Title

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Contents

1 Undergraduate Analysis: Uniform Convergence

1.1 Fall 2018 # 1 ✨

Let $f(x) = \frac{1}{x}$. Show that f is uniformly continuous on $(1, \infty)$ but not on $(0, \infty)$.

Solution:

Concepts Used:

- Uniform continuity:

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \quad \text{such that} \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

- Negating uniform continuity: $\exists \varepsilon > 0$ such that $\forall \delta(\varepsilon)$ there exist x, y such that $|x - y| < \delta$ *and* $|f(x) - f(y)| > \varepsilon$.

Claim: $f(x) = \frac{1}{x}$ is uniformly continuous on (c, ∞) for any $c > 0$.

- Note that

$$|x|, |y| > c > 0 \implies |xy| = |x||y| > c^2 \implies \frac{1}{|xy|} < \frac{1}{c^2}.$$

- Letting ε be arbitrary, choose $\delta < \varepsilon c^2$.
  - Note that δ does not depend on x, y.
- Then

$$\begin{aligned}
|f(x) - f(y)| &= \left| \frac{1}{x} - \frac{1}{y} \right| \\
&= \frac{|x - y|}{xy} \\
&\leq \frac{\delta}{xy} \\
&< \frac{\delta}{c^2} \\
&< \varepsilon,
\end{aligned}$$

  which shows uniform continuity.

Claim: f is *not* uniformly continuous when $c = 0$.

- Toward a contradiction, let $\varepsilon < 1$.
- Let $x_n = \frac{1}{n}$ for $n \geq 1$.
- Choose n large enough such that $|x_n - x_{n+1}| = \frac{1}{n} - \frac{1}{n+1} < \delta$.
  - Why this can be done: by the archimedean property of $\mathbb{R}$, choose n such that $\frac{1}{n} < \varepsilon$.
  - Then

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \leq \frac{1}{n} < \varepsilon \quad \text{since } n + 1 > 1.$$

- Note $f(x_n) = n$ and thus

$$|f(x_{n+1}) - f(x_n)| = (n+1) - n = 1 > \varepsilon,$$

a contradiction.

1.2 Fall 2017 # 1 ✨

Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which f does and does not converge uniformly.

Solution:

Concepts Used:

- $f_N \longrightarrow f$ uniformly $\iff$ $\|f_N - f\|_\infty \longrightarrow 0$.
- $\displaystyle\sum_{n=0}^{\infty} c_n x^n := \lim_{N \longrightarrow \infty} \sum_{n=0}^{N} c_n x^n$
  - I.e. an infinite sum is defined as the pointwise limit of its partial sums.
- If $\displaystyle\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on a set A, then $\displaystyle\sup_{x \in A} |f_n(x)| \longrightarrow 0$.

- Set $f_N(x) = \displaystyle\sum_{n=1}^{N} \frac{x^n}{n!}$.
  - Then by definition, $f_N(x) \longrightarrow f(x)$ pointwise on $\mathbb{R}$.
- For any compact interval $[-M, M]$, we have

$$\begin{aligned}
\|f_N(x) - f(x)\|_\infty &= \sup_{-M \leq x \leq M} \left| \sum_{n=N+1}^{\infty} \frac{x^n}{n!} \right| \\
&\leq \sup_{-M \leq x \leq M} \sum_{n=N+1}^{\infty} \left| \frac{x^n}{n!} \right| \\
&\leq \sum_{n=N+1}^{\infty} \frac{M^n}{n!} \\
&\leq \sum_{n=0}^{\infty} \frac{M^n}{n!} \quad \text{since all additional terms are positive} \\
&= e^M \\
&< \infty,
\end{aligned}$$

so $f_N \longrightarrow f$ uniformly on $[-M, M]$ by the M-test.

Here we've used that e^x is equal to its power series expansion.

- Thus f converges on any bounded interval, since any bounded interval is contained in some larger compact interval.

