

Algebra Qualifying Exam Questions

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1 Group Theory

1.1 Spring 2020 #1

- Show that any group of order 2020 is solvable.
- Give (without proof) a classification of all abelian groups of order 2020.
- Describe one nonabelian group of order 2020.

1.2 Spring 2020 #2

Let H be a normal subgroup of a finite group G where the order of H and the index of H in G are relatively prime. Prove that no other subgroup of G has the same order as H .

1.3 Fall 2019 #1

Let G be a finite group with n distinct conjugacy classes. Let $g_1 \cdots g_n$ be representatives of the conjugacy classes of G .

Prove that if $g_i g_j = g_j g_i$ for all i, j then G is abelian.

1.4 Fall 2019 #2

Let G be a group of order 105 and let P, Q, R be Sylow 3, 5, 7 subgroups respectively.

- (a) Prove that at least one of Q and R is normal in G .
- (b) Prove that G has a cyclic subgroup of order 35.
- (c) Prove that both Q and R are normal in G .
- (d) Prove that if P is normal in G then G is cyclic.

1.5 Spring 2019 #3

How many isomorphism classes are there of groups of order 45?

Describe a representative from each class.

1.6 Spring 2019 #4

For a finite group G , let $c(G)$ denote the number of conjugacy classes of G .

- (a) Prove that if two elements of G are chosen uniformly at random, then the probability they commute is precisely

$$\frac{c(G)}{|G|}.$$

- (b) State the class equation for a finite group.
- (c) Using the class equation (or otherwise) show that the probability in part (a) is at most

$$\frac{1}{2} + \frac{1}{2[G : Z(G)]}.$$

Here, as usual, $Z(G)$ denotes the center of G .

1.7 Fall 2012 #1

Let G be a finite group and X a set on which G acts.

- a. Let $x \in X$ and $G_x := \{g \in G \mid g \cdot x = x\}$. Show that G_x is a subgroup of G .
- b. Let $x \in X$ and $G \cdot x := \{g \cdot x \mid g \in G\}$. Prove that there is a bijection between elements in $G \cdot x$ and the left cosets of G_x in G .

1.8 Fall 2012 #2

Let G be a group of order 30.

- Show that G contains normal subgroups of orders 3, 5, and 15.
- Give all possible presentations and relations for G .
- Determine how many groups of order 30 there are up to isomorphism.

1.9 Spring 2012 #2

Let G be a finite group and p a prime number such that there is a normal subgroup $H \trianglelefteq G$ with $|H| = p^i > 1$.

- Show that H is a subgroup of any Sylow p -subgroup of G .
- Show that G contains a nonzero abelian normal subgroup of order divisible by p .

1.10 Spring 2012 #3

Let G be a group of order 70.

- Show that G is not simple.
- Exhibit 3 nonisomorphic groups of order 70 and prove that they are not isomorphic.

1.11 Fall 2018 #1

Let G be a finite group whose order is divisible by a prime number p . Let P be a normal p -subgroup of G (so $|P| = p^c$ for some c).

- Show that P is contained in every Sylow p -subgroup of G .
- Let M be a maximal proper subgroup of G . Show that either $P \subseteq M$ or $|G/M| = p^b$ for some $b \leq c$.

1.12 Fall 2018 #2

- Suppose the group G acts on the set X . Show that the stabilizers of elements in the same orbit are conjugate.
- Let G be a finite group and let H be a proper subgroup. Show that the union of the conjugates of H is strictly smaller than G , i.e.

$$\bigcup_{g \in G} gHg^{-1} \subsetneq G$$

- Suppose G is a finite group acting transitively on a set S with at least 2 elements. Show that there is an element of G with no fixed points in S .

1.13 Spring 2018 #1

- (a) Use the Class Equation (equivalently, the conjugation action of a group on itself) to prove that any p -group (a group whose order is a positive power of a prime integer p) has a nontrivial center.
- (b) Prove that any group of order p^2 (where p is prime) is abelian.
- (c) Prove that any group of order $5^2 \cdot 7^2$ is abelian.
- (d) Write down exactly one representative in each isomorphism class of groups of order $5^2 \cdot 7^2$.

