## **Complex Analysis Qualifying Exam Review**

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A great deal of content borrowed from the following: https://web.stanford.edu/~chriseur/notes\_pdf/Eur\_ComplexAnalysis\_Notes.pdf

# 1 | Useful Techniques

## 1.1 Notation

Notation	Definition
$\mathbb{D} \coloneqq \left\{ z \mid  z  \le 1 \right\}$	The unit disc
$\mathbb{H} := \left\{ x + iy \mid y > 0 \right\}$	The upper half-plane
$X_{rac{1}{2}}$	A "half version of $X$ ", see examples
$\mathbb{H}_{\frac{1}{2}}$ $\mathbb{D}_{\frac{1}{2}}$	The first quadrant  The portion of the first quadrant inside the unit disc
$S := \left\{ x + iy \mid x \in \mathbb{R}, \ 0 < y < \pi \right\}$	The horizontal strip

## 1.2 Greatest Hits



Things to know well:

- Estimates for derivatives, mean value theorem
- ??CauchyTheorem|Cauchy's Theorem
- ??CauchyIntegral|Cauchy's Integral Formula
- ??CauchyInequality]Cauchy's Inequality
- ??Morera]Morera's Theorem
- ??Liouville]Liouville's Theorem
- ??MaximumModulus|Maximum Modulus Principle
- ??Rouche]Rouché's Theorem

1.2 Greatest Hits

- $\bullet \enskip \enskip ?? Schwarz Reflection Principle$
- ??SchwarzLemma|The Schwarz Lemma
- ??Casorati|Casorati-Weierstrass Theorem
- Properties of linear fractional transformations
- Automorphisms of  $\mathbb{D}, \mathbb{C}, \mathbb{CP}^1$ .

#### 1.2.1 Common Tricks

• Virtually any time: consider 1/f(z) and f(1/z).

Remark 1.2.1 (Showing a function is constant): If you want to show that a function f is constant, try one of the following:

- Write f = u + iv and use Cauchy-Riemann to show  $u_x, u_y = 0$ , etc.
- Show that f is entire and bounded.

If you additionally want to show f is zero, try one of these:

• Show f is entire, bounded, and  $\lim_{z\to\infty} f(z) = 0$ .

## 1.3 Basic but Useful Facts

Fact 1.3.1 (Some useful facts about basic complex algebra)

• 
$$z\bar{z} = |z|^2$$

$$\Re(z) = \frac{z + \bar{z}}{2}$$

$$\Im(z) = \frac{z - \bar{z}}{2i}.$$

- $\operatorname{Arg}(z/w) = \operatorname{Arg}(z) \operatorname{Arg}(w)$ .
- Exponential forms of cosine and sine:

$$\cos(\theta) = \frac{1}{2} \left( e^{i\theta} + e^{-i\theta} \right) \qquad \sin(\theta) = \frac{1}{2i} \left( e^{i\theta} - e^{-i\theta} \right).$$

• Various differentials:

$$dz = dx + i dy$$
$$d\bar{z} = dx - i dy$$
$$f_z = f_x = f_y/i.$$

• Integral of a complex exponential:

$$\int_0^{2\pi} e^{i\ell x} dx = \left\{ \begin{array}{ll} 2\pi & \ell = 0 \\ 0 & \text{else} \end{array} \right..$$

Fact 1.3.2 (Generalized Binomial Theorem)

Define  $(n)_k$  to be the falling factorial  $\prod_{j=0}^{k-1} (n-k) = n(n-1)\cdots(n-k+1)$  and set  $\binom{n}{k} \coloneqq (n)_k/k!$ , then

$$(x+y)^n = \sum_{k>0} \binom{n}{k} x^k y^{n-k}.$$

Fact 1.3.3 (Some useful series)

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^{2} = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^{3} = \frac{n^{2}(n+1)^{2}}{4}$$

$$\log(1-x) = \sum_{n\geq 0} \frac{x^{n}}{n}$$

$$x$$

$$\frac{\partial}{\partial z} \sum_{j=0}^{\infty} a_{j}z^{j} = \sum_{j=0}^{\infty} a_{j+1}z^{j}\sqrt{1+x} = (1+x)^{1/2} = 1 + (1/2)x + \frac{(1/2)(-1/2)}{2!}x^{2} + \frac{(1/2)(-1/2)(-3/2)}{3!}x^{3} + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^{2} + \frac{1}{16}x^{3} - \cdots$$

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Fact 1.3.4

1.3 Basic but Useful Facts

Useful trick for expanding square roots:

$$\sqrt{z} = \sqrt{z_0 + z - z_0} = \sqrt{z_0 \left(1 + \frac{z - z_0}{z}\right)} = \sqrt{z_0} \sqrt{1 + u}, \quad u := \frac{z - z_0}{z}$$

$$\implies \sqrt{z} = \sqrt{z_0} \sum_{k > 0} \binom{1/2}{k} \left(\frac{z - z_0}{z}\right)^k.$$

# $\mathbf{2}\mid$ Calculus Preliminaries

## 2.1 Definitions

**Definition 2.1.1** (Locally uniform convergence)

A sequence of functions  $f_n$  is said to converge **locally uniformly** on  $\Omega \subseteq \mathbb{C}$  iff  $f_n \to f$  uniformly on every compact subset  $K \subseteq \Omega$ .

**Definition 2.1.2** (Equicontinuous Family)

A family of functions  $f_n$  is **equicontinuous** iff for every  $\varepsilon$  there exists a  $\delta = \delta(\varepsilon)$  (not depending on n or  $f_n$ ) such that  $|x - y| < \varepsilon \implies |f_n(x) - f_n(y)| < \varepsilon$  for all n.

Remark 2.1.3: Recall Arzelà-Ascoli, an analog of Heine-Borel: for X compact Hausdorff, consider the Banach space  $C(X;\mathbb{R})$  equipped with the uniform norm  $\|f\|_{\infty,X} := \sup_{x \in X} |f(x)|$ . Then a subset  $A \subseteq X$  is compact iff A is closed, uniformly bounded, and equicontinuous. As a consequence, if A is a sequence, it contains a subsequence converging uniformly to a continuous function. The proof is an  $\varepsilon/3$  argument.

**Definition 2.1.4** (Normal Family)

Remark 2.1.5: A continuous function on a compact set is uniformly continuous.

#### 2.2 Theorems

Theorem 2.2.1 (Implicit Function Theorem).

Theorem 2.2.2 (Inverse Function Theorem).

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#### Theorem 2.2.3 (Green's Theorem).

If  $\Omega \subseteq \mathbb{C}$  is bounded with  $\partial \Omega$  piecewise smooth and  $f, g \in C^1(\overline{\Omega})$ , then

$$\int_{\partial\Omega} f \, dx + g \, dy = \iint_{\Omega} \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dA.$$

## 2.3 Convergence

Remark 2.3.1: Recall that absolutely convergent implies convergent, but not conversely:  $\sum k^{-1} = \infty$  but  $\sum (-1)^k k^{-1} < \infty$ . This converges because the even (odd) partial sums are monotone increasing/decreasing respectively and in (0,1), so they converge to a finite number. Their difference converges to 0, and their common limit is the limit of the sum.

## Proposition 2.3.2 (Uniform Convergence of Series).

A series of functions  $\sum_{n=1}^{\infty} f_n(x)$  converges uniformly iff

$$\lim_{n \to \infty} \left\| \sum_{k \ge n} f_k \right\|_{\infty} = 0.$$

#### Theorem 2.3.3 (Weierstrass M-Test).

If  $\{f_n\}$  with  $f_n: \Omega \to \mathbb{C}$  and there exists a sequence  $\{M_n\}$  with  $\|f_n\|_{\infty} \leq M_n$  and  $\sum_{n \in \mathbb{N}} M_n < \infty$ ,

then  $f(x) := \sum_{n \in \mathbb{N}} f_n(x)$  converges absolutely and uniformly on  $\Omega$ . Moreover, if the  $f_n$  are continuous, by the uniform limit theorem, f is again continuous.

## 2.4 Series and Sequences

**Remark 2.4.1:** Note that if a power series converges uniformly, then summing commutes with integrating or differentiating.

#### Fact 2.4.2

Consider  $\sum c_k z^k$ , set  $R = \lim \left| \frac{c_{k+1}}{c_k} \right|$ , and recall the **ratio test**:

- $R \in (0,1) \implies$  convergence.
- $R \in (1, \infty] \implies$  divergence.
- R = 1 yields no information.

2.3 Convergence

## Proposition 2.4.3 (Radius of Convergence by the Root Test).

For 
$$f(z) = \sum_{k \in \mathbb{N}} c_k z^k$$
, defining

$$\frac{1}{R} \coloneqq \limsup_{k} |a_k|^{\frac{1}{k}},$$

then f converges absolutely and uniformly for  $D_R := |z| < R$  and diverges for |z| > R. Moreover f is holomorphic in  $D_R$ , can be differentiated term-by-term, and  $f' = \sum_{k \in \mathbb{N}} nc_k z^k$ .

#### Fact 2.4.4

Recall the p-test:

$$\sum n^{-p} < \infty \iff p \in (1, \infty).$$

#### Fact 2.4.5

The product of two sequences is given by the Cauchy product

$$\sum a_k z^k \cdot \sum b_k z^k = \sum c_k z^k, \quad c_k := \sum_{j \le k} a_k b_{k-j}.$$

#### Fact 2.4.6

Recall how to carry out polynomial long division:

Polynomial long division

#### Fact 2.4.7 (Partial Fraction Decomposition)

- For every root  $r_i$  of multiplicity 1, include a term  $A/(x-r_i)$ .
- For any factors g(x) of multiplicity k, include terms  $A_1/g(x), A_2/g(x)^2, \cdots, A_k/g(x)^k$ .
- For irreducible quadratic factors  $h_i(x)$ , include terms of the form  $\frac{Ax+B}{h_i(x)}$ .

#### 2.5 Exercises

#### Exercise 2.5.1 (?)

Find the radius of convergences for the power series expansion of  $\sqrt{z}$  about  $z_0 = 4 + 3i$ .

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# 3 | Preliminaries

## 3.1 Complex Arithmetic

Fact 3.1.1 (Complex roots of a number)

The complex nth roots of  $z := re^{i\theta}$  are given by

$$\left\{\omega_k := r^{1/n} e^{i\left(\frac{\theta + 2k\pi}{n}\right)} \mid 0 \le k \le n - 1\right\}.$$

Note that one root is  $r^{1/n} \in \mathbb{R}$ , and the rest are separated by angles of  $2\pi/n$ . Mnemonic:

$$z = re^{i\theta} = re^{i(\theta + 2k\pi)} \implies z^{1/n} = \cdots$$

#### Fact 3.1.2

Common trick:

$$f^{1/n} = e^{\frac{1}{n}\log(f)},$$

taking (say) a principal branch of log given by  $\mathbb{C} \setminus (-\infty, 0] \times 0$ .

