

# Real Analysis Review Notes

D. Zack Garza

Saturday 15<sup>th</sup> August, 2020

## Contents

<b>1</b>	<b>Basics</b>	<b>2</b>
1.1	Useful Techniques . . . . .	2
1.2	Definitions . . . . .	3
1.3	Theorems . . . . .	4
1.3.1	Topology / Sets . . . . .	4
1.3.2	Functions . . . . .	5
1.4	Uniform Convergence . . . . .	5
1.4.1	Series . . . . .	6
<b>2</b>	<b>Measure Theory</b>	<b>6</b>
2.1	Useful Techniques . . . . .	6
2.2	Definitions . . . . .	7
2.3	Theorems . . . . .	7
<b>3</b>	<b>Integration</b>	<b>9</b>
3.1	Useful Techniques . . . . .	9
3.2	Definitions . . . . .	9
3.3	Theorems . . . . .	10
3.3.1	Convergence Theorems . . . . .	11
3.4	$L^1$ Facts . . . . .	14
3.5	$L^p$ Spaces . . . . .	16
<b>4</b>	<b>Fourier Transform and Convolution</b>	<b>17</b>
4.1	The Fourier Transform . . . . .	17
4.2	Approximate Identities . . . . .	19
<b>5</b>	<b>Functional Analysis</b>	<b>20</b>
5.1	Definitions . . . . .	20
5.2	Theorems . . . . .	21
<b>6</b>	<b>Extra Problems</b>	<b>24</b>
6.1	Greatest Hits . . . . .	24
6.2	By Topic . . . . .	25
6.2.1	Topology . . . . .	25
6.2.2	Continuity . . . . .	25

---

6.2.3	Differentiation . . . . .	25
6.2.4	Advanced Limitology . . . . .	25
<b>7</b>	<b>Practice Exam (November 2014)</b>	<b>27</b>
7.1	1: Fubini-Tonelli . . . . .	27
7.1.1	a . . . . .	27
7.1.2	b . . . . .	28
7.2	2: Convolutions and the Fourier Transform . . . . .	28
7.2.1	a . . . . .	28
7.2.2	b . . . . .	28
7.2.3	c . . . . .	28
7.3	3: Hilbert Spaces . . . . .	28
7.3.1	a . . . . .	28
7.3.2	b . . . . .	29
7.3.3	c . . . . .	29
7.4	4: $L^p$ Spaces . . . . .	30
7.4.1	a . . . . .	30
7.4.2	b . . . . .	30
7.4.3	c . . . . .	30
7.5	5: Dual Spaces . . . . .	30
7.5.1	a . . . . .	30
7.5.2	b . . . . .	31
7.5.3	c . . . . .	31
<b>8</b>	<b>Midterm Exam 2 (November 2018)</b>	<b>31</b>
8.1	1 (Integration by Parts) . . . . .	31
8.2	2 . . . . .	31
8.3	3 . . . . .	31
<b>9</b>	<b>Midterm Exam 2 (December 2014)</b>	<b>31</b>
9.1	1 . . . . .	31
9.2	2 . . . . .	32
9.3	3 . . . . .	32
9.4	4 (Weierstrass Approximation Theorem) . . . . .	32
<b>10</b>	<b>Inequalities and Equalities</b>	<b>32</b>
10.1	Less Explicitly Used Inequalities . . . . .	35

## 1 Basics

### 1.1 Useful Techniques

- General advice: try swapping the orders of limits, sums, integrals, etc.
- Limits:
  - Take the  $\limsup$  or  $\liminf$ , which always exist, and aim for an inequality like

$$c \leq \liminf a_n \leq \limsup a_n \leq c.$$

- $\lim f_n = \limsup f_n = \liminf f_n$  iff the limit exists, so to show some  $g$  is a limit, show

$$\limsup f_n \leq g \leq \liminf f_n \quad (\implies g = \lim f).$$

- A limit does *not* exist if  $\liminf a_n > \limsup a_n$ .

- Sequences and Series

- If  $f_n$  has a global maximum (computed using  $f'_n$  and the first derivative test)  $M_n \rightarrow 0$ , then  $f_n \rightarrow 0$  uniformly.
- For a fixed  $x$ , if  $f = \sum f_n$  converges *uniformly* on some  $B_r(x)$  and each  $f_n$  is continuous at  $x$ , then  $f$  is also continuous at  $x$ .

- Equalities

- Split into upper and lower bounds:

$$a = b \iff a \leq b \text{ and } a \geq b.$$

- Use an epsilon of room:

$$a < b + \epsilon \forall \epsilon \implies a \leq b.$$

- Showing something is zero:

$$|a| \leq \epsilon \forall \epsilon \implies a = 0.$$

- Simplifications:

- To show something for a measurable set, show it for bounded/compact/elementary sets/
- To show something for a function, show it for continuous, bounded, compactly supported, simple, indicator functions,  $L^1$ , etc
- Replace a continuous sequence ( $\epsilon \rightarrow 0$ ) with an arbitrary countable sequence ( $x_n \rightarrow 0$ )
- Intersect with a ball  $B_r(\mathbf{0}) \subset \mathbb{R}^n$ .

- Integrals

- Break up  $\mathbb{R}^n = \{|x| \leq 1\} \coprod \{|x| > 1\}$ .

## 1.2 Definitions

**Definition 1.0.1** (Uniform Continuity).

$f$  is uniformly continuous iff

$$\begin{aligned} \forall \epsilon \quad \exists \delta(\epsilon) \mid \quad \forall x, y, \quad |x - y| < \delta \implies |f(x) - f(y)| < \epsilon \\ \iff \forall \epsilon \quad \exists \delta(\epsilon) \mid \quad \forall x, y, \quad |y| < \delta \implies |f(x - y) - f(y)| < \epsilon. \end{aligned}$$

**Definition (Nowhere Dense Sets)** A set  $S$  is **nowhere dense** iff the closure of  $S$  has empty interior  
iff every interval contains a subinterval that does not intersect  $S$ .

**Definition (Meager Sets)** A set is **meager** if it is a *countable* union of nowhere dense sets.

**Definition 1.0.2** ( $F_\sigma$  and  $G_\delta$  Sets).

An  $F_\sigma$  set is a union of closed sets, and a  $G_\delta$  set is an intersection of opens.

Mnemonic: “F” stands for *ferme*, which is “closed” in French, and  $\sigma$  corresponds to a “sum”,  
i.e. a union.

**Theorem (Heine-Cantor)** Every continuous function on a compact space is uniformly continuous.

**Definition 1.0.3** (Limsup/Liminf).

$$\begin{aligned}\limsup_n a_n &= \lim_{n \rightarrow \infty} \sup_{j \geq n} a_j = \inf_{n \geq 0} \sup_{j \geq n} a_j \\ \liminf_n a_n &= \lim_{n \rightarrow \infty} \inf_{j \geq n} a_j = \sup_{n \geq 0} \inf_{j \geq n} a_j.\end{aligned}$$

## 1.3 Theorems

### 1.3.1 Topology / Sets

**Lemma** Metric spaces are compact iff they are sequentially compact, (i.e. every sequence has a convergent subsequence).

**Proposition** The unit ball in  $C([0, 1])$  with the sup norm is not compact.

**Proof** Take  $f_k(x) = x^n$ , which converges to a dirac delta at 1. The limit is not continuous, so no subsequence can converge.

**Proposition** A *finite* union of nowhere dense is again nowhere dense.

**Lemma (Convergent Sums Have Small Tails)**

$$\sum a_n < \infty \implies a_n \rightarrow 0 \quad \text{and} \quad \sum_{k=N}^{\infty} a_n \xrightarrow{N \rightarrow \infty} 0$$

**Theorem (Heine-Borel)**  $X \subseteq \mathbb{R}^n$  is compact  $\iff X$  is closed and bounded.

**Lemma (Geometric Series)**

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \iff |x| < 1.$$

*Corollary:*  $\sum_{k=0}^{\infty} \frac{1}{2^k} = 1.$

**Lemma** The Cantor set is closed with empty interior.

**Proof** Its complement is a union of open intervals, and can't contain an interval since intervals have positive measure and  $m(C_n)$  tends to zero.

**Corollary** The Cantor set is nowhere dense.

**Lemma** Singleton sets in  $\mathbb{R}$  are closed, and thus  $\mathbb{Q}$  is an  $F_\sigma$  set.

**Theorem (Baire)**  $\mathbb{R}$  is a **Baire space** (countable intersections of open, dense sets are still dense). Thus  $\mathbb{R}$  can not be written as a countable union of nowhere dense sets.

**Lemma** Any nonempty set which is bounded from above (resp. below) has a well-defined supremum (resp. infimum).

### 1.3.2 Functions

**Proposition (Existence of Smooth Compactly Supported Functions)** There exist smooth compactly supported functions, e.g. take

$$f(x) = e^{-\frac{1}{x^2}} \chi_{(0,\infty)}(x).$$

**Lemma** There is a function discontinuous precisely on  $\mathbb{Q}$ .

**Proof**  $f(x) = \frac{1}{n}$  if  $x = r_n \in \mathbb{Q}$  is an enumeration of the rationals, and zero otherwise. The limit at every point is 0.

**Lemma** There *do not* exist functions that are discontinuous precisely on  $\mathbb{R} \setminus \mathbb{Q}$ .

**Proof**  $D_f$  is always an  $F_\sigma$  set, which follows by considering the oscillation  $\omega_f$ .  $\omega_f(x) = 0 \iff f$  is continuous at  $x$ , and  $D_f = \bigcup_n A_{\frac{1}{n}}$  where  $A_\varepsilon = \{\omega_f \geq \varepsilon\}$  is closed.

**Proposition** A function  $f : (a, b) \rightarrow \mathbb{R}$  is Lipschitz  $\iff f$  is differentiable and  $f'$  is bounded. In this case,  $|f'(x)| \leq C$ , the Lipschitz constant.