1.14 Fall 2017 #1

Suppose the group G acts on the set A . Assume this action is faithful (recall that this means that the kernel of the homomorphism from G to $\text{Sym}(A)$ which gives the action is trivial) and transitive (for all a, b in A , there exists g in G such that $g \cdot a = b$.)

- (a) For $a \in A$, let G_a denote the stabilizer of a in G . Prove that for any $a \in A$,

$$\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = \{1\}.$$

- (b) Suppose that G is abelian. Prove that $|G| = |A|$. Deduce that every abelian transitive subgroup of S_n has order n .

1.15 Fall 2017 #2

- (a) Classify the abelian groups of order 36.

For the rest of the problem, assume that G is a non-abelian group of order 36.

You may assume that the only subgroup of order 12 in S_4 is A_4 and that A_4 has no subgroup of order 6.

- (b) Prove that if the 2-Sylow subgroup of G is normal, G has a normal subgroup N such that G/N is isomorphic to A_4 .
- (c) Show that if G has a normal subgroup N such that G/N is isomorphic to A_4 and a subgroup H isomorphic to A_4 it must be the direct product of N and H .
- (d) Show that the dihedral group of order 36 is a non-abelian group of order 36 whose Sylow-2 subgroup is not normal.

1.16 Spring 2017 #1

Let G be a finite group and $\pi : G \rightarrow \text{Sym}(G)$ the Cayley representation. (Recall that this means that for an element $x \in G$, $\pi(x)$ acts by left translation on G .)

Prove that $\pi(x)$ is an odd permutation \iff the order $|\pi(x)|$ of $\pi(x)$ is even and $|G|/|\pi(x)|$ is odd.

1.17 Spring 2017 #2

- How many isomorphism classes of abelian groups of order 56 are there? Give a representative for one of each class.
- Prove that if G is a group of order 56, then either the Sylow-2 subgroup or the Sylow-7 subgroup is normal.
- Give two non-isomorphic groups of order 56 where the Sylow-7 subgroup is normal and the Sylow-2 subgroup is *not* normal. Justify that these two groups are not isomorphic.

1.18 Fall 2016 #1

Let G be a finite group and $s, t \in G$ be two distinct elements of order 2. Show that subgroup of G generated by s and t is a dihedral group.

Recall that the dihedral groups of order $2m$ for $m \geq 2$ are of the form

$$D_{2m} = \langle \sigma, \tau \mid \sigma^m = 1 = \tau^2, \tau\sigma = \sigma^{-1}\tau \rangle.$$

1.19 Fall 2016 #3

How many groups are there up to isomorphism of order pq where $p < q$ are prime integers?

1.20 ★ Fall 2016 #7

- Define what it means for a group G to be *solvable*.
- Show that every group G of order 36 is solvable.

Hint: you can use that S_4 is solvable.

1.21 Spring 2016 #3

- State the three Sylow theorems.
- Prove that any group of order 1225 is abelian.
- Write down exactly one representative in each isomorphism class of abelian groups of order 1225.

1.22 Spring 2016 #5

Let G be a finite group acting on a set X . For $x \in X$, let G_x be the stabilizer of x and $G \cdot x$ be the orbit of x .

- Prove that there is a bijection between the left cosets G/G_x and $G \cdot x$.
- Prove that the center of every finite p -group G is nontrivial by considering that action of G on $X = G$ by conjugation.

1.23 Fall 2015 #1

Let G be a group containing a subgroup H not equal to G of finite index. Prove that G has a normal subgroup which is contained in every conjugate of H which is of finite index.

1.24 Fall 2015 #2

Let G be a finite group, H a p -subgroup, and P a Sylow p -subgroup for p a prime. Let H act on the left cosets of P in G by left translation.

Prove that this is an orbit under this action of length 1.

Prove that xP is an orbit of length 1 $\iff H$ is contained in xPx^{-1} .

1.25 Spring 2015 #1

For a prime p , let G be a finite p -group and let N be a normal subgroup of G of order p . Prove that N is contained in the center of G .

1.26 Spring 2015 #4

Let N be a positive integer, and let G be a finite group of order N .

- a. Let $\text{Sym}G$ be the set of all bijections from $G \rightarrow G$ viewed as a group under composition. Note that $\text{Sym}G \cong S_N$. Prove that the Cayley map

$$\begin{aligned} C : G &\longrightarrow \text{Sym}G \\ g &\mapsto (x \mapsto gx) \end{aligned}$$

is an injective homomorphism.