Claim: f does not converge on $\mathbb{R}$.

- If $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on a set A, then $\sup_{x\in A} |f_n(x)| \longrightarrow 0$.
- But taking $A = \mathbb{R}$ and $g_n(x) = \frac{x^n}{n!}$, we have

$$\sup_{x\in\mathbb{R}} |g_n(x)| = \sup_{x\in\mathbb{R}} \frac{x^n}{n!} = \infty.$$

1.3 Fall 2014 # 1 ✨

Let $\{f_n\}$ be a sequence of continuous functions such that $\sum f_n$ converges uniformly.

Prove that $\sum f_n$ is also continuous.

Solution:

Claim: If $F_N \longrightarrow F$ uniformly with each F_N continuous, then F is continuous.

- Follows from an $\varepsilon/3$ argument:

$$|F(x) - F(y| \leq |F(x) - F_N(x)| + |F_N(x) - F_N(y)| + |F_N(y) - F(y)| \leq \varepsilon \longrightarrow 0.$$

  - The first and last $\varepsilon/3$ come from uniform convergence of $F_N \longrightarrow F$.
  - The middle $\varepsilon/3$ comes from continuity of each F_N.
- Now setting $F_N := \sum_{n=1}^{N} f_n$ yields a finite sum of continuous functions, which is continuous.
- Each F_N is continuous and $F_N \longrightarrow F$ uniformly, so applying the claim yields the desired result.

1.4 Spring 2017 # 4 🚩

Let $f(x,y)$ on $[-1,1]^2$ be defined by

$$f(x,y) = \begin{cases} \frac{xy}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Determine if f is integrable.

Solution:

Concepts Used:

- Just Calculus.

Switching to polar coordinates and integrating over one quarter of the unit disc $D \subseteq I^2$, we have

$$
\begin{aligned}
\int_{I^2} f\,dA &\geq \int_D f\,dA \\
&\geq \int_0^{\pi/2} \int_0^1 \frac{\cos(\theta)\sin(\theta)}{r^4}\ r\ dr\ d\theta \\
&= \int_0^{\pi/2} \cos(\theta)\sin(\theta) \int_0^1 \frac{1}{r^3}\ dr\ d\theta \\
&= \left(\int_0^1 \frac{1}{r^3}\,dr\right)\left(\int_0^{\pi/2} \cos(\theta)\sin(\theta)\,d\theta\right) \\
&= \left(\int_0^1 \frac{1}{r^3}\,dr\right)\left(-\frac{1}{2}\cos^2(\theta)\Big|_0^{\pi/2}\right) \\
&= -\frac{1}{2r^2}\Big|_0^1 \left(\frac{1}{2}\right) \\
&= \left(\frac{1}{4}\right)\left(-1 + \lim_{r \longrightarrow 0} \frac{1}{r^2}\right) \\
&= \infty,
\end{aligned}
$$

so f is not integrable.

1.5 Spring 2015 # 1 ✨

Let (X, d) and (Y, ρ) be metric spaces, $f : X \longrightarrow Y$, and $x_0 \in X$.

Prove that the following statements are equivalent:

1. For every $\varepsilon > 0 \quad \exists \delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$.
2. The sequence $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow f(x_0)$ for every sequence $\{x_n\} \longrightarrow x_0$ in X.

1.6 Fall 2014 # 2 🚩

Let I be an index set and $\alpha : I \longrightarrow (0, \infty)$.

1. Show that

$$
\sum_{i \in I} a(i) := \sup_{\substack{J \subset I \\ J \text{ finite}}} \sum_{i \in J} a(i) < \infty \implies I \text{ is countable.}
$$

2. Suppose $I = \mathbb{Q}$ and $\sum_{q \in \mathbb{Q}} a(q) < \infty$. Define

$$
f(x) := \sum_{\substack{q \in \mathbb{Q} \\ q \leq x}} a(q).
$$

Show that f is continuous at $x \iff x \notin \mathbb{Q}$.