## 1.4 Uniform Convergence

**Theorem (Weierstrass Approximation)** If  $[a, b] \subset \mathbb{R}$  is a closed interval and  $f$  is continuous, then for every  $\varepsilon > 0$  there exists a polynomial  $p_\varepsilon$  such that  $\|f - p_\varepsilon\|_{L^\infty([a,b])} \xrightarrow{\varepsilon \rightarrow 0} 0$ .

**Theorem (Egorov)** Let  $E \subseteq \mathbb{R}^n$  be measurable with  $m(E) > 0$  and  $\{f_k : E \rightarrow \mathbb{R}\}$  be measurable functions such that

$$f(x) := \lim_{k \rightarrow \infty} f_k(x) < \infty$$

exists almost everywhere.

Then  $f_k \rightarrow f$  *almost uniformly*, i.e.

$$\forall \varepsilon > 0, \exists F \subseteq E \text{ closed such that } m(E \setminus F) < \varepsilon \text{ and } f_k \xrightarrow{u} f \text{ on } F.$$

**Proposition** The space  $X = C([0, 1])$ , continuous functions  $f : [0, 1] \rightarrow \mathbb{R}$ , equipped with the norm  $\|f\| = \sup_{x \in [0,1]} |f(x)|$ , is a **complete** metric space.

**Proof**

1. Let  $\{f_k\}$  be Cauchy in  $X$ .
2. Define a candidate limit using pointwise convergence:

Fix an  $x$ ; since

$$|f_k(x) - f_j(x)| \leq \|f_k - f_j\| \rightarrow 0$$

the sequence  $\{f_k(x)\}$  is Cauchy in  $\mathbb{R}$ . So define  $f(x) := \lim_k f_k(x)$ .

3. Show that  $\|f_k - f\| \rightarrow 0$ :

$$|f_k(x) - f_j(x)| < \varepsilon \quad \forall x \implies \lim_j |f_k(x) - f_j(x)| < \varepsilon \quad \forall x$$

Alternatively,  $\|f_k - f\| \leq \|f_k - f_N\| + \|f_N - f_j\|$ , where  $N, j$  can be chosen large enough to bound each term by  $\varepsilon/2$ .

---

4. Show that  $f \in X$ :

The uniform limit of continuous functions is continuous.

Note: in other cases, you may need to show the limit is bounded, or has bounded derivative, or whatever other conditions define  $X$ .

**Theorem (Uniform Limits of Continuous Functions are Continuous)** A uniform limit of continuous functions is continuous.

**Lemma (Testing Uniform Convergence)**  $f_n \rightarrow f$  uniformly iff there exists an  $M_n$  such that  $\|f_n - f\|_\infty \leq M_n \rightarrow 0$ .

**Negating:** find an  $x$  which depends on  $n$  for which the norm is bounded below.

**Lemma (Uniform Limits Commute with Integrals)** If  $f_n \rightarrow f$  uniformly, then  $\int f_n = \int f$ .

**Lemma (Uniform Convergence and Derivatives)** If  $f'_n \rightarrow g$  uniformly for some  $g$  and  $f_n \rightarrow f$  pointwise (or at least at one point), then  $g = f'$ .

### 1.4.1 Series

**Lemma (Pointwise Convergence for a Series of Functions)** If  $f_n(x) \leq M_n$  for a fixed  $x$  where  $\sum M_n < \infty$ , then the series  $f(x) = \sum f_n(x)$  converges pointwise.

**Lemma (Small Tails for Series of Functions)** If  $\sum f_n$  converges then  $f_n \rightarrow 0$  uniformly.

**Lemma (M-test for Series)** If  $|f_n(x)| \leq M_n$  which does not depend on  $x$ , then  $\sum f_n$  converges uniformly.

**Lemma (p-tests)** Let  $n$  be a fixed dimension and set  $B = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ .

$$\begin{aligned}\sum \frac{1}{n^p} < \infty &\iff p > 1 \\ \int_\varepsilon^\infty \frac{1}{x^p} < \infty &\iff p > 1 \\ \int_0^1 \frac{1}{x^p} < \infty &\iff p < 1 \\ \int_B \frac{1}{|x|^p} < \infty &\iff p < n \\ \int_{B^c} \frac{1}{|x|^p} < \infty &\iff p > n\end{aligned}$$

## 2 Measure Theory

### 2.1 Useful Techniques

- $s = \inf \{x \in X\} \implies$  for every  $\varepsilon$  there is an  $x \in X$  such that  $x \leq s + \varepsilon$ .
- Always consider bounded sets, and if  $E$  is unbounded write  $E = \bigcup_n B_n(0) \cap E$  and use countable subadditivity or continuity of measure.

## 2.2 Definitions

**Definition (Outer Measure)** The outer measure of a set is given by

$$m_*(E) := \inf_{\substack{\{Q_i\} \supset E \\ \text{closed cubes}}} \sum |Q_i|.$$

**Definition (Limsup and Liminf of Sets)**

$$\begin{aligned} \limsup_n A_n &:= \bigcap_n \bigcup_{j \geq n} A_j = \left\{ x \mid x \in A_n \text{ for inf. many } n \right\} \\ \liminf_n A_n &:= \bigcup_n \bigcap_{j \geq n} A_j = \left\{ x \mid x \in A_n \text{ for all except fin. many } n \right\} \end{aligned}$$

**Definition (Lebesgue Measurable Set)** A subset  $E \subseteq \mathbb{R}^n$  is *Lebesgue measurable* iff for every  $\varepsilon > 0$  there exists an open set  $O \supseteq E$  such that  $m_*(O \setminus E) < \varepsilon$ . In this case, we define  $m(E) := m_*(E)$ .

## 2.3 Theorems

**Lemma** Every open subset of  $\mathbb{R}$  (resp  $\mathbb{R}^n$ ) can be written as a unique countable union of disjoint (resp. almost disjoint) intervals (resp. cubes).

**Lemma (Properties of Outer Measure)**

- Monotonicity:  $E \subseteq F \implies m_*(E) \leq m_*(F)$ .
- Countable Subadditivity:  $m_*(\bigcup E_i) \leq \sum m_*(E_i)$ .
- Approximation: For all  $E$  there exists a  $G \supseteq E$  such that  $m_*(G) \leq m_*(E) + \varepsilon$ .
- Disjoint<sup>1</sup> Additivity:  $m_*(A \amalg B) = m_*(A) + m_*(B)$ .

**Lemma (Subtraction of Measure)**

$$m(A) = m(B) + m(C) \quad \text{and} \quad m(C) < \infty \implies m(A) - m(C) = m(B).$$

**Lemma (Continuity of Measure)**

$$\begin{aligned} E_i \nearrow E &\implies m(E_i) \longrightarrow m(E) \\ m(E_1) < \infty \text{ and } E_i \searrow E &\implies m(E_i) \longrightarrow m(E). \end{aligned}$$

**Proof** 1. Break into disjoint annuli  $A_2 = E_2 \setminus E_1$ , etc then apply countable disjoint additivity to  $E = \amalg A_i$ .

2. Use  $E_1 = (\amalg E_j \setminus E_{j+1}) \amalg (\bigcap E_j)$ , taking measures yields a telescoping sum, and use countable disjoint additivity.

---

<sup>1</sup>This holds for outer measure **iff**  $\text{dist}(A, B) > 0$ .

**Theorem** Suppose  $E$  is measurable; then for every  $\varepsilon > 0$ ,

1. There exists an open  $O \supset E$  with  $m(O \setminus E) < \varepsilon$
2. There exists a closed  $F \subset E$  with  $m(E \setminus F) < \varepsilon$
3. There exists a compact  $K \subset E$  with  $m(E \setminus K) < \varepsilon$ .

**Proof**

- (1): Take  $\{Q_i\} \rightrightarrows E$  and set  $O = \bigcup Q_i$ .
- (2): Since  $E^c$  is measurable, produce  $O \supset E^c$  with  $m(O \setminus E^c) < \varepsilon$ .
  - Set  $F = O^c$ , so  $F$  is closed.
  - Then  $F \subset E$  by taking complements of  $O \supset E^c$
  - $E \setminus F = O \setminus E^c$  and taking measures yields  $m(E \setminus F) < \varepsilon$
- (3): Pick  $F \subset E$  with  $m(E \setminus F) < \varepsilon/2$ .
  - Set  $K_n = F \cap \mathbb{D}_n$ , a ball of radius  $n$  about 0.
  - Then  $E \setminus K_n \searrow E \setminus F$
  - Since  $m(E) < \infty$ , there is an  $N$  such that  $n \geq N \implies m(E \setminus K_n) < \varepsilon$ .

**Lemma** Lebesgue measure is translation and dilation invariant.

**Proof** Obvious for cubes; if  $Q_i \rightrightarrows E$  then  $Q_i + k \rightrightarrows E + k$ , etc.

Flesh out this proof.

**Theorem (Non-Measurable Sets)** There is a non-measurable set.

**Proof**

- Use AOC to choose one representative from every coset of  $\mathbb{R}/\mathbb{Q}$  on  $[0, 1)$ , which is countable, and assemble them into a set  $N$
- Enumerate the rationals in  $[0, 1]$  as  $q_j$ , and define  $N_j = N + q_j$ . These intersect trivially.
- Define  $M := \bigsqcup N_j$ , then  $[0, 1) \subseteq M \subseteq [-1, 2)$ , so the measure must be between 1 and 3. By translation invariance,  $m(N_j) = m(N)$ , and disjoint additivity forces  $m(M) = 0$ , a contradiction.

**Proposition (Borel Characterization of Measurable Sets)** If  $E$  is Lebesgue measurable, then  $E = H \bigsqcup N$  where  $H \in \mathcal{F}_\sigma$  and  $N$  is null.

**Useful technique:**  $\mathcal{F}_\sigma$  sets are Borel, so establish something for Borel sets and use this to extend it to Lebesgue.