## 3.2 Complex Log

Fact 3.2.1 (Complex Log)

For  $z = re^{i\theta} \neq 0$ ,  $\theta$  is of the form  $\Theta + 2k\pi$  where  $\Theta = \operatorname{Arg} z$ 

#### Proposition 3.2.2 (Existence of complex log).

Suppose  $\Omega$  is a simply-connected region such that  $1 \in \Omega, 0 \notin \Omega$ . Then there exists a branch of F(z) := Log(z) such that

- F is holomorphic on  $\Omega$ ,
- $e^{F(z)} = z$  for all  $z \in \Omega$
- $F(x) = \log(x)$  for  $x \in \mathbb{R}$  in a neighborhood of 1.

**Definition 3.2.3** (Principal branch and exponential)

Take  $\mathbb{C}$  and delete  $\mathbb{R}^{\leq 0}$  to obtain the **principal branch** of the logarithm, defined as

$$Log(z) := log(r) + i\theta$$
  $|\theta| < \pi$ .

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Similarly define

$$z^{\alpha} \coloneqq e^{\alpha \operatorname{Log}(z)}.$$

#### Theorem 3.2.4 (Existence of log).

If f is holomorphic and nonvanishing on a simply-connected region  $\Omega$ , then there exists a holomorphic G on  $\Omega$  such that

$$f(z) = e^{G(z)}.$$

## 3.3 Complex Calculus

**Remark 3.3.1:** When parameterizing integrals  $\int_{\gamma} f(z) dz$ , parameterize  $\gamma$  by  $\theta$  and write  $z = re^{i\theta}$  so  $dz = ire^{i\theta} d\theta$ .

## **⚠** Warning 3.3.2

 $f(z) = \sin(z), \cos(z)$  are unbounded on  $\mathbb{C}$ ! An easy way to see this: they are nonconstant and entire, thus unbounded by Liouville.

**Example 3.3.3(?):** You can show  $f(z) = \sqrt{z}$  is not holomorphic by showing its integral over  $S^1$  is nonzero. This is a direct computation:

$$\begin{split} \int_{S^1} z^{1/2} \, dz &= \int_0^{2\pi} (e^{i\theta})^{1/2} i e^{i\theta} \, d\theta \\ &= i \int_0^{2\pi} e^{\frac{i3\theta}{2}} \, d\theta \\ &= i \left(\frac{2}{3i}\right) e^{\frac{i3\theta}{2}} \Big|_0^{2\pi} \\ &= \frac{2}{3} \left(e^{3\pi i - 1}\right) \\ &= -\frac{4}{3}. \end{split}$$

Note an issue: a different parameterization yields a different (still nonzero) number

$$\dots = \int_{-\pi}^{\pi} (e^{i\theta})^{1/2} i e^{i\theta} d\theta$$
$$= \frac{2}{3} \left( e^{\frac{3\pi i}{2}} - e^{\frac{-3\pi i}{2}} \right)$$
$$= -\frac{4i}{3}.$$

This is these are paths that don't lift to closed loops on the Riemann surface defined by  $z \mapsto z^2$ .

#### 3.3.1 Holomorphy and Cauchy-Riemann

#### **Definition 3.3.4** (Analytic)

A function  $f: \Omega \to \mathbb{C}$  is analytic at  $z_0 \in \Omega$  iff there exists a power series  $g(z) = \sum a_n (z - z_0)^n$  with radius of convergence R > 0 and a neighborhood  $U \ni z_0$  such that f(z) = g(z) on U.

## **Definition 3.3.5** (Complex differentiable / holomorphic /entire)

A function  $f: \mathbb{C} \to \mathbb{C}$  is **complex differentiable** or **holomorphic** at  $z_0$  iff the following limit exists:

$$\lim_{h\to 0}\frac{f(z_0+h)-f(h)}{h}.$$

A function that is holomorphic on  $\mathbb{C}$  is said to be **entire**.

Equivalently, there exists an  $\alpha \in \mathbb{C}$  such that

$$f(z_0 + h) - f(z_0) = \alpha h + R(h)$$
  $R(h) \xrightarrow{h \to 0} 0.$ 

In this case,  $\alpha = f'(z_0)$ .

#### Example 3.3.6 (Holomorphic vs non-holomorphic):

- $f(z) = \frac{1}{z}$  is holomorphic on  $\mathbb{C} \setminus \{0\}$ .
- $f(z) = \bar{z}$  is not holomorphic, since  $\frac{\bar{h}}{h}$  does not converge (but is real differentiable).

## Definition 3.3.7 (Real (multivariate) differentiable)

A function  $F: \mathbb{R}^n \to \mathbb{R}^m$  is **real-differentiable** at **p** iff there exists a linear transformation A such that

$$\frac{\|F(\mathbf{p} + \mathbf{h}) - F(\mathbf{p}) - A(\mathbf{h})\|}{\|\mathbf{h}\|} \stackrel{\|\mathbf{h}\| \to 0}{\longrightarrow} 0.$$

Rewriting,

$$||F(\mathbf{p} + \mathbf{h}) - F(\mathbf{p}) - A(\mathbf{h})|| = ||\mathbf{h}|| ||R(\mathbf{h})|| \qquad ||R(\mathbf{h})|| \stackrel{||\mathbf{h}|| \to 0}{\longrightarrow} 0.$$

Equivalently,

$$F(\mathbf{p} + \mathbf{h}) - F(\mathbf{p}) = A(\mathbf{h}) + \|\mathbf{h}\| R(\mathbf{h}) \qquad \|R(\mathbf{h})\| \stackrel{\|\mathbf{h}\| \to 0}{\longrightarrow} 0.$$

Or in a slightly more useful form,

$$F(\mathbf{p} + \mathbf{h}) = F(\mathbf{p}) + A(\mathbf{h}) + R(\mathbf{h}) \qquad \qquad R \in o(\|\mathbf{h}\|), \text{ i.e. } \frac{\|R(\mathbf{h})\|}{\|\mathbf{h}\|} \xrightarrow{\mathbf{h} \to 0} 0.$$

#### Proposition 3.3.8 (Complex differentiable implies Cauchy-Riemann).

If f is differentiable at  $z_0$ , then the limit defining  $f'(z_0)$  must exist when approaching from

any direction. Identify f(z) = f(x, y) and write  $z_0 = x + iy$ , then first consider  $h \in RR$ , so  $h = h_1 + ih_2$  with  $h_2 = 0$ . Then

$$f'(z_0) = \lim_{h_1 \to 0} \frac{f(x + h_1, y) - f(x, y)}{h_1} := \frac{\partial f}{\partial x}(x, y).$$

Taking  $h \in i\mathbb{R}$  purely imaginary, so  $h = ih_2$ ,

$$f'(z_0) = \lim_{ih_2 \to 0} \frac{f(x, y + h_2) - f(x, y)}{ih_2} := \frac{1}{i} \frac{\partial f}{\partial y}(x, y).$$

Equating,

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} \,,$$

and writing f = u + iv and 1/i = -i yields

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} 
\frac{1}{i} \frac{\partial f}{\partial y} = \frac{1}{i} \left( \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

Thus

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \,.$$

Proposition 3.3.9 (Polar Cauchy-Riemann equations).

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$
 and  $\frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$ .

Proof.

Setting

$$z = re^{i\theta} = r(\cos(\theta) + i\sin(\theta)) = x + iy$$

yields  $x = r\cos(\theta), y = r\sin(\theta)$ , one can identify

$$x_r = \cos(\theta), x_\theta = -r\sin(\theta)$$
  
 $y_r = \sin(\theta), y_\theta = r\cos(\theta).$ 

Now apply the chain rule:

$$u_r = u_x x_r + u_y y_r$$

$$= v_y x_r - v_x y_r$$

$$= v_y \cos(\theta) - v_x \sin(\theta)$$

$$= \frac{1}{r} (v_y r \cos(\theta) - v_x r \sin(\theta))$$

$$= \frac{1}{r} (v_y y_\theta + v_x x_\theta)$$

$$= \frac{1}{r} v_\theta.$$
CR

Similarly,

$$\begin{aligned} v_r &= v_x x_r + v_y y_r \\ &= v_x \cos(\theta) + v_y \sin(\theta) \\ &= -u_y \cos(\theta) + u_x \sin(\theta) \end{aligned} \qquad \text{CR}$$
 
$$= \frac{1}{r} \left( -u_y r \cos(\theta) + u_x r \sin(\theta) \right) \\ &= \frac{1}{r} \left( -u_y y_\theta - u_x x_0 \right) \\ &= -\frac{1}{r} u_\theta. \end{aligned}$$

Thus

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$
 and  $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$ 

 ${\bf Proposition}~3.3.10 (Holomorphic~functions~are~continuous.).$ 

f is holomorphic at  $z_0$  iff there exists an  $a \in \mathbb{C}$  such that

$$f(z_0 + h) - f(z_0) - ah = h\psi(h), \quad \psi(h) \stackrel{h \to 0}{\to} 0.$$

In this case,  $a = f'(z_0)$ .

Prove

#### 3.3.2 Delbar and the Laplacian

**Definition 3.3.11** (del and delbar operators)

$$\partial := \partial_z := \frac{1}{2} (\partial_x - i \partial_y)$$
 and  $\bar{\partial} := \partial_{\bar{z}} = \frac{1}{2} (\partial_x + i \partial_y)$ .

Moreover,  $f' = \partial f + \bar{\partial} f$ .

Proposition 3.3.12 (Holomorphic iff delbar vanishes).

f is holomorphic at  $z_0$  iff  $\bar{\partial} f(z_0) = 0$ :

$$2\bar{\partial}f := (\partial_x + i\partial_y)(u + iv)$$

$$= u_x + iv_x + iu_y - v_y$$

$$= (u_x - v_y) + i(u_y + v_x)$$

$$= 0$$

by Cauchy-Riemann.

#### 3.3.3 Harmonic Functions and the Laplacian

**Definition 3.3.13** (Laplacian and Harmonic Functions)

A real function of two variables u(x,y) is **harmonic** iff it is in the kernel of the Laplacian operator:

$$\Delta u := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0.$$

Proposition 3.3.14 (Cauchy-Riemann implies holomorphic).

If f = u + iv with  $u, v \in C^1(\mathbb{R})$  satisfying the Cauchy-Riemann equations on  $\Omega$ , then f is holomorphic on  $\Omega$  and

$$f'(z) = \partial f = \frac{1}{2} (u_x + iv_x).$$

Proposition 3.3.15 (Holomorphic functions have harmonic components).

