# Real Analysis Qualifying Exam Questions

# D. Zack Garza

# Thursday 13<sup>th</sup> August, 2020

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## **Todo list**

Walk through.	. 7
Add concepts	
Redo, may just be wrong.	
Add concepts	
Add concepts	
Need to justify removing floor function and cancellation	
No clue how to show this	
Not sure. Approximate $f$ by simple functions?	
Add concepts	
Walk through.	
Not complete	
Add concepts	
Walk through.	
<u> </u>	
Add concepts	
?	
Walk through.	
Add concepts	
Actually true? Need conditions?	
Add concepts	
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How to prove the hint?	
Add concepts	
All messed up!	
Move this to review notes to clean things up	
Try to construct the set	
Walk through.	
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Add concepts	
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Add concepts	
What theorem is this using?	
Is this assumed? Or did we show it?	
Add concepts	
-	

show that $\int_0^1  f_n(x) - f_m(x)  dx \longrightarrow 0$ rigorously, show that no $g \in L^1([0,1])$ can converge to
this indicator function
Add concepts
Add concepts
Shouldn't this be a supremum? The max may not exist?
Review and clean up

### 1 Undergraduate Analysis: Uniform Convergence

### 1.1 Fall 2018 # 1 💝

Let  $f(x) = \frac{1}{x}$ . Show that f is uniformly continuous on  $(1, \infty)$  but not on  $(0, \infty)$ .

#### **Solution:**

### Concepts Used:

- Uniform continuity.
- Show a stronger statement:  $f(x) = \frac{1}{x}$  is uniformly continuous on any interval of the form  $(c, \infty)$  where c > 0.
- Note that

$$|x|, |y| > c > 0 \implies |xy| = |x||y| > c^2 \implies \frac{1}{|xy|} < \frac{1}{c^2}.$$

- Letting  $\varepsilon$  be arbitrary, choose  $\delta < \varepsilon c^2$ .
- Note that  $\delta$  does not depend on x, y.
- Then

$$|f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right|$$

$$= \frac{|x - y|}{xy}$$

$$\leq \frac{\delta}{xy}$$

$$< \frac{\delta}{c^2}$$

$$< \varepsilon,$$

which shows uniform continuity.

To see that f is not uniformly continuous when c = 0:

Note: negating uniform continuity says  $\exists \varepsilon > 0$  such that  $\forall \delta(\varepsilon)$  there exist x, y such that  $|x-y| < \delta \text{ and } |f(x)-f(y)| > \varepsilon.$ 

- Let  $\varepsilon < 1$ . Let  $x_n = \frac{1}{n}$  for  $n \ge 1$ .
- Choose n large enough such that  $|x_n x_{n+1}| = \frac{1}{n} \frac{1}{n+1} < \delta$ .

- Why this can be done: by the archimedean property of  $\mathbb{R}$ , choose n such that  $\frac{1}{n} < \varepsilon$ .
- Then

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \le \frac{1}{n} < \varepsilon \quad \text{since } n+1 > 1.$$

• Note  $f(x_n) = n$  and thus

$$|f(x_n) - f(x_{n+1})| = n - (n+1) = 1 > \varepsilon.$$

### 1.2 Fall 2017 # 1 🐪

Let

$$f(x) = s \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which f does and does not converge uniformly.

Walk through

Add concepts.

#### Solution:

Concepts Used:

• ?

Note that  $f(x) = e^x$  is entire and thus equal to its power series. So  $f(x) = \sum_{j=0}^{\infty} \frac{1}{j!} x^j$ .

Letting  $f_N(x) = \sum_{i=1}^N \frac{1}{j!} x^j$ , we have  $f_N(x) \longrightarrow f(x)$  pointwise on  $(-\infty, \infty)$ .

For any compact interval [-M, M], we have

$$||f_N(x) - f(x)||_{\infty} = \sup_{-M \le x \le M} \left| \sum_{j=N+1}^{\infty} \frac{1}{j!} x^j \right|$$

$$\le \sup_{-M \le x \le M} \sum_{j=N+1}^{\infty} \frac{1}{j!} |x|^j$$

$$\le \sum_{j=N+1}^{\infty} \frac{1}{j!} M^j$$

$$\le \sum_{j=0}^{\infty} \frac{1}{j!} M^j$$

$$= e^M$$

$$< \infty,$$

so  $f_N \longrightarrow f$  uniformly on [-M, M] by the M-test. Thus it converges on any bounded interval. It does not converge on  $\mathbb{R}$ , since  $x^N$  is unbounded.

## 1.3 Fall 2014 # 1

Let  $\{f_n\}$  be a sequence of continuous functions such that  $\sum f_n$  converges uniformly.

Prove that  $\sum f_n$  is also continuous.

## 1.4 Spring 2017 # 4 😽

Let f(x,y) on  $[-1,1]^2$  be defined by

$$f(x,y) = \begin{cases} \frac{xy}{(x^2 + y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Determine if f is integrable.

Redo, may just be wrong

Add concepts.

#### **Solution:**

Concepts Used:

• ?

Switching to polar coordinates and integrating over a half-circle contained in  $I^2$ , we have

$$\int_{I^2} f \ge \int_0^{\pi} \int_0^1 \frac{\cos(\theta)\sin(\theta)}{r^2} dr d\theta = \infty,$$

so f is not integrable.

### 1.5 Spring 2015 # 1 💝

Let (X, d) and  $(Y, \rho)$  be metric spaces,  $f: X \longrightarrow Y$ , and  $x_0 \in X$ .

Prove that the following statements are equivalent:

- 1. For every  $\varepsilon > 0$   $\exists \delta > 0$  such that  $\rho(f(x), f(x_0)) < \varepsilon$  whenever  $d(x, x_0) < \delta$ .
- 2. The sequence  $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow f(x_0)$  for every sequence  $\{x_n\} \longrightarrow x_0$  in X.

## 1.6 Fall 2014 # 2

Let I be an index set and  $\alpha: I \longrightarrow (0, \infty)$ .

1. Show that

$$\sum_{i \in I} a(i) := \sup_{\substack{J \subset I \\ J \text{ finite}}} \sum_{i \in J} a(i) < \infty \implies I \text{ is countable.}$$

2. Suppose  $I=\mathbb{Q}$  and  $\sum_{q\in\mathbb{Q}}a(q)<\infty.$  Define

$$f(x) := \sum_{\substack{q \in \mathbb{Q} \\ q \le x}} a(q).$$

Show that f is continuous at  $x \iff x \notin \mathbb{Q}$ .

### 1.7 Spring 2014 # 2

Let  $\{a_n\}$  be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that  $\sum a_n^2 < \infty$ .

Note: Assume  $a_n, b_n$  are all non-negative.

### 2 General Analysis

### 2.1 Spring 2020 # 1 💝

Prove that if  $f:[0,1] \longrightarrow \mathbb{R}$  is continuous then

$$\lim_{k \to \infty} \int_0^1 kx^{k-1} f(x) \, dx = f(1).$$

### Solution:

Concepts Used:

- DCT
- Weierstrass Approximation Theorem
- Suppose p is a polynomial, then

$$\lim_{k \to \infty} \int_0^1 kx^{k-1} p(x) \, dx = \lim_{k \to \infty} \int_0^1 \left( \frac{\partial}{\partial x} x^k \right) p(x) \, dx$$

$$= \lim_{k \to \infty} \left[ x^k p(x) \Big|_0^1 - \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) \, dx \right] \quad \text{integrating by parts}$$

$$= p(1) - \lim_{k \to \infty} \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) \, dx,$$

• Thus it suffices to show that

$$\lim_{k \to \infty} \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx = 0.$$

• Integrating by parts a second time yields

$$\lim_{k \to \infty} \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx = \lim_{k \to \infty} \frac{x^{k+1}}{k+1} p'(x) \Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1} \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx$$

$$= -\lim_{k \to \infty} \int_0^1 \frac{x^{k+1}}{k+1} \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx$$

$$= -\int_0^1 \lim_{k \to \infty} \frac{x^{k+1}}{k+1} \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx \quad \text{by DCT}$$

$$= -\int_0^1 0 \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx$$

$$= 0.$$

– The DCT can be applied here because f'' is continuous and [0,1] is compact, so f'' is bounded on [0,1] by a constant M and

$$\int_0^1 \left| x^k f''(x) \right| \le \int_0^1 1 \cdot M = M < \infty.$$

- Now use the Weierstrass approximation theorem:
  - If  $f:[a,b] \longrightarrow \mathbb{R}$  is continuous, then for every  $\varepsilon > 0$  there exists a polynomial  $p_{\varepsilon}(x)$  such that  $||f p_{\varepsilon}||_{\infty} < \varepsilon$ .
- Thus

$$\left| \int_0^1 kx^{k-1} p_{\varepsilon}(x) \, dx - \int_0^1 kx^{k-1} f(x) \, dx \right| = \left| \int_0^1 kx^{k-1} (p_{\varepsilon}(x) - f(x)) \, dx \right|$$

$$\leq \left| \int_0^1 kx^{k-1} || p_{\varepsilon} - f ||_{\infty} \, dx \right|$$

$$= || p_{\varepsilon} - f ||_{\infty} \cdot \left| \int_0^1 kx^{k-1} \, dx \right|$$

$$= || p_{\varepsilon} - f ||_{\infty} \cdot x^k \right|_0^1$$

$$= || p_{\varepsilon} - f ||_{\infty} \xrightarrow{\varepsilon \longrightarrow 0} 0$$

and the integrals are equal.

• By the first argument,

$$\int_0^1 kx^{k-1} p_{\varepsilon}(x) dx = p_{\varepsilon}(1) \text{ for each } \varepsilon$$

• Since uniform convergence implies pointwise convergence,  $p_{\varepsilon}(1) \stackrel{\varepsilon \longrightarrow 0}{\longrightarrow} f(1)$ .

## 2.2 Fall 2019 # 1 😽

Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of real numbers.

### 2.2.1 a

Prove that if  $\lim_{n \to \infty} a_n = 0$ , then

$$\lim_{n \to \infty} \frac{a_1 + \dots + a_n}{n} = 0$$

#### 2.2.2 b

Prove that if  $\sum_{n=1}^{\infty} \frac{a_n}{n}$  converges, then

$$\lim_{n\to\infty}\frac{a_1+\cdots+a_n}{n}=0$$

#### Solution:

Concepts Used:

- Cesaro mean/summation.
- Break series apart into pieces that can be handled separately.

### 2.2.3 a

Prove a stronger result:

$$a_k \longrightarrow S \implies S_N \coloneqq \frac{1}{N} \sum_{k=1}^N a_k \longrightarrow S.$$

Idea: once N is large enough,  $a_k \approx S$ , and all smaller terms will die off as  $N \longrightarrow \infty$ . See this MSE answer.

• Use convergence  $a_k \longrightarrow S$ : choose M large enough such that

$$k \ge M + 1 \implies |a_k - S| < \varepsilon.$$

Then

$$\left| \left( \frac{1}{N} \sum_{k=1}^{N} a_k \right) - S \right| = \frac{1}{N} \left| \left( \sum_{k=1}^{N} a_k \right) - NS \right|$$

$$= \frac{1}{N} \left| \left( \sum_{k=1}^{N} a_k \right) - \sum_{k=1}^{N} S \right|$$

$$= \frac{1}{N} \left| \sum_{k=1}^{N} (a_k - S) \right|$$

$$\leq \frac{1}{N} \sum_{k=1}^{N} |a_k - S|$$

$$= \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} |a_k - S|$$

$$\leq \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} \frac{\varepsilon}{2}$$

$$= \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + (N - M) \frac{\varepsilon}{2}$$

$$\stackrel{\varepsilon}{\Longrightarrow} \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + 0$$

$$\stackrel{N \longrightarrow \infty}{\Longrightarrow} 0 + 0.$$

Note: M is fixed, so the last sum is some constant c, and  $c/N \longrightarrow 0$  as  $N \longrightarrow \infty$  for any constant. To be more careful, choose M first to get  $\varepsilon/2$  for the tail, then choose N(M) > M for the remaining truncated part of the sum.

#### 2.2.4 b

• Define

$$\Gamma_n := \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

- $\Gamma_1 = \sum_{k=1}^n \frac{a_k}{k}$  is the original series and each  $\Gamma_n$  is a tail of  $\Gamma_1$ , so by assumption  $\Gamma_n \stackrel{n \longrightarrow \infty}{\longrightarrow} 0$ .
- Compute

$$\frac{1}{n}\sum_{k=1}^{n}a_k=\frac{1}{n}(\Gamma_1+\Gamma_2+\cdots+\Gamma_n-\mathbf{\Gamma_{n+1}})$$

• This comes from consider the following summation:

$$\Gamma_1: \qquad a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots$$

$$\Gamma_2: \qquad \frac{a_2}{2} + \frac{a_3}{3} + \cdots$$

$$\Gamma_3: \qquad \frac{a_3}{3} + \cdots$$

$$\sum_{i=1}^n \Gamma_i: \qquad a_1 + a_2 + a_3 + \cdots + a_n + \frac{a_{n+1}}{n+1} + \cdots$$

- Use part (a): since  $\Gamma_n \stackrel{n \to \infty}{\longrightarrow} 0$ , we have  $\frac{1}{n} \sum_{k=1}^n \Gamma_k \stackrel{n \to \infty}{\longrightarrow} 0$ .
- Also a minor check:  $\Gamma_n \longrightarrow 0 \implies \frac{1}{n}\Gamma_n \longrightarrow 0$ .
- Then

$$\frac{1}{n} \sum_{k=1}^{n} a_k = \frac{1}{n} (\Gamma_1 + \Gamma_2 + \dots + \Gamma_n - \Gamma_{n+1})$$
$$= \left(\frac{1}{n} \sum_{k=0}^{n} \Gamma_k\right) - \left(\frac{1}{n} \Gamma_{n+1}\right)$$
$$\stackrel{n \to \infty}{\longrightarrow} 0.$$

## 2.3 Fall 2018 # 4 🐪

Let  $f \in L^1([0,1])$ . Prove that

$$\lim_{n \to \infty} \int_0^1 f(x) |\sin nx| \ dx = \frac{2}{\pi} \int_0^1 f(x) \ dx$$

> Hint: Begin with the case that f is the characteristic function of an interval.

Add concepts.

#### **Solution:**

### Concepts Used:

• ?

Case of characteristic function

- First suppose  $f(x) = \chi_{[0,1]}(x)$ .
- Note that  $\sin(nx)$  has a period of  $2\pi/n$ , and thus  $\left|\frac{n}{2\pi}\right|$  full periods in [0,1].
- Taking the absolute value yields a new function with half the period, so a period of  $\pi/n$ and  $|\pi/n|$  full periods in [0,1].

• We can compute the integral over one full period (which is independent of which period is chosen), and since  $\sin(x)$  is positive and agrees with  $|\sin(nx)|$  on the first period, we have

$$\int_{\text{One Period}} |\sin(nx)| \, dx = \int_0^{\pi/n} \sin(nx) \, dx$$

$$= \frac{1}{n} \int_0^{\pi} \sin(u) \, du \quad u = nx$$

$$= \frac{1}{n} - \cos(u) \Big|_0^{\pi}$$

$$= \frac{2}{n}.$$

• Then break the integral up into integrals over periods  $P_1, P_2, \dots, P_N$  where  $N := \lfloor n/\pi \rfloor$ :

$$\int_{0}^{1} |\sin(nx)| dx = \left(\sum_{j=1}^{N} \int_{P_{j}} |\sin(nx)| dx\right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| dx$$

$$= \left(\sum_{j=1}^{N} \frac{2}{n}\right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| dx$$

$$= N\left(\frac{2}{n}\right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| dx$$

$$\coloneqq \left\lfloor \frac{n}{\pi} \right\rfloor \frac{2}{n} + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| dx$$

$$= \frac{2}{\pi} + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| dx$$

$$\coloneqq \frac{2}{\pi} + R(n)$$

so it suffices to show that  $R(n) \stackrel{n \longrightarrow \infty}{\longrightarrow} 0$ .

Need to justify removing floor function and cancellation.

• Showing this: ??????????

No clue how to show this

General case

Not sure. Approximate f by simple functions...?

## 2.4 Fall 2017 # 4 차

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

1. Show that  $f_n \longrightarrow 0$  pointwise but not uniformly on [0,1].

Hint: Consider the maximum of  $f_n$ .

2.

$$\lim_{n \to \infty} \int_0^1 n(1-x)^n \sin x \, dx = 0$$

Add concepts.

Walk through

#### Solution:

Concepts Used:

• ?

#### 2.4.1 a

Let  $G(x) = \sum_{n=1}^{\infty} nx(1-x)^n$ . Applying the ratio test, we have

$$\left| \frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n} \right| = \frac{n+1}{n} |1-x| \xrightarrow{n \to \infty} |1-x| < 1 \iff 0 \le x \le 2,$$

and in particular, this series converges on [0,2]. Thus its terms go to zero, and  $nx(1-x)^n \longrightarrow 0$  on  $[0,1] \subset [0,2]$ .

To see that the convergence is not uniform, let  $x_n = \frac{1}{n}$  and  $\varepsilon > \frac{1}{e}$ , then

$$\sup_{x \in [0,1]} |nx(1-x)^n - 0| \ge |nx_n(1-x_n)^n| = \left| \left(1 - \frac{1}{n}\right)^n \right| \stackrel{n \to \infty}{\longrightarrow} e^{-1} > \varepsilon.$$

#### 2.4.2 b

Note: could use the first part with  $\sin(x) \le x$ , but then integral ends up more complicated. Noting that  $\sin(x) \le 1$ , we have

$$\left| \int_0^1 n(1-x)^n \sin(x) \right| \le \int_0^1 |n(1-x)^n \sin(x)|$$

$$\le \int_0^1 |n(1-x)^n|$$

$$= n \int_0^1 (1-x)^n$$

$$= -\frac{n(1-x)^{n+1}}{n+1}$$

$$\stackrel{n \to \infty}{\longrightarrow} 0.$$

## 2.5 Spring 2017 # 3 🔭

Let

$$f_n(x) = ae^{-nax} - be^{-nbx}$$
 where  $0 < a < b$ .

Show that

a. 
$$\sum_{n=1}^{\infty} |f_n| \text{ is not in } L^1([0,\infty),m)$$

Hint:  $f_n(x)$  has a root  $x_n$ .

b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0,\infty),m) \quad \text{and} \quad \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x) \, dm = \ln \frac{b}{a}$$

Not complete

Add concepts

Walk through.

### Solution:

Concepts Used:

• ?

### 2.5.1 a

Letting  $x_n := \frac{1}{n}$ , we have

$$\sum_{k=1}^{\infty} |f_k(x)| \ge |f_n(x_n)| = \left| ae^{-ax} - be^{-bx} \right| := M.$$

In particular,  $\sup_{x} |f_n(x)| \not\longrightarrow 0$ , so the terms do not go to zero and the sum can not converge.

2.5.2 b

?

## 2.6 Fall 2016 # 1 😽

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that f converges to a differentiable function on  $(1, \infty)$  and that

$$f'(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n^x}\right)'.$$

Hint:

$$\left(\frac{1}{n^x}\right)' = -\frac{1}{n^x} \ln n$$

Add concepts

#### Solution:

#### Concepts Used:

• ?

- Set  $f_N(x) := \sum_{n=1}^N n^{-x}$ , so  $f(x) = \lim_{N \to \infty} f_N(x)$ .
- If an interchange of limits is justified, we have

$$\frac{\partial}{\partial x} \lim_{N \to \infty} \sum_{n=1}^{N} n^{-x} = \lim_{h \to 0} \lim_{N \to \infty} \frac{1}{h} \left[ \left( \sum_{n=1}^{N} n^{-x} \right) - \left( \sum_{n=1}^{N} n^{-(x+h)} \right) \right]$$

$$\stackrel{?}{=} \lim_{N \to \infty} \lim_{h \to 0} \frac{1}{h} \left[ \left( \sum_{n=1}^{N} n^{-x} \right) - \left( \sum_{n=1}^{N} n^{-(x+h)} \right) \right]$$

$$= \lim_{N \to \infty} \lim_{h \to 0} \frac{1}{h} \left[ \sum_{n=1}^{N} n^{-x} - n^{-(x+h)} \right] \quad (1)$$

$$= \lim_{N \to \infty} \sum_{n=1}^{N} \lim_{h \to 0} \frac{1}{h} \left[ n^{-x} - n^{-(x+h)} \right] \quad \text{since this is a finite sum}$$

$$:= \lim_{N \to \infty} \sum_{n=1}^{N} \frac{\partial}{\partial x} \left( \frac{1}{n^x} \right)$$

$$= \lim_{N \to \infty} \sum_{n=1}^{N} -\frac{\ln(n)}{n^x},$$

where the combining of sums in (1) is valid because  $\sum n^{-x}$  is absolutely convergent for x > 1 by the *p*-test.

