Title

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0.1 Sylow Theorems

Definition 0.0.1.
A p-group is a group G such that every element is order p^k for some k. If G is a finite p-group, then $|G| = p^j$ for some j.

Write

- $|G| = p^k m$ where $(p, m) = 1$,
- S_p a Sylow-p subgroup, and
- n_p the number of Sylow-p subgroups.

Some useful facts:

- Coprime order subgroups are disjoint, or more generally $\mathbb{Z}_p, \mathbb{Z}_q \subset G \implies \mathbb{Z}_p \bigcap \mathbb{Z}_q = \mathbb{Z}_{(p,q)}$.
- The Chinese Remainder theorem: $(p, q) = 1 \implies \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$

0.1.1 Sylow 1 (Cauchy for Prime Powers)

$$\forall p^n \text{ dividing } |G| \text{ there exists a subgroup of size } p^n$$

Idea: Sylow p-subgroups exist for any p dividing $|G|$, and are maximal in the sense that every p-subgroup of G is contained in a Sylow p-subgroup.

If $|G| = \prod p_i^{\alpha_i}$, then there exist subgroups of order $p_i^{\beta_i}$ for every i and every $0 \leq \beta_i \leq \alpha_i$. In particular, Sylow p-subgroups always exist.

0.1.2 Sylow 2 (Sylows are Conjugate)

All sylow-p subgroups S_p are conjugate, i.e.

$$S_p^i, S_p^j \in \mathrm{Syl}_p(G) \implies \exists g \text{ such that } gS_p^i g^{-1} = S_p^j$$

Corollary 0.1.
$n_p = 1 \iff S_p \trianglelefteq G$

0.1.3 Sylow 3 (Numerical Constraints)

1. $n_p \mathrel{\Big|} m$ (in particular, $n_p \leq m$),
2. $n_p \equiv 1 \mod p$,
3. $n_p = [G : N_G(S_p)]$ where N_G is the normalizer.

Corollary 0.2.
p does not divide n_p.

Proposition 0.3.
Every p-subgroup of G is contained in a Sylow p-subgroup.

Something proof title="Something"

Proof .
Let $H \leq G$ be a p-subgroup. If H is not *properly* contained in any other p-subgroup, it is a Sylow p-subgroup by definition.
Otherwise, it is contained in some p-subgroup H^1. Inductively this yields a chain $H \subsetneq H^1 \subsetneq \cdots$, and by Zorn's lemma $H := \bigcup_i H^i$ is maximal and thus a Sylow p-subgroup.

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Theorem 0.4 ***(Fratini's Argument).***
If $H \trianglelefteq G$ and $P \in \mathrm{Syl}_p(G)$, then $HN_G(P) = G$ and $[G : H]$ divides $|N_G(P)|$.

0.2 Products

Theorem 0.5 ***(Recognizing Direct Products).***
We have $G \cong H \times K$ when

- $H, K \trianglelefteq G$
- $G = HK$.
- $H \bigcap K = \{e\} \subset G$

Note: can relax to $[h, k] = 1$ for all h, k.

Theorem 0.6 ***(Recognizing Generalized Direct Products).***
We have $G = \prod_{i=1}^{n} H_i$ when

- $H_i \trianglelefteq G$ for all i.
- $G = H_1 \cdots H_n$
- $H_k \bigcap H_1 \cdots \widehat{H_k} \cdots H_n = \emptyset$

Note on notation: intersect H_k with the amalgam *leaving out* H_k.

Theorem 0.7 ***(Recognizing Semidirect Products).***
We have $G = N \rtimes_\psi H$ when

- $G = NH$
- $N \trianglelefteq G$
- $H \curvearrowright N$ by conjugation via a map

$$\begin{aligned} \psi : H &\longrightarrow \mathrm{Aut}(N) \\ h &\mapsto h(\,\cdot\,)h^{-1}. \end{aligned}$$

Note relaxed conditions compared to direct product: $H \trianglelefteq G$ and $K \leq G$ to get a semidirect product instead

Useful Facts

- If $\sigma \in \mathrm{Aut}(H)$, then $N \rtimes_\psi H \cong N \rtimes_{\psi\circ\sigma} H$.
- $\mathrm{Aut}((\mathbb{Z}/(p)^n) \cong \mathrm{GL}(n, \mathbb{F}_p)$, which has size
$$|\mathrm{Aut}(\mathbb{Z}/(p)^n)| = (p^n - 1)(p^n - p) \cdots (p^n - p^{n-1}).$$
  - If this occurs in a semidirect product, it suffices to consider similarity classes of matrices (i.e. just use canonical forms)

$$\mathrm{Aut}(\mathbb{Z}/(n)) \cong \mathbb{Z}/(n)^\times \cong \mathbb{Z}/(\varphi(n))$$

where φ is the totient function.

- - $\varphi(p^k) = p^{k-1}(p-1)$
- If G, H have coprime order then $\mathrm{Aut}(G \oplus H) \cong \mathrm{Aut}(G) \oplus \mathrm{Aut}(H)$.

0.3 Isomorphism Theorems

Theorem 0.8 ***(1st Isomorphism Theorem).***
If $\varphi : G \longrightarrow H$ is a group morphism then

$$G/\ker\varphi \cong \mathrm{im}\ \varphi.$$

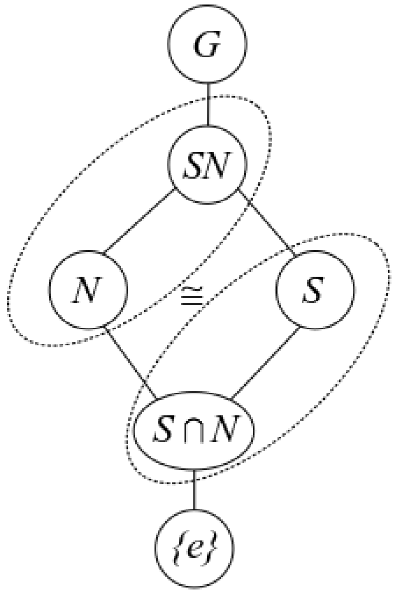

Figure 1: The 2nd “Diamond” Isomorphism Theorem

Note: for this to make sense, we also have

- $\ker\varphi \trianglelefteq G$
- $\operatorname{im}\varphi \leq G$

Corollary 0.9.
If $\varphi : G \longrightarrow H$ is surjective then $H \cong G/\ker\varphi$.

