

Real Analysis Qualifying Exam Questions

D. Zack Garza

Tuesday 11th August, 2020

Contents

1 Undergraduate Analysis: Uniform Convergence	4
1.1 Fall 2018 # 1	4
1.2 1	4
1.3 Fall 2017 # 1	5
1.4 Fall 2014 # 1	5
1.5 Spring 2017 # 4	5
1.6 Spring 2015 # 1	5
1.7 Fall 2014 # 2	5
1.8 Spring 2014 # 2	6
2 General Analysis	6
2.1 Spring 2020 # 1	6
2.2 Fall 2019 # 1.	7
2.2.1 a	8
2.2.2 b	9
2.3 Fall 2018 # 4	10
2.4 Fall 2017 # 4	10
2.5 Spring 2017 # 3	11
2.6 Fall 2016 # 1	11
2.7 Fall 2016 # 5	11
2.8 Fall 2016 # 6	12
2.9 Spring 2016 # 1	12
2.10 Fall 2015 # 1	12
3 Measure Theory: Sets	12
3.1 Spring 2020 # 2	12
3.1.1 a.	12
3.1.2 b.	13
3.1.3 a	13
3.1.4 b	14
3.2 Fall 2019 # 3.	15
3.2.1 a	15
3.2.2 b	15
3.2.3 c	16

3.3	Spring 2019 # 2	16
3.3.1	a	17
3.3.2	b	17
3.3.3	a	17
3.3.4	b	18
3.4	Fall 2018 # 2	18
3.4.1	Indirect Proof	20
3.4.2	Direct Proof (Todo)	20
3.5	Spring 2018 # 1	21
3.6	Fall 2017 # 2	21
3.7	Spring 2017 # 2	21
3.8	Fall 2016 # 4	21
3.9	Spring 2016 # 3	22
3.10	Spring 2016 # 5	22
3.11	Fall 2015 # 2	22
3.12	Spring 2015 # 3	22
3.13	Spring 2014 # 3	22
3.14	Spring 2014 # 4	23
3.15	Spring 2017 # 1	23
3.16	Spring 2016 # 2	23
4	Measure Theory: Functions	23
4.1	Fall 2016 # 2	23
4.2	Spring 2016 # 4	23
5	Integrals: Convergence	24
5.1	Fall 2019 # 2	24
5.2	Spring 2020 # 5	26
5.3	Spring 2019 # 3	27
5.4	Fall 2018 # 6	29
5.5	Fall 2018 # 3	29
5.6	Spring 2018 # 5	30
5.7	Spring 2018 # 2	30
5.8	Fall 2016 # 3	31
5.9	Fall 2015 # 3	31
5.10	Fall 2015 # 4	31
6	Integrals: Approximation	31
6.1	Spring 2018 # 3	31
6.2	Spring 2018 # 4	31
6.3	Spring 2015 # 2	31
6.4	Fall 2014 # 4	32
7	L^1	32
7.1	Spring 2020 # 3	32
7.1.1	a	32
7.1.2	b	33
7.1.3	c	36

7.2	Fall 2019 # 5	36
7.2.1	a	36
7.2.2	b	36
7.3	a	36
7.4	b	37
7.5	Fall 2017 # 3	37
7.6	Spring 2015 # 4	38
7.7	Fall 2014 # 3	38
7.8	Spring 2014 # 1	38
8	Fubini-Tonelli	38
8.1	Spring 2020 # 4	38
8.2	Spring 2019 # 4	40
8.2.1	a	40
8.2.2	b	40
8.3	Fall 2018 # 5	41
8.4	Fall 2015 # 5	41
8.5	Spring 2014 # 5	41
9	L^2 and Fourier Analysis	42
9.1	Spring 2020 # 6	42
9.1.1	a	42
9.1.2	b	42
9.1.3	a	42
9.2	Fall 2017 # 5	44
9.3	Spring 2017 # 5	44
9.4	Spring 2015 # 6	44
9.5	Fall 2014 # 5	44
10	Functional Analysis: General	45
10.1	Fall 2019 # 4	45
10.1.1	a	45
10.1.2	b	45
10.1.3	a	45
10.1.4	b	46
10.2	Spring 2019 # 5	46
10.2.1	a	46
10.2.2	b	46
10.2.3	a	47
10.2.4	b	47
10.3	Spring 2016 # 6	49
10.4	Spring 2015 # 5	49
10.5	Fall 2015 # 6	50
10.6	Fall 2014 # 6	50
11	Functional Analysis: Banach Spaces	50
11.1	Spring 2019 # 1	50
11.1.1	a	50

11.1.2 b	51
11.2 Spring 2017 # 5	52
11.3 Fall 2017 # 6	52

1 Undergraduate Analysis: Uniform Convergence

1.1 Fall 2018 # 1

Let $f(x) = \frac{1}{x}$. Show that f is uniformly continuous on $(1, \infty)$ but not on $(0, \infty)$.

Solution.

1.2 1

Concepts used:

- Uniform continuity.

Show a stronger statement: $f(x) = \frac{1}{x}$ is uniformly continuous on any interval of the form (c, ∞) where $c > 0$.

- Note that

$$|x|, |y| > c > 0 \implies |xy| = |x||y| > c^2 \implies \frac{1}{|xy|} < \frac{1}{c^2}.$$

- Letting ε be arbitrary, choose $\delta < \varepsilon c^2$.
- Note that δ does not depend on x, y .
- Then

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{x} - \frac{1}{y} \right| \\ &= \frac{|x - y|}{xy} \\ &\leq \frac{\delta}{xy} \\ &< \frac{\delta}{c^2} \\ &< \varepsilon, \end{aligned}$$

which shows uniform continuity.

To see that f is not uniformly continuous when $c = 0$:

Note: negating uniform continuity says $\exists \varepsilon > 0$ such that $\forall \delta(\varepsilon)$ there exist x, y such that $|x - y| < \delta$ and $|f(x) - f(y)| > \varepsilon$.

- Let $\varepsilon < 1$.
- Let $x_n = \frac{1}{n}$ for $n \geq 1$.
- Choose n large enough such that $|x_n - x_{n+1}| = \frac{1}{n} - \frac{1}{n+1} < \delta$.
 - Why this can be done: by the archimedean property of \mathbb{R} , choose n such that $\frac{1}{n} < \varepsilon$.
 - Then

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \leq \frac{1}{n} < \varepsilon \quad \text{since } n+1 > 1.$$

- Note $f(x_n) = n$ and thus

$$|f(x_n) - f(x_{n+1})| = n - (n + 1) = 1 > \varepsilon.$$

1.3 Fall 2017 # 1

Let

$$f(x) = s \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which f does and does not converge uniformly.

1.4 Fall 2014 # 1

Let $\{f_n\}$ be a sequence of continuous functions such that $\sum f_n$ converges uniformly.

Prove that $\sum f_n$ is also continuous.

1.5 Spring 2017 # 4

Let $f(x, y)$ on $[-1, 1]^2$ be defined by

$$f(x, y) = \begin{cases} \frac{xy}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Determine if f is integrable.

1.6 Spring 2015 # 1

Let (X, d) and (Y, ρ) be metric spaces, $f : X \rightarrow Y$, and $x_0 \in X$.

Prove that the following statements are equivalent:

1. For every $\varepsilon > 0$ $\exists \delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$.
2. The sequence $\{f(x_n)\}_{n=1}^{\infty} \rightarrow f(x_0)$ for every sequence $\{x_n\} \rightarrow x_0$ in X .

1.7 Fall 2014 # 2

Let I be an index set and $\alpha : I \rightarrow (0, \infty)$.

1. Show that

$$\sum_{i \in I} \alpha(i) < \infty \implies I \text{ is countable.}$$

2. Suppose $I = \mathbb{Q}$ and $\sum_{q \in \mathbb{Q}} a(q) < \infty$. Define

$$f(x) := \sum_{\substack{q \in \mathbb{Q} \\ q \leq x}} a(q).$$

Show that f is continuous at $x \iff x \notin \mathbb{Q}$.

1.8 Spring 2014 # 2

Let $\{a_n\}$ be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that $\sum a_n^2 < \infty$.

Note: Assume a_n, b_n are all non-negative.

2 General Analysis

2.1 Spring 2020 # 1

Prove that if $f : [0, 1] \rightarrow \mathbb{R}$ is continuous then

$$\lim_{k \rightarrow \infty} \int_0^1 kx^{k-1} f(x) dx = f(1).$$

Solution.

Concepts used:

- DCT
- Weierstrass Approximation Theorem

Solution:

- Suppose p is a polynomial, then

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_0^1 kx^{k-1} p(x) dx &= \lim_{k \rightarrow \infty} \int_0^1 \left(\frac{\partial}{\partial x} x^k \right) p(x) dx \\ &= \lim_{k \rightarrow \infty} \left[x^k p(x) \Big|_0^1 - \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx \right] \quad \text{integrating by parts} \\ &= p(1) - \lim_{k \rightarrow \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx, \end{aligned}$$

- Thus it suffices to show that

$$\lim_{k \rightarrow \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx = 0.$$

- Integrating by parts a second time yields

$$\begin{aligned}
\lim_{k \rightarrow \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx &= \lim_{k \rightarrow \infty} \frac{x^{k+1}}{k+1} p'(x) \Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \\
&= - \lim_{k \rightarrow \infty} \int_0^1 \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \\
&= - \int_0^1 \lim_{k \rightarrow \infty} \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \quad \text{by DCT} \\
&= - \int_0^1 0 \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \\
&= 0.
\end{aligned}$$

- The DCT can be applied here because f'' is continuous and $[0, 1]$ is compact, so f'' is bounded on $[0, 1]$ by a constant M and

$$\int_0^1 |x^k f''(x)| \leq \int_0^1 1 \cdot M = M < \infty.$$

- Now use the Weierstrass approximation theorem:
 - If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then for every $\varepsilon > 0$ there exists a polynomial $p_\varepsilon(x)$ such that $\|f - p_\varepsilon\|_\infty < \varepsilon$.
- Thus

$$\begin{aligned}
\left| \int_0^1 kx^{k-1} p_\varepsilon(x) dx - \int_0^1 kx^{k-1} f(x) dx \right| &= \left| \int_0^1 kx^{k-1} (p_\varepsilon(x) - f(x)) dx \right| \\
&\leq \left| \int_0^1 kx^{k-1} \|p_\varepsilon - f\|_\infty dx \right| \\
&= \|p_\varepsilon - f\|_\infty \cdot \left| \int_0^1 kx^{k-1} dx \right| \\
&= \|p_\varepsilon - f\|_\infty \cdot x^k \Big|_0^1 \\
&= \|p_\varepsilon - f\|_\infty \xrightarrow{\varepsilon \rightarrow 0} 0
\end{aligned}$$

and the integrals are equal.

- By the first argument,

$$\int_0^1 kx^{k-1} p_\varepsilon(x) dx = p_\varepsilon(1) \text{ for each } \varepsilon$$

- Since uniform convergence implies pointwise convergence, $p_\varepsilon(1) \xrightarrow{\varepsilon \rightarrow 0} f(1)$.

2.2 Fall 2019 # 1.

Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers.

a. Prove that if $\lim_{n \rightarrow \infty} a_n = 0$, then

$$\lim_{n \rightarrow \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

b. Prove that if $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges, then

$$\lim_{n \rightarrow \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

Solution.

Cesaro mean/summation. Break series apart into pieces that can be handled separately.

2.2.1 a

Prove a stronger result:

$$a_k \rightarrow S \implies S_N := \frac{1}{N} \sum_{k=1}^N a_k \rightarrow S.$$

Idea: once N is large enough, $a_k \approx S$, and all smaller terms will die off as $N \rightarrow \infty$.
See this MSE answer.

- Use convergence $a_k \rightarrow S$: choose M large enough such that

$$k \geq M + 1 \implies |a_k - S| < \varepsilon.$$

Then

$$\begin{aligned}
\left| \left(\frac{1}{N} \sum_{k=1}^N a_k \right) - S \right| &= \frac{1}{N} \left| \left(\sum_{k=1}^N a_k \right) - NS \right| \\
&= \frac{1}{N} \left| \left(\sum_{k=1}^N a_k \right) - \sum_{k=1}^N S \right| \\
&= \frac{1}{N} \left| \sum_{k=1}^N (a_k - S) \right| \\
&\leq \frac{1}{N} \sum_{k=1}^N |a_k - S| \\
&= \frac{1}{N} \sum_{k=1}^M |a_k - S| + \sum_{k=M+1}^N |a_k - S| \\
&\leq \frac{1}{N} \sum_{k=1}^M |a_k - S| + \sum_{k=M+1}^N \frac{\varepsilon}{2} \\
&= \frac{1}{N} \sum_{k=1}^M |a_k - S| + (N - M) \frac{\varepsilon}{2} \\
&\xrightarrow{\varepsilon \rightarrow 0} \frac{1}{N} \sum_{k=1}^M |a_k - S| + 0 \\
&\xrightarrow{N \rightarrow \infty} 0 + 0.
\end{aligned}$$

Note: M is fixed, so the last sum is some constant c , and $c/N \rightarrow 0$ as $N \rightarrow \infty$ for any constant. To be more careful, choose M first to get $\varepsilon/2$ for the tail, then choose $N(M) > M$ for the remaining truncated part of the sum.

