Topology Qualifying Exam Solutions

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1 Problems to Revisit

- 4
- 6 (Without Heine-Borel)
- 8
- 10
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- 14

2 General Topology

2.1 1 (Todo)

2.2 2

i. See definition section of review doc.

ii. Use Heine-Borel theorem: a set $U \subset \mathbb{R}^n$ is compact $\iff$ U is *closed* and *bounded*.

- X is closed in $\mathbb{R}$, since we can write its complement as an arbitrary union of open intervals:

$$X^c = (-\infty, 0) \bigcup \left(\bigcup_{n \in \mathbb{Z}^+} \left(\frac{1}{n}, \frac{1}{n+1} \right) \right) \bigcup (1, \infty)$$

- X is *bounded*, since we can pick $r = 1$, then $x, y \in X \implies d(x, y) \leq r = 1$.

iii. Use Heine-Borel again: X is not closed because it does not contain all of its limit points, e.g. the sequence $\left\{ x_n := \frac{1}{n} \;\middle|\; n \in \mathbb{Z}^{\geq 1} \right\} \subset X$ but $x_n \overset{n \to \infty}{\longrightarrow} 0 \in X^c$. Thus is is **not** compact.

2.2.1 Alternate Proof of (ii)

See Munkres p.164

- Let $\left\{ U_i \;\middle|\; j \in J \right\} \rightrightarrows X$; then $0 \in U_j$ for some $j \in J$.
- In the subspace topology, U_i is given by some $V \in \tau(\mathbb{R})$ such that $V \bigcap X = U_i$
  - A basis for the subspace topology on $\mathbb{R}$ is open intervals, so write V as a union of open intervals $V = \bigcup_{k \in K} I_k$.
  - Since $0 \in U_j$, $0 \in I_k$ for some k.
- Since I_k is an interval, it contains infinitely many points of the form $x_n = \frac{1}{n} \in X$
- Then $I_k \bigcap X \subset U_j$ contains infinitely many such points.
- So there are only *finitely* many points in $X \setminus U_j$, each of which is in $U_{j(n)}$ for some $j(n) \in J$ depending on n.
- So U_j and the *finitely* many $U_{j(n)}$ form a finite subcover of X. $\blacksquare$

2.3 3 (Todo)

