UGA Real Analysis Qualifying Exam Questions and Solutions

D. Zack Garza

Sunday 16th August, 2020

Contents

1 Undergraduate Analysis: Uniform Convergence **6**
- 1.1 Fall 2018 # 1 ✨ . . . 6
- 1.2 Fall 2017 # 1 ✨ . . . 7
- 1.3 Fall 2014 # 1 🚩 . . . 8
- 1.4 Spring 2017 # 4 ✨ . . . 8
- 1.5 Spring 2015 # 1 ✨ . . . 9
- 1.6 Fall 2014 # 2 🚩 . . . 9
- 1.7 Spring 2014 # 2 🚩 . . . 9

2 General Analysis **9**
- 2.1 Spring 2020 # 1 ✨ . . . 9
- 2.2 Fall 2019 # 1 ✨ . . . 11
  - 2.2.1 a . . . 11
  - 2.2.2 b . . . 11
  - 2.2.3 a . . . 12
  - 2.2.4 b . . . 12
- 2.3 Fall 2018 # 4 ✨ . . . 13
- 2.4 Fall 2017 # 4 ✨ . . . 15
  - 2.4.1 a . . . 15
  - 2.4.2 b . . . 15
- 2.5 Spring 2017 # 3 ✨ . . . 16
  - 2.5.1 a . . . 16
  - 2.5.2 b . . . 17
- 2.6 Fall 2016 # 1 ✨ . . . 17
- 2.7 Fall 2016 # 5 ✨ . . . 18
- 2.8 Fall 2016 # 6 ✨ . . . 20
- 2.9 Spring 2016 # 1 🚩 . . . 20
- 2.10 Fall 2015 # 1 🚩 . . . 21

3 Measure Theory: Sets **21**
- 3.1 Spring 2020 # 2 ✨ . . . 21
  - 3.1.1 a. . . . 21
  - 3.1.2 b. . . . 21

3.1.3 a . . . 22
3.1.4 b . . . 23
3.2 Fall 2019 # 3. ✨ . . . 23
3.2.1 a . . . 24
3.2.2 b . . . 24
3.2.3 c . . . 25
3.3 Spring 2019 # 2 ✨ . . . 25
3.3.1 a . . . 25
3.3.2 b . . . 25
3.3.3 a . . . 26
3.3.4 b . . . 26
3.4 Fall 2018 # 2 ✨ . . . 27
3.4.1 Indirect Proof . . . 29
3.4.2 Direct Proof (Todo) . . . 29
3.5 Spring 2018 # 1 ✨ . . . 29
3.6 Fall 2017 # 2 ✨ . . . 31
3.6.1 a . . . 31
3.6.2 b . . . 32
3.7 Spring 2017 # 2 ✨ . . . 32
3.7.1 a . . . 32
3.7.2 b . . . 32
3.7.3 a . . . 33
3.7.4 b . . . 33
3.8 Fall 2016 # 4 ✨ . . . 34
3.9 Spring 2016 # 3 🚩 . . . 35
3.10 Spring 2016 # 5 🚩 . . . 35
3.11 Fall 2015 # 2 🚩 . . . 35
3.12 Spring 2015 # 3 🚩 . . . 36
3.13 Spring 2014 # 3 🚩 . . . 36
3.14 Spring 2014 # 4 🚩 . . . 36
3.15 Spring 2017 # 1 ✨ . . . 36
3.16 Spring 2016 # 2 🚩 . . . 38

4 Measure Theory: Functions **39**
4.1 Fall 2016 # 2 ✨ . . . 39
4.2 Spring 2016 # 4 🚩 . . . 39

5 Integrals: Convergence **40**
5.1 Fall 2019 # 2 ✨ . . . 40
5.2 Spring 2020 # 5 ✨ . . . 42
5.3 Spring 2019 # 3 ✨ . . . 43
5.4 Fall 2018 # 6 ✨ . . . 45
5.5 Fall 2018 # 3 ✨ . . . 45
5.6 Spring 2018 # 5 ✨ . . . 46
5.7 Spring 2018 # 2 ✨ . . . 48
5.7.1 a . . . 48
5.7.2 b . . . 49
5.8 Fall 2016 # 3 ✨ . . . 49

5.9 Fall 2015 # 3 . . . 51
5.10 Fall 2015 # 4 . . . 51

6 Integrals: Approximation **51**
6.1 Spring 2018 # 3 . . . 51
6.2 Spring 2018 # 4 . . . 52
6.3 Spring 2015 # 2 . . . 53
6.3.1 Proof 1: Using Fourier Transforms . . . 53
6.3.2 Alternative Proof . . . 54
6.4 Fall 2014 # 4 . . . 55

7 L^1 **55**
7.1 Spring 2020 # 3 . . . 55
7.1.1 a . . . 55
7.1.2 b . . . 56
7.1.3 c . . . 59
7.2 Fall 2019 # 5. . . . 59
7.2.1 a . . . 59
7.2.2 b . . . 59
7.2.3 a . . . 59
7.2.4 b . . . 60
7.3 Fall 2017 # 3 . . . 61
7.4 Spring 2015 # 4 . . . 61
7.5 Fall 2014 # 3 . . . 62
7.6 Spring 2014 # 1 . . . 62

8 Fubini-Tonelli **62**
8.1 Spring 2020 # 4 . . . 62
8.2 Spring 2019 # 4 . . . 63
8.2.1 a . . . 64
8.2.2 b . . . 64
8.3 Fall 2018 # 5 . . . 65
8.4 Fall 2015 # 5 . . . 66
8.5 Spring 2014 # 5 . . . 67

9 L^2 and Fourier Analysis **67**
9.1 Spring 2020 # 6 . . . 67
9.1.1 a . . . 67
9.1.2 b . . . 67
9.1.3 a . . . 68
9.2 Fall 2017 # 5 . . . 69
9.2.1 a . . . 70
9.2.2 b . . . 71
9.3 Spring 2017 # 5 . . . 72
9.4 Spring 2015 # 6 . . . 72
9.5 Fall 2014 # 5 . . . 72

10 Functional Analysis: General **72**
10.1 Fall 2019 # 4 ✨ . . . 72
10.1.1 a . . . 72
10.1.2 b . . . 72
10.1.3 a . . . 73
10.1.4 b . . . 73
10.2 Spring 2019 # 5 ✨ . . . 74
10.2.1 a . . . 74
10.2.2 b . . . 74
10.2.3 a . . . 75
10.2.4 b . . . 75
10.3 Spring 2016 # 6 🚩 . . . 77
10.4 Spring 2015 # 5 🚩 . . . 77
10.5 Fall 2015 # 6 🚩 . . . 77
10.6 Fall 2014 # 6 🚩 . . . 77

11 Functional Analysis: Banach Spaces **78**
11.1 Spring 2019 # 1 ✨ . . . 78
11.1.1 a . . . 78
11.1.2 b . . . 79
11.2 Spring 2017 # 6 ✨ . . . 79
11.3 Fall 2017 # 6 ✨ . . . 80

Todo list

Walk through. . . . 7
Add concepts. . . . 7
Redo, may just be wrong. . . . 8
Add concepts. . . . 8
Add concepts. . . . 13
Need to justify removing floor function and cancellation. . . . 14
No clue how to show this. . . . 14
Not sure. Approximate f by simple functions...? . . . 14
Add concepts. . . . 15
Walk through. . . . 15
Not complete. . . . 16
Add concepts. . . . 16
Walk through. . . . 16
Add concepts. . . . 17
? . . . 18
Walk through. . . . 18
Add concepts. . . . 18
Actually true? Need conditions? . . . 19
Add concepts. . . . 20
How to prove small tails in L^p? . . . 20
How to prove the hint..? . . . 25
Add concepts. . . . 25
All messed up! . . . 27
Move this to review notes to clean things up. . . . 27
Try to construct the set. . . . 29
Walk through. . . . 31
What is the final step in this approximation? . . . 33
Add concepts. . . . 34
Last step seems wrong! . . . 35
Add concepts. . . . 39
Not finished, flesh out. . . . 42
Walk through. . . . 42
What is the "right" proof here that uses the first part? . . . 44
Add concepts. . . . 45
Walk through. . . . 46
Add concepts. . . . 48
Missing some stuff. . . . 49
Add concepts. . . . 53
Walk through. . . . 59
Walk through. . . . 61
Add concepts. . . . 64
Why is FT justified. . . . 65
Add concepts. . . . 69
What theorem is this using? . . . 75
Is this assumed..? Or did we show it..? . . . 76
Add concepts. . . . 78

show that $\int_0^1 |f_n(x) - f_m(x)|\,dx \longrightarrow 0$ rigorously, show that no $g \in L^1([0,1])$ can converge to this indicator function. 79
Add concepts. 80
Add concepts. 80
Shouldn't this be a supremum? The max may not exist? 81
Review and clean up. 81

Preface

I'd like to extend my gratitude to Peter Woolfitt for supplying many solutions and checking many proofs of the rest in problem sessions.

1 Undergraduate Analysis: Uniform Convergence

1.1 Fall 2018 # 1 ✨

Let $f(x) = \dfrac{1}{x}$. Show that f is uniformly continuous on $(1, \infty)$ but not on $(0, \infty)$.

Solution:

Concepts Used:

- Uniform continuity:

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \quad \text{such that} \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

- Negating uniform continuity: $\exists \varepsilon > 0$ such that $\forall \delta(\varepsilon)$ there exist x, y such that $|x - y| < \delta$ *and* $|f(x) - f(y)| > \varepsilon$.

Claim: $f(x) = \dfrac{1}{x}$ is uniformly continuous on (c, ∞) for any $c > 0$.

- Note that

$$|x|, |y| > c > 0 \implies |xy| = |x||y| > c^2 \implies \frac{1}{|xy|} < \frac{1}{c^2}.$$

- Letting ε be arbitrary, choose $\delta < \varepsilon c^2$.
  - Note that δ does not depend on x, y.
- Then

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{x} - \frac{1}{y} \right| \\ &= \frac{|x - y|}{xy} \\ &\leq \frac{\delta}{xy} \\ &< \frac{\delta}{c^2} \\ &< \varepsilon, \end{aligned}$$

which shows uniform continuity.

Claim: f is *not* uniformly continuous when $c = 0$.

- Toward a contradiction, let $\varepsilon < 1$.
- Let $x_n = \dfrac{1}{n}$ for $n \geq 1$.
- Choose n large enough such that $|x_n - x_{n+1}| = \dfrac{1}{n} - \dfrac{1}{n+1} < \delta$.
  - Why this can be done: by the archimedean property of $\mathbb{R}$, choose n such that $\dfrac{1}{n} < \varepsilon$.
  - Then

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \leq \frac{1}{n} < \varepsilon \quad \text{since } n+1 > 1.$$

- Note $f(x_n) = n$ and thus

$$|f(x_{n+1}) - f(x_n)| = (n+1) - n = 1 > \varepsilon,$$

  a contradiction.

1.2 Fall 2017 # 1 ✨

Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which f does and does not converge uniformly.

Walk through.

Add concepts.

Solution:

Concepts Used:

- $f_N \longrightarrow f$ uniformly $\iff \|f_N - f\| \longrightarrow 0$.
- $\displaystyle\sum_{n=0}^{\infty} c_n x^n \coloneqq \lim_{N \longrightarrow \infty} \sum_{n=0}^{N} c_n x^n$, i.e. an infinite sum is defined pointwise as the limit of its partial sums.

- Set $f_N(x) = \displaystyle\sum_{n=1}^{N} \frac{x^n}{n!}$.
  - Then by definition, $f_N(x) \longrightarrow f(x)$ pointwise on $\mathbb{R}$.

- For any compact interval $[-M, M]$, we have

$$\begin{aligned}
\|f_N(x) - f(x)\|_\infty &= \sup_{-M\le x\le M} \left| \sum_{n=N+1}^{\infty} \frac{x^n}{n!} \right| \\
&\le \sup_{-M\le x\le M} \sum_{n=N+1}^{\infty} \left| \frac{x^n}{n!} \right| \\
&\le \sum_{n=N+1}^{\infty} \frac{M^n}{n!} \\
&\le \sum_{n=0}^{\infty} \frac{M^n}{n!} \\
&= e^M \\
&< \infty,
\end{aligned}$$

  so $f_N \longrightarrow f$ uniformly on $[-M, M]$ by the M-test.
- Thus f converges on any bounded interval.

It does not converge on $\mathbb{R}$, since x^N is unbounded.

1.3 Fall 2014 # 1

Let $\{f_n\}$ be a sequence of continuous functions such that $\sum f_n$ converges uniformly.

Prove that $\sum f_n$ is also continuous.

1.4 Spring 2017 # 4 ✨

Let $f(x, y)$ on $[-1, 1]^2$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Determine if f is integrable.

Redo, may just be wrong.

Add concepts.

Solution:

Concepts Used:

- ?

Switching to polar coordinates and integrating over a half-circle contained in I^2, we have

$$\int_{I^2} f \ge \int_0^\pi \int_0^1 \frac{\cos(\theta)\sin(\theta)}{r^2}\, dr\, d\theta = \infty,$$

so f is not integrable.

1.5 Spring 2015 # 1 ✨

Let (X, d) and (Y, ρ) be metric spaces, $f : X \longrightarrow Y$, and $x_0 \in X$.

Prove that the following statements are equivalent:

1. For every $\varepsilon > 0 \quad \exists \delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$.
2. The sequence $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow f(x_0)$ for every sequence $\{x_n\} \longrightarrow x_0$ in X.

1.6 Fall 2014 # 2 🚩

Let I be an index set and $\alpha : I \longrightarrow (0, \infty)$.

1. Show that

$$\sum_{i \in I} a(i) := \sup_{\substack{J \subset I \\ J \text{ finite}}} \sum_{i \in J} a(i) < \infty \implies I \text{ is countable.}$$

2. Suppose $I = \mathbb{Q}$ and $\sum_{q \in \mathbb{Q}} a(q) < \infty$. Define

$$f(x) := \sum_{\substack{q \in \mathbb{Q} \\ q \leq x}} a(q).$$

   Show that f is continuous at $x \iff x \notin \mathbb{Q}$.

1.7 Spring 2014 # 2 🚩

Let $\{a_n\}$ be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that $\sum a_n^2 < \infty$.

Note: Assume a_n, b_n are all non-negative.

2 General Analysis

2.1 Spring 2020 # 1 ✨

Prove that if $f : [0, 1] \longrightarrow \mathbb{R}$ is continuous then

$$\lim_{k \longrightarrow \infty} \int_0^1 k x^{k-1} f(x)\, dx = f(1).$$

Solution:

Concepts Used:

- DCT
- Weierstrass Approximation Theorem

- Suppose p is a polynomial, then

$$\begin{aligned}\lim_{k\longrightarrow\infty}\int_0^1 kx^{k-1}p(x)\,dx &= \lim_{k\longrightarrow\infty}\int_0^1 \left(\frac{\partial}{\partial x}\,x^k\right)p(x)\,dx \\ &= \lim_{k\longrightarrow\infty}\left[x^k p(x)\Big|_0^1 - \int_0^1 x^k\left(\frac{\partial}{\partial x}\,p(x)\right)dx\right] \quad \text{integrating by parts} \\ &= p(1) - \lim_{k\longrightarrow\infty}\int_0^1 x^k\left(\frac{\partial}{\partial x}\,p(x)\right)dx,\end{aligned}$$

- Thus it suffices to show that

$$\lim_{k\longrightarrow\infty}\int_0^1 x^k\left(\frac{\partial}{\partial x}\,p(x)\right)dx = 0.$$

- Integrating by parts a second time yields

$$\begin{aligned}\lim_{k\longrightarrow\infty}\int_0^1 x^k\left(\frac{\partial}{\partial x}\,p(x)\right)dx &= \lim_{k\longrightarrow\infty}\frac{x^{k+1}}{k+1}p'(x)\Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1}\left(\frac{\partial^2}{\partial x^2}\,p(x)\right)dx \\ &= -\lim_{k\longrightarrow\infty}\int_0^1 \frac{x^{k+1}}{k+1}\left(\frac{\partial^2}{\partial x^2}\,p(x)\right)dx \\ &= -\int_0^1 \lim_{k\longrightarrow\infty}\frac{x^{k+1}}{k+1}\left(\frac{\partial^2}{\partial x^2}\,p(x)\right)dx \quad \text{by DCT} \\ &= -\int_0^1 0\left(\frac{\partial^2}{\partial x^2}\,p(x)\right)dx \\ &= 0.\end{aligned}$$

  - The DCT can be applied here because f'' is continuous and $[0,1]$ is compact, so f'' is bounded on $[0,1]$ by a constant M and

$$\int_0^1 \left|x^k f''(x)\right| \leq \int_0^1 1\cdot M = M < \infty.$$

- Now use the Weierstrass approximation theorem:
  - If $f:[a,b]\longrightarrow\mathbb{R}$ is continuous, then for every $\varepsilon>0$ there exists a polynomial $p_\varepsilon(x)$ such that $\|f-p_\varepsilon\|_\infty<\varepsilon$.

- Thus

$$\begin{aligned}\left|\int_0^1 kx^{k-1}p_\varepsilon(x)\,dx - \int_0^1 kx^{k-1}f(x)\,dx\right| &= \left|\int_0^1 kx^{k-1}(p_\varepsilon(x)-f(x))\,dx\right| \\ &\leq \left|\int_0^1 kx^{k-1}\|p_\varepsilon - f\|_\infty\,dx\right| \\ &= \|p_\varepsilon - f\|_\infty \cdot \left|\int_0^1 kx^{k-1}\,dx\right| \\ &= \|p_\varepsilon - f\|_\infty \cdot x^k\Big|_0^1 \\ &= \|p_\varepsilon - f\|_\infty \overset{\varepsilon\longrightarrow 0}{\longrightarrow} 0\end{aligned}$$

  and the integrals are equal.
- By the first argument,

$$\int_0^1 kx^{k-1}p_\varepsilon(x)\,dx = p_\varepsilon(1) \text{ for each } \varepsilon$$

- Since uniform convergence implies pointwise convergence, $p_\varepsilon(1) \overset{\varepsilon\longrightarrow 0}{\longrightarrow} f(1)$.

2.2 Fall 2019 # 1 ✨

Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers.

2.2.1 a

Prove that if $\lim_{n\longrightarrow\infty} a_n = 0$, then

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

2.2.2 b

Prove that if $\sum_{n=1}^\infty \frac{a_n}{n}$ converges, then

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

Solution:

Concepts Used:

- Cesaro mean/summation.
- Break series apart into pieces that can be handled separately.

2.2.3 a

Prove a stronger result:

$$a_k \longrightarrow S \implies S_N \coloneqq \frac{1}{N}\sum_{k=1}^{N} a_k \longrightarrow S.$$

Idea: once N is large enough, $a_k \approx S$, and all smaller terms will die off as $N \longrightarrow \infty$. See this MSE answer.

- Use convergence $a_k \longrightarrow S$: choose M large enough such that

$$k \geq M+1 \implies |a_k - S| < \varepsilon.$$

Then

$$\begin{aligned}
\left|\left(\frac{1}{N}\sum_{k=1}^{N} a_k\right) - S\right| &= \frac{1}{N}\left|\left(\sum_{k=1}^{N} a_k\right) - NS\right| \\
&= \frac{1}{N}\left|\left(\sum_{k=1}^{N} a_k\right) - \sum_{k=1}^{N} S\right| \\
&= \frac{1}{N}\left|\sum_{k=1}^{N} (a_k - S)\right| \\
&\leq \frac{1}{N}\sum_{k=1}^{N} |a_k - S| \\
&= \frac{1}{N}\sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} |a_k - S| \\
&\leq \frac{1}{N}\sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} \frac{\varepsilon}{2} \\
&= \frac{1}{N}\sum_{k=1}^{M} |a_k - S| + (N-M)\frac{\varepsilon}{2} \\
&\overset{\varepsilon \longrightarrow 0}{\longrightarrow} \frac{1}{N}\sum_{k=1}^{M} |a_k - S| + 0 \\
&\overset{N \longrightarrow \infty}{\longrightarrow} 0 + 0.
\end{aligned}$$

Note: M is fixed, so the last sum is some constant c, and $c/N \longrightarrow 0$ as $N \longrightarrow \infty$ for any constant. To be more careful, choose M first to get $\varepsilon/2$ for the tail, then choose $N(M) > M$ for the remaining truncated part of the sum.

2.2.4 b

- Define

$$\Gamma_n \coloneqq \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

- $\Gamma_1 = \sum_{k=1}^{n} \frac{a_k}{k}$ is the original series and each Γ_n is a tail of Γ_1, so by assumption $\Gamma_n \overset{n\to\infty}{\longrightarrow} 0$.
- Compute

$$\frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n}(\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \mathbf{\Gamma_{n+1}})$$

.