- Thus it suffices to justify the interchange of limits and show that the last sum converges on  $(1, \infty)$ .
- Claim:  $\sum n^{-x} \ln(n)$  converges.
  - Use the fact that for any fixed  $\varepsilon > 0$ ,

$$\lim_{n \to \infty} \frac{\ln(n)}{n^{\varepsilon}} \stackrel{\text{L.H.}}{=} \lim_{n \to \infty} \frac{1/n}{\varepsilon n^{\varepsilon - 1}} = \lim_{n \to \infty} \frac{1}{\varepsilon n^{\varepsilon}} = 0,$$

- This implies that for a fixed  $\varepsilon > 0$  and for any constant c > 0 there exists an N large enough such that  $n \geq N$  implies  $\ln(n)/n^{\varepsilon} < c$ , i.e.  $\ln(n) < cn^{\varepsilon}$ .
- Taking c = 1, we have  $n \ge N \implies \ln(n) < n^{\varepsilon}$

- We thus break up the sum:

$$\sum_{n \in \mathbb{N}} \frac{\ln(n)}{n^x} = \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{\ln(n)}{n^x}$$

$$\leq \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{n^{\varepsilon}}{n^x}$$

$$\coloneqq C_{\varepsilon} + \sum_{n=N}^{\infty} \frac{n^{\varepsilon}}{n^x} \quad \text{with } C_{\varepsilon} < \infty \text{ a constant}$$

$$= C_{\varepsilon} + \sum_{n=N}^{\infty} \frac{1}{n^{x-\varepsilon}},$$

where the last term converges by the p-test if  $x - \varepsilon > 1$ .

- But  $\varepsilon$  can depend on x, and if  $x \in (1, \infty)$  is fixed we can choose  $\varepsilon < |x 1|$  to ensure this.
- Claim: the interchange of limits is justified.

# 2.7 Fall 2016 # 5 🐆

Let  $\varphi \in L^{\infty}(\mathbb{R})$ . Show that the following limit exists and satisfies the equality

$$\lim_{n \to \infty} \left( \int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1 + x^2} \, dx \right)^{\frac{1}{n}} = \|\varphi\|_{\infty}.$$

Walk through

Add concepts.

Solution:

Concepts Used:

• ?

Let L be the LHS and R be the RHS.

Claim:  $L \leq R$ . - Since  $|\varphi| \leq ||\varphi||_{\infty}$  a.e., we can write

$$\begin{split} L^{\frac{1}{n}} &\coloneqq \int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} \\ &\le \int_{\mathbb{R}} \frac{\|\varphi\|_{\infty}^n}{1+x^2} \\ &= \|\varphi\|_{\infty}^n \int_{\mathbb{R}} \frac{1}{1+x^2} \\ &= \|\varphi\|_{\infty}^n \arctan(x)\Big|_{-\infty}^{\infty} \\ &= \|\varphi\|_{\infty}^n \left(\frac{\pi}{2} - \frac{-\pi}{2}\right) \\ &= \pi \|\varphi\|_{\infty}^n \end{split}$$

$$\implies L^{\frac{1}{n}} \leq \sqrt[n]{\pi \|\varphi\|_{\infty}^{n}}$$

$$\implies L \leq \pi^{\frac{1}{n}} \|\varphi\|_{\infty}$$

$$\stackrel{n \to \infty}{\longrightarrow} \|\varphi\|_{\infty},$$

where we've used the fact that  $c^{\frac{1}{n}} \stackrel{n \longrightarrow \infty}{\longrightarrow} 1$  for any constant c.

Actually true? Need conditions?

Claim:  $R \leq L$ .

- We will show that  $R \leq L + \varepsilon$  for every  $\varepsilon > 0$ .
- Set

$$S_{\varepsilon} := \left\{ x \in \mathbb{R}^n \mid |\varphi(x)| \ge ||\varphi||_{\infty} - \varepsilon \right\}.$$

• Then we have

$$\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} dx \ge \int_{S_{\varepsilon}} \frac{|\varphi(x)|^n}{1+x^2} dx \quad S_{\varepsilon} \subset \mathbb{R}$$

$$\ge \int_{S_{\varepsilon}} \frac{(\|\varphi\|_{\infty} - \varepsilon)^n}{1+x^2} dx \quad \text{by definition of } S_{\varepsilon}$$

$$= (\|\varphi\|_{\infty} - \varepsilon)^n \int_{S_{\varepsilon}} \frac{1}{1+x^2} dx$$

$$= (\|\varphi\|_{\infty} - \varepsilon)^n C_{\varepsilon} \quad \text{where } C_{\varepsilon} \text{ is some constant}$$

$$\implies \left( \int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} \, dx \right)^{\frac{1}{n}} \ge (\|\varphi\|_{\infty} - \varepsilon) C_{\varepsilon}^{\frac{1}{n}}$$

$$\stackrel{n \longrightarrow \infty}{\longrightarrow} (\|\varphi\|_{\infty} - \varepsilon) \cdot 1$$

$$\stackrel{\varepsilon \longrightarrow 0}{\longrightarrow} \|\varphi\|_{\infty},$$

where we've again used the fact that  $c^{\frac{1}{n}} \longrightarrow 1$  for any constant.

## 2.8 Fall 2016 # 6 💝

Let  $f, g \in L^2(\mathbb{R})$ . Show that

$$\lim_{n \to \infty} \int_{\mathbb{R}} f(x)g(x+n) \, dx = 0$$

Add concepts

Solution:

Concepts Used:

• ?

• Use the fact that  $L^p$  has small tails: if  $h \in L^2(\mathbb{R})$ , then for any  $\varepsilon > 0$ ,

$$\forall \varepsilon, \exists N \in \mathbb{N} \text{ such that } \int_{|x| \ge N} |h(x)|^2 dx < \varepsilon.$$

How to prove small tails in  $L^p$ ?

- So choose n large enough so the tails of both f and g are smaller than  $\varepsilon$ .
- Apply Cauchy-Schwarz:

$$\left| \int_{\mathbb{R}} f(x)g(x+n) \, dx \right| \le \int_{\mathbb{R}} |f(x)g(x+n)| \, dx$$
$$\le \int_{\mathbb{R}}.$$

## 2.9 Spring 2016 # 1

For  $n \in \mathbb{N}$ , define

$$]e_n = \left(1 + \frac{1}{n}\right)^n$$
 and  $E_n = \left(1 + \frac{1}{n}\right)^{n+1}$ 

Show that  $e_n < E_n$ , and prove Bernoulli's inequality:

$$(1+x)^n \ge 1 + nx$$
 for  $-1 < x < \infty$  and  $n \in \mathbb{N}$ 

Use this to show the following:

- 1. The sequence  $e_n$  is increasing.
- 2. The sequence  $E_n$  is decreasing.
- 3.  $2 < e_n < E_n < 4$ .
- $4. \lim_{n \to \infty} e_n = \lim_{n \to \infty} E_n.$

## 2.10 Fall 2015 # 1

Define

$$f(x) = c_0 + c_1 x^1 + c_2 x^2 + \ldots + c_n x^n$$
 with  $n$  even and  $c_n > 0$ .

Show that there is a number  $x_m$  such that  $f(x_m) \leq f(x)$  for all  $x \in \mathbb{R}$ .

### 3 Measure Theory: Sets

### 3.1 Spring 2020 # 2 💝

Let  $m_*$  denote the Lebesgue outer measure on  $\mathbb{R}$ .

### 3.1.1 a.

Prove that for every  $E \subseteq \mathbb{R}$  there exists a Borel set B containing E such that

$$m_*(B) = m_*(E).$$

#### 3.1.2 b.

Prove that if  $E \subseteq \mathbb{R}$  has the property that

$$m_*(A) = m_*(A \cap E) + m_*(A \cap E^c)$$

for every set  $A \subseteq \mathbb{R}$ , then there exists a Borel set  $B \subseteq \mathbb{R}$  such that  $E = B \setminus N$  with  $m_*(N) = 0$ . Be sure to address the case when  $m_*(E) = \infty$ .

#### **Solution:**

### Concepts Used:

• Definition of outer measure:

$$m_*(E) = \inf_{\{Q_j\} \rightrightarrows E} \sum |Q_j|$$

where  $\{Q_j\}$  is a countable collection of closed cubes.

- Break  $\mathbb{R}$  into  $\coprod_{n\in\mathbb{Z}}[n,n+1)$ , each with finite measure.
- Theorem:  $m_*(Q) = |Q|$  for Q a closed cube (i.e. the outer measure equals the volume).

Proof.

- $m_*(Q) \le |Q|$ :
- Since  $Q \subseteq Q$ ,  $Q \rightrightarrows Q$  and  $m_*(Q) \leq |Q|$  since  $m_*$  is an infimum over such coverings.
- $|Q| \le m_*(Q)$ :
- Fix  $\varepsilon > 0$ .
- Let  $\{Q_i\}_{i=1}^{\infty} \rightrightarrows Q$  be arbitrary, it suffices to show that

$$|Q| \le \left(\sum_{i=1}^{\infty} |Q_i|\right) + \varepsilon.$$

- Pick open cubes  $S_i$  such that  $Q_i \subseteq S_i$  and  $|Q_i| \le |S_i| \le (1+\varepsilon)|Q_i|$ .
- Then  $\{S_i\} \rightrightarrows Q$ , so by compactness of Q pick a finite subcover with N elements.
- Note

$$Q \subseteq \bigcup_{i=1}^{N} S_i \implies |Q| \le \sum_{i=1}^{N} |S_i| \le \sum_{i=1}^{N} (1+\varepsilon)|Q_i| \le (1+\varepsilon) \sum_{i=1}^{\infty} |Q_i|.$$

• Taking an infimum over coverings on the RHS preserves the inequality, so

$$|Q| \le (1+\varepsilon)m_*(Q)$$

• Take  $\varepsilon \longrightarrow 0$  to obtain final inequality.

3.1.3 a

- If  $m_*(E) = \infty$ , then take  $B = \mathbb{R}^n$  since  $m(\mathbb{R}^n) = \infty$ .
- Suppose  $N := m_*(E) < \infty$ .
- Since  $m_*(E)$  is an infimum, by definition, for every  $\varepsilon > 0$  there exists a covering by closed cubes  $\{Q_i(\varepsilon)\}_{i=1}^{\infty} \rightrightarrows E$  depending on  $\varepsilon$  such that

$$\sum_{i=1}^{\infty} |Q_i(\varepsilon)| < N + \varepsilon.$$

- For each fixed n, set  $\varepsilon_n = \frac{1}{n}$  to produce such a covering  $\{Q_i(\varepsilon_n)\}_{i=1}^{\infty}$  and set  $B_n := \bigcup_{i=1}^{\infty} Q_i(\varepsilon_n)$ .
- The outer measure of cubes is equal to the sum of their volumes, so

$$m_*(B_n) = \sum_{i=1}^{\infty} |Q_i(\varepsilon_n)| < N + \varepsilon_n = N + \frac{1}{n}.$$

- Now set  $B := \bigcap^{\infty} B_n$ .
  - Since  $E \subseteq B_n$  for every  $n, E \subseteq B$
  - Since B is a countable intersection of countable unions of closed sets, B is Borel.

- Since  $B_n \subseteq B$  for every n, we can apply subadditivity to obtain the inequality

$$E \subseteq B \subseteq B_n \implies N \le m_*(B) \le m_*(B_n) < N + \frac{1}{n} \text{ for all } n \in \mathbb{Z}^{\ge 1}.$$

• This forces  $m_*(E) = m_*(B)$ .

#### 3.1.4 b

Suppose  $m_*(E) < \infty$ .

- By (a), find a Borel set  $B \supseteq E$  such that  $m_*(B) = m_*(E)$
- Note that  $E \subseteq B \implies B \cap E = E$  and  $B \cap E^c = B \setminus E$ .
- By assumption,

$$m_*(B) = m_*(B \cap E) + m_*(B \cap E^c)$$

$$m_*(E) = m_*(E) + m_*(B \setminus E)$$

$$m_*(E) - m_*(E) = m_*(B \setminus E) \quad \text{since } m_*(E) < \infty$$

$$\implies m_*(B \setminus E) = 0.$$

• So take  $N = B \setminus E$ ; this shows  $m_*(N) = 0$  and  $E = B \setminus (B \setminus E) = B \setminus N$ . If  $m_*(E) = \infty$ :

- Apply result to  $E_R := E \bigcap [R, R+1)^n \subset \mathbb{R}^n$  for  $R \in \mathbb{Z}$ , so  $E = \coprod_R E_R$
- Obtain  $B_R$ ,  $N_R$  such that  $E_R = B_R \setminus N_R$ ,  $m_*(E_R) = m_*(B_R)$ , and  $m_*(N_R) = 0$ .
- - $-B := \bigcup_{R} B_R$  is a union of Borel sets and thus still Borel
  - $-E = \bigcup E_R$

  - $-N := B \setminus E$   $-N' := \bigcup_{i=1}^{R} N_{R} \text{ is a union of null sets and thus still null}$
- Since  $E_R \subset B_R$  for every R, we have  $E \subset B$
- We can compute

$$N = B \setminus E = \left(\bigcup_R B_R\right) \setminus \left(\bigcup_R E_R\right) \subseteq \bigcup_R \left(B_R \setminus E_R\right) = \bigcup_R N_R := N'$$

where  $m_*(N') = 0$  since N' is null, and thus subadditivity forces  $m_*(N) = 0$ .

## 3.2 Fall 2019 # 3. 💝

Let  $(X, \mathcal{B}, \mu)$  be a measure space with  $\mu(X) = 1$  and  $\{B_n\}_{n=1}^{\infty}$  be a sequence of  $\mathcal{B}$ -measurable subsets of X, and

$$B := \left\{ x \in X \mid x \in B_n \text{ for infinitely many } n \right\}.$$

- a. Argue that B is also a  $\mathcal{B}$ -measurable subset of X.
- b. Prove that if  $\sum_{n=0}^{\infty} \mu(B_n) < \infty$  then  $\mu(B) = 0$ .

c. Prove that if  $\sum_{n=1}^{\infty} \mu(B_n) = \infty$  and the sequence of set complements  $\{B_n^c\}_{n=1}^{\infty}$  satisfies

$$\mu\left(\bigcap_{n=k}^{K} B_{n}^{c}\right) = \prod_{n=k}^{K} \left(1 - \mu\left(B_{n}\right)\right)$$

for all positive integers k and K with k < K, then  $\mu(B) = 1$ .

Hint: Use the fact that  $1 - x \le e^{-x}$  for all x.

#### Solution:

### Concepts Used:

• Borel-Cantelli: for a sequence of sets  $X_n$ ,

$$\limsup_n X_n = \Big\{ x \; \Big| \; x \in X_n \text{ for infinitely many } n \Big\} \qquad = \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} X_n$$
$$\liminf_n X_n = \Big\{ x \; \Big| \; x \in X_n \text{ for all but finitely many } n \Big\} \qquad = \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} X_n.$$

• Properties of logs and exponentials:

$$\prod_{n} e^{x_n} = e^{\sum_{n} x_n} \quad \text{and} \quad \sum_{n} \log(x_n) = \log\left(\prod_{n} x_n\right).$$

- Tails of convergent sums vanish.
- Continuity of measure:  $B_n \searrow B$  and  $\mu(B_0) < \infty$  implies  $\lim_n \mu(B_n) = \mu(B)$ , and  $B_n \nearrow B \implies \lim_n \mu(B_n) = \mu(B)$ .

### 3.2.1 a

- The Borel  $\sigma$ -algebra is closed under countable unions/intersections/complements,
- $B = \limsup_{n} B_n$  is an intersection of unions of measurable sets.

#### 3.2.2 b

- Tails of convergent sums go to zero, so  $\sum_{n\geq M} \mu(B_n) \xrightarrow{M\longrightarrow\infty} 0$ ,
- $B_M := \bigcap_{m=1}^M \bigcup_{n \ge m} B_n \searrow B$ .

$$\mu(B_M) = \mu\left(\bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} B_n\right)$$

$$\leq \mu\left(\bigcup_{n \geq m} B_n\right) \quad \text{for all } m \in \mathbb{N} \text{ by countable subadditivity}$$

$$\longrightarrow 0.$$

• The result follows by continuity of measure.

#### 3.2.3 c

• To show  $\mu(B) = 1$ , we'll show  $\mu(B^c) = 0$ .

• Let 
$$B_k = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{K} B_n$$
. Then

$$\mu(B_K^c) = \mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^K B_n^c\right)$$

$$\leq \sum_{m=1}^{\infty} \mu\left(\bigcap_{n=m}^K B_n^c\right) \quad \text{by subadditivity}$$

$$= \sum_{m=1}^{\infty} \prod_{n=m}^K \left(1 - \mu(B_n)\right) \quad \text{by assumption}$$

$$\leq \sum_{m=1}^{\infty} \prod_{n=m}^K e^{-\mu(B_n^c)} \quad \text{by hint}$$

$$= \sum_{m=1}^{\infty} \exp\left(-\sum_{n=m}^K \mu(B_n^c)\right)$$

$$K \xrightarrow{\longrightarrow} 0$$

since 
$$\sum_{n=m}^{K} \mu(B_n^c) \stackrel{K \longrightarrow \infty}{\longrightarrow} \infty$$
 by assumption

• We can apply continuity of measure since  $B_K^c \xrightarrow{K \longrightarrow \infty} B^c$ .

How to prove the hint..?

## 3.3 Spring 2019 # 2 💝

Let  $\mathcal{B}$  denote the set of all Borel subsets of  $\mathbb{R}$  and  $\mu: \mathcal{B} \longrightarrow [0, \infty)$  denote a finite Borel measure on  $\mathbb{R}$ .

#### 3.3.1 a

Prove that if  $\{F_k\}$  is a sequence of Borel sets for which  $F_k \supseteq F_{k+1}$  for all k, then

$$\lim_{k \to \infty} \mu(F_k) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right)$$

#### 3.3.2 b

Suppose  $\mu$  has the property that  $\mu(E) = 0$  for every  $E \in \mathcal{B}$  with Lebesgue measure m(E) = 0.

Prove that for every  $\varepsilon > 0$  there exists  $\delta > 0$  so that if  $E \in \mathcal{B}$  with  $m(E) < \delta$ , then  $\mu(E) < \varepsilon$ .

#### Add concepts

#### **Solution:**

Concepts Used:

• ?

#### 3.3.3 a

See Folland p.26

• Lemma 1: 
$$\mu(\coprod_{k=1}^{\infty} E_k) = \lim_{N \to \infty} \sum_{k=1}^{N} \mu(E_k)$$
.

- Suppose  $F_0 \supseteq F_1 \supseteq \cdots$ .
- Let  $A_k = F_k \setminus F_{k+1}$ , since the  $F_k$  are nested the  $A_k$  are disjoint

• Set 
$$A := \coprod_{k=1}^{\infty} A_k$$
 and  $F := \bigcap_{k=1}^{\infty} F_k$ .

