# Real Analysis Qualifying Exam Notes

D. Zack Garza

Monday 18th May, 2020

## Contents

# 1 Inequalities and Equalities

**AM-GM Inequality**:

$$\sqrt{ab} \le \frac{a+b}{2}.$$

**Reverse Triangle Inequality**

$$\left| \|x\| - \|y\| \right| \le \|x - y\|.$$

**Chebyshev's Inequality**

$$\mu(\{x : |f(x)| > \alpha\}) \le \left( \frac{\|f\|_p}{\alpha} \right)^p.$$

**Holder's Inequality:**

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \|fg\|_1 \le \|f\|_p \|g\|_q.$$

*Application:* For finite measure spaces,

$$1 \le p < q \le \infty \implies L^q \subset L^p \quad (\text{ and } \ell^p \subset \ell^q)$$

> *Proof .*
> Fix $p, q$, let $r = \dfrac{q}{p}$ and $s = \dfrac{r}{r-1}$ so $r^{-1} + s^{-1} = 1$. Then let $h = |f|^p$:
>
> $$\|f\|_p^p = \|h \cdot 1\|_1 \le \|1\|_s \|h\|_r = \mu(X)^{\frac{1}{s}} \|f\|_q^{\frac{q}{r}} \implies \|f\|_p \le \mu(X)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q.$$
>
> $\blacksquare$

> Note: doesn't work for $\ell$ spaces, but just note that $\sum |x_n| < \infty \implies x_n < 1$ for large enough $n$, and thus $p < q \implies |x_n|^q \le |x_n|^q$.

**Cauchy-Schwarz**:

$$|\langle f, \ g \rangle| = \|fg\|_1 \le \|f\|_2 \|g\|_2 \quad \text{with equality} \quad \iff \quad f = \lambda g.$$

> Relates inner product to norm, and only happens to relate norms in $L^1$.

**Minkowski's Inequality:**

$$1 \le p < \infty \implies \|f + g\|_p \le \|f\|_p + \|g\|_p$$

Note: does not handle $p = \infty$ case. Use to prove $L^p$ is a normed space.

**Young's Inequality**:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \implies \|f * g\|_r \le \|f\|_p \|g\|_q.$$

**Application**: Some useful specific cases:

$$\|f * g\|_1 \le \|f\|_1 \|g\|_1$$
$$\|f * g\|_p \le \|f\|_1 \|g\|_p,$$
$$\|f * g\|_\infty \le \|f\|_2 \|g\|_2$$
$$\|f * g\|_\infty \le \|f\|_p \|g\|_q.$$

**Bessel's Inequality:**

For $x \in H$ a Hilbert space and $\{e_k\}$ an orthonormal sequence,

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \le \|x\|^2.$$

Note: this does not need to be a basis.

**Parseval's Identity:**

Equality in Bessel's inequality, attained when $\{e_k\}$ is a *basis*, i.e. it is complete, i.e. the span of its closure is all of $H$.

## 2 Basics

**Useful Technique:** $\lim f_n = \limsup f_n = \liminf f_n$ iff the limit exists, so $\limsup f_n \le g \le \liminf f_n$ implies that $g = \lim f$. Similarly, a limit does not exist iff $\liminf f_n > \limsup f_n$.

> **Lemma 2.1** *(Convergent Sums Have Small Tails).*
>
> $$\sum a_n < \infty \implies a_n \longrightarrow 0 \quad \text{and} \quad \sum_{k=N}^{\infty} \overset{N \longrightarrow \infty}{\longrightarrow} 0$$

**Theorem 2.2** *(Heine-Borel).*
$X \subseteq \mathbb{R}^n$ is compact $\iff$ $X$ is closed and bounded.

**Lemma 2.3** *(Geometric Series).*

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \iff |x| < 1.$$

*Corollary:* $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k} = 1.$

**Definition 2.3.1.**
A set $S$ is **nowhere dense** iff the closure of $S$ has empty interior iff every interval contains a subinterval that does not intersect $S$.

**Definition 2.3.2.**
A set is **meager** if it is a *countable* union of nowhere dense sets.

> Note that a *finite* union of nowhere dense is still nowhere dense.

**Lemma 2.4.**
The Cantor set is closed with empty interior.

> Proof: Its complement is a union of open intervals, and can't contain an interval since intervals have positive measure and $m(C_n)$ tends to zero.

> **Corollary:** The Cantor set is nowhere dense.

**Definition:** An $F_\sigma$ set is a union of closed sets, and a $G_\delta$ set is an intersection of opens.

> Mnemonic: "F" stands for *ferme*, which is "closed" in French, and $\sigma$ corresponds to a "sum", i.e. a union.

**Lemma 2.5.**
Singleton sets in $\mathbb{R}$ are closed, and thus $\mathbb{Q}$ is an $F_\sigma$ set.

**Theorem (Baire):** $\mathbb{R}$ is a Baire space, i.e. countable intersections of open, dense sets are still dense. Thus $\mathbb{R}$ can not be written as a countable union of nowhere dense sets.

**Lemma 2.6.**
There is a function discontinuous precisely on $\mathbb{Q}$.

> *Proof:* $f(x) = \dfrac{1}{n}$ if $x = r_n \in \mathbb{Q}$ is an enumeration of the rationals, and zero otherwise. The limit at every point is 0.

**Lemma 2.7.**
There *do not* exist functions that are discontinuous precisely on $\mathbb{R} \setminus \mathbb{Q}$.

*Proof:* $D_f$ is always an $F_\sigma$ set, which follows by considering the oscillation $\omega_f$. $\omega_f(x) = 0 \iff f$ is continuous at $x$, and $D_f = \bigcup_n A_{\frac{1}{n}}$ where $A_\varepsilon = \{\omega_f \geq \varepsilon\}$ is closed.

**Lemma 2.8.**
Any nonempty set which is bounded from above (resp. below) has a well-defined supremum (resp. infimum).

# 3 Uniform Convergence

**Theorem (Egorov)**:

Let $E \subseteq \mathbb{R}^n$ be measurable with $m(E) > 0$ and $\{f_k : E \longrightarrow \mathbb{R}\}$ be measurable functions such that $f(x) := \lim_{k \longrightarrow \infty} f_k(x) < \infty$ exists almost everywhere.

Then $f_k \longrightarrow f$ *almost uniformly*, i.e.

$$\forall \varepsilon > 0, \ \exists F \subseteq E \text{ closed such that } m(E \setminus F) < \varepsilon \ \text{ and } \ f_k \xrightarrow{u} f \text{ on } F.$$

**Theorem (Important Example):** The space $X = C([0, 1])$, continuous functions $f : [0, 1] \longrightarrow \mathbb{R}$, equipped with the norm $\|f\| = \sup_{x \in [0,1]} |f(x)|$, is a **complete** metric space.

