability of slices and equality of iterated integrals
- Fubini:  $f(x, y) \in L^1$  yields *integrable* slices and equality of iterated integrals
- F/T: apply Tonelli to  $|f|$ ; if finite,  $f \in L^1$  and apply Fubini to  $f$

$$\begin{aligned}
\|H(x)\|_1 &= \int_{\mathbb{R}} |H(x, y)| dx \\
&= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y)g(x-y) dy \right| dx \\
&\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)g(x-y)| dy \right) dx \\
&= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)g(x-y)| dx \right) dy \quad \text{by Tonelli} \\
&= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)g(t)| dt \right) dy \quad \text{setting } t = x - y, dt = -dx \\
&= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)| \cdot |g(t)| dt \right) dy \\
&= \int_{\mathbb{R}} |f(y)| \cdot \left( \int_{\mathbb{R}} |g(t)| dt \right) dy \\
&:= \int_{\mathbb{R}} |f(y)| \cdot \|g\|_1 dy \\
&= \|g\|_1 \int_{\mathbb{R}} |f(y)| dy \\
&:= \|g\|_1 \|f\|_1 \\
&< \infty \quad \text{by assumption} .
\end{aligned}$$

- $H$  is measurable on  $\mathbb{R}^2$ :
  - If we can show  $\tilde{f}(x, y) := f(y)$  and  $\tilde{g}(x, y) := g(x - y)$  are both measurable on  $\mathbb{R}^2$ , then  $H = \tilde{f} \cdot \tilde{g}$  is a product of measurable functions and thus measurable.
  - $f \in L^1$ , and  $L^1$  functions are measurable by definition.
  - The function  $(x, y) \mapsto g(x - y)$  is measurable on  $\mathbb{R}^2$ :
    - \* Let  $g$  be measurable on  $\mathbb{R}$ , then the cylinder function  $G(x, y) = g(x)$  on  $\mathbb{R}^2$  is always measurable
    - \* Define a linear transformation  $T := [1, -1; 0, 1]$  which sends  $(x, y) \rightarrow (x - y, y)$ , then  $T \in \text{GL}(2, \mathbb{R})$  is linear and thus measurable.
    - \* Then  $(G \circ T)(x, y) = G(x - y, y) = \tilde{g}(x - y)$ , so  $\tilde{g}$  is a composition of measurable functions and thus measurable.
- Apply **Tonelli** to  $|H|$ 
  - $H$  measurable implies  $|H|$  is measurable
  - $|H|$  is non-negative
  - So the iterated integrals are equal in the extended sense
  - The calculation shows the iterated integral is finite, to  $\int |H|$  is finite and  $H$  is thus integrable on  $\mathbb{R}^2$ .

Note: Fubini is not needed, since we're not calculating the actual integral, just showing  $H$  is integrable.

## 8.2 Spring 2019 # 4

Let  $f$  be a non-negative function on  $\mathbb{R}^n$  and  $\mathcal{A} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq f(x)\}$ .

Prove the validity of the following two statements:

- a.  $f$  is a Lebesgue measurable function on  $\mathbb{R}^n \iff \mathcal{A}$  is a Lebesgue measurable subset of  $\mathbb{R}^{n+1}$
- b. If  $f$  is a Lebesgue measurable function on  $\mathbb{R}^n$ , then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x) dx = \int_0^\infty m(\{x \in \mathbb{R}^n : f(x) \geq t\}) dt$$

*Solution.*

See S&S p.82.

### 8.2.1 a

$\implies$  :

- Suppose  $f$  is a measurable function.
- Note that  $\mathcal{A} = \{f(x) - t \geq 0\} \cap \{t \geq 0\}$ .
- Define  $F(x, t) = f(x)$ ,  $G(x, t) = t$ , which are cylinders on measurable functions and thus measurable.
- Define  $H(x, y) = F(x, t) - G(x, t)$ , which are linear combinations of measurable functions and thus measurable.
- Then  $\mathcal{A} = \{H \geq 0\} \cap \{G \geq 0\}$  as a countable intersection of measurable sets, which is again measurable.

$\impliedby$  :

- Suppose  $\mathcal{A}$  is a measurable set.
- Then FT on  $\chi_{\mathcal{A}}$  implies that for almost every  $x \in \mathbb{R}^n$ , the  $x$ -slices  $\mathcal{A}_x$  are measurable and

$$\mathcal{A}_x := \{t \in \mathbb{R} \mid (x, t) \in \mathcal{A}\} = [0, f(x)] \implies m(\mathcal{A}_x) = f(x) - 0 = f(x)$$

- But  $x \mapsto m(\mathcal{A}_x)$  is a measurable function, and is exactly the function  $x \mapsto f(x)$ , so  $f$  is measurable.

### 8.2.2 b

- Note

$$\begin{aligned} \mathcal{A} &= \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid 0 \leq t \leq f(x)\} \\ \mathcal{A}_t &= \{x \in \mathbb{R}^n \mid t \leq f(x)\}. \end{aligned}$$

- Then

$$\begin{aligned} \int_{\mathbb{R}^n} f(x) dx &= \int_{\mathbb{R}^n} \int_0^{f(x)} 1 dt dx \\ &= \int_{\mathbb{R}^n} \int_0^\infty \chi_{\mathcal{A}} dt dx \\ &\stackrel{F.T.}{=} \int_0^\infty \int_{\mathbb{R}^n} \chi_{\mathcal{A}} dx dt \\ &= \int_0^\infty m(\mathcal{A}_t) dt, \end{aligned}$$

where we just use that  $\int \int \chi_A = m(A)$

- By F.T., all of these integrals are equal.

