

# Complex Analysis Qualifying Exam Notes

D. Zack Garza

Wednesday 5<sup>th</sup> August, 2020

## Contents

<b>1</b>	<b>Theorems</b>	<b>2</b>
<b>2</b>	<b>Stuff</b>	<b>4</b>
2.0.1	Fundamental Theorem of Algebra: Argument Principle . . . . .	4
2.0.2	Fundamental Theorem of Algebra: Rouché's Theorem . . . . .	4
2.0.3	Fundamental Theorem of Algebra: Liouville's Theorem . . . . .	5
2.0.4	Fundamental Theorem of Algebra: Open Mapping Theorem . . . . .	5
<b>3</b>	<b>Conformal Maps</b>	<b>5</b>
3.1	Plane to Disc . . . . .	5
3.2	Sector to Disc . . . . .	6
3.3	Strip to Disc . . . . .	6
<b>4</b>	<b>Appendix</b>	<b>6</b>
4.1	Things to know well: . . . . .	7
4.2	Theorems . . . . .	7
4.2.1	The Argument Principle . . . . .	7
4.2.2	Rouche . . . . .	7
4.3	Misc Prereq . . . . .	7
4.4	Useful Techniques . . . . .	8
4.5	Residues . . . . .	8
4.6	Pithy Statements . . . . .	9
4.7	Precise Refinements . . . . .	9

## Preface

### References

- Simon

# 1 Theorems

**Theorem 1.1 (Summation by Parts).**

Define the forward difference operator  $\Delta f_k = f_{k+1} - f_k$ , then

$$\sum_{k=m}^n f_k \Delta g_k + \sum_{k=m}^{n-1} g_{k+1} \Delta f_k = f_n g_{n+1} - f_m g_m.$$

Note: compare to  $\int_a^b f dg + \int_a^b g df = f(b)g(b) - f(a)g(a)$ .

**Theorem 1.2 (Morera's Theorem).**

If  $f$  is continuous on a domain  $\Omega$  and  $\int_T f = 0$  for every triangle  $T \subset \Omega$ , then  $f$  is holomorphic.

**Theorem 1.3 (Cauchy Integral Formula).**

Suppose  $f$  is holomorphic on  $\Omega$ , then

$$f(z) = \frac{1}{2\pi i} \oint_{\partial\Omega}$$

and

$$\frac{\partial^n f}{\partial z^n}(z) = \frac{n!}{2\pi i} \oint_{\partial\Omega} \frac{f\xi}{(\xi - z)^{n+1}} d\xi.$$

**Theorem 1.4 (Cauchy's Inequality).**

For  $z_o \in D_R(z_o) \subset \Omega$ , we have

$$|f^{(n)}(z_o)| \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{\|f\|_{C_R}}{R^{n+1}} R d\theta = \frac{n! \|f\|_{C_R}}{R^n}.$$

**Theorem 1.5 (Liouville).**

If  $f$  is entire and bounded,  $f$  is constant.

**Theorem 1.6 (Argument Principle).**

Todo

Argument principle.

**Theorem 1.7 (Green's Theorem).**

If  $\Omega \subseteq \mathbb{C}$  is bounded with  $\partial\Omega$  piecewise smooth and  $f, g \in C^1(\bar{\Omega})$ , then

$$\int_{\partial\Omega} f dx + g dy = \iint_{\Omega} \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA.$$

**Theorem 1.8 (Rouche).**

If  $f, g$  are analytic on a domain  $\Omega$  with finitely many zeros in  $\Omega$  and  $\gamma \subset \Omega$  is a closed curve

---

surrounding each point exactly once, where  $|g| < |f|$  on  $\gamma$ , then  $f$  and  $f + g$  have the same number of zeros.

