

# Algebra Qualifying Exam Notes

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## 1 Study Guide for Algebra Qualifying Exam

### References:

- [1]. David Dummit and Richard Foote, Abstract Algebra, Wiley, 2003.
- [2]. Kenneth Hoffman and Ray Kunze, Linear Algebra, Prentice-Hall, 1971.
- [3]. Thomas W. Hungerford, Algebra, Springer, 1974.

[4]. Roy Smith, Algebra Course Notes (843-1 through 845-3), <http://www.math.uga.edu/~roy/>,

As a general rule, students are responsible for knowing both the theory (proofs) and practical applications (e.g. **how to find the Jordan or rational canonical form** of a given matrix, **or the Galois group of a given polynomial**) of the topics mentioned.

A supplement to this study guide is available at:

<http://www.math.uga.edu/sites/default/files/PDFs/Graduate/QualsStudyGuides/AlgebraPhDQualremarks.pdf>

## 1.1 Group Theory

- Subgroups and quotient groups
- Lagrange's Theorem
- Fundamental homomorphism theorems
- Group actions with applications to the structure of groups such as
  - The Sylow Theorems
- Group constructions such as:
  - Direct and semi-direct products
- Structures of special types of groups such as:
  - p-groups
  - Dihedral,
  - Symmetric and Alternating groups
    - \* Cycle decompositions
- The simplicity of  $A_n$ , for  $n \geq 5$
- Free groups, generators and relations
- Solvable groups

References: [1,3,4]

## 1.2 Linear Algebra

- Determinants
- Eigenvalues and eigenvectors
- Cayley-Hamilton Theorem
- Canonical forms for matrices
- Linear groups ( $GL_n, SL_n, O_n, U_n$ )
- Duality
  - Dual spaces,
  - Dual bases,
  - Induced dual map,

- Double duals
- Finite-dimensional spectral theorem

References: [1,2,4]

### 1.3 Rings and Modules

- Zorn's Lemma
  - Every vector space has a basis
  - Maximal ideals exist
- Properties of ideals and quotient rings
- Fundamental homomorphism theorems for rings and modules
- Characterizations and properties of special domains such as:
  - Euclidean  $\implies$  PID  $\implies$  UFD
- Classification of finitely generated modules over PIDs (*with emphasis on Euclidean Domains*)
- Applications to the structure of:
  - Finitely generated abelian groups
  - Canonical forms of matrices

References: [1,3,4]

### 1.4 Field Theory

- Algebraic extensions of fields
- Fundamental theorem of Galois theory
- Properties of finite fields
- Separable extensions
- Computations of Galois groups of polynomials of small degree and cyclotomic
- Polynomials
- Solvability of polynomials by radicals

References: [1,3,4]

## 2 Remarks

Adapted from remark written by Roy Smith, August 2006

### 2.1 Group theory:

The first 6 chapters (220 pages) of DF are excellent.

All the definitions and proofs of these theorems on groups are given in Smith's web based lecture notes for math 843 part 1.

#### Key topics:

- Sylow theorems
- Simplicity of  $A_n$  for  $n > 4$ .
- The first isomorphism theorem,
- The Jordan Holder theorem,

The last two (one easy, one hard) are left as exercises.

**The proof JH is seldom tested on the qual**, but proofs are always of interest.

- Fundamental theorem of finite abelian groups

*DF Exercises 12.1.16-19*

- The simple groups of order between 60 and 168 have prime order

### 2.2 Rings:

- DF Chapters 7,8,9.
- Gauss's important theorem on unique factorization of polynomials:
  - $\mathbb{Z}[x]$  is a UFD
  - $R[x]$  is a UFD when  $R$  is a UFD
- The fundamental isomorphism theorems for rings (easy and useful exercise)
- How to use Zorn's lemma
  - To find maximal ideals
  - Construct algebraic field closures
  - Why it is unnecessary in countable or noetherian rings.

Smith discusses extensively in 844-1.

- Results about PIDs

(DF Section 8.2)

- Example of a PID that is not a Euclidean domain  
(*DF p.277*)
- Proof that a Euclidean domain is a PID and hence a UFD
- Proof that  $\mathbb{Z}$  and  $k[x]$  are UFDs  
(*p.289 Smith, p.300 DF*)

- A polynomial ring in infinitely many variables over a UFD is still a ufd  
(*Easy, DF, p.305*)
- Eisenstein's criterion  
(*DF p.309*)
  - Stated only for monic polynomials – proof of general case identical.
  - See Smith's notes for the full version.
- Cyclic product structure of  $(\mathbb{Z}/n\mathbb{Z})^\times$   
(*exercise in DF, Smith 844-2, section 18*)
- Grobner bases and division algorithms for polynomials in several variables  
(*DF 9.6.*)
- Modules over pid's and Canonical forms of matrices.  
*DF sections 10.1, 10.2, 10.3, and 12.1, 12.2, 12.3.*
  - Constructive proof of decomposition: DF Exercises 12.1.16-19
  - Smith 845-1 and 845-2: Detailed discussion of the constructive proof.

## 2.3 Field Theory / Galois Theory.

- DF chapters 13,14 (about 145 pages).
- Smith:
  - 843-2, sections 11,12, and 16-21 (39 pages)
  - 844-1, sections 7-9 (20 pages)
  - 844-2, sections 10-16, (37 pages)

## 3 Outline of Topics: UCSD Qual Algebra, Fall 2018

Chapters 1-9 of Dummit and Foote

- Groups
  - Left and right cosets
  - Lagrange's theorem
  - Isomorphism theorems
  - Group generated by a subset
  - Structure of cyclic groups
  - Composite groups
  - Normalizer
  - Symmetric groups
  - Cayley's theorem
  - Orbit stabilizer theorem
  - Orbits act on left cosets of subgroups
  - Subgroups of index  $p$ , the smallest prime dividing  $|G|$ , are normal

- 
- Action of  $G$  on itself by conjugation
  - Class equation
  - $p$ -groups
  - $p^2$  groups are abelian
  - Automorphisms
    - \* Inner automorphisms
  - Proof of Sylow theorems
  - $A_n$  is simple for  $n \geq 5$
  - Recognition of internal direct product
  - Recognition of semi-direct product
  - Classification of groups of order  $pq$
  - Free group & presentations
  - Commutator subgroup
  - Solvable groups
  - Derived series
  - Nilpotent groups
  - Upper central series
  - Lower central series
  - Frattini's argument
  - Rings
    - $I$  maximal iff  $R/I$  is a field
    - Zorn's lemma
    - Chinese Remainder Theorem
    - Localization of a domain
    - Field of fractions
    - Factorization in domains
    - Euclidean algorithm
    - Gaussian integers
    - Primes and irreducibles
    - Domains
      - \* Primes are irreducible
    - UFDs
      - \* Have GCDs
      - \* Sometimes PIDs
    - PIDs
      - \* Noetherian
      - \* Irreducibles are prime
      - \* Are UFDs
      - \* Have GCDs
    - Euclidean domains
      - \* Are PIDs
    - Factorization in  $\mathbb{Z}[i]$
    - Polynomial rings
    - Gauss' lemma
    - Remainder and factor theorem
    - Polynomials
    - Reducibility
    - Rational root test

## 4 Group Theory

### 4.1 Random References

### 4.2 Big List of Notation

$C_G(x) =$	$\{g \in G \mid [g, x] = 1\}$	$\subseteq G$	Centralizer (Element)
$C_G(H) =$	$\{g \in G \mid [g, h] = 1 \ \forall h \in H\} = \bigcap_{h \in H} C_G(h)$	$\leq G$	Centralizer (Subgroup)
$? =$	$\{ghg^{-1} \mid g \in G\}$	$\subseteq G$	Conjugacy Class
$\mathcal{O}_x, G \cdot x =$	$\{g.x \mid x \in X\}$	$\subseteq X$	Orbit
$\text{Stab}_G(x), G_x =$	$\{g \in G \mid g.x = x\}$	$\subseteq G$	Stabilizer
$X^g =$	$\{x \in X \mid \forall g \in G, g.x = x\}$	$\subseteq X$	Fixed Points
$Z(G) =$	$\{x \in G \mid \forall g \in G, gxg^{-1} = x\}$	$\subseteq G$	Center
$N_G(H) =$	$\{g \in G \mid gHg^{-1} = H\}$	$\subseteq G$	Normalizer
$\text{Inn}(G) =$	$\{\varphi_g(x) = gxg^{-1}\}$	$\subseteq \text{Aut}(G)$	Inner Aut.
$\text{Out}(G) =$	$\text{Aut}(G)/\text{Inn}(G)$	$\hookrightarrow \text{Aut}(G)$	Outer Aut.

- $[g, h] = ghg^{-1}h^{-1}$
- $[G, H] = \langle [g, h] : g \in G, h \in H \rangle$  (Subgroup generated by commutators)

**Definition 4.0.1** (Normal Closure of a subgroup).

- $H^G = \{gHg^{-1} : g \in G\}$
- Equivalently,

$$H^G = \bigcap \{N : H \leq N \trianglelefteq G\}$$

– (The smallest normal subgroup of  $G$  containing  $H$ )

**Definition 4.0.2** (Normal Core of a subgroup).

- 

$$H_G = \bigcap_{g \in G} gHg^{-1}$$

- Equivalently,  $H_G = \langle N : N \trianglelefteq G \text{ \& } N \leq H \rangle$   
– Largest normal subgroup that contains  $H$
- Equivalently,  $H_G = \ker \psi$  where  $\psi : G \rightarrow \text{Sym}(G/H)$ ;  $g \sim (xH) = (gx)H$



- Facts:
  - $H_G \trianglelefteq G$  and is an idempotent operation

**Definition 4.0.3** (Characteristic subgroup).

- $H \text{ char } G \iff \forall \varphi \in \text{Aut}(G), \varphi(H) = H$ 
  - i.e.,  $H$  is fixed by all automorphisms of  $G$ .

**Definition 4.0.4** (Subgroup Generated by a Subset).

$\langle A \rangle = \{a_1^{\pm 1}, a_2^{\pm 1}, \dots, a_n^{\pm 1} : a_i \in A, n \in \mathbb{N}\}$ , or equivalently the intersection of all subgroups  $H$  such that  $A \subseteq H \leq G$ .

### 4.3 Basics

**Definition 4.0.5** (Centralizer:).

$$C_G(H) = \{g \in G \mid ghg^{-1} = h \ \forall h \in H\}$$

**Definition 4.0.6** (Normalizer).

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}$$

- Equivalently,  $\bigcup \{K : H \trianglelefteq K \leq G\}$  (the largest  $K \leq G$  for which  $H \trianglelefteq K$ )
- Equivalently, the stabilizer of  $H$  under  $G$  acting on its subgroups via conjugation

**Lemma 4.1.**

$$C_G(H) \trianglelefteq N_G(H)$$

**Lemma 4.2.**

The size of the conjugacy class of  $H$  is the index of its centralizer, i.e.

$$\left| \{gHg^{-1} \mid g \in G\} \right| = [G : C_G(H)].$$

Proof: Orbit-stabilizer.

**Theorem 4.3** (*The Fundamental Theorem of Cosets*).

$$aH = bH \iff a^{-1}b \in H \text{ or } aH \cap bH = \emptyset$$

**Definition 4.3.1** (Commutator).

$[x, y] = x^{-1}y^{-1}xy$  is the **commutator**, and  $[G, G] := \{[x, y] \mid x, y \in G\}$  is the **commutator**

### subgroup.

**Lemma 4.4.**

$[G, G] \leq H$  and  $H \trianglelefteq G \implies G/H$  is abelian.

