UGA Real Analysis Qualifying Exam Questions and Solutions

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Preface

I'd like to extend my gratitude to Peter Woolfitt for supplying many solutions and checking many proofs of the rest in problem sessions.

1 Undergraduate Analysis: Uniform Convergence

1.1 Fall 2018 # 1 🔆

Let $f(x) = \frac{1}{x}$. Show that f is uniformly continuous on $(1, \infty)$ but not on $(0, \infty)$.

Solution:

Concepts Used:

• Uniform continuity:

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$$
 such that $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$.

• Negating uniform continuity: $\exists \varepsilon > 0$ such that $\forall \delta(\varepsilon)$ there exist x, y such that $|x - y| < \delta$ and $|f(x) - f(y)| > \varepsilon$.

Claim: $f(x) = \frac{1}{x}$ is uniformly continuous on (c, ∞) for any c > 0.

• Note that

$$|x|, |y| > c > 0 \implies |xy| = |x||y| > c^2 \implies \frac{1}{|xy|} < \frac{1}{c^2}.$$

- Letting ε be arbitrary, choose $\delta < \varepsilon c^2$.
 - Note that δ does not depend on x, y.
- Then

$$|f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right|$$

$$= \frac{|x - y|}{xy}$$

$$\leq \frac{\delta}{xy}$$

$$< \frac{\delta}{c^2}$$

$$< \varepsilon.$$

which shows uniform continuity.

Claim: f is not uniformly continuous when c = 0.

• Toward a contradiction, let $\varepsilon < 1$.

• Let
$$x_n = \frac{1}{n}$$
 for $n \ge 1$.

• Choose *n* large enough such that
$$|x_n - x_{n+1}| = \frac{1}{n} - \frac{1}{n+1} < \delta$$
.

- Why this can be done: by the archimedean property of
$$\mathbb{R}$$
, choose n such that $\frac{1}{n} < \varepsilon$.

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \le \frac{1}{n} < \varepsilon \quad \text{since } n+1 > 1.$$

• Note $f(x_n) = n$ and thus

$$|f(x_{n+1}) - f(x_n)| = (n+1) - n = 1 > \varepsilon,$$

a contradiction.

1.2 Fall 2017 # 1 🙀

Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which f does and does not converge uniformly.

Solution:

Concepts Used:

•
$$f_N \longrightarrow f$$
 uniformly $\iff ||f_N - f||_{\infty} \longrightarrow 0.$

•
$$\sum_{n=0}^{\infty} c_n x^n := \lim_{N \to \infty} \sum_{n=0}^{N} c_n x^n$$

• $f_N \longrightarrow f$ uniformly $\iff \|f_N - f\|_{\infty} \longrightarrow 0$. • $\sum_{n=0}^{\infty} c_n x^n \coloneqq \lim_{N \longrightarrow \infty} \sum_{n=0}^{N} c_n x^n$ — I.e. an infinite sum is defined as the pointwise limit of its partial sums.

• If
$$\sum_{n=0}^{\infty} g_n(x)$$
 converges uniformly on a set A , then $\sup_{x \in A} |f_n(x)| \longrightarrow 0$.

• Set
$$f_N(x) = \sum_{n=1}^N \frac{x^n}{n!}$$
.

- Then by definition, $f_N(x) \longrightarrow f(x)$ pointwise on \mathbb{R} .

• For any compact interval [-M, M], we have

$$||f_N(x) - f(x)||_{\infty} = \sup_{-M \le x \le M} \left| \sum_{n=N+1}^{\infty} \frac{x^n}{n!} \right|$$

$$\leq \sup_{-M \le x \le M} \sum_{n=N+1}^{\infty} \left| \frac{x^n}{n!} \right|$$

$$\leq \sum_{n=N+1}^{\infty} \frac{M^n}{n!}$$

$$\leq \sum_{n=0}^{\infty} \frac{M^n}{n!} \quad \text{since all additional terms are positive}$$

$$= e^M$$

$$< \infty,$$

so $f_N \longrightarrow f$ uniformly on [-M, M] by the M-test.

Here we've used that e^x is equal to its power series expansion.

• Thus f converges on any bounded interval, since any bounded interval is contained in some larger compact interval.

Claim: f does not converge on \mathbb{R} .

- If $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on a set A, then $\sup_{x \in A} |f_n(x)| \longrightarrow 0$.
- But taking $A = \mathbb{R}$ and $g_n(x) = \frac{x^n}{n!}$, we have

$$\sup_{x \in \mathbb{R}} |g_n(x)| = \sup_{x \in \mathbb{R}} \frac{x^n}{n!} = \infty.$$

1.3 Fall 2014 # 1 💝

Let $\{f_n\}$ be a sequence of continuous functions such that $\sum f_n$ converges uniformly.

Prove that $\sum f_n$ is also continuous.

Solution:

Claim: If $F_N \longrightarrow F$ uniformly with each F_N continuous, then F is continuous.

• Follows from an $\varepsilon/3$ argument:

$$|F(x) - F(y)| \le |F(x) - F_N(x)| + |F_N(x) - F_N(y)| + |F_N(y) - F(y)| \le \varepsilon \longrightarrow 0.$$

- The first and last $\varepsilon/3$ come from uniform convergence of $F_N \longrightarrow F$.
- The middle $\varepsilon/3$ comes from continuity of each F_N .
- Now setting $F_N := \sum_{n=1}^N f_n$ yields a finite sum of continuous functions, which is continuous.
- Each F_N is continuous and $F_N \longrightarrow F$ uniformly, so applying the claim yields the desired result.

1.4 Spring 2017 # 4 🙀

Let f(x,y) on $[-1,1]^2$ be defined by

$$f(x,y) = \begin{cases} \frac{xy}{(x^2 + y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Determine if f is integrable.

Solution:

Concepts Used:

• Just Calculus.

Switching to polar coordinates and integrating over one quarter of the unit disc $D \subseteq I^2$, we have

$$\int_{I^2} f \, dA \ge \int_D f \, dA$$

$$\ge \int_0^{\pi/2} \int_0^1 \frac{\cos(\theta) \sin(\theta)}{r^4} r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \cos(\theta) \sin(\theta) \int_0^1 \frac{1}{r^3} \, dr \, d\theta$$

$$= \left(\int_0^1 \frac{1}{r^3} \, dr \right) \left(\int_0^{\pi/2} \cos(\theta) \sin(\theta) \, d\theta \right)$$

$$= \left(\int_0^1 \frac{1}{r^3} \, dr \right) \left(-\frac{1}{2} \cos^2(\theta) \Big|_0^{\pi/2} \right)$$

$$= -\frac{1}{2r^2} \Big|_0^1 \left(\frac{1}{2} \right)$$

$$= \left(\frac{1}{4} \right) \left(-1 + \lim_{r \to 0} \frac{1}{r^2} \right)$$

$$= \infty.$$

so f is not integrable.

1.5 Spring 2015 # 1

Let (X, d) and (Y, ρ) be metric spaces, $f: X \longrightarrow Y$, and $x_0 \in X$.

Prove that the following statements are equivalent:

- 1. For every $\varepsilon > 0$ $\exists \delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$.
- 2. The sequence $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow f(x_0)$ for every sequence $\{x_n\} \longrightarrow x_0$ in X.

Solution:

Concepts Used:

- What it means for a sequence to converge.
- - Let $\{x_n\} \stackrel{n \longrightarrow \infty}{\longrightarrow} x_0$; we want to show $\{f(x_n)\} \stackrel{n \longrightarrow \infty}{\longrightarrow} f(x_0)$. - We thus want to show that for every $\varepsilon > 0$, there exists an $N(\varepsilon)$ such that

$$n \ge N(\varepsilon) \implies \rho(f(x_n), f(x_0)) < \varepsilon.$$

- Let $\varepsilon > 0$ be arbitrary, then by (1) choose δ such that $\rho(f(x), f(x_0)) < \varepsilon$ when $d(x, x_0) < \varepsilon$
- Since $x_n \longrightarrow x$, there is some N such that $n \ge N \implies |x_n x_0| < \delta$ Then for $n \ge N$, $d(x_n, x_0) < \delta \implies \rho(f(x), f(x_0)) < \varepsilon$, so f^*
- - Suppose $x_n \longrightarrow x_0 \implies f(x_n) \longrightarrow f(x_0)$ and let $\varepsilon > 0$ be arbitrary.

1.6 Fall 2014 # 2

Let I be an index set and $\alpha: I \longrightarrow (0, \infty)$.

1. Show that

$$\sum_{i \in I} a(i) := \sup_{\substack{J \subset I \\ J \text{ finite}}} \sum_{i \in J} a(i) < \infty \implies I \text{ is countable.}$$

2. Suppose $I = \mathbb{Q}$ and $\sum_{q \in \mathbb{Q}} a(q) < \infty$. Define

$$f(x) := \sum_{\substack{q \in \mathbb{Q} \\ q \le x}} a(q).$$

Show that f is continuous at $x \iff x \notin \mathbb{Q}$

1.7 Spring 2014 # 2

Let $\{a_n\}$ be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that $\sum a_n^2 < \infty$.

Note: Assume a_n, b_n are all non-negative.

2 General Analysis

2.1 Spring 2020 # 1 😽

Prove that if $f:[0,1] \longrightarrow \mathbb{R}$ is continuous then

$$\lim_{k \to \infty} \int_0^1 kx^{k-1} f(x) \, dx = f(1).$$

Solution:

Concepts Used:

- DCT
- Weierstrass Approximation Theorem
- Suppose p is a polynomial, then

$$\lim_{k \to \infty} \int_0^1 kx^{k-1} p(x) \, dx = \lim_{k \to \infty} \int_0^1 \left(\frac{\partial}{\partial x} x^k \right) p(x) \, dx$$

$$= \lim_{k \to \infty} \left[x^k p(x) \Big|_0^1 - \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) \, dx \right] \quad \text{integrating by parts}$$

$$= p(1) - \lim_{k \to \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) \, dx,$$

• Thus it suffices to show that

$$\lim_{k \to \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx = 0.$$

• Integrating by parts a second time yields

$$\lim_{k \to \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx = \lim_{k \to \infty} \frac{x^{k+1}}{k+1} p'(x) \Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx$$

$$= -\lim_{k \to \infty} \int_0^1 \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx$$

$$= -\int_0^1 \lim_{k \to \infty} \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \quad \text{by DCT}$$

$$= -\int_0^1 0 \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx$$

$$= 0$$

– The DCT can be applied here because f'' is continuous and [0,1] is compact, so f'' is bounded on [0,1] by a constant M and

$$\int_0^1 \left| x^k f''(x) \right| \le \int_0^1 1 \cdot M = M < \infty.$$

• Now use the Weierstrass approximation theorem:

- If $f:[a,b] \longrightarrow \mathbb{R}$ is continuous, then for every $\varepsilon > 0$ there exists a polynomial $p_{\varepsilon}(x)$ such that $||f - p_{\varepsilon}||_{\infty} < \varepsilon$.

• Thus

$$\left| \int_0^1 kx^{k-1} p_{\varepsilon}(x) \, dx - \int_0^1 kx^{k-1} f(x) \, dx \right| = \left| \int_0^1 kx^{k-1} (p_{\varepsilon}(x) - f(x)) \, dx \right|$$

$$\leq \left| \int_0^1 kx^{k-1} || p_{\varepsilon} - f ||_{\infty} \, dx \right|$$

$$= || p_{\varepsilon} - f ||_{\infty} \cdot \left| \int_0^1 kx^{k-1} \, dx \right|$$

$$= || p_{\varepsilon} - f ||_{\infty} \cdot x^k \Big|_0^1$$

$$= || p_{\varepsilon} - f ||_{\infty} \xrightarrow{\varepsilon \longrightarrow 0} 0$$

and the integrals are equal.

• By the first argument,

$$\int_0^1 kx^{k-1} p_{\varepsilon}(x) dx = p_{\varepsilon}(1) \text{ for each } \varepsilon$$

• Since uniform convergence implies pointwise convergence, $p_{\varepsilon}(1) \stackrel{\varepsilon \longrightarrow 0}{\longrightarrow} f(1)$.

2.2 Fall 2019 # 1 💝

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

2.2.1 a

Prove that if $\lim_{n\to\infty} a_n = 0$, then

$$\lim_{n \to \infty} \frac{a_1 + \dots + a_n}{n} = 0$$

2.2.2 b

Prove that if $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges, then

$$\lim_{n \to \infty} \frac{a_1 + \dots + a_n}{n} = 0$$

Solution:

Concepts Used:

- Cesaro mean/summation.
- Break series apart into pieces that can be handled separately.

2.2.3 a

Prove a stronger result:

$$a_k \longrightarrow S \implies S_N := \frac{1}{N} \sum_{k=1}^N a_k \longrightarrow S.$$

Idea: once N is large enough, $a_k \approx S$, and all smaller terms will die off as $N \longrightarrow \infty$. See this MSE answer.

• Use convergence $a_k \longrightarrow S$: choose M large enough such that

$$k \ge M + 1 \implies |a_k - S| < \varepsilon.$$

Then

$$\left| \left(\frac{1}{N} \sum_{k=1}^{N} a_k \right) - S \right| = \frac{1}{N} \left| \left(\sum_{k=1}^{N} a_k \right) - NS \right|$$

$$= \frac{1}{N} \left| \left(\sum_{k=1}^{N} a_k \right) - \sum_{k=1}^{N} S \right|$$

$$= \frac{1}{N} \left| \sum_{k=1}^{N} (a_k - S) \right|$$

$$\leq \frac{1}{N} \sum_{k=1}^{N} |a_k - S|$$

$$= \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} |a_k - S|$$

$$\leq \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} \frac{\varepsilon}{2}$$

$$= \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + (N - M) \frac{\varepsilon}{2}$$

$$\stackrel{\varepsilon}{\longrightarrow} \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + 0$$

$$\stackrel{N}{\longrightarrow} 0 + 0.$$

Note: M is fixed, so the last sum is some constant c, and $c/N \longrightarrow 0$ as $N \longrightarrow \infty$ for any constant. To be more careful, choose M first to get $\varepsilon/2$ for the tail, then choose N(M) > M for the remaining truncated part of the sum.

2.2.4 b

• Define

$$\Gamma_n \coloneqq \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

- $\Gamma_1 = \sum_{k=1}^n \frac{a_k}{k}$ is the original series and each Γ_n is a tail of Γ_1 , so by assumption $\Gamma_n \xrightarrow{n \to \infty} 0$.
- Compute

$$\frac{1}{n}\sum_{k=1}^{n}a_{k}=\frac{1}{n}(\Gamma_{1}+\Gamma_{2}+\cdots+\Gamma_{n}-\mathbf{\Gamma_{n+1}})$$

• This comes from consider the following summation:

$$\Gamma_1:$$
 $a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots$
 $\Gamma_2:$ $\frac{a_2}{2} + \frac{a_3}{3} + \cdots$
 $\Gamma_3:$ $\frac{a_3}{3} + \cdots$
 $\sum_{i=1}^n \Gamma_i:$ $a_1 + a_2 + a_3 + \cdots + a_n + \frac{a_{n+1}}{n+1} + \cdots$

- Use part (a): since $\Gamma_n \stackrel{n \longrightarrow \infty}{\longrightarrow} 0$, we have $\frac{1}{n} \sum_{k=1}^n \Gamma_k \stackrel{n \longrightarrow \infty}{\longrightarrow} 0$.
- Also a minor check: $\Gamma_n \longrightarrow 0 \implies \frac{1}{n}\Gamma_n \longrightarrow 0$.
- Then

$$\frac{1}{n} \sum_{k=1}^{n} a_k = \frac{1}{n} (\Gamma_1 + \Gamma_2 + \dots + \Gamma_n - \mathbf{\Gamma_{n+1}})$$
$$= \left(\frac{1}{n} \sum_{k=0}^{n} \Gamma_k\right) - \left(\frac{1}{n} \Gamma_{n+1}\right)$$
$$\stackrel{n \to \infty}{\longrightarrow} 0$$

2.3 Fall 2018 # 4 😽

Let $f \in L^1([0,1])$. Prove that

$$\lim_{n \to \infty} \int_0^1 f(x) |\sin nx| \ dx = \frac{2}{\pi} \int_0^1 f(x) \ dx$$

> Hint: Begin with the case that f is the characteristic function of an interval.

Add concepts

Solution:

Concepts Used:

• ?

Case of characteristic function

- First suppose $f(x) = \chi_{[0,1]}(x)$.
- Note that $\sin(nx)$ has a period of $2\pi/n$, and thus $\left|\frac{n}{2\pi}\right|$ full periods in [0,1].
- Taking the absolute value yields a new function with half the period, so a period of π/n and $|\pi/n|$ full periods in [0,1].
- We can compute the integral over one full period (which is independent of which period is chosen), and since $\sin(x)$ is positive and agrees with $|\sin(nx)|$ on the first period, we have

$$\int_{\text{One Period}} |\sin(nx)| \, dx = \int_0^{\pi/n} \sin(nx) \, dx$$

$$= \frac{1}{n} \int_0^{\pi} \sin(u) \, du \quad u = nx$$

$$= \frac{1}{n} - \cos(u) \Big|_0^{\pi}$$

$$= \frac{2}{n}.$$

• Then break the integral up into integrals over periods P_1, P_2, \dots, P_N where $N := \lfloor n/\pi \rfloor$:

$$\int_{0}^{1} |\sin(nx)| \, dx = \left(\sum_{j=1}^{N} \int_{P_{j}} |\sin(nx)| \, dx\right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| \, dx$$

$$= \left(\sum_{j=1}^{N} \frac{2}{n}\right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| \, dx$$

$$= N\left(\frac{2}{n}\right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| \, dx$$

$$\coloneqq \left\lfloor \frac{n}{\pi} \right\rfloor \frac{2}{n} + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| \, dx$$

$$= \frac{2}{\pi} + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| \, dx$$

$$\coloneqq \frac{2}{\pi} + R(n)$$

so it suffices to show that $R(n) \stackrel{n \longrightarrow \infty}{\longrightarrow} 0$.

Need to justify removing floor function and cancellation

• Showing this: ??????????

General case

Not sure. Approximate f by simple functions...

2.4 Fall 2017 # 4 🐪

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

1. Show that $f_n \longrightarrow 0$ pointwise but not uniformly on [0,1].

Hint: Consider the maximum of f_n .

2.

$$\lim_{n \to \infty} \int_0^1 n(1-x)^n \sin x \, dx = 0$$

Add concepts

Walk through.

Solution:

Concepts Used:

. 7

2.4.1 a

Let $G(x) = \sum_{n=1}^{\infty} nx(1-x)^n$. Applying the ratio test, we have

$$\left| \frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n} \right| = \frac{n+1}{n} |1-x| \stackrel{n \longrightarrow \infty}{\longrightarrow} |1-x| < 1 \iff 0 \le x \le 2,$$

and in particular, this series converges on [0,2]. Thus its terms go to zero, and $nx(1-x)^n \longrightarrow 0$ on $[0,1] \subset [0,2]$.

To see that the convergence is not uniform, let $x_n = \frac{1}{n}$ and $\varepsilon > \frac{1}{e}$, then

$$\sup_{x \in [0,1]} |nx(1-x)^n - 0| \ge |nx_n(1-x_n)^n| = \left| \left(1 - \frac{1}{n}\right)^n \right| \stackrel{n \to \infty}{\longrightarrow} e^{-1} > \varepsilon.$$

2.4.2 b

Note: could use the first part with $\sin(x) \le x$, but then integral ends up more complicated.

