Topology Qualifying Exam Notes

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Tuesday 9^{th} June, 2020

Contents

1 Definitions

- Topology: Closed under arbitrary unions and finite intersections.
- Basis: A subset $\{B_i\}$ is a basis iff
  - $x \in X \implies x \in B_i$ for some i.
  - $x \in B_i \bigcap B_j \implies x \in B_k \subset B_i \bigcap B_k$.
  - Topology generated by this basis: $x \in N_x \implies x \in B_i \subset N_x$ for some i.
- Dense: A subset $Q \subset X$ is dense iff $y \in N_y \subset X \implies N_y \bigcap Q \neq \emptyset$ iff $\overline{Q} = X$.
- Neighborhood: A neighborhood of a point x is any open set containing x.
- Hausdorff
- Second Countable: admits a countable basis.
- Closed (several characterizations)
- Closure in a subspace: $Y \subset X \implies \mathrm{cl}_Y(A) \coloneqq \mathrm{cl}_X(A) \bigcap Y$.
- Bounded
- Compact: A topological space (X, τ) is **compact** if every open cover has a *finite* subcover.

  That is, if $\left\{U_j \;\middle|\; j \in J\right\} \subset \tau$ is a collection of open sets such that $X \subseteq \bigcup_{j \in J} U_j$, then there exists a *finite* subset $J' \subset J$ such that $X \subseteq \bigcup_{j \in J'} U_j$.
- Locally compact For every $x \in X$, there exists a $K_x \ni x$ such that K_x is compact.
- Connected: There does not exist a disconnecting set $X = A \coprod B$ such that $\emptyset \neq A, B \subsetneq$, i.e. X is the union of two proper disjoint nonempty sets.

  Equivalently, X contains no proper nonempty clopen sets.
  - Additional condition for a subspace $Y \subset X$: $\mathrm{cl}_Y(A) \bigcap V = A \bigcap \mathrm{cl}_Y(B) = \emptyset$.
- Locally connected: A space is locally connected at a point x iff $\forall N_x \ni x$, there exists a $U \subset N_x$ containing x that is connected.
- Retract: A subspace $A \subset X$ is a *retract* of X iff there exists a continuous map $f : X \longrightarrow A$ such that $f \Big|_A = \mathrm{id}_A$. Equivalently it is a *left* inverse to the inclusion.
- Uniform Continuity: For $f : (X, d_x) \longrightarrow (Y, d_Y)$ metric spaces,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \quad d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

- Lebesgue number: For (X, d) a compact metric space and $\{U_\alpha\} \rightrightarrows X$, there exist $\delta_L > 0$ such that

$$A \subset X,\ \mathrm{diam}(A) < \delta_L \implies A \subseteq U_\alpha \text{ for some } \alpha.$$

- Paracompact

- Components: Set $x \sim y$ iff there exists a connected set $U \ni x, y$ and take equivalence classes.
- Path Components: Set $x \sim y$ iff there exists a path-connected set $U \ni x, y$ and take equivalence classes.
- Separable: Contains a countable dense subset.
- Limit Point: For $A \subset X$, x is a limit point of A if every punctured neighborhood P_x of x satisfies $P_x \bigcap A \neq \emptyset$, i.e. every neighborhood of x intersects A in some point other than x itself.

  Equivalently, x is a limit point of A iff $x \in \mathrm{cl}_X(A \setminus \{x\})$.

1.1 Algebraic

1.1.1 Homotopy

Todo: Merge the two van Kampen theorems.

Theorem 1.1 ***(Van Kampen).***
The pushout is the northwest colimit of the following diagram

$$\begin{tikzcd} A\coprod_Z B & A \arrow[l] \\ B \arrow[u] & Z \arrow[l, hook', "\iota_B"] \arrow[u, hook', "\iota_A"'] \end{tikzcd}$$

For groups, the pushout is given by the amalgamated free product: if $A = \left\langle G_A \mid R_A \right\rangle, B = \left\langle G_B \mid R_B \right\rangle$, then

$$A *_Z B = \left\langle G_A, G_B \mid R_A, R_B, T \right\rangle$$

where T is a set of relations given by

$$T = \left\{ \iota_A(z)\iota_B(z)^{-1} \mid z \in Z \right\}.$$

Suppose $X = U_1 \bigcup U_2$ such that $U_1 \bigcap U_2 \neq \emptyset$ is **path connected** (necessary condition). Then taking $x_0 \in U := U_1 \bigcap U_2$ yields a pushout of fundamental groups

$$\pi_1(X; x_0) = \pi_1(U_1; x_0) *_{\pi_1(U; x_0)} \pi_1(U_2; x_0).$$

Theorem 1.2 ***(Van Kampen).***
If $X = U \bigcup V$ where $U, V, U \bigcap V$ are all path-connected then

$$\pi_1(X) = \pi_1 U *_{\pi_1(U \bigcap V)} \pi_1 V,$$

where the amalgamated product can be computed as follows: If we have presentations

$$\pi_1(U,w) = \left\langle u_1, \cdots, u_k \,\middle|\, \alpha_1, \cdots, \alpha_l \right\rangle$$
$$\pi_1(V,w) = \left\langle v_1, \cdots, v_m \,\middle|\, \beta_1, \cdots, \beta_n \right\rangle$$
$$\pi_1(U\cap V,w) = \left\langle w_1, \cdots, w_p \,\middle|\, \gamma_1, \cdots, \gamma_q \right\rangle$$

then

$$\begin{aligned}\pi_1(X,w) &= \langle u_1, \cdots, u_k, v_1, \cdots, v_m \rangle \\ &\qquad \bmod \left\langle \alpha_1, \cdots, \alpha_l, \beta_1, \cdots, \beta_n, I\left(w_1\right) J\left(w_1\right)^{-1}, \cdots, I\left(w_p\right) J\left(w_p\right)^{-1} \right\rangle \\ &= \frac{\pi_1(U) * \pi_1(B)}{\left\langle \left\{ I(w_i) J(w_i)^{-1} \,\middle|\, 1 \le i \le p \right\} \right\rangle}\end{aligned}$$

where

$$I : \pi_1(U\cap V, w) \to \pi_1(U,w)$$
$$J : \pi_1(U\cap V, w) \to \pi_1(V,w).$$