**Proof** For every  $\frac{1}{n}$  there exists a closed set  $K_n \subset E$  such that  $m(E \setminus K_n) \leq \frac{1}{n}$ . Take  $K = \bigcup K_n$ , wlog  $K_n \nearrow K$  so  $m(K) = \lim m(K_n) = m(E)$ . Take  $N := E \setminus K$ , then  $m(N) = 0$ .

**Lemma** If  $A_n$  are all measurable,  $\limsup A_n$  and  $\liminf A_n$  are measurable.

**Proof** Measurable sets form a sigma algebra, and these are expressed as countable unions/intersections of measurable sets.

**Theorem (Borel-Cantelli)** Let  $\{E_k\}$  be a countable collection of measurable sets. Then

$$\sum_k m(E_k) < \infty \implies \text{almost every } x \in \mathbb{R} \text{ is in at most finitely many } E_k.$$

**Proof**



- 
- If  $E = \limsup_j E_j$  with  $\sum m(E_j) < \infty$  then  $m(E) = 0$ .
  - If  $E_j$  are measurable, then  $\limsup_j E_j$  is measurable.
  - If  $\sum_j m(E_j) < \infty$ , then  $\sum_{j=N}^{\infty} m(E_j) \xrightarrow{N \rightarrow \infty} 0$  as the tail of a convergent sequence.
  - $E = \limsup_j E_j = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j \implies E \subseteq \bigcup_{j=k}^{\infty} E_j$  for all  $k$
  - $E \subseteq \bigcup_{j=k}^{\infty} E_j \implies m(E) \leq \sum_{j=k}^{\infty} m(E_j) \xrightarrow{k \rightarrow \infty} 0$ .

#### Lemma

- Characteristic functions are measurable
- If  $f_n$  are measurable, so are  $|f_n|$ ,  $\limsup f_n$ ,  $\liminf f_n$ ,  $\lim f_n$ ,
- Sums and differences of measurable functions are measurable,
- Cones  $F(x, y) = f(x)$  are measurable,
- Compositions  $f \circ T$  for  $T$  a linear transformation are measurable,
- “Convolution-ish” transformations  $(x, y) \mapsto f(x - y)$  are measurable

**Proof (Convolution)** Take the cone on  $f$  to get  $F(x, y) = f(x)$ , then compose  $F$  with the linear transformation  $T = [1, -1; 1, 0]$ .

## 3 Integration

Notation:

- “ $f$  vanishes at infinity” means  $f(x) \xrightarrow{|x| \rightarrow \infty} 0$ .
- “ $f$  has small tails” means  $\int_{|x| \geq N} f \xrightarrow{N \rightarrow \infty} 0$ .

### 3.1 Useful Techniques

- Break integration domain up into disjoint annuli.
- Break integrals or sums into  $x < 1$  and  $x \geq 1$ .
- Calculus techniques: Taylor series, IVT, ...
- Approximate by dense subsets of functions
- Useful facts about compactly supported continuous functions:
  - Uniformly continuous
  - Bounded

### 3.2 Definitions

**Definition (\$L^+\$)**  $f \in L^+$  iff  $f$  is measurable and non-negative.

**Definition (Integrable)** A measurable function is integrable iff  $\|f\|_1 < \infty$ .

**Definition (The Infinity Norm)**

$$\|f\|_\infty := \inf_{\alpha \geq 0} \left\{ \alpha \mid m\{|f| \geq \alpha\} = 0 \right\}.$$

**Definition (Essentially Bounded Functions)** A function  $f : X \rightarrow \mathbb{C}$  is *essentially bounded* iff there exists a real number  $c$  such that  $\mu(\{|f| > c\}) = 0$ , i.e.  $\|f\|_\infty < \infty$ .

If  $f \in L^\infty(X)$ , then  $f$  is equal to some bounded function  $g$  almost everywhere.

**Definition (L infty)**

$$L^\infty(X) := \left\{ f : X \rightarrow \mathbb{C} \mid f \text{ is essentially bounded} \right\} := \left\{ f : X \rightarrow \mathbb{C} \mid \|f\|_\infty < \infty \right\},$$

Example:

- $f(x) = x\chi_{\mathbb{Q}}(x)$  is essentially bounded but not bounded.

**3.3 Theorems**

Useful facts about  $C_c$  functions:

- Bounded almost everywhere
- Uniformly continuous

**Theorem (p-Test for Integrals in  $\mathbb{R}$ )**

$$\begin{aligned} \int_0^1 \frac{1}{x^p} < \infty &\iff p < 1 \\ \int_1^\infty \frac{1}{x^p} < \infty &\iff p > 1. \end{aligned}$$

Slogan: big powers of  $x$  help us in neighborhoods of infinity and hurt around zero

Some integrable functions:

- $\int \frac{1}{1+x^2} = \arctan(x) \xrightarrow{x \rightarrow \infty} \pi/2 < \infty$
- Any bounded function (or continuous on a compact set, by EVT)
- $\int_0^1 \frac{1}{\sqrt{x}} < \infty$
- $\int_0^1 \frac{1}{x^{1-\varepsilon}} < \infty$
- $\int_1^\infty \frac{1}{x^{1+\varepsilon}} < \infty$

Some non-integrable functions:

- $\int_0^1 \frac{1}{x} = \infty$ .

- $\int_1^\infty \frac{1}{x} = \infty.$
- $\int_1^\infty \frac{1}{\sqrt{x}} = \infty$
- $\int_1^\infty \frac{1}{x^{1-\varepsilon}} = \infty$
- $\int_0^1 \frac{1}{x^{1+\varepsilon}} = \infty$

### 3.3.1 Convergence Theorems

**Theorem (Monotone Convergence)** If  $f_n \in L^+$  and  $f_n \nearrow f$  a.e., then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \quad \int f_n \longrightarrow \int f.$$

Needs to be positive and increasing.

**Theorem (Dominated Convergence)** If  $f_n \in L^1$  and  $f_n \longrightarrow f$  a.e. with  $|f_n| \leq g$  for some  $g \in L^1$ , then  $f \in L^1$  and

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \quad \int f_n \longrightarrow \int f < \infty,$$

and more generally,

$$\int |f_n - f| \longrightarrow 0.$$

Positivity *not* needed.

**Theorem (Generalized DCT)** If

- $f_n \in L^1$  with  $f_n \longrightarrow f$  a.e.,
- There exist  $g_n \in L^1$  with  $|f_n| \leq g_n$ ,  $g_n \geq 0$ .
- $g_n \longrightarrow g$  a.e. with  $g \in L^1$ , and
- $\lim \int g_n = \int g$ ,

then  $f \in L^1$  and  $\lim \int f_n = \int f < \infty$ .

Note that this is the DCT with  $|f_n| < |g|$  relaxed to  $|f_n| < g_n \longrightarrow g \in L^1$ .

**Proof (Generalized DCT)** Proceed by showing  $\limsup \int f_n \leq \int f \leq \liminf \int f_n$ :

- $\int f \geq \limsup \int f_n$ :

$$\begin{aligned}
\int g - \int f &= \int (g - f) \\
&\leq \liminf \int (g_n - f_n) \quad \text{Fatou} \\
&= \lim \int g_n + \liminf \int (-f_n) \\
&= \lim \int g_n - \limsup \int f_n \\
&= \int g - \limsup \int f_n
\end{aligned}$$

$$\implies \int f \geq \limsup \int f_n.$$

- Here we use  $g_n - f_n \xrightarrow{n \rightarrow \infty} g - f$  with  $0 \leq |f_n| - f_n \leq g_n - f_n$ , so  $g_n - f_n$  are nonnegative (and measurable) and Fatou's lemma applies.

- $\int f \leq \liminf \int f_n$ :

$$\begin{aligned}
\int g + \int f &= \int (g + f) \\
&\leq \liminf \int (g_n + f_n) \\
&= \lim \int g_n + \liminf \int f_n \\
&= \int g + \liminf \int f_n
\end{aligned}$$

$$\int f \leq \liminf \int f_n.$$

- Here we use that  $g_n + f_n \rightarrow g + f$  with  $0 \leq |f_n| + f_n \leq g_n + f_n$  so Fatou's lemma again applies.

**Lemma (Converges in  $L^1$  implies convergence of  $L^1$  norms)** If  $f \in L^1$ , then

$$\int |f_n - f| \rightarrow 0 \iff \int |f_n| \rightarrow \int |f|.$$

**Proof** Let  $g_n = |f_n| - |f_n - f|$ , then  $g_n \rightarrow |f|$  and

$$|g_n| = ||f_n| - |f_n - f|| \leq |f_n - (f_n - f)| = |f| \in L^1,$$

so the DCT applies to  $g_n$  and

$$\begin{aligned}
\|f_n - f\|_1 &= \int |f_n - f| + |f_n| - |f_n| = \int |f_n| - g_n \\
&\xrightarrow{DCT} \lim \int |f_n| - \int |f|.
\end{aligned}$$

**Theorem (Fatou's Lemma)** If  $f_n$  is a sequence of nonnegative measurable functions, then

$$\begin{aligned} \int \liminf_n f_n &\leq \liminf_n \int f_n \\ \limsup_n \int f_n &\leq \int \limsup_n f_n. \end{aligned}$$

**Theorem (Tonelli)** For  $f(x, y)$  **non-negative and measurable**, for almost every  $x \in \mathbb{R}^n$ ,

- $f_x(y)$  is a **measurable** function
- $F(x) = \int f(x, y) dy$  is a **measurable** function,
- For  $E$  measurable, the slices  $E_x := \{y \mid (x, y) \in E\}$  are measurable.
- $\int f = \int \int F$ , i.e. any iterated integral is equal to the original.