Why is FT justified.

### 8.3 Fall 2018 # 5

Let  $f \geq 0$  be a measurable function on  $\mathbb{R}$ . Show that

$$\int_{\mathbb{R}} f = \int_0^\infty m(\{x : f(x) > t\}) dt$$

*Solution.*

Concepts used:

- Claim: If  $E \subseteq \mathbb{R}^a \times \mathbb{R}^b$  is a measurable set, then for almost every  $y \in \mathbb{R}^b$ , the slice  $E^y$  is measurable and

$$m(E) = \int_{\mathbb{R}^b} m(E^y) dy.$$

- Set  $g = \chi_E$ , which is non-negative and measurable, so apply Tonelli.
- Conclude that  $g^y = \chi_{E^y}$  is measurable, the function  $y \mapsto \int g^y(x) dx$  is measurable, and  $\int \int g^y(x) dx dy = \int g$ .
- But  $\int g = m(E)$  and  $\int \int g^y(x) dx dy = \int m(E^y) dy$ .

**Solution**

Note:  $f$  is a function  $\mathbb{R} \rightarrow \mathbb{R}$  in the original problem, but here I've assumed  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ .

- Since  $f \geq 0$ , set

$$E := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) > t\} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid 0 \leq t < f(x)\}.$$

- Claim: since  $f$  is measurable,  $E$  is measurable and thus  $m(E)$  makes sense.
  - Since  $f$  is measurable,  $F(x, t) := t - f(x)$  is measurable on  $\mathbb{R}^n \times \mathbb{R}$ .
  - Then write  $E = \{F < 0\} \cap \{t \geq 0\}$  as an intersection of measurable sets.
- We have slices

$$E^t := \{x \in \mathbb{R}^n \mid (x, t) \in E\} = \{x \in \mathbb{R}^n \mid 0 \leq t < f(x)\}$$

$$E^x := \{t \in \mathbb{R} \mid (x, t) \in E\} = \{t \in \mathbb{R} \mid 0 \leq t < f(x)\} = [0, f(x)).$$

- $E_t$  is precisely the set that appears in the original RHS integrand.
- $m(E^x) = f(x)$ .
- Claim:  $\chi_E$  satisfies the conditions of Tonelli, and thus  $m(E) = \int \chi_E$  is equal to any iterated integral.
  - Non-negative: clear since  $0 \leq \chi_E \leq 1$
  - Measurable: characteristic functions of measurable sets are measurable.
- Conclude:

1. For almost every  $x$ ,  $E^x$  is a measurable set,  $x \mapsto m(E^x)$  is a measurable function, and  $m(E) = \int_{\mathbb{R}^n} m(E^x) dx$
  2. For almost every  $t$ ,  $E^t$  is a measurable set,  $t \mapsto m(E^t)$  is a measurable function, and  $m(E) = \int_{\mathbb{R}} m(E^t) dt$
- On one hand,

$$\begin{aligned}
 m(E) &= \int_{\mathbb{R}^{n+1}} \chi_E(x, t) \\
 &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_E(x, t) dt dx \quad \text{by Tonelli} \\
 &= \int_{\mathbb{R}^n} m(E^x) dx \quad \text{first conclusion} \\
 &= \int_{\mathbb{R}^n} f(x) dx.
 \end{aligned}$$

- On the other hand,

$$\begin{aligned}
 m(E) &= \int_{\mathbb{R}^{n+1}} \chi_E(x, t) \\
 &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_E(x, t) dx dt \quad \text{by Tonelli} \\
 &= \int_{\mathbb{R}} m(E^t) dt \quad \text{second conclusion.}
 \end{aligned}$$

- Thus

$$\int_{\mathbb{R}^n} f dx = m(E) = \int_{\mathbb{R}} m(E^t) dt = \int_{\mathbb{R}} m(\{x \mid f(x) > t\}).$$

## 8.4 Fall 2015 # 5

Let  $f, g \in L^1(\mathbb{R})$  be Borel measurable.

1. Show that
  - The function

$$F(x, y) := f(x - y)g(y)$$

is Borel measurable on  $\mathbb{R}^2$ , and

- For almost every  $y \in \mathbb{R}$ ,

$$F_y(x) := f(x - y)g(y)$$

is integrable with respect to  $y$ .