Noting that $sin(x) \leq 1$, we have We have

$$\left| \int_0^1 n(1-x)^n \sin(x) \right| \le \int_0^1 |n(1-x)^n \sin(x)|$$

$$\le \int_0^1 |n(1-x)^n|$$

$$= n \int_0^1 (1-x)^n$$

$$= -\frac{n(1-x)^{n+1}}{n+1}$$

$$\xrightarrow{n \to \infty} 0.$$

2.5 Spring 2017 # 3 💝

Let

$$f_n(x) = ae^{-nax} - be^{-nbx}$$
 where $0 < a < b$.

Show that

a.
$$\sum_{n=1}^{\infty} |f_n| \text{ is not in } L^1([0,\infty),m)$$

Hint: $f_n(x)$ has a root x_n .

b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0,\infty),m) \quad \text{and} \quad \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x) \, dm = \ln \frac{b}{a}$$

Not complete

Add concepts

Walk through.

Solution:

Concepts Used:

• (

2.5.1 a

Letting $x_n := \frac{1}{n}$, we have

$$\sum_{k=1}^{\infty} |f_k(x)| \ge |f_n(x_n)| = \left| ae^{-ax} - be^{-bx} \right| := M.$$

In particular, $\sup_{x} |f_n(x)| \not\longrightarrow 0$, so the terms do not go to zero and the sum can not converge.

2.5.2 b

2.6 Fall 2016 # 1 😽

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that f converges to a differentiable function on $(1, \infty)$ and that

$$f'(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n^x}\right)'.$$

Hint:

$$\left(\frac{1}{n^x}\right)' = -\frac{1}{n^x} \ln n$$

Add concepts.

Solution:

Concepts Used:

• Set
$$f_N(x) := \sum_{n=1}^N n^{-x}$$
, so $f(x) = \lim_{N \to \infty} f_N(x)$.

• If an interchange of limits is justified, we have

$$\frac{\partial}{\partial x} \lim_{N \to \infty} \sum_{n=1}^{N} n^{-x} = \lim_{h \to 0} \lim_{N \to \infty} \frac{1}{h} \left[\left(\sum_{n=1}^{N} n^{-x} \right) - \left(\sum_{n=1}^{N} n^{-(x+h)} \right) \right]$$

$$\stackrel{?}{=} \lim_{N \to \infty} \lim_{h \to 0} \frac{1}{h} \left[\left(\sum_{n=1}^{N} n^{-x} \right) - \left(\sum_{n=1}^{N} n^{-(x+h)} \right) \right]$$

$$= \lim_{N \to \infty} \lim_{h \to 0} \frac{1}{h} \left[\sum_{n=1}^{N} n^{-x} - n^{-(x+h)} \right] \quad (1)$$

$$= \lim_{N \to \infty} \sum_{n=1}^{N} \lim_{h \to 0} \frac{1}{h} \left[n^{-x} - n^{-(x+h)} \right] \quad \text{since this is a finite sum}$$

$$:= \lim_{N \to \infty} \sum_{n=1}^{N} \frac{\partial}{\partial x} \left(\frac{1}{n^x} \right)$$

$$= \lim_{N \to \infty} \sum_{n=1}^{N} -\frac{\ln(n)}{n^x},$$

where the combining of sums in (1) is valid because $\sum n^{-x}$ is absolutely convergent for x > 1 by the *p*-test.

- Thus it suffices to justify the interchange of limits and show that the last sum converges on $(1, \infty)$.
- Claim: $\sum n^{-x} \ln(n)$ converges.
 - Use the fact that for any fixed $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{\ln(n)}{n^{\varepsilon}} \stackrel{\text{L.H.}}{=} \lim_{n \to \infty} \frac{1/n}{\varepsilon n^{\varepsilon - 1}} = \lim_{n \to \infty} \frac{1}{\varepsilon n^{\varepsilon}} = 0,$$

- This implies that for a fixed $\varepsilon > 0$ and for any constant c > 0 there exists an N large enough such that $n \geq N$ implies $\ln(n)/n^{\varepsilon} < c$, i.e. $\ln(n) < cn^{\varepsilon}$.
- Taking c = 1, we have $n \ge N \implies \ln(n) < n^{\varepsilon}$
- We thus break up the sum:

$$\sum_{n \in \mathbb{N}} \frac{\ln(n)}{n^x} = \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{\ln(n)}{n^x}$$

$$\leq \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{n^{\varepsilon}}{n^x}$$

$$\coloneqq C_{\varepsilon} + \sum_{n=N}^{\infty} \frac{n^{\varepsilon}}{n^x} \quad \text{with } C_{\varepsilon} < \infty \text{ a constant}$$

$$= C_{\varepsilon} + \sum_{n=N}^{\infty} \frac{1}{n^{x-\varepsilon}},$$

where the last term converges by the p-test if $x - \varepsilon > 1$.

- But ε can depend on x, and if $x \in (1, \infty)$ is fixed we can choose $\varepsilon < |x 1|$ to ensure this.
- Claim: the interchange of limits is justified.

2.7 Fall 2016 # 5 💝

Let $\varphi \in L^{\infty}(\mathbb{R})$. Show that the following limit exists and satisfies the equality

$$\lim_{n \to \infty} \left(\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} \, dx \right)^{\frac{1}{n}} = \|\varphi\|_{\infty}.$$

Walk through

Add concepts.

Solution:

Concepts Used:

• ?

Let L be the LHS and R be the RHS.

Claim: $L \leq R$. - Since $|\varphi| \leq ||\varphi||_{\infty}$ a.e., we can write

$$\begin{split} L^{\frac{1}{n}} &\coloneqq \int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} \\ &\le \int_{\mathbb{R}} \frac{\|\varphi\|_{\infty}^n}{1+x^2} \\ &= \|\varphi\|_{\infty}^n \int_{\mathbb{R}} \frac{1}{1+x^2} \\ &= \|\varphi\|_{\infty}^n \arctan(x)\Big|_{-\infty}^{\infty} \\ &= \|\varphi\|_{\infty}^n \left(\frac{\pi}{2} - \frac{-\pi}{2}\right) \\ &= \pi \|\varphi\|_{\infty}^n \end{split}$$

$$\implies L^{\frac{1}{n}} \leq \sqrt[n]{\pi \|\varphi\|_{\infty}^{n}}$$

$$\implies L \leq \pi^{\frac{1}{n}} \|\varphi\|_{\infty}$$

$$\stackrel{n \longrightarrow \infty}{\longrightarrow} \|\varphi\|_{\infty},$$

where we've used the fact that $c^{\frac{1}{n}} \xrightarrow{n \to \infty} 1$ for any constant c.

Actually true? Need conditions

Claim: $R \leq L$.

- We will show that $R \leq L + \varepsilon$ for every $\varepsilon > 0$.
- Set

$$S_{\varepsilon} := \left\{ x \in \mathbb{R}^n \mid |\varphi(x)| \ge ||\varphi||_{\infty} - \varepsilon \right\}.$$

• Then we have

$$\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} dx \ge \int_{S_{\varepsilon}} \frac{|\varphi(x)|^n}{1+x^2} dx \quad S_{\varepsilon} \subset \mathbb{R}$$

$$\ge \int_{S_{\varepsilon}} \frac{(\|\varphi\|_{\infty} - \varepsilon)^n}{1+x^2} dx \quad \text{by definition of } S_{\varepsilon}$$

$$= (\|\varphi\|_{\infty} - \varepsilon)^n \int_{S_{\varepsilon}} \frac{1}{1+x^2} dx$$

$$= (\|\varphi\|_{\infty} - \varepsilon)^n C_{\varepsilon} \quad \text{where } C_{\varepsilon} \text{ is some constant}$$

$$\implies \left(\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} \, dx \right)^{\frac{1}{n}} \ge (\|\varphi\|_{\infty} - \varepsilon) C_{\varepsilon}^{\frac{1}{n}}$$

$$\stackrel{n \longrightarrow \infty}{\longrightarrow} (\|\varphi\|_{\infty} - \varepsilon) \cdot 1$$

$$\stackrel{\varepsilon \longrightarrow 0}{\longrightarrow} \|\varphi\|_{\infty},$$

where we've again used the fact that $c^{\frac{1}{n}} \longrightarrow 1$ for any constant.

2.8 Fall 2016 # 6 💝

Let $f, g \in L^2(\mathbb{R})$. Show that

$$\lim_{n \to \infty} \int_{\mathbb{R}} f(x)g(x+n) \, dx = 0$$

Add concepts.

Solution:

Concepts Used:

- ?
- Use the fact that L^p has small tails: if $h \in L^2(\mathbb{R})$, then for any $\varepsilon > 0$,

$$\forall \varepsilon, \exists N \in \mathbb{N} \text{ such that } \int_{|x|>N} |h(x)|^2 dx < \varepsilon.$$

How to prove small tails in L^p ?

- So choose n large enough so the tails of both f and g are smaller than ε .
- Apply Cauchy-Schwarz:

$$\left| \int_{\mathbb{R}} f(x)g(x+n) \, dx \right| \le \int_{\mathbb{R}} |f(x)g(x+n)| \, dx$$
$$\le \int_{\mathbb{R}}.$$

2.9 Spring 2016 # 1

For $n \in \mathbb{N}$, define

$$e_n = \left(1 + \frac{1}{n}\right)^n$$
 and $E_n = \left(1 + \frac{1}{n}\right)^{n+1}$

Show that $e_n < E_n$, and prove Bernoulli's inequality:

$$(1+x)^n \ge 1 + nx$$
 for $-1 < x < \infty$ and $n \in \mathbb{N}$

Use this to show the following:

- 1. The sequence e_n is increasing.
- 2. The sequence E_n is decreasing.
- 3. $2 < e_n < E_n < 4$.
- $4. \lim_{n \to \infty} e_n = \lim_{n \to \infty} E_n.$

2.10 Fall 2015 # 1

Define

$$f(x) = c_0 + c_1 x^1 + c_2 x^2 + \ldots + c_n x^n$$
 with n even and $c_n > 0$.

Show that there is a number x_m such that $f(x_m) \leq f(x)$ for all $x \in \mathbb{R}$.

3 Measure Theory: Sets

3.1 Spring 2020 # 2 💝

Let m_* denote the Lebesgue outer measure on \mathbb{R} .

3.1.1 a.

Prove that for every $E \subseteq \mathbb{R}$ there exists a Borel set B containing E such that

$$m_*(B) = m_*(E).$$

3.1.2 b.

Prove that if $E \subseteq \mathbb{R}$ has the property that

$$m_*(A) = m_*(A \cap E) + m_*(A \cap E^c)$$

for every set $A \subseteq \mathbb{R}$, then there exists a Borel set $B \subseteq \mathbb{R}$ such that $E = B \setminus N$ with $m_*(N) = 0$.

Be sure to address the case when $m_*(E) = \infty$.

Solution:

Concepts Used:

• Definition of outer measure:

$$m_*(E) = \inf_{\{Q_j\} \rightrightarrows E} \sum |Q_j|$$

where $\{Q_j\}$ is a countable collection of closed cubes.

- Break \mathbb{R} into $\coprod_{n\in\mathbb{Z}}[n,n+1)$, each with finite measure.
- Theorem: $m_*(Q) = |Q|$ for Q a closed cube (i.e. the outer measure equals the volume).

Proof.

- $m_*(Q) \le |Q|$:
- Since $Q \subseteq Q$, $Q \Rightarrow Q$ and $m_*(Q) \leq |Q|$ since m_* is an infimum over such coverings.
- $|Q| \le m_*(Q)$:
- Fix $\varepsilon > 0$.
- Let $\{Q_i\}_{i=1}^{\infty} \rightrightarrows Q$ be arbitrary, it suffices to show that

$$|Q| \le \left(\sum_{i=1}^{\infty} |Q_i|\right) + \varepsilon.$$

- Pick open cubes S_i such that $Q_i \subseteq S_i$ and $|Q_i| \le |S_i| \le (1+\varepsilon)|Q_i|$.
- Then $\{S_i\} \rightrightarrows Q$, so by compactness of Q pick a finite subcover with N elements.
- Note

$$Q \subseteq \bigcup_{i=1}^{N} S_i \implies |Q| \le \sum_{i=1}^{N} |S_i| \le \sum_{i=1}^{N} (1+\varepsilon)|Q_j| \le (1+\varepsilon) \sum_{i=1}^{\infty} |Q_i|.$$

• Taking an infimum over coverings on the RHS preserves the inequality, so

$$|Q| \leq (1+\varepsilon)m_*(Q)$$

• Take $\varepsilon \longrightarrow 0$ to obtain final inequality.

3.1.3 a

- If $m_*(E) = \infty$, then take $B = \mathbb{R}^n$ since $m(\mathbb{R}^n) = \infty$.
- Suppose $N := m_*(E) < \infty$.
- Since $m_*(E)$ is an infimum, by definition, for every $\varepsilon > 0$ there exists a covering by closed cubes $\{Q_i(\varepsilon)\}_{i=1}^{\infty} \rightrightarrows E$ depending on ε such that

$$\sum_{i=1}^{\infty} |Q_i(\varepsilon)| < N + \varepsilon.$$

- For each fixed n, set $\varepsilon_n = \frac{1}{n}$ to produce such a covering $\{Q_i(\varepsilon_n)\}_{i=1}^{\infty}$ and set $B_n :=$ $\bigcup_{i=1}^{\infty} Q_i(\varepsilon_n).$
- The outer measure of cubes is equal to the sum of their volumes, so

$$m_*(B_n) = \sum_{i=1}^{\infty} |Q_i(\varepsilon_n)| < N + \varepsilon_n = N + \frac{1}{n}.$$

- Now set $B := \bigcap^{\infty} B_n$.
 - Since $E \subseteq B_n$ for every $n, E \subseteq B$
 - Since B is a countable intersection of countable unions of closed sets, B is Borel.
 - Since $B_n \subseteq B$ for every n, we can apply subadditivity to obtain the inequality

$$E \subseteq B \subseteq B_n \implies N \le m_*(B) \le m_*(B_n) < N + \frac{1}{n} \text{ for all } n \in \mathbb{Z}^{\ge 1}.$$

This forces $m_*(E) = m_*(B)$.

3.1.4 b

Suppose $m_*(E) < \infty$.

- By (a), find a Borel set $B \supset E$ such that $m_*(B) = m_*(E)$
- Note that $E \subseteq B \implies B \cap E = E$ and $B \cap E^c = B \setminus E$.
- By assumption,

$$m_*(B) = m_*(B \cap E) + m_*(B \cap E^c)$$

$$m_*(E) = m_*(E) + m_*(B \setminus E)$$

$$m_*(E) - m_*(E) = m_*(B \setminus E) \quad \text{since } m_*(E) < \infty$$

$$\implies m_*(B \setminus E) = 0.$$

- So take $N = B \setminus E$; this shows $m_*(N) = 0$ and $E = B \setminus (B \setminus E) = B \setminus N$.
 - Apply result to E_R := E ∩ [R, R + 1)ⁿ ⊂ ℝⁿ for R ∈ ℤ, so E = ∐_RE_R
 Obtain B_R, N_R such that E_R = B_R \ N_R, m_{*}(E_R) = m_{*}(B_R), and m_{*}(N_R) = 0.

 - - $-B := \bigcup B_R$ is a union of Borel sets and thus still Borel
 - $-E = \bigcup_{-R}^{R} E_{R}$

 - $-N := \stackrel{R}{B} \setminus E$ $-N' := \bigcup N_R \text{ is a union of null sets and thus still null}$
 - Since $E_R \subset B_R$ for every R, we have $E \subset B$
 - We can compute

$$N = B \setminus E = \left(\bigcup_R B_R\right) \setminus \left(\bigcup_R E_R\right) \subseteq \bigcup_R \left(B_R \setminus E_R\right) = \bigcup_R N_R := N'$$

where $m_*(N') = 0$ since N' is null, and thus subadditivity forces $m_*(N) = 0$.

3.2 Fall 2019 # 3. 😽

Let (X, \mathcal{B}, μ) be a measure space with $\mu(X) = 1$ and $\{B_n\}_{n=1}^{\infty}$ be a sequence of \mathcal{B} -measurable subsets of X, and

$$B := \left\{ x \in X \mid x \in B_n \text{ for infinitely many } n \right\}.$$

- a. Argue that B is also a \mathcal{B} -measurable subset of X.
- b. Prove that if $\sum_{n=1}^{\infty} \mu(B_n) < \infty$ then $\mu(B) = 0$.
- c. Prove that if $\sum_{n=1}^{\infty} \mu(B_n) = \infty$ and the sequence of set complements $\{B_n^c\}_{n=1}^{\infty}$ satisfies

$$\mu\left(\bigcap_{n=k}^{K} B_{n}^{c}\right) = \prod_{n=k}^{K} \left(1 - \mu\left(B_{n}\right)\right)$$

for all positive integers k and K with k < K, then $\mu(B) = 1$.

Hint: Use the fact that $1 - x \le e^{-x}$ for all x.

Solution:

Concepts Used:

• Borel-Cantelli: for a sequence of sets X_n ,

$$\limsup_{n} X_{n} = \left\{ x \mid x \in X_{n} \text{ for infinitely many } n \right\} = \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} X_{n}$$

$$\liminf_{n} X_{n} = \left\{ x \mid x \in X_{n} \text{ for all but finitely many } n \right\} = \prod_{m \in \mathbb{N}} \bigcap_{n \geq m} X_{n}.$$

 $\liminf_{n} X_{n} = \left\{ x \mid x \in X_{n} \text{ for all but finitely many } n \right\} \qquad = \bigcup_{m \in \mathbb{N}} \bigcap_{n > m} X_{n}.$

• Properties of logs and exponentials:

$$\prod_{n} e^{x_n} = e^{\sum_{n} x_n} \quad \text{and} \quad \sum_{n} \log(x_n) = \log\left(\prod_{n} x_n\right).$$

- Tails of convergent sums vanish.
- Continuity of measure: $B_n \searrow B$ and $\mu(B_0) < \infty$ implies $\lim_n \mu(B_n) = \mu(B)$, and $B_n \nearrow B \implies \lim_n \mu(B_n) = \mu(B).$

3.2.1 a

- The Borel σ -algebra is closed under countable unions/intersections/complements,
- $B = \limsup B_n$ is an intersection of unions of measurable sets.

3.2.2 b

• Tails of convergent sums go to zero, so $\sum_{n>M} \mu(B_n) \xrightarrow{M \longrightarrow \infty} 0$,

•
$$B_M := \bigcap_{m=1}^M \bigcup_{n>m} B_n \searrow B$$
.

$$\mu(B_M) = \mu\left(\bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} B_n\right)$$

$$\leq \mu\left(\bigcup_{n \geq m} B_n\right) \quad \text{for all } m \in \mathbb{N} \text{ by countable subadditivity}$$

$$\longrightarrow 0,$$

The result follows by continuity of measure.

3.2.3 c

• To show $\mu(B) = 1$, we'll show $\mu(B^c) = 0$.

• Let
$$B_k = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{K} B_n$$
. Then

$$\mu(B_K^c) = \mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^K B_n^c\right)$$

$$\leq \sum_{m=1}^{\infty} \mu\left(\bigcap_{n=m}^K B_n^c\right) \quad \text{by subadditivity}$$

$$= \sum_{m=1}^{\infty} \prod_{n=m}^K \left(1 - \mu(B_n)\right) \quad \text{by assumption}$$

$$\leq \sum_{m=1}^{\infty} \prod_{n=m}^K e^{-\mu(B_n^c)} \quad \text{by hint}$$

$$= \sum_{m=1}^{\infty} \exp\left(-\sum_{n=m}^K \mu(B_n^c)\right)$$

$$\stackrel{K \to \infty}{\longrightarrow} 0$$

since
$$\sum_{n=m}^K \mu(B_n^c) \stackrel{K \longrightarrow \infty}{\longrightarrow} \infty$$
 by assumption

• We can apply continuity of measure since $B_K^c \xrightarrow{K \longrightarrow \infty} B^c$.

