Real Analysis Qualifying Exam Questions

D. Zack Garza

Monday 27^{th} July, 2020

Contents

1 Spring 2020

1.1 1

Prove that if $f : [0,1] \longrightarrow \mathbb{R}$ is continuous then

$$\lim_{k \longrightarrow \infty} \int_0^1 k x^{k-1} f(x)\, dx = f(1).$$

1.2 2

Let m_* denote the Lebesgue outer measure on $\mathbb{R}$.

a. Prove that for every $E \subseteq \mathbb{R}$ there exists a Borel set B containing E such that

$$m_*(B) = m_*(E).$$

b. Prove that if $E \subseteq \mathbb{R}$ has the property that

$$m_*(A) = m_*(A \bigcap E) + m_*(A \bigcap E^c)$$

for every set $A \subseteq \mathbb{R}$, then there exists a Borel set $B \subseteq \mathbb{R}$ such that $E = B \setminus N$ with $m_*(N) = 0$. Be sure to address the case when $m_*(E) = \infty$.

1.3 3

a. Prove that if $g \in L^1(\mathbb{R})$ then

$$\lim_{N \longrightarrow \infty} \int_{|x| \geq N} |f(x)|\, dx = 0,$$

and demonstrate that it is not necessarily the case that $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.

b. Prove that if $f \in L^1([1,\infty])$ and is decreasing, then $\lim_{x \longrightarrow \infty} f(x) = 0$ and in fact $\lim_{x \longrightarrow \infty} x f(x) = 0$.

c. If $f : [1,\infty) \longrightarrow [0,\infty)$ is decreasing with $\lim_{x \longrightarrow \infty} x f(x) = 0$, does this ensure that $f \in L^1([1,\infty))$?

1.4 4

Let $f, g \in L^1(\mathbb{R})$. Argue that $H(x,y) \coloneqq f(y)g(x-y)$ defines a function in $L^1(\mathbb{R}^2)$ and deduce from this fact that

$$(f * g)(x) \coloneqq \int_{\mathbb{R}} f(y)g(x-y)\,dy$$

defines a function in $L^1(\mathbb{R})$ that satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

1.5 5

Compute the following limit and justify your calculations:

$$\lim_{n\longrightarrow\infty} \int_0^n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx.$$

1.6 6

a. Show that

$$L^2([0,1]) \subseteq L^1([0,1]) \quad \text{and} \quad \ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z}).$$

b. For $f \in L^1([0,1])$ define

$$\widehat{f}(n) \coloneqq \int_0^1 f(x)e^{-2\pi inx}\,dx.$$

Prove that if $f \in L^1([0,1])$ and $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$ then

$$S_N f(x) \coloneqq \sum_{|n|\leq N} \widehat{f}(n)e^{2\pi inx}.$$

converges uniformly on $[0,1]$ to a continuous function g such that $g = f$ almost everywhere.

Hint: One approach is to argue that if $f \in L^1([0,1])$ with $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$ then $f \in L^2([0,1])$.

2 Fall 2019

2.1 1.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

a. Prove that if $\lim_{n\longrightarrow\infty} a_n = 0$, then

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

b. Prove that if $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges, then

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

2.2 2.

Prove that

$$\left| \frac{d^n}{dx^n} \frac{\sin x}{x} \right| \leq \frac{1}{n}$$

for all $x \neq 0$ and positive integers n.

> Hint: Consider $\int_0^1 \cos(tx)dt$

2.3 3.

Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) = 1$ and $\{B_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{B}$-measurable subsets of X, and

$$B := \left\{ x \in X \;\middle|\; x \in B_n \text{ for infinitely many } n \right\}.$$

a. Argue that B is also a $\mathcal{B}$-measurable subset of X.

b. Prove that if $\sum_{n=1}^{\infty} \mu(B_n) < \infty$ then $\mu(B) = 0$.

c. Prove that if $\sum_{n=1}^{\infty} \mu(B_n) = \infty$ **and** the sequence of set complements $\{B_n^c\}_{n=1}^{\infty}$ satisfies

$$\mu\left(\bigcap_{n=k}^{K} B_n^c\right) = \prod_{n=k}^{K} \left(1 - \mu\left(B_n\right)\right)$$

for all positive integers k and K with $k < K$, then $\mu(B) = 1$.

> Hint: Use the fact that $1 - x \leq e^{-x}$ for all x.

2.4 4.