1.7 Spring 2014 # 2 🚩

Let $\{a_n\}$ be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that $\sum a_n^2 < \infty$.

Note: Assume a_n, b_n are all non-negative.

2 General Analysis

2.1 Spring 2020 # 1 ✨

Prove that if $f : [0,1] \longrightarrow \mathbb{R}$ is continuous then

$$\lim_{k \longrightarrow \infty} \int_0^1 kx^{k-1} f(x)\, dx = f(1).$$

Solution:

Concepts Used:

- DCT
- Weierstrass Approximation Theorem

- Suppose p is a polynomial, then

$$\begin{aligned} \lim_{k \longrightarrow \infty} \int_0^1 kx^{k-1} p(x)\, dx &= \lim_{k \longrightarrow \infty} \int_0^1 \left(\frac{\partial}{\partial x} x^k \right) p(x)\, dx \\ &= \lim_{k \longrightarrow \infty} \left[x^k p(x) \Big|_0^1 - \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx \right] && \text{integrating by parts} \\ &= p(1) - \lim_{k \longrightarrow \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx, \end{aligned}$$

- Thus it suffices to show that

$$\lim_{k \longrightarrow \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx = 0.$$

- Integrating by parts a second time yields

$$\begin{aligned}
\lim_{k\longrightarrow\infty}\int_0^1 x^k\left(\frac{\partial}{\partial x}p(x)\right)dx &= \lim_{k\longrightarrow\infty}\frac{x^{k+1}}{k+1}p'(x)\Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1}\left(\frac{\partial^2}{\partial x^2}p(x)\right)dx \\
&= -\lim_{k\longrightarrow\infty}\int_0^1 \frac{x^{k+1}}{k+1}\left(\frac{\partial^2}{\partial x^2}p(x)\right)dx \\
&= -\int_0^1 \lim_{k\longrightarrow\infty}\frac{x^{k+1}}{k+1}\left(\frac{\partial^2}{\partial x^2}p(x)\right)dx \quad \text{by DCT} \\
&= -\int_0^1 0\left(\frac{\partial^2}{\partial x^2}p(x)\right)dx \\
&= 0.
\end{aligned}$$

  - The DCT can be applied here because f'' is continuous and $[0,1]$ is compact, so f'' is bounded on $[0,1]$ by a constant M and

$$\int_0^1 \left|x^k f''(x)\right| \leq \int_0^1 1\cdot M = M < \infty.$$

- Now use the Weierstrass approximation theorem:
  - If $f:[a,b]\longrightarrow\mathbb{R}$ is continuous, then for every $\varepsilon>0$ there exists a polynomial $p_\varepsilon(x)$ such that $\|f-p_\varepsilon\|_\infty<\varepsilon$.
- Thus

$$\begin{aligned}
\left|\int_0^1 kx^{k-1}p_\varepsilon(x)\,dx - \int_0^1 kx^{k-1}f(x)\,dx\right| &= \left|\int_0^1 kx^{k-1}(p_\varepsilon(x)-f(x))\,dx\right| \\
&\leq \left|\int_0^1 kx^{k-1}\|p_\varepsilon - f\|_\infty\,dx\right| \\
&= \|p_\varepsilon - f\|_\infty\cdot\left|\int_0^1 kx^{k-1}\,dx\right| \\
&= \|p_\varepsilon - f\|_\infty\cdot x^k\Big|_0^1 \\
&= \|p_\varepsilon - f\|_\infty \overset{\varepsilon\longrightarrow 0}{\Longrightarrow} 0
\end{aligned}$$

  and the integrals are equal.
- By the first argument,

$$\int_0^1 kx^{k-1}p_\varepsilon(x)\,dx = p_\varepsilon(1)\text{ for each }\varepsilon$$

- Since uniform convergence implies pointwise convergence, $p_\varepsilon(1)\overset{\varepsilon\longrightarrow 0}{\Longrightarrow} f(1)$.