How to prove the hint..?

3.3 Spring 2019 # 2 😽

Let \mathcal{B} denote the set of all Borel subsets of \mathbb{R} and $\mu : \mathcal{B} \longrightarrow [0, \infty)$ denote a finite Borel measure on \mathbb{R} .

3.3.1 a

Prove that if $\{F_k\}$ is a sequence of Borel sets for which $F_k \supseteq F_{k+1}$ for all k, then

$$\lim_{k \to \infty} \mu(F_k) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right)$$

3.3.2 b

Suppose μ has the property that $\mu(E) = 0$ for every $E \in \mathcal{B}$ with Lebesgue measure m(E) = 0.

Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $E \in \mathcal{B}$ with $m(E) < \delta$, then $\mu(E) < \varepsilon$.

Add concepts.

Solution:

Concepts Used:

• (

3.3.3 a

See Folland p.26 $\,$

- Lemma 1: $\mu(\coprod_{k=1}^{\infty} E_k) = \lim_{N \to \infty} \sum_{k=1}^{N} \mu(E_k)$.
- Suppose $F_0 \supseteq F_1 \supseteq \cdots$.
- Let $A_k = F_k \setminus F_{k+1}$, since the F_k are nested the A_k are disjoint
- Set $A := \coprod_{k=1}^{\infty} A_k$ and $F := \bigcap_{k=1}^{\infty} F_k$.
- Note $X = X \setminus Y \coprod X \cap Y$ for any two sets (just write $X \setminus Y := X \cap Y^c$)
- Note that A contains anything that was removed from F_0 when passing from any F_j to F_{j+1} , while F contains everything that is never removed at any stage, and these are disjoint possibilities.

• Thus $F_0 = F \coprod A$, so

$$\mu(F_0) = \mu(F) + \mu(A)$$

$$= \mu(F) + \mu(\coprod_{k=1}^{\infty} A_k)$$

$$= \mu(F) + \lim_{n \to \infty} \sum_{k=0}^{n} \mu(A_k) \text{ by countable additivity}$$

$$= \mu(F) + \lim_{n \to \infty} \sum_{k=0}^{n} \mu(F_k) - \mu(F_{k+1})$$

$$= \mu(F) + \lim_{n \to \infty} (\mu(F_1) - \mu(F_n)) \text{ (Telescoping)}$$

$$= \mu(F) + \mu(F_1) - \lim_{N \to \infty} \mu(F_n),$$

• Since μ is a finite measure, $\mu(F_1) < \infty$ and can be subtracted, yielding

$$\mu(F_1) = \mu(F) + \mu(F_1) - \lim_{n \to \infty} \mu(F_n)$$

$$\implies \mu(F) = \lim_{n \to \infty} \mu(F_n)$$

$$\implies \mu\left(\bigcap_{k=1}^{\infty} F_k\right) = \lim_{n \to \infty} \mu(F_n).$$

3.3.4 b

- Toward a contradiction, negate the implication: suppose there exists an $\varepsilon > 0$ such that
- for all δ , we have $m(E) < \delta$ but $\mu(E) > \varepsilon$.

 The sequence $\left\{ \delta_n := \frac{1}{2^n} \right\}_{n \in \mathbb{N}}$ and produce sets $A_n \in \mathcal{B}$ such $m(A_n) < \frac{1}{2^n}$ but $\mu(A_n) > \varepsilon$.

$$F_n := \bigcup_{j \ge n} A_j$$

$$C_m := \bigcap_{k=1}^m F_k$$

$$A := C_\infty := \bigcap_{k=1}^\infty F_k.$$

- Note that $F_1 \supseteq F_2 \supseteq \cdots$, since each increase in index unions fewer sets.
- By continuity for the Lebesgue measure,

$$m(A) = m\left(\bigcap_{k=1}^{\infty} F_k\right) = \lim_{k \to \infty} m(F_k) = \lim_{k \to \infty} m\left(\bigcup_{j \ge k} A_j\right) \le \lim_{k \to \infty} \sum_{j \ge k} m(A_j) = \lim_{k \to \infty} \sum_{j \ge k} \frac{1}{2^n} = 0,$$

which follows because this is the tail of a convergent sum

- Thus m(A) = 0 and by assumption, this implies $\mu(A) = 0$.
- However, by part (a),

$$\mu(A) = \lim \mu\left(\bigcup_{k=0}^{\infty} A_k\right) \ge \lim \mu(A_n) = \lim \varepsilon = \varepsilon > 0.$$

All messed up!

3.4 Fall 2018 # 2 🔭

Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Show that there is a Borel set $B \subset E$ such that $m(E \setminus B) = 0$.

Move this to review notes to clean things up.

Solution:

Concepts Used:

- Definition of measurability: there exists an open $O \supset E$ such that $m_*(O \setminus E) < \varepsilon$
- Theorem: E is Lebesgue measurable iff there exists a closed set $F \subseteq E$ such that $m_*(E \setminus F) < \varepsilon \text{ for all } \varepsilon > 0.$
- Every F_{σ}, G_{δ} is Borel.
- Claim: E is measurable \iff for every ε there exist $F_{\varepsilon} \subset E \subset G_{\varepsilon}$ with F_{ε} closed and G_{ε} open and $m(G_{\varepsilon} \setminus E) < \varepsilon$ and $m(E \setminus F_{\varepsilon}) < \varepsilon$.
 - Proof: existence of G_{ε} is the definition of measurability.
 - Existence of F_{ε} :?
- Claim: E is measurable \implies there exists an open $O \supseteq E$ such that $m(O \setminus E) = 0$.
 - Since E is measurable, for each $n \in \mathbb{N}$ choose $G_n \supseteq E$ such that $m_*(G_n \setminus E) <$
 - Set $O_N := \bigcap_{n=1}^N G_n$ and $O := \bigcap_{n=1}^\infty G_n$. Suppose E is bounded.
 - - * Note $O_N \setminus O$ and $m_*(O_1) < \infty$ if E is bounded, since in this case

$$m_*(G_n \setminus E) = m_*(G_1) - m_*(E) < 1 \iff m_*(G_1) < m_*(E) + \frac{1}{n} < \infty.$$

- * Note $O_N \setminus E \searrow O \setminus E$ since $O_N \setminus E := O_N \cap E^c \supseteq O_{N+1} \cap E^c$ for all N, and again $m_*(O_1 \setminus E) < \infty$.
- * So it's valid to apply continuity of measure from above:

$$m_*(O \setminus E) = \lim_{N \to \infty} m_*(O_N \setminus E)$$

$$\leq \lim_{N \to \infty} m_*(G_N \setminus E)$$

$$= \lim_{N \to \infty} \frac{1}{N} = 0,$$

where the inequality uses subadditivity on $\bigcap^{N} G_n \subseteq G_N$

- Suppose E is unbounded.
 - * Write $E^k = E \cap [k, k+1]^d \subset \mathbb{R}^d$ as the intersection of E with an annulus, and note that $E = \coprod_{k \in \mathbb{N}} E_k$.
 - * Each E_k is bounded, so apply the previous case to obtain $O_k \supseteq E_k$ with $m(O_k \setminus E_k) = 0.$
 - * So write $O_k = E_k \prod N_k$ where $N_k := O_k \setminus E_k$ is a null set.
 - * Define $O = \bigcup O_k$, note that $E \subseteq O$.
 - * Now note

$$O \setminus E = \left(\coprod_{k} O_{k}\right) \setminus \left(\coprod_{K} E_{k}\right)$$

$$\subseteq \coprod_{k} (O_{k} \setminus E_{k})$$

$$\implies m_{*}(O \setminus E) \le m_{*}\left(\coprod (O_{k} \setminus E_{k})\right) = 0,$$

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since any countable union of null sets is again null. - So $O \supseteq E$ with $m(O \setminus E) = 0$.

- Theorem: since E is measurable, E^c is measurable
 - Proof: It suffices to write E^c as the union of two measurable sets, E^c $S \mid (E^c - S)$, where S is to be determined.
 - We'll produce an S such that $m(E^c S) = 0$ and use the fact that any subset

3.4.1 Indirect Proof

- Since E is measurable, E^c is measurable.
- Since E^c is measurable exists an open $O \supseteq E^c$ such that $m(O \setminus E^c) = 0$.
- Set $B := O^c$, then $O \supseteq E^c \iff \mathcal{O}^c \subseteq E \iff B \subseteq E$.
- Computing measures yields

$$E \setminus B := E \setminus \mathcal{O}^c := E \bigcap (\mathcal{O}^c)^c = E \bigcap \mathcal{O} = \mathcal{O} \bigcap (E^c)^c := \mathcal{O} \setminus E^c,$$

thus $m(E \setminus B) = m(\mathcal{O} \setminus E^c) = 0$.

• Since \mathcal{O} is open, B is closed and thus Borel.

3.4.2 Direct Proof (Todo)

Try to construct the set

3.5 Spring 2018 # 1 😽

Define

$$E := \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N} \right\}.$$

Prove that m(E) = 0.

Solution:

Concepts Used:

- Borel-Cantelli: If $\{E_k\}_{k\in\mathbb{Z}}\subset 2^{\mathbb{R}}$ is a countable collection of Lebesgue measurable sets with $\sum_{k\in\mathbb{Z}} m(E_k) < \infty$, then almost every $x\in\mathbb{R}$ is in at most finitely many E_k .
 - Equivalently (?), $m(\limsup_{k\to\infty} E_k) = 0$, where $\limsup_{k\to\infty} E_k = \bigcap_{k=1}^{\infty} \bigcup_{j\geq k} E_j$, the elements which are in E_k for infinitely many k.
- Strategy: Borel-Cantelli.
- We'll show that $m(E) \bigcap [n, n+1] = 0$ for all $n \in \mathbb{Z}$; then the result follows from

$$m(E) = m\left(\bigcup_{n \in \mathbb{Z}} E \bigcap [n, n+1]\right) \le \sum_{n=1}^{\infty} m(E \bigcap [n, n+1]) = 0.$$

- By translation invariance of measure, it suffices to show $m(E \cap [0,1]) = 0$.
 - So WLOG, replace E with $E \cap [0,1]$.
- Define

$$E_j := \left\{ x \in [0,1] \mid \exists p \in \mathbb{Z}^{\geq 0} \text{ s.t. } \left| x - \frac{p}{j} \right| < \frac{1}{j^3} \right\}.$$

- Note that $E_j \subseteq \coprod_{p \in \mathbb{Z}^{\geq 0}} B_{j^{-3}}\left(\frac{p}{j}\right)$, i.e. a union over integers p of intervals of radius $1/j^3$ around the points p/j. Since $1/j^3 < 1/j$, this union is in fact disjoint.
- Importantly, note that

$$\limsup_{j \to \infty} E_j \coloneqq \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j = E$$

since

 $x \in \limsup_{j} E_{j} \iff x \in E_{j}$ for infinitely many j $\iff \text{ there are infinitely many } j \text{ for which there exist a } p \text{ such that } \left| x - \frac{p}{j} \right| < j^{-3}$ $\iff x \in E.$

• Intersecting with [0,1], we can write E_i as a union of intervals:

$$E_{j} = \left(0, j^{-3}\right) \quad \coprod \quad B_{j^{-3}}\left(\frac{1}{j}\right) \coprod B_{j^{-3}}\left(\frac{2}{j}\right) \coprod \cdots \coprod B_{j^{-3}}\left(\frac{j-1}{j}\right) \quad \coprod \quad (1-j^{-3}, 1),$$

where we've separated out the "boundary" terms to emphasize that they are balls about 0 and 1 intersected with [0,1].

- Since E_i is a union of open sets, it is Borel and thus Lebesgue measurable.
- Computing the measure of E_i :
 - For a fixed j, there are exactly j+1 possible choices for a numerator $(0,1,\cdots,j)$, thus there are exactly j+1 sets appearing in the above decomposition.
 - The first and last intervals are length $\frac{1}{i^3}$
 - The remaining (j+1)-2=j-1 intervals are twice this length, $\frac{2}{i^3}$
 - Thus

$$m(E_j) = 2\left(\frac{1}{j^3}\right) + (j-1)\left(\frac{2}{j^3}\right) = \frac{2}{j^2}$$

• Note that

$$\sum_{j \in \mathbb{N}} m(E_j) = 2 \sum_{j \in \mathbb{N}} \frac{1}{j^2} < \infty,$$

which converges by the p-test for sums.

• But then

$$m(E) = m(\limsup_{j \in \mathbb{N}} E_j)$$

$$= m(\bigcap_{n \in \mathbb{N}} \bigcup_{j \geq n} E_j)$$

$$\leq m(\bigcup_{j \geq N} E_j) \text{ for every } N$$

$$\leq \sum_{j \geq N} m(E_j)$$

$$\stackrel{N \longrightarrow \infty}{\longrightarrow} 0$$

• Thus E is measurable as a subset of a null set and m(E) = 0.

3.6 Fall 2017 # 2 처

Let $f(x) = x^2$ and $E \subset [0, \infty) := \mathbb{R}^+$.

1. Show that

$$m^*(E) = 0 \iff m^*(f(E)) = 0.$$

2. Deduce that the map

$$\varphi: \mathcal{L}(\mathbb{R}^+) \longrightarrow \mathcal{L}(\mathbb{R}^+)$$

$$E \mapsto f(E)$$

is a bijection from the class of Lebesgue measurable sets of $[0, \infty)$ to itself.

Walk through.

Solution:

Concepts Used:

• ?

3.6.1 a

It suffices to consider the bounded case, i.e. $E \subseteq B_M(0)$ for some M. Then write $E_n = B_n(0) \cap E$ and apply the theorem to E_n , and by subadditivity, $m^*(E) = m^*(\bigcup_n E_n) \le 1$

$$\sum_{n} m^*(E_n) = 0.$$

Lemma: $f(x) = x^2, f^{-1}(x) = \sqrt{x}$ are Lipschitz on any compact subset of $[0, \infty)$. Proof: Let g = f or f^{-1} . Then $g \in C^1([0, M])$ for any M, so g is differentiable and g' is

Proof: Let g = f or f^{-1} . Then $g \in C^1([0, M])$ for any M, so g is differentiable and g' is continuous. Since g' is continuous on a compact interval, it is bounded, so $|g'(x)| \leq L$ for all x. Applying the MVT,

$$|f(x) - f(y)| = f'(c)|x - y| \le L|x - y|.$$

Lemma: If g is Lipschitz on \mathbb{R}^n , then $m(E) = 0 \implies m(g(E)) = 0$.

Proof: If g is Lipschitz, then

$$g(B_r(x)) \subseteq B_{Lr}(x),$$

which is a dilated ball/cube, and so

$$m^*(B_{Lr}(x)) \le L^n \cdot m^*(B_r(x)).$$

Now choose $\{Q_j\} \rightrightarrows E$; then $\{g(Q_j)\} \rightrightarrows g(E)$.

By the above observation,

$$|g(Q_j)| \le L^n |Q_j|,$$

and so

$$m^*(g(E)) \le \sum_j |g(Q_j)| \le \sum_j L^n |Q_j| = L^n \sum_j |Q_j| \longrightarrow 0.$$

Now just take $g(x) = x^2$ for one direction, and $g(x) = f^{-1}(x) = \sqrt{x}$ for the other.

3.6.2 b

Lemma: E is measurable iff $E = K \coprod N$ for some K compact, N null.

Write $E = K \coprod N$ where K is compact and N is null.

Then
$$\varphi^{-1}(E) = \varphi^{-1}(K \coprod N) = \varphi^{-1}(K) \coprod \varphi^{-1}(N)$$
.

Since $\varphi^{-1}(N)$ is null by part (a) and $\varphi^{-1}(K)$ is the preimage of a compact set under a continuous map and thus compact, $\varphi^{-1}(E) = K' \coprod N'$ where K' is compact and N' is null, so $\varphi^{-1}(E)$ is measurable.

So φ is a measurable function, and thus yields a well-defined map $\mathcal{L}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathbb{R})$ since it preserves measurable sets. Restricting to $[0, \infty)$, f is bijection, and thus so is φ .

3.7 Spring 2017 # 2 처

3.7.1 a

Let μ be a measure on a measurable space (X, \mathcal{M}) and f a positive measurable function.

Define a measure λ by

$$\lambda(E) := \int_{E} f \ d\mu, \quad E \in \mathcal{M}$$

Show that for g any positive measurable function,

$$\int_X g \ d\lambda = \int_X fg \ d\mu$$

3.7.2 b

Let $E \subset \mathbb{R}$ be a measurable set such that

$$\int_{E} x^2 \ dm = 0.$$

Show that m(E) = 0.

Solution:

Concepts Used:

- Absolute continuity of measures: $\lambda \ll \mu \iff E \in \mathcal{M}, \mu(E) = 0 \implies \lambda(E) = 0.$
- Radon-Nikodym: if $\lambda \ll \mu$, then there exists a measurable function $\frac{\partial \lambda}{\partial \mu} := f$ where

$$\lambda(E) = \int_E f \, d\mu.$$
 • Chebyshev's inequality:

$$A_c := \left\{ x \in X \mid |f(x)| \ge c \right\} \implies \mu(A_c) \le c^{-p} \int_{A_c} |f|^p d\mu \quad \forall 0$$

3.7.3 a

- Strategy: use approximation by simple functions to show absolute continuity and apply Radon-Nikodym
- Claim: $\lambda \ll \mu$, i.e. $\mu(E) = 0 \implies \lambda(E) = 0$.
 - Note that if this holds, by Radon-Nikodym, $f = \frac{\partial \lambda}{\partial \mu} \implies d\lambda = f d\mu$, which would

$$\int g \ d\lambda = \int g f \ d\mu.$$

- So let E be measurable and suppose $\mu(E) = 0$.
- Then

$$\lambda(E) \coloneqq \int_{E} f \ d\mu = \lim_{n \to \infty} \left\{ \int_{E} s_n \, d\mu \mid s_n \coloneqq \sum_{j=1}^{\infty} c_j \mu(E_j), \, s_n \nearrow f \right\}$$

where we take a sequence of simple functions increasing to f.

• But since each $E_i \subseteq E$, we must have $\mu(E_i) = 0$ for any such E_i , so every such s_n must be zero and thus $\lambda(E) = 0$.

What is the final step in this approximation

3.7.4 b

- Set $g(x) = x^2$, note that g is positive and measurable.
- By part (a), there exists a positive f such that for any $E \subseteq \mathbb{R}$,

$$\int_{E} g \ dm = \int_{E} g f \ d\mu$$

- The LHS is zero by assumption and thus so is the RHS.
- $-m \ll \mu$ by construction.
- Note that gf is positive.
- Define $A_k = \left\{ x \in X \mid gf \cdot \chi_E > \frac{1}{k} \right\}$, for $k \in \mathbb{Z}^{\geq 0}$
- Then by Chebyshev with p = 1, for every k we have

$$\mu(A_k) \le k \int_E gf \ d\mu = 0$$

- Then noting that $A_k \searrow A := \{x \in X \mid gf \cdot \chi_E(x) > 0\}$, we have $\mu(A) = 0$.
- Since gf is positive, we have

$$x \in E \iff gf\chi_E(x) > 0 \iff x \in A$$

so E = A and $\mu(E) = \mu(A)$.