**Example 1.1.** • Take  $P(z) = z^4 + 6z + 3$ .

- On  $|z| < 2$ :
  - Set  $f(z) = z^4$  and  $g(z) = 6z + 3$ , then  $|g(z)| \leq 6|z| + 3 = 15 < 16 = |f(z)|$ .
  - So  $P$  has 4 zeros here.
- On  $|z| < 1$ :
  - Set  $f(z) = 6z$  and  $g(z) = z^4 + 3$ .
  - Check  $|g(z)| \leq |z|^4 + 3 = 4 < 6 = |f(z)|$ .
  - So  $P$  has 1 zero here.

**Example 1.2.** • Claim: the equation  $\alpha ze^z = 1$  where  $|\alpha| > e$  has exactly one solution in  $\mathbb{D}$ .

- Set  $f(z) = \alpha z$  and  $g(z) = e^{-z}$ .
- Estimate at  $|z| = 1$  we have  $|g| = |e^{-z}| = e^{-\Re(z)} \leq e^1 < |\alpha| = |f(z)|$
- $f$  has one zero at  $z_0 = 0$ , thus so does  $f + g$ .

**Theorem 1.9 (Open Mapping).**

Any holomorphic non-constant map is an open map.

**Theorem 1.10 (Maximum Modulus).**

If  $f$  is holomorphic and nonconstant on an open region  $\Omega$ , then  $|f|$  can not attain a maximum on  $\Omega$ .

If  $\Omega$  is bounded and  $f$  is continuous on  $\bar{\Omega}$ , then  $\max_{\bar{\Omega}} |f|$  occurs on  $\partial\Omega$ .

Conversely, if  $f$  attains a local maximum at  $z_0 \in \bar{\Omega}$ , then  $f$  is constant on  $\Omega$ .

**Theorem 1.11 (Casorati-Weierstrass).**

If  $f$  is holomorphic on  $\Omega \setminus \{z_0\}$  where  $z_0$  is an essential singularity, then for every  $V \subset \Omega \setminus \{z_0\}$ ,  $f(V)$  is dense in  $\mathbb{C}$ .

**Theorem 1.12 (Cayley Transform).**

The fractional linear transformation given by  $F(z) = \frac{i - z}{i + z}$  maps  $\mathbb{D} \longrightarrow \mathbb{H}$  with inverse  $G(w) = i \frac{1 - w}{1 + w}$ .

**Theorem 1.13 (Continuation Principle).**

If  $f$  is holomorphic on a bounded connected domain  $\Omega$  and there exists a sequence  $\{z_i\}$  with a limit point in  $\Omega$  such that  $f(z_i) = 0$ , then  $f \equiv 0$  on  $\Omega$ .

**Theorem 1.14 (Schwarz Reflection).**

If  $f$  is continuous and holomorphic on  $\mathbb{H}^+$  and real-valued on  $\mathbb{R}$ , then the extension defined by  $F(z) = \overline{f(\bar{z})}$  for  $z \in \mathbb{H}^-$  is a well-defined holomorphic function on  $\mathbb{C}$ .

---

**Theorem (Schwarz Lemma)** If  $f : \mathbb{D} \rightarrow \mathbb{D}$  is holomorphic with  $f(0) = 0$ , then

1.  $|f(z)| \leq |z|$  for all  $z \in \mathbb{D}$
2.  $|f'(0)| \leq 1$ .

Moreover, if  $|f(z_0)| = |z_0|$  for any  $z_0 \in \mathbb{D}$  or  $|f'(0)| = 1$ , then  $f$  is a rotation

**Theorem (Riemann Mapping)** If  $\Omega$  is simply connected, nonempty, and not  $\mathbb{C}$ , then for every  $z_0 \in \Omega$  there exists a unique conformal map  $F : \Omega \rightarrow \mathbb{D}$  such that  $F(z_0) = 0$  and  $F'(z_0) > 0$ .

Thus any two such sets  $\Omega_1, \Omega_2$  are conformally equivalent.