2.2 Fall 2019 # 1 ✨

Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers.

2.2.1 a

Prove that if $\lim_{n \longrightarrow \infty} a_n = 0$, then

$$\lim_{n \to \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

2.2.2 b

Prove that if $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges, then

$$\lim_{n \to \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

Solution:

Concepts Used:

- Cesaro mean/summation.
- Break series apart into pieces that can be handled separately.

2.2.3 a

Prove a stronger result:

$$a_k \longrightarrow S \implies S_N := \frac{1}{N} \sum_{k=1}^{N} a_k \longrightarrow S.$$

Idea: once N is large enough, $a_k \approx S$, and all smaller terms will die off as $N \longrightarrow \infty$. See this MSE answer.

- Use convergence $a_k \longrightarrow S$: choose M large enough such that

$$k \geq M + 1 \implies |a_k - S| < \varepsilon.$$

Then

$$\begin{aligned}
\left|\left(\frac{1}{N}\sum_{k=1}^N a_k\right) - S\right| &= \frac{1}{N}\left|\left(\sum_{k=1}^N a_k\right) - NS\right| \\
&= \frac{1}{N}\left|\left(\sum_{k=1}^N a_k\right) - \sum_{k=1}^N S\right| \\
&= \frac{1}{N}\left|\sum_{k=1}^N (a_k - S)\right| \\
&\leq \frac{1}{N}\sum_{k=1}^N |a_k - S| \\
&= \frac{1}{N}\sum_{k=1}^M |a_k - S| + \sum_{k=M+1}^N |a_k - S| \\
&\leq \frac{1}{N}\sum_{k=1}^M |a_k - S| + \sum_{k=M+1}^N \frac{\varepsilon}{2} \\
&= \frac{1}{N}\sum_{k=1}^M |a_k - S| + (N-M)\frac{\varepsilon}{2} \\
&\overset{\varepsilon\longrightarrow 0}{\longrightarrow} \frac{1}{N}\sum_{k=1}^M |a_k - S| + 0 \\
&\overset{N\longrightarrow\infty}{\longrightarrow} 0 + 0.
\end{aligned}$$

> Note: M is fixed, so the last sum is some constant c, and $c/N \longrightarrow 0$ as $N \longrightarrow \infty$ for any constant. To be more careful, choose M first to get $\varepsilon/2$ for the tail, then choose $N(M) > M$ for the remaining truncated part of the sum.

2.2.4 b

- Define

$$\Gamma_n := \sum_{k=n}^\infty \frac{a_k}{k}.$$

- $\Gamma_1 = \sum_{k=1}^n \frac{a_k}{k}$ is the original series and each Γ_n is a tail of Γ_1, so by assumption $\Gamma_n \overset{n\longrightarrow\infty}{\longrightarrow} 0$.
- Compute

$$\frac{1}{n}\sum_{k=1}^n a_k = \frac{1}{n}(\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \mathbf{\Gamma_{n+1}})$$

.

- This comes from consider the following summation:

$$
\begin{array}{r|lllllll}
\Gamma_1: & a_1 & +\frac{a_2}{2} & +\frac{a_3}{3} & +\cdots & & & \\
\Gamma_2: & & \frac{a_2}{2} & +\frac{a_3}{3} & +\cdots & & & \\
\Gamma_3: & & & \frac{a_3}{3} & +\cdots & & & \\
\hline
\sum_{i=1}^{n} \Gamma_i: & a_1 & +a_2 & +a_3 & +\cdots & a_n & +\frac{a_{n+1}}{n+1} & +\cdots
\end{array}
$$