- Note  $X = X \setminus Y \coprod X \bigcap Y$  for any two sets (just write  $X \setminus Y \coloneqq X \bigcap Y^c$ )
- Note that A contains anything that was removed from  $F_0$  when passing from any  $F_j$  to  $F_{j+1}$ , while F contains everything that is never removed at any stage, and these are disjoint possibilities.
- Thus  $F_0 = F \prod A$ , so

$$\begin{split} \mu(F_0) &= \mu(F) + \mu(A) \\ &= \mu(F) + \mu(\coprod_{k=1}^{\infty} A_k) \\ &= \mu(F) + \lim_{n \to \infty} \sum_{k=0}^{n} \mu(A_k) \quad \text{by countable additivity} \\ &= \mu(F) + \lim_{n \to \infty} \sum_{k=0}^{n} \mu(F_k) - \mu(F_{k+1}) \\ &= \mu(F) + \lim_{n \to \infty} (\mu(F_1) - \mu(F_n)) \quad \text{(Telescoping)} \\ &= \mu(F) + \mu(F_1) - \lim_{N \to \infty} \mu(F_n), \end{split}$$

• Since  $\mu$  is a finite measure,  $\mu(F_1) < \infty$  and can be subtracted, yielding

$$\mu(F_1) = \mu(F) + \mu(F_1) - \lim_{n \to \infty} \mu(F_n)$$

$$\implies \mu(F) = \lim_{n \to \infty} \mu(F_n)$$

$$\implies \mu\left(\bigcap_{k=1}^{\infty} F_k\right) = \lim_{n \to \infty} \mu(F_n).$$

#### 3.3.4 b

- Toward a contradiction, negate the implication: suppose there exists an  $\varepsilon > 0$  such that for all  $\delta$ , we have  $m(E) < \delta$  but  $\mu(E) > \varepsilon$ .
- The sequence  $\left\{\delta_n := \frac{1}{2^n}\right\}_{n \in \mathbb{N}}$  and produce sets  $A_n \in \mathcal{B}$  such  $m(A_n) < \frac{1}{2^n}$  but  $\mu(A_n) > \varepsilon$ .

• Define

$$F_n := \bigcup_{j \ge n} A_j$$

$$C_m := \bigcap_{k=1}^m F_k$$

$$A := C_\infty := \bigcap_{k=1}^\infty F_k.$$

- Note that  $F_1 \supseteq F_2 \supseteq \cdots$ , since each increase in index unions fewer sets.
- By continuity for the Lebesgue measure,

$$m(A) = m\left(\bigcap_{k=1}^{\infty} F_k\right) = \lim_{k \to \infty} m(F_k) = \lim_{k \to \infty} m\left(\bigcup_{j \ge k} A_j\right) \le \lim_{k \to \infty} \sum_{j \ge k} m(A_j) = \lim_{k \to \infty} \sum_{j \ge k} \frac{1}{2^n} = 0,$$

which follows because this is the tail of a convergent sum

- Thus m(A) = 0 and by assumption, this implies  $\mu(A) = 0$ .
- However, by part (a),

$$\mu(A) = \lim_{n} \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \ge \lim_{n} \mu(A_n) = \lim_{n} \varepsilon = \varepsilon > 0.$$

All messed up!

### 3.4 Fall 2018 # 2 🦙

Let  $E \subset \mathbb{R}$  be a Lebesgue measurable set. Show that there is a Borel set  $B \subset E$  such that  $m(E \setminus B) = 0$ .

Move this to review notes to clean things up

Solution:

### Concepts Used:

- Definition of measurability: there exists an open  $O \supset E$  such that  $m_*(O \setminus E) < \varepsilon$
- Theorem: E is Lebesgue measurable iff there exists a closed set  $F \subseteq E$  such that  $m_*(E \setminus F) < \varepsilon \text{ for all } \varepsilon > 0.$
- Every  $F_{\sigma}, G_{\delta}$  is Borel.
- Claim: E is measurable  $\iff$  for every  $\varepsilon$  there exist  $F_{\varepsilon} \subset E \subset G_{\varepsilon}$  with  $F_{\varepsilon}$  closed and  $G_{\varepsilon}$  open and  $m(G_{\varepsilon} \setminus E) < \varepsilon$  and  $m(E \setminus F_{\varepsilon}) < \varepsilon$ .
  - Proof: existence of  $G_{\varepsilon}$  is the definition of measurability.
  - Existence of  $F_{\varepsilon}$ :?
- Claim: E is measurable  $\implies$  there exists an open  $O \supseteq E$  such that  $m(O \setminus E) = 0$ .
  - Since E is measurable, for each  $n \in \mathbb{N}$  choose  $G_n \supseteq E$  such that  $m_*(G_n \setminus E) <$
  - Set  $O_N := \bigcap_{n=1}^N G_n$  and  $O := \bigcap_{n=1}^\infty G_n$ . Suppose E is bounded.
  - - \* Note  $O_N \setminus O$  and  $m_*(O_1) < \infty$  if E is bounded, since in this case

$$m_*(G_n \setminus E) = m_*(G_1) - m_*(E) < 1 \iff m_*(G_1) < m_*(E) + \frac{1}{n} < \infty.$$

- \* Note  $O_N \setminus E \searrow O \setminus E$  since  $O_N \setminus E := O_N \cap E^c \supseteq O_{N+1} \cap E^c$  for all N, and again  $m_*(O_1 \setminus E) < \infty$ .
- \* So it's valid to apply continuity of measure from above:

$$m_*(O \setminus E) = \lim_{N \to \infty} m_*(O_N \setminus E)$$

$$\leq \lim_{N \to \infty} m_*(G_N \setminus E)$$

$$= \lim_{N \to \infty} \frac{1}{N} = 0,$$

where the inequality uses subadditivity on  $\bigcap^{N} G_n \subseteq G_N$ 

- Suppose E is unbounded.
  - \* Write  $E^k = E \cap [k, k+1]^d \subset \mathbb{R}^d$  as the intersection of E with an annulus, and note that  $E = \coprod_{k \in \mathbb{N}} E_k$ .
  - \* Each  $E_k$  is bounded, so apply the previous case to obtain  $O_k \supseteq E_k$  with  $m(O_k \setminus E_k) = 0.$
  - \* So write  $O_k = E_k \prod N_k$  where  $N_k := O_k \setminus E_k$  is a null set.
  - \* Define  $O = \bigcup O_k$ , note that  $E \subseteq O$ .
  - \* Now note

$$O \setminus E = \left(\coprod_{k} O_{k}\right) \setminus \left(\coprod_{K} E_{k}\right)$$

$$\subseteq \coprod_{k} (O_{k} \setminus E_{k})$$

$$\implies m_{*}(O \setminus E) \le m_{*}\left(\coprod (O_{k} \setminus E_{k})\right) = 0,$$

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since any countable union of null sets is again null. - So  $O \supseteq E$  with  $m(O \setminus E) = 0$ .

- Theorem: since E is measurable,  $E^c$  is measurable
  - Proof: It suffices to write  $E^c$  as the union of two measurable sets,  $E^c$  $S \mid (E^c - S)$ , where S is to be determined.
  - We'll produce an S such that  $m(E^c S) = 0$  and use the fact that any subset

#### 3.4.1 Indirect Proof

- Since E is measurable,  $E^c$  is measurable.
- Since  $E^c$  is measurable exists an open  $O \supseteq E^c$  such that  $m(O \setminus E^c) = 0$ .
- Set  $B := O^c$ , then  $O \supseteq E^c \iff \mathcal{O}^c \subseteq E \iff B \subseteq E$ .
- Computing measures yields

$$E \setminus B := E \setminus \mathcal{O}^c := E \cap (\mathcal{O}^c)^c = E \cap \mathcal{O} = \mathcal{O} \cap (E^c)^c := \mathcal{O} \setminus E^c$$

thus  $m(E \setminus B) = m(\mathcal{O} \setminus E^c) = 0$ .

• Since  $\mathcal{O}$  is open, B is closed and thus Borel.

#### 3.4.2 Direct Proof (Todo)

Try to construct the set

### 3.5 Spring 2018 # 1 😽

Define

$$E := \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N} \right\}.$$

Prove that m(E) = 0.

#### **Solution:**

#### Concepts Used:

- Borel-Cantelli: If  $\{E_k\}_{k\in\mathbb{Z}}\subset 2^{\mathbb{R}}$  is a countable collection of Lebesgue measurable sets with  $\sum_{k\in\mathbb{Z}} m(E_k) < \infty$ , then almost every  $x\in\mathbb{R}$  is in at most finitely many  $E_k$ .
  - Equivalently (?),  $m(\limsup_{k\to\infty} E_k) = 0$ , where  $\limsup_{k\to\infty} E_k = \bigcap_{k=1}^{\infty} \bigcup_{j\geq k} E_j$ , the elements which are in  $E_k$  for infinitely many k.
- Strategy: Borel-Cantelli.
- We'll show that  $m(E) \cap [n, n+1] = 0$  for all  $n \in \mathbb{Z}$ ; then the result follows from

$$m(E) = m\left(\bigcup_{n \in \mathbb{Z}} E \bigcap [n, n+1]\right) \le \sum_{n=1}^{\infty} m(E \bigcap [n, n+1]) = 0.$$

- By translation invariance of measure, it suffices to show  $m(E \cap [0,1]) = 0$ .
  - So WLOG, replace E with  $E \cap [0,1]$ .
- Define

$$E_j := \left\{ x \in [0, 1] \mid \exists p \in \mathbb{Z}^{\geq 0} \text{ s.t. } \left| x - \frac{p}{j} \right| < \frac{1}{j^3} \right\}.$$

- Note that  $E_j \subseteq \coprod_{p \in \mathbb{Z}^{\geq 0}} B_{j^{-3}}\left(\frac{p}{j}\right)$ , i.e. a union over integers p of intervals of radius

 $1/j^3$  around the points p/j. Since  $1/j^3 < 1/j$ , this union is in fact disjoint.

• Importantly, note that

$$\lim_{j \to \infty} \sup E_j := \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j = E$$

since

 $x \in \limsup_{j} E_{j} \iff x \in E_{j}$  for infinitely many j  $\iff$  there are infinitely many j for which there exist a p such that  $\left|x - \frac{p}{j}\right| < j^{-3}$   $\iff$  there are infinitely many such pairs p, j  $\iff x \in E$ .

• Intersecting with [0,1], we can write  $E_i$  as a union of intervals:

$$E_{j} = (0, j^{-3}) \coprod B_{j^{-3}}(\frac{1}{i}) \coprod B_{j^{-3}}(\frac{2}{i}) \coprod \cdots \coprod B_{j^{-3}}(\frac{j-1}{i}) \coprod (1-j^{-3}, 1),$$

where we've separated out the "boundary" terms to emphasize that they are balls about 0 and 1 intersected with [0,1].

- Since  $E_i$  is a union of open sets, it is Borel and thus Lebesgue measurable.
- Computing the measure of  $E_i$ :
  - For a fixed j, there are exactly j+1 possible choices for a numerator  $(0,1,\cdots,j)$ , thus there are exactly j+1 sets appearing in the above decomposition.
  - The first and last intervals are length  $\frac{1}{i^3}$
  - The remaining (j+1)-2=j-1 intervals are twice this length,  $\frac{2}{i^3}$
  - Thus

$$m(E_j) = 2\left(\frac{1}{j^3}\right) + (j-1)\left(\frac{2}{j^3}\right) = \frac{2}{j^2}$$

• Note that

$$\sum_{j \in \mathbb{N}} m(E_j) = 2 \sum_{j \in \mathbb{N}} \frac{1}{j^2} < \infty,$$

which converges by the p-test for sums.

• But then

$$m(E) = m(\limsup_{j \to \infty} E_j)$$

$$= m(\bigcap_{n \in \mathbb{N}} \bigcup_{j \ge n} E_j)$$

$$\leq m(\bigcup_{j \ge N} E_j) \text{ for every } N$$

$$\leq \sum_{j \ge N} m(E_j)$$

$$\stackrel{N \to \infty}{\longrightarrow} 0$$

• Thus E is measurable as a subset of a null set and m(E) = 0.

### 3.6 Fall 2017 # 2 처

Let  $f(x) = x^2$  and  $E \subset [0, \infty) := \mathbb{R}^+$ .

1. Show that

$$m^*(E) = 0 \iff m^*(f(E)) = 0.$$

2. Deduce that the map

$$\varphi: \mathcal{L}(\mathbb{R}^+) \longrightarrow \mathcal{L}(\mathbb{R}^+)$$

$$E \mapsto f(E)$$

is a bijection from the class of Lebesgue measurable sets of  $[0, \infty)$  to itself.

Walk through

**Solution:** 

Concepts Used:

• ?

#### 3.6.1 a

It suffices to consider the bounded case, i.e.  $E \subseteq B_M(0)$  for some M. Then write  $E_n =$  $B_n(0) \cap E$  and apply the theorem to  $E_n$ , and by subadditivity,  $m^*(E) = m^*(\bigcup E_n) \le$ 

$$\sum m^*(E_n) = 0.$$

**Lemma:**  $f(x) = x^2, f^{-1}(x) = \sqrt{x}$  are Lipschitz on any compact subset of  $[0, \infty)$ . Proof: Let g = f or  $f^{-1}$ . Then  $g \in C^1([0, M])$  for any M, so g is differentiable and g' is continuous. Since g' is continuous on a compact interval, it is bounded, so  $|g'(x)| \leq L$  for all x. Applying the MVT,

$$|f(x) - f(y)| = f'(c)|x - y| \le L|x - y|.$$

**Lemma:** If g is Lipschitz on  $\mathbb{R}^n$ , then  $m(E) = 0 \implies m(g(E)) = 0$ . *Proof:* If g is Lipschitz, then

$$g(B_r(x)) \subseteq B_{Lr}(x),$$

which is a dilated ball/cube, and so

$$m^*(B_{Lr}(x)) \le L^n \cdot m^*(B_r(x)).$$

Now choose  $\{Q_i\} \rightrightarrows E$ ; then  $\{g(Q_i)\} \rightrightarrows g(E)$ .

By the above observation,

$$|g(Q_j)| \le L^n |Q_j|,$$

and so

$$m^*(g(E)) \le \sum_j |g(Q_j)| \le \sum_j L^n |Q_j| = L^n \sum_j |Q_j| \longrightarrow 0.$$

Now just take  $g(x) = x^2$  for one direction, and  $g(x) = f^{-1}(x) = \sqrt{x}$  for the other.

#### 3.6.2 b

Lemma: E is measurable iff  $E = K \coprod N$  for some K compact, N null.

Write  $E = K \coprod N$  where K is compact and N is null.

Then 
$$\varphi^{-1}(E) = \varphi^{-1}(K \coprod N) = \varphi^{-1}(K) \coprod \varphi^{-1}(N)$$
.

Since  $\varphi^{-1}(N)$  is null by part (a) and  $\varphi^{-1}(K)$  is the preimage of a compact set under a continuous map and thus compact,  $\varphi^{-1}(E) = K' \coprod N'$  where K' is compact and N' is null, so  $\varphi^{-1}(E)$  is measurable.

So  $\varphi$  is a measurable function, and thus yields a well-defined map  $\mathcal{L}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathbb{R})$  since it preserves measurable sets. Restricting to  $[0, \infty)$ , f is bijection, and thus so is  $\varphi$ .

### 3.7 Spring 2017 # 2 😽

#### 3.7.1 a

Let  $\mu$  be a measure on a measurable space  $(X, \mathcal{M})$  and f a positive measurable function.

Define a measure  $\lambda$  by

$$\lambda(E) := \int_{E} f \ d\mu, \quad E \in \mathcal{M}$$

Show that for g any positive measurable function,

$$\int_X g \ d\lambda = \int_X fg \ d\mu$$

#### 3.7.2 b

Let  $E \subset \mathbb{R}$  be a measurable set such that

$$\int_{E} x^2 \ dm = 0.$$

Show that m(E) = 0.

Solution:

### Concepts Used:

- Absolute continuity of measures:  $\lambda \ll \mu \iff E \in \mathcal{M}, \mu(E) = 0 \implies \lambda(E) = 0.$
- Radon-Nikodym: if  $\lambda \ll \mu$ , then there exists a measurable function  $\frac{\partial \lambda}{\partial \mu} := f$  where

$$\lambda(E) = \int_E f \, d\mu.$$
 • Chebyshev's inequality:

$$A_c := \left\{ x \in X \mid |f(x)| \ge c \right\} \implies \mu(A_c) \le c^{-p} \int_{A_c} |f|^p d\mu \quad \forall 0$$

#### 3.7.3 a

- Strategy: use approximation by simple functions to show absolute continuity and apply Radon-Nikodym
- Claim:  $\lambda \ll \mu$ , i.e.  $\mu(E) = 0 \implies \lambda(E) = 0$ .
  - Note that if this holds, by Radon-Nikodym,  $f = \frac{\partial \lambda}{\partial \mu} \implies d\lambda = f d\mu$ , which would

$$\int g \ d\lambda = \int g f \ d\mu.$$

- So let E be measurable and suppose  $\mu(E) = 0$ .
- Then

$$\lambda(E) \coloneqq \int_{E} f \ d\mu = \lim_{n \to \infty} \left\{ \int_{E} s_n \, d\mu \mid s_n \coloneqq \sum_{j=1}^{\infty} c_j \mu(E_j), \, s_n \nearrow f \right\}$$

where we take a sequence of simple functions increasing to f.

• But since each  $E_i \subseteq E$ , we must have  $\mu(E_i) = 0$  for any such  $E_i$ , so every such  $s_n$  must be zero and thus  $\lambda(E) = 0$ .

What is the final step in this approximation

#### 3.7.4 b

- Set  $g(x) = x^2$ , note that g is positive and measurable.
- By part (a), there exists a positive f such that for any  $E \subseteq \mathbb{R}$ ,

$$\int_{F} g \ dm = \int_{F} g f \ d\mu$$

- The LHS is zero by assumption and thus so is the RHS.
- $-m \ll \mu$  by construction.
- Note that gf is positive.
- Define  $A_k = \left\{ x \in X \mid gf \cdot \chi_E > \frac{1}{k} \right\}$ , for  $k \in \mathbb{Z}^{\geq 0}$
- Then by Chebyshev with p = 1, for every k we have

$$\mu(A_k) \le k \int_E gf \ d\mu = 0$$

- Then noting that  $A_k \searrow A := \{x \in X \mid gf \cdot \chi_E(x) > 0\}$ , we have  $\mu(A) = 0$ .
- Since gf is positive, we have

$$x \in E \iff gf\chi_E(x) > 0 \iff x \in A$$

so E = A and  $\mu(E) = \mu(A)$ .

• But  $m \ll \mu$  and  $\mu(E) = 0$ , so we can conclude that m(E) = 0.

### 3.8 Fall 2016 # 4 🐪

Let  $(X, \mathcal{M}, \mu)$  be a measure space and suppose  $\{E_n\} \subset \mathcal{M}$  satisfies

$$\lim_{n\to\infty}\mu\left(X\backslash E_n\right)=0.$$

Define

$$G := \{ x \in X \mid x \in E_n \text{ for only finitely many } n \}.$$

Show that  $G \in \mathcal{M}$  and  $\mu(G) = 0$ .

#### Add concepts

#### Solution:

Concepts Used:

- . 7
- Claim:  $G \in \mathcal{M}$ .
  - Claim:

$$G = \left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n\right)^c = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n^c.$$

- \* This follows because x is in the RHS  $\iff x \in E_n^c$  for all but finitely many  $n \iff x \in E_n$  for at most finitely many n.
- But  $\mathcal{M}$  is a  $\sigma$ -algebra, and this shows G is obtained by countable unions/intersections/complements of measurable sets, so  $G \in \mathcal{M}$ .
- Claim:  $\mu(G) = 0$ .

- We have

$$\mu(G) = \mu\left(\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n^c\right)$$

$$\leq \sum_{N=1}^{\infty} \mu\left(\bigcap_{n=N}^{\infty} E_n^c\right)$$

$$\leq \sum_{N=1}^{\infty} \mu(E_M^c)$$

$$:= \sum_{N=1}^{\infty} \mu(X \setminus E_N)$$

$$\stackrel{N \longrightarrow \infty}{\longrightarrow} 0.$$

Last step seems wrong

## 3.9 Spring 2016 # 3

Let f be Lebesgue measurable on  $\mathbb R$  and  $E\subset\mathbb R$  be measurable such that

$$0 < A = \int_{E} f(x)dx < \infty.$$

Show that for every 0 < t < 1, there exists a measurable set  $E_t \subset E$  such that

$$\int_{E_{+}} f(x)dx = tA.$$

## 3.10 Spring 2016 # 5

Let  $(X, \mathcal{M}, \mu)$  be a measure space. For  $f \in L^1(\mu)$  and  $\lambda > 0$ , define

$$\varphi(\lambda) = \mu(\{x \in X | f(x) > \lambda\}) \quad \text{ and } \quad \psi(\lambda) = \mu(\{x \in X | f(x) < -\lambda\})$$

Show that  $\varphi, \psi$  are Borel measurable and

$$\int_{X} |f| \ d\mu = \int_{0}^{\infty} [\varphi(\lambda) + \psi(\lambda)] \ d\lambda$$

## 3.11 Fall 2015 # 2

Let  $f: \mathbb{R} \longrightarrow \mathbb{R}$  be Lebesgue measurable.