## 2 Stuff

### 2.0.1 Fundamental Theorem of Algebra: Argument Principle

- Let  $P(z) = a_n z^n + \cdots + a_0$  and  $g(z) = P'(z)/P(z)$ , note  $P$  is holomorphic
- Since  $\lim_{|z| \rightarrow \infty} P(z) = \infty$ , there exist an  $R > 0$  such that  $P$  has no roots in  $\{|z| \geq R\}$ .
- Apply the argument principle:

$$N(0) = \frac{1}{2\pi i} \oint_{|\xi|=R} g(\xi) d\xi.$$

- Check that  $\lim_{|z| \rightarrow \infty} zg(z) = n$ , so  $g$  has a simple pole at  $\infty$
- Then  $g$  has a Laurent series  $\frac{n}{z} + \frac{c_2}{z^2} + \cdots$
- Integrate term-by-term to get  $N(0) = n$ .

### 2.0.2 Fundamental Theorem of Algebra: Rouché's Theorem

- Let  $P(z) = a_n z^n + \cdots + a_0$
- Set  $f(z) = a_n z^n$  and  $g(z) = P(z) - f(z) = a_{n-1} z^{n-1} + \cdots + a_0$ , so  $f + g = P$ .
- Choose  $R > \max \left( \frac{|a_{n-1}| + \cdots + |a_0|}{|a_n|}, 1 \right)$ , then

$$\begin{aligned}
 |g(z)| &:= |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| \\
 &\leq |a_{n-1} z^{n-1}| + \cdots + |a_1 z| + |a_0| \quad \text{by the triangle inequality} \\
 &= |a_{n-1}| \cdot |z|^{n-1} + \cdots + |a_1| \cdot |z| + |a_0| \\
 &= |a_{n-1}| \cdot R^{n-1} + \cdots + |a_1| R + |a_0| \\
 &\leq |a_{n-1}| \cdot R^{n-1} + |a_{n-2}| \cdot R^{n-1} + \cdots + |a_1| \cdot R^{n-1} + |a_0| \cdot R^{n-1} \quad \text{since } R > 1 \implies R^{a+b} \geq R^a \\
 &= R^{n-1} (|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0|) \\
 &\leq R^{n-1} (|a_n| \cdot R) \quad \text{by choice of } R \\
 &= R^n |a_n| \\
 &= |a_n z^n| \\
 &:= |f(z)|
 \end{aligned}$$

- Then  $a_n z^n$  has  $n$  zeros in  $|z| < R$ , so  $f + g$  also has  $n$  zeros.

---

### 2.0.3 Fundamental Theorem of Algebra: Liouville's Theorem

- Suppose  $p$  is nonconstant and has no roots, then  $\frac{1}{p}$  is entire
- Write  $g(z) := \frac{p(z)}{z^n} = a_n \left( \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right)$
- Outside a disc:
  - Note  $\lim_{z \rightarrow \infty} = 0$  for the parenthesized terms, so there exists an  $R$  large enough such that  $|g(z)| \geq \frac{1}{2}|a_n|$
  - Then  $|p(z)| \geq \frac{R^n}{2}|a_n|$  implies  $\frac{1}{p}$  is bounded in  $|z| > R$
- Inside a disc:
  - $p$  is continuous with no roots so  $p$  is bounded below on  $|z| < R$ .
  - $p$  is continuous on a compact set and thus achieves a min  $A$
  - Set  $B = \min(A, \frac{R^n}{2}|a_n|)$ , then  $p \geq B$  on  $|z| < R$ .
- Thus  $p$  is bounded below everywhere and thus  $\frac{1}{p}$  is bounded above everywhere, thus bounded.
- Thus  $\frac{1}{p}$  is constant, forcing  $p$  to be constant.