2.2.2 b

- Define

$$\Gamma_n := \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

- $\Gamma_1 = \sum_{k=1}^{\infty} \frac{a_k}{k}$ is the original series and each Γ_n is a tail of Γ_1 , so by assumption $\Gamma_n \xrightarrow{n \rightarrow \infty} 0$.
- Compute

$$\frac{1}{n} \sum_{k=1}^n a_k = \frac{1}{n} (\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \Gamma_{n+1})$$

- This comes from consider the following summation:

$\Gamma_1 :$	a_1	$+\frac{a_2}{2}$	$+\frac{a_3}{3}$	$+\cdots$
$\Gamma_2 :$		$\frac{a_2}{2}$	$+\frac{a_3}{3}$	$+\cdots$
$\Gamma_3 :$			$\frac{a_3}{3}$	$+\cdots$
<hr/>				
$\sum_{i=1}^n \Gamma_i :$	a_1	$+a_2$	$+a_3$	$+\cdots$
				a_n
				$+\frac{a_{n+1}}{n+1}$
				$+\cdots$

- Use part (a): since $\Gamma_n \xrightarrow{n \rightarrow \infty} 0$, we have $\frac{1}{n} \sum_{k=1}^n \Gamma_k \xrightarrow{n \rightarrow \infty} 0$.
- Also a minor check: $\Gamma_n \rightarrow 0 \implies \frac{1}{n} \Gamma_n \rightarrow 0$.
- Then

$$\begin{aligned}
 \frac{1}{n} \sum_{k=1}^n a_k &= \frac{1}{n} (\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \Gamma_{n+1}) \\
 &= \left(\frac{1}{n} \sum_{k=0}^n \Gamma_k \right) - \left(\frac{1}{n} \Gamma_{n+1} \right) \\
 &\xrightarrow{n \rightarrow \infty} 0.
 \end{aligned}$$

2.3 Fall 2018 # 4

Let $f \in L^1([0, 1])$. Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) |\sin nx| \, dx = \frac{2}{\pi} \int_0^1 f(x) \, dx$$

Hint: Begin with the case that f is the characteristic function of an interval.

2.4 Fall 2017 # 4

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

1. Show that $f_n \rightarrow 0$ pointwise but not uniformly on $[0, 1]$.

Hint: Consider the maximum of f_n .

2.

$$\lim_{n \rightarrow \infty} \int_0^1 n(1-x)^n \sin x \, dx = 0$$

2.5 Spring 2017 # 3

Let

$$f_n(x) = ae^{-nax} - be^{-nbx} \quad \text{where } 0 < a < b.$$

Show that

a. $\sum_{n=1}^{\infty} |f_n|$ is not in $L^1([0, \infty), m)$

Hint: $f_n(x)$ has a root x_n .

b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0, \infty), m) \quad \text{and} \quad \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x) \, dm = \ln \frac{b}{a}$$

2.6 Fall 2016 # 1

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that f converges to a differentiable function on $(1, \infty)$ and that

$$f'(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n^x} \right)'.$$

Hint:

$$\left(\frac{1}{n^x} \right)' = -\frac{1}{n^x} \ln n$$

2.7 Fall 2016 # 5

Let $\varphi \in L^\infty(\mathbb{R})$. Show that the following limit exists and satisfies the equality

$$\lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} \, dx \right)^{\frac{1}{n}} = \|\varphi\|_\infty.$$

2.8 Fall 2016 # 6

Let $f, g \in L^2(\mathbb{R})$. Show that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x)g(x+n) dx = 0$$

2.9 Spring 2016 # 1

For $n \in \mathbb{N}$, define

$$e_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad E_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

Show that $e_n < E_n$, and prove Bernoulli's inequality:

$$(1+x)^n \geq 1+nx \text{ for } -1 < x < \infty \text{ and } n \in \mathbb{N}$$

Use this to show the following:

1. The sequence e_n is increasing.
2. The sequence E_n is decreasing.
3. $2 < e_n < E_n < 4$.
4. $\lim_{n \rightarrow \infty} e_n = \lim_{n \rightarrow \infty} E_n$.

2.10 Fall 2015 # 1

Define

$$f(x) = c_0 + c_1x^1 + c_2x^2 + \dots + c_nx^n \text{ with } n \text{ even and } c_n > 0.$$

Show that there is a number x_m such that $f(x_m) \leq f(x)$ for all $x \in \mathbb{R}$.

3 Measure Theory: Sets**3.1 Spring 2020 # 2**

Let m_* denote the Lebesgue outer measure on \mathbb{R} .

3.1.1 a.

Prove that for every $E \subseteq \mathbb{R}$ there exists a Borel set B containing E such that

$$m_*(B) = m_*(E).$$

3.1.2 b.

Prove that if $E \subseteq \mathbb{R}$ has the property that

$$m_*(A) = m_*(A \cap E) + m_*(A \cap E^c)$$

for every set $A \subseteq \mathbb{R}$, then there exists a Borel set $B \subseteq \mathbb{R}$ such that $E = B \setminus N$ with $m_*(N) = 0$.

Be sure to address the case when $m_*(E) = \infty$.

Solution.

Concepts used:

- Definition of outer measure: $m_*(E) = \inf_{\{Q_j\} \Rightarrow E} \sum |Q_j|$ where $\{Q_j\}$ is a countable collection of closed cubes.
- Break \mathbb{R} into $\prod_{n \in \mathbb{Z}} [n, n+1)$, each with finite measure.
- Theorem: $m_*(Q) = |Q|$ for Q a closed cube (i.e. the outer measure equals the volume).

Proof (of Theorem) Statement: if Q is a closed cube, then $m_*(Q) = |Q|$, the usual volume.

- $m_*(Q) \leq |Q|$:
 - Since $Q \subseteq Q$, $Q \Rightarrow Q$ and $m_*(Q) \leq |Q|$ since m_* is an infimum over such coverings.
- $|Q| \leq m_*(Q)$:
 - Fix $\varepsilon > 0$.
 - Let $\{Q_i\}_{i=1}^\infty \Rightarrow Q$ be arbitrary, it suffices to show that

$$|Q| \leq \left(\sum_{i=1}^\infty |Q_i| \right) + \varepsilon.$$

- Pick open cubes S_i such that $Q_i \subseteq S_i$ and $|Q_i| \leq |S_i| \leq (1 + \varepsilon)|Q_i|$.
- Then $\{S_i\} \Rightarrow Q$, so by compactness of Q pick a finite subcover with N elements.
- Note

$$Q \subseteq \bigcup_{i=1}^N S_i \implies |Q| \leq \sum_{i=1}^N |S_i| \leq \sum_{i=1}^N (1 + \varepsilon)|Q_i| \leq (1 + \varepsilon) \sum_{i=1}^\infty |Q_i|.$$

- Taking an infimum over coverings on the RHS preserves the inequality, so

$$|Q| \leq (1 + \varepsilon)m_*(Q)$$

- Take $\varepsilon \rightarrow 0$ to obtain final inequality.

3.1.3 a

- If $m_*(E) = \infty$, then take $B = \mathbb{R}^n$ since $m(\mathbb{R}^n) = \infty$.
- Suppose $N := m_*(E) < \infty$.
- Since $m_*(E)$ is an infimum, by definition, for every $\varepsilon > 0$ there exists a covering by closed cubes $\{Q_i(\varepsilon)\}_{i=1}^\infty \Rightarrow E$ depending on ε such that

$$\sum_{i=1}^\infty |Q_i(\varepsilon)| < N + \varepsilon.$$

- For each fixed n , set $\varepsilon_n = \frac{1}{n}$ to produce such a covering $\{Q_i(\varepsilon_n)\}_{i=1}^\infty$ and set $B_n := \bigcup_{i=1}^\infty Q_i(\varepsilon_n)$.
- The outer measure of cubes is *equal* to the sum of their volumes, so

$$m_*(B_n) = \sum_{i=1}^\infty |Q_i(\varepsilon_n)| < N + \varepsilon_n = N + \frac{1}{n}.$$

- Now set $B := \bigcap_{n=1}^\infty B_n$.
 - Since $E \subseteq B_n$ for every n , $E \subseteq B$
 - Since B is a countable intersection of countable unions of closed sets, B is Borel.
 - Since $B_n \subseteq B$ for every n , we can apply subadditivity to obtain the inequality

$$E \subseteq B \subseteq B_n \implies N \leq m_*(B) \leq m_*(B_n) < N + \frac{1}{n} \quad \text{for all } n \in \mathbb{Z}^{\geq 1}.$$

- This forces $m_*(E) = m_*(B)$.

3.1.4 b

Suppose $m_*(E) < \infty$.

- By (a), find a Borel set $B \supseteq E$ such that $m_*(B) = m_*(E)$
- Note that $E \subseteq B \implies B \cap E = E$ and $B \cap E^c = B \setminus E$.
- By assumption,

$$\begin{aligned} m_*(B) &= m_*(B \cap E) + m_*(B \cap E^c) \\ m_*(E) &= m_*(E) + m_*(B \setminus E) \\ m_*(E) - m_*(E) &= m_*(B \setminus E) \quad \text{since } m_*(E) < \infty \\ \implies m_*(B \setminus E) &= 0. \end{aligned}$$

- So take $N = B \setminus E$; this shows $m_*(N) = 0$ and $E = B \setminus (B \setminus E) = B \setminus N$.

If $m_*(E) = \infty$:

- Apply result to $E_R := E \cap [R, R+1)^n \subset \mathbb{R}^n$ for $R \in \mathbb{Z}$, so $E = \bigsqcup_R E_R$
- Obtain B_R, N_R such that $E_R = B_R \setminus N_R$, $m_*(E_R) = m_*(B_R)$, and $m_*(N_R) = 0$.
- Note that
 - $B := \bigcup_R B_R$ is a union of Borel sets and thus still Borel
 - $E = \bigcup_R E_R$
 - $N := B \setminus E$
 - $N' := \bigcup_R N_R$ is a union of null sets and thus still null
- Since $E_R \subset B_R$ for every R , we have $E \subset B$
- We can compute

$$N = B \setminus E = \left(\bigcup_R B_R \right) \setminus \left(\bigcup_R E_R \right) \subseteq \bigcup_R (B_R \setminus E_R) = \bigcup_R N_R := N'$$

where $m_*(N') = 0$ since N' is null, and thus subadditivity forces $m_*(N) = 0$.

3.2 Fall 2019 # 3.

Let (X, \mathcal{B}, μ) be a measure space with $\mu(X) = 1$ and $\{B_n\}_{n=1}^\infty$ be a sequence of \mathcal{B} -measurable subsets of X , and

$$B := \left\{x \in X \mid x \in B_n \text{ for infinitely many } n\right\}.$$

- Argue that B is also a \mathcal{B} -measurable subset of X .
- Prove that if $\sum_{n=1}^\infty \mu(B_n) < \infty$ then $\mu(B) = 0$.
- Prove that if $\sum_{n=1}^\infty \mu(B_n) = \infty$ **and** the sequence of set complements $\{B_n^c\}_{n=1}^\infty$ satisfies

$$\mu\left(\bigcap_{n=k}^K B_n^c\right) = \prod_{n=k}^K (1 - \mu(B_n))$$

for all positive integers k and K with $k < K$, then $\mu(B) = 1$.

Hint: Use the fact that $1 - x \leq e^{-x}$ for all x .

Solution.

Concepts used:

- Borel-Cantelli: for a sequence of sets X_n ,

$$\begin{aligned} \limsup_n X_n &= \left\{x \mid x \in X_n \text{ for infinitely many } n\right\} &= \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} X_n \\ \liminf_n X_n &= \left\{x \mid x \in X_n \text{ for all but finitely many } n\right\} &= \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} X_n. \end{aligned}$$

- Properties of logs and exponentials:

$$\prod_n e^{x_n} = e^{\sum_n x_n} \quad \text{and} \quad \sum_n \log(x_n) = \log\left(\prod_n x_n\right).$$

- Tails of convergent sums vanish.
- Continuity of measure: $B_n \searrow B$ and $\mu(B_0) < \infty$ implies $\lim_n \mu(B_n) = \mu(B)$, and $B_n \nearrow B \implies \lim_n \mu(B_n) = \mu(B)$.

3.2.1 a

- The Borel σ -algebra is closed under countable unions/intersections/complements,
- $B = \limsup_n B_n$ is an intersection of unions of measurable sets.

