

Real Analysis Qualifying Exam Solutions

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Contents

1	Spring 2020	3
1.1	1	3
1.2	2	4
	1.2.1 a	4
	1.2.2 b	5
1.3	3	5
	1.3.1 a	6
	1.3.2 b	6
	1.3.3 c	7
1.4	4	8
1.5	5	9
1.6	6	9
2	Spring 2019	10
2.1	1	10
	2.1.1 a	10
	2.1.2 b	10
2.2	2	11
	2.2.1 a	11
	2.2.2 b	11
2.3	3	12
2.4	4	12
	2.4.1 a	12
	2.4.2 b	12
2.5	5	13
	2.5.1 a	13
	2.5.2 b	13
3	Fall 2019	15
3.1	1	15
3.2	a	15
3.3	b	16
3.4	2	17

3.5	3	17
3.5.1	a	17
3.5.2	b	18
3.5.3	c	18
3.6	4	18
3.6.1	a	19
3.6.2	b	19
3.7	5	20
3.8	a	20
3.9	b	20
4	Spring 2018	21
4.1	1	21
4.2	2	22
4.2.1	a	22
4.2.2	b	22
4.3	3	23
4.4	4	23
4.5	5	24
5	Fall 2018	25
5.1	1	25
5.2	2	25
5.3	3	26
5.4	4	26
5.5	5	26
6	Spring 2017	27
6.1	1	27
6.2	2	28
6.2.1	a	28
6.2.2	b	28
6.3	3	29
6.3.1	a	29
6.3.2	b	29
6.4	4	29
6.5	5	29
7	Fall 2017	30
7.1	1	30
7.2	2	30
7.2.1	a	30
7.2.2	b	31
7.3	3	31
7.4	4	32
7.4.1	a	32
7.4.2	b	32

7.5	5	...	32
7.5.1	a	...	32
7.5.2	b	...	34
7.6	6	...	35
8	Spring 2016		35
8.1	1	...	35
9	Fall 2016		35
9.1	1	...	35
10	Spring 2014		35
10.1	1	...	35

1 Spring 2020

1.1 1

Concepts used:

- DCT
- Weierstrass Approximation Theorem

Suppose p is a polynomial, then

$$\begin{aligned}
 \lim_{k \rightarrow \infty} \int_0^1 k x^{k-1} p(x) dx &= \lim_{k \rightarrow \infty} \int_0^1 \left(\frac{\partial}{\partial x} x^k \right) p(x) dx \\
 &= \lim_{k \rightarrow \infty} \left[x^k p(x) \Big|_0^1 - \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx \right] \quad \text{integrating by parts} \\
 &= p(1) - \lim_{k \rightarrow \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx,
 \end{aligned}$$

and thus it suffices to show that

$$\lim_{k \rightarrow \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx = 0.$$

Integrating by parts a second time yields

$$\begin{aligned}
 \lim_{k \rightarrow \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx &= \lim_{k \rightarrow \infty} \frac{x^{k+1}}{k+1} p'(x) \Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \\
 &= - \lim_{k \rightarrow \infty} \int_0^1 \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \\
 &= - \int_0^1 \lim_{k \rightarrow \infty} \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \quad \text{by DCT} \\
 &= - \int_0^1 0 \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \\
 &= 0.
 \end{aligned}$$

The DCT can be applied here because f'' is continuous and $[0, 1]$ is compact, so f'' is bounded on $[0, 1]$ by a constant M and $\int_0^1 |x^k f''(x)| \leq \int_0^1 1 \cdot M = M < \infty$.

We now use the Weierstrass approximation theorem: if $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then for every $\varepsilon > 0$ there exists a polynomial $p_\varepsilon(x)$ such that $\|f - p_\varepsilon\|_\infty < \varepsilon$. Thus

$$\begin{aligned} \left| \int_0^1 kx^{k-1} p_\varepsilon(x) dx - \int_0^1 kx^{k-1} f(x) dx \right| &= \left| \int_0^1 kx^{k-1} (p_\varepsilon(x) - f(x)) dx \right| \\ &\leq \left| \int_0^1 kx^{k-1} \|p_\varepsilon - f\|_\infty dx \right| \\ &= \|p_\varepsilon - f\|_\infty \cdot \left| \int_0^1 kx^{k-1} dx \right| \\ &= \|p_\varepsilon - f\|_\infty \cdot x^k \Big|_0^1 \\ &= \|p_\varepsilon - f\|_\infty \xrightarrow{\varepsilon \rightarrow 0} 0 \end{aligned}$$

and the integrals are equal. Finally, the first integral is equal to $p_\varepsilon(1)$ for each ε , which converges to $f(1)$ since uniform convergence implies pointwise convergence.

1.2 2

Concepts used:

- Definition of outer measure: $m_*(E) = \inf_{\{Q_j\} \Rightarrow E} \sum |Q_j|$ where $\{Q_j\}$ is a countable collection of closed cubes.
- Break \mathbb{R} into $\coprod_{n \in \mathbb{Z}} [n, n+1)$, each with finite measure.

1.2.1 a

Suppose $m_*(E) = N < \infty$.

Since $m_*(E)$ is an infimum, by definition, for every $\varepsilon > 0$ there exists a covering by closed cubes $\{Q_i(\varepsilon)\}_{i=1}^\infty \Rightarrow E$ such that

$$\sum_{i=1}^\infty |Q_i(\varepsilon)| < N + \varepsilon.$$

Set $\varepsilon_n = \frac{1}{n}$ to produce such a collection $\{Q_i(\varepsilon_n)\}$ and set $B_n := \bigcup_{i=1}^\infty Q_i(\varepsilon_n)$. Then (theorem) the outer measure of cubes is *equal* to the sum of their volumes, so

$$m_*(B_n) = \sum_{i=1}^\infty |Q_i(\varepsilon_n)| < N + \varepsilon_n = N + \frac{1}{n}.$$

Now set $B := \bigcap_{n=1}^\infty B_n$.

- Since $E \subseteq B_n$ for every n , $E \subseteq B$
- Since B is a countable intersection of countable unions of closed sets, B is Borel.
- Since $B_n \subseteq B$ for every n , we can apply subadditivity to obtain the inequality

$$E \subseteq B \subseteq B_n \implies N \leq m_*(B) \leq m_*(B_n) < N + \frac{1}{n} \quad \text{for all } n \in \mathbb{Z}^{\geq 1}.$$

This forces $m_*(E) = m_*(B)$.

If $m_*(E) = \infty$, then take $B = \mathbb{R}^n$ since $m(\mathbb{R}^n) = \infty$.

1.2.2 b

Suppose $m_*(E) < \infty$.

- By (a), find a Borel set $B \supseteq E$ such that $m_*(B) = m_*(E)$
- Note that $E \subseteq B \implies B \cap E = E$ and $B \cap E^c = B \setminus E$.
- By assumption,

$$\begin{aligned} m_*(B) &= m_*(B \cap E) + m_*(B \cap E^c) \\ m_*(E) &= m_*(E) + m_*(B \setminus E) \\ m_*(E) - m_*(E) &= m_*(B \setminus E) \quad \text{since } m_*(E) < \infty \\ \implies m_*(B \setminus E) &= 0. \end{aligned}$$

- So take $N = B \setminus E$; this shows $m_*(N) = 0$ and $E = B \setminus (B \setminus E) = B \setminus N$.
- If $m_*(E) = \infty$
 - Apply result to $E_R := E \cap [R, R+1)^n \subset \mathbb{R}^n$ for $R \in \mathbb{Z}$, so $E = \coprod_R E_R$
 - Obtain B_R, N_R such that $E_R = B_R \setminus N_R$, $m_*(E_R) = m_*(B_R)$, and $m_*(N_R) = 0$.
 - Note that
 - * $B := \bigcup_R B_R$, which is a union of Borel sets and thus still Borel
 - * $E = \bigcup_R E_R$
 - * $N := B \setminus E$
 - * $N' := \bigcup_R N_R$ which is a union of null sets and thus still null
 - Since $E_R \subset B_R$ for every R , we have $E \subset B$
 - We can compute

$$N = B \setminus E = \left(\bigcup_R B_R \right) \setminus \left(\bigcup_R E_R \right) \subseteq \bigcup_R (B_R \setminus E_R) = N'$$

where $m_*(N) = 0$ and thus subadditivity forces $m_*(N) = 0$.