### 2.0.4 Fundamental Theorem of Algebra: Open Mapping Theorem

- $p$  induces a continuous map  $\mathbb{CP}^1 \rightarrow \mathbb{CP}^1$
- The continuous image of compact space is compact;
- Since the codomain is Hausdorff space, the image is closed.
- $p$  is holomorphic and non-constant, so by the Open Mapping Theorem, the image is open.
- Thus the image is clopen in  $\mathbb{CP}^1$ .
- The image is nonempty, since  $p(1) = \sum a_i \in \mathbb{C}$
- $\mathbb{CP}^1$  is connected
- But the only nonempty clopen subset of a connected space is the entire space.
- So  $p$  is surjective, and  $p^{-1}(0)$  is nonempty.
- So  $p$  has a root.

## 3 Conformal Maps

Conformal maps  $\mathbb{D} \rightarrow \mathbb{D}$  have the form

$$g(z) = \lambda \frac{1-a}{1-\bar{a}z}, \quad |a| < 1, \quad |\lambda| = 1.$$

### 3.1 Plane to Disc

$$\begin{aligned} \varphi : \mathbb{H} &\rightarrow \mathbb{D} \\ \varphi(z) &= \frac{z-i}{z+i} \quad f^{-1}(z) = i \left( \frac{1+w}{1-w} \right). \end{aligned}$$

### 3.2 Sector to Disc

For  $S_\alpha := \{z \in \mathbb{C} \mid 0 < \arg(z) < \alpha\}$  an open sector for  $\alpha$  some angle, first map the sector to the half-plane:

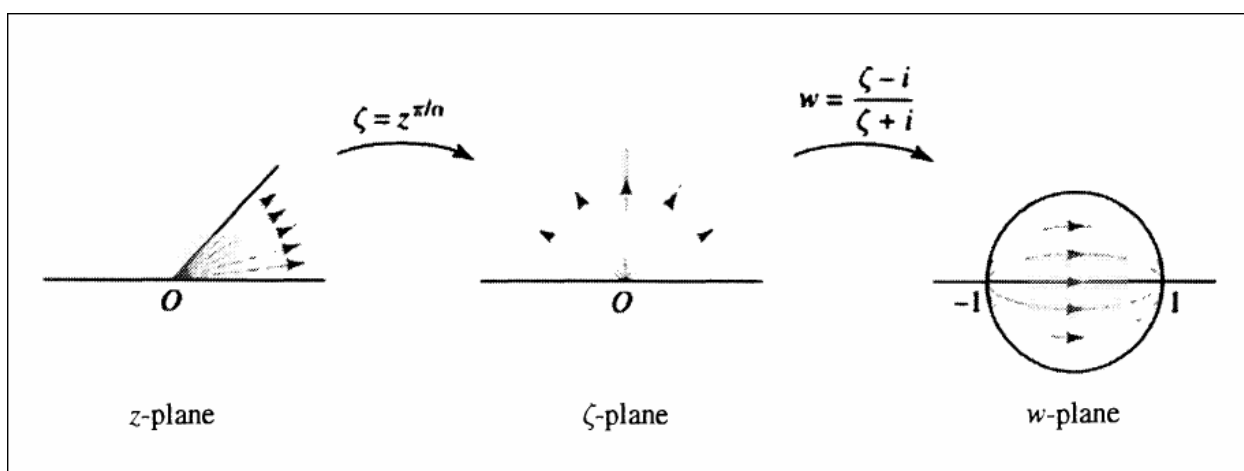
$$g : S_\alpha \longrightarrow \mathbb{H}$$

$$g(z) = z^{\frac{\pi}{\alpha}}.$$

Then compose with a map  $\mathbb{H} \longrightarrow \mathbb{D}$ :

$$f : S_\alpha \longrightarrow \mathbb{D}$$

$$f(z) = (\varphi \circ g)(z) = \frac{z^{\frac{\pi}{\alpha}} - i}{z^{\frac{\pi}{\alpha}} + i}.$$



### 3.3 Strip to Disc

- Map to horizontal strip by rotation  $z \mapsto \lambda z$ .
- Map horizontal strip to sector by  $z \mapsto e^z$ .
- Map sector to  $\mathbb{H}$  by  $z \mapsto z^{\frac{\pi}{\alpha}}$ .
- Map  $\mathbb{H} \longrightarrow \mathbb{D}$ .