2. Show that  $f * g \in L^1(\mathbb{R})$  and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

**8.5 Spring 2014 # 5**

Let  $f, g \in L^1([0, 1])$  and for all  $x \in [0, 1]$  define

$$F(x) := \int_0^x f(y) dy \quad \text{and} \quad G(x) := \int_0^x g(y) dy.$$

Prove that

$$\int_0^1 F(x)g(x) dx = F(1)G(1) - \int_0^1 f(x)G(x) dx$$

**9  $L^2$  and Fourier Analysis****9.1 Spring 2020 # 6****9.1.1 a**

Show that

$$L^2([0, 1]) \subseteq L^1([0, 1]) \quad \text{and} \quad \ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z}).$$

**9.1.2 b**

For  $f \in L^1([0, 1])$  define

$$\hat{f}(n) := \int_0^1 f(x)e^{-2\pi i n x} dx.$$

Prove that if  $f \in L^1([0, 1])$  and  $\{\hat{f}(n)\} \in \ell^1(\mathbb{Z})$  then

$$S_N f(x) := \sum_{|n| \leq N} \hat{f}(n)e^{2\pi i n x}.$$

converges uniformly on  $[0, 1]$  to a continuous function  $g$  such that  $g = f$  almost everywhere.

Hint: One approach is to argue that if  $f \in L^1([0, 1])$  with  $\{\hat{f}(n)\} \in \ell^1(\mathbb{Z})$  then  $f \in L^2([0, 1])$ .

*Solution.*

Concepts used:

- For  $e_n(x) := e^{2\pi i n x}$ , the set  $\{e_n\}$  is an orthonormal basis for  $L^2([0, 1])$ .
- For any orthonormal sequence in a Hilbert space, we have Bessel's inequality:

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

- When  $\{e_n\}$  is a basis, the above is an *equality* (Parseval)
- Arguing uniform convergence: since  $\{\hat{f}(n)\} \in \ell^1(\mathbb{Z})$ , we should be able to apply the  $M$  test.



**9.1.3 a**

Claim:  $\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$ .

- Set  $\mathbf{c} = \{c_k \mid k \in \mathbb{Z}\} \in \ell^1(\mathbb{Z})$ .
- It suffices to show that if  $\sum_{k \in \mathbb{Z}} |c_k| < \infty$  then  $\sum_{k \in \mathbb{Z}} |c_k|^2 < \infty$ .
- Let  $S = \{c_k \mid |c_k| \leq 1\}$ , then  $c_k \in S \implies |c_k|^2 \leq |c_k|$
- Claim:  $S^c$  can only contain finitely many elements, all of which are finite.
  - If not, either  $S^c := \{c_j\}_{j=1}^\infty$  is infinite with every  $|c_j| > 1$ , which forces

$$\sum_{c_k \in S^c} |c_k| = \sum_{j=1}^\infty |c_j| > \sum_{j=1}^\infty 1 = \infty.$$

- If any  $c_j = \infty$ , then  $\sum_{k \in \mathbb{Z}} |c_k| \geq c_j = \infty$ .

- So  $S^c$  is a finite set of finite integers, let  $N = \max\{|c_j|^2 \mid c_j \in S^c\} < \infty$ .
- Rewrite the sum

$$\begin{aligned} \sum_{k \in \mathbb{Z}} |c_k|^2 &= \sum_{c_k \in S} |c_k|^2 + \sum_{c_k \in S^c} |c_k|^2 \\ &\leq \sum_{c_k \in S} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \\ &\leq \sum_{k \in \mathbb{Z}} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \quad \text{since the } |c_k| \text{ are all positive} \\ &= \|\mathbf{c}\|_{\ell^1} + \sum_{c_k \in S^c} |c_k|^2 \\ &\leq \|\mathbf{c}\|_{\ell^1} + |S^c| \cdot N \\ &< \infty. \end{aligned}$$

Claim:  $L^2([0, 1]) \subseteq L^1([0, 1])$ .

- It suffices to show that  $\int |f|^2 < \infty \implies \int |f| < \infty$ .
- Define  $S = \{x \in [0, 1] \mid |f(x)| \leq 1\}$ , then  $x \in S^c \implies |f(x)|^2 \geq |f(x)|$ .
- Break up the integral:

$$\begin{aligned} \int_{\mathbb{R}} |f| &= \int_S |f| + \int_{S^c} |f| \\ &\leq \int_S |f| + \int_{S^c} |f|^2 \\ &\leq \int_S |f| + \|f\|_2 \\ &\leq \sup_{x \in S} \{|f(x)|\} \cdot \mu(S) + \|f\|_2 \\ &= 1 \cdot \mu(S) + \|f\|_2 \quad \text{by definition of } S \\ &\leq 1 \cdot \mu([0, 1]) + \|f\|_2 \quad \text{since } S \subseteq [0, 1] \\ &= 1 + \|f\|_2 \\ &< \infty. \end{aligned}$$

Note: this proof shows  $L^2(X) \subseteq L^1(X)$  whenever  $\mu(X) < \infty$ .

## 9.2 Fall 2017 # 5

Let  $\varphi$  be a compactly supported smooth function that vanishes outside of an interval  $[-N, N]$  such that  $\int_{\mathbb{R}} \varphi(x) dx = 1$ .

