# Title

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## 1 Undergraduate Analysis: Uniform Convergence

## 1.1 Fall 2018 # 1 💝

Let  $f(x) = \frac{1}{x}$ . Show that f is uniformly continuous on  $(1, \infty)$  but not on  $(0, \infty)$ .

#### **Solution:**

#### Concepts Used:

• Uniform continuity:

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$$
 such that  $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$ .

• Negating uniform continuity:  $\exists \varepsilon > 0$  such that  $\forall \delta(\varepsilon)$  there exist x, y such that  $|x-y| < \delta$  and  $|f(x) - f(y)| > \varepsilon$ .

Claim:  $f(x) = \frac{1}{x}$  is uniformly continuous on  $(c, \infty)$  for any c > 0.

• Note that

$$|x|, |y| > c > 0 \implies |xy| = |x||y| > c^2 \implies \frac{1}{|xy|} < \frac{1}{c^2}.$$

• Letting  $\varepsilon$  be arbitrary, choose  $\delta < \varepsilon c^2$ .

- Note that  $\delta$  does not depend on x, y.

• Then

$$|f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right|$$

$$= \frac{|x - y|}{xy}$$

$$\leq \frac{\delta}{xy}$$

$$< \frac{\delta}{c^2}$$

$$< \varepsilon,$$

which shows uniform continuity.

Claim: f is not uniformly continuous when c = 0.

• Toward a contradiction, let  $\varepsilon < 1$ .

• Let  $x_n = \frac{1}{n}$  for  $n \ge 1$ .

• Choose n large enough such that  $|x_n - x_{n+1}| = \frac{1}{n} - \frac{1}{n+1} < \delta$ .

- Why this can be done: by the archimedean property of  $\mathbb{R}$ , choose n such that  $\frac{1}{n} < \varepsilon$ .

- Then

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \le \frac{1}{n} < \varepsilon \quad \text{since } n+1 > 1.$$

• Note  $f(x_n) = n$  and thus

$$|f(x_{n+1}) - f(x_n)| = (n+1) - n = 1 > \varepsilon,$$

a contradiction.

## 1.2 Fall 2017 # 1 🔭

Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which f does and does not converge uniformly.

#### **Solution:**

#### Concepts Used:

• 
$$f_N \longrightarrow f$$
 uniformly  $\iff \|f_N - f\|_{\infty} \longrightarrow 0$ .  
•  $\sum_{n=0}^{\infty} c_n x^n := \lim_{N \longrightarrow \infty} \sum_{n=0}^{N} c_n x^n$ 

I.e. an infinite sum is defined as the pointwise limit of its partial sums.

• If  $\sum_{n=0}^{\infty} g_n(x)$  converges uniformly on a set A, then  $\sup_{x \in A} |f_n(x)| \longrightarrow 0$ .

• Set 
$$f_N(x) = \sum_{n=1}^N \frac{x^n}{n!}$$
.

- Then by definition,  $f_N(x) \longrightarrow f(x)$  pointwise on  $\mathbb{R}$ .

• For any compact interval [-M, M], we have

$$\begin{split} \|f_N(x) - f(x)\|_{\infty} &= \sup_{-M \le x \le M} \ \left| \sum_{n=N+1}^{\infty} \frac{x^n}{n!} \right| \\ &\le \sup_{-M \le x \le M} \ \sum_{n=N+1}^{\infty} \left| \frac{x^n}{n!} \right| \\ &\le \sum_{n=N+1}^{\infty} \frac{M^n}{n!} \\ &\le \sum_{n=0}^{\infty} \frac{M^n}{n!} \quad \text{since all additional terms are positive} \\ &= e^M \\ &< \infty, \end{split}$$

so  $f_N \longrightarrow f$  uniformly on [-M, M] by the M-test.

Here we've used that  $e^x$  is equal to its power series expansion.

Thus f converges on any bounded interval, since any bounded interval is contained in

some larger compact interval.

Claim: f does not converge on  $\mathbb{R}$ .

- If  $\sum_{n=0}^{\infty} g_n(x)$  converges uniformly on a set A, then  $\sup_{x \in A} |f_n(x)| \longrightarrow 0$ .
- But taking  $A = \mathbb{R}$  and  $g_n(x) = \frac{x^n}{n!}$ , we have

$$\sup_{x \in \mathbb{R}} |g_n(x)| = \sup_{x \in \mathbb{R}} \frac{x^n}{n!} = \infty.$$

# 1.3 Fall 2014 # 1 🔆

Let  $\{f_n\}$  be a sequence of continuous functions such that  $\sum f_n$  converges uniformly.