3.2.2 b

- Tails of convergent sums go to zero, so $\sum_{n \geq M} \mu(B_n) \xrightarrow{M \rightarrow \infty} 0$,
- $B_M := \bigcap_{m=1}^M \bigcup_{n \geq m} B_n \searrow B$.

$$\begin{aligned}
\mu(B_M) &= \mu\left(\bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} B_n\right) \\
&\leq \mu\left(\bigcup_{n \geq m} B_n\right) \quad \text{for all } m \in \mathbb{N} \text{ by countable subadditivity} \\
&\longrightarrow 0,
\end{aligned}$$

- The result follows by continuity of measure.

3.2.3 c

- To show $\mu(B) = 1$, we'll show $\mu(B^c) = 0$.
- Let $B_K = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^K B_n$. Then

$$\begin{aligned}
\mu(B_K^c) &= \mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^K B_n^c\right) \\
&\leq \sum_{m=1}^{\infty} \mu\left(\bigcap_{n=m}^K B_n^c\right) \quad \text{by subadditivity} \\
&= \sum_{m=1}^{\infty} \prod_{n=m}^K (1 - \mu(B_n)) \quad \text{by assumption} \\
&\leq \sum_{m=1}^{\infty} \prod_{n=m}^K e^{-\mu(B_n^c)} \quad \text{by hint} \\
&= \sum_{m=1}^{\infty} \exp\left(-\sum_{n=m}^K \mu(B_n^c)\right) \\
&\xrightarrow{K \rightarrow \infty} 0
\end{aligned}$$

since $\sum_{n=m}^K \mu(B_n^c) \xrightarrow{K \rightarrow \infty} \infty$ by assumption

- We can apply continuity of measure since $B_K^c \xrightarrow{K \rightarrow \infty} B^c$.

Proving the hint: ?

3.3 Spring 2019 # 2

Let \mathcal{B} denote the set of all Borel subsets of \mathbb{R} and $\mu : \mathcal{B} \rightarrow [0, \infty)$ denote a finite Borel measure on \mathbb{R} .

3.3.1 a

Prove that if $\{F_k\}$ is a sequence of Borel sets for which $F_k \supseteq F_{k+1}$ for all k , then

$$\lim_{k \rightarrow \infty} \mu(F_k) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right)$$

3.3.2 b

Suppose μ has the property that $\mu(E) = 0$ for every $E \in \mathcal{B}$ with Lebesgue measure $m(E) = 0$.

Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $E \in \mathcal{B}$ with $m(E) < \delta$, then $\mu(E) < \varepsilon$.

Solution.

3.3.3 a

See Folland p.26

- Lemma 1: $\mu(\coprod_{k=1}^{\infty} E_k) = \lim_{N \rightarrow \infty} \sum_{k=1}^N \mu(E_k)$.
- Suppose $F_0 \supseteq F_1 \supseteq \dots$.
- Let $A_k = F_k \setminus F_{k+1}$, since the F_k are nested the A_k are disjoint
- Set $A := \coprod_{k=1}^{\infty} A_k$ and $F := \bigcap_{k=1}^{\infty} F_k$.
- Note $X = X \setminus Y \coprod X \cap Y$ for any two sets (just write $X \setminus Y := X \cap Y^c$)
- Note that A contains anything that was removed from F_0 when passing from any F_j to F_{j+1} , while F contains everything that is never removed at any stage, and these are disjoint possibilities.
- Thus $F_0 = F \coprod A$, so

$$\begin{aligned} \mu(F_0) &= \mu(F) + \mu(A) \\ &= \mu(F) + \mu(\coprod_{k=1}^{\infty} A_k) \\ &= \mu(F) + \lim_{n \rightarrow \infty} \sum_{k=0}^n \mu(A_k) \quad \text{by countable additivity} \\ &= \mu(F) + \lim_{n \rightarrow \infty} \sum_{k=0}^n (\mu(F_k) - \mu(F_{k+1})) \\ &= \mu(F) + \lim_{n \rightarrow \infty} (\mu(F_1) - \mu(F_n)) \quad (\text{Telescoping}) \\ &= \mu(F) + \mu(F_1) - \lim_{n \rightarrow \infty} \mu(F_n), \end{aligned}$$

- Since μ is a finite measure, $\mu(F_1) < \infty$ and can be subtracted, yielding

$$\begin{aligned} \mu(F_1) &= \mu(F) + \mu(F_1) - \lim_{n \rightarrow \infty} \mu(F_n) \\ \implies \mu(F) &= \lim_{n \rightarrow \infty} \mu(F_n) \\ \implies \mu\left(\bigcap_{k=1}^{\infty} F_k\right) &= \lim_{n \rightarrow \infty} \mu(F_n). \end{aligned}$$

3.3.4 b

- Toward a contradiction, negate the implication: suppose there exists an $\varepsilon > 0$ such that for all δ , we have $m(E) < \delta$ but $\mu(E) > \varepsilon$.
- The sequence $\left\{\delta_n := \frac{1}{2^n}\right\}_{n \in \mathbb{N}}$ and produce sets $A_n \in \mathcal{B}$ such $m(A_n) < \frac{1}{2^n}$ but $\mu(A_n) > \varepsilon$.
- Define

$$F_n := \bigcup_{j \geq n} A_j$$

$$C_m := \bigcap_{k=1}^m F_k$$

$$A := C_\infty := \bigcap_{k=1}^{\infty} F_k.$$

- Note that $F_1 \supseteq F_2 \supseteq \dots$, since each increase in index unions fewer sets.
- By continuity for the Lebesgue measure,

$$m(A) = m\left(\bigcap_{k=1}^{\infty} F_k\right) = \lim_{k \rightarrow \infty} m(F_k) = \lim_{k \rightarrow \infty} m\left(\bigcup_{j \geq k} A_j\right) \leq \lim_{k \rightarrow \infty} \sum_{j \geq k} m(A_j) = \lim_{k \rightarrow \infty} \sum_{j \geq k} \frac{1}{2^j} = 0,$$

which follows because this is the tail of a convergent sum

- Thus $m(A) = 0$ and by assumption, this implies $\mu(A) = 0$.
- However, by part (a),

$$\mu(A) = \lim_n \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \geq \lim_n \mu(A_n) = \lim_n \varepsilon = \varepsilon > 0.$$

All messed up.

3.4 Fall 2018 # 2

Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Show that there is a Borel set $B \subset E$ such that $m(E \setminus B) = 0$.

Solution.

Concepts used:

- Definition of measurability: there exists an open $O \supset E$ such that $m_*(O \setminus E) < \varepsilon$ for all $\varepsilon > 0$.
- Theorem: E is Lebesgue measurable iff there exists a closed set $F \subseteq E$ such that $m_*(E \setminus F) < \varepsilon$ for all $\varepsilon > 0$.
- Every F_σ, G_δ is Borel.
- Claim: E is measurable \iff for every ε there exist $F_\varepsilon \subset E \subset G_\varepsilon$ with F_ε closed and G_ε open and $m(G_\varepsilon \setminus E) < \varepsilon$ and $m(E \setminus F_\varepsilon) < \varepsilon$.
 - Proof: existence of G_ε is the definition of measurability.
 - Existence of F_ε : ?
- Claim: E is measurable \implies there exists an open $O \supseteq E$ such that $m(O \setminus E) = 0$.

- Since E is measurable, for each $n \in \mathbb{N}$ choose $G_n \supseteq E$ such that $m_*(G_n \setminus E) < \frac{1}{n}$.
- Set $O_N := \bigcap_{n=1}^N G_n$ and $O := \bigcap_{n=1}^{\infty} G_n$.
- Suppose E is bounded.
 - * Note $O_N \searrow O$ and $m_*(O_1) < \infty$ if E is bounded, since in this case

$$m_*(G_n \setminus E) = m_*(G_1) - m_*(E) < 1 \iff m_*(G_1) < m_*(E) + \frac{1}{n} < \infty.$$

- * Note $O_N \setminus E \searrow O \setminus E$ since $O_N \setminus E := O_N \cap E^c \supseteq O_{N+1} \cap E^c$ for all N , and again $m_*(O_1 \setminus E) < \infty$.
- * So it's valid to apply continuity of measure from above:

$$\begin{aligned} m_*(O \setminus E) &= \lim_{N \rightarrow \infty} m_*(O_N \setminus E) \\ &\leq \lim_{N \rightarrow \infty} m_*(G_N \setminus E) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} = 0, \end{aligned}$$

where the inequality uses subadditivity on $\bigcap_{n=1}^N G_n \subseteq G_N$

- Suppose E is unbounded.
 - * Write $E^k = E \cap [k, k+1]^d \subset \mathbb{R}^d$ as the intersection of E with an annulus, and note that $E = \bigcup_{k \in \mathbb{N}} E_k$.
 - * Each E_k is bounded, so apply the previous case to obtain $O_k \supseteq E_k$ with $m(O_k \setminus E_k) = 0$.
 - * So write $O_k = E_k \sqcup N_k$ where $N_k := O_k \setminus E_k$ is a null set.
 - * Define $O = \bigcup_{k \in \mathbb{N}} O_k$, note that $E \subseteq O$.
 - * Now note

$$\begin{aligned} O \setminus E &= \left(\bigcup_k O_k \right) \setminus \left(\bigcup_k E_k \right) \\ &\subseteq \bigcup_k (O_k \setminus E_k) \\ \implies m_*(O \setminus E) &\leq m_*\left(\bigcup_k (O_k \setminus E_k)\right) = 0, \end{aligned}$$

since any countable union of null sets is again null.

- So $O \supseteq E$ with $m(O \setminus E) = 0$.
- Theorem: since E is measurable, E^c is measurable
 - Proof: It suffices to write E^c as the union of two measurable sets, $E^c = S \cup (E^c - S)$, where S is to be determined.
 - We'll produce an S such that $m_*(E^c - S) = 0$ and use the fact that any subset of a null set is measurable.
 - Since E is measurable, for every $\varepsilon > 0$ there exists an open $\mathcal{O}_\varepsilon \supseteq E$ such that $m_*(\mathcal{O}_\varepsilon \setminus E) < \varepsilon$.
 - Take the sequence $\left\{ \varepsilon_n := \frac{1}{n} \right\}$ to produce a sequence of sets \mathcal{O}_n .

- Note that each \mathcal{O}_n^c is closed and

$$\mathcal{O}_n \supseteq E \iff \mathcal{O}_n^c \subseteq E^c.$$

- Set $S := \bigcup_n \mathcal{O}_n^c$, which is a union of closed sets, thus an F_σ set, thus Borel, thus measurable.
- Note that $S \subseteq E^c$ since each $\mathcal{O}_n \subseteq E^c$.
- Note that

$$\begin{aligned} E^c \setminus S &:= E^c \setminus \left(\bigcup_{n=1}^{\infty} \mathcal{O}_n^c \right) \\ &:= E^c \cap \left(\bigcup_{n=1}^{\infty} \mathcal{O}_n^c \right)^c \quad \text{definition of set minus} \\ &= E^c \cap \left(\bigcap_{n=1}^{\infty} \mathcal{O}_n \right)^c \quad \text{De Morgan's law} \\ &= E^c \cup \left(\bigcap_{n=1}^{\infty} \mathcal{O}_n \right) \\ &:= \left(\bigcap_{n=1}^{\infty} \mathcal{O}_n \right) \setminus E \\ &\subseteq \mathcal{O}_N \setminus E \quad \text{for every } N \in \mathbb{N}. \end{aligned}$$

- Then by subadditivity,

$$m_*(E^c \setminus S) \leq m_*(\mathcal{O}_N \setminus E) \leq \frac{1}{N} \quad \forall N \implies m_*(E^c \setminus S) = 0.$$

- Thus $E^c \setminus S$ is measurable.

3.4.1 Indirect Proof

- Since E is measurable, E^c is measurable.
- Since E^c is measurable exists an open $O \supseteq E^c$ such that $m(O \setminus E^c) = 0$.
- Set $B := O^c$, then $O \supseteq E^c \iff O^c \subseteq E \iff B \subseteq E$.
- Computing measures yields

$$E \setminus B := E \setminus \mathcal{O}^c := E \cap (\mathcal{O}^c)^c = E \cap \mathcal{O} = \mathcal{O} \cap (E^c)^c := \mathcal{O} \setminus E^c,$$

thus $m(E \setminus B) = m(\mathcal{O} \setminus E^c) = 0$.

- Since \mathcal{O} is open, B is closed and thus Borel.

3.4.2 Direct Proof (Todo)

?

Try to construct the set.

3.5 Spring 2018 # 1

Define

$$E := \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N} \right\}.$$

Prove that $m(E) = 0$.

3.6 Fall 2017 # 2

Let $f(x) = x^2$ and $E \subset [0, \infty) := \mathbb{R}^+$.