1.3 3

Concepts used:

- Limits
- Cauchy Criterion for Integrals: $\int_a^\infty f(x) dx$ converges iff for every $\varepsilon > 0$ there exists an M_0 such that $A, B \geq M_0$ implies $\left| \int_A^B f \right| < \varepsilon$.
- Mean Value Theorem for Integrals: $\int_a^b f(t) dt = (b-a)f(c)$ for some $c \in [a, b]$.

1.3.1 a

Stated integral equality:

- Let $\varepsilon > 0$
- $C_c(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$ is dense so choose $\{f_n\} \rightarrow f$ with $\|f_n - f\|_1 \rightarrow 0$.
- Since $\{f_n\}$ are compactly supported, choose $N_0 \gg 1$ such that f_n is zero outside of $B_{N_0}(\mathbf{0})$.
- Then

$$\begin{aligned}
 N \geq N_0 &\implies \int_{|x|>N} |f| = \int_{|x|>N} |f - f_n + f_n| \\
 &\leq \int_{|x|>N} |f - f_n| + \int_{|x|>N} |f_n| \\
 &= \int_{|x|>N} |f - f_n| \\
 &\leq \int_{|x|>N} \|f - f_n\|_1 \\
 &= \|f_n - f\|_1 \left(\int_{|x|>N} 1 \right) \\
 &\xrightarrow{n \rightarrow \infty} 0 \left(\int_{|x|>N} 1 \right) \\
 &= 0 \\
 &\xrightarrow{N \rightarrow \infty} 0.
 \end{aligned}$$

To see that this doesn't force $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$:

- Take $f(x)$ to be a train of boxes of height 1 and area $1/2^j$ centered on even integers.
- Then the

$$\int_{|x|>N} |f| = \sum_{j=N}^{\infty} 1/2^j \xrightarrow{N \rightarrow \infty} 0$$

as the tail of a convergent sum.

- However $f(x) = 1$ for infinitely many even integers $x > N$, so $f(x) \not\rightarrow 0$ as $|x| \rightarrow \infty$.

1.3.2 b

- Since f is decreasing on $[1, \infty)$, for any $t \in [x-n, x]$ we have

$$x-n \leq t \leq x \implies f(x) \leq f(t) \leq f(x-n).$$

- Integrate over $[x, 2x]$, using monotonicity of the integral:

$$\begin{aligned} \int_x^{2x} f(x) dt &\leq \int_x^{2x} f(t) dt \leq \int_x^{2x} f(x-n) dt \\ \implies f(x) \int_x^{2x} dt &\leq \int_x^{2x} f(t) dt \leq f(x-n) \int_x^{2x} dt \\ \implies xf(x) &\leq \int_x^{2x} f(t) dt \leq xf(x-n). \end{aligned}$$

- By the Cauchy Criterion for integrals, $\lim_{x \rightarrow \infty} \int_x^{2x} f(t) dt = 0$.
- So the LHS term $\lim_{x \rightarrow \infty} xf(x) = 0$.
- Since $x > 1$, $|f(x)| \leq |xf(x)|$
- Thus $f(x) \rightarrow 0$ as well.

Alternatively, use mean value theorem for integrals:

$$\int_x^{2x} f(t) dt = xf(c_x) \quad \text{for some } c_x \in [x, 2x] \text{ depending on } x.$$

- Since f is decreasing,

$$\begin{aligned} x \leq c_x \leq 2x &\implies f(2x) \leq f(c_x) \leq f(x) \\ &\implies xf(2x) \leq xf(c_x) \leq xf(x) \\ &\implies xf(2x) \leq x \int_x^{2x} f(t) dt \leq xf(x) \end{aligned}$$

- By Cauchy Criterion, $\int_x^{2x} f \rightarrow 0$.

Alternatively, *just* showing $f(x) \xrightarrow{x \rightarrow \infty} 0$:

- Toward a contradiction, suppose not.
- If $f(x) \rightarrow -\infty$, then $f \notin L^1(\mathbb{R})$: choose $x_0 \gg 1$ so that $x \geq x_0 \implies f(x) < -1$, then

$$\int_{\mathbb{R}} |f| \geq \int_{x_0}^{\infty} |f(t)| dt \geq \int_{x_0}^{\infty} 1 = \infty.$$

1.3.3 c

- No: take $f(x) = \frac{1}{x \ln x}$
- Then by a u -substitution,

$$\int_0^x f = \ln(\ln(x)) \xrightarrow{x \rightarrow \infty} \infty$$

is unbounded, so $f \notin L^1([1, \infty))$.

- But

$$xf(x) = \frac{1}{\ln(x)} \xrightarrow{x \rightarrow \infty} 0.$$

1.4 4

Relevant concepts:

- Tonelli: non-negative and measurable yields measurability of slices and equality of iterated integrals
- Fubini: $f(x, y) \in L^1$ yields *integrable* slices and equality of iterated integrals
- F/T: apply Tonelli to $|f|$; if finite, $f \in L^1$ and apply Fubini to f

$$\begin{aligned}
 \|(f * g)(x)\|_1 &= \left\| \int_{\mathbb{R}} H(x, y) dy \right\|_1 \\
 &:= \left\| \int_{\mathbb{R}} f(y)g(x - y) dy \right\|_1 \\
 &\leq \int_{\mathbb{R}} \|f(y)g(x - y)\|_1 dy \\
 &\leq \int_{\mathbb{R}} |f(y)| \cdot \|g(x - y)\|_1 dy \\
 &\leq \int_{\mathbb{R}} |f(y)| \cdot \|g\|_1 dy \\
 &\leq \|g\|_1 \int_{\mathbb{R}} |f(y)| dy \\
 &\leq \|g\|_1 \|f\|_1 \\
 &< \infty \quad \text{by assumption} \quad .
 \end{aligned}$$

$$\begin{aligned}
 \|H(x)\|_1 &= \int_{\mathbb{R}} |H(x, y)| dx \\
 &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y)g(x - y) dy \right| dx \\
 &\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(x - y)| dy \right) dx \\
 &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(x - y)| dx \right) dy \quad \text{by Tonelli} \\
 &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(t)| dt \right) dy \quad \text{setting } t = x - y, dt = -dx \\
 &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)||g(t)| dt \right) dy \\
 &= \int_{\mathbb{R}} |f(y)| \left(\int_{\mathbb{R}} |g(t)| dt \right) dy \\
 &:= \int_{\mathbb{R}} |f(y)| \|g\|_1 dy \\
 &= \|g\|_1 \int_{\mathbb{R}} |f(y)| dy \\
 &:= \|g\|_1 \|f\|_1 .
 \end{aligned}$$

Now to justify the interchange of integrals:

- We first apply Fubini to f :
 - Checking measurability: $f \in L^1$ implies that f is measurable (by how L^1 is defined)
 - The function $\tau_y(x) = x - y$ for a fixed y is measurable

Todo:

- Change of variables for $x - y$?
- How to justify F/T?

1.5 5

Note that

$$\begin{aligned}\lim_n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} &= \frac{1}{\lim_n \left(1 + \frac{x^2}{n}\right)^1 \left(1 + \frac{x^2}{n}\right)^n} \\ &= \frac{1}{1 \cdot e^{x^2}} \\ &= e^{-x^2}.\end{aligned}$$

If passing the limit through the integral is justified, we will have $\int_0^\infty e^{-x^2} = \frac{\sqrt{\pi}}{2}$.

Computing the integral:

$$\begin{aligned}\left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2 &= \left(\int_{\mathbb{R}} e^{-x^2} dx\right) \left(\int_{\mathbb{R}} e^{-y^2} dx\right) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(x+y)^2} dx \\ &= \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta \quad u = r^2 \\ &= \frac{1}{2} \int_0^{2\pi} \int_0^\infty e^{-u} du d\theta \\ &= \frac{1}{2} \int_0^{2\pi} 1 \\ &= \pi,\end{aligned}$$

and now use the fact that the function is even.

Todo:

- Justify, MCT? Possible have

$$f_n(x) := \frac{1}{\left(1 + \frac{x^2}{n}\right)^{n+1}} \nearrow e^{-x^2}.$$

1.6 6

Concepts used:

-
- For $e_n(x) := 2^{2\pi i n x}$, the set $\{e_n\}$ is an orthonormal basis for $L^2([0, 1])$.
 - For any orthonormal set in a Hilbert space, Bessel's inequality:

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

- When a basis, the above is an inequality (Parseval)
- Arguing uniform convergence: since $\{\hat{f}(n)\} \in \ell^1(\mathbb{Z})$, we should be able to apply the M test.