Prove that  $\sum f_n$  is also continuous.

#### Solution:

**Claim:** If  $F_N \longrightarrow F$  uniformly with each  $F_N$  continuous, then F is continuous.

• Follows from an  $\varepsilon/3$  argument:

$$|F(x) - F(y)| \le |F(x) - F_N(x)| + |F_N(x) - F_N(y)| + |F_N(y) - F(y)| \le \varepsilon \longrightarrow 0.$$

- The first and last  $\varepsilon/3$  come from uniform convergence of  $F_N \longrightarrow F$ .
- The middle  $\varepsilon/3$  comes from continuity of each  $F_N$ .
- Now setting  $F_N := \sum_{n=1}^N f_n$  yields a finite sum of continuous functions, which is continuous.
- Each  $F_N$  is continuous and  $F_N \longrightarrow F$  uniformly, so applying the claim yields the desired result.

# 1.4 Spring 2017 # 4

Let f(x,y) on  $[-1,1]^2$  be defined by

$$f(x,y) = \begin{cases} \frac{xy}{(x^2 + y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Determine if f is integrable.

Redo, may just be wrong

Add concepts.

Solution:

#### Concepts Used:

• ?

Switching to polar coordinates and integrating over half of the unit disc  $\mathbb{D} \subseteq I^2$ , we have

$$\begin{split} \int_{I^2} f \, dA &\geq \int_{\mathbb{D}} f \, dA \\ &\geq \int_0^{\pi/2} \int_0^1 \frac{\cos(\theta) \sin(\theta)}{r^4} \, r \, dr \, d\theta \\ &= \int_0^{\pi/2} \cos(\theta) \sin(\theta) \int_0^1 \frac{1}{r^3} \, dr \, d\theta \\ &= \left( \int_0^1 \frac{1}{r^3} \, dr \right) \left( \int_0^{\pi/2} \cos(\theta) \sin(\theta) \, d\theta \right) \\ &= \left( \int_0^1 \frac{1}{r^3} \, dr \right) = \infty, \end{split}$$

so f is not integrable.

## 1.5 Spring 2015 # 1 😽

Let (X, d) and  $(Y, \rho)$  be metric spaces,  $f: X \longrightarrow Y$ , and  $x_0 \in X$ .

Prove that the following statements are equivalent:

- 1. For every  $\varepsilon > 0$   $\exists \delta > 0$  such that  $\rho(f(x), f(x_0)) < \varepsilon$  whenever  $d(x, x_0) < \delta$ .
- 2. The sequence  $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow f(x_0)$  for every sequence  $\{x_n\} \longrightarrow x_0$  in X.

## 1.6 Fall 2014 # 2

Let I be an index set and  $\alpha: I \longrightarrow (0, \infty)$ .

1. Show that

$$\sum_{i \in I} a(i) := \sup_{\substack{J \subset I \\ J \text{ finite}}} \sum_{i \in J} a(i) < \infty \implies I \text{ is countable.}$$

2. Suppose  $I = \mathbb{Q}$  and  $\sum_{q \in \mathbb{Q}} a(q) < \infty$ . Define

$$f(x) := \sum_{\substack{q \in \mathbb{Q} \\ q \le x}} a(q).$$

Show that f is continuous at  $x \iff x \notin \mathbb{Q}$ .

# 1.7 Spring 2014 # 2

Let  $\{a_n\}$  be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that  $\sum a_n^2 < \infty$ .

Note: Assume  $a_n, b_n$  are all non-negative.