**Theorem (Fubini)** For  $f(x, y)$  **integrable**, for almost every  $x \in \mathbb{R}^n$ ,

- $f_x(y)$  is an **integrable** function
- $F(x) := \int f(x, y) dy$  is an **integrable** function,
- For  $E$  measurable, the slices  $E_x := \{y \mid (x, y) \in E\}$  are measurable.
- $\int f = \int \int f(x, y)$ , i.e. any iterated integral is equal to the original

**Theorem (Fubini/Tonelli)** If any iterated integral is **absolutely integrable**, i.e.  $\int \int |f(x, y)| < \infty$ , then  $f$  is integrable and  $\int f$  equals any iterated integral.

**Corollary (Measurable Slices)** Let  $E$  be a measurable subset of  $\mathbb{R}^n$ . Then

- For almost every  $x \in \mathbb{R}^{n_1}$ , the slice  $E_x := \{y \in \mathbb{R}^{n_2} \mid (x, y) \in E\}$  is measurable in  $\mathbb{R}^{n_2}$ .
- The function

$$\begin{aligned} F : \mathbb{R}^{n_1} &\longrightarrow \mathbb{R} \\ x &\mapsto m(E_x) = \int_{\mathbb{R}^{n_2}} \chi_{E_x} dy \end{aligned}$$

is measurable and

$$m(E) = \int_{\mathbb{R}^{n_1}} m(E_x) dx = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{E_x} dy dx$$

**Proof (Measurable Slices)**

$\implies :$

- Let  $f$  be measurable on  $\mathbb{R}^n$ .
- Then the cylinders  $F(x, y) = f(x)$  and  $G(x, y) = f(y)$  are both measurable on  $\mathbb{R}^{n+1}$ .
- Write  $\mathcal{A} = \{G \leq F\} \cap \{G \geq 0\}$ ; both are measurable.

$\impliedby :$

- Let  $A$  be measurable in  $\mathbb{R}^{n+1}$ .
- Define  $A_x = \{y \in \mathbb{R} \mid (x, y) \in A\}$ , then  $m(A_x) = f(x)$ .
- By the corollary,  $A_x$  is measurable set,  $x \mapsto A_x$  is a measurable function, and  $m(A) = \int f(x) dx$ .
- Then explicitly,  $f(x) = \chi_A$ , which makes  $f$  a measurable function.

**Proposition (Differentiating Under an Integral)** If  $\left| \frac{\partial}{\partial t} f(x, t) \right| \leq g(x) \in L^1$ , then letting  $F(t) = \int f(x, t) dt$ ,

$$\begin{aligned} \frac{\partial}{\partial t} F(t) &:= \lim_{h \rightarrow 0} \int \frac{f(x, t+h) - f(x, t)}{h} dx \\ &\stackrel{DCT}{=} \int \frac{\partial}{\partial t} f(x, t) dx. \end{aligned}$$

To justify passing the limit, let  $h_k \rightarrow 0$  be any sequence and define

$$f_k(x, t) = \frac{f(x, t + h_k) - f(x, t)}{h_k},$$

$$\text{so } f_k \xrightarrow{\text{pointwise}} \frac{\partial}{\partial t} f.$$

Apply the MVT to  $f_k$  to get  $f_k(x, t) = f_k(\xi, t)$  for some  $\xi \in [0, h_k]$ , and show that  $f_k(\xi, t) \in L^1$ .

**Proposition (Swapping Sum and Integral)** If  $f_n$  are non-negative and  $\sum \int |f|_n < \infty$ , then  $\sum \int f_n = \int \sum f_n$ .

**Proof** MCT. Let  $F_N = \sum_{n=1}^N f_n$  be a finite partial sum; then there are simple functions  $\varphi_n \nearrow f_n$  and so  $\sum_{n=1}^N \varphi_n \nearrow F_N$ , so apply MCT.

**Lemma** If  $f_k \in L^1$  and  $\sum \|f_k\|_1 < \infty$  then  $\sum f_k$  converges almost everywhere and in  $L^1$ .

**Proof** Define  $F_N = \sum_{k=1}^N f_k$  and  $F = \lim_N F_N$ , then  $\|F_N\|_1 \leq \sum_{k=1}^N \|f_k\|_1 < \infty$  so  $F \in L^1$  and  $\|F_N - F\|_1 \rightarrow 0$  so the sum converges in  $L^1$ . Almost everywhere convergence: ?

### 3.4 $L^1$ Facts

**Lemma (Translation Invariance)** The Lebesgue integral is translation invariant, i.e.  $\int f(x) dx = \int f(x+h) dx$  for any  $h$ .

**Proof**

- For characteristic functions,  $\int_E f(x+h) = \int_{E+h} f(x) = m(E+h) = m(E) = \int_E f$  by translation invariance of measure.
- So this also holds for simple functions by linearity

- For  $f \in L^+$ , choose  $\varphi_n \nearrow f$  so  $\int \varphi_n \rightarrow \int f$ .
- Similarly,  $\tau_h \varphi_n \nearrow \tau_h f$  so  $\int \tau_h f \rightarrow \int f$
- Finally  $\left\{ \int \tau_h \varphi \right\} = \left\{ \int \varphi \right\}$  by step 1, and the suprema are equal by uniqueness of limits.

**Lemma (Integrals Distribute Over Disjoint Sets)** If  $X \subseteq A \cup B$ , then  $\int_X f \leq \int_A f + \int_{A^c} f$  with equality iff  $X = A \sqcup B$ .

**Lemma (Unif. Cts.  $L^1$  Functions Vanish at Infinity)** If  $f \in L^1$  and  $f$  is uniformly continuous, then  $f(x) \xrightarrow{|x| \rightarrow \infty} 0$ .

Doesn't hold for general  $L^1$  functions, take any train of triangles with height 1 and summable areas.

**Lemma ( $L^1$  Functions Have Small Tails)** If  $f \in L^1$ , then for every  $\varepsilon$  there exists a radius  $R$  such that if  $A = B_R(0)^c$ , then  $\int_A |f| < \varepsilon$ .

**Proof** Approximate with compactly supported functions. Take  $g \xrightarrow{L^1} f$  with  $g \in C_c$ , then choose  $N$  large enough so that  $g = 0$  on  $E := B_N(0)^c$ , then  $\int_E |f| \leq \int_E |f - g| + \int_E |g|$ .

**Lemma ( $L^1$  Functions Have Absolutely Continuity)**  $m(E) \rightarrow 0 \implies \int_E f \rightarrow 0$ .

**Proof** Approximate with compactly supported functions. Take  $g \xrightarrow{L^1} f$ , then  $g \leq M$  so  $\int_E f \leq \int_E f - g + \int_E g \rightarrow 0 + M \cdot m(E) \rightarrow 0$ .

**Lemma ( $L^1$  Functions Are Finite Almost Everywhere)** If  $f \in L^1$ , then  $m(\{f(x) = \infty\}) = 0$ .

**Proof** Idea: Split up domain Let  $A = \{f(x) = \infty\}$ , then  $\infty > \int f = \int_A f + \int_{A^c} f = \infty \cdot m(A) + \int_{A^c} f \implies m(X) = 0$ .

**Proposition (Continuity in  $L^1$ )**

$$\|\tau_h f - f\|_1 \xrightarrow{h \rightarrow 0} 0$$

**Proof** Approximate with compactly supported functions. Take  $g \xrightarrow{L^1} f$  with  $g \in C_c$ .

$$\begin{aligned} \int f(x+h) - f(x) &\leq \int f(x+h) - g(x+h) + \int g(x+h) - g(x) + \int g(x) - f(x) \\ &\xrightarrow{??} 2\varepsilon + \int g(x+h) - g(x) \\ &= \int_K g(x+h) - g(x) + \int_{K^c} g(x+h) - g(x) \\ &\xrightarrow{??} 0, \end{aligned}$$

which follows because we can enlarge the support of  $g$  to  $K$  where the integrand is zero on  $K^c$ , then apply uniform continuity on  $K$ .

**Proposition (Integration by Parts, Special Case)**

$$F(x) := \int_0^x f(y)dy \quad \text{and} \quad G(x) := \int_0^x g(y)dy$$

$$\implies \int_0^1 F(x)g(x)dx = F(1)G(1) - \int_0^1 f(x)G(x)dx.$$

**Proof** Fubini-Tonelli, and sketch region to change integration bounds.**Theorem (Lebesgue Density)**

$$A_h(f)(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(y)dy \implies \|A_h(f) - f\| \xrightarrow{h \rightarrow 0} 0.$$

**Proof** Fubini-Tonelli, and sketch region to change integration bounds, and continuity in  $L^1$ .**3.5  $L^p$  Spaces****Lemma** The following are dense subspaces of  $L^2([0, 1])$ :

- Simple functions
- Step functions
- $C_0([0, 1])$
- Smoothly differentiable functions  $C_0^\infty([0, 1])$
- Smooth compactly supported functions  $C_c^\infty$  Theorem :

$$m(X) < \infty \implies \lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty.$$

**Proof**

- Let  $M = \|f\|_\infty$ .
- For any  $L < M$ , let  $S = \{|f| \geq L\}$ .
- Then  $m(S) > 0$  and

$$\begin{aligned} \|f\|_p &= \left( \int_X |f|^p \right)^{\frac{1}{p}} \\ &\geq \left( \int_S |f|^p \right)^{\frac{1}{p}} \\ &\geq L m(S)^{\frac{1}{p}} \xrightarrow{p \rightarrow \infty} L \\ &\implies \liminf_p \|f\|_p \geq M. \end{aligned}$$

We also have

$$\begin{aligned} \|f\|_p &= \left( \int_X |f|^p \right)^{\frac{1}{p}} \\ &\leq \left( \int_X M^p \right)^{\frac{1}{p}} \\ &= M m(X)^{\frac{1}{p}} \xrightarrow{p \rightarrow \infty} M \\ &\implies \limsup_p \|f\|_p \leq M \blacksquare. \end{aligned}$$



---

**Theorem (Dual Lp Spaces)** For  $p \neq \infty$ ,  $(L^p)^\vee \cong L^q$ .

**Proof (p=1) ?**

**Proof (p=2)** Use Riesz Representation for Hilbert spaces.

*Proof .*

$L^1 \subset (L^\infty)^\vee$ , since the isometric mapping is always injective, but *never* surjective. ■

## 4 Fourier Transform and Convolution

### 4.1 The Fourier Transform

**Definition (Convolution)**

$$f * g(x) = \int f(x-y)g(y)dy.$$

**Definition (The Fourier Transform)**

$$\hat{f}(\xi) = \int f(x) e^{2\pi i x \cdot \xi} dx.$$

**Lemma** If  $\hat{f} = \hat{g}$  then  $f = g$  almost everywhere.

**Lemma (Riemann-Lebesgue: Fourier transforms have small tails)**

$$f \in L^1 \implies \hat{f}(\xi) \rightarrow 0 \text{ as } |\xi| \rightarrow \infty,$$

if  $f \in L^1$ , then  $\hat{f}$  is continuous and bounded.