• But $m \ll \mu$ and $\mu(E) = 0$, so we can conclude that m(E) = 0.

3.8 Fall 2016 # 4 💝

Let (X, \mathcal{M}, μ) be a measure space and suppose $\{E_n\} \subset \mathcal{M}$ satisfies

$$\lim_{n \to \infty} \mu\left(X \backslash E_n\right) = 0.$$

Define

$$G := \left\{ x \in X \mid x \in E_n \text{ for only finitely many } n \right\}.$$

Show that $G \in \mathcal{M}$ and $\mu(G) = 0$.

Add concepts

Solution:

Concepts Used:

- ?
- Claim: $G \in \mathcal{M}$.
 - Claim:

$$G = \left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n\right)^c = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n^c.$$

- * This follows because x is in the RHS $\iff x \in E_n^c$ for all but finitely many $n \iff x \in E_n$ for at most finitely many n.
- But \mathcal{M} is a σ -algebra, and this shows G is obtained by countable unions/intersections/complements of measurable sets, so $G \in \mathcal{M}$.
- Claim: $\mu(G) = 0$.

- We have

$$\mu(G) = \mu \left(\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n^c \right)$$

$$\leq \sum_{N=1}^{\infty} \mu \left(\bigcap_{n=N}^{\infty} E_n^c \right)$$

$$\leq \sum_{N=1}^{\infty} \mu(E_M^c)$$

$$\coloneqq \sum_{N=1}^{\infty} \mu(X \setminus E_N)$$

$$\stackrel{N \longrightarrow \infty}{\longrightarrow} 0$$

Last step seems wrong

3.9 Spring 2016 # 3

Let f be Lebesgue measurable on $\mathbb R$ and $E\subset\mathbb R$ be measurable such that

$$0 < A = \int_{E} f(x)dx < \infty.$$

Show that for every 0 < t < 1, there exists a measurable set $E_t \subset E$ such that

$$\int_{E_{+}} f(x)dx = tA.$$

3.10 Spring 2016 # 5

Let (X, \mathcal{M}, μ) be a measure space. For $f \in L^1(\mu)$ and $\lambda > 0$, define

$$\varphi(\lambda) = \mu(\{x \in X | f(x) > \lambda\}) \quad \text{ and } \quad \psi(\lambda) = \mu(\{x \in X | f(x) < -\lambda\})$$

Show that φ, ψ are Borel measurable and

$$\int_{X} |f| \ d\mu = \int_{0}^{\infty} [\varphi(\lambda) + \psi(\lambda)] \ d\lambda$$

3.11 Fall 2015 # 2

Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be Lebesgue measurable.

- 1. Show that there is a sequence of simple functions $s_n(x)$ such that $s_n(x) \longrightarrow f(x)$ for all $x \in \mathbb{R}$.
- 2. Show that there is a Borel measurable function g such that g = f almost everywhere.

3.12 Spring 2015 # 3

Let μ be a finite Borel measure on $\mathbb R$ and $E \subset \mathbb R$ Borel. Prove that the following statements are equivalent:

1. $\forall \varepsilon > 0$ there exists G open and F closed such that

$$F \subseteq E \subseteq G$$
 and $\mu(G \setminus F) < \varepsilon$.

2. There exists a $V \in G_{\delta}$ and $H \in F_{\sigma}$ such that

$$H \subseteq E \subseteq V$$
 and $\mu(V \setminus H) = 0$

3.13 Spring 2014 # 3

Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ and suppose

$$\forall x \in \mathbb{R}, \quad f(x) \ge \limsup_{y \to x} f(y)$$

Prove that f is Borel measurable.

3.14 Spring 2014 # 4

Let (X, \mathcal{M}, μ) be a measure space and suppose f is a measurable function on X. Show that

$$\lim_{n \to \infty} \int_X f^n \ d\mu = \begin{cases} \infty & \text{or} \\ \mu(f^{-1}(1)), \end{cases}$$

and characterize the collection of functions of each type.

3.15 Spring 2017 # 1 🔆

Let K be the set of numbers in [0,1] whose decimal expansions do not use the digit 4.

We use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 with $399\cdots$. For example, $0.8754 = 0.8753999\cdots$.

Show that K is a compact, nowhere dense set without isolated points, and find the Lebesgue measure m(K).

Solution:

Concepts Used:

• Definition: A is nowhere dense \iff every interval I contains a subinterval $S \subseteq A^c$.

— Equivalently, the interior of the closure is empty, $\left(\overline{K}\right)^{\circ} = \emptyset$.

Claim: K is compact.

• It suffices to show that $K^c := [0,1] \setminus K$ is open; Then K will be a closed and bounded

subset of \mathbb{R} and thus compact by Heine-Borel.

- Strategy: write K^c as the union of open balls (since these form a basis for the Euclidean topology on \mathbb{R}).
 - Do this by showing every point $x \in K^c$ is an interior point, i.e. x admits a neighborhood N_x such that $N_x \subseteq K^c$.
- Identify K^c as the set of real numbers in [0,1] whose decimal expansion **does** contain a 4.
 - We will show that there exists a neighborhood small enough such that all points in it contain a 4 in their decimal expansions.
- Let $x \in K^c$, suppose a 4 occurs as the kth digit, and write

$$x = 0.d_1 d_2 \cdots d_{k-1} \ 4 \ d_{k+1} \cdots = \left(\sum_{j=1}^k d_j 10^{-j} \right) + \left(4 \cdot 10^{-k} \right) + \left(\sum_{j=k+1}^\infty d_j 10^{-j} \right).$$

• Set $r_x < 10^{-k}$ and let $y \in [0,1] \cap B_{r_x}(x)$ be arbitrary and write

$$y = \sum_{j=1}^{\infty} c_j 10^{-j}.$$

- Thus $|x y| < r_x < 10^{-k}$, and the first k digits of x and y must agree:
 - We first compute the difference:

$$x - y = \sum_{i=1}^{\infty} d_i 10^{-i} - \sum_{i=1}^{\infty} c_i 10^{-i} = \sum_{i=1}^{\infty} (d_i - c_i) 10^{-i}$$

- Thus (claim)

$$|x - y| \le \sum_{j=1}^{\infty} |d_j - c_j| 10^j < 10^{-k} \iff |d_j - c_j| = 0 \quad \forall j \le k.$$

- Otherwise we can note that any term $|d_j - c_j| \ge 1$ and there is a contribution to |x - y| of at least $1 \cdot 10^{-j}$ for some j < k, whereas

$$j < k \iff 10^{-j} > 10^{-k}$$

a contradiction.

- This means that for all $j \leq k$ we have $d_j = c_j$, and in particular $d_k = 4 = c_k$, so y has a 4 in its decimal expansion.
- But then $K^c = \bigcup B_{r_x}(x)$ is a union of open sets and thus open.

Claim: K is nowhere dense and m(K) = 0:

- Strategy: Show $(\overline{K})^{\circ} = \emptyset$.
- Since K is closed, $\overline{K} = K$, so it suffices to show that K does not properly contain any interval.
- It suffices to show $m(K^c) = 1$, since this implies m(K) = 0 and since any interval has strictly positive measure, this will mean K can not contain an interval.

- As in the construction of the Cantor set, let
 - K_0 denote [0,1] with 1 interval $\left(\frac{4}{10},\frac{5}{10}\right)$ of length $\frac{1}{10}$ deleted, so

$$m(K_0^c) = \frac{1}{10}.$$

 $-K_1$ denote K_0 with 9 intervals $\left(\frac{1}{100}, \frac{5}{100}\right), \left(\frac{14}{100}, \frac{15}{100}\right), \cdots \left(\frac{94}{100}, \frac{95}{100}\right)$ of length $\frac{1}{100}$ deleted, so

$$m(K_1^c) = \frac{1}{10} + \frac{9}{100}.$$

- K_n denote K_{n-1} with 9^n such intervals of length $\frac{1}{10^{n+1}}$ deleted, so

$$m(K_n^c) = \frac{1}{10} + \frac{9}{100} + \dots + \frac{9^n}{10^{n+1}}.$$

• Then compute

$$m(K^c) = \sum_{j=0}^{\infty} \frac{9^n}{10^{n+1}} = \frac{1}{10} \sum_{j=0}^{\infty} \left(\frac{9}{10}\right)^n = \frac{1}{10} \left(\frac{1}{1 - \frac{9}{10}}\right) = 1.$$

Claim: *K* has no isolated points:

- A point $x \in K$ is isolated iff there there is an open ball $B_r(x)$ containing x such that $B_r(x) \subseteq K^c$.
 - So every point in this ball **should** have a 4 in its decimal expansion.
- Strategy: show that if $x \in K$, every neighborhood of x intersects K.
- Note that $m(K_n) = \left(\frac{9}{10}\right)^n \xrightarrow{n \to \infty} 0$
- Also note that we deleted open intervals, and the endpoints of these intervals are never deleted.
 - Thus endpoints of deleted intervals are elements of K.
- Fix x. Then for every ε , by the Archimedean property of \mathbb{R} , choose n such that $\left(\frac{9}{10}\right)^n < \varepsilon$.
- Then there is an endpoint x_n of some deleted interval I_n satisfying

$$|x - x_n| \le \left(\frac{9}{10}\right)^n < \varepsilon.$$

• So every ball containing x contains some endpoint of a removed interval, and thus an element of K.

3.16 Spring 2016 # 2

Let $0 < \lambda < 1$ and construct a Cantor set C_{λ} by successively removing middle intervals of length λ . Prove that $m(C_{\lambda}) = 0$.

4 Measure Theory: Functions

4.1 Fall 2016 # 2 처

Let $f, g: [a, b] \longrightarrow \mathbb{R}$ be measurable with

$$\int_a^b f(x) \ dx = \int_a^b g(x) \ dx.$$

Show that either

- 1. f(x) = g(x) almost everywhere, or
- 2. There exists a measurable set $E \subset [a, b]$ such that

$$\int_{E} f(x) dx > \int_{E} g(x) dx$$

Add concepts

Solution:

Concepts Used:

- ?
- Suppose it is *not* the case that f = g almost everywhere; then letting $A := \{x \in [a,b] \mid f(x) \neq g(x)\}$, we have m(A) > 0.
- . Write

$$A = A_1 \coprod A_2 := \{f > g\} \coprod \{f < g\},\,$$

then $m(A_1) > 0$ or $m(A_2) > 0$ (or both).

- Wlog (by relabeling f, g if necessary), suppose $m(A_1) > 0$, and take $E := A_1$.
- Then on E, we have f(x) > g(x) pointwise. This is preserved by monotonicity of the integral, thus

$$f(x) > g(x)$$
 on $E \implies \int_E f(x) dx > \int_E g(x) dx$.

4.2 Spring 2016 # 4

Let $E \subset \mathbb{R}$ be measurable with $m(E) < \infty$. Define

$$f(x) = m(E \cap (E + x)).$$

Show that

- 1. $f \in L^1(\mathbb{R})$.
- 2. f is uniformly continuous.
- $3. \lim_{|x| \to \infty} f(x) = 0.$

Hint:

$$\chi_{E\cap(E+x)}(y) = \chi_E(y)\chi_E(y-x)$$

5 Integrals: Convergence

5.1 Fall 2019 # 2 🔆

Prove that

$$\left| \frac{d^n}{dx^n} \frac{\sin x}{x} \right| \le \frac{1}{n}$$

for all $x \neq 0$ and positive integers n.

Hint: Consider
$$\int_0^1 \cos(tx) dt$$

Solution:

Concepts Used:

- DCT
- Bounding in the right place. Don't evaluate the actual integral!
- By induction on the number of limits we can pass through the integral.
- For n = 1 we first pass one derivative into the integral: let $x_n \longrightarrow x$ be any sequence converging to x, then

$$\frac{\partial}{\partial x} \frac{\sin(x)}{x} = \frac{\partial}{\partial x} \int_0^1 \cos(tx) dt$$

$$= \lim_{x_n \to x} \frac{1}{x_n - x} \left(\int_0^1 \cos(tx_n) dt - \int_0^1 \cos(tx) dt \right)$$

$$= \lim_{x_n \to x} \left(\int_0^1 \frac{\cos(tx_n) - \cos(tx)}{x_n - x} dt \right)$$

$$= \lim_{x_n \to x} \left(\int_0^1 \left(t \sin(tx) \Big|_{x = \xi_n} \right) dt \right) \quad \text{where} \quad \xi_n \in [x_n, x] \text{ by MVT}, \xi_n \to x$$

$$= \lim_{\xi_n \to x} \left(\int_0^1 t \sin(t\xi_n) dt \right)$$

$$= \int_0^1 t \sin(tx) dt$$

$$= \int_0^1 t \sin(tx) dt$$

5 INTEGRALS: CONVERGENCE

• Taking absolute values we obtain an upper bound

$$\left| \frac{\partial}{\partial x} \frac{\sin(x)}{x} \right| = \left| \int_0^1 t \sin(tx) dt \right|$$

$$\leq \int_0^1 |t \sin(tx)| dt$$

$$\leq \int_0^1 1 dt = 1,$$

since $t \in [0,1] \implies |t| < 1$, and $|\sin(xt)| \le 1$ for any x and t.

• Note that this bound also justifies the DCT, since the functions $f_n(t) = t \sin(t\xi_n)$ are uniformly dominated by g(t) = 1 on $L^1([0,1])$.

Note: integrating by parts here yields the actual formula:

$$\int_{0}^{1} t \sin(tx) dt =_{IBP} \left(\frac{-t \cos(tx)}{x} \right) \Big|_{t=0}^{t=1} - \int_{0}^{1} \frac{\cos(tx)}{x} dt$$
$$= \frac{-\cos(x)}{x} - \frac{\sin(x)}{x^{2}}$$
$$= \frac{x \cos(x) - \sin(x)}{x^{2}}.$$

• For the inductive step, we assume that we can pass n-1 limits through the integral and show we can pass the nth through as well.

$$\frac{\partial^n}{\partial x^n} \frac{\sin(x)}{x} = \frac{\partial^n}{\partial x^n} \int_0^1 \cos(tx) \, dt$$
$$= \frac{\partial}{\partial x} \int_0^1 \frac{\partial^{n-1}}{\partial x^{n-1}} \cos(tx) \, dt$$
$$= \frac{\partial}{\partial x} \int_0^1 t^{n-1} f_{n-1}(x,t) \, dt$$

- Note that $f_n(x,t) = \pm \sin(tx)$ when n is odd and $f_n(x,t) = \pm \cos(tx)$ when n is even, and a constant factor of t is multiplied when each derivative is taken.
- We continue as in the base case:

$$\frac{\partial}{\partial x} \int_0^1 t^{n-1} f_{n-1}(x,t) dt = \lim_{x_k \to x} \int_0^1 t^{n-1} \left(\frac{f_{n-1}(x_n,t) - f_{n-1}(x,t)}{x_n - x} \right) dt$$

$$=_{\text{IVT}} \lim_{x_k \to x} \int_0^1 t^{n-1} \frac{\partial f_{n-1}}{\partial x} \left(\xi_k, t \right) dt \quad \text{where } \xi_k \in [x_k, x], \, \xi_k \to x$$

$$=_{\text{DCT}} \int_0^1 \lim_{x_k \to x} t^{n-1} \frac{\partial f_{n-1}}{\partial x} \left(\xi_k, t \right) dt$$

$$\coloneqq \int_0^1 \lim_{x_k \to x} t^n f_n(\xi_k, t) dt$$

$$\coloneqq \int_0^1 t^n f_n(x, t) dt.$$

- We've used the fact that $f_0(x) = \cos(tx)$ is smooth as a function of x, and in particular continuous
- The DCT is justified because the functions $h_{n,k}(x,t) = t^n f_n(\xi_k,t)$ are again uniformly (in k) bounded by 1 since $t \leq 1 \implies t^n \leq 1$ and each f_n is a sin or cosine.

• Now take absolute values

$$\left| \frac{\partial^n}{\partial x^n} \frac{\sin(x)}{x} \right| = \left| \int_0^1 -t^n f_n(x, t) \, dt \right|$$

$$\leq \int_0^1 |t^n f_n(x, t)| \, dt$$

$$\leq \int_0^1 |t^n| |f_n(x, t)| \, dt$$

$$\leq \int_0^1 |t^n| \cdot 1 \, dt$$

$$\leq \int_0^1 t^n \, dt \quad \text{since } t \text{ is positive}$$

$$= \frac{1}{n+1}$$

$$< \frac{1}{n}.$$

- We've again used the fact that $f_n(x,t)$ is of the form $\pm \cos(tx)$ or $\pm \sin(tx)$, both of which are bounded by 1.

5.2 Spring 2020 # 5 💝

Compute the following limit and justify your calculations:

$$\lim_{n \to \infty} \int_0^n \left(1 + \frac{x^2}{n} \right)^{-(n+1)} dx.$$

Not finished, flesh out

Walk through.

Solution:

Concepts Used:

- DCT
- Passing limits through products and quotients

Note that

$$\lim_{n} \left(1 + \frac{x^2}{n} \right)^{-(n+1)} = \frac{1}{\lim_{n} \left(1 + \frac{x^2}{n} \right)^1 \left(1 + \frac{x^2}{n} \right)^n}$$
$$= \frac{1}{1 \cdot e^{x^2}}$$
$$= e^{-x^2}.$$

If passing the limit through the integral is justified, we will have

$$\lim_{n \to \infty} \int_0^n \left(1 + \frac{x^2}{n} \right)^{-(n+1)} dx = \lim_{n \to \infty} \int_{\mathbb{R}} \chi_{[0,n]} \left(1 + \frac{x^2}{n} \right)^{-(n+1)} dx$$

$$= \int_{\mathbb{R}} \lim_{n \to \infty} \chi_{[0,n]} \left(1 + \frac{x^2}{n} \right)^{-(n+1)} dx \quad \text{by the DCT}$$

$$= \int_{\mathbb{R}} \lim_{n \to \infty} \left(1 + \frac{x^2}{n} \right)^{-(n+1)} dx$$

$$= \int_0^\infty e^{-x^2}$$

$$= \frac{\sqrt{\pi}}{2}.$$

Computing the last integral:

$$\left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2 = \left(\int_{\mathbb{R}} e^{-x^2} dx\right) \left(\int_{\mathbb{R}} e^{-y^2} dx\right)$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(x+y)^2} dx$$

$$= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta \qquad u = r^2$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^{\infty} e^{-u} du d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} 1$$

$$= \pi.$$

and now use the fact that the function is even so $\int_0^\infty f = \frac{1}{2} \int_{\mathbb{R}} f$. Justifying the DCT:

• Apply Bernoulli's inequality:

$$1 + \frac{x^2}{n}^{n+1} \ge 1 + \frac{x^2}{n} (1 + x^2) \ge 1 + x^2,$$

where the last inequality follows from the fact that $1 + \frac{x^2}{n} \ge 1$

5.3 Spring 2019 # 3 🔭

Let $\{f_k\}$ be any sequence of functions in $L^2([0,1])$ satisfying $\|f_k\|_2 \leq M$ for all $k \in \mathbb{N}$. Prove that if $f_k \longrightarrow f$ almost everywhere, then $f \in L^2([0,1])$ with $\|f\|_2 \leq M$ and

$$\lim_{k \to \infty} \int_0^1 f_k(x) dx = \int_0^1 f(x) dx$$

Hint: Try using Fatou's Lemma to show that $||f||_2 \leq M$ and then try applying Egorov's

Solution:

Concepts Used:

- Definition of L^+ : space of measurable function $X \longrightarrow [0, \infty]$.
- Fatou: For any sequence of L⁺ functions, ∫ lim inf f_n ≤ lim inf ∫ f_n.
 Egorov's Theorem: If E ⊆ ℝⁿ is measurable, m(E) > 0, f_k: E → ℝ a sequence of
- measurable functions where $\lim_{n \to \infty} f_n(x)$ exists and is finite a.e., then $f_n \to f$ almost uniformly: for every $\varepsilon > 0$ there exists a closed subset $F_{\varepsilon} \subseteq E$ with $m(E \setminus F) < \varepsilon$ and $f_n \longrightarrow f$ uniformly on F.