*Proof:*
*Step 0:* Let $\{f_k\}$ be Cauchy in $X$.
*Step 1:* Define a candidate limit using pointwise convergence:
Fix an $x$; since

$$|f_k(x) - f_j(x)| \leq \|f_k - f_k\| \longrightarrow 0,$$

the sequence $\{f_k(x)\}$ is Cauchy in $\mathbb{R}$. So define $f(x) := \lim_k f_k(x)$.
*Step 2:* Show that $\|f_k - f\| \longrightarrow 0$:

$$|f_k(x) - f_j(x)| < \varepsilon \ \forall x \implies \lim_j |f_k(x) - f_j(x)| < \varepsilon \ \forall x$$

Alternatively, $\|f_k - f\| \leq \|f_k - f_N\| + \|f_N - f_j\|$, where $N, j$ can be chosen large enough to bound each term by $\varepsilon/2$.
*Step 3:* Show that $f \in X$:
The uniform limit of continuous functions is continuous. (Note: in other cases, you may need to show the limit is bounded, or has bounded derivative, or whatever other conditions define $X$.) ∎

**Lemma 3.1.**
Metric spaces are compact iff they are sequentially compact, (i.e. every sequence has a convergent subsequence).

**Corollary:** The unit ball in $C([0, 1])$ with the sup norm is not compact.

*Proof:* Take $f_k(x) = x^n$, which converges to a dirac delta at 1. The limit is not continuous, so no subsequence can converge.

> **Lemma 3.2.**
> A uniform limit of continuous functions is continuous.

**Lemma (Testing Uniform Convergence):** $f_n \longrightarrow f$ uniformly iff there exists an $M_n$ such that $\|f_n - f\|_\infty \leq M_n \longrightarrow 0$.

> Negating: find an $x$ which depends on $n$ for which the norm is bounded below.

**Useful Technique**: If $f_n$ has a global maximum (computed using $f_n'$ and the first derivative test) $M_n \longrightarrow 0$, then $f_n \longrightarrow 0$ uniformly.

**Lemma (Baby Commuting Limits with Integrals):** If $f_n \longrightarrow f$ uniformly, then $\int f_n = \int f$.

**Lemma (Uniform Convergence and Derivatives)** If $f_n' \longrightarrow g$ uniformly for some $g$ and $f_n \longrightarrow f$ pointwise (or at least at one point), then $g = f'$.

**Lemma (Uniform Convergence of Series):** If $f_n(x) \leq M_n$ **for a fixed** $x$ where $\sum M_n < \infty$, then the series $f(x) = \sum f_n(x)$ converges pointwise.

> **Lemma 3.3.**
> If $\sum f_n$ converges then $f_n \longrightarrow 0$ uniformly.

**Useful Technique:** For a fixed $x$, if $f = \sum f_n$ converges *uniformly* on some $B_r(x)$ and each $f_n$ is continuous at $x$, then $f$ is also continuous at $x$ .

**Lemma (M-test for Series):** If $|f_n(x)| \leq M_n$ which does not depend on $x$, then $\sum f_n$ converges uniformly.

**Lemma (p-tests):** Let $n$ be a fixed dimension and set $B = \left\{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \right\}$.

$$\sum \frac{1}{n^p} < \infty \iff p > 1$$
$$\int_\varepsilon^\infty \frac{1}{x^p} < \infty \iff p > 1$$
$$\int_0^1 \frac{1}{x^p} < \infty \iff p < 1$$
$$\int_B \frac{1}{|x|^p} < \infty \iff p < n$$
$$\int_{B^c} \frac{1}{|x|^p} < \infty \iff p > n$$

.

# 4 Measure Theory

**Useful Technique:** $s = \inf \{x \in X\} \implies$ for every $\varepsilon$ there is an $x \in X$ such that $x \leq s + \varepsilon$.

**Useful Techniques**: Always consider bounded sets, and if $E$ is unbounded write $E = \bigcup_n B_n(0) \bigcap E$ and use countable subadditivity or continuity of measure.

**Lemma 4.1.**
Every open subset of $\mathbb{R}$ (resp $\mathbb{R}^n$) can be written as a unique countable union of disjoint (resp. almost disjoint) intervals (resp. cubes).

**Definition**: The outer measure of a set is given by

$$m_*(E) = \inf_{\substack{\{Q_i\} \rightrightarrows E \\ \text{closed cubes}}} \sum |Q_i|.$$

**Lemma 4.2** *(Properties of [Outer).*
Measure]

- Montonicity: $E \subseteq F \implies m_*(E) \leq m_*(F)$.
- Countable Subadditivity: $m_*(\bigcup E_i) \leq \sum m_*(E_i)$.
- Approximation: For all $E$ there exists a $G \supseteq E$ such that $m_*(G) \leq m_*(E) + \varepsilon$.
- Disjoint* Additivity: $m_*(A \coprod B) = m_*(A) + m_*(B)$.

Note: this holds for outer measure **iff** $\text{dist}(A, B) > 0$.

**Lemma 4.3** *(Subtraction of Measure):).*
$m(A) = m(B) + m(C)$ and $m(C) < \infty$ implies that $m(A) - m(C) = m(B)$.

**Lemma 4.4** *(Continuity of Measure).*

$$E_i \nearrow E \implies m(E_i) \longrightarrow m(E)$$
$$m(E_1) < \infty \text{ and } E_i \searrow E \implies m(E_i) \longrightarrow m(E).$$

*Proof .*

1. Break into disjoint annuli $A_2 = E_2 \setminus E_1$, etc then apply countable disjoint additivity to $E = \coprod A_i$.
   ∎

2. Use $E_1 = (\coprod E_j \setminus E_{j+1}) \coprod (\bigcap E_j)$, taking measures yields a telescoping sum, and use countable disjoint additivity.

**Lemma 4.5.**
Lebesgue measure is translation and dilation invariant.

*Proof .*
Obvious for cubes; if $Q_i \rightrightarrows E$ then $Q_i + k \rightrightarrows E + k$, etc.
   ∎

**Theorem 4.6** *(Non-Measurable Sets).*

There is a non-measurable set.

*Proof .*

- Use AOC to choose one representative from every coset of $\mathbb{R}/\mathbb{Q}$ on $[0,1)$, which is countable, and assemble them into a set $N$
- Enumerate the rationals in $[0,1]$ as $q_j$, and define $N_j = N + q_j$. These intersect trivially.
- Define $M := \coprod N_j$, then $[0,1) \subseteq M \subseteq [-1,2)$, so the measure must be between 1 and 3. By translation invariance, $m(N_j) = m(N)$, and disjoint additivity forces $m(M) = 0$, a contradiction. ∎

**Lemma (Borel Characterization of Measurable Sets)**

If $E$ is Lebesgue measurable, then $E = H \coprod N$ where $H \in F_\sigma$ and $N$ is null.

> **Useful technique:** $F_\sigma$ sets are Borel, so establish something for Borel sets and use this to extend it to Lebesgue.

**Proof:** For every $\dfrac{1}{n}$ there exists a closed set $K_n \subset E$ such that $m(E \setminus K_n) \leq \dfrac{1}{n}$. Take $K = \bigcup K_n$, wlog $K_n \nearrow K$ so $m(K) = \lim m(K_n) = m(E)$. Take $N := E \setminus K$, then $m(N) = 0$.

**Lemma 4.7.**

$$\limsup_n A_n = \bigcap_n \bigcup_{j \geq n} A_j = \left\{ x \;\middle|\; x \in A_n \text{ for inf. many } n \right\}$$

$$\liminf_n A_n = \bigcup_n \bigcap_{j \geq n} A_j = \left\{ x \;\middle|\; x \in A_n \text{ for all except fin. many } n \right\}$$

.