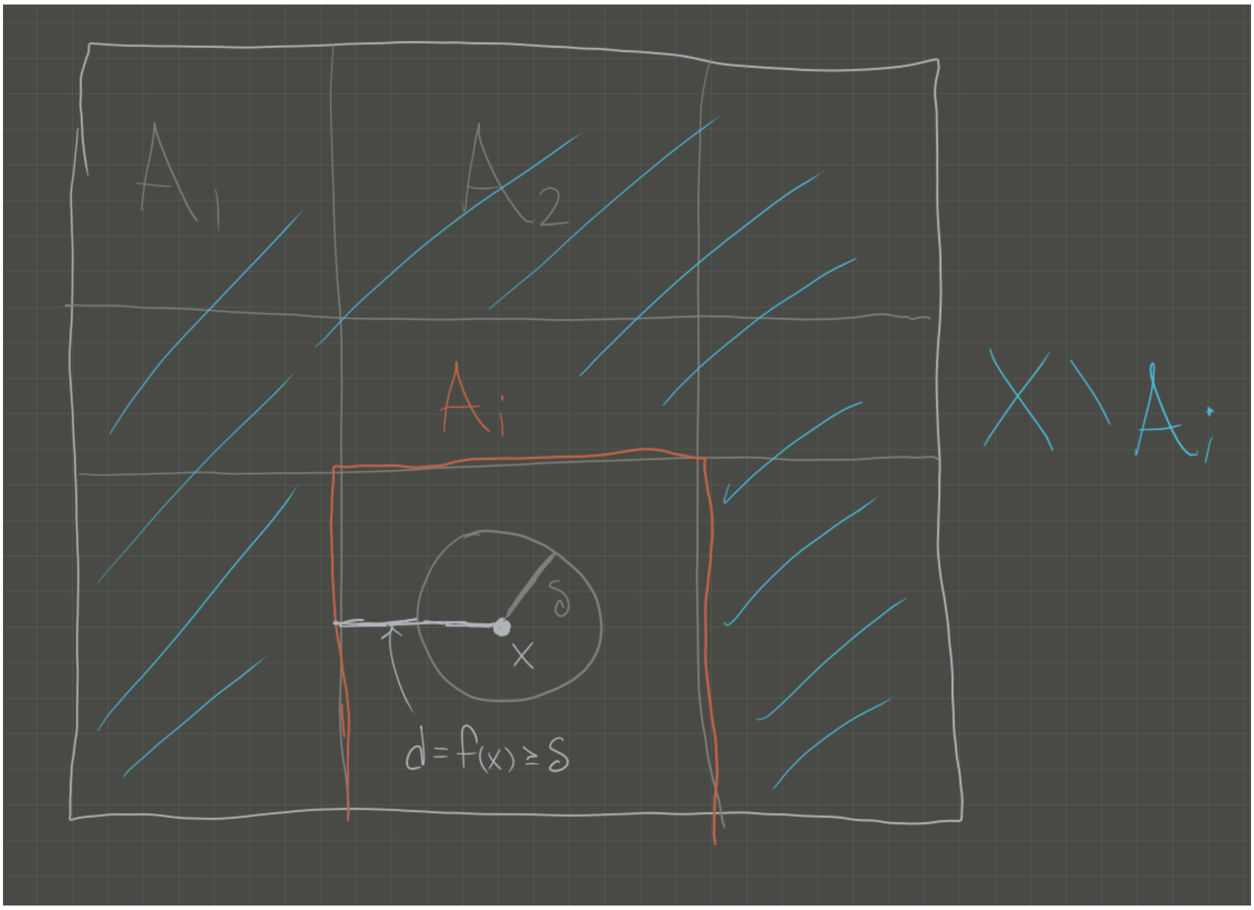
2.4 4

Statement: show that the *Lebesgue number* is well-defined for compact metric spaces.

Note: this is a question about the *Lebesgue Number*. See Wikipedia for detailed proof.

- Write $U = \left\{ U_i \;\middle|\; i \in I \right\}$, then $X \subseteq \bigcup_{i \in I} U_i$. Need to construct a $\delta > 0$.
- By compactness of X, choose a finite subcover $U_1, \cdots, U_n$.
- Define the distance between a point x and a set $Y \subset X$: $d(x, Y) = \inf_{y \in Y} d(x, y)$.
  - **Claim**: the function $d(\,\cdot\,, Y) : X \longrightarrow \mathbb{R}$ is continuous for a fixed set.

- Proof: Todo, not obvious.



- Define a function

$$f : X \longrightarrow \mathbb{R}$$
$$x \mapsto \frac{1}{n}\sum_{i=1}^{n} d(x, X \setminus U_i).$$

  - Note this is a sum of continuous functions and thus continuous.
- **Claim**:

$$\delta := \inf_{x\in X} f(x) = \min_{x\in X} f(x) = f(x_{\min}) > 0$$

  suffices.
  - That the infimum is a minimum: f is a continuous function on a compact set, apply the extreme value theorem: it attains its minimum.
  - That $\delta > 0$: otherwise, $\delta = 0 \implies \exists x_0$ such that $d(x_0, X \setminus U_i) = 0$ for all i.
    * Forces $x_0 \in X \setminus U_i$ for all i, but $X \setminus \bigcup U_i = \emptyset$ since the U_i cover X.
  - That it satisfies the Lebesgue condition:

$$\forall x \in X, \exists i \quad \text{such that} \quad B_\delta(x) \subset U_i$$

    * Let $B_\delta(x) \ni x$; then by minimality $f(x) \geq \delta$.