Want to show

$$\int_0^1 |f(x)|^2 dx \leq \int_0^1 |f(x)| dx$$

$$\sum_{n=1}^{\infty} |c_n|^2 \leq \sum_{n=1}^{\infty} |c_n|.$$

- First inequality: ?
- Second inequality: break into 2 pieces?

2 Spring 2019

2.1 1

2.1.1 a

Let $\{f_k\}$ be a Cauchy sequence in $C(I)$. For each fixed $x \in [0, 1]$, the sequence of real numbers $\{f_k(x)\}$ is Cauchy in \mathbb{R} , which is complete, since

$$x_0 \in I \implies |f_k(x_0) - f_j(x_0)| \leq \sup_{x \in I} |f_k(x) - f_j(x)| = \|f_k - f_j\|_{\infty} \longrightarrow 0,$$

so we can define $f(x) := \lim_k f_k(x)$.

We also have

$$\|f_k - f\|_{\infty} = \left\| f_k - \lim_{j \rightarrow \infty} f_j \right\|_{\infty} = \lim_{j \rightarrow \infty} \|f_k - f_j\|_{\infty} \longrightarrow 0.$$

Finally, f is the uniform limit of continuous functions and thus continuous.

■

2.1.2 b

It suffices to produce a Cauchy sequence that does not converge to a continuous function. Take

$$f_k(x) = \begin{cases} (x + \frac{1}{2})^k & x \in [0, \frac{1}{2}) \\ 1 & x \in [\frac{1}{2}, 1] \end{cases} \xrightarrow{k \rightarrow \infty} f(x) = \begin{cases} 0 & x \in [0, \frac{1}{2}) \\ 1 & x \in [\frac{1}{2}, 1] \end{cases},$$

which is Cauchy, but there is no $g \in L^1$ that is continuous such that $\|f - g\|_1 = 0$.

2.2 2

2.2.1 a

Lemma 1: $\mu(\prod_{k=1}^{\infty} E_k) = \lim_{N \rightarrow \infty} \sum_{k=1}^N \mu(E_k)$.

Lemma 2: $A = A \setminus B \coprod A \cap B$.

Let $A_k = F_k \setminus F_{k+1}$, so the A_k are disjoint, and let $A = \prod_k A_k$.

Let $F = \bigcap_k F_k$. Then $F_1 = F \coprod A$ by lemma 2, so

$$\begin{aligned} \mu(F_1) &= \mu(F) + \mu(A) \\ &= \mu(F) + \lim_{N \rightarrow \infty} \sum_k^N \mu(A_k) \quad \text{by Lemma 1} \\ &= \mu(F) + \lim_{N \rightarrow \infty} \sum_k^N (\mu(F_k) - \mu(F_{k+1})) \\ &= \mu(F) + \lim_{N \rightarrow \infty} (\mu(F_1) - \mu(F_N)) \quad (\text{Telescoping}) \\ &= \mu(F) + \mu(F_1) - \lim_{N \rightarrow \infty} \mu(F_N), \end{aligned}$$

and since the measure is finite, $\mu(F_1) < \infty$ and can be subtracted, yielding

$$\begin{aligned} \mu(F_1) &= \mu(F) + \mu(F_1) - \lim_{N \rightarrow \infty} \mu(F_N) \\ \implies \mu(F) &= \lim_{N \rightarrow \infty} \mu(F_N). \end{aligned}$$

2.2.2 b

Suppose toward a contradiction that there is some $\varepsilon > 0$ for which no such δ exists.

This means that we can take any sequence $\delta_n \rightarrow 0$ and produce sets A_n such $m(A) < \delta_n$ but $\mu(A) > \varepsilon$.

So choose the sequence $\delta_n = \frac{1}{2^n}$ and define A_n accordingly, and let

$$A = \limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

Since

$$\mu\left(\bigcup_{k=n}^{\infty} A_k\right) \leq \sum_{k=n}^{\infty} \mu(A_k) \approx \frac{1}{2^n} \rightarrow 0,$$

by part (a) we have $m(A) = 0$. Now by assumption, we should thus have $\mu(A) = 0$ as well.

However, again by part (a), we have

$$\mu(A) = \lim_n \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \geq \lim_n \mu(A_n) = \lim_n \varepsilon = \varepsilon > 0.$$

2.3 3

Since $f_k \rightarrow f$ almost everywhere, we have $\liminf_k f_k(x) = f(x)$ and since $|f|^2 \in L^+$ we can apply Fatou:

$$\begin{aligned}\|f\|_2^2 &= \int |f(x)|^2 \\ &= \int \liminf_k |f_k(x)|^2 \\ &\leq \liminf_k \int |f_k(x)|^2 \\ &= M^2,\end{aligned}$$

so $\|f\| \leq M < \infty$ and $f \in L^2$.

Let $I = [0, 1]$. Applying Egorov's theorem to produce sets F_ε such that $f_k \xrightarrow{u} f$ on F_ε and taking $F = \bigcap F_\varepsilon$, we have

$$\int_I f_k = \int_{F_\varepsilon} f_k + \int_{F_\varepsilon^c} f_k \xrightarrow{\varepsilon \rightarrow 0} \int_F f_k + 0 \xrightarrow{k \rightarrow \infty} \int_F f,$$

using that fact that uniform converges allows commuting limits and integrals.

2.4 4

2.4.1 a

$\Rightarrow :$

Idea: $\mathcal{A} = \{f(x) - t \geq 0\} \cap \{t \geq 0\}$.

Define $F(x, t) = f(x)$, $G(x, t) = t$, and $H(x, y) = F(x, t) - G(x, t)$, which are all measurable functions.

Then $\mathcal{A} = \{H \geq 0\} \cap \{G \geq 0\}$ which is an intersection of measurable sets.

$\Leftarrow :$

By F.T., for almost every $x \in \mathbb{R}^n$, the x -slices are measurable, so

$$\mathcal{A}_x := \{t \in \mathbb{R} \mid (x, t) \in \mathcal{A}\} = [0, f(x)] \implies m(\mathcal{A}_x) = f(x)$$

But $x \mapsto m(\mathcal{A}_x)$ is a measurable function, and is exactly to $x \mapsto f(x)$, so f is measurable.

2.4.2 b

We first note

$$\begin{aligned}\mathcal{A} &= \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid 0 \leq t \leq f(x)\} \\ \mathcal{A}_t &= \{x \in \mathbb{R}^n \mid t \leq f(x)\}.\end{aligned}$$

Then,

$$\begin{aligned}
 \int_{\mathbb{R}^n} f(x) \, dx &= \int_{\mathbb{R}^n} \int_0^{f(x)} 1 \, dt \, dx \\
 &= \int_{\mathbb{R}^n} \int_{\mathbb{R}} \chi_{\mathcal{A}} \, dt \, dx \\
 &\stackrel{F.T.}{=} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_{\mathcal{A}} \, dx \, dt \\
 &= \int_0^\infty \int_{\mathbb{R}^n} \chi_{\mathcal{A}} \, dx \, dt \\
 &= \int_0^\infty m(\mathcal{A}_t) \, dt,
 \end{aligned}$$

where we just note that $\int \chi_{\mathcal{A}} = m(\mathcal{A})$, and by F.T., all of these integrals are equal.

2.5 5

2.5.1 a

By Holder's inequality with $p = q = 2$, we have

$$\|f\|_1 = \|f \cdot 1\|_1 \leq \|f\|_2 \|1\|_2 = \|f\|_2 m(X)^{\frac{1}{2}} = \|f\|_2,$$

since $X = [0, 1] \implies m(X) = 1$.

So $L^2(X) \subseteq L^1(X)$, and since simple functions are dense in both spaces, L^2 is dense in L^1 .

2.5.2 b

Step 1 Let $\Lambda \in L^1(X)^\vee$; we'll show that in fact $\Lambda \in L^2(X)^\vee$, and by Riesz Representation for L^2 there will be a $g \in L^2$ such that $\Lambda(f) = \langle f, g \rangle$.

Lemma: $m(X) < \infty \implies L^p(X) \subset L^2(X)$.

Proof: Write Holder's inequality as $\|fg\|_1 \leq \|f\|_a \|g\|_b$ where $\frac{1}{a} + \frac{1}{b} = 1$, then

$$\|f\|_p^p = \| |f|^p \|_1 \leq \| |f|^p \|_a \|1\|_b.$$

Now take $a = \frac{2}{p}$ and this reduces to

$$\begin{aligned}
 \|f\|_p^p &\leq \|f\|_2^p m(X)^{\frac{1}{b}} \\
 \implies \|f\|_p &\leq \|f\|_2 \cdot O(m(X)) < \infty.
 \end{aligned}$$

Let $f \in L^2$ be arbitrary – by the lemma, $\|f\|_1 \leq C\|f\|_2$ for some constant $C = O(m(X))$.