- This comes from consider the following summation:

$$\begin{array}{lccccccc}
\Gamma_1: & a_1 & +\frac{a_2}{2} & +\frac{a_3}{3} & +\cdots & & & \\
\Gamma_2: & & \frac{a_2}{2} & +\frac{a_3}{3} & +\cdots & & & \\
\Gamma_3: & & & \frac{a_3}{3} & +\cdots & & & \\
\hline
\sum_{i=1}^{n}\Gamma_i: & a_1 & +a_2 & +a_3 & +\cdots & a_n & +\frac{a_{n+1}}{n+1} & +\cdots
\end{array}$$

- Use part (a): since $\Gamma_n \overset{n\to\infty}{\longrightarrow} 0$, we have $\frac{1}{n}\sum_{k=1}^{n}\Gamma_k \overset{n\to\infty}{\longrightarrow} 0$.
- Also a minor check: $\Gamma_n \longrightarrow 0 \implies \frac{1}{n}\Gamma_n \longrightarrow 0$.
- Then

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} a_k &= \frac{1}{n}(\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \mathbf{\Gamma_{n+1}}) \\
&= \left(\frac{1}{n}\sum_{k=0}^{n}\Gamma_k\right) - \left(\frac{1}{n}\Gamma_{n+1}\right) \\
&\overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

2.3 Fall 2018 # 4 ✨

Let $f \in L^1([0,1])$. Prove that

$$\lim_{n\to\infty}\int_0^1 f(x)|\sin nx|\,dx = \frac{2}{\pi}\int_0^1 f(x)\,dx$$

> Hint: Begin with the case that f is the characteristic function of an interval.

Add concepts.

Solution:

Concepts Used:

- ?

Case of characteristic function

- First suppose $f(x) = \chi_{[0,1]}(x)$.
- Note that $\sin(nx)$ has a period of $2\pi/n$, and thus $\left\lfloor \frac{n}{2\pi} \right\rfloor$ full periods in $[0,1]$.
- Taking the absolute value yields a new function with half the period, so a period of π/n and $\lfloor \pi/n \rfloor$ full periods in $[0,1]$.
- We can compute the integral over one full period (which is independent of *which* period is chosen), and since $\sin(x)$ is positive and agrees with $|\sin(nx)|$ on the first period, we have

$$\begin{aligned}
\int_{\text{One Period}} |\sin(nx)|\,dx &= \int_0^{\pi/n} \sin(nx)\,dx \\
&= \frac{1}{n}\int_0^{\pi} \sin(u)\,du \quad u = nx \\
&= \frac{1}{n} - \cos(u)\,\Big|_0^{\pi} \\
&= \frac{2}{n}.
\end{aligned}$$

- Then break the integral up into integrals over periods $P_1, P_2, \cdots, P_N$ where $N \coloneqq \lfloor n/\pi \rfloor$:

$$\begin{aligned}
\int_0^1 |\sin(nx)|\,dx &= \left(\sum_{j=1}^{N} \int_{P_j} |\sin(nx)|\,dx\right) + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)|\,dx \\
&= \left(\sum_{j=1}^{N} \frac{2}{n}\right) + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)|\,dx \\
&= N\left(\frac{2}{n}\right) + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)|\,dx \\
&\coloneqq \left\lfloor \frac{n}{\pi} \right\rfloor \frac{2}{n} + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)|\,dx \\
&= \frac{2}{\pi} + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)|\,dx \\
&\coloneqq \frac{2}{\pi} + R(n)
\end{aligned}$$

  so it suffices to show that $R(n) \overset{n\to\infty}{\longrightarrow} 0$.

  Need to justify removing floor function and cancellation.

- Showing this: ????????????

  No clue how to show this.

General case

Not sure. Approximate f by simple functions...?

2.4 Fall 2017 # 4 ✨

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

1. Show that $f_n \longrightarrow 0$ pointwise but not uniformly on $[0,1]$.

Hint: Consider the maximum of f_n.

2.

$$\lim_{n\longrightarrow\infty} \int_0^1 n(1-x)^n \sin x \, dx = 0$$

Add concepts.

Walk through.

Solution:

Concepts Used:

- ?

2.4.1 a

Let $G(x) = \sum_{n=1}^{\infty} nx(1-x)^n$. Applying the ratio test, we have

$$\left|\frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n}\right| = \frac{n+1}{n}|1-x| \overset{n\longrightarrow\infty}{\longrightarrow} |1-x| < 1 \iff 0 \leq x \leq 2,$$

and in particular, this series converges on $[0,2]$. Thus its terms go to zero, and $nx(1-x)^n \longrightarrow 0$ on $[0,1] \subset [0,2]$.

To see that the convergence is not uniform, let $x_n = \frac{1}{n}$ and $\varepsilon > \frac{1}{e}$, then

$$\sup_{x\in[0,1]} |nx(1-x)^n - 0| \geq |nx_n(1-x_n)^n| = \left|\left(1-\frac{1}{n}\right)^n\right| \overset{n\longrightarrow\infty}{\longrightarrow} e^{-1} > \varepsilon.$$

2.4.2 b

Note: could use the first part with $\sin(x) \leq x$, but then integral ends up more complicated.

Noting that $\sin(x) \leq 1$, we have We have

$$\begin{aligned}
\left|\int_0^1 n(1-x)^n \sin(x)\right| &\leq \int_0^1 |n(1-x)^n \sin(x)| \\
&\leq \int_0^1 |n(1-x)^n| \\
&= n\int_0^1 (1-x)^n \\
&= -\frac{n(1-x)^{n+1}}{n+1} \\
&\xrightarrow{n\to\infty} 0.
\end{aligned}$$

2.5 Spring 2017 # 3 ✨

Let

$$f_n(x) = ae^{-nax} - be^{-nbx} \quad \text{where } 0 < a < b.$$

Show that

a. $\sum_{n=1}^{\infty} |f_n|$ is not in $L^1([0,\infty), m)$

Hint: $f_n(x)$ has a root x_n.

b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0,\infty), m) \quad \text{and} \quad \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)\, dm = \ln\frac{b}{a}$$

Not complete.

Add concepts.

Walk through.

Solution:

Concepts Used:

- ?

2.5.1 a

Letting $x_n \coloneqq \frac{1}{n}$, we have

$$\sum_{k=1}^{\infty} |f_k(x)| \geq |f_n(x_n)| = \left|ae^{-ax} - be^{-bx}\right| \coloneqq M.$$

In particular, $\sup_x |f_n(x)| \not\longrightarrow 0$, so the terms do not go to zero and the sum can not converge.

2.5.2 b

?

2.6 Fall 2016 # 1 ✨

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that f converges to a differentiable function on $(1, \infty)$ and that

$$f'(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n^x}\right)'.$$

Hint:

$$\left(\frac{1}{n^x}\right)' = -\frac{1}{n^x} \ln n$$

Add concepts.

Solution:

Concepts Used:

- ?

- Set $f_N(x) \coloneqq \sum_{n=1}^{N} n^{-x}$, so $f(x) = \lim_{N \to \infty} f_N(x)$.
- If an interchange of limits is justified, we have

$$\begin{aligned}
\frac{\partial}{\partial x} \lim_{N \to \infty} \sum_{n=1}^{N} n^{-x} &= \lim_{h \to 0} \lim_{N \to \infty} \frac{1}{h} \left[\left(\sum_{n=1}^{N} n^{-x} \right) - \left(\sum_{n=1}^{N} n^{-(x+h)} \right) \right] \\
&\overset{?}{=} \lim_{N \to \infty} \lim_{h \to 0} \frac{1}{h} \left[\left(\sum_{n=1}^{N} n^{-x} \right) - \left(\sum_{n=1}^{N} n^{-(x+h)} \right) \right] \\
&= \lim_{N \to \infty} \lim_{h \to 0} \frac{1}{h} \left[\sum_{n=1}^{N} n^{-x} - n^{-(x+h)} \right] \quad (1) \\
&= \lim_{N \to \infty} \sum_{n=1}^{N} \lim_{h \to 0} \frac{1}{h} \left[n^{-x} - n^{-(x+h)} \right] \quad \text{since this is a finite sum} \\
&\coloneqq \lim_{N \to \infty} \sum_{n=1}^{N} \frac{\partial}{\partial x} \left(\frac{1}{n^x} \right) \\
&= \lim_{N \to \infty} \sum_{n=1}^{N} -\frac{\ln(n)}{n^x},
\end{aligned}$$

where the combining of sums in (1) is valid because $\sum n^{-x}$ is absolutely convergent for $x > 1$ by the p-test.

- Thus it suffices to justify the interchange of limits and show that the last sum converges on $(1, \infty)$.
- Claim: $\sum n^{-x} \ln(n)$ converges.
  - Use the fact that for any fixed $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{\ln(n)}{n^\varepsilon} \overset{L'H.}{=} \lim_{n \to \infty} \frac{1/n}{\varepsilon n^{\varepsilon - 1}} = \lim_{n \to \infty} \frac{1}{\varepsilon n^\varepsilon} = 0,$$

  - This implies that for a fixed $\varepsilon > 0$ and for any constant $c > 0$ there exists an N large enough such that $n \geq N$ implies $\ln(n)/n^\varepsilon < c$, i.e. $\ln(n) < cn^\varepsilon$.
  - Taking $c = 1$, we have $n \geq N \implies \ln(n) < n^\varepsilon$
  - We thus break up the sum:

$$\begin{aligned}
\sum_{n \in \mathbb{N}} \frac{\ln(n)}{n^x} &= \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{\ln(n)}{n^x} \\
&\leq \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{n^\varepsilon}{n^x} \\
&:= C_\varepsilon + \sum_{n=N}^{\infty} \frac{n^\varepsilon}{n^x} \quad \text{with } C_\varepsilon < \infty \text{ a constant} \\
&= C_\varepsilon + \sum_{n=N}^{\infty} \frac{1}{n^{x-\varepsilon}},
\end{aligned}$$

    where the last term converges by the p-test if $x - \varepsilon > 1$.
  - But ε can depend on x, and if $x \in (1, \infty)$ is fixed we can choose $\varepsilon < |x - 1|$ to ensure this.
- Claim: the interchange of limits is justified.

?

2.7 Fall 2016 # 5 ✨

Let $\varphi \in L^\infty(\mathbb{R})$. Show that the following limit exists and satisfies the equality

$$\lim_{n \to \infty} \left(\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1 + x^2} \, dx \right)^{\frac{1}{n}} = \|\varphi\|_\infty.$$

Walk through.

Add concepts.

Solution:

Concepts Used:

- ?

Let L be the LHS and R be the RHS.
Claim: $L \leq R$. - Since $|\varphi| \leq \|\varphi\|_\infty$ a.e., we can write

$$\begin{aligned}
L^{\frac{1}{n}} &:= \int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} \\
&\leq \int_{\mathbb{R}} \frac{\|\varphi\|_\infty^n}{1+x^2} \\
&= \|\varphi\|_\infty^n \int_{\mathbb{R}} \frac{1}{1+x^2} \\
&= \|\varphi\|_\infty^n \arctan(x)\Big|_{-\infty}^{\infty} \\
&= \|\varphi\|_\infty^n \left(\frac{\pi}{2} - \frac{-\pi}{2}\right) \\
&= \pi \|\varphi\|_\infty^n
\end{aligned}$$

$$\begin{aligned}
&\implies L^{\frac{1}{n}} \leq \sqrt[n]{\pi \|\varphi\|_\infty^n} \\
&\implies L \leq \pi^{\frac{1}{n}} \|\varphi\|_\infty \\
&\qquad \overset{n \to \infty}{\longrightarrow} \|\varphi\|_\infty,
\end{aligned}$$

where we've used the fact that $c^{\frac{1}{n}} \overset{n \to \infty}{\longrightarrow} 1$ for any constant c.

Actually true? Need conditions?

Claim: $R \leq L$.

- We will show that $R \leq L + \varepsilon$ for every $\varepsilon > 0$.
- Set

$$S_\varepsilon := \left\{ x \in \mathbb{R}^n \;\middle|\; |\varphi(x)| \geq \|\varphi\|_\infty - \varepsilon \right\}.$$

- Then we have

$$\begin{aligned}
\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2}\, dx &\geq \int_{S_\varepsilon} \frac{|\varphi(x)|^n}{1+x^2}\, dx \quad S_\varepsilon \subset \mathbb{R} \\
&\geq \int_{S_\varepsilon} \frac{(\|\varphi\|_\infty - \varepsilon)^n}{1+x^2}\, dx \qquad \text{by definition of } S_\varepsilon \\
&= (\|\varphi\|_\infty - \varepsilon)^n \int_{S_\varepsilon} \frac{1}{1+x^2}\, dx \\
&= (\|\varphi\|_\infty - \varepsilon)^n C_\varepsilon \qquad \text{where } C_\varepsilon \text{ is some constant}
\end{aligned}$$

$$\begin{aligned}
\implies \left(\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2}\, dx\right)^{\frac{1}{n}} &\geq (\|\varphi\|_\infty - \varepsilon) C_\varepsilon^{\frac{1}{n}} \\
&\overset{n \to \infty}{\longrightarrow} (\|\varphi\|_\infty - \varepsilon) \cdot 1 \\
&\overset{\varepsilon \to 0}{\longrightarrow} \|\varphi\|_\infty,
\end{aligned}$$

where we've again used the fact that $c^{\frac{1}{n}} \longrightarrow 1$ for any constant.

2.8 Fall 2016 # 6 ✨

Let $f, g \in L^2(\mathbb{R})$. Show that

$$\lim_{n \longrightarrow \infty} \int_{\mathbb{R}} f(x)g(x+n)\,dx = 0$$

Add concepts.

Solution:

Concepts Used:

- ?

- Use the fact that L^p has small tails: if $h \in L^2(\mathbb{R})$, then for any $\varepsilon > 0$,

$$\forall \varepsilon,\ \exists N \in \mathbb{N} \quad \text{such that} \quad \int_{|x| \geq N} |h(x)|^2\,dx < \varepsilon.$$

How to prove small tails in L^p?

- So choose n large enough so the tails of both f and g are smaller than ε.
- Apply Cauchy-Schwarz:

$$\begin{aligned} \left| \int_{\mathbb{R}} f(x)g(x+n)\,dx \right| &\leq \int_{\mathbb{R}} |f(x)g(x+n)|\,dx \\ &\leq \int_{\mathbb{R}} . \end{aligned}$$

2.9 Spring 2016 # 1 🚩

For $n \in \mathbb{N}$, define

$$]e_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad E_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

Show that $e_n < E_n$, and prove Bernoulli's inequality:

$$(1+x)^n \geq 1 + nx \text{ for } -1 < x < \infty \text{ and } n \in \mathbb{N}$$

Use this to show the following:

1. The sequence e_n is increasing.
2. The sequence E_n is decreasing.
3. $2 < e_n < E_n < 4$.
4. $\lim_{n \longrightarrow \infty} e_n = \lim_{n \longrightarrow \infty} E_n$.

2.10 Fall 2015 # 1 🚩

Define

$$f(x) = c_0 + c_1 x^1 + c_2 x^2 + \ldots + c_n x^n \text{ with } n \text{ even and } c_n > 0.$$

Show that there is a number x_m such that $f(x_m) \leq f(x)$ for all $x \in \mathbb{R}$.

3 Measure Theory: Sets

3.1 Spring 2020 # 2 ✨

Let m_* denote the Lebesgue outer measure on $\mathbb{R}$.

3.1.1 a.

Prove that for every $E \subseteq \mathbb{R}$ there exists a Borel set B containing E such that

$$m_*(B) = m_*(E).$$

3.1.2 b.

Prove that if $E \subseteq \mathbb{R}$ has the property that

$$m_*(A) = m_*(A \bigcap E) + m_*(A \bigcap E^c)$$

for every set $A \subseteq \mathbb{R}$, then there exists a Borel set $B \subseteq \mathbb{R}$ such that $E = B \setminus N$ with $m_*(N) = 0$.

Be sure to address the case when $m_*(E) = \infty$.

Solution:

Concepts Used:

- Definition of outer measure:

$$m_*(E) = \inf_{\{Q_j\} \rightrightarrows E} \sum |Q_j|$$

  where $\{Q_j\}$ is a countable collection of closed cubes.
- Break $\mathbb{R}$ into $\coprod_{n \in \mathbb{Z}} [n, n+1)$, each with finite measure.
- Theorem: $m_*(Q) = |Q|$ for Q a closed cube (i.e. the outer measure equals the volume).

Proof .

- $m_*(Q) \leq |Q|$:
- Since $Q \subseteq Q$, $Q \rightrightarrows Q$ and $m_*(Q) \leq |Q|$ since m_* is an infimum over such coverings.
- $|Q| \leq m_*(Q)$:
- Fix $\varepsilon > 0$.
- Let $\{Q_i\}_{i=1}^{\infty} \rightrightarrows Q$ be arbitrary, it suffices to show that

$$|Q| \leq \left(\sum_{i=1}^{\infty} |Q_i|\right) + \varepsilon.$$

- Pick open cubes S_i such that $Q_i \subseteq S_i$ and $|Q_i| \leq |S_i| \leq (1+\varepsilon)|Q_i|$.
- Then $\{S_i\} \rightrightarrows Q$, so by compactness of Q pick a finite subcover with N elements.
- Note

$$Q \subseteq \bigcup_{i=1}^{N} S_i \implies |Q| \leq \sum_{i=1}^{N} |S_i| \leq \sum_{i=1}^{N} (1+\varepsilon)|Q_j| \leq (1+\varepsilon) \sum_{i=1}^{\infty} |Q_i|.$$

- Taking an infimum over coverings on the RHS preserves the inequality, so

$$|Q| \leq (1+\varepsilon) m_*(Q)$$

- Take $\varepsilon \longrightarrow 0$ to obtain final inequality.

■

3.1.3 a

- If $m_*(E) = \infty$, then take $B = \mathbb{R}^n$ since $m(\mathbb{R}^n) = \infty$.
- Suppose $N \coloneqq m_*(E) < \infty$.
- Since $m_*(E)$ is an infimum, by definition, for every $\varepsilon > 0$ there exists a covering by closed cubes $\{Q_i(\varepsilon)\}_{i=1}^{\infty} \rightrightarrows E$ depending on ε such that

$$\sum_{i=1}^{\infty} |Q_i(\varepsilon)| < N + \varepsilon.$$

- For each fixed n, set $\varepsilon_n = \dfrac{1}{n}$ to produce such a covering $\{Q_i(\varepsilon_n)\}_{i=1}^{\infty}$ and set $B_n \coloneqq \bigcup_{i=1}^{\infty} Q_i(\varepsilon_n)$.
- The outer measure of cubes is *equal* to the sum of their volumes, so

$$m_*(B_n) = \sum_{i=1}^{\infty} |Q_i(\varepsilon_n)| < N + \varepsilon_n = N + \frac{1}{n}.$$

- Now set $B \coloneqq \bigcap_{n=1}^{\infty} B_n$.
  - Since $E \subseteq B_n$ for every n, $E \subseteq B$
  - Since B is a countable intersection of countable unions of closed sets, B is Borel.

- Since $B_n \subseteq B$ for every n, we can apply subadditivity to obtain the inequality

$$E \subseteq B \subseteq B_n \implies N \leq m_*(B) \leq m_*(B_n) < N + \frac{1}{n} \quad \text{for all} \quad n \in \mathbb{Z}^{\geq 1}.$$

- This forces $m_*(E) = m_*(B)$.

3.1.4 b

Suppose $m_*(E) < \infty$.

- By (a), find a Borel set $B \supseteq E$ such that $m_*(B) = m_*(E)$
- Note that $E \subseteq B \implies B \bigcap E = E$ and $B \bigcap E^c = B \setminus E$.
- By assumption,

$$\begin{aligned} m_*(B) &= m_*(B \bigcap E) + m_*(B \bigcap E^c) \\ m_*(E) &= m_*(E) + m_*(B \setminus E) \\ m_*(E) - m_*(E) &= m_*(B \setminus E) \qquad \text{since } m_*(E) < \infty \\ \implies m_*(B \setminus E) &= 0. \end{aligned}$$

- So take $N = B \setminus E$; this shows $m_*(N) = 0$ and $E = B \setminus (B \setminus E) = B \setminus N$.