- 1. Show that there is a sequence of simple functions  $s_n(x)$  such that  $s_n(x) \longrightarrow f(x)$  for all  $x \in \mathbb{R}$ .
- 2. Show that there is a Borel measurable function g such that g = f almost everywhere.

## 3.12 Spring 2015 # 3

Let  $\mu$  be a finite Borel measure on  $\mathbb R$  and  $E \subset \mathbb R$  Borel. Prove that the following statements are equivalent:

1.  $\forall \varepsilon > 0$  there exists G open and F closed such that

$$F \subseteq E \subseteq G$$
 and  $\mu(G \setminus F) < \varepsilon$ .

2. There exists a  $V \in G_{\delta}$  and  $H \in F_{\sigma}$  such that

$$H \subseteq E \subseteq V$$
 and  $\mu(V \setminus H) = 0$ 

## 3.13 Spring 2014 # 3

Let  $f: \mathbb{R} \longrightarrow \mathbb{R}$  and suppose

$$\forall x \in \mathbb{R}, \quad f(x) \ge \limsup_{y \to x} f(y)$$

Prove that f is Borel measurable.

## 3.14 Spring 2014 # 4

Let  $(X, \mathcal{M}, \mu)$  be a measure space and suppose f is a measurable function on X. Show that

$$\lim_{n \to \infty} \int_X f^n \ d\mu = \begin{cases} \infty & \text{or} \\ \mu(f^{-1}(1)), \end{cases}$$

and characterize the collection of functions of each type.

## 3.15 Spring 2017 # 1 🔆

Let K be the set of numbers in [0,1] whose decimal expansions do not use the digit 4.

We use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 with  $399\cdots$ . For example,  $0.8754 = 0.8753999\cdots$ .

Show that K is a compact, nowhere dense set without isolated points, and find the Lebesgue measure m(K).

#### Solution:

#### Concepts Used:

• Definition: A is nowhere dense  $\iff$  every interval I contains a subinterval  $S \subseteq A^c$ .

- Equivalently, the interior of the closure is empty,  $(\overline{K})^{\circ} = \emptyset$ .

Claim: K is compact.

• It suffices to show that  $K^c := [0,1] \setminus K$  is open; Then K will be a closed and bounded subset of  $\mathbb{R}$  and thus compact by Heine-Borel.

- Strategy: write  $K^c$  as the union of open balls (since these form a basis for the Euclidean topology on  $\mathbb{R}$ ).
  - Do this by showing every point  $x \in K^c$  is an interior point, i.e. x admits a neighborhood  $N_x$  such that  $N_x \subseteq K^c$ .
- Identify  $K^c$  as the set of real numbers in [0,1] whose decimal expansion **does** contain a
  - We will show that there exists a neighborhood small enough such that all points in it contain a 4 in their decimal expansions.
- Let  $x \in K^c$ , suppose a 4 occurs as the kth digit, and write

$$x = 0.d_1 d_2 \cdots d_{k-1} \ 4 \ d_{k+1} \cdots = \left( \sum_{j=1}^k d_j 10^{-j} \right) + \left( 4 \cdot 10^{-k} \right) + \left( \sum_{j=k+1}^\infty d_j 10^{-j} \right).$$

• Set  $r_x < 10^{-k}$  and let  $y \in [0,1] \cap B_{r_x}(x)$  be arbitrary and write

$$y = \sum_{j=1}^{\infty} c_j 10^{-j}.$$

- Thus  $|x-y| < r_x < 10^{-k}$ , and the first k digits of x and y must agree:
  - We first compute the difference:

$$x - y = \sum_{i=1}^{\infty} d_i 10^{-i} - \sum_{i=1}^{\infty} c_i 10^{-i} = \sum_{i=1}^{\infty} (d_i - c_i) 10^{-i}$$

- Thus (claim)

$$|x - y| \le \sum_{j=1}^{\infty} |d_j - c_j| 10^j < 10^{-k} \iff |d_j - c_j| = 0 \quad \forall j \le k.$$

- Otherwise we can note that any term  $|d_j - c_j| \ge 1$  and there is a contribution to |x - y| of at least  $1 \cdot 10^{-j}$  for some j < k, whereas

$$j < k \iff 10^{-j} > 10^{-k},$$

a contradiction.

- This means that for all  $j \leq k$  we have  $d_j = c_j$ , and in particular  $d_k = 4 = c_k$ , so y has a 4 in its decimal expansion.
- But then  $K^c = \bigcup_{x} B_{r_x}(x)$  is a union of open sets and thus open.

Claim: K is nowhere dense and m(K) = 0:

- Strategy: Show  $(\overline{K})^{\circ} = \emptyset$
- Since K is closed,  $\overline{K} = K$ , so it suffices to show that K does not properly contain any interval.
- It suffices to show  $m(K^c) = 1$ , since this implies m(K) = 0 and since any interval has strictly positive measure, this will mean K can not contain an interval.
- As in the construction of the Cantor set, let

-  $K_0$  denote [0, 1] with 1 interval  $\left(\frac{4}{10}, \frac{5}{10}\right)$  of length  $\frac{1}{10}$  deleted, so

$$m(K_0^c) = \frac{1}{10}.$$

 $-K_1$  denote  $K_0$  with 9 intervals  $\left(\frac{1}{100}, \frac{5}{100}\right), \left(\frac{14}{100}, \frac{15}{100}\right), \cdots, \left(\frac{94}{100}, \frac{95}{100}\right)$  of length  $\frac{1}{100}$  deleted, so

$$m(K_1^c) = \frac{1}{10} + \frac{9}{100}.$$

-  $K_n$  denote  $K_{n-1}$  with  $9^n$  such intervals of length  $\frac{1}{10^{n+1}}$  deleted, so

$$m(K_n^c) = \frac{1}{10} + \frac{9}{100} + \dots + \frac{9^n}{10^{n+1}}.$$

Then compute

$$m(K^c) = \sum_{i=0}^{\infty} \frac{9^n}{10^{n+1}} = \frac{1}{10} \sum_{i=0}^{\infty} \left(\frac{9}{10}\right)^n = \frac{1}{10} \left(\frac{1}{1 - \frac{9}{10}}\right) = 1.$$

Claim: K has no isolated points:

- A point  $x \in K$  is isolated iff there there is an open ball  $B_r(x)$  containing x such that  $B_r(x) \subseteq K^c$ .
  - So every point in this ball **should** have a 4 in its decimal expansion.
- Strategy: show that if x ∈ K, every neighborhood of x intersects K.
  Note that m(K<sub>n</sub>) = (<sup>9</sup>/<sub>10</sub>)<sup>n</sup> <sup>n→∞</sup>/<sub>n→∞</sub> 0
- Also note that we deleted open intervals, and the endpoints of these intervals are never
  - Thus endpoints of deleted intervals are elements of K.
- Fix x. Then for every  $\varepsilon$ , by the Archimedean property of  $\mathbb{R}$ , choose n such that
- Then there is an endpoint  $x_n$  of some deleted interval  $I_n$  satisfying

$$|x - x_n| \le \left(\frac{9}{10}\right)^n < \varepsilon.$$

• So every ball containing x contains some endpoint of a removed interval, and thus an element of K.

### 3.16 Spring 2016 # 2

Let  $0 < \lambda < 1$  and construct a Cantor set  $C_{\lambda}$  by successively removing middle intervals of length  $\lambda$ . Prove that  $m(C_{\lambda}) = 0$ .

### 4 Measure Theory: Functions

### 4.1 Fall 2016 # 2 처

Let  $f, g: [a, b] \longrightarrow \mathbb{R}$  be measurable with

$$\int_{a}^{b} f(x) \ dx = \int_{a}^{b} g(x) \ dx.$$

Show that either

- 1. f(x) = g(x) almost everywhere, or
- 2. There exists a measurable set  $E \subset [a, b]$  such that

$$\int_{E} f(x) dx > \int_{E} g(x) dx$$

Add concepts

Solution:

Concepts Used:

• ?

- Suppose it is *not* the case that f = g almost everywhere; then letting  $A := \{x \in [a,b] \mid f(x) \neq g(x)\}$ , we have m(A) > 0.
- Write

$$A = A_1 \coprod A_2 := \{f > g\} \coprod \{f < g\},\,$$

then  $m(A_1) > 0$  or  $m(A_2) > 0$  (or both).

- Wlog (by relabeling f, g if necessary), suppose  $m(A_1) > 0$ , and take  $E := A_1$ .
- Then on E, we have f(x) > g(x) pointwise. This is preserved by monotonicity of the integral, thus

$$f(x) > g(x)$$
 on  $E \implies \int_E f(x) dx > \int_E g(x) dx$ .

# 4.2 Spring 2016 # 4

Let  $E \subset \mathbb{R}$  be measurable with  $m(E) < \infty$ . Define

$$f(x) = m(E \cap (E + x)).$$

Show that

- 1.  $f \in L^1(\mathbb{R})$ .
- 2. f is uniformly continuous.
- $3. \lim_{|x| \to \infty} f(x) = 0.$

Hint:

$$\chi_{E\cap(E+x)}(y) = \chi_E(y)\chi_E(y-x)$$

### 5 Integrals: Convergence

### 5.1 Fall 2019 # 2 💝

Prove that

$$\left| \frac{d^n}{dx^n} \frac{\sin x}{x} \right| \le \frac{1}{n}$$

for all  $x \neq 0$  and positive integers n.

Hint: Consider  $\int_0^1 \cos(tx) dt$ 

#### Solution:

Concepts Used:

• DCT

• Bounding in the right place. Don't evaluate the actual integral!

• By induction on the number of limits we can pass through the integral.

• For n = 1 we first pass one derivative into the integral: let  $x_n \longrightarrow x$  be any sequence converging to x, then

$$\begin{split} \frac{\partial}{\partial x} \frac{\sin(x)}{x} &= \frac{\partial}{\partial x} \int_0^1 \cos(tx) \, dt \\ &= \lim_{x_n \longrightarrow x} \frac{1}{x_n - x} \left( \int_0^1 \cos(tx_n) \, dt - \int_0^1 \cos(tx) \, dt \right) \\ &= \lim_{x_n \longrightarrow x} \left( \int_0^1 \frac{\cos(tx_n) - \cos(tx)}{x_n - x} \, dt \right) \\ &= \lim_{x_n \longrightarrow x} \left( \int_0^1 \left( t \sin(tx) \Big|_{x = \xi_n} \right) dt \right) \quad \text{where} \quad \xi_n \in [x_n, x] \text{ by MVT}, \xi_n \longrightarrow x \\ &= \lim_{\xi_n \longrightarrow x} \left( \int_0^1 t \sin(t\xi_n) \, dt \right) \\ &=_{\text{DCT}} \int_0^1 \lim_{\xi_n \longrightarrow x} t \sin(t\xi_n) \, dt \\ &= \int_0^1 t \sin(tx) \, dt \end{split}$$

• Taking absolute values we obtain an upper bound

$$\left| \frac{\partial}{\partial x} \frac{\sin(x)}{x} \right| = \left| \int_0^1 t \sin(tx) dt \right|$$

$$\leq \int_0^1 |t \sin(tx)| dt$$

$$\leq \int_0^1 1 dt = 1,$$

since  $t \in [0,1] \implies |t| < 1$ , and  $|\sin(xt)| \le 1$  for any x and t.

• Note that this bound also justifies the DCT, since the functions  $f_n(t) = t \sin(t\xi_n)$  are uniformly dominated by g(t) = 1 on  $L^1([0,1])$ .

Note: integrating by parts here yields the actual formula:

$$\int_{0}^{1} t \sin(tx) dt =_{IBP} \left( \frac{-t \cos(tx)}{x} \right) \Big|_{t=0}^{t=1} - \int_{0}^{1} \frac{\cos(tx)}{x} dt$$
$$= \frac{-\cos(x)}{x} - \frac{\sin(x)}{x^{2}}$$
$$= \frac{x \cos(x) - \sin(x)}{x^{2}}.$$

• For the inductive step, we assume that we can pass n-1 limits through the integral and show we can pass the nth through as well.

$$\frac{\partial^n}{\partial x^n} \frac{\sin(x)}{x} = \frac{\partial^n}{\partial x^n} \int_0^1 \cos(tx) dt$$
$$= \frac{\partial}{\partial x} \int_0^1 \frac{\partial^{n-1}}{\partial x^{n-1}} \cos(tx) dt$$
$$= \frac{\partial}{\partial x} \int_0^1 t^{n-1} f_{n-1}(x, t) dt$$

- Note that  $f_n(x,t) = \pm \sin(tx)$  when n is odd and  $f_n(x,t) = \pm \cos(tx)$  when n is even, and a constant factor of t is multiplied when each derivative is taken.
- We continue as in the base case:

$$\frac{\partial}{\partial x} \int_0^1 t^{n-1} f_{n-1}(x,t) dt = \lim_{x_k \to x} \int_0^1 t^{n-1} \left( \frac{f_{n-1}(x_n,t) - f_{n-1}(x,t)}{x_n - x} \right) dt$$

$$=_{\text{IVT}} \lim_{x_k \to x} \int_0^1 t^{n-1} \frac{\partial f_{n-1}}{\partial x} \left( \xi_k, t \right) dt \quad \text{where } \xi_k \in [x_k, x], \, \xi_k \to x$$

$$=_{\text{DCT}} \int_0^1 \lim_{x_k \to x} t^{n-1} \frac{\partial f_{n-1}}{\partial x} \left( \xi_k, t \right) dt$$

$$\coloneqq \int_0^1 \lim_{x_k \to x} t^n f_n(\xi_k, t) dt$$

$$\coloneqq \int_0^1 t^n f_n(x, t) dt.$$

- We've used the fact that  $f_0(x) = \cos(tx)$  is smooth as a function of x, and in particular continuous
- The DCT is justified because the functions  $h_{n,k}(x,t) = t^n f_n(\xi_k,t)$  are again uniformly (in k) bounded by 1 since  $t \leq 1 \implies t^n \leq 1$  and each  $f_n$  is a sin or cosine.

• Now take absolute values

$$\left| \frac{\partial^n}{\partial x^n} \frac{\sin(x)}{x} \right| = \left| \int_0^1 -t^n f_n(x, t) \, dt \right|$$

$$\leq \int_0^1 |t^n f_n(x, t)| \, dt$$

$$\leq \int_0^1 |t^n| |f_n(x, t)| \, dt$$

$$\leq \int_0^1 |t^n| \cdot 1 \, dt$$

$$\leq \int_0^1 t^n \, dt \quad \text{since } t \text{ is positive}$$

$$= \frac{1}{n+1}$$

$$< \frac{1}{n}.$$

- We've again used the fact that  $f_n(x,t)$  is of the form  $\pm \cos(tx)$  or  $\pm \sin(tx)$ , both of which are bounded by 1.

### 5.2 Spring 2020 # 5 💝

Compute the following limit and justify your calculations:

$$\lim_{n \to \infty} \int_0^n \left( 1 + \frac{x^2}{n} \right)^{-(n+1)} dx.$$

Not finished, flesh out.

Walk through.

#### Solution:

Concepts Used:

- DCT
- Passing limits through products and quotients

Note that

$$\lim_{n} \left( 1 + \frac{x^2}{n} \right)^{-(n+1)} = \frac{1}{\lim_{n} \left( 1 + \frac{x^2}{n} \right)^1 \left( 1 + \frac{x^2}{n} \right)^n}$$
$$= \frac{1}{1 \cdot e^{x^2}}$$
$$= e^{-x^2}.$$

If passing the limit through the integral is justified, we will have

$$\lim_{n \to \infty} \int_0^n \left( 1 + \frac{x^2}{n} \right)^{-(n+1)} dx = \lim_{n \to \infty} \int_{\mathbb{R}} \chi_{[0,n]} \left( 1 + \frac{x^2}{n} \right)^{-(n+1)} dx$$

$$= \int_{\mathbb{R}} \lim_{n \to \infty} \chi_{[0,n]} \left( 1 + \frac{x^2}{n} \right)^{-(n+1)} dx \quad \text{by the DCT}$$

$$= \int_{\mathbb{R}} \lim_{n \to \infty} \left( 1 + \frac{x^2}{n} \right)^{-(n+1)} dx$$

$$= \int_0^\infty e^{-x^2}$$

$$= \frac{\sqrt{\pi}}{2}.$$

Computing the last integral:

$$\left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2 = \left(\int_{\mathbb{R}} e^{-x^2} dx\right) \left(\int_{\mathbb{R}} e^{-y^2} dx\right)$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(x+y)^2} dx$$

$$= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta \qquad u = r^2$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^{\infty} e^{-u} du d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} 1$$

$$= \pi.$$

and now use the fact that the function is even so  $\int_0^\infty f = \frac{1}{2} \int_{\mathbb{R}} f$ . Justifying the DCT:

• Apply Bernoulli's inequality:

$$1 + \frac{x^2}{n}^{n+1} \ge 1 + \frac{x^2}{n} (1 + x^2) \ge 1 + x^2,$$

where the last inequality follows from the fact that  $1 + \frac{x^2}{n} \ge 1$ 

# 5.3 Spring 2019 # 3 💝

Let  $\{f_k\}$  be any sequence of functions in  $L^2([0,1])$  satisfying  $\|f_k\|_2 \leq M$  for all  $k \in \mathbb{N}$ . Prove that if  $f_k \longrightarrow f$  almost everywhere, then  $f \in L^2([0,1])$  with  $\|f\|_2 \leq M$  and

$$\lim_{k \to \infty} \int_0^1 f_k(x) dx = \int_0^1 f(x) dx$$

Hint: Try using Fatou's Lemma to show that  $||f||_2 \leq M$  and then try applying Egorov's

#### Solution:

### Concepts Used:

- Definition of  $L^+$ : space of measurable function  $X \longrightarrow [0, \infty]$ .
- Fatou: For any sequence of L<sup>+</sup> functions, ∫ lim inf f<sub>n</sub> ≤ lim inf ∫ f<sub>n</sub>.
  Egorov's Theorem: If E ⊆ ℝ<sup>n</sup> is measurable, m(E) > 0, f<sub>k</sub>: E → ℝ a sequence of measurable functions where  $\lim_{n \to \infty} f_n(x)$  exists and is finite a.e., then  $f_n \to f$  almost uniformly: for every  $\varepsilon > 0$  there exists a closed subset  $F_{\varepsilon} \subseteq E$  with  $m(E \setminus F) < \varepsilon$ and  $f_n \longrightarrow f$  uniformly on F.

### $L^2$ bound:

- Since  $f_k \longrightarrow f$  almost everywhere,  $\liminf_{n \to \infty} f_n(x) = f(x)$  a.e.
- $||f_n||_2 < \infty$  implies each  $f_n$  is measurable and thus  $|f_n|^2 \in L^+$ , so we can apply Fatou:

$$||f||_2^2 = \int |f(x)|^2$$

$$= \int \liminf_n |f_n(x)|^2$$

$$\leq \liminf_n \int |f_n(x)|^2$$

$$\leq \liminf_n M$$

$$= M.$$

• Thus  $\|f\|_2 \leq \sqrt{M} < \infty$  implying  $f \in L^2$ . What is the "right" proof here that uses the first part?

