

# Complex Analysis Qualifying Exam Solutions

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## Contents

|          |   |          |
|----------|---|----------|
| <b>1</b> | <b>Topology and Functions of One Variable (8155a)</b> | <b>2</b> |
| <b>2</b> | <b>Several Variables (8155h)</b>                      | <b>2</b> |
| <b>3</b> | <b>Conformal Maps (8155c)</b>                         | <b>2</b> |
| <b>4</b> | <b>Integrals and Cauchy's Theorem (8155d)</b>         | <b>2</b> |
| 4.1      | 5 . . . . .   | 2        |
| 4.2      | 9 . . . . .   | 2        |
| 4.3      | 10 . . . . .  | 3        |
| <b>5</b> | <b>Liouville's Theorem, Power Series (8155e)</b>      | <b>3</b> |
| 5.1      | 1 . . . . .   | 3        |
| 5.2      | 5 . . . . .   | 4        |
| 5.3      | 6 . . . . .   | 4        |
| <b>6</b> | <b>Laurent Expansions and Singularities (8155f)</b>   | <b>5</b> |
| 6.1      | 1 . . . . .   | 5        |
| 6.2      | 2 . . . . .   | 5        |
| 6.3      | 8 . . . . .   | 6        |
| <b>7</b> | <b>Residues (8155g)</b>                               | <b>6</b> |
| 7.1      | 6 . . . . .   | 6        |
| <b>8</b> | <b>Rouche's Theorem (8155h)</b>                       | <b>6</b> |
| 8.1      | 1 . . . . .   | 6        |
| 8.2      | 2 . . . . .   | 6        |
| 8.3      | 4 . . . . .   | 7        |
| <b>9</b> | <b>Schwarz Lemma and Reflection Principle (8155i)</b> | <b>7</b> |

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## 1 Topology and Functions of One Variable (8155a)

## 2 Several Variables (8155h)

## 3 Conformal Maps (8155c)

## 4 Integrals and Cauchy's Theorem (8155d)

### 4.1 5

Show that there is no sequence of polynomials converging uniformly to  $f(z) = 1/z$  on  $S^1$ .

Solution

- By Cauchy's integral formula,  $\int_{S^1} f = 2\pi i$
- If  $p_j$  is any polynomial, then  $p_j$  is holomorphic in  $\mathbb{D}$ , so  $\int_{S^1} p_j = 0$ .
- Contradiction: compact sets in  $\mathbb{C}$  are bounded, so

$$\left| \int f - \int p_j \right| \leq \int |p_j - f| \leq \int \|p_j - f\|_\infty = \|p_j - f\|_\infty \int_{S^1} 1 dz = \|p_j - f\|_\infty \cdot 2\pi \rightarrow 0$$

which forces  $\int f = \int p_j = 0$ .

### 4.2 9

- Note  $f$  is continuous on  $\mathbb{C}$  since analytic implies continuous ( $f$  equals its power series, where the partials sums uniformly converge to it, and uniform limit of continuous is continuous).
- Strategy: take  $D$  a disc centered at a point  $x \in \mathbb{R}$ , show  $f$  is holomorphic in  $D$  by Morera's theorem.
- Let  $\Delta \subset D$  be a triangle in  $D$ .
- Case 1: If  $\Delta \cap \mathbb{R} = \emptyset$ , then  $f$  is holomorphic on  $\Delta$  and  $\int_{\Delta} f = 0$ .
- Case 2: one side or vertex of  $\Delta$  intersects  $\mathbb{R}$ , and wlog the rest of  $\Delta$  is in  $\mathbb{H}^+$ .
  - Then let  $\Delta_\varepsilon$  be the perturbation  $\Delta + i\varepsilon = \{z + i\varepsilon \mid z \in \Delta\}$ ; then  $\Delta_\varepsilon \cap \mathbb{R} = \emptyset$  and  $\int_{\Delta_\varepsilon} f = 0$ .
  - Now let  $\varepsilon \rightarrow 0$  and conclude by continuity of  $f$  (???)
    - \* We want

$$\int_{\Delta_\varepsilon} f = \int_a^b f(\gamma_\varepsilon(t)) \gamma'_\varepsilon(t) dt \xrightarrow{\varepsilon \rightarrow 0} \int_a^b f(\gamma(t)) \gamma'_\varepsilon(t) dt = \int_{\Delta} f$$

where  $\gamma_\varepsilon, \gamma$  are curves parametrizing  $\Delta_\varepsilon, \Delta$  respectively.

- \* Since  $\gamma, \gamma_\varepsilon$  are closed and bounded in  $\mathbb{C}$ , they are compact subsets. Thus it suffices to show that  $f(\gamma_\varepsilon(t)) \gamma'_\varepsilon(t)$  converges uniformly to  $f(\gamma(t)) \gamma'(t)$ .

\* ??

- Case 3:  $\Delta$  intersects both  $\mathbb{H}^+$  and  $\mathbb{H}^-$ .
  - Break into smaller triangles, each of which falls into one of the previous two cases.