If f(z) = u(x, y) + iv(x, y) is holomorphic, then u, v are harmonic.

Proof (?).

• By CR,

$$u_x = v_y u_y = -v_x.$$

• Differentiate with respect to x:

$$u_{xx} = v_{yx} u_{yx} = -v_{xx}.$$

• Differentiate with respect to y:

$$u_{xy} = v_{yy} u_{yy} = -v_{xy}.$$

• Clairaut's theorem: partials are equal, so

$$u_{xx} - v_{yx} = 0 \implies u_{xx} + u_{yy} = 0$$

$$v_{xx} + u_{yx} = 0 \implies v_{xx} + v_{yy} = 0$$

.

#### 3.3.4 Exercises

#### Proposition 3.3.16 (Injectivity Relates to Derivatives).

If  $z_0$  is a zero of f' of order n, then f is (n+1)-to-one in a neighborhood of  $z_0$ .

 $\frac{Proof}{?}$ 

proof

#### Exercise 3.3.17 (Zero derivative implies constant)

Show that if f'=0 on a domain  $\Omega$ , then f is constant on  $\Omega$ 

#### Solution:

Write f = u + iv, then  $0 = 2f' = u_x + iv_x = u_y - iu_y$ , so grad u = grad v = 0. Show f is constant along every straight line segment L by computing the directional derivative grad  $u \cdot \mathbf{v} = 0$  along L connecting p, q. Then u(p) = u(q) = a some constant, and v(p) = v(q) = b, so f(z) = a + bi everywhere.

#### Exercise 3.3.18 (f and fbar holomorphic implies constant)

Show that if f and  $\bar{f}$  are both holomorphic on a domain  $\Omega$ , then f is constant on  $\Omega$ .

#### Solution:

- Strategy: show f' = 0.
- Write f = u + iv. Since f is analytic, it satisfies CR, so

$$u_x = v_y u_y = -v_x.$$

• Similarly write  $\bar{f} = U + iV$  where U = u and V = -v. Since  $\bar{f}$  is analytic, it also satisfies CR, so

$$U_x = V_y U_y = -V_x$$

$$\implies u_x = -v_y$$
  $u_y = v_x$ 

- Add the LHS of these two equations to get  $2u_x = 0 \implies u_x = 0$ . Subtract the right-hand side to get  $-2v_x = 0 \implies v_x = 0$
- Since f is analytic, it is holomorphic, so f' exists and satisfies  $f' = u_x + iv_x$ . But by above, this is zero.
- By the previous exercise,  $f' = 0 \implies f$  is constant.

Exercise 3.3.19 (SS 1.13: Constant real/imaginary/magnitude implies constant) If f is holomorphic on  $\Omega$  and any of the following hold, then f is constant:

- 1.  $\Re(f)$  is constant.
- 2.  $\Im(f)$  is constant.
- 3. |f| is constant.

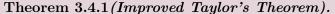
#### Solution:

#### Part 3:

- Write  $|f| = c \in \mathbb{R}$ .
- If c = 0, done, so suppose c > 0.
   Use f\(\bar{f} = |f|^2 = c^2\) to write \(\bar{f} = c^2/f\).
- Since  $|f(z)| = 0 \iff f(z) = 0$ , we have  $f \neq 0$  on  $\Omega$ , so  $\overline{f}$  is analytic.
- Similarly f is analytic, and  $f, \bar{f}$  analytic implies f' = 0 implies f is constant.

Finish

#### 3.4 Power Series



If f is holomorphic on a region  $\Omega$  with  $D_R(z_0) \subseteq \Omega$ , and for every  $z \in D_r(z_0)$ , f has a power series expansion of the following form:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$
 where  $a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta$ .

## Proposition 3.4.2 (Power Series are Smooth).

Any power series is smooth (and thus holomorphic) on its disc of convergence, and its derivatives can be obtained using term-by-term differentiation:

$$\frac{\partial}{\partial z} f(z) = \frac{\partial}{\partial z} \sum_{k>0} c_k (z - z_0)^k = \sum_{k>1} k c_k (z - z_0)^k.$$

Moreover, the coefficients are given by

$$c_k = \frac{f^{(n)}(z_0)}{n!}.$$

**Remark 3.4.3:** By an application of the Cauchy integral formula (see S&S 7.1) if f is holomorphic on  $D_R(z_0)$  there is a formula for all  $k \ge 0$  and all 0 < r < R:

$$c_k = \frac{1}{2\pi r^k} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta.$$

### Proposition 3.4.4 (Exponential is uniformly convergent in discs).

 $f(z) = e^z$  is uniformly convergent in any disc in  $\mathbb{C}$ .

Proof.

Apply the estimate

$$|e^z| \le \sum \frac{|z|^n}{n!} = e^{|z|}.$$

Now by the M-test,

$$|z| \le R < \infty \implies \left| \sum \frac{z^n}{n!} \right| \le e^R < \infty.$$

#### Lemma 3.4.5 (Dirichlet's Test).

Given two sequences of real numbers  $\{a_k\}, \{b_k\}$  which satisfy

- 1. The sequence of partial sums  $\{A_n\}$  is bounded,
- $b_k \searrow 0.$

then

$$\sum_{k\geq 1} a_k b_k < \infty.$$

Proof (?).

 $See \ http://www.math.uwaterloo.ca/~krdavids/Comp/Abel.pdf$ 

Use summation by parts. For a fixed  $\sum a_k b_k$ , write

$$\sum_{n=1}^{m} x_n Y_n + \sum_{n=1}^{m} X_n y_{n+1} = X_m Y_{m+1}.$$

Set  $x_n := a_n, y_N := b_n - b_{n-1}$ , so  $X_n = A_n$  and  $Y_n = b_n$  as a telescoping sum. Importantly, all  $y_n$  are negative, so  $|y_n| = |b_n - b_{n-1}| = b_{n-1} - b_n$ , and moreover  $a_n b_n = x_n Y_n$  for all n. We have

$$\begin{split} \sum_{n\geq 1} a_n b_n &= \lim_{N\to\infty} \sum_{n\leq N} x_n Y_n \\ &= \lim_{N\to\infty} \sum_{n\leq N} X_N Y_N - \sum_{n\leq N} X_n y_{n+1} \\ &= -\sum_{n\geq 1} X_n y_{n+1}, \end{split}$$

where in the last step we've used that

$$|X_N| = |A_N| \le M \implies |X_N Y_N| = |X_N| |b_{n+1}| \le M b_{n+1} \to 0.$$

So it suffices to bound the latter sum:

$$\begin{split} \sum_{k \geq n} |X_k y_{k+1}| &\leq M \sum_{k \geq 1} |y_{k+1}| \\ &\leq M \sum_{k \geq 1} b_k - b_{k+1} \\ &\leq 2M (b_1 - b_{n+1}) \\ &< 2M b_1. \end{split}$$

Theorem 3.4.6 (Abel's Theorem).

If  $\sum_{k=1}^{\infty} c_k z^j$  converges on |z| < 1 then

$$\lim_{z \to 1^{-}} \sum_{k \in \mathbb{N}} c_k z^k = \sum_{k \in \mathbb{N}} c_k.$$

Lemma 3.4.7 (Abel's Test).

If  $f(z) := \sum_{k} c_k z^k$  is a power series with  $c_k \in \mathbb{R}^{\geq 0}$  and  $c_k \searrow 0$ , then f converges on  $S^1$  except possibly at z = 1.

**Example 3.4.8** (application of Abel's theorem): What is the value of the alternating harmonic series? Integrate a geometric series to obtain

$$\sum_{n} \frac{(-1)^k z^k}{n} = \log(z+1)$$
 |z| < 1.

Since  $c_k := (-1)^k/k \searrow 0$ , this converges at z = 1, and by Abel's theorem  $f(1) = \log(2)$ .

**Remark 3.4.9:** The converse to Abel's theorem is false: take  $f(z) = \sum (-z)^n = 1/(1+z)$ . Then  $f(1) = 1 - 1 + 1 - \cdots$  diverges at 1, but 1/1 + 1 = 1/2. So the limit  $s := \lim_{x \to 1^-} f(x)1/2$ , but  $\sum a_n$  doesn't converge to s.

#### Proposition 3.4.10 (Summation by Parts).

Setting  $A_n := \sum_{k=1}^n b_k$  and  $B_0 := 0$ ,

$$\sum_{k=m}^{n} a_k b_k = A_n b_n - A_{m-1} b_m - \sum_{k=m}^{n-1} A_k (b_{k+1} - b_k).$$

Compare this to integrating by parts:

$$\int_a^b fg = F(b)g(b) - F(a)g(a) - \int_a^b Fg'.$$

Note there is a useful form for taking the product of sums:

$$A_n B_n = \sum_{k=1}^n A_k b_k + \sum_{k=1}^n a_k B_{k-1}.$$

Proof (?).

An inelegant proof: define  $A_n := \sum_{k \le n} a_k$ , use that  $a_k = A_k - A_{k-1}$ , reindex, and peel a top/bottom term off of each sum to pattern-match.

Behold:

$$\begin{split} \sum_{m \leq k \leq n} a_k b_k &= \sum_{m \leq k \leq n} (A_k - A_{k-1}) b_k \\ &= \sum_{m \leq k \leq n} A_k b_k - \sum_{m \leq k \leq n} A_{k-1} b_k \\ &= \sum_{m \leq k \leq n} A_k b_k - \sum_{m-1 \leq k \leq n-1} A_k b_{k+1} \\ &= A_n b_n + \sum_{m \leq k \leq n-1} A_k b_k - \sum_{m-1 \leq k \leq n-1} A_k b_{k+1} \\ &= A_n b_n - A_{m-1} b_m + \sum_{m \leq k \leq n-1} A_k b_k - \sum_{m \leq k \leq n-1} A_k b_{k+1} \\ &= A_n b_n - A_{m-1} b_m + \sum_{m \leq k \leq n-1} A_k (b_k - b_{k+1}) \\ &= A_n b_n - A_{m-1} b_m - \sum_{m \leq k \leq n-1} A_k (b_{k+1} - b_k). \end{split}$$

**Theorem 2.2 (Root Test).** Suppose  $\sum a_n(z-z_0)^n$  is a formal power series. Let

$$R = \liminf_{n \to \infty} |a_n|^{-\frac{1}{n}} = \frac{1}{\limsup_{n \to \infty} |a_n|^{\frac{1}{n}}} \in [0, +\infty].$$

Then  $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ 

- (a) converges absolutely in  $\{z : |z z_0| < R\}$ ,
- (b) converges uniformly in  $\{z : |z z_0| \le r\}$  for all r < R, and
- (c) diverges in  $\{z : |z z_0| > R\}$ .