**Lemma 4.8.**

If $A_n$ are all measurable, $\limsup A_n$ and $\liminf A_n$ are measurable.

> *Proof:* Measurable sets form a sigma algebra, and these are expressed as countable unions/intersections of measurable sets.

**Theorem 4.9** *(Borel-Cantelli).*

Let $\{E_k\}$ be a countable collection of measurable sets. Then

$$\sum_k m(E_k) < \infty \implies \text{ almost every } x \in \mathbb{R} \text{ is in at most finitely many } E_k.$$

**Application:**

$$m\left(\left\{x \text{ such that } \exists \text{ inf. many } \frac{p}{q} \text{ with } \left|x - \frac{p}{q}\right| \le \frac{1}{q^3}\right\}\right) = 0.$$

*Proof.*
Idea: write $E_j$ to be the above set with $p, q$ replaced by $p_j, q_j$ where $r_j = \dfrac{p_j}{q_j}$ is an enumeration

of $\mathbb{Q}$, then $m(E_j) \le \dfrac{2}{q^3}$ and $\sum \dfrac{1}{q^3} < \infty$.

- If $E = \limsup\limits_{j} E_j$ with $\sum m(E_j) < \infty$ then $m(E) = 0$.
- If $E_j$ are measurable, then $\limsup\limits_{j} E_j$ is measurable.
- If $\sum\limits_{j} m(E_j) < \infty$, then $\sum\limits_{j=N}^{\infty} m(E_j) \overset{N \to \infty}{\longrightarrow} 0$ as the tail of a convergent sequence.
- $E = \limsup\limits_{j} E_j = \bigcap\limits_{k=1}^{\infty} \bigcup\limits_{j=k}^{\infty} E_j \implies E \subseteq \bigcup\limits_{j=k}^{\infty}$ for all $k$
- $E \subset \bigcup\limits_{j=k}^{\infty} \implies m(E) \le \sum\limits_{j=k}^{\infty} m(E_j) \overset{k \to \infty}{\longrightarrow} 0.$   ∎

**Lemma 4.10.**

- Characteristic functions are measurable
- If $f_n$ are measurable, so are $|f_n|, \limsup f_n, \liminf f_n, \lim f_n,$
- Sums and differences of measurable functions are measurable,
- Cones $F(x, y) = f(x)$ are measurable,
- Compositions $f \circ T$ for $T$ a linear transformation are measurable,
- "Convolution-ish" transformations $(x, y) \mapsto f(x - y)$ are measurable

**Proof (Convolution):** Take the cone on $f$ to get $F(x, y) = f(x)$, then compose $F$ with the linear transformation $T = [1, -1; 1, 0]$.

# 5 Integration

**Definition:** $f \in L^+$ iff $f$ is measurable and non-negative.

Useful techniques:

- Break integration domain up into disjoint annuli.
- Break real integrals up into $x < 1$ and $x > 1$.

**Definition:** A measurable function is integrable iff $\|f\|_1 < \infty$.

Useful facts about $C_c$ functions:

- Bounded almost everywhere
- Uniformly continuous

## 5.1 Convergence Theorems

**Monotone Convergence Theorem (MCT):**

If $f_n \in L^+$ and $f_n \nearrow f$ a.e., then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \quad \int f_n \longrightarrow \int f.$$

> Needs to be positive and increasing.

**Dominated Convergence Theorem (DCT):**

If $f_n \in L^1$ and $f_n \longrightarrow f$ a.e. with $|f_n| \le g$ for some $g \in L^1$, then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \quad \int f_n \longrightarrow \int f,$$

and more generally,

$$\int |f_n - f| \longrightarrow 0.$$

> Positivity *not* needed.

> Generalized DCT: can relax $|f_n| < g$ to $|f_n| < g_n \longrightarrow g \in L^1$.

**Lemma:** If $f \in L^1$, then

$$\int |f_n - f| \longrightarrow 0 \iff \int |f_n| \longrightarrow |f|.$$

> *Proof:* Let $g_n = |f_n| - |f_n - f|$, then $g_n \longrightarrow |f|$ and
>
> $$|g_n| = ||f_n| - |f_n - f|| \le |f_n - (f_n - f)| = |f| \in L^1,$$
>
> so the DCT applies to $g_n$ and
>
> $$\|f_n - f\|_1 = \int |f_n - f| + |f_n| - |f_n| = \int |f_n| - g_n$$
> $$\longrightarrow_{DCT} \lim \int |f_n| - \int |f|.$$

**Fatou's Lemma:**

If $f_n \in L^+$, then

$$\int \liminf_n f_n \le \liminf_n \int f_n$$
$$\limsup_n \int f_n \le \int \limsup_n f_n.$$

> Only need positivity.

**Theorem (Tonelli):** For $f(x,y)$ **non-negative and measurable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is a **measurable** function
- $F(x) = \int f(x, y) \, dy$ is a **measurable** function,
- For $E$ measurable, the slices $E_x := \left\{ y \mid (x, y) \in E \right\}$ are measurable.
- $\int f = \int \int F$, i.e. any iterated integral is equal to the original.

**Theorem (Fubini):** For $f(x, y)$ **integrable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is an **integrable** function
- $F(x) = \int f(x, y) \, dy$ is an **integrable** function,
- For $E$ measurable, the slices $E_x := \left\{ y \mid (x, y) \in E \right\}$ are measurable.
- $\int f = \int \int f(x, y)$, i.e. any iterated integral is equal to the original

**Theorem (Fubini/Tonelli):** If any iterated integral is **absolutely integrable**, i.e. $\int \int |f(x, y)| < \infty$, then $f$ is integrable and $\int f$ equals any iterated integral.

**Differentiating under the integral**:

If $\left| \dfrac{\partial}{\partial t} f(x, t) \right| \leq g(x) \in L^1$, then letting $F(t) = \int f(x, t) \, dt$,

$$\frac{\partial}{\partial t} F(t) := \lim_{h \to 0} \int \frac{f(x, t + h) - f(x, t)}{h} dx$$
$$\overset{\text{DCT}}{=\!=} \int \frac{\partial}{\partial t} f(x, t) \, dx.$$

To justify passing the limit, let $h_k \longrightarrow 0$ be any sequence and define

$$f_k(x, t) = \frac{f(x, t + h_k) - f(x, t)}{h_k},$$

so $f_k \overset{\text{pointwise}}{\longrightarrow} \dfrac{\partial}{\partial t} f$.

Apply the MVT to $f_k$ to get $f_k(x, t) = f_k(\xi, t)$ for some $\xi \in [0, h_k]$, and show that $f_k(\xi, t) \in L_1$.

**Lemma (Swapping Sum and Integral)** If $f_n$ are non-negative and $\sum \int |f|_n < \infty$, then $\sum \int f_n = \int \sum f_n$.

> *Proof: MCT.* Let $F_N = \sum^N f_n$ be a finite partial sum; then there are simple functions $\phi_n \nearrow f_n$ and so $\sum^N \phi_n \nearrow F_N$, so apply MCT.