Let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space $\mathcal{H}$.

a. Prove that for every $x \in \mathcal{H}$ one has

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

b. Prove that for any sequence $\{a_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$ there exists an element $x \in \mathcal{H}$ such that

$$a_n = \langle x,\ u_n\rangle \text{ for all } n \in \mathbb{N}$$

and

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2$$

2.5 5.

a. Show that if f is continuous with compact support on $\mathbb{R}$, then

$$\lim_{y\to 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dx = 0$$

b. Let $f \in L^1(\mathbb{R})$ and for each $h > 0$ let

$$\mathcal{A}_h f(x) := \frac{1}{2h} \int_{|y|\le h} f(x-y) dy$$

c. Prove that $\|\mathcal{A}_h f\|_1 \le \|f\|_1$ for all $h > 0$.

ii. Prove that $\mathcal{A}_h f \longrightarrow f$ in $L^1(\mathbb{R})$ as $h \longrightarrow 0^+$.

3 Spring 2019

3.1 1

Let $C([0,1])$ denote the space of all continuous real-valued functions on $[0,1]$.

a. Prove that $C([0,1])$ is complete under the uniform norm $\|f\|_u := \sup\limits_{x\in[0,1]} |f(x)|$.

b. Prove that $C([0,1])$ is not complete under the L^1-norm $\|f\|_1 = \displaystyle\int_0^1 |f(x)|\, dx$.

3.2 2

Let $\mathcal{B}$ denote the set of all Borel subsets of $\mathbb{R}$ and $\mu : \mathcal{B} \longrightarrow [0,\infty)$ denote a finite Borel measure on $\mathbb{R}$.

a. Prove that if $\{F_k\}$ is a sequence of Borel sets for which $F_k \supseteq F_{k+1}$ for all k, then

$$\lim_{k\to\infty} \mu(F_k) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right)$$

b. Suppose μ has the property that $\mu(E) = 0$ for every $E \in \mathcal{B}$ with Lebesgue measure $m(E) = 0$. Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $E \in \mathcal{B}$ with $m(E) < \delta$, then $\mu(E) < \varepsilon$.

3.3 3

Let $\{f_k\}$ be any sequence of functions in $L^2([0,1])$ satisfying $\|f_k\|_2 \leq M$ for all $k \in \mathbb{N}$.

Prove that if $f_k \longrightarrow f$ almost everywhere, then $f \in L^2([0,1])$ with $\|f\|_2 \leq M$ and

$$\lim_{k\to\infty} \int_0^1 f_k(x)dx = \int_0^1 f(x)dx$$

> Hint: Try using Fatou's Lemma to show that $\|f\|_2 \leq M$ and then try applying Egorov's Theorem.

3.4 4

Let f be a non-negative function on $\mathbb{R}^n$ and $\mathcal{A} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq f(x)\}$.

Prove the validity of the following two statements:

a. f is a Lebesgue measurable function on $\mathbb{R}^n$ $\iff$ $\mathcal{A}$ is a Lebesgue measurable subset of $\mathbb{R}^{n+1}$

b. If f is a Lebesgue measurable function on $\mathbb{R}^n$, then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)dx = \int_0^\infty m\left(\{x \in \mathbb{R}^n : f(x) \geq t\}\right) dt$$

3.5 5

a. Show that $L^2([0,1]) \subseteq L^1([0,1])$ and argue that $L^2([0,1])$ in fact forms a dense subset of $L^1([0,1])$.

b. Let Λ be a continuous linear functional on $L^1([0,1])$.

   Prove the Riesz Representation Theorem for $L^1([0,1])$ by following the steps below:

   i. Establish the existence of a function $g \in L^2([0,1])$ which represents Λ in the sense that

   $$\Lambda(f) = f(x)g(x)dx \text{ for all } f \in L^2([0,1]).$$

   > Hint: You may use, without proof, the Riesz Representation Theorem for $L^2([0,1])$.

   ii. Argue that the g obtained above must in fact belong to $L^\infty([0,1])$ and represent Λ in the sense that

   $$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}dx \qquad \text{for all } f \in L^1([0,1])$$

   with

   $$\|g\|_{L^\infty([0,1])} = \|\Lambda\|_{L^1([0,1])^\vee}$$

4 Fall 2018

4.1 1

Let $f(x) = \dfrac{1}{x}$. Show that f is uniformly continuous on $(1,\infty)$ but not on $(0,\infty)$.

4.2 2

Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Show that there is a Borel set $B \subset E$ such that $m(E \setminus B) = 0$.

