Complex Analysis Qualifying Exam Solutions

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1 Week 1

1.1 Integrals and Cauchy's Theorem

1.1.1 5

Show that there is no sequence of polynomials converging uniformly to $f(z) = 1/z$ on S^1.

Solution

- By Cauchy's integral formula, $\displaystyle\int_{S^1} f = 2\pi i$
- If p_j is any polynomial, then p_j is holomorphic in $\mathbb{D}$, so $\displaystyle\int_{S^1} p_j = 0$.
- Contradiction: compact sets in $\mathbb{C}$ are bounded, so

$$\left| \int f - \int p_j \right| \leq \int |p_j - f| \leq \int \|p_j - f\|_\infty = \|p_j - f\|_\infty \int_{S^1} 1 \, dz = \|p_j - f\|_\infty \cdot 2\pi \longrightarrow 0$$

  which forces $\displaystyle\int f = \int p_j = 0$.

1.1.2 10

Suppose $f : \mathbb{C} \longrightarrow \mathbb{C}$ is entire and bounded, and use Cauchy's theorem to prove that $f' \equiv 0$ and thus f is constant.

Solution

- Suffices to prove $f' = 0$ because $\mathbb{C}$ is connected (see Stein Ch 1, 3.4)
- Fix $z_0 \in \mathbb{C}$, let B be the bound for f, so $|f(z)| \leq B$ for all z.
- Apply Cauchy inequalities: if f is holomorphic on $U \supset \overline{D}_R(z_0)$ then setting $\|f\|_C \coloneqq \sup_{z \in C} |f(z)|$,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!\|f\|_C}{R^n}.$$

  - Yields $|f'(z_0)| \leq B/R$
- Take $R \longrightarrow \infty$, QED.

1.2 Liouville. The Fundamental Theorem of Algebra, Power Series

1.2.1 1

Suppose f is analytic on $\Omega \supseteq \mathbb{D}$ whose power series $\sum a_n z^n$ has radius of convergence 1.

a. Give an example of an f which converges at every point on S^1.
b. Give an example of an f which is analytic at $z = 1$ but $\sum a_n$ diverges.
c. Prove that f can not be analytic at every point of S^1.

Solution:

a. Take $\sum \frac{z^n}{n^2}$; then $|z| \leq 1 \implies \left|\frac{z^n}{n^2}\right| \leq \frac{1}{n^2}$ which is summable, so the series converges for $|z| \leq 1$.

b. Take $\sum \frac{z^n}{n}$; then $z = 1$ yields the harmonic series, which diverges.

   - For $z \in S^1 \setminus \{1\}$, we have $z = e^{2\pi i t}$ for $0 < t < 2\pi$.
   - So fix t.
   - Toward applying the Dirichlet test, set $a_n = 1/n, b_n = z^n$.
   - Then for all N,

$$\left|\sum_{n=1}^{N} b_n\right| = \left|\sum_{n=1}^{N} b_n\right| = \left|\sum_{n=1}^{N} z^n\right| = \left|\frac{z - z^{N+1}}{|1-z|}\right| \leq \frac{2}{1-z} < \infty.$$

   - Thus $\sum a_n b_n < \infty$ and $\sum z^n/n$ converges.

c. ?

1.2.2 5

Prove the Fundamental Theorem of Algebra: every non-constant polynomial $p(z) = a_n z^n + \cdots + a_0 \in \mathbb{C}[x]$ has a root in $\mathbb{C}$.

Solution:

- Strategy: By contradiction with Liouville's Theorem
- Suppose p is non-constant and has no roots.

- Claim: $1/p(z)$ is a bounded holomorphic function on $\mathbb{C}$.
  - Holomorphic: clear? Since p has no roots.
  - Bounded: for $z \neq 0$, write

$$\frac{P(z)}{z^n} = a_n + \left(\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right).$$

  - The term in parentheses goes to 0 as $|z| \longrightarrow \infty$
  - Thus there exists an $R > 0$ such that

$$|z| > R \implies \left|\frac{P(z)}{z^n}\right| \geq c := \frac{|a_n|}{2}.$$

  - So p is bounded below when $|z| > R$
  - Since p is continuous and has no roots in $|z| \leq R$, it is bounded below when $|z| \leq R$.
  - Thus p is bounded below on $\mathbb{C}$ and thus $1/p$ is bounded above on $\mathbb{C}$.
- By Liouville's theorem, $1/p$ is constant and thus p is constant, a contradiction.

1.2.3 6

Find all entire functions f which satisfy the following inequality, and prove the list is complete:

$$|f(z)| \geq z.$$

Solution:

- Suppose f is entire and define $g(z) := \dfrac{z}{f(z)}$.
- By the inequality, $|g(z)| \leq 1$, so g is bounded.
- g potentially has singularities at the zeros $Z_f := f^{-1}(0)$, but since f is entire, g is holomorphic on $\mathbb{C} \setminus Z_f$.
- Claim: $Z_f \subset \mathbb{C}$ is closed and discrete
  - ???
- Thus the singularities Z_f are isolated
- By Riemann's removable singularity theorem, the singularities Z_f are removable and g has an extension to an entire function $\tilde{g}$.
- By continuity, we have $|\tilde{g}(z)| \leq 1$ on all of $\mathbb{C}$
- By Liouville, $\tilde{g}$ is constant, so $\tilde{g}(z) = c_0$ with $|c_0| \leq 1$
- Thus $f(z) = c_o^{-1} z$

Thus all such functions are of the form $f(z) = cz$ for some $c \neq 0 \in \mathbb{C}$.