Figure 1: image\_2021-05-27-15-40-58

#### Proposition 3.4.11(?).

If f is non-constant, then f' is analytic and the zeros of f' are isolated. If f, g are analytic with f' = g', then f - g is constant.

#### 3.4.1 Exercises: Series

Exercise 3.4.12 (Application of summation by parts)

Use summation by parts to show that  $\sin(n)/n$  converges.

Exercise 3.4.13 (1.20: Series convergence on the circle)

Show that

- 1.  $\sum kz^k$  diverges on  $S^1$ . 2.  $\sum k^{-2}z^k$  converges on  $S^1$ . 3.  $\sum k^{-1}z^k$  converges on  $S^1\setminus\{1\}$  and diverges at 1.

1. Use that  $|z^k| = 1$  and  $\sum c_k z^k < \infty \implies |c_k| \to 0$ , but  $|kz^k| = |k| \to \infty$  here.

- 2. Use that absolutely convergent implies convergent, and  $\sum |k^{-2}z^k| = \sum |k^{-2}|$  converges by the p-test.
- 3. If z=1, this is the harmonic series. Otherwise take  $a_k=1/k, b_k=e^{ik\theta}$  where  $\theta\in(0,2\pi)$ is some constant, and apply Dirichlet's test. It suffices to bound the partial sums of the

 $b_k$ . Recalling that  $\sum_{k \leq N} r^k = (1 - r^{N+1})/(1 - r)$ ,

$$\left\| \sum_{k \le m} e^{ik\theta} \right\| = \left\| \frac{1 - e^{i(m+1)\theta}}{1 - e^{i\theta}} \right\| \le \frac{2}{\|1 - e^{i\theta}\|} \coloneqq M,$$

which is a constant. Here we've used that two points on  $S^1$  are at most distance 2 from each other.

Exercise 3.4.14 (Laurent expansions inside and outside of a disc)

Expand  $f(z) = \frac{1}{z(z-1)}$  in both

- |z| < 1
- |z| > 1

Solution:

$$\frac{1}{z(z-1)} = -\frac{1}{z}\frac{1}{1-z} = -\frac{1}{z}\sum z^k.$$

and

$$\frac{1}{z(z-1)} = \frac{1}{z^2(1-\frac{1}{z})} = \frac{1}{z^2} \sum_{k=1}^{\infty} \left(\frac{1}{z}\right)^k.$$

Exercise 3.4.15 (Laurent expansions about different points)

Find the Laurent expansion about z = 0 and z = 1 respectively of the following function:

$$f(z) \coloneqq \frac{z+1}{z(z-1)}.$$

Solutions

Note: once you see that everything is in terms of powers of  $(z - z_0)$ , you're essentially done. For z = 0:

$$\frac{z+1}{z(z-1)} = \frac{1}{z} \frac{z+1}{z-1}$$

$$= -\frac{z+1}{z} \frac{1}{1-z}$$

$$= -\left(1 + \frac{1}{z}\right) \sum_{k>0} z^k.$$

For z = 1:

$$\frac{z+1}{z(z-1)} = \frac{1}{z-1} \left( 1 + \frac{1}{z} \right)$$

$$= \frac{1}{z-1} \left( 1 + \frac{1}{1-(1-z)} \right)$$

$$= \frac{1}{z-1} \left( 1 + \sum_{k \ge 0} (1-z)^k \right)$$

$$= \frac{1}{z-1} \left( 1 + \sum_{k \ge 0} (-1)^k (z-1)^k \right).$$

# 4 Cauchy's Theorem

## 4.1 Complex Integrals

**Definition 4.1.1** (Complex Integral)

$$\int_{\gamma} f dz := \int_{I} f(\gamma(t)) \gamma'(t) dt = \int_{\gamma} (u + iv) dx \wedge (-v + iu) dy.$$

Theorem 4.1.2 (Cauchy-Goursat Theorem).

If f is holomorphic on a region  $\Omega$  with  $\pi_1\Omega=1$ , then for any closed path  $\gamma\subseteq\Omega$ ,

$$\int_{\gamma} f(z) \, dz = 0.$$

#### Slogan 4.1.3

Closed path integrals of holomorphic functions vanish.

## 4.2 Applications of Cauchy's Theorem

#### 4.2.1 Integral Formulas and Estimates

See reference

Cauchy's Theorem 23

#### Theorem 4.2.1 (Cauchy Integral Formula).

Suppose f is holomorphic on  $\Omega$ , then for any  $z_0 \in \Omega$  and any open disc  $\overline{D_R(z_0)}$  such that  $\gamma := \partial \overline{D_R(z_0)} \subseteq \Omega$ ,

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z_0} d\xi$$

and

$$\frac{\partial^n f}{\partial z^n}(z_0) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi.$$

*Proof.* It follows from a consequence of Cauchy's theorem (see above) that if  $C(z_0, r)$  denotes the circle of radius r around  $z_0$  for a sufficiently small r > 0 then

$$\left| \frac{1}{2\pi i} \int_{C} \frac{f(z)}{z - z_{0}} dz - f(z_{0}) \right| = \left| \frac{1}{2\pi i} \int_{C(z_{0}, r)} \frac{f(z) - f(z_{0})}{z - z_{0}} dz \right| 
= \left| \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{f(z_{0} + re^{i\theta}) - f(z_{0})}{re^{i\theta}} ire^{i\theta} d\theta \right| 
\leq \frac{1}{2\pi} 2\pi \times \sup_{\theta \in [0, 2\pi)} |f(z_{0} + re^{i\theta}) - f(z_{0})| 
\text{(by } ML \text{ inequality)}.$$

As f is continuous it follows that the righthand side goes to zero as r tends to zero. This completes the proof.

Figure 2: image 2021-05-27-16-54-06

Proof (?).

Proof (?).

*Proof.* (\*) Using Cauchy's integral formula we can write that

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \to 0} \frac{1}{2\pi i h} \int_C (\frac{f(z)}{z - z_0 - h} - \frac{f(z)}{z - z_0}) dz$$

$$(C \text{ is so chosen that the point } z_0 + h \text{ is enclosed by } C)$$

$$= \lim_{h \to 0} \frac{1}{2\pi i h} \int_C \frac{f(z)h}{(z - z_0 - h)(z - z_0)} dz.$$

So we need to prove that

$$\left| \int_{C} \frac{f(z)}{(z - z_0 - h)(z - z_0)} dz - \int_{C} \frac{f(z)}{(z - z_0)^2} dz \right|$$

$$= \left| \int_{C} \frac{f(z)h}{(z - z_0 - h)(z - z_0)^2} dz \right| \to 0, \text{ as } h \to 0.$$

We will basically use ML inequality to prove this. Note that, as f is continuous it is bounded on C by M (say). Let  $\alpha = \min\{|z - z_0| : z \in C\}$ . Then  $|z - z_0|^2 \ge \alpha^2$  and  $\alpha \le |z - z_0| = |z - z_0 - h + h| \le |z - z_0 - h| + |h|$  and hence for  $|h| \le \frac{\alpha}{2}$  (after all h is going to be small) we get  $|z - z_0 - h| \ge \alpha - |h| \ge \frac{\alpha}{2}$ . Therefore

$$\left| \int_C \frac{f(z)h}{(z-z_0-h)(z-z_0)^2} dz \right| \le \frac{M|h|l}{\frac{\alpha}{2}\alpha^2} = \frac{2M|h|l}{\alpha^3} \to 0,$$

as  $h \to 0$ . By repeating exactly the same technique we get  $f^2(z_0) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^3} dz$  and so on.

Theorem 4.2.2 (Cauchy's Inequality / Cauchy's Estimate).

For  $z_0 \in D_R(z_0) \subset \Omega$ , setting  $M := \sup_{z \in \gamma} |f(z)|$  so  $|f(z)| \leq M$  on  $\gamma$ 

$$\left| f^{(n)}(z_0) \right| \le \frac{n!}{2\pi} \int_0^{2\pi} \frac{M}{R^{n+1}} R \, d\theta = \frac{Mn!}{R^n}.$$

Proof (of Cauchy's inequality).

- Given  $z_0 \in \Omega$ , pick the largest disc  $D_R(z_0) \subset \Omega$  and let  $C = \partial D_R$ .
- Then apply the integral formula.

$$\left| f^{(n)}(z_0) \right| = \left| \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right| 
= \left| \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f\left(z_0 + re^{i\theta}\right) rie^{i\theta}}{(re^{i\theta})^{n+1}} d\theta \right| 
\leq \frac{n!}{2\pi} \int_0^{2\pi} \left| \frac{f\left(z_0 + re^{i\theta}\right) rie^{i\theta}}{(re^{i\theta})^{n+1}} \right| d\theta 
= \frac{n!}{2\pi} \int_0^{2\pi} \frac{\left| f\left(z_0 + re^{i\theta}\right) \right|}{r^n} d\theta 
\leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{M}{r^n} d\theta 
= \frac{Mn!}{r^n}.$$

#### **Slogan 4.2.3**

The *n*th Taylor coefficient of an analytic function is at most  $\sup_{|z|=R} |f|/R^n$ .

#### Theorem 4.2.4 (Mean Value Property for Holomorphic Functions).

If f is holomorphic on  $D_r(z_0)$ 

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta = \frac{1}{\pi r^2} \iint_{D_r(z_0)} f(z) dA.$$

Taking the real part of both sides, one can replace f = u + iv with u.

#### 4.2.2 Liouville

#### Theorem 4.2.5 (Liouville's Theorem).

If f is entire and bounded, f is constant.

Proof (of Liouville).

- Since f is bounded,  $f(z) \leq M$  uniformly on  $\mathbb{C}$ .
- Apply Cauchy's estimate for the 1st derivative:

$$|f'(z)| \le \frac{1! ||f||_{C_R}}{R} \le \frac{M}{R} \stackrel{R \to \infty}{\longrightarrow} 0,$$

so f'(z) = 0 for all z.

**Exercise 2.E.** [SSh03, 2.15] Suppose f is continuous and non-zero on  $\overline{\mathbb{D}}$  and holomorphic on  $\mathbb{D}$  such that |f(z)| = 1 for all |z| = 1. Show that f is then constant.

Figure 3: image\_2021-05-17-11-54-14

#### Exercise 4.2.6 (?)

#### 4.2.3 Continuation Principle

## Theorem 4.2.7 (Continuation Principle / Identity Theorem).

If f is holomorphic on a bounded connected domain  $\Omega$  and there exists a sequence  $\{z_i\}$  with a limit point in  $\Omega$  such that  $f(z_i) = 0$ , then  $f \equiv 0$  on  $\Omega$ .

#### **Slogan 4.2.8**

Two functions agreeing on a set with a limit point are equal on a domain.

Proof (?).