Theorem 1.3 ***(Seifert-van Kampen Theorem).***
Suppose $X = U_1 \bigcup U_2$ where $U \coloneqq U_1 \bigcap U_2 \neq \emptyset$ is path-connected, and let $\{\mathrm{pt}\} \in U$. Then the maps $i_1 : U_1 \longrightarrow X$ and $i_2 : U_2 \longrightarrow X$ induce the following group homomorphisms:

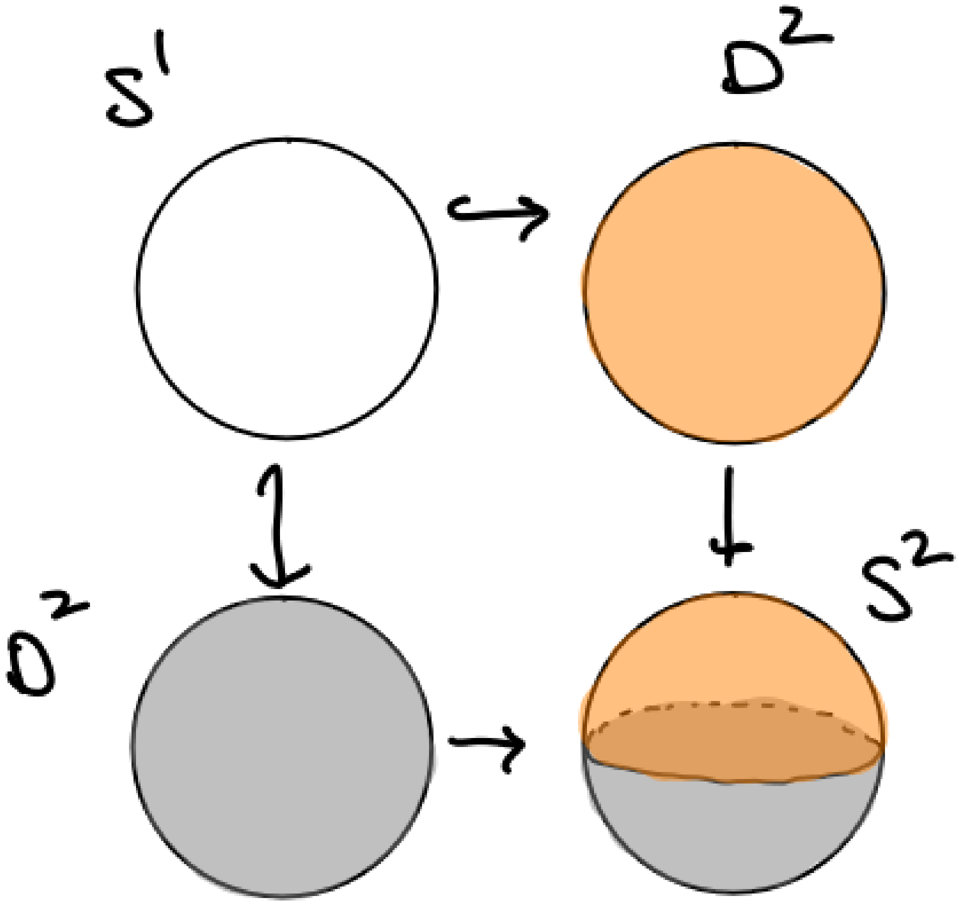
$$i_1^* : \pi_1(U_1, \{\mathrm{pt}\}) \longrightarrow \pi_1(X, \{\mathrm{pt}\})$$
$$i_2^* : \pi_1(U_2, \{\mathrm{pt}\}) \longrightarrow \pi_1(X, \{\mathrm{pt}\})$$

and letting $P = \pi_1(U), \{\mathrm{pt}\}$, there is a natural isomorphism

$$\pi_1(X, \{\mathrm{pt}\}) \cong \pi_1(U_1, \{\mathrm{pt}\}) *_P \pi_1(U_2, \{\mathrm{pt}\})$$

where $*_P$ is the amalgamated free product over P.

(Todo: formulate in terms of pushouts)



Examples

Example 1.1.
$A = \mathbb{Z}/4\mathbb{Z} = \left\langle x \;\middle|\; x^4 \right\rangle, B = \mathbb{Z}/6\mathbb{Z} = \left\langle y \;\middle|\; x^6 \right\rangle, Z = \mathbb{Z}/2\mathbb{Z} = \left\langle z \;\middle|\; z^2 \right\rangle$. Then we can identify Z as a subgroup of A, B using $\iota_A(z) = x^2$ and $\iota_B(z) = y^3$. So

$$A *_Z B = \left\langle x, y \;\middle|\; x^4, y^6, x^2 y^{-3} \right\rangle$$

.

- Computing $\pi_1(S^1 \vee S^1)$
- Computing $\pi_1(S^1 \times S^1)$
- Counterexample when $U \cap V$ isn't path-connected: S^1 with U, V neighborhoods of the poles.

1.1.2 Homology

Useful fact: since $\mathbb{Z}$ is free, any exact sequence of the form $0 \longrightarrow \mathbb{Z}^n \longrightarrow A \longrightarrow \mathbb{Z}^m \longrightarrow 0$ splits and $A \cong \mathbb{Z}^n \times \mathbb{Z}^m$.

Useful fact: $\tilde{H}_*(A \vee B) \cong H_*(A) \times H_*(B)$.

Theorem 1.4 ***(Mayer Vietoris).***
Let $X = A^\circ \bigcup B^\circ$; then there is a SES of chain complexes

$$0 \longrightarrow C_n(A \bigcap B) \xrightarrow{x \mapsto (x,-x)} C_n(A) \oplus C_n(B) \xrightarrow{(x,y) \mapsto x+y} C_n(A+B) \longrightarrow 0$$

where $C_n(A+B)$ denotes the chains that are sums of chains in A and chains in B. This yields a LES in homology:

$$\cdots \longrightarrow H_n(A \bigcap B) \xrightarrow{x \mapsto (x,-x)} H_n(A) \oplus H_n(B) \xrightarrow{(x,y) \mapsto x+y} H_n(X) \longrightarrow \cdots.$$

2 Theorems

2.1 Point-Set

Theorem 2.1.
$U \subset X$ a Hausdorff spaces is closed $\iff$ it is compact.

Theorem 2.2 ***(Cantor's Intersection Theorem).***
A bounded collection of nested closed sets $C_1 \supset C_2 \supset \cdots$ in a metric space X is nonempty $\iff$ X is complete.