#### Equality of Integrals:

- Take the sequence  $\varepsilon_n = \frac{1}{n}$  Apply Egorov's theorem: obtain a set  $F_{\varepsilon}$  such that  $f_n \longrightarrow f$  uniformly on  $F_{\varepsilon}$  and  $m(I \setminus F_{\varepsilon}) < \varepsilon$ .

$$\lim_{n \to \infty} \left| \int_0^1 f_n - f \right| \le \lim_{n \to \infty} \int_0^1 |f_n - f|$$

$$= \lim_{n \to \infty} \left( \int_{F_{\varepsilon}} |f_n - f| + \int_{I \setminus F_{\varepsilon}} |f_n - f| \right)$$

$$= \int_{F_{\varepsilon}} \lim_{n \to \infty} |f_n - f| + \lim_{n \to \infty} \int_{I \setminus F_{\varepsilon}} |f_n - f| \quad \text{by uniform convergence}$$

$$= 0 + \lim_{n \to \infty} \int_{I \setminus F_{\varepsilon}} |f_n - f|,$$

so it suffices to show  $\int_{I\setminus F_c} |f_n - f| \stackrel{n\longrightarrow\infty}{\longrightarrow} 0.$ 

• We can obtain a bound using Holder's inequality with p = q = 2:

$$\int_{I \setminus F_{\varepsilon}} |f_n - f| \le \left( \int_{I \setminus F_{\varepsilon}} |f_n - f|^2 \right) \left( \int_{I \setminus F_{\varepsilon}} |1|^2 \right)$$

$$= \left( \int_{I \setminus F_{\varepsilon}} |f_n - f|^2 \right) \mu(F_{\varepsilon})$$

$$\le \|f_n - f\|_2 \mu(F_{\varepsilon})$$

$$\le (\|f_n\|_2 + \|f\|_2) \mu(F_{\varepsilon})$$

$$\le 2M \cdot \mu(F_{\varepsilon})$$

where M is now a constant not depending on  $\varepsilon$  or n.

• Now take a nested sequence of sets  $F_{\varepsilon}$  with  $\mu(F_{\varepsilon}) \longrightarrow 0$  and applying continuity of measure yields the desired statement.

### 5.4 Fall 2018 # 6 💝

Compute the following limit and justify your calculations:

$$\lim_{n \to \infty} \int_1^n \frac{dx}{\left(1 + \frac{x}{n}\right)^n \sqrt[n]{x}}$$

Add concepts

**Solution:** 

Concepts Used:

- Note that x<sup>1/n</sup> n→∞ 1 for any 0 < x < ∞.</li>
  Thus the integrand converges to 1/ex, which is integrable on (0,∞) and integrates to 1.
- Break the integrand up:

$$\int_0^\infty \frac{1}{\left(1 + \frac{x}{n}\right)^n x^{\frac{1}{n}}} \, dx = \int_0^1 \frac{1}{\left(1 + \frac{x}{n}\right)^n x^{\frac{1}{n}}} \, dx = \int_1^\infty \frac{1}{\left(1 + \frac{x}{n}\right)^n x^{\frac{1}{n}}} \, dx.$$

# 5.5 Fall 2018 # 3 💝

Suppose f(x) and xf(x) are integrable on  $\mathbb{R}$ . Define F by

$$F(t) := \int_{-\infty}^{\infty} f(x) \cos(xt) dx$$

Show that

$$F'(t) = -\int_{-\infty}^{\infty} x f(x) \sin(xt) dx.$$

Walk through

#### **Solution:**

### Concepts Used:

- Mean Value Theorem
- DCT

$$\frac{\partial}{\partial t} F(t) = \frac{\partial}{\partial t} \int_{\mathbb{R}} f(x) \cos(xt) dx$$

$$\stackrel{DCT}{=} \int_{\mathbb{R}} f(x) \frac{\partial}{\partial t} \cos(xt) dx$$

$$= \int_{\mathbb{R}} x f(x) \cos(xt) dx,$$

so it only remains to justify the DCT.

- Fix t, then let  $t_n \longrightarrow t$  be arbitrary.
- Define

$$h_n(x,t) = f(x) \left( \frac{\cos(tx) - \cos(t_n x)}{t_n - t} \right) \stackrel{n \to \infty}{\longrightarrow} \frac{\partial}{\partial t} \left( f(x) \cos(xt) \right)$$

since  $\cos(tx)$  is differentiable in t and this is the limit definition of differentiability.

• Note that

$$\frac{\partial}{\partial t} \cos(tx) := \lim_{t_n \to t} \frac{\cos(tx) - \cos(t_n x)}{t_n - t}$$

$$\stackrel{MVT}{=} \frac{\partial}{\partial t} \cos(tx) \Big|_{t = \xi_n} \qquad \text{for some } \xi_n \in [t, t_n] \text{ or } [t_n, t]$$

$$= x \sin(\xi_n x)$$

where  $\xi_n \stackrel{n \longrightarrow \infty}{\longrightarrow} t$  since wlog  $t_n \le \xi_n \le t$  and  $t_n \nearrow t$ .

• We then have

$$|h_n(x)| = |f(x)x\sin(\xi_n x)| \le |xf(x)|$$
 since  $|\sin(\xi_n x)| \le 1$ 

for every x and every n.

• Since  $xf(x) \in L^1(\mathbb{R})$  by assumption, the DCT applies.

# 5.6 Spring 2018 # 5 처

Suppose that

- $f_n, f \in L^1$ ,  $f_n \longrightarrow f$  almost everywhere, and  $\int |f_n| \to \int |f|$ .

Show that  $\int f_n \to \int f$ .

#### Solution:

### Concepts Used:

• 
$$\int |f_n - f| \longrightarrow \iff \int f_n = \int f$$
.  
• Fatou:

$$\int \liminf f_n \le \liminf \int f_n$$
$$\int \limsup f_n \ge \limsup \int f_n.$$

• Since  $\int |f_n| \stackrel{n \to \infty}{\longrightarrow} \int |f|$ , define

$$h_n = |f_n - f|$$
  $\xrightarrow{n \longrightarrow \infty} 0 \ a.e.$   $g_n = |f_n| + |f|$   $\xrightarrow{n \longrightarrow \infty} 2|f| \ a.e.$ 

- Note that  $g_n - h_n \xrightarrow{n \to \infty} 2|f| - 0 = 2|f|$ .

• Then

$$\int 2|f| = \int \liminf_n (g_n - h_n)$$

$$= \int \liminf_n (g_n) + \int \liminf_n (-h_n)$$

$$= \int \liminf_n (g_n) - \int \limsup_n (h_n)$$

$$= \int 2|f| - \int \limsup_n (h_n)$$

$$\leq \int 2|f| - \limsup_n \int h_n \quad \text{by Fatou,}$$

• Since  $f \in L^1$ ,  $\int 2|f| = 2||f||_1 < \infty$  and it makes sense to subtract it from both sides,

$$0 \le -\limsup_{n} \int h_{n}$$

$$:= -\limsup_{n} \int |f_{n} - f|.$$

which forces  $\limsup_{n} \int |f_n - f| = 0$ , since

– The integral of a nonnegative function is nonnegative, so  $\int |f_n - f| \ge 0$ .

$$- \operatorname{So}\left(-\int |f_n - f|\right) \le 0.$$

– But the above inequality shows  $\left(-\int |f_n - f|\right) \ge 0$  as well.

• Since  $\liminf_{n} \int h_n \leq \limsup_{n} \int h_n = 0$ ,  $\lim_{n} \int h_n$  exists and is equal to zero.

• But then

$$\left| \int f_n - \int f \right| = \left| \int f_n - f \right| \le \int |f_n - f|,$$

and taking  $\lim_{n \to \infty}$  on both sides yields

$$\lim_{n \to \infty} \left| \int f_n - \int f \right| \le \lim_{n \to \infty} \int |f_n - f| = 0,$$

so 
$$\lim_{n \to \infty} \int f_n = \int f$$
.

### 5.7 Spring 2018 # 2 💝

Let

$$f_n(x) := \frac{x}{1 + x^n}, \quad x \ge 0.$$

- a. Show that this sequence converges pointwise and find its limit. Is the convergence uniform on  $[0,\infty)$ ?
- b. Compute

$$\lim_{n\to\infty}\int_0^\infty f_n(x)dx$$

Add concepts

#### Solution:

Concepts Used:

• ?

#### 5.7.1 a

Claim:  $f_n$  does not converge uniformly to its limit.

- Note each  $f_n(x)$  is clearly continuous on  $(0, \infty)$ , since it is a quotient of continuous functions where the denominator is never zero.
- Note

$$x < 1 \implies x^n \stackrel{n \longrightarrow \infty}{\longrightarrow} 0$$
 and  $x > 1 \implies x^n \stackrel{n \longrightarrow \infty}{\longrightarrow} \infty$ .

• Thus

$$f_n(x) = \frac{x}{1+x^n} \xrightarrow{n \longrightarrow \infty} f(x) := \begin{cases} x, & 0 \le x < 1 \\ \frac{1}{2}, & x = 1 \\ 0, & x > 1 \end{cases}.$$

- If  $f_n \longrightarrow f$  uniformly on  $[0, \infty)$ , it would converge uniformly on every subset and thus uniformly on  $(0, \infty)$ .
  - Then f would be a uniform limit of continuous functions on  $(0, \infty)$  and thus continuous on  $(0, \infty)$ .
  - By uniqueness of limits,  $f_n$  would converge to the pointwise limit f above, which is not continuous at x = 1, a contradiction.

#### 5.7.2 b

• If the DCT applies, interchange the limit and integral:

$$\lim_{n \to \infty} \int_0^\infty f_n(x) \, dx = \int_0^\infty \lim_{n \to \infty} f_n(x) \, dx \quad \text{DCT}$$

$$= \int_0^\infty f(x) \, dx$$

$$= \int_0^1 x \, dx + \int_1^\infty 0 \, dx$$

$$= \frac{1}{2} x^2 \Big|_0^1$$

$$= \frac{1}{2}.$$

• To justify the DCT, write

$$\int_0^\infty f_n(x) = \int_0^1 f_n(x) + \int_1^\infty f_n(x).$$

•  $f_n$  restricted to (0,1) is uniformly bounded by  $g_0(x) = 1$  in the first integral, since

$$x \in [0,1] \implies \frac{x}{1+x^n} < \frac{1}{1+x^n} < 1 := g(x)$$

so

$$\int_0^1 f_n(x) \, dx \le \int_0^1 1 \, dx = 1 < \infty.$$

Also note that  $g_0 \cdot \chi_{(0,1)} \in L^1((0,\infty))$ .

• The  $f_n$  restricted to  $(1,\infty)$  are uniformly bounded by  $g_1(x) = \frac{1}{x^2}$  on  $[1,\infty)$ , since

$$x \in (1, \infty) \implies \frac{x}{1 + x^n} \le \frac{x}{x^n} = \frac{1}{x^{n-1}} \le \frac{1}{x^2} \in L^1([1, \infty) \text{ when } n \ge 3,$$

by the p-test for integrals.

• So set

$$g := g_0 \cdot \chi_{(0,1)} + g_1 \cdot \chi_{[1,\infty)},$$

then by the above arguments  $g \in L^1((0,\infty))$  and  $f_n \leq g$  everywhere, so the DCT applies.

# 5.8 Fall 2016 # 3 💝

Let  $f \in L^1(\mathbb{R})$ . Show that

$$\lim_{x \to 0} \int_{\mathbb{R}} |f(y - x) - f(y)| \, dy = 0$$

Missing some stuff

#### **Solution:**

#### Concepts Used:

- $C_c^{\infty} \hookrightarrow L^p$  is dense. If  $f \dots$ ?
- Fixing notation, set  $\tau_x f(y) := f(y-x)$ ; we then want to show

$$\|\tau_x f - f\|_{L^1} \stackrel{x \longrightarrow 0}{\longrightarrow} 0.$$

- Claim: by an  $\varepsilon/3$  argument, it suffices to show this for compactly supported functions:
  - Since  $f \in L^1$ , choose  $g_n \subset C_c^{\infty}(\mathbb{R}^1)$  smooth and compactly supported so that

$$||f - g||_{L^1} < \varepsilon.$$

- Claim:  $\|\tau_x f \tau_x g\| < \varepsilon$ .
  - \* Proof 1: translation invariance of the integral.
  - \* Proof 2: Apply a change of variables:

$$\|\tau_x f - \tau_x g\|_1 := \int_{\mathbb{R}} |\tau_x f(y) - \tau_x g(y)| \, dy$$

$$= \int_{\mathbb{R}} |f(y - x) - g(y - x)| \, dy$$

$$= \int_{\mathbb{R}} |f(u) - g(u)| \, du \qquad (u = y - x, \, du = dy)$$

$$= \|f - g\|_1$$

$$< \varepsilon.$$

- Then

$$\begin{aligned} \|\tau_x f - f\|_1 &= \|\tau_x f - \tau_x g + \tau_x g - g + g - f\|_1 \\ &\leq \|\tau_x f - \tau_x g\|_1 + \|\tau_x g - g\|_1 + \|g - f\|_1 \\ &\leq 2\varepsilon + \|\tau_x g - g\|_1. \end{aligned}$$

- To show this for compactly supported functions:
  - Let  $g \in C_c^{\infty}(\mathbb{R}^1)$ , let E = supp(g), and write

$$\|\tau_x g - g\|_1 = \int_{\mathbb{R}} |g(y - x) - g(y)| \, dy$$

$$= \int_{E} |g(y - x) - g(y)| \, dy + \int_{E^c} |g(y - x) - g(y)| \, dy$$

$$= \int_{E} |g(y - x) - g(y)| \, dy.$$

- But g is smooth and compactly supported on E, and thus uniformly continuous on E, so

$$\lim_{x \to 0} \int_{E} |g(y-x) - g(y)| dy = \int_{E} \lim_{x \to 0} |g(y-x) - g(y)| dy$$
$$= \int_{E} 0 dy$$
$$= 0.$$

# 5.9 Fall 2015 # 3

Compute the following limit:

$$\lim_{n \to \infty} \int_1^n \frac{ne^{-x}}{1 + nx^2} \sin\left(\frac{x}{n}\right) dx$$

# 5.10 Fall 2015 # 4

Let  $f:[1,\infty)\longrightarrow \mathbb{R}$  such that f(1)=1 and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Show that the following limit exists and satisfies the equality

$$\lim_{x \to \infty} f(x) \le 1 + \frac{\pi}{4}$$

### 6 Integrals: Approximation

### 6.1 Spring 2018 # 3 😽

Let f be a non-negative measurable function on [0, 1].

Show that

$$\lim_{p \to \infty} \left( \int_{[0,1]} f(x)^p dx \right)^{\frac{1}{p}} = \|f\|_{\infty}.$$

#### Solution:

Concepts Used:

- $||f||_{\infty} := \inf_{t} \{t \mid m(\{x \in \mathbb{R}^n \mid f(x) > t\}) = 0\}$ , i.e. this is the lowest upper bound that holds almost everywhere.
- $\|f\|_p \le \|f\|_\infty$ : - Note  $|f(x)| \le \|f\|_\infty$  almost everywhere and taking pth powers preserves this inequality.

- Thus

$$\begin{split} |f(x)| &\leq \|f\|_{\infty} \quad \text{a.e. by definition} \\ \Longrightarrow |f(x)|^p &\leq \|f\|_{\infty}^p \quad \text{for } p \geq 0 \\ \Longrightarrow \|f\|_p^p &= \int_X |f(x)|^p \ dx \\ &\leq \int_X \|f\|_{\infty}^p \ dx \\ &= \|f\|_{\infty}^p \int_X 1 \ dx \\ &= \|f\|_{\infty}^p \cdot m(X) \quad \text{since the norm doesn't depend on } x \\ &= \|f\|_{\infty}^p \quad \text{since } m(X) = 1. \end{split}$$

- \* Thus  $\|f\|_p \leq \|f\|_{\infty}$  for all p and taking  $\lim_{p \longrightarrow \infty}$  preserves this inequality.
- $||f||_p \ge ||f||_\infty$ : Fix  $\varepsilon > 0$ .

  - Define

$$S_{\varepsilon} := \left\{ x \in \mathbb{R}^n \mid |f(x)| \ge ||f||_{\infty} - \varepsilon \right\}.$$

- \* Note that  $m(S_{\varepsilon}) > 0$ ; otherwise if  $m(S_{\varepsilon}) = 0$ , then  $t := ||f||_{\infty} \varepsilon < ||f||_{\varepsilon}$ . But this produces a smaller upper bound almost everywhere than  $||f||_{\varepsilon}$ , contradicting the definition of  $||f||_{\varepsilon}$  as an infimum over such bounds.
- Then

$$\begin{split} \|f\|_p^p &= \int_X |f(x)|^p \ dx \\ &\geq \int_{S_\varepsilon} |f(x)|^p \ dx \quad \text{since } S_\varepsilon \subseteq X \\ &\geq \int_{S_\varepsilon} |\|f\|_\infty - \varepsilon|^p \ dx \quad \text{since on } S_\varepsilon, |f| \geq \|f\|_\infty - \varepsilon \\ &= |\|f\|_\infty - \varepsilon|^p \cdot m(S_\varepsilon) \quad \text{since the integrand is independent of } x \\ &\geq 0 \quad \text{since } m(S_\varepsilon) > 0 \end{split}$$

- Taking pth roots for  $p \ge 1$  preserves the inequality, so

$$\implies \|f\|_p \ge \|\|f\|_{\infty} - \varepsilon\| \cdot m(S_{\varepsilon})^{\frac{1}{p}} \stackrel{p \longrightarrow \infty}{\longrightarrow} \|\|f\|_{\infty} - \varepsilon\| \stackrel{\varepsilon \longrightarrow 0}{\longrightarrow} \|f\|_{\infty}$$

where we've used the fact that above arguments work

- Thus  $||f||_p \ge ||f||_{\infty}$ .

# 6.2 Spring 2018 # 4 처

Let  $f \in L^2([0,1])$  and suppose

$$\int_{[0,1]} f(x)x^n dx = 0 \text{ for all integers } n \ge 0.$$

Show that f = 0 almost everywhere.

### 6.3 Spring 2015 # 2 💝

Let  $f: \mathbb{R} \longrightarrow \mathbb{C}$  be continuous with period 1. Prove that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_{0}^{1} f(t)dt \quad \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

Hint: show this first for the functions  $f(t) = e^{2\pi i kt}$  for  $k \in \mathbb{Z}$ .

Add concepts

#### Solution:

Concepts Used:

• 1

#### 6.3.1 Proof 1: Using Fourier Transforms

### Concepts Used:

- Weierstrass Approximation: A uniformly continuous function on a compact set can be uniformly approximated by polynomials.
- Fix  $k \in \mathbb{Z}$ .
- Since  $e^{2\pi ikx}$  is continuous on the compact interval [0, 1], it is uniformly continuous.
- Thus there is a sequence of polynomials  $P_{\ell}$  such that

$$P_{\ell,k} \stackrel{\ell \longrightarrow \infty}{\longrightarrow} e^{2\pi i k x}$$
 uniformly on [0, 1].

• Note applying linearity to the assumption  $\int f(x) x^n$ , we have

$$\int f(x)x^n dx = 0 \ \forall n \implies \int f(x)p(x) dx = 0$$

for any polynomial p(x), and in particular for  $P_{\ell,k}(x)$  for every  $\ell$  and every k.

• But then

$$\begin{split} \langle f,\ e_k \rangle &= \int_0^1 f(x) e^{-2\pi i k x}\ dx \\ &= \int_0^1 f(x) \lim_{\ell \to \infty} P_\ell(x) \\ &= \lim_{\ell \to \infty} \int_0^1 f(x) P_\ell(x) \quad \text{by uniform convergence on a compact interval} \\ &= \lim_{\ell \to \infty} 0 \quad \text{by assumption} \\ &= 0 \quad \forall k \in \mathbb{Z}, \end{split}$$

- so f is orthogonal to every  $e_k$ . Thus  $f \in S^{\perp} := \operatorname{span}_{\mathbb{C}} \{e_k\}_{k \in \mathbb{Z}}^{\perp} \subseteq L^2([0,1])$ , but since this is a basis, S is dense and thus
- Thus  $f \equiv 0$  in  $L^2([0,1])$ , which implies that f is zero almost everywhere.