Apply Improved Taylor Theorem?

todo

**Exercise 2.D.** [SSh03, 2.13] If f is holomorphic on a region  $\Omega$  and for each  $z_0 \in \Omega$  at least one coefficient in the power series expansion  $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$  is zero. Then show that f is a polynomial.

Figure 4: image\_2021-05-17-11-53-33

Exercise 4.2.9 (?)

#### 4.3 Exercises

Exercise 4.3.1 (Primitives imply vanishing integral)

Show that if f has a primitive F on  $\Omega$  then  $\int_{\gamma} f = 0$  for every closed curve  $\gamma \subseteq \Omega$ .

#### Exercise 4.3.2 (?)

Prove the uniform limit theorem for holomorphic functions: if  $f_n \to f$  locally uniformly and each  $f_n$  is holomorphic then f is holomorphic.

#### Solution:

This is S&S Theorem 5.2. Statement: if  $f_n \to f$  uniformly locally uniformly on  $\Omega$  then f is holomorphic on  $\Omega$ .

• Let  $D \subset \Omega$  with  $\overline{\mathbb{D}} \subset \Omega$  and  $\Delta \subset D$  be a triangle.

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- Apply Goursat: ∫<sub>Δ</sub> f<sub>n</sub> = 0.
   f<sub>n</sub> → f uniformly on Δ since it is closed and bounded and thus compact by Heine-Borel, so f is continuous and

$$\lim_{n} \int_{\Delta} f_n = \int_{\Delta} \lim_{n} f_n := \int_{\Delta} f.$$

• Apply Morera's theorem:  $\int_{\Lambda} f$  vanishes on every triangle in  $\Omega$ , so f is holomorphic on

#### Exercise 4.3.3 (?)

Prove that if  $f_n \to f$  locally uniformly with  $f_n$  holomorphic, then  $f'_n \to f'$  locally uniformly and f' is holomorphic.

#### Solution:

- Simplifying step: for some reason, it suffices to assume  $f_n \to f$  uniformly on all of  $\Omega$ ?
- Take  $\Omega_R$  to be  $\Omega$  with a buffer of R, so  $d(z,\partial\Omega) > R$  for every  $z \in \overline{\Omega_R}$ .
- It suffices to show the following bound for F any holomorphic function on  $\Omega$ :

$$\sup_{z \in \Omega_R} |F'(z)| \le \frac{1}{R} \sup_{\zeta \in \Omega} |F(\zeta)| \qquad \forall R,$$

where on the right we take the sup over all  $\Omega$ .

- Then take  $F := f_n f$  and  $R \to 0$  to conclude, since the right-hand side is a constant not depending on  $\Omega_R$ .
- For any  $z \in \Omega_R$ , we have  $\overline{D_R(z)} \subseteq \Omega_R$ , so Cauchy's integral formula can be applied:

$$\begin{split} |F'(z)| &= \left| \frac{1}{2\pi i} \int_{\partial D_R(z)} \frac{F(\xi)}{(\xi - z)^2} d\xi \right| \\ &\leq \frac{1}{2\pi} \int_{\partial D_R(z)} \frac{|F(\xi)|}{|\xi - z|^2} d\xi \\ &\leq \frac{1}{2\pi} \int_{\partial D_R(z)} \frac{\sup_{\zeta \in \Omega} |F(\zeta)|}{|\xi - z|^2} d\xi \\ &= \frac{1}{2\pi} \sup_{\zeta \in \Omega} |F(\zeta)| \int_{\partial D_R(z)} \frac{1}{R^2} d\xi \\ &= \frac{1}{2\pi} \sup_{\zeta \in \Omega} |F(\zeta)| \frac{1}{R^2} \int_{\partial D_R(z)} d\xi \\ &= \frac{1}{2\pi} \sup_{\zeta \in \Omega} |F(\zeta)| \frac{1}{R^2} 2\pi R \\ &\leq \frac{1}{2\pi} \sup_{\zeta \in \Omega} |F(\zeta)| \frac{1}{R^2} (2\pi R) \\ &= \frac{1}{R} \sup_{\zeta \in \Omega} |F(\zeta)|. \end{split}$$

Now

$$||f'_n - f'||_{\infty,\Omega_R} \le \frac{1}{R} ||f_n - f||_{\infty,\Omega},$$

where if R is fixed then by uniform convergence of  $f_n \to f$ , for n large enough  $||f_n - f|| < \varepsilon/R$ .

#### 4.4 Morera's Theorem

Theorem 4.4.1 (Morera's Theorem).

If f is continuous on a domain  $\Omega$  and  $\int_T f = 0$  for every triangle  $T \subset \Omega$ , then f is holomorphic.

#### Slogan 4.4.2

If every integral along a triangle vanishes, implies holomorphic.

Corollary 4.4.3 (Sufficient condition for a sequence to converge to a holomorphic function).

If  $\{f_n\}_{n\in\mathbb{N}}$  is a holomorphic sequence on a region  $\Omega$  which uniformly converges to f on every compact subset  $K\subseteq\Omega$ , then f is holomorphic, and  $f'_n\to f'$  uniformly on every such compact subset K.

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Proof (?).

Commute limit with integral and apply Morera's theorem.

**Remark 4.4.4:** This can be applied to series of the form  $\sum_{k} f_k(z)$ .

#### 4.4.1 Symmetric Regions

In this section, take  $\Omega$  to be a region symmetric about the real axis, so  $z \in \Omega \iff \bar{z} \in \Omega$ . Partition this set as  $\Omega^+ \subseteq \mathbb{H}, I \subseteq \mathbb{R}, \Omega^- \subseteq \overline{\mathbb{H}}$ .

#### Theorem 4.4.5 (Symmetry Principle).

Suppose that  $f^+$  is holomorphic on  $\Omega^+$  and  $f^-$  is holomorphic on  $\Omega^-$ , and f extends continuously to I with  $f^+(x) = f^-(x)$  for  $x \in I$ . Then the following piecewise-defined function is holomorphic on  $\Omega$ :

$$f(z) := \begin{cases} f^{+}(z) & z \in \Omega^{+} \\ f^{-}(z) & z \in \Omega^{-} \\ f^{+}(z) = f^{-}(z) & z \in I. \end{cases}$$

Proof (?).

Apply Morera?

#### Theorem 4.4.6 (Schwarz Reflection).

If f is continuous and holomorphic on  $\mathbb{H}^+$  and real-valued on  $\mathbb{R}$ , then the extension defined by  $F^-(z) = \overline{f(\overline{z})}$  for  $z \in \mathbb{H}^-$  is a well-defined holomorphic function on  $\mathbb{C}$ .

Proof (?).

Apply the symmetry principle.

**Remark 4.4.7:**  $\mathbb{H}^+$ ,  $\mathbb{H}^-$  can be replaced with any region symmetric about a line segment  $L \subseteq \mathbb{R}$ .

# **5** Zeros and Singularities

#### **Definition 5.0.1** (Singularity)

A point  $z_0$  is an **isolated singularity** if  $f(z_0)$  is undefined but f(z) is defined in a punctured neighborhood  $D(z_0) \setminus \{z_0\}$  of  $z_0$ .

There are three types of isolated singularities:

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- Removable singularities
- Poles
- Essential singularities

**Example 5.0.2**(?): The singularities of a rational function are always isolated, since there are finitely many zeros of any polynomial. The function F(z) := Log(z) has a singularity at z = 0 that is **not** isolated, since every neighborhood intersects the branch cut  $(-\infty, 0) \times \{0\}$ , where F is not even defined. The function  $G(z) := 1/\sin(\pi/z)$  has a non-isolated singularity at 0 and isolated singularities at 1/n for all n.

#### **Definition 5.0.3** (Removable Singularities)

If  $z_0$  is a singularity of f, then  $z_0$  is a **removable singularity** iff there exists a holomorphic function g such that f(z) = g(z) in a punctured neighborhood of  $z_0$ . Equivalently,

$$\lim_{z \to z_0} (z - z_0) f(z) = 0.$$

Equivalently, f is bounded on a neighborhood of  $z_0$ .

#### Theorem 5.0.4 (Extension over removable singularities).

If f is holomorphic on  $\Omega \setminus \{z_0\}$  where  $z_0$  is a removable singularity, then there is a unique holomorphic extension of f to all of  $\Omega$ .

Proof (?).

Take  $\gamma$  to be a circle centered at  $z_0$  and use

$$f(z) \coloneqq \int_{\gamma} \frac{f(\xi)}{\xi - z} dx.$$

This is valid for  $z \neq z_0$ , but the right-hand side is analytic. (?)

Revisit

#### Theorem 5.0.5 (Improved Taylor Remainder Theorem).

If f is analytic on a region  $\Omega$  containing  $z_0$ , then f can be written as

$$f(z) = \left(\sum_{k=0}^{n-1} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k\right) + R_n(z) (z - z_0)^n,$$

where  $R_n$  is analytic.

#### **Definition 5.0.6** (Zeros)

If f is analytic and not identically zero on  $\Omega$  with  $f(z_0) = 0$ , then there exists a nonvanishing holomorphic function g such that

$$f(z) = (z - z_0)^n q(z).$$

We refer to  $z_0$  as a **zero of order** n.

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## **Definition 5.0.7** (Poles (and associated terminology))

A pole  $z_0$  of a function f(z) is a zero of  $g(z) := \frac{1}{f(z)}$ . Equivalently,  $\lim_{z \to z_0} f(z) = \infty$ . In this case there exists a minimal n and a holomorphic h such that

$$f(z) = (z - z_0)^{-n} h(z).$$

Such an n is the order of the pole. A pole of order 1 is said to be a simple pole.

## **Definition 5.0.8** (Principal Part and Residue)

If f has a pole of order n at  $z_0$ , then there exist a holomorphic G in a neighborhood of  $z_0$  such that

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \dots + \frac{a_{-1}}{z - z_0} + G(z) := P(z) + G(z).$$

The term P(z) is referred to as the *principal part of f at z*<sub>0</sub> consists of terms with negative degree, and the *residue* of f at  $z_0$  is the coefficient  $a_{-1}$ .

#### **Definition 5.0.9** (Essential Singularity)

A singularity  $z_0$  is essential iff it is neither removable nor a pole. Equivalently, a Laurent series expansion about  $z_0$  has a principal part with infinitely many terms.

#### Theorem 5.0.10 (Casorati-Weierstrass).

If f is holomorphic on  $\Omega \setminus \{z_0\}$  where  $z_0$  is an essential singularity, then for every  $V \subset \Omega \setminus \{z_0\}$ , f(V) is dense in  $\mathbb{C}$ .

#### Slogan 5.0.11

The image of a punctured disc at an essential singularity is dense in  $\mathbb{C}$ .

Proof (of Casorati-Weierstrass).