* Thus it can *not* be the case that $d(x, X \setminus U_i) < \delta$ for *every* i, otherwise

$$f(x) \leq \frac{1}{n}(\delta + \cdots + \delta) = \frac{n\delta}{n} = \delta$$

* So there is some particular i such that $d(x, X \setminus U_i) \geq \delta$.
* But then $B_\delta \subseteq U_i$ as desired.

2.5 5 (Todo)

2.6 6

Facts used:

- Cantor's Intersection Theorem
- Bases for standard topology on $\mathbb{R}$.
- Definition of compactness

- Toward a contradiction, let $\{U_\alpha\} \rightrightarrows [0,1]$ be an open cover with no finite subcover.
- Then either $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$ has no finite subcover; WLOG assume it is $[0, \frac{1}{2}]$.
- Then either $[0, \frac{1}{4}]$ or $[\frac{1}{4}, \frac{1}{2}]$ has no finite subcover
- Inductively defining $[a_n, b_n]$ this way yields a sequence of closed, bounded, nested intervals (each with no finite subcover) with $\mathrm{diam}([a_n, b_n]) \leq \frac{1}{2^n} \longrightarrow 0$, so Cantor's Nested Interval theorem applies and the intersection contains exactly one point $p \in [0,1]$.
- Since $p \in [0,1]$, $p \in U_\alpha$ for some α.
- Since a basis for $\tau(\mathbb{R})$ is given by open intervals, we can find an $\varepsilon > 0$ such that $(p-\varepsilon, p+\varepsilon) \subseteq U_\alpha$
- Then if $\frac{1}{2^N} < \varepsilon$, for $n \geq N$ we have

$$[a_n, b_n] \subseteq (p - \varepsilon, p + \varepsilon) \subseteq U_\alpha.$$

- But then $U_\alpha \rightrightarrows [a_n, b_n]$, yielding a finite subcover of $[a_n, b_n]$, a contradiction.

2.7 7 (Todo)

2.8 8

Topic: proof of the tube lemma.

Statement: show $X, Y \in \mathrm{Top}_{\text{compact}} \iff X \times Y \in \mathrm{Top}_{\text{compact}}$

2.8.1 Proof 1

$\Longleftarrow$:

- By universal properties, the product $X \times Y$ is equipped with continuous projections
- The continuous image of a compact set is compact, and $\pi_1(X \times Y) = X, p_2(X \times Y) = Y$
- So X, Y are compact.

$\Longrightarrow$:

Proof of Tube Lemma:

- Let $\left\{U_j \times V_j \;\middle|\; j \in J\right\} \rightrightarrows X \times Y$.
- Fix a point $x_0 \in X$, then $\{x_0\} \times Y \subset N$ for some open set N.
- By the tube lemma, there is a $U^x \subset X$ such that the tube $U^x \times Y \subset N$.
- Since $\{x_0\} \times Y \cong Y$ which is compact, there is a finite subcover $\left\{U_j \times V_j \;\middle|\; j \leq n\right\} \rightrightarrows \{x_0\} \times Y$.
- "Integrate the X": write

  $$W = \bigcap_{j=1}^{n} U_j,$$

  then $x_0 \in W$ and W is a finite intersection of open sets and thus open.
- Claim: $\left\{U_j \times V_j \;\middle|\; j \leq n\right\} \rightrightarrows W \times Y$
  - Let $(x, y) \in W \times Y$; want to show $(x, y) \in U_j \times V_j$ for some $j \leq n$.
  - Then $(x_0, y) \in \{x_0\} \times Y$ is on the same horizontal line
  - $(x_0, y) \in U_j \times V_j$ for some j by construction
  - So $y \in V_j$ for this j
  - Since $x \in W$, $x \in U_j$ for *every* j, thus $x \in U_j$.
  - So $(x, y) \in U_j \times V_j$

Actual Proof:

- Let $\left\{U_j \;\middle|\; j \in J\right\} \rightrightarrows X \times Y$.
- Fix $x_0 \in X$, the slice $\{x_0\} \times Y$ is compact and can be covered by finitely many elements $\left\{U_j \;\middle|\; j \leq m\right\} \rightrightarrows \{x_0\} \times Y$.
  - Sum: write $N = \bigcup_{j=1}^{m} U_j$; then $\{x_0\} \times Y \subset N$.
  - Apply the tube lemma to N: produce $\{x_0\} \times Y \in W \times Y \subset N$; then $\left\{U_j \;\middle|\; j \leq m\right\} \rightrightarrows W \times Y$.
- Now let $x \in X$ vary: for each $x \in X$, produce $W_x \times Y$ as above, then $\left\{W_x \times Y \;\middle|\; x \in X\right\} \rightrightarrows X$.
  - By above argument, every tube $W_x \times Y$ can be covered by *finitely* many U_j.
- Since $\left\{W_x \;\middle|\; x \in X\right\} \rightrightarrows X$ and X is compact, produce a finite subset $\left\{W_k \;\middle|\; k \leq m'\right\} \rightrightarrows X$.
- Then $\left\{W_k \times Y \;\middle|\; k \leq m'\right\} \rightrightarrows X \times Y$; the claim is that it is a finite cover.
  - Finitely many k
  - For each k, the tube $W_k \times Y$ is covered by finitely by U_j
  - And finite $\times$ finite $=$ finite. ■

Shorter mnemonic:

19.U It is sufficient to consider a cover consisting of elementary sets. Since Y is compact, each fiber $x \times Y$ has a finite subcovering $\{U_i^x \times V_i^x\}$. Put $W^x = \cap U_i^x$. Since X is compact, the cover $\{W^x\}_{x \in X}$ has a finite subcovering W^{x_j}. Then $\{U_i^{x_j} \times V_i^{x_j}\}$ is the required finite subcovering.

2.9 9

2.10 10

X is connected:

- Write $X = L\coprod G$ where $L = \{0\} \times [-1,1]$ and $G = \left\{\Gamma(\sin(x)) \;\middle|\; x \in (0,1]\right\}$ is the graph of $\sin(x)$.
- $L \cong [0,1]$ which is connected
  - Claim: Every interval is connected (todo)
- Claim: G is connected (i.e. as the graph of a continuous function on a connected set)
  - The function
$$\begin{aligned} f : (0,1] &\longrightarrow [-1,1] \\ x &\mapsto \sin(x) \end{aligned}$$
    is continuous (how to prove?)
  - Products of continuous functions are continuous iff all of the components are continuous.
  - Claim: The diagonal map $\Delta : Y \longrightarrow Y \times Y$ where $\Delta(t) = (t,t)$ is continuous for any Y since $\Delta = (\mathrm{id}, \mathrm{id})$
    - Product of identity functions, which are continuous.
  - The composition of continuous function is continuous, therefore
$$\begin{aligned} F : (0,1] &\xrightarrow{\Delta} (0,1]^2 \xrightarrow{(\mathrm{id},f)} (0,1] \times [-1,1] \\ t &\mapsto (t,t) \mapsto (t, f(t)) \end{aligned}$$
  - Then $G = F((0,1])$ is the continuous image of a connected set and thus connected.
- Claim: X is connected
  - Suppose there is a disconnecting cover $X = A\coprod B$ such that $\overline{A}\bigcap B = A\bigcap \overline{B} = \emptyset$ and $A, B \neq \emptyset$.
  - WLOG let $(x, \sin(x)) \in B$ for $x > 0$ (otherwise just relabeling A, B)
  - Claim: $B = G$
    - It can't be the case that A intersects G: otherwise
$$X = A\coprod B \implies G = (A\bigcap G)\coprod(B\bigcap V)$$
      disconnects G. So $A\bigcap G = \emptyset$, forcing $A \subseteq L$
    - Similarly L can not be disconnected, so $B\bigcap L = \emptyset$ forcing $B \subset G$
    - So $A \subset L$ and $B \subset G$, and since $X = A\coprod B$, this forces $A = L$ and $B = G$.
  - But any open set U in the subspace topology $L \subset \mathbb{R}^2$ (generated by open balls) containing $(0,0) \in L$ is the restriction of a ball $V \subset \mathbb{R}^2$ of radius $r > 0$, i.e. $U = V\bigcap X$.
    - But any such ball contains points of G:
$$n \gg 0 \implies \frac{1}{n\pi} < r \implies \exists g \in G \text{ s.t. } g \in U.$$
    - So $U\bigcap L\bigcap G \neq \emptyset$, contradicting $L\bigcap G = \emptyset$.
- Claim: X is *not* path-connected.
  - Todo: "can't get from L to G in finite time".