- b. Let $\Phi : \text{Sym}G \rightarrow S_N$ be an isomorphism. For $a \in G$ define $\varepsilon(a) \in \{\pm 1\}$ to be the sign of the permutation $\Phi(C(a))$. Suppose that a has order d . Prove that $\varepsilon(a) = -1 \iff d$ is even and N/d is odd.
- c. Suppose $N > 2$ and $n \equiv 2 \pmod{4}$. Prove that G is not simple.

Hint: use part (b).

1.27 Fall 2014 #2

Let G be a group of order 96.

- a. Show that G has either one or three 2-Sylow subgroups.
- b. Show that either G has a normal subgroup of order 32, or a normal subgroup of order 16.

1.28 Fall 2014 #6

Let G be a group and $H, K < G$ be subgroups of finite index. Show that

$$[G : H \cap K] \leq [G : H] [G : K].$$

1.29 Spring 2014 #1

Let p, n be integers such that p is prime and p does not divide n . Find a real number $k = k(p, n)$ such that for every integer $m \geq k$, every group of order $p^m n$ is not simple.

1.30 Spring 2014 #2

Let $G \subset S_9$ be a Sylow-3 subgroup of the symmetric group on 9 letters.

- Show that G contains a subgroup H isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ by exhibiting an appropriate set of cycles.
- Show that H is normal in G .
- Give generators and relations for G as an abstract group, such that all generators have order 3. Also exhibit elements of S_9 in cycle notation corresponding to these generators.
- Without appealing to the previous parts of the problem, show that G contains an element of order 9.

1.31 Fall 2013 #1

Let p, q be distinct primes.

- Let $\bar{q} \in \mathbb{Z}_p$ be the class of $q \pmod p$ and let k denote the order of \bar{q} as an element of \mathbb{Z}_p^\times . Prove that no group of order pq^k is simple.
- Let G be a group of order pq , and prove that G is not simple.

1.32 Fall 2013 #2

Let G be a group of order 30.

- Show that G has a subgroup of order 15.
- Show that every group of order 15 is cyclic.
- Show that G is isomorphic to some semidirect product $\mathbb{Z}_{15} \rtimes \mathbb{Z}_2$.
- Exhibit three nonisomorphic groups of order 30 and prove that they are not isomorphic. You are not required to use your answer to (c).

1.33 Spring 2013 #3

Let P be a finite p -group. Prove that every nontrivial normal subgroup of P intersects the center of P nontrivially.

1.34 Spring 2013 #4

Define a *simple group*. Prove that a group of order 56 can not be simple.

1.35 Fall 2019 Midterm #1

Let G be a group of order p^2q for p, q prime. Show that G has a nontrivial normal subgroup.

1.36 Fall 2019 Midterm #2

Let G be a finite group and let P be a Sylow p -subgroup for p prime. Show that $N(N(P)) = N(P)$ where N is the normalizer in G .

1.37 Fall 2019 Midterm #3

Show that there exist no simple groups of order 148.

1.38 Fall 2019 Midterm #4

Let p be a prime. Show that $S_p = \langle \tau, \sigma \rangle$ where τ is a transposition and σ is a p -cycle.

1.39 Fall 2019 Midterm #5

Let G be a nonabelian group of order p^3 for p prime. Show that $Z(G) = [G, G]$

2 Commutative Algebra**2.1 Spring 2020 #5**

Let R be a ring and $f : M \rightarrow N$ and $g : N \rightarrow M$ be R -module homomorphisms such that $g \circ f = \text{id}_M$. Show that $N \cong \text{im } f \oplus \ker g$.

2.2 Fall 2019 #3

Let R be a ring with the property that for every $a \in R$, $a^2 = a$.

- (a) Prove that R has characteristic 2.
- (b) Prove that R is commutative.

2.3 Fall 2019 #6

Let R be a commutative ring with multiplicative identity. Assume Zorn's Lemma.

- (a) Show that

$$N = \{r \in R \mid r^n = 0 \text{ for some } n > 0\}$$

is an ideal which is contained in any prime ideal.