1. Show that

$$m^*(E) = 0 \iff m^*(f(E)) = 0.$$

2. Deduce that the map

$$\begin{aligned} \varphi : \mathcal{L}(\mathbb{R}^+) &\longrightarrow \mathcal{L}(\mathbb{R}^+) \\ E &\mapsto f(E) \end{aligned}$$

is a bijection from the class of Lebesgue measurable sets of $[0, \infty)$ to itself.

3.7 Spring 2017 # 2

- a. Let μ be a measure on a measurable space (X, \mathcal{M}) and f a positive measurable function.

Define a measure λ by

$$\lambda(E) := \int_E f \, d\mu, \quad E \in \mathcal{M}$$

Show that for g any positive measurable function,

$$\int_X g \, d\lambda = \int_X fg \, d\mu$$

- b. Let $E \subset \mathbb{R}$ be a measurable set such that

$$\int_E x^2 \, dm = 0.$$

Show that $m(E) = 0$.

3.8 Fall 2016 # 4

Let (X, \mathcal{M}, μ) be a measure space and suppose $\{E_n\} \subset \mathcal{M}$ satisfies

$$\lim_{n \rightarrow \infty} \mu(X \setminus E_n) = 0.$$

Define

$$G := \left\{ x \in X \mid x \in E_n \text{ for only finitely many } n \right\}.$$

Show that $G \in \mathcal{M}$ and $\mu(G) = 0$.

3.9 Spring 2016 # 3

Let f be Lebesgue measurable on \mathbb{R} and $E \subset \mathbb{R}$ be measurable such that

$$0 < A = \int_E f(x) dx < \infty.$$

Show that for every $0 < t < 1$, there exists a measurable set $E_t \subset E$ such that

$$\int_{E_t} f(x) dx = tA.$$

3.10 Spring 2016 # 5

Let (X, \mathcal{M}, μ) be a measure space. For $f \in L^1(\mu)$ and $\lambda > 0$, define

$$\varphi(\lambda) = \mu(\{x \in X | f(x) > \lambda\}) \quad \text{and} \quad \psi(\lambda) = \mu(\{x \in X | f(x) < -\lambda\})$$

Show that φ, ψ are Borel measurable and

$$\int_X |f| d\mu = \int_0^\infty [\varphi(\lambda) + \psi(\lambda)] d\lambda$$

3.11 Fall 2015 # 2

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be Lebesgue measurable.

1. Show that there is a sequence of simple functions $s_n(x)$ such that $s_n(x) \rightarrow f(x)$ for all $x \in \mathbb{R}$.
2. Show that there is a Borel measurable function g such that $g = f$ almost everywhere.

3.12 Spring 2015 # 3

Let μ be a finite Borel measure on \mathbb{R} and $E \subset \mathbb{R}$ Borel. Prove that the following statements are equivalent:

1. $\forall \varepsilon > 0$ there exists G open and F closed such that

$$F \subseteq E \subseteq G \quad \text{and} \quad \mu(G \setminus F) < \varepsilon.$$

2. There exists a $V \in G_\delta$ and $H \in F_\sigma$ such that

$$H \subseteq E \subseteq V \quad \text{and} \quad \mu(V \setminus H) = 0$$

3.13 Spring 2014 # 3

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and suppose

$$\forall x \in \mathbb{R}, \quad f(x) \geq \limsup_{y \rightarrow x} f(y)$$

Prove that f is Borel measurable.

3.14 Spring 2014 # 4

Let (X, \mathcal{M}, μ) be a measure space and suppose f is a measurable function on X . Show that

$$\lim_{n \rightarrow \infty} \int_X f^n d\mu = \begin{cases} \infty \\ \mu(f^{-1}(1)), \end{cases} \quad \text{or}$$

and characterize the collection of functions of each type.

3.15 Spring 2017 # 1

Let K be the set of numbers in $[0, 1]$ whose decimal expansions do not use the digit 4.

We use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 with $399\cdots$. For example, $0.8754 = 0.8753999\cdots$.

Show that K is a compact, nowhere dense set without isolated points, and find the Lebesgue measure $m(K)$.

3.16 Spring 2016 # 2

Let $0 < \lambda < 1$ and construct a Cantor set C_λ by successively removing middle intervals of length λ .

Prove that $m(C_\lambda) = 0$.

4 Measure Theory: Functions**4.1 Fall 2016 # 2**

Let $f, g : [a, b] \rightarrow \mathbb{R}$ be measurable with

$$\int_a^b f(x) dx = \int_a^b g(x) dx.$$

Show that either

1. $f(x) = g(x)$ almost everywhere, or
2. There exists a measurable set $E \subset [a, b]$ such that

$$\int_E f(x) dx > \int_E g(x) dx$$

4.2 Spring 2016 # 4

Let $E \subset \mathbb{R}$ be measurable with $m(E) < \infty$. Define

$$f(x) = m(E \cap (E + x)).$$

Show that

1. $f \in L^1(\mathbb{R})$.

2. f is uniformly continuous.
3. $\lim_{|x| \rightarrow \infty} f(x) = 0$.

Hint:

$$\chi_{E \cap (E+x)}(y) = \chi_E(y) \chi_E(y-x)$$

5 Integrals: Convergence

5.1 Fall 2019 # 2.

Prove that

$$\left| \frac{d^n \sin x}{dx^n} \right| \leq \frac{1}{n}$$

for all $x \neq 0$ and positive integers n .

Hint: Consider $\int_0^1 \cos(tx) dt$

Solution.

Concepts used:

- DCT
- Bounding in the right place. Don't evaluate the actual integral!

Solution:

- By induction on the number of limits we can pass through the integral.
- For $n = 1$ we first pass one derivative into the integral: let $x_n \rightarrow x$ be any sequence converging to x , then

$$\begin{aligned} \frac{\partial}{\partial x} \frac{\sin(x)}{x} &= \frac{\partial}{\partial x} \int_0^1 \cos(tx) dt \\ &= \lim_{x_n \rightarrow x} \frac{1}{x_n - x} \left(\int_0^1 \cos(tx_n) dt - \int_0^1 \cos(tx) dt \right) \\ &= \lim_{x_n \rightarrow x} \left(\int_0^1 \frac{\cos(tx_n) - \cos(tx)}{x_n - x} dt \right) \\ &= \lim_{x_n \rightarrow x} \left(\int_0^1 \left(t \sin(tx) \Big|_{x=\xi_n} \right) dt \right) \quad \text{where } \xi_n \in [x_n, x] \text{ by MVT, } \xi_n \rightarrow x \\ &= \lim_{\xi_n \rightarrow x} \left(\int_0^1 t \sin(t\xi_n) dt \right) \\ &=_{\text{DCT}} \int_0^1 \lim_{\xi_n \rightarrow x} t \sin(t\xi_n) dt \\ &= \int_0^1 t \sin(tx) dt \end{aligned}$$

- Taking absolute values we obtain an upper bound

$$\begin{aligned} \left| \frac{\partial}{\partial x} \frac{\sin(x)}{x} \right| &= \left| \int_0^1 t \sin(tx) dt \right| \\ &\leq \int_0^1 |t \sin(tx)| dt \\ &\leq \int_0^1 1 dt = 1, \end{aligned}$$

since $t \in [0, 1] \implies |t| < 1$, and $|\sin(xt)| \leq 1$ for any x and t .

- Note that this bound also justifies the DCT, since the functions $f_n(t) = t \sin(t\xi_n)$ are uniformly dominated by $g(t) = 1$ on $L^1([0, 1])$.

Note: integrating by parts here yields the actual formula:

$$\begin{aligned} \int_0^1 t \sin(tx) dt &=_{\text{IBP}} \left(\frac{-t \cos(tx)}{x} \right) \Big|_{t=0}^{t=1} - \int_0^1 \frac{\cos(tx)}{x} dt \\ &= \frac{-\cos(x)}{x} - \frac{\sin(x)}{x^2} \\ &= \frac{x \cos(x) - \sin(x)}{x^2}. \end{aligned}$$

- For the inductive step, we assume that we can pass $n - 1$ limits through the integral and show we can pass the n th through as well.

$$\begin{aligned} \frac{\partial^n}{\partial x^n} \frac{\sin(x)}{x} &= \frac{\partial^n}{\partial x^n} \int_0^1 \cos(tx) dt \\ &= \frac{\partial}{\partial x} \int_0^1 \frac{\partial^{n-1}}{\partial x^{n-1}} \cos(tx) dt \\ &= \frac{\partial}{\partial x} \int_0^1 t^{n-1} f_{n-1}(x, t) dt \end{aligned}$$

- Note that $f_n(x, t) = \pm \sin(tx)$ when n is odd and $f_n(x, t) = \pm \cos(tx)$ when n is even, and a constant factor of t is multiplied when each derivative is taken.

- We continue as in the base case:

$$\begin{aligned} \frac{\partial}{\partial x} \int_0^1 t^{n-1} f_{n-1}(x, t) dt &= \lim_{x_k \rightarrow x} \int_0^1 t^{n-1} \left(\frac{f_{n-1}(x_k, t) - f_{n-1}(x, t)}{x_k - x} \right) dt \\ &=_{\text{IVT}} \lim_{x_k \rightarrow x} \int_0^1 t^{n-1} \frac{\partial f_{n-1}}{\partial x}(\xi_k, t) dt \quad \text{where } \xi_k \in [x_k, x], \xi_k \rightarrow x \\ &=_{\text{DCT}} \int_0^1 \lim_{x_k \rightarrow x} t^{n-1} \frac{\partial f_{n-1}}{\partial x}(\xi_k, t) dt \\ &:= \int_0^1 \lim_{x_k \rightarrow x} t^n f_n(\xi_k, t) dt \\ &:= \int_0^1 t^n f_n(x, t) dt. \end{aligned}$$

- We've used the fact that $f_0(x) = \cos(tx)$ is smooth as a function of x , and in particular continuous
- The DCT is justified because the functions $h_{n,k}(x, t) = t^n f_n(\xi_k, t)$ are again uniformly (in k) bounded by 1 since $t \leq 1 \implies t^n \leq 1$ and each f_n is a sin or cosine.

- Now take absolute values

$$\begin{aligned}
 \left| \frac{\partial^n}{\partial x^n} \frac{\sin(x)}{x} \right| &= \left| \int_0^1 -t^n f_n(x, t) \, dt \right| \\
 &\leq \int_0^1 |t^n f_n(x, t)| \, dt \\
 &\leq \int_0^1 |t^n| |f_n(x, t)| \, dt \\
 &\leq \int_0^1 |t^n| \cdot 1 \, dt \\
 &\leq \int_0^1 t^n \, dt \quad \text{since } t \text{ is positive} \\
 &= \frac{1}{n+1} \\
 &< \frac{1}{n}.
 \end{aligned}$$

- We've again used the fact that $f_n(x, t)$ is of the form $\pm \cos(tx)$ or $\pm \sin(tx)$, both of which are bounded by 1.

5.2 Spring 2020 # 5

Compute the following limit and justify your calculations:

$$\lim_{n \rightarrow \infty} \int_0^n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx.$$

Not finished, flesh out.

Solution.

Concepts used:

- DCT
- Passing limits through products and quotients

Note that

$$\begin{aligned}
 \lim_n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} &= \frac{1}{\lim_n \left(1 + \frac{x^2}{n}\right)^1 \left(1 + \frac{x^2}{n}\right)^n} \\
 &= \frac{1}{1 \cdot e^{x^2}} \\
 &= e^{-x^2}.
 \end{aligned}$$

If passing the limit through the integral is justified, we will have

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \int_0^n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \chi_{[0,n]} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \\
 &= \int_{\mathbb{R}} \lim_{n \rightarrow \infty} \chi_{[0,n]} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \quad \text{by the DCT} \\
 &= \int_{\mathbb{R}} \lim_{n \rightarrow \infty} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \\
 &= \int_0^\infty e^{-x^2} dx \\
 &= \frac{\sqrt{\pi}}{2}.
 \end{aligned}$$

Computing the last integral:

$$\begin{aligned}
 \left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2 &= \left(\int_{\mathbb{R}} e^{-x^2} dx\right) \left(\int_{\mathbb{R}} e^{-y^2} dx\right) \\
 &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(x+y)^2} dx \\
 &= \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta \quad u = r^2 \\
 &= \frac{1}{2} \int_0^{2\pi} \int_0^\infty e^{-u} du d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} 1 d\theta \\
 &= \pi,
 \end{aligned}$$

and now use the fact that the function is even so $\int_0^\infty f = \frac{1}{2} \int_{\mathbb{R}} f$.

Justifying the DCT:

- Apply Bernoulli's inequality:

$$1 + \frac{x^{2n+1}}{n} \geq 1 + \frac{x^2}{n} (1 + x^2) \geq 1 + x^2,$$

where the last inequality follows from the fact that $1 + \frac{x^2}{n} \geq 1$

5.3 Spring 2019 # 3

Let $\{f_k\}$ be any sequence of functions in $L^2([0, 1])$ satisfying $\|f_k\|_2 \leq M$ for all $k \in \mathbb{N}$.