L^2 bound:

- Since $f_k \longrightarrow f$ almost everywhere, $\liminf_{n \to \infty} f_n(x) = f(x)$ a.e.
- $||f_n||_2 < \infty$ implies each f_n is measurable and thus $|f_n|^2 \in L^+$, so we can apply Fatou:

$$||f||_2^2 = \int |f(x)|^2$$

$$= \int \liminf_n |f_n(x)|^2$$

$$\leq \lim_n \inf \int |f_n(x)|^2$$

$$\leq \liminf_n M$$

$$= M.$$

• Thus $||f||_2 \le \sqrt{M} < \infty$ implying $f \in L^2$.

What is the "right" proof here that uses the first part

Equality of Integrals:

- Take the sequence $\varepsilon_n = \frac{1}{n}$ Apply Egorov's theorem: obtain a set F_{ε} such that $f_n \longrightarrow f$ uniformly on F_{ε} and $m(I \setminus F_{\varepsilon}) < \varepsilon$.

$$\lim_{n \to \infty} \left| \int_0^1 f_n - f \right| \le \lim_{n \to \infty} \int_0^1 |f_n - f|$$

$$= \lim_{n \to \infty} \left(\int_{F_{\varepsilon}} |f_n - f| + \int_{I \setminus F_{\varepsilon}} |f_n - f| \right)$$

$$= \int_{F_{\varepsilon}} \lim_{n \to \infty} |f_n - f| + \lim_{n \to \infty} \int_{I \setminus F_{\varepsilon}} |f_n - f| \quad \text{by uniform convergence}$$

$$= 0 + \lim_{n \to \infty} \int_{I \setminus F} |f_n - f|,$$

so it suffices to show $\int_{I\setminus F_{-}} |f_{n}-f| \stackrel{n\longrightarrow\infty}{\longrightarrow} 0.$

• We can obtain a bound using Holder's inequality with p = q = 2:

$$\int_{I \setminus F_{\varepsilon}} |f_n - f| \le \left(\int_{I \setminus F_{\varepsilon}} |f_n - f|^2 \right) \left(\int_{I \setminus F_{\varepsilon}} |1|^2 \right)$$

$$= \left(\int_{I \setminus F_{\varepsilon}} |f_n - f|^2 \right) \mu(F_{\varepsilon})$$

$$\le \|f_n - f\|_2 \mu(F_{\varepsilon})$$

$$\le (\|f_n\|_2 + \|f\|_2) \mu(F_{\varepsilon})$$

$$\le 2M \cdot \mu(F_{\varepsilon})$$

where M is now a constant not depending on ε or n.

• Now take a nested sequence of sets F_{ε} with $\mu(F_{\varepsilon}) \longrightarrow 0$ and applying continuity of measure yields the desired statement.

5.4 Fall 2018 # 6 💝

Compute the following limit and justify your calculations:

$$\lim_{n\to\infty} \int_1^n \frac{dx}{\left(1+\frac{x}{n}\right)^n \sqrt[n]{x}}$$

Add concepts

Solution:

Concepts Used:

- ?
- Note that $x^{\frac{1}{n}} \xrightarrow{n \to \infty} 1$ for any $0 < x < \infty$.
- Thus the integrand converges to $\frac{1}{e^x}$, which is integrable on $(0,\infty)$ and integrates to 1.
- Break the integrand up:

$$\int_0^\infty \frac{1}{\left(1 + \frac{x}{n}\right)^n x^{\frac{1}{n}}} \, dx = \int_0^1 \frac{1}{\left(1 + \frac{x}{n}\right)^n x^{\frac{1}{n}}} \, dx = \int_1^\infty \frac{1}{\left(1 + \frac{x}{n}\right)^n x^{\frac{1}{n}}} \, dx.$$

5.5 Fall 2018 # 3 🐪

Suppose f(x) and xf(x) are integrable on \mathbb{R} . Define F by

$$F(t) := \int_{-\infty}^{\infty} f(x) \cos(xt) dx$$

Show that

$$F'(t) = -\int_{-\infty}^{\infty} x f(x) \sin(xt) dx.$$

Walk through

Solution:

Concepts Used:

- Mean Value Theorem
- DCT

$$\frac{\partial}{\partial t} F(t) = \frac{\partial}{\partial t} \int_{\mathbb{R}} f(x) \cos(xt) dx$$

$$\stackrel{DCT}{=} \int_{\mathbb{R}} f(x) \frac{\partial}{\partial t} \cos(xt) dx$$

$$= \int_{\mathbb{R}} x f(x) \cos(xt) dx,$$

so it only remains to justify the DCT.

- Fix t, then let $t_n \longrightarrow t$ be arbitrary.
- Define

$$h_n(x,t) = f(x) \left(\frac{\cos(tx) - \cos(t_n x)}{t_n - t} \right) \stackrel{n \to \infty}{\longrightarrow} \frac{\partial}{\partial t} \left(f(x) \cos(xt) \right)$$

since $\cos(tx)$ is differentiable in t and this is the limit definition of differentiability.

• Note that

$$\frac{\partial}{\partial t} \cos(tx) := \lim_{t_n \to t} \frac{\cos(tx) - \cos(t_n x)}{t_n - t}$$

$$\stackrel{MVT}{=} \frac{\partial}{\partial t} \cos(tx) \Big|_{t = \xi_n} \qquad \text{for some } \xi_n \in [t, t_n] \text{ or } [t_n, t]$$

$$= x \sin(\xi_n x)$$

where $\xi_n \stackrel{n \longrightarrow \infty}{\longrightarrow} t$ since wlog $t_n \le \xi_n \le t$ and $t_n \nearrow t$.

• We then have

$$|h_n(x)| = |f(x)x\sin(\xi_n x)| \le |xf(x)|$$
 since $|\sin(\xi_n x)| \le 1$

for every x and every n.

• Since $x f(x) \in L^1(\mathbb{R})$ by assumption, the DCT applies.

5.6 Spring 2018 # 5 🔭

Suppose that

- $f_n, f \in L^1$,
- $f_n \longrightarrow f$ almost everywhere, and $\int |f_n| \to \int |f|$.

Show that $\int f_n \to \int f$.

Solution:

Concepts Used:

•
$$\int |f_n - f| \longrightarrow \iff \int f_n = \int f$$
.
• Fatou:

$$\int \liminf f_n \le \liminf \int f_n$$
$$\int \limsup f_n \ge \limsup \int f_n.$$

• Since $\int |f_n| \stackrel{n \to \infty}{\longrightarrow} \int |f|$, define

$$h_n = |f_n - f|$$
 $\xrightarrow{n \to \infty} 0 \ a.e.$ $g_n = |f_n| + |f|$ $\xrightarrow{n \to \infty} 2|f| \ a.e.$

- Note that $g_n h_n \xrightarrow{n \to \infty} 2|f| 0 = 2|f|$.
- Then

$$\int 2|f| = \int \liminf_n (g_n - h_n)$$

$$= \int \liminf_n (g_n) + \int \liminf_n (-h_n)$$

$$= \int \liminf_n (g_n) - \int \limsup_n (h_n)$$

$$= \int 2|f| - \int \limsup_n (h_n)$$

$$\leq \int 2|f| - \limsup_n \int h_n \quad \text{by Fatou,}$$

• Since $f \in L^1$, $\int 2|f| = 2||f||_1 < \infty$ and it makes sense to subtract it from both sides,

$$0 \le -\limsup_{n} \int h_{n}$$
$$:= -\limsup_{n} \int |f_{n} - f|.$$

which forces $\limsup_{n} \int |f_n - f| = 0$, since

- The integral of a nonnegative function is nonnegative, so $\int |f_n - f| \ge 0$.

$$- \operatorname{So}\left(-\int |f_n - f|\right) \le 0.$$

– But the above inequality shows $\left(-\int |f_n - f|\right) \ge 0$ as well.

- Since $\liminf_{n} \int h_n \leq \limsup_{n} \int h_n = 0$, $\lim_{n} \int h_n$ exists and is equal to zero.
- But then

$$\left| \int f_n - \int f \right| = \left| \int f_n - f \right| \le \int |f_n - f|,$$

and taking $\lim_{n \to \infty}$ on both sides yields

$$\lim_{n \to \infty} \left| \int f_n - \int f \right| \le \lim_{n \to \infty} \int |f_n - f| = 0,$$

so
$$\lim_{n \to \infty} \int f_n = \int f$$
.

5.7 Spring 2018 # 2 😽

Let

$$f_n(x) := \frac{x}{1+x^n}, \quad x \ge 0.$$

- a. Show that this sequence converges pointwise and find its limit. Is the convergence uniform on $[0,\infty)$?
- b. Compute

$$\lim_{n\to\infty}\int_0^\infty f_n(x)dx$$

Add concepts.

Solution:

Concepts Used:

• ?

5.7.1 a

Claim: f_n does not converge uniformly to its limit.

- Note each $f_n(x)$ is clearly continuous on $(0, \infty)$, since it is a quotient of continuous functions where the denominator is never zero.
- Note

$$x < 1 \implies x^n \stackrel{n \longrightarrow \infty}{\longrightarrow} 0$$
 and $x > 1 \implies x^n \stackrel{n \longrightarrow \infty}{\longrightarrow} \infty$.

• Thus

$$f_n(x) = \frac{x}{1+x^n} \xrightarrow{n \to \infty} f(x) := \begin{cases} x, & 0 \le x < 1 \\ \frac{1}{2}, & x = 1 \\ 0, & x > 1 \end{cases}.$$

- If $f_n \longrightarrow f$ uniformly on $[0, \infty)$, it would converge uniformly on every subset and thus uniformly on $(0, \infty)$.
 - Then f would be a uniform limit of continuous functions on $(0, \infty)$ and thus continuous on $(0, \infty)$.
 - By uniqueness of limits, f_n would converge to the pointwise limit f above, which is not continuous at x = 1, a contradiction.

5.7.2 b

• If the DCT applies, interchange the limit and integral:

$$\lim_{n \to \infty} \int_0^\infty f_n(x) \, dx = \int_0^\infty \lim_{n \to \infty} f_n(x) \, dx \quad \text{DCT}$$

$$= \int_0^\infty f(x) \, dx$$

$$= \int_0^1 x \, dx + \int_1^\infty 0 \, dx$$

$$= \frac{1}{2} x^2 \Big|_0^1$$

$$= \frac{1}{2}.$$

• To justify the DCT, write

$$\int_{0}^{\infty} f_n(x) = \int_{0}^{1} f_n(x) + \int_{1}^{\infty} f_n(x).$$

• f_n restricted to (0,1) is uniformly bounded by $g_0(x) = 1$ in the first integral, since

$$x \in [0,1] \implies \frac{x}{1+x^n} < \frac{1}{1+x^n} < 1 := g(x)$$

so

$$\int_0^1 f_n(x) \, dx \le \int_0^1 1 \, dx = 1 < \infty.$$

Also note that $g_0 \cdot \chi_{(0,1)} \in L^1((0,\infty))$.

• The f_n restricted to $(1,\infty)$ are uniformly bounded by $g_1(x) = \frac{1}{x^2}$ on $[1,\infty)$, since

$$x \in (1, \infty) \implies \frac{x}{1+x^n} \le \frac{x}{x^n} = \frac{1}{x^{n-1}} \le \frac{1}{x^2} \in L^1([1, \infty) \text{ when } n \ge 3,$$

by the p-test for integrals.

• So set

$$g := g_0 \cdot \chi_{(0,1)} + g_1 \cdot \chi_{[1,\infty)},$$

then by the above arguments $g \in L^1((0,\infty))$ and $f_n \leq g$ everywhere, so the DCT applies.

5.8 Fall 2016 # 3 🙀

Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{x \to 0} \int_{\mathbb{R}} |f(y - x) - f(y)| \, dy = 0$$

Missing some stuff

Solution:

Concepts Used:

- $C_c^{\infty} \hookrightarrow L^p$ is dense. If $f \dots$?
- Fixing notation, set $\tau_x f(y) := f(y-x)$; we then want to show

$$\|\tau_x f - f\|_{L^1} \stackrel{x \longrightarrow 0}{\longrightarrow} 0.$$

- Claim: by an $\varepsilon/3$ argument, it suffices to show this for compactly supported functions:
 - Since $f \in L^1$, choose $g_n \subset C_c^{\infty}(\mathbb{R}^1)$ smooth and compactly supported so that

$$||f - g||_{L^1} < \varepsilon.$$

- Claim: $\|\tau_x f \tau_x g\| < \varepsilon$.
 - * Proof 1: translation invariance of the integral.
 - * Proof 2: Apply a change of variables:

$$\begin{aligned} \|\tau_x f - \tau_x g\|_1 &\coloneqq \int_{\mathbb{R}} |\tau_x f(y) - \tau_x g(y)| \, dy \\ &= \int_{\mathbb{R}} |f(y - x) - g(y - x)| \, dy \\ &= \int_{\mathbb{R}} |f(u) - g(u)| \, du \qquad (u = y - x, \, du = dy) \\ &= \|f - g\|_1 \\ &< \varepsilon. \end{aligned}$$

- Then

$$\|\tau_x f - f\|_1 = \|\tau_x f - \tau_x g + \tau_x g - g + g - f\|_1$$

$$\leq \|\tau_x f - \tau_x g\|_1 + \|\tau_x g - g\|_1 + \|g - f\|_1$$

$$\leq 2\varepsilon + \|\tau_x g - g\|_1.$$

- To show this for compactly supported functions:
 - Let $g \in C_c^{\infty}(\mathbb{R}^1)$, let E = supp(g), and write

$$\|\tau_x g - g\|_1 = \int_{\mathbb{R}} |g(y - x) - g(y)| \, dy$$

$$= \int_E |g(y - x) - g(y)| \, dy + \int_{E^c} |g(y - x) - g(y)| \, dy$$

$$= \int_E |g(y - x) - g(y)| \, dy.$$

– But g is smooth and compactly supported on E, and thus uniformly continuous on E, so

$$\lim_{x \to 0} \int_{E} |g(y-x) - g(y)| dy = \int_{E} \lim_{x \to 0} |g(y-x) - g(y)| dy$$
$$= \int_{E} 0 dy$$
$$= 0.$$

5.9 Fall 2015 # 3

Compute the following limit:

$$\lim_{n \to \infty} \int_{1}^{n} \frac{ne^{-x}}{1 + nx^{2}} \sin\left(\frac{x}{n}\right) dx$$

5.10 Fall 2015 # 4

Let $f:[1,\infty)\longrightarrow \mathbb{R}$ such that f(1)=1 and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Show that the following limit exists and satisfies the equality

$$\lim_{x \to \infty} f(x) \le 1 + \frac{\pi}{4}$$

6 Integrals: Approximation

6.1 Spring 2018 # 3 💝

Let f be a non-negative measurable function on [0, 1].

Show that

$$\lim_{p \to \infty} \left(\int_{[0,1]} f(x)^p dx \right)^{\frac{1}{p}} = \|f\|_{\infty}.$$

Solution:

Concepts Used:

- $||f||_{\infty} := \inf_{t} \{t \mid m(\{x \in \mathbb{R}^n \mid f(x) > t\}) = 0\}$, i.e. this is the lowest upper bound that holds almost everywhere.
- $||f||_p \le ||f||_{\infty}$:

- Note $|f(x)| \leq ||f||_{\infty}$ almost everywhere and taking pth powers preserves this inequality.
- Thus

$$|f(x)| \le ||f||_{\infty}$$
 a.e. by definition
 $\implies |f(x)|^p \le ||f||_{\infty}^p$ for $p \ge 0$
 $\implies ||f||_p^p = \int_X |f(x)|^p dx$
 $\le \int_X ||f||_{\infty}^p dx$
 $= ||f||_{\infty}^p \int_X 1 dx$
 $= ||f||_{\infty}^p \cdot m(X)$ since the norm doesn't depend on x
 $= ||f||_{\infty}^p$ since $m(X) = 1$.

- * Thus $\|f\|_p \leq \|f\|_{\infty}$ for all p and taking $\lim_{n \longrightarrow \infty}$ preserves this inequality.
- $||f||_p \ge ||f||_\infty$: Fix $\varepsilon > 0$.

 - Define

$$S_{\varepsilon} := \left\{ x \in \mathbb{R}^n \mid |f(x)| \ge ||f||_{\infty} - \varepsilon \right\}.$$

- * Note that $m(S_{\varepsilon}) > 0$; otherwise if $m(S_{\varepsilon}) = 0$, then $t := ||f||_{\infty} \varepsilon < ||f||_{\varepsilon}$. But this produces a smaller upper bound almost everywhere than $||f||_{\varepsilon}$, contradicting the definition of $||f||_{\varepsilon}$ as an infimum over such bounds.
- Then

$$\begin{split} \|f\|_p^p &= \int_X |f(x)|^p \ dx \\ &\geq \int_{S_\varepsilon} |f(x)|^p \ dx \quad \text{since } S_\varepsilon \subseteq X \\ &\geq \int_{S_\varepsilon} |\|f\|_\infty - \varepsilon|^p \ dx \quad \text{since on } S_\varepsilon, |f| \geq \|f\|_\infty - \varepsilon \\ &= |\|f\|_\infty - \varepsilon|^p \cdot m(S_\varepsilon) \quad \text{since the integrand is independent of } x \\ &\geq 0 \quad \text{since } m(S_\varepsilon) > 0 \end{split}$$

- Taking pth roots for $p \ge 1$ preserves the inequality, so

$$\implies \|f\|_p \ge \|\|f\|_{\infty} - \varepsilon\| \cdot m(S_{\varepsilon})^{\frac{1}{p}} \stackrel{p \to \infty}{\longrightarrow} \|\|f\|_{\infty} - \varepsilon\| \stackrel{\varepsilon \to 0}{\longrightarrow} \|f\|_{\infty}$$

where we've used the fact that above arguments work

- Thus $||f||_p \ge ||f||_{\infty}$.

6.2 Spring 2018 # 4 😽

Let $f \in L^2([0,1])$ and suppose

$$\int_{[0,1]} f(x)x^n dx = 0 \text{ for all integers } n \ge 0.$$

Show that f = 0 almost everywhere.

6.3 Spring 2015 # 2 처

Let $f: \mathbb{R} \longrightarrow \mathbb{C}$ be continuous with period 1. Prove that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_{0}^{1} f(t)dt \quad \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

Hint: show this first for the functions $f(t) = e^{2\pi i kt}$ for $k \in \mathbb{Z}$.

Add concepts.

Solution:

Concepts Used:

. 7

6.3.1 Proof 1: Using Fourier Transforms

Concepts Used:

- Weierstrass Approximation: A uniformly continuous function on a compact set can be uniformly approximated by polynomials.
- Fix $k \in \mathbb{Z}$.
- Since $e^{2\pi ikx}$ is continuous on the compact interval [0, 1], it is uniformly continuous.
- Thus there is a sequence of polynomials P_ℓ such that

$$P_{\ell,k} \stackrel{\ell \longrightarrow \infty}{\longrightarrow} e^{2\pi i k x}$$
 uniformly on [0, 1].