### 2 General Analysis

### 2.1 Spring 2020 # 1 😽

Prove that if  $f:[0,1]\longrightarrow \mathbb{R}$  is continuous then

$$\lim_{k \to \infty} \int_0^1 kx^{k-1} f(x) \, dx = f(1).$$

#### **Solution:**

#### Concepts Used:

- DCT
- Weierstrass Approximation Theorem
- Suppose p is a polynomial, then

$$\lim_{k \to \infty} \int_0^1 kx^{k-1} p(x) \, dx = \lim_{k \to \infty} \int_0^1 \left( \frac{\partial}{\partial x} x^k \right) p(x) \, dx$$

$$= \lim_{k \to \infty} \left[ x^k p(x) \Big|_0^1 - \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) \, dx \right] \quad \text{integrating by parts}$$

$$= p(1) - \lim_{k \to \infty} \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx,$$

• Thus it suffices to show that

$$\lim_{k \to \infty} \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx = 0.$$

• Integrating by parts a second time yields

$$\lim_{k \to \infty} \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx = \lim_{k \to \infty} \frac{x^{k+1}}{k+1} p'(x) \Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1} \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx$$

$$= -\lim_{k \to \infty} \int_0^1 \frac{x^{k+1}}{k+1} \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx$$

$$= -\int_0^1 \lim_{k \to \infty} \frac{x^{k+1}}{k+1} \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx \quad \text{by DCT}$$

$$= -\int_0^1 0 \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx$$

$$= 0$$

– The DCT can be applied here because f'' is continuous and [0,1] is compact, so f''

is bounded on [0,1] by a constant M and

$$\int_0^1 \left| x^k f''(x) \right| \le \int_0^1 1 \cdot M = M < \infty.$$

- Now use the Weierstrass approximation theorem:
  - If  $f:[a,b] \longrightarrow \mathbb{R}$  is continuous, then for every  $\varepsilon > 0$  there exists a polynomial  $p_{\varepsilon}(x)$  such that  $||f p_{\varepsilon}||_{\infty} < \varepsilon$ .
- Thus

$$\left| \int_0^1 kx^{k-1} p_{\varepsilon}(x) \, dx - \int_0^1 kx^{k-1} f(x) \, dx \right| = \left| \int_0^1 kx^{k-1} (p_{\varepsilon}(x) - f(x)) \, dx \right|$$

$$\leq \left| \int_0^1 kx^{k-1} || p_{\varepsilon} - f ||_{\infty} \, dx \right|$$

$$= || p_{\varepsilon} - f ||_{\infty} \cdot \left| \int_0^1 kx^{k-1} \, dx \right|$$

$$= || p_{\varepsilon} - f ||_{\infty} \cdot x^k \right|_0^1$$

$$= || p_{\varepsilon} - f ||_{\infty} \xrightarrow{\varepsilon \longrightarrow 0} 0$$

and the integrals are equal.

• By the first argument,

$$\int_0^1 kx^{k-1} p_{\varepsilon}(x) dx = p_{\varepsilon}(1) \text{ for each } \varepsilon$$

• Since uniform convergence implies pointwise convergence,  $p_{\varepsilon}(1) \stackrel{\varepsilon \longrightarrow 0}{\longrightarrow} f(1)$ .

# 2.2 Fall 2019 # 1 💝

Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of real numbers.

#### 2.2.1 a

Prove that if  $\lim_{n \to \infty} a_n = 0$ , then

$$\lim_{n \to \infty} \frac{a_1 + \dots + a_n}{n} = 0$$

#### 2.2.2 b

Prove that if  $\sum_{n=1}^{\infty} \frac{a_n}{n}$  converges, then

$$\lim_{n \to \infty} \frac{a_1 + \dots + a_n}{n} = 0$$

#### Solution:

#### Concepts Used:

- Cesaro mean/summation.
- Break series apart into pieces that can be handled separately.

#### 2.2.3 a

Prove a stronger result:

$$a_k \longrightarrow S \implies S_N \coloneqq \frac{1}{N} \sum_{k=1}^N a_k \longrightarrow S.$$

Idea: once N is large enough,  $a_k \approx S$ , and all smaller terms will die off as  $N \longrightarrow \infty$ . See this MSE answer.