Since $\|\Lambda\|_{1^\vee} := \sup_{\|f\|_1=1} |\Lambda(f)|$, given an arbitrary $f \in L^1$, we can define $\hat{f} = f/\|f\|_1$, so $\|\hat{f}\|_1 = 1$, and obtain

$$|\Lambda(\widehat{f})| \leq \|\Lambda\|_{1^\vee},$$

since $\|\Lambda\|_{1^\vee}$ is the *least* such bound over all $f \in L^1$, and thus

$$\begin{aligned} \frac{|\Lambda(f)|}{\|f\|_1} &= |\Lambda(\widehat{f})| \leq \|\Lambda\|_{1^\vee} \\ \implies |\Lambda(f)| &\leq \|\Lambda\|_{1^\vee} \cdot \|f\|_1 \\ &\leq \|\Lambda\|_{1^\vee} \cdot C\|f\|_2, \end{aligned}$$

which is finite by assumption. So $\Lambda \in (L^2)^\vee$ since it is bounded and thus continuous.

By Riesz Representation for L^2 , there is a $g \in L^2$ such that for all $f \in L^2$, $\Lambda(f) = \langle f, g \rangle$

Step 2 By Holder, we already have

$$\begin{aligned} \|\Lambda\|_{1^\vee} &= \sup_{\|f\|_1=1} |\Lambda(f)| \\ &= \sup_{\|f\|_1=1} \left| \int_X fg \right| \\ &\leq \sup_{\|f\|_1=1} \|fg\|_1 \\ &\leq \sup_{\|f\|_1=1} \|f\|_1 \|g\|_\infty \\ &= \|g\|_\infty, \end{aligned}$$

so it just remains to show that $\|g\|_\infty \leq \|\Lambda\|_{1^\vee}$.

Suppose otherwise, so $\|g\|_\infty > \|\Lambda\|_{1^\vee}$.

Then there exists some $E \subseteq X$ with $m(E) > 0$ such that $x \in E \implies |g(x)| > \|\Lambda\|_{1^\vee}$.

Define

$$h = \frac{1}{m(E)} \frac{\bar{g}}{|g|} \chi_E.$$

$$\begin{aligned}
\Lambda(h) &= \int_X hg \\
&= \int_X \frac{1}{m(E)} \frac{g\bar{g}}{|g|} \chi_E \\
&= \frac{1}{m(E)} \int_E |g| \\
&\geq \frac{1}{m(E)} \|g\|_\infty m(E) \\
&= \|g\|_\infty \\
&> \|\Lambda\|_{1^\vee},
\end{aligned}$$

a contradiction. ■

3 Fall 2019

3.1 1

Cesaro mean/summation. Break series apart into pieces that can be handled separately.

3.2 a

Prove a stronger result:

$$a_n \longrightarrow S \implies \frac{1}{N} \sum_{k=1}^N a_k \longrightarrow S.$$

Idea: once N is large enough, $a_k \approx S$, and all smaller terms will die off as $N \longrightarrow \infty$.
See this MSE answer.

Suppose $S_k \longrightarrow S$. Choose ℓ large enough such that

$$k \geq \ell \implies |S_k - S| < \varepsilon.$$

With ℓ fixed, choose N large enough such that

$$k \leq \ell \implies \frac{|S_k - S|}{N} < \varepsilon.$$

Then

$$\begin{aligned}
 \left| \left(\frac{1}{N} \sum_{k=1}^N S_k \right) - S \right| &= \frac{1}{N} \left| \sum_{k=1}^N (S_k - S) \right| \\
 &\leq \frac{1}{N} \sum_{k=1}^N |S_k - S| \\
 &= \sum_{k=1}^{\ell} \frac{|S_k - S|}{N} + \sum_{k=\ell+1}^N \frac{|S_k - S|}{N} \\
 &\longrightarrow 0.
 \end{aligned}$$

3.3 b

Define

$$\Gamma_n := \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

Then $\Gamma_1 = \sum_k \frac{a_k}{k}$ and each Γ_n is a tail of this series, so by assumption $\Gamma_n \longrightarrow 0$.

Then

$$\begin{aligned}
 \frac{1}{n} \sum_{k=1}^n a_k &= \frac{1}{n} (\Gamma_0 + \Gamma_1 + \cdots + \Gamma_n - \Gamma_{n+1}) \\
 &\longrightarrow 0.
 \end{aligned}$$

This comes from consider the following summation:

$\Gamma_1 :$	a_1	$+\frac{a_2}{2}$	$+\frac{a_3}{3}$	$+\cdots$	
$\Gamma_2 :$		$\frac{a_2}{2}$	$+\frac{a_3}{3}$	$+\cdots$	
$\Gamma_3 :$			$\frac{a_3}{3}$	$+\cdots$	
$\sum_{i=1}^n \Gamma_i :$	a_1	$+a_2$	$+a_3$	$+\cdots$	$a_n + \frac{a_{n+1}}{n+1} + \cdots$

■

3.4 2

DCT, and bounding in the right place. Don't evaluate the actual integral!

Use the fact that $\int_0^1 \cos(tx) \, dt = \sin(x)/x$, then

$$\begin{aligned}
 \left| \frac{\partial^n}{\partial x} \sin(x)/x \right| &= \left| \frac{\partial^n}{\partial x} \int_0^1 \cos(tx) \, dt \right| \\
 &= ? \left| \int_0^1 \frac{\partial^n}{\partial x} \cos(tx) \, dt \right| \\
 &= \left| \int_0^1 -t^n \sin(tx) \, dt \right| \quad \text{for } n \text{ odd} \\
 &\leq \int_0^1 |t^n \sin(tx)| \, dt \\
 &\leq \int_0^1 t^n \, dt \\
 &= \frac{1}{n+1} \\
 &< \frac{1}{n}.
 \end{aligned}$$

Where the DCT is justified by noting that $f(t) = \cos(tx)$ is dominated by $g(t) = 1$ on $[0, 1]$, which integrates to 1. ■

3.5 3

Borel-Cantelli.

Use the following observation: for a sequence of sets X_n ,

$$\begin{aligned}
 \limsup_n X_n &= \left\{ x \mid x \in X_n \text{ for infinitely many } n \right\} &= \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} X_n \\
 \liminf_n X_n &= \left\{ x \mid x \in X_n \text{ for all but finitely many } n \right\} &= \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} X_n.
 \end{aligned}$$

And recall

$$\prod_n e^{x_n} = e^{\sum_n x_n} \quad \text{and} \quad \sum_n \log(x_n) = \log \left(\prod_n x_n \right).$$

3.5.1 a

The Borel σ -algebra is closed under countable unions/intersections/complements, and $B = \limsup_n B_n$ is an intersection of unions of measurable sets.

3.5.2 b

We'll use the fact that tails of convergent sums go to zero, so $\sum_{n \geq M} \mu(B_n) \xrightarrow{M \rightarrow \infty} 0$, and $B_M :=$

$$\bigcap_{m=1}^M \bigcup_{n \geq m} B_n \searrow B.$$

$$\begin{aligned} \mu(B_M) &= \mu \left(\bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} B_n \right) \\ &\leq \mu \left(\bigcup_{n \geq m} B_n \right) \quad \text{for all } m \in \mathbb{N} \\ &\longrightarrow 0, \end{aligned}$$

and the result follows by continuity of measure.

3.5.3 c

To show $\mu(B) = 1$, we'll show $\mu(B^c) = 0$.

Let $B_k = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^K B_n$. Then

$$\begin{aligned} \mu(B_K^c) &= \mu \left(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^K B_n^c \right) \\ &\leq \sum_{m=1}^{\infty} \mu \left(\bigcap_{n=m}^K B_n^c \right) \quad \text{by subadditivity} \\ &= \sum_{m=1}^{\infty} \prod_{n=m}^K (1 - \mu(B_n)) \\ &\leq \sum_{m=1}^{\infty} \prod_{n=m}^K e^{-\mu(B_n)} \quad \text{by hint} \\ &= \sum_{m=1}^{\infty} e^{-\sum_{n=m}^K \mu(B_n)} \\ &\longrightarrow 0 \end{aligned}$$

since $\sum_{n=m}^K \mu(B_n) \longrightarrow \infty$, and we can apply continuity of measure since $B_K^c \xrightarrow{K \rightarrow \infty} B^c$. ■

3.6 4

Bessel's Inequality, surjectivity of Riesz map, and Parseval's Identity.
Trick – remember to write out finite sum S_N , and consider $\|x - S_N\|$.