**Lemma:** If $f_k \in L^1$ and $\sum \|f_k\|_1 < \infty$ then $\sum f_k$ converges almost everywhere and in $L^1$.

> *Proof:* Define $F_N = \sum^N f_k$ and $F = \lim_N F_N$, then $\|F_N\|_1 \leq \sum^N \|f_k\| < \infty$ so $F \in L^1$ and $\|F_N - F\|_1 \longrightarrow 0$ so the sum converges in $L^1$. Almost everywhere convergence: ?

## 5.2 $L^1$ **Facts**

**Lemma (Translation Invariance):** The Lebesgue integral is translation invariant, i.e. $\int f(x)\, dx = \int f(x+h)\, dx$ for any $h$.

> *Proof:*
>
> - For characteristic functions, $\int_E f(x+h) = \int_{E+h} f(x) = m(E+h) = m(E) = \int_E f$ by translation invariance of measure.
> - So this also holds for simple functions by linearity
> - For $f \in L^+$, choose $\phi_n \nearrow f$ so $\int \phi_n \longrightarrow \int f$.
> - Similarly, $\tau_h \phi_n \nearrow \tau_h f$ so $\int \tau_h f \longrightarrow \int f$
> - Finally $\left\{ \int \tau_h \phi \right\} = \left\{ \int \phi \right\}$ by step 1, and the suprema are equal by uniqueness of limits.

**Lemma (Integrals Distribute Over Disjoint Sets):**

If $X \subseteq A \bigcup B$, then $\int_X f \le \int_A f + \int_{A^c} f$ with equality iff $X = A \coprod B$.

**Lemma ($L^1$ functions may Decay Rapidly):**

If $f \in L^1$ and $f$ is uniformly continuous, then $f(x) \overset{|x| \longrightarrow \infty}{\longrightarrow} 0$.

> Doesn't hold for general $L^1$ functions, take any train of triangles with height 1 and summable areas.

**Lemma ($L^1$ functions have Small Tails):**

If $f \in L^1$, then for every $\varepsilon$ there exists a radius $R$ such that if $A = B_R(0)^c$, then $\int_A |f| < \varepsilon$.

> *Proof: Approximate with compactly supported functions. Take $g \overset{L_1}{\longrightarrow} f$ with $g \in C_c$, then choose $N$ large enough so that $g = 0$ on $E := B_N(0)^c$, then $\int_E |f| \le \int_E |f - g| + \int_E |g|$.*

**Lemma ($L^1$ functions have absolutely continuity):**

$m(E) \longrightarrow 0 \implies \int_E f \longrightarrow 0$.

> *Proof: Approximate with compactly supported functions. Take $g \overset{L_1}{\longrightarrow} f$, then $g \le M$ so $\int_E f \le \int_E f - g + \int_E g \longrightarrow 0 + M \cdot m(E) \longrightarrow 0$.*

**Lemma ($L^1$ functions are finite almost everywhere):**

If $f \in L^1$, then $m(\{f(x) = \infty\}) = 0$.

> *Proof (Split up domain2):* Let $A = \{f(x) = \infty\}$, then $\infty > \int f = \int_A f + \int_{A^c} f = \infty \cdot m(A) + \int_{A^c} f \implies m(X) = 0$.

**Lemma (Continuity in $L^1$):** $\|\tau_h f - f\|_1 \longrightarrow 0$ as $h \longrightarrow 0$.

*Proof: Approximate with compactly supported functions.* Take $g \xrightarrow{L_1} f$ with $g \in C_c$.

$$\int f(x+h) - f(x) \leq$$

$$\int f(x+h) - g(x+h) + \int g(x+h) - g(x) + \int g(x) - f(x)$$

$$\longrightarrow 2\varepsilon + \int g(x+h) - g(x)$$

$$= \int_K g(x+h) - g(x) + \int_{K^c} g(x+h) - g(x) \longrightarrow 0,$$

which follows because we can enlarge the support of $g$ to $K$ where the integrand is zero on $K^c$, then apply uniform continuity on $K$.

**Theorem (Integration by Parts, Special Case):**

$$F(x) := \int_0^x f(y) dy \quad \text{and} \quad G(x) := \int_0^x g(y) dy$$

$$\implies \int_0^1 F(x) g(x) dx = F(1)G(1) - \int_0^1 f(x) G(x) dx.$$

*Proof:* Fubini-Tonelli, and sketch region to change integration bounds.

**Theorem (Lebesgue Density):**

$$A_h(f)(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(y) dy \implies \|A_h(f) - f\| \xrightarrow{h \to 0} 0.$$

*Proof*: Fubini-Tonelli, and sketch region to change integration bounds, and continuity in $L^1$.

## 5.3  $L^p$ Spaces

**Lemma:** The following are dense subspaces of $L^2([0,1])$:

- Simple functions
- Step functions
- $C_0([0,1])$
- Smoothly differentiable functions $C_0^\infty([0,1])$
- Smooth compactly supported functions $C_c^\infty$

**Dual Spaces:** In general, $(L^p)^\vee \cong L^q$

- For qual, supposed to know the $p = 1$ case, i.e. $(L^1)^\vee \cong L^\infty$
  - For the analogous $p = \infty$ case: $L^1 \subset (L^\infty)^\vee$, since the isometric mapping is always injective, but *never* surjective. So this containment is always proper (requires Hahn-Banach Theorem).
- The $p = 2$ case: Easy by the Riesz Representation for Hilbert spaces.

# 6 Fourier Series and Convolution

**Definition (Convolution)**

$$f * g(x) = \int f(x - y)g(y)dy.$$

**Definition (Dilation)**

$$\phi_t(x) = t^{-n}\phi\left(t^{-1}x\right).$$

**Definition (The Fourier Transform):**

$$\widehat{f}(\xi) = \int f(x)\ e^{2\pi i x \cdot \xi}\ dx.$$

**Lemma:** $\widehat{f} = \widehat{g} \implies f = g$ almost everywhere.

**Lemma (Riemann-Lebesgue)**

$$f \in L^1 \implies \widehat{f}(\xi) \to 0 \text{ as } |\xi| \to \infty.$$

> Motto: Fourier transforms decay.

**Lemma:** If $f \in L^1$, then $\widehat{f}$ is continuous and bounded.

> *Proof:* $\left|\widehat{f}\right| \le \int |f| \cdot |e^{\cdots}| \le \|f\|_1$, and the DCT shows that $\left|\widehat{f}(\xi_n) - \widehat{f}(\xi)\right| \longrightarrow 0$.

Todo: search qual alerts.

# 7 Exam 2 (Practice)

Link to PDF File

Proving uniform continuity: show

$$\|f - \tau_h f\|_1 \xrightarrow{h \longrightarrow 0} 0$$

Notation: $C_0$ is the set of functions that vanish at infinity.

## 7.1 1: Fubini-Tonelli

**Theorem (Fubini):**

Suppose

- $f : \mathbb{R}^{n_1 + n_2} \longrightarrow \mathbb{R}$ is measurable on its domain
- $f$ is non-negative

Then for almost every $x \in \mathbb{R}^{n_1}$,

1. Every slice

$$f_x : \mathbb{R}^{n_2} \longrightarrow \mathbb{R}$$
$$y \mapsto f(x, y)$$

is measurable on $\mathbb{R}^{n_2}$.