• Use convergence  $a_k \longrightarrow S$ : choose M large enough such that

$$k \ge M + 1 \implies |a_k - S| < \varepsilon.$$

Then

$$\left| \left( \frac{1}{N} \sum_{k=1}^{N} a_k \right) - S \right| = \frac{1}{N} \left| \left( \sum_{k=1}^{N} a_k \right) - NS \right|$$

$$= \frac{1}{N} \left| \left( \sum_{k=1}^{N} a_k \right) - \sum_{k=1}^{N} S \right|$$

$$= \frac{1}{N} \left| \sum_{k=1}^{N} (a_k - S) \right|$$

$$\leq \frac{1}{N} \sum_{k=1}^{N} |a_k - S|$$

$$= \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} |a_k - S|$$

$$\leq \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} \frac{\varepsilon}{2}$$

$$= \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + (N - M) \frac{\varepsilon}{2}$$

$$\stackrel{\varepsilon}{\longrightarrow} \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + 0$$

$$\stackrel{N}{\longrightarrow} 0 + 0.$$

Note: M is fixed, so the last sum is some constant c, and  $c/N \longrightarrow 0$  as  $N \longrightarrow \infty$  for any constant. To be more careful, choose M first to get  $\varepsilon/2$  for the tail, then choose N(M) > M for the remaining truncated part of the sum.

#### 2.2.4 b

• Define

$$\Gamma_n \coloneqq \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

- $\Gamma_1 = \sum_{k=1}^n \frac{a_k}{k}$  is the original series and each  $\Gamma_n$  is a tail of  $\Gamma_1$ , so by assumption  $\Gamma_n \xrightarrow{n \to \infty} 0$ .
- Compute

$$\frac{1}{n}\sum_{k=1}^{n}a_{k}=\frac{1}{n}(\Gamma_{1}+\Gamma_{2}+\cdots+\Gamma_{n}-\mathbf{\Gamma_{n+1}})$$

• This comes from consider the following summation:

$$\Gamma_1:$$
  $a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots$ 
 $\Gamma_2:$   $\frac{a_2}{2} + \frac{a_3}{3} + \cdots$ 
 $\Gamma_3:$   $\frac{a_3}{3} + \cdots$ 
 $\sum_{i=1}^n \Gamma_i:$   $a_1 + a_2 + a_3 + \cdots + a_n + \frac{a_{n+1}}{n+1} + \cdots$ 

- Use part (a): since  $\Gamma_n \stackrel{n \longrightarrow \infty}{\longrightarrow} 0$ , we have  $\frac{1}{n} \sum_{k=1}^n \Gamma_k \stackrel{n \longrightarrow \infty}{\longrightarrow} 0$ .
- Also a minor check:  $\Gamma_n \longrightarrow 0 \implies \frac{1}{n}\Gamma_n \longrightarrow 0$ .
- Then

$$\frac{1}{n} \sum_{k=1}^{n} a_k = \frac{1}{n} (\Gamma_1 + \Gamma_2 + \dots + \Gamma_n - \Gamma_{n+1})$$
$$= \left(\frac{1}{n} \sum_{k=0}^{n} \Gamma_k\right) - \left(\frac{1}{n} \Gamma_{n+1}\right)$$
$$\stackrel{n \to \infty}{\longrightarrow} 0$$

## 2.3 Fall 2018 # 4 🙀

Let  $f \in L^1([0,1])$ . Prove that

$$\lim_{n \to \infty} \int_0^1 f(x) |\sin nx| \ dx = \frac{2}{\pi} \int_0^1 f(x) \ dx$$

> Hint: Begin with the case that f is the characteristic function of an interval.

#### Add concepts

#### Solution:

#### Concepts Used:

• ?

Case of characteristic function

- First suppose  $f(x) = \chi_{[0,1]}(x)$ .
- Note that  $\sin(nx)$  has a period of  $2\pi/n$ , and thus  $\left|\frac{n}{2\pi}\right|$  full periods in [0,1].
- Taking the absolute value yields a new function with half the period, so a period of  $\pi/n$  and  $|\pi/n|$  full periods in [0,1].
- We can compute the integral over one full period (which is independent of which period is chosen), and since  $\sin(x)$  is positive and agrees with  $|\sin(nx)|$  on the first period, we have

$$\int_{\text{One Period}} |\sin(nx)| \, dx = \int_0^{\pi/n} \sin(nx) \, dx$$

$$= \frac{1}{n} \int_0^{\pi} \sin(u) \, du \quad u = nx$$

$$= \frac{1}{n} - \cos(u) \Big|_0^{\pi}$$

$$= \frac{2}{n}.$$

• Then break the integral up into integrals over periods  $P_1, P_2, \dots, P_N$  where  $N := \lfloor n/\pi \rfloor$ :

$$\int_{0}^{1} |\sin(nx)| dx = \left(\sum_{j=1}^{N} \int_{P_{j}} |\sin(nx)| dx\right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| dx$$

$$= \left(\sum_{j=1}^{N} \frac{2}{n}\right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| dx$$

$$= N\left(\frac{2}{n}\right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| dx$$

$$\coloneqq \left\lfloor \frac{n}{\pi} \right\rfloor \frac{2}{n} + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| dx$$

$$= \frac{2}{\pi} + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| dx$$

$$\coloneqq \frac{2}{\pi} + R(n)$$

so it suffices to show that  $R(n) \stackrel{n \longrightarrow \infty}{\longrightarrow} 0$ .