Prove that if $f_k \rightarrow f$ almost everywhere, then $f \in L^2([0, 1])$ with $\|f\|_2 \leq M$ and

$$\lim_{k \rightarrow \infty} \int_0^1 f_k(x) dx = \int_0^1 f(x) dx$$

Hint: Try using Fatou's Lemma to show that $\|f\|_2 \leq M$ and then try applying Egorov's Theorem.

Solution.

Concepts used:

- Definition of L^+ : space of measurable function $X \rightarrow [0, \infty]$.
- Fatou: For any sequence of L^+ functions, $\int \liminf f_n \leq \liminf \int f_n$.
- Egorov's Theorem: If $E \subseteq \mathbb{R}^n$ is measurable, $m(E) > 0$, $f_k : E \rightarrow \mathbb{R}$ a sequence of measurable functions where $\lim_{n \rightarrow \infty} f_n(x)$ exists and is finite a.e., then $f_n \rightarrow f$ *almost uniformly*: for every $\varepsilon > 0$ there exists a closed subset $F_\varepsilon \subseteq E$ with $m(E \setminus F_\varepsilon) < \varepsilon$ and $f_n \rightarrow f$ uniformly on F_ε .

L^2 bound:

- Since $f_k \rightarrow f$ almost everywhere, $\liminf_n f_n(x) = f(x)$ a.e.
- $\|f_n\|_2 < \infty$ implies each f_n is measurable and thus $|f_n|^2 \in L^+$, so we can apply Fatou:

$$\begin{aligned} \|f\|_2^2 &= \int |f(x)|^2 \\ &= \int \liminf_n |f_n(x)|^2 \\ &\leq \liminf_n \int |f_n(x)|^2 \\ &\leq \liminf_n M \\ &= M. \end{aligned}$$

- Thus $\|f\|_2 \leq \sqrt{M} < \infty$ implying $f \in L^2$.

What is the "right" proof here that uses the first part?

Equality of Integrals:

- Take the sequence $\varepsilon_n = \frac{1}{n}$
- Apply Egorov's theorem: obtain a set F_ε such that $f_n \rightarrow f$ uniformly on F_ε and $m(I \setminus F_\varepsilon) < \varepsilon$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \int_0^1 f_n - f \right| &\leq \lim_{n \rightarrow \infty} \int_0^1 |f_n - f| \\ &= \lim_{n \rightarrow \infty} \left(\int_{F_\varepsilon} |f_n - f| + \int_{I \setminus F_\varepsilon} |f_n - f| \right) \\ &= \int_{F_\varepsilon} \lim_{n \rightarrow \infty} |f_n - f| + \lim_{n \rightarrow \infty} \int_{I \setminus F_\varepsilon} |f_n - f| \quad \text{by uniform convergence} \\ &= 0 + \lim_{n \rightarrow \infty} \int_{I \setminus F_\varepsilon} |f_n - f|, \end{aligned}$$

so it suffices to show $\int_{I \setminus F_\varepsilon} |f_n - f| \xrightarrow{n \rightarrow \infty} 0$.

- We can obtain a bound using Holder's inequality with $p = q = 2$:

$$\begin{aligned}
 \int_{I \setminus F_\varepsilon} |f_n - f| &\leq \left(\int_{I \setminus F_\varepsilon} |f_n - f|^2 \right)^{1/2} \left(\int_{I \setminus F_\varepsilon} |1|^2 \right)^{1/2} \\
 &= \left(\int_{I \setminus F_\varepsilon} |f_n - f|^2 \right)^{1/2} \mu(F_\varepsilon)^{1/2} \\
 &\leq \|f_n - f\|_2 \mu(F_\varepsilon)^{1/2} \\
 &\leq (\|f_n\|_2 + \|f\|_2) \mu(F_\varepsilon)^{1/2} \\
 &\leq 2M \cdot \mu(F_\varepsilon)^{1/2}
 \end{aligned}$$

where M is now a constant not depending on ε or n .

- Now take a nested sequence of sets F_ε with $\mu(F_\varepsilon) \rightarrow 0$ and applying continuity of measure yields the desired statement.

5.4 Fall 2018 # 6

Compute the following limit and justify your calculations:

$$\lim_{n \rightarrow \infty} \int_1^n \frac{dx}{\left(1 + \frac{x}{n}\right)^n \sqrt[n]{x}}$$

5.5 Fall 2018 # 3

Suppose $f(x)$ and $xf(x)$ are integrable on \mathbb{R} . Define F by

$$F(t) := \int_{-\infty}^{\infty} f(x) \cos(xt) dx$$

Show that

$$F'(t) = - \int_{-\infty}^{\infty} xf(x) \sin(xt) dx.$$

Solution.

Concepts used:

- Mean Value Theorem
- DCT

$$\begin{aligned}
 \frac{\partial}{\partial t} F(t) &= \frac{\partial}{\partial t} \int_{\mathbb{R}} f(x) \cos(xt) dx \\
 &\stackrel{DCT}{=} \int_{\mathbb{R}} f(x) \frac{\partial}{\partial t} \cos(xt) dx \\
 &= \int_{\mathbb{R}} xf(x) \cos(xt) dx,
 \end{aligned}$$

so it only remains to justify the DCT.

- Fix t , then let $t_n \rightarrow t$ be arbitrary.

- Define

$$h_n(x, t) = f(x) \left(\frac{\cos(tx) - \cos(t_n x)}{t_n - t} \right) \xrightarrow{n \rightarrow \infty} \frac{\partial}{\partial t} (f(x) \cos(xt))$$

since $\cos(tx)$ is differentiable in t and this is the limit definition of differentiability.

- Note that

$$\begin{aligned} \frac{\partial}{\partial t} \cos(tx) &:= \lim_{t_n \rightarrow t} \frac{\cos(tx) - \cos(t_n x)}{t_n - t} \\ &\stackrel{MVT}{=} \frac{\partial}{\partial t} \cos(tx) \Big|_{t=\xi_n} \quad \text{for some } \xi_n \in [t, t_n] \text{ or } [t_n, t] \\ &= x \sin(\xi_n x) \end{aligned}$$

where $\xi_n \xrightarrow{n \rightarrow \infty} t$ since wlog $t_n \leq \xi_n \leq t$ and $t_n \nearrow t$.

- We then have

$$|h_n(x)| = |f(x)x \sin(\xi_n x)| \leq |xf(x)| \quad \text{since } |\sin(\xi_n x)| \leq 1$$

for every x and every n .

- Since $xf(x) \in L^1(\mathbb{R})$ by assumption, the DCT applies.

5.6 Spring 2018 # 5

Suppose that

- $f_n, f \in L^1$,
- $f_n \rightarrow f$ almost everywhere, and
- $\int |f_n| \rightarrow \int |f|$.

Show that $\int f_n \rightarrow \int f$

5.7 Spring 2018 # 2

Let

$$f_n(x) := \frac{x}{1 + x^n}, \quad x \geq 0.$$

- Show that this sequence converges pointwise and find its limit. Is the convergence uniform on $[0, \infty)$?
- Compute

$$\lim_{n \rightarrow \infty} \int_0^\infty f_n(x) dx$$

5.8 Fall 2016 # 3

Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{x \rightarrow 0} \int_{\mathbb{R}} |f(y-x) - f(y)| dy = 0$$

5.9 Fall 2015 # 3

Compute the following limit:

$$\lim_{n \rightarrow \infty} \int_1^n \frac{ne^{-x}}{1+nx^2} \sin\left(\frac{x}{n}\right) dx$$

5.10 Fall 2015 # 4

Let $f : [1, \infty) \rightarrow \mathbb{R}$ such that $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Show that the following limit exists and satisfies the equality

$$\lim_{x \rightarrow \infty} f(x) \leq 1 + \frac{\pi}{4}$$

6 Integrals: Approximation**6.1 Spring 2018 # 3**

Let f be a non-negative measurable function on $[0, 1]$.

Show that

$$\lim_{p \rightarrow \infty} \left(\int_{[0,1]} f(x)^p dx \right)^{\frac{1}{p}} = \|f\|_{\infty}.$$

6.2 Spring 2018 # 4

Let $f \in L^2([0, 1])$ and suppose

$$\int_{[0,1]} f(x)x^n dx = 0 \text{ for all integers } n \geq 0.$$

Show that $f = 0$ almost everywhere.

6.3 Spring 2015 # 2

Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be continuous with period 1. Prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(n\alpha) = \int_0^1 f(t) dt \quad \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

Hint: show this first for the functions $f(t) = e^{2\pi i k t}$ for $k \in \mathbb{Z}$.

6.4 Fall 2014 # 4

Let $g \in L^\infty([0, 1])$ Prove that

$$\int_{[0,1]} f(x)g(x) dx = 0 \quad \text{for all continuous } f : [0, 1] \longrightarrow \mathbb{R} \implies g(x) = 0 \text{ almost everywhere.}$$

7 L^1 **7.1 Spring 2020 # 3**

a. Prove that if $g \in L^1(\mathbb{R})$ then

$$\lim_{N \rightarrow \infty} \int_{|x| \geq N} |f(x)| dx = 0,$$

and demonstrate that it is not necessarily the case that $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

b. Prove that if $f \in L^1([1, \infty))$ and is decreasing, then $\lim_{x \rightarrow \infty} f(x) = 0$ and in fact $\lim_{x \rightarrow \infty} xf(x) = 0$.

c. If $f : [1, \infty) \rightarrow [0, \infty)$ is decreasing with $\lim_{x \rightarrow \infty} xf(x) = 0$, does this ensure that $f \in L^1([1, \infty))$?

Solution.

Concepts used:

- Limits
- Cauchy Criterion for Integrals: $\int_a^\infty f(x) dx$ converges iff for every $\varepsilon > 0$ there exists an M_0 such that $A, B \geq M_0$ implies $\left| \int_A^B f \right| < \varepsilon$, i.e. $\left| \int_A^B f \right| \xrightarrow{A \rightarrow \infty} 0$.
- Integrals of L^1 functions have vanishing tails: $\int_N^\infty |f| \xrightarrow{N \rightarrow \infty} 0$.
- Mean Value Theorem for Integrals: $\int_a^b f(t) dt = (b - a)f(c)$ for some $c \in [a, b]$.

7.1.1 a

Stated integral equality:

- Let $\varepsilon > 0$
- $C_c(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$ is dense so choose $\{f_n\} \rightarrow f$ with $\|f_n - f\|_1 \rightarrow 0$.
- Since $\{f_n\}$ are compactly supported, choose $N_0 \gg 1$ such that f_n is zero outside of $B_{N_0}(\mathbf{0})$.

- Then

$$\begin{aligned}
N \geq N_0 &\implies \int_{|x|>N} |f| = \int_{|x|>N} |f - f_n + f_n| \\
&\leq \int_{|x|>N} |f - f_n| + \int_{|x|>N} |f_n| \\
&= \int_{|x|>N} |f - f_n| \\
&\leq \int_{|x|>N} \|f - f_n\|_1 \\
&= \|f_n - f\|_1 \left(\int_{|x|>N} 1 \right) \\
&\stackrel{n \rightarrow \infty}{\longrightarrow} 0 \left(\int_{|x|>N} 1 \right) \\
&= 0 \\
&\stackrel{N \rightarrow \infty}{\longrightarrow} 0.
\end{aligned}$$

To see that this doesn't force $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$:

- Take $f(x)$ to be a train of rectangles of height 1 and area $1/2^j$ centered on even integers.
- Then

$$\int_{|x|>N} |f| = \sum_{j=N}^{\infty} 1/2^j \stackrel{N \rightarrow \infty}{\longrightarrow} 0$$

as the tail of a convergent sum.

- However $f(x) = 1$ for infinitely many even integers $x > N$, so $f(x) \not\rightarrow 0$ as $|x| \rightarrow \infty$.

7.1.2 b

Solution 1 ("Trick")

- Since f is decreasing on $[1, \infty)$, for any $t \in [x - n, x]$ we have

$$x - n \leq t \leq x \implies f(x) \leq f(t) \leq f(x - n).$$

- Integrate over $[x, 2x]$, using monotonicity of the integral:

$$\begin{aligned}
&\int_x^{2x} f(x) dt \leq \int_x^{2x} f(t) dt \leq \int_x^{2x} f(x - n) dt \\
\implies f(x) \int_x^{2x} dt &\leq \int_x^{2x} f(t) dt \leq f(x - n) \int_x^{2x} dt \\
&\implies xf(x) \leq \int_x^{2x} f(t) dt \leq xf(x - n).
\end{aligned}$$

- By the Cauchy Criterion for integrals, $\lim_{x \rightarrow \infty} \int_x^{2x} f(t) dt = 0$.
- So the LHS term $xf(x) \stackrel{x \rightarrow \infty}{\longrightarrow} 0$.
- Since $x > 1$, $|f(x)| \leq |xf(x)|$
- Thus $f(x) \stackrel{x \rightarrow \infty}{\longrightarrow} 0$ as well.