Proposition 0.10.
If $H, K \leq G$ and $H \leq N_G(K)$ (or $K \trianglelefteq G$) then $HK \leq G$ is a subgroup.

Theorem 0.11 ***(Diamond Theorem / 2nd Isomorphism Theorem).***
If $S \leq G$ and $N \trianglelefteq G$, then

$$\frac{SN}{N} \cong \frac{S}{S\bigcap N} \quad \text{and} \quad |SN| = \frac{|S||N|}{|S\bigcap N|}.$$

Note: for this to make sense, we also have

- $SN \leq G$,
- $S\bigcap N \trianglelefteq S$,

Corollary 0.12.
If we relax the conditions to $S, N \leq G$ with $S \in N_G(N)$, then $S\bigcap N \trianglelefteq S$ (but is not normal in G) and the theorem still applies.

Theorem 0.13 ***(Cancellation / 3rd Isomorphism Theorem).***
Suppose $N, K \leq G$ with $N \trianglelefteq G$ and $N \subseteq K \subseteq G$.

1. If $K \leq G$ then $K/N \leq G/N$ is a subgroup
2. If $K \trianglelefteq G$ then $K/N \trianglelefteq G/N$.
3. Every subgroup of G/N is of the form K/N for some such $K \leq G$.
4. Every *normal* subgroup of G/N is of the form K/N for some such $K \trianglelefteq G$.
5. If $K \trianglelefteq G$, then we can cancel normal subgroups:

$$\frac{G/N}{K/N} \cong \frac{G}{K}.$$

Theorem 0.14 ***(The Correspondence Theorem / 4th Isomorphism Theorem).***
Suppose $N \trianglelefteq G$, then there exists a correspondence:

$$\left\{H < G \;\middle|\; N \subseteq H\right\} \iff \left\{H \;\middle|\; H < \frac{G}{N}\right\}$$
$$\left\{\begin{matrix}\text{Subgroups of } G \\ \text{containing } N\end{matrix}\right\} \iff \left\{\begin{matrix}\text{Subgroups of the} \\ \text{quotient } G/N\end{matrix}\right\}.$$

In words, subgroups of G containing N correspond to subgroups of the quotient group G/N. This is given by the map $H \mapsto H/N$.

Note: $N \trianglelefteq G$ and $N \subseteq H < G \implies N \trianglelefteq H$.

0.4 Special Classes of Groups

Definition 0.14.1 (2 out of 3 Property)**.**
The **"2 out of 3 property"** is satisfied by a class of groups $\mathcal{C}$ iff whenever $G \in \mathcal{C}$, then $N, G/N \in \mathcal{C}$ for any $N \trianglelefteq G$.

Definition 0.14.2 (p-groups)**.**
If $|G| = p^k$, then G is a **p-group.**

Definition 0.14.3 (Normalizers Grow)**.**
If for every proper $H < G$, $H \trianglelefteq N_G(H)$ is again proper, then "normalizers grow" in G.

0.5 Classification of Groups

General strategy: find a normal subgroup (usually a Sylow) and use recognition of semidirect products.

- Keith Conrad: Classifying Groups of Order 12
- Order p: cyclic.
- Order p^2q: ?

0.6 Finitely Generated Abelian Groups

Definition 0.14.4 (Invariant Factor Decomposition)**.**

$$G \cong \mathbb{Z}^r \times \prod_{j=1}^{m} \mathbb{Z}/n_j\mathbb{Z} \quad \text{where } n_1 \,\Big|\, \cdots \,\Big|\, n_m.$$

Invariant factors $\longrightarrow$ Elementary Divisors:

- Take prime factorization of each factor
- Split into coprime pieces

Example 0.1.

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^3 \cdot 5^2 \cdot 7} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^3} \times \mathbb{Z}_{5^2} \times \mathbb{Z}_7$$

Going from elementary divisors to invariant factors:

- Bin up by primes occurring (keeping exponents)
- Take highest power from each prime as *last* invariant factor
- Take highest power from all remaining primes as next, etc

Example 0.2.
Given the invariant factor decomposition

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{5^2}$$

$p = 2$	$p = 3$	$p = 5$
$2, 2, 2$	$3, 3$	5^2

$\implies n_m = 5^2 \cdot 3 \cdot 2$

$p = 2$	$p = 3$	$p = 5$
$2, 2$	3	$\emptyset$

$\implies n_{m-1} = 3 \cdot 2$

$p = 2$	$p = 3$	$p = 5$
2	$\emptyset$	$\emptyset$

$\implies n_{m-2} = 2$

and thus

$$G \cong \mathbb{Z}_2 \times \mathbb{Z}_{3\cdot 2} \times \mathbb{Z}_{5^2\cdot 3\cdot 2}$$

Classifying Abelian Groups of a Given Order:

Let $p(x)$ be the integer partition function.

Example: $p(6) = 11$, given by $6, 5+1, 4+2, \cdots$.

Write $G = p_1^{k_1} p_2^{k_2} \cdots$; then there are $p(k_1)p(k_2)\cdots$ choices, each yielding a distinct group.