2. The function

$$F : \mathbb{R}^{n_1} \longrightarrow \mathbb{R}$$
$$x \mapsto \int_{\mathbb{R}^{n_2}} f_x(y) \ dy$$

is measurable on $\mathbb{R}^{n_1}$

3. The function

$$G(y) = \int_{\mathbb{R}^n} F(x) \ dx$$

is measurable and

$$G(y) = \int_{\mathbb{R}^n} f = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} f(x, y) \ dy \ dx$$

for any iterated version of this integral.

**Corollary (Measurable Slices):**

Let $E$ be a measurable subset of $\mathbb{R}^n$. Then

- For almost every $x \in \mathbb{R}^{n_1}$, the slice $E_x := \left\{ y \in \mathbb{R}^{n_2} \ \middle| \ (x, y) \in E \right\}$ is measurable in $\mathbb{R}^{n_2}$.
- The function

$$F : \mathbb{R}^{n_1} \longrightarrow \mathbb{R}$$
$$x \mapsto m(E_x) = \int_{\mathbb{R}^{n_2}} \chi_{E_x} \ dy$$

is measurable and

$$m(E) = \int_{\mathbb{R}^{n_1}} m(E_x) \ dx = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{E_x} \ dy \ dx$$

$\Longrightarrow$ :

- Let $f$ be measurable on $\mathbb{R}^n$.
- Then the cylinders $F(x, y) = f(x)$ and $G(x, y) = f(y)$ are both measurable on $\mathbb{R}^{n+1}$.
- Write $\mathcal{A} = \{G \leq F\} \bigcap \{G \geq 0\}$; both are measurable.

$\Longleftarrow$ :

- Let $A$ be measurable in $\mathbb{R}^{n+1}$.
- Define $A_x = \left\{ y \in \mathbb{R} \ \middle| \ (x, y) \in \mathcal{A} \right\}$, then $m(A_x) = f(x)$.
- By the corollary, $A_x$ is measurable set, $x \mapsto A_x$ is a measurable function, and $m(A) = \int f(x) \ dx$.
- Then explicitly, $f(x) = \chi_A$, which makes $f$ a measurable function.

---

### 7.1.1 b

- Define $A_y = \left\{ x \in \mathbb{R}^n \mid (x, y) \in A \right\}$, and notice that $A_y = \left\{ x \in \mathbb{R}^n \mid 0 \leq y \leq f(x) \right\}$.
- By the corollary, $A_y$ is measurable and

$$m(\mathcal{A}) = \int m\left(\mathcal{A}_y\right) dy = \int_0^y m\left(\{x \in \mathbb{R}^n \ni f(x) \geq y\}\right) dy$$

## 7.2  2: Convolutions and the Fourier Transform

### 7.2.1 a

**Definition (Convolution):**

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y) \ dy.$$

**Facts:**

- $f, g \in L^1 \implies f * g \in L^1$
- $f \in L^1, g \leq M \implies f * g \leq M'$ and is uniformly continuous.
- $f, g \in L^1, f \leq M, g \leq M' \implies f * g \xrightarrow{x \longrightarrow \infty} 0.2$
- $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$
- $f \in L^1, g' \text{ exists}, \dfrac{\partial g}{\partial x_i} \text{ all bounded} \implies \dfrac{\partial}{\partial x_i}(f * g) = f * \dfrac{\partial g}{\partial x_i}$
- $f, g \in C_c^\infty \implies f * g \in C^\infty$ and $f * g \xrightarrow{x \longrightarrow \infty} 0.$

### 7.2.2 b

**Definition (Approximation to the Identity):**

$$\phi(x) = e^{-\pi x^2}$$
$$\phi_t(x) = t^{-n} \phi\left(\frac{x}{t}\right).$$

**Facts:**

- $\displaystyle\int \phi = \int \phi_t = 1$
- $f$ bounded and uniformly continuous $\implies f * \phi_t \rightrightarrows f$

**Theorem (Norm Convergence of Approximate Identities):**

$$\|f * \phi_t - f\|_1 \xrightarrow{t \longrightarrow 0} 0.$$

*Proof:*

$$\|f - f * \phi_t\|_1 = \int f(x) - \int f(x-y)\phi_t(y) \; dy dx$$

$$= \int f(x) \int \phi_t(y) \; dy - \int f(x-y)\phi_t(y) \; dy dx$$

$$= \int \int \phi_t(y)[f(x) - f(x-y)] \; dy dx$$

$$=_{FT} \int \int \phi_t(y)[f(x) - f(x-y)] \; dx dy$$

$$= \int \phi_t(y) \int f(x) - f(x-y) \; dx dy$$

$$= \int \phi_t(y) \|f - \tau_y f\|_1 dy$$

$$= \int_{y < \delta} \phi_t(y) \|f - \tau_y f\|_1 dy + \int_{y \geq \delta} \phi_t(y) \|f - \tau_y f\|_1 dy$$

$$\leq \int_{y < \delta} \phi_t(y)\varepsilon + \int_{y \geq \delta} \phi_t(y) \left( \|f\|_1 + \|\tau_y f\|_1 \right) dy \quad \text{by continuity in } L^1$$

$$\leq \varepsilon + 2\|f\|_1 \int_{y \geq \delta} \phi_t(y) dy$$

$$\leq \varepsilon + 2\|f\|_1 \varepsilon \quad \text{since } \phi_t \text{ has small tails}$$

$$\longrightarrow 0 \blacksquare.$$

## Theorem (Convolutions Vanish at Infinity)

$$f, g \in L^1 \text{ and bounded} \implies \lim_{|x| \to \infty} (f * g)(x) = 0.$$

*Proof:*

- Choose $M \geq f, g$.

- By small tails, choose $N$ such that $\int_{B_N^c} |f|, \int_{B_n^c} |g| < \varepsilon$

- Note

$$|f * g| \leq \int |f(x-y)| \, |g(y)| \; dy \coloneqq I$$

- Use $|x| \leq |x - y| + |y|$, take $|x| \geq 2N$ so either

  –

  $$|x - y| \geq N \implies I \leq \int_{\{x-y \geq N\}} |f(x-y)| M \; dy \leq \varepsilon M \longrightarrow 0$$

  –

  $$|y| \geq N \implies I \leq \int_{\{y \geq N\}} M |g(y)| \; dy \leq M\varepsilon \longrightarrow 0$$

$\blacksquare$

### 7.2.3 c

**Definition (The Fourier Transform):**

$$\widehat{f}(\xi) = \int f(x)e^{-2\pi i x \cdot \xi} \, dx.$$

**Facts:**

- *Riemann-Lebesgue lemma:* $\widehat{f}$ vanishes at infinity
- $f \in L^1 \implies \widehat{f}$ is bounded and uniformly continuous
- $f, \widehat{f} \in L^1 \implies f$ is bounded, uniformly continuous, and vanishes at infinity
- $f \in L^1$ and $\widehat{f} = 0$ almost everywhere $\implies f = 0$ almost everywhere.

**Theorem (Fourier Inversion):**

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(x)e^{2\pi i x \cdot \xi} d\xi.$$

*Proof:* Idea: Fubini-Tonelli doesn't work directly, so introduce a convergence factor, take limits, and use uniqueness of limits.