Need to justify removing floor function and cancellation

• Showing this: ??????????

#### General case

Not sure. Approximate f by simple functions...

## 2.4 Fall 2017 # 4 💝

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

1. Show that  $f_n \longrightarrow 0$  pointwise but not uniformly on [0,1].

Hint: Consider the maximum of  $f_n$ .

2.

$$\lim_{n \to \infty} \int_0^1 n(1-x)^n \sin x \, dx = 0$$

Add concepts

Walk through.

#### Solution:

Concepts Used:

. 7

#### 2.4.1 a

Let  $G(x) = \sum_{n=1}^{\infty} nx(1-x)^n$ . Applying the ratio test, we have

$$\left| \frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n} \right| = \frac{n+1}{n} |1-x| \stackrel{n \longrightarrow \infty}{\longrightarrow} |1-x| < 1 \iff 0 \le x \le 2,$$

and in particular, this series converges on [0,2]. Thus its terms go to zero, and  $nx(1-x)^n \longrightarrow 0$  on  $[0,1] \subset [0,2]$ .

To see that the convergence is not uniform, let  $x_n = \frac{1}{n}$  and  $\varepsilon > \frac{1}{e}$ , then

$$\sup_{x \in [0,1]} |nx(1-x)^n - 0| \ge |nx_n(1-x_n)^n| = \left| \left(1 - \frac{1}{n}\right)^n \right| \stackrel{n \to \infty}{\longrightarrow} e^{-1} > \varepsilon.$$

#### 2.4.2 b

Note: could use the first part with  $\sin(x) \le x$ , but then integral ends up more complicated.

Noting that  $sin(x) \leq 1$ , we have We have

$$\left| \int_0^1 n(1-x)^n \sin(x) \right| \le \int_0^1 |n(1-x)^n \sin(x)|$$

$$\le \int_0^1 |n(1-x)^n|$$

$$= n \int_0^1 (1-x)^n$$

$$= -\frac{n(1-x)^{n+1}}{n+1}$$

$$\stackrel{n \to \infty}{\longrightarrow} 0.$$

# 2.5 Spring 2017 # 3 💝

Let

$$f_n(x) = ae^{-nax} - be^{-nbx}$$
 where  $0 < a < b$ .

Show that

a. 
$$\sum_{n=1}^{\infty} |f_n| \text{ is not in } L^1([0,\infty),m)$$

Hint:  $f_n(x)$  has a root  $x_n$ .

b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0,\infty),m) \quad \text{and} \quad \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x) \, dm = \ln \frac{b}{a}$$

Not complete

Add concepts

Walk through.

#### **Solution:**

Concepts Used:

• (

#### 2.5.1 a

Letting  $x_n := \frac{1}{n}$ , we have

$$\sum_{k=1}^{\infty} |f_k(x)| \ge |f_n(x_n)| = \left| ae^{-ax} - be^{-bx} \right| := M.$$

In particular,  $\sup_{x} |f_n(x)| \not\longrightarrow 0$ , so the terms do not go to zero and the sum can not converge.

2.5.2 b

# 2.6 Fall 2016 # 1 🙀

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that f converges to a differentiable function on  $(1, \infty)$  and that

$$f'(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n^x}\right)'.$$

Hint:

$$\left(\frac{1}{n^x}\right)' = -\frac{1}{n^x} \ln n$$

Add concepts.

#### Solution:

Concepts Used:

- Set  $f_N(x) := \sum_{n=1}^N n^{-x}$ , so  $f(x) = \lim_{N \to \infty} f_N(x)$ .

