Title

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Monday 27th July, 2020

Contents

1 Inequalities and Equalities

Proposition (Reverse Triangle Inequality)

$$\left| \|x\| - \|y\| \right| \leq \|x - y\|.$$

Proposition (Chebyshev's Inequality)

$$\mu(\{x : |f(x)| > \alpha\}) \leq \left(\frac{\|f\|_p}{\alpha} \right)^p.$$

Proposition (Holder's Inequality When Surjective)

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Application: For finite measure spaces,

$$1 \leq p < q \leq \infty \implies L^q \subset L^p \quad (\text{ and } \ell^p \subset \ell^q).$$

Proof (Holder's Inequality) Fix p, q, let $r = \dfrac{q}{p}$ and $s = \dfrac{r}{r-1}$ so $r^{-1} + s^{-1} = 1$. Then let $h = |f|^p$:

$$\|f\|_p^p = \|h \cdot 1\|_1 \leq \|1\|_s \|h\|_r = \mu(X)^{\frac{1}{s}} \|f\|_q^{\frac{q}{r}} \implies \|f\|_p \leq \mu(X)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q.$$

> Note: doesn't work for ℓ_p spaces, but just note that $\sum |x_n| < \infty \implies x_n < 1$ for large enough n, and thus $p < q \implies |x_n|^q \leq |x_n|^q$.

Proof (Holder's Inequality) It suffices to show this when $\|f\|_p = \|g\|_q = 1$, since

$$\|fg\|_1 \leq \|f\|_p \|f\|_q \iff \int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \leq 1.$$

Using $AB \leq \frac{1}{p}A^p + \frac{1}{q}B^q$, we have

$$\int |f||g| \leq \int \frac{|f|^p}{p} \frac{|g|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1.$$

Proposition (Cauchy-Schwarz Inequality)

$$|\langle f,\ g \rangle| = \|fg\|_1 \leq \|f\|_2 \|g\|_2 \quad \text{with equality} \quad \iff f = \lambda g.$$

Note: Relates inner product to norm, and only happens to relate norms in L^1.

Proof ?

Proposition (Minkowski's Inequality:)

$$1 \leq p < \infty \implies \|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Note: does not handle $p = \infty$ case. Use to prove L^p is a normed space.

Proof

- We first note

$$|f+g|^p = |f+g||f+g|^{p-1} \leq (|f| + |g|)\,|f+g|^{p-1}.$$

- Note that if p, q are conjugate exponents then

$$\begin{aligned} \frac{1}{q} &= 1 - \frac{1}{p} = \frac{p-1}{p} \\ q &= \frac{p}{p-1}. \end{aligned}$$

- Then taking integrals yields

$$\begin{aligned}
\|f+g\|_p^p &= \int |f+g|^p \\
&\leq \int (|f|+|g|)\,|f+g|^{p-1} \\
&= \int |f||f+g|^{p-1} + \int |g||f+g|^{p-1} \\
&= \left\| f(f+g)^{p-1} \right\|_1 + \left\| g(f+g)^{p-1} \right\|_1 \\
&\leq \|f\|_p \left\| (f+g)^{p-1}) \right\|_q + \|g\|_p \left\| (f+g)^{p-1}) \right\|_q \\
&= \left(\|f\|_p + \|g\|_p \right) \left\| (f+g)^{p-1}) \right\|_q \\
&= \left(\|f\|_p + \|g\|_p \right) \left(\int |f+g|^{(p-1)q} \right)^{\frac{1}{q}} \\
&= \left(\|f\|_p + \|g\|_p \right) \left(\int |f+g|^{p} \right)^{1-\frac{1}{p}} \\
&= \left(\|f\|_p + \|g\|_p \right) \frac{\int |f+g|^p}{(\int |f+g|^p)^{\frac{1}{p}}} \\
&= \left(\|f\|_p + \|g\|_p \right) \frac{\|f+g\|_p^p}{\|f+g\|_p}.
\end{aligned}$$

- Cancelling common terms yields

$$\begin{aligned}
1 &\leq \left(\|f\|_p + \|g\|_p \right) \frac{1}{\|f+g\|_p} \\
&\implies \|f+g\|_p \leq \|f\|_p + \|g\|_p.
\end{aligned}$$

Proposition (Young's Inequality*)

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \implies \|f * g\|_r \leq \|f\|_p \|g\|_q$$

Application: Some useful specific cases:

$$\begin{aligned}
\|f * g\|_1 &\leq \|f\|_1 \|g\|_1 \\
\|f * g\|_p &\leq \|f\|_1 \|g\| p, \\
\|f * g\|_\infty &\leq \|f\|_2 \|g\|_2 \\
\|f * g\|_\infty &\leq \|f\|_p \|g\|_q.
\end{aligned}$$

Proposition (Bezel's Inequality:)

For $x \in H$ a Hilbert space and $\{e_k\}$ an orthonormal sequence,

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

Note: this does not need to be a basis.

Proposition (Parseval's Identity:) Equality in Bessel's inequality, attained when $\{e_k\}$ is a *basis*, i.e. it is complete, i.e. the span of its closure is all of H.

1.1 Less Explicitly Used Inequalities

Proposition (AM-GM Inequality)

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Proposition (Jensen's Inequality)

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Proposition (???) :

$$AB \leq \frac{A^p}{p} + \frac{B^q}{q}.$$

Proposition (? Inequality)

$$(a+b)^p \leq 2^p(a^p + b^p).$$

Proposition (Bernoulli's Inequality)

$$(1+x)^n \geq 1 + nx \quad x \geq -1, \mathrm{or} n \in 2\mathbb{Z} \text{ and } \forall x.$$