## 4 Appendix

$$dz = dx + i dy$$

$$d\bar{z} = dx - i dy$$

$$f_z = f_x = i^{-1} f_y$$

$$\int_0^{2\pi} e^{i\ell x} dx = \begin{cases} 2\pi & (\ell = 0) \\ 0 & (\ell \neq 0) \end{cases}.$$

- Holomorphic: once complex differentiable in neighborhoods of every point.
- Analytic: equal to its Taylor series expansion

Collection of facts used on problem sets

**4.1 Things to know well:**

- Cauchy Integral Formula
- Estimates for derivatives, mean value theorem
- Rouché's theorem
- Casorati-Weierstrass
- The 8 types of conformal maps

**4.2 Theorems****4.2.1 The Argument Principle**

**Theorem (Statement 1)** For  $f$  meromorphic in  $\gamma^\circ$ ,

$$\Delta_\gamma \arg f(z) = 2\pi(Z_f - P_f).$$

**4.2.2 Rouché**

**Theorem (Statement 1)** Suppose  $f = g + h$  with  $g \neq 0, \infty$  on  $\gamma$  with  $|g| > |h|$  on  $\gamma$ . Then

$$\Delta_\gamma \arg(f) = \Delta_\gamma \arg(h) \quad \text{and} \quad Z_f - P_f = Z_g - P_g.$$

**4.3 Misc Prereq**

**Standard forms of conic sections:**

- Circle:  $x^2 + y^2 = r^2$
- Ellipse:  $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$
- Hyperbola:  $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$ 
  - Rectangular Hyperbola:  $xy = \frac{c^2}{2}$ .
- Parabola:  $-4ax + y^2 = 0$ .

Mnemonic: Write  $f(x, y) = Ax^2 + Bxy + Cy^2 + \dots$ , then consider the discriminant  $\Delta = B^2 - 4AC$ :

- $\Delta < 0 \iff$  ellipse
  - $\Delta < 0$  and  $A = C, B = 0 \iff$  circle
- $\Delta = 0 \iff$  parabola
- $\Delta > 0 \iff$  hyperbola

**Completing the square:**

$$x^2 - bx = (x - s)^2 - s^2 \quad \text{where } s = \frac{b}{2}$$

$$x^2 + bx = (x + s)^2 - s^2 \quad \text{where } s = \frac{b}{2}.$$

**Useful Properties**

- $\Re(z) = \frac{1}{2}(z + \bar{z})$  and  $\Im(z) = \frac{1}{2i}(z - \bar{z})$ .
- $z\bar{z} = |z|^2$
- $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$
- $\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ .

**Useful Series**

$$\begin{aligned}\sum_{k=1}^n k &= \frac{n(n+1)}{2} \\ \sum_{k=1}^n k^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{k=1}^n k^3 &= \frac{n^2(n+1)^2}{4} \\ \log(z) &= \sum_{j=0}^{\infty} (-1)^j \frac{(z-a)^j}{j}\end{aligned}$$

**Cauchy-Riemann Equations**

$$\begin{aligned}u_x &= v_y \quad \text{and} \quad u_y = -v_x \\ \frac{\partial u}{\partial r} &= \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}\end{aligned}$$

**4.4 Useful Techniques****Showing a function is constant:**

- Write  $f = u + iv$  and use Cauchy-Riemann to show  $u_x, u_y = 0$ , etc.
- Show that  $f$  is entire and bounded.

**Showing a function is zero:** Show  $f$  is entire, bounded, and  $\lim_{z \rightarrow \infty} f(z) = 0$ .

**Deriving Polar Cauchy-Riemann:** See walkthrough here. Take derivative along two paths, along a ray with constant angle  $\theta_0$  and along a circular arc of constant radius  $r_0$ . Then equate real and imaginary parts. See problem set 1.