4.3 3

Suppose $f(x)$ and $xf(x)$ are integrable on $\mathbb{R}$. Define F by

$$F(t) := \int_{-\infty}^{\infty} f(x)\cos(xt)dx$$

Show that

$$F'(t) = -\int_{-\infty}^{\infty} xf(x)\sin(xt)dx.$$

4.4 4

Let $f \in L^1([0,1])$. Prove that

$$\lim_{n\longrightarrow\infty} \int_0^1 f(x)|\sin nx|\ dx = \frac{2}{\pi}\int_0^1 f(x)\ dx$$

Hint: Begin with the case that f is the characteristic function of an interval.

4.5 5

Let $f \geq 0$ be a measurable function on $\mathbb{R}$. Show that

$$\int_{\mathbb{R}} f = \int_0^{\infty} m(\{x : f(x) > t\})dt$$

4.6 6

Compute the following limit and justify your calculations:

$$\lim_{n\to\infty} \int_1^n \frac{dx}{(1+\frac{x}{n})^n\ \sqrt[n]{x}}$$

5 Spring 2018

5.1 1

Define

$$E := \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N} \right\}.$$

Prove that $m(E) = 0$.

5.2 2

Let

$$f_n(x) := \frac{x}{1+x^n}, \quad x \geq 0.$$

a. Show that this sequence converges pointwise and find its limit. Is the convergence uniform on $[0, \infty)$?

b. Compute

$$\lim_{n \to \infty} \int_0^\infty f_n(x) dx$$

5.3 3

Let f be a non-negative measurable function on $[0,1]$.

Show that

$$\lim_{p \to \infty} \left(\int_{[0,1]} f(x)^p dx \right)^{\frac{1}{p}} = \|f\|_\infty.$$

5.4 4

Let $f \in L^2([0,1])$ and suppose

$$\int_{[0,1]} f(x) x^n dx = 0 \text{ for all integers } n \geq 0.$$

Show that $f = 0$ almost everywhere.

5.5 5

Suppose that

- $f_n, f \in L^1$,
- $f_n \longrightarrow f$ almost everywhere, and
- $\displaystyle\int |f_n| \to \int |f|$.

Show that $\displaystyle\int f_n \to \int f$

6 Fall 2017

6.1 1

Let

$$f(x) = s \sum_{n=0}^\infty \frac{x^n}{n!}.$$

Describe the intervals on which f does and does not converge uniformly.

6.2 2

Let $f(x) = x^2$ and $E \subset [0, \infty) := \mathbb{R}^+$.

1. Show that

$$m^*(E) = 0 \iff m^*(f(E)) = 0.$$

2. Deduce that the map

$$\begin{aligned} \varphi : \mathcal{L}(\mathbb{R}^+) &\longrightarrow \mathcal{L}(\mathbb{R}^+) \\ E &\mapsto f(E) \end{aligned}$$

is a bijection from the class of Lebesgue measurable sets of $[0, \infty)$ to itself.

6.3 3

Let

$$S = \operatorname{span}_{\mathbb{C}} \left\{ \chi_{(a,b)} \;\middle|\; a, b \in \mathbb{R} \right\},$$

the complex linear span of characteristic functions of intervals of the form (a, b).

Show that for every $f \in L^1(\mathbb{R})$, there exists a sequence of functions $\{f_n\} \subset S$ such that

$$\lim_{n \to \infty} \|f_n - f\|_1 = 0$$

6.4 4

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

1. Show that $f_n \longrightarrow 0$ pointwise but not uniformly on $[0, 1]$.

   Hint: Consider the maximum of f_n.

2. $$\lim_{n \to \infty} \int_0^1 n(1-x)^n \sin x dx = 0$$

6.5 5

Let φ be a compactly supported smooth function that vanishes outside of an interval $[-N, N]$ such that $\int_{\mathrm{R}} \varphi(x) dx = 1$.

For $f \in L^1(\mathbb{R})$, define

$$K_j(x) := j\varphi(jx), \qquad f * K_j(x) := \int_{\mathbb{R}} f(x-y) K_j(y) \, dy$$

and prove the following:

1. Each $f * K_j$ is smooth and compactly supported.
2. $$\lim_{j\to\infty} \|f * K_j - f\|_1 = 0$$

Hint:

$$\lim_{y\to 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dy = 0$$

6.6 6

Let X be a complete metric space and define a norm

$$\|f\| := \max\{|f(x)| : x \in X\}.$$

Show that $(C^0(\mathbb{R}), \|\cdot\|)$ (the space of continuous functions $f : X \longrightarrow \mathbb{R}$) is complete.

