Title

D. Zack Garza

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Contents

1 Group Theory

- 2^X denotes the powerset of X.
- For any p dividing the order of G, $\mathrm{Syl}_p(G)$ denotes the *set* of Sylow-p subgroups of G.

1.1 Big List of Notation

$C_G(x) =$	$\left\{ g \in G \;\middle\vert\; [g, x] = 1 \right\}$	$\subseteq G$	Centralizer (Element)
$C_G(H) =$	$\left\{ g \in G \;\middle\vert\; [g, h] = 1 \;\forall h \in H \right\} = \bigcap_{h \in H} C_G(h)$	$\leq G$	Centralizer (Subgroup)
$C(h) =$	$\left\{ ghg^{-1} \;\middle\vert\; g \in G \right\}$	$\subseteq G$	Conjugacy Class
$Z(G) =$	$\left\{ x \in G \;\middle\vert\; \forall g \in G,\; gxg^{-1} = x \right\}$	$\subseteq G$	Center
$N_G(H) =$	$\left\{ g \in G \;\middle\vert\; gHg^{-1} = H \right\}$	$\subseteq G$	Normalizer
$\mathrm{Inn}(G) =$	$\left\{ \varphi_g(x) = gxg^{-1} \right\}$	$\subseteq \mathrm{Aut}(G)$	Inner Aut.
$\mathrm{Out}(G) =$	$\mathrm{Aut}(G)/\mathrm{Inn}(G)$	$\hookrightarrow \mathrm{Aut}(G)$	Outer Aut.
$[g, h] =$	$ghgh^{-1}$	$\in G$	Commutator (Element)
$[G, H] =$	$\left\langle \left\{ [g, h] \;\middle\vert\; g \in G, h \in H \right\} \right\rangle$	$\leq G$	Commutator (Subgroup)
$\mathcal{O}_x$ or $G \cdot x =$	$\left\{ g.x \;\middle\vert\; x \in X \right\}$	$\subseteq X$	Orbit
$\mathrm{Stab}_G(x)$ or $G_x =$	$\left\{ g \in G \;\middle\vert\; g.x = x \right\}$	$\subseteq G$	Stabilizer
$X/G =$	$\left\{ G_x \;\middle\vert\; x \in X \right\}$	$\subseteq 2^X$	Set of Orbits
$X^g =$	$\left\{ x \in X \;\middle\vert\; \forall g \in G,\; g.x = x \right\}$	$\subseteq X$	Fixed Points

Definition 1.0.1 (Normal Closure of a Subgroup)**.**
The smallest normal subgroup of G containing H:

$$H^G \coloneqq \{gHg^{-1} : g \in G\} = \bigcap \{N : H \leq N \trianglelefteq G\}.$$

Definition 1.0.2 (Normal Core of a subgroup)**.**
The largest normal subgroup of G containing H:

$$H_G = \bigcap_{g \in G} gHg^{-1} = \langle N : N \trianglelefteq G \;\&\; N \leq H \rangle = \ker \psi.$$

where

$$\begin{aligned} \psi : G &\longrightarrow \mathrm{Aut}(G/H) \\ g &\mapsto (xH \mapsto gxH) \end{aligned}$$

Definition 1.0.3 (Characteristic subgroup)**.**
$H \leq G$ is *characteristic* iff H is fixed by every element of $\mathrm{Aut}(G)$.

Definition 1.0.4 (Subgroup Generated by a Subset)**.**
If $H \subset G$, then $\langle H \rangle$ is the smallest subgroup containing H:

$$\langle H \rangle = \bigcap \left\{ H \;\middle|\; H \subseteq M \leq G \right\} M = \left\{ h_1^{\pm 1} \cdots h_n^{\pm 1} \;\middle|\; n \geq 0, h_i \in H \right\}$$

Definition 1.0.5 (Centralizer)**.**

$$C_G(H) = \left\{ g \in G \;\middle|\; ghg^{-1} = h \; \forall h \in H \right\}$$