Pick  $w \in \mathbb{C}$  and suppose toward a contradiction that  $D_R(w) \cap f(V)$  is empty. Consider

$$g(z) \coloneqq \frac{1}{f(z) - w},$$

and use that it's bounded to conclude that  $z_0$  is either removable or a pole for f.

#### **Definition 5.0.12** (Singularities at infinity)

For any f holomorphic on an unbounded region, we say  $z = \infty$  is a singularity (of any of the above types) of f if g(z) := f(1/z) has a corresponding singularity at z = 0.

#### **Definition 5.0.13** (Meromorphic)

A function  $f:\Omega\to\mathbb{C}$  is meromorphic iff there exists a sequence  $\{z_n\}$  such that

•  $\{z_n\}$  has no limit points in  $\Omega$ .

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- f is holomorphic in  $\Omega \setminus \{z_n\}$ .
- f has poles at the points  $\{z_n\}$ .

Equivalently, f is holomorphic on  $\Omega$  with a discrete set of points delete which are all poles of f.

#### Theorem 5.0.14 (Meromorphic implies rational).

Meromorphic functions on  $\mathbb{C}$  are rational functions.

Proof (?).

Consider f(z) - P(z), subtracting off the principal part at each pole  $z_0$ , to get a bounded entire function and apply Liouville.

#### Theorem 5.0.15 (Riemann Extension Theorem).

A singularity of a holomorphic function is removable if and only if the function is bounded in some punctured neighborhood of the singular point.

# 6 Residues

Proposition 6.0.1 (Length bound).

$$\left| \int_{\gamma} f \right| \le \sup_{z \in \gamma} |f| \cdot \operatorname{length}(\gamma).$$

This follows from

$$\left| \int_{\gamma} f(z) dz \right| \leq \sup_{t \in [a,b]} |f(z(t))| \int_{a}^{b} |z'(t)| dt \leq \sup_{z \in \gamma} |f(z)| \cdot \operatorname{length}(\gamma).$$

Proposition 6.0.2 (Jordan's Lemma).

todo

#### Fact 6.0.3

Some useful facts:

$$|e^z| = e^{\Re(z)}.$$

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On  $S^1$ ,

$$1/z = e^{-i\theta}$$
$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + 1/z}{2}$$
$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - 1/z}{2i}.$$

#### Fact 6.0.4

The major fact that reduces integrals to residues:

$$\int_{\gamma} z^k dz = \int_0^{2\pi} e^{ik\theta} i e^{i\theta} d\theta = \int_0^{2\pi} e^{i(k+1)\theta} d\theta = \begin{cases} 2\pi i & k = -1 \\ 0 & \text{else.} \end{cases}.$$

Thus

$$\int \sum_{k>-M} c_k z^k = \sum_{k>-M} \int c_k z^k = 2\pi i c_{-1},$$

i.e. the integral picks out the  $c_{-1}$  coefficient in a Laurent series expansion.

Remark 6.0.5: Check: do you need residues at all?? You may be able to just compute an integral!

• Directly by parameterization:

$$\int_{\gamma} f = \int_{a}^{b} f(z(t)) z'(t) \qquad \text{for } z(t) \text{ a parameterization of } \gamma,$$

- Finding a primitive F,
- Writing  $z = z_0 + re^{i\theta}$

## 6.1 Computing Residues

**Definition 6.1.1** (Toy contour)

A closed Jordan curve that separates  $\mathbb{C}$  into an exterior and interior region is referred to as a **toy contour**.

**Definition 6.1.2** (Winding Number)

For  $\gamma \subseteq \Omega$  a closed curve not passing through a point  $z_0$ , the winding number of  $\gamma$  about

 $z_0$  is defined as

$$n_{\gamma}(z_0) := \frac{1}{2\pi i} \int_{\gamma} \frac{1}{\xi - z_0} d\xi.$$

Proposition 6.1.3 (Residue formula for higher order poles).

If f has a pole  $z_0$  of order n, then

Res 
$$f = \lim_{z \to z_0} \frac{1}{(n-1)!} \left( \frac{\partial}{\partial z} \right)^{n-1} (z - z_0)^n f(z).$$

Proposition 6.1.4 (Residue formula for simple poles).

As a special case, if  $z_0$  is a simple pole of f, then

$$\operatorname{Res}_{z=z_0} f = \lim_{z \to z_0} (z - z_0) f(z).$$

Corollary 6.1.5 (Better derivative formula that sometimes works for simple poles). If additionally f = g/h where  $h(z_0) = 0$  and  $h'(z_0) \neq 0$ , we can apply L'Hopital's rule to compute this residue:

$$(z-z_0)\frac{g(z)}{h(z)} = \frac{(z-z_0)g(z)}{h(z)} \stackrel{LH}{=} \frac{g(z) + (z-z_0)g'(z)}{h'(z)} \stackrel{z \to z_0}{\longrightarrow} \frac{g(z_0)}{h'(z_0)}.$$

Example 6.1.6 (Residue of a simple pole (order 1)): Let  $f(z) = \frac{1}{1+z^2}$ , then  $Res(i, f) = \frac{1}{2i}$ .

Theorem 6.1.7(The Residue Theorem).

Let f be meromorphic on a region  $\Omega$  with poles  $\{z_1, z_2, \dots, z_N\}$ . Then for any  $\gamma \in \Omega \setminus \{z_1, z_2, \dots, z_N\}$ ,

$$\frac{1}{2\pi i} \int_{\gamma} f(z) dz = \sum_{j=1}^{N} n_{\gamma}(z_j) \operatorname{Res}_{z=z_j} f.$$

If  $\gamma$  is a toy contour, then

$$\frac{1}{2\pi i} \int_{\gamma} f \, dz = \sum_{j=1}^{N} \operatorname{Res}_{z=z_{j}} f.$$

#### 6.1.1 Exercises

Some good computations here.

Problem 6.1.1 (?)

Show that the complex zeros of  $f(z) := \sin(\pi z)$  are exactly  $\mathbb{Z}$ , and each is order 1. Calculate the residue of  $1/\sin(\pi x)$  at  $z = n \in \mathbb{Z}$ .

Exerci

Exe

**Exercise 3.A.** [SSh03, 3.1] Show that the complex zeros of  $\sin \pi z$  are exactly at the integers, and are each of order 1. Calculate the residue of  $1/\sin \pi x$  are  $z=n\in\mathbb{Z}$ .

Exercise 3.C. [SSh03, 3.8] Prove that

 $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$ 

Exercise 6.1.8 (?)

$$\int_{\mathbb{R}} \frac{1}{(1+x^2)^2} \, dx.$$

Solution:

- Factor  $(1+z^2)^2 = ((z-i)(z+i))^2$ , so f has poles at  $\pm i$  of order 2.
- Take a semicircular contour  $\gamma := I_R \cup D_R$ , then  $f(z) \approx 1/z^4 \to 0$  for large R and  $\int_{D_R} f \to 0$ .
- Note  $\int_{I_R} f \to \int_{\mathbb{R}} f$ , so  $\int_{\gamma} f \to \int_{\mathbb{R}} f$ .
- $\int_{\gamma} f = 2\pi i \sum_{z_0} \underset{z=z_0}{\text{Res }} f$ , and  $z_0 = i$  is the only pole in this region.
- Compute

$$\operatorname{Res}_{z=i} f = \lim_{z \to i} \frac{1}{(2-1)!} \frac{\partial}{\partial z} (z-i)^2 f(z)$$

$$= \lim_{z \to i} \frac{\partial}{\partial z} \frac{1}{(z+i)^2}$$

$$= \lim_{z \to i} \frac{-2}{(z+i)^3}$$

$$= -\frac{2}{(2i)^3}$$

$$= \frac{1}{4i}$$

$$\implies \int_{\gamma} f = \frac{2\pi i}{4i} = \pi/2,$$

6.1 Computing Residues

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6 Residues

### 6.2 Argument Principle

### Theorem 6.2.1 (Argument Principle).

For f meromorphic in  $\gamma^{\circ}$  with zeros  $\{z_j\}$  and poles  $\{p_k\}$  repeated with multiplicity where  $\gamma$  does not intersect any zeros or poles, then

$$\Delta_{\gamma} \arg f(z) := \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{i} n_{\gamma}(z_i) - \sum_{k} n_{\gamma}(p_k) = Z_f - P_f,$$

where  $Z_f$  and  $P_f$  are the number of zeros and poles respectively enclosed by  $\gamma$ , counted with multiplicity.

### Proof (?).

Residue formula applied to  $\frac{f'}{f}$ ?

### Corollary 6.2.2 (Rouché's Theorem).

If f, g are analytic on a domain  $\Omega$  with finitely many zeros in  $\Omega$  and  $\gamma \subset \Omega$  is a closed curve surrounding each point exactly once, where |g| < |f| on  $\gamma$ , then f and f + g have the same number of zeros.

Alternatively:

Suppose f = g + h with  $g \neq 0, \infty$  on  $\gamma$  with |g| > |h| on  $\gamma$ . Then

$$\Delta_{\gamma} \arg(f) = \Delta_{\gamma} \arg(h)$$
 and  $Z_f - P_f = Z_g - P_g$ .

Prove

### Corollary 6.2.3 (Open Mapping).

Any holomorphic non-constant map is an open map.

Prove

### Corollary 6.2.4 (Maximum Modulus).

If f is holomorphic and nonconstant on an open connected region  $\Omega$ , then |f| can not attain a maximum on  $\Omega$ . If  $\Omega$  is bounded and f is continuous on  $\overline{\Omega}$ , then  $\max_{\overline{\Omega}} |f|$  occurs on  $\partial\Omega$ .

Conversely, if f attains a local supremum at  $z_0 \in \Omega$ , then f is constant on  $\Omega$ .

Prove

### Corollary 6.2.5(?).

If f is nonzero on  $\Omega$ , then f attains a minimum on  $\partial\Omega$ . This follows from applying the MMP to 1/f.

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### 6.2.1 Exercises

**Exercise 3.E.** [SSh03, 3.14] Prove that all entire functions that are also injective take the form f(z) = az + b with  $a, b \in \mathbb{C}$  and  $a \neq 0$ .

Figure 5: image\_2021-05-17-13-33-55

# **7** Rouche

### 7.1 Counting Zeros

### Example 7.1.1:

- Take  $P(z) = z^4 + 6z + 3$ .
- On |z| < 2:
  - Set  $f(z) = z^4$  and g(z) = 6z + 3, then  $|g(z)| \le 6|z| + 3 = 15 < 16 = |f(z)|$ .
  - So P has 4 zeros here.
- On |z| < 1:
  - Set f(z) = 6z and  $g(z) = z^4 + 3$ .
  - Check  $|g(z)| \le |z|^4 + 3 = 4 < 6 = |f(z)|$ .
  - So P has 1 zero here.