**Proof**

- Boundedness:

$$|\hat{f}(\xi)| \leq \int |f| \cdot |e^{2\pi i x \cdot \xi}| = \|f\|_1.$$

- Continuity:

$$\begin{aligned} & - |\hat{f}(\xi_n) - \hat{f}(\xi)| \\ & - \text{Apply DCT to show } a \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

**Theorem (Fourier Inversion)**

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

**Proof** Idea: Fubini-Tonelli doesn't work directly, so introduce a convergence factor, take limits, and use uniqueness of limits.

- Take the modified integral:

$$\begin{aligned}
 I_t(x) &= \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi} e^{-\pi t^2 |\xi|^2} \\
 &= \int \widehat{f}(\xi) \varphi(\xi) \\
 &= \int f(\xi) \widehat{\varphi}(\xi) \\
 &= \int f(\xi) \widehat{g}_t(\xi - x) \\
 &= \int f(\xi) g_t(x - \xi) d\xi \\
 &= \int f(y - x) g_t(y) dy \quad (\xi = y - x) \\
 &= (f * g_t) \\
 &\longrightarrow f \text{ in } L^1 \text{ as } t \longrightarrow 0.
 \end{aligned}$$

- We also have

$$\begin{aligned}
 \lim_{t \rightarrow 0} I_t(x) &= \lim_{t \rightarrow 0} \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi} e^{-\pi t^2 |\xi|^2} \\
 &= \lim_{t \rightarrow 0} \int \widehat{f}(\xi) \varphi(\xi) \\
 &=_{DCT} \int \widehat{f}(\xi) \lim_{t \rightarrow 0} \varphi(\xi) \\
 &= \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi}
 \end{aligned}$$

- So

$$I_t(x) \longrightarrow \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi} \text{ pointwise and } \|I_t(x) - f(x)\|_1 \longrightarrow 0.$$

- So there is a subsequence  $I_{t_n}$  such that  $I_{t_n}(x) \longrightarrow f(x)$  almost everywhere
- Thus  $f(x) = \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi}$  almost everywhere by uniqueness of limits.

**Proposition (Eigenfunction of the Fourier Transform)**

$$g(x) := e^{-\pi |x|^2} \implies \widehat{g}(\xi) = g(\xi) \quad \text{and} \quad \widehat{g}_t(x) = g(tx) = e^{-\pi t^2 |x|^2}.$$

**Proposition (Properties of the Fourier Transform)**

?????.

## 4.2 Approximate Identities

### Definition (Dilation)

$$\varphi_t(x) = t^{-n} \varphi(t^{-1}x).$$

**Definition (Approximation to the Identity)** For  $\varphi \in L^1$ , the dilations satisfy  $\int \varphi_t = \int \varphi$ , and if  $\int \varphi = 1$  then  $\varphi$  is an *approximate identity*.

Example:  $\varphi(x) = e^{-\pi x^2}$

### Theorem (Convolution Against Approximate Identities Converge in $L^1$ )

$$\|f * \varphi_t - f\|_1 \xrightarrow{t \rightarrow 0} 0.$$

#### Proof

$$\begin{aligned} \|f - f * \varphi_t\|_1 &= \int |f(x) - \int f(x-y)\varphi_t(y) dy| dx \\ &= \int |f(x)| \int \varphi_t(y) dy - \int |f(x-y)\varphi_t(y)| dy dx \\ &= \int \int \varphi_t(y) |f(x) - f(x-y)| dy dx \\ &=_{FT} \int \int \varphi_t(y) |f(x) - f(x-y)| dx dy \\ &= \int \varphi_t(y) \int |f(x) - f(x-y)| dx dy \\ &= \int \varphi_t(y) \|f - \tau_y f\|_1 dy \\ &= \int_{y < \delta} \varphi_t(y) \|f - \tau_y f\|_1 dy + \int_{y \geq \delta} \varphi_t(y) \|f - \tau_y f\|_1 dy \\ &\leq \int_{y < \delta} \varphi_t(y) \varepsilon + \int_{y \geq \delta} \varphi_t(y) (\|f\|_1 + \|\tau_y f\|_1) dy \quad \text{by continuity in } L^1 \\ &\leq \varepsilon + 2\|f\|_1 \int_{y \geq \delta} \varphi_t(y) dy \\ &\leq \varepsilon + 2\|f\|_1 \cdot \varepsilon \quad \text{since } \varphi_t \text{ has small tails} \\ &\xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

### Theorem (Convolutions Vanish at Infinity)

$$f, g \in L^1 \text{ and bounded} \implies \lim_{|x| \rightarrow \infty} (f * g)(x) = 0.$$

#### Proof

- Choose  $M \geq f, g$ .
- By small tails, choose  $N$  such that  $\int_{B_N^c} |f|, \int_{B_N^c} |g| < \varepsilon$

- Note

$$|f * g| \leq \int |f(x-y)| |g(y)| dy := I.$$

- Use  $|x| \leq |x-y| + |y|$ , take  $|x| \geq 2N$  so either

$$|x-y| \geq N \implies I \leq \int_{\{x-y \geq N\}} |f(x-y)| M dy \leq \varepsilon M \longrightarrow 0$$

then

$$|y| \geq N \implies I \leq \int_{\{y \geq N\}} M |g(y)| dy \leq M\varepsilon \longrightarrow 0.$$

Proposition (Young's Inequality?) :

$$\frac{1}{r} := \frac{1}{p} + \frac{1}{q} - 1 \implies \|f * g\|_r \leq \|f\|_p \|g\|_q.$$

**Corollary** Take  $q = 1$  to obtain

$$\|f * g\|_p \leq \|f\|_p \|g\|_1.$$

**Corollary** If  $f, g \in L^1$  then  $f * g \in L^1$ .

## 5 Functional Analysis

### 5.1 Definitions

Notation:  $H$  denotes a Hilbert space.

**Definition (Orthonormal Sequence) ?**

**Definition (Basis)** A set  $\{u_n\}$  is a *basis* for a Hilbert space  $\mathcal{H}$  iff it is dense in  $\mathcal{H}$ .

**Definition (Complete)** A collection of vectors  $\{u_n\} \subset H$  is *complete* iff  $\langle x, u_n \rangle = 0$  for all  $n \iff x = 0$  in  $H$ .

**Definition (Dual Space)**

$$X^\vee := \left\{ L : X \longrightarrow \mathbb{C} \mid L \text{ is continuous} \right\}.$$

**Definition** A map  $L : X \longrightarrow \mathbb{C}$  is a linear functional iff

$$L(\alpha \mathbf{x} + \mathbf{y}) = \alpha L(\mathbf{x}) + L(\mathbf{y}).$$

**Definition (Operator Norm)**

$$\|L\|_{X^\vee} := \sup_{\substack{x \in X \\ \|x\|=1}} |L(x)|.$$

**Definition (Banach Space)** A complete normed vector space.

**Definition (Hilbert Space)** An inner product space which is a Banach space under the induced norm.

## 5.2 Theorems

**Theorem (Bessel's Inequality)** For any orthonormal set  $\{u_n\} \subseteq \mathcal{H}$  a Hilbert space (not necessarily a basis),

$$\left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2$$

and thus

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2.$$

### Proof

- Let  $S_N = \sum_{n=1}^N \langle x, u_n \rangle u_n$

$$\begin{aligned} \|x - S_N\|^2 &= \langle x - S_N, x - S_N \rangle \\ &= \|x\|^2 + \|S_N\|^2 - 2\Re \langle x, S_N \rangle \\ &= \|x\|^2 + \|S_N\|^2 - 2\Re \left\langle x, \sum_{n=1}^N \langle x, u_n \rangle u_n \right\rangle \\ &= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^N \langle x, \langle x, u_n \rangle u_n \rangle \\ &= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^N \overline{\langle x, u_n \rangle} \langle x, u_n \rangle \\ &= \|x\|^2 + \left\| \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|^2 - 2 \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\ &= \|x\|^2 + \sum_{n=1}^N |\langle x, u_n \rangle|^2 - 2 \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\ &= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2. \end{aligned}$$

- By continuity of the norm and inner product, we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \|x - S_N\|^2 &= \lim_{N \rightarrow \infty} \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\ \implies \left\| x - \lim_{N \rightarrow \infty} S_N \right\|^2 &= \|x\|^2 - \lim_{N \rightarrow \infty} \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\ \implies \left\| x - \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2. \end{aligned}$$

- Then noting that  $0 \leq \|x - S_N\|^2$ ,

$$\begin{aligned} 0 &\leq \|x\|^2 - \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \\ \implies \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 &\leq \|x\|^2 \blacksquare. \end{aligned}$$

**Theorem (Riesz Representation for Hilbert Spaces)** If  $\Lambda$  is a continuous linear functional on a Hilbert space  $H$ , then there exists a unique  $y \in H$  such that

$$\forall x \in H, \quad \Lambda(x) = \langle x, y \rangle.$$

**Proof**

- Define  $M := \ker \Lambda$ .
- Then  $M$  is a closed subspace and so  $H = M \oplus M^\perp$ .
- There is some  $z \in M^\perp$  such that  $\|z\| = 1$ .
- Set  $u := \Lambda(x)z - \Lambda(z)x$ .
- Check

$$\Lambda(u) = \Lambda(\Lambda(x)z - \Lambda(z)x) = \Lambda(x)\Lambda(z) - \Lambda(z)\Lambda(x) = 0 \implies u \in M$$