- Use part (a): since $\Gamma_n \overset{n\longrightarrow\infty}{\longrightarrow} 0$, we have $\frac{1}{n}\sum_{k=1}^{n}\Gamma_k \overset{n\longrightarrow\infty}{\longrightarrow} 0$.
- Also a minor check: $\Gamma_n \longrightarrow 0 \implies \frac{1}{n}\Gamma_n \longrightarrow 0$.
- Then

$$
\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} a_k &= \frac{1}{n}(\Gamma_1+\Gamma_2+\cdots+\Gamma_n - \mathbf{\Gamma_{n+1}}) \\
&= \left(\frac{1}{n}\sum_{k=0}^{n}\Gamma_k\right) - \left(\frac{1}{n}\Gamma_{n+1}\right) \\
&\overset{n\longrightarrow\infty}{\longrightarrow} 0.
\end{aligned}
$$

2.3 Fall 2018 # 4 ✨

Let $f \in L^1([0,1])$. Prove that

$$\lim_{n\longrightarrow\infty} \int_0^1 f(x)|\sin nx|\, dx = \frac{2}{\pi}\int_0^1 f(x)\, dx$$

> Hint: Begin with the case that f is the characteristic function of an interval.

Add concepts.

Solution:

Concepts Used:

- ?

Case of characteristic function

- First suppose $f(x) = \chi_{[0,1]}(x)$.
- Note that $\sin(nx)$ has a period of $2\pi/n$, and thus $\left\lfloor \frac{n}{2\pi} \right\rfloor$ full periods in $[0,1]$.
- Taking the absolute value yields a new function with half the period, so a period of π/n

and $\lfloor \pi/n \rfloor$ full periods in $[0,1]$.

- We can compute the integral over one full period (which is independent of *which* period is chosen), and since $\sin(x)$ is positive and agrees with $|\sin(nx)|$ on the first period, we have

$$\begin{aligned}
\int_{\text{One Period}} |\sin(nx)|\, dx &= \int_0^{\pi/n} \sin(nx)\, dx \\
&= \frac{1}{n} \int_0^{\pi} \sin(u)\, du \quad u = nx \\
&= \frac{1}{n} - \cos(u) \Big|_0^{\pi} \\
&= \frac{2}{n}.
\end{aligned}$$

- Then break the integral up into integrals over periods $P_1, P_2, \cdots, P_N$ where $N := \lfloor n/\pi \rfloor$:

$$\begin{aligned}
\int_0^1 |\sin(nx)|\, dx &= \left(\sum_{j=1}^{N} \int_{P_j} |\sin(nx)|\, dx \right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)|\, dx \\
&= \left(\sum_{j=1}^{N} \frac{2}{n} \right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)|\, dx \\
&= N\left(\frac{2}{n}\right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)|\, dx \\
&:= \left\lfloor \frac{n}{\pi} \right\rfloor \frac{2}{n} + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)|\, dx \\
&= \frac{2}{\pi} + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)|\, dx \\
&:= \frac{2}{\pi} + R(n)
\end{aligned}$$

so it suffices to show that $R(n) \overset{n\to\infty}{\longrightarrow} 0$.

Need to justify removing floor function and cancellation.

- Showing this: ????????????

No clue how to show this.

General case

Not sure. Approximate f by simple functions...?

2.4 Fall 2017 # 4 ✨

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

1. Show that $f_n \longrightarrow 0$ pointwise but not uniformly on $[0,1]$.

Hint: Consider the maximum of f_n.

2.

$$\lim_{n \longrightarrow \infty} \int_0^1 n(1-x)^n \sin x \, dx = 0$$

Add concepts.

Walk through.

Solution:

Concepts Used:

- ?

2.4.1 a

Let $G(x) = \sum_{n=1}^{\infty} nx(1-x)^n$. Applying the ratio test, we have

$$\left| \frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n} \right| = \frac{n+1}{n} |1-x| \overset{n \longrightarrow \infty}{\longrightarrow} |1-x| < 1 \iff 0 \leq x \leq 2,$$

and in particular, this series converges on $[0,2]$. Thus its terms go to zero, and $nx(1-x)^n \longrightarrow 0$ on $[0,1] \subset [0,2]$.