Definition 1.0.6 (Normalizer)**.**

$$N_G(H) = \left\{ g \in G \;\middle|\; gHg^{-1} = H \right\} = \bigcup \left\{ H \;\middle|\; H \trianglelefteq M \leq G \right\} M$$

Theorem 1.1 ***(The Fundamental Theorem of Cosets).***
$aH = bH \iff a^{-1}b \in H$ or $aH \bigcap bH = \emptyset$

Definition 1.1.1 (The Quaternion Group)**.**
The *Quaternion group* of order 8 is given by

$$\begin{aligned} Q &= \left\langle x, y, z \;\middle|\; x^2 = y^2 = z^2 = xyz = -1 \right\rangle \\ &= \left\langle x, y \;\middle|\; x^4 = y^4, x^2 = y^2, yxy^{-1} = x^{-1} \right\rangle \end{aligned}$$

Definition 1.1.2 (The Dihedral Group)**.**
A *dihedral group* of order $2n$ is given by

$$D_n = \left\langle r, s \;\middle|\; r^n, s^2, rsr^{-1} = s^{-1} \right\rangle$$

1.2 The Symmetric Group

Definition 1.1.3 (Parity of a Cycle)**.**

- A cycle is **even** $\iff$ product of an *even* number of transpositions.
- A cycle of even *length* is **odd**
- A cycle of odd *length* is **even**

Mnemonic: the parity of a k-cycle is the parity of $k - 1$.

Definition 1.1.4 (Alternating Group)**.**
The **alternating group** is the subgroup of **even** permutations, i.e.

$$A_n := \left\{ \sigma \in S_n \;\middle|\; \operatorname{sign}(\sigma) = 1 \right\}$$

where $\operatorname{sign}(\sigma) = (-1)^m$ and m is the number of cycles of even length.

Corollary 1.2 ***(Alternating Group).***
Every $\sigma \in A_n$ has an even number of *odd* cycles (i.e. an even number of *even-length* cycles).

Example 1.1.

$$\begin{aligned} A_4 = \{&\mathrm{id}, \\ &(1,3)(2,4), (1,2)(3,4), (1,4)(2,3), \\ &(1,2,3), (1,3,2), \\ &(1,2,4), (1,4,2), \\ &(1,3,4), (1,4,3), \\ &(2,3,4), (2,4,3)\} \end{aligned}$$

Definition 1.2.1 (Transitive Subgroup)**.**
A subgroup of S_n is **transitive** iff its action on $\{1, 2, \cdots, n\}$ is transitive.

Useful Facts:

- $\sigma \circ (a_1 \cdots a_k) \circ \sigma^{-1} = (\sigma(a_1), \cdots \sigma(a_k))$
- Conjugacy classes are determined by cycle type
- The order of a cycle is its length.
- The order of an element is the least common multiple of the sizes of its cycles.
- $A_{n\geq 5}$ is *simple*.

1.3 Counting Theorems

Theorem 1.3 ***(Lagrange's Theorem).***

$$H \leq G \implies |H| \,\Big|\, |G|.$$

Corollary 1.4.
The order of every element divides the size of G, i.e.

$$g \in G \implies o(g) \,\Big|\, o(G) \implies g^{|G|} = e.$$

⚠**Warning:** There does **not** necessarily exist $H \leq G$ with $|H| = n$ for every $n \,\Big|\, |G|$. Counterexample: $|A_4| = 12$ but has no subgroup of order 6.

Theorem 1.5 ***(Cauchy's Theorem).***
For every prime p dividing $|G|$. there is an element (and thus a subgroup) of order p.

This is a partial converse to Lagrange's theorem, and strengthened by Sylow's theorem.