#### 6.3.2 Alternative Proof

#### Concepts Used:

- $C^1([0,1])$  is dense in  $L^2([0,1])$
- Polynomials are dense in  $L^p(X, \mathcal{M}, \mu)$  for any  $X \subseteq \mathbb{R}^n$  compact and  $\mu$  a finite measure, for all  $1 \le p < \infty$ .
  - Use Weierstrass Approximation, then uniform convergence implies  $L^p(\mu)$  convergence by DCT.
- By density of polynomials, for  $f \in L^2([0,1])$  choose  $p_{\varepsilon}(x)$  such that  $||f p_{\varepsilon}|| < \varepsilon$  by Weierstrass approximation.
- Then on one hand,

$$||f(f - p_{\varepsilon})||_1 = ||f^2||_1 - ||f \cdot p_{\varepsilon}||_1$$
$$= ||f^2||_1 - 0 \quad \text{by assumption}$$
$$= ||f||_2^2.$$

- Where we've used that  $||f^2||_1 = \int |f^2| = \int |f|^2 = ||f||_2^2$ .
- On the other hand

$$\begin{split} \|f(f-p_{\varepsilon})\| &\leq \|f\|_1 \|f-p_{\varepsilon}\|_{\infty} \quad \text{by Holder} \\ &\leq \varepsilon \|f\|_1 \\ &\leq \varepsilon \|f\|_2 \sqrt{m(X)} \\ &= \varepsilon \|f\|_2 \quad \text{since } m(X) = 1. \end{split}$$

- Where we've used that  $||fg||_1 = \int |fg| = \int |f||g| \le \int ||f||_{\infty} |g| = ||f||_{\infty} ||g||_1$ .
- Combining these,

$$||f||_2^2 \le ||f||_2 \varepsilon \implies ||f||_2 < \varepsilon \longrightarrow 0,.$$

so  $||f||_2 = 0$ , which implies f = 0 almost everywhere.

### 6.4 Fall 2014 # 4

Let  $g \in L^{\infty}([0,1])$  Prove that

 $\int_{[0,1]} f(x)g(x) dx = 0$  for all continuous  $f:[0,1] \longrightarrow \mathbb{R} \implies g(x) = 0$  almost everywhere.

### **7** $L^{1}$

### 7.1 Spring 2020 # 3 😽

a. Prove that if  $g \in L^1(\mathbb{R})$  then

$$\lim_{N \to \infty} \int_{|x| \ge N} |f(x)| \, dx = 0,$$

and demonstrate that it is not necessarily the case that  $f(x) \longrightarrow 0$  as  $|x| \longrightarrow \infty$ .

b. Prove that if  $f \in L^1([1,\infty])$  and is decreasing, then  $\lim_{x \to \infty} f(x) = 0$  and in fact  $\lim_{x \to \infty} x f(x) = 0$ .

c. If  $f:[1,\infty) \longrightarrow [0,\infty)$  is decreasing with  $\lim_{x \to \infty} x f(x) = 0$ , does this ensure that  $f \in L^1([1,\infty))$ ?

#### Solution:

### Concepts Used:

Limits

• Cauchy Criterion for Integrals:  $\int_{a}^{\infty} f(x) dx$  converges iff for every  $\varepsilon > 0$  there exists an  $M_0$  such that  $A, B \geq M_0$  implies  $\left| \int_{A}^{B} f \right| < \varepsilon$ , i.e.  $\left| \int_{A}^{B} f \right| \stackrel{A \longrightarrow \infty}{\longrightarrow} 0$ .

• Integrals of  $L^1$  functions have vanishing tails:  $\int_N^\infty |f| \stackrel{N \longrightarrow \infty}{\longrightarrow} 0$ .

• Mean Value Theorem for Integrals:  $\int_a^b f(t) dt = (b-a)f(c)$  for some  $c \in [a,b]$ .

#### 7.1.1 a

Stated integral equality:

- Let  $\varepsilon > 0$
- $C_c(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$  is dense so choose  $\{f_n\} \longrightarrow f$  with  $||f_n f||_1 \longrightarrow 0$ .
- Since  $\{f_n\}$  are compactly supported, choose  $N_0 \gg 1$  such that  $f_n$  is zero outside of  $B_{N_0}(\mathbf{0})$ .

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• Then

$$N \ge N_0 \implies \int_{|x|>N} |f| = \int_{|x|>N} |f - f_n + f_n|$$

$$\le \int_{|x|>N} |f - f_n| + \int_{|x|>N} |f_n|$$

$$= \int_{|x|>N} |f - f_n|$$

$$\le \int_{|x|>N} ||f - f_n||_1$$

$$= ||f_n - f||_1 \left( \int_{|x|>N} 1 \right)$$

$$\stackrel{n \longrightarrow \infty}{\longrightarrow} 0 \left( \int_{|x|>N} 1 \right)$$

$$= 0$$

$$\stackrel{N \longrightarrow \infty}{\longrightarrow} 0.$$

To see that this doesn't force  $f(x) \longrightarrow 0$  as  $|x| \longrightarrow \infty$ :

- Take f(x) to be a train of rectangles of height 1 and area  $1/2^{j}$  centered on even integers.

$$\int_{|x|>N} |f| = \sum_{j=N}^{\infty} 1/2^j \stackrel{N \longrightarrow \infty}{\longrightarrow} 0$$

as the tail of a convergent sum.

• However f(x) = 1 for infinitely many even integers x > N, so  $f(x) \not\longrightarrow 0$  as  $|x| \longrightarrow \infty$ .

### 7.1.2 b

#### Solution 1 ("Trick")

• Since f is decreasing on  $[1, \infty)$ , for any  $t \in [x - n, x]$  we have

$$x - n < t < x \implies f(x) < f(t) < f(x - n).$$

• Integrate over [x, 2x], using monotonicity of the integral:

$$\int_{x}^{2x} f(x) dt \le \int_{x}^{2x} f(t) dt \le \int_{x}^{2x} f(x-n) dt$$

$$\implies f(x) \int_{x}^{2x} dt \le \int_{x}^{2x} f(t) dt \le f(x-n) \int_{x}^{2x} dt$$

$$\implies x f(x) \le \int_{x}^{2x} f(t) dt \le x f(x-n).$$

- By the Cauchy Criterion for integrals,  $\lim_{x \to \infty} \int_{x}^{2x} f(t) dt = 0$ .
- So the LHS term  $xf(x) \stackrel{x \longrightarrow \infty}{\longrightarrow} 0$ .
- Since x > 1,  $|f(x)| \le |xf(x)|$  Thus  $f(x) \xrightarrow{x \to \infty} 0$  as well.

### Solution 2 (Variation on the Trick)

• Use mean value theorem for integrals:

$$\int_{x}^{2x} f(t) dt = x f(c_x) \quad \text{for some } c_x \in [x, 2x] \text{ depending on } x.$$

• Since f is decreasing,

$$x \le c_x \le 2x \implies f(2x) \le f(c_x) \le f(x)$$

$$\implies 2xf(2x) \le 2xf(c_x) \le 2xf(x)$$

$$\implies 2xf(2x) \le 2x \int_x^{2x} f(t) dt \le 2xf(x)$$

- By Cauchy Criterion,  $\int_{x}^{2x} f \longrightarrow 0$ .
- So  $2xf(2x) \longrightarrow 0$ , which by a change of variables gives  $uf(u) \longrightarrow 0$ .
- Since  $u \ge 1$ ,  $f(u) \le u f(u)$  so  $f(u) \longrightarrow 0$  as well.

### **Solution 3 (Contradiction)** Just showing $f(x) \stackrel{x \longrightarrow \infty}{\longrightarrow} 0$ :

- Toward a contradiction, suppose not.
- Since f is decreasing, it can not diverge to  $+\infty$
- If  $f(x) \longrightarrow -\infty$ , then  $f \notin L^1(\mathbb{R})$ : choose  $x_0 \gg 1$  so that  $t \geq x_0 \implies f(t) < -1$ , then
- Then  $t \ge x_0 \implies |f(t)| \ge 1$ , so

$$\int_{1}^{\infty} |f| \ge \int_{x_0}^{\infty} |f(t)| dt \ge \int_{x_0}^{\infty} 1 = \infty.$$

- Otherwise  $f(x) \longrightarrow L \neq 0$ , some finite limit.
- If L > 0:
  - Fix  $\varepsilon > 0$ , choose  $x_0 \gg 1$  such that  $t \geq x_0 \implies L \varepsilon \leq f(t) \leq L$
  - Then

$$\int_{1}^{\infty} f \ge \int_{x_0}^{\infty} f \ge \int_{x_0}^{\infty} (L - \varepsilon) dt = \infty$$

- If L < 0:
  - Fix  $\varepsilon > 0$ , choose  $x_0 \gg 1$  such that  $t \geq x_0 \implies L \leq f(t) \leq L + \varepsilon$ .
  - Then

$$\int_{1}^{\infty} f \ge \int_{x_0}^{\infty} f \ge \int_{x_0}^{\infty} (L) dt = \infty$$

Showing  $xf(x) \stackrel{x \longrightarrow \infty}{\longrightarrow} 0$ .

- Toward a contradiction, suppose not.
- (How to show that  $xf(x) \not\longrightarrow +\infty$ ?)
- If  $x f(x) \longrightarrow -\infty$ 
  - Choose a sequence  $\Gamma = \{\hat{x}_i\}$  such that  $x_i \longrightarrow \infty$  and  $x_i f(x_i) \longrightarrow -\infty$ .
  - Choose a subsequence  $\Gamma' = \{x_i\}$  such that  $x_i f(x_i) \leq -1$  for all i and  $x_i \leq x_{i+1}$ .
  - Choose a further subsequence  $S = \{x_i \in \Gamma' \mid 2x_i < x_{i+1}\}.$

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- Then since f is always decreasing, for  $t \ge x_0$ , |f| is increasing, and  $|f(x_i)| \le |f(2x_i)|$ , so

$$\int_{1}^{\infty} |f| \ge \int_{x_0}^{\infty} |f| \ge \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(t)| \, dt \ge \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(x_i)| = \sum_{x_i \in S} x_i f(x_i) \longrightarrow \infty.$$

• If  $xf(x) \longrightarrow L \neq 0$  for  $0 < L < \infty$ :

- Fix  $\varepsilon > 0$ , choose an infinite sequence  $\{x_i\}$  such that  $L - \varepsilon \leq x_i f(x_i) \leq L$  for all i.

$$\int_{1}^{\infty} |f| \ge \sum_{S} \int_{x_i}^{2x_i} |f(t)| dt \ge \sum_{S} \int_{x_i}^{2x_i} f(x_i) dt = \sum_{S} x_i f(x_i) \ge \sum_{S} (L - \varepsilon) \longrightarrow \infty.$$

• If  $xf(x) \longrightarrow L \neq 0$  for  $-\infty < L < 0$ :

- Fix  $\varepsilon > 0$ , choose an infinite sequence  $\{x_i\}$  such that  $L \leq x_i f(x_i) \leq L + \varepsilon$  for all i.

$$\int_{1}^{\infty} |f| \ge \sum_{S} \int_{x_i}^{2x_i} |f(t)| dt \ge \sum_{S} \int_{x_i}^{2x_i} f(x_i) dt = \sum_{S} x_i f(x_i) \ge \sum_{S} (L) \longrightarrow \infty.$$

Solution 4 (Akos's Suggestion) For  $x \ge 1$ .

$$|xf(x)| = \left| \int_{x}^{2x} f(x) dt \right| \le \int_{x}^{2x} |f(x)| dt \le \int_{x}^{2x} |f(t)| dt \le \int_{x}^{\infty} |f(t)| dt \xrightarrow{x \to \infty} 0$$

where we've used

- Since f is decreasing and  $\lim_{x \to \infty} f(x) = 0$  from part (a), f is non-negative.
- Since f is positive and decreasing, for every  $t \in [a, b]$  we have  $|f(a)| \le |f(t)|$ .
- By part (a), the last integral goes to zero.

#### Solution 5 (Peter's)

• Toward a contradiction, produce a sequence  $x_i \longrightarrow \infty$  with  $x_i f(x_i) \longrightarrow \infty$  and  $x_i f(x_i) > \varepsilon > 0$ , then

$$\int f(x) dx \ge \sum_{i=1}^{\infty} \int_{x_i}^{x_{i+1}} f(x) dx$$

$$\ge \sum_{i=1}^{\infty} \int_{x_i}^{x_{i+1}} f(x_{i+1}) dx$$

$$= \sum_{i=1}^{\infty} f(x_{i+1}) \int_{x_i}^{x_{i+1}} dx$$

$$\ge \sum_{i=1}^{\infty} (x_{i+1} - x_i) f(x_{i+1})$$

$$\ge \sum_{i=1}^{\infty} (x_{i+1} - x_i) \frac{\varepsilon}{x_{i+1}}$$

$$= \varepsilon \sum_{i=1}^{\infty} \left( 1 - \frac{x_{i-1}}{x_i} \right) \longrightarrow \infty$$

which can be ensured by passing to a subsequence where  $\sum \frac{x_{i-1}}{x_i} < \infty$ .

7.1.3 c

• No: take 
$$f(x) = \frac{1}{x \ln x}$$
  
• Then by a *u*-substitution,

$$\int_0^x f = \ln\left(\ln(x)\right) \stackrel{x \longrightarrow \infty}{\longrightarrow} \infty$$

is unbounded, so  $f \notin L^1([1,\infty))$ .

• But

$$xf(x) = \frac{1}{\ln(x)} \stackrel{x \to \infty}{\longrightarrow} 0.$$

# 7.2 Fall 2019 # 5. 💝

#### 7.2.1 a

Show that if f is continuous with compact support on  $\mathbb{R}$ , then

$$\lim_{y \to 0} \int_{\mathbb{R}} |f(x - y) - f(x)| dx = 0$$

#### 7.2.2 b

Let  $f \in L^1(\mathbb{R})$  and for each h > 0 let

$$\mathcal{A}_h f(x) := \frac{1}{2h} \int_{|y| \le h} f(x - y) dy$$

i. Prove that  $\|\mathcal{A}_h f\|_1 \leq \|f\|_1$  for all h > 0.

ii. Prove that  $\mathcal{A}_h f \longrightarrow f$  in  $L^1(\mathbb{R})$  as  $h \longrightarrow 0^+$ .

Walk through

#### **Solution:**

Concepts Used:

- Continuity in  $L^1$  (recall that DCT won't work! Notes 19.4, prove it for a dense
- Lebesgue differentiation in 1-dimensional case. See HW 5.6.

#### 7.2.3 a

Choose  $g \in C_c^0$  such that  $||f - g||_1 \longrightarrow 0$ . By translation invariance,  $||\tau_h f - \tau_h g||_1 \longrightarrow 0$ .

 $7 L^1$ 59 Write

$$\|\tau f - f\|_{1} = \|\tau_{h} f - g + g - \tau_{h} g + \tau_{h} g - f\|_{1}$$

$$\leq \|\tau_{h} f - \tau_{h} g\| + \|g - f\| + \|\tau_{h} g - g\|$$

$$\longrightarrow \|\tau_{h} g - g\|,$$

so it suffices to show that  $\|\tau_h g - g\| \longrightarrow 0$  for  $g \in C_c^0$ . Fix  $\varepsilon > 0$ . Enlarge the support of g to K such that

$$|h| \le 1$$
 and  $x \in K^c \implies |g(x-h) - g(x)| = 0$ .

By uniform continuity of g, pick  $\delta \leq 1$  small enough such that

$$x \in K$$
,  $|h| \le \delta \implies |g(x-h) - g(x)| < \varepsilon$ ,

then

$$\int_{K} |g(x-h) - g(x)| \le \int_{K} \varepsilon = \varepsilon \cdot m(K) \longrightarrow 0.$$

#### 7.2.4 b

We have

$$\int_{\mathbb{R}} |A_h(f)(x)| \ dx = \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y) \ dy \right| \ dx$$

$$\leq \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)| \ dy \ dx$$

$$=_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)| \ d\mathbf{x} \ d\mathbf{y}$$

$$= \int_{\mathbb{R}} |f(y)| \ dy$$

$$= ||f||_{1},$$

and (rough sketch)

$$\int_{\mathbb{R}} |A_h(f)(x) - f(x)| \ dx = \int_{\mathbb{R}} \left| \left( \frac{1}{2h} \int_{B(h,x)} f(y) \ dy \right) - f(x) \right| \ dx$$

$$= \int_{\mathbb{R}} \left| \left( \frac{1}{2h} \int_{B(h,x)} f(y) \ dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x) \ dy \right| \ dx$$

$$\leq_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{B(h,x)} |f(y - x) - f(x)| \ \mathbf{dx} \ \mathbf{dy}$$

$$\leq \frac{1}{2h} \int_{\mathbb{R}} \|\tau_x f - f\|_1 \ dy$$

$$\longrightarrow 0 \quad \text{by (a)}.$$

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### 7.3 Fall 2017 # 3 🙀

Let

$$S = \operatorname{span}_{\mathbb{C}} \left\{ \chi_{(a,b)} \mid a, b \in \mathbb{R} \right\},$$

the complex linear span of characteristic functions of intervals of the form (a, b).

Show that for every  $f \in L^1(\mathbb{R})$ , there exists a sequence of functions  $\{f_n\} \subset S$  such that

$$\lim_{n \to \infty} ||f_n - f||_1 = 0$$

Walk through

#### Solution:

#### Concepts Used:

• From homework: E is Lebesgue measurable iff there exists a finite union of closed cubes A such that  $m(E\Delta A) < \varepsilon$ .

It suffices to show that S is dense in simple functions, and since simple functions are *finite* linear combinations of characteristic functions, it suffices to show this for  $\chi_A$  for A a measurable set.

Let  $s = \chi_A$ . By regularity of the Lebesgue measure, choose an open set  $O \supseteq A$  such that  $m(O \setminus A) < \varepsilon$ .

O is an open subset of  $\mathbb{R}$ , and thus  $O = \coprod_{j \in \mathbb{N}} I_j$  is a disjoint union of countably many open intervals.

Now choose N large enough such that  $m(O\Delta I_{N,n}) < \varepsilon = \frac{1}{n}$  where we define  $I_{N,n} := \coprod_{j=1}^{N} I_j$ . Now define  $f_n = \chi_{I_{N,n}}$ , then

$$\|s - f_n\|_1 = \int |\chi_A - \chi_{I_{N,n}}| = m(A\Delta I_{N,n}) \xrightarrow{n \to \infty} 0.$$

Since any simple function is a finite linear combination of  $\chi_{A_i}$ , we can do this for each i to extend this result to all simple functions. But simple functions are dense in  $L^1$ , so S is dense in  $L^1$ .

### 7.4 Spring 2015 # 4

Define

$$f(x,y) := \begin{cases} \frac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0 \le x \le y\\ 0 & \text{otherwise} \end{cases}$$

Carefully show that  $f \in L^1(\mathbb{R}^2)$ .

