

Title

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Contents

1	Spring 2017	1
1.1	1	1
1.2	2	3
	1.2.1 a	3
	1.2.2 b	3
1.3	3	4
	1.3.1 a	4
	1.3.2 b	4
1.4	4	4
1.5	5	5

1 Spring 2017

1.1 1

Concepts used:

- Definition: A is *nowhere dense* \iff every interval I contains a subinterval $S \subseteq A^c$.
 - Equivalently, the interior of the closure is empty, $(\overline{K})^\circ = \emptyset$.

Solution

Claim: K is **compact**.

- It suffices to show that $K^c := [0, 1] \setminus K$ is open; Then K will be a closed and bounded subset of \mathbb{R} and thus compact by Heine-Borel.
- Strategy: write K^c as the union of open balls (since these form a basis for the Euclidean topology on \mathbb{R}).
 - Do this by showing every point $x \in K^c$ is an interior point, i.e. x admits a neighborhood N_x such that $N_x \subseteq K^c$.
- Identify K^c as the set of real numbers in $[0, 1]$ whose decimal expansion **does** contain a 4.
 - We will show that there exists a neighborhood small enough such that all points in it contain a 4 in their decimal expansions.

- Let $x \in K^c$, suppose a 4 occurs as the k th digit, and write

$$x = 0.d_1d_2 \cdots d_{k-1} 4 d_{k+1} \cdots = \left(\sum_{j=1}^k d_j 10^{-j} \right) + (4 \cdot 10^{-k}) + \left(\sum_{j=k+1}^{\infty} d_j 10^{-j} \right).$$

- Set $r_x < 10^{-k}$ and let $y \in [0, 1] \cap B_{r_x}(x)$ be arbitrary and write

$$y = \sum_{j=1}^{\infty} c_j 10^{-j}.$$

- Thus $|x - y| < r_x < 10^{-k}$, and the first k digits of x and y must agree:

– We first compute the difference:

$$x - y = \sum_{i=1}^{\infty} d_i 10^{-i} - \sum_{i=1}^{\infty} c_i 10^{-i} = \sum_{i=1}^{\infty} (d_i - c_i) 10^{-i}$$

– Thus (claim)

$$|x - y| \leq \sum_{j=1}^{\infty} |d_j - c_j| 10^j < 10^{-k} \iff |d_j - c_j| = 0 \quad \forall j \leq k.$$

– Otherwise we can note that any term $|d_j - c_j| \geq 1$ and there is a contribution to $|x - y|$ of at least $1 \cdot 10^{-j}$ for some $j < k$, whereas

$$j < k \iff 10^{-j} > 10^{-k},$$

a contradiction.

- This means that for all $j \leq k$ we have $d_j = c_j$, and in particular $d_k = 4 = c_k$, so y has a 4 in its decimal expansion.
- But then $K^c = \bigcup_x B_{r_x}(x)$ is a union of open sets and thus open.

Claim: K is nowhere dense and $m(K) = 0$:

- Strategy: Show $(\overline{K})^\circ = \emptyset$.
- Since K is closed, $\overline{K} = K$, so it suffices to show that K does not properly contain any interval.
- As in the construction of the Cantor set, let
 - K_1 denote $[0, 1]$ with 1 interval $\left[\frac{4}{10}, \frac{5}{10}\right]$ of length $\frac{1}{10}$ deleted
 - K_2 denote K_1 with 9 intervals $\left[\frac{1}{100}, \frac{5}{100}\right], \left[\frac{14}{100}, \frac{15}{100}\right], \dots, \left[\frac{94}{100}, \frac{95}{100}\right]$ length $\frac{1}{100}$ deleted
 - K_n denote K_{n-1} with 9^{n-1} such intervals of length 10^{-n} deleted.

- By construction, $K = \bigcap K_n$, and

$$m(K) = 1 - m(K^c) = 1 - \sum_{j=0}^{\infty} \frac{9^j}{10^{j+1}} = 1 - \frac{1}{10} \left(\frac{1}{1 - \frac{9}{10}} \right) = 0,$$

and since any interval has strictly positive measure, K can not contain any interval.