- Tube lemma
- Properties pushed forward through continuous maps:
  - Compactness?
  - Connectedness (when surjective)
  - Separability
  - Density **only when** f is surjective
  - **Not** openness
  - **Not** closedness
- A retract of a Hausdorff/connected/compact space is closed/connected/compact respectively.

Proposition 2.3.
A continuous function on a compact set is uniformly continuous.

Proof .
Take $\left\{ B_{\frac{\varepsilon}{2}}(y) \;\middle|\; y \in Y \right\} \rightrightarrows Y$, pull back to an open cover of X, has Lebesgue number $\delta_L > 0$, then $x' \in B_{\delta_L}(x) \implies f(x), f(x') \in B_{\frac{\varepsilon}{2}}(y)$ for some y. ■

Corollary 2.4.
Lipschitz continuity implies uniform continuity (take $\delta = \varepsilon/C$)

Counterexample to converse: $f(x) = \sqrt{x}$ on $[0,1]$ has unbounded derivative.

Theorem 2.5 ***(Extreme Value Theorem).***
For $f : X \longrightarrow Y$ continuous with X compact and Y ordered in the order topology, there exist points $c, d \in X$ such that $f(x) \in [f(c), f(d)]$ for every x.

Theorem 2.6.
Points are closed in T_1 spaces.

Theorem 2.7.
A metric space X is sequentially compact iff it is complete and totally bounded.

Theorem 2.8.
A metric space is totally bounded iff every sequence has a Cauchy subsequence.

Theorem 2.9.
A metric space is compact iff it is complete and totally bounded.

Theorem 2.10 ***(Baire).***
If X is a complete metric space, then the intersection of countably many dense open sets is dense in X.

Theorem 2.11.
A continuous bijective open map is a homeomorphism.

Theorem 2.12.
A closed subset A of a compact set B is compact.

Proof .

- Let $\{A_i\} \rightrightarrows A$ be a covering of A by sets open in A.
- Each $A_i = B_i \bigcap A$ for some B_i open in B (definition of subspace topology)
- Define $V = \{B_i\}$, then $V \rightrightarrows A$ is an open cover.
- Since A is closed, $W := B \setminus A$ is open
- Then $V \bigcup W$ is an open cover of B, and has a finite subcover $\{V_i\}$
- Then $\left\{V_i \bigcap A\right\}$ is a finite open cover of A.

■

Theorem 2.13.
The continuous image of a compact set is compact.

Theorem 2.14.
A closed subset of a Hausdorff space is compact.

Theorem 2.15.
A continuous bijection $f : X \longrightarrow Y$ where X is compact and Y is Hausdorff is an open map and hence a homeomorphism.

3 Examples

3.1 Common Spaces and Operations

Point-Set:

- Finite discrete sets with the discrete topology
- Subspaces of $\mathbb{R}$: $(a, b), (a, b], (a, \infty)$, etc.
  - $\{0\} \bigcup \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{Z}^{\geq 1} \right\}$
- $\mathbb{Q}$
- The topologist's sine curve
- One-point compactifications
- $\mathbb{R}^{\omega}$
- Hawaiian earring
- Cantor set

Non-Hausdorff spaces:

- The cofinite topology on any infinite set.
- $\mathbb{R}/\mathbb{Q}$
- The line with two origins.

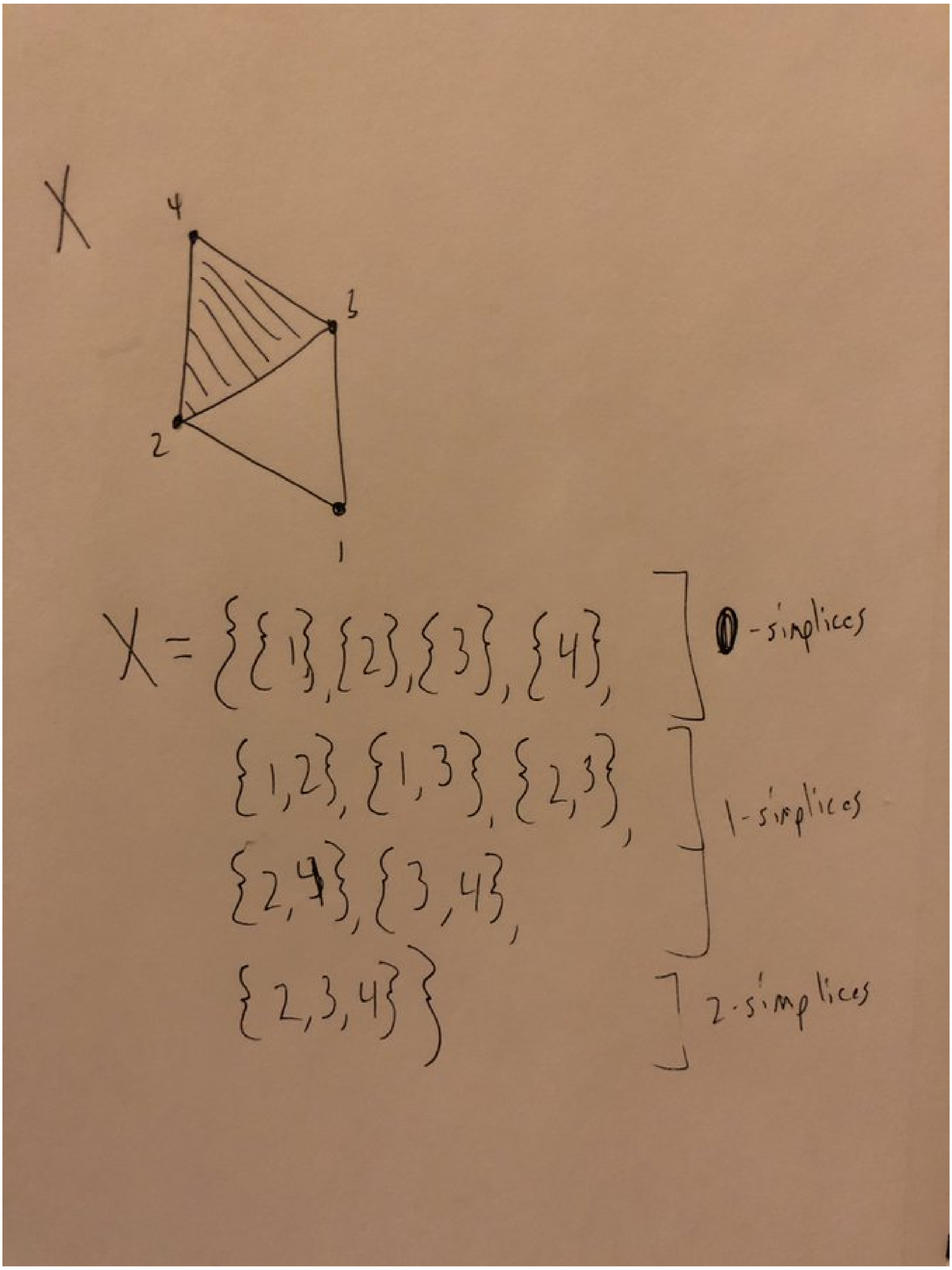
General Spaces:

$$S^n, \mathbb{D}^n, T^n, \mathbb{RP}^n, \mathbb{CP}^n, \mathbb{M}, \mathbb{K}, \Sigma_g, \mathbb{RP}^{\infty}, \mathbb{CP}^{\infty}.$$

"Constructed" Spaces

- Knot complements in S^3
- Covering spaces (hyperbolic geometry)
- Lens spaces
- Matrix groups
- Prism spaces
- Pair of pants
- Seifert surfaces
- Surgery
- Simplicial Complexes

– Nice minimal example:



Exotic/Pathological Spaces

- $\mathbb{HP}^n$

- Dunce Cap
- Horned sphere

Operations

- Cartesian product $A \times B$
- Wedge product $A \vee B$
- Connect Sum $A \# B$
- Quotienting A/B
- Puncturing $A \setminus \{a_i\}$
- Smash product
- Join
- Cones
- Suspension
- Loop space
- Identifying a finite number of points

3.2 Alternative Topologies

- Discrete
- Cofinite
- Discrete and Indiscrete
- Uniform

The cofinite topology:

- Non-Hausdorff
- Compact

The discrete topology:

- Discrete iff points are open
- Always Hausdorff
- Compact iff finite
- Totally disconnected
- If the domain, every map is continuous

The indiscrete topology:

- Only open sets are $\emptyset, X$
- Non-Hausdorff
- If the codomain, every map is continuous
- Compact

4 AT Summary

4.1 Conventions

- $\pi_0(X)$ is the set of path components of X, and I write $\pi_0(X) = \mathbb{Z}$ if X is path-connected (although it is not a group). Similarly, $H_0(X)$ is a free abelian group on the set of path components of X.

- Lists start at entry 1, since all spaces are connected here and thus $\pi_0 = H_0 = \mathbb{Z}$. That is,
  - $\pi_*(X) = [\pi_1(X), \pi_2(X), \pi_3(X), \cdots]$
  - $H_*(X) = [H_1(X), H_2(X), H_3(X), \cdots]$
- For a finite index set I, $\prod_I G = \bigoplus_I G$ in **Grp**, i.e. the finite direct product and finite direct sum coincide.

  Otherwise, if I is infinite, the direct sum requires cofinitely many zero entries (i.e. finitely many nonzero entries), so here we always use $\prod$.

  In other words, there is an injective map

$$\bigoplus_I G \hookrightarrow \prod_I G$$

  which is an isomorphism when $|I| < \infty$
- The free abelian group of rank n:

$$\mathbb{Z}^n := \prod_{i=1}^n \mathbb{Z} = \mathbb{Z} \times \mathbb{Z} \times \ldots \mathbb{Z}.$$

  - $x \in \mathbb{Z}^n = \langle a_1, \cdots, a_n \rangle \implies x = \sum_n c_i a_i$ for some $c_i \in \mathbb{Z}$, i.e. a_i form a basis.
  - Example: $x = 2a_1 + 4a_2 + a_1 - a_2 = 3a_1 + 3a_2$.
- The **free product** of n free abelian groups:

$$\mathbb{Z}^{*n} := \mathop{\ast}_{i=1}^n \mathbb{Z} = \mathbb{Z} * \mathbb{Z} * \ldots \mathbb{Z}$$

  This is a free *nonabelian* group on n generators.
  - $x \in \mathbb{Z}^{*n} = \langle a_1, \ldots, a_n \rangle$ implies that x is a finite word in the noncommuting symbols a_i^k for $k \in \mathbb{Z}$.
  - Example: $x = a_1^2 a_2^4 a_1 a_2^{-2}$
- $K(G, n)$ is an Eilenberg-MacLane space, the homotopy-unique space satisfying

$$\pi_k(K(G, n)) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

  - $K(\mathbb{Z}, 1) = S^1$
  - $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$
  - $K(\mathbb{Z}/2\mathbb{Z}, 1) = \mathbb{RP}^\infty$
- $M(G, n)$ is a Moore space, the homotopy-unique space satisfying

$$H_k(M(G, n); G) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

  - $M(\mathbb{Z}, n) = S^n$
  - $M(\mathbb{Z}/2\mathbb{Z}, 1) = \mathbb{RP}^2$

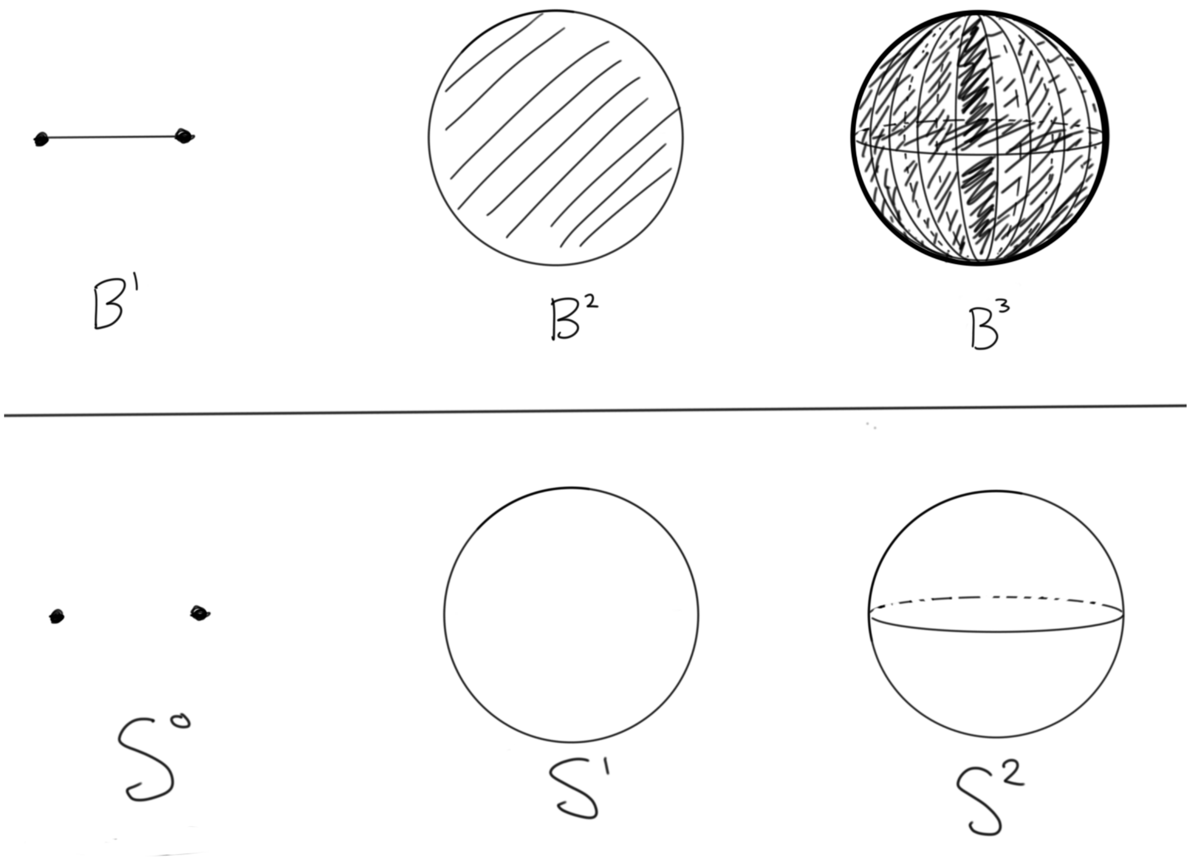


Figure 1: Low-Dimensional Spheres/Discs/Balls

  - $M(\mathbb{Z}/p\mathbb{Z}, n)$ is made by attaching e^{n+1} to S^n via a degree p map.

- $B^n = \left\{ \mathbf{v} \in \mathbb{R}^n \;\middle|\; \|\mathbf{v}\| \leq 1 \right\} \subset \mathbb{R}^n$
- $S^{n-1} = \partial B^n = \left\{ \mathbf{v} \in \mathbb{R}^n \;\middle|\; \|\mathbf{v}\| = 1 \right\} \subset \mathbb{R}^n$
- $\mathbb{RP}^n = S^n/S^0 = S^n/\mathbb{Z}/2\mathbb{Z}$
- $\mathbb{CP}^n = S^{2n+1}/S^1$
- $T^n = \prod_n S^1$ is the n-torus
- $D(k, X)$ is the space X with $k \in \mathbb{N}$ distinct points deleted, i.e. the punctured space $X - \{x_1, x_2, \ldots x_k\}$ where each $x_i \in X$.

4.2 Table of Homotopy and Homology Structures

X	$\pi_*(X)$	$H_*(X)$	CW Structure	$H^*(X)$
$\mathbb{R}^1$	0	0	$\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot x$	0

X	$\pi_*(X)$	$H_*(X)$	CW Structure	$H^*(X)$
$\mathbb{R}^n$	0	0	$(\mathbb{Z}\cdot 1+\mathbb{Z}\cdot x)^n$	0
$D(k,\mathbb{R}^n)$	$\pi_* \bigvee^k S^1$	$\bigoplus_k H_*M(\mathbb{Z},1)$	$1+kx$	?
B^n	$\pi_*(\mathbb{R}^n)$	$H_*(\mathbb{R}^n)$	$1+x^n+x^{n+1}$	0
S^n	$[0\ldots,\mathbb{Z},?\ldots]$	$H_*M(\mathbb{Z},n)$	$1+x^n$ or $\sum_{i=0}^n 2x^i$	$\mathbb{Z}[_n x]/(x^2)$
$D(k,S^n)$	$\pi_* \bigvee^{k-1} S^1$	$\bigoplus_{k-1} H_*M(\mathbb{Z},1)$	$1+(k-1)x^1$	?
T^2	$\pi_*S^1\times\pi_*S^1$	$(H_*M(\mathbb{Z},1))^2\times H_*M(\mathbb{Z},2)$	$1+2x+x^2$	$\Lambda(_1x_1,{}_1x_2)$
T^n	$\prod^n \pi_*S^1$	$\prod_{i=1}^n (H_*M(\mathbb{Z},i))^{\binom{n}{i}}$	$(1+x)^n$	$\Lambda(_1x_1,{}_1x_2,\ldots{}_1x_n)$
$D(k,T^n)$	$[0,0,0,0,\ldots]$?	$[0,0,0,0,\ldots]$?	$1+x$	?
$S^1\vee S^1$	$\pi_*S^1 * \pi_*S^1$	$(H_*M(\mathbb{Z},1))^2$	$1+2x$	?
$\bigvee^n S^1$	$*^n\pi_*S^1$	$\prod H_*M(\mathbb{Z},1)$	$1+x$	?
$\mathbb{RP}^1$	π_*S^1	$H_*M(\mathbb{Z},1)$	$1+x$	$_0\mathbb{Z}\times{}_1\mathbb{Z}$
$\mathbb{RP}^2$	$\pi_*K(\mathbb{Z}/2\mathbb{Z},1)+\pi_*S^2$	$H_*M(\mathbb{Z}/2\mathbb{Z},1)$	$1+x+x^2$	$_0\mathbb{Z}\times{}_2\mathbb{Z}/2\mathbb{Z}$
$\mathbb{RP}^3$	$\pi_*K(\mathbb{Z}/2\mathbb{Z},1)+\pi_*S^3$	$H_*M(\mathbb{Z}/2\mathbb{Z},1)+H_*M(\mathbb{Z},3)$	$1+x+x^2+x^3$	$_0\mathbb{Z}\times{}_2\mathbb{Z}/2\mathbb{Z}\times{}_3\mathbb{Z}$
$\mathbb{RP}^4$	$\pi_*K(\mathbb{Z}/2\mathbb{Z},1)+\pi_*S^4$	$H_*M(\mathbb{Z}/2\mathbb{Z},1)+H_*M(\mathbb{Z}/2\mathbb{Z},3)$	$1+x+x^2+x^3+x^4$	$_0\mathbb{Z}\times({}_2\mathbb{Z}/2\mathbb{Z})^2$
$\mathbb{RP}^n, n\geq 4$ even	$\pi_*K(\mathbb{Z}/2\mathbb{Z},1)+\pi_*S^n$	$\prod_{\text{odd } i<n} H_*M(\mathbb{Z}/2\mathbb{Z},i)$	$\sum_{i=1}^n x^i$	$_0\mathbb{Z}\times\prod_{i=1}^{n/2}{}_2\mathbb{Z}/2\mathbb{Z}$
$\mathbb{RP}^n, n\geq 4$ odd	$\pi_*K(\mathbb{Z}/2\mathbb{Z},1)+\pi_*S^n$	$\prod_{\text{odd } i\leq n-2} H_*M(\mathbb{Z}/2\mathbb{Z},i)\times H_*S^n$	$\sum_{i=1}^n x^i$	$H^*(\mathbb{RP}^{n-1})\times{}_n\mathbb{Z}$
$\mathbb{CP}^1$	$\pi_*K(\mathbb{Z},2)+\pi_*S^3$	H_*S^2	x^0+x^2	$\mathbb{Z}[_2x]/(_2x^2)$
$\mathbb{CP}^2$	$\pi_*K(\mathbb{Z},2)+\pi_*S^5$	$H_*S^2\times H_*S^4$	$x^0+x^2+x^4$	$\mathbb{Z}[_2x]/(_2x^3)$
$\mathbb{CP}^n, n\geq 2$	$\pi_*K(\mathbb{Z},2)+\pi_*S^{2n+1}$	$\prod_{i=1}^n H_*S^{2i}$	$\sum_{i=1}^n x^{2i}$	$\mathbb{Z}[_2x]/(_2x^{n+1})$
Mobius Band	π_*S^1	H_*S^1	$1+x$	?
Klein Bottle	$K(\mathbb{Z}\rtimes_{-1}\mathbb{Z},1)$	$H_*S^1\times H_*\mathbb{RP}^\infty$	$1+2x+x^2$	?

Facts used to compute the above table:

- $\mathbb{R}^n$ is a contractible space, and so $[S^m,\mathbb{R}^n]=0$ for all n,m which makes its homotopy groups all zero.
- $D(k,\mathbb{R}^n)=\mathbb{R}^n-\{x_1\ldots x_k\}\simeq\bigvee_{i=1}^k S^1$ by a deformation retract.
- $S^n\cong B^n/\partial B^n$ and employs an attaching map

$$\begin{aligned}\varphi:(D^n,\partial D^n)&\longrightarrow S^n\\ (D^n,\partial D^n)&\mapsto(e^n,e^0).\end{aligned}$$

- $B^n\simeq\mathbb{R}^n$ by normalizing vectors.
- Use the inclusion $S^n\hookrightarrow B^{n+1}$ as the attaching map.
- $\mathbb{CP}^1\cong S^2$.

- $\mathbb{RP}^1 \cong S^1$.
- Use $\left[\pi_1, \prod\right] = 0$ and the universal cover $\mathbb{R}^1 \twoheadrightarrow S^1$ to yield the cover $\mathbb{R}^n \twoheadrightarrow T^n$.
- Take the universal double cover $S^n \twoheadrightarrow^{\times 2} \mathbb{RP}^n$ to get equality in $\pi_{i\geq 2}$.
- Use $\mathbb{CP}^n = S^{2n+1}/S^1$
- Alternatively, the fundamental group is $\mathbb{Z} * \mathbb{Z}/bab^{-1}a$. Use the fact the $\tilde{K} = \mathbb{R}^2$.
- $M \simeq S^1$ by deformation-retracting onto the center circle.
- $D(1, S^n) \cong \mathbb{R}^n$ and thus $D(k, S^n) \cong D(k-1, \mathbb{R}^n) \cong \bigvee^{k-1} S^1$

4.3 Euler Characteristics

- Only surfaces with positive χ:
  - $\chi S^2 = 2$
  - $\chi \mathbb{RP}^2 = 1$
  - $\chi B^2 = 1$
- Manifolds with zero χ
  - $T^2, K, M, S^1 \times I$
- Manifolds with negative χ
  - $\Sigma_{g\geq 2}$ by $\chi(X) = 2 - 2g$.

4.4 Useful Facts and Techniques

- Homotopy Groups
  - Hurewicz map
- Homology
  - Mayer-Vietoris
    - $(X = A \bigcup B) \mapsto (\bigcap, \oplus, \bigcup)$ in homology
  - LES of a pair
    - $(A \hookrightarrow X) \mapsto (A, X, X/A)$
  - Excision
- $\pi_{i\geq 2}(X)$ is always abelian.
- The ranks of π_0 and H_0 are the number of path components, and $\pi_0(X) = \mathbb{Z}$ iff X is simply connected.
  - X simply connected $\implies \pi_k(X) \cong H_k(X)$ up to and including the first nonvanishing H_k
  - $H_1(X) = \mathrm{Ab}(\pi_1 X)$, the abelianization.
- General mantra: homotopy plays nicely with products, homology with wedge products.[1]

[1] More generally, in **Top**, we can look at $A \leftarrow \{\mathrm{pt}\} \longrightarrow B$ – then $A \times B$ is the pullback and $A \vee B$ is the pushout. In this case, homology $h : \mathbf{Top} \longrightarrow \mathbf{Grp}$ takes pushouts to pullbacks but doesn't behave well with pullbacks. Similarly, while π takes pullbacks to pullbacks, it doesn't behave nicely with pushouts.

In general, homotopy groups behave nicely under homotopy pull-backs (e.g., fibrations and products), but not homotopy push-outs (e.g., cofibrations and wedges). Homology is the opposite.

- $\pi_k \prod X = \prod \pi_k X$ by LES.[2]
- $H_k \prod X \neq \prod H_k X$ due to torsion.
  - Nice case: $H_k(A \times B) = \displaystyle\prod_{i+j=k} H_i A \otimes H_j B$ by Kunneth when all groups are torsion-free.[3]
- $H_k \bigvee X = \prod H_k X$ by Mayer-Vietoris.[4]
- $\pi_k \bigvee X \neq \prod \pi_k X$ (counterexample: $S^1 \vee S^2$)
  - Nice case: $\pi_1 \bigvee X = *\pi_1 X$ by Van Kampen.
- $\pi_i(\widehat{X}) \cong \pi_i(X)$ for $i \geq 2$ whenever $\widehat{X} \twoheadrightarrow X$ is a universal cover.
- Groups and Group Actions
  - $\pi_0(G) = G$ for G a discrete topological group.
  - $\pi_k(G/H) = \pi_k(G)$ if $\pi_k(H) = \pi_{k-1}(H) = 0$.
  - $\pi_1(X/G) = \pi_0(G)$ when G acts freely/transitively on X.
- Manifolds
  - $H^n(M^n) = \mathbb{Z}$ if M^n is orientable and zero if M^n is nonorientable.
  - Poincare Duality: $H_i M^n =\cong H^{n-i} M^n$ iff M^n is closed and orientable.

4.5 Other Interesting Things To Consider

- The "generalized uniform bouquet"? $\mathcal{B}^n(m) = \displaystyle\bigvee_{i=1}^{n} S^m$
- Lie Groups
  - The real general linear group, $\mathrm{GL}_n(\mathbb{R})$
    - The real special linear group $SL_n(\mathbb{R})$
    - The real orthogonal group, $O_n(\mathbb{R})$
      - The real special orthogonal group, $SO_n(\mathbb{R})$
    - The real unitary group, $U_n(\mathbb{R})$
      - The real special unitary group, $SU_n(\mathbb{R})$

[2] This follows because $X \times Y \twoheadrightarrow X$ is a fiber bundle, so use LES in homotopy and the fact that $\pi_{i \geq 2} \in \mathbf{Ab}$.

[3] The generalization of Kunneth is as follows: write $\mathcal{P}(n,k)$ be the set of partitions of n into k parts, i.e. $\mathbf{x} \in \mathcal{P}(n,k) \implies \mathbf{x} = (x_1, x_2, \ldots, x_k)$ where $\sum x_i = n$. Then

$$H_n\left(\prod_{j=1}^{k} X_j\right) = \bigoplus_{\mathbf{x} \in \mathcal{P}(n,k)} \bigotimes_{i=1}^{k} H_{x_i}(X_i).$$

[4] $\bigvee$ is the coproduct in the category $\mathbf{Top}_0$ of pointed topological spaces, and alternatively, $X \vee Y$ is the pushout in $\mathbf{Top}$ of $X \leftarrow \{\mathrm{pt}\} \longrightarrow Y$

    * The real symplectic group $Sp(n)$

- "Geometric" Stuff
  - Affine n-space over a field $\mathbb{A}^n(k) = k^n \rtimes GL_n(k)$
  - The projective space $\mathbb{P}^n(k)$
    * The projective linear group over a ring R, $PGL_n(R)$
    * The projective special linear group over a ring R, $PSL_n(R)$
    * The modular groups $PSL_n(\mathbb{Z})$
      · Specifically $PSL_2(\mathbb{Z})$
- The real Grassmannian, $Gr(n, k, \mathbb{R})$, i.e. the set of k dimensional subspaces of $\mathbb{R}^n$
- The Stiefel manifold $V_n(k)$
- Possible modifications to a space X:
  - Remove k points by taking $D(k, X)$
  - Remove a line segment
  - Remove an entire line/axis
  - Remove a hole
  - Quotient by a group action (e.g. antipodal map, or rotation)
  - Remove a knot
  - Take complement in ambient space
- Assorted info about other Lie Groups:
- $O_n, U_n, SO_n, SU_n, Sp_n$
- $\pi_k(U_n) = \mathbb{Z} \cdot \mathbb{1}\,[k \text{ odd}]$
  - $\pi_1(U_n) = 1$
- $\pi_k(SU_n) = \mathbb{Z} \cdot \mathbb{1}\,[k \text{ odd}]$
  - $\pi_1(SU_n) = 0$
- $\pi_k(U_n) = \mathbb{Z}/2\mathbb{Z} \cdot \mathbb{1}\,[k = 0, 1 \mod 8] + \mathbb{Z} \cdot \mathbb{1}\,[k = 3, 7 \mod 8]$
- $\pi_k(SP_n) = \mathbb{Z}/2\mathbb{Z} \cdot \mathbb{1}\,[k = 4, 5 \mod 8] + \mathbb{Z} \cdot \mathbb{1}\,[k = 3, 7 \mod 8]$

4.6 Spheres

- $\pi_i(S^n) = 0$ for $i < n$, $\pi_n(S^n) = \mathbb{Z}$
  - Not necessarily true that $\pi_i(S^n) = 0$ when $i > n$!!!
    * E.g. $\pi_3(S^2) = \mathbb{Z}$ by Hopf fibration
- $H_i(S^n) = \mathbb{1}\,[i \in \{0, n\}]$
- $H_n(\bigvee_i X_i) \cong \prod_i H_n(X_i)$ for "good pairs"
  - Corollary: $H_n(\bigvee_k S^n) = \mathbb{Z}^k$
- $S^n/S^k \simeq S^n \vee \Sigma S^k$

– $\Sigma S^n = S^{n+1}$

- S^n has the CW complex structure of 2 k-cells for each $0 \leq k \leq n$.

5 Fall 2014

5.1 1. Let $X = \mathbb{R}^3 - \Delta^{(1)}$, the complement of the skeleton of regular tetrahedron, and compute $\pi_1(X)$ and $H_*(X)$.

Lay the graph out flat in the plane, then take a maximal tree - these leaves 3 edges, and so $\pi_1(X) = \mathbb{Z}^{*3}$.

Moreover $X \simeq S^1 \vee S^1 \vee S^1$ which has only a 1-skeleton, thus $H_*(X) = [\mathbb{Z}, \mathbb{Z}^3, 0 \rightarrow]$.

5.2 2. Let $X = S^1 \times B^2 - L$ where L is two linked solid torii inside a larger solid torus. Compute $H_*(X)$.

?

5.3 3. Let L be a 3-manifold with homology $[\mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z}, \ldots]$ and let $X = L \times \Sigma L$. Compute $H_*(X), H^*(X)$.

Useful facts:

- $H_k(X \times Y) \cong \bigoplus_{i+j=k} H_i(X) \otimes H_j(Y) \bigoplus_{i+j=k-1} \mathrm{Tor}(H_i(X), H_j(Y))$
- $\tilde{H}_i(\Sigma X) = \tilde{H}_{i-1}(X)$

We will use the fact that $H_*(\Sigma L) = [\mathbb{Z}, \mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z}]$.