- Toward a contradiction, choose a continuous function $f : I \longrightarrow X$ with $f(0) \in G$ and $f(1) \in L$.
  - Since $L \cong [0,1]$, use path-connectedness to create a path $f(1) \longrightarrow (0,1)$
  - Concatenate paths and reparameterize to obtain $f(1) = (0,1) \in L \subset \mathbb{R}^2$.
- Let $\varepsilon = \dfrac{1}{2}$; by continuity there exists a $\delta \in I$ such that

$$t \in B_\delta(1) \subset I \implies f(t) \in B_\varepsilon(\mathbf{0}) \in X$$

- Using the fact that $[1-\delta, 1]$ is connected, $f([1-\delta, 1]) \subset X$ is connected.
- Let $f(1-\delta) = \mathbf{x}_0 = (x_0, y_0) \subset X \subset \mathbb{R}^2$.
- Define a composite map

$$F : [0,1] \longrightarrow \mathbb{R}F \qquad := \mathfrak{p}_{x\text{-axis}} \circ f.$$

  - F is continuous as a composition of continuous functions.
- Then $F([1-\delta, 1]) \subset \mathbb{R}$ is connected and thus must be an interval (a,b)
- Since $f(1) = \mathbf{0}$ which has x-component zero, $[0,b] \subset (a,b)$.
- Since $f(1-\delta) = \mathbf{x}$, $F(\mathbf{x}) = x_0$ and this $[0, x_0] \subset (a,b)$.
- Thus for all $x \in (0, x_0]$ there exists a $t \in [1-\delta, 1]$ such that $f(t) = (x, \sin\left(\dfrac{1}{x}\right))$.
- Now toward the contradiction, choose $x = \dfrac{1}{2n\pi - \pi/2} \in \mathbb{R}$ with n large enough such that $x \in (0, x_0)$.
  - Note that $\sin\left(\dfrac{1}{x}\right) = -1$ by construction.
  - Apply the previous statement: there exists a t such that $f(t) = (x, \sin\left(\dfrac{1}{x}\right)) = (x, -1)$.
  - But then

$$\|f(t) - f(x)\| = \|(x,-1) - (0,1)\| = \|(x,2)\| > \frac{1}{2},$$

    contradicting continuity of f.

2.11 11 (Todo)

2.12 12 (Todo)

- Using the fact that $[0,\infty) \subset \mathbb{R}$ is Hausdorff, any retract must be closed, so any closed interval $[\varepsilon, N]$ for $0 \leq \varepsilon \leq N \leq \infty$.
  - Note that $\varepsilon = N$ yields all one point sets $\{x_0\}$ for $x_0 \geq 0$.
- No finite discrete sets occur, since the retract of a connected set is connected.
- ?

2.13 13 (Todo)

2.14 14

- Take two connected sets X, Y; then there exists $p \in X \bigcap Y$.
- Toward a contradiction: write $X \bigcup Y = A \coprod B$ with both $A, B \subset A \coprod B$ open.