**Lemmas:**

- Every subgroup of a cyclic group is itself cyclic.
- Intersections of subgroups are still subgroups
  - Intersections of distinct coprime-order subgroups are trivial
  - Intersections of subgroups of the same prime order are either trivial or equality
- The Quaternion group has only one element of order 2, namely  $-1$ .
  - They also have the presentation

$$\begin{aligned} Q &= \langle x, y, z \mid x^2 = y^2 = z^2 = xyz = -1 \rangle \\ &= \langle x, y \mid x^4 = y^4 = e, x^2 = y^2, yxy^{-1} = x^{-1} \rangle. \end{aligned}$$

- A dihedral group always has a presentation of the form

$$D_n = \langle x, y \mid x^n = y^2 = (xy)^2 = e \rangle,$$

yielding at least 2 distinct elements of order 2.

## 4.4 Finitely Generated Abelian Groups

Invariant factor decomposition:

$$G \cong \mathbb{Z}^r \times \prod_{j=1}^m \mathbb{Z}/(n_j) \quad \text{where } n_1 \mid \cdots \mid n_m.$$

### Going from invariant divisors to elementary divisors:

- Take prime factorization of each factor
- Split into coprime pieces

*Example:*

$$\begin{aligned} &\mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(2^3 \cdot 5^2 \cdot 7) \\ &\cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(2^3) \oplus \mathbb{Z}/(5^2) \oplus \mathbb{Z}/(7) \end{aligned}$$

### Going from elementary divisors to invariant factors:

- Bin up by primes occurring (keeping exponents)
- Take highest power from each prime as *last* invariant factor
- Take highest power from all remaining primes as next, etc

*Example:* Given the invariant factor decomposition

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}, .$$

$p = 2$	$p = 3$	$p = 5$
2, 2, 2	3, 3	$5^2$

$$\implies n_m = 5^2 \cdot 3 \cdot 2$$

$p = 2$	$p = 3$	$p = 5$
2, 2	3	$\emptyset$

$$\implies n_{m-1} = 3 \cdot 2$$

$p = 2$	$p = 3$	$p = 5$
2	$\emptyset$	$\emptyset$

$$\implies n_{m-2} = 2$$

and thus

$$G \cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(3 \cdot 2) \oplus \mathbb{Z}/(5^2 \cdot 3 \cdot 2).$$

### Classifying Abelian Groups of a Given Order:

Let  $p(x)$  be the integer partition function.  
 Example:  $p(6) = 11$ , given by  $6, 5 + 1, 4 + 2, \dots$ .

Write  $G = p_1^{k_1} p_2^{k_2} \dots$ ; then there are  $p(k_1)p(k_2) \dots$  choices, each yielding a distinct group.

## 4.5 The Symmetric Group

### Definitions:

- A cycle is **even**  $\iff$  product of an *even* number of transpositions.
  - A cycle of even *length* is **odd**
  - A cycle of odd *length* is **even**

Mnemonic: the parity of a  $k$ -cycle is the parity of  $k - 1$ .

**Definition** The **alternating group** is the subgroup of **even** permutations, i.e.  $A_n := \left\{ \sigma \in S_n \mid \text{sign}(\sigma) = 1 \right\}$  where  $\text{sign}(\sigma) = (-1)^m$  where  $m$  is the number of cycles of even length.

*Corollary:* Every  $\sigma \in A_n$  has an even number of *odd* cycles (i.e. an even number of *even-length* cycles).

*Example:*

$$A_4 = \{\text{id}, (1, 3)(2, 4), (1, 2)(3, 4), (1, 4)(2, 3), (1, 2, 3), (1, 3, 2), (1, 2, 4), (1, 4, 2), (1, 3, 4), (1, 4, 3), (2, 3, 4), (2, 4, 3)\}.$$

**Definition 4.4.1** (Dihedral Groups).

$$\langle a, b \mid a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle \cong \langle r, s \rangle$$

Useful Facts:

- Conjugacy classes are determined by cycle type
- The order of a cycle is its length.
- The order of an element is the least common multiple of the sizes of its cycles.
- The transitive subgroups of  $S_3$  are  $S_3, A_3$
- The transitive subgroups of  $S_4$  are  $S_4, A_4, D_4, \mathbb{Z}_2^2, \mathbb{Z}_4$ .
- $S_4$  has two normal subgroups:  $A_4, \mathbb{Z}_2^2$ .
- $S_{n \geq 5}$  has one normal subgroup:  $A_n$ .
- $Z(S_n) = 1$  for  $n \geq 3$
- $Z(A_n) = 1$  for  $n \geq 4$
- $[S_n, S_n] = A_n$
- $[A_4, A_4] \cong \mathbb{Z}_2^2$
- $[A_n, A_n] = A_n$  for  $n \geq 5$ , so  $A_{n \geq 5}$  is nonabelian.
- $A_{n \geq 5}$  is *simple*.
- $\sigma \circ (a_1 \cdots a_k) \circ \sigma^{-1} = (\sigma(a_1), \dots, \sigma(a_k))$

## 4.6 Counting Theorems

**Theorem 4.5** (*Lagrange's Theorem*).

$$H \leq G \implies |H| \mid |G|.$$

**Corollary 4.6.**

The order of every element divides the size of  $G$ , i.e.

$$g \in G \implies o(g) \mid o(G) \implies g^{|G|} = e.$$

**Warning:** There does **not** necessarily exist  $H \leq G$  with  $|H| = n$  for every  $n \mid |G|$ .

Counterexample:  $|A_4| = 12$  but has no subgroup of order 6.

**Theorem 4.7** (*Cauchy's Theorem*).

For every prime  $p$  dividing  $|G|$ , there is an element (and thus a subgroup) of order  $p$ .

This is a partial converse to Lagrange's theorem, and strengthened by Sylow's theorem.

### 4.6.1 Group Actions

**Definition 4.7.1** (Group Action).

An action of  $G$  on  $X$  is a group morphism

$$\begin{aligned}\varphi : G \times X &\rightarrow X \\ (g, x) &\mapsto g \cdot x\end{aligned}$$

or equivalently

$$\begin{aligned}\varphi : G &\longrightarrow \text{Aut}(X) \\ g &\mapsto (x \mapsto \varphi_g(x) := g \cdot x)\end{aligned}$$

satisfying

1.  $e \cdot x = x$
2.  $g \cdot (h \cdot x) = (gh) \cdot x$

Note that  $\ker \psi = \bigcap_{x \in X} G_x$  is the intersection of all stabilizers.

**Definition 4.7.2** (Transitive).

A group action  $G \curvearrowright X$  is *transitive* iff for all  $x, y \in X$  there exists a  $g \in G$  such that  $g \cdot x = y$ . Equivalently, the action has a single orbit.

**Notation:** For a group  $G$  acting on a set  $X$ ,

- $G \cdot x = \{g \cdot x \mid g \in G\} \subseteq X$  is the orbit
- $G_x = \{g \in G \mid g \cdot x = x\} \subseteq G$  is the stabilizer
- $X/G \subset \mathcal{P}(X)$  is the set of orbits
- $X^g = \{x \in X \mid g \cdot x = x\} \subseteq X$  are the fixed points

Note that being in the same orbit is an equivalence relation which partitions  $X$ , and  $G$  acts transitively if restricted to any single orbit.

**Orbit-Stabilizer:**

$$|G \cdot x| = [G : G_x] = |G|/|G_x| \quad \text{if } G \text{ is finite}$$

Mnemonic:  $G/G_x \cong G \cdot x$ .

### 4.6.2 Examples of Orbit-Stabilizer

1. Let  $G$  act on itself by left translation, where  $g \mapsto (h \mapsto gh)$ .
  - The orbit  $G \cdot x = G$  is the entire group

- The stabilizer  $G_x$  is only the identity.
  - The fixed points  $X^g$  are only the identity.
1. Let  $G$  act on *itself* by conjugation.
    - $G \cdot x$  is the **conjugacy class** of  $x$  (so not generally transitive)
    - $G_x = Z(x) := C_G(x) = \{g \mid [g, x] = e\}$ , the **centralizer** of  $x$ .
    - $G^g$  (the fixed points) is the **center**  $Z(G)$ .

**Corollary 4.8.**

The number of conjugates of an element (i.e. the size of its conjugacy class) is the index of its centralizer,  $[G : C_G(x)]$ .

**Corollary 4.9 (Class Equation).**

$$|G| = |Z(G)| + \sum_{\substack{\text{One } x_i \text{ from} \\ \text{each conjugacy} \\ \text{class}}} [G : C_G(x_i)]$$

Note that  $[G : C_G(x_i)]$  is the number of elements in the conjugacy class of  $x_i$ , and each  $x_i \in Z(G)$  has a singleton conjugacy class.

1. Let  $G$  act on  $X$ , its set of *subgroups*, by conjugation.
  - $G \cdot H = \{gHg^{-1}\}$  is the **set of conjugate subgroups** of  $H$
  - $G_H = N_G(H)$  is the **normalizer** of in  $G$  of  $H$
  - $X^g$  is the set of **normal subgroups** of  $G$

Corollary: Given  $H \leq G$ , the number of conjugate subgroups is  $[G : N_G(H)]$ .

1. For a fixed proper subgroup  $H < G$ , let  $G$  act on its cosets  $G/H = \{gH \mid g \in G\}$  by left translation.
  - $G \cdot gH = G/H$ , i.e. this is a *transitive* action.
  - $G_{gH} = gHg^{-1}$  is a *conjugate subgroup* of  $H$
  - $(G/H)^G = \emptyset$

*Application:* If  $G$  is simple,  $H < G$  proper, and  $[G : H] = n$ , then there exists an injective map  $\varphi : G \hookrightarrow S_n$ .

*Proof:* This action induces  $\varphi$ ; it is nontrivial since  $gH = H$  for all  $g$  implies  $H = G$ ;  $\ker \varphi \trianglelefteq G$  and  $G$  simple implies  $\ker \varphi = 1$ .

**Theorem 4.10 (Burnside's Formula).**

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

### 4.6.3 Sylow Theorems

**Notation:** For any  $p$ , let  $\text{Syl}_p(G)$  be the set of Sylow- $p$  subgroups of  $G$ .

Write

- $|G| = p^k m$  where  $(p, m) = 1$ ,
- $S_p$  a Sylow- $p$  subgroup, and
- $n_p$  the number of Sylow- $p$  subgroups.

#### Definition 4.10.1.

A  $p$ -group is a group  $G$  such that every element is order  $p^k$  for some  $k$ . If  $G$  is a finite  $p$ -group, then  $|G| = p^j$  for some  $j$ .

Some useful facts:

- Coprime order subgroups are disjoint, or more generally  $\mathbb{Z}_p, \mathbb{Z}_q \subset G \implies \mathbb{Z}_p \cap \mathbb{Z}_q = \mathbb{Z}_{(p,q)}$ .
- The Chinese Remainder theorem:  $(p, q) = 1 \implies \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$

### 4.6.4 Sylow 1 (Cauchy for Prime Powers)

Idea: Sylow  $p$ -subgroups exist for any  $p$  dividing  $|G|$ , and are maximal in the sense that every  $p$ -subgroup of  $G$  is contained in a Sylow  $p$ -subgroup.

$\forall p^n$  dividing  $|G|$  there exists a subgroup of size  $p^n$ .

If  $|G| = \prod p_i^{\alpha_i}$ , then there exist subgroups of order  $p_i^{\beta_i}$  for every  $i$  and every  $0 \leq \beta_i \leq \alpha_i$ .

In particular, Sylow  $p$ -subgroups always exist.

### 4.6.5 Sylow 2 (Sylows are Conjugate)

All sylow- $p$  subgroups  $S_p$  are conjugate, i.e.

$$S_p^1, S_p^2 \in \text{Syl}_p(G) \implies \exists g \text{ such that } gS_p^1g^{-1} = S_p^2.$$

**Corollary:**  $n_p = 1 \iff S_p \trianglelefteq G$

### 4.6.6 Sylow 3 (Numerical Constraints)

1.  $n_p \mid m$  (in particular,  $n_p \leq m$ ),
2.  $n_p \equiv 1 \pmod{p}$ ,
3.  $n_p = [G : N_G(S_p)]$  where  $N_G$  is the normalizer.