To see that the convergence is not uniform, let $x_n = \frac{1}{n}$ and $\varepsilon > \frac{1}{e}$, then

$$\sup_{x \in [0,1]} |nx(1-x)^n - 0| \geq |nx_n(1-x_n)^n| = \left| \left(1 - \frac{1}{n}\right)^n \right| \overset{n \longrightarrow \infty}{\longrightarrow} e^{-1} > \varepsilon.$$

2.4.2 b

Note: could use the first part with $\sin(x) \leq x$, but then integral ends up more complicated.

Noting that $\sin(x) \leq 1$, we have We have

$$\begin{aligned}
\left| \int_0^1 n(1-x)^n \sin(x) \right| &\leq \int_0^1 |n(1-x)^n \sin(x)| \\
&\leq \int_0^1 |n(1-x)^n| \\
&= n \int_0^1 (1-x)^n \\
&= -\frac{n(1-x)^{n+1}}{n+1} \\
&\overset{n \longrightarrow \infty}{\longrightarrow} 0.
\end{aligned}$$

2.5 Spring 2017 # 3 ✨

Let

$$f_n(x) = ae^{-nax} - be^{-nbx} \quad \text{where } 0 < a < b.$$

Show that

a. $\displaystyle\sum_{n=1}^{\infty} |f_n|$ is not in $L^1([0,\infty), m)$

Hint: $f_n(x)$ has a root x_n.

b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0,\infty), m) \quad \text{and} \quad \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)\, dm = \ln \frac{b}{a}$$

Not complete.

Add concepts.

Walk through.

Solution:

Concepts Used:

- ?

2.5.1 a

Letting $x_n := \dfrac{1}{n}$, we have

$$\sum_{k=1}^{\infty} |f_k(x)| \geq |f_n(x_n)| = \left| ae^{-ax} - be^{-bx} \right| := M.$$

In particular, $\sup_x |f_n(x)| \not\longrightarrow 0$, so the terms do not go to zero and the sum can not converge.

2.5.2 b

?

2.6 Fall 2016 # 1 ✨

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that f converges to a differentiable function on $(1, \infty)$ and that

$$f'(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n^x} \right)'.$$

Hint:

$$\left(\frac{1}{n^x} \right)' = -\frac{1}{n^x} \ln n$$

Add concepts.

Solution:

Concepts Used:

- ?

- Set $f_N(x) \coloneqq \sum_{n=1}^{N} n^{-x}$, so $f(x) = \lim_{N \longrightarrow \infty} f_N(x)$.
- If an interchange of limits is justified, we have

$$\begin{aligned}
\frac{\partial}{\partial x} \lim_{N \longrightarrow \infty} \sum_{n=1}^{N} n^{-x} &= \lim_{h \longrightarrow 0} \lim_{N \longrightarrow \infty} \frac{1}{h} \left[\left(\sum_{n=1}^{N} n^{-x} \right) - \left(\sum_{n=1}^{N} n^{-(x+h)} \right) \right] \\
&\overset{?}{=} \lim_{N \longrightarrow \infty} \lim_{h \longrightarrow 0} \frac{1}{h} \left[\left(\sum_{n=1}^{N} n^{-x} \right) - \left(\sum_{n=1}^{N} n^{-(x+h)} \right) \right] \\
&= \lim_{N \longrightarrow \infty} \lim_{h \longrightarrow 0} \frac{1}{h} \left[\sum_{n=1}^{N} n^{-x} - n^{-(x+h)} \right] \quad (1) \\
&= \lim_{N \longrightarrow \infty} \sum_{n=1}^{N} \lim_{h \longrightarrow 0} \frac{1}{h} \left[n^{-x} - n^{-(x+h)} \right] \quad \text{since this is a finite sum} \\
&\coloneqq \lim_{N \longrightarrow \infty} \sum_{n=1}^{N} \frac{\partial}{\partial x} \left(\frac{1}{n^x} \right) \\
&= \lim_{N \longrightarrow \infty} \sum_{n=1}^{N} -\frac{\ln(n)}{n^x},
\end{aligned}$$