- (b) Let r be an element of R not in N . Let S be the collection of all proper ideals of R not containing any positive power of r . Use Zorn's Lemma to prove that there is a prime ideal in S .

- (c) Suppose that R has exactly one prime ideal P . Prove that every element r of R is either nilpotent or a unit.

2.4 Spring 2019 #6

Let R be a commutative ring with 1.

Recall that $x \in R$ is nilpotent iff $x^n = 0$ for some positive integer n .

- (a) Show that every proper ideal of R is contained within a maximal ideal.
 (b) Let $J(R)$ denote the intersection of all maximal ideals of R .

Show that $x \in J(R) \iff 1 + rx$ is a unit for all $r \in R$.

- (c) Suppose now that R is finite. Show that in this case $J(R)$ consists precisely of the nilpotent elements in R .

2.5 Fall 2018 #7

Let R be a commutative ring.

- (a) Let $r \in R$. Show that the map

$$\begin{aligned} r \bullet : R &\longrightarrow R \\ x &\mapsto rx. \end{aligned}$$

is an R -module endomorphism of R .

- (b) We say that r is a **zero-divisor** if $r \bullet$ is not injective. Show that if r is a zero-divisor and $r \neq 0$, then the kernel and image of R each consist of zero-divisors.
 (c) Let $n \geq 2$ be an integer. Show: if R has exactly n zero-divisors, then $\#R \leq n^2$.
 (d) Show that up to isomorphism there are exactly two commutative rings R with precisely 2 zero-divisors.

You may use without proof the following fact: every ring of order 4 is isomorphic to exactly one of the following:

$$\frac{\mathbb{Z}}{4\mathbb{Z}}, \quad \frac{\frac{\mathbb{Z}}{2\mathbb{Z}}[t]}{(t^2 + t + 1)}, \quad \frac{\frac{\mathbb{Z}}{2\mathbb{Z}}[t]}{(t^2 - t)}, \quad \frac{\frac{\mathbb{Z}}{2\mathbb{Z}}[t]}{(t^2)}.$$

2.6 Spring 2018 #5

Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} x & u \\ -y & -v \end{pmatrix}$$

over a commutative ring R , where b and x are units of R . Prove that

$$MN = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \implies MN = 0.$$

2.7 Spring 2018 #8

Let $R = C[0, 1]$ be the ring of continuous real-valued functions on the interval $[0, 1]$. Let I be an ideal of R .

- (a) Show that if $f \in I$, $a \in [0, 1]$ are such that $f(a) \neq 0$, then there exists $g \in I$ such that $g(x) \geq 0$ for all $x \in [0, 1]$, and $g(x) > 0$ for all x in some open neighborhood of a .
- (b) If $I \neq R$, show that the set $Z(I) = \{x \in [0, 1] \mid f(x) = 0 \text{ for all } f \in I\}$ is nonempty.
- (c) Show that if I is maximal, then there exists $x_0 \in [0, 1]$ such that $I = \{f \in R \mid f(x_0) = 0\}$.

2.8 Fall 2017 #5

A ring R is called *simple* if its only two-sided ideals are 0 and R .

- (a) Suppose R is a commutative ring with 1 . Prove R is simple if and only if R is a field.
- (b) Let k be a field. Show the ring $M_n(k)$, $n \times n$ matrices with entries in k , is a simple ring.

2.9 Fall 2017 #6

For a ring R , let $U(R)$ denote the multiplicative group of units in R . Recall that in an integral domain R , $r \in R$ is called *irreducible* if r is not a unit in R , and the only divisors of r have the form ru with u a unit in R .

We call a non-zero, non-unit $r \in R$ *prime* in R if $r \mid ab \implies r \mid a$ or $r \mid b$. Consider the ring $R = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$.

- (a) Prove R is an integral domain.
- (b) Show $U(R) = \{\pm 1\}$.
- (c) Show 3 , $2 + \sqrt{-5}$, and $2 - \sqrt{-5}$ are irreducible in R .
- (d) Show 3 is not prime in R .
- (e) Conclude R is not a PID.

2.10 Spring 2017 #3

Let R be a commutative ring with 1 . Suppose that M is a free R -module with a finite basis X .