Use the following facts:

- $f, g \in L^1 \implies \int \widehat{f}g = \int f\widehat{g}.$
- $g(x) := e^{-\pi|t|^2} \implies \widehat{g}(\xi) = g(\xi)$
- $g_t(x) = t^{-n}g(x/t) = t^{-n}e^{-\pi|x|^2/t^2}$
- $\widehat{g_t}(x) = g(tx) = e^{-\pi t^2 |x|^2}$
- $\phi(\xi) := e^{2\pi i x \cdot \xi} \, \widehat{g_t}(\xi)$
- $\widehat{\phi}(\xi) = \mathcal{F}(\widehat{g_t}(\xi - x)) = g_t(x - \xi)$
- $\lim\limits_{t \longrightarrow 0} \phi(\xi) = e^{2\pi i x \cdot \xi}$

Take the modified integral:

$$\begin{aligned}
I_t(x) &= \int \widehat{f}(\xi) \, e^{2\pi i x \cdot \xi} \, e^{-\pi t^2 |\xi|^2} \\
&= \int \widehat{f}(\xi)\phi(\xi) \\
&= \int f(\xi)\widehat{\phi}(\xi) \\
&= \int f(\xi)\widehat{\widehat{g}}(\xi - x) \\
&= \int f(\xi)g_t(x - \xi) \, d\xi \\
&= \int f(y - x)g_t(y) \, dy \quad (\xi = y - x) \\
&= (f * g_t) \\
&\longrightarrow f \text{ in } L^1 \text{ as } t \longrightarrow 0,
\end{aligned}$$

---

but we also have

$$\lim_{t \longrightarrow 0} I_t(x) = \lim_{t \longrightarrow 0} \int \widehat{f}(\xi) \, e^{2\pi i x \cdot \xi} \, e^{-\pi t^2 |\xi|^2}$$

$$= \lim_{t \longrightarrow 0} \int \widehat{f}(\xi) \phi(\xi)$$

$$=_{DCT} \int \widehat{f}(\xi) \lim_{t \to 0} \phi(\xi)$$

$$= \int \widehat{f}(\xi) \, e^{2\pi i x \cdot \xi}$$

.

So

$$I_t(x) \longrightarrow \int \widehat{f}(\xi) \, e^{2\pi i x \cdot \xi} \quad \text{pointwise and} \quad \|I_t(x) - f(x)\|_1 \longrightarrow 0.$$

So there is a subsequence $I_{t_n}$ such that $I_{t_n}(x) \longrightarrow f(x)$ almost everywhere, so $f(x) = \int \widehat{f}(\xi) \, e^{2\pi i x \cdot \xi}$ almost everywhere by uniqueness of limits. $\blacksquare$

## 7.3 3: Hilbert Spaces

### 7.3.1 a

**Theorem (Bessel's Inequality):**

$$\left\| x - \sum_{n=1}^{N} \langle x, u_n \rangle \, u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n \rangle|^2$$

and thus

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \le \|x\|^2$$

*Proof:* Let $S_N = \sum\limits_{n=1}^{N} \langle x, \ u_n \rangle u_n$

$$
\begin{aligned}
\|x - S_N\|^2 &= \langle x - S_n, \ x - S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\langle x, \ S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\left\langle x, \ \sum_{n=1}^{N} \langle x, \ u_n \rangle u_n \right\rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^{N} \langle x, \ \langle x, \ u_n \rangle u_n \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^{N} \overline{\langle x, \ u_n \rangle} \langle x, \ u_n \rangle \\
&= \|x\|^2 + \left\|\sum_{n=1}^{N} \langle x, \ u_n \rangle u_n\right\|^2 - 2 \sum_{n=1}^{N} |\langle x, \ u_n \rangle|^2 \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x, \ u_n \rangle|^2 - 2 \sum_{n=1}^{N} |\langle x, \ u_n \rangle|^2 \\
&= \|x\|^2 - \sum_{n=1}^{N} |\langle x, \ u_n \rangle|^2.
\end{aligned}
$$

And by continuity of the norm and inner product, we have

$$
\lim_{N \to \infty} \|x - S_N\|^2 = \lim_{N \to \infty} \|x\|^2 - \sum_{n=1}^{N} |\langle x, \ u_n \rangle|^2
$$

$$
\implies \left\|x - \lim_{N \to \infty} S_N\right\|^2 = \|x\|^2 - \lim_{N \to \infty} \sum_{n=1}^{N} |\langle x, \ u_n \rangle|^2
$$

$$
\implies \left\|x - \sum_{n=1}^{\infty} \langle x, \ u_n \rangle u_n\right\|^2 = \|x\|^2 - \sum_{n=1}^{\infty} |\langle x, \ u_n \rangle|^2.
$$

Then noting that $0 \le \|x - S_N\|^2$, we have

$$
0 \le \|x\|^2 - \sum_{n=1}^{\infty} |\langle x, \ u_n \rangle|^2
$$

$$
\implies \sum_{n=1}^{\infty} |\langle x, \ u_n \rangle|^2 \le \|x\|^2 \blacksquare.
$$

### 7.3.2  b

- Take $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.

- Define $x = \lim\limits_{N \to \infty} S_N$ where $S_N = \sum\limits_{k=1}^{N} a_k u_k$

- $\{S_N\}$ is Cauchy and $H$ is complete, so $x \in H$.

- By construction, $\langle x,\ u_n \rangle = \left\langle \sum_k a_k u_k,\ u_n \right\rangle = \sum_k a_k \langle u_k,\ u_n \rangle = a_n$ since the $u_k$ are all orthogonal.

- $\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$ by Pythagoras since the $u_k$ are normal.

### 7.3.3 c

Let $x$ and $u_n$ be arbitrary. Then

$$\left\langle x - \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k,\ u_n \right\rangle = \langle x,\ u_n \rangle - \left\langle \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k,\ u_n \right\rangle$$

$$= \langle x,\ u_n \rangle - \sum_{k=1}^{\infty} \langle \langle x,\ u_k \rangle u_k,\ u_n \rangle$$

$$= \langle x,\ u_n \rangle - \sum_{k=1}^{\infty} \langle x,\ u_k \rangle \langle u_k,\ u_n \rangle$$

$$= \langle x,\ u_n \rangle - \langle x,\ u_n \rangle = 0$$

$$\implies x - \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k = 0 \quad \text{by completeness.}$$

So

$$x = \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k \implies \|x\|^2 = \sum_{k=1}^{\infty} |\langle x,\ u_k \rangle|^2. \blacksquare$$

## 7.4  4: Lp Spaces

p-test for integrals:

$$\int_0^1 x^{-p} < \infty \iff p < 1$$
$$\int_1^{\infty} x^{-p} < \infty \iff p > 1.$$

Yields a general technique: break integrals apart at $x = 1$.