For  $f \in L^1(\mathbb{R})$ , define

$$K_j(x) := j\varphi(jx), \quad f * K_j(x) := \int_{\mathbb{R}} f(x-y)K_j(y) dy$$

and prove the following:

1. Each  $f * K_j$  is smooth and compactly supported.
- 2.

$$\lim_{j \rightarrow \infty} \|f * K_j - f\|_1 = 0$$

Hint:

$$\lim_{y \rightarrow 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dy = 0$$

*Solution.*

### 9.2.1 a

**Lemma:** If  $\varphi \in C_c^1$ , then  $(f * \varphi)' = f * \varphi'$  almost everywhere.

*Silly Proof:*

$$\begin{aligned} \mathcal{F}((f * \varphi)') &= 2\pi i \xi \mathcal{F}(f * \varphi) \\ &= 2\pi i \xi \mathcal{F}(f) \mathcal{F}(\varphi) \\ &= \mathcal{F}(f) \cdot (2\pi i \xi \mathcal{F}(\varphi)) \\ &= \mathcal{F}(f) \cdot \mathcal{F}(\varphi') \\ &= \mathcal{F}(f * \varphi'). \end{aligned}$$

*Actual proof:*

$$\begin{aligned}
(f * \varphi)'(x) &= (\varphi * f)'(x) \\
&= \lim_{h \rightarrow 0} \frac{(\varphi * f)'(x+h) - (\varphi * f)'(x)}{h} \\
&= \lim_{h \rightarrow 0} \int \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \\
&\stackrel{DCT}{=} \int \lim_{h \rightarrow 0} \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \\
&= \int \varphi'(x-y) f(y) \\
&= (\varphi' * f)(x) \\
&= (f * \varphi')(x).
\end{aligned}$$

To see that the DCT is justified, we can apply the MVT on the interval  $[0, h]$  to  $f$  to obtain

$$\frac{\varphi(x+h-y) - \varphi(x-y)}{h} = \varphi'(c) \quad c \in [0, h],$$

and since  $\varphi'$  is continuous and compactly supported,  $\varphi'$  is bounded by some  $M < \infty$  by the extreme value theorem and thus

$$\begin{aligned}
\int \left| \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \right| &= \int |\varphi'(c) f(y)| \\
&\leq \int |M| |f| \\
&= |M| \int |f| < \infty,
\end{aligned}$$

since  $f \in L^1$  by assumption, so we can take  $g := |M||f|$  as the dominating function.

Applying this theorem infinitely many times shows that  $f * \varphi$  is smooth.

To see that  $f * \varphi$  is compactly supported, approximate  $f$  by a *continuous* compactly supported function  $h$ , so  $\|h - f\|_1 \xrightarrow{L^1} 0$ .

Now let  $g_x(y) = \varphi(x-y)$ , and note that  $\text{supp}(g) = x - \text{supp}(\varphi)$  which is still compact.

But since  $\text{supp}(h)$  is bounded, there is some  $N$  such that

$$|x| > N \implies A_x := \text{supp}(h) \cap \text{supp}(g_x) = \emptyset$$

and thus

$$\begin{aligned}
(h * \varphi)(x) &= \int_{\mathbb{R}} \varphi(x-y) h(y) dy \\
&= \int_{A_x} g_x(y) h(y) \\
&= 0,
\end{aligned}$$

so  $\{x \mid f * g(x) = 0\}$  is open, and its complement is closed and bounded and thus compact.

**9.2.2 b**

$$\begin{aligned}
\|f * K_j - f\|_1 &= \int \left| \int f(x-y)K_j(y) dy - f(x) \right| dx \\
&= \int \left| \int f(x-y)K_j(y) dy - \int f(x)K_j(y) dy \right| dx \\
&= \int \left| \int (f(x-y) - f(x))K_j(y) dy \right| dx \\
&\leq \int \int |(f(x-y) - f(x))| \cdot |K_j(y)| dy dx \\
&\stackrel{FT}{=} \int \int |(f(x-y) - f(x))| \cdot |K_j(y)| \mathbf{dx} \mathbf{dy} \\
&= \int |K_j(y)| \left( \int |(f(x-y) - f(x))| dx \right) dy \\
&= \int |K_j(y)| \cdot \|f - \tau_y f\|_1 dy.
\end{aligned}$$

We now split the integral up into pieces.

1. Chose  $\delta$  small enough such that  $|y| < \delta \implies \|f - \tau_y f\|_1 < \varepsilon$  by continuity of translation in  $L^1$ , and
2. Since  $\varphi$  is compactly supported, choose  $J$  large enough such that

$$j > J \implies \int_{|y| \geq \delta} |K_j(y)| dy = \int_{|y| \geq \delta} |j\varphi(jy)| dy = 0$$

Then

$$\begin{aligned}
\|f * K_j - f\|_1 &\leq \int |K_j(y)| \cdot \|f - \tau_y f\|_1 dy \\
&= \int_{|y| < \delta} |K_j(y)| \cdot \|f - \tau_y f\|_1 dy + \int_{|y| \geq \delta} |K_j(y)| \cdot \|f - \tau_y f\|_1 dy \\
&= \varepsilon \int_{|y| \geq \delta} |K_j(y)| dy + 0 \\
&\leq \varepsilon(1) \longrightarrow 0.
\end{aligned}$$

**9.3 Spring 2017 # 5**

Let  $f, g \in L^2(\mathbb{R})$ . Prove that the formula

$$h(x) := \int_{-\infty}^{\infty} f(t)g(x-t) dt$$

defines a uniformly continuous function  $h$  on  $\mathbb{R}$ .