7 Spring 2017

7.1 1

Let K be the set of numbers in $[0,1]$ whose decimal expansions do not use the digit 4.

We use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 with $399\cdots$. For example, $0.8754 = 0.8753999\cdots$.

Show that K is a compact, nowhere dense set without isolated points, and find the Lebesgue measure $m(K)$.

7.2 2

a. Let μ be a measure on a measurable space $(X, \mathcal{M})$ and f a positive measurable function.

   Define a measure λ by

$$\lambda(E) := \int_E f \, d\mu, \quad E \in \mathcal{M}$$

   Show that for g any positive measurable function,

$$\int_X g \, d\lambda = \int_X fg \, d\mu$$

b. Let $E \subset \mathbb{R}$ be a measurable set such that

$$\int_E x^2 \, dm = 0.$$

   Show that $m(E) = 0$.

7.3 3

Let

$$f_n(x) = ae^{-nax} - be^{-nbx} \qquad \text{where } 0 < a < b.$$

Show that

a. $\displaystyle\sum_{n=1}^{\infty} |f_n|$ is not in $L^1([0,\infty), m)$

Hint: $f_n(x)$ has a root x_n.

b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0,\infty), m) \quad \text{and} \quad \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)\, dm = \ln \frac{b}{a}$$

7.4 4

Let $f(x,y)$ on $[-1,1]^2$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Determine if f is integrable.

7.5 5

Let $f, g \in L^2(\mathbb{R})$. Prove that the formula

$$h(x) := \int_{-\infty}^{\infty} f(t) g(x-t) dt$$

defines a uniformly continuous function h on $\mathbb{R}$.

7.6 5

Show that the space $C^1([a,b])$ is a Banach space when equipped with the norm

$$\|f\| := \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |f'(x)|.$$

8 Fall 2016 (Neil-ish)

8.1 1

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that f converges to a differentiable function on $(1,\infty)$ and that

$$f'(x) = \sum_{n=1}^{\infty}\left(\frac{1}{n^x}\right)'.$$

Hint:

$$\left(\frac{1}{n^x}\right)' = -\frac{1}{n^x}\ln n$$

8.2 2

Let $f, g : [a,b] \longrightarrow \mathbb{R}$ be measurable with

$$\int_a^b f(x)\ dx = \int_a^b g(x)\ dx.$$

Show that either

1. $f(x) = g(x)$ almost everywhere, or
2. There exists a measurable set $E \subset [a,b]$ such that

$$\int_E f(x)\ dx > \int_E g(x)\ dx$$

8.3 3

Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{x\to 0}\int_{\mathbb{R}} |f(y-x) - f(y)|dy = 0$$

8.4 4

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\{E_n\} \subset \mathcal{M}$ satisfies

$$\lim_{n\to\infty}\mu\left(X \backslash E_n\right) = 0.$$

Define

$$G := \left\{x \in X \;\middle|\; x \in E_n \text{ for only finitely many } n\right\}.$$

Show that $G \in \mathcal{M}$ and $\mu(G) = 0$.

8.5 5

Let $\varphi \in L^\infty(\mathbb{R})$. Show that the following limit exists and satisfies the equality

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2}dx\right)^{\frac{1}{n}} = \|\varphi\|_\infty.$$

8.6 6

Let $f, g \in L^2(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)g(x+n)dx = 0$$

9 Spring 2016 (Neil-ish)

9.1 1

For $n \in \mathbb{N}$, define

$$e_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad E_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

Show that $e_n < E_n$, and prove Bernoulli's inequality:

$$(1+x)^n \geq 1 + nx \text{ for } -1 < x < \infty \text{ and } n \in \mathbb{N}$$

Use this to show the following:

1. The sequence e_n is increasing.
2. The sequence E_n is decreasing.
3. $2 < e_n < E_n < 4$.
4. $\lim_{n\to\infty} e_n = \lim_{n\to\infty} E_n$.

9.2 2

Let $0 < \lambda < 1$ and construct a Cantor set C_λ by successively removing middle intervals of length λ.

Prove that $m(C_\lambda) = 0$.