3.6.1 a

Claim:

$$\begin{aligned} 0 \leq \left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\ &\implies \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2. \end{aligned}$$

Proof: Let $S_N = \sum_{n=1}^N \langle x, u_n \rangle u_n$. Then

$$\begin{aligned} 0 &\leq \|x - S_N\|^2 \\ &= \langle x - S_N, x - S_N \rangle \\ &= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\ &\xrightarrow{N \rightarrow \infty} \|x\|^2 - \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2. \end{aligned}$$

3.6.2 b

1. Fix $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
2. Define

$$x := \lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \sum_{k=1}^N a_k u_k$$

3. $\{S_N\}$ Cauchy (by 1) and H complete $\implies x \in H$.
- 4.

$$\langle x, u_n \rangle = \left\langle \sum_k a_k u_k, u_n \right\rangle = \sum_k a_k \langle u_k, u_n \rangle = a_n \quad \forall n \in \mathbb{N}$$

since the u_k are all orthogonal.

- 5.

$$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$$

by Pythagoras since the u_k are normal.

Bonus: We didn't use completeness here, so the Fourier series may not actually converge to x . If $\{u_n\}$ is **complete** (so $x = 0 \iff \langle x, u_n \rangle = 0 \forall n$) then the Fourier series *does* converge to x and $\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 = \|x\|^2$ for all $x \in H$.

■

3.7 5

Continuity in L^1 (recall that DCT won't work! Notes 19.4, prove it for a dense subset first).
Lebesgue differentiation in 1-dimensional case. See HW 5.6.

3.8 a

Choose $g \in C_c^0$ such that $\|f - g\|_1 \rightarrow 0$.

By translation invariance, $\|\tau_h f - \tau_h g\|_1 \rightarrow 0$.

Write

$$\begin{aligned} \|\tau f - f\|_1 &= \|\tau_h f - g + g - \tau_h g + \tau_h g - f\|_1 \\ &\leq \|\tau_h f - \tau_h g\| + \|g - f\| + \|\tau_h g - g\| \\ &\rightarrow \|\tau_h g - g\|, \end{aligned}$$

so it suffices to show that $\|\tau_h g - g\| \rightarrow 0$ for $g \in C_c^0$.

Fix $\varepsilon > 0$. Enlarge the support of g to K such that

$$|h| \leq 1 \text{ and } x \in K^c \implies |g(x-h) - g(x)| = 0.$$

By uniform continuity of g , pick $\delta \leq 1$ small enough such that

$$x \in K, |h| \leq \delta \implies |g(x-h) - g(x)| < \varepsilon,$$

then

$$\int_K |g(x-h) - g(x)| \leq \int_K \varepsilon = \varepsilon \cdot m(K) \rightarrow 0.$$

3.9 b

We have

$$\begin{aligned} \int_{\mathbb{R}} |A_h(f)(x)| \, dx &= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y) \, dy \right| \, dx \\ &\leq \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)| \, dy \, dx \\ &=_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)| \, dx \, dy \\ &= \int_{\mathbb{R}} |f(y)| \, dy \\ &= \|f\|_1. \end{aligned}$$

and (rough sketch)

$$\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x) - f(x)| \, dx &= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y) \, dy \right) - f(x) \right| \, dx \\
&= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y) \, dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x) \, dy \right| \, dx \\
&\leq_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{B(h,x)} |f(y-x) - f(x)| \, \mathbf{d}\mathbf{x} \, dy \\
&\leq \frac{1}{2h} \int_{\mathbb{R}} \|\tau_x f - f\|_1 \, dy \\
&\longrightarrow 0 \quad \text{by (a).}
\end{aligned}$$

■

4 Spring 2018

4.1 1

We'll show that $m(E) \cap [n, n+1] = 0$ for all $n \in \mathbb{Z}$; then the result will follow from that fact that

$$m(E) = m\left(\bigcup_{n \in \mathbb{Z}} E \cap [n, n+1]\right) \leq \sum m(E \cap [n, n+1]) = 0$$

By translation invariance of measure, it suffices to show $m(E \cap [0, 1]) = 0$.

Define

$$E_j := \left\{ x \in [0, 1] \mid \exists p \in \mathbb{Z}^{\geq 0} \text{ s.t. } \left| x - \frac{p}{j} \right| < \frac{1}{j^3} \right\}.$$

Note that we can write E_j is a union of intervals

$$\begin{aligned}
E_j &= (1, \frac{1}{j^3}) \\
&\quad \amalg B_{\frac{1}{j^3}}\left(\frac{1}{j}\right) \amalg B_{\frac{1}{j^3}}\left(\frac{2}{j}\right) \amalg \cdots \amalg B_{\frac{1}{j^3}}\left(\frac{j-1}{j}\right) \\
&\quad \amalg (1 - \frac{1}{j^3}, 1),
\end{aligned}$$

from which we can conclude that E_j is Borel and thus Lebesgue measurable, and that for each j , there are exactly $j+1$ possible choices for a numerator (corresponding to the $j+1$ sets appearing above.)

The first and last intervals are length $\frac{1}{j^3}$ and the remaining $(j+1) - 2 = j-1$ intervals are length $\frac{2}{j^3}$, so we find that

$$m(E_j) = 2\left(\frac{1}{j^3}\right) + (j-1)\left(\frac{2}{j^3}\right) = \frac{2}{j^2}$$

We can then note that

$$\sum_{j \in \mathbb{N}} m(E_j) \leq 2 \sum_{j \in \mathbb{N}} \frac{1}{j^2} < \infty,$$

which converges by the p -test for sums.

Since $\{E_j\}$ is a countable collection of measurable sets such that $\sum m(E_j) < \infty$, Borel-Cantelli applies and $m(\limsup_j E_j) = 0$, where we can just note that $\limsup_j E_j = E \cap [0, 1]$. ■

4.2 2

4.2.1 a

Since $x < 1 \implies x^n \longrightarrow 0$ and $x > 1 \implies x^n \longrightarrow \infty$, we have

$$f_n(x) = \frac{x}{1+x^n} \xrightarrow{n \rightarrow \infty} f(x) = \begin{cases} 0, & x = 0 \\ x, & x < 1 \\ \frac{1}{2}, & x = 1 \\ 0, & x > 1 \end{cases}.$$

If $f_n \longrightarrow f$ uniformly on $[0, \infty)$, it would converge uniformly on every subset.

But $f_n(x)$ is clearly continuous on $(0, \infty)$, and if the convergence was uniform then f would be continuous. However f has a clear discontinuity at $x = 1$.

4.2.2 b

If the DCT applies, we can interchange the limit and integral, and the value would be the area under the graph of f which is $\int_0^1 x \, dx = \frac{1}{2}$.

To justify the DCT, write

$$\int_0^\infty f_n(x) \, dx = \int_0^1 f_n(x) \, dx + \int_1^\infty f_n(x) \, dx.$$

Then

$$x \in [0, 1] \implies \frac{x}{1+x^n} < \frac{1}{1+x^n} < 1$$

and $\int_0^1 1 \, dx = 1 < \infty$.

On the other hand,

$$x \in (1, \infty) \implies \frac{x}{1+x^n} \approx O\left(\frac{1}{x^{n-1}}\right),$$

and so for $n > 2$ the integral will converge by the p -test.

4.3 3

Since $|f(x)| \leq \|f\|_\infty$ almost everywhere, we have

$$\|f\|_p^p = \int_X |f(x)|^p dx \leq \int_X \|f\|_\infty^p dx = \|f\|_\infty^p \cdot m(X) = \|f\|_\infty^p,$$

so $\|f\|_p \leq \|f\|_\infty$ for all p and taking $\lim_{p \rightarrow \infty}$ preserves this inequality.

Conversely, let $\varepsilon > 0$. Define

$$S_\varepsilon := \left\{ x \in \mathbb{R} \mid |f(x)| \geq \|f\|_\infty - \varepsilon \right\}.$$

Then

$$\begin{aligned} \|f\|_p^p &= \int_X |f(x)|^p dx \\ &\geq \int_{S_\varepsilon} |f(x)|^p dx \\ &\geq \int_{S_\varepsilon} (\|f\|_\infty - \varepsilon)^p dx \\ &= (\|f\|_\infty - \varepsilon)^p \cdot m(S_\varepsilon) \\ \implies \|f\|_p &\geq (\|f\|_\infty - \varepsilon) \cdot m(S_\varepsilon)^{\frac{1}{p}} \\ &\xrightarrow{p \rightarrow \infty} (\|f\|_\infty - \varepsilon) \\ &\xrightarrow{\varepsilon \rightarrow 0} \|f\|_\infty. \end{aligned}$$

So $\|f\|_p \geq \|f\|_\infty$.