- Compute

$$\begin{aligned} 0 &= \langle u, z \rangle \\ &= \langle \Lambda(x)z - \Lambda(z)x, z \rangle \\ &= \langle \Lambda(x)z, z \rangle - \langle \Lambda(z)x, z \rangle \\ &= \Lambda(x)\langle z, z \rangle - \Lambda(z)\langle x, z \rangle \\ &= \Lambda(x)\|z\|^2 - \Lambda(z)\langle x, z \rangle \\ &= \Lambda(x) - \Lambda(z)\langle x, z \rangle \\ &= \Lambda(x) - \langle x, \overline{\Lambda(z)}z \rangle, \end{aligned}$$

- Choose  $y := \overline{\Lambda(z)}z$ .
- Check uniqueness:

$$\begin{aligned} \langle x, y \rangle &= \langle x, y' \rangle \quad \forall x \\ \implies \langle x, y - y' \rangle &= 0 \quad \forall x \\ \implies \langle y - y', y - y' \rangle &= 0 \\ \implies \|y - y'\| &= 0 \\ \implies y - y' &= \mathbf{0} \implies y = y'. \end{aligned}$$

**Theorem (Continuous iff Bounded)** Let  $L : X \longrightarrow \mathbb{C}$  be a linear functional, then the following are equivalent:

1.  $L$  is continuous
2.  $L$  is continuous at zero
3.  $L$  is bounded, i.e.  $\exists c \geq 0 \mid |L(x)| \leq c\|x\|$  for all  $x \in H$

**Proof**

2  $\implies$  3: Choose  $\delta < 1$  such that

$$\|x\| \leq \delta \implies |L(x)| < 1.$$

Then

$$\begin{aligned} |L(x)| &= \left| L\left(\frac{\|x\|}{\delta} \frac{\delta}{\|x\|} x\right) \right| \\ &= \frac{\|x\|}{\delta} \left| L\left(\delta \frac{x}{\|x\|}\right) \right| \\ &\leq \frac{\|x\|}{\delta} 1, \end{aligned}$$

so we can take  $c = \frac{1}{\delta}$ . ■

3  $\implies$  1:

We have  $|L(x - y)| \leq c\|x - y\|$ , so given  $\varepsilon \geq 0$  simply choose  $\delta = \frac{\varepsilon}{c}$ .

**Theorem (Operator Norm is a Norm)** If  $H$  is a Hilbert space, then  $(H^\vee, \|\cdot\|_{\text{op}})$  is a normed space.

**Proof** The only nontrivial property is the triangle inequality, but

$$\|L_1 + L_2\|_{\text{op}} = \sup |L_1(x) + L_2(x)| \leq \sup |L_1(x)| + \sup |L_2(x)| = \|L_1\|_{\text{op}} + \|L_2\|_{\text{op}}.$$

**Theorem (Completeness in Operator Norm)** If  $X$  is a normed vector space, then  $(X^\vee, \|\cdot\|_{\text{op}})$  is a Banach space.

**Proof**

- Let  $\{L_n\}$  be Cauchy in  $X^\vee$ .
- Then for all  $x \in C$ ,  $\{L_n(x)\} \subset \mathbb{C}$  is Cauchy and converges to something denoted  $L(x)$ .
- Need to show  $L$  is continuous and  $\|L_n - L\| \longrightarrow 0$ .
- Since  $\{L_n\}$  is Cauchy in  $X^\vee$ , choose  $N$  large enough so that

$$n, m \geq N \implies \|L_n - L_m\| < \varepsilon \implies |L_m(x) - L_n(x)| < \varepsilon \quad \forall x \mid \|x\| = 1.$$

- Take  $n \longrightarrow \infty$  to obtain

$$\begin{aligned} m \geq N &\implies |L_m(x) - L(x)| < \varepsilon \quad \forall x \mid \|x\| = 1 \\ &\implies \|L_m - L\| < \varepsilon \longrightarrow 0. \end{aligned}$$

- Continuity:

$$\begin{aligned}
|L(x)| &= |L(x) - L_n(x) + L_n(x)| \\
&\leq |L(x) - L_n(x)| + |L_n(x)| \\
&\leq \varepsilon \|x\| + c \|x\| \\
&= (\varepsilon + c) \|x\| \blacksquare.
\end{aligned}$$

**Theorem (Riesz-Fischer)** Let  $U = \{u_n\}_{n=1}^\infty$  be an orthonormal set (not necessarily a basis), then

1. There is an isometric surjection

$$\begin{aligned}
\mathcal{H} &\longrightarrow \ell^2(\mathbb{N}) \\
\mathbf{x} &\mapsto \{\langle \mathbf{x}, \mathbf{u}_n \rangle\}_{n=1}^\infty
\end{aligned}$$

i.e. if  $\{a_n\} \in \ell^2(\mathbb{N})$ , so  $\sum |a_n|^2 < \infty$ , then there exists a  $\mathbf{x} \in \mathcal{H}$  such that

$$a_n = \langle \mathbf{x}, \mathbf{u}_n \rangle \quad \forall n.$$

2.  $\mathbf{x}$  can be chosen such that

$$\|\mathbf{x}\|^2 = \sum |a_n|^2$$

Note: the choice of  $\mathbf{x}$  is unique  $\iff \{u_n\}$  is **complete**, i.e.  $\langle \mathbf{x}, \mathbf{u}_n \rangle = 0$  for all  $n$  implies  $\mathbf{x} = \mathbf{0}$ .

**Proof**

- Given  $\{a_n\}$ , define  $S_N = \sum_{n=1}^N a_n \mathbf{u}_n$ .
- $S_N$  is Cauchy in  $\mathcal{H}$  and so  $S_N \longrightarrow \mathbf{x}$  for some  $\mathbf{x} \in \mathcal{H}$ .
- $\langle x, u_n \rangle = \langle x - S_N, u_n \rangle + \langle S_N, u_n \rangle \longrightarrow a_n$
- By construction,  $\|x - S_N\|^2 = \|x\|^2 - \sum_{n=1}^N |a_n|^2 \longrightarrow 0$ , so  $\|x\|^2 = \sum_{n=1}^\infty |a_n|^2$ .

## 6 Extra Problems

### 6.1 Greatest Hits

- $\star$ : Show that for  $E \subseteq \mathbb{R}^n$ , TFAE:
  1.  $E$  is measurable
  2.  $E = H \bigcup Z$  here  $H$  is  $F_\sigma$  and  $Z$  is null
  3.  $E = V \setminus Z'$  where  $V \in G_\delta$  and  $Z'$  is null.
- $\star$ : Show that if  $E \subseteq \mathbb{R}^n$  is measurable then  $m(E) = \sup \left\{ m(K) \mid K \subset E \text{ compact} \right\}$  iff for all  $\varepsilon > 0$  there exists a compact  $K \subseteq E$  such that  $m(K) \geq m(E) - \varepsilon$ .
- $\star$ : Show that cylinder functions are measurable, i.e. if  $f$  is measurable on  $\mathbb{R}^s$ , then  $F(x, y) := f(x)$  is measurable on  $\mathbb{R}^s \times \mathbb{R}^t$  for any  $t$ .



- ★: Prove that the Lebesgue integral is translation invariant, i.e. if  $\tau_h(x) = x + h$  then  $\int \tau_h f = \int f$ .
- ★: Prove that the Lebesgue integral is dilation invariant, i.e. if  $f_\delta(x) = \frac{f(\frac{x}{\delta})}{\delta^n}$  then  $\int f_\delta = \int f$ .
- ★: Prove continuity in  $L^1$ , i.e.

$$f \in L^1 \implies \lim_{h \rightarrow 0} \int |f(x+h) - f(x)| = 0.$$

- ★: Show that

$$f, g \in L^1 \implies f * g \in L^1 \quad \text{and} \quad \|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

- ★: Show that if  $X \subseteq \mathbb{R}$  with  $\mu(X) < \infty$  then

$$\|f\|_p \xrightarrow{p \rightarrow \infty} \|f\|_\infty.$$

## 6.2 By Topic

### 6.2.1 Topology

- Show that every compact set is closed and bounded.
- Show that if a subset of a metric space is complete and totally bounded, then it is compact.
- Show that if  $K$  is compact and  $F$  is closed with  $K, F$  disjoint then  $\text{dist}(K, F) > 0$ .

### 6.2.2 Continuity

- Show that a continuous function on a compact set is uniformly continuous.

### 6.2.3 Differentiation

- Show that if  $f \in C^1(\mathbb{R})$  and both  $\lim_{x \rightarrow \infty} f(x)$  and  $\lim_{x \rightarrow \infty} f'(x)$  exist, then  $\lim_{x \rightarrow \infty} f'(x)$  must be zero.

### 6.2.4 Advanced Limitology

- If  $f$  is continuous, is it necessarily the case that  $f'$  is continuous?
- If  $f_n \rightarrow f$ , is it necessarily the case that  $f'_n$  converges to  $f'$  (or at all)?
- Is it true that the sum of differentiable functions is differentiable?
- Is it true that the limit of integrals equals the integral of the limit?
- Is it true that a limit of continuous functions is continuous?
- Show that a subset of a metric space is closed iff it is complete.
- Show that if  $m(E) < \infty$  and  $f_n \rightarrow f$  uniformly, then  $\lim \int_E f_n = \int_E f$ .

#### Uniform Convergence

- Show that a uniform limit of bounded functions is bounded.