**Corollary:**  $p$  does not divide  $n_p$ .

**Lemma:** Every  $p$ -subgroup of  $G$  is contained in a Sylow  $p$ -subgroup.

*Proof:* Let  $H \leq G$  be a  $p$ -subgroup. If  $H$  is not *properly* contained in any other  $p$ -subgroup, it is a Sylow  $p$ -subgroup by definition.

Otherwise, it is contained in some  $p$ -subgroup  $H^1$ . Inductively this yields a chain  $H \subsetneq H^1 \subsetneq \dots$ , and by Zorn's lemma  $H := \bigcup_i H^i$  is maximal and thus a Sylow  $p$ -subgroup.

#### Theorem 4.11 (*Fratini's Argument*).

If  $H \trianglelefteq G$  and  $P \in \text{Syl}_p(G)$ , then  $HN_G(P) = G$  and  $[G : H]$  divides  $|N_G(P)|$ .

## 4.7 Products

#### Theorem 4.12 (*Recognizing Direct Products*).

We have  $G \cong H \times K$  when

- $H, K \trianglelefteq G$
- $G = HK$ .
- $H \cap K = \{e\} \subset G$

Note: can relax to  $[h, k] = 1$  for all  $h, k$ .

#### Theorem 4.13 (*Recognizing Generalized Direct Products*).

We have  $G = \prod_{i=1}^n H_i$  when

- $H_i \trianglelefteq G$  for all  $i$ .
- $G = H_1 \cdots H_n$
- $H_k \cap H_1 \cdots \widehat{H_k} \cdots H_n = \emptyset$

Note on notation: intersect  $H_k$  with the amalgam *leaving out*  $H_k$ .

#### Theorem 4.14 (*Recognizing Semidirect Products*).

We have  $G = N \rtimes_\psi H$  when

- $G = NH$
- $N \trianglelefteq G$
- $H \curvearrowright N$  by conjugation via a map

$$\begin{aligned} \psi : H &\longrightarrow \text{Aut}(N) \\ h &\mapsto h(\cdot)h^{-1}. \end{aligned}$$



Note relaxed conditions compared to direct product:  $H \trianglelefteq G$  and  $K \leq G$  to get a semidirect product instead

### Useful Facts

- If  $\sigma \in \text{Aut}(H)$ , then  $N \rtimes_{\psi} H \cong N \rtimes_{\psi \circ \sigma} H$ .
- $\text{Aut}(\mathbb{Z}/(p)^n) \cong \text{GL}(n, \mathbb{F}_p)$ , which has size  $|\text{Aut}(\mathbb{Z}/(p)^n)| = (p^n - 1)(p^n - p) \cdots (p^n - p^{n-1})$ .
  - If this occurs in a semidirect product, it suffices to consider similarity classes of matrices (i.e. just use canonical forms)
- $\text{Aut}(\mathbb{Z}/(n)) \cong \mathbb{Z}/(n)^{\times} \cong \mathbb{Z}/(\varphi(n))$  where  $\varphi$  is the totient function.
  - $\varphi(p^k) = p^{k-1}(p - 1)$
- If  $G, H$  have coprime order then  $\text{Aut}(G \oplus H) \cong \text{Aut}(G) \oplus \text{Aut}(H)$ .

## 4.8 Isomorphism Theorems

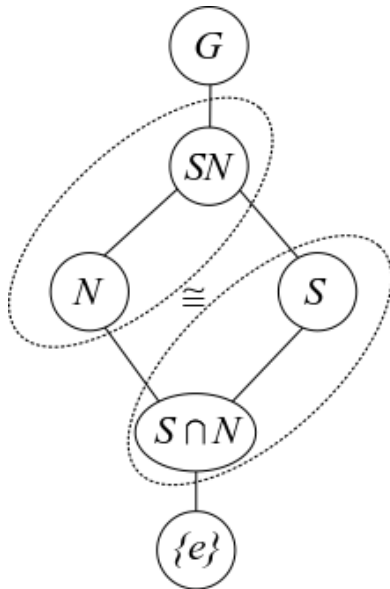
### Lemma 4.15.

If  $H, K \leq G$  and  $H \leq N_G(K)$  (or  $K \trianglelefteq G$ ) then  $HK \leq G$  is a subgroup.

### Diamond Theorem / 2nd Isomorphism Theorem:

If  $S \leq G$  and  $N \trianglelefteq G$ , then

$$\frac{SN}{N} \cong \frac{S}{S \cap N} \quad \text{and} \quad |SN| = \frac{|S||N|}{|S \cap N|}$$



Mnemonic:

Note: for this to make sense, we also have

- $SN \leq G$ ,
- $S \cap N \leq S$ ,

**Corollary 4.16.**

If we relax the conditions to  $S, N \leq G$  with  $S \in N_G(N)$ , then  $S \cap N \trianglelefteq S$  (but is not normal in  $G$ ) and the theorem still applies.

**Cancellation / 3rd Isomorphism Theorem**

Suppose  $N, K \leq G$  with  $N \trianglelefteq G$  and  $N \subseteq K \subseteq G$ .

1. If  $K \leq G$  then  $K/N \leq G/N$  is a subgroup
2. If  $K \trianglelefteq G$  then  $K/N \trianglelefteq G/N$ .
3. Every subgroup of  $G/N$  is of the form  $K/N$  for some such  $K \leq G$ .
4. Every *normal* subgroup of  $G/N$  is of the form  $K/N$  for some such  $K \trianglelefteq G$ .
5. If  $K \trianglelefteq G$ , then we can cancel normal subgroups:

$$\frac{G/N}{K/N} \cong \frac{G}{K}.$$

**Theorem 4.17 (The Correspondence Theorem / 4th Isomorphism Theorem).**

Suppose  $N \trianglelefteq G$ , then there exists a correspondence:

$$\begin{aligned} \left\{ H < G \mid N \subseteq H \right\} &\iff \left\{ H \mid H < \frac{G}{N} \right\} \\ \left\{ \begin{array}{c} \text{Subgroups of } G \\ \text{containing } N \end{array} \right\} &\iff \left\{ \begin{array}{c} \text{Subgroups of the} \\ \text{quotient } G/N \end{array} \right\}. \end{aligned}$$

In words, subgroups of  $G$  containing  $N$  correspond to subgroups of the quotient group  $G/N$ . This is given by the map  $H \mapsto H/N$ .

Note:  $N \trianglelefteq G$  and  $N \subseteq H < G \implies N \trianglelefteq H$ .

## 4.9 Special Classes of Groups

**Definition 4.17.1** (2 out of 3 Property).

The “**2 out of 3 property**” is satisfied by a class of groups  $\mathcal{C}$  iff whenever  $G \in \mathcal{C}$ , then  $N, G/N \in \mathcal{C}$  for any  $N \trianglelefteq G$ .

**Definition 4.17.2** (p-groups).

If  $|G| = p^k$ , then  $G$  is a **p-group**.

**Definition 4.17.3** (Normalizers Grow).

If for every proper  $H < G$ ,  $H \trianglelefteq N_G(H)$  is again proper, then “normalizers grow” in  $G$ .

- Normalizers grow

**Facts about other special order groups:**

General strategy: find a normal subgroup (usually a Sylow) and use recognition of semidirect products.

- $|G| = pq$ : Two possibilities. By cases:
    - If  $p$  divides  $q - 1$ , two cases:
      - \*  $G \cong \mathbb{Z}/(pq)$  or  $\mathbb{Z}(p) \times \mathbb{Z}/(q)$
    - Otherwise,  $G \cong \mathbb{Z}/(pq)$
- Proof: Sylow theorems. Note: Such groups are never simple.
- $|G| = p^2q$ :
    - $q \mid p^2 - 1$ : Two abelian possibilities,  $\mathbb{Z}/(p) \times \mathbb{Z}/(q^2)$ , or  $\mathbb{Z}/(pq) \times \mathbb{Z}/(q)$ .
    - Otherwise, the sylow- $q$  subgroup  $H$  is normal and order  $q^2$ , so either  $\mathbb{Z}/(q)^2$  or  $\mathbb{Z}/(q^2)$ .
      - \* Case 2:  $|\text{Aut}(\mathbb{Z}/(q)^2)| = q(q - 1)$ , so only trivial action
      - \* Case 1:  $|\text{Aut}(\mathbb{Z}/(q^2))| = q(q - 1)^2(q + 1)$ 
        - If  $p$  doesn't divide  $q + 1$ , nothing new
        - Otherwise, a nontrivial semidirect product.

## 4.10 Classification of Groups

- Keith Conrad: Classifying Groups of Order 12
- Order  $p$ : cyclic.
- Order  $pq$ : ?
- Order  $p^2q$ : ?

## 4.11 Groups of Small Order

## 4.12 Series of Groups

### Definition 4.17.4.

A **normal series** of a group  $G$  is a sequence  $G \longrightarrow G^1 \longrightarrow G^2 \longrightarrow \cdots$  such that  $G^{i+1} \trianglelefteq G_i$  for every  $i$ .

### Definition 4.17.5.

A **central series** for a group  $G$  is a terminating normal series  $G \longrightarrow G^1 \longrightarrow \cdots \longrightarrow \{e\}$  such that each quotient is **central**, i.e.  $[G, G^i] \leq G^{i-1}$  for all  $i$ .

### Definition 4.17.6 (Composition Series).

A **composition series** of a group  $G$  is a finite normal series such that  $G^{i+1}$  is a *maximal proper* normal subgroup of  $G^i$ .

### Theorem 4.18 (Jordan-Holder).

Any two composition series of a group have the same length and isomorphic composition factors (up to permutation).

### Definition 4.18.1 (Simple Groups).

A group  $G$  is **simple** iff  $H \trianglelefteq G \implies H = \{e\}, G$ , i.e. it has no non-trivial proper subgroups.

**Lemma 4.19.**

If  $G$  is *not* simple, then for any  $N \trianglelefteq G$ , it is the case that  $G \cong E$  for an extension of the form  $N \longrightarrow E \longrightarrow G/N$ .

**Definition 4.19.1** (Lower Central Series).

Set  $G^0 = G$  and  $G^{i+1} = [G, G^i]$ , then  $G^0 \geq G^1 \geq \cdots$  is the *lower central series* of  $G$ .

Mnemonic: “lower” because the chain is descending. Iterate the adjoint map  $[\cdot, G]$ , if this terminates then the map is nilpotent, so call  $G$  nilpotent!

**Definition 4.19.2** (Upper Central Series).

Set  $Z_0 = 1$ ,  $Z_1 = Z(G)$ , and  $Z_{i+1} \leq G$  to be the subgroup satisfying  $Z_{i+1}/Z_i = Z(G/Z_i)$ . Then  $Z_0 \leq Z_1 \leq \cdots$  is the *upper central series* of  $G$ .

Equivalently, since  $Z_i \trianglelefteq G$ , there is a quotient map  $\pi : G \longrightarrow G/Z_i$ , so define  $Z_{i+1} := \pi^{-1}(Z(G/Z_i))$  (?).

Mnemonic: “upper” because the chain is ascending. “Take higher centers”.

**Definition 4.19.3** (Derived Series).

Set  $G^{(0)} = G$  and  $G^{(i+1)} = [G^{(i)}, G^{(i)}]$ , then  $G^{(0)} \geq G^{(1)} \geq \cdots$  is the *derived series* of  $G$ .