■

4.4 4

Fix $k \in \mathbb{Z}$. Since $e^{2\pi i k x}$ is continuous on the compact interval $[0, 1]$, it is uniformly continuous, and is thus there is a sequence of polynomials P_ℓ such that

$$P_{\ell,k} \xrightarrow{\ell \rightarrow \infty} e^{2\pi i k x} \text{ uniformly on } [0, 1].$$

Note that by linearity,

$$\int f(x) x^n = 0 \quad \forall n \implies \int f(x) P_{\ell,k}(x) = 0 \quad \forall \ell \in \mathbb{N}$$

But then the k th Fourier coefficient of f is given by

$$\begin{aligned}
 \langle f, e_k \rangle &= \int_0^1 f(x) e^{-2\pi i k x} dx \\
 &= \int_0^1 f(x) \lim_{\ell \rightarrow \infty} P_\ell(x) \\
 &= \lim_{\ell \rightarrow \infty} \int_0^1 f(x) P_\ell(x) \quad \text{by uniform convergence} \\
 &= \lim_{\ell \rightarrow \infty} 0 \\
 &= 0 \quad \forall k \in \mathbb{Z},
 \end{aligned}$$

so \hat{f} is the zero function, and $\hat{f} = 0 \iff f = 0$ almost everywhere. ■

4.5 5

$$\text{Moral: } \int |f_n - f| \rightarrow 0 \iff \int f_n = \int f.$$

Since if $\int |f_n| \rightarrow \int |f|$ then we can define

$$\begin{aligned}
 h_n &= |f_n - f| && \rightarrow 0 \text{ a.e.} \\
 g_n &= |f_n| + |f| && \rightarrow 2|f| \text{ a.e.}
 \end{aligned}$$

$$\begin{aligned}
 \int 2|f| &= \int \liminf (g_n - h_n) \\
 &= \int \liminf g_n - \int \liminf h_n \\
 &= \int 2|f| - \int \liminf h_n \\
 &\stackrel{\text{Fatou}}{\leq} \int 2|f| + \limsup \int h_n,
 \end{aligned}$$

which forces $\int h_n = \int |f_n - f| \rightarrow 0$.

But then

$$\left| \int f_n - \int f \right| = \left| \int f_n - f \right| \leq \int |f_n - f| \rightarrow 0,$$

so $\int f_n \rightarrow \int f$. ■

5 Fall 2018

Note: this is considered...not the most useful or representative exam of all time.

5.1 1

We'll show a stronger statement: $f(x) = \frac{1}{x}$ is uniformly continuous on any interval of the form (c, ∞) where $c > 0$.

We can use that fact that $x, y > c \implies xy > c^2 \implies \frac{1}{xy} < \frac{1}{c^2}$.

Letting ε be arbitrary, choose $\delta < \varepsilon c^2$. Then

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{x} - \frac{1}{y} \right| \\ &= \frac{|x - y|}{xy} \\ &\leq \frac{\delta}{xy} \\ &< \frac{\delta}{c^2} \\ &< \varepsilon, \end{aligned}$$

which shows uniform continuity since δ does not depend on x or y .

To see that f is not uniformly continuous when $c = 0$, let $\varepsilon < 1$ be arbitrary.

Let $x_n = \frac{1}{n}$. Then choose n large enough such that $|x_n - x_{n+1}| = \frac{1}{n} - \frac{1}{n+1} < \delta$. Then just note that $f(x_n) = n$ and thus $|f(x_n) - f(x_{n+1})| = n - (n+1) = 1 > \varepsilon$.

5.2 2

First consider the bounded case where $m(E) < \infty$.

E is measurable \iff for every ε there exist $F_\varepsilon \subset E \subset G_\varepsilon$ with F_ε closed and G_ε open and $m(G_\varepsilon \setminus E) < \varepsilon$ and $m(E \setminus F_\varepsilon) < \varepsilon$.

So take the sequence $\varepsilon_n = \frac{1}{n} \rightarrow 0$ to produce a sequence of closed sets F_n such that $m(E \setminus F_n) < \frac{1}{n}$ for all n , and let $F := \bigcup_n F_n$, which is clearly an F_σ and thus Borel set.

Since $F_n \subseteq F_{n+1}$, we have $F_n \nearrow F$ and so by continuity of measure,

$$m(F) = \lim_n m(F_n) < \lim_n \left(\frac{1}{n} \right) \rightarrow 0.$$

If E is not bounded, let $E_N = B_N(0) \cap E$ which is bounded. Then $E_N \nearrow E$, and for each N we can find an F_N by the previous case such that $m(E_N \setminus F_N) = 0$.

So take $F := \bigcup_N F_N$ so $F_N \nearrow F$. Then

$$E_N \setminus F_N \nearrow E \setminus F \implies m(E \setminus F) = \lim_N m(E_N \setminus F_N) = 0.$$

5.3 3

$$\begin{aligned} \frac{\partial}{\partial t} F(t) &= \frac{\partial}{\partial t} \int_{\mathbb{R}} f(x) \cos(xt) \, dx \\ &\stackrel{DCT}{=} \int_{\mathbb{R}} f(x) \frac{\partial}{\partial t} \cos(xt) \, dx \\ &= \int_{\mathbb{R}} x f(x) \cos(xt) \, dx, \end{aligned}$$

so it only remains to justify the DCT.

Fix t , then let $t_n \rightarrow t$ be any sequence. Then

$$\begin{aligned} \frac{\partial}{\partial t} \cos(tx) &:= \lim_{t_n \rightarrow t} \frac{\cos(tx) - \cos(t_n x)}{t_n - t} \\ &\stackrel{MVT}{=} \frac{\partial}{\partial t} \cos(tx) \Big|_{t=\xi_n} \quad \text{for some } \xi_n \in [t, t_n] \text{ or } [t_n, t] \\ &= x \sin(\xi_n x). \end{aligned}$$

So we can define

$$h_n(x, t) = f(x) \left(\frac{\cos(tx) - \cos(t_n x)}{t_n - t} \right)$$

and note that $h_n \rightarrow \frac{\partial}{\partial t} [f(x) \cos(xt)]$ pointwise.

We then have $|h_n| = |f(x)x \sin(\xi_n x)| \leq |xf(x)|$ for every n by the above argument, and since $g(x) := xf(x) \in L^1(\mathbb{R})$ by assumption, the DCT can be applied.

5.4 4

???

Apparently “easy” part: let $f(x) = \chi_{[0, \pi]}$, then $\int_{\mathbb{R}} f(x) |\sin(nx)| \, dx = \int_0^\pi |\sin(nx)| \, dx = 2$, and so $\int_0^1 |\sin(nx)| \, dx = \frac{2}{\pi}$, none of which depend on n .

Now approximate f by step functions.

5.5 5

???

6 Spring 2017

6.1 1

A is nowhere dense \iff every interval I contains a subinterval $S \subseteq A^c$.

K is compact:

It suffices to show that $K^c := [0, 1] \setminus K$ is open; then K will be a closed and bounded subset of \mathbb{R} and thus compact by Heine-Borel.

We can identify K^c as the set of real numbers in $[0, 1]$ whose decimal expansion **does** use a 4. Let $x \in K^c$, and suppose a 4 occurs as the k th digit and write

$$x = 0.d_1d_2\cdots d_{k-1}4d_{k+1}\cdots = \sum_{j=1}^k d_j 10^{-j} + 4 \cdot 10^{-k} + \sum_{j=k+1}^{\infty} d_j 10^{-j}.$$

Then if we set $r < 10^{-k}$ and pick any $y \in [0, 1]$ such that $y \in B_r(x)$, then $|x - y| < 10^{-k}$. If we write $y = \sum_{j=1}^{\infty} c_j 10^{-j}$, this means that for all $j \leq k$ we have $d_j = c_j$, and in particular $d_k = 4 = c_k$, so y has a 4 in its decimal expansion.

But then $K^c = \bigcup_x B_r(x)$ is a union of open sets and thus open.

K is nowhere dense and $m(K) = 0$:

Since K is closed, we'll show that K can not properly contain any interval, so $(\overline{K})^\circ = \emptyset$.