Inclusions and relationships:

$$m(X) < \infty \implies L^{\infty} \subset L^2 \subset L^1$$
$$\ell^2(\mathbb{Z}) \subset \ell^1(\mathbb{Z}) \subset \ell^{\infty}(\mathbb{Z}).$$

### 7.4.1 a

**Theorem (Holder's Inequality):**

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

*Proof:*

It suffices to show this when $\|f\|_p = \|g\|_q = 1$, since

$$\|fg\|_1 \leq \|f\|_p \|f\|_q \iff \int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \leq 1.$$

Using $AB \leq \frac{1}{p} A^p + \frac{1}{q} B^q$, we have

$$\int |f||g| \leq \int \frac{|f|^p}{p} \frac{|g|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1 \blacksquare.$$

**Theorem (Minkowski's Inequality):**

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

*Proof:*

We first note

$$|f + g|^p = |f + g||f + g|^{p-1} \leq (|f| + |g|) |f + g|^{p-1}.$$

Note that if $p, q$ are conjugate exponents then

$$\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$$
$$q = \frac{p}{p-1}.$$

Then taking integrals yields

$$
\begin{aligned}
\|f + g\|_p^p &= \int |f + g|^p \\
&\leq \int (|f| + |g|) |f + g|^{p-1} \\
&= \int |f||f + g|^{p-1} + \int |g||f + g|^{p-1} \\
&= \left\| f(f + g)^{p-1} \right\|_1 + \left\| g(f + g)^{p-1} \right\|_1 \\
&\leq \|f\|_p \left\| (f + g)^{p-1} \right\|_q + \|g\|_p \left\| (f + g)^{p-1} \right\|_q \\
&= \left( \|f\|_p + \|g\|_p \right) \left\| (f + g)^{p-1} \right\|_q \\
&= \left( \|f\|_p + \|g\|_p \right) \left( \int |f + g|^{(p-1)q} \right)^{\frac{1}{q}} \\
&= \left( \|f\|_p + \|g\|_p \right) \left( \int |f + g|^p \right)^{1 - \frac{1}{p}} \\
&= \left( \|f\|_p + \|g\|_p \right) \frac{\int |f + g|^p}{\left( \int |f + g|^p \right)^{\frac{1}{p}}} \\
&= \left( \|f\|_p + \|g\|_p \right) \frac{\|f + g\|_p^p}{\|f + g\|_p}
\end{aligned}
$$

and canceling common terms yields

$$
1 \leq \left( \|f\|_p + \|g\|_p \right) \frac{1}{\|f + g\|_p}
$$
$$
\implies \|f + g\|_p \leq \|f\|_p + \|g\|_p \blacksquare.
$$

### 7.4.2 c

**Definition (Infinity Norm):**

$$
L^\infty(X) = \left\{ f : X \longrightarrow \mathbb{C} \ \middle| \ \|f\|_\infty < \infty \right\}
$$
$$
\text{where}
$$
$$
\|f\|_\infty = \inf_{\alpha \geq 0} \left\{ \alpha \ \middle| \ m\{|f| \geq \alpha\} = 0 \right\}.
$$

**Theorem:**

$$
m(X) < \infty \implies \lim_{p \longrightarrow \infty} \|f\|_p = \|f\|_\infty.
$$

*Proof:* Let $M = \|f\|_\infty$. For any $L < M$, let $S = \{|f| \geq L\}$. Then $m(S) > 0$ and

$$\|f\|_p = \left( \int_X |f|^p \right)^{\frac{1}{p}}$$

$$\geq \left( \int_S |f|^p \right)^{\frac{1}{p}}$$

$$\geq L \, m(S)^{\frac{1}{p}} \xrightarrow{\ p \to \infty\ } L$$

$$\implies \liminf_p \|f\|_p \geq M.$$

We also have

$$\|f\|_p = \left( \int_X |f|^p \right)^{\frac{1}{p}}$$

$$\leq \left( \int_X M^p \right)^{\frac{1}{p}}$$

$$= M \, m(X)^{\frac{1}{p}} \xrightarrow{\ p \to \infty\ } M$$

$$\implies \limsup_p \|f\|_p \leq M \blacksquare.$$

Note: this doesn't help with this problem at all.

Solution:

$$\int f^p = \int_{x \leq 1} f^p + \int_{x=1} f^p + \int_{x \geq 1} f^p$$

$$= \int_{x \leq 1} f^p + \int_{x=1} 1 + \int_{x \geq 1} f^p$$

$$= \int_{x \leq 1} f^p + m(\{f = 1\}) + \int_{x \geq 1} f^p$$

$$\xrightarrow{\ p \to \infty\ } 0 + m(\{f = 1\}) + \begin{cases} 0 & m(\{x \geq 1\}) = 0 \\ \infty & m(\{x \geq 1\}) > 0. \end{cases}$$

## 7.5  5: Dual Spaces

**Definition:** A map $L : X \longrightarrow \mathbb{C}$ is a linear functional iff

$$L(\alpha \mathbf{x} + \mathbf{y}) = \alpha L(\mathbf{x}) + L(\mathbf{y}).$$

**Theorem (Riesz Representation for Hilbert Spaces):** If $\Lambda$ is a continuous linear functional on a Hilbert space $H$, then there exists a unique $y \in H$ such that

$$\forall x \in H, \quad \Lambda(x) = \langle x, \, y \rangle.$$

*Proof:*

- Define $M := \ker \Lambda$.
- Then $M$ is a closed subspace and so $H = M \oplus M^\perp$
- There is some $z \in M^\perp$ such that $\|z\| = 1$.
- Set $u := \Lambda(x)z - \Lambda(z)x$
- Check

$$\Lambda(u) = \Lambda(\Lambda(x)z - \Lambda(z)x) = \Lambda(x)\Lambda(z) - \Lambda(z)\Lambda(x) = 0 \implies u \in M$$

- Compute

$$
\begin{aligned}
0 &= \langle u, \ z \rangle \\
&= \langle \Lambda(x)z - \Lambda(z)x, \ z \rangle \\
&= \langle \Lambda(x)z, \ z \rangle - \langle \Lambda(z)x, \ z \rangle \\
&= \Lambda(x)\langle z, \ z \rangle - \Lambda(z)\langle x, \ z \rangle \\
&= \Lambda(x)\|z\|^2 - \Lambda(z)\langle x, \ z \rangle \\
&= \Lambda(x) - \Lambda(z)\langle x, \ z \rangle \\
&= \Lambda(x) - \left\langle x, \ \overline{\Lambda(z)}z \right\rangle,
\end{aligned}
$$

- Choose $y := \overline{\Lambda(z)}z$.
- Check uniqueness:

$$
\begin{aligned}
\langle x, \ y \rangle &= \langle x, \ y' \rangle \quad \forall x \\
\implies \langle x, \ y - y' \rangle &= 0 \quad \forall x \\
\implies \langle y - y', \ y - y' \rangle &= 0 \\
\implies \|y - y'\| &= 0 \\
\implies y - y' = \mathbf{0} &\implies y = y'.
\end{aligned}
$$

### 7.5.1  b

**Theorem (Continuous iff Bounded):** Let $L : X \longrightarrow \mathbb{C}$ be a linear functional, then the following are equivalent:

1. $L$ is continuous
2. $L$ is continuous at zero
3. $L$ is bounded, i.e. $\exists c \geq 0 \ \big| \ |L(x)| \leq c\|x\|$ for all $x \in H$

$2 \implies 3$: Choose $\delta < 1$ such that

$$\|x\| \leq \delta \implies |L(x)| < 1.$$

Then

$$|L(x)| = \left| L\left( \frac{\|x\|}{\delta} \frac{\delta}{\|x\|} x \right) \right|$$
$$= \frac{\|x\|}{\delta} \left| L\left( \delta \frac{x}{\|x\|} \right) \right|$$
$$\leq \frac{\|x\|}{\delta} 1,$$

so we can take $c = \dfrac{1}{\delta}$. ∎

$3 \implies 1$:

We have $|L(x - y)| \leq c\|x - y\|$, so given $\varepsilon \geq 0$ simply choose $\delta = \dfrac{\varepsilon}{c}$.