**Definition 4.19.4** (Solvable).

A group  $G$  is **solvable** iff  $G$  has a terminating normal series with abelian composition factors, i.e.

$$G \longrightarrow G^1 \longrightarrow \cdots \longrightarrow \{e\} \text{ with } G^i/G^{i+1} \text{ abelian for all } i.$$

**Theorem 4.20.**

A group  $G$  is solvable iff its derived series terminates.

**Theorem 4.21.**

If  $n \geq 4$  then  $S_n$  is solvable.

**Lemmas**

- $G$  is solvable iff  $G$  has a terminating *derived series*.
- Solvable groups satisfy the 2 out of 3 property
- Abelian  $\implies$  solvable
- Every group of order less than 60 is solvable.

**Definition 4.21.1** (Nilpotent).

A group  $G$  is **nilpotent** iff  $G$  has a terminating upper central series.

Moral: the adjoint map is nilpotent.

---

**Theorem 4.22.**

A group  $G$  is nilpotent iff all of its Sylow  $p$ -subgroups are normal for every  $p$  dividing  $|G|$ .

**Theorem 4.23.**

A group  $G$  is nilpotent iff every maximal subgroup is normal.

**Theorem 4.24.**

$G$  is nilpotent iff  $G$  has an upper central series terminating at  $G$ .

**Theorem 4.25.**

$G$  is nilpotent iff  $G$  has a lower central series terminating at 1.

**Lemma:** For  $G$  a finite group, TFAE:

- $G$  is nilpotent
- Normalizers grow (i.e.  $H < N_G(H)$  whenever  $H$  is proper)
- Every Sylow- $p$  subgroup is normal
- $G$  is the direct product of its Sylow  $p$ -subgroups
- Every maximal subgroup is normal
- $G$  has a terminating *Lower* Central Series
- $G$  has a terminating *Upper* Central Series

**Lemmas:**

- $G$  nilpotent  $\implies G$  solvable
- Nilpotent groups satisfy the 2 out of 3 property.
- $G$  has normal subgroups of order  $d$  for *every*  $d$  dividing  $|G|$
- $G$  nilpotent  $\implies Z(G) \neq 0$
- Abelian  $\implies$  nilpotent
- $p$ -groups  $\implies$  nilpotent

## 5 Rings

### 5.1 Definitions

**Definition 5.0.1** (Irreducible Element).

An element  $r \in R$  is **irreducible** iff  $r = ab \implies a$  is a unit or  $b$  is a unit.

**Definition 5.0.2** (Prime Element).

An element  $r \in R$  is **prime** iff  $ab \mid r \implies a \mid r$  or  $b \mid r$  whenever  $a, b$  are nonzero and not units.

**Definition 5.0.3** (Integral Domain).

?

**Definition 5.0.4** (Principal Ideal Domain).

?

**Definition 5.0.5** (Unique Factorization Domain).  
?

**Definition 5.0.6** (Noetherian).

A ring  $R$  is Noetherian if the ACC holds: every ascending chain of ideals  $I_1 \leq I_2 \cdots$  stabilizes.

**Theorem 5.1** (*Zorn's Lemma*).

If  $P$  is a poset in which every chain has an upper bound, then  $P$  has a maximal element.

**Definition 5.1.1** (Principal Ideals).

$I \trianglelefteq R$  *principal* when  $\exists a \in R : I = \langle a \rangle$

**Definition 5.1.2** (Irreducible Ideal).

$I \trianglelefteq R$  *irreducible* when  $\nexists \{J \trianglelefteq R : I \subset J\} : I = \bigcap J$

**Definition 5.1.3** (Primary Ideal).

An ideal  $I \trianglelefteq R$  is *primary* iff whenever  $pq \in I$ ,  $p \in I$  and  $q^n \in I$  for some  $n$ .

**Definition 5.1.4** (Simple Ring).

A ring  $R$  is **simple** iff every ideal  $I \trianglelefteq R$  is either 0 or  $R$ .

**Definition 5.1.5** (Local Ring).

A ring  $R$  is *local* iff it contains a unique maximal ideal.

**Definition 5.1.6** (Prime Ideal).

$\mathfrak{p}$  is a **prime** ideal  $\iff ab \in \mathfrak{p} \implies a \in \mathfrak{p}$  or  $b \in \mathfrak{p}$ .

**Definition 5.1.7** (Prime Spectrum).

$\text{Spec}(R) = \{\mathfrak{p} \trianglelefteq R \mid \mathfrak{p} \text{ is prime}\}$  is the **spectrum** of  $R$ .

**Definition 5.1.8** (Maximal Ideal).

$\mathfrak{m}$  is **maximal**  $\iff I \triangleleft R \implies I \subseteq \mathfrak{m}$ .

Examples:

- Maximal ideals of  $R[x]$  are of the form  $I = (x - a_i)$  for some  $a_i \in R$ .

**Definition 5.1.9** (Max Spectrum).

$\text{maxSpec}(R) = \{\mathfrak{m} \trianglelefteq R \mid \mathfrak{m} \text{ is maximal}\}$  is the **max-spectrum** of  $R$ .

**Definition 5.1.10** (Nilradical).

$\mathfrak{N}(R) := \{x \in R \mid x^n = 0 \text{ for some } n\}$  is the **nilradical** of  $R$ .

**Definition 5.1.11** (Jacobson Radical).

The **Jacobson radical**  $\mathfrak{J}(R)$  is the intersection of all maximal ideals, i.e.

$$\mathfrak{J}(R) = \bigcap_{\mathfrak{m} \in \text{Spec}_{\max}} \mathfrak{m}$$

Definition (Semisimple)

A nonzero unital ring  $R$  is **semisimple** iff  $R \cong \bigoplus_{i=1}^n M_i$  with each  $M_i$  a simple module.

**Definition 5.1.12** (Radical of an Ideal).

For an ideal  $I \trianglelefteq R$ , the radical  $\text{rad}(I) := \{r \in R \mid r^n \in I \text{ for some } n \geq 0\}$ , so  $x^n \in I \iff x \in I$ .

**Definition 5.1.13** (Radical Ideal).

An ideal is *radical* iff  $\text{rad}(I) = I$ .

**Lemma (Characterizations of Rings):**

- $R$  a commutative division ring  $\implies R$  is a field
- $R$  a finite integral domain  $\implies R$  is a field.
- $\mathbb{F}$  a field  $\implies \mathbb{F}[x]$  is a Euclidean domain.
- $\mathbb{F}$  a field  $\implies \mathbb{F}[x]$  is a PID.
- $\mathbb{F}$  is a field  $\iff \mathbb{F}$  is a commutative simple ring.
- $R$  is a UFD  $\iff R[x]$  is a UFD.
- $R$  a PID  $\implies R[x]$  is a UFD
- $R$  a PID  $\implies R$  Noetherian
- $R[x]$  a PID  $\implies R$  is a field.

**Lemma:** Fields  $\subset$  Euclidean domains  $\subset$  PIDs  $\subset$  UFDs  $\subset$  Integral Domains  $\subset$  Rings

- A Euclidean Domain that is not a field:  $\mathbb{F}[x]$  for  $\mathbb{F}$  a field
  - *Proof:* Use previous lemma, and  $x$  is not invertible
- A PID that is not a Euclidean Domain:  $\mathbb{Z}\left[\frac{1 + \sqrt{-19}}{2}\right]$ .
  - *Proof:* complicated.
- A UFD that is not a PID:  $\mathbb{F}[x, y]$ .
  - *Proof:*  $\langle x, y \rangle$  is not principal
- An integral domain that is not a UFD:  $\mathbb{Z}[\sqrt{-5}]$ 
  - *Proof:*  $(2 + \sqrt{-5})(2 - \sqrt{-5}) = 9 = 3 \cdot 3$ , where all factors are irreducible (check norm).
- A ring that is not an integral domain:  $\mathbb{Z}/(4)$ 
  - *Proof:*  $2 \bmod 4$  is a zero divisor.

**Lemma 5.2.**

In  $R$  a UFD, an element  $r \in R$  is prime  $\iff r$  is irreducible.

Note: For  $R$  an integral domain, prime  $\implies$  irreducible, but generally not the converse.

*Example of a prime that is not irreducible:*  $x^2 \bmod (x^2 + x) \in \mathbb{Q}[x]/(x^2 + x)$ . Check that  $x$  is prime

directly, but  $x = x \cdot x$  and  $x$  is not a unit.

*Example of an irreducible that is not prime:*  $3 \in \mathbb{Z}[\sqrt{-5}]$ . Check norm to see irreducibility, but  $3 \mid 9 = (2 + \sqrt{-5})(2 - \sqrt{-5})$  and doesn't divide either factor.

**Lemma 5.3.**

If  $R$  is a PID, then every element in  $R$  has a unique prime factorization.

**Theorem 5.4 (Krull).**

Every ring has proper maximal ideals, and any proper ideal is contained in a maximal ideal.

**Theorem 5.5 (Artin-Wedderburn).**

If  $R$  is a nonzero, unital, *semisimple* ring then  $R \cong \bigoplus_{i=1}^m \text{Mat}(n_i, D_i)$ , a finite sum of matrix rings over division rings.

**Corollary 5.6.**

If  $M$  is a simple ring over  $R$  a division ring, the  $M$  is isomorphic to a matrix ring.

### 5.1.1 Zorn's Lemma

**Lemma 5.7.**

Fields are simple rings.

**Lemma 5.8.**

If  $I \subseteq R$  is a proper ideal  $\iff I$  contains no units.

*Proof.*

$$r \in R^\times \cap I \implies r^{-1}r \in I \implies 1 \in I \implies x \cdot 1 \in I \quad \forall x \in R.$$

■

**Lemma 5.9.**

If  $I_1 \subseteq I_2 \subseteq \dots$  are ideals then  $\bigcup_j I_j$  is an ideal.

**Example Application:** Every proper ideal is contained in a maximal ideal.

*Proof.*

Let  $0 < I < R$  be a proper ideal, and consider the set

$$S = \{J \mid I \subseteq J < R\}.$$

Note  $I \in S$ , so  $S$  is nonempty. The claim is that  $S$  contains a maximal element  $M$ .

$S$  is a poset, ordered by set inclusion, so if we can show that every chain has an upper bound, we can apply Zorn's lemma to produce  $M$ .



Let  $C \subseteq S$  be a chain in  $S$ , so  $C = \{C_1 \subseteq C_2 \subseteq \dots\}$  and define  $\widehat{C} = \bigcup_i C_i$ .

**$\widehat{C}$  is an upper bound for  $C$ :** This follows because every  $C_i \subseteq \widehat{C}$ .

**$\widehat{C}$  is in  $S$ :** Use the fact that  $I \subseteq C_i < R$  for every  $C_i$  and since no  $C_i$  contains a unit,  $\widehat{C}$  doesn't contain a unit, and is thus proper. ■

## 6 Fields

Let  $k$  denote a field.

**Lemmas:**

- The characteristic of any field  $k$  is either 0 or  $p$  a prime.
- All fields are simple rings (no proper nontrivial ideals).
- If  $L/k$  is algebraic, then  $\min(\alpha, L)$  divides  $\min(\alpha, k)$ .
- Every field morphism is either zero or injective.

**Theorem 6.1.**

Every finite extension is algebraic.

*Proof.*

Todo? ■

**Theorem 6.2 (Gauss' Lemma).**

Let  $R$  be a UFD and  $F$  its field of fractions. Then a primitive  $p \in R[x]$  is irreducible in  $R[x] \iff p$  is irreducible in  $F[x]$ .

**Corollary 6.3.**

A primitive polynomial  $p \in \mathbb{Q}[x]$  is irreducible  $\iff p$  is irreducible in  $\mathbb{Z}[x]$ .