As in the construction of the Cantor set, let

- K_1 denote $[0, 1]$ with 1 interval $[0.4, 0.5]$ of length $\frac{1}{10}$ deleted
- K_2 denote K_1 with 9 intervals $[0.04, 0.05], [0.14, 0.15], \dots [0.94, 0.95]$ length $\frac{1}{100}$ deleted
- K_n denote K_{n-1} with 9^{n-1} such intervals of length 10^{-n} deleted.

Then $K = \bigcap K_n$, and

$$m(K) = 1 - m(K^c) = 1 - \sum_{j=0}^{\infty} \frac{9^j}{10^{j+1}} = 1 - \frac{1}{10} \left(\frac{1}{1 - \frac{9}{10}} \right) = 0,$$

and since any interval has strictly positive measure, K can not contain any interval.

K has no isolated points:

A point $x \in K$ is isolated iff there is an open ball $B_r(x)$ containing x such that $B_r(x) \cap K = \{x\}$, so every point in this ball has a 4 in its decimal expansion.

Note that $m(K_n) = \left(\frac{9}{10}\right)^n \rightarrow 0$ and that the endpoints of intervals are never removed and are thus elements of K . Then for every ε , we can choose n such that $\left(\frac{9}{10}\right)^n < \varepsilon$; then there is an endpoint of a removed interval e_n satisfying $|x - e_n| \leq \left(\frac{9}{10}\right)^n < \varepsilon$.

So every ball containing x contains some endpoint of a removed interval, and thus an element of K . ■

6.2 2

$$\lambda \ll \mu \iff E \in \mathcal{M}, \mu(E) = 0 \implies \lambda(E) = 0.$$

6.2.1 a

By Radon-Nikodym, if $\lambda \ll \mu$ then $d\lambda = f d\mu$, which would yield

$$\int g d\lambda = \int gf d\mu.$$

So let E be measurable and suppose $\mu(E) = 0$. Then

$$\lambda(E) := \int_E f d\mu = \lim_n \left\{ \varphi_n := \sum_j c_j \mu(E_j) \right\},$$

where we take a sequence of simple functions increasing to f .

But since each $E_j \subseteq E$, we must have $\mu(E_j) = 0$ for any such E_j , so every such φ_n must be zero and thus $\lambda(E) = 0$.

6.2.2 b

By Radon-Nikodym, there exists a positive f such that

$$\int g dm = \int gf d\mu,$$

where we can take $g(x) = x^2$, then the LHS is zero by assumption and thus so is the RHS.

Note that gf is positive.

Define $A_k = \left\{ x \in X \mid gf\chi_E > \frac{1}{k} \right\}$, then by Chebyshev

$$\mu(A_k) \leq k \int_E gf d\mu = 0,$$

which holds for every k .

Then noting that $A_k \searrow A := \left\{ x \in E \mid x^2 > 0 \right\}$, and gf is positive, we have

$$x \in E \iff gf\chi_E(x) > 0 \iff x \in A,$$

so $E = A$ and $\mu(E) = \mu(A)$.

But since $m \ll \mu$ by construction, we can conclude that $m(E) = 0$. ■

6.3 3**6.3.1 a**

Letting $x_n := \frac{1}{n}$, we have

$$\sum_{k=1}^{\infty} |f_k(x)| \geq |f_n(x_n)| = |ae^{-ax} - be^{-bx}| := M.$$

In particular, $\sup_x |f_n(x)| \not\rightarrow 0$, so the terms do not go to zero and the sum can not converge.

6.3.2 b

?

6.4 4

Switching to polar coordinates and integrating over a half-circle contained in I^2 , we have

$$\int_{I^2} f \geq \int_0^\pi \int_0^1 \frac{\cos(\theta) \sin(\theta)}{r^2} dr d\theta = \infty,$$

so f is not integrable.

6.5 5

See <https://math.stackexchange.com/questions/507263/prove-that-c1a-b-with-the-c1-norm-is-a-banach-space>

This is clearly a norm, which we'll write $\|\cdot\|_u$

Let f_n be a Cauchy sequence and define a candidate limit $f(x) = \lim_n f_n(x)$.

Then noting that $\|f_n\|_\infty, \|f'_n\|_\infty \leq \|f_n\|_u < \infty$, both f_n, f'_n are Cauchy sequences in $C^0([a, b], \|\cdot\|_\infty)$, which is a Banach space.

So $f_n \rightarrow f$ uniformly, and $f'_n \rightarrow g$ uniformly for some g , and moreover $f, g \in C^0([a, b])$.

We thus have

$$\begin{aligned} f_n(x) - f_n(a) &\xrightarrow{u} f(x) - f(a) \\ \int_a^x f'_n &\xrightarrow{u} \int_a^x g, \end{aligned}$$

and by the FTC, the left-hand sides are equal, and by uniqueness of limits so are the right-hand sides, so $f' = g$.

Since $f, f' \in C^0([a, b])$, they are bounded, and so $\|f\|_u < \infty$. This means that $\|f_n - f\|_u \rightarrow 0$, so f_n converges to f , which is in the same space.

■

7 Fall 2017

7.1 1

Note that $f(x) = e^x$ is entire and thus equal to its power series. So $f(x) = \sum_{j=0}^{\infty} \frac{1}{j!} x^j$.

Letting $f_N(x) = \sum_{j=1}^N \frac{1}{j!} x^j$, we have $f_N(x) \rightarrow f(x)$ pointwise on $(-\infty, \infty)$.

For any compact interval $[-M, M]$, we have

$$\begin{aligned} \|f_N(x) - f(x)\|_{\infty} &= \sup_{-M \leq x \leq M} \left| \sum_{j=N+1}^{\infty} \frac{1}{j!} x^j \right| \\ &\leq \sup_{-M \leq x \leq M} \sum_{j=N+1}^{\infty} \frac{1}{j!} |x|^j \\ &\leq \sum_{j=N+1}^{\infty} \frac{1}{j!} M^j \\ &\leq \sum_{j=0}^{\infty} \frac{1}{j!} M^j \\ &= e^M \\ &< \infty, \end{aligned}$$

so $f_N \rightarrow f$ uniformly on $[-M, M]$ by the M-test. Thus it converges on any bounded interval.

It does not converge on \mathbb{R} , since x^N is unbounded.

7.2 2

7.2.1 a

It suffices to consider the bounded case, i.e. $E \subseteq B_M(0)$ for some M . Then write $E_n = B_n(0) \cap E$ and apply the theorem to E_n , and by subadditivity, $m^*(E) = m^*\left(\bigcup_n E_n\right) \leq \sum_n m^*(E_n) = 0$.

Lemma: $f(x) = x^2, f^{-1}(x) = \sqrt{x}$ are Lipschitz on any compact subset of $[0, \infty)$.

Proof: Let $g = f$ or f^{-1} . Then $g \in C^1([0, M])$ for any M , so g is differentiable and g' is continuous. Since g' is continuous on a compact interval, it is bounded, so $|g'(x)| \leq L$ for all x . Applying the MVT,

$$|f(x) - f(y)| = |f'(c)| |x - y| \leq L |x - y|.$$

Lemma: If g is Lipschitz on \mathbb{R}^n , then $m(E) = 0 \implies m(g(E)) = 0$.

Proof: If g is Lipschitz, then

$$g(B_r(x)) \subseteq B_{Lr}(g(x)),$$

which is a dilated ball/cube, and so

$$m^*(B_{Lr}(x)) \leq L^n \cdot m^*(B_r(x)).$$

Now choose $\{Q_j\} \rightrightarrows E$; then $\{g(Q_j)\} \rightrightarrows g(E)$.

By the above observation,

$$|g(Q_j)| \leq L^n |Q_j|,$$

and so

$$m^*(g(E)) \leq \sum_j |g(Q_j)| \leq \sum_j L^n |Q_j| = L^n \sum_j |Q_j| \rightarrow 0.$$

Now just take $g(x) = x^2$ for one direction, and $g(x) = f^{-1}(x) = \sqrt{x}$ for the other. ■

7.2.2 b

Lemma: E is measurable iff $E = K \coprod N$ for some K compact, N null.

Write $E = K \coprod N$ where K is compact and N is null.

Then $\varphi^{-1}(E) = \varphi^{-1}(K \coprod N) = \varphi^{-1}(K) \coprod \varphi^{-1}(N)$.

Since $\varphi^{-1}(N)$ is null by part (a) and $\varphi^{-1}(K)$ is the preimage of a compact set under a continuous map and thus compact, $\varphi^{-1}(E) = K' \coprod N'$ where K' is compact and N' is null, so $\varphi^{-1}(E)$ is measurable.