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# 7.5 Fall 2014 # 3

Let  $f \in L^1(\mathbb{R})$ . Show that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that} \qquad m(E) < \delta \implies \int_E |f(x)| \, dx < \varepsilon$$

# 7.6 Spring 2014 # 1

- 1. Give an example of a continuous  $f \in L^1(\mathbb{R})$  such that  $f(x) \not\longrightarrow 0$  as  $|x| \longrightarrow \infty$ .
- 2. Show that if f is uniformly continuous, then

$$\lim_{|x| \to \infty} f(x) = 0.$$

### 8 Fubini-Tonelli

### 8.1 Spring 2020 # 4 💝

Let  $f,g\in L^1(\mathbb{R})$ . Argue that  $H(x,y)\coloneqq f(y)g(x-y)$  defines a function in  $L^1(\mathbb{R}^2)$  and deduce from this fact that

$$(f * g)(x) \coloneqq \int_{\mathbb{R}} f(y)g(x - y) \, dy$$

defines a function in  $L^1(\mathbb{R})$  that satisfies

$$\|f*g\|_1 \leq \|f\|_1 \|g\|_1.$$

#### **Solution:**

### Concepts Used:

- Tonelli: non-negative and measurable yields measurability of slices and equality of iterated integrals
- Fubini:  $f(x,y) \in L^1$  yields integrable slices and equality of iterated integrals
- F/T: apply Tonelli to |f|; if finite,  $f \in L^1$  and apply Fubini to f

$$\begin{split} \|H(x)\|_1 &= \int_{\mathbb{R}} \left| f(y)g(x-y) \, dy \right| dx \\ &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y)g(x-y) \, dy \right| dx \\ &\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)g(x-y)| \, dy \right) dx \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)g(x-y)| \, dx \right) dy \quad \text{by Tonelli} \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)g(t)| \, dt \right) dy \quad \text{setting } t = x - y, \, dt = -dx \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)| \cdot |g(t)| \, dt \right) dy \\ &= \int_{\mathbb{R}} |f(y)| \cdot \left( \int_{\mathbb{R}} |g(t)| \, dt \right) dy \\ &\coloneqq \int_{\mathbb{R}} |f(y)| \cdot \|g\|_1 \, dy \\ &= \|g\|_1 \int_{\mathbb{R}} |f(y)| \, dy \\ &\coloneqq \|g\|_1 \|f\|_1 \\ &< \infty \quad \text{by assumption} \quad . \end{split}$$

- H is measurable on  $\mathbb{R}^2$ :
  - If we can show  $\tilde{f}(x,y) := f(y)$  and  $\tilde{g}(x,y) := g(x-y)$  are both measurable on  $\mathbb{R}^2$ , then  $H = \tilde{f} \cdot \tilde{g}$  is a product of measurable functions and thus measurable.
  - $-f \in L^1$ , and  $L^1$  functions are measurable by definition.
  - The function  $(x,y) \mapsto g(x-y)$  is measurable on  $\mathbb{R}^2$ :
    - \* Let g be measurable on  $\mathbb{R}$ , then the cylinder function G(x,y)=g(x) on  $\mathbb{R}^2$  is always measurable
    - \* Define a linear transformation T := [1, -1; 0, 1] which sends  $(x, y) \longrightarrow (x y, y)$ , then  $T \in GL(2, \mathbb{R})$  is linear and thus measurable.
    - \* Then  $(G \circ T)(x,y) = G(x-y,y) = \tilde{g}(x-y)$ , so  $\tilde{g}$  is a composition of measurable functions and thus measurable.
- Apply **Tonelli** to |H|
  - -H measurable implies |H| is measurable
  - -|H| is non-negative
  - So the iterated integrals are equal in the extended sense
  - The calculation shows the iterated integral is finite, to  $\int |H|$  is finite and H is thus integrable on  $\mathbb{R}^2$ .

Note: Fubini is not needed, since we're not calculating the actual integral, just showing H is integrable.

# 8.2 Spring 2019 # 4 🦙

Let f be a non-negative function on  $\mathbb{R}^n$  and  $\mathcal{A} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : 0 \le t \le f(x)\}.$ 

Prove the validity of the following two statements:

- a. f is a Lebesgue measurable function on  $\mathbb{R}^n \iff \mathcal{A}$  is a Lebesgue measurable subset of  $\mathbb{R}^{n+1}$
- b. If f is a Lebesgue measurable function on  $\mathbb{R}^n$ , then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)dx = \int_0^\infty m\left(\left\{x \in \mathbb{R}^n : f(x) \ge t\right\}\right)dt$$

Add concepts.

#### Solution:

#### Concepts Used:

• See S&S p.82.

#### 8.2.1 a

 $\Longrightarrow$ 

- Suppose f is a measurable function.
- Note that  $A = \{f(x) t \ge 0\} \cap \{t \ge 0\}.$
- Define F(x,t) = f(x), G(x,t) = t, which are cylinders on measurable functions and thus measurable.
- Define H(x,y) = F(x,t) G(x,t), which are linear combinations of measurable functions and thus measurable.
- Then  $A = \{H \ge 0\} \bigcap \{G \ge 0\}$  as a countable intersection of measurable sets, which is again measurable.

 $\Leftarrow$ :

- Suppose  $\mathcal{A}$  is a measurable set.
- Then FT on  $\chi_{\mathcal{A}}$  implies that for almost every  $x \in \mathbb{R}^n$ , the x-slices  $\mathcal{A}_x$  are measurable and \$

$$\mathcal{A}_x := \left\{ t \in \mathbb{R} \mid (x, t) \in \mathcal{A} \right\} = [0, f(x)] \implies m(\mathcal{A}_x) = f(x) - 0 = f(x)$$

• But  $x \mapsto m(A_x)$  is a measurable function, and is exactly the function  $x \mapsto f(x)$ , so f is measurable.

#### 8.2.2 b

Note

$$\mathcal{A} = \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid 0 \le t \le f(x) \right\}$$
$$\mathcal{A}_t = \left\{ x \in \mathbb{R}^n \mid t \le f(x) \right\}.$$

• Then

$$\int_{\mathbb{R}^n} f(x) \ dx = \int_{\mathbb{R}^n} \int_0^{f(x)} 1 \ dt \ dx$$

$$= \int_{\mathbb{R}^n} \int_0^{\infty} \chi_{\mathcal{A}} \ dt \ dx$$

$$\stackrel{F.T.}{=} \int_0^{\infty} \int_{\mathbb{R}^n} \chi_{\mathcal{A}} \ dx \ dt$$

$$= \int_0^{\infty} m(\mathcal{A}_t) \ dt,$$

where we just use that  $\int \int \chi_{\mathcal{A}} = m(\mathcal{A})$ 

• By F.T., all of these integrals are equal. Why is FT justified

# 8.3 Fall 2018 # 5 🙀

Let  $f \geq 0$  be a measurable function on  $\mathbb{R}$ . Show that

$$\int_{\mathbb{R}} f = \int_0^\infty m(\{x : f(x) > t\}) dt$$

#### Solution:

### Concepts Used:

• Claim: If  $E \subseteq \mathbb{R}^a \times \mathbb{R}^b$  is a measurable set, then for almost every  $y \in \mathbb{R}^b$ , the slice  $E^{y}$  is measurable and

$$m(E) = \int_{\mathbb{R}^b} m(E^y) \, dy.$$

- Set  $g = \chi_E$ , which is non-negative and measurable, so apply Tonelli.
- Conclude that  $g^y = \chi_{E^y}$  is measurable, the function  $y \mapsto \int g^y(x) dx$  is measurable, and  $\int \int g^y(x) dx dy = \int g$ .
- But  $\int g = m(E)$  and  $\int \int g^y(x) dx dy = \int m(E^y) dy$ .

Note: f is a function  $\mathbb{R} \longrightarrow \mathbb{R}$  in the original problem, but here I've assumed  $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ .

• Since  $f \geq 0$ , set

$$E := \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) > t \right\} = \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid 0 \le t < f(x) \right\}.$$

- Claim: since f is measurable, E is measurable and thus m(E) makes sense.
  - Since f is measurable, F(x,t) := t f(x) is measurable on  $\mathbb{R}^n \times \mathbb{R}$ .
  - Then write  $E = \{F < 0\} \cap \{t \ge 0\}$  as an intersection of measurable sets.

• We have slices

$$E^{t} := \left\{ x \in \mathbb{R}^{n} \mid (x, t) \in E \right\} = \left\{ x \in \mathbb{R}^{n} \mid 0 \le t < f(x) \right\}$$
$$E^{x} := \left\{ t \in \mathbb{R} \mid (x, t) \in E \right\} = \left\{ t \in \mathbb{R} \mid 0 \le t \le f(x) \right\} = [0, f(x)].$$

- $E_t$  is precisely the set that appears in the original RHS integrand.
- $m(E^x) = f(x).$
- Claim:  $\chi_E$  satisfies the conditions of Tonelli, and thus  $m(E) = \int \chi_E$  is equal to any iterated integral.
  - Non-negative: clear since  $0 \le \chi_E \le 1$
  - Measurable: characteristic functions of measurable sets are measurable.
- Conclude
  - 1. For almost every x,  $E^x$  is a measurable set,  $x \mapsto m(E^x)$  is a measurable function, and  $m(E) = \int_{\mathbb{R}^n} m(E^x) dx$
  - 2. For almost every t,  $E^t$  is a measurable set,  $t \mapsto m(E^t)$  is a measurable function, and  $m(E) = \int_{\mathbb{R}} m(E^t) dt$
- On one hand,

$$\begin{split} m(E) &= \int_{\mathbb{R}^{n+1}} \chi_E(x,t) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_E(x,t) \, dt \, dx \quad \text{by Tonelli} \\ &= \int_{\mathbb{R}^n} m(E^x) \, dx \quad \text{first conclusion} \\ &= \int_{\mathbb{R}^n} f(x) \, dx. \end{split}$$

• On the other hand,

$$m(E) = \int_{\mathbb{R}^{n+1}} \chi_E(x, t)$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_E(x, t) dx dt \text{ by Tonelli}$$

$$= \int_{\mathbb{R}} m(E^t) dt \text{ second conclusion.}$$

• Thus

$$\int_{\mathbb{R}^n} f \, dx = m(E) = \int_{\mathbb{R}} m(E^t) \, dt = \int_{\mathbb{R}} m\Big(\Big\{x \mid f(x) > t\Big\}\Big).$$

# 8.4 Fall 2015 # 5

Let  $f, g \in L^1(\mathbb{R})$  be Borel measurable.

1. Show that

• The function

$$F(x,y) \coloneqq f(x-y)g(y)$$

is Borel measurable on  $\mathbb{R}^2$ , and

• For almost every  $y \in \mathbb{R}$ ,

$$F_{y}(x) := f(x - y)g(y)$$

is integrable with respect to y.

2. Show that  $f * g \in L^1(\mathbb{R})$  and

$$||f * g||_1 \le ||f||_1 ||g||_1$$

### 8.5 Spring 2014 # 5

Let  $f, g \in L^1([0,1])$  and for all  $x \in [0,1]$  define

$$F(x) := \int_0^x f(y) \, dy$$
 and  $G(x) := \int_0^x g(y) \, dy$ .

Prove that

$$\int_0^1 F(x)g(x) \, dx = F(1)G(1) - \int_0^1 f(x)G(x) \, dx$$

# **9** $L^2$ and Fourier Analysis

# 9.1 Spring 2020 # 6 💝

#### 9.1.1 a

Show that

$$L^2([0,1]) \subseteq L^1([0,1])$$
 and  $\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$ .

#### 9.1.2 b

For  $f \in L^1([0,1])$  define

$$\widehat{f}(n) := \int_0^1 f(x)e^{-2\pi i nx} dx.$$

Prove that if  $f \in L^1([0,1])$  and  $\{\widehat{f}(n)\} \in \ell^1(\mathbb{Z})$  then

$$S_N f(x) := \sum_{|n| < N} \widehat{f}(n) e^{2\pi i n x}.$$

converges uniformly on [0,1] to a continuous function g such that g=f almost everywhere.

Hint: One approach is to argue that if  $f \in L^1([0,1])$  with  $\{\widehat{f}(n)\} \in \ell^1(\mathbb{Z})$  then  $f \in L^2([0,1])$ .

#### Solution:

### Concepts Used:

- For  $e_n(x) := e^{2\pi i n x}$ , the set  $\{e_n\}$  is an orthonormal basis for  $L^2([0,1])$ . For any orthonormal sequence in a Hilbert space, we have Bessel's inequality:

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \le ||x||^2.$$

- When  $\{e_n\}$  is a basis, the above is an equality (Parseval)
- Arguing uniform convergence: since  $\{\widehat{f}(n)\}\in\ell^1(\mathbb{Z})$ , we should be able to apply the M test.

#### 9.1.3 a

Claim:  $\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$ .

- Set  $\mathbf{c} = \left\{ c_k \mid k \in \mathbb{Z} \right\} \in \ell^1(\mathbb{Z}).$
- It suffices to show that if  $\sum_{k\in\mathbb{Z}}|c_k|<\infty$  then  $\sum_{k\in\mathbb{Z}}|c_k|^2<\infty$ .
- Let  $S = \{c_k \mid |c_k| \le 1\}$ , then  $c_k \in S \implies |c_k|^2 \le |c_k|$
- Claim:  $S^c$  can only contain finitely many elements, all of which are finite.
  - If not, either  $S^c := \{c_j\}_{j=1}^{\infty}$  is infinite with every  $|c_j| > 1$ , which forces

$$\sum_{c_k \in S^c} |c_k| = \sum_{j=1}^{\infty} |c_j| > \sum_{j=1}^{\infty} 1 = \infty.$$

- If any  $c_j = \infty$ , then  $\sum_{k \in \mathbb{Z}} |c_k| \ge c_j = \infty$ .
- So  $S^c$  is a finite set of finite integers, let  $N = \max \left\{ |c_j|^2 \mid c_j \in S^c \right\} < \infty$ .
- Rewrite the sum

$$\sum_{k \in \mathbb{Z}} |c_k|^2 = \sum_{c_k \in S} |c_k|^2 + \sum_{c_k \in S^c} |c_k|^2$$

$$\leq \sum_{c_k \in S} |c_k| + \sum_{c_k \in S^c} |c_k|^2$$

$$\leq \sum_{k \in \mathbb{Z}} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \quad \text{since the } |c_k| \text{ are all positive}$$

$$= \|\mathbf{c}\|_{\ell^1} + \sum_{c_k \in S^c} |c_k|^2$$

$$\leq \|\mathbf{c}\|_{\ell^1} + |S^c| \cdot N$$

$$\leq \infty$$

Claim:  $L^2([0,1]) \subseteq L^1([0,1])$ .

• It suffices to show that  $\int |f|^2 < \infty \implies \int |f| < \infty$ .

- Define  $S = \left\{ x \in [0,1] \mid |f(x)| \le 1 \right\}$ , then  $x \in S^c \implies |f(x)|^2 \ge |f(x)|$ .
- Break up the integral:

$$\begin{split} \int_{\mathbb{R}} |f| &= \int_{S} |f| + \int_{S^{c}} |f| \\ &\leq \int_{S} |f| + \int_{S^{c}} |f|^{2} \\ &\leq \int_{S} |f| + \|f\|_{2} \\ &\leq \sup_{x \in S} \{|f(x)|\} \cdot \mu(S) + \|f\|_{2} \\ &= 1 \cdot \mu(S) + \|f\|_{2} \quad \text{by definition of } S \\ &\leq 1 \cdot \mu([0,1]) + \|f\|_{2} \quad \text{since } S \subseteq [0,1] \\ &= 1 + \|f\|_{2} \\ &< \infty. \end{split}$$

Note: this proof shows  $L^2(X) \subseteq L^1(X)$  whenever  $\mu(X) < \infty$ .

# 9.2 Fall 2017 # 5

Let  $\varphi$  be a compactly supported smooth function that vanishes outside of an interval [-N,N] such that  $\int_{\mathbb{D}} \varphi(x) dx = 1$ .

For  $f \in L^1(\mathbb{R})$ , define

$$K_j(x) \coloneqq j\varphi(jx), \qquad f * K_j(x) \coloneqq \int_{\mathbb{R}} f(x-y)K_j(y) \, dy$$

and prove the following:

1. Each  $f * K_j$  is smooth and compactly supported.

2.

$$\lim_{j \to \infty} \|f * K_j - f\|_1 = 0$$

Hint:

$$\lim_{y \to 0} \int_{\mathbb{R}} |f(x - y) - f(x)| dy = 0$$

Add concepts.

Solution:

Concepts Used:

• ?

#### 9.2.1 a

**Lemma:** If  $\varphi \in C_c^1$ , then  $(f * \varphi)' = f * \varphi'$  almost everywhere. Silly Proof:

$$\mathcal{F}((f * \varphi)') = 2\pi i \xi \ \mathcal{F}(f * \varphi)$$

$$= 2\pi i \xi \ \mathcal{F}(f) \ \mathcal{F}(\varphi)$$

$$= \mathcal{F}(f) \cdot (2\pi i \xi \ \mathcal{F}(\varphi))$$

$$= \mathcal{F}(f) \cdot \mathcal{F}(\varphi')$$

$$= \mathcal{F}(f * \varphi').$$

Actual proof:

$$(f * \varphi)'(x) = (\varphi * f)'(x)$$

$$= \lim_{h \to 0} \frac{(\varphi * f)'(x+h) - (\varphi * f)'(x)}{h}$$

$$= \lim_{h \to 0} \int \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y)$$

$$\stackrel{DCT}{=} \int \lim_{h \to 0} \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y)$$

$$= \int \varphi'(x-y) f(y)$$

$$= (\varphi' * f)(x)$$

$$= (f * \varphi')(x).$$

To see that the DCT is justified, we can apply the MVT on the interval [0, h] to f to obtain

$$\frac{\varphi(x+h-y)-\varphi(x-y)}{h}=\varphi'(c)\quad c\in[0,h],$$

and since  $\varphi'$  is continuous and compactly supported,  $\varphi'$  is bounded by some  $M < \infty$  by the extreme value theorem and thus

$$\int \left| \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \right| = \int \left| \varphi'(c) f(y) \right|$$

$$\leq \int |M| |f|$$

$$= |M| \int |f| < \infty,$$

since  $f \in L^1$  by assumption, so we can take g := |M||f| as the dominating function.

Applying this theorem infinitely many times shows that  $f * \varphi$  is smooth.

To see that  $f * \varphi$  is compactly supported, approximate f by a continuous compactly supported function h, so  $||h - f||_1 \xrightarrow{L^1} 0$ . Now let  $g_x(y) = \varphi(x - y)$ , and note that  $\operatorname{supp}(g) = x - \operatorname{supp}(\varphi)$  which is still compact.

But since supp(h) is bounded, there is some N such that

$$|x| > N \implies A_x := \operatorname{supp}(h) \bigcap \operatorname{supp}(g_x) = \emptyset$$

and thus

$$(h * \varphi)(x) = \int_{\mathbb{R}} \varphi(x - y)h(y) \ dy$$
$$= \int_{A_x} g_x(y)h(y)$$
$$= 0.$$

so  $\{x \mid f * g(x) = 0\}$  is open, and its complement is closed and bounded and thus compact.

#### 9.2.2 b

$$||f * K_{j} - f||_{1} = \int \left| \int f(x - y) K_{j}(y) \, dy - f(x) \right| \, dx$$

$$= \int \left| \int f(x - y) K_{j}(y) \, dy - \int f(x) K_{j}(y) \, dy \right| \, dx$$

$$= \int \left| \int (f(x - y) - f(x)) K_{j}(y) \, dy \right| \, dx$$

$$\leq \int \int |(f(x - y) - f(x))| \cdot |K_{j}(y)| \, dy \, dx$$

$$\stackrel{FT}{=} \int \int |(f(x - y) - f(x))| \cdot |K_{j}(y)| \, dx \, dy$$

$$= \int |K_{j}(y)| \left( \int |(f(x - y) - f(x))| \, dx \right) \, dy$$

$$= \int |K_{j}(y)| \cdot ||f - \tau_{y} f||_{1} \, dy.$$

We now split the integral up into pieces.

- 1. Chose  $\delta$  small enough such that  $|y| < \delta \implies ||f \tau_y f||_1 < \varepsilon$  by continuity of translation in  $L^1$ , and
- 2. Since  $\varphi$  is compactly supported, choose J large enough such that

$$j > J \implies \int_{|y| > \delta} |K_j(y)| \ dy = \int_{|y| > \delta} |j\varphi(jy)| = 0$$

Then

$$||f * K_{j} - f||_{1} \leq \int |K_{j}(y)| \cdot ||f - \tau_{y}f||_{1} dy$$

$$= \int_{|y| < \delta} |K_{j}(y)| \cdot ||f - \tau_{y}f||_{1} dy + \int_{|y| \ge \delta} |K_{j}(y)| \cdot ||f - \tau_{y}f||_{1} dy$$

$$= \varepsilon \int_{|y| \ge \delta} |K_{j}(y)| + 0$$

$$\leq \varepsilon(1) \longrightarrow 0.$$

# 9.3 Spring 2017 # 5

Let  $f, g \in L^2(\mathbb{R})$ . Prove that the formula

$$h(x) := \int_{-\infty}^{\infty} f(t)g(x-t) dt$$

defines a uniformly continuous function h on  $\mathbb{R}$ .

# 9.4 Spring 2015 # 6

Let  $f \in L^1(\mathbb{R})$  and g be a bounded measurable function on  $\mathbb{R}$ .