**9.4 Spring 2015 # 6**

Let  $f \in L^1(\mathbb{R})$  and  $g$  be a bounded measurable function on  $\mathbb{R}$ .

1. Show that the convolution  $f * g$  is well-defined, bounded, and uniformly continuous on  $\mathbb{R}$ .

2. Prove that one further assumes that  $g \in C^1(\mathbb{R})$  with bounded derivative, then  $f * g \in C^1(\mathbb{R})$  and

$$\frac{d}{dx}(f * g) = f * \left(\frac{d}{dx}g\right)$$

## 9.5 Fall 2014 # 5

1. Let  $f \in C_c^0(\mathbb{R}^n)$ , and show

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^n} |f(x+t) - f(x)| dx = 0.$$

2. Extend the above result to  $f \in L^1(\mathbb{R}^n)$  and show that

$$f \in L^1(\mathbb{R}^n), \quad g \in L^\infty(\mathbb{R}^n) \quad \implies \quad f * g \text{ is bounded and uniformly continuous.}$$

## 10 Functional Analysis: General

### 10.1 Fall 2019 # 4.

Let  $\{u_n\}_{n=1}^\infty$  be an orthonormal sequence in a Hilbert space  $\mathcal{H}$ .

#### 10.1.1 a

Prove that for every  $x \in \mathcal{H}$  one has

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

#### 10.1.2 b

Prove that for any sequence  $\{a_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$  there exists an element  $x \in \mathcal{H}$  such that

$$a_n = \langle x, u_n \rangle \text{ for all } n \in \mathbb{N}$$

and

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$$

*Solution.*

Concepts used:

- Bessel's Inequality
- Pythagoras
- Surjectivity of the Riesz map
- Parseval's Identity
- Trick – remember to write out finite sum  $S_N$ , and consider  $\|x - S_N\|$ .

**10.1.3 a****Claim:**

$$\begin{aligned}
0 \leq \left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\
&\implies \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2.
\end{aligned}$$

*Proof:* Let  $S_N = \sum_{n=1}^N \langle x, u_n \rangle u_n$ . Then

$$\begin{aligned}
0 &\leq \|x - S_N\|^2 \\
&= \langle x - S_N, x - S_N \rangle \\
&= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\
&\xrightarrow{N \rightarrow \infty} \|x\|^2 - \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2.
\end{aligned}$$

**10.1.4 b**

1. Fix  $\{a_n\} \in \ell^2$ , then note that  $\sum |a_n|^2 < \infty \implies$  the tails vanish.
2. Define

$$x := \lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \sum_{k=1}^N a_k u_k$$

3.  $\{S_N\}$  Cauchy (by 1) and  $H$  complete  $\implies x \in H$ .
- 4.

$$\langle x, u_n \rangle = \left\langle \sum_k a_k u_k, u_n \right\rangle = \sum_k a_k \langle u_k, u_n \rangle = a_n \quad \forall n \in \mathbb{N}$$

since the  $u_k$  are all orthogonal.

- 5.

$$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$$

by Pythagoras since the  $u_k$  are normal.

Bonus: We didn't use completeness here, so the Fourier series may not actually converge to  $x$ . If  $\{u_n\}$  is **complete** (so  $x = 0 \iff \langle x, u_n \rangle = 0 \forall n$ ) then the Fourier series *does* converge to  $x$  and  $\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 = \|x\|^2$  for all  $x \in H$ .

**10.2 Spring 2019 # 5****10.2.1 a**

Show that  $L^2([0, 1]) \subseteq L^1([0, 1])$  and argue that  $L^2([0, 1])$  in fact forms a dense subset of  $L^1([0, 1])$ .

**10.2.2 b**

Let  $\Lambda$  be a continuous linear functional on  $L^1([0, 1])$ .

Prove the Riesz Representation Theorem for  $L^1([0, 1])$  by following the steps below:

- i. Establish the existence of a function  $g \in L^2([0, 1])$  which represents  $\Lambda$  in the sense that

$$\Lambda(f) = \int_0^1 f(x)g(x)dx \text{ for all } f \in L^2([0, 1]).$$

Hint: You may use, without proof, the Riesz Representation Theorem for  $L^2([0, 1])$ .

- ii. Argue that the  $g$  obtained above must in fact belong to  $L^\infty([0, 1])$  and represent  $\Lambda$  in the sense that

$$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}dx \quad \text{for all } f \in L^1([0, 1])$$

with

$$\|g\|_{L^\infty([0,1])} = \|\Lambda\|_{L^1([0,1])^\vee}$$

*Solution.*

Concepts used:

- Holders' inequality:  $\|fg\|_1 \leq \|f\|_p \|g\|_q$
- Riesz Representation for  $L^2$ : If  $\Lambda \in (L^2)^\vee$  then there exists a unique  $g \in L^2$  such that  $\Lambda(f) = \int fg$ .
- $\|f\|_{L^\infty(X)} := \inf \left\{ t \geq 0 \mid |f(x)| \leq t \text{ almost everywhere} \right\}$ .
- **Lemma:**  $m(X) < \infty \implies L^p(X) \subset L^2(X)$ .