If $m_*(E) = \infty$:

- Apply result to $E_R \coloneqq E \bigcap [R, R+1)^n \subset \mathbb{R}^n$ for $R \in \mathbb{Z}$, so $E = \coprod_R E_R$
- Obtain B_R, N_R such that $E_R = B_R \setminus N_R$, $m_*(E_R) = m_*(B_R)$, and $m_*(N_R) = 0$.
- Note that
  - $B \coloneqq \bigcup_R B_R$ is a union of Borel sets and thus still Borel
  - $E = \bigcup_R E_R$
  - $N \coloneqq B \setminus E$
  - $N' \coloneqq \bigcup_R N_R$ is a union of null sets and thus still null
- Since $E_R \subset B_R$ for every R, we have $E \subset B$
- We can compute

$$N = B \setminus E = \left(\bigcup_R B_R\right) \setminus \left(\bigcup_R E_R\right) \subseteq \bigcup_R (B_R \setminus E_R) = \bigcup_R N_R \coloneqq N'$$

where $m_*(N') = 0$ since N' is null, and thus subadditivity forces $m_*(N) = 0$.

3.2 Fall 2019 # 3. ✨

Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) = 1$ and $\{B_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{B}$-measurable subsets of X, and

$$B \coloneqq \left\{ x \in X \;\middle|\; x \in B_n \text{ for infinitely many } n \right\}.$$

a. Argue that B is also a $\mathcal{B}$-measurable subset of X.

b. Prove that if $\sum_{n=1}^{\infty} \mu(B_n) < \infty$ then $\mu(B) = 0$.

c. Prove that if $\sum_{n=1}^{\infty} \mu(B_n) = \infty$ **and** the sequence of set complements $\{B_n^c\}_{n=1}^{\infty}$ satisfies

$$\mu\left(\bigcap_{n=k}^{K} B_n^c\right) = \prod_{n=k}^{K} \left(1 - \mu\left(B_n\right)\right)$$

for all positive integers k and K with $k < K$, then $\mu(B) = 1$.

Hint: Use the fact that $1 - x \leq e^{-x}$ for all x.

Solution:

Concepts Used:

- Borel-Cantelli: for a sequence of sets X_n,

$$\limsup_n X_n = \left\{x \;\middle|\; x \in X_n \text{ for infinitely many } n\right\} = \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} X_n$$

$$\liminf_n X_n = \left\{x \;\middle|\; x \in X_n \text{ for all but finitely many } n\right\} = \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} X_n.$$

- Properties of logs and exponentials:

$$\prod_n e^{x_n} = e^{\Sigma_n x_n} \quad \text{and} \quad \sum_n \log(x_n) = \log\left(\prod_n x_n\right).$$

- Tails of convergent sums vanish.
- Continuity of measure: $B_n \searrow B$ and $\mu(B_0) < \infty$ implies $\lim_n \mu(B_n) = \mu(B)$, and $B_n \nearrow B \implies \lim_n \mu(B_n) = \mu(B)$.

3.2.1 a

- The Borel σ-algebra is closed under countable unions/intersections/complements,
- $B = \limsup_n B_n$ is an intersection of unions of measurable sets.

3.2.2 b

- Tails of convergent sums go to zero, so $\sum_{n\geq M} \mu(B_n) \xrightarrow{M\to\infty} 0$,
- $B_M \coloneqq \bigcap_{m=1}^{M} \bigcup_{n\geq m} B_n \searrow B$.

$$\begin{aligned}
\mu(B_M) &= \mu\left(\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} B_n\right) \\
&\leq \mu\left(\bigcup_{n\geq m} B_n\right) \quad \text{for all } m \in \mathbb{N} \text{ by countable subadditivity} \\
&\longrightarrow 0,
\end{aligned}$$

- The result follows by continuity of measure.

3.2.3 c

- To show $\mu(B) = 1$, we'll show $\mu(B^c) = 0$.
- Let $B_k = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{K} B_n$. Then

$$\begin{aligned}
\mu(B_K^c) &= \mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{K} B_n^c\right) \\
&\leq \sum_{m=1}^{\infty} \mu\left(\bigcap_{n=m}^{K} B_n^c\right) \quad \text{by subadditivity} \\
&= \sum_{m=1}^{\infty} \prod_{n=m}^{K} (1 - \mu(B_n)) \quad \text{by assumption} \\
&\leq \sum_{m=1}^{\infty} \prod_{n=m}^{K} e^{-\mu(B_n^c)} \quad \text{by hint} \\
&= \sum_{m=1}^{\infty} \exp\left(-\sum_{n=m}^{K} \mu(B_n^c)\right) \\
&\overset{K \longrightarrow \infty}{\longrightarrow} 0
\end{aligned}$$

since $\sum_{n=m}^{K} \mu(B_n^c) \overset{K \longrightarrow \infty}{\longrightarrow} \infty$ by assumption

- We can apply continuity of measure since $B_K^c \xrightarrow{K \longrightarrow \infty} B^c$.

How to prove the hint..?

3.3 Spring 2019 # 2 ✨

Let $\mathcal{B}$ denote the set of all Borel subsets of $\mathbb{R}$ and $\mu : \mathcal{B} \longrightarrow [0, \infty)$ denote a finite Borel measure on $\mathbb{R}$.

3.3.1 a

Prove that if $\{F_k\}$ is a sequence of Borel sets for which $F_k \supseteq F_{k+1}$ for all k, then

$$\lim_{k \to \infty} \mu\left(F_k\right) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right)$$

3.3.2 b

Suppose μ has the property that $\mu(E) = 0$ for every $E \in \mathcal{B}$ with Lebesgue measure $m(E) = 0$.

Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $E \in \mathcal{B}$ with $m(E) < \delta$, then $\mu(E) < \varepsilon$.

Add concepts.

Solution:

Concepts Used:

- ?

3.3.3 a

See Folland p.26

- Lemma 1: $\mu(\coprod_{k=1}^{\infty} E_k) = \lim_{N\longrightarrow\infty} \sum_{k=1}^{N} \mu(E_k)$.
- Suppose $F_0 \supseteq F_1 \supseteq \cdots$.
- Let $A_k = F_k \setminus F_{k+1}$, since the F_k are nested the A_k are disjoint
- Set $A \coloneqq \coprod_{k=1}^{\infty} A_k$ and $F \coloneqq \bigcap_{k=1}^{\infty} F_k$.
- Note $X = X \setminus Y \coprod X \bigcap Y$ for any two sets (just write $X \setminus Y \coloneqq X \bigcap Y^c$)
- Note that A contains anything that was removed from F_0 when passing from any F_j to F_{j+1}, while F contains everything that is never removed at any stage, and these are disjoint possibilities.
- Thus $F_0 = F \coprod A$, so

$$
\begin{aligned}
\mu(F_0) &= \mu(F) + \mu(A) \\
&= \mu(F) + \mu(\coprod_{k=1}^{\infty} A_k) \\
&= \mu(F) + \lim_{n\longrightarrow\infty} \sum_{k=0}^{n} \mu(A_k) \quad \text{by countable additivity} \\
&= \mu(F) + \lim_{n\longrightarrow\infty} \sum_{k=0}^{n} \mu(F_k) - \mu(F_{k+1}) \\
&= \mu(F) + \lim_{n\longrightarrow\infty} (\mu(F_1) - \mu(F_n)) \quad \text{(Telescoping)} \\
&= \mu(F) + \mu(F_1) - \lim_{N\longrightarrow\infty} \mu(F_n),
\end{aligned}
$$

- Since μ is a finite measure, $\mu(F_1) < \infty$ and can be subtracted, yielding

$$
\begin{aligned}
\mu(F_1) &= \mu(F) + \mu(F_1) - \lim_{n\longrightarrow\infty} \mu(F_n) \\
\implies \mu(F) &= \lim_{n\longrightarrow\infty} \mu(F_n) \\
\implies \mu\left(\bigcap_{k=1}^{\infty} F_k\right) &= \lim_{n\longrightarrow\infty} \mu(F_n).
\end{aligned}
$$

3.3.4 b

- Toward a contradiction, negate the implication: suppose there exists an $\varepsilon > 0$ such that for all δ, we have $m(E) < \delta$ but $\mu(E) > \varepsilon$.
- The sequence $\left\{\delta_n \coloneqq \frac{1}{2^n}\right\}_{n\in\mathbb{N}}$ and produce sets $A_n \in \mathcal{B}$ such $m(A_n) < \frac{1}{2^n}$ but $\mu(A_n) > \varepsilon$.

- Define

$$F_n \coloneqq \bigcup_{j \geq n} A_j$$
$$C_m \coloneqq \bigcap_{k=1}^{m} F_k$$
$$A \coloneqq C_\infty \coloneqq \bigcap_{k=1}^{\infty} F_k.$$

- Note that $F_1 \supseteq F_2 \supseteq \cdots$, since each increase in index unions fewer sets.
- By continuity for the Lebesgue measure,

$$m(A) = m\left(\bigcap_{k=1}^{\infty} F_k\right) = \lim_{k \to \infty} m(F_k) = \lim_{k \to \infty} m\left(\bigcup_{j \geq k} A_j\right) \leq \lim_{k \to \infty} \sum_{j \geq k} m(A_j) = \lim_{k \to \infty} \sum_{j \geq k} \frac{1}{2^n} = 0,$$

  which follows because this is the tail of a convergent sum
- Thus $m(A) = 0$ and by assumption, this implies $\mu(A) = 0$.
- However, by part (a),

$$\mu(A) = \lim_n \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \geq \lim_n \mu(A_n) = \lim_n \varepsilon = \varepsilon > 0.$$

All messed up!

3.4 Fall 2018 # 2 ✨

Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Show that there is a Borel set $B \subset E$ such that $m(E \setminus B) = 0$.

Move this to review notes to clean things up.

Solution:

Concepts Used:

- Definition of measurability: there exists an open $O \supset E$ such that $m_*(O \setminus E) < \varepsilon$ for all $\varepsilon > 0$.
- Theorem: E is Lebesgue measurable iff there exists a closed set $F \subseteq E$ such that $m_*(E \setminus F) < \varepsilon$ for all $\varepsilon > 0$.
- Every F_σ, G_δ is Borel.
- Claim: E is measurable $\iff$ for every ε there exist $F_\varepsilon \subset E \subset G_\varepsilon$ with F_ε closed and G_ε open and $m(G_\varepsilon \setminus E) < \varepsilon$ and $m(E \setminus F_\varepsilon) < \varepsilon$.
  - Proof: existence of G_ε is the definition of measurability.
  - Existence of F_ε: ?
- Claim: E is measurable $\implies$ there exists an open $O \supseteq E$ such that $m(O \setminus E) = 0$.
  - Since E is measurable, for each $n \in \mathbb{N}$ choose $G_n \supseteq E$ such that $m_*(G_n \setminus E) < \frac{1}{n}$.
  - Set $O_N \coloneqq \bigcap_{n=1}^{N} G_n$ and $O \coloneqq \bigcap_{n=1}^{\infty} G_n$.
  - Suppose E is bounded.
    - Note $O_N \searrow O$ and $m_*(O_1) < \infty$ if E is bounded, since in this case

$$m_*(G_n \setminus E) = m_*(G_1) - m_*(E) < 1 \iff m_*(G_1) < m_*(E) + \frac{1}{n} < \infty.$$

    - Note $O_N \setminus E \searrow O \setminus E$ since $O_N \setminus E \coloneqq O_N \bigcap E^c \supseteq O_{N+1} \bigcap E^c$ for all N, and again $m_*(O_1 \setminus E) < \infty$.
    - So it's valid to apply continuity of measure from above:

$$\begin{aligned} m_*(O \setminus E) &= \lim_{N \to \infty} m_*(O_N \setminus E) \\ &\leq \lim_{N \to \infty} m_*(G_N \setminus E) \\ &= \lim_{N \to \infty} \frac{1}{N} = 0, \end{aligned}$$

    where the inequality uses subadditivity on $\bigcap_{n=1}^{N} G_n \subseteq G_N$
  - Suppose E is unbounded.
    - Write $E^k = E \bigcap [k, k+1]^d \subset \mathbb{R}^d$ as the intersection of E with an annulus, and note that $E = \coprod_{k \in \mathbb{N}} E_k$.
    - Each E_k is bounded, so apply the previous case to obtain $O_k \supseteq E_k$ with $m(O_k \setminus E_k) = 0$.
    - So write $O_k = E_k \coprod N_k$ where $N_k \coloneqq O_k \setminus E_k$ is a null set.
    - Define $O = \bigcup_{k \in \mathbb{N}} O_k$, note that $E \subseteq O$.
    - Now note

$$\begin{aligned} O \setminus E &= \left(\coprod_k O_k\right) \setminus \left(\coprod_K E_k\right) \\ &\subseteq \coprod_k (O_k \setminus E_k) \\ \implies m_*(O \setminus E) &\leq m_*\left(\coprod (O_k \setminus E_k)\right) = 0, \end{aligned}$$

    since any countable union of null sets is again null.
  - So $O \supseteq E$ with $m(O \setminus E) = 0$.
- Theorem: since E is measurable, E^c is measurable
  - Proof: It suffices to write E^c as the union of two measurable sets, $E^c = S \bigcup (E^c - S)$, where S is to be determined.
  - We'll produce an S such that $m_*(E^c - S) = 0$ and use the fact that any subset

3.4.1 Indirect Proof

- Since E is measurable, E^c is measurable.
- Since E^c is measurable exists an open $O \supseteq E^c$ such that $m(O \setminus E^c) = 0$.
- Set $B \coloneqq O^c$, then $O \supseteq E^c \iff \mathcal{O}^c \subseteq E \iff B \subseteq E$.
- Computing measures yields

$$E \setminus B \coloneqq E \setminus \mathcal{O}^c \coloneqq E \bigcap (\mathcal{O}^c)^c = E \bigcap \mathcal{O} = \mathcal{O} \bigcap (E^c)^c \coloneqq \mathcal{O} \setminus E^c,$$

  thus $m(E \setminus B) = m(\mathcal{O} \setminus E^c) = 0$.
- Since $\mathcal{O}$ is open, B is closed and thus Borel.

3.4.2 Direct Proof (Todo)

Try to construct the set.

3.5 Spring 2018 # 1 ✨

Define

$$E \coloneqq \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N} \right\}.$$

Prove that $m(E) = 0$.

Solution:

Concepts Used:

- Borel-Cantelli: If $\{E_k\}_{k \in \mathbb{Z}} \subset 2^{\mathbb{R}}$ is a countable collection of Lebesgue measurable sets with $\sum_{k \in \mathbb{Z}} m(E_k) < \infty$, then almost every $x \in \mathbb{R}$ is in *at most finitely* many E_k.
  - Equivalently (?), $m(\limsup_{k \to \infty} E_k) = 0$, where $\limsup_{k \to \infty} E_k = \bigcap_{k=1}^{\infty} \bigcup_{j \geq k} E_j$, the elements which are in E_k for infinitely many k.

- Strategy: Borel-Cantelli.
- We'll show that $m(E) \bigcap [n, n+1] = 0$ for all $n \in \mathbb{Z}$; then the result follows from

$$m(E) = m\left(\bigcup_{n \in \mathbb{Z}} E \bigcap [n, n+1] \right) \leq \sum_{n=1}^{\infty} m(E \bigcap [n, n+1]) = 0.$$

- By translation invariance of measure, it suffices to show $m(E \bigcap [0,1]) = 0$.
  - So WLOG, replace E with $E \bigcap [0,1]$.
- Define

$$E_j \coloneqq \left\{ x \in [0,1] \;\middle|\; \exists p \in \mathbb{Z}^{\geq 0} \text{ s.t. } \left| x - \frac{p}{j} \right| < \frac{1}{j^3} \right\}.$$

  - Note that $E_j \subseteq \coprod_{p \in \mathbb{Z}^{\geq 0}} B_{j^{-3}}\left(\frac{p}{j}\right)$, i.e. a union over integers p of intervals of radius

$1/j^3$ around the points p/j. Since $1/j^3 < 1/j$, this union is in fact disjoint.

- Importantly, note that

$$\limsup_{j\to\infty} E_j \coloneqq \bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty} E_j = E$$

since

$$\begin{aligned}
x \in \limsup_j E_j &\iff x \in E_j \text{ for infinitely many } j \\
&\iff \text{there are infinitely many } j \text{ for which there exist a } p \text{ such that } \left|x - \frac{p}{j}\right| < j^{-3} \\
&\iff \text{there are infinitely many such pairs } p, j \\
&\iff x \in E.
\end{aligned}$$

- Intersecting with $[0,1]$, we can write E_j as a union of intervals:

$$E_j = \left(0, j^{-3}\right) \coprod B_{j^{-3}}\left(\frac{1}{j}\right) \coprod B_{j^{-3}}\left(\frac{2}{j}\right) \coprod \cdots \coprod B_{j^{-3}}\left(\frac{j-1}{j}\right) \coprod (1 - j^{-3}, 1),$$

where we've separated out the "boundary" terms to emphasize that they are balls about 0 and 1 intersected with $[0,1]$.

- Since E_j is a union of open sets, it is Borel and thus Lebesgue measurable.
- Computing the measure of E_j:
  - For a fixed j, there are exactly $j+1$ possible choices for a numerator $(0, 1, \cdots, j)$, thus there are exactly $j+1$ sets appearing in the above decomposition.
  - The first and last intervals are length $\frac{1}{j^3}$
  - The remaining $(j+1) - 2 = j - 1$ intervals are twice this length, $\frac{2}{j^3}$
  - Thus

$$m(E_j) = 2\left(\frac{1}{j^3}\right) + (j-1)\left(\frac{2}{j^3}\right) = \frac{2}{j^2}$$

- Note that

$$\sum_{j\in\mathbb{N}} m(E_j) = 2\sum_{j\in\mathbb{N}} \frac{1}{j^2} < \infty,$$

which converges by the p-test for sums.

- But then

$$\begin{aligned}
m(E) &= m(\limsup_j E_j) \\
&= m(\bigcap_{n\in\mathbb{N}}\bigcup_{j\geq n} E_j) \\
&\leq m(\bigcup_{j\geq N} E_j) \quad \text{for every } N \\
&\leq \sum_{j\geq N} m(E_j) \\
&\xrightarrow{N\to\infty} 0 \quad .
\end{aligned}$$

- Thus E is measurable as a subset of a null set and $m(E) = 0$.

3.6 Fall 2017 # 2 ✨

Let $f(x) = x^2$ and $E \subset [0, \infty) := \mathbb{R}^+$.

1. Show that

$$m^*(E) = 0 \iff m^*(f(E)) = 0.$$

2. Deduce that the map

$$\begin{aligned} \varphi : \mathcal{L}(\mathbb{R}^+) &\longrightarrow \mathcal{L}(\mathbb{R}^+) \\ E &\mapsto f(E) \end{aligned}$$

is a bijection from the class of Lebesgue measurable sets of $[0, \infty)$ to itself.

Walk through.

Solution:

Concepts Used:

- ?

3.6.1 a

It suffices to consider the bounded case, i.e. $E \subseteq B_M(0)$ for some M. Then write $E_n = B_n(0) \bigcap E$ and apply the theorem to E_n, and by subadditivity, $m^*(E) = m^*(\bigcup_n E_n) \leq \sum_n m^*(E_n) = 0$.

Lemma: $f(x) = x^2, f^{-1}(x) = \sqrt{x}$ are Lipschitz on any compact subset of $[0, \infty)$.
Proof: Let $g = f$ or f^{-1}. Then $g \in C^1([0, M])$ for any M, so g is differentiable and g' is continuous. Since g' is continuous on a compact interval, it is bounded, so $|g'(x)| \leq L$ for all x. Applying the MVT,

$$|f(x) - f(y)| = f'(c)|x - y| \leq L|x - y|.$$

Lemma: If g is Lipschitz on $\mathbb{R}^n$, then $m(E) = 0 \implies m(g(E)) = 0$.
Proof: If g is Lipschitz, then

$$g(B_r(x)) \subseteq B_{Lr}(x),$$

which is a dilated ball/cube, and so

$$m^*(B_{Lr}(x)) \leq L^n \cdot m^*(B_r(x)).$$

Now choose $\{Q_j\} \rightrightarrows E$; then $\{g(Q_j)\} \rightrightarrows g(E)$.
By the above observation,

$$|g(Q_j)| \leq L^n |Q_j|,$$

and so

$$m^*(g(E)) \leq \sum_j |g(Q_j)| \leq \sum_j L^n |Q_j| = L^n \sum_j |Q_j| \longrightarrow 0.$$

Now just take $g(x) = x^2$ for one direction, and $g(x) = f^{-1}(x) = \sqrt{x}$ for the other. ■

3.6.2 b

Lemma: E is measurable iff $E = K \coprod N$ for some K compact, N null.

Write $E = K \coprod N$ where K is compact and N is null.
Then $\varphi^{-1}(E) = \varphi^{-1}(K \coprod N) = \varphi^{-1}(K) \coprod \varphi^{-1}(N)$.
Since $\varphi^{-1}(N)$ is null by part (a) and $\varphi^{-1}(K)$ is the preimage of a compact set under a continuous map and thus compact, $\varphi^{-1}(E) = K' \coprod N'$ where K' is compact and N' is null, so $\varphi^{-1}(E)$ is measurable.
So φ is a measurable function, and thus yields a well-defined map $\mathcal{L}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathbb{R})$ since it preserves measurable sets. Restricting to $[0, \infty)$, f is bijection, and thus so is φ.