- 1. Show that the convolution f \* g is well-defined, bounded, and uniformly continuous on  $\mathbb{R}$ .
- 2. Prove that one further assumes that  $g \in C^1(\mathbb{R})$  with bounded derivative, then  $f * g \in C^1(\mathbb{R})$  and

$$\frac{d}{dx}(f*g) = f*\left(\frac{d}{dx}g\right)$$

# 9.5 Fall 2014 # 5

1. Let  $f \in C_c^0(\mathbb{R}^n)$ , and show

$$\lim_{t \to 0} \int_{\mathbb{R}^n} |f(x+t) - f(x)| \, dx = 0.$$

2. Extend the above result to  $f \in L^1(\mathbb{R}^n)$  and show that

 $f \in L^1(\mathbb{R}^n), \quad g \in L^\infty(\mathbb{R}^n) \implies f * g$  is bounded and uniformly continuous.

# 10 Functional Analysis: General

### 10.1 Fall 2019 # 4 🐈

Let  $\{u_n\}_{n=1}^{\infty}$  be an orthonormal sequence in a Hilbert space  $\mathcal{H}$ .

#### 10.1.1 a

Prove that for every  $x \in \mathcal{H}$  one has

$$\sum_{n=1}^{\infty} \left| \langle x, u_n \rangle \right|^2 \le \|x\|^2$$

#### 10.1.2 b

Prove that for any sequence  $\{a_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$  there exists an element  $x \in \mathcal{H}$  such that

$$a_n = \langle x, u_n \rangle$$
 for all  $n \in \mathbb{N}$ 

and

$$||x||^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$$

### Solution:

### Concepts Used:

- Bessel's Inequality
- Pythagoras
- Surjectivity of the Riesz map
- Parseval's Identity
- Trick remember to write out finite sum  $S_N$ , and consider  $||x S_N||$ .

#### 10.1.3 a

Claim:

$$0 \le \left\| x - \sum_{n=1}^{N} \langle x, u_n \rangle u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n \rangle|^2$$
$$\implies \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \le \|x\|^2.$$

*Proof:* Let 
$$S_N = \sum_{n=1}^N \langle x, u_n \rangle u_n$$
. Then

$$0 \le \|x - S_N\|^2$$

$$= \langle x - S_n, x - S_N \rangle$$

$$= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2$$

$$\xrightarrow{N \to \infty} \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2.$$

#### 10.1.4 b

- 1. Fix  $\{a_n\} \in \ell^2$ , then note that  $\sum |a_n|^2 < \infty \implies$  the tails vanish.
- 2. Define

$$x := \lim_{N \to \infty} S_N = \lim_{N \to \infty} \sum_{k=1}^N a_k u_k$$

- 3.  $\{S_N\}$  Cauchy (by 1) and H complete  $\implies x \in H$ .
- 4.

$$\langle x, u_n \rangle = \left\langle \sum_k a_k u_k, u_n \right\rangle = \sum_k a_k \langle u_k, u_n \rangle = a_n \quad \forall n \in \mathbb{N}$$

since the  $u_k$  are all orthogonal.

5.

$$||x||^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k ||a_k u_k||^2 = \sum_k |a_k|^2$$

by Pythagoras since the  $u_k$  are normal.

Bonus: We didn't use completeness here, so the Fourier series may not actually converge to x. If  $\{u_n\}$  is **complete** (so  $x=0\iff \langle x,\ u_n\rangle=0\ \forall n$ ) then the Fourier series does converge to x and  $\sum_{n=1}^{\infty}|\langle x,u_n\rangle|^2=\|x\|^2$  for all  $x\in H$ .

### 10.2 Spring 2019 # 5 🦙

#### 10.2.1 a

Show that  $L^2([0,1]) \subseteq L^1([0,1])$  and argue that  $L^2([0,1])$  in fact forms a dense subset of  $L^1([0,1])$ .

#### 10.2.2 b

Let  $\Lambda$  be a continuous linear functional on  $L^1([0,1])$ .

Prove the Riesz Representation Theorem for  $L^1([0,1])$  by following the steps below:

i. Establish the existence of a function  $g \in L^2([0,1])$  which represents  $\Lambda$  in the sense that

$$\Lambda(f) = f(x)g(x)dx \text{ for all } f \in L^2([0,1]).$$

Hint: You may use, without proof, the Riesz Representation Theorem for  $L^2([0,1])$ .

ii. Argue that the g obtained above must in fact belong to  $L^{\infty}([0,1])$  and represent  $\Lambda$  in the sense that

$$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}dx \quad \text{ for all } f \in L^1([0,1])$$

with

$$||g||_{L^{\infty}([0,1])} = ||\Lambda||_{L^{1}([0,1])}$$

Solution:

### Concepts Used:

- Holders' inequality:  $\|fg\|_1 \leq \|f\|_p \|f\|_q$  Riesz Representation for  $L^2$ : If  $\Lambda \in (L^2)^\vee$  then there exists a unique  $g \in L^2$  such that  $\Lambda(f) = \int fg$
- $||f||_{L^{\infty}(X)} := \inf \{ t \ge 0 \mid |f(x)| \le t \text{ almost everywhere } \}.$
- Lemma:  $m(X) < \infty \implies L^p(X) \subset L^2(X)$ .

Proof.

– Write Holder's inequality as  $||fg||_1 \le ||f||_a ||g||_b$  where  $\frac{1}{a} + \frac{1}{b} = 1$ , then

$$||f||_p^p = |||f|^p||_1 \le |||f|^p||_a ||1||_b.$$

- Now take  $a = \frac{2}{n}$  and this reduces to

$$\begin{split} \|f\|_p^p &\leq \|f\|_2^p \ m(X)^{\frac{1}{b}} \\ \Longrightarrow \ \|f\|_p &\leq \|f\|_2 \cdot O(m(X)) < \infty. \end{split}$$

#### 10.2.3 a

- Note  $X = [0,1] \implies m(X) = 1$ .
- By Holder's inequality with p = q = 2,

$$\|f\|_1 = \|f \cdot 1\|_1 \leq \|f\|_2 \cdot \|1\|_2 = \|f\|_2 \cdot m(X)^{\frac{1}{2}} = \|f\|_2,$$

- Thus  $L^2(X) \subseteq L^1(X)$
- Since they share a common dense subset (simple functions)  $L^2$  is dense in  $L^1$

#### 10.2.4 b

Let  $\Lambda \in L^1(X)^{\vee}$  be arbitrary.

1: Existence of g Representing  $\Lambda$ . Let  $f \in L^2 \subseteq L^1$  be arbitrary.

Claim:  $\Lambda \in L^1(X)^{\vee} \implies \Lambda \in L^2(X)^{\vee}$ .

- Suffices to show that  $\|\Gamma\|_{L^2(X)^\vee} := \sup_{\|f\|_2=1} |\Gamma(f)| < \infty$ , since bounded implies continuous.
- By the lemma,  $||f||_1 \le C||f||_2$  for some constant  $C \approx m(X)$ .
- Note

$$\|\Lambda\|_{L^1(X)^\vee}\coloneqq \sup_{\|f\|_1=1} |\Lambda(f)|$$

• Define 
$$\hat{f} = \frac{f}{\|f\|_1}$$
 so  $\|\hat{f}\|_1 = 1$ 

• Since  $\|\Lambda\|_{1^{\vee}}$  is a supremum over all  $f \in L^1(X)$  with  $\|f\|_1 = 1$ ,

$$\left|\Lambda(\widehat{f})\right| \le \|\Lambda\|_{(L^1(X))^\vee},$$

• Then

$$\begin{split} \frac{|\Lambda(f)|}{\|f\|_1} &= \left|\Lambda(\widehat{f})\right| \leq \|\Lambda\|_{L^1(X)^\vee} \\ \Longrightarrow & |\Lambda(f)| \leq \|\Lambda\|_{1^\vee} \cdot \|f\|_1 \\ &\leq \|\Lambda\|_{1^\vee} \cdot C\|f\|_2 < \infty \quad \text{by assumption,} \end{split}$$

• So  $\Lambda \in (L^2)^{\vee}$ .

Now apply Riesz Representation for  $L^2$ : there is a  $g \in L^2$  such that

$$f \in L^2 \implies \Lambda(f) = \langle f, g \rangle := \int_0^1 f(x) \overline{g(x)} \, dx.$$

**2:** g is in  $L^{\infty}$ 

• It suffices to show  $||g||_{L^{\infty}(X)} < \infty$ .

• Since we're assuming  $\|\Gamma\|_{L^1(X)^{\vee}} < \infty$ , it suffices to show the stated equality.

Is this assumed..? Or did we show it..?

• Claim:  $\|\Lambda\|_{L^1(X)^{\vee}} = \|g\|_{L^{\infty}(X)}$ 

- The result will follow since  $\Lambda$  was assumed to be in  $L^1(X)^\vee$ , so  $\|\Lambda\|_{L^1(X)^\vee} < \infty$ .

- ≤:

$$\begin{split} \|\Lambda\|_{L^1(X)^\vee} &= \sup_{\|f\|_1 = 1} |\Lambda(f)| \\ &= \sup_{\|f\|_1 = 1} \left| \int_X f\bar{g} \right| \quad \text{by (i)} \\ &= \sup_{\|f\|_1 = 1} \int_X |f\bar{g}| \\ &\coloneqq \sup_{\|f\|_1 = 1} \|fg\|_1 \\ &\leq \sup_{\|f\|_1 = 1} \|f\|_1 \|g\|_\infty \quad \text{by Holder with } p = 1, q = \infty \\ &= \|g\|_\infty, \end{split}$$

- >:

\* Suppose toward a contradiction that  $\|g\|_{\infty} > \|\Lambda\|_{1^{\vee}}.$ 

\* Then there exists some  $E \subseteq X$  with m(E) > 0 such that

$$x \in E \implies |g(x)| > \|\Lambda\|_{L^1(X)^\vee}.$$

\* Define

$$h = \frac{1}{m(E)} \frac{\overline{g}}{|g|} \chi_E.$$

- \* Note  $||h||_{L^1(X)} = 1$ .
- \* Then

$$\begin{split} \Lambda(h) &= \int_X hg \\ &\coloneqq \int_X \frac{1}{m(E)} \frac{g\overline{g}}{|g|} \chi_E \\ &= \frac{1}{m(E)} \int_E |g| \\ &\ge \frac{1}{m(E)} \|g\|_\infty m(E) \\ &= \|g\|_\infty \\ &> \|\Lambda\|_{L^1(X)^\vee}, \end{split}$$

a contradiction since  $\|\Lambda\|_{L^1(X)^{\vee}}$  is the supremum over all  $h_{\alpha}$  with  $\|h_{\alpha}\|_{L^1(X)} = 1$ .

# 10.3 Spring 2016 # 6

Without using the Riesz Representation Theorem, compute

$$\sup \left\{ \left| \int_0^1 f(x)e^x dx \right| \mid f \in L^2([0,1], m), \|f\|_2 \le 1 \right\}$$

# 10.4 Spring 2015 # 5

Let  $\mathcal{H}$  be a Hilbert space.

1. Let  $x \in \mathcal{H}$  and  $\{u_n\}_{n=1}^N$  be an orthonormal set. Prove that the best approximation to x in  $\mathcal{H}$  by an element in  $\operatorname{span}_{\mathbb{C}} \{u_n\}$  is given by

$$\widehat{x} := \sum_{n=1}^{N} \langle x, u_n \rangle u_n.$$

2. Conclude that finite dimensional subspaces of  $\mathcal{H}$  are always closed.

### 10.5 Fall 2015 # 6

Let  $f:[0,1] \longrightarrow \mathbb{R}$  be continuous. Show that

$$\sup \left\{ \|fg\|_1 \mid g \in L^1[0,1], \|g\|_1 \le 1 \right\} = \|f\|_{\infty}$$

# 10.6 Fall 2014 # 6

Let  $1 \leq p, q \leq \infty$  be conjugate exponents, and show that

$$f \in L^p(\mathbb{R}^n) \implies ||f||_p = \sup_{\|g\|_q = 1} \left| \int f(x)g(x)dx \right|$$

### 11 Functional Analysis: Banach Spaces

### 11.1 Spring 2019 # 1 🙀

Let C([0,1]) denote the space of all continuous real-valued functions on [0,1].

- a. Prove that C([0,1]) is complete under the uniform norm  $||f||_u := \sup_{x \in [0,1]} |f(x)|$ .
- b. Prove that C([0,1]) is not complete under the  $L^1$ -norm  $||f||_1 = \int_0^1 |f(x)| dx$ .

Add concepts

#### Solution:

Concepts Used:

• ?

#### 11.1.1 a

- Let  $\{f_n\}$  be a Cauchy sequence in  $C(I, \|\cdot\|_{\infty})$ , so  $\lim_{n} \lim_{m} \|f_m f_n\|_{\infty} = 0$ , we will show it converges to some f in this space.
- For each fixed  $x_0 \in [0,1]$ , the sequence of real numbers  $\{f_n(x_0)\}$  is Cauchy in  $\mathbb{R}$  since

$$x_0 \in I \implies |f_m(x_0) - f_n(x_0)| \le \sup_{x \in I} |f_m(x) - f_n(x)| := ||f_m - f_n||_{\infty} \xrightarrow{m > n \longrightarrow \infty} 0,$$

- Since  $\mathbb{R}$  is complete, this sequence converges and we can define  $f(x) := \lim_{n \to \infty} f_n(x)$ .
- Thus  $f_n \longrightarrow f$  pointwise by construction Claim:  $||f f_n|| \stackrel{n \longrightarrow \infty}{\longrightarrow} 0$ , so  $f_n$  converges to f in  $C([0, 1], ||\cdot||_{\infty})$ .
  - - \* Fix  $\varepsilon > 0$ ; we will show there exists an N such that  $n \geq N \implies ||f_n f|| < \varepsilon$
    - \* Fix an  $x_0 \in I$ . Since  $f_n \longrightarrow f$  pointwise, choose  $N_1$  large enough so that

$$n \ge N_1 \implies |f_n(x_0) - f(x_0)| < \varepsilon/2.$$

\* Since  $||f_n - f_m||_{\infty} \longrightarrow 0$ , choose and  $N_2$  large enough so that

$$n, m \geq N_2 \implies ||f_n - f_m||_{\infty} < \varepsilon/2.$$

\* Then for  $n, m \ge \max(N_1, N_2)$ , we have

$$|f_n(x_0) - f(x_0)| = |f_n(x_0) - f(x_0) + f_m(x_0) - f_m(x_0)|$$

$$= |f_n(x_0) - f_m(x_0) + f_m(x_0) - f(x_0)|$$

$$\leq |f_n(x_0) - f_m(x_0)| + |f_m(x_0) - f(x_0)|$$

$$< |f_n(x_0) - f_m(x_0)| + \frac{\varepsilon}{2}$$

$$\leq \sup_{x \in I} |f_n(x) - f_m(x)| + \frac{\varepsilon}{2}$$

$$< ||f_n - f_m||_{\infty} + \frac{\varepsilon}{2}$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\implies |f_n(x_0) - f(x_0)| < \varepsilon$$

$$\implies \sup_{x \in I} |f_n(x_0) - f(x_0)| \leq \sup_{x \in I} \varepsilon \text{ by order limit laws}$$

$$\implies ||f_n - f|| \leq \varepsilon$$

• f is the uniform limit of continuous functions and thus continuous, so  $f \in C([0,1])$ .

#### 11.1.2 b

- It suffices to produce a Cauchy sequence that does not converge to a continuous function.
- Take the following sequence of functions:
  - $f_1$  increases linearly from 0 to 1 on [0, 1/2] and is 1 on [1/2, 1]
  - $-f_2$  is 0 on [0,1/4] increases linearly from 0 to 1 on [1/4,1/2] and is 1 on [1/2,1]
  - $-f_3$  is 0 on [0,3/8] increases linearly from 0 to 1 on [3/8,1/2] and is 1 on [1/2,1]
  - $-f_3$  is 0 on [0, (1/2 3/8)/2] increases linearly from 0 to 1 on [(1/2 3/8)/2, 1/2] and is 1 on [1/2, 1]

Idea: take sequence starting points for the triangles:  $0, 0 + \frac{1}{4}, 0 + \frac{1}{4} + \frac{1}{8}, \cdots$  which converges to 1/2 since  $\sum_{k=1}^{\infty} \frac{1}{2^k} = -\frac{1}{2} + \sum_{k=0}^{\infty} \frac{1}{2^k}$ .

- Then each  $f_n$  is clearly integrable, since its graph is contained in the unit square.
- $\{f_n\}$  is Cauchy: geometrically subtracting areas yields a single triangle whose area tends to 0.
- But  $f_n$  converges to  $\chi_{\left[\frac{1}{2},1\right]}$  which is discontinuous.

show that  $\int_0^1 |f_n(x) - f_m(x)| dx \longrightarrow 0$  rigorously, show that no  $g \in L^1([0,1])$  can converge to this indicator function.

# 11.2 Spring 2017 # 6 🙀

Show that the space  $C^1([a,b])$  is a Banach space when equipped with the norm

$$||f|| := \sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |f'(x)|.$$

#### Add concepts

### **Solution:**

### Concepts Used:

- See https://math.stackexchange.com/questions/507263/ prove-that-c1a-b-with-the-c1-norm-is-a-banach-space/
- Denote this norm  $\|\cdot\|_u$
- Let  $f_n$  be a Cauchy sequence in this space, so  $||f_n||_u < \infty$  for every n and  $||f_j - f_k||_u \xrightarrow{j,k} \infty 0.$

and define a candidate limit: for each  $x \in I$ , set

$$f(x) := \lim_{n \to \infty} f_n(x).$$

• Note that

$$||f_n||_{\infty} \le ||f_n||_u < \infty$$
$$||f_n'||_{\infty} \le ||f_n||_u < \infty.$$

- Thus both  $f_n, f'_n$  are Cauchy sequences in  $C^0([a, b], \|\cdot\|_{\infty})$ , which is a Banach space, so they converge.
- So
  - $-f_n \longrightarrow f$  uniformly (by uniqueness of limits),
  - $-f'_n \longrightarrow g$  uniformly for some g, and  $-f, g \in C^0([a, b]).$
- Claim: g = f'
  - For any fixed  $a \in I$ , we have

$$f_n(x) - f_n(a) \xrightarrow{u} f(x) - f(a)$$
  
$$\int_a^x f'_n \xrightarrow{u} \int_a^x g.$$

- By the FTC, the left-hand sides are equal.
- By uniqueness of limits so are the right-hand sides, so f' = g.
- Claim: the limit f is an element in this space.
  - Since  $f, f' \in C^0([a, b])$ , they are bounded, and so  $||f||_u < \infty$ .
- Claim:  $||f_n f||_u \xrightarrow{n \to \infty} 0$
- Thus the Cauchy sequence  $\{f_n\}$  converges to a function f in the u-norm where f is an element of this space, making it complete.

# 11.3 Fall 2017 # 6 처

Let X be a complete metric space and define a norm

$$||f|| := \max\{|f(x)| : x \in X\}.$$

Show that  $(C^0(\mathbb{R}), \|\cdot\|)$  (the space of continuous functions  $f: X \longrightarrow \mathbb{R}$ ) is complete.

Add concepts.

Shouldn't this be a supremum? The max may not exist

Review and clean up.

#### Solution:

#### Concepts Used:

• ?

Let  $\{f_k\}$  be a Cauchy sequence, so  $||f_k|| < \infty$  for all k. Then for a fixed x, the sequence  $f_k(x)$  is Cauchy in  $\mathbb{R}$  and thus converges to some f(x), so define f by  $f(x) := \lim_{k \to \infty} f_k(x)$ .

Then  $||f_k - f|| = \max_{x \in X} |f_k(x) - f(x)| \stackrel{k \to \infty}{\longrightarrow} 0$ , and thus  $f_k \to f$  uniformly and thus f is continuous. It just remains to show that f has bounded norm.

Choose N large enough so that  $||f - f_N|| < \varepsilon$ , and write  $||f_N|| := M < \infty$ 

$$||f|| \le ||f - f_N|| + ||f_N|| < \varepsilon + M < \infty.$$