*Proof:* Write Holder's inequality as  $\|fg\|_1 \leq \|f\|_a \|g\|_b$  where  $\frac{1}{a} + \frac{1}{b} = 1$ , then

$$\|f\|_p^p = \| |f|^p \|_1 \leq \| |f|^p \|_a \|1\|_b.$$

Now take  $a = \frac{2}{p}$  and this reduces to

$$\begin{aligned} \|f\|_p^p &\leq \|f\|_2^p m(X)^{\frac{1}{p}} \\ \implies \|f\|_p &\leq \|f\|_2 \cdot O(m(X)) < \infty. \end{aligned}$$

**10.2.3 a**

- Note  $X = [0, 1] \implies m(X) = 1$ .

- By Holder's inequality with  $p = q = 2$ ,

$$\|f\|_1 = \|f \cdot 1\|_1 \leq \|f\|_2 \cdot \|1\|_2 = \|f\|_2 \cdot m(X)^{\frac{1}{2}} = \|f\|_2,$$

- Thus  $L^2(X) \subseteq L^1(X)$
- Since they share a common dense subset (simple functions)  $L^2$  is dense in  $L^1$

What theorem is this using?

### 10.2.4 b

Let  $\Lambda \in L^1(X)^\vee$  be arbitrary.

#### (i): Existence of $g$ Representing $\Lambda$ .

- Let  $f \in L^2 \subseteq L^1$  be arbitrary
- Claim:  $\Lambda \in L^1(X)^\vee \implies \Lambda \in L^2(X)^\vee$ .
  - Suffices to show that  $\|\Lambda\|_{L^2(X)^\vee} := \sup_{\|f\|_2=1} |\Lambda(f)| < \infty$ , since bounded implies continuous.
  - By the lemma,  $\|f\|_1 \leq C\|f\|_2$  for some constant  $C \approx m(X)$ .
  - Note

$$\|\Lambda\|_{L^1(X)^\vee} := \sup_{\|f\|_1=1} |\Lambda(f)|$$

- Define  $\hat{f} = \frac{f}{\|f\|_1}$  so  $\|\hat{f}\|_1 = 1$
- Since  $\|\Lambda\|_{1^\vee}$  is a supremum over all  $f \in L^1(X)$  with  $\|f\|_1 = 1$ ,

$$|\Lambda(\hat{f})| \leq \|\Lambda\|_{(L^1(X))^\vee},$$

- Then

$$\begin{aligned} \frac{|\Lambda(f)|}{\|f\|_1} &= |\Lambda(\hat{f})| \leq \|\Lambda\|_{L^1(X)^\vee} \\ \implies |\Lambda(f)| &\leq \|\Lambda\|_{1^\vee} \cdot \|f\|_1 \\ &\leq \|\Lambda\|_{1^\vee} \cdot C\|f\|_2 < \infty \quad \text{by assumption,} \end{aligned}$$

- So  $\Lambda \in (L^2)^\vee$ .
- Now apply Riesz Representation for  $L^2$ : there is a  $g \in L^2$  such that

$$f \in L^2 \implies \Lambda(f) = \langle f, g \rangle := \int_0^1 f(x) \overline{g(x)} dx.$$

#### (ii): $g$ is in $L^\infty$

- It suffices to show  $\|g\|_{L^\infty(X)} < \infty$ .
- Since we're assuming  $\|\Lambda\|_{L^1(X)^\vee} < \infty$ , it suffices to show the stated equality.

Is this assumed..? Or did we show it..?

- Claim:  $\|\Lambda\|_{L^1(X)^\vee} = \|g\|_{L^\infty(X)}$ 
  - The result follows because  $\Lambda$  was assumed to be in  $L^1(X)^\vee$ , so  $\|\Lambda\|_{L^1(X)^\vee} < \infty$ .



–  $\leq$ :

$$\begin{aligned}
 \|\Lambda\|_{L^1(X)^\vee} &= \sup_{\|f\|_1=1} |\Lambda(f)| \\
 &= \sup_{\|f\|_1=1} \left| \int_X f \bar{g} \right| \quad \text{by (i)} \\
 &= \sup_{\|f\|_1=1} \int_X |f \bar{g}| \\
 &:= \sup_{\|f\|_1=1} \|fg\|_1 \\
 &\leq \sup_{\|f\|_1=1} \|f\|_1 \|g\|_\infty \quad \text{by Holder with } p=1, q=\infty \\
 &= \|g\|_\infty,
 \end{aligned}$$

–  $\geq$ :

- \* Suppose toward a contradiction that  $\|g\|_\infty > \|\Lambda\|_{L^1(X)^\vee}$ .
- \* Then there exists some  $E \subseteq X$  with  $m(E) > 0$  such that

$$x \in E \implies |g(x)| > \|\Lambda\|_{L^1(X)^\vee}.$$

- \* Define

$$h = \frac{1}{m(E)} \frac{\bar{g}}{|g|} \chi_E.$$

- \* Note  $\|h\|_{L^1(X)} = 1$ .
- \* Then

$$\begin{aligned}
 \Lambda(h) &= \int_X hg \\
 &:= \int_X \frac{1}{m(E)} \frac{g\bar{g}}{|g|} \chi_E \\
 &= \frac{1}{m(E)} \int_E |g| \\
 &\geq \frac{1}{m(E)} \|g\|_\infty m(E) \\
 &= \|g\|_\infty \\
 &> \|\Lambda\|_{L^1(X)^\vee},
 \end{aligned}$$

a contradiction since  $\|\Lambda\|_{L^1(X)^\vee}$  is the supremum over all  $h_\alpha$  with  $\|h_\alpha\|_{L^1(X)} = 1$ .