• Note applying linearity to the assumption $\int f(x) x^n$, we have

$$\int f(x)x^n dx = 0 \ \forall n \implies \int f(x)p(x) dx = 0$$

for any polynomial p(x), and in particular for $P_{\ell,k}(x)$ for every ℓ and every k.

• But then

$$\langle f, e_k \rangle = \int_0^1 f(x) e^{-2\pi i k x} dx$$

$$= \int_0^1 f(x) \lim_{\ell \to \infty} P_{\ell}(x)$$

$$= \lim_{\ell \to \infty} \int_0^1 f(x) P_{\ell}(x) \quad \text{by uniform convergence on a compact interval}$$

$$= \lim_{\ell \to \infty} 0 \quad \text{by assumption}$$

$$= 0 \quad \forall k \in \mathbb{Z},$$

- so f is orthogonal to every e_k . Thus $f \in S^{\perp} \coloneqq \operatorname{span}_{\mathbb{C}} \{e_k\}_{k \in \mathbb{Z}}^{\perp} \subseteq L^2([0,1])$, but since this is a basis, S is dense and thus $S^{\perp} = \{0\} \text{ in } L^2([0,1]).$
- Thus $f \equiv 0$ in $L^2([0,1])$, which implies that f is zero almost everywhere.

6.3.2 Alternative Proof

Concepts Used:

- $C^1([0,1])$ is dense in $L^2([0,1])$
- Polynomials are dense in $L^p(X, \mathcal{M}, \mu)$ for any $X \subseteq \mathbb{R}^n$ compact and μ a finite measure, for all $1 \le p < \infty$.
 - Use Weierstrass Approximation, then uniform convergence implies $L^p(\mu)$ convergence by DCT.
- By density of polynomials, for $f \in L^2([0,1])$ choose $p_{\varepsilon}(x)$ such that $||f p_{\varepsilon}|| < \varepsilon$ by Weierstrass approximation.
- Then on one hand,

$$||f(f - p_{\varepsilon})||_1 = ||f^2||_1 - ||f \cdot p_{\varepsilon}||_1$$
$$= ||f^2||_1 - 0 \quad \text{by assumption}$$
$$= ||f||_2^2.$$

- Where we've used that $||f^2||_1 = \int |f^2| = \int |f|^2 = ||f||_2^2$.
- On the other hand

$$\begin{split} \|f(f-p_{\varepsilon})\| &\leq \|f\|_1 \|f-p_{\varepsilon}\|_{\infty} \quad \text{by Holder} \\ &\leq \varepsilon \|f\|_1 \\ &\leq \varepsilon \|f\|_2 \sqrt{m(X)} \\ &= \varepsilon \|f\|_2 \quad \text{since } m(X) = 1. \end{split}$$

- Where we've used that $||fg||_1 = \int |fg| = \int |f||g| \le \int ||f||_{\infty} |g| = ||f||_{\infty} ||g||_1$.

• Combining these.

$$||f||_2^2 \le ||f||_2 \varepsilon \implies ||f||_2 < \varepsilon \longrightarrow 0,$$

so $||f||_2 = 0$, which implies f = 0 almost everywhere.

6.4 Fall 2014 # 4

Let $g \in L^{\infty}([0,1])$ Prove that

 $\int_{[0,1]} f(x)g(x) dx = 0 \quad \text{for all continuous } f:[0,1] \longrightarrow \mathbb{R} \implies g(x) = 0 \text{ almost everywhere.}$

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7.1 Spring 2020 # 3

a. Prove that if $g \in L^1(\mathbb{R})$ then

$$\lim_{N \to \infty} \int_{|x| > N} |f(x)| \, dx = 0,$$

and demonstrate that it is not necessarily the case that $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.

- b. Prove that if $f \in L^1([1,\infty])$ and is decreasing, then $\lim_{x \to \infty} f(x) = 0$ and in fact $\lim_{x \to \infty} x f(x) = 0$.
- c. If $f:[1,\infty) \longrightarrow [0,\infty)$ is decreasing with $\lim_{x \to \infty} x f(x) = 0$, does this ensure that $f \in L^1([1,\infty))$?

Solution:

Concepts Used:

- Limits
- Cauchy Criterion for Integrals: $\int_{a}^{\infty} f(x) dx$ converges iff for every $\varepsilon > 0$ there exists an M_0 such that $A, B \ge M_0$ implies $\left| \int_{A}^{B} f \right| < \varepsilon$, i.e. $\left| \int_{A}^{B} f \right| \stackrel{A \longrightarrow \infty}{\longrightarrow} 0$.
- Integrals of L^1 functions have vanishing tails: $\int_N^\infty |f| \stackrel{N \longrightarrow \infty}{\longrightarrow} 0$.
- Mean Value Theorem for Integrals: $\int_a^b f(t) dt = (b-a)f(c)$ for some $c \in [a,b]$.

7.1.1 a

Stated integral equality:

- Let $\varepsilon > 0$
- $C_c(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$ is dense so choose $\{f_n\} \longrightarrow f$ with $\|f_n f\|_1 \longrightarrow 0$.

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- Since $\{f_n\}$ are compactly supported, choose $N_0 \gg 1$ such that f_n is zero outside of $B_{N_0}(\mathbf{0})$.
- Then

$$N \ge N_0 \implies \int_{|x|>N} |f| = \int_{|x|>N} |f - f_n + f_n|$$

$$\le \int_{|x|>N} |f - f_n| + \int_{|x|>N} |f_n|$$

$$= \int_{|x|>N} |f - f_n|$$

$$\le \int_{|x|>N} ||f - f_n||_1$$

$$= ||f_n - f||_1 \left(\int_{|x|>N} 1 \right)$$

$$\stackrel{n \to \infty}{\longrightarrow} 0 \left(\int_{|x|>N} 1 \right)$$

$$= 0$$

$$\stackrel{N \to \infty}{\longrightarrow} 0.$$

To see that this doesn't force $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$:

- Take f(x) to be a train of rectangles of height 1 and area $1/2^{j}$ centered on even integers.
- Then

$$\int_{|x|>N} |f| = \sum_{j=N}^{\infty} 1/2^j \stackrel{N \longrightarrow \infty}{\longrightarrow} 0$$

as the tail of a convergent sum.

• However f(x) = 1 for infinitely many even integers x > N, so $f(x) \not\longrightarrow 0$ as $|x| \longrightarrow \infty$.

7.1.2 b

Solution 1 ("Trick")

• Since f is decreasing on $[1, \infty)$, for any $t \in [x - n, x]$ we have

$$x - n \le t \le x \implies f(x) \le f(t) \le f(x - n).$$

• Integrate over [x, 2x], using monotonicity of the integral:

$$\int_{x}^{2x} f(x) dt \le \int_{x}^{2x} f(t) dt \le \int_{x}^{2x} f(x-n) dt$$

$$\implies f(x) \int_{x}^{2x} dt \le \int_{x}^{2x} f(t) dt \le f(x-n) \int_{x}^{2x} dt$$

$$\implies x f(x) \le \int_{x}^{2x} f(t) dt \le x f(x-n).$$

- By the Cauchy Criterion for integrals, $\lim_{x \to \infty} \int_x^{2x} f(t) dt = 0$.
- So the LHS term $xf(x) \stackrel{x \longrightarrow \infty}{\longrightarrow} 0$.
- Since x > 1, $|f(x)| \le |xf(x)|$
- Thus $f(x) \xrightarrow{x \to \infty} 0$ as well.

Solution 2 (Variation on the Trick)

• Use mean value theorem for integrals:

$$\int_{x}^{2x} f(t) dt = x f(c_x) \quad \text{for some } c_x \in [x, 2x] \text{ depending on } x.$$

• Since f is decreasing,

$$x \le c_x \le 2x \implies f(2x) \le f(c_x) \le f(x)$$

$$\implies 2xf(2x) \le 2xf(c_x) \le 2xf(x)$$

$$\implies 2xf(2x) \le 2x \int_x^{2x} f(t) dt \le 2xf(x)$$

- By Cauchy Criterion, $\int_{x}^{2x} f \longrightarrow 0$.
- So $2xf(2x) \longrightarrow 0$, which by a change of variables gives $uf(u) \longrightarrow 0$.
- Since $u \ge 1$, $f(u) \le u f(u)$ so $f(u) \longrightarrow 0$ as well.

Solution 3 (Contradiction) Just showing $f(x) \stackrel{x \longrightarrow \infty}{\longrightarrow} 0$:

- Toward a contradiction, suppose not.
- Since f is decreasing, it can not diverge to $+\infty$
- If $f(x) \longrightarrow -\infty$, then $f \notin L^1(\mathbb{R})$: choose $x_0 \gg 1$ so that $t \geq x_0 \implies f(t) < -1$, then
- Then $t \ge x_0 \implies |f(t)| \ge 1$, so

$$\int_{1}^{\infty} |f| \ge \int_{x_0}^{\infty} |f(t)| dt \ge \int_{x_0}^{\infty} 1 = \infty.$$

- Otherwise $f(x) \longrightarrow L \neq 0$, some finite limit.
- If L > 0
 - Fix $\varepsilon > 0$, choose $x_0 \gg 1$ such that $t \geq x_0 \implies L \varepsilon \leq f(t) \leq L$
 - Then

$$\int_{1}^{\infty} f \ge \int_{x_0}^{\infty} f \ge \int_{x_0}^{\infty} (L - \varepsilon) dt = \infty$$

- If L < 0:
 - Fix $\varepsilon > 0$, choose $x_0 \gg 1$ such that $t \geq x_0 \implies L \leq f(t) \leq L + \varepsilon$.
 - Then

$$\int_{1}^{\infty} f \ge \int_{x_0}^{\infty} f \ge \int_{x_0}^{\infty} (L) dt = \infty$$

Showing $xf(x) \stackrel{x \longrightarrow \infty}{\longrightarrow} 0$.

- Toward a contradiction, suppose not.
- (How to show that $xf(x) \not\longrightarrow +\infty$?)
- If $xf(x) \longrightarrow -\infty$
 - Choose a sequence $\Gamma = \{\hat{x}_i\}$ such that $x_i \longrightarrow \infty$ and $x_i f(x_i) \longrightarrow -\infty$.
 - Choose a subsequence $\Gamma' = \{x_i\}$ such that $x_i f(x_i) \leq -1$ for all i and $x_i \leq x_{i+1}$.
 - Choose a further subsequence $S = \{x_i \in \Gamma' \mid 2x_i < x_{i+1}\}.$

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- Then since f is always decreasing, for $t \ge x_0$, |f| is increasing, and $|f(x_i)| \le |f(2x_i)|$, so

$$\int_{1}^{\infty} |f| \ge \int_{x_0}^{\infty} |f| \ge \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(t)| \, dt \ge \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(x_i)| = \sum_{x_i \in S} x_i f(x_i) \longrightarrow \infty.$$

• If $xf(x) \longrightarrow L \neq 0$ for $0 < L < \infty$:

- Fix $\varepsilon > 0$, choose an infinite sequence $\{x_i\}$ such that $L - \varepsilon \leq x_i f(x_i) \leq L$ for all i.

$$\int_{1}^{\infty} |f| \ge \sum_{S} \int_{x_i}^{2x_i} |f(t)| dt \ge \sum_{S} \int_{x_i}^{2x_i} f(x_i) dt = \sum_{S} x_i f(x_i) \ge \sum_{S} (L - \varepsilon) \longrightarrow \infty.$$

• If $xf(x) \longrightarrow L \neq 0$ for $-\infty < L < 0$:

- Fix $\varepsilon > 0$, choose an infinite sequence $\{x_i\}$ such that $L \leq x_i f(x_i) \leq L + \varepsilon$ for all i.

$$\int_{1}^{\infty} |f| \ge \sum_{S} \int_{x_i}^{2x_i} |f(t)| dt \ge \sum_{S} \int_{x_i}^{2x_i} f(x_i) dt = \sum_{S} x_i f(x_i) \ge \sum_{S} (L) \longrightarrow \infty.$$

Solution 4 (Akos's Suggestion) For $x \ge 1$.

$$|xf(x)| = \left| \int_{x}^{2x} f(x) dt \right| \le \int_{x}^{2x} |f(x)| dt \le \int_{x}^{2x} |f(t)| dt \le \int_{x}^{\infty} |f(t)| dt \xrightarrow{x \to \infty} 0$$

where we've used

- Since f is decreasing and $\lim_{x \to \infty} f(x) = 0$ from part (a), f is non-negative.
- Since f is positive and decreasing, for every $t \in [a, b]$ we have $|f(a)| \leq |f(t)|$.
- By part (a), the last integral goes to zero.

Solution 5 (Peter's)

• Toward a contradiction, produce a sequence $x_i \longrightarrow \infty$ with $x_i f(x_i) \longrightarrow \infty$ and $x_i f(x_i) > \varepsilon > 0$, then

$$\int f(x) dx \ge \sum_{i=1}^{\infty} \int_{x_i}^{x_{i+1}} f(x) dx$$

$$\ge \sum_{i=1}^{\infty} \int_{x_i}^{x_{i+1}} f(x_{i+1}) dx$$

$$= \sum_{i=1}^{\infty} f(x_{i+1}) \int_{x_i}^{x_{i+1}} dx$$

$$\ge \sum_{i=1}^{\infty} (x_{i+1} - x_i) f(x_{i+1})$$

$$\ge \sum_{i=1}^{\infty} (x_{i+1} - x_i) \frac{\varepsilon}{x_{i+1}}$$

$$= \varepsilon \sum_{i=1}^{\infty} \left(1 - \frac{x_{i-1}}{x_i} \right) \longrightarrow \infty$$

which can be ensured by passing to a subsequence where $\sum \frac{x_{i-1}}{x_i} < \infty$.

7.1.3 c

• No: take $f(x) = \frac{1}{x \ln x}$ • Then by a *u*-substitution,

$$\int_0^x f = \ln\left(\ln(x)\right) \stackrel{x \longrightarrow \infty}{\longrightarrow} \infty$$

is unbounded, so $f \notin L^1([1,\infty))$.

• But

$$xf(x) = \frac{1}{\ln(x)} \stackrel{x \to \infty}{\longrightarrow} 0.$$

7.2 Fall 2019 # 5. 💝

7.2.1 a

Show that if f is continuous with compact support on \mathbb{R} , then

$$\lim_{y \to 0} \int_{\mathbb{R}} |f(x - y) - f(x)| dx = 0$$

7.2.2 b

Let $f \in L^1(\mathbb{R})$ and for each h > 0 let

$$\mathcal{A}_h f(x) := \frac{1}{2h} \int_{|y| \le h} f(x - y) dy$$

i. Prove that $\|\mathcal{A}_h f\|_1 \leq \|f\|_1$ for all h > 0.

ii. Prove that $A_h f \longrightarrow f$ in $L^1(\mathbb{R})$ as $h \longrightarrow 0^+$.

Walk through

Solution:

Concepts Used:

- Continuity in L^1 (recall that DCT won't work! Notes 19.4, prove it for a dense
- Lebesgue differentiation in 1-dimensional case. See HW 5.6.

7.2.3 a

Choose $g \in C_c^0$ such that $||f - g||_1 \longrightarrow 0$. By translation invariance, $||\tau_h f - \tau_h g||_1 \longrightarrow 0$.

Write

$$\begin{aligned} \|\tau f - f\|_1 &= \|\tau_h f - g + g - \tau_h g + \tau_h g - f\|_1 \\ &\leq \|\tau_h f - \tau_h g\| + \|g - f\| + \|\tau_h g - g\| \\ &\longrightarrow \|\tau_h g - g\|, \end{aligned}$$

so it suffices to show that $\|\tau_h g - g\| \longrightarrow 0$ for $g \in C_c^0$. Fix $\varepsilon > 0$. Enlarge the support of g to K such that

$$|h| \le 1$$
 and $x \in K^c \implies |g(x-h) - g(x)| = 0$.

By uniform continuity of g, pick $\delta \leq 1$ small enough such that

$$x \in K, |h| \le \delta \implies |g(x-h) - g(x)| < \varepsilon,$$

then

$$\int_{K} |g(x-h) - g(x)| \le \int_{K} \varepsilon = \varepsilon \cdot m(K) \longrightarrow 0.$$

7.2.4 b

We have

$$\int_{\mathbb{R}} |A_h(f)(x)| \ dx = \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y) \ dy \right| \ dx$$

$$\leq \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)| \ dy \ dx$$

$$=_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)| \ d\mathbf{x} \ d\mathbf{y}$$

$$= \int_{\mathbb{R}} |f(y)| \ dy$$

$$= ||f||_{1},$$

and (rough sketch)

$$\int_{\mathbb{R}} |A_h(f)(x) - f(x)| \ dx = \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y) \ dy \right) - f(x) \right| \ dx$$

$$= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y) \ dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x) \ dy \right| \ dx$$

$$\leq_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{B(h,x)} |f(y - x) - f(x)| \ \mathbf{dx} \ \mathbf{dy}$$

$$\leq \frac{1}{2h} \int_{\mathbb{R}} ||\tau_x f - f||_1 \ dy$$

$$\to 0 \quad \text{by (a)}.$$

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7.3 Fall 2017 # 3 🙀

Let

$$S = \operatorname{span}_{\mathbb{C}} \left\{ \chi_{(a,b)} \mid a, b \in \mathbb{R} \right\},$$

the complex linear span of characteristic functions of intervals of the form (a, b).

Show that for every $f \in L^1(\mathbb{R})$, there exists a sequence of functions $\{f_n\} \subset S$ such that

$$\lim_{n \to \infty} ||f_n - f||_1 = 0$$

Walk through

Solution:

Concepts Used:

• From homework: E is Lebesgue measurable iff there exists a finite union of closed cubes A such that $m(E\Delta A) < \varepsilon$.

It suffices to show that S is dense in simple functions, and since simple functions are *finite* linear combinations of characteristic functions, it suffices to show this for χ_A for A a measurable set.

Let $s = \chi_A$. By regularity of the Lebesgue measure, choose an open set $O \supseteq A$ such that $m(O \setminus A) < \varepsilon$.

O is an open subset of \mathbb{R} , and thus $O = \coprod_{j \in \mathbb{N}} I_j$ is a disjoint union of countably many open intervals

Now choose N large enough such that $m(O\Delta I_{N,n}) < \varepsilon = \frac{1}{n}$ where we define $I_{N,n} := \coprod_{j=1}^{N} I_j$. Now define $f_n = \chi_{I_{N,n}}$, then

$$||s - f_n||_1 = \int |\chi_A - \chi_{I_{N,n}}| = m(A\Delta I_{N,n}) \xrightarrow{n \longrightarrow \infty} 0.$$

Since any simple function is a finite linear combination of χ_{A_i} , we can do this for each i to extend this result to all simple functions. But simple functions are dense in L^1 , so S is dense in L^1 .

7.4 Spring 2015 # 4

Define

$$f(x,y) := \begin{cases} \frac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0 \le x \le y\\ 0 & \text{otherwise} \end{cases}$$

Carefully show that $f \in L^1(\mathbb{R}^2)$.