Solution 2 (Variation on the Trick)

- Use mean value theorem for integrals:

$$\int_x^{2x} f(t) dt = xf(c_x) \quad \text{for some } c_x \in [x, 2x] \text{ depending on } x.$$

- Since f is decreasing,

$$\begin{aligned} x \leq c_x \leq 2x &\implies f(2x) \leq f(c_x) \leq f(x) \\ &\implies 2xf(2x) \leq 2xf(c_x) \leq 2xf(x) \\ &\implies 2xf(2x) \leq 2x \int_x^{2x} f(t) dt \leq 2xf(x) \end{aligned}$$

- By Cauchy Criterion, $\int_x^{2x} f \rightarrow 0$.
- So $2xf(2x) \rightarrow 0$, which by a change of variables gives $uf(u) \rightarrow 0$.
- Since $u \geq 1$, $f(u) \leq uf(u)$ so $f(u) \rightarrow 0$ as well.

Solution 3 (Contradiction)

Just showing $f(x) \xrightarrow{x \rightarrow \infty} 0$:

- Toward a contradiction, suppose not.
- Since f is decreasing, it can not diverge to $+\infty$
- If $f(x) \rightarrow -\infty$, then $f \notin L^1(\mathbb{R})$: choose $x_0 \gg 1$ so that $t \geq x_0 \implies f(t) < -1$, then
- Then $t \geq x_0 \implies |f(t)| \geq 1$, so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f(t)| dt \geq \int_{x_0}^\infty 1 = \infty.$$

- Otherwise $f(x) \rightarrow L \neq 0$, some finite limit.
- If $L > 0$:
 - Fix $\varepsilon > 0$, choose $x_0 \gg 1$ such that $t \geq x_0 \implies L - \varepsilon \leq f(t) \leq L$
 - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L - \varepsilon) dt = \infty$$

- If $L < 0$:
 - Fix $\varepsilon > 0$, choose $x_0 \gg 1$ such that $t \geq x_0 \implies L \leq f(t) \leq L + \varepsilon$.
 - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L) dt = \infty$$

Showing $xf(x) \xrightarrow{x \rightarrow \infty} 0$.

- Toward a contradiction, suppose not.
- (How to show that $xf(x) \not\rightarrow +\infty$?)
- If $xf(x) \rightarrow -\infty$
 - Choose a sequence $\Gamma = \{\hat{x}_i\}$ such that $x_i \rightarrow \infty$ and $x_i f(x_i) \rightarrow -\infty$.
 - Choose a subsequence $\Gamma' = \{x_i\}$ such that $x_i f(x_i) \leq -1$ for all i and $x_i \leq x_{i+1}$.

- Choose a further subsequence $S = \{x_i \in \Gamma' \mid 2x_i < x_{i+1}\}$.
- Then since f is always decreasing, for $t \geq x_0$, $|f|$ is increasing, and $|f(x_i)| \leq |f(2x_i)|$, so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f| \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(t)| dt \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(x_i)| = \sum_{x_i \in S} x_i f(x_i) \rightarrow \infty.$$

- If $xf(x) \rightarrow L \neq 0$ for $0 < L < \infty$:
 - Fix $\varepsilon > 0$, choose an infinite sequence $\{x_i\}$ such that $L - \varepsilon \leq x_i f(x_i) \leq L$ for all i .

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)| dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i) dt = \sum_S x_i f(x_i) \geq \sum_S (L - \varepsilon) \rightarrow \infty.$$

- If $xf(x) \rightarrow L \neq 0$ for $-\infty < L < 0$:
 - Fix $\varepsilon > 0$, choose an infinite sequence $\{x_i\}$ such that $L \leq x_i f(x_i) \leq L + \varepsilon$ for all i .

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)| dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i) dt = \sum_S x_i f(x_i) \geq \sum_S (L) \rightarrow \infty.$$

Solution 4 (Akos's Suggestion) For $x \geq 1$,

$$|xf(x)| = \left| \int_x^{2x} f(x) dt \right| \leq \int_x^{2x} |f(x)| dt \leq \int_x^{2x} |f(t)| dt \leq \int_x^\infty |f(t)| dt \xrightarrow{x \rightarrow \infty} 0$$

where we've used

- Since f is decreasing and $\lim_{x \rightarrow \infty} f(x) = 0$ from part (a), f is non-negative.
- Since f is positive and decreasing, for every $t \in [a, b]$ we have $|f(a)| \leq |f(t)|$.
- By part (a), the last integral goes to zero.

Solution 5 (Peter's)

- Toward a contradiction, produce a sequence $x_i \rightarrow \infty$ with $x_i f(x_i) \rightarrow \infty$ and $x_i f(x_i) > \varepsilon > 0$, then

$$\begin{aligned} \int f(x) dx &\geq \sum_{i=1}^{\infty} \int_{x_i}^{x_{i+1}} f(x) dx \\ &\geq \sum_{i=1}^{\infty} \int_{x_i}^{x_{i+1}} f(x_{i+1}) dx \\ &= \sum_{i=1}^{\infty} f(x_{i+1}) \int_{x_i}^{x_{i+1}} dx \\ &\geq \sum_{i=1}^{\infty} (x_{i+1} - x_i) f(x_{i+1}) \\ &\geq \sum_{i=1}^{\infty} (x_{i+1} - x_i) \frac{\varepsilon}{x_{i+1}} \\ &= \varepsilon \sum_{i=1}^{\infty} \left(1 - \frac{x_{i-1}}{x_i}\right) \rightarrow \infty \end{aligned}$$

which can be ensured by passing to a subsequence where $\sum \frac{x_{i-1}}{x_i} < \infty$.

7.1.3 c

- No: take $f(x) = \frac{1}{x \ln x}$
- Then by a u -substitution,

$$\int_0^x f = \ln(\ln(x)) \xrightarrow{x \rightarrow \infty} \infty$$

is unbounded, so $f \notin L^1([1, \infty))$.

- But

$$xf(x) = \frac{1}{\ln(x)} \xrightarrow{x \rightarrow \infty} 0.$$

7.2 Fall 2019 # 5.**7.2.1 a**

Show that if f is continuous with compact support on \mathbb{R} , then

$$\lim_{y \rightarrow 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dx = 0$$

7.2.2 b

Let $f \in L^1(\mathbb{R})$ and for each $h > 0$ let

$$\mathcal{A}_h f(x) := \frac{1}{2h} \int_{|y| \leq h} f(x-y) dy$$

- Prove that $\|\mathcal{A}_h f\|_1 \leq \|f\|_1$ for all $h > 0$.
- Prove that $\mathcal{A}_h f \rightarrow f$ in $L^1(\mathbb{R})$ as $h \rightarrow 0^+$.

Solution.

Continuity in L^1 (recall that DCT won't work! Notes 19.4, prove it for a dense subset first).

Lebesgue differentiation in 1-dimensional case. See HW 5.6.

7.3 a

Choose $g \in C_c^0$ such that $\|f - g\|_1 \rightarrow 0$.

By translation invariance, $\|\tau_h f - \tau_h g\|_1 \rightarrow 0$.

Write

$$\begin{aligned} \|\tau f - f\|_1 &= \|\tau_h f - g + g - \tau_h g + \tau_h g - f\|_1 \\ &\leq \|\tau_h f - \tau_h g\| + \|g - f\| + \|\tau_h g - g\| \\ &\rightarrow \|\tau_h g - g\|, \end{aligned}$$

so it suffices to show that $\|\tau_h g - g\| \rightarrow 0$ for $g \in C_c^0$.

Fix $\varepsilon > 0$. Enlarge the support of g to K such that

$$|h| \leq 1 \text{ and } x \in K^c \implies |g(x-h) - g(x)| = 0.$$

By uniform continuity of g , pick $\delta \leq 1$ small enough such that

$$x \in K, |h| \leq \delta \implies |g(x-h) - g(x)| < \varepsilon,$$

then

$$\int_K |g(x-h) - g(x)| \leq \int_K \varepsilon = \varepsilon \cdot m(K) \longrightarrow 0.$$

7.4 b

We have

$$\begin{aligned} \int_{\mathbb{R}} |A_h(f)(x)| \, dx &= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y) \, dy \right| \, dx \\ &\leq \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)| \, dy \, dx \\ &=_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)| \, \mathbf{d}x \, dy \\ &= \int_{\mathbb{R}} |f(y)| \, dy \\ &= \|f\|_1. \end{aligned}$$

and (rough sketch)

$$\begin{aligned} \int_{\mathbb{R}} |A_h(f)(x) - f(x)| \, dx &= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y) \, dy \right) - f(x) \right| \, dx \\ &= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y) \, dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x) \, dy \right| \, dx \\ &\leq_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{B(h,x)} |f(y-x) - f(x)| \, \mathbf{d}x \, dy \\ &\leq \frac{1}{2h} \int_{\mathbb{R}} \|\tau_x f - f\|_1 \, dy \\ &\longrightarrow 0 \quad \text{by (a).} \end{aligned}$$

7.5 Fall 2017 # 3

Let

$$S = \text{span}_{\mathbb{C}} \left\{ \chi_{(a,b)} \mid a, b \in \mathbb{R} \right\},$$

the complex linear span of characteristic functions of intervals of the form (a, b) .

Show that for every $f \in L^1(\mathbb{R})$, there exists a sequence of functions $\{f_n\} \subset S$ such that

$$\lim_{n \rightarrow \infty} \|f_n - f\|_1 = 0$$

7.6 Spring 2015 # 4

Define

$$f(x, y) := \begin{cases} \frac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Carefully show that $f \in L^1(\mathbb{R}^2)$.

7.7 Fall 2014 # 3

Let $f \in L^1(\mathbb{R})$. Show that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } m(E) < \delta \implies \int_E |f(x)| dx < \varepsilon$$

7.8 Spring 2014 # 1

1. Give an example of a continuous $f \in L^1(\mathbb{R})$ such that $f(x) \not\rightarrow 0$ as $|x| \rightarrow \infty$.
2. Show that if f is *uniformly* continuous, then

$$\lim_{|x| \rightarrow \infty} f(x) = 0.$$

8 Fubini-Tonelli

8.1 Spring 2020 # 4

Let $f, g \in L^1(\mathbb{R})$. Argue that $H(x, y) := f(y)g(x - y)$ defines a function in $L^1(\mathbb{R}^2)$ and deduce from this fact that

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x - y) dy$$

defines a function in $L^1(\mathbb{R})$ that satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

Solution.

Relevant concepts:

- Tonelli: non-negative and measurable yields measurability of slices and equality of iterated integrals
- Fubini: $f(x, y) \in L^1$ yields *integrable* slices and equality of iterated integrals

- F/T: apply Tonelli to $|f|$; if finite, $f \in L^1$ and apply Fubini to f

$$\begin{aligned}
\|H(x)\|_1 &= \int_{\mathbb{R}} |H(x, y)| dx \\
&= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y)g(x-y) dy \right| dx \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(x-y)| dy \right) dx \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(x-y)| dx \right) dy \quad \text{by Tonelli} \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(t)| dt \right) dy \quad \text{setting } t = x - y, dt = -dx \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)| \cdot |g(t)| dt \right) dy \\
&= \int_{\mathbb{R}} |f(y)| \cdot \left(\int_{\mathbb{R}} |g(t)| dt \right) dy \\
&:= \int_{\mathbb{R}} |f(y)| \cdot \|g\|_1 dy \\
&= \|g\|_1 \int_{\mathbb{R}} |f(y)| dy \\
&:= \|g\|_1 \|f\|_1 \\
&< \infty \quad \text{by assumption} \quad .
\end{aligned}$$

- H is measurable on \mathbb{R}^2 :
 - If we can show $\tilde{f}(x, y) := f(y)$ and $\tilde{g}(x, y) := g(x - y)$ are both measurable on \mathbb{R}^2 , then $H = \tilde{f} \cdot \tilde{g}$ is a product of measurable functions and thus measurable.
 - $f \in L^1$, and L^1 functions are measurable by definition.
 - The function $(x, y) \mapsto g(x - y)$ is measurable on \mathbb{R}^2 :
 - * Let g be measurable on \mathbb{R} , then the cylinder function $G(x, y) = g(x)$ on \mathbb{R}^2 is always measurable
 - * Define a linear transformation $T := [1, -1; 0, 1]$ which sends $(x, y) \longrightarrow (x - y, y)$, then $T \in \text{GL}(2, \mathbb{R})$ is linear and thus measurable.
 - * Then $(G \circ T)(x, y) = G(x - y, y) = \tilde{g}(x - y)$, so \tilde{g} is a composition of measurable functions and thus measurable.
- Apply **Tonelli** to $|H|$
 - H measurable implies $|H|$ is measurable
 - $|H|$ is non-negative
 - So the iterated integrals are equal in the extended sense
 - The calculation shows the iterated integral is finite, to $\int |H|$ is finite and H is thus integrable on \mathbb{R}^2 .