3.7 Spring 2017 # 2 ✨

3.7.1 a

Let μ be a measure on a measurable space $(X, \mathcal{M})$ and f a positive measurable function.

Define a measure λ by

$$\lambda(E) := \int_E f \, d\mu, \quad E \in \mathcal{M}$$

Show that for g any positive measurable function,

$$\int_X g \, d\lambda = \int_X fg \, d\mu$$

3.7.2 b

Let $E \subset \mathbb{R}$ be a measurable set such that

$$\int_E x^2 \, dm = 0.$$

Show that $m(E) = 0$.

Solution:

Concepts Used:

- Absolute continuity of measures: $\lambda \ll \mu \iff E \in \mathcal{M}, \mu(E) = 0 \implies \lambda(E) = 0$.
- Radon-Nikodym: if $\lambda \ll \mu$, then there exists a measurable function $\dfrac{\partial \lambda}{\partial \mu} \coloneqq f$ where $\lambda(E) = \displaystyle\int_E f \, d\mu$.
- Chebyshev's inequality:

$$A_c \coloneqq \left\{ x \in X \;\middle|\; |f(x)| \geq c \right\} \implies \mu(A_c) \leq c^{-p} \int_{A_c} |f|^p \, d\mu \quad \forall 0 < p < \infty.$$

3.7.3 a

- Strategy: use approximation by simple functions to show absolute continuity and apply Radon-Nikodym
- Claim: $\lambda \ll \mu$, i.e. $\mu(E) = 0 \implies \lambda(E) = 0$.
  - Note that if this holds, by Radon-Nikodym, $f = \dfrac{\partial \lambda}{\partial \mu} \implies d\lambda = f d\mu$, which would yield

$$\int g \, d\lambda = \int gf \, d\mu.$$

- So let E be measurable and suppose $\mu(E) = 0$.
- Then

$$\lambda(E) \coloneqq \int_E f \, d\mu = \lim_{n \to \infty} \left\{ \int_E s_n \, d\mu \;\middle|\; s_n \coloneqq \sum_{j=1}^{\infty} c_j \mu(E_j),\ s_n \nearrow f \right\}$$

  where we take a sequence of simple functions increasing to f.
- But since each $E_j \subseteq E$, we must have $\mu(E_j) = 0$ for any such E_j, so every such s_n must be zero and thus $\lambda(E) = 0$.

What is the final step in this approximation?

3.7.4 b

- Set $g(x) = x^2$, note that g is positive and measurable.
- By part (a), there exists a positive f such that for any $E \subseteq \mathbb{R}$,

$$\int_E g \, dm = \int_E gf \, d\mu$$

  - The LHS is zero by assumption and thus so is the RHS.
  - $m \ll \mu$ by construction.
  - Note that gf is positive.
- Define $A_k = \left\{ x \in X \;\middle|\; gf \cdot \chi_E > \dfrac{1}{k} \right\}$, for $k \in \mathbb{Z}^{\geq 0}$
- Then by Chebyshev with $p = 1$, for every k we have

$$\mu(A_k) \leq k \int_E gf \, d\mu = 0$$

- Then noting that $A_k \searrow A := \left\{x \in X \;\middle|\; gf \cdot \chi_E(x) > 0\right\}$, we have $\mu(A) = 0$.
- Since gf is positive, we have

$$x \in E \iff gf\chi_E(x) > 0 \iff x \in A$$

  so $E = A$ and $\mu(E) = \mu(A)$.
- But $m \ll \mu$ and $\mu(E) = 0$, so we can conclude that $m(E) = 0$.

3.8 Fall 2016 # 4 ✨

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\{E_n\} \subset \mathcal{M}$ satisfies

$$\lim_{n\to\infty} \mu\left(X \backslash E_n\right) = 0.$$

Define

$$G := \left\{x \in X \;\middle|\; x \in E_n \text{ for only finitely many } n\right\}.$$

Show that $G \in \mathcal{M}$ and $\mu(G) = 0$.

Add concepts.

Solution:

Concepts Used:

- ?

- Claim: $G \in \mathcal{M}$.
  - Claim:

$$G = \left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n\right)^c = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n^c.$$

    * This follows because x is in the RHS $\iff$ $x \in E_n^c$ for all but finitely many n $\iff$ $x \in E_n$ for at most finitely many n.
  - But $\mathcal{M}$ is a σ-algebra, and this shows G is obtained by countable unions/intersections/complements of measurable sets, so $G \in \mathcal{M}$.
- Claim: $\mu(G) = 0$.

– We have

$$\begin{aligned}
\mu(G) &= \mu\left(\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty} E_n^c\right) \\
&\leq \sum_{N=1}^{\infty} \mu\left(\bigcap_{n=N}^{\infty} E_n^c\right) \\
&\leq \sum_{N=1}^{\infty} \mu(E_M^c) \\
&:= \sum_{N=1}^{\infty} \mu(X \setminus E_N) \\
&\overset{N\longrightarrow\infty}{\longrightarrow} 0.
\end{aligned}$$

Last step seems wrong!

3.9 Spring 2016 # 3

Let f be Lebesgue measurable on $\mathbb{R}$ and $E \subset \mathbb{R}$ be measurable such that

$$0 < A = \int_E f(x)dx < \infty.$$

Show that for every $0 < t < 1$, there exists a measurable set $E_t \subset E$ such that

$$\int_{E_t} f(x)dx = tA.$$

3.10 Spring 2016 # 5

Let $(X, \mathcal{M}, \mu)$ be a measure space. For $f \in L^1(\mu)$ and $\lambda > 0$, define

$$\varphi(\lambda) = \mu(\{x \in X | f(x) > \lambda\}) \quad \text{and} \quad \psi(\lambda) = \mu(\{x \in X | f(x) < -\lambda\})$$

Show that φ, ψ are Borel measurable and

$$\int_X |f| \, d\mu = \int_0^{\infty} [\varphi(\lambda) + \psi(\lambda)] \, d\lambda$$

3.11 Fall 2015 # 2

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be Lebesgue measurable.

1. Show that there is a sequence of simple functions $s_n(x)$ such that $s_n(x) \longrightarrow f(x)$ for all $x \in \mathbb{R}$.
2. Show that there is a Borel measurable function g such that $g = f$ almost everywhere.

3.12 Spring 2015 # 3

Let μ be a finite Borel measure on $\mathbb{R}$ and $E \subset \mathbb{R}$ Borel. Prove that the following statements are equivalent:

1. $\forall \varepsilon > 0$ there exists G open and F closed such that

$$F \subseteq E \subseteq G \quad \text{and} \quad \mu(G \setminus F) < \varepsilon.$$

2. There exists a $V \in G_\delta$ and $H \in F_\sigma$ such that

$$H \subseteq E \subseteq V \quad \text{and} \quad \mu(V \setminus H) = 0$$

3.13 Spring 2014 # 3

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and suppose

$$\forall x \in \mathbb{R}, \quad f(x) \geq \limsup_{y \to x} f(y)$$

Prove that f is Borel measurable.

3.14 Spring 2014 # 4

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose f is a measurable function on X. Show that

$$\lim_{n \to \infty} \int_X f^n \, d\mu = \begin{cases} \infty & \text{or} \\ \mu(f^{-1}(1)), \end{cases}$$

and characterize the collection of functions of each type.

3.15 Spring 2017 # 1

Let K be the set of numbers in $[0, 1]$ whose decimal expansions do not use the digit 4.

> We use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 with $399\cdots$. For example, $0.8754 = 0.8753999\cdots$.

Show that K is a compact, nowhere dense set without isolated points, and find the Lebesgue measure $m(K)$.

Solution:

Concepts Used:

- Definition: A is *nowhere dense* $\iff$ every interval I contains a subinterval $S \subseteq A^c$.
  - Equivalently, the interior of the closure is empty, $\left(\overline{K}\right)^\circ = \emptyset$.

Claim: K **is compact**.

- It suffices to show that $K^c := [0, 1] \setminus K$ is open; Then K will be a closed and bounded subset of $\mathbb{R}$ and thus compact by Heine-Borel.

- Strategy: write K^c as the union of open balls (since these form a basis for the Euclidean topology on $\mathbb{R}$).
  - Do this by showing every point $x \in K^c$ is an interior point, i.e. x admits a neighborhood N_x such that $N_x \subseteq K^c$.
- Identify K^c as the set of real numbers in $[0,1]$ whose decimal expansion **does** contain a 4.
  - We will show that there exists a neighborhood small enough such that all points in it contain a 4 in their decimal expansions.
- Let $x \in K^c$, suppose a 4 occurs as the kth digit, and write

$$x = 0.d_1 d_2 \cdots d_{k-1}\ 4\ d_{k+1} \cdots = \left(\sum_{j=1}^{k} d_j 10^{-j}\right) + \left(4 \cdot 10^{-k}\right) + \left(\sum_{j=k+1}^{\infty} d_j 10^{-j}\right).$$

- Set $r_x < 10^{-k}$ and let $y \in [0,1] \bigcap B_{r_x}(x)$ be arbitrary and write

$$y = \sum_{j=1}^{\infty} c_j 10^{-j}.$$

- Thus $|x-y| < r_x < 10^{-k}$, and the first k digits of x and y must agree:
  - We first compute the difference:

$$x - y = \sum_{i=1}^{\infty} d_j 10^{-j} - \sum_{i=1}^{\infty} c_j 10^{-j} = \sum_{i=1}^{\infty} (d_j - c_j) 10^{-j}$$

  - Thus (claim)

$$|x-y| \leq \sum_{j=1}^{\infty} |d_j - c_j| 10^{j} < 10^{-k} \iff |d_j - c_j| = 0 \quad \forall j \leq k.$$

  - Otherwise we can note that any term $|d_j - c_j| \geq 1$ and there is a contribution to $|x-y|$ of at least $1 \cdot 10^{-j}$ for some $j < k$, whereas

$$j < k \iff 10^{-j} > 10^{-k},$$

    a contradiction.
- This means that for all $j \leq k$ we have $d_j = c_j$, and in particular $d_k = 4 = c_k$, so y has a 4 in its decimal expansion.
- But then $K^c = \bigcup_x B_{r_x}(x)$ is a union of open sets and thus open.

Claim: K **is nowhere dense and** $m(K) = 0$**:**

- Strategy: Show $\left(\overline{K}\right)^{\circ} = \emptyset$.
- Since K is closed, $\overline{K} = K$, so it suffices to show that K does not properly contain any interval.
- It suffices to show $m(K^c) = 1$, since this implies $m(K) = 0$ and since any interval has strictly positive measure, this will mean K can not contain an interval.
- As in the construction of the Cantor set, let

- K_0 denote $[0,1]$ with 1 interval $\left(\frac{4}{10}, \frac{5}{10}\right)$ of length $\frac{1}{10}$ deleted, so

$$m(K_0^c) = \frac{1}{10}.$$

- K_1 denote K_0 with 9 intervals $\left(\frac{1}{100}, \frac{5}{100}\right), \left(\frac{14}{100}, \frac{15}{100}\right), \cdots \left(\frac{94}{100}, \frac{95}{100}\right)$ of length $\frac{1}{100}$ deleted, so

$$m(K_1^c) = \frac{1}{10} + \frac{9}{100}.$$

- K_n denote K_{n-1} with 9^n such intervals of length $\frac{1}{10^{n+1}}$ deleted, so

$$m(K_n^c) = \frac{1}{10} + \frac{9}{100} + \cdots + \frac{9^n}{10^{n+1}}.$$

- Then compute

$$m(K^c) = \sum_{j=0}^{\infty} \frac{9^n}{10^{n+1}} = \frac{1}{10} \sum_{j=0}^{\infty} \left(\frac{9}{10}\right)^n = \frac{1}{10}\left(\frac{1}{1-\frac{9}{10}}\right) = 1.$$

Claim: K **has no isolated points**:

- A point $x \in K$ is isolated iff there there is an open ball $B_r(x)$ containing x such that $B_r(x) \subsetneq K^c$.
  - So every point in this ball **should** have a 4 in its decimal expansion.
- Strategy: show that if $x \in K$, every neighborhood of x intersects K.
- Note that $m(K_n) = \left(\frac{9}{10}\right)^n \xrightarrow{n\to\infty} 0$
- Also note that we deleted open intervals, and the endpoints of these intervals are never deleted.
  - Thus endpoints of deleted intervals are elements of K.
- Fix x. Then for every ε, by the Archimedean property of $\mathbb{R}$, choose n such that $\left(\frac{9}{10}\right)^n < \varepsilon$.
- Then there is an endpoint x_n of some deleted interval I_n satisfying

$$|x - x_n| \leq \left(\frac{9}{10}\right)^n < \varepsilon.$$

- So every ball containing x contains some endpoint of a removed interval, and thus an element of K.

3.16 Spring 2016 # 2

Let $0 < \lambda < 1$ and construct a Cantor set C_λ by successively removing middle intervals of length λ.

Prove that $m(C_\lambda) = 0$.

4 Measure Theory: Functions

4.1 Fall 2016 # 2 ✨

Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be measurable with

$$\int_a^b f(x)\ dx = \int_a^b g(x)\ dx.$$

Show that either

1. $f(x) = g(x)$ almost everywhere, or
2. There exists a measurable set $E \subset [a, b]$ such that

$$] \int_E f(x)\, dx > \int_E g(x)\, dx$$

Add concepts.

Solution:

Concepts Used:

- ?

- Suppose it is *not* the case that $f = g$ almost everywhere; then letting $A \coloneqq \left\{x \in [a, b] \;\middle|\; f(x) \neq g(x)\right\}$, we have $m(A) > 0$.
- Write

$$A = A_1 \coprod A_2 \coloneqq \{f > g\} \coprod \{f < g\},$$

  then $m(A_1) > 0$ or $m(A_2) > 0$ (or both).
- Wlog (by relabeling f, g if necessary), suppose $m(A_1) > 0$, and take $E \coloneqq A_1$.
- Then on E, we have $f(x) > g(x)$ pointwise. This is preserved by monotonicity of the integral, thus

$$f(x) > g(x) \text{ on } E \implies \int_E f(x)\, dx > \int_E g(x)\, dx.$$

4.2 Spring 2016 # 4 🚩

Let $E \subset \mathbb{R}$ be measurable with $m(E) < \infty$. Define

$$f(x) = m(E \cap (E + x)).$$

Show that

1. $f \in L^1(\mathbb{R})$.
2. f is uniformly continuous.
3. $\lim_{|x| \longrightarrow \infty} f(x) = 0$.

Hint:

$$\chi_{E\cap(E+x)}(y) = \chi_E(y)\chi_E(y-x)$$

5 Integrals: Convergence

5.1 Fall 2019 # 2 ✨

Prove that

$$\left|\frac{d^n}{dx^n}\frac{\sin x}{x}\right| \leq \frac{1}{n}$$

for all $x \neq 0$ and positive integers n.

Hint: Consider $\int_0^1 \cos(tx)dt$

Solution:

Concepts Used:

- DCT
- Bounding in the right place. Don't evaluate the actual integral!

- By induction on the number of limits we can pass through the integral.
- For $n = 1$ we first pass one derivative into the integral: let $x_n \longrightarrow x$ be any sequence converging to x, then

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\sin(x)}{x} &= \frac{\partial}{\partial x}\int_0^1 \cos(tx)\,dt \\
&= \lim_{x_n\longrightarrow x}\frac{1}{x_n - x}\left(\int_0^1 \cos(tx_n)\,dt - \int_0^1 \cos(tx)\,dt\right) \\
&= \lim_{x_n\longrightarrow x}\left(\int_0^1 \frac{\cos(tx_n)-\cos(tx)}{x_n - x}\,dt\right) \\
&= \lim_{x_n\longrightarrow x}\left(\int_0^1 \left(t\sin(tx)\Big|_{x=\xi_n}\right)dt\right) \quad \text{where} \quad \xi_n \in [x_n, x] \text{ by MVT}, \xi_n \longrightarrow x \\
&= \lim_{\xi_n\longrightarrow x}\left(\int_0^1 t\sin(t\xi_n)\,dt\right) \\
&=_{\text{DCT}} \int_0^1 \lim_{\xi_n\longrightarrow x} t\sin(t\xi_n)\,dt \\
&= \int_0^1 t\sin(tx)\,dt
\end{aligned}$$

.

- Taking absolute values we obtain an upper bound

$$\begin{aligned}\left|\frac{\partial}{\partial x}\frac{\sin(x)}{x}\right| &= \left|\int_0^1 t\sin(tx)\,dt\right| \\ &\leq \int_0^1 |t\sin(tx)|\,dt \\ &\leq \int_0^1 1\,dt = 1,\end{aligned}$$

  since $t \in [0,1] \implies |t| < 1$, and $|\sin(xt)| \leq 1$ for any x and t.
- Note that this bound also justifies the DCT, since the functions $f_n(t) = t\sin(t\xi_n)$ are uniformly dominated by $g(t) = 1$ on $L^1([0,1])$.

Note: integrating by parts here yields the actual formula:

$$\begin{aligned}\int_0^1 t\sin(tx)\,dt &=_{\text{IBP}} \left(\frac{-t\cos(tx)}{x}\right)\Big|_{t=0}^{t=1} - \int_0^1 \frac{\cos(tx)}{x}\,dt \\ &= \frac{-\cos(x)}{x} - \frac{\sin(x)}{x^2} \\ &= \frac{x\cos(x) - \sin(x)}{x^2}.\end{aligned}$$

- For the inductive step, we assume that we can pass $n-1$ limits through the integral and show we can pass the nth through as well.

$$\begin{aligned}\frac{\partial^n}{\partial x^n}\frac{\sin(x)}{x} &= \frac{\partial^n}{\partial x^n}\int_0^1 \cos(tx)\,dt \\ &= \frac{\partial}{\partial x}\int_0^1 \frac{\partial^{n-1}}{\partial x^{n-1}}\cos(tx)\,dt \\ &= \frac{\partial}{\partial x}\int_0^1 t^{n-1} f_{n-1}(x,t)\,dt\end{aligned}$$

  - Note that $f_n(x,t) = \pm\sin(tx)$ when n is odd and $f_n(x,t) = \pm\cos(tx)$ when n is even, and a constant factor of t is multiplied when each derivative is taken.
- We continue as in the base case:

$$\begin{aligned}\frac{\partial}{\partial x}\int_0^1 t^{n-1} f_{n-1}(x,t)\,dt &= \lim_{x_k \to x}\int_0^1 t^{n-1}\left(\frac{f_{n-1}(x_n,t) - f_{n-1}(x,t)}{x_n - x}\right)dt \\ &=_{\text{IVT}} \lim_{x_k \to x}\int_0^1 t^{n-1}\frac{\partial f_{n-1}}{\partial x}(\xi_k,t)\,dt \quad \text{where } \xi_k \in [x_k, x],\ \xi_k \longrightarrow x \\ &=_{\text{DCT}} \int_0^1 \lim_{x_k \to x} t^{n-1}\frac{\partial f_{n-1}}{\partial x}(\xi_k,t)\,dt \\ &\coloneqq \int_0^1 \lim_{x_k \to x} t^n f_n(\xi_k,t)\,dt \\ &\coloneqq \int_0^1 t^n f_n(x,t)\,dt.\end{aligned}$$

  - We've used the fact that $f_0(x) = \cos(tx)$ is smooth as a function of x, and in particular continuous
  - The DCT is justified because the functions $h_{n,k}(x,t) = t^n f_n(\xi_k,t)$ are again uniformly (in k) bounded by 1 since $t \leq 1 \implies t^n \leq 1$ and each f_n is a sin or cosine.

- Now take absolute values

$$\begin{aligned}
\left| \frac{\partial^n}{\partial x^n} \frac{\sin(x)}{x} \right| &= \left| \int_0^1 -t^n f_n(x,t)\ dt \right| \\
&\leq \int_0^1 |t^n f_n(x,t)|\ dt \\
&\leq \int_0^1 |t^n||f_n(x,t)|\ dt \\
&\leq \int_0^1 |t^n| \cdot 1\ dt \\
&\leq \int_0^1 t^n\ dt \quad \text{since } t \text{ is positive} \\
&= \frac{1}{n+1} \\
&< \frac{1}{n}.
\end{aligned}$$

  - We've again used the fact that $f_n(x,t)$ is of the form $\pm\cos(tx)$ or $\pm\sin(tx)$, both of which are bounded by 1.

5.2 Spring 2020 # 5 ✨

Compute the following limit and justify your calculations:

$$\lim_{n\longrightarrow\infty} \int_0^n \left(1+\frac{x^2}{n}\right)^{-(n+1)} dx.$$

Not finished, flesh out.