- Since $p \in X \bigcup Y = A \coprod B$, WLOG $p \in A$. We will show B must be empty.
- Claim: $A \bigcap X$ is clopen in X.
  - $A \bigcap X$ is open in X: ?
  - $A \bigcap X$ is closed in X: ?
- The only clopen sets of a connected set are empty or the entire thing, and since $p \in A$, we must have $A \bigcap X = X$.
- By the same argument, $A \bigcap Y = Y$.
- So $A \bigcap \left(X \bigcup Y\right) = \left(A \bigcap X\right) \bigcup \left(A \bigcap Y\right) = X \bigcup Y$
- Since $A \subset X \bigcup Y$, $A \bigcap \left(X \bigcup Y\right) = A$
- Thus $A = X \bigcup Y$, forcing $B = \emptyset$.

2.15 15 (Todo)

2.16 16

Topic: closure and connectedness in the subspace topology. See Munkres p.148

- $S \subset X$ is **not** connected if S with the subspace topology is not connected.
  - I.e. there exist $A, B \subset S$ such that
    - $A, B \neq \emptyset$,
    - $A \bigcap B = \emptyset$,
    - $A \coprod B = S$.
- Or equivalently, there exists a nontrivial $A \subset S$ that is clopen in S.

Show stronger statement: this is an iff.

$\Longrightarrow$:

- Suppose S is not connected; we then have sets $A \bigcup B = S$ from above and it suffices to show $\mathrm{cl}_Y(A) \bigcap B = A \bigcap \mathrm{cl}_X(B) = \emptyset$.
- A is open by assumption and $Y \setminus A = B$ is closed in Y, so A is clopen.
- Write $\mathrm{cl}_Y(A) := \mathrm{cl}_X(A) \bigcap Y$.
- Since A is closed in Y, $A = \mathrm{cl}_Y(A)$ by definition, so $A = \mathrm{cl}_Y(A) = \mathrm{cl}_X(A) \bigcap Y$.
- Since $A \bigcap B = \emptyset$, we then have $\mathrm{cl}_Y(A) \bigcap B = \emptyset$.
- The same argument applies to B, so $\mathrm{cl}_Y(B) \bigcap A = \emptyset$.

$\Longleftarrow$:

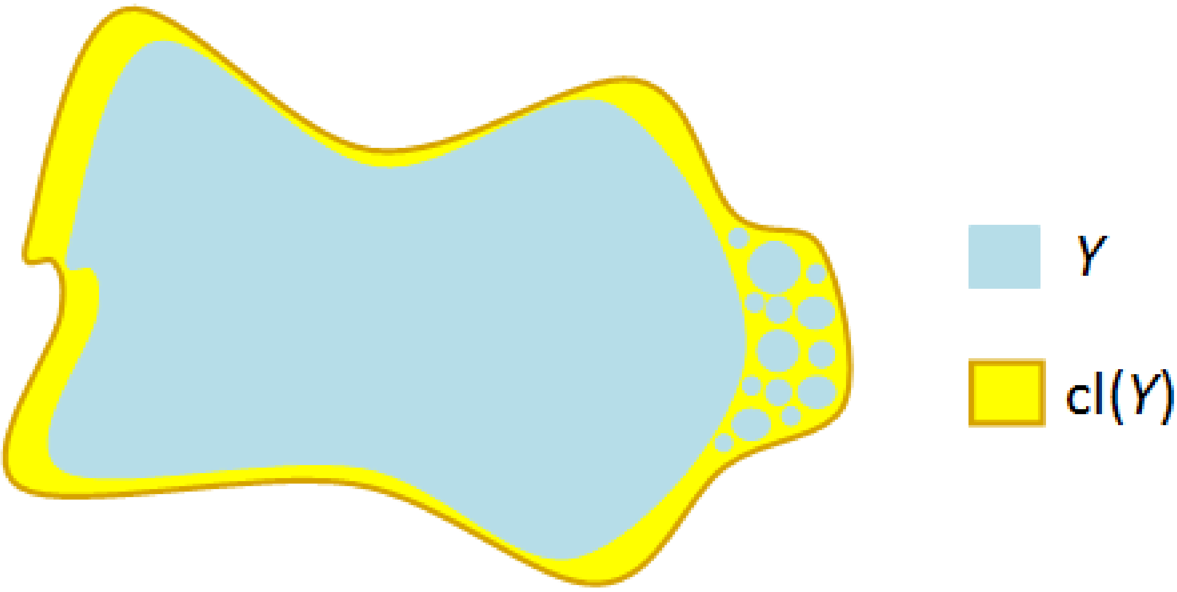
- Suppose displayed condition holds; given such A, B we will show they are clopen in Y.
- Since $\mathrm{cl}_Y(A) \bigcap B = \emptyset$, (claim) we have $\mathrm{cl}_Y(A) = A$ and thus A is closed in Y.