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7.5 Fall 2014 # 3

Let $f \in L^1(\mathbb{R})$. Show that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that} \qquad m(E) < \delta \implies \int_E |f(x)| \, dx < \varepsilon$$

7.6 Spring 2014 # 1

- 1. Give an example of a continuous $f \in L^1(\mathbb{R})$ such that $f(x) \not\longrightarrow 0$ as $|x| \longrightarrow \infty$.
- 2. Show that if f is uniformly continuous, then

$$\lim_{|x| \to \infty} f(x) = 0.$$

8 Fubini-Tonelli

8.1 Spring 2020 # 4 💝

Let $f,g\in L^1(\mathbb{R})$. Argue that $H(x,y)\coloneqq f(y)g(x-y)$ defines a function in $L^1(\mathbb{R}^2)$ and deduce from this fact that

$$(f * g)(x) \coloneqq \int_{\mathbb{R}} f(y)g(x - y) \, dy$$

defines a function in $L^1(\mathbb{R})$ that satisfies

$$\|f*g\|_1 \leq \|f\|_1 \|g\|_1.$$

Solution:

Concepts Used:

- Tonelli: non-negative and measurable yields measurability of slices and equality of iterated integrals
- Fubini: $f(x,y) \in L^1$ yields integrable slices and equality of iterated integrals
- F/T: apply Tonelli to |f|; if finite, $f \in L^1$ and apply Fubini to f

$$\begin{split} \|H(x)\|_1 &= \int_{\mathbb{R}} |H(x,y)| \, dx \\ &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y) g(x-y) \, dy \right| \, dx \\ &\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y) g(x-y)| \, dy \right) \, dx \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y) g(x-y)| \, dx \right) \, dy \quad \text{by Tonelli} \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y) g(t)| \, dt \right) \, dy \quad \text{setting } t = x - y, \, dt = -dx \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)| \cdot |g(t)| \, dt \right) \, dy \\ &= \int_{\mathbb{R}} |f(y)| \cdot \left(\int_{\mathbb{R}} |g(t)| \, dt \right) \, dy \\ &\coloneqq \int_{\mathbb{R}} |f(y)| \cdot \|g\|_1 \, dy \\ &= \|g\|_1 \int_{\mathbb{R}} |f(y)| \, dy \\ &\coloneqq \|g\|_1 \|f\|_1 \\ &< \infty \quad \text{by assumption} \quad . \end{split}$$

- H is measurable on \mathbb{R}^2 :
 - If we can show $\tilde{f}(x,y) := f(y)$ and $\tilde{g}(x,y) := g(x-y)$ are both measurable on \mathbb{R}^2 , then $H = \tilde{f} \cdot \tilde{g}$ is a product of measurable functions and thus measurable.
 - $-f \in L^1$, and L^1 functions are measurable by definition.
 - The function $(x,y) \mapsto g(x-y)$ is measurable on \mathbb{R}^2 :
 - * Let g be measurable on \mathbb{R} , then the cylinder function G(x,y)=g(x) on \mathbb{R}^2 is always measurable
 - * Define a linear transformation T := [1, -1; 0, 1] which sends $(x, y) \longrightarrow (x y, y)$, then $T \in GL(2, \mathbb{R})$ is linear and thus measurable.
 - * Then $(G \circ T)(x,y) = G(x-y,y) = \tilde{g}(x-y)$, so \tilde{g} is a composition of measurable functions and thus measurable.
- Apply **Tonelli** to |H|
 - -H measurable implies |H| is measurable
 - |H| is non-negative
 - So the iterated integrals are equal in the extended sense
 - The calculation shows the iterated integral is finite, to $\int |H|$ is finite and H is thus integrable on \mathbb{R}^2 .

Note: Fubini is not needed, since we're not calculating the actual integral, just showing H is integrable.

8.2 Spring 2019 # 4 🦙

Let f be a non-negative function on \mathbb{R}^n and $\mathcal{A} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : 0 \le t \le f(x)\}.$

Prove the validity of the following two statements:

- a. f is a Lebesgue measurable function on $\mathbb{R}^n \iff \mathcal{A}$ is a Lebesgue measurable subset of \mathbb{R}^{n+1}
- b. If f is a Lebesgue measurable function on \mathbb{R}^n , then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)dx = \int_0^\infty m\left(\left\{x \in \mathbb{R}^n : f(x) \ge t\right\}\right)dt$$

Add concepts.

Solution:

Concepts Used:

• See S&S p.82.

8.2.1 a

 \Longrightarrow :

- Suppose f is a measurable function.
- Note that $A = \{f(x) t \ge 0\} \cap \{t \ge 0\}.$
- Define F(x,t) = f(x), G(x,t) = t, which are cylinders on measurable functions and thus measurable
- Define H(x,y) = F(x,t) G(x,t), which are linear combinations of measurable functions and thus measurable.
- Then $A = \{H \ge 0\} \bigcap \{G \ge 0\}$ as a countable intersection of measurable sets, which is again measurable.

 $\Leftarrow=$:

- Suppose A is a measurable set.
- Then FT on $\chi_{\mathcal{A}}$ implies that for almost every $x \in \mathbb{R}^n$, the x-slices \mathcal{A}_x are measurable and \$

$$\mathcal{A}_x := \left\{ t \in \mathbb{R} \mid (x, t) \in \mathcal{A} \right\} = [0, f(x)] \implies m(\mathcal{A}_x) = f(x) - 0 = f(x)$$

• But $x \mapsto m(A_x)$ is a measurable function, and is exactly the function $x \mapsto f(x)$, so f is measurable.

8.2.2 b

• Note

$$\mathcal{A} = \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid 0 \le t \le f(x) \right\}$$
$$\mathcal{A}_t = \left\{ x \in \mathbb{R}^n \mid t \le f(x) \right\}.$$

• Then

$$\int_{\mathbb{R}^n} f(x) \ dx = \int_{\mathbb{R}^n} \int_0^{f(x)} 1 \ dt \ dx$$

$$= \int_{\mathbb{R}^n} \int_0^{\infty} \chi_{\mathcal{A}} \ dt \ dx$$

$$\stackrel{F.T.}{=} \int_0^{\infty} \int_{\mathbb{R}^n} \chi_{\mathcal{A}} \ dx \ dt$$

$$= \int_0^{\infty} m(\mathcal{A}_t) \ dt,$$

where we just use that $\int \int \chi_{\mathcal{A}} = m(\mathcal{A})$

• By F.T., all of these integrals are equal.

Why is FT justified

8.3 Fall 2018 # 5 🙀

Let $f \geq 0$ be a measurable function on \mathbb{R} . Show that

$$\int_{\mathbb{R}} f = \int_0^\infty m(\{x : f(x) > t\}) dt$$

Solution:

Concepts Used:

• Claim: If $E \subseteq \mathbb{R}^a \times \mathbb{R}^b$ is a measurable set, then for almost every $y \in \mathbb{R}^b$, the slice E^y is measurable and

$$m(E) = \int_{\mathbb{R}^b} m(E^y) \, dy.$$

- Set $g = \chi_E$, which is non-negative and measurable, so apply Tonelli.
- Conclude that $g^y = \chi_{E^y}$ is measurable, the function $y \mapsto \int g^y(x) dx$ is measurable, and $\int \int g^y(x) dx dy = \int g$.
- But $\int g = m(E)$ and $\int \int g^y(x) dx dy = \int m(E^y) dy$.

Note: f is a function $\mathbb{R} \longrightarrow \mathbb{R}$ in the original problem, but here I've assumed $f: \mathbb{R}^n \longrightarrow \mathbb{R}$.

• Since $f \geq 0$, set

$$E := \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) > t \right\} = \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid 0 \le t < f(x) \right\}.$$

- Claim: since f is measurable, E is measurable and thus m(E) makes sense.
 - Since f is measurable, F(x,t) := t f(x) is measurable on $\mathbb{R}^n \times \mathbb{R}$.
 - Then write $E = \{F < 0\} \bigcap \{t \ge 0\}$ as an intersection of measurable sets.

• We have slices

$$E^{t} := \left\{ x \in \mathbb{R}^{n} \mid (x, t) \in E \right\} = \left\{ x \in \mathbb{R}^{n} \mid 0 \le t < f(x) \right\}$$
$$E^{x} := \left\{ t \in \mathbb{R} \mid (x, t) \in E \right\} = \left\{ t \in \mathbb{R} \mid 0 \le t \le f(x) \right\} = [0, f(x)].$$

- E_t is precisely the set that appears in the original RHS integrand.
- $m(E^x) = f(x).$
- Claim: χ_E satisfies the conditions of Tonelli, and thus $m(E) = \int \chi_E$ is equal to any iterated integral.
 - Non-negative: clear since $0 \le \chi_E \le 1$
 - Measurable: characteristic functions of measurable sets are measurable.
- Conclude
 - 1. For almost every x, E^x is a measurable set, $x \mapsto m(E^x)$ is a measurable function, and $m(E) = \int_{\mathbb{R}^n} m(E^x) dx$
 - 2. For almost every t, E^t is a measurable set, $t \mapsto m(E^t)$ is a measurable function, and $m(E) = \int_{\mathbb{R}} m(E^t) dt$
- On one hand,

$$m(E) = \int_{\mathbb{R}^{n+1}} \chi_E(x,t)$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_E(x,t) dt dx \text{ by Tonelli}$$

$$= \int_{\mathbb{R}^n} m(E^x) dx \text{ first conclusion}$$

$$= \int_{\mathbb{R}^n} f(x) dx.$$

• On the other hand,

$$m(E) = \int_{\mathbb{R}^{n+1}} \chi_E(x, t)$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_E(x, t) dx dt \text{ by Tonelli}$$

$$= \int_{\mathbb{R}} m(E^t) dt \text{ second conclusion.}$$

• Thus

$$\int_{\mathbb{R}^n} f \, dx = m(E) = \int_{\mathbb{R}} m(E^t) \, dt = \int_{\mathbb{R}} m\Big(\Big\{x \mid f(x) > t\Big\}\Big).$$

8.4 Fall 2015 # 5

Let $f, g \in L^1(\mathbb{R})$ be Borel measurable.

1. Show that

• The function

$$F(x,y) \coloneqq f(x-y)g(y)$$

is Borel measurable on \mathbb{R}^2 , and

• For almost every $y \in \mathbb{R}$,

$$F_y(x) \coloneqq f(x-y)g(y)$$

is integrable with respect to y.

2. Show that $f * g \in L^1(\mathbb{R})$ and

$$||f * g||_1 \le ||f||_1 ||g||_1$$

8.5 Spring 2014 # 5

Let $f, g \in L^1([0,1])$ and for all $x \in [0,1]$ define

$$F(x) := \int_0^x f(y) \, dy$$
 and $G(x) := \int_0^x g(y) \, dy$.

Prove that

$$\int_0^1 F(x)g(x) \, dx = F(1)G(1) - \int_0^1 f(x)G(x) \, dx$$

9 L^2 and Fourier Analysis

9.1 Spring 2020 # 6 💝

9.1.1 a

Show that

$$L^2([0,1]) \subseteq L^1([0,1])$$
 and $\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$.

9.1.2 b

For $f \in L^1([0,1])$ define

$$\widehat{f}(n) := \int_0^1 f(x)e^{-2\pi i nx} dx.$$

Prove that if $f \in L^1([0,1])$ and $\{\widehat{f}(n)\} \in \ell^1(\mathbb{Z})$ then

$$S_N f(x) := \sum_{|n| < N} \widehat{f}(n) e^{2\pi i n x}.$$

converges uniformly on [0,1] to a continuous function g such that g=f almost everywhere.

Hint: One approach is to argue that if $f \in L^1([0,1])$ with $\{\widehat{f}(n)\} \in \ell^1(\mathbb{Z})$ then $f \in L^2([0,1])$.

Solution:

Concepts Used:

- For $e_n(x) := e^{2\pi i n x}$, the set $\{e_n\}$ is an orthonormal basis for $L^2([0,1])$.
- For any orthonormal sequence in a Hilbert space, we have Bessel's inequality:

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \le ||x||^2.$$

- When $\{e_n\}$ is a basis, the above is an equality (Parseval)
- Arguing uniform convergence: since $\{\widehat{f}(n)\}\in\ell^1(\mathbb{Z})$, we should be able to apply the M test.

9.1.3 a

Claim: $\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$.

- Set $\mathbf{c} = \{c_k \mid k \in \mathbb{Z}\} \in \ell^1(\mathbb{Z}).$
- It suffices to show that if $\sum_{k\in\mathbb{Z}}|c_k|<\infty$ then $\sum_{k\in\mathbb{Z}}|c_k|^2<\infty$.
- Let $S = \{c_k \mid |c_k| \le 1\}$, then $c_k \in S \implies |c_k|^2 \le |c_k|$
- Claim: S^{c} can only contain finitely many elements, all of which are finite.
 - If not, either $S^c := \{c_j\}_{j=1}^{\infty}$ is infinite with every $|c_j| > 1$, which forces

$$\sum_{c_k \in S^c} |c_k| = \sum_{j=1}^{\infty} |c_j| > \sum_{j=1}^{\infty} 1 = \infty.$$

- If any $c_j = \infty$, then $\sum_{k \in \mathbb{Z}} |c_k| \ge c_j = \infty$.
- So S^c is a finite set of finite integers, let $N = \max\left\{|c_j|^2 \mid c_j \in S^c\right\} < \infty$.
- Rewrite the sum

$$\sum_{k \in \mathbb{Z}} |c_k|^2 = \sum_{c_k \in S} |c_k|^2 + \sum_{c_k \in S^c} |c_k|^2$$

$$\leq \sum_{c_k \in S} |c_k| + \sum_{c_k \in S^c} |c_k|^2$$

$$\leq \sum_{k \in \mathbb{Z}} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \quad \text{since the } |c_k| \text{ are all positive}$$

$$= \|\mathbf{c}\|_{\ell^1} + \sum_{c_k \in S^c} |c_k|^2$$

$$\leq \|\mathbf{c}\|_{\ell^1} + |S^c| \cdot N$$

$$< \infty.$$

Claim: $L^2([0,1]) \subseteq L^1([0,1])$.

• It suffices to show that $\int |f|^2 < \infty \implies \int |f| < \infty$.

- Define $S = \{x \in [0,1] \mid |f(x)| \le 1\}$, then $x \in S^c \implies |f(x)|^2 \ge |f(x)|$.
- Break up the integral:

$$\begin{split} \int_{\mathbb{R}} |f| &= \int_{S} |f| + \int_{S^{c}} |f| \\ &\leq \int_{S} |f| + \int_{S^{c}} |f|^{2} \\ &\leq \int_{S} |f| + \|f\|_{2} \\ &\leq \sup_{x \in S} \{|f(x)|\} \cdot \mu(S) + \|f\|_{2} \\ &= 1 \cdot \mu(S) + \|f\|_{2} \quad \text{by definition of } S \\ &\leq 1 \cdot \mu([0,1]) + \|f\|_{2} \quad \text{since } S \subseteq [0,1] \\ &= 1 + \|f\|_{2} \\ &< \infty. \end{split}$$

Note: this proof shows $L^2(X) \subseteq L^1(X)$ whenever $\mu(X) < \infty$.

9.2 Fall 2017 # 5

Let φ be a compactly supported smooth function that vanishes outside of an interval [-N, N] such that $\int_{\mathbb{R}} \varphi(x) dx = 1$.

For $f \in L^1(\mathbb{R})$, define

$$K_j(x) := j\varphi(jx), \qquad f * K_j(x) := \int_{\mathbb{R}} f(x-y)K_j(y) \, dy$$

and prove the following:

- 1. Each $f * K_j$ is smooth and compactly supported.
- 2.

$$\lim_{j \to \infty} \|f * K_j - f\|_1 = 0$$

Hint:

$$\lim_{y \to 0} \int_{\mathbb{R}} |f(x - y) - f(x)| dy = 0$$

Add concepts.

Solution:

Concepts Used:

• ?

9.2.1 a

Lemma: If $\varphi \in C_c^1$, then $(f * \varphi)' = f * \varphi'$ almost everywhere. Silly Proof:

$$\mathcal{F}((f * \varphi)') = 2\pi i \xi \ \mathcal{F}(f * \varphi)$$

$$= 2\pi i \xi \ \mathcal{F}(f) \ \mathcal{F}(\varphi)$$

$$= \mathcal{F}(f) \cdot (2\pi i \xi \ \mathcal{F}(\varphi))$$

$$= \mathcal{F}(f) \cdot \mathcal{F}(\varphi')$$

$$= \mathcal{F}(f * \varphi').$$

Actual proof:

$$(f * \varphi)'(x) = (\varphi * f)'(x)$$

$$= \lim_{h \to 0} \frac{(\varphi * f)'(x+h) - (\varphi * f)'(x)}{h}$$

$$= \lim_{h \to 0} \int \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y)$$

$$\stackrel{DCT}{=} \int \lim_{h \to 0} \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y)$$

$$= \int \varphi'(x-y) f(y)$$

$$= (\varphi' * f)(x)$$

$$= (f * \varphi')(x).$$

To see that the DCT is justified, we can apply the MVT on the interval [0, h] to f to obtain

$$\frac{\varphi(x+h-y)-\varphi(x-y)}{h}=\varphi'(c)\quad c\in[0,h],$$

and since φ' is continuous and compactly supported, φ' is bounded by some $M<\infty$ by the extreme value theorem and thus

$$\int \left| \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \right| = \int |\varphi'(c)f(y)|$$

$$\leq \int |M||f|$$

$$= |M| \int |f| < \infty,$$

since $f \in L^1$ by assumption, so we can take g := |M||f| as the dominating function.

Applying this theorem infinitely many times shows that $f * \varphi$ is smooth.

To see that $f * \varphi$ is compactly supported, approximate f by a *continuous* compactly supported function h, so $||h - f||_1 \xrightarrow{L^1} 0$.

Now let $g_x(y) = \varphi(x - y)$, and note that $\operatorname{supp}(g) = x - \operatorname{supp}(\varphi)$ which is still compact. But since $\operatorname{supp}(h)$ is bounded, there is some N such that

$$|x| > N \implies A_x := \operatorname{supp}(h) \bigcap \operatorname{supp}(g_x) = \emptyset$$

and thus

$$(h * \varphi)(x) = \int_{\mathbb{R}} \varphi(x - y)h(y) \ dy$$
$$= \int_{A_x} g_x(y)h(y)$$
$$= 0,$$

so $\{x \mid f * g(x) = 0\}$ is open, and its complement is closed and bounded and thus compact.

9.2.2 b

$$||f * K_{j} - f||_{1} = \int \left| \int f(x - y)K_{j}(y) dy - f(x) \right| dx$$

$$= \int \left| \int f(x - y)K_{j}(y) dy - \int f(x)K_{j}(y) dy \right| dx$$

$$= \int \left| \int (f(x - y) - f(x))K_{j}(y) dy \right| dx$$

$$\leq \int \int \left| (f(x - y) - f(x)) \right| \cdot |K_{j}(y)| dy dx$$

$$\stackrel{FT}{=} \int \int \left| (f(x - y) - f(x)) \right| \cdot |K_{j}(y)| dx dy$$

$$= \int |K_{j}(y)| \left(\int \left| (f(x - y) - f(x)) \right| dx \right) dy$$

$$= \int |K_{j}(y)| \cdot ||f - \tau_{y}f||_{1} dy.$$

We now split the integral up into pieces.

- 1. Chose δ small enough such that $|y| < \delta \implies ||f \tau_y f||_1 < \varepsilon$ by continuity of translation in L^1 , and
- 2. Since φ is compactly supported, choose J large enough such that

$$j > J \implies \int_{|y| \ge \delta} |K_j(y)| \ dy = \int_{|y| \ge \delta} |j\varphi(jy)| = 0$$

Then

$$||f * K_{j} - f||_{1} \leq \int |K_{j}(y)| \cdot ||f - \tau_{y}f||_{1} dy$$

$$= \int_{|y| < \delta} |K_{j}(y)| \cdot ||f - \tau_{y}f||_{1} dy + \int_{|y| \ge \delta} |K_{j}(y)| \cdot ||f - \tau_{y}f||_{1} dy$$

$$= \varepsilon \int_{|y| \ge \delta} |K_{j}(y)| + 0$$

$$\leq \varepsilon(1) \longrightarrow 0.$$

9.3 Spring 2017 # 5

Let $f, g \in L^2(\mathbb{R})$. Prove that the formula

$$h(x) := \int_{-\infty}^{\infty} f(t)g(x-t) dt$$

defines a uniformly continuous function h on \mathbb{R} .