Walk through.

Solution:

Concepts Used:

- DCT
- Passing limits through products and quotients

Note that

$$\begin{aligned}
\lim_n \left(1+\frac{x^2}{n}\right)^{-(n+1)} &= \frac{1}{\lim_n \left(1+\frac{x^2}{n}\right)^1 \left(1+\frac{x^2}{n}\right)^n} \\
&= \frac{1}{1 \cdot e^{x^2}} \\
&= e^{-x^2}.
\end{aligned}$$

If passing the limit through the integral is justified, we will have

$$\begin{aligned}
\lim_{n\to\infty}\int_0^n \left(1+\frac{x^2}{n}\right)^{-(n+1)} dx &= \lim_{n\to\infty}\int_{\mathbb{R}} \chi_{[0,n]}\left(1+\frac{x^2}{n}\right)^{-(n+1)} dx \\
&= \int_{\mathbb{R}} \lim_{n\to\infty}\chi_{[0,n]}\left(1+\frac{x^2}{n}\right)^{-(n+1)} dx \quad \text{by the DCT} \\
&= \int_{\mathbb{R}} \lim_{n\to\infty}\left(1+\frac{x^2}{n}\right)^{-(n+1)} dx \\
&= \int_0^\infty e^{-x^2} \\
&= \frac{\sqrt{\pi}}{2}.
\end{aligned}$$

Computing the last integral:

$$\begin{aligned}
\left(\int_{\mathbb{R}} e^{-x^2}\,dx\right)^2 &= \left(\int_{\mathbb{R}} e^{-x^2}\,dx\right)\left(\int_{\mathbb{R}} e^{-y^2}\,dx\right) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(x+y)^2}\,dx \\
&= \int_0^{2\pi}\int_0^\infty e^{-r^2} r\,dr\,d\theta \qquad u = r^2 \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^\infty e^{-u}\,du\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 1 \\
&= \pi,
\end{aligned}$$

and now use the fact that the function is even so $\int_0^\infty f = \frac{1}{2}\int_{\mathbb{R}} f$.

Justifying the DCT:

- Apply Bernoulli's inequality:

$$1+\frac{x^2}{n}^{n+1} \geq 1+\frac{x^2}{n}\left(1+x^2\right) \geq 1+x^2,$$

where the last inequality follows from the fact that $1+\frac{x^2}{n} \geq 1$

5.3 Spring 2019 # 3 ✨

Let $\{f_k\}$ be any sequence of functions in $L^2([0,1])$ satisfying $\|f_k\|_2 \leq M$ for all $k \in \mathbb{N}$.

Prove that if $f_k \longrightarrow f$ almost everywhere, then $f \in L^2([0,1])$ with $\|f\|_2 \leq M$ and

$$\lim_{k\to\infty}\int_0^1 f_k(x)dx = \int_0^1 f(x)dx$$

Hint: Try using Fatou's Lemma to show that $\|f\|_2 \leq M$ and then try applying Egorov's Theorem.

Solution:

Concepts Used:

- Definition of L^+: space of measurable function $X \longrightarrow [0, \infty]$.
- Fatou: For any sequence of L^+ functions, $\int \liminf f_n \leq \liminf \int f_n$.
- Egorov's Theorem: If $E \subseteq \mathbb{R}^n$ is measurable, $m(E) > 0$, $f_k : E \longrightarrow \mathbb{R}$ a sequence of measurable functions where $\lim_{n \longrightarrow \infty} f_n(x)$ exists and is finite a.e., then $f_n \longrightarrow f$ *almost uniformly*: for every $\varepsilon > 0$ there exists a closed subset $F_\varepsilon \subseteq E$ with $m(E \setminus F) < \varepsilon$ and $f_n \longrightarrow f$ uniformly on F.

L^2 bound:

- Since $f_k \longrightarrow f$ almost everywhere, $\liminf_n f_n(x) = f(x)$ a.e.
- $\|f_n\|_2 < \infty$ implies each f_n is measurable and thus $|f_n|^2 \in L^+$, so we can apply Fatou:

$$\begin{aligned}
\|f\|_2^2 &= \int |f(x)|^2 \\
&= \int \liminf_n |f_n(x)|^2 \\
&\underset{\text{Fatou}}{\leq} \liminf_n \int |f_n(x)|^2 \\
&\leq \liminf_n M \\
&= M.
\end{aligned}$$

- Thus $\|f\|_2 \leq \sqrt{M} < \infty$ implying $f \in L^2$.

What is the "right" proof here that uses the first part?

Equality of Integrals:

- Take the sequence $\varepsilon_n = \dfrac{1}{n}$
- Apply Egorov's theorem: obtain a set F_ε such that $f_n \longrightarrow f$ uniformly on F_ε and $m(I \setminus F_\varepsilon) < \varepsilon$.

$$\begin{aligned}
\lim_{n \longrightarrow \infty} \left| \int_0^1 f_n - f \right| &\leq \lim_{n \longrightarrow \infty} \int_0^1 |f_n - f| \\
&= \lim_{n \longrightarrow \infty} \left(\int_{F_\varepsilon} |f_n - f| + \int_{I \setminus F_\varepsilon} |f_n - f| \right) \\
&= \int_{F_\varepsilon} \lim_{n \longrightarrow \infty} |f_n - f| + \lim_{n \longrightarrow \infty} \int_{I \setminus F_\varepsilon} |f_n - f| \quad \text{by uniform convergence} \\
&= 0 + \lim_{n \longrightarrow \infty} \int_{I \setminus F_\varepsilon} |f_n - f|,
\end{aligned}$$

so it suffices to show $\int_{I \setminus F_\varepsilon} |f_n - f| \overset{n \longrightarrow \infty}{\longrightarrow} 0$.

- We can obtain a bound using Holder's inequality with $p = q = 2$:

$$\begin{aligned}\int_{I\setminus F_\varepsilon} |f_n - f| &\leq \left(\int_{I\setminus F_\varepsilon} |f_n - f|^2\right)\left(\int_{I\setminus F_\varepsilon} |1|^2\right) \\ &= \left(\int_{I\setminus F_\varepsilon} |f_n - f|^2\right)\mu(F_\varepsilon) \\ &\leq \|f_n - f\|_2 \mu(F_\varepsilon) \\ &\leq (\|f_n\|_2 + \|f\|_2)\mu(F_\varepsilon) \\ &\leq 2M \cdot \mu(F_\varepsilon)\end{aligned}$$

  where M is now a constant not depending on ε or n.
- Now take a nested sequence of sets F_ε with $\mu(F_\varepsilon) \longrightarrow 0$ and applying continuity of measure yields the desired statement.

5.4 Fall 2018 # 6 ✨

Compute the following limit and justify your calculations:

$$\lim_{n\to\infty} \int_1^n \frac{dx}{(1+\frac{x}{n})^n \sqrt[n]{x}}$$

Add concepts.

Solution:

Concepts Used:

- ?

- Note that $x^{\frac{1}{n}} \overset{n\longrightarrow\infty}{\longrightarrow} 1$ for any $0 < x < \infty$.
- Thus the integrand converges to $\dfrac{1}{e^x}$, which is integrable on $(0,\infty)$ and integrates to 1.
- Break the integrand up:

$$\int_0^\infty \frac{1}{(1+\frac{x}{n})^n x^{\frac{1}{n}}}\,dx = \int_0^1 \frac{1}{(1+\frac{x}{n})^n x^{\frac{1}{n}}}\,dx = \int_1^\infty \frac{1}{(1+\frac{x}{n})^n x^{\frac{1}{n}}}\,dx.$$

5.5 Fall 2018 # 3 ✨

Suppose $f(x)$ and $xf(x)$ are integrable on $\mathbb{R}$. Define F by

$$F(t) \coloneqq \int_{-\infty}^\infty f(x)\cos(xt)dx$$

Show that

$$F'(t) = -\int_{-\infty}^\infty xf(x)\sin(xt)dx.$$

Walk through.

Solution:

Concepts Used:

- Mean Value Theorem
- DCT

$$\begin{aligned}\frac{\partial}{\partial t} F(t) &= \frac{\partial}{\partial t} \int_{\mathbb{R}} f(x)\cos(xt)\ dx \\ &\overset{DCT}{=} \int_{\mathbb{R}} f(x)\frac{\partial}{\partial t}\cos(xt)\ dx \\ &= \int_{\mathbb{R}} xf(x)\cos(xt)\ dx,\end{aligned}$$

so it only remains to justify the DCT.

- Fix t, then let $t_n \longrightarrow t$ be arbitrary.
- Define

$$h_n(x,t) = f(x)\left(\frac{\cos(tx)-\cos(t_n x)}{t_n - t}\right) \overset{n\longrightarrow\infty}{\longrightarrow} \frac{\partial}{\partial t}\left(f(x)\cos(xt)\right)$$

  since $\cos(tx)$ is differentiable in t and this is the limit definition of differentiability.
- Note that

$$\begin{aligned}\frac{\partial}{\partial t}\cos(tx) &:= \lim_{t_n \longrightarrow t} \frac{\cos(tx)-\cos(t_n x)}{t_n - t} \\ &\overset{MVT}{=} \frac{\partial}{\partial t}\cos(tx)\Big|_{t=\xi_n} && \text{for some } \xi_n \in [t, t_n] \text{ or } [t_n, t] \\ &= x\sin(\xi_n x)\end{aligned}$$

  where $\xi_n \overset{n\longrightarrow\infty}{\longrightarrow} t$ since wlog $t_n \leq \xi_n \leq t$ and $t_n \nearrow t$.
- We then have

$$|h_n(x)| = |f(x)x\sin(\xi_n x)| \leq |xf(x)| \quad \text{since } |\sin(\xi_n x)| \leq 1$$

  for every x and every n.
- Since $xf(x) \in L^1(\mathbb{R})$ by assumption, the DCT applies.

5.6 Spring 2018 # 5 ✨

Suppose that

- $f_n, f \in L^1$,
- $f_n \longrightarrow f$ almost everywhere, and
- $\int |f_n| \to \int |f|$.

Show that $\int f_n \to \int f$.

Solution:

Concepts Used:

- $\int |f_n - f| \longrightarrow \iff \int f_n = \int f$.
- Fatou:

$$\int \liminf f_n \leq \liminf \int f_n$$
$$\int \limsup f_n \geq \limsup \int f_n.$$

- Since $\int |f_n| \overset{n\longrightarrow\infty}{\longrightarrow} \int |f|$, define

$$h_n = |f_n - f| \qquad \overset{n\longrightarrow\infty}{\longrightarrow} 0 \ a.e.$$
$$g_n = |f_n| + |f| \qquad \overset{n\longrightarrow\infty}{\longrightarrow} 2|f| \ a.e.$$

  - Note that $g_n - h_n \overset{n\longrightarrow\infty}{\longrightarrow} 2|f| - 0 = 2|f|$.
- Then

$$\begin{aligned}
\int 2|f| &= \int \liminf_n (g_n - h_n) \\
&= \int \liminf_n (g_n) + \int \liminf_n (-h_n) \\
&= \int \liminf_n (g_n) - \int \limsup_n (h_n) \\
&= \int 2|f| - \int \limsup_n (h_n) \\
&\leq \int 2|f| - \limsup_n \int h_n \quad \text{by Fatou,}
\end{aligned}$$

- Since $f \in L^1$, $\int 2|f| = 2\|f\|_1 < \infty$ and it makes sense to subtract it from both sides, thus

$$\begin{aligned}
0 &\leq -\limsup_n \int h_n \\
&\coloneqq -\limsup_n \int |f_n - f|.
\end{aligned}$$

  which forces $\limsup_n \int |f_n - f| = 0$, since
  - The integral of a nonnegative function is nonnegative, so $\int |f_n - f| \geq 0$.
  - So $\left(-\int |f_n - f|\right) \leq 0$.
  - But the above inequality shows $\left(-\int |f_n - f|\right) \geq 0$ as well.
- Since $\liminf_n \int h_n \leq \limsup_n \int h_n = 0$, $\lim_n \int h_n$ exists and is equal to zero.
- But then

$$\left|\int f_n - \int f\right| = \left|\int f_n - f\right| \leq \int |f_n - f|,$$

and taking $\lim_{n\to\infty}$ on both sides yields

$$\lim_{n\to\infty}\left|\int f_n - \int f\right| \le \lim_{n\to\infty}\int |f_n - f| = 0,$$

so $\lim_{n\to\infty}\int f_n = \int f$.

5.7 Spring 2018 # 2 ✨

Let

$$f_n(x) := \frac{x}{1+x^n}, \quad x \ge 0.$$

a. Show that this sequence converges pointwise and find its limit. Is the convergence uniform on $[0,\infty)$?

b. Compute

$$\lim_{n\to\infty}\int_0^\infty f_n(x)dx$$

Add concepts.

Solution:

Concepts Used:

- ?

5.7.1 a

Claim: f_n does not converge uniformly to its limit.

- Note each $f_n(x)$ is clearly continuous on $(0,\infty)$, since it is a quotient of continuous functions where the denominator is never zero.
- Note

$$x < 1 \implies x^n \xrightarrow{n\to\infty} 0 \quad \text{and} \quad x > 1 \implies x^n \xrightarrow{n\to\infty} \infty.$$

- Thus

$$f_n(x) = \frac{x}{1+x^n} \xrightarrow{n\to\infty} f(x) := \begin{cases} x, & 0 \le x < 1 \\ \frac{1}{2}, & x = 1 \\ 0, & x > 1 \end{cases}.$$

- If $f_n \longrightarrow f$ uniformly on $[0,\infty)$, it would converge uniformly on every subset and thus uniformly on $(0,\infty)$.
  - Then f would be a uniform limit of continuous functions on $(0,\infty)$ and thus continuous on $(0,\infty)$.
  - By uniqueness of limits, f_n would converge to the pointwise limit f above, which is not continuous at $x = 1$, a contradiction.

5.7.2 b

- If the DCT applies, interchange the limit and integral:

$$\begin{aligned}\lim_{n\longrightarrow\infty}\int_0^\infty f_n(x)\,dx &= \int_0^\infty \lim_{n\longrightarrow\infty} f_n(x)\,dx \quad \text{DCT}\\ &= \int_0^\infty f(x)\,dx\\ &= \int_0^1 x\,dx + \int_1^\infty 0\,dx\\ &= \frac{1}{2}x^2\Big|_0^1\\ &= \frac{1}{2}.\end{aligned}$$

- To justify the DCT, write

$$\int_0^\infty f_n(x) = \int_0^1 f_n(x) + \int_1^\infty f_n(x).$$

- f_n restricted to $(0,1)$ is uniformly bounded by $g_0(x) = 1$ in the first integral, since

$$x \in [0,1] \implies \frac{x}{1+x^n} < \frac{1}{1+x^n} < 1 \coloneqq g(x)$$

  so

$$\int_0^1 f_n(x)\,dx \leq \int_0^1 1\,dx = 1 < \infty.$$

  Also note that $g_0 \cdot \chi_{(0,1)} \in L^1((0,\infty))$.
- The f_n restricted to $(1,\infty)$ are uniformly bounded by $g_1(x) = \dfrac{1}{x^2}$ on $[1,\infty)$, since

$$x \in (1,\infty) \implies \frac{x}{1+x^n} \leq \frac{x}{x^n} = \frac{1}{x^{n-1}} \leq \frac{1}{x^2} \in L^1([1,\infty) \text{ when } n \geq 3,$$

  by the p-test for integrals.
- So set

$$g \coloneqq g_0 \cdot \chi_{(0,1)} + g_1 \cdot \chi_{[1,\infty)},$$

  then by the above arguments $g \in L^1((0,\infty))$ and $f_n \leq g$ everywhere, so the DCT applies.

5.8 Fall 2016 # 3 ✨

Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{x\longrightarrow 0}\int_{\mathbb{R}} |f(y-x) - f(y)|\,dy = 0$$

Missing some stuff.

Solution:

Concepts Used:

- $C_c^\infty \hookrightarrow L^p$ is dense.
- If $f \dots ?$

- Fixing notation, set $\tau_x f(y) \coloneqq f(y-x)$; we then want to show

$$\|\tau_x f - f\|_{L^1} \overset{x\longrightarrow 0}{\longrightarrow} 0.$$

- Claim: by an $\varepsilon/3$ argument, it suffices to show this for compactly supported functions:
  - Since $f \in L^1$, choose $g_n \subset C_c^\infty(\mathbb{R}^1)$ smooth and compactly supported so that

$$\|f-g\|_{L^1} < \varepsilon.$$

  - Claim: $\|\tau_x f - \tau_x g\| < \varepsilon$.
    - Proof 1: translation invariance of the integral.
    - Proof 2: Apply a change of variables:

$$\begin{aligned}
\|\tau_x f - \tau_x g\|_1 &\coloneqq \int_{\mathbb{R}} |\tau_x f(y) - \tau_x g(y)|\, dy \\
&= \int_{\mathbb{R}} |f(y-x) - g(y-x)|\, dy \\
&= \int_{\mathbb{R}} |f(u) - g(u)|\, du \qquad (u = y-x,\ du = dy) \\
&= \|f-g\|_1 \\
&< \varepsilon.
\end{aligned}$$

  - Then

$$\begin{aligned}
\|\tau_x f - f\|_1 &= \|\tau_x f - \tau_x g + \tau_x g - g + g - f\|_1 \\
&\leq \|\tau_x f - \tau_x g\|_1 + \|\tau_x g - g\|_1 + \|g - f\|_1 \\
&\leq 2\varepsilon + \|\tau_x g - g\|_1.
\end{aligned}$$

- To show this for compactly supported functions:
  - Let $g \in C_c^\infty(\mathbb{R}^1)$, let $E = \operatorname{supp}(g)$, and write

$$\begin{aligned}
\|\tau_x g - g\|_1 &= \int_{\mathbb{R}} |g(y-x) - g(y)|\, dy \\
&= \int_E |g(y-x) - g(y)|\, dy + \int_{E^c} |g(y-x) - g(y)|\, dy \\
&= \int_E |g(y-x) - g(y)|\, dy.
\end{aligned}$$

  - But g is smooth and compactly supported on E, and thus uniformly continuous on E, so

$$\begin{aligned}
\lim_{x\longrightarrow 0} \int_E |g(y-x) - g(y)|\, dy &= \int_E \lim_{x\longrightarrow 0} |g(y-x) - g(y)|\, dy \\
&= \int_E 0\, dy \\
&= 0.
\end{aligned}$$

5.9 Fall 2015 # 3 🚩

Compute the following limit:

$$\lim_{n\to\infty} \int_1^n \frac{ne^{-x}}{1+nx^2} \sin\left(\frac{x}{n}\right) dx$$

5.10 Fall 2015 # 4 🚩

Let $f : [1, \infty) \longrightarrow \mathbb{R}$ such that $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Show that the following limit exists and satisfies the equality

$$\lim_{x\to\infty} f(x) \leq 1 + \frac{\pi}{4}$$

6 Integrals: Approximation

6.1 Spring 2018 # 3 ✨

Let f be a non-negative measurable function on $[0, 1]$.

Show that

$$\lim_{p\to\infty} \left(\int_{[0,1]} f(x)^p dx \right)^{\frac{1}{p}} = \|f\|_\infty.$$

Solution:

Concepts Used:

- $\|f\|_\infty \coloneqq \inf_t \left\{ t \;\middle|\; m\left(\left\{ x \in \mathbb{R}^n \;\middle|\; f(x) > t \right\}\right) = 0 \right\}$, i.e. this is the lowest upper bound that holds almost everywhere.

- $\|f\|_p \leq \|f\|_\infty$:
  - Note $|f(x)| \leq \|f\|_\infty$ almost everywhere and taking pth powers preserves this inequality.

- Thus

$$\begin{aligned}
&|f(x)| \leq \|f\|_\infty \quad \text{a.e. by definition} \\
\implies &|f(x)|^p \leq \|f\|_\infty^p \quad \text{for } p \geq 0 \\
\implies &\|f\|_p^p = \int_X |f(x)|^p \, dx \\
&\leq \int_X \|f\|_\infty^p \, dx \\
&= \|f\|_\infty^p \int_X 1 \, dx \\
&= \|f\|_\infty^p \cdot m(X) \quad \text{since the norm doesn't depend on } x \\
&= \|f\|_\infty^p \qquad \text{since } m(X) = 1.
\end{aligned}$$

  - Thus $\|f\|_p \leq \|f\|_\infty$ for all p and taking $\lim_{p \longrightarrow \infty}$ preserves this inequality.