Claim: K has no isolated points:

- A point $x \in K$ is isolated iff there is an open ball $B_r(x)$ containing x such that $B_r(x) \cap K = \emptyset$, so every point in this ball has a 4 in its decimal expansion.
- Note that $m(K_n) = \left(\frac{9}{10}\right)^n \rightarrow 0$ and that the endpoints of intervals are never removed and are thus elements of K . Then for every ε , we can choose n such that $\left(\frac{9}{10}\right)^n < \varepsilon$; then there is an endpoint of a removed interval e_n satisfying $|x - e_n| \leq \left(\frac{9}{10}\right)^n < \varepsilon$.
- So every ball containing x contains some endpoint of a removed interval, and thus an element of K .

1.2 2

$$\lambda \ll \mu \iff E \in \mathcal{M}, \mu(E) = 0 \implies \lambda(E) = 0.$$

1.2.1 a

By Radon-Nikodym, if $\lambda \ll \mu$ then $d\lambda = f d\mu$, which would yield

$$\int g d\lambda = \int g f d\mu.$$

So let E be measurable and suppose $\mu(E) = 0$. Then

$$\lambda(E) := \int_E f d\mu = \lim_n \left\{ \varphi_n := \sum_j c_j \mu(E_j) \right\},$$

where we take a sequence of simple functions increasing to f .

But since each $E_j \subseteq E$, we must have $\mu(E_j) = 0$ for any such E_j , so every such φ_n must be zero and thus $\lambda(E) = 0$.

1.2.2 b

By Radon-Nikodym, there exists a positive f such that

$$\int g dm = \int g f d\mu,$$

where we can take $g(x) = x^2$, then the LHS is zero by assumption and thus so is the RHS.

Note that gf is positive.

Define $A_k = \left\{ x \in X \mid gf\chi_E > \frac{1}{k} \right\}$, then by Chebyshev

$$\mu(A_k) \leq k \int_E gf \, d\mu = 0,$$

which holds for every k .

Then noting that $A_k \searrow A := \left\{ x \in E \mid x^2 > 0 \right\}$, and gf is positive, we have

$$x \in E \iff gf\chi_E(x) > 0 \iff x \in A,$$

so $E = A$ and $\mu(E) = \mu(A)$.

But since $m \ll \mu$ by construction, we can conclude that $m(E) = 0$.

■

1.3 3

1.3.1 a

Letting $x_n := \frac{1}{n}$, we have

$$\sum_{k=1}^{\infty} |f_k(x)| \geq |f_n(x_n)| = \left| ae^{-ax} - be^{-bx} \right| := M.$$

In particular, $\sup_x |f_n(x)| \not\rightarrow 0$, so the terms do not go to zero and the sum can not converge.

1.3.2 b

?

1.4 4

Switching to polar coordinates and integrating over a half-circle contained in I^2 , we have

$$\int_{I^2} f \geq \int_0^\pi \int_0^1 \frac{\cos(\theta) \sin(\theta)}{r^2} \, dr \, d\theta = \infty,$$

so f is not integrable.

1.5 5

See <https://math.stackexchange.com/questions/507263/prove-that-c1a-b-with-the-c1-norm-is-a-banach-space>

This is clearly a norm, which we'll write $\|\cdot\|_u$

Let f_n be a Cauchy sequence and define a candidate limit $f(x) = \lim_n f_n(x)$.

Then noting that $\|f_n\|_\infty, \|f'_n\|_\infty \leq \|f_n\|_u < \infty$, both f_n, f'_n are Cauchy sequences in $C^0([a, b], \|\cdot\|_\infty)$, which is a Banach space.

So $f_n \rightarrow f$ uniformly, and $f'_n \rightarrow g$ uniformly for some g , and moreover $f, g \in C^0([a, b])$.

We thus have

$$\begin{aligned} f_n(x) - f_n(a) &\xrightarrow{u} f(x) - f(a) \\ \int_a^x f'_n &\xrightarrow{u} \int_a^x g, \end{aligned}$$

and by the FTC, the left-hand sides are equal, and by uniqueness of limits so are the right-hand sides, so $f' = g$.

Since $f, f' \in C^0([a, b])$, they are bounded, and so $\|f\|_u < \infty$. This means that $\|f_n - f\|_u \rightarrow 0$, so f_n converges to f , which is in the same space.

■