• If an interchange of limits is justified, we have

$$\frac{\partial}{\partial x} \lim_{N \to \infty} \sum_{n=1}^{N} n^{-x} = \lim_{h \to 0} \lim_{N \to \infty} \frac{1}{h} \left[ \left( \sum_{n=1}^{N} n^{-x} \right) - \left( \sum_{n=1}^{N} n^{-(x+h)} \right) \right]$$

$$\stackrel{?}{=} \lim_{N \to \infty} \lim_{h \to 0} \frac{1}{h} \left[ \left( \sum_{n=1}^{N} n^{-x} \right) - \left( \sum_{n=1}^{N} n^{-(x+h)} \right) \right]$$

$$= \lim_{N \to \infty} \lim_{h \to 0} \frac{1}{h} \left[ \sum_{n=1}^{N} n^{-x} - n^{-(x+h)} \right] \quad (1)$$

$$= \lim_{N \to \infty} \sum_{n=1}^{N} \lim_{h \to 0} \frac{1}{h} \left[ n^{-x} - n^{-(x+h)} \right] \quad \text{since this is a finite sum}$$

$$:= \lim_{N \to \infty} \sum_{n=1}^{N} \frac{\partial}{\partial x} \left( \frac{1}{n^x} \right)$$

$$= \lim_{N \to \infty} \sum_{n=1}^{N} -\frac{\ln(n)}{n^x},$$

where the combining of sums in (1) is valid because  $\sum n^{-x}$  is absolutely convergent for x > 1 by the *p*-test.

- Thus it suffices to justify the interchange of limits and show that the last sum converges on  $(1, \infty)$ .
- Claim:  $\sum n^{-x} \ln(n)$  converges.
  - Use the fact that for any fixed  $\varepsilon > 0$ ,

$$\lim_{n \to \infty} \frac{\ln(n)}{n^{\varepsilon}} \stackrel{\text{L.H.}}{=} \lim_{n \to \infty} \frac{1/n}{\varepsilon n^{\varepsilon - 1}} = \lim_{n \to \infty} \frac{1}{\varepsilon n^{\varepsilon}} = 0,$$

- This implies that for a fixed  $\varepsilon > 0$  and for any constant c > 0 there exists an N large enough such that  $n \geq N$  implies  $\ln(n)/n^{\varepsilon} < c$ , i.e.  $\ln(n) < cn^{\varepsilon}$ .
- Taking c=1, we have  $n \geq N \implies \ln(n) < n^{\varepsilon}$
- We thus break up the sum:

$$\sum_{n \in \mathbb{N}} \frac{\ln(n)}{n^x} = \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{\ln(n)}{n^x}$$

$$\leq \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{n^{\varepsilon}}{n^x}$$

$$\coloneqq C_{\varepsilon} + \sum_{n=N}^{\infty} \frac{n^{\varepsilon}}{n^x} \quad \text{with } C_{\varepsilon} < \infty \text{ a constant}$$

$$= C_{\varepsilon} + \sum_{n=N}^{\infty} \frac{1}{n^{x-\varepsilon}},$$

where the last term converges by the p-test if  $x - \varepsilon > 1$ .

- But  $\varepsilon$  can depend on x, and if  $x \in (1, \infty)$  is fixed we can choose  $\varepsilon < |x 1|$  to ensure this.
- Claim: the interchange of limits is justified.

# 2.7 Fall 2016 # 5 🔭

Let  $\varphi \in L^{\infty}(\mathbb{R})$ . Show that the following limit exists and satisfies the equality

$$\lim_{n \to \infty} \left( \int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} \, dx \right)^{\frac{1}{n}} = \|\varphi\|_{\infty}.$$

Walk through

Add concepts.

#### **Solution:**

Concepts Used:

• ?

Let L be the LHS and R be the RHS.

Claim:  $L \leq R$ . - Since  $|\varphi| \leq ||\varphi||_{\infty}$  a.e., we can write

$$\begin{split} L^{\frac{1}{n}} &\coloneqq \int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} \\ &\le \int_{\mathbb{R}} \frac{\|\varphi\|_{\infty}^n}{1+x^2} \\ &= \|\varphi\|_{\infty}^n \int_{\mathbb{R}} \frac{1}{1+x^2} \\ &= \|\varphi\|_{\infty}^n \arctan(x)\Big|_{-\infty}^{\infty} \\ &= \|\varphi\|_{\infty}^n \left(\frac{\pi}{2} - \frac{-\pi}{2}\right) \\ &= \pi \|\varphi\|_{\infty}^n \end{split}$$

$$\implies L^{\frac{1}{n}} \leq \sqrt[n]{\pi \|\varphi\|_{\infty}^{n}}$$

$$\implies L \leq \pi^{\frac{1}{n}} \|\varphi\|_{\infty}$$

$$\stackrel{n \longrightarrow \infty}{\longrightarrow} \|\varphi\|_{\infty},$$

where we've used the fact that  $c^{\frac{1}{n}} \stackrel{n \longrightarrow \infty}{\longrightarrow} 1$  for any constant c.