- $\|f\|_p \geq \|f\|_\infty$:
  - Fix $\varepsilon > 0$.
  - Define

$$S_\varepsilon \coloneqq \left\{ x \in \mathbb{R}^n \;\middle|\; |f(x)| \geq \|f\|_\infty - \varepsilon \right\}.$$

    - Note that $m(S_\varepsilon) > 0$; otherwise if $m(S_\varepsilon) = 0$, then $t \coloneqq \|f\|_\infty - \varepsilon < \|f\|_\varepsilon$. But this produces a *smaller* upper bound almost everywhere than $\|f\|_\varepsilon$, contradicting the definition of $\|f\|_\varepsilon$ as an infimum over such bounds.
  - Then

$$\begin{aligned}
\|f\|_p^p &= \int_X |f(x)|^p \, dx \\
&\geq \int_{S_\varepsilon} |f(x)|^p \, dx \quad \text{since } S_\varepsilon \subseteq X \\
&\geq \int_{S_\varepsilon} |\|f\|_\infty - \varepsilon|^p \, dx \quad \text{since on } S_\varepsilon, |f| \geq \|f\|_\infty - \varepsilon \\
&= |\|f\|_\infty - \varepsilon|^p \cdot m(S_\varepsilon) \quad \text{since the integrand is independent of } x \\
&\geq 0 \quad \text{since } m(S_\varepsilon) > 0
\end{aligned}$$

  - Taking pth roots for $p \geq 1$ preserves the inequality, so

$$\implies \|f\|_p \geq |\|f\|_\infty - \varepsilon| \cdot m(S_\varepsilon)^{\frac{1}{p}} \overset{p \longrightarrow \infty}{\longrightarrow} |\|f\|_\infty - \varepsilon| \overset{\varepsilon \longrightarrow 0}{\longrightarrow} \|f\|_\infty$$

    where we've used the fact that above arguments work
  - Thus $\|f\|_p \geq \|f\|_\infty$.

6.2 Spring 2018 # 4 ✨

Let $f \in L^2([0,1])$ and suppose

$$\int_{[0,1]} f(x) x^n dx = 0 \text{ for all integers } n \geq 0.$$

Show that $f = 0$ almost everywhere.

6.3 Spring 2015 # 2 ✨

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be continuous with period 1. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(t)dt \quad \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

Hint: show this first for the functions $f(t) = e^{2\pi ikt}$ for $k \in \mathbb{Z}$.

Add concepts.

Solution:

Concepts Used:

- ?

6.3.1 Proof 1: Using Fourier Transforms

Concepts Used:

- Weierstrass Approximation: A uniformly continuous function on a compact set can be uniformly approximated by polynomials.

- Fix $k \in \mathbb{Z}$.
- Since $e^{2\pi ikx}$ is continuous on the compact interval $[0, 1]$, it is uniformly continuous.
- Thus there is a sequence of polynomials P_ℓ such that

$$P_{\ell,k} \overset{\ell\longrightarrow\infty}{\longrightarrow} e^{2\pi ikx} \text{ uniformly on } [0, 1].$$

- Note applying linearity to the assumption $\int f(x)\, x^n$, we have

$$\int f(x)x^n\, dx = 0 \ \forall n \implies \int f(x)p(x)\, dx = 0$$

  for any polynomial $p(x)$, and in particular for $P_{\ell,k}(x)$ for every ℓ and every k.
- But then

$$\begin{aligned}
\langle f,\ e_k\rangle &= \int_0^1 f(x)e^{-2\pi ikx}\, dx \\
&= \int_0^1 f(x) \lim_{\ell\longrightarrow\infty} P_\ell(x) \\
&= \lim_{\ell\longrightarrow\infty} \int_0^1 f(x)P_\ell(x) \qquad \text{by uniform convergence on a compact interval} \\
&= \lim_{\ell\longrightarrow\infty} 0 \quad \text{by assumption} \\
&= 0 \quad \forall k \in \mathbb{Z},
\end{aligned}$$

so f is orthogonal to every e_k.

- Thus $f \in S^\perp \coloneqq \operatorname{span}_{\mathbb{C}} \{e_k\}_{k\in\mathbb{Z}}^\perp \subseteq L^2([0,1])$, but since this is a basis, S is dense and thus $S^\perp = \{0\}$ in $L^2([0,1])$.
- Thus $f \equiv 0$ in $L^2([0,1])$, which implies that f is zero almost everywhere.

■

6.3.2 Alternative Proof

Concepts Used:

- $C^1([0,1])$ is dense in $L^2([0,1])$
- Polynomials are dense in $L^p(X, \mathcal{M}, \mu)$ for any $X \subseteq \mathbb{R}^n$ compact and μ a finite measure, for all $1 \leq p < \infty$.
  - Use Weierstrass Approximation, then uniform convergence implies $L^p(\mu)$ convergence by DCT.

- By density of polynomials, for $f \in L^2([0,1])$ choose $p_\varepsilon(x)$ such that $\|f - p_\varepsilon\| < \varepsilon$ by Weierstrass approximation.
- Then on one hand,

$$\begin{aligned}
\|f(f-p_\varepsilon)\|_1 &= \left\|f^2\right\|_1 - \|f\cdot p_\varepsilon\|_1 \\
&= \left\|f^2\right\|_1 - 0 \quad \text{by assumption} \\
&= \|f\|_2^2.
\end{aligned}$$

  - Where we've used that $\left\|f^2\right\|_1 = \int \left|f^2\right| = \int |f|^2 = \|f\|_2^2$.

- On the other hand

$$\begin{aligned}
\|f(f-p_\varepsilon)\| &\leq \|f\|_1 \|f-p_\varepsilon\|_\infty \quad \text{by Holder} \\
&\leq \varepsilon \|f\|_1 \\
&\leq \varepsilon \|f\|_2 \sqrt{m(X)} \\
&= \varepsilon \|f\|_2 \quad \text{since } m(X) = 1.
\end{aligned}$$

  - Where we've used that $\|fg\|_1 = \int |fg| = \int |f||g| \leq \int \|f\|_\infty |g| = \|f\|_\infty \|g\|_1$.

- Combining these,

$$\|f\|_2^2 \leq \|f\|_2 \varepsilon \implies \|f\|_2 < \varepsilon \longrightarrow 0, .$$

so $\|f\|_2 = 0$, which implies $f = 0$ almost everywhere.

6.4 Fall 2014 # 4

Let $g \in L^\infty([0,1])$ Prove that

$$\int_{[0,1]} f(x)g(x)\,dx = 0 \quad \text{for all continuous } f : [0,1] \longrightarrow \mathbb{R} \implies g(x) = 0 \text{ almost everywhere.}$$

7 L^1

7.1 Spring 2020 # 3 ✨

a. Prove that if $g \in L^1(\mathbb{R})$ then

$$\lim_{N \longrightarrow \infty} \int_{|x| \geq N} |f(x)|\,dx = 0,$$

and demonstrate that it is not necessarily the case that $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.

b. Prove that if $f \in L^1([1,\infty])$ and is decreasing, then $\lim_{x \longrightarrow \infty} f(x) = 0$ and in fact $\lim_{x \longrightarrow \infty} xf(x) = 0$.

c. If $f : [1,\infty) \longrightarrow [0,\infty)$ is decreasing with $\lim_{x \longrightarrow \infty} xf(x) = 0$, does this ensure that $f \in L^1([1,\infty))$?

Solution:

Concepts Used:

- Limits
- Cauchy Criterion for Integrals: $\int_a^\infty f(x)\,dx$ converges iff for every $\varepsilon > 0$ there exists an M_0 such that $A, B \geq M_0$ implies $\left|\int_A^B f\right| < \varepsilon$, i.e. $\left|\int_A^B f\right| \overset{A \longrightarrow \infty}{\longrightarrow} 0$.
- Integrals of L^1 functions have vanishing tails: $\int_N^\infty |f| \overset{N \longrightarrow \infty}{\longrightarrow} 0$.
- Mean Value Theorem for Integrals: $\int_a^b f(t)\,dt = (b-a)f(c)$ for some $c \in [a,b]$.

7.1.1 a

Stated integral equality:

- Let $\varepsilon > 0$
- $C_c(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$ is dense so choose $\{f_n\} \longrightarrow f$ with $\|f_n - f\|_1 \longrightarrow 0$.
- Since $\{f_n\}$ are compactly supported, choose $N_0 \gg 1$ such that f_n is zero outside of $B_{N_0}(\mathbf{0})$.

- Then

$$\begin{aligned}
N \geq N_0 \implies \int_{|x|>N} |f| &= \int_{|x|>N} |f - f_n + f_n| \\
&\leq \int_{|x|>N} |f - f_n| + \int_{|x|>N} |f_n| \\
&= \int_{|x|>N} |f - f_n| \\
&\leq \int_{|x|>N} \|f - f_n\|_1 \\
&= \|f_n - f\|_1 \left(\int_{|x|>N} 1 \right) \\
&\overset{n\to\infty}{\longrightarrow} 0 \left(\int_{|x|>N} 1 \right) \\
&= 0 \\
&\overset{N\to\infty}{\longrightarrow} 0.
\end{aligned}$$

To see that this doesn't force $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$:

- Take $f(x)$ to be a train of rectangles of height 1 and area $1/2^j$ centered on even integers.
- Then

$$\int_{|x|>N} |f| = \sum_{j=N}^{\infty} 1/2^j \overset{N\to\infty}{\longrightarrow} 0$$

  as the tail of a convergent sum.
- However $f(x) = 1$ for infinitely many even integers $x > N$, so $f(x) \not\longrightarrow 0$ as $|x| \longrightarrow \infty$.

7.1.2 b

Solution 1 ("Trick")

- Since f is decreasing on $[1, \infty)$, for any $t \in [x - n, x]$ we have

$$x - n \leq t \leq x \implies f(x) \leq f(t) \leq f(x - n).$$

- Integrate over $[x, 2x]$, using monotonicity of the integral:

$$\begin{aligned}
\int_x^{2x} f(x)\, dt \leq \int_x^{2x} f(t)\, dt &\leq \int_x^{2x} f(x - n)\, dt \\
\implies f(x) \int_x^{2x} dt \leq \int_x^{2x} f(t)\, dt &\leq f(x - n) \int_x^{2x} dt \\
\implies xf(x) \leq \int_x^{2x} f(t)\, dt &\leq xf(x - n).
\end{aligned}$$

- By the Cauchy Criterion for integrals, $\lim_{x\to\infty} \int_x^{2x} f(t)\ dt = 0$.
- So the LHS term $xf(x) \overset{x\to\infty}{\longrightarrow} 0$.
- Since $x > 1$, $|f(x)| \leq |xf(x)|$
- Thus $f(x) \overset{x\to\infty}{\longrightarrow} 0$ as well.

Solution 2 (Variation on the Trick)

- Use mean value theorem for integrals:

$$\int_x^{2x} f(t)\,dt = xf(c_x) \quad \text{for some } c_x \in [x, 2x] \text{ depending on } x.$$

- Since f is decreasing,

$$\begin{aligned} x \leq c_x \leq 2x &\implies f(2x) \leq f(c_x) \leq f(x) \\ &\implies 2xf(2x) \leq 2xf(c_x) \leq 2xf(x) \\ &\implies 2xf(2x) \leq 2x\int_x^{2x} f(t)\,dt \leq 2xf(x) \end{aligned}$$

.

- By Cauchy Criterion, $\int_x^{2x} f \longrightarrow 0$.
- So $2xf(2x) \longrightarrow 0$, which by a change of variables gives $uf(u) \longrightarrow 0$.
- Since $u \geq 1$, $f(u) \leq uf(u)$ so $f(u) \longrightarrow 0$ as well.

Solution 3 (Contradiction) Just showing $f(x) \overset{x\longrightarrow\infty}{\longrightarrow} 0$:

- Toward a contradiction, suppose not.
- Since f is decreasing, it can not diverge to $+\infty$
- If $f(x) \longrightarrow -\infty$, then $f \notin L^1(\mathbb{R})$: choose $x_0 \gg 1$ so that $t \geq x_0 \implies f(t) < -1$, then
- Then $t \geq x_0 \implies |f(t)| \geq 1$, so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f(t)|\,dt \geq \int_{x_0}^\infty 1 = \infty.$$

- Otherwise $f(x) \longrightarrow L \neq 0$, some finite limit.
- If $L > 0$:
  - Fix $\varepsilon > 0$, choose $x_0 \gg 1$ such that $t \geq x_0 \implies L - \varepsilon \leq f(t) \leq L$
  - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L - \varepsilon)\,dt = \infty$$

- If $L < 0$:
  - Fix $\varepsilon > 0$, choose $x_0 \gg 1$ such that $t \geq x_0 \implies L \leq f(t) \leq L + \varepsilon$.
  - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L)\,dt = \infty$$

Showing $xf(x) \overset{x\longrightarrow\infty}{\longrightarrow} 0$.

- Toward a contradiction, suppose not.
- (How to show that $xf(x) \not\longrightarrow +\infty$?)
- If $xf(x) \longrightarrow -\infty$
  - Choose a sequence $\Gamma = \{\widehat{x}_i\}$ such that $x_i \longrightarrow \infty$ and $x_i f(x_i) \longrightarrow -\infty$.
  - Choose a subsequence $\Gamma' = \{x_i\}$ such that $x_i f(x_i) \leq -1$ for all i and $x_i \leq x_{i+1}$.
  - Choose a further subsequence $S = \left\{x_i \in \Gamma' \;\middle|\; 2x_i < x_{i+1}\right\}$.

- Then since f is always decreasing, for $t \geq x_0$, $|f|$ is increasing, and $|f(x_i)| \leq |f(2x_i)|$, so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f| \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(t)|\,dt \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(x_i)| = \sum_{x_i \in S} x_i f(x_i) \longrightarrow \infty.$$

- If $xf(x) \longrightarrow L \neq 0$ for $0 < L < \infty$:
  - Fix $\varepsilon > 0$, choose an infinite sequence $\{x_i\}$ such that $L - \varepsilon \leq x_i f(x_i) \leq L$ for all i.

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)|\,dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i)\,dt = \sum_S x_i f(x_i) \geq \sum_S (L - \varepsilon) \longrightarrow \infty.$$

- If $xf(x) \longrightarrow L \neq 0$ for $-\infty < L < 0$:
  - Fix $\varepsilon > 0$, choose an infinite sequence $\{x_i\}$ such that $L \leq x_i f(x_i) \leq L + \varepsilon$ for all i.

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)|\,dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i)\,dt = \sum_S x_i f(x_i) \geq \sum_S (L) \longrightarrow \infty.$$

Solution 4 (Akos's Suggestion) For $x \geq 1$,

$$|xf(x)| = \left| \int_x^{2x} f(x)\,dt \right| \leq \int_x^{2x} |f(x)|\,dt \leq \int_x^{2x} |f(t)|\,dt \leq \int_x^\infty |f(t)|\,dt \overset{x \longrightarrow \infty}{\longrightarrow} 0$$

where we've used

- Since f is decreasing and $\lim_{x \longrightarrow \infty} f(x) = 0$ from part (a), f is non-negative.
- Since f is positive and decreasing, for every $t \in [a, b]$ we have $|f(a)| \leq |f(t)|$.
- By part (a), the last integral goes to zero.

Solution 5 (Peter's)

- Toward a contradiction, produce a sequence $x_i \longrightarrow \infty$ with $x_i f(x_i) \longrightarrow \infty$ and $x_i f(x_i) > \varepsilon > 0$, then

$$\begin{aligned}
\int f(x)\,dx &\geq \sum_{i=1}^\infty \int_{x_i}^{x_{i+1}} f(x)\,dx \\
&\geq \sum_{i=1}^\infty \int_{x_i}^{x_{i+1}} f(x_{i+1})\,dx \\
&= \sum_{i=1}^\infty f(x_{i+1}) \int_{x_i}^{x_{i+1}} dx \\
&\geq \sum_{i=1}^\infty (x_{i+1} - x_i) f(x_{i+1}) \\
&\geq \sum_{i=1}^\infty (x_{i+1} - x_i) \frac{\varepsilon}{x_{i+1}} \\
&= \varepsilon \sum_{i=1}^\infty \left(1 - \frac{x_{i-1}}{x_i}\right) \longrightarrow \infty
\end{aligned}$$

  which can be ensured by passing to a subsequence where $\sum \frac{x_{i-1}}{x_i} < \infty$.

7.1.3 c

- No: take $f(x) = \dfrac{1}{x \ln x}$
- Then by a u-substitution,

$$\int_0^x f = \ln\left(\ln(x)\right) \overset{x \longrightarrow \infty}{\longrightarrow} \infty$$

  is unbounded, so $f \notin L^1([1, \infty))$.
- But

$$xf(x) = \frac{1}{\ln(x)} \overset{x \longrightarrow \infty}{\longrightarrow} 0.$$

7.2 Fall 2019 # 5. ✨

7.2.1 a

Show that if f is continuous with compact support on $\mathbb{R}$, then

$$\lim_{y \to 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dx = 0$$

7.2.2 b

Let $f \in L^1(\mathbb{R})$ and for each $h > 0$ let

$$\mathcal{A}_h f(x) := \frac{1}{2h} \int_{|y| \leq h} f(x-y) dy$$

i. Prove that $\|\mathcal{A}_h f\|_1 \leq \|f\|_1$ for all $h > 0$.

ii. Prove that $\mathcal{A}_h f \longrightarrow f$ in $L^1(\mathbb{R})$ as $h \longrightarrow 0^+$.

Walk through.

Solution:

Concepts Used:

- Continuity in L^1 (recall that DCT won't work! Notes 19.4, prove it for a dense subset first).
- Lebesgue differentiation in 1-dimensional case. See HW 5.6.

7.2.3 a

Choose $g \in C_c^0$ such that $\|f - g\|_1 \longrightarrow 0$.
By translation invariance, $\|\tau_h f - \tau_h g\|_1 \longrightarrow 0$.

Write

$$\begin{aligned}\|\tau f - f\|_1 &= \|\tau_h f - g + g - \tau_h g + \tau_h g - f\|_1 \\ &\leq \|\tau_h f - \tau_h g\| + \|g - f\| + \|\tau_h g - g\| \\ &\longrightarrow \|\tau_h g - g\|,\end{aligned}$$

so it suffices to show that $\|\tau_h g - g\| \longrightarrow 0$ for $g \in C_c^0$.
Fix $\varepsilon > 0$. Enlarge the support of g to K such that

$$|h| \leq 1 \text{ and } x \in K^c \implies |g(x-h) - g(x)| = 0.$$

By uniform continuity of g, pick $\delta \leq 1$ small enough such that

$$x \in K,\ |h| \leq \delta \implies |g(x-h) - g(x)| < \varepsilon,$$

then

$$\int_K |g(x-h) - g(x)| \leq \int_K \varepsilon = \varepsilon \cdot m(K) \longrightarrow 0.$$

7.2.4 b

We have

$$\begin{aligned}\int_{\mathbb{R}} |A_h(f)(x)|\ dx &= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y)\ dy \right| dx \\ &\leq \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)|\ dy\ dx \\ &=_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)|\ \mathbf{dx\ dy} \\ &= \int_{\mathbb{R}} |f(y)|\ dy \\ &= \|f\|_1,\end{aligned}$$

and (rough sketch)

$$\begin{aligned}\int_{\mathbb{R}} |A_h(f)(x) - f(x)|\ dx &= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y)\ dy \right) - f(x) \right| dx \\ &= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y)\ dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x)\ dy \right| dx \\ &\leq_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{B(h,x)} |f(y-x) - f(x)|\ \mathbf{dx\ dy} \\ &\leq \frac{1}{2h} \int_{\mathbb{R}} \|\tau_x f - f\|_1\ dy \\ &\longrightarrow 0 \quad \text{by (a).}\end{aligned}$$

7.3 Fall 2017 # 3 ✨

Let

$$S = \text{span}_{\mathbb{C}} \left\{ \chi_{(a,b)} \;\middle|\; a, b \in \mathbb{R} \right\},$$

the complex linear span of characteristic functions of intervals of the form (a, b).

Show that for every $f \in L^1(\mathbb{R})$, there exists a sequence of functions $\{f_n\} \subset S$ such that

$$\lim_{n\to\infty} \|f_n - f\|_1 = 0$$

Walk through.

Solution:

Concepts Used:

- From homework: E is Lebesgue measurable iff there exists a finite union of closed cubes A such that $m(E \Delta A) < \varepsilon$.

It suffices to show that S is dense in simple functions, and since simple functions are *finite* linear combinations of characteristic functions, it suffices to show this for χ_A for A a measurable set.
Let $s = \chi_A$. By regularity of the Lebesgue measure, choose an open set $O \supseteq A$ such that $m(O \setminus A) < \varepsilon$.
O is an open subset of $\mathbb{R}$, and thus $O = \coprod_{j\in\mathbb{N}} I_j$ is a disjoint union of countably many open intervals.
Now choose N large enough such that $m(O \Delta I_{N,n}) < \varepsilon = \dfrac{1}{n}$ where we define $I_{N,n} \coloneqq \coprod_{j=1}^{N} I_j$.
Now define $f_n = \chi_{I_{N,n}}$, then

$$\|s - f_n\|_1 = \int \left| \chi_A - \chi_{I_{N,n}} \right| = m(A \Delta I_{N,n}) \overset{n\to\infty}{\longrightarrow} 0.$$

Since any simple function is a finite linear combination of χ_{A_i}, we can do this for each i to extend this result to all simple functions. But simple functions are dense in L^1, so S is dense in L^1.