**Computing Arguments:**  $\text{Arg}(z/w) = \text{Arg}(z) - \text{Arg}(w)$ .

The sum of the interior angles of an  $n$ -gon is  $(n-2)\pi$ , where each angle is  $\frac{n-2}{n}\pi$ .

**4.5 Residues**

If  $p$  is a simple pole,  $\text{Res}(p, f) = \lim_{z \rightarrow p} (z-p)f(z)$ . Example: Let  $f(z) = \frac{1}{1+z^2}$ , then  $\text{Res}(i, f) = \frac{1}{2i}$ .



Green's Theorem: Todo

$$\frac{\partial}{\partial z} \sum_{j=0}^{\infty} a_j z^j = \sum_{j=0}^{\infty} a_{j+1} z^j.$$

## 4.6 Pithy Statements

- Little Picard:  $f$  misses at most one point and is a homeomorphism onto its image.
- Baire's Theorem: The intersection of open dense sets is open.
- Casorati-Weierstrass: The image of a disc punctured at an essential singularity is dense in  $\mathbb{C}$ .
- Open Mapping: Holomorphic functions preserve open sets.
- Argument Principle: The logarithmic derivative measures the difference of zeros and poles.
- Liouville: Bounded entire functions are constant.
- Maximum Modulus: Holomorphic functions take extrema only on boundaries.
- Cauchy Inequalities: The  $n$ th Taylor coefficient is at most  $\sup_{|z|=R} |f|/R^n$ .
- Cauchy's Theorem: Integrals of holomorphic functions vanish.
- Morera: Integrals vanishing along every rectangle implies holomorphic.
- Schwarz Reflection: ???
- Identity Theorem: Two functions agreeing on a set with a limit point are equal on a domain.
- The ring of holomorphic functions on a domain in  $\mathbb{C}$  has no zero divisors (by the identity principle).

## 4.7 Precise Refinements

**Cauchy Inequality:** Given  $z_0 \in \Omega$ , pick the largest disc  $D_R(z_0) \subset \Omega$  and let  $C_R = \partial D_R$ . Using the integral formula, defining  $\|f\|_{C_R} = \max_{|z-z_0|=R} |f(z)|$

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{\|f\|_{C_R}}{R^{n+1}} R d\theta = \frac{n! \|f\|_{C_R}}{R^n}.$$

Basics

- Show that  $\frac{1}{z} \sum_{k=1}^{\infty} \frac{z^k}{k}$  converges on  $S^1 \setminus \{1\}$  using summation by parts.
- Show that any power series is continuous on its domain of convergence.
- Show that a uniform limit of continuous functions is continuous.

??

- Show that if  $f$  is holomorphic on  $\mathbb{D}$  then  $f$  has a power series expansion that converges uniformly on every compact  $K \subset \mathbb{D}$ .
- Show that any holomorphic function  $f$  can be uniformly approximated by polynomials.
- Show that if  $f$  is holomorphic on a connected region  $\Omega$  and  $f' \equiv 0$  on  $\Omega$ , then  $f$  is constant on  $\Omega$ .
- Show that if  $|f| = 0$  on  $\partial\Omega$  then either  $f$  is constant or  $f$  has a zero in  $\Omega$ .

- Show that if  $\{f_n\}$  is a sequence of holomorphic functions converging uniformly to a function  $f$  on every compact subset of  $\Omega$ , then  $f$  is holomorphic on  $\Omega$  and  $\{f'_n\}$  converges uniformly to  $f'$  on every such compact subset.
- Show that if each  $f_n$  is holomorphic on  $\Omega$  and  $F := \sum f_n$  converges uniformly on every compact subset of  $\Omega$ , then  $F$  is holomorphic.
- Show that if  $f$  is once complex differentiable at each point of  $\Omega$ , then  $f$  is holomorphic.