**Theorem 6.4 (Eisenstein's Criterion).**

If  $f(x) = \sum_{i=0}^n \alpha_i x^i \in \mathbb{Q}[x]$  and  $\exists p$  such that

- $p$  divides every coefficient *except*  $a_n$  and
- $p^2$  does not divide  $a_0$ ,

then  $f$  is irreducible over  $\mathbb{Q}[x]$ , and by Gauss' lemma, over  $\mathbb{Z}[x]$ .

**Definition 6.4.1 (Primitive).**

For  $R$  a UFD, a polynomial  $p \in R[x]$  is **primitive** iff the greatest common divisors of its coefficients is a unit.

## 6.1 Finite Fields

### Definition 6.4.2.

The **prime subfield** of a field  $F$  is the subfield generated by 1.

### Lemma 6.5 (*Characterization of Prime Subfields*).

The prime subfield of any field is isomorphic to either  $\mathbb{Q}$  or  $\mathbb{F}_p$  for some  $p$ .

### Proposition 6.6 (*Freshman's Dream*).

If  $\text{char } k = p$  then  $(a + b)^p = a^p + b^p$  and  $(ab)^p = a^p b^p$ .

*Proof.*

Todo

■

### Theorem 6.7 (*Construction of Finite Fields*).

$\mathbb{GF}(p^n) \cong \frac{\mathbb{F}_p[x]}{(f)}$  where  $f \in \mathbb{F}_p[x]$  is any irreducible of degree  $n$ , and  $\mathbb{GF}(p^n) \cong \mathbb{F}[\alpha] \cong \text{span}_{\mathbb{F}} \{1, \alpha, \dots, \alpha^{n-1}\}$  for any root  $\alpha$  of  $f$ .

### Lemma 6.8 (*Prime Subfields of Finite Fields*).

Every finite field  $F$  is isomorphic to a unique field of the form  $\mathbb{GF}(p^n)$  and if  $\text{char } F = p$ , it has prime subfield  $\mathbb{F}_p$ .

### Lemma 6.9 (*Containment of Finite Fields*).

$\mathbb{GF}(p^\ell) \leq \mathbb{GF}(p^k) \iff \ell \text{ divides } k$ .

### Lemma 6.10 (*Identification of Finite Fields as Splitting Fields*).

$\mathbb{GF}(p^n)$  is the splitting field of  $\rho(x) = x^{p^n} - x$ , and the elements are exactly the roots of  $\rho$ .

*Proof.*

Todo. Every element is a root by Cauchy's theorem, and the  $p^n$  roots are distinct since its derivative is identically  $-1$ .

■

### Lemma 6.11 (*Splits Product of Irreducibles*).

Let  $\rho_n := x^{p^n} - x$ . Then  $f(x) \mid \rho_n(x) \iff \deg f \mid n$  and  $f$  is irreducible.

### Corollary 6.12.

$x^{p^n} - x = \prod f_i(x)$  over all irreducible monic  $f_i \in \mathbb{F}_p[x]$  of degree  $d$  dividing  $n$ .

*Proof.*

$\Leftarrow$  : Suppose  $f$  is irreducible of degree  $d$ . Then  $f \mid x^{p^d} - x$  (consider  $F[x]/\langle f \rangle$ ) and  $x^{p^d} - x \mid x^{p^n} - x \iff d \mid n$ .

$\Rightarrow$  :

- $\alpha \in \mathbb{GF}(p^n) \iff \alpha^{p^n} - \alpha = 0$ , so every element is a root of  $\varphi_n$  and  $\deg \min(\alpha, \mathbb{F}_p) \mid n$  since  $\mathbb{F}_p(\alpha)$  is an intermediate extension.
- So if  $f$  is an irreducible factor of  $\varphi_n$ ,  $f$  is the minimal polynomial of some root  $\alpha$  of  $\varphi_n$ , so  $\deg f \mid n$ .  $\varphi'_n(x) = p^n x^{p^n-1} \neq 0$ , so  $\varphi_n$  has distinct roots and thus no repeated factors. So  $\varphi_n$  is the product of all such irreducible  $f$ .

■

### Lemma 6.13.

No finite field is algebraically closed.

*Proof.*

Todo?

■

## 6.2 Galois Theory

### Definition 6.13.1.

A field extension  $L/k$  is **algebraic** iff every  $\alpha \in L$  is the root of some polynomial  $f \in k[x]$ .

### Definition 6.13.2.

Let  $L/k$  be a finite extension. Then TFAE:

- $L/k$  is **normal**.
- Every irreducible  $f \in k[x]$  that has one root in  $L$  has *all* of its roots in  $L$   
– i.e. every polynomial splits into linear factors
- Every embedding  $\sigma : L \hookrightarrow \bar{k}$  that is a lift of the identity on  $k$  satisfies  $\sigma(L) = L$ .
- If  $L$  is separable:  $L$  is the splitting field of some irreducible  $f \in k[x]$ .

### Definition 6.13.3.

Let  $L/k$  be a field extension,  $\alpha \in L$  be arbitrary, and  $f(x) := \min(\alpha, k)$ . TFAE:

- $L/k$  is **separable**
- $f$  has no repeated factors/roots
- $\gcd(f, f') = 1$ , i.e.  $f$  is coprime to its derivative
- $f' \not\equiv 0$

### Lemma 6.14.

If  $\text{char } k = 0$  or  $k$  is finite, then every *algebraic* extension  $L/k$  is separable.

### Definition 6.14.1.

$\text{Aut}(L/k) = \left\{ \sigma : L \longrightarrow L \mid \sigma|_k = \text{id}_k \right\}$ .

**Lemma 6.15.**

If  $L/k$  is algebraic, then  $\text{Aut}(L/k)$  permutes the roots of irreducible polynomials.

**Lemma 6.16.**

$|\text{Aut}(L/k)| \leq [L : k]$  with equality precisely when  $L/k$  is normal.

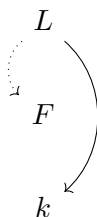
**Definition 6.16.1.**

If  $L/k$  is Galois, we define  $\text{Gal}(L/k) := \text{Aut}(L/k)$ .

**6.2.1 Lemmas About Towers**

Let  $L/F/k$  be a finite tower of field extensions

- Multiplicativity:  $[L : k] = [L : F][F : k]$
- $L/k$  normal/algebraic/Galois  $\implies L/F$  normal/algebraic/Galois.
  - *Proof (normal):*  $\min(\alpha, F) \mid \min(\alpha, k)$ , so if the latter splits in  $L$  then so does the former.
  - *Corollary:*  $\alpha \in L$  algebraic over  $k \implies \alpha$  algebraic over  $F$ .
  - *Corollary:*  $E_1/k$  normal and  $E_2/k$  normal  $\implies E_1E_2/k$  normal and  $E_1 \cap E_2/k$  normal.



- $F/k$  algebraic and  $L/F$  algebraic  $\implies L/k$  algebraic.
- If  $L/k$  is algebraic, then  $F/k$  separable and  $L/F$  separable  $\iff L/k$  separable



- $F/k$  Galois and  $L/F$  Galois  $\implies F/k$  Galois **only if**  $\text{Gal}(L/F) \leq \text{Gal}(L/k)$ 
  - $\implies \text{Gal}(F/k) \cong \frac{\text{Gal}(L/k)}{\text{Gal}(L/F)}$



**Common Counterexamples:**

- $\mathbb{Q}(\zeta_3, 2^{1/3})$  is normal but  $\mathbb{Q}(2^{1/3})$  is not since the irreducible polynomial  $x^3 - 2$  has only one root in it.

**Definition 6.16.2** (Characterizations of Galois Extensions).

Let  $L/k$  be a finite field extension. TFAE:

- $L/k$  is **Galois**
- $L/k$  is finite, normal, and separable.
- $L/k$  is the splitting field of a separable polynomial
- $|\text{Aut}(L/k)| = [L : k]$
- The fixed field of  $\text{Aut}(L/k)$  is exactly  $k$ .

**Theorem 6.17** (*Fundamental Theorem of Galois Theory*).

Let  $L/k$  be a Galois extension, then there is a correspondence:

$$\begin{aligned} \{\text{Subgroups } H \leq \text{Gal}(L/k)\} &\longleftrightarrow \left\{ \begin{array}{l} \text{Fields } F \text{ such} \\ \text{that } L/F/k \end{array} \right\} \\ H &\rightarrow \{E^H := \text{The fixed field of } H\} \\ \left\{ \text{Gal}(L/F) := \left\{ \sigma \in \text{Gal}(L/k) \mid \sigma(F) = F \right\} \right\} &\leftarrow F. \end{aligned}$$

- This is contravariant with respect to subgroups/subfields.
- $[F : k] = [G : H]$ , so degrees of extensions over the base field correspond to indices of subgroups.
- $[K : F] = |H|$
- $L/F$  is Galois and  $\text{Gal}(K/F) = H$
- $F/k$  is Galois  $\iff H$  is normal, and  $\text{Gal}(F/k) = \text{Gal}(L/k)/H$ .
- The compositum  $F_1 F_2$  corresponds to  $H_1 \cap H_2$ .
- The subfield  $F_1 \cap F_2$  corresponds to  $H_1 H_2$ .

**6.2.2 Examples**

1.  $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong \mathbb{Z}/(n)^\times$  and is generated by maps of the form  $\zeta_n \mapsto \zeta_n^j$  where  $(j, n) = 1$ .

I.e., the following map is an isomorphism:

$$\begin{aligned} \mathbb{Z}/(n)^\times &\longrightarrow \text{Gal}(\mathbb{Q}(\zeta_n), \mathbb{Q}) \\ r \pmod n &\mapsto (\varphi_r : \zeta_n \mapsto \zeta_n^r). \end{aligned}$$

2.  $\text{Gal}(\mathbb{GF}(p^n)/\mathbb{GF}(p)) \cong \mathbb{Z}/(n)$ , a cyclic group generated by powers of the Frobenius automorphism:

$$\begin{aligned} \varphi_p : \mathbb{GF}(p^n) &\longrightarrow \mathbb{GF}(p^n) \\ x &\mapsto x^p. \end{aligned}$$

**Lemma 6.18.**

Every quadratic extension is Galois.

**Lemma 6.19.**

If  $K$  is the splitting field of an irreducible polynomial of degree  $n$ , then  $\text{Gal}(K/\mathbb{Q}) \leq S_n$  is a transitive subgroup.

**Corollary 6.20.**

$n$  divides the order  $|\text{Gal}(K/\mathbb{Q})|$ .

**Definition 6.20.1.**

TFAE:

- $k$  is a **perfect** field.
- Every irreducible polynomial  $p \in k[x]$  is separable
- Every finite extension  $F/k$  is separable.
- If  $\text{char } k > 0$ , the Frobenius is an automorphism of  $k$ .

**Theorem 6.21.**

- If  $\text{char } k = 0$  or  $k$  is finite, then  $k$  is perfect.
- $k = \mathbb{Q}, \mathbb{F}_p$  are perfect, and any finite normal extension is Galois.
- Every splitting field of a polynomial over a perfect field is Galois.

**Proposition 6.22 (Composite Extensions).**

If  $F/k$  is finite and Galois and  $L/k$  is arbitrary, then  $FL/L$  is Galois and

$$\text{Gal}(FL/L) = \text{Gal}(F/F \cap L) \subset \text{Gal}(F/k).$$

## 6.3 Cyclotomic Polynomials

**Definition 6.22.1 (Cyclotomic Polynomials).**

Let  $\zeta_n = e^{2\pi i/n}$ , then the  $n$ th cyclotomic polynomial is given by

$$\Phi_n(x) = \prod_{\substack{k=1 \\ (k,n)=1}}^n (x - \zeta_n^k),$$

which is a product over primitive roots of unity. It is the unique irreducible polynomial which is a divisor of  $x^n - 1$  but *not* a divisor of  $x^k - 1$  for any  $k < n$ .

**Proposition 6.23.**

$\deg \Phi_n(x) = \varphi(n)$  for  $\varphi$  the totient function.