### 7.5.2 c

**Definition (Dual Space):**

$$X^\vee := \left\{ L : X \longrightarrow \mathbb{C} \;\middle|\; L \text{ is continuous} \right\}$$

**Definition (Operator Norm):**

$$\|L\|_{X^\vee} := \sup_{\substack{x \in X \\ \|x\| = 1}} |L(x)|$$

**Theorem: (Operator Norm is a Norm)**

*Proof:* The only nontrivial property is the triangle inequality, but

$$\|L_1 + L_2\| = \sup |L_1(x) + L_2(x)| \leq \sup L_1(x) + \sup L_2(x) = \|L_1\| + \|L_2\|.$$

**Theorem (Completeness in Operator Norm):**

$X^\vee$ equipped with the operator norm is a Banach space.

*Proof:*

- Let $\{L_n\}$ be Cauchy in $X^\vee$.
- Then for all $x \in C$, $\{L_n(x)\} \subset \mathbb{C}$ is Cauchy and converges to something denoted $L(x)$.
- Need to show $L$ is continuous and $\|L_n - L\| \longrightarrow 0$.
- Since $\{L_n\}$ is Cauchy in $X^\vee$, choose $N$ large enough so that

$$n, m \geq N \implies \|L_n - L_m\| < \varepsilon \implies |L_m(x) - L_n(x)| < \varepsilon \quad \forall x \;\middle|\; \|x\| = 1.$$

- Take $n \longrightarrow \infty$ to obtain

$$m \geq N \implies |L_m(x) - L(x)| < \varepsilon \quad \forall x \;\middle|\; \|x\| = 1$$
$$\implies \|L_m - L\| < \varepsilon \longrightarrow 0.$$

- Continuity:

$$\begin{aligned}
|L(x)| &= |L(x) - L_n(x) + L_n(x)| \\
&\leq |L(x) - L_n(x)| + |L_n(x)| \\
&\leq \varepsilon \|x\| + c\|x\| \\
&= (\varepsilon + c)\|x\| \blacksquare.
\end{aligned}$$

# 8 Exam 2 (2018)

Link to PDF File

# 9 Exam 2 (2014)

Link to PDF File

# 10 Qual: Fall 2019

## 10.1 1

See phone photo?

## 10.2 2

DCT?

## 10.3 3

"Follow your nose."

## 10.4 4

See Problem Set 8.

**Bessel's Inequality:** For any orthonormal set in a Hilbert space (not necessarily a basis), we have

$$\sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2 \leq \|x\|^2$$

*Proof:*

$$0 \leq \left\| x - \sum_{k=1}^{n} \langle x, e_k \rangle \, e_k \right\|^2$$

**Corollary (Parseval's Identity):** If span $\{u_n\}$ is dense in $\mathcal{H}$, so it is a basis, then this is an equality.

**Riesz-Fischer:** Let $U = \{u_n\}_{n=1}^{\infty}$ be an orthonormal set (not necessarily a basis), then

1. There is an isometric surjection

$$\mathcal{H} \longrightarrow \ell^2(\mathbb{N})$$
$$\mathbf{x} \mapsto \{\langle \mathbf{x}, \ \mathbf{u}_n \rangle\}_{n=1}^{\infty}$$

i.e. if $\{a_n\} \in \ell^2(\mathbb{N})$, so $\sum |a_n|^2 < \infty$, then there exists a $\mathbf{x} \in \mathcal{H}$ such that

$$a_n = \langle \mathbf{x}, \ \mathbf{u}_n \rangle \quad \forall n.$$

2. $\mathbf{x}$ can be chosen such that

$$\|\mathbf{x}\|^2 = \sum |a_n|^2$$

Note: the choice of $\mathbf{x}$ is unique $\iff$ $\{u_n\}$ is **complete**, i.e. $\langle \mathbf{x}, \ \mathbf{u}_n \rangle = 0$ for all $n$ implies $\mathbf{x} = \mathbf{0}$.

*Proof:*

- Given $\{a_n\}$, define $S_N = \sum\limits^{N} a_n \mathbf{u}_n$.
- $S_N$ is Cauchy in $\mathcal{H}$ and so $S_N \longrightarrow \mathbf{x}$ for some $\mathbf{x} \in \mathcal{H}$.
- $\langle x, \ u_n \rangle = \langle x - S_N, \ u_n \rangle + \langle S_N, \ u_n \rangle \longrightarrow a_n$
- By construction, $\|x - S_N\|^2 = \|x\|^2 - \sum\limits^{N} |a_n|^2 \longrightarrow 0$, so $\|x\|^2 = \sum\limits^{\infty} |a_n|^2$.

## 10.5  5

See Problem Set 5.

**Heine-Cantor theorem:** Every continuous function on a compact set is uniformly continuous.

Uniform continuity:

$$\forall \varepsilon \quad \exists \delta(\varepsilon) \ \bigg| \quad \forall x, y, \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$
$$\iff \forall \varepsilon \quad \exists \delta(\varepsilon) \ \bigg| \quad \forall x, y, \quad |y| < \delta \implies |f(x - y) - f(y)| < \varepsilon$$

.

Fubini-Tonelli interchange of integrals, where the change of bounds becomes very important.

Continuity in $L^1$:

$$\lim_{y \longrightarrow 0} \|\tau_y f - f\|_1 = 0.$$

## 10.6  Definitions

- Banach Space
- $L^p$

## 10.7 Useful Results

- Cauchy-Schwarz
- Young's Inequality
- Holder's Inequality
- Minkowski's Inequality
- Jensen's Inequality:

$$r^{-1} := p^{-1} + q^{-1} - 1 \implies \|f * g\|_r \leq \|f\|_p \|g\|_q$$

  - Useful variant - take $q = 1$ to get $\|f * g\|_p \leq \|f\|_p \|g\|_1$
  - Take $p = 1$ to show $L_1$ is closed under $*$.
- The Riemann-Lebesgue Lemma
- Proving that $\delta \notin L_1(\mathbb{R})$ and that there is no such identity
  - Rather, is a distribution or measure that *acts* on $f$ and satisfies $f(x) \int_{\mathbb{R}} f(t)\delta(t - x) \, dt$
- Fubini's Theorem
- Density Results:
  - $C_c(\mathbb{R}) \subset C_0(\mathbb{R})$
- $C_c(\mathbb{R}) \bigcap C^\infty(\mathbb{R}) \neq \emptyset$, e.g. take $f(x) = e^{\frac{-1}{x^2}} \chi_{(0,\infty)}(x)$.
- The Banach Algebra $L^1(\mathbb{R})$ is not a principal ideal domain.
- Every locally compact abelian group $G$ has a unique Borel measure (up to scaling) that is positive, regular, translation-invariant (the Haar measure).
  - For $\mathbb{R}, (S_1)^2$, equal to the Lebesgue measure. For $\mathbb{Z}$, the counting measure.