### 10.3 Spring 2016 # 6

Without using the Riesz Representation Theorem, compute

$$\sup \left\{ \left| \int_0^1 f(x) e^x dx \right| \mid f \in L^2([0, 1], m), \|f\|_2 \leq 1 \right\}$$

**10.4 Spring 2015 # 5**

Let  $\mathcal{H}$  be a Hilbert space.

1. Let  $x \in \mathcal{H}$  and  $\{u_n\}_{n=1}^N$  be an orthonormal set. Prove that the best approximation to  $x$  in  $\mathcal{H}$  by an element in  $\text{span}_{\mathbb{C}} \{u_n\}$  is given by

$$\hat{x} := \sum_{n=1}^N \langle x, u_n \rangle u_n.$$

2. Conclude that finite dimensional subspaces of  $\mathcal{H}$  are always closed.

**10.5 Fall 2015 # 6**

Let  $f : [0, 1] \rightarrow \mathbb{R}$  be continuous. Show that

$$\sup \left\{ \|fg\|_1 \mid g \in L^1[0, 1], \|g\|_1 \leq 1 \right\} = \|f\|_{\infty}$$

**10.6 Fall 2014 # 6**

Let  $1 \leq p, q \leq \infty$  be conjugate exponents, and show that

$$f \in L^p(\mathbb{R}^n) \implies \|f\|_p = \sup_{\|g\|_q=1} \left| \int f(x)g(x)dx \right|$$

**11 Functional Analysis: Banach Spaces****11.1 Spring 2019 # 1**

Let  $C([0, 1])$  denote the space of all continuous real-valued functions on  $[0, 1]$ .

- a. Prove that  $C([0, 1])$  is complete under the uniform norm  $\|f\|_{\infty} := \sup_{x \in [0, 1]} |f(x)|$ .
- b. Prove that  $C([0, 1])$  is not complete under the  $L^1$ -norm  $\|f\|_1 = \int_0^1 |f(x)| dx$ .

*Solution.*

**11.1.1 a**

- Let  $\{f_n\}$  be a Cauchy sequence in  $C(I, \|\cdot\|_{\infty})$ , so  $\lim_n \lim_m \|f_m - f_n\|_{\infty} = 0$ , we will show it converges to some  $f$  in this space.
- For each fixed  $x_0 \in [0, 1]$ , the sequence of real numbers  $\{f_n(x_0)\}$  is Cauchy in  $\mathbb{R}$  since

$$x_0 \in I \implies |f_m(x_0) - f_n(x_0)| \leq \sup_{x \in I} |f_m(x) - f_n(x)| := \|f_m - f_n\|_{\infty} \xrightarrow{m, n \rightarrow \infty} 0,$$

- Since  $\mathbb{R}$  is complete, this sequence converges and we can define  $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ .
- Thus  $f_n \rightarrow f$  pointwise by construction
- Claim:  $\|f - f_n\| \xrightarrow{n \rightarrow \infty} 0$ , so  $f_n$  converges to  $f$  in  $C([0, 1], \|\cdot\|_{\infty})$ .

– Proof:

\* Fix  $\varepsilon > 0$ ; we will show there exists an  $N$  such that  $n \geq N \implies \|f_n - f\| < \varepsilon$

- \* Fix an  $x_0 \in I$ . Since  $f_n \rightarrow f$  pointwise, choose  $N_1$  large enough so that

$$n \geq N_1 \implies |f_n(x_0) - f(x_0)| < \varepsilon/2.$$

- \* Since  $\|f_n - f_m\|_\infty \rightarrow 0$ , choose and  $N_2$  large enough so that

$$n, m \geq N_2 \implies \|f_n - f_m\|_\infty < \varepsilon/2.$$

- \* Then for  $n, m \geq \max(N_1, N_2)$ , we have

$$\begin{aligned} |f_n(x_0) - f(x_0)| &= |f_n(x_0) - f(x_0) + f_m(x_0) - f_m(x_0)| \\ &= |f_n(x_0) - f_m(x_0) + f_m(x_0) - f(x_0)| \\ &\leq |f_n(x_0) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \\ &< |f_n(x_0) - f_m(x_0)| + \frac{\varepsilon}{2} \\ &\leq \sup_{x \in I} |f_n(x) - f_m(x)| + \frac{\varepsilon}{2} \\ &< \|f_n - f_m\|_\infty + \frac{\varepsilon}{2} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ \implies |f_n(x_0) - f(x_0)| &< \varepsilon \\ \implies \sup_{x \in I} |f_n(x_0) - f(x_0)| &\leq \sup_{x \in I} \varepsilon \quad \text{by order limit laws} \\ \implies \|f_n - f\| &\leq \varepsilon \end{aligned}$$

- $f$  is the uniform limit of continuous functions and thus continuous, so  $f \in C([0, 1])$ .