*Proof .*

$\deg \Phi_n(x)$  is the number of  $n$ th primitive roots, which is the number of numbers less than and coprime to  $n$ . ■

---

## Computing $\Phi_n$ :

1.

$$\Phi_n(z) = \prod_{d|n, d>0} (z^d - 1)^{\mu(\frac{n}{d})}$$

where

$$\mu(n) \equiv \begin{cases} 0 & \text{if } n \text{ has one or more repeated prime factors} \\ 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes,} \end{cases}$$

2.

$$x^n - 1 = \prod_{d|n} \Phi_d(x) \implies \Phi_n(x) = \frac{x^n - 1}{\prod_{\substack{d|n \\ d < n}} \Phi_d(x)},$$

so just use polynomial long division.

**Lemma 6.24.**

$$\begin{aligned} \Phi_p(x) &= x^{p-1} + x^{p-2} + \cdots + x + 1 \\ \Phi_{2p}(x) &= x^{p-1} - x^{p-2} + \cdots - x + 1. \end{aligned}$$

**Lemma 6.25.**

$$k \mid n \implies \Phi_{nk}(x) = \Phi_n(x^k)$$

**Definition 6.25.1.**

An extension  $F/k$  is **simple** if  $F = k[\alpha]$  for a single element  $\alpha$ .

**Theorem 6.26 (Primitive Element).**

Every finite separable extension is simple.

**Corollary 6.27.**

$\mathbb{GF}(p^n)$  is a simple extension over  $\mathbb{F}_p$ .

## 7 Modules

### 7.1 General Modules

**Definition:** A module is **simple** iff it has no nontrivial proper submodules.

**Definition:** A **free** module is a module with a basis (i.e. a spanning, linearly independent set).

*Example:*  $\mathbb{Z}/(6)$  is a  $\mathbb{Z}$ -module that is *not* free.

**Definition:** A module  $M$  is **projective** iff  $M$  is a direct summand of a free module  $F = M \oplus \cdots$ .

Free implies projective, but not the converse.

**Definition:** A sequence of homomorphisms  $0 \xrightarrow{d_1} A \xrightarrow{d_2} B \xrightarrow{d_3} C \longrightarrow 0$  is *exact* iff  $\text{im } d_i = \ker d_{i+1}$ .

**Lemma:** If  $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$  is a short exact sequence, then

- $C$  free  $\implies$  the sequence splits
- $C$  projective  $\implies$  the sequence splits
- $A$  injective  $\implies$  the sequence splits

Moreover, if this sequence splits, then  $B \cong A \oplus C$ .

## 7.2 Classification of Modules over a PID

Let  $M$  be a finitely generated modules over a PID  $R$ . Then there is an invariant factor decomposition

$$M \cong F \bigoplus R/(r_i) \quad \text{where } r_1 \mid r_2 \mid \cdots,$$

and similarly an elementary divisor decomposition.

## 7.3 Minimal / Characteristic Polynomials

Fix some notation:

$\min_A(x)$  : The minimal polynomial of  $A$

$\chi_A(x)$  : The characteristic polynomial of  $A$ .

**Definition:** The minimal polynomial is the unique polynomial  $\min_A(x)$  of minimal degree such that  $\min_A(A) = 0$ .

**Definition:** The **characteristic polynomial** of  $A$  is given by

$$\chi_A(x) = \det(A - xI) = \det(SNF(A - xI)).$$

*Useful lemma:* If  $A$  is upper triangular, then  $\det(A) = \prod_i a_{ii}$

**Theorem (Cayley-Hamilton):** The minimal polynomial divides the characteristic polynomial, and in particular  $\chi_A(A) = 0$ .

**Lemma:** Writing

$$\begin{aligned} \min_A(x) &= \prod (x - \lambda_i)^{a_i} \\ \chi_A(x) &= \prod (x - \lambda_i)^{b_i} \end{aligned}$$



- $a_i \leq b_i$
- The roots both polynomials are precisely the eigenvalues of  $A$ .

*Proof:* By Cayley-Hamilton,  $\min_A$  divides  $\chi_A$ . Every  $\lambda_i$  is a root of  $\mu_M$ :

Let  $(\mathbf{v}_i, \lambda_i)$  be a nontrivial eigenpair. Then by linearity,

$$\min_A(\lambda_i)\mathbf{v}_i = \min_A(A)\mathbf{v}_i = \mathbf{0},$$

which forces  $\min_A(\lambda_i) = 0$ .

**Definition:** Two matrices  $A, B$  are **similar** (i.e.  $A = PBP^{-1}$ )  $\iff A, B$  have the same Jordan Canonical Form (JCF).

**Definition:** Two matrices  $A, B$  are **equivalent** (i.e.  $A = PBQ$ )  $\iff$

- They have the same rank,
- They have the same invariant factors, *and*
- They have the same (JCF)

### Finding the minimal polynomial:

Let  $m(x)$  denote the minimal polynomial  $A$ .

1. Find the characteristic polynomial  $\chi(x)$ ; this annihilates  $A$  by Cayley-Hamilton. Then  $m(x) \mid \chi(x)$ , so just test the finitely many products of irreducible factors.
2. Pick any  $\mathbf{v}$  and compute  $T\mathbf{v}, T^2\mathbf{v}, \dots, T^k\mathbf{v}$  until a linear dependence is introduced. Write this as  $p(T) = 0$ ; then  $\min_A(x) \mid p(x)$ .

**Definition:** Given a monic  $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + x^n$ , the **companion matrix** of  $p$  is given by

$$C_p := \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}.$$

## 7.4 Canonical Forms

### 7.4.1 Rational Canonical Form

Corresponds to the **Invariant Factor Decomposition** of  $T$ .

**Lemma:**  $RCF(A)$  is a block matrix where each block is the companion matrix of an invariant factor of  $A$ .

**Derivation:**

- Let  $k[x] \curvearrowright V$  using  $T$ , take invariant factors  $a_i$ ,
- Note that  $T \curvearrowright V$  by multiplication by  $x$
- Write  $\bar{x} = \pi(x)$  where  $F[x] \xrightarrow{\pi} F[x]/(a_i)$ ; then  $\text{span}\{\bar{x}\} = F[x]/(a_i)$ .
- Write  $a_i(x) = \sum b_i x^i$ , note that  $V \longrightarrow F[x]$  pushes  $T \curvearrowright V$  to  $T \curvearrowright k[x]$  by multiplication by  $\bar{x}$
- WRT the basis  $\bar{x}$ ,  $T$  then acts via the companion matrix on this summand.
- Each invariant factor corresponds to a block of the RCF.

### 7.4.2 Jordan Canonical Form

Corresponds to the **Elementary Divisor Decomposition** of  $T$ .

**Lemma:** The elementary divisors of  $A$  are the minimal polynomials of the Jordan blocks.

**Lemma:** Writing

$$\begin{aligned}\min_A(x) &= \prod (x - \lambda_i)^{a_i} \\ \chi_A(x) &= \prod (x - \lambda_i)^{b_i}\end{aligned}$$

- $a_i \leq b_i$
- $a_i$  tells you the size of the **largest** Jordan block associated to  $\lambda_i$ ,
- $b_i$  is the **sum of sizes** of all Jordan blocks associated to  $\lambda_i$
- $\dim E_{\lambda_i}$  is the **number of Jordan blocks** associated to  $\lambda_i$

## 7.5 Using Canonical Forms

**Lemma:** The characteristic polynomial is the *product of the invariant factors*, i.e.

$$\chi_A(x) = \prod_{j=1}^n f_j(x).$$

**Lemma:** The minimal polynomial of  $A$  is the *invariant factor of highest degree*, i.e.

$$\min_A(x) = f_n(x).$$

**Lemma:** For a linear operator on a vector space of nonzero finite dimension, TFAE:

- The minimal polynomial is equal to the characteristic polynomial.
- The list of invariant factors has length one.
- The Rational Canonical Form has a single block.
- The operator has a matrix similar to a companion matrix.
- There exists a *cyclic vector*  $\mathbf{v}$  such that  $\text{span}_k \{T^j \mathbf{v} \mid j = 1, 2, \dots\} = V$ .
- $T$  has  $\dim V$  distinct eigenvalues

## 7.6 Diagonalizability

*Notation:*  $A^*$  denotes the conjugate transpose of  $A$ .

**Lemma:** Let  $V$  be a vector space over  $k$  an algebraically closed and  $A \in \text{End}(V)$ . Then if  $W \subseteq V$  is an invariant subspace, so  $A(W) \subseteq W$ , the  $A$  has an eigenvector in  $W$ .

**Theorem (The Spectral Theorem):**

1. Hermitian matrices (i.e.  $A^* = A$ ) are diagonalizable over  $\mathbb{C}$ .
2. Symmetric matrices (i.e.  $A^t = A$ ) are diagonalizable over  $\mathbb{R}$ .

*Proof:* Suppose  $A$  is Hermitian. Since  $V$  itself is an invariant subspace,  $A$  has an eigenvector  $\mathbf{v}_1 \in V$ . Let  $W_1 = \text{span}_k \{\mathbf{v}_1\}^\perp$ . Then for any  $\mathbf{w}_1 \in W_1$ ,

$$\langle \mathbf{v}_1, A\mathbf{w}_1 \rangle = \langle A\mathbf{v}_1, \mathbf{w}_1 \rangle = \lambda \langle \mathbf{v}_1, \mathbf{w}_1 \rangle = 0,$$

so  $A(W_1) \subseteq W_1$  is an invariant subspace, etc.

Suppose now that  $A$  is symmetric. Then there is an eigenvector of norm 1,  $\mathbf{v} \in V$ .

$$\lambda = \lambda \langle \mathbf{v}, \mathbf{v} \rangle = \langle A\mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{v}, A\mathbf{v} \rangle = \bar{\lambda} \implies \lambda \in \mathbb{R}.$$

**Lemma:**  $\{A_i\}$  pairwise commute  $\iff$  they are all simultaneously diagonalizable.

*Proof:* By induction on number of operators

- $A_n$  is diagonalizable, so  $V = \bigoplus E_i$  a sum of eigenspaces
- Restrict all  $n - 1$  operators  $A$  to  $E_n$ .
- The commute in  $V$  so they commute in  $E_n$
- **(Lemma)** They were diagonalizable in  $V$ , so they're diagonalizable in  $E_n$
- So they're simultaneously diagonalizable by I.H.
- But these eigenvectors for the  $A_i$  are all in  $E_n$ , so they're eigenvectors for  $A_n$  too.
- Can do this for each eigenspace. ■

Full details here

**Theorem (Characterizations of Diagonalizability)**

$M$  is diagonalizable over  $\mathbb{F} \iff \min_M(x, \mathbb{F})$  splits into distinct linear factors over  $\mathbb{F}$ , or equivalently iff all of the roots of  $\min_M$  lie in  $\mathbb{F}$ .

*Proof:*  $\implies$  : If  $\min_A$  factors into linear factors, so does each invariant factor, so every elementary divisor is linear and  $JCF(A)$  is diagonal.

$\impliedby$  : If  $A$  is diagonalizable, every elementary divisor is linear, so every invariant factor factors into linear pieces. But the minimal polynomial is just the largest invariant factor.

## 7.7 Matrix Counterexamples

1. A matrix that is:
  - Not diagonalizable over  $\mathbb{R}$  but diagonalizable over  $\mathbb{C}$
  - No eigenvalues in  $\mathbb{R}$  but distinct eigenvalues over  $\mathbb{C}$
  - $\min_M(x) = \chi_M(x) = x^2 + 1$

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \sim \left[ \begin{array}{c|c} -1\sqrt{-1} & 0 \\ \hline 0 & 1\sqrt{-1} \end{array} \right].$$

2.