7.4 Spring 2015 # 4 🚩

Define

$$f(x,y) := \begin{cases} \dfrac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0 \le x \le y \\ 0 & \text{otherwise} \end{cases}$$

Carefully show that $f \in L^1(\mathbb{R}^2)$.

7.5 Fall 2014 # 3 🚩

Let $f \in L^1(\mathbb{R})$. Show that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } \qquad m(E) < \delta \implies \int_E |f(x)|\,dx < \varepsilon$$

7.6 Spring 2014 # 1 🚩

1. Give an example of a continuous $f \in L^1(\mathbb{R})$ such that $f(x) \not\longrightarrow 0$ as$|x| \longrightarrow \infty$.
2. Show that if f is *uniformly* continuous, then

$$\lim_{|x| \longrightarrow \infty} f(x) = 0.$$

8 Fubini-Tonelli

8.1 Spring 2020 # 4 ✨

Let $f, g \in L^1(\mathbb{R})$. Argue that $H(x,y) \coloneqq f(y)g(x-y)$ defines a function in $L^1(\mathbb{R}^2)$ and deduce from this fact that

$$(f * g)(x) \coloneqq \int_{\mathbb{R}} f(y)g(x-y)\,dy$$

defines a function in $L^1(\mathbb{R})$ that satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

Solution:

Concepts Used:

- Tonelli: non-negative and measurable yields measurability of slices and equality of iterated integrals
- Fubini: $f(x,y) \in L^1$ yields *integrable* slices and equality of iterated integrals
- F/T: apply Tonelli to $|f|$; if finite, $f \in L^1$ and apply Fubini to f

$$
\begin{aligned}
\|H(x)\|_1 &= \int_{\mathbb{R}} |H(x,y)|\,dx \\
&= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y)g(x-y)\,dy \right| dx \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(x-y)|\,dy \right) dx \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(x-y)|\,dx \right) dy \quad \text{by Tonelli} \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(t)|\,dt \right) dy \quad \text{setting } t = x-y,\ dt = -dx \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)| \cdot |g(t)|\,dt \right) dy \\
&= \int_{\mathbb{R}} |f(y)| \cdot \left(\int_{\mathbb{R}} |g(t)|\,dt \right) dy \\
&\coloneqq \int_{\mathbb{R}} |f(y)| \cdot \|g\|_1\,dy \\
&= \|g\|_1 \int_{\mathbb{R}} |f(y)|\,dy \\
&\coloneqq \|g\|_1 \|f\|_1 \\
&< \infty \quad \text{by assumption} \quad .
\end{aligned}
$$

- H is measurable on $\mathbb{R}^2$:
  - If we can show $\tilde{f}(x,y) \coloneqq f(y)$ and $\tilde{g}(x,y) \coloneqq g(x-y)$ are both measurable on $\mathbb{R}^2$, then $H = \tilde{f} \cdot \tilde{g}$ is a product of measurable functions and thus measurable.
  - $f \in L^1$, and L^1 functions are measurable by definition.
  - The function $(x,y) \mapsto g(x-y)$ is measurable on $\mathbb{R}^2$:
    - Let g be measurable on $\mathbb{R}$, then the cylinder function $G(x,y) = g(x)$ on $\mathbb{R}^2$ is always measurable
    - Define a linear transformation $T \coloneqq [1,-1;0,1]$ which sends $(x,y) \longrightarrow (x-y,y)$, then $T \in \mathrm{GL}(2,\mathbb{R})$ is linear and thus measurable.
    - Then $(G \circ T)(x,y) = G(x-y,y) = \tilde{g}(x-y)$, so $\tilde{g}$ is a composition of measurable functions and thus measurable.
- Apply **Tonelli** to $|H|$
  - H measurable implies $|H|$ is measurable
  - $|H|$ is non-negative
  - So the iterated integrals are equal in the extended sense
  - The calculation shows the iterated integral is finite, to $\int |H|$ is finite and H is thus integrable on $\mathbb{R}^2$.

Note: Fubini is not needed, since we're not calculating the actual integral, just showing H is integrable.

8.2 Spring 2019 # 4 ✨

Let f be a non-negative function on $\mathbb{R}^n$ and $\mathcal{A} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq f(x)\}$.

Prove the validity of the following two statements:

a. f is a Lebesgue measurable function on $\mathbb{R}^n \iff \mathcal{A}$ is a Lebesgue measurable subset of $\mathbb{R}^{n+1}$

b. If f is a Lebesgue measurable function on $\mathbb{R}^n$, then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)dx = \int_0^\infty m\left(\{x \in \mathbb{R}^n : f(x) \geq t\}\right) dt$$

Add concepts.

Solution:

Concepts Used:

- See S&S p.82.

8.2.1 a

$\implies$:

- Suppose f is a measurable function.
- Note that $\mathcal{A} = \{f(x) - t \geq 0\} \bigcap \{t \geq 0\}$.
- Define $F(x,t) = f(x)$, $G(x,t) = t$, which are cylinders on measurable functions and thus measurable.
- Define $H(x,y) = F(x,t) - G(x,t)$, which are linear combinations of measurable functions and thus measurable.
- Then $\mathcal{A} = \{H \geq 0\} \bigcap \{G \geq 0\}$ as a countable intersection of measurable sets, which is again measurable.

$\impliedby$:

- Suppose $\mathcal{A}$ is a measurable set.
- Then FT on $\chi_{\mathcal{A}}$ implies that for almost every $x \in \mathbb{R}^n$, the x-slices $\mathcal{A}_x$ are measurable and \$

$$\mathcal{A}_x \coloneqq \left\{t \in \mathbb{R} \;\middle|\; (x,t) \in \mathcal{A}\right\} = [0, f(x)] \implies m(\mathcal{A}_x) = f(x) - 0 = f(x)$$

- But $x \mapsto m(\mathcal{A}_x)$ is a measurable function, and is exactly the function $x \mapsto f(x)$, so f is measurable.

8.2.2 b

- Note

$$\mathcal{A} = \left\{(x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; 0 \leq t \leq f(x)\right\}$$
$$\mathcal{A}_t = \left\{x \in \mathbb{R}^n \;\middle|\; t \leq f(x)\right\}.$$

- Then

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)\ dx &= \int_{\mathbb{R}^n} \int_0^{f(x)} 1\ dt\ dx \\ &= \int_{\mathbb{R}^n} \int_0^{\infty} \chi_{\mathcal{A}}\ dt\ dx \\ &\overset{F.T.}{=} \int_0^{\infty} \int_{\mathbb{R}^n} \chi_{\mathcal{A}}\ dx\ dt \\ &= \int_0^{\infty} m(\mathcal{A}_t)\ dt,\end{aligned}$$

  where we just use that $\int \int \chi_{\mathcal{A}} = m(\mathcal{A})$
- By F.T., all of these integrals are equal.

  Why is FT justified.

8.3 Fall 2018 # 5 ✨

Let $f \geq 0$ be a measurable function on $\mathbb{R}$. Show that

$$\int_{\mathbb{R}} f = \int_0^{\infty} m(\{x : f(x) > t\})dt$$

Solution:

Concepts Used:

- Claim: If $E \subseteq \mathbb{R}^a \times \mathbb{R}^b$ is a measurable set, then for almost every $y \in \mathbb{R}^b$, the slice E^y is measurable and

$$m(E) = \int_{\mathbb{R}^b} m(E^y)\, dy.$$

  - Set $g = \chi_E$, which is non-negative and measurable, so apply Tonelli.
  - Conclude that $g^y = \chi_{E^y}$ is measurable, the function $y \mapsto \int g^y(x)\, dx$ is measurable, and $\int \int g^y(x)\, dx\, dy = \int g$.
  - But $\int g = m(E)$ and $\int \int g^y(x)\, dx\, dy = \int m(E^y)\, dy$.

Note: f is a function $\mathbb{R} \longrightarrow \mathbb{R}$ in the original problem, but here I've assumed $f : \mathbb{R}^n \longrightarrow \mathbb{R}$.

- Since $f \geq 0$, set

$$E \coloneqq \left\{(x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; f(x) > t\right\} = \left\{(x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; 0 \leq t < f(x)\right\}.$$

- Claim: since f is measurable, E is measurable and thus $m(E)$ makes sense.
  - Since f is measurable, $F(x,t) \coloneqq t - f(x)$ is measurable on $\mathbb{R}^n \times \mathbb{R}$.
  - Then write $E = \{F < 0\} \bigcap \{t \geq 0\}$ as an intersection of measurable sets.

- We have slices

$$E^t := \left\{x \in \mathbb{R}^n \;\middle|\; (x,t) \in E\right\} = \left\{x \in \mathbb{R}^n \;\middle|\; 0 \le t < f(x)\right\}$$
$$E^x := \left\{t \in \mathbb{R} \;\middle|\; (x,t) \in E\right\} = \left\{t \in \mathbb{R} \;\middle|\; 0 \le t \le f(x)\right\} = [0, f(x)].$$

  - E_t is precisely the set that appears in the original RHS integrand.
  - $m(E^x) = f(x)$.
- Claim: χ_E satisfies the conditions of Tonelli, and thus $m(E) = \int \chi_E$ is equal to any iterated integral.
  - Non-negative: clear since $0 \le \chi_E \le 1$
  - Measurable: characteristic functions of measurable sets are measurable.
- Conclude:
  1. For almost every x, E^x is a measurable set, $x \mapsto m(E^x)$ is a measurable function, and $m(E) = \int_{\mathbb{R}^n} m(E^x)\,dx$
  2. For almost every t, E^t is a measurable set, $t \mapsto m(E^t)$ is a measurable function, and $m(E) = \int_{\mathbb{R}} m(E^t)\,dt$
- On one hand,

$$\begin{aligned} m(E) &= \int_{\mathbb{R}^{n+1}} \chi_E(x,t) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_E(x,t)\,dt\,dx \quad \text{by Tonelli} \\ &= \int_{\mathbb{R}^n} m(E^x)\,dx \quad \text{first conclusion} \\ &= \int_{\mathbb{R}^n} f(x)\,dx. \end{aligned}$$

- On the other hand,

$$\begin{aligned} m(E) &= \int_{\mathbb{R}^{n+1}} \chi_E(x,t) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_E(x,t)\,dx\,dt \quad \text{by Tonelli} \\ &= \int_{\mathbb{R}} m(E^t)\,dt \quad \text{second conclusion.} \end{aligned}$$

- Thus

$$\int_{\mathbb{R}^n} f\,dx = m(E) = \int_{\mathbb{R}} m(E^t)\,dt = \int_{\mathbb{R}} m\Big(\Big\{x \;\Big|\; f(x) > t\Big\}\Big).$$

8.4 Fall 2015 # 5

Let $f, g \in L^1(\mathbb{R})$ be Borel measurable.

1. Show that

- The function

$$F(x,y) \coloneqq f(x-y)g(y)$$

  is Borel measurable on $\mathbb{R}^2$, and
- For almost every $y \in \mathbb{R}$,

$$F_y(x) \coloneqq f(x-y)g(y)$$

  is integrable with respect to y.

2. Show that $f * g \in L^1(\mathbb{R})$ and

$$\|f*g\|_1 \leq \|f\|_1 \|g\|_1$$

8.5 Spring 2014 # 5

Let $f, g \in L^1([0,1])$ and for all $x \in [0,1]$ define

$$F(x) \coloneqq \int_0^x f(y)\,dy \quad \text{and} \quad G(x) \coloneqq \int_0^x g(y)\,dy.$$

Prove that

$$\int_0^1 F(x)g(x)\,dx = F(1)G(1) - \int_0^1 f(x)G(x)\,dx$$

9 L^2 and Fourier Analysis

9.1 Spring 2020 # 6

9.1.1 a

Show that

$$L^2([0,1]) \subseteq L^1([0,1]) \quad \text{and} \quad \ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z}).$$

9.1.2 b

For $f \in L^1([0,1])$ define

$$\widehat{f}(n) \coloneqq \int_0^1 f(x)e^{-2\pi i n x}\,dx.$$

Prove that if $f \in L^1([0,1])$ and $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$ then

$$S_N f(x) \coloneqq \sum_{|n|\leq N} \widehat{f}(n)e^{2\pi i n x}.$$

converges uniformly on $[0,1]$ to a continuous function g such that $g = f$ almost everywhere.

Hint: One approach is to argue that if $f \in L^1([0,1])$ with $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$ then $f \in L^2([0,1])$.

Solution:

Concepts Used:

- For $e_n(x) \coloneqq e^{2\pi i n x}$, the set $\{e_n\}$ is an orthonormal basis for $L^2([0,1])$.
- For any orthonormal sequence in a Hilbert space, we have Bessel's inequality:

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

- When $\{e_n\}$ is a basis, the above is an *equality* (Parseval)
- Arguing uniform convergence: since $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$, we should be able to apply the M test.

9.1.3 a

Claim: $\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$.

- Set $\mathbf{c} = \left\{c_k \;\middle|\; k \in \mathbb{Z}\right\} \in \ell^1(\mathbb{Z})$.
- It suffices to show that if $\sum_{k\in\mathbb{Z}} |c_k| < \infty$ then $\sum_{k\in\mathbb{Z}} |c_k|^2 < \infty$.
- Let $S = \left\{c_k \;\middle|\; |c_k| \leq 1\right\}$, then $c_k \in S \implies |c_k|^2 \leq |c_k|$
- Claim: S^c can only contain finitely many elements, all of which are finite.
  - If not, either $S^c \coloneqq \{c_j\}_{j=1}^{\infty}$ is infinite with every $|c_j| > 1$, which forces

$$\sum_{c_k \in S^c} |c_k| = \sum_{j=1}^{\infty} |c_j| > \sum_{j=1}^{\infty} 1 = \infty.$$

  - If any $c_j = \infty$, then $\sum_{k\in\mathbb{Z}} |c_k| \geq c_j = \infty$.
- So S^c is a finite set of finite integers, let $N = \max\left\{|c_j|^2 \;\middle|\; c_j \in S^c\right\} < \infty$.
- Rewrite the sum

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |c_k|^2 &= \sum_{c_k\in S} |c_k|^2 + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \sum_{c_k\in S} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \sum_{k\in\mathbb{Z}} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \quad \text{since the } |c_k| \text{ are all positive} \\
&= \|\mathbf{c}\|_{\ell^1} + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \|\mathbf{c}\|_{\ell^1} + |S^c| \cdot N \\
&< \infty.
\end{aligned}$$

Claim: $L^2([0,1]) \subseteq L^1([0,1])$.

- It suffices to show that $\int |f|^2 < \infty \implies \int |f| < \infty$.

- Define $S = \left\{ x \in [0,1] \;\middle|\; |f(x)| \leq 1 \right\}$, then $x \in S^c \implies |f(x)|^2 \geq |f(x)|$.
- Break up the integral:

$$\begin{aligned}
\int_{\mathbb{R}} |f| &= \int_S |f| + \int_{S^c} |f| \\
&\leq \int_S |f| + \int_{S^c} |f|^2 \\
&\leq \int_S |f| + \|f\|_2 \\
&\leq \sup_{x \in S} \{|f(x)|\} \cdot \mu(S) + \|f\|_2 \\
&= 1 \cdot \mu(S) + \|f\|_2 \quad \text{by definition of } S \\
&\leq 1 \cdot \mu([0,1]) + \|f\|_2 \quad \text{since } S \subseteq [0,1] \\
&= 1 + \|f\|_2 \\
&< \infty.
\end{aligned}$$

Note: this proof shows $L^2(X) \subseteq L^1(X)$ whenever $\mu(X) < \infty$.

9.2 Fall 2017 # 5 ✨

Let φ be a compactly supported smooth function that vanishes outside of an interval $[-N, N]$ such that $\int_{\mathbb{R}} \varphi(x)\, dx = 1$.

For $f \in L^1(\mathbb{R})$, define

$$K_j(x) \coloneqq j\varphi(jx), \qquad f * K_j(x) \coloneqq \int_{\mathbb{R}} f(x-y) K_j(y)\, dy$$

and prove the following:

1. Each $f * K_j$ is smooth and compactly supported.
2. $$\lim_{j \longrightarrow \infty} \|f * K_j - f\|_1 = 0$$

Hint:

$$\lim_{y \longrightarrow 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dy = 0$$

Add concepts.

Solution:

Concepts Used:

- ?

9.2.1 a

Lemma: If $\varphi \in C_c^1$, then $(f * \varphi)' = f * \varphi'$ almost everywhere.
Silly Proof:

$$\begin{aligned}
\mathcal{F}((f*\varphi)') &= 2\pi i\xi\ \mathcal{F}(f*\varphi) \\
&= 2\pi i\xi\ \mathcal{F}(f)\ \mathcal{F}(\varphi) \\
&= \mathcal{F}(f)\cdot(2\pi i\xi\ \mathcal{F}(\varphi)) \\
&= \mathcal{F}(f)\cdot\mathcal{F}(\varphi') \\
&= \mathcal{F}(f*\varphi').
\end{aligned}$$

Actual proof:

$$\begin{aligned}
(f*\varphi)'(x) &= (\varphi * f)'(x) \\
&= \lim_{h\to 0} \frac{(\varphi*f)'(x+h) - (\varphi*f)'(x)}{h} \\
&= \lim_{h\to 0} \int \frac{\varphi(x+h-y)-\varphi(x-y)}{h} f(y) \\
&\overset{DCT}{=} \int \lim_{h\to 0} \frac{\varphi(x+h-y)-\varphi(x-y)}{h} f(y) \\
&= \int \varphi'(x-y) f(y) \\
&= (\varphi' * f)(x) \\
&= (f*\varphi')(x).
\end{aligned}$$

To see that the DCT is justified, we can apply the MVT on the interval $[0,h]$ to f to obtain

$$\frac{\varphi(x+h-y)-\varphi(x-y)}{h} = \varphi'(c) \quad c\in[0,h],$$

and since φ' is continuous and compactly supported, φ' is bounded by some $M<\infty$ by the extreme value theorem and thus

$$\begin{aligned}
\int \left| \frac{\varphi(x+h-y)-\varphi(x-y)}{h} f(y) \right| &= \int |\varphi'(c) f(y)| \\
&\le \int |M||f| \\
&= |M| \int |f| < \infty,
\end{aligned}$$

since $f\in L^1$ by assumption, so we can take $g \coloneqq |M||f|$ as the dominating function.
Applying this theorem infinitely many times shows that $f*\varphi$ is smooth.
To see that $f*\varphi$ is compactly supported, approximate f by a *continuous* compactly supported function h, so $\|h-f\|_1 \xrightarrow{L^1} 0$.
Now let $g_x(y) = \varphi(x-y)$, and note that $\operatorname{supp}(g) = x - \operatorname{supp}(\varphi)$ which is still compact.

But since $\text{supp}(h)$ is bounded, there is some N such that

$$|x| > N \implies A_x \coloneqq \text{supp}(h) \bigcap \text{supp}(g_x) = \emptyset$$

and thus

$$\begin{aligned}(h * \varphi)(x) &= \int_{\mathbb{R}} \varphi(x-y)h(y)\ dy \\ &= \int_{A_x} g_x(y)h(y) \\ &= 0,\end{aligned}$$

so $\left\{x \;\middle|\; f * g(x) = 0\right\}$ is open, and its complement is closed and bounded and thus compact.

9.2.2 b

$$\begin{aligned}\|f * K_j - f\|_1 &= \int \left| \int f(x-y)K_j(y)\ dy - f(x) \right| dx \\ &= \int \left| \int f(x-y)K_j(y)\ dy - \int f(x)K_j(y)\ dy \right| dx \\ &= \int \left| \int (f(x-y) - f(x))K_j(y)\ dy \right| dx \\ &\leq \int \int |(f(x-y) - f(x))| \cdot |K_j(y)|\ dy\ dx \\ &\overset{FT}{=} \int \int |(f(x-y) - f(x))| \cdot |K_j(y)|\ \mathbf{dx\ dy} \\ &= \int |K_j(y)| \left(\int |(f(x-y) - f(x))|\ dx \right)\ dy \\ &= \int |K_j(y)| \cdot \|f - \tau_y f\|_1\ dy.\end{aligned}$$

We now split the integral up into pieces.