- a. Let $I \trianglelefteq R$ be a proper ideal. Prove that M/IM is a free R/I -module with basis X' , where X' is the image of X under the canonical map $M \rightarrow M/IM$.
- b. Prove that any two bases of M have the same number of elements. You may assume that the result is true when R is a field.

2.11 Spring 2017 #4

- a. Let R be an integral domain with quotient field F . Suppose that $p(x), a(x), b(x)$ are monic polynomials in $F[x]$ with $p(x) = a(x)b(x)$ and with $p(x) \in R[x]$, $a(x)$ not in $R[x]$, and both $a(x), b(x)$ not constant. Prove that R is not a UFD. (You may assume Gauss' lemma)
- b. Prove that $\mathbb{Z}[2\sqrt{2}]$ is not a UFD.

Hint: let $p(x) = x^2 - 2$.

2.12 Spring 2016 #8

Let R be a simple rng (a nonzero ring which is not assumed to have a 1, whose only two-sided ideals are (0) and R) satisfying the following two conditions:

- i. R has no zero divisors, and
ii. If $x \in R$ with $x \neq 0$ then $2x \neq 0$, where $2x := x + x$.

Prove the following:

- a. For each $x \in R$ there is one and only one element $y \in R$ such that $x = 2y$.
b. Suppose $x, y \in R$ such that $x \neq 0$ and $2(xy) = x$, then $yz = zy$ for all $z \in R$.

You can get partial credit for (b) by showing it in the case R has a 1.

2.13 Fall 2015 #3

Let R be a rng (a ring without 1) which contains an element u such that for all $y \in R$, there exists an $x \in R$ such that $xu = y$.

Prove that R contains a maximal left ideal.

Hint: imitate the proof (using Zorn's lemma) in the case where R does have a 1.

2.14 Fall 2015 #4

Let R be a PID and $(a_1) < (a_2) < \cdots$ be an ascending chain of ideals in R . Prove that for some n , we have $(a_j) = (a_n)$ for all $j \geq n$.

2.15 Spring 2015 #7

Let R be a commutative ring, and $S \subset R$ be a nonempty subset that does not contain 0 such that for all $x, y \in S$ we have $xy \in S$. Let \mathcal{I} be the set of all ideals $I \trianglelefteq R$ such that $I \cap S = \emptyset$.

Show that for every ideal $I \in \mathcal{I}$, there is an ideal $J \in \mathcal{I}$ such that $I \subset J$ and J is not properly contained in any other ideal in \mathcal{I} .

Prove that every such ideal J is prime.

2.16 Fall 2014 #7

Give a careful proof that $\mathbb{C}[x, y]$ is not a PID.

2.17 Fall 2014 #8

Let R be a nonzero commutative ring without unit such that R does not contain a proper maximal ideal. Prove that for all $x \in R$, the ideal xR is proper. You may assume the axiom of choice.

2.18 Spring 2014 #5

Let R be a commutative ring and $a \in R$. Prove that a is not nilpotent \iff there exists a commutative ring S and a ring homomorphism $\varphi : R \rightarrow S$ such that $\varphi(a)$ is a unit.

Note: by definition, a is nilpotent \iff there is a natural number n such that $a^n = 0$.

2.19 Spring 2014 #6

Let R be a commutative ring with identity and let n be a positive integer.

- Prove that every surjective R -linear endomorphism $T : R^n \rightarrow R^n$ is injective.
- Show that an injective R -linear endomorphism of R^n need not be surjective.

2.20 Fall 2013 #3

- Define *prime ideal*, give an example of a nontrivial ideal in the ring \mathbb{Z} that is not prime, and prove that it is not prime.
- Define *maximal ideal*, give an example of a nontrivial maximal ideal in \mathbb{Z} and prove that it is maximal.

2.21 Fall 2013 #4

Let R be a commutative ring with $1 \neq 0$. Recall that $x \in R$ is *nilpotent* iff $x^n = 0$ for some positive integer n .

- Show that the collection of nilpotent elements in R forms an ideal.
- Show that if x is nilpotent, then x is contained in every prime ideal of R .
- Suppose $x \in R$ is not nilpotent and let $S = \{x^n \mid n \in \mathbb{N}\}$. There is at least one ideal of R disjoint from S , namely (0) . By Zorn's lemma the set of ideals disjoint from S has a maximal element with respect to inclusion, say I . In other words, I is disjoint from S and if J is any ideal disjoint from S with $I \subseteq J \subseteq R$ then $J = I$ or $J = R$.