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

- Not diagonalizable over  $\mathbb{C}$
  - Eigenvalues  $[1, 1]$  (repeated, multiplicity 2)
  - $\min_M(x) = \chi_M(x) = x^2 - 2x + 1$
3. Non-similar matrices with the same characteristic polynomial

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

4. A full-rank matrix that is not diagonalizable:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. Matrix roots of unity:

$$\sqrt{I_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$\sqrt{-I_2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

## 7.8 Miscellaneous

**Lemma:**  $I \trianglelefteq R$  is a free  $R$ -module iff  $I$  is a principal ideal.

Proof:  $\Rightarrow$  :

Suppose  $I$  is free as an  $R$ -module, and let  $B = \{\mathbf{m}_j\}_{j \in J} \subseteq I$  be a basis so we can write  $M = \langle B \rangle$ .

Suppose that  $|B| \geq 2$ , so we can pick at least 2 basis elements  $\mathbf{m}_1 \neq \mathbf{m}_2$ , and consider

$$\mathbf{c} = \mathbf{m}_1\mathbf{m}_2 - \mathbf{m}_2\mathbf{m}_1,$$

which is also an element of  $M$ .

Since  $R$  is an integral domain,  $R$  is commutative, and so

$$\mathbf{c} = \mathbf{m}_1\mathbf{m}_2 - \mathbf{m}_2\mathbf{m}_1 = \mathbf{m}_1\mathbf{m}_2 - \mathbf{m}_1\mathbf{m}_2 = \mathbf{0}_M$$

However, this exhibits a linear dependence between  $\mathbf{m}_1$  and  $\mathbf{m}_2$ , namely that there exist  $\alpha_1, \alpha_2 \neq 0_R$  such that  $\alpha_1\mathbf{m}_1 + \alpha_2\mathbf{m}_2 = \mathbf{0}_M$ ; this follows because  $M \subset R$  means that we can take  $\alpha_1 = -m_2, \alpha_2 = m_1$ . This contradicts the assumption that  $B$  was a basis, so we must have  $|B| = 1$  and so  $B = \{\mathbf{m}\}$  for some  $\mathbf{m} \in I$ . But then  $M = \langle B \rangle = \langle \mathbf{m} \rangle$  is generated by a single element, so  $M$  is principal.

$\Leftarrow$  :

Suppose  $M \trianglelefteq R$  is principal, so  $M = \langle \mathbf{m} \rangle$  for some  $\mathbf{m} \neq \mathbf{0}_M \in M \subset R$ .

Then  $x \in M \Rightarrow x = \alpha\mathbf{m}$  for some element  $\alpha \in R$  and we just need to show that  $\alpha\mathbf{m} = \mathbf{0}_M \Rightarrow \alpha = 0_R$  in order for  $\{\mathbf{m}\}$  to be a basis for  $M$ , making  $M$  a free  $R$ -module.

But since  $M \subset R$ , we have  $\alpha, m \in R$  and  $\mathbf{0}_M = 0_R$ , and since  $R$  is an integral domain, we have  $\alpha m = 0_R \Rightarrow \alpha = 0_R$  or  $m = 0_R$ .

Since  $m \neq 0_R$ , this forces  $\alpha = 0_R$ , which allows  $\{\mathbf{m}\}$  to be a linearly independent set and thus a basis for  $M$  as an  $R$ -module. ■

### Lemma 7.1.

Every  $a \in R$  for a finite ring is either a unit or a zero divisor.

*Proof.*

Let  $a \in R$  and define  $\varphi(x) = ax$ . If  $\varphi$  is injective, then it is surjective, so  $1 = ax$  for some  $x \Rightarrow x^{-1} = a$ . Otherwise,  $ax_1 = ax_2$  with  $x_1 \neq x_2 \Rightarrow a(x_1 - x_2) = 0$  and  $x_1 - x_2 \neq 0$ , so  $a$  is a zero divisor. ■

### Lemma 7.2.

Maximal  $\Rightarrow$  prime, but generally not the converse.

*Proof.*

Suppose  $\mathfrak{m}$  is maximal,  $ab \in \mathfrak{m}$ , and  $b \notin \mathfrak{m}$ . Then there is a containment of ideals  $\mathfrak{m} \subsetneq \mathfrak{m} + (b) \Rightarrow \mathfrak{m} + (b) = R$ .

So

$$1 = m + rb \Rightarrow a = am + r(ab),$$

but  $am \in \mathfrak{m}$  and  $ab \in \mathfrak{m} \Rightarrow a \in \mathfrak{m}$ . ■

*Counterexample:*  $(0) \in \mathbb{Z}$  is prime since  $\mathbb{Z}$  is a domain, but not maximal since it is properly contained in any other ideal.

---

**Lemma 7.3.**

The nilradical is the intersection of all prime ideals, i.e.

$$\mathfrak{N}(R) = \bigcap_{\mathfrak{p} \in \text{Spec}(R)} \mathfrak{p}$$

*Proof.*

$$\mathfrak{N} \subseteq \bigcap \mathfrak{p}: x \in \mathfrak{N} \implies x^n = 0 \in \mathfrak{p} \implies x \in \mathfrak{p} \text{ or } x^{n-1} \in \mathfrak{p}.$$

$\mathfrak{N}^c \subseteq \bigcup \mathfrak{p}^c$ : Define  $S = \{I \trianglelefteq R \mid a^n \notin I \text{ for any } n\}$ . Then apply Zorn's lemma to get a maximal ideal  $\mathfrak{m}$ , and maximal  $\implies$  prime. ■

**Lemma 7.4.**

$R/\mathfrak{N}(R)$  has no nonzero nilpotent elements.

*Proof.*

$$\begin{aligned} a + \mathfrak{N}(R) \text{ nilpotent} &\implies (a + \mathfrak{N}(R))^n := a^n + \mathfrak{N}(R) = \mathfrak{N}(R) \\ &\implies a^n \in \mathfrak{N}(R) \\ &\implies \exists \ell \text{ such that } (a^n)^\ell = 0 \\ &\implies a \in \mathfrak{N}(R). \end{aligned}$$
 ■

**Lemma 7.5.**

$\mathfrak{N}(R) \subseteq \mathfrak{J}(R)$ .

*Proof.*

Maximal  $\implies$  prime, and so if  $x$  is in every prime ideal, it is necessarily in every maximal ideal as well. ■

## 8 Extra Problems

### 8.1 Group Theory

#### 8.1.1 Basic Structure

- Show that any cyclic group is abelian.
- Show that if  $G/Z(G)$  is cyclic then  $G$  is abelian.
- Show that the intersection of two subgroups is again a subgroup.
- Show that if  $G \curvearrowright X$  is a group action, then the stabilizer  $G_x$  of a point is a subgroup.

- Show that  $G = H \times K$  iff the conditions for recognizing direct products hold.
- Show that if  $H, K \leq G$  and  $H \cap K = \emptyset$ , then  $hk = kh$  for all  $h \in H, k \in K$ .
- Show that every normal subgroup of  $G$  is contained in  $Z(G)$ .
- Show that  $|G|/|H| = [G : H]$ .
- Show that the order of any element in a group divides the order of the group.
- Show that

$$\varphi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right).$$

- Show that  $Z(G) \subseteq C_G(H) \subseteq N_G(H)$ .
- Show that  $G/N$  is abelian iff  $[G, G] \leq N$ .
- Give a counterexample where  $H, K \leq G$  but  $HK$  is not a subgroup of  $G$ .
- Show that if  $H, K \leq G$  are normal subgroups that intersect trivially, then  $[H, K] = 1$  (so  $hk = kh$  for all  $k$  and  $h$ ).
- Give an example showing that normality is not transitive: i.e.  $H \leq K \leq G$  with  $H$  *not* normal in  $G$ .
- Show that the size of a conjugacy class divides the order of a group.
- Show that  $C_G(H) \subseteq N_G(H) \leq G$ .
- Given  $H \subseteq G$ , let  $S(H) = \bigcup_{g \in G} gHg^{-1}$ , so  $|S(H)|$  is the number of conjugates to  $H$ . Show that  $|S(H)| = [G : N_G(H)]$ .
  - That is, the number of subgroups conjugate to  $H$  equals the index of the normalizer of  $H$ .
- Show that  $Z(G) = \bigcap_{a \in G} C_G(a)$ .
- Show that the centralizer  $C_G(H)$  of a subgroup is again a subgroup.
- Show that  $C_G(H) \leq N_G(H)$  is a normal subgroup.
- Show that  $C_G(G) = Z(G)$ .
- Show that for  $H \leq G$ ,  $C_H(x) = H \cap C_G(x)$ .
- Let  $H, K \leq G$  a finite group, and without using the normalizers of  $H$  or  $K$ , show that  $|HK| = |H||K|/|H \cap K|$ .
- Show that if  $H \leq N_G(K)$  then  $HK \leq H$ , and give a counterexample showing that this condition is necessary.
- Show that  $HK$  is a subgroup of  $G$  iff  $HK = KH$ .

**8.1.2 Primes in Group Theory**

- Show that any group of prime order is cyclic and simple.
- Analyze groups of order  $pq$  with  $q < p$ .

Hint: consider the cases when  $p$  does or does not divide  $q - 1$ .

- Show that if  $q$  does not divide  $p - 1$ , then  $G$  is cyclic.
- Show that  $G$  is never simple.
- Analyze groups of order  $p^2q$ .
- Show that no group of order  $p^2q^2$  is simple for  $p < q$  primes.
- Show that a group of order  $p^2q^2$  has a normal Sylow subgroup.
- Show that a group of order  $p^2q^2$  where  $q$  does not divide  $p^2 - 1$  and  $p$  does not divide  $q^2 - 1$  is abelian.
- Show that every group of order  $pqr$  with  $p < q < r$  primes contains a normal Sylow subgroup.
  - Show that  $G$  is never simple.
- Show that any normal  $p$ -subgroup is contained in every Sylow  $p$ -subgroup of  $G$ .

**8.1.3  $p$ -Groups**

- Show that every  $p$ -group has a nontrivial center.
- Show that every  $p$ -group is nilpotent.
- Show that every  $p$ -group is solvable.
- Show that every maximal subgroup of a  $p$ -group has index  $p$ .
- Show that every maximal subgroup of a  $p$ -group is normal.
- Show that every group of order  $p$  is cyclic.
- Show that every group of order  $p^2$  is abelian and classify them.
- Show that every normal subgroup of a  $p$ -group is contained in the center.

Hint: Consider  $G/Z(G)$ .

- Let  $O_p(G)$  be the intersection of all Sylow  $p$ -subgroups of  $G$ . Show that  $O_p(G) \trianglelefteq G$ , is maximal among all normal  $p$ -subgroups of  $G$ .
- Let  $P \in \text{Syl}_p(H)$  where  $H \trianglelefteq G$  and show that  $P \cap H \in \text{Syl}_p(H)$ .
- Show that Sylow  $p_i$ -subgroups  $S_{p_1}, S_{p_2}$  for distinct primes  $p_1 \neq p_2$  intersect trivially.

**8.1.4 Symmetric, Alternating, Dihedral Groups**

- Show that the center of  $S_3$  is trivial.
- Show that  $\text{Aut}(S_3) = \text{Inn}(S_3) \cong S_3$ .