- Show that a uniform limit of continuous function is continuous.
  - I.e. if  $f_n \rightarrow f$  uniformly with each  $f_n$  continuous then  $f$  is continuous.
- Show that if  $f_n \rightarrow f$  pointwise,  $f'_n \rightarrow g$  uniformly for some  $f, g$ , then  $f$  is differentiable and  $g = f'$ .
- Prove that uniform convergence implies pointwise convergence implies a.e. convergence, but none of the implications may be reversed.
- Show that  $\sum \frac{x^n}{n!}$  converges uniformly on any compact subset of  $\mathbb{R}$ .

#### Measure Theory

- Show that continuity of measure from above/below holds for outer measures.
- Show that a countable union of null sets is null.

#### Measurability

- Show that  $f = 0$  a.e. iff  $\int_E f = 0$  for every measurable set  $E$ .

#### Integrability

- Show that if  $f$  is a measurable function, then  $f = 0$  a.e. iff  $\int f = 0$ .
- Show that a bounded function is Lebesgue integrable iff it is measurable.
- Show that simple functions are dense in  $L^1$ .
- Show that step functions are dense in  $L^1$ .
- Show that smooth compactly supported functions are dense in  $L^1$ .

#### Convergence

- Prove Fatou's lemma using the Monotone Convergence Theorem.
- Show that if  $\{f_n\}$  is in  $L^1$  and  $\sum \int |f_n| < \infty$  then  $\sum f_n$  converges to an  $L^1$  function and

$$\int \sum f_n = \sum \int f_n.$$

#### Convolution

- Show that if  $f \in L^1$  and  $g$  is bounded, then  $f * g$  is bounded and uniformly continuous.
- If  $f, g$  are compactly supported, is it necessarily the case that  $f * g$  is compactly supported?
- Show that under any of the following assumptions,  $f * g$  vanishes at infinity:
  - $f, g \in L^1$  are both bounded.
  - $f, g \in L^1$  with just  $g$  bounded.
  - $f, g$  smooth and compactly supported (and in fact  $f * g$  is smooth)
  - $f \in L^1$  and  $g$  smooth and compactly supported (and in fact  $f * g$  is smooth)
- Show that if  $f \in L^1$  and  $g'$  exists with  $\frac{\partial g}{\partial x_i}$  all bounded, then

$$\frac{\partial}{\partial x_i} (f * g) = f * \frac{\partial g}{\partial x_i}$$

#### Fourier Analysis

- Show that if  $f \in L^1$  then  $\hat{f}$  is bounded and uniformly continuous.
- Is it the case that  $f \in L^1$  implies  $\hat{f} \in L^1$ ?

- 
- Show that if  $f, \widehat{f} \in L^1$  then  $f$  is bounded, uniformly continuous, and vanishes at infinity.
    - Show that this is not true for arbitrary  $L^1$  functions.
  - Show that if  $f \in L^1$  and  $\widehat{f} = 0$  almost everywhere then  $f = 0$  almost everywhere.
    - Prove that  $\widehat{f} = \widehat{g}$  implies that  $f = g$  a.e.
  - Show that if  $f, g \in L^1$  then

$$\int \widehat{f}g = \int f\widehat{g}.$$

- Give an example showing that this fails if  $g$  is not bounded.
- Show that if  $f \in C^1$  then  $f$  is equal to its Fourier series.

#### Approximate Identities

- Show that if  $\varphi$  is an approximate identity, then

$$\|f * \varphi_t - f\|_1 \xrightarrow{t \rightarrow 0} 0.$$

- Show that if additionally  $|\varphi(x)| \leq c(1 + |x|)^{-n-\varepsilon}$  for some  $c, \varepsilon > 0$ , then this converges almost everywhere.
- Show that if  $f$  is bounded and uniformly continuous and  $\varphi_t$  is an approximation to the identity, then  $f * \varphi_t$  uniformly converges to  $f$ .

#### $L^p$ Spaces

- Show that if  $E \subseteq \mathbb{R}^n$  is measurable with  $\mu(E) < \infty$  and  $f \in L^p(X)$  then

$$\|f\|_{L^p(X)} \xrightarrow{p \rightarrow \infty} \|f\|_\infty.$$

- Is it true that the converse to the DCT holds? I.e. if  $\int f_n \rightarrow \int f$ , is there a  $g \in L^p$  such that  $f_n < g$  a.e. for every  $n$ ?
- Prove continuity in  $L^p$ : If  $f$  is uniformly continuous then for all  $p$ ,

$$\|\tau_h f - f\|_p \xrightarrow{h \rightarrow 0} 0.$$

- Prove the following inclusions of  $L^p$  spaces for  $m(X) < \infty$ :

$$L^\infty(X) \subset L^2(X) \subset L^1(X)$$

$$\ell^2(\mathbb{Z}) \subset \ell^1(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z}).$$

## 7 Practice Exam (November 2014)

### 7.1 1: Fubini-Tonelli

#### 7.1.1 a

Carefully state Tonelli's theorem for a nonnegative function  $F(x, t)$  on  $\mathbb{R}^n \times \mathbb{R}$ .

**7.1.2 b**

Let  $f : \mathbb{R}^n \rightarrow [0, \infty]$  and define

$$\mathcal{A} := \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid 0 \leq t \leq f(x) \right\}.$$

Prove the validity of the following two statements:

1.  $f$  is Lebesgue measurable on  $\mathbb{R}^n \iff \mathcal{A}$  is a Lebesgue measurable subset of  $\mathbb{R}^{n+1}$ .
2. If  $f$  is Lebesgue measurable on  $\mathbb{R}^n$  then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x) dx = \int_0^\infty m\left(\left\{x \in \mathbb{R}^n \mid f(x) \geq t\right\}\right) dt.$$

**7.2 2: Convolutions and the Fourier Transform****7.2.1 a**

Let  $f, g \in L^1(\mathbb{R}^n)$  and give a definition of  $f * g$ .

**7.2.2 b**

Prove that if  $f, g$  are integrable and bounded, then

$$(f * g)(x) \xrightarrow{|x| \rightarrow \infty} 0.$$

**7.2.3 c**

1. Define the *Fourier transform* of an integrable function  $f$  on  $\mathbb{R}^n$ .
2. Give an outline of the proof of the Fourier inversion formula.
3. Give an example of a function  $f \in L^1(\mathbb{R}^n)$  such that  $\hat{f}$  is not in  $L^1(\mathbb{R}^n)$ .

**7.3 3: Hilbert Spaces**

Let  $\{u_n\}_{n=1}^\infty$  be an orthonormal sequence in a Hilbert space  $H$ .

**7.3.1 a**

Let  $x \in H$  and verify that

$$\left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|_H^2 = \|x\|_H^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2.$$

for any  $N \in \mathbb{N}$  and deduce that

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|_H^2.$$

## 7.3.2 b

Let  $\{a_n\}_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$  and prove that there exists an  $x \in H$  such that  $a_n = \langle x, u_n \rangle$  for all  $n \in \mathbb{N}$ , and moreover  $x$  may be chosen such that

$$\|x\|_H = \left( \sum_{n \in \mathbb{N}} |a_n|^2 \right)^{\frac{1}{2}}.$$

**Proof**

- Take  $\{a_n\} \in \ell^2$ , then note that  $\sum |a_n|^2 < \infty \implies$  the tails vanish.
- Define  $x := \lim_{N \rightarrow \infty} S_N$  where  $S_N = \sum_{k=1}^N a_k u_k$
- $\{S_N\}$  is Cauchy and  $H$  is complete, so  $x \in H$ .
- By construction,

$$\langle x, u_n \rangle = \left\langle \sum_k a_k u_k, u_n \right\rangle = \sum_k a_k \langle u_k, u_n \rangle = a_n$$

since the  $u_k$  are all orthogonal.

- By Pythagoras since the  $u_k$  are normal,

$$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2.$$

## 7.3.3 c

Prove that if  $\{u_n\}$  is *complete*, Bessel's inequality becomes an equality.

**Proof** Let  $x$  and  $u_n$  be arbitrary.

$$\begin{aligned} \left\langle x - \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k, u_n \right\rangle &= \langle x, u_n \rangle - \left\langle \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k, u_n \right\rangle \\ &= \langle x, u_n \rangle - \sum_{k=1}^{\infty} \langle \langle x, u_k \rangle u_k, u_n \rangle \\ &= \langle x, u_n \rangle - \sum_{k=1}^{\infty} \langle x, u_k \rangle \langle u_k, u_n \rangle \\ &= \langle x, u_n \rangle - \langle x, u_n \rangle = 0 \\ \implies x - \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k &= 0 \quad \text{by completeness.} \end{aligned}$$

So

$$x = \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k \implies \|x\|^2 = \sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2. \blacksquare$$

**7.4 4:  $L^p$  Spaces****7.4.1 a**

Prove Hölder's inequality: let  $f \in L^p, g \in L^q$  with  $p, q$  conjugate, and show that

$$\|fg\|_p \leq \|f\|_p \cdot \|g\|_q.$$

**7.4.2 b**

Prove Minkowski's Inequality:

$$1 \leq p < \infty \implies \|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Conclude that if  $f, g \in L^p(\mathbb{R}^n)$  then so is  $f + g$ .

**7.4.3 c**

Let  $X = [0, 1] \subset \mathbb{R}$ .

1. Give a definition of the Banach space  $L^\infty(X)$  of essentially bounded functions of  $X$ .
2. Let  $f$  be non-negative and measurable on  $X$ , prove that

$$\int_X f(x)^p dx \xrightarrow{p \rightarrow \infty} \begin{cases} \infty & \text{or} \\ m(\{f^{-1}(1)\}) \end{cases},$$

and characterize the functions of each type

**Proof**

$$\begin{aligned} \int f^p &= \int_{x < 1} f^p + \int_{x=1} f^p + \int_{x > 1} f^p \\ &= \int_{x < 1} f^p + \int_{x=1} 1 + \int_{x > 1} f^p \\ &= \int_{x < 1} f^p + m(\{f = 1\}) + \int_{x > 1} f^p \\ &\xrightarrow{p \rightarrow \infty} 0 + m(\{f = 1\}) + \begin{cases} 0 & m(\{x \geq 1\}) = 0 \\ \infty & m(\{x \geq 1\}) > 0. \end{cases} \end{aligned}$$

Justify passing  
limit into integrals.