1.3.1 Group Actions

Definition 1.5.1 (Group Action)**.**
An action of G on X is a group morphism

$$\begin{aligned}\varphi : G \times X &\to X \\ (g, x) &\mapsto g \cdot x\end{aligned}$$

or equivalently

$$\begin{aligned}\varphi : G &\longrightarrow \operatorname{Aut}(X) \\ g &\mapsto (x \mapsto \varphi_g(x) \coloneqq g \cdot x)\end{aligned}$$

satisfying

1. $e \cdot x = x$
2. $g \cdot (h \cdot x) = (gh) \cdot x$

Useful fact: $\ker \psi = \bigcap_{x \in X} G_x$ is the intersection of all stabilizers.

Definition 1.5.2 (Transitive Group Action)**.**
A group action $G \curvearrowright X$ is *transitive* iff for all $x, y \in X$ there exists a $g \in G$ such that $g \cdot x = x$. Equivalently, the action has a single orbit.

Reminder of notation: for a group G acting on a set X,

- $G \cdot x = \left\{ g \cdot x \;\middle|\; g \in G \right\} \subseteq X$ is the orbit
- $G_x = \left\{ g \in G \;\middle|\; g \cdot x = x \right\} \subseteq G$ is the stabilizer
- $X/G \subset 2^X$ is the set of orbits
- $X^g = \left\{ x \in X \;\middle|\; g \cdot x = x \right\} \subseteq X$ are the fixed points

Note that being in the same orbit is an equivalence relation which partitions X, and G acts transitively if restricted to any single orbit.

Theorem 1.6 ***(Orbit-Stabilizer).***

$$|G \cdot x| = [G : G_x] = |G|/|G_x| \quad \text{if } G \text{ is finite}$$

Mnemonic: $G/G_x \cong G \cdot x$.

1.3.2 Examples of Orbit-Stabilizer

1. Let G act on itself by left translation, where $g \mapsto (h \mapsto gh)$.

- The orbit $G \cdot x = G$ is the entire group
- The stabilizer G_x is only the identity.

- The fixed points X^g are only the identity.

2. Let G act on *itself* by conjugation.

- $G \cdot x$ is the **conjugacy class** of x (so not generally transitive)
- $G_x = Z(x) \coloneqq C_G(x) = \left\{ g \;\middle|\; [g, x] = e \right\}$, the **centralizer** of x.
- G^g (the fixed points) is the **center** $Z(G)$.

Corollary 1.7.
The number of conjugates of an element (i.e. the size of its conjugacy class) is the index of its centralizer, $[G : C_G(x)]$.

1.3.3 The Class Equation

$$|G| = |Z(G)| + \sum_{\substack{\text{One } x_i \text{ from} \\ \text{each conjugacy} \\ \text{class}}} [G : C_G(x_i)]$$

Note that $[G : C_G(x_i)]$ is the number of elements in the conjugacy class of x_i, and each $x_i \in Z(G)$ has a singleton conjugacy class.

Examples

1. Let G act on X, its set of *subgroups*, by conjugation.

- $G \cdot H = \left\{ gHg^{-1} \right\}$ is the **set of conjugate subgroups** of H
- $G_H = N_G(H)$ is the **normalizer** of in G of H
- X^g is the set of **normal subgroups** of G

Corollary 1.8.
Given $H \leq G$, the number of conjugate subgroups is $[G : N_G(H)]$.

2. For a fixed proper subgroup $H < G$, let G act on its cosets $G/H = \left\{ gH \;\middle|\; g \in G \right\}$ by left translation.

- $G \cdot gH = G/H$, i.e. this is a *transitive* action.
- $G_{gH} = gHg^{-1}$ is a *conjugate subgroup* of H
- $(G/H)^G = \emptyset$

Proposition 1.9 ***(Application of the Class Equation).***
If G is simple, $H < G$ proper, and $[G : H] = n$, then there exists an injective map $\varphi : G \hookrightarrow S_n$.

Proof .
This action induces φ; it is nontrivial since $gH = H$ for all g implies $H = G$; $\ker \varphi \trianglelefteq G$ and G simple implies $\ker \varphi = 1$.

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Theorem 1.10 ***(Burnside's Formula).***
Slogan: the number of orbits is equal to the average number of fixed points, i.e.

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$