Actually true? Need conditions

Claim:  $R \leq L$ .

- We will show that  $R \leq L + \varepsilon$  for every  $\varepsilon > 0$ .
- Set

$$S_{\varepsilon} := \left\{ x \in \mathbb{R}^n \mid |\varphi(x)| \ge ||\varphi||_{\infty} - \varepsilon \right\}.$$

• Then we have

$$\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} dx \ge \int_{S_{\varepsilon}} \frac{|\varphi(x)|^n}{1+x^2} dx \quad S_{\varepsilon} \subset \mathbb{R}$$

$$\ge \int_{S_{\varepsilon}} \frac{(\|\varphi\|_{\infty} - \varepsilon)^n}{1+x^2} dx \quad \text{by definition of } S_{\varepsilon}$$

$$= (\|\varphi\|_{\infty} - \varepsilon)^n \int_{S_{\varepsilon}} \frac{1}{1+x^2} dx$$

$$= (\|\varphi\|_{\infty} - \varepsilon)^n C_{\varepsilon} \quad \text{where } C_{\varepsilon} \text{ is some constant}$$

$$\implies \left( \int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} \, dx \right)^{\frac{1}{n}} \ge (\|\varphi\|_{\infty} - \varepsilon) C_{\varepsilon}^{\frac{1}{n}}$$

$$\stackrel{n \longrightarrow \infty}{\longrightarrow} (\|\varphi\|_{\infty} - \varepsilon) \cdot 1$$

$$\stackrel{\varepsilon \longrightarrow 0}{\longrightarrow} \|\varphi\|_{\infty},$$

where we've again used the fact that  $c^{\frac{1}{n}} \longrightarrow 1$  for any constant.

### 2.8 Fall 2016 # 6 💝

Let  $f, g \in L^2(\mathbb{R})$ . Show that

$$\lim_{n \to \infty} \int_{\mathbb{R}} f(x)g(x+n) \, dx = 0$$

Add concepts.

#### Solution:

Concepts Used:

- ?
- Use the fact that  $L^p$  has small tails: if  $h \in L^2(\mathbb{R})$ , then for any  $\varepsilon > 0$ ,

$$\forall \varepsilon, \exists N \in \mathbb{N} \text{ such that } \int_{|x|>N} |h(x)|^2 dx < \varepsilon.$$

How to prove small tails in  $L^p$ ?

- So choose n large enough so the tails of both f and g are smaller than  $\varepsilon$ .
- Apply Cauchy-Schwarz:

$$\left| \int_{\mathbb{R}} f(x)g(x+n) \, dx \right| \le \int_{\mathbb{R}} |f(x)g(x+n)| \, dx$$
$$\le \int_{\mathbb{R}}.$$

# 2.9 Spring 2016 # 1

For  $n \in \mathbb{N}$ , define

$$]e_n = \left(1 + \frac{1}{n}\right)^n$$
 and  $E_n = \left(1 + \frac{1}{n}\right)^{n+1}$ 

Show that  $e_n < E_n$ , and prove Bernoulli's inequality:

$$(1+x)^n \ge 1 + nx$$
 for  $-1 < x < \infty$  and  $n \in \mathbb{N}$ 

Use this to show the following:

- 1. The sequence  $e_n$  is increasing.
- 2. The sequence  $E_n$  is decreasing.
- 3.  $2 < e_n < E_n < 4$ .
- 4.  $\lim_{n \to \infty} e_n = \lim_{n \to \infty} E_n.$

# 2.10 Fall 2015 # 1

Define

$$f(x) = c_0 + c_1 x^1 + c_2 x^2 + \ldots + c_n x^n$$
 with  $n$  even and  $c_n > 0$ .

Show that there is a number  $x_m$  such that  $f(x_m) \leq f(x)$  for all  $x \in \mathbb{R}$ .