### Example 7.1.2:

- Claim: the equation  $\alpha z e^z = 1$  where  $|\alpha| > e$  has exactly one solution in  $\mathbb{D}$ .
- Set  $f(z) = \alpha z$  and  $g(z) = e^{-z}$ .
- Estimate at |z|=1 we have  $|g|=|e^{-z}|=e^{-\Re(z)}\leq e^1<|\alpha|=|f(z)|$
- f has one zero at  $z_0 = 0$ , thus so does f + g.

# 8 | Conformal Maps

### 8.1 Linear Fractional Transformations

Rouche 38

### **Definition 8.1.1** (Conformal Map / Biholomorphism)

A map f is **conformal** on  $\Omega$  iff f is complex-differentiable,  $f'(z) \neq 0$  for  $z \in \Omega$ , and f preserves signed angles (so f is orientation-preserving). Conformal implies holomorphic, and a bijective conformal map has a holomorphic inverse. A bijective conformal map  $f: U \to V$  is called a **biholomorphism**, and we say U and V are **biholomorphic**. Self-biholomorphisms of a domain  $\Omega$  form a group  $\operatorname{Aut}_{\mathbb{C}}(\Omega)$ .

**Remark 8.1.2:** There is an oft-used weaker condition that  $f'(z) \neq 0$  for any point. Note that that this condition alone doesn't necessarily imply f is holomorphic, since anti-holomorphic maps may be nonzero derivative. For example, take  $f(z) = \bar{z}$ , so f(x+iy) = x - iy – this does not satisfy the Cauchy-Riemann equations.

**Remark 8.1.3:** A bijective holomorphic map automatically has a holomorphic inverse. This can be weakened: an injective holomorphic map satisfies  $f'(z) \neq 0$  and  $f^{-1}$  is well-defined on its range and holomorphic.

**Definition 8.1.4** (Linear fractional transformation / Mobius transformation) A map of the following form is a *linear fractional transformation*:

$$T(z) = \frac{az+b}{cz+d},$$

where the denominator is assumed to not be a multiple of the numerator. These have inverses given by

$$T^{-1}(w) = \frac{dw - b}{-cw + a}.$$

### Proposition 8.1.5(?).

Given any three points  $z_1, z_2, z_3$ , the following Mobius transformation sends them to  $1, 0, \infty$  respectively:

$$f(z) := \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}.$$

Such a map is sometimes denoted  $(z, z_1, z_2, z_3)$ .

### Example 8.1.6(?):

- $(z, i, 1, -1) : \mathbb{D} \to \mathbb{H}$
- $(z, 0, -1, 1) : \mathbb{D} \cap \mathbb{H} \to Q_1$ .

### Theorem 8.1.7 (Cayley Transform).

The fractional linear transformation given by  $F(z) = \frac{i-z}{i+z}$  maps  $\mathbb{D} \to \mathbb{H}$  with inverse G(w) =

$$i\frac{1-w}{1+w}.$$

### Theorem 8.1.8 (Classification of Conformal Maps).

There are 8 major types of conformal maps:

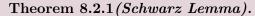
Type/Domains	Formula
Translation/Dilation/Rotation	$z \mapsto e^{i\theta}(cz+h)$
Sectors to sectors	$z \mapsto z^n$
$\mathbb{D}_{\frac{1}{2}} \to \mathbb{H}_{\frac{1}{2}}$ , the first quadrant	$z \mapsto \frac{1+z}{1-z}$
$\mathbb{H} \to S$	$z\mapsto \log(z)$
$\mathbb{D}_{rac{1}{2}}  o S_{rac{1}{2}}$	$z \mapsto \log(z)$
$\begin{array}{c} \mathbb{D}_{\frac{1}{2}} \to S_{\frac{1}{2}} \\ S_{\frac{1}{2}} \to \mathbb{D}_{\frac{1}{2}} \end{array}$	$z \mapsto e^{iz}$
$\mathbb{D}_{rac{1}{2}} o\mathbb{H}$	$z\mapsto \frac{1}{2}\left(z+\frac{1}{z}\right)$
$S_{rac{1}{2}}  o \mathbb{H}$	$z\mapsto \sin(z)$

### Theorem 8.1.9 (Characterization of conformal maps).

Conformal maps  $\mathbb{D} \to \mathbb{D}$  have the form

$$g(z) = \lambda \frac{1-a}{1-\bar{a}z}, \quad |a| < 1, \quad |\lambda| = 1.$$

### 8.2 Schwarz



If  $f: \mathbb{D} \to \mathbb{D}$  is holomorphic with f(0) = 0, then

- 1.  $|f(z)| \le |z|$  for all  $z \in \mathbb{D}$ 2.  $|f'(0)| \le 1$ .

Moreover, if

- $|f(z_0)| = |z_0|$  for any  $z_0 \in \mathbb{D}$ , or |f'(0)| = 1,

then f is a rotation.

Proof (?).

Apply the maximum modulus principle to f(z)/z.

8.2 Schwarz

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Exercise 8.2.2 (?)

Show that  $\operatorname{Aut}_{\mathbb{C}}(\mathbb{C}) = \{z \mapsto az + b \mid a \in \mathbb{C}^{\times}, b \in \mathbb{C}\}.$ 

Theorem  $8.2.3 (Biholomorphisms\ of\ the\ disc).$ 

$$\operatorname{Aut}_{\mathbb{C}}(\mathbb{D}) = \left\{ z \mapsto e^{i\theta} \left( \frac{\alpha - z}{1 - \overline{\alpha}z} \right) \right\}.$$

Proof(?).

Schwarz lemma.

Theorem 8.2.4(?).

$$\operatorname{Aut}_{\mathbb{C}}(\mathbb{H}) = \left\{ z \mapsto \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{C}, ad-bc = 1 \right\} \cong \operatorname{PSL}_2(\mathbb{R}).$$

### 8.3 By Type

### 8.3.1 Plane to Disc

$$\varphi : \mathbb{H} \to \mathbb{D}$$

$$\varphi(z) = \frac{z - i}{z + i} \qquad f^{-1}(z) = i \left(\frac{1 + w}{1 - w}\right).$$

### 8.3.2 Sector to Disc

For  $S_{\alpha} \coloneqq \left\{ z \in \mathbb{C} \mid 0 < \arg(z) < \alpha \right\}$  an open sector for  $\alpha$  some angle, first map the sector to the half-plane:

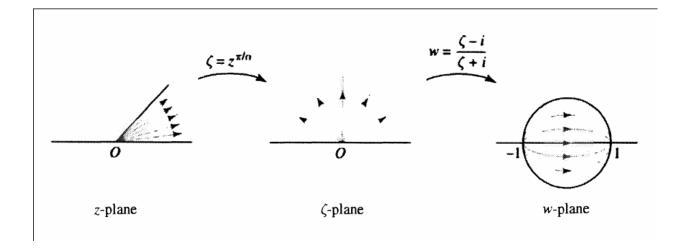
$$g: S_{\alpha} \to \mathbb{H}$$
$$g(z) = z^{\frac{\pi}{\alpha}}.$$

Then compose with a map  $\mathbb{H} \to \mathbb{D}$ :

$$\begin{split} f: S_{\alpha} &\to \mathbb{D} \\ f(z) &= (\varphi \circ g)(z) = \frac{z^{\frac{\pi}{\alpha}} - i}{z^{\frac{\pi}{\alpha}} + i}. \end{split}$$

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8.3 By Type



### 8.3.3 Strip to Disc

- Map to horizontal strip by rotation  $z \mapsto \lambda z$ .
- Map horizontal strip to sector by  $z \mapsto e^z$
- Map sector to  $\mathbb{H}$  by  $z \mapsto z^{\frac{\pi}{\alpha}}$ .
- Map  $\mathbb{H} \to \mathbb{D}$ .

$$e^z : \mathbb{R} \times (0, \pi) \to \mathbb{R} \times (0, \infty).$$

# 9 | Schwarz Reflection

# 10 | Schwarz Lemma

?

Montel's theorem

Normal families

Schwarz lemma

Equicontinuity

Schwarz Reflection 42

### 11 | Linear Fractional Transformations

# 12 | Montel's Theorem

# 13 Unsorted Theorems

### Theorem 13.0.1 (Riemann Mapping).

If  $\Omega$  is simply connected, nonempty, and not  $\mathbb{C}$ , then for every  $z_0 \in \Omega$  there exists a unique conformal map  $F: \Omega \to \mathbb{D}$  such that  $F(z_0) = 0$  and  $F'(z_0) > 0$ .

Thus any two such sets  $\Omega_1, \Omega_2$  are conformally equivalent.

### Theorem 13.0.2 (Riemann's Removable Singularity Theorem).

If f is holomorphic on  $\Omega$  except possibly at  $z_0$  and f is bounded on  $\Omega \setminus \{z_0\}$ , then  $z_0$  is a removable singularity.

### Theorem 13.0.3 (Little Picard).

If  $f: \mathbb{C} \to \mathbb{C}$  is entire and nonconstant, then  $\operatorname{im}(f)$  is either  $\mathbb{C}$  or  $\mathbb{C} \setminus \{z_0\}$  for some point  $z_0$ .

### Corollary 13.0.4.

The ring of holomorphic functions on a domain in  $\mathbb{C}$  has no zero divisors.

Proof. ???

Find the proof!

### Morera

Proposition 13.0.5 (Bounded Complex Analytic Functions form a Banach Space). For  $\Omega \subseteq \mathbb{C}$ , show that  $A(\mathbb{C}) := \{ f : \Omega \to \mathbb{C} \mid f \text{ is bounded} \}$  is a Banach space.

Proof.

Apply Morera's Theorem and Cauchy's Theorem

# Appendix: Proofs of the Fundamental Theorem of Algebra

### 14.0.1 Fundamental Theorem of Algebra: Argument Principle

- Let  $P(z) = a_n z^n + \cdots + a_0$  and g(z) = P'(z)/P(z), note P is holomorphic
- Since  $\lim_{|z|\to\infty} P(z) = \infty$ , there exist an R > 0 such that P has no roots in  $\{|z| \ge R\}$ .
- Apply the argument principle:

$$N(0) = \frac{1}{2\pi i} \oint_{|\xi|=R} g(\xi) d\xi.$$

- Check that  $\lim_{|z \to \infty|} zg(z) = n$ , so g has a simple pole at  $\infty$
- Then g has a Laurent series  $\frac{n}{z} + \frac{c_2}{z^2} + \cdots$
- Integrate term-by-term to get N(0) = n.