9.3 3

Let f be Lebesgue measurable on $\mathbb{R}$ and $E \subset \mathbb{R}$ be measurable such that

$$0 < A = \int_E f(x)dx < \infty.$$

Show that for every $0 < t < 1$, there exists a measurable set $E_t \subset E$ such that

$$\int_{E_t} f(x)dx = tA.$$

9.4 4

Let $E \subset \mathbb{R}$ be measurable with $m(E) < \infty$. Define

$$f(x) = m(E \cap (E + x)).$$

Show that

1. $f \in L^1(\mathbb{R})$.
2. f is uniformly continuous.
3. $\lim\limits_{|x|\to\infty} f(x) = 0$

Hint:

$$\chi_{E\cap(E+x)}(y) = \chi_E(y)\chi_E(y-x)$$

9.5 5

Let $(X, \mathcal{M}, \mu)$ be a measure space. For $f \in L^1(\mu)$ and $\lambda > 0$, define

$$\varphi(\lambda) = \mu(\{x \in X | f(x) > \lambda\}) \quad \text{and} \quad \psi(\lambda) = \mu(\{x \in X | f(x) < -\lambda\})$$

Show that φ, ψ are Borel measurable and

$$\int_X |f|\ d\mu = \int_0^\infty [\varphi(\lambda) + \psi(\lambda)]\ d\lambda$$

9.6 6

Without using the Riesz Representation Theorem, compute

$$\sup\left\{\left|\int_0^1 f(x)e^x dx\right| \ \middle|\ f \in L^2([0,1], m),\ \|f\|_2 \leq 1\right\}$$

10 Fall 2015

10.1 1

Define

$$f(x) = c_0 + c_1x^1 + c_2x^2 + \ldots + c_nx^n \text{ with } n \text{ even and } c_n > 0.$$

Show that there is a number x_m such that $f(x_m) \leq f(x)$ for all $x \in \mathbb{R}$.

10.2 2

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be Lebesgue measurable.

1. Show that there is a sequence of simple functions $s_n(x)$ such that $s_n(x) \longrightarrow f(x)$ for all $x \in \mathbb{R}$.
2. Show that there is a Borel measurable function g such that $g = f$ almost everywhere.

10.3 3

Compute the following limit:

$$\lim_{n\to\infty} \int_1^n \frac{ne^{-x}}{1+nx^2}\ \sin\left(\frac{x}{n}\right)\ dx$$

10.4 4

Let $f : [1, \infty) \longrightarrow \mathbb{R}$ such that $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Show that the following limit exists and satisfies the equality

$$\lim_{x \to \infty} f(x) \leq 1 + \frac{\pi}{4}$$

10.5 5

Let $f, g \in L^1(\mathbb{R})$ be Borel measurable.

1. Show that

- The function

$$F(x, y) \coloneqq f(x - y)g(y)$$

  is Borel measurable on $\mathbb{R}^2$, and
- For almost every $y \in \mathbb{R}$,

$$F_y(x) \coloneqq f(x - y)g(y)$$

  is integrable with respect to y.

2. Show that $f * g \in L^1(\mathbb{R})$ and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

10.6 6

Let $f : [0, 1] \longrightarrow \mathbb{R}$ be continuous. Show that

$$\sup \left\{ \|fg\|_1 \;\middle|\; g \in L^1[0, 1],\; \|g\|_1 \leq 1 \right\} = \|f\|_\infty$$

11 Spring 2015

11.1 1

Let (X, d) and (Y, ρ) be metric spaces, $f : X \longrightarrow Y$, and $x_0 \in X$.

Prove that the following statements are equivalent:

1. For every $\varepsilon > 0 \quad \exists \delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$.
2. The sequence $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow f(x_0)$ for every sequence $\{x_n\} \longrightarrow x_0$ in X.

11.2 2

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be continuous with period 1. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(t)dt \quad \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

Hint: show this first for the functions $f(t) = e^{2\pi ikt}$ for $k \in \mathbb{Z}$.

11.3 3

Let μ be a finite Borel measure on $\mathbb{R}$ and $E \subset \mathbb{R}$ Borel. Prove that the following statements are equivalent:

1. $\forall \varepsilon > 0$ there exists G open and F closed such that
$$F \subseteq E \subseteq G \quad \text{and} \quad \mu(G \setminus F) < \varepsilon.$$
2. There exists a $V \in G_\delta$ and $H \in F_\sigma$ such that
$$H \subseteq E \subseteq V \quad \text{and} \quad \mu(V \setminus H) = 0$$

11.4 4

Define

$$f(x, y) := \begin{cases} \dfrac{x^{1/3}}{(1 + xy)^{3/2}} & \text{if } 0 \le x \le y \\ 0 & \text{otherwise} \end{cases}$$

Carefully show that $f \in L^1(\mathbb{R}^2)$.