So φ is a measurable function, and thus yields a well-defined map $\mathcal{L}(\mathbb{R}) \rightarrow \mathcal{L}(\mathbb{R})$ since it preserves measurable sets. Restricting to $[0, \infty)$, f is bijection, and thus so is φ . ■

7.3 3

From homework: E is Lebesgue measurable iff there exists a finite union of closed cubes A such that $m(E \Delta A) < \varepsilon$.

It suffices to show that S is dense in simple functions, and since simple functions are *finite* linear combinations of characteristic functions, it suffices to show this for χ_A for A a measurable set.

Let $s = \chi_A$. By regularity of the Lebesgue measure, choose an open set $O \supseteq A$ such that $m(O \setminus A) < \varepsilon$.

O is an open subset of \mathbb{R} , and thus $O = \coprod_{j \in \mathbb{N}} I_j$ is a disjoint union of countably many open intervals.

Now choose N large enough such that $m(O \Delta I_{N,n}) < \varepsilon = \frac{1}{n}$ where we define $I_{N,n} := \coprod_{j=1}^N I_j$.

Now define $f_n = \chi_{I_{N,n}}$, then

$$\|s - f_n\|_1 = \int |\chi_A - \chi_{I_{N,n}}| = m(A \Delta I_{N,n}) \xrightarrow{n \rightarrow \infty} 0.$$

Since any simple function is a finite linear combination of χ_{A_i} , we can do this for each i to extend this result to all simple functions. But simple functions are dense in L^1 , so S is dense in L^1 .

7.4 4

7.4.1 a

Let $G(x) = \sum_{n=1}^{\infty} nx(1-x)^n$. Applying the ratio test, we have

$$\left| \frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n} \right| = \frac{n+1}{n} |1-x| \xrightarrow{n \rightarrow \infty} |1-x| < 1 \iff 0 \leq x \leq 2,$$

and in particular, this series converges on $[0, 2]$. Thus its terms go to zero, and $nx(1-x)^n \rightarrow 0$ on $[0, 1] \subset [0, 2]$.

To see that the convergence is not uniform, let $x_n = \frac{1}{n}$ and $\varepsilon > \frac{1}{e}$, then

$$\sup_{x \in [0, 1]} |nx(1-x)^n - 0| \geq |nx_n(1-x_n)^n| = \left| \left(1 - \frac{1}{n}\right)^n \right| \xrightarrow{n \rightarrow \infty} e^{-1} > \varepsilon.$$

7.4.2 b

Note: could use the first part with $\sin(x) \leq x$, but then integral ends up more complicated.

Noting that $\sin(x) \leq 1$, we have We have

$$\begin{aligned} \left| \int_0^1 n(1-x)^n \sin(x) \right| &\leq \int_0^1 |n(1-x)^n \sin(x)| \\ &\leq \int_0^1 |n(1-x)^n| \\ &= n \int_0^1 (1-x)^n \\ &= -\frac{n(1-x)^{n+1}}{n+1} \\ &\xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

7.5 5

7.5.1 a

Lemma: If $\varphi \in C_c^1$, then $(f * \varphi)' = f * \varphi'$ almost everywhere.

Silly Proof:

$$\begin{aligned}
\mathcal{F}((f * \varphi)') &= 2\pi i \xi \mathcal{F}(f * \varphi) \\
&= 2\pi i \xi \mathcal{F}(f) \mathcal{F}(\varphi) \\
&= \mathcal{F}(f) \cdot (2\pi i \xi \mathcal{F}(\varphi)) \\
&= \mathcal{F}(f) \cdot \mathcal{F}(\varphi') \\
&= \mathcal{F}(f * \varphi').
\end{aligned}$$

Actual proof:

$$\begin{aligned}
(f * \varphi)'(x) &= (\varphi * f)'(x) \\
&= \lim_{h \rightarrow 0} \frac{(\varphi * f)'(x+h) - (\varphi * f)'(x)}{h} \\
&= \lim_{h \rightarrow 0} \int \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \\
&\stackrel{DCT}{=} \int \lim_{h \rightarrow 0} \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \\
&= \int \varphi'(x-y) f(y) \\
&= (\varphi' * f)(x) \\
&= (f * \varphi')(x).
\end{aligned}$$

To see that the DCT is justified, we can apply the MVT on the interval $[0, h]$ to f to obtain

$$\frac{\varphi(x+h-y) - \varphi(x-y)}{h} = \varphi'(c) \quad c \in [0, h],$$

and since φ' is continuous and compactly supported, φ' is bounded by some $M < \infty$ by the extreme value theorem and thus

$$\begin{aligned}
\int \left| \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \right| &= \int |\varphi'(c) f(y)| \\
&\leq \int |M| |f| \\
&= |M| \int |f| < \infty,
\end{aligned}$$

since $f \in L^1$ by assumption, so we can take $g := |M||f|$ as the dominating function.

Applying this theorem infinitely many times shows that $f * \varphi$ is smooth.

To see that $f * \varphi$ is compactly supported, approximate f by a *continuous* compactly supported function h , so $\|h - f\|_1 \xrightarrow{L^1} 0$.

Now let $g_x(y) = \varphi(x - y)$, and note that $\text{supp}(g) = x - \text{supp}(\varphi)$ which is still compact.

But since $\text{supp}(h)$ is bounded, there is some N such that

$$|x| > N \implies A_x := \text{supp}(h) \cap \text{supp}(g_x) = \emptyset$$

and thus

$$\begin{aligned} (h * \varphi)(x) &= \int_{\mathbb{R}} \varphi(x-y)h(y) dy \\ &= \int_{A_x} g_x(y)h(y) dy \\ &= 0, \end{aligned}$$

so $\{x \mid f * g(x) = 0\}$ is open, and its complement is closed and bounded and thus compact.

7.5.2 b

$$\begin{aligned} \|f * K_j - f\|_1 &= \int \left| \int f(x-y)K_j(y) dy - f(x) \right| dx \\ &= \int \left| \int f(x-y)K_j(y) dy - \int f(x)K_j(y) dy \right| dx \\ &= \int \left| \int (f(x-y) - f(x))K_j(y) dy \right| dx \\ &\leq \int \int |(f(x-y) - f(x))| \cdot |K_j(y)| dy dx \\ &\stackrel{FT}{=} \int \int |(f(x-y) - f(x))| \cdot |K_j(y)| \mathbf{dx} \mathbf{dy} \\ &= \int |K_j(y)| \left(\int |(f(x-y) - f(x))| dx \right) dy \\ &= \int |K_j(y)| \cdot \|f - \tau_y f\|_1 dy. \end{aligned}$$

We now split the integral up into pieces.

1. Chose δ small enough such that $|y| < \delta \implies \|f - \tau_y f\|_1 < \varepsilon$ by continuity of translation in L^1 , and
2. Since φ is compactly supported, choose J large enough such that

$$j > J \implies \int_{|y| \geq \delta} |K_j(y)| dy = \int_{|y| \geq \delta} |j\varphi(jy)| = 0$$

Then

$$\begin{aligned} \|f * K_j - f\|_1 &\leq \int |K_j(y)| \cdot \|f - \tau_y f\|_1 dy \\ &= \int_{|y| < \delta} |K_j(y)| \cdot \|f - \tau_y f\|_1 dy + \int_{|y| \geq \delta} |K_j(y)| \cdot \|f - \tau_y f\|_1 dy \\ &= \varepsilon \int_{|y| < \delta} |K_j(y)| + 0 \\ &\leq \varepsilon(1) \longrightarrow 0. \end{aligned}$$

■

7.6 6

Should be supremum maybe..?

Let $\{f_k\}$ be a Cauchy sequence, so $\|f_k\| < \infty$ for all k . Then for a fixed x , the sequence $f_k(x)$ is Cauchy in \mathbb{R} and thus converges to some $f(x)$, so define f by $f(x) := \lim_{k \rightarrow \infty} f_k(x)$.

Then $\|f_k - f\| = \max_{x \in X} |f_k(x) - f(x)| \xrightarrow{k \rightarrow \infty} 0$, and thus $f_k \rightarrow f$ uniformly and thus f is continuous. It just remains to show that f has bounded norm.

Choose N large enough so that $\|f - f_N\| < \varepsilon$, and write $\|f_N\| := M < \infty$

$$\|f\| \leq \|f - f_N\| + \|f_N\| < \varepsilon + M < \infty.$$

8 Spring 2016**8.1 1****9 Fall 2016****9.1 1****10 Spring 2014****10.1 1**