  where the combining of sums in (1) is valid because $\sum n^{-x}$ is absolutely convergent for $x > 1$ by the p-test.
- Thus it suffices to justify the interchange of limits and show that the last sum converges on $(1, \infty)$.
- Claim: $\sum n^{-x} \ln(n)$ converges.
  - Use the fact that for any fixed $\varepsilon > 0$,

$$\lim_{n \longrightarrow \infty} \frac{\ln(n)}{n^{\varepsilon}} \overset{L.H.}{=} \lim_{n \longrightarrow \infty} \frac{1/n}{\varepsilon n^{\varepsilon - 1}} = \lim_{n \longrightarrow \infty} \frac{1}{\varepsilon n^{\varepsilon}} = 0,$$

  - This implies that for a fixed $\varepsilon > 0$ and for any constant $c > 0$ there exists an N large enough such that $n \geq N$ implies $\ln(n)/n^{\varepsilon} < c$, i.e. $\ln(n) < cn^{\varepsilon}$.

- Taking $c = 1$, we have $n \geq N \implies \ln(n) < n^{\varepsilon}$
- We thus break up the sum:

$$\begin{aligned}\sum_{n\in\mathbb{N}} \frac{\ln(n)}{n^x} &= \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{\ln(n)}{n^x} \\ &\leq \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{n^{\varepsilon}}{n^x} \\ &:= C_{\varepsilon} + \sum_{n=N}^{\infty} \frac{n^{\varepsilon}}{n^x} \quad \text{with } C_{\varepsilon} < \infty \text{ a constant} \\ &= C_{\varepsilon} + \sum_{n=N}^{\infty} \frac{1}{n^{x-\varepsilon}},\end{aligned}$$

  where the last term converges by the p-test if $x - \varepsilon > 1$.
- But ε can depend on x, and if $x \in (1, \infty)$ is fixed we can choose $\varepsilon < |x - 1|$ to ensure this.

- Claim: the interchange of limits is justified.

  ?

2.7 Fall 2016 # 5 ✨

Let $\varphi \in L^{\infty}(\mathbb{R})$. Show that the following limit exists and satisfies the equality

$$\lim_{n\longrightarrow\infty} \left(\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2}\,dx \right)^{\frac{1}{n}} = \|\varphi\|_{\infty}.$$

Walk through.

Add concepts.

Solution:

Concepts Used:

- ?

Let L be the LHS and R be the RHS.

Claim: $L \leq R$. - Since $|\varphi| \leq \|\varphi\|_\infty$ a.e., we can write

$$
\begin{aligned}
L^{\frac{1}{n}} &:= \int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} \\
&\leq \int_{\mathbb{R}} \frac{\|\varphi\|_\infty^n}{1+x^2} \\
&= \|\varphi\|_\infty^n \int_{\mathbb{R}} \frac{1}{1+x^2} \\
&= \|\varphi\|_\infty^n \arctan(x) \Big|_{-\infty}^{\infty} \\
&= \|\varphi\|_\infty^n \left(\frac{\pi}{2} - \frac{-\pi}{2} \right) \\
&= \pi \|\varphi\|_\infty^n
\end{aligned}
$$

$$
\begin{aligned}
\implies L^{\frac{1}{n}} &\leq \sqrt[n]{\pi \|\varphi\|_\infty^n} \\
\implies L &\leq \pi^{\frac{1}{n}} \|\varphi\|_\infty \\
&\overset{n \longrightarrow \infty}{\longrightarrow} \|\varphi\|_\infty,
\end{aligned}
$$

where we've used the fact that $c^{\frac{1}{n}} \overset{n \longrightarrow \infty}{\longrightarrow} 1$ for any constant c.