11.5 5

Let $\mathcal{H}$ be a Hilbert space.

1. Let $x \in \mathcal{H}$ and $\{u_n\}_{n=1}^N$ be an orthonormal set. Prove that the best approximation to x in $\mathcal{H}$ by an element in $\operatorname{span}_{\mathbb{C}} \{u_n\}$ is given by
$$\widehat{x} := \sum_{n=1}^{N} \langle x,\ u_n\rangle u_n.$$
2. Conclude that finite dimensional subspaces of $\mathcal{H}$ are always closed.

11.6 6

Let $f \in L^1(\mathbb{R})$ and g be a bounded measurable function on $\mathbb{R}$.

1. Show that the convolution $f * g$ is well-defined, bounded, and uniformly continuous on $\mathbb{R}$.
2. Prove that one further assumes that $g \in C^1(\mathbb{R})$ with bounded derivative, then $f * g \in C^1(\mathbb{R})$ and
$$\frac{d}{dx}(f * g) = f * \left(\frac{d}{dx} g\right)$$

12 Fall 2014

12.1 1

Let $\{f_n\}$ be a sequence of continuous functions such that $\sum f_n$ converges uniformly.

Prove that $\sum f_n$ is also continuous.

12.2 2

Let I be an index set and $\alpha : I \longrightarrow (0, \infty)$.

1. Show that

$$\sum_{i \in I} a(i) := \sup_{\substack{J \subset I \\ J \text{ finite}}} \sum_{i \in J} a(i) < \infty \implies I \text{ is countable.}$$

2. Suppose $I = \mathbb{Q}$ and $\sum_{q \in \mathbb{Q}} a(q) < \infty$. Define

$$f(x) := \sum_{\substack{q \in \mathbb{Q} \\ q \leq x}} a(q).$$

Show that f is continuous at $x \iff x \notin \mathbb{Q}$.

12.3 3

Let $f \in L^1(\mathbb{R})$. Show that

$$\forall \varepsilon > 0 \; \exists \delta > 0 \text{ such that } m(E) < \delta \implies \int_E |f(x)| dx < \varepsilon$$

12.4 4

Let $g \in L^\infty([0,1])$ Prove that

$$\int_{[0,1]} f(x) g(x) dx = 0 \quad \text{for all continuous } f : [0,1] \longrightarrow \mathbb{R} \implies g(x) = 0 \text{ almost everywhere.}$$

12.5 5

1. Let $f \in C_c^0(\mathbb{R}^n)$, and show

$$\lim_{t \to 0} \int_{\mathbb{R}^n} |f(x+t) - f(x)| dx = 0.$$

2. Extend the above result to $f \in L^1(\mathbb{R}^n)$ and show that

$$f \in L^1(\mathbb{R}^n), \; g \in L^\infty(\mathbb{R}^n) \implies f * g \text{ is bounded and uniformly continuous.}$$

12.6 6

Let $1 \leq p, q \leq \infty$ be conjugate exponents, and show that

$$f \in L^p(\mathbb{R}^n) \implies \|f\|_p = \sup_{\|g\|_q = 1} \left| \int f(x)g(x)dx \right|$$

13 Spring 2014

13.1 1

1. Give an example of a continuous $f \in L^1(\mathbb{R})$ such that $f(x) \not\longrightarrow 0$ as$|x| \longrightarrow \infty$.
2. Show that if f is *uniformly* continuous, then

$$\lim_{|x| \longrightarrow \infty} f(x) = 0.$$

13.2 2

Let $\{a_n\}$ be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that $\sum a_n^2 < \infty$.

Note: Assume a_n, b_n are all non-negative.

13.3 3

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and suppose

$$\forall x \in \mathbb{R}, \quad f(x) \geq \limsup_{y \to x} f(y)$$

Prove that f is Borel measurable.

13.4 4

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose f is a measurable function on X. Show that

$$\lim_{n \to \infty} \int_X f^n \, d\mu = \begin{cases} \infty & \text{or} \\ \mu(f^{-1}(1)), & \end{cases}$$

and characterize the collection of functions of each type.

13.5 5

Let $f, g \in L^1([0,1])$ and for all $x \in [0,1]$ define

$$F(x) := \int_0^x f(y)dy \quad \text{and} \quad G(x) := \int_0^x g(y)dy.$$

Prove that

$$\int_0^1 F(x)g(x)dx = F(1)G(1) - \int_0^1 f(x)G(x)dx$$