**7.5 5: Dual Spaces**

Let  $X$  be a normed vector space.

**7.5.1 a**

Give the definition of what it means for a map  $L : X \rightarrow \mathbb{C}$  to be a *linear functional*.

---

### 7.5.2 b

Define what it means for  $L$  to be *bounded* and show  $L$  is bounded  $\iff L$  is continuous.

### 7.5.3 c

Prove that  $(X^\vee, \|\cdot\|_{\text{op}})$  is a Banach space.

## 8 Midterm Exam 2 (November 2018)

### 8.1 1 (Integration by Parts)

Let  $f, g \in L^1([0, 1])$ , define  $F(x) = \int_0^x f$  and  $G(x) = \int_0^x g$ , and show

$$\int_0^1 F(x)g(x) dx = F(1)G(1) - \int_0^1 f(x)G(x) dx.$$

### 8.2 2

Let  $\varphi \in L^1(\mathbb{R}^n)$  such that  $\int \varphi = 1$  and define  $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$ .

Show that if  $f$  is bounded and uniformly continuous then  $f * \varphi_t \xrightarrow{t \rightarrow 0} f$  uniformly.

### 8.3 3

Let  $g \in L^\infty([0, 1])$ .

a. Prove

$$\|g\|_{L^p([0,1])} \xrightarrow{p \rightarrow \infty} \|g\|_{L^\infty([0,1])}.$$

b. Prove that the map

$$\begin{aligned} \Lambda_g : L^1([0, 1]) &\longrightarrow \mathbb{C} \\ f &\mapsto \int_0^1 fg \end{aligned}$$

defines an element of  $L^1([0, 1])^\vee$  with  $\|\Lambda_g\|_{L^1([0,1])^\vee} = \|g\|_{L^\infty([0,1])}$ .

Note: 4 is a repeat.

## 9 Midterm Exam 2 (December 2014)

### 9.1 1

Note: (a) is a repeat.

- Let  $\Lambda \in L^2(X)^\vee$ .

- Show that  $M := \{f \in L^2(X) \mid \Lambda(f) = 0\} \subseteq L^2(X)$  is a closed subspace, and  $L^2(X) = M \oplus M^\perp$ .
- Prove that there exists a unique  $g \in L^2(X)$  such that  $\Lambda(f) = \int_X g \bar{f}$ .

## 9.2 2

a. In parts:

- Given a definition of  $L^\infty(\mathbb{R}^n)$ .
- Verify that  $\|\cdot\|_\infty$  defines a norm on  $L^\infty(\mathbb{R}^n)$ .
- Carefully prove that  $(L^\infty(\mathbb{R}^n), \|\cdot\|_\infty)$  is a Banach space.

b. Prove that for any measurable  $f : \mathbb{R}^n \rightarrow \mathbb{C}$ ,

$$L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \quad \text{and} \quad \|f\|_2 \leq \|f\|_1^{\frac{1}{2}} \cdot \|f\|_\infty^{\frac{1}{2}}.$$

## 9.3 3

- a. Prove that if  $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$  is both measurable then  $F(x, y) := f(x)$  and  $h(x, y) := f(x-y)g(y)$  is measurable on  $\mathbb{R}^n \times \mathbb{R}^n$ .
- b. Show that if  $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$  and  $g \in L^1(\mathbb{R}^n)$ , then  $f * g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$  is well defined, and carefully show that it satisfies the following properties:

$$\|f * g\|_\infty \leq \|g\|_1 \|f\|_\infty \|f * g\|_1 \leq \|g\|_1 \|f\|_1 \|f * g\|_2 \leq \|g\|_1 \|f\|_2.$$

Hint: first show  $|f * g|^2 \leq \|g\|_1 (|f|^2 * |g|)$ .

## 9.4 4 (Weierstrass Approximation Theorem)

Note: (a) is a repeat.

Let  $f : [0, 1] \rightarrow \mathbb{R}$  be continuous, and prove the Weierstrass approximation theorem: for any  $\varepsilon > 0$  there exists a polynomial  $P$  such that  $\|f - P\|_\infty < \varepsilon$ .

## 10 Inequalities and Equalities

### Proposition (Reverse Triangle Inequality)

$$||x| - |y|| \leq \|x - y\|.$$

### Proposition (Chebyshev's Inequality)

$$\mu(\{x : |f(x)| > \alpha\}) \leq \left( \frac{\|f\|_p}{\alpha} \right)^p.$$



---

**Proposition (Holder's Inequality When Surjective)**

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \|fg\|_1 \leq \|f\|_p \|g\|_q.$$

*Application:* For finite measure spaces,

$$1 \leq p < q \leq \infty \implies L^q \subset L^p \quad (\text{and } \ell^p \subset \ell^q).$$

**Proof (Holder's Inequality)** Fix  $p, q$ , let  $r = \frac{q}{p}$  and  $s = \frac{r}{r-1}$  so  $r^{-1} + s^{-1} = 1$ . Then let  $h = |f|^p$ :

$$\|f\|_p^p = \|h \cdot 1\|_1 \leq \|1\|_s \|h\|_r = \mu(X)^{\frac{1}{s}} \|f\|_q^{\frac{q}{r}} \implies \|f\|_p \leq \mu(X)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q.$$

Note: doesn't work for  $\ell_p$  spaces, but just note that  $\sum |x_n| < \infty \implies x_n < 1$  for large enough  $n$ , and thus  $p < q \implies |x_n|^q \leq |x_n|^p$ .

**Proof (Holder's Inequality)** It suffices to show this when  $\|f\|_p = \|g\|_q = 1$ , since

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \iff \int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \leq 1.$$

Using  $AB \leq \frac{1}{p}A^p + \frac{1}{q}B^q$ , we have

$$\int |f| |g| \leq \int \frac{|f|^p}{p} \frac{|g|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1.$$

**Proposition (Cauchy-Schwarz Inequality)**

$$|\langle f, g \rangle| = \|fg\|_1 \leq \|f\|_2 \|g\|_2 \quad \text{with equality} \iff f = \lambda g.$$

Note: Relates inner product to norm, and only happens to relate norms in  $L^1$ .

**Proof ?**

**Proposition (Minkowski's Inequality:)**

$$1 \leq p < \infty \implies \|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Note: does not handle  $p = \infty$  case. Use to prove  $L^p$  is a normed space.

**Proof**

- We first note

$$|f + g|^p = |f + g| |f + g|^{p-1} \leq (|f| + |g|) |f + g|^{p-1}.$$

- Note that if  $p, q$  are conjugate exponents then

$$\begin{aligned} \frac{1}{q} &= 1 - \frac{1}{p} = \frac{p-1}{p} \\ q &= \frac{p}{p-1}. \end{aligned}$$

- Then taking integrals yields

$$\begin{aligned} \|f + g\|_p^p &= \int |f + g|^p \\ &\leq \int (|f| + |g|) |f + g|^{p-1} \\ &= \int |f| |f + g|^{p-1} + \int |g| |f + g|^{p-1} \\ &= \|f(f + g)^{p-1}\|_1 + \|g(f + g)^{p-1}\|_1 \\ &\leq \|f\|_p \|(f + g)^{p-1}\|_q + \|g\|_p \|(f + g)^{p-1}\|_q \\ &= (\|f\|_p + \|g\|_p) \|(f + g)^{p-1}\|_q \\ &= (\|f\|_p + \|g\|_p) \left( \int |f + g|^{(p-1)q} \right)^{\frac{1}{q}} \\ &= (\|f\|_p + \|g\|_p) \left( \int |f + g|^p \right)^{1 - \frac{1}{p}} \\ &= (\|f\|_p + \|g\|_p) \frac{\int |f + g|^p}{(\int |f + g|^p)^{\frac{1}{p}}} \\ &= (\|f\|_p + \|g\|_p) \frac{\|f + g\|_p^p}{\|f + g\|_p}. \end{aligned}$$

- Cancelling common terms yields

$$\begin{aligned} 1 &\leq (\|f\|_p + \|g\|_p) \frac{1}{\|f + g\|_p} \\ \implies \|f + g\|_p &\leq \|f\|_p + \|g\|_p. \end{aligned}$$

**Proposition (Young's Inequality\*)**

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \implies \|f * g\|_r \leq \|f\|_p \|g\|_q$$

**Application:** Some useful specific cases:

$$\begin{aligned} \|f * g\|_1 &\leq \|f\|_1 \|g\|_1 \\ \|f * g\|_p &\leq \|f\|_1 \|g\|_p, \\ \|f * g\|_\infty &\leq \|f\|_2 \|g\|_2 \\ \|f * g\|_\infty &\leq \|f\|_p \|g\|_q. \end{aligned}$$

**Proposition (Bessel's Inequality:)**

For  $x \in H$  a Hilbert space and  $\{e_k\}$  an orthonormal sequence,

$$\sum_{k=1}^{\infty} \|\langle x, e_k \rangle\|^2 \leq \|x\|^2.$$

Note: this does not need to be a basis.

**Proposition (Parseval's Identity:)** Equality in Bessel's inequality, attained when  $\{e_k\}$  is a *basis*, i.e. it is complete, i.e. the span of its closure is all of  $H$ .

**10.1 Less Explicitly Used Inequalities****Proposition (AM-GM Inequality)**

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

**Proposition (Jensen's Inequality)**

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Proposition (???) :

$$AB \leq \frac{A^p}{p} + \frac{B^q}{q}.$$

**Proposition (? Inequality)**

$$(a+b)^p \leq 2^p(a^p + b^p).$$

**Proposition (Bernoulli's Inequality)**

$$(1+x)^n \geq 1+nx \quad x \geq -1, \text{ or } n \in 2\mathbb{Z} \text{ and } \forall x.$$