Note: Fubini is not needed, since we're not calculating the actual integral, just showing H is integrable.

8.2 Spring 2019 # 4

Let f be a non-negative function on \mathbb{R}^n and $\mathcal{A} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq f(x)\}$.

Prove the validity of the following two statements:

- f is a Lebesgue measurable function on $\mathbb{R}^n \iff \mathcal{A}$ is a Lebesgue measurable subset of \mathbb{R}^{n+1}
- If f is a Lebesgue measurable function on \mathbb{R}^n , then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x) dx = \int_0^\infty m(\{x \in \mathbb{R}^n : f(x) \geq t\}) dt$$

Solution.

See S&S p.82.

8.2.1 a

$\implies :$

- Suppose f is a measurable function.
- Note that $\mathcal{A} = \{f(x) - t \geq 0\} \cap \{t \geq 0\}$.
- Define $F(x, t) = f(x)$, $G(x, t) = t$, which are cylinders on measurable functions and thus measurable.
- Define $H(x, y) = F(x, t) - G(x, t)$, which are linear combinations of measurable functions and thus measurable.
- Then $\mathcal{A} = \{H \geq 0\} \cap \{G \geq 0\}$ as a countable intersection of measurable sets, which is again measurable.

$\impliedby :$

- Suppose \mathcal{A} is a measurable set.
- Then F.T on $\chi_{\mathcal{A}}$ implies that for almost every $x \in \mathbb{R}^n$, the x -slices \mathcal{A}_x are measurable and

$$\mathcal{A}_x := \{t \in \mathbb{R} \mid (x, t) \in \mathcal{A}\} = [0, f(x)] \implies m(\mathcal{A}_x) = f(x) - 0 = f(x)$$

- But $x \mapsto m(\mathcal{A}_x)$ is a measurable function, and is exactly the function $x \mapsto f(x)$, so f is measurable.

8.2.2 b

- Note

$$\begin{aligned} \mathcal{A} &= \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid 0 \leq t \leq f(x)\} \\ \mathcal{A}_t &= \{x \in \mathbb{R}^n \mid t \leq f(x)\}. \end{aligned}$$

- Then

$$\begin{aligned}
 \int_{\mathbb{R}^n} f(x) \, dx &= \int_{\mathbb{R}^n} \int_0^{f(x)} 1 \, dt \, dx \\
 &= \int_{\mathbb{R}^n} \int_0^\infty \chi_{\mathcal{A}} \, dt \, dx \\
 &\stackrel{F.T.}{=} \int_0^\infty \int_{\mathbb{R}^n} \chi_{\mathcal{A}} \, dx \, dt \\
 &= \int_0^\infty m(\mathcal{A}_t) \, dt,
 \end{aligned}$$

where we just use that $\int \chi_{\mathcal{A}} = m(\mathcal{A})$

- By F.T., all of these integrals are equal.

Why is FT justified.

8.3 Fall 2018 # 5

Let $f \geq 0$ be a measurable function on \mathbb{R} . Show that

$$\int_{\mathbb{R}} f = \int_0^\infty m(\{x : f(x) > t\}) dt$$

8.4 Fall 2015 # 5

Let $f, g \in L^1(\mathbb{R})$ be Borel measurable.

1. Show that

- The function

$$F(x, y) := f(x - y)g(y)$$

is Borel measurable on \mathbb{R}^2 , and

- For almost every $y \in \mathbb{R}$,

$$F_y(x) := f(x - y)g(y)$$

is integrable with respect to y .

2. Show that $f * g \in L^1(\mathbb{R})$ and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

8.5 Spring 2014 # 5

Let $f, g \in L^1([0, 1])$ and for all $x \in [0, 1]$ define

$$F(x) := \int_0^x f(y) \, dy \quad \text{and} \quad G(x) := \int_0^x g(y) \, dy.$$

Prove that

$$\int_0^1 F(x)g(x) \, dx = F(1)G(1) - \int_0^1 f(x)G(x) \, dx$$

9 L^2 and Fourier Analysis

9.1 Spring 2020 # 6

9.1.1 a

Show that

$$L^2([0, 1]) \subseteq L^1([0, 1]) \quad \text{and} \quad \ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z}).$$

9.1.2 b

For $f \in L^1([0, 1])$ define

$$\hat{f}(n) := \int_0^1 f(x) e^{-2\pi i n x} dx.$$

Prove that if $f \in L^1([0, 1])$ and $\{\hat{f}(n)\} \in \ell^1(\mathbb{Z})$ then

$$S_N f(x) := \sum_{|n| \leq N} \hat{f}(n) e^{2\pi i n x}.$$

converges uniformly on $[0, 1]$ to a continuous function g such that $g = f$ almost everywhere.

Hint: One approach is to argue that if $f \in L^1([0, 1])$ with $\{\hat{f}(n)\} \in \ell^1(\mathbb{Z})$ then $f \in L^2([0, 1])$.

Solution.

Concepts used:

- For $e_n(x) := e^{2\pi i n x}$, the set $\{e_n\}$ is an orthonormal basis for $L^2([0, 1])$.
- For any orthonormal sequence in a Hilbert space, we have Bessel's inequality:

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

- When $\{e_n\}$ is a basis, the above is an *equality* (Parseval)
- Arguing uniform convergence: since $\{\hat{f}(n)\} \in \ell^1(\mathbb{Z})$, we should be able to apply the M test.

9.1.3 a

Claim: $\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$.

- Set $\mathbf{c} = \{c_k \mid k \in \mathbb{Z}\} \in \ell^1(\mathbb{Z})$.
- It suffices to show that if $\sum_{k \in \mathbb{Z}} |c_k| < \infty$ then $\sum_{k \in \mathbb{Z}} |c_k|^2 < \infty$.
- Let $S = \{c_k \mid |c_k| \leq 1\}$, then $c_k \in S \implies |c_k|^2 \leq |c_k|$
- Claim: S^c can only contain finitely many elements, all of which are finite.

- If not, either $S^c := \{c_j\}_{j=1}^\infty$ is infinite with every $|c_j| > 1$, which forces

$$\sum_{c_k \in S^c} |c_k| = \sum_{j=1}^\infty |c_j| > \sum_{j=1}^\infty 1 = \infty.$$

- If any $c_j = \infty$, then $\sum_{k \in \mathbb{Z}} |c_k| \geq c_j = \infty$.

- So S^c is a finite set of finite integers, let $N = \max \{|c_j|^2 \mid c_j \in S^c\} < \infty$.
- Rewrite the sum

$$\begin{aligned} \sum_{k \in \mathbb{Z}} |c_k|^2 &= \sum_{c_k \in S} |c_k|^2 + \sum_{c_k \in S^c} |c_k|^2 \\ &\leq \sum_{c_k \in S} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \\ &\leq \sum_{k \in \mathbb{Z}} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \quad \text{since the } |c_k| \text{ are all positive} \\ &= \|\mathbf{c}\|_{\ell^1} + \sum_{c_k \in S^c} |c_k|^2 \\ &\leq \|\mathbf{c}\|_{\ell^1} + |S^c| \cdot N \\ &< \infty. \end{aligned}$$

Claim: $L^2([0, 1]) \subseteq L^1([0, 1])$.

- It suffices to show that $\int |f|^2 < \infty \implies \int |f| < \infty$.
- Define $S = \{x \in [0, 1] \mid |f(x)| \leq 1\}$, then $x \in S^c \implies |f(x)|^2 \geq |f(x)|$.
- Break up the integral:

$$\begin{aligned} \int_{\mathbb{R}} |f| &= \int_S |f| + \int_{S^c} |f| \\ &\leq \int_S |f| + \int_{S^c} |f|^2 \\ &\leq \int_S |f| + \|f\|_2 \\ &\leq \sup_{x \in S} \{|f(x)|\} \cdot \mu(S) + \|f\|_2 \\ &= 1 \cdot \mu(S) + \|f\|_2 \quad \text{by definition of } S \\ &\leq 1 \cdot \mu([0, 1]) + \|f\|_2 \quad \text{since } S \subseteq [0, 1] \\ &= 1 + \|f\|_2 \\ &< \infty. \end{aligned}$$

Note: this proof shows $L^2(X) \subseteq L^1(X)$ whenever $\mu(X) < \infty$.

9.2 Fall 2017 # 5

Let φ be a compactly supported smooth function that vanishes outside of an interval $[-N, N]$ such that $\int_{\mathbb{R}} \varphi(x) dx = 1$.

For $f \in L^1(\mathbb{R})$, define

$$K_j(x) := j\varphi(jx), \quad f * K_j(x) := \int_{\mathbb{R}} f(x-y)K_j(y) dy$$

and prove the following:

1. Each $f * K_j$ is smooth and compactly supported.
- 2.

$$\lim_{j \rightarrow \infty} \|f * K_j - f\|_1 = 0$$

Hint:

$$\lim_{y \rightarrow 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dy = 0$$

9.3 Spring 2017 # 5

Let $f, g \in L^2(\mathbb{R})$. Prove that the formula

$$h(x) := \int_{-\infty}^{\infty} f(t)g(x-t) dt$$

defines a uniformly continuous function h on \mathbb{R} .

9.4 Spring 2015 # 6

Let $f \in L^1(\mathbb{R})$ and g be a bounded measurable function on \mathbb{R} .

1. Show that the convolution $f * g$ is well-defined, bounded, and uniformly continuous on \mathbb{R} .
2. Prove that one further assumes that $g \in C^1(\mathbb{R})$ with bounded derivative, then $f * g \in C^1(\mathbb{R})$ and

$$\frac{d}{dx}(f * g) = f * \left(\frac{d}{dx}g \right)$$

9.5 Fall 2014 # 5

1. Let $f \in C_c^0(\mathbb{R}^n)$, and show

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^n} |f(x+t) - f(x)| dx = 0.$$

2. Extend the above result to $f \in L^1(\mathbb{R}^n)$ and show that

$$f \in L^1(\mathbb{R}^n), \quad g \in L^\infty(\mathbb{R}^n) \implies f * g \text{ is bounded and uniformly continuous.}$$

10 Functional Analysis: General

10.1 Fall 2019 # 4.

Let $\{u_n\}_{n=1}^\infty$ be an orthonormal sequence in a Hilbert space \mathcal{H} .

10.1.1 a

Prove that for every $x \in \mathcal{H}$ one has

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

10.1.2 b

Prove that for any sequence $\{a_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$ there exists an element $x \in \mathcal{H}$ such that

$$a_n = \langle x, u_n \rangle \text{ for all } n \in \mathbb{N}$$

and

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$$

Solution.

Concepts used:

- Bessel's Inequality
- Pythagoras
- Surjectivity of the Riesz map
- Parseval's Identity
- Trick – remember to write out finite sum S_N , and consider $\|x - S_N\|$.

10.1.3 a

Claim:

$$\begin{aligned} 0 \leq \left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\ &\implies \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2. \end{aligned}$$

Proof: Let $S_N = \sum_{n=1}^N \langle x, u_n \rangle u_n$. Then

$$\begin{aligned} 0 &\leq \|x - S_N\|^2 \\ &= \langle x - S_N, x - S_N \rangle \\ &= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\ &\xrightarrow{N \rightarrow \infty} \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2. \end{aligned}$$

10.1.4 b

1. Fix $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
2. Define

$$x := \lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \sum_{k=1}^N a_k u_k$$

3. $\{S_N\}$ Cauchy (by 1) and H complete $\implies x \in H$.
- 4.

$$\langle x, u_n \rangle = \left\langle \sum_k a_k u_k, u_n \right\rangle = \sum_k a_k \langle u_k, u_n \rangle = a_n \quad \forall n \in \mathbb{N}$$

since the u_k are all orthogonal.

- 5.

$$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$$

by Pythagoras since the u_k are normal.

Bonus: We didn't use completeness here, so the Fourier series may not actually converge to x . If $\{u_n\}$ is **complete** (so $x = 0 \iff \langle x, u_n \rangle = 0 \forall n$) then the Fourier series *does* converge to x and $\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 = \|x\|^2$ for all $x \in H$.

10.2 Spring 2019 # 5

10.2.1 a

Show that $L^2([0, 1]) \subseteq L^1([0, 1])$ and argue that $L^2([0, 1])$ in fact forms a dense subset of $L^1([0, 1])$.

10.2.2 b

Let Λ be a continuous linear functional on $L^1([0, 1])$.

Prove the Riesz Representation Theorem for $L^1([0, 1])$ by following the steps below:

- i. Establish the existence of a function $g \in L^2([0, 1])$ which represents Λ in the sense that

$$\Lambda(f) = \int_0^1 f(x)g(x)dx \text{ for all } f \in L^2([0, 1]).$$

Hint: You may use, without proof, the Riesz Representation Theorem for $L^2([0, 1])$.