– Why?

$$\begin{aligned}
\mathrm{cl}_Y(A) &:= \mathrm{cl}_X(A) \bigcap Y \\
&= \mathrm{cl}_X(A) \bigcap \left(A \coprod B\right) \\
&= \left(\mathrm{cl}_X(A) \bigcap A\right) \coprod \left(\mathrm{cl}_X(A) \bigcap B\right) \\
&= A \coprod \left(\mathrm{cl}_X(A) \bigcap B\right) \quad \text{since } A \subset \mathrm{cl}_Y(A) \\
&= A \coprod \left(\mathrm{cl}_Y(A) \bigcap B\right) \quad \text{since } B \subset Y \\
&= A \coprod \emptyset \quad \text{using the assumption} \\
&= A.
\end{aligned}$$

- But $A = Y \setminus B$ where B is closed, so A is open and thus a nontrivial clopen subset.

■



2.17 17 (Todo)

2.18 18

- Define a new function

$$\begin{aligned}
g : X &\longrightarrow \mathbb{R} \\
x &\mapsto d_X(x, f(x)).
\end{aligned}$$

- Attempt to minimize. Claim: g is a continuous function.
- Given claim, a continuous function on a compact space attains its infimum, so set

$$m := \inf_{x \in X} g(x)$$

and produce $x_0 \in X$ such that $g(x) = m$.

- Then

$$m > 0 \iff d(x_0, f(x_0)) > 0 \iff x_0 \neq f(x_0).$$

- Now apply f and use the assumption that f is a contraction to contradict minimality of m:

$$\begin{aligned} d(f(f(x_0)), f(x_0)) &\leq C \cdot d(f(x_0), x_0) \\ &< d(f(x_0), x_0) \quad \text{since } C < 1 \\ &\leq m \end{aligned}$$

- Proof that g is continuous: use the definition of g, the triangle inequality, and that f is a contraction:

$$\begin{aligned} d(x, f(x)) &\leq d(x, y) + d(y, f(y)) + d(f(x), f(y)) \\ \implies d(x, f(x)) - d(y, f(y)) &\leq d(x, y) + d(f(x), f(y)) \\ \implies g(x) - g(y) &\leq d(x, y) + C \cdot d(x, y) = (C + 1) \cdot d(x, y) \end{aligned}$$

  - This shows that g is Lipschitz continuous with constant $C + 1$ (implies uniformly continuous, but not used).

2.19 19 (Todo)

2.20 20

Space	Connected	Locally Connected
$\mathbb{R}$	✓	✓
$[0, 1] \bigcup [2, 3]$		✓
Sine Curve	✓	
$\mathbb{Q}$		

a. See definitions in intro.

b. Claim: the Topologist's sine curve X suffices.

Proof:

- Claim 1: X is connected.
  - Intervals and graphs of cts functions are connected, so the only problem point is 0.
- Claim 2: X is **not** locally path connected.
  - Take any $B_\varepsilon(0) \in \mathbb{R}^2$; then $\pi_X B_\varepsilon(0)$ yields infinitely many arcs, each intersecting the graph at two points on $\partial B_\varepsilon(0)$.
  - These are homeomorphic to a collection of disjoint embedded open intervals, and any disjoint union of intervals is clearly not connected. ■

Todo: what's the picture?