### 11.1.2 b

- It suffices to produce a Cauchy sequence that does not converge to a continuous function.
- Take the following sequence of functions:
  - $f_1$  increases linearly from 0 to 1 on  $[0, 1/2]$  and is 1 on  $[1/2, 1]$
  - $f_2$  is 0 on  $[0, 1/4]$  increases linearly from 0 to 1 on  $[1/4, 1/2]$  and is 1 on  $[1/2, 1]$
  - $f_3$  is 0 on  $[0, 3/8]$  increases linearly from 0 to 1 on  $[3/8, 1/2]$  and is 1 on  $[1/2, 1]$
  - $f_3$  is 0 on  $[0, (1/2 - 3/8)/2]$  increases linearly from 0 to 1 on  $[(1/2 - 3/8)/2, 1/2]$  and is 1 on  $[1/2, 1]$

Idea: take sequence starting points for the triangles:  $0, 0 + \frac{1}{4}, 0 + \frac{1}{4} + \frac{1}{8}, \dots$  which converges to  $1/2$  since  $\sum_{k=1}^{\infty} \frac{1}{2^k} = -\frac{1}{2} + \sum_{k=0}^{\infty} \frac{1}{2^k}$ .

- Then each  $f_n$  is clearly integrable, since its graph is contained in the unit square.
- $\{f_n\}$  is Cauchy: geometrically subtracting areas yields a single triangle whose area tends to 0.
- But  $f_n$  converges to  $\chi_{[1/2, 1]}$  which is discontinuous.

show that  $\int_0^1 |f_n(x) - f_m(x)| dx \rightarrow 0$  rigorously, show that no  $g \in L^1([0, 1])$  can converge to this indicator function.

## 11.2 Spring 2017 # 6

Show that the space  $C^1([a, b])$  is a Banach space when equipped with the norm

$$\|f\| := \sup_{x \in [a, b]} |f(x)| + \sup_{x \in [a, b]} |f'(x)|.$$

*Solution.*

See <https://math.stackexchange.com/questions/507263/prove-that-c1a-b-with-the-c1-norm-is-a-banach-space>

- Denote this norm  $\|\cdot\|_u$
- Let  $f_n$  be a Cauchy sequence in this space, so  $\|f_n\|_u < \infty$  for every  $n$  and  $\|f_j - f_k\|_u \xrightarrow{j, k \rightarrow \infty} 0$ .

and define a candidate limit: for each  $x \in I$ , set

$$f(x) := \lim_{n \rightarrow \infty} f_n(x).$$

- Note that

$$\begin{aligned} \|f_n\|_\infty &\leq \|f_n\|_u < \infty \\ \|f'_n\|_\infty &\leq \|f_n\|_u < \infty. \end{aligned}$$

- Thus both  $f_n, f'_n$  are Cauchy sequences in  $C^0([a, b], \|\cdot\|_\infty)$ , which is a Banach space, so they converge.
- So
  - $f_n \rightarrow f$  uniformly (by uniqueness of limits),
  - $f'_n \rightarrow g$  uniformly for some  $g$ , and
  - $f, g \in C^0([a, b])$ .
- Claim:  $g = f'$ 
  - For any fixed  $a \in I$ , we have

$$\begin{aligned} f_n(x) - f_n(a) &\xrightarrow{u} f(x) - f(a) \\ \int_a^x f'_n &\xrightarrow{u} \int_a^x g. \end{aligned}$$

- By the FTC, the left-hand sides are equal.
- By uniqueness of limits so are the right-hand sides, so  $f' = g$ .
- Claim: the limit  $f$  is an element in this space.
  - Since  $f, f' \in C^0([a, b])$ , they are bounded, and so  $\|f\|_u < \infty$ .
- Claim:  $\|f_n - f\|_u \xrightarrow{n \rightarrow \infty} 0$
- Thus the Cauchy sequence  $\{f_n\}$  converges to a function  $f$  in the  $u$ -norm where  $f$  is an element of this space, making it complete.

## 11.3 Fall 2017 # 6

Let  $X$  be a complete metric space and define a norm

$$\|f\| := \max\{|f(x)| : x \in X\}.$$

Show that  $(C^0(\mathbb{R}), \|\cdot\|)$  (the space of continuous functions  $f : X \rightarrow \mathbb{R}$ ) is complete.

*Solution.*

Should be supremum maybe..?

Let  $\{f_k\}$  be a Cauchy sequence, so  $\|f_k\| < \infty$  for all  $k$ . Then for a fixed  $x$ , the sequence  $f_k(x)$  is Cauchy in  $\mathbb{R}$  and thus converges to some  $f(x)$ , so define  $f$  by  $f(x) := \lim_{k \rightarrow \infty} f_k(x)$ .

Then  $\|f_k - f\| = \max_{x \in X} |f_k(x) - f(x)| \xrightarrow{k \rightarrow \infty} 0$ , and thus  $f_k \rightarrow f$  uniformly and thus  $f$  is continuous. It just remains to show that  $f$  has bounded norm.

Choose  $N$  large enough so that  $\|f - f_N\| < \varepsilon$ , and write  $\|f_N\| := M < \infty$

$$\|f\| \leq \|f - f_N\| + \|f_N\| < \varepsilon + M < \infty.$$