- ii. Argue that the g obtained above must in fact belong to $L^\infty([0, 1])$ and represent Λ in the sense that

$$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}dx \quad \text{for all } f \in L^1([0, 1])$$

with

$$\|g\|_{L^\infty([0,1])} = \|\Lambda\|_{L^1([0,1])^\vee}$$

Solution.

Concepts used:

- Holders' inequality: $\|fg\|_1 \leq \|f\|_p \|g\|_q$
- Riesz Representation for L^2 : If $\Lambda \in (L^2)^\vee$ then there exists a unique $g \in L^2$ such that $\Lambda(f) = \int fg$.
- $\|f\|_{L^\infty(X)} := \inf \{t \geq 0 \mid |f(x)| \leq t \text{ almost everywhere}\}$.
- **Lemma:** $m(X) < \infty \implies L^p(X) \subset L^2(X)$.

Proof: Write Holder's inequality as $\|fg\|_1 \leq \|f\|_a \|g\|_b$ where $\frac{1}{a} + \frac{1}{b} = 1$, then

$$\|f\|_p^p = \| |f|^p \|_1 \leq \| |f|^p \|_a \|1\|_b.$$

Now take $a = \frac{2}{p}$ and this reduces to

$$\begin{aligned} \|f\|_p^p &\leq \|f\|_2^p m(X)^{\frac{1}{p}} \\ \implies \|f\|_p &\leq \|f\|_2 \cdot O(m(X)) < \infty. \end{aligned}$$

10.2.3 a

- Note $X = [0, 1] \implies m(X) = 1$.
- By Holder's inequality with $p = q = 2$,

$$\|f\|_1 = \|f \cdot 1\|_1 \leq \|f\|_2 \cdot \|1\|_2 = \|f\|_2 \cdot m(X)^{\frac{1}{2}} = \|f\|_2,$$

- Thus $L^2(X) \subseteq L^1(X)$
- Since they share a common dense subset (simple functions) L^2 is dense in L^1

What theorem is this using?

10.2.4 b

Let $\Lambda \in L^1(X)^\vee$ be arbitrary.

(i): Existence of g Representing Λ .

- Let $f \in L^2 \subseteq L^1$ be arbitrary
- Claim: $\Lambda \in L^1(X)^\vee \implies \Lambda \in L^2(X)^\vee$.
 - Suffices to show that $\|\Gamma\|_{L^2(X)^\vee} := \sup_{\|f\|_2=1} |\Gamma(f)| < \infty$, since bounded implies continuous.
 - By the lemma, $\|f\|_1 \leq C\|f\|_2$ for some constant $C \approx m(X)$.
 - Note

$$\|\Lambda\|_{L^1(X)^\vee} := \sup_{\|f\|_1=1} |\Lambda(f)|$$

- Define $\hat{f} = \frac{f}{\|f\|_1}$ so $\|\hat{f}\|_1 = 1$
- Since $\|\Lambda\|_{1^\vee}$ is a supremum over all $f \in L^1(X)$ with $\|f\|_1 = 1$,

$$|\Lambda(\hat{f})| \leq \|\Lambda\|_{(L^1(X))^\vee},$$

- Then

$$\begin{aligned} \frac{|\Lambda(f)|}{\|f\|_1} &= |\Lambda(\hat{f})| \leq \|\Lambda\|_{L^1(X)^\vee} \\ \implies |\Lambda(f)| &\leq \|\Lambda\|_{1^\vee} \cdot \|f\|_1 \\ &\leq \|\Lambda\|_{1^\vee} \cdot C\|f\|_2 < \infty \quad \text{by assumption,} \end{aligned}$$

- So $\Lambda \in (L^2)^\vee$.
- Now apply Riesz Representation for L^2 : there is a $g \in L^2$ such that

$$f \in L^2 \implies \Lambda(f) = \langle f, g \rangle := \int_0^1 f(x) \overline{g(x)} dx.$$

(ii): g is in L^∞

- It suffices to show $\|g\|_{L^\infty(X)} < \infty$.
- Since we're assuming $\|\Gamma\|_{L^1(X)^\vee} < \infty$, it suffices to show the stated equality.

Is this assumed..? Or did we show it..?

- Claim: $\|\Lambda\|_{L^1(X)^\vee} = \|g\|_{L^\infty(X)}$
 - The result follows because Λ was assumed to be in $L^1(X)^\vee$, so $\|\Lambda\|_{L^1(X)^\vee} < \infty$.
 - \leq :

$$\begin{aligned} \|\Lambda\|_{L^1(X)^\vee} &= \sup_{\|f\|_1=1} |\Lambda(f)| \\ &= \sup_{\|f\|_1=1} \left| \int_X f \bar{g} \right| \quad \text{by (i)} \\ &= \sup_{\|f\|_1=1} \int_X |f \bar{g}| \\ &:= \sup_{\|f\|_1=1} \|fg\|_1 \\ &\leq \sup_{\|f\|_1=1} \|f\|_1 \|g\|_\infty \quad \text{by Holder with } p=1, q=\infty \\ &= \|g\|_\infty, \end{aligned}$$

– \geq :

- * Suppose toward a contradiction that $\|g\|_\infty > \|\Lambda\|_{L^1(X)^\vee}$.
- * Then there exists some $E \subseteq X$ with $m(E) > 0$ such that

$$x \in E \implies |g(x)| > \|\Lambda\|_{L^1(X)^\vee}.$$

- * Define

$$h = \frac{1}{m(E)} \frac{\bar{g}}{|g|} \chi_E.$$

- * Note $\|h\|_{L^1(X)} = 1$.
- * Then

$$\begin{aligned} \Lambda(h) &= \int_X hg \\ &:= \int_X \frac{1}{m(E)} \frac{g\bar{g}}{|g|} \chi_E \\ &= \frac{1}{m(E)} \int_E |g| \\ &\geq \frac{1}{m(E)} \|g\|_\infty m(E) \\ &= \|g\|_\infty \\ &> \|\Lambda\|_{L^1(X)^\vee}, \end{aligned}$$

a contradiction since $\|\Lambda\|_{L^1(X)^\vee}$ is the supremum over all h_α with $\|h_\alpha\|_{L^1(X)} = 1$.

10.3 Spring 2016 # 6

Without using the Riesz Representation Theorem, compute

$$\sup \left\{ \left| \int_0^1 f(x) e^x dx \right| \mid f \in L^2([0, 1], m), \|f\|_2 \leq 1 \right\}$$

10.4 Spring 2015 # 5

Let \mathcal{H} be a Hilbert space.

1. Let $x \in \mathcal{H}$ and $\{u_n\}_{n=1}^N$ be an orthonormal set. Prove that the best approximation to x in \mathcal{H} by an element in $\text{span}_{\mathbb{C}} \{u_n\}$ is given by

$$\hat{x} := \sum_{n=1}^N \langle x, u_n \rangle u_n.$$

2. Conclude that finite dimensional subspaces of \mathcal{H} are always closed.

10.5 Fall 2015 # 6

Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous. Show that

$$\sup \left\{ \|fg\|_1 \mid g \in L^1[0, 1], \|g\|_1 \leq 1 \right\} = \|f\|_\infty$$

10.6 Fall 2014 # 6

Let $1 \leq p, q \leq \infty$ be conjugate exponents, and show that

$$f \in L^p(\mathbb{R}^n) \implies \|f\|_p = \sup_{\|g\|_q=1} \left| \int f(x)g(x)dx \right|$$

11 Functional Analysis: Banach Spaces**11.1 Spring 2019 # 1**

Let $C([0, 1])$ denote the space of all continuous real-valued functions on $[0, 1]$.

- Prove that $C([0, 1])$ is complete under the uniform norm $\|f\|_\infty := \sup_{x \in [0, 1]} |f(x)|$.
- Prove that $C([0, 1])$ is not complete under the L^1 -norm $\|f\|_1 = \int_0^1 |f(x)| dx$.

Solution.

11.1.1 a

- Let $\{f_n\}$ be a Cauchy sequence in $C(I, \|\cdot\|_\infty)$, so $\lim_n \lim_m \|f_m - f_n\|_\infty = 0$, we will show it converges to some f in this space.
- For each fixed $x_0 \in [0, 1]$, the sequence of real numbers $\{f_n(x_0)\}$ is Cauchy in \mathbb{R} since

$$x_0 \in I \implies |f_m(x_0) - f_n(x_0)| \leq \sup_{x \in I} |f_m(x) - f_n(x)| := \|f_m - f_n\|_\infty \xrightarrow{m, n \rightarrow \infty} 0,$$

- Since \mathbb{R} is complete, this sequence converges and we can define $f(x) := \lim_{k \rightarrow \infty} f_k(x)$.
- Thus $f_n \rightarrow f$ pointwise by construction
- Claim: $\|f - f_n\| \xrightarrow{n \rightarrow \infty} 0$, so f_n converges to f in $C([0, 1], \|\cdot\|_\infty)$.

– Proof:

- * Fix $\varepsilon > 0$; we will show there exists an N such that $n \geq N \implies \|f_n - f\| < \varepsilon$
- * Fix an $x_0 \in I$. Since $f_n \rightarrow f$ pointwise, choose N_1 large enough so that

$$n \geq N_1 \implies |f_n(x_0) - f(x_0)| < \varepsilon/2.$$

- * Since $\|f_n - f_m\|_\infty \rightarrow 0$, choose and N_2 large enough so that

$$n, m \geq N_2 \implies \|f_n - f_m\|_\infty < \varepsilon/2.$$

* Then for $n, m \geq \max(N_1, N_2)$, we have

$$\begin{aligned}
 |f_n(x_0) - f(x_0)| &= |f_n(x_0) - f(x_0) + f_m(x_0) - f_m(x_0)| \\
 &= |f_n(x_0) - f_m(x_0) + f_m(x_0) - f(x_0)| \\
 &\leq |f_n(x_0) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \\
 &< |f_n(x_0) - f_m(x_0)| + \frac{\varepsilon}{2} \\
 &\leq \sup_{x \in I} |f_n(x) - f_m(x)| + \frac{\varepsilon}{2} \\
 &< \|f_n - f_m\|_\infty + \frac{\varepsilon}{2} \\
 &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
 \implies |f_n(x_0) - f(x_0)| &< \varepsilon \\
 \implies \sup_{x \in I} |f_n(x_0) - f(x_0)| &\leq \sup_{x \in I} \varepsilon \quad \text{by order limit laws} \\
 \implies \|f_n - f\| &\leq \varepsilon
 \end{aligned}$$

- f is the uniform limit of continuous functions and thus continuous, so $f \in C([0, 1])$.

11.1.2 b

- It suffices to produce a Cauchy sequence that does not converge to a continuous function.
- Take the following sequence of functions:
 - f_1 increases linearly from 0 to 1 on $[0, 1/2]$ and is 1 on $[1/2, 1]$
 - f_2 is 0 on $[0, 1/4]$ increases linearly from 0 to 1 on $[1/4, 1/2]$ and is 1 on $[1/2, 1]$
 - f_3 is 0 on $[0, 3/8]$ increases linearly from 0 to 1 on $[3/8, 1/2]$ and is 1 on $[1/2, 1]$
 - f_3 is 0 on $[0, (1/2 - 3/8)/2]$ increases linearly from 0 to 1 on $[(1/2 - 3/8)/2, 1/2]$ and is 1 on $[1/2, 1]$

Idea: take sequence starting points for the triangles: $0, 0 + \frac{1}{4}, 0 + \frac{1}{4} + \frac{1}{8}, \dots$ which converges to $1/2$ since $\sum_{k=1}^{\infty} \frac{1}{2^k} = -\frac{1}{2} + \sum_{k=0}^{\infty} \frac{1}{2^k}$.

- Then each f_n is clearly integrable, since its graph is contained in the unit square.
- $\{f_n\}$ is Cauchy: geometrically subtracting areas yields a single triangle whose area tends to 0.
- But f_n converges to $\chi_{[1/2, 1]}$ which is discontinuous.

Todo: show that $\int_0^1 |f_n(x) - f_m(x)| dx \rightarrow 0$ rigorously, show that no $g \in L^1([0, 1])$ can converge to this indicator function.

11.2 Spring 2017 # 5

Show that the space $C^1([a, b])$ is a Banach space when equipped with the norm

$$\|f\| := \sup_{x \in [a, b]} |f(x)| + \sup_{x \in [a, b]} |f'(x)|.$$

11.3 Fall 2017 # 6

Let X be a complete metric space and define a norm

$$\|f\| := \max\{|f(x)| : x \in X\}.$$

Show that $(C^0(\mathbb{R}), \|\cdot\|)$ (the space of continuous functions $f : X \rightarrow \mathbb{R}$) is complete.