1. Chose δ small enough such that $|y| < \delta \implies \|f - \tau_y f\|_1 < \varepsilon$ by continuity of translation in L^1, and
2. Since φ is compactly supported, choose J large enough such that

$$j > J \implies \int_{|y| \geq \delta} |K_j(y)|\ dy = \int_{|y| \geq \delta} |j\varphi(jy)| = 0$$

Then

$$\begin{aligned}\|f * K_j - f\|_1 &\leq \int |K_j(y)| \cdot \|f - \tau_y f\|_1\ dy \\ &= \int_{|y| < \delta} |K_j(y)| \cdot \|f - \tau_y f\|_1\ dy + \int_{|y| \geq \delta} |K_j(y)| \cdot \|f - \tau_y f\|_1\ dy \\ &= \varepsilon \int_{|y| \geq \delta} |K_j(y)| + 0 \\ &\leq \varepsilon(1) \longrightarrow 0.\end{aligned}$$

9.3 Spring 2017 # 5

Let $f, g \in L^2(\mathbb{R})$. Prove that the formula

$$h(x) := \int_{-\infty}^{\infty} f(t)g(x-t)\,dt$$

defines a uniformly continuous function h on $\mathbb{R}$.

9.4 Spring 2015 # 6

Let $f \in L^1(\mathbb{R})$ and g be a bounded measurable function on $\mathbb{R}$.

1. Show that the convolution $f * g$ is well-defined, bounded, and uniformly continuous on $\mathbb{R}$.
2. Prove that one further assumes that $g \in C^1(\mathbb{R})$ with bounded derivative, then $f * g \in C^1(\mathbb{R})$ and

$$\frac{d}{dx}(f * g) = f * \left(\frac{d}{dx} g\right)$$

9.5 Fall 2014 # 5

1. Let $f \in C_c^0(\mathbb{R}^n)$, and show

$$\lim_{t \longrightarrow 0} \int_{\mathbb{R}^n} |f(x+t) - f(x)|\,dx = 0.$$

2. Extend the above result to $f \in L^1(\mathbb{R}^n)$ and show that

$$f \in L^1(\mathbb{R}^n), \quad g \in L^\infty(\mathbb{R}^n) \quad \Longrightarrow \; f * g \text{ is bounded and uniformly continuous.}$$

10 Functional Analysis: General

10.1 Fall 2019 # 4

Let $\{u_n\}_{n=1}^\infty$ be an orthonormal sequence in a Hilbert space $\mathcal{H}$.

10.1.1 a

Prove that for every $x \in \mathcal{H}$ one has

$$\sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 \leq \|x\|^2$$

10.1.2 b

Prove that for any sequence $\{a_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$ there exists an element $x \in \mathcal{H}$ such that

$$a_n = \langle x,\ u_n\rangle \text{ for all } n \in \mathbb{N}$$

and

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2$$

Solution:

Concepts Used:

- Bessel's Inequality
- Pythagoras
- Surjectivity of the Riesz map
- Parseval's Identity
- Trick – remember to write out finite sum S_N, and consider $\|x - S_N\|$.

10.1.3 a

Claim:

$$0 \leq \left\| x - \sum_{n=1}^{N} \langle x, u_n\rangle\, u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n\rangle|^2$$
$$\implies \sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 \leq \|x\|^2.$$

Proof: Let $S_N = \sum_{n=1}^{N} \langle x,\ u_n\rangle u_n$. Then

$$\begin{aligned}
0 &\leq \|x - S_N\|^2 \\
&= \langle x - S_n,\ x - S_N\rangle \\
&= \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n\rangle|^2 \\
&\xrightarrow{N\to\infty} \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n\rangle|^2.
\end{aligned}$$

10.1.4 b

1. Fix $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
2. Define
$$x \coloneqq \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{k=1}^{N} a_k u_k$$
3. $\{S_N\}$ Cauchy (by 1) and H complete $\implies x \in H$.
4.
$$\langle x,\ u_n\rangle = \left\langle \sum_k a_k u_k,\ u_n \right\rangle = \sum_k a_k \langle u_k,\ u_n\rangle = a_n \quad \forall n \in \mathbb{N}$$

since the u_k are all orthogonal.

5.

$$\|x\|^2 = \left\|\sum_k a_k u_k\right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$$

by Pythagoras since the u_k are normal.

Bonus: We didn't use completeness here, so the Fourier series may not actually converge to x. If $\{u_n\}$ is **complete** (so $x = 0 \iff \langle x,\ u_n\rangle = 0\ \forall n$) then the Fourier series *does* converge to x and $\sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 = \|x\|^2$ for all $x \in H$.

10.2 Spring 2019 # 5 ✨

10.2.1 a

Show that $L^2([0,1]) \subseteq L^1([0,1])$ and argue that $L^2([0,1])$ in fact forms a dense subset of $L^1([0,1])$.

10.2.2 b

Let Λ be a continuous linear functional on $L^1([0,1])$.

Prove the Riesz Representation Theorem for $L^1([0,1])$ by following the steps below:

i. Establish the existence of a function $g \in L^2([0,1])$ which represents Λ in the sense that

$$\Lambda(f) = f(x)g(x)dx \text{ for all } f \in L^2([0,1]).$$

Hint: You may use, without proof, the Riesz Representation Theorem for $L^2([0,1])$.

ii. Argue that the g obtained above must in fact belong to $L^\infty([0,1])$ and represent Λ in the sense that

$$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}dx \quad \text{for all } f \in L^1([0,1])$$

with

$$\|g\|_{L^\infty([0,1])} = \|\Lambda\|_{L^1([0,1])^\vee}$$

Solution:

Concepts Used:

- Holders' inequality: $\|fg\|_1 \leq \|f\|_p \|f\|_q$
- Riesz Representation for L^2: If $\Lambda \in (L^2)^\vee$ then there exists a unique $g \in L^2$ such that $\Lambda(f) = \int fg$.
- $\|f\|_{L^\infty(X)} \coloneqq \inf \left\{ t \geq 0 \;\middle|\; |f(x)| \leq t \text{ almost everywhere} \right\}$.
- **Lemma**: $m(X) < \infty \implies L^p(X) \subset L^2(X)$.

  Proof .

  - Write Holder's inequality as $\|fg\|_1 \leq \|f\|_a \|g\|_b$ where $\frac{1}{a} + \frac{1}{b} = 1$, then

    $$\|f\|_p^p = \||f|^p\|_1 \leq \||f|^p\|_a \, \|1\|_b.$$

  - Now take $a = \frac{2}{p}$ and this reduces to

    $$\begin{aligned} &\|f\|_p^p \leq \|f\|_2^p \, m(X)^{\frac{1}{b}} \\ \implies &\|f\|_p \leq \|f\|_2 \cdot O(m(X)) < \infty. \end{aligned}$$

  ■

10.2.3 a

- Note $X = [0,1] \implies m(X) = 1$.
- By Holder's inequality with $p = q = 2$,

$$\|f\|_1 = \|f \cdot 1\|_1 \leq \|f\|_2 \cdot \|1\|_2 = \|f\|_2 \cdot m(X)^{\frac{1}{2}} = \|f\|_2,$$

- Thus $L^2(X) \subseteq L^1(X)$
- Since they share a common dense subset (simple functions) L^2 is dense in L^1

What theorem is this using?

10.2.4 b

Let $\Lambda \in L^1(X)^\vee$ be arbitrary.

1: Existence of g Representing Λ. Let $f \in L^2 \subseteq L^1$ be arbitrary.
Claim: $\Lambda \in L^1(X)^\vee \implies \Lambda \in L^2(X)^\vee$.

- Suffices to show that $\|\Gamma\|_{L^2(X)^\vee} \coloneqq \sup_{\|f\|_2 = 1} |\Gamma(f)| < \infty$, since bounded implies continuous.
- By the lemma, $\|f\|_1 \leq C \|f\|_2$ for some constant $C \approx m(X)$.
- Note

$$\|\Lambda\|_{L^1(X)^\vee} \coloneqq \sup_{\|f\|_1 = 1} |\Lambda(f)|$$

- Define $\widehat{f} = \dfrac{f}{\|f\|_1}$ so $\left\|\widehat{f}\right\|_1 = 1$
- Since $\|\Lambda\|_{1^\vee}$ is a supremum over *all* $f \in L^1(X)$ with $\|f\|_1 = 1$,

$$\left|\Lambda(\widehat{f})\right| \leq \|\Lambda\|_{(L^1(X))^\vee},$$

- Then

$$\begin{aligned} \frac{|\Lambda(f)|}{\|f\|_1} = \left|\Lambda(\widehat{f})\right| &\leq \|\Lambda\|_{L^1(X)^\vee} \\ \implies |\Lambda(f)| &\leq \|\Lambda\|_{1^\vee} \cdot \|f\|_1 \\ &\leq \|\Lambda\|_{1^\vee} \cdot C\|f\|_2 < \infty \quad \text{by assumption,} \end{aligned}$$

- So $\Lambda \in (L^2)^\vee$.
  Now apply Riesz Representation for L^2: there is a $g \in L^2$ such that

$$f \in L^2 \implies \Lambda(f) = \langle f,\ g\rangle \coloneqq \int_0^1 f(x)\overline{g(x)}\,dx.$$

2: g is in L^∞

- It suffices to show $\|g\|_{L^\infty(X)} < \infty$.
- Since we're assuming $\|\Gamma\|_{L^1(X)^\vee} < \infty$, it suffices to show the stated equality.

Is this assumed..? Or did we show it..?

- Claim: $\|\Lambda\|_{L^1(X)^\vee} = \|g\|_{L^\infty(X)}$
  - The result will follow since Λ was assumed to be in $L^1(X)^\vee$, so $\|\Lambda\|_{L^1(X)^\vee} < \infty$.
  - $\leq$:

$$\begin{aligned} \|\Lambda\|_{L^1(X)^\vee} &= \sup_{\|f\|_1=1} |\Lambda(f)| \\ &= \sup_{\|f\|_1=1} \left|\int_X f\bar{g}\right| \quad \text{by (i)} \\ &= \sup_{\|f\|_1=1} \int_X |f\bar{g}| \\ &\coloneqq \sup_{\|f\|_1=1} \|fg\|_1 \\ &\leq \sup_{\|f\|_1=1} \|f\|_1\|g\|_\infty \quad \text{by Holder with } p=1, q=\infty \\ &= \|g\|_\infty, \end{aligned}$$

  - $\geq$:
    - Suppose toward a contradiction that $\|g\|_\infty > \|\Lambda\|_{1^\vee}$.
    - Then there exists some $E \subseteq X$ with $m(E) > 0$ such that

$$x \in E \implies |g(x)| > \|\Lambda\|_{L^1(X)^\vee}.$$

    - Define

$$h = \frac{1}{m(E)}\frac{\overline{g}}{|g|}\chi_E.$$

* Note $\|h\|_{L^1(X)} = 1$.
* Then

$$\begin{aligned}
\Lambda(h) &= \int_X hg \\
&:= \int_X \frac{1}{m(E)} \frac{g\overline{g}}{|g|} \chi_E \\
&= \frac{1}{m(E)} \int_E |g| \\
&\geq \frac{1}{m(E)} \|g\|_\infty m(E) \\
&= \|g\|_\infty \\
&> \|\Lambda\|_{L^1(X)^\vee},
\end{aligned}$$

a contradiction since $\|\Lambda\|_{L^1(X)^\vee}$ is the supremum over all h_α with $\|h_\alpha\|_{L^1(X)} = 1$.

10.3 Spring 2016 # 6

Without using the Riesz Representation Theorem, compute

$$\sup \left\{ \left| \int_0^1 f(x) e^x dx \right| \;\middle|\; f \in L^2([0,1], m),\ \|f\|_2 \leq 1 \right\}$$

10.4 Spring 2015 # 5

Let $\mathcal{H}$ be a Hilbert space.

1. Let $x \in \mathcal{H}$ and $\{u_n\}_{n=1}^N$ be an orthonormal set. Prove that the best approximation to x in $\mathcal{H}$ by an element in $\operatorname{span}_{\mathbb{C}} \{u_n\}$ is given by

$$\widehat{x} := \sum_{n=1}^N \langle x,\ u_n \rangle u_n.$$

2. Conclude that finite dimensional subspaces of $\mathcal{H}$ are always closed.

10.5 Fall 2015 # 6

Let $f : [0,1] \longrightarrow \mathbb{R}$ be continuous. Show that

$$\sup \left\{ \|fg\|_1 \;\middle|\; g \in L^1[0,1],\ \|g\|_1 \leq 1 \right\} = \|f\|_\infty$$

10.6 Fall 2014 # 6

Let $1 \leq p, q \leq \infty$ be conjugate exponents, and show that

$$f \in L^p(\mathbb{R}^n) \implies \|f\|_p = \sup_{\|g\|_q = 1} \left| \int f(x) g(x) dx \right|$$

11 Functional Analysis: Banach Spaces

11.1 Spring 2019 # 1 ✨

Let $C([0,1])$ denote the space of all continuous real-valued functions on $[0,1]$.

a. Prove that $C([0,1])$ is complete under the uniform norm $\|f\|_u := \sup_{x\in[0,1]} |f(x)|$.

b. Prove that $C([0,1])$ is not complete under the L^1-norm $\|f\|_1 = \int_0^1 |f(x)|\, dx$.

Add concepts.

Solution:

Concepts Used:

- ?

11.1.1 a

- Let $\{f_n\}$ be a Cauchy sequence in $C(I, \|\cdot\|_\infty)$, so $\lim_n \lim_m \|f_m - f_n\|_\infty = 0$, we will show it converges to some f in this space.
- For each fixed $x_0 \in [0,1]$, the sequence of real numbers $\{f_n(x_0)\}$ is Cauchy in $\mathbb{R}$ since

$$x_0 \in I \implies |f_m(x_0) - f_n(x_0)| \leq \sup_{x\in I} |f_m(x) - f_n(x)| := \|f_m - f_n\|_\infty \overset{m>n\to\infty}{\longrightarrow} 0,$$

- Since $\mathbb{R}$ is complete, this sequence converges and we can define $f(x) := \lim_{k\to\infty} f_n(x)$.
- Thus $f_n \longrightarrow f$ pointwise by construction
- Claim: $\|f - f_n\| \overset{n\to\infty}{\longrightarrow} 0$, so f_n converges to f in $C([0,1], \|\cdot\|_\infty)$.
  - Proof:
    - Fix $\varepsilon > 0$; we will show there exists an N such that $n \geq N \implies \|f_n - f\| < \varepsilon$
    - Fix an $x_0 \in I$. Since $f_n \longrightarrow f$ pointwise, choose N_1 large enough so that

$$n \geq N_1 \implies |f_n(x_0) - f(x_0)| < \varepsilon/2.$$

    - Since $\|f_n - f_m\|_\infty \longrightarrow 0$, choose and N_2 large enough so that

$$n, m \geq N_2 \implies \|f_n - f_m\|_\infty < \varepsilon/2.$$

* Then for $n, m \geq \max(N_1, N_2)$, we have

$$\begin{aligned}
|f_n(x_0) - f(x_0)| &= |f_n(x_0) - f(x_0) + f_m(x_0) - f_m(x_0)| \\
&= |f_n(x_0) - f_m(x_0) + f_m(x_0) - f(x_0)| \\
&\leq |f_n(x_0) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \\
&< |f_n(x_0) - f_m(x_0)| + \frac{\varepsilon}{2} \\
&\leq \sup_{x \in I} |f_n(x) - f_m(x)| + \frac{\varepsilon}{2} \\
&< \|f_n - f_m\|_\infty + \frac{\varepsilon}{2} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
\implies |f_n(x_0) - f(x_0)| &< \varepsilon \\
\implies \sup_{x \in I} |f_n(x_0) - f(x_0)| &\leq \sup_{x \in I} \varepsilon \quad \text{by order limit laws} \\
\implies \|f_n - f\| &\leq \varepsilon
\end{aligned}$$

.

- f is the uniform limit of continuous functions and thus continuous, so $f \in C([0,1])$.

11.1.2 b

- It suffices to produce a Cauchy sequence that does not converge to a continuous function.
- Take the following sequence of functions:
  - f_1 increases linearly from 0 to 1 on $[0, 1/2]$ and is 1 on $[1/2, 1]$
  - f_2 is 0 on $[0, 1/4]$ increases linearly from 0 to 1 on $[1/4, 1/2]$ and is 1 on $[1/2, 1]$
  - f_3 is 0 on $[0, 3/8]$ increases linearly from 0 to 1 on $[3/8, 1/2]$ and is 1 on $[1/2, 1]$
  - f_3 is 0 on $[0, (1/2 - 3/8)/2]$ increases linearly from 0 to 1 on $[(1/2 - 3/8)/2, 1/2]$ and is 1 on $[1/2, 1]$

  Idea: take sequence starting points for the triangles: $0, 0 + \frac{1}{4}, 0 + \frac{1}{4} + \frac{1}{8}, \cdots$ which converges to $1/2$ since $\sum_{k=1}^{\infty} \frac{1}{2^k} = -\frac{1}{2} + \sum_{k=0}^{\infty} \frac{1}{2^k}$.
- Then each f_n is clearly integrable, since its graph is contained in the unit square.
- $\{f_n\}$ is Cauchy: geometrically subtracting areas yields a single triangle whose area tends to 0.
- But f_n converges to $\chi_{[\frac{1}{2}, 1]}$ which is discontinuous.

show that $\int_0^1 |f_n(x) - f_m(x)|\, dx \longrightarrow 0$ rigorously, show that no $g \in L^1([0,1])$ can converge to this indicator function.

11.2 Spring 2017 # 6 ✨

Show that the space $C^1([a,b])$ is a Banach space when equipped with the norm

$$\|f\| := \sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |f'(x)| .$$

Add concepts.

Solution:

Concepts Used:

- See https://math.stackexchange.com/questions/507263/prove-that-c1a-b-with-the-c1-norm-is-a-banach-space/

- Denote this norm $\|\cdot\|_u$
- Let f_n be a Cauchy sequence in this space, so $\|f_n\|_u < \infty$ for every n and $\|f_j - f_k\|_u \overset{j,k\longrightarrow\infty}{\longrightarrow} 0$.

and define a candidate limit: for each $x \in I$, set

$$f(x) \coloneqq \lim_{n\longrightarrow\infty} f_n(x).$$

- Note that

$$\|f_n\|_\infty \leq \|f_n\|_u < \infty$$
$$\|f_n'\|_\infty \leq \|f_n\|_u < \infty.$$

  - Thus both f_n, f_n' are Cauchy sequences in $C^0([a,b], \|\cdot\|_\infty)$, which is a Banach space, so they converge.
- So
  - $f_n \longrightarrow f$ uniformly (by uniqueness of limits),
  - $f_n' \longrightarrow g$ uniformly for some g, and
  - $f, g \in C^0([a,b])$.
- Claim: $g = f'$
  - For any fixed $a \in I$, we have

$$f_n(x) - f_n(a) \quad \overset{u}{\longrightarrow} f(x) - f(a)$$
$$\int_a^x f_n' \quad \overset{u}{\longrightarrow} \int_a^x g.$$

  - By the FTC, the left-hand sides are equal.
  - By uniqueness of limits so are the right-hand sides, so $f' = g$.
- Claim: the limit f is an element in this space.
  - Since $f, f' \in C^0([a,b])$, they are bounded, and so $\|f\|_u < \infty$.
- Claim: $\|f_n - f\|_u \overset{n\longrightarrow\infty}{\longrightarrow} 0$
- Thus the Cauchy sequence $\{f_n\}$ converges to a function f in the u-norm where f is an element of this space, making it complete.

11.3 Fall 2017 # 6 ✨

Let X be a complete metric space and define a norm

$$\|f\| := \max\{|f(x)| : x \in X\}.$$

Show that $(C^0(\mathbb{R}), \|\cdot\|)$ (the space of continuous functions $f : X \longrightarrow \mathbb{R}$) is complete.

Add concepts.

Shouldn't this be a supremum? The max may not exist?

Review and clean up.

Solution:

Concepts Used:

- ?

Let $\{f_k\}$ be a Cauchy sequence, so $\|f_k\| < \infty$ for all k. Then for a fixed x, the sequence $f_k(x)$ is Cauchy in $\mathbb{R}$ and thus converges to some $f(x)$, so define f by $f(x) \coloneqq \lim_{k \longrightarrow \infty} f_k(x)$.

Then $\|f_k - f\| = \max_{x \in X} |f_k(x) - f(x)| \overset{k \longrightarrow \infty}{\longrightarrow} 0$, and thus $f_k \longrightarrow f$ uniformly and thus f is continuous. It just remains to show that f has bounded norm.
Choose N large enough so that $\|f - f_N\| < \varepsilon$, and write $\|f_N\| \coloneqq M < \infty$

$$\|f\| \leq \|f - f_N\| + \|f_N\| < \varepsilon + M < \infty.$$