## 4.3 10

Suppose  $f : \mathbb{C} \rightarrow \mathbb{C}$  is entire and bounded, and use Cauchy's theorem to prove that  $f' \equiv 0$  and thus  $f$  is constant.

Solution

- Suffices to prove  $f' = 0$  because  $\mathbb{C}$  is connected (see Stein Ch 1, 3.4)
  - Idea: Fix  $w_0$ , show  $f(w) = f(w_0)$  for any  $w \neq w_0$
  - Connected = Path connected in  $\mathbb{C}$ , so take  $\gamma$  joining  $w$  to  $w_0$ .
  - $f$  is a primitive for  $f'$ , and  $\int_{\gamma} f' = f(w) - f(w_0)$ , but  $f' = 0$ .
- Fix  $z_0 \in \mathbb{C}$ , let  $B$  be the bound for  $f$ , so  $|f(z)| \leq B$  for all  $z$ .
- Apply Cauchy inequalities: if  $f$  is holomorphic on  $U \supset \bar{D}_R(z_0)$  then setting  $\|f\|_C := \sup_{z \in C} |f(z)|$ ,

$$|f^{(n)}(z_0)| \leq \frac{n! \|f\|_C}{R^n}.$$

- Yields  $|f'(z_0)| \leq B/R$
- Take  $R \rightarrow \infty$ , QED.

## 5 Liouville's Theorem, Power Series (8155e)

## 5.1 1

Suppose  $f$  is analytic on  $\Omega \supseteq \mathbb{D}$  whose power series  $\sum a_n z^n$  has radius of convergence 1.

- Give an example of an  $f$  which converges at every point on  $S^1$ .
- Give an example of an  $f$  which is analytic at  $z = 1$  but  $\sum a_n$  diverges.
- Prove that  $f$  can not be analytic at every point of  $S^1$ .

Solution:

- Take  $\sum \frac{z^n}{n^2}$ ; then  $|z| \leq 1 \implies \left| \frac{z^n}{n^2} \right| \leq \frac{1}{n^2}$  which is summable, so the series converges for  $|z| \leq 1$ .
- Take  $\sum \frac{z^n}{n}$ ; then  $z = 1$  yields the harmonic series, which diverges.
  - For  $z \in S^1 \setminus \{1\}$ , we have  $z = e^{2\pi it}$  for  $0 < t < 2\pi$ .
  - So fix  $t$ .
  - Toward applying the Dirichlet test, set  $a_n = 1/n, b_n = z^n$ .
  - Then for all  $N$ ,

$$\left| \sum_{n=1}^N b_n \right| = \left| \sum_{n=1}^N \frac{1}{n} z^n \right| = \left| \sum_{n=1}^N z^n \right| = \left| \frac{z - z^{N+1}}{1 - z} \right| \leq \frac{2}{1 - z} < \infty.$$

- Thus  $\sum a_n b_n < \infty$  and  $\sum z^n/n$  converges.
- c. ?

## 5.2 5

Prove the Fundamental Theorem of Algebra: every non-constant polynomial  $p(z) = a_n z^n + \cdots + a_0 \in \mathbb{C}[x]$  has a root in  $\mathbb{C}$ .

Solution:

- Strategy: By contradiction with Liouville's Theorem
- Suppose  $p$  is non-constant and has no roots.
- Claim:  $1/p(z)$  is a bounded holomorphic function on  $\mathbb{C}$ .
  - Holomorphic: clear? Since  $p$  has no roots.
  - Bounded: for  $z \neq 0$ , write

$$\frac{P(z)}{z^n} = a_n + \left( \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right).$$

- The term in parentheses goes to 0 as  $|z| \rightarrow \infty$
- Thus there exists an  $R > 0$  such that

$$|z| > R \implies \left| \frac{P(z)}{z^n} \right| \geq c := \frac{|a_n|}{2}.$$

- So  $p$  is bounded below when  $|z| > R$
- Since  $p$  is continuous and has no roots in  $|z| \leq R$ , it is bounded below when  $|z| \leq R$ .
- Thus  $p$  is bounded below on  $\mathbb{C}$  and thus  $1/p$  is bounded above on  $\mathbb{C}$ .
- By Liouville's theorem,  $1/p$  is constant and thus  $p$  is constant, a contradiction.

## 5.3 6

Find all entire functions  $f$  which satisfy the following inequality, and prove the list is complete:

$$|f(z)| \geq |z|.$$

Solution:

- Suppose  $f$  is entire and define  $g(z) := \frac{z}{f(z)}$ .
- By the inequality,  $|g(z)| \leq 1$ , so  $g$  is bounded.
- $g$  potentially has singularities at the zeros  $Z_f := f^{-1}(0)$ , but since  $f$  is entire,  $g$  is holomorphic on  $\mathbb{C} \setminus Z_f$ .
- Claim:  $Z_f = \{0\}$ .
  - If  $f(z) = 0$ , then  $|z| \leq |f(z)| = 0$  which forces  $z = 0$ .
- We can now apply Riemann's removable singularity theorem:
  - Check  $g$  is bounded on some open subset  $D \setminus \{0\}$ , clear since it's bounded everywhere
  - Check  $g$  is holomorphic on  $D \setminus \{0\}$ , clear since the only singularity of  $g$  is  $z = 0$ .
- By Riemann's removable singularity theorem, the singularity  $z = 0$  is removable and  $g$  has an extension to an entire function  $\tilde{g}$ .
- By continuity, we have  $|\tilde{g}(z)| \leq 1$  on all of  $\mathbb{C}$ 
  - If not, then  $|\tilde{g}(0)| = 1 + \varepsilon > 1$ , but then there would be a domain  $\Omega \subseteq \mathbb{C} \setminus \{0\}$  such that  $1 < |\tilde{g}(z)| \leq 1 + \varepsilon$  on  $\Omega$ , a contradiction.
- By Liouville,  $\tilde{g}$  is constant, so  $\tilde{g}(z) = c_0$  with  $|c_0| \leq 1$
- Thus  $f(z) = c_0^{-1}z := cz$  where  $|c| \geq 1$

Thus all such functions are of the form  $f(z) = cz$  for some  $c \in \mathbb{C}$  with  $|c| \geq 1$ .

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## 6 Laurent Expansions and Singularities (8155f)

### 6.1 1

Let  $f(z) = \frac{z+1}{z(z-1)}$ .

About  $z = 0$ :

$$\begin{aligned} f(z) &= (z+1) \left( -\frac{1}{z} + \frac{1}{z-1} \right) \\ &= -(z+1) \left( \frac{1}{z} + \sum_{n=0}^{\infty} z^n \right) \\ &= -(z+1) \sum_{n=-1}^{\infty} z^n \\ &= -\frac{1}{z} + 2 \sum_{n=0}^{\infty} z^n \\ &= -\frac{1}{z} - 2 - 2z - 2z^2 - \dots \end{aligned}$$

About  $z = 1$ :

$$\begin{aligned} f(z) &= \left( \frac{(1-z)-2}{1-z} \right) \left( \frac{1}{1-(1-z)} \right) \\ &= \left( 1 - \frac{2}{1-z} \right) \sum_{n=0}^{\infty} (1-z)^n \\ &= \sum_{n=0}^{\infty} (1-z)^n - 2 \sum_{n=-1}^{\infty} (1-z)^n \\ &= -\frac{2}{1-z} - \sum_{n=0}^{\infty} (1-z)^n \\ &= \frac{2}{z-1} + \sum_{n=0}^{\infty} (-1)^{n+1} (z-1)^n \\ &= \frac{2}{z-1} - 1 + (z-1) - (z-1)^2 + \dots \end{aligned}$$

### 6.2 2

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{z} \right)^n = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \dots$$

$$\begin{aligned}
\cos\left(\frac{1}{z}\right) &= \frac{1}{2}\left(e^{\frac{i}{z}} + e^{-\frac{i}{z}}\right) \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \left(\frac{i}{z}\right)^n + \left(\frac{-i}{z}\right)^n \right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{z}\right)^{2n}.
\end{aligned}$$

### 6.3 8

Idea: show their  $f - g$  is analytic by taking away all of the negative powers, and bounded by (c).

## 7 Residues (8155g)

### 7.1 6

Difficult!

## 8 Rouché's Theorem (8155h)

### 8.1 1

Note

- $f_1(z) = 1 + z$ , which has the single root  $z = -1$  which is not inside  $|z| < 1$ .
- $f_2(z) = 1 + z + \frac{1}{2}z^2 = (z - (1 + i))(z - (1 - i))$ , and  $|1 \pm i| = \sqrt{2} > 1$ .
- Note that  $p_n(z) \xrightarrow{n \rightarrow \infty} e^z$  uniformly on any compact set.
- Let  $r$  be arbitrary and fix  $N := \mathbb{D}_r(0)$ , then  $p_n(z) \rightarrow e^z$  uniformly on  $\overline{N}$ .
- Set  $g_n(z) := p_n(z)/e^z$ , then  $g_n \rightarrow 1$  uniformly on  $\overline{N}$ .
- Choose  $n \gg 0$  so that  $|f(z) - 1| < \varepsilon < 1$  for all  $z \in \overline{N}$ .
- So take  $h(z) = 1$ , then on  $\partial N$ ,

### 8.2 2

Multiple versions of Rouches theorem!

- Set  $h(z) = 3z^2$  and  $g(z) = z^3 + bz + b^2$ .
- Then on  $|z| = 1$ ,

$$|g(z)| \leq 1 + b + b^2 < 3 = 3|z|^2 = |3z^2| = |h|,$$

so  $g, h$  have the same number of roots in  $|z| \leq 1$ .

- But  $h$  evidently has two roots in this region.

**8.3 4**

- Set  $h(z) = -4z^3$  and  $g(z) = z^7 - 1$ , then on  $|z| = 1$ ,

$$|g(z)| = |z^7 - 1| \leq 1 + 1 = 2 < 4 = |-4z^3| = |h(z)|.$$

- So  $h$  and  $h + g$  have the same number of roots, but  $h$  has three roots here.

**9 Schwarz Lemma and Reflection Principle (8155i)**