Show that I is a prime ideal.

- Deduce from (a) and (b) that the set of nilpotent elements of R is the intersection of all prime ideals of R .

2.22 Spring 2013 #1

Let R be a commutative ring.

- Define a *maximal ideal* and prove that R has a maximal ideal.

- b. Show that an element $r \in R$ is not invertible $\iff r$ is contained in a maximal ideal.
- c. Let M be an R -module, and recall that for $0 \neq \mu \in M$, the *annihilator* of μ is the set

$$\text{Ann}(\mu) = \{r \in R \mid r\mu = 0\}.$$

Suppose that I is an ideal in R which is maximal with respect to the property that there exists an element $\mu \in M$ such that $I = \text{Ann}(\mu)$ for some $\mu \in M$. In other words, $I = \text{Ann}(\mu)$ but there does not exist $\nu \in M$ with $J = \text{Ann}(\nu) \subsetneq R$ such that $I \subsetneq J$.

Prove that I is a prime ideal.

2.23 Spring 2013 #2

- a. Define a *Euclidean domain*.
- b. Define a *unique factorization domain*.
- c. Is a Euclidean domain an UFD? Give either a proof or a counterexample with justification.
- d. Is a UFD a Euclidean domain? Give either a proof or a counterexample with justification.

3 Fields and Galois Theory

3.1 Spring 2020 #3

Let E be an extension field of F and $\alpha \in E$ be algebraic of odd degree over F .

- a. Show that $F(\alpha) = F(\alpha^2)$.
- b. Prove that α^{2020} is algebraic of odd degree over F .

3.2 Spring 2020 #4

Let $f(x) = x^4 - 2 \in \mathbb{Q}[x]$.

- a. Define what it means for a finite extension field E of a field F to be a Galois extension.
- b. Determine the Galois group $\text{Gal}(E/\mathbb{Q})$ for the polynomial $f(x)$, and justify your answer carefully.
- c. Exhibit a subfield K in (b) such that $\mathbb{Q} \leq K \leq E$ with K not a Galois extension over \mathbb{Q} . Explain.

3.3 Fall 2019 #4

Let F be a finite field with q elements.

Let n be a positive integer relatively prime to q and let ω be a primitive n th root of unity in an extension field of F .

Let $E = F[\omega]$ and let $k = [E : F]$.

- (a) Prove that n divides $q^k - 1$.

- (b) Let m be the order of q in $\mathbb{Z}/n\mathbb{Z}^\times$. Prove that m divides k .
- (c) Prove that $m = k$.

3.4 Fall 2019 #7

Let ζ_n denote a primitive n th root of $1 \in \mathbb{Q}$. You may assume the roots of the minimal polynomial $p_n(x)$ of ζ_n are exactly the primitive n th roots of 1.

Show that the field extension $\mathbb{Q}(\zeta_n)$ over \mathbb{Q} is Galois and prove its Galois group is $(\mathbb{Z}/n\mathbb{Z})^\times$.

How many subfields are there of $\mathbb{Q}(\zeta_{20})$?

3.5 Spring 2019 #2

Let $F = \mathbb{F}_p$, where p is a prime number.

- (a) Show that if $\pi(x) \in F[x]$ is irreducible of degree d , then $\pi(x)$ divides $x^{p^d} - x$.
- (b) Show that if $\pi(x) \in F[x]$ is an irreducible polynomial that divides $x^{p^n} - x$, then $\deg \pi(x)$ divides n .

3.6 Spring 2019 #8

Let $\zeta = e^{2\pi i/8}$.

- (a) What is the degree of $\mathbb{Q}(\zeta)/\mathbb{Q}$?
- (b) How many quadratic subfields of $\mathbb{Q}(\zeta)$ are there?
- (c) What is the degree of $\mathbb{Q}(\zeta, \sqrt[4]{2})$ over \mathbb{Q} ?

3.7 Fall 2018 #3

Let $F \subset K \subset L$ be finite degree field extensions. For each of the following assertions, give a proof or a counterexample.