9.4 Spring 2015 # 6

Let $f \in L^1(\mathbb{R})$ and g be a bounded measurable function on \mathbb{R} .

- 1. Show that the convolution f * g is well-defined, bounded, and uniformly continuous on \mathbb{R} .
- 2. Prove that one further assumes that $g \in C^1(\mathbb{R})$ with bounded derivative, then $f * g \in C^1(\mathbb{R})$ and

$$\frac{d}{dx}(f*g) = f*\left(\frac{d}{dx}g\right)$$

9.5 Fall 2014 # 5

1. Let $f \in C_c^0(\mathbb{R}^n)$, and show

$$\lim_{t \to 0} \int_{\mathbb{R}^n} |f(x+t) - f(x)| \, dx = 0.$$

2. Extend the above result to $f \in L^1(\mathbb{R}^n)$ and show that

 $f \in L^1(\mathbb{R}^n), \quad g \in L^\infty(\mathbb{R}^n) \implies f * g$ is bounded and uniformly continuous.

10 Functional Analysis: General

10.1 Fall 2019 # 4 🐈

Let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space \mathcal{H} .

10.1.1 a

Prove that for every $x \in \mathcal{H}$ one has

$$\sum_{n=1}^{\infty} \left| \langle x, u_n \rangle \right|^2 \le \|x\|^2$$

10.1.2 b

Prove that for any sequence $\{a_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$ there exists an element $x \in \mathcal{H}$ such that

$$a_n = \langle x, u_n \rangle$$
 for all $n \in \mathbb{N}$

and

$$||x||^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$$

Solution:

Concepts Used:

- Bessel's Inequality
- Pythagoras Surjectivity of the Riesz map
- Trick remember to write out finite sum S_N , and consider $||x S_N||$.

10.1.3 a

Claim:

$$0 \le \left\| x - \sum_{n=1}^{N} \langle x, u_n \rangle u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n \rangle|^2$$
$$\implies \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \le \|x\|^2.$$

Proof: Let
$$S_N = \sum_{n=1}^N \langle x, u_n \rangle u_n$$
. Then

$$0 \le \|x - S_N\|^2$$

$$= \langle x - S_n, x - S_N \rangle$$

$$= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2$$

$$\xrightarrow{N \to \infty} \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2.$$

10.1.4 b

- 1. Fix $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.

$$x := \lim_{N \to \infty} S_N = \lim_{N \to \infty} \sum_{k=1}^N a_k u_k$$

3. $\{S_N\}$ Cauchy (by 1) and H complete $\implies x \in H$.

$$\langle x, u_n \rangle = \left\langle \sum_k a_k u_k, u_n \right\rangle = \sum_k a_k \langle u_k, u_n \rangle = a_n \quad \forall n \in \mathbb{N}$$

since the u_k are all orthogonal.

5.

$$||x||^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k ||a_k u_k||^2 = \sum_k |a_k|^2$$

by Pythagoras since the u_k are normal.

Bonus: We didn't use completeness here, so the Fourier series may not actually converge to x. If $\{u_n\}$ is **complete** (so $x=0 \iff \langle x, u_n \rangle = 0 \ \forall n$) then the Fourier series does converge to x and $\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 = ||x||^2$ for all $x \in H$.

10.2 Spring 2019 # 5 🔆

10.2.1 a

Show that $L^2([0,1]) \subseteq L^1([0,1])$ and argue that $L^2([0,1])$ in fact forms a dense subset of $L^1([0,1])$.

10.2.2 b

Let Λ be a continuous linear functional on $L^1([0,1])$.

Prove the Riesz Representation Theorem for $L^1([0,1])$ by following the steps below:

i. Establish the existence of a function $g \in L^2([0,1])$ which represents Λ in the sense that

$$\Lambda(f) = f(x)g(x)dx \text{ for all } f \in L^2([0,1]).$$

Hint: You may use, without proof, the Riesz Representation Theorem for $L^2([0,1])$.

ii. Argue that the g obtained above must in fact belong to $L^{\infty}([0,1])$ and represent Λ in the sense that

$$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}dx \quad \text{ for all } f \in L^1([0,1])$$

with

$$||g||_{L^{\infty}([0,1])} = ||\Lambda||_{L^{1}([0,1])}$$

Solution:

Concepts Used:

- Holders' inequality: $\|fg\|_1 \leq \|f\|_p \|f\|_q$ Riesz Representation for L^2 : If $\Lambda \in (L^2)^\vee$ then there exists a unique $g \in L^2$ such that $\Lambda(f) = \int fg$
- $||f||_{L^{\infty}(X)} := \inf \{ t \ge 0 \mid |f(x)| \le t \text{ almost everywhere } \}.$
- Lemma: $m(X) < \infty \implies L^p(X) \subset L^2(X)$.

Proof.

– Write Holder's inequality as $||fg||_1 \le ||f||_a ||g||_b$ where $\frac{1}{a} + \frac{1}{b} = 1$, then

$$||f||_p^p = |||f|^p||_1 \le |||f|^p||_a ||1||_b.$$

- Now take $a = \frac{2}{n}$ and this reduces to

$$\begin{split} \|f\|_p^p &\leq \|f\|_2^p \ m(X)^{\frac{1}{b}} \\ \Longrightarrow \ \|f\|_p &\leq \|f\|_2 \cdot O(m(X)) < \infty. \end{split}$$

10.2.3 a

- Note $X = [0,1] \implies m(X) = 1$.
- By Holder's inequality with p = q = 2,

$$\|f\|_1 = \|f \cdot 1\|_1 \leq \|f\|_2 \cdot \|1\|_2 = \|f\|_2 \cdot m(X)^{\frac{1}{2}} = \|f\|_2,$$

- Thus $L^2(X) \subseteq L^1(X)$
- Since they share a common dense subset (simple functions) L^2 is dense in L^1

10.2.4 b

Let $\Lambda \in L^1(X)^{\vee}$ be arbitrary.

1: Existence of g Representing Λ . Let $f \in L^2 \subseteq L^1$ be arbitrary.

Claim: $\Lambda \in L^1(X)^{\vee} \implies \Lambda \in L^2(X)^{\vee}$.

- Suffices to show that $\|\Gamma\|_{L^2(X)^\vee} := \sup_{\|f\|_2=1} |\Gamma(f)| < \infty$, since bounded implies continuous.
- By the lemma, $||f||_1 \le C||f||_2$ for some constant $C \approx m(X)$.
- Note

$$\|\Lambda\|_{L^1(X)^\vee}\coloneqq \sup_{\|f\|_1=1} |\Lambda(f)|$$

- Define $\hat{f} = \frac{f}{\|f\|_1}$ so $\|\hat{f}\|_1 = 1$
- Since $\|\Lambda\|_{1^{\vee}}$ is a supremum over all $f \in L^1(X)$ with $\|f\|_1 = 1$,

$$\left|\Lambda(\widehat{f})\right| \le \|\Lambda\|_{(L^1(X))^\vee},$$

• Then

$$\begin{split} \frac{|\Lambda(f)|}{\|f\|_1} &= \left|\Lambda(\widehat{f})\right| \leq \|\Lambda\|_{L^1(X)^\vee} \\ \Longrightarrow & |\Lambda(f)| \leq \|\Lambda\|_{1^\vee} \cdot \|f\|_1 \\ &\leq \|\Lambda\|_{1^\vee} \cdot C\|f\|_2 < \infty \quad \text{by assumption,} \end{split}$$

• So $\Lambda \in (L^2)^{\vee}$. Now apply Riesz Representation for L^2 : there is a $g \in L^2$ such that

$$f \in L^2 \implies \Lambda(f) = \langle f, g \rangle := \int_0^1 f(x) \overline{g(x)} \, dx.$$

2: g is in L^{∞}

- It suffices to show $||g||_{L^{\infty}(X)} < \infty$.
- Since we're assuming $\|\Gamma\|_{L^1(X)^{\vee}} < \infty$, it suffices to show the stated equality.

Is this assumed..? Or did we show it..?

- Claim: $\|\Lambda\|_{L^1(X)^{\vee}} = \|g\|_{L^{\infty}(X)}$
 - The result will follow since Λ was assumed to be in $L^1(X)^\vee$, so $\|\Lambda\|_{L^1(X)^\vee} < \infty$.
 - ≤:

$$\begin{split} \|\Lambda\|_{L^1(X)^\vee} &= \sup_{\|f\|_1 = 1} |\Lambda(f)| \\ &= \sup_{\|f\|_1 = 1} \left| \int_X f\bar{g} \right| \quad \text{by (i)} \\ &= \sup_{\|f\|_1 = 1} \int_X |f\bar{g}| \\ &\coloneqq \sup_{\|f\|_1 = 1} \|fg\|_1 \\ &\leq \sup_{\|f\|_1 = 1} \|f\|_1 \|g\|_\infty \quad \text{by Holder with } p = 1, q = \infty \\ &= \|g\|_\infty, \end{split}$$

- >:

- * Suppose toward a contradiction that $\|g\|_{\infty} > \|\Lambda\|_{1^{\vee}}.$
- * Then there exists some $E \subseteq X$ with m(E) > 0 such that

$$x \in E \implies |g(x)| > \|\Lambda\|_{L^1(X)^\vee}.$$

* Define

$$h = \frac{1}{m(E)} \frac{\overline{g}}{|g|} \chi_E.$$

- * Note $||h||_{L^1(X)} = 1$.
- * Then

$$\begin{split} \Lambda(h) &= \int_X hg \\ &\coloneqq \int_X \frac{1}{m(E)} \frac{g\overline{g}}{|g|} \chi_E \\ &= \frac{1}{m(E)} \int_E |g| \\ &\ge \frac{1}{m(E)} \|g\|_\infty m(E) \\ &= \|g\|_\infty \\ &> \|\Lambda\|_{L^1(X)^\vee}, \end{split}$$

a contradiction since $\|\Lambda\|_{L^1(X)^{\vee}}$ is the supremum over all h_{α} with $\|h_{\alpha}\|_{L^1(X)} = 1$.

10.3 Spring 2016 # 6

Without using the Riesz Representation Theorem, compute

$$\sup \left\{ \left| \int_0^1 f(x)e^x dx \right| \mid f \in L^2([0,1], m), \|f\|_2 \le 1 \right\}$$

10.4 Spring 2015 # 5

Let \mathcal{H} be a Hilbert space.

1. Let $x \in \mathcal{H}$ and $\{u_n\}_{n=1}^N$ be an orthonormal set. Prove that the best approximation to x in \mathcal{H} by an element in $\operatorname{span}_{\mathbb{C}} \{u_n\}$ is given by

$$\widehat{x} := \sum_{n=1}^{N} \langle x, u_n \rangle u_n.$$

2. Conclude that finite dimensional subspaces of \mathcal{H} are always closed.

10.5 Fall 2015 # 6

Let $f:[0,1] \longrightarrow \mathbb{R}$ be continuous. Show that

$$\sup \left\{ \|fg\|_1 \mid g \in L^1[0,1], \|g\|_1 \le 1 \right\} = \|f\|_{\infty}$$

10.6 Fall 2014 # 6

Let $1 \leq p, q \leq \infty$ be conjugate exponents, and show that

$$f \in L^p(\mathbb{R}^n) \implies ||f||_p = \sup_{\|g\|_q = 1} \left| \int f(x)g(x)dx \right|$$

11 Functional Analysis: Banach Spaces

11.1 Spring 2019 # 1 🙀

Let C([0,1]) denote the space of all continuous real-valued functions on [0,1].

- a. Prove that C([0,1]) is complete under the uniform norm $||f||_u := \sup_{x \in [0,1]} |f(x)|$.
- b. Prove that C([0,1]) is not complete under the L^1 -norm $||f||_1 = \int_0^1 |f(x)| dx$.

Add concepts

Solution:

Concepts Used:

• ?

11.1.1 a

- Let $\{f_n\}$ be a Cauchy sequence in $C(I, \|\cdot\|_{\infty})$, so $\lim_{n \to \infty} \lim_{m \to \infty} \|f_m f_n\|_{\infty} = 0$, we will show it converges to some f in this space.
- For each fixed $x_0 \in [0,1]$, the sequence of real numbers $\{f_n(x_0)\}$ is Cauchy in \mathbb{R} since

$$x_0 \in I \implies |f_m(x_0) - f_n(x_0)| \le \sup_{x \in I} |f_m(x) - f_n(x)| := ||f_m - f_n||_{\infty} \xrightarrow{m > n \longrightarrow \infty} 0,$$

- Since \mathbb{R} is complete, this sequence converges and we can define $f(x) := \lim_{k \to \infty} f_n(x)$.
- Thus $f_n \longrightarrow f$ pointwise by construction Claim: $||f f_n|| \stackrel{n \longrightarrow \infty}{\longrightarrow} 0$, so f_n converges to f in $C([0, 1], ||\cdot||_{\infty})$.
 - Proof:
 - * Fix $\varepsilon > 0$; we will show there exists an N such that $n \geq N \implies ||f_n f|| < \varepsilon$
 - * Fix an $x_0 \in I$. Since $f_n \longrightarrow f$ pointwise, choose N_1 large enough so that

$$n \ge N_1 \implies |f_n(x_0) - f(x_0)| < \varepsilon/2.$$

* Since $||f_n - f_m||_{\infty} \longrightarrow 0$, choose and N_2 large enough so that

$$n, m \geq N_2 \implies ||f_n - f_m||_{\infty} < \varepsilon/2.$$

* Then for $n, m \ge \max(N_1, N_2)$, we have

$$|f_n(x_0) - f(x_0)| = |f_n(x_0) - f(x_0) + f_m(x_0) - f_m(x_0)|$$

$$= |f_n(x_0) - f_m(x_0) + f_m(x_0) - f(x_0)|$$

$$\leq |f_n(x_0) - f_m(x_0)| + |f_m(x_0) - f(x_0)|$$

$$< |f_n(x_0) - f_m(x_0)| + \frac{\varepsilon}{2}$$

$$\leq \sup_{x \in I} |f_n(x) - f_m(x)| + \frac{\varepsilon}{2}$$

$$< ||f_n - f_m||_{\infty} + \frac{\varepsilon}{2}$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\implies |f_n(x_0) - f(x_0)| < \varepsilon$$

$$\implies \sup_{x \in I} |f_n(x_0) - f(x_0)| \leq \sup_{x \in I} \varepsilon \text{ by order limit laws}$$

$$\implies ||f_n - f|| \leq \varepsilon$$

• f is the uniform limit of continuous functions and thus continuous, so $f \in C([0,1])$.

11.1.2 b

- It suffices to produce a Cauchy sequence that does not converge to a continuous function.
- Take the following sequence of functions:
 - f_1 increases linearly from 0 to 1 on [0, 1/2] and is 1 on [1/2, 1]
 - $-f_2$ is 0 on [0,1/4] increases linearly from 0 to 1 on [1/4,1/2] and is 1 on [1/2,1]
 - $-f_3$ is 0 on [0,3/8] increases linearly from 0 to 1 on [3/8,1/2] and is 1 on [1/2,1]
 - $-f_3$ is 0 on [0, (1/2 3/8)/2] increases linearly from 0 to 1 on [(1/2 3/8)/2, 1/2] and is 1 on [1/2, 1]

Idea: take sequence starting points for the triangles: $0, 0 + \frac{1}{4}, 0 + \frac{1}{4} + \frac{1}{8}, \cdots$ which converges to 1/2 since $\sum_{k=1}^{\infty} \frac{1}{2^k} = -\frac{1}{2} + \sum_{k=0}^{\infty} \frac{1}{2^k}$.

- Then each f_n is clearly integrable, since its graph is contained in the unit square.
- $\{f_n\}$ is Cauchy: geometrically subtracting areas yields a single triangle whose area tends to 0.
- But f_n converges to $\chi_{[\frac{1}{2},1]}$ which is discontinuous.

show that $\int_0^1 |f_n(x) - f_m(x)| dx \longrightarrow 0$ rigorously, show that no $g \in L^1([0,1])$ can converge to this indicator function.

11.2 Spring 2017 # 6 💝

Show that the space $C^1([a,b])$ is a Banach space when equipped with the norm

$$||f|| := \sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |f'(x)|$$

Add concepts

Solution:

Concepts Used:

- See https://math.stackexchange.com/questions/507263/ prove-that-c1a-b-with-the-c1-norm-is-a-banach-space/
- Denote this norm $\|\cdot\|_u$
- Let f_n be a Cauchy sequence in this space, so $||f_n||_u < \infty$ for every n and $||f_j - f_k||_u \stackrel{j,k \longrightarrow \infty}{\longrightarrow} 0.$

and define a candidate limit: for each $x \in I$, set

$$f(x) := \lim_{n \to \infty} f_n(x).$$

• Note that

$$||f_n||_{\infty} \le ||f_n||_u < \infty$$
$$||f_n'||_{\infty} \le ||f_n||_u < \infty.$$

- Thus both f_n, f'_n are Cauchy sequences in $C^0([a, b], \|\cdot\|_{\infty})$, which is a Banach space, so they converge.
- $-f_n \longrightarrow f$ uniformly (by uniqueness of limits),
- $-f'_n \longrightarrow g$ uniformly for some g, and $-f, g \in C^0([a, b])$.
- Claim: g = f'
 - For any fixed $a \in I$, we have

$$f_n(x) - f_n(a) \xrightarrow{u} f(x) - f(a)$$

$$\int_a^x f'_n \xrightarrow{u} \int_a^x g.$$

- By the FTC, the left-hand sides are equal.
- By uniqueness of limits so are the right-hand sides, so f' = g.
- Claim: the limit f is an element in this space.
- Since $f, f' \in C^0([a, b])$, they are bounded, and so $||f||_u < \infty$.

 Claim: $||f_n f||_u \stackrel{n \to \infty}{\longrightarrow} 0$
- Thus the Cauchy sequence $\{f_n\}$ converges to a function f in the u-norm where f is an element of this space, making it complete.

11.3 Fall 2017 # 6 🦙

Let X be a complete metric space and define a norm

$$||f|| := \max\{|f(x)| : x \in X\}.$$

Show that $(C^0(\mathbb{R}), \|\cdot\|)$ (the space of continuous functions $f: X \longrightarrow \mathbb{R}$) is complete.

Add concepts.

Shouldn't this be a supremum? The max may not exist?

Review and clean up.

Solution:

Concepts Used:

• ?

Let $\{f_k\}$ be a Cauchy sequence, so $||f_k|| < \infty$ for all k. Then for a fixed x, the sequence $f_k(x)$ is Cauchy in \mathbb{R} and thus converges to some f(x), so define f by $f(x) := \lim_{k \to \infty} f_k(x)$.

Then $||f_k - f|| = \max_{x \in X} |f_k(x) - f(x)| \stackrel{k \to \infty}{\longrightarrow} 0$, and thus $f_k \to f$ uniformly and thus f is continuous. It just remains to show that f has bounded norm.

Choose N large enough so that $||f - f_N|| < \varepsilon$, and write $||f_N|| := M < \infty$

$$||f|| \le ||f - f_N|| + ||f_N|| < \varepsilon + M < \infty.$$