Real Analysis Qualifying Exam Notes

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Contents

1 Practice Exam 2 (November 2014)

1.1 1: Fubini-Tonelli

1.1.1 a

Carefully state Tonelli's theorem for a nonnegative function $F(x,t)$ on $\mathbb{R}^n \times \mathbb{R}$.

1.1.2 b

Let $f : \mathbb{R}^n \longrightarrow [0,\infty]$ and define

$$\mathcal{A} \coloneqq \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; 0 \leq t \leq f(x) \right\}.$$

Prove the validity of the following two statements:

1. f is Lebesgue measurable on $\mathbb{R}^n \iff \mathcal{A}$ is a Lebesgue measurable subset of $\mathbb{R}^{n+1}$.
2. If f is Lebesgue measurable on $\mathbb{R}^n$ then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)dx = \int_0^\infty m\left(\left\{ x \in \mathbb{R}^n \;\middle|\; f(x) \geq t \right\} \right) dt.$$

1.2 2: Convolutions and the Fourier Transform

1.2.1 a

Let $f, g \in L^1(\mathbb{R}^n)$ and give a definition of $f * g$.

1.2.2 b

Prove that if f, g are integrable and bounded, then

$$(f*g)(x) \overset{|x| \longrightarrow \infty}{\longrightarrow} 0.$$

1.2.3 c

1. Define the *Fourier transform* of an integrable function f on $\mathbb{R}^n$.
2. Give an outline of the proof of the Fourier inversion formula.
3. Give an example of a function $f \in L^1(\mathbb{R}^n)$ such that $\widehat{f}$ is not in $L^1(\mathbb{R}^n)$.

1.3 3: Hilbert Spaces

Let $\{u_n\}_{n=1}^\infty$ be an orthonormal sequence in a Hilbert space H.

1.3.1 a

Let $x \in H$ and verify that

$$\left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|_H^2 = \|x\|_H^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2.$$

for any $N \in \mathbb{N}$ and deduce that

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|_H^2.$$

1.3.2 b

Let $\{a_n\}_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$ and prove that there exists an $x \in H$ such that $a_n = \langle x,\ u_n \rangle$ for all $n \in \mathbb{N}$, and moreover x may be chosen such that

$$\|x\|_H = \left(\sum_{n \in \mathbb{N}} |a_n|^2 \right)^{\frac{1}{2}}.$$

Proof .

- Take $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
- Define $x := \lim_{N \longrightarrow \infty} S_N$ where $S_N = \sum_{k=1}^{N} a_k u_k$
- $\{S_N\}$ is Cauchy and H is complete, so $x \in H$.
- By construction,

  $$\langle x,\ u_n \rangle = \left\langle \sum_k a_k u_k,\ u_n \right\rangle = \sum_k a_k \langle u_k,\ u_n \rangle = a_n$$

  since the u_k are all orthogonal.
- By Pythagoras since the u_k are normal,

  $$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2.$$

$\blacksquare$

1.3.3 c

Prove that if $\{u_n\}$ is *complete*, Bessel's inequality becomes an equality.

Proof .
Let x and u_n be arbitrary.

$$\begin{aligned}
\left\langle x - \sum_{k=1}^{\infty} \langle x,\ u_k\rangle u_k,\ u_n \right\rangle &= \langle x,\ u_n\rangle - \left\langle \sum_{k=1}^{\infty} \langle x,\ u_k\rangle u_k,\ u_n \right\rangle \\
&= \langle x,\ u_n\rangle - \sum_{k=1}^{\infty} \langle\langle x,\ u_k\rangle u_k,\ u_n\rangle \\
&= \langle x,\ u_n\rangle - \sum_{k=1}^{\infty} \langle x,\ u_k\rangle\langle u_k,\ u_n\rangle \\
&= \langle x,\ u_n\rangle - \langle x,\ u_n\rangle = 0 \\
\implies x - \sum_{k=1}^{\infty} \langle x,\ u_k\rangle u_k = 0 \quad \text{by completeness.}
\end{aligned}$$

So

$$x = \sum_{k=1}^{\infty} \langle x,\ u_k\rangle u_k \implies \|x\|^2 = \sum_{k=1}^{\infty} |\langle x,\ u_k\rangle|^2.\blacksquare.$$

$\blacksquare$

1.4 4: Lp Spaces

1.4.1 a

Prove Holder's inequality: let $f \in L^p, g \in L^q$ with p, q conjugate, and show that

$$\|fg\|_p \leq \|f\|_p \cdot \|g\|_q.$$

1.4.2 b

Prove Minkowski's Inequality:

$$1 \leq p < \infty \implies \|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Conclude that if $f, g \in L^p(\mathbb{R}^n)$ then so is $f+g$.

1.4.3 c

Let $X = [0,1] \subset \mathbb{R}$.