Actually true? Need conditions?

Claim: $R \leq L$.

- We will show that $R \leq L + \varepsilon$ for every $\varepsilon > 0$.
- Set

$$
S_\varepsilon := \left\{ x \in \mathbb{R}^n \;\middle|\; |\varphi(x)| \geq \|\varphi\|_\infty - \varepsilon \right\}.
$$

- Then we have

$$
\begin{aligned}
\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2}\, dx &\geq \int_{S_\varepsilon} \frac{|\varphi(x)|^n}{1+x^2}\, dx \quad S_\varepsilon \subset \mathbb{R} \\
&\geq \int_{S_\varepsilon} \frac{(\|\varphi\|_\infty - \varepsilon)^n}{1+x^2}\, dx \qquad \text{by definition of } S_\varepsilon \\
&= (\|\varphi\|_\infty - \varepsilon)^n \int_{S_\varepsilon} \frac{1}{1+x^2}\, dx \\
&= (\|\varphi\|_\infty - \varepsilon)^n C_\varepsilon \qquad \text{where } C_\varepsilon \text{ is some constant}
\end{aligned}
$$

$$
\begin{aligned}
\implies \left(\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2}\, dx \right)^{\frac{1}{n}} &\geq (\|\varphi\|_\infty - \varepsilon) C_\varepsilon^{\frac{1}{n}} \\
&\overset{n \longrightarrow \infty}{\longrightarrow} (\|\varphi\|_\infty - \varepsilon) \cdot 1 \\
&\overset{\varepsilon \longrightarrow 0}{\longrightarrow} \|\varphi\|_\infty,
\end{aligned}
$$

where we've again used the fact that $c^{\frac{1}{n}} \longrightarrow 1$ for any constant.

2.8 Fall 2016 # 6 ✨

Let $f, g \in L^2(\mathbb{R})$. Show that

$$\lim_{n \longrightarrow \infty} \int_{\mathbb{R}} f(x) g(x+n)\, dx = 0$$

Add concepts.

Solution:

Concepts Used:

- ?

- Use the fact that L^p has small tails: if $h \in L^2(\mathbb{R})$, then for any $\varepsilon > 0$,

$$\forall \varepsilon,\ \exists N \in \mathbb{N} \quad \text{such that} \quad \int_{|x| \geq N} |h(x)|^2\, dx < \varepsilon.$$

How to prove small tails in L^p?

- So choose n large enough so the tails of both f and g are smaller than ε.
- Apply Cauchy-Schwarz:

$$\begin{aligned} \left| \int_{\mathbb{R}} f(x) g(x+n)\, dx \right| &\leq \int_{\mathbb{R}} |f(x) g(x+n)|\, dx \\ &\leq \int_{\mathbb{R}} . \end{aligned}$$

2.9 Spring 2016 # 1 🚩

For $n \in \mathbb{N}$, define

$$]e_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad E_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

Show that $e_n < E_n$, and prove Bernoulli's inequality:

$$(1+x)^n \geq 1 + nx \text{ for } -1 < x < \infty \text{ and } n \in \mathbb{N}$$

Use this to show the following:

1. The sequence e_n is increasing.
2. The sequence E_n is decreasing.
3. $2 < e_n < E_n < 4$.
4. $\lim_{n \longrightarrow \infty} e_n = \lim_{n \longrightarrow \infty} E_n$.

2.10 Fall 2015 # 1 🚩

Define

$$f(x) = c_0 + c_1 x^1 + c_2 x^2 + \ldots + c_n x^n \text{ with } n \text{ even and } c_n > 0.$$

Show that there is a number x_m such that $f(x_m) \leq f(x)$ for all $x \in \mathbb{R}$.