### 14.0.2 Fundamental Theorem of Algebra: Rouche's Theorem

- Let  $P(z) = a_n z^n + \cdots + a_0$
- Set  $f(z) = a_n z^n$  and  $g(z) = P(z) f(z) = a_{n-1} z^{n-1} + \dots + a_0$ , so f + g = P. Choose  $R > \max\left(\frac{|a_{n-1}| + \dots + |a_0|}{|a_n|}, 1\right)$ , then

$$\begin{split} |g(z)| &\coloneqq |a_{n-1}z^{n-1} + \dots + a_1z + a_0| \\ &\le |a_{n-1}z^{n-1}| + \dots + |a_1z| + |a_0| \quad \text{by the triangle inequality} \\ &= |a_{n-1}| \cdot |z^{n-1}| + \dots + |a_1| \cdot |z| + |a_0| \\ &= |a_{n-1}| \cdot R^{n-1} + \dots + |a_1|R + |a_0| \\ &\le |a_{n-1}| \cdot R^{n-1} + |a_{n-2}| \cdot R^{n-1} + \dots + |a_1| \cdot R^{n-1} + |a_0| \cdot R^{n-1} \quad \text{since } R > 1 \implies R^{a+b} \ge R^a \\ &= R^{n-1} \left( |a_{n-1}| + |a_{n-2}| + \dots + |a_1| + |a_0| \right) \\ &\le R^{n-1} \left( |a_n| \cdot R \right) \quad \text{by choice of } R \\ &= R^n |a_n| \\ &= |a_n z^n| \\ &\coloneqq |f(z)| \end{split}$$

• Then  $a_n z^n$  has n zeros in |z| < R, so f + g also has n zeros.

115 Appendix

### 14.0.3 Fundamental Theorem of Algebra: Liouville's Theorem

• Suppose p is nonconstant and has no roots, then  $\frac{1}{p}$  is entire. We will show it is also bounded and thus constant, a contradiction.

• Write  $p(z) = z^n \left( a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n} \right)$ 

- Outside a disc:
  - Note that  $p(z) \stackrel{z \to \infty}{\to} \infty$ . so there exists an R large enough such that  $|p(z)| \ge \frac{1}{A}$  for any fixed chosen constant A.
  - Then  $|1/p(z)| \le A$  outside of |z| > R, i.e. 1/p(z) is bounded there.
- Inside a disc:
  - -p is continuous with no roots and thus must be bounded below on |z| < R.
  - p is entire and thus continuous, and since  $\overline{D}_r(0)$  is a compact set, p achieves a min A there
  - Set  $C := \min(A, B)$ , then  $|p(z)| \ge C$  on all of  $\mathbb C$  and thus  $|1/p(z)| \le C$  everywhere.
  - So 1/p(z) is bounded an entire and thus constant by Liouville's theorem but this forces p to be constant.  $\mathcal{I}$

### 14.0.4 Fundamental Theorem of Algebra: Open Mapping Theorem

- p induces a continuous map  $\mathbb{CP}^1 \to \mathbb{CP}^1$
- The continuous image of compact space is compact;
- Since the codomain is Hausdorff space, the image is closed.
- p is holomorphic and non-constant, so by the Open Mapping Theorem, the image is open.
- Thus the image is clopen in  $\mathbb{CP}^1$ .
- The image is nonempty, since  $p(1) = \sum a_i \in \mathbb{C}$
- $\mathbb{CP}^1$  is connected
- But the only nonempty clopen subset of a connected space is the entire space.
- So p is surjective, and  $p^{-1}(0)$  is nonempty.
- So p has a root.

# 15 | Appendix

### 15.1 Misc Basic Algebra



### Fact 15.1.1 (Standard forms of conic sections)

• Circle:  $x^2 + y^2 = r^2$ 

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• Ellipse: 
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$
  
• Hyperbola:  $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$ 

• Hyperbola: 
$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$

- Rectangular Hyperbola:  $xy = \frac{c^2}{2}$ .
- Parabola:  $-4ax + y^2 = 0$ .

Mnemonic: Write  $f(x,y) = Ax^2 + Bxy + Cy^2 + \cdots$ , then consider the discriminant  $\Delta = B^2 - 4AC$ :

- $\Delta < 0 \iff \text{ellipse}$  $-\Delta < 0$  and  $A = C, B = 0 \iff$  circle
- $\Delta = 0 \iff \text{parabola}$
- $\Delta > 0 \iff \text{hyperbola}$

Fact 15.1.2 (Completing the square)

$$x^{2} - bx = (x - s)^{2} - s^{2}$$
 where  $s = \frac{b}{2}$   
 $x^{2} + bx = (x + s)^{2} - s^{2}$  where  $s = \frac{b}{2}$ .

### Fact 15.1.3

The sum of the interior angles of an *n*-gon is  $(n-2)\pi$ , where each angle is  $\frac{n-2}{n}\pi$ .

### **Definition 15.1.4** (The Dirichlet Problem)

Given a bounded piecewise continuous function  $u: S^1 \to \mathbb{R}$ , is there a unique extension to a continuous harmonic function  $\tilde{u}: \mathbb{D} \to \mathbb{R}$ ?

**Remark 15.1.5:** More generally, this is a boundary value problem for a region where the *values* of the function on the boundary are given. Compare to prescribing conditions on the normal vector on the boundary, which would be a Neumann BVP. Why these show up: a harmonic function on a simply connected region has a harmonic conjugate, and solutions of BVPs are always analytic functions with harmonic real/imaginary parts.

Example 15.1.6 (Dirichlet problem on the strip): See section 27, example 1 in Brown and Churchill. On the strip  $(x,y) \in (0,\pi) \times (0,\infty)$ , set up the BVP for temperature on a thin plate

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with no sinks/sources:

$$\Delta T = 0 \qquad T(0, y) = 0, T(\pi, y) = 0 \ \forall y$$

$$T(x,0) = \sin(x)$$
 
$$T(x,y) \stackrel{y \to \infty}{\longrightarrow} 0.$$

Then the following function is harmonic on  $\mathbb{R}^2$  and satisfies that Dirichlet problem:

$$T(x,y) = e^{-y}\sin(x) = \Re(-ie^{iz}) = \Im(e^{iz}).$$

**Definition 15.1.7** (Logarithmic Derivative)

The **logarithmic derivative** of f is  $(\ln f)' = f'/f$ .

**Remark 15.1.8:** Why this is useful: deriving the argument principle. If f has a pole of order n at  $z_0$ , then write  $f(z) = (z - z_0)^{-n} g(z)$  with g analytic in a neighborhood of  $z_0$ . Then a direct computation of the derivatives will show

$$(\ln f)' := \frac{f'(z)}{f(z)} = -\frac{n}{z - z_0} + \frac{g'(z)}{g(z)}.$$

Thus

$$\frac{1}{2\pi i} \int_{\gamma} (\ln f)' = -n,$$

for  $\gamma$  a small circle about  $z_0$ . A similar argument for  $z_0$  a **zero** of f yields

$$\frac{1}{2\pi i} \int_{\gamma} h = +n.$$

### Exercise 15.1.9 (?)

Show that there is no continuous square root function defined on all of  $\mathbb{C}$ .

#### Solutions

Suppose  $f(z)^2 = z$ . Then f is a section to the covering map

$$p: \mathbb{C}^{\times} \to \mathbb{C}^{\times}$$
$$z \mapsto z^2.$$

so  $p \circ f = \text{id}$ . Using  $\pi_1(\mathbb{C}^\times) = \mathbb{Z}$ , the induced maps are  $p_*(1) = 2$  and  $f_*(1) = n$  for some  $n \in \mathbb{Z}$ . But then  $p_* \circ f_*$  is multiplication by 2n, contradicting  $p_* \circ f_* = \text{id}$  by functoriality.

### Theorem 15.1.10 (Uniformization).

Every Riemann surface S is the quotient of a free proper holomorphic action of a group G on the universal cover  $\tilde{S}$  of S, so  $S \cong \tilde{S}/G$  is a biholomorphism. Moreover,  $\tilde{S}$  is biholomorphic to either

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- $\mathbb{CP}^1$
- $\mathbb{C}$

### **Basics**

- Show that  $\frac{1}{z}\sum_{k=1}^{\infty}\frac{z^k}{k}$  converges on  $S^1\setminus\{1\}$  using summation by parts.
- Show that any power series is continuous on its domain of convergence.
- Show that a uniform limit of continuous functions is continuous.

??

- Show that if f is holomorphic on  $\mathbb{D}$  then f has a power series expansion that converges uniformly on every compact  $K \subset \mathbb{D}$ .
- Show that any holomorphic function f can be uniformly approximated by polynomials.
- Show that if f is holomorphic on a connected region  $\Omega$  and  $f' \equiv 0$  on  $\Omega$ , then f is constant on  $\Omega$ .
- Show that if |f| = 0 on  $\partial\Omega$  then either f is constant or f has a zero in  $\Omega$ .
- Show that if  $\{f_n\}$  is a sequence of holomorphic functions converging uniformly to a function f on every compact subset of  $\Omega$ , then f is holomorphic on  $\Omega$  and  $\{f'_n\}$  converges uniformly to f' on every such compact subset.
- Show that if each  $f_n$  is holomorphic on  $\Omega$  and  $F\coloneqq \sum f_n$  converges uniformly on every compact subset of  $\Omega$ , then F is holomorphic.
- Show that if f is once complex differentiable at each point of  $\Omega$ , then f is holomorphic.

# **Draft of Problem Book**

- Prove the triangle inequality
- Prove the reverse triangle inequality Show that  $\sum z^{k-1}/k$  converges for all  $z \in S^1$  except z=1.
- What is an example of a noncontinuous limit of continuous functions?
- Show that the uniform limit of continuous functions is continuous.
- Show that f is holomorphic if and only if  $\bar{\partial} f = 0$ .
- Show  $n^{\frac{1}{n}} \stackrel{n \to \infty}{\to} 1$ .
- Show that if f is holomorphic with f' = 0 on  $\Omega$  then f is constant.
- Show that holomorphic implies analytic.
- Use Cauchy's inequality to prove Liouville's theorem

Draft of Problem Book 48 Problem 16.0.1 (?)

What is a pair of conformal equivalences between  $\mathbb{H}$  and  $\mathbb{D}$ ?

Solution:

$$F: HH \to \mathbb{D}$$
 
$$z \mapsto \frac{i-z}{i+z}$$

$$G: \mathbb{D} \to \mathbb{H}$$
 
$$w \mapsto i\frac{1-w}{1+w}.$$

Mnemonic: any point in  $\mathbb{H}$  is closer to i than -i, so |F(z)| < 1.

• Maps  $\mathbb{R} \to S^1 \setminus \{-1\}$ .

Problem 16.0.2 (?)

What is conformal equivalence  $\mathbb{H} \rightleftharpoons S \coloneqq \left\{ w \in \mathbb{C} \ \middle| \ 0 < \arg(w) < \alpha \pi \right\}$ ?

Solution:

$$f(z) = z^{\alpha}.$$