1. Give a definition of the Banach space $L^\infty(X)$ of essentially bounded functions of X.
2. Let f be non-negative and measurable on X, prove that

$$\int_X f(x)^p \, dx \overset{p\longrightarrow\infty}{\longrightarrow} \begin{cases} \infty & \text{or} \\ m\left(\left\{f^{-1}(1)\right\}\right) \end{cases},$$

   and characterize the functions of each type

Proof :

$$\begin{aligned}
\int f^p &= \int_{x\leq 1} f^p + \int_{x=1} f^p + \int_{x\geq 1} f^p \\
&= \int_{x\leq 1} f^p + \int_{x=1} 1 + \int_{x\geq 1} f^p \\
&= \int_{x\leq 1} f^p + m(\{f=1\}) + \int_{x\geq 1} f^p \\
&\overset{p\longrightarrow\infty}{\longrightarrow} 0 + m(\{f=1\}) + \begin{cases} 0 & m(\{x\geq 1\}) = 0 \\ \infty & m(\{x\geq 1\}) > 0. \end{cases}
\end{aligned}$$

1.5 5: Dual Spaces

Let X be a normed vector space.

1.5.1 a

Give the definition of what it means for a map $L : X \longrightarrow \mathbb{C}$ to be a *linear functional.*

1.5.2 b

Define what it means for L to be *bounded* and show L is bounded $\iff$ L is continuous.

1.5.3 c

Prove that $(X^\vee, \|\cdot\|_{\text{op}})$ is a Banach space.

2 Qual: Fall 2019

2.1 1

See phone photo?

2.2 2

DCT?

2.3 3

“Follow your nose.”

2.4 4

See Problem Set 8.

Bessel's Inequality: For any orthonormal set in a Hilbert space (not necessarily a basis), we have

$$\sum_{n=1}^{\infty} |\langle x,\ u_n\rangle|^2 \leq \|x\|^2$$

Proof:

$$0 \leq \left\| x - \sum_{k=1}^{n} \langle x, e_k\rangle\, e_k \right\|^2$$

Corollary (Parseval's Identity): If span $\{u_n\}$ is dense in $\mathcal{H}$, so it is a basis, then this is an equality.

Riesz-Fischer: Let $U = \{u_n\}_{n=1}^{\infty}$ be an orthonormal set (not necessarily a basis), then

1. There is an isometric surjection

$$\begin{aligned} \mathcal{H} &\longrightarrow \ell^2(\mathbb{N}) \\ \mathbf{x} &\mapsto \{\langle \mathbf{x},\ \mathbf{u}_n\rangle\}_{n=1}^{\infty} \end{aligned}$$

i.e. if $\{a_n\} \in \ell^2(\mathbb{N})$, so $\sum |a_n|^2 < \infty$, then there exists a $\mathbf{x} \in \mathcal{H}$ such that

$$a_n = \langle \mathbf{x},\ \mathbf{u}_n\rangle \quad \forall n.$$

2. $\mathbf{x}$ can be chosen such that

$$\|\mathbf{x}\|^2 = \sum |a_n|^2$$

> Note: the choice of $\mathbf{x}$ is unique $\iff$ $\{u_n\}$ is **complete**, i.e. $\langle \mathbf{x},\ \mathbf{u}_n\rangle = 0$ for all n implies $\mathbf{x} = \mathbf{0}$.

Proof:

- Given $\{a_n\}$, define $S_N = \sum^{N} a_n \mathbf{u}_n$.
- S_N is Cauchy in $\mathcal{H}$ and so $S_N \longrightarrow \mathbf{x}$ for some $\mathbf{x} \in \mathcal{H}$.
- $\langle x,\ u_n\rangle = \langle x - S_N,\ u_n\rangle + \langle S_N,\ u_n\rangle \longrightarrow a_n$
- By construction, $\|x - S_N\|^2 = \|x\|^2 - \sum^{N} |a_n|^2 \longrightarrow 0$, so $\|x\|^2 = \sum^{\infty} |a_n|^2$.

2.5 5

See Problem Set 5.

Heine-Cantor theorem: Every continuous function on a compact set is uniformly continuous.

Uniform continuity:

$$\begin{aligned} &\forall \varepsilon \quad \exists \delta(\varepsilon) \Big| \quad \forall x, y, \quad |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon \\ \iff &\forall \varepsilon \quad \exists \delta(\varepsilon) \Big| \quad \forall x, y, \quad |y| < \delta \implies |f(x-y) - f(y)| < \varepsilon \\ &. \end{aligned}$$

Fubini-Tonelli interchange of integrals, where the change of bounds becomes very important.

Continuity in L^1:

$$\lim_{y \longrightarrow 0} \|\tau_y f - f\|_1 = 0.$$