- Show that  $\text{Out}(A_4)$  is nontrivial.
- Show that an  $m$ -cycle is an odd permutation iff  $m$  is an even number.
- Show that a permutation is odd iff it has an odd number of even cycles.
- Show that the center of  $S_n$  for  $n \geq 4$  is nontrivial.
- Show that disjoint cycles commute.
- Show that  $S_n$  is generated by any of the following types of cycles:

Group	Generating Set	Size
$S_n, n \geq 2$	$(ij)$ 's	$\frac{n(n-1)}{2}$
	$(12), (13), \dots, (1n)$	$n - 1$
	$(12), (23), \dots, (n-1 \ n)$	$n - 1$
	$(12), (12 \dots n)$ if $n \geq 3$	2
	$(12), (23 \dots n)$ if $n \geq 3$	2
	$(ab), (12 \dots n)$ if $(b-a, n) = 1$	2
$A_n, n \geq 3$	3-cycles	$\frac{n(n-1)(n-2)}{3}$
	$(1ij)$ 's	$(n-1)(n-2)$
	$(12i)$ 's	$n-2$
	$(i \ i+1 \ i+2)$ 's	$n-2$
	$(123), (12 \dots n)$ if $n \geq 4$ odd	2
	$(123), (23 \dots n)$ if $n \geq 4$ even	2

- Show directly that any  $k$ -cycle is a product of transpositions, and determine how many transpositions are needed.
- Show that  $S_n$  is generated by transpositions.
- Show that  $S_n$  is generated by *adjacent* transpositions.
- Show that  $S_n$  is generated by  $\{(12), (12 \dots n)\}$  for  $n \geq 2$
- Show that  $S_n$  is generated by  $\{(12), (23 \dots n)\}$  for  $n \geq 3$
- Show that  $S_n$  is generated by  $\{(ab), (12 \dots n)\}$  where  $1 \leq a < b \leq n$  iff  $\gcd(b-a, n) = 1$ .
- Show that  $S_p$  is generated by any arbitrary transposition and any arbitrary  $p$ -cycle.
- Show that  $A_n$  is generated 3-cycles.
- Show that  $\mathbb{Q}$  is not finitely generated as a group.
- Show that if  $N \trianglelefteq D_n$  is a normal subgroup of a dihedral group, then  $D_n/N$  is again a dihedral group.
- Prove that  $A_n$  is normal in  $S_n$ .
- Argue that  $A_n$  is simple for  $n \geq 5$ .

- Compute  $\text{Aut}(\mathbb{Z}/n\mathbb{Z})$  for  $n$  composite.
- Compute  $\text{Aut}((\mathbb{Z}/p\mathbb{Z})^n)$ .

### 8.1.5 Classification

- Show that no group of order 36 is simple.
- Show that no group of order 90 is simple.
- Show that all groups of order 45 are abelian.
- Classify all groups of order 10.
- Classify the five groups of order 12.
- Classify the four groups of order 28.

### 8.1.6 Group Actions

- Show that the stabilizer of an element  $G_x$  is a subgroup of  $G$ .
- Show that if  $x, y$  are in the same orbit, then their stabilizers are conjugate.
- Show that the stabilizer of an element need not be a normal subgroup?

### 8.1.7 Series of Groups

- Show that  $A_n$  is simple for  $n \geq 5$
- Give a necessary and sufficient condition for a cyclic group to be solvable.
- Prove that every simple abelian group is cyclic.
- Show that  $S_n$  is generated by disjoint cycles.
- Show that  $S_n$  is generated by transpositions.
- Show if  $G$  is finite, then  $G$  is solvable  $\iff$  all of its composition factors are of prime order.
- Show that if  $N$  and  $G/N$  are solvable, then  $G$  is solvable.
- Show that if  $G$  is finite and solvable then every composition factor has prime order.
- Show that  $G$  is solvable iff its derived series terminates.
- Show that  $S_3$  is not nilpotent.

### 8.1.8 Misc

- Prove Burnside's theorem.
- Show that  $\text{Inn}(G) \trianglelefteq \text{Aut}(G)$
- Show that  $\text{Inn}(G) \cong G/Z(G)$
- Show that the kernel of the map  $G \longrightarrow \text{Aut}(G)$  given by  $g \mapsto (h \mapsto ghg^{-1})$  is  $Z(G)$ .
- Show that  $N_G(H)/C_G(H) \cong A \leq \text{Aut}(H)$
- Show that if  $|G| = 12$  and has a normal subgroup of order 4, then  $G \cong A_4$ .

### 8.1.9 Nonstandard Topics

- Show that  $H \text{ char } G \Rightarrow H \trianglelefteq G$

Thus “characteristic” is a strictly stronger condition than normality

- Show that  $H \text{ char } K \text{ char } G \Rightarrow H \text{ char } G$

So “characteristic” is a transitive relation for subgroups.

- Show that if  $H \leq G$ ,  $K \trianglelefteq G$  is a normal subgroup, and  $H \text{ char } K$  then  $H$  is normal in  $G$ .

So normality is not transitive, but strengthening one to “characteristic” gives a weak form of transitivity.

## 8.2 Ring Theory

### Basic Structure

- Show that if an ideal  $I \trianglelefteq R$  contains a unit then  $I = R$ .
- Show that  $R^\times$  need not be closed under addition.

### Ideals

- Show that every proper ideal is contained in a maximal ideal
- Show that if  $x \in R$  a PID, then  $x$  is irreducible  $\iff \langle x \rangle \trianglelefteq R$  is maximal.
- Show that intersections, products, and sums of ideals are ideals.
- Show that the union of two ideals need not be an ideal.
- Show that every ring has a proper maximal ideal.
- Show that  $I \trianglelefteq R$  is maximal iff  $R/I$  is a field.
- Show that  $I \trianglelefteq R$  is prime iff  $R/I$  is an integral domain.
- Show that  $\bigcup_{\mathfrak{m} \in \max\text{Spec}(R)} \mathfrak{m} = R \setminus R^\times$ .
- Show that  $\max\text{Spec}(R) \subsetneq \text{Spec}(R)$  but the containment is strict.
- Show that if  $x$  is not a unit, then  $x$  is contained in some maximal ideal.
- Show that if  $R$  is a finite ring then every  $a \in R$  is either a unit or a zero divisor.
- Show that  $R/\mathfrak{N}(R)$  has no nonzero nilpotent elements.
- Show that the nilradical is contained in the Jacobson radical.
- Show that every prime ideal is radical.
- Show that the nilradical is given by  $\mathfrak{N}(R) = \text{rad}(0)$ .
- Show that  $\text{rad}(IJ) = \text{rad}(I) \cap \text{rad}(J)$
- Show that if  $\text{Spec}(R) \subseteq \max\text{Spec}(R)$  then  $R$  is a UFD.
- Show that if  $R$  is Noetherian then every ideal is finitely generated.

### Characterizing Certain Ideals

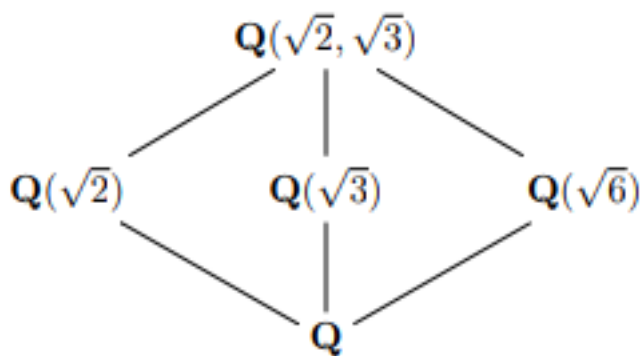
- Show that the nilradical is the intersection of all prime ideals.
- Show that for an ideal  $I \trianglelefteq R$ , its radical is the intersection of all prime ideals containing  $I$ .
- Show that  $\text{rad}(I)$  is the intersection of all prime ideals containing  $I$ .

### Misc

- Show that localizing a ring at a prime ideal produces a local ring.
- Show that  $R$  is a local ring iff for every  $x \in R$ , either  $x$  or  $1 - x$  is a unit.
- Show that if  $R$  is a local ring then  $R \setminus R^\times$  is a proper ideal that is contained in  $\mathfrak{J}(R)$ .
- Show that if  $R \neq 0$  is a ring in which every non-unit is nilpotent then  $R$  is local.
- Show that every prime ideal is primary.

## 8.3 Field Theory

- What is  $[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}]$ ?
- What is  $[\mathbb{Q}(2^{\frac{3}{2}}) : \mathbb{Q}]$ ?
- Show that every field is simple.
- Show that any field morphism is either 0 or injective.
- Show that if  $p \in \mathbb{Q}[x]$  and  $r \in \mathbb{Q}$  is a rational root, then in fact  $r \in \mathbb{Z}$ .
- If  $\{\alpha_i\}_{i=1}^n \subset F$  are algebraic over  $K$ , show that  $K[\alpha_1, \dots, \alpha_n] = K(\alpha_1, \dots, \alpha_n)$ .
- Show that the Galois group of  $x^n - 2$  is  $D_n$ , the dihedral group on  $n$  vertices.
- Compute all intermediate field extensions of  $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ , show it is equal to  $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ , and find a corresponding minimal polynomial.



- Compute all intermediate field extensions of  $\mathbb{Q}(2^{\frac{1}{4}}, \zeta_8)$ .
- Show that  $\mathbb{Q}(2^{\frac{1}{3}})$  and  $\mathbb{Q}(\zeta_3 2^{\frac{1}{3}})$
- Show that if  $L/K$  is separable, then  $L$  is normal  $\iff$  there exists a polynomial  $p(x) = \prod_{i=1}^n (x - \alpha_i) \in K[x]$  such that  $L = K(\alpha_1, \dots, \alpha_n)$  (so  $L$  is the splitting field of  $p$ ).
- Is  $\mathbb{Q}(2^{\frac{1}{3}})/\mathbb{Q}$  normal?
- Show that any finite integral domain is a field.
- Prove that if  $R$  is an integral domain, then  $R[t]$  is again an integral domain.
- Show that  $f(R[t]) = f(R)(t)$ .
- Prove that  $x^{p^n} - x$  is the product of all monic irreducible polynomials in  $\mathbb{F}_p[x]$  with degree dividing  $n$ .
- Prove that an irreducible  $\pi(x) \in \mathbb{F}_p[x]$  divides  $x^{p^n} - x \iff \deg \pi(x)$  divides  $n$ .
- Show that a field with  $p^n$  elements has exactly one subfield of size  $p^d$  for every  $d$  dividing  $n$ .
- Show that  $\mathbb{GF}(p^n)$  is the splitting field of  $x^{p^n} - x \in \mathbb{F}_p[x]$ .
- Show that  $x^{p^d} - x \mid x^{p^n} - x \iff d \mid n$
- Show that  $\mathbb{GF}(p^d) \leq \mathbb{GF}(p^n) \iff d \mid n$
- Show that  $x^{p^n} - x = \prod f_i(x)$  over all irreducible monic  $f_i$  of degree  $d$  dividing  $n$ .
- Compute the Galois group of  $x^n - 1 \in \mathbb{Q}[x]$  as a function of  $n$ .
- Identify all of the elements of the Galois group of  $x^p - 2$  for  $p$  an odd prime (note: this has a complicated presentation).
- Show that  $\text{Gal}(x^{15} + 2)/\mathbb{Q} \cong S_2 \times \mathbb{Z}/15\mathbb{Z}$  for  $S_2$  a Sylow 2-subgroup.
- Show that  $\text{Gal}(x^3 + 4x + 2)/\mathbb{Q} \cong S_3$ , a symmetric group.

**8.4 Modules and Linear Algebra**

- Prove the Cayley-Hamilton theorem.
- Prove that the minimal polynomial divides the characteristic polynomial.
- Prove that the cokernel of  $A \in \text{Mat}(n \times n, \mathbb{Z})$  is finite  $\iff \det A \neq 0$ , and show that in this case  $|\text{coker}(A)| = |\det(A)|$ .
- Show that a nilpotent operator is diagonalizable.
- Show that if  $A, B$  are diagonalizable and  $[A, B] = 0$  then  $A, B$  are simultaneously diagonalizable.
- Does diagonalizable imply invertible? The converse?

**8.5 Commutative Algebra**

- Show that a finitely generated module over a Noetherian local ring is flat iff it is free using Nakayama and Tor.