Represent $H_*(L)$ by $p(x,y) = 1 + yx + x^3$ and $H_*(\Sigma L)$ by $q(x,y) = 1 + x + yx^2 + x^4$, we can extract the free part of $H_*(X)$ by multiplying

$$p(x,y)q(x,y) = 1 + (1+y)x + 2yx^2 + (y^2+1)x^3 + 2x^4 + 2yx^5 + x^7$$

where multiplication corresponds to the tensor product, addition to the direct sum/product.

So the free portion is

$$\begin{aligned} H_*(X) &= [\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}_3 \otimes \mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z}_3 \otimes \mathbb{Z}_3, \mathbb{Z}^2, \mathbb{Z}_3^2, 0, \mathbb{Z}] \\ &= [\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}^2, \mathbb{Z}_3^2, 0, \mathbb{Z}] \end{aligned}$$

We can add in the correction from torsion by noting that only terms of the form $\mathrm{Tor}(\mathbb{Z}_3, \mathbb{Z}_3) = \mathbb{Z}_3$ survive. These come from the terms $i = 1, j = 2$, so $i + j = k - 1 \implies k = 1 + 2 + 1 = 4$ and there is thus an additional torsion term appearing in dimension 4. So we have

$$\begin{aligned} H_*(X) &= [\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z} \times \mathbb{Z}_3, \mathbb{Z}^2 \times \mathbb{Z}_3, \mathbb{Z}_3^2, 0, \mathbb{Z}] \\ &= [\mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}^2, 0, 0, \mathbb{Z}] \times [0, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3^2, 0, 0] \end{aligned}$$

and

$$\begin{aligned} H^*(X) &= [\mathbb{Z},\mathbb{Z},0,\mathbb{Z},\mathbb{Z}^2,0,0,\mathbb{Z}] \times [0,0,\mathbb{Z}_3,\mathbb{Z}_3,\mathbb{Z}_3,\mathbb{Z}_3,\mathbb{Z}_3^2,0] \\ &= [\mathbb{Z},\mathbb{Z},\mathbb{Z}_3,\mathbb{Z}\times\mathbb{Z}_3,\mathbb{Z}^2\times\mathbb{Z}_3,\mathbb{Z}_3,\mathbb{Z}_3^2,\mathbb{Z}]. \end{aligned}$$

■

5.4 4. Let M be a closed, connected, oriented 4-manifold such that $H_2(M;\mathbb{Z})$ has rank 1. Show that there is not a free $\mathbb{Z}_2$ action on M.

Useful facts:

- $X \twoheadrightarrow_{\times p} Y$ induces $\chi(X) = p\chi(Y)$
- Moral: always try a simple Euler characteristic argument first!

We know that $H_*(M) = [\mathbb{Z}, A, \mathbb{Z}\times G, A, \mathbb{Z}]$ for some group A and some torsion group G. Letting $n = \operatorname{rank}(A)$ and taking the Euler characteristic, we have $\chi(M) = (1)1+(-1)n+(1)1+(-1)n+(1)1 = 3-2n$. Note that this is odd for any n.

However, a free action of $\mathbb{Z}_2 \curvearrowright M$ would produce a double covering $M \twoheadrightarrow_{\times 2} M/\mathbb{Z}_2$, and multiplicativity of Euler characteristics would force $\chi(M) = 2\chi(M/\mathbb{Z}_2)$ and thus $3-2n = 2k$ for some integer k. This would require $3-2n$ to be even, so we have a contradiction. ■

5.5 5. Let X be T^2 with a 2-cell attached to the interior along a longitude. Compute $\pi_2(X)$.

Useful facts:

- $T^2 = e^0 + e^1_1 + e^1_2 + e^2$ as a CW complex.
- $S^2/(x_0 \sim x_1) \simeq S^2 \wedge S^1$ when x_0, x_1 are two distinct points. (Picture: sphere with a string handle connecting north/south poles.)
- $\pi_{\geq 2}(\tilde{X}) \cong \pi_{\geq 2}(X)$ for $\tilde{X} \twoheadrightarrow X$ the universal cover.

Write $T^2 = e^0 + e^1_1 + e^1_2 + e^2$, where the first and second 1-cells denote the longitude and meridian respectively. By symmetry, we could have equivalently attached a disk to the meridian instead of the longitude, filling the center hole in the torus. Contract this disk to a point, then pull it vertically in both directions to obtain S^2 with two points identified, which is homotopy-equivalent to $S^2 \vee S_1$.

Take the universal cover, which is $\mathbb{R}^1 \bigcup\limits_{\mathbb{Z}} S^2$ and has the same π_2. This is homotopy-equivalent to $\bigvee\limits_{i\in\mathbb{Z}} S^2$ and so $\pi_2(X) = \prod\limits_{i\in\mathbb{Z}} \mathbb{Z}$ generated by each distinct copy of S^2. (Alternatively written as $\mathbb{Z}[t,t^{-1}]$).

6 Extra Problems

1. Compute $\pi_1(X)$ where $X \coloneqq S^2/\sim$, where $x \sim -x$ only for x on the equator $S^1 \hookrightarrow S^2$.

- Hint: try cellular homology. Should yield $[\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}, 0, \cdots]$.

3. Show that a local homeomorphism between compact Hausdorff spaces is a covering space.

4. Describe all connected covering spaces of $\mathbb{RP}^2 \vee \mathbb{RP}^2$.
5. Compute the homology of the Klein bottle using the Mayer-Vietoris sequence and a decomposition $K = M \coprod_f M$
6. Show that if $X = S^2 \coprod_{\mathrm{id}} S^2$ is a pushout along the equators, then $H_n(X) = [\mathbb{Z}, 0, \mathbb{Z}^3, 0, \cdots]$.
7. Use the Kunneth formula to compute $H^*(S^2 \times S^2; \mathbb{Z})$.

- Known to be $[\mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z}, 0, 0, \cdots]$.

9. Compute $H^*(S^2 \vee S^2 \vee S^4)$

- Known to be $[\mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z}, 0, 0, \cdots]$.

10. Show that $\chi(\Sigma_g + \Sigma_h) = \chi(\Sigma_g) + \chi(\Sigma_h) - 2$.

Suggested by Ernest

1. Let X be a compact space and let A be a closed subspace. Show that A is compact.
2. Let $f : X- > Y$ be a continuous function, with X compact. Show that $f(X)$ is compact.
3. Let A be a compact subspace of a Hausdorff space X. Show that A is closed.
4. Show that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.