- (a) If L/F is Galois, then so is K/F .
- (b) If L/F is Galois, then so is L/K .
- (c) If K/F and L/K are both Galois, then so is L/F .

3.8 Spring 2018 #2

Let $f(x) = x^4 - 4x^2 + 2 \in \mathbb{Q}[x]$.

- (a) Find the splitting field K of f , and compute $[K : \mathbb{Q}]$.
- (b) Find the Galois group G of f , both as an explicit group of automorphisms, and as a familiar abstract group to which it is isomorphic.
- (c) Exhibit explicitly the correspondence between subgroups of G and intermediate fields between \mathbb{Q} and K .

3.9 Spring 2018 #3

Let K be a Galois extension of \mathbb{Q} with Galois group G , and let E_1, E_2 be intermediate fields of K which are the splitting fields of irreducible $f_i(x) \in \mathbb{Q}[x]$.

Let $E = E_1 E_2 \subset K$.

Let $H_i = \text{Gal}(K/E_i)$ and $H = \text{Gal}(K/E)$.

- (a) Show that $H = H_1 \cap H_2$.
- (b) Show that $H_1 H_2$ is a subgroup of G .
- (c) Show that

$$\text{Gal}(K/(E_1 \cap E_2)) = H_1 H_2.$$

3.10 Fall 2017 #3

Let F be a field. Let $f(x)$ be an irreducible polynomial in $F[x]$ of degree n and let $g(x)$ be any polynomial in $F[x]$. Let $p(x)$ be an irreducible factor (of degree m) of the polynomial $f(g(x))$.

Prove that n divides m . Use this to prove that if r is an integer which is not a perfect square, and n is a positive integer then every irreducible factor of $x^{2n} - r$ over $\mathbb{Q}[x]$ has even degree.

3.11 Fall 2017 #4

- (a) Let $f(x)$ be an irreducible polynomial of degree 4 in $\mathbb{Q}[x]$ whose splitting field K over \mathbb{Q} has Galois group $G = S_4$.

Let θ be a root of $f(x)$. Prove that $\mathbb{Q}[\theta]$ is an extension of \mathbb{Q} of degree 4 and that there are no intermediate fields between \mathbb{Q} and $\mathbb{Q}[\theta]$.

- (b) Prove that if K is a Galois extension of \mathbb{Q} of degree 4, then there is an intermediate subfield between K and \mathbb{Q} .

3.12 Spring 2017 #7

Let F be a field and let $f(x) \in F[x]$.

- a. Define what a splitting field of $f(x)$ over F is.
- b. Let F now be a finite field with q elements. Let E/F be a finite extension of degree $n > 0$. Exhibit an explicit polynomial $g(x) \in F[x]$ such that E/F is a splitting field of $g(x)$ over F . Fully justify your answer.
- c. Show that the extension E/F in (b) is a Galois extension.

3.13 Spring 2017 #8

- a. Let K denote the splitting field of $x^5 - 2$ over \mathbb{Q} . Show that the Galois group of K/\mathbb{Q} is isomorphic to the group of invertible matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \quad \text{where } a \in \mathbb{F}_5^\times \text{ and } b \in \mathbb{F}_5.$$

- b. Determine all intermediate fields between K and \mathbb{Q} which are Galois over \mathbb{Q} .

3.14 Fall 2016 #4

Set $f(x) = x^3 - 5 \in \mathbb{Q}[x]$.

- Find the splitting field K of $f(x)$ over \mathbb{Q} .
- Find the Galois group G of K over \mathbb{Q} .
- Exhibit explicitly the correspondence between subgroups of G and intermediate fields between \mathbb{Q} and K .

3.15 ★ Fall 2016 #5

How many monic irreducible polynomials over \mathbb{F}_p of prime degree ℓ are there? Justify your answer.

3.16 Spring 2016 #2

Let $K = \mathbb{Q}[\sqrt{2} + \sqrt{5}]$.

- Find $[K : \mathbb{Q}]$.
- Show that K/\mathbb{Q} is Galois, and find the Galois group G of K/\mathbb{Q} .
- Exhibit explicitly the correspondence between subgroups of G and intermediate fields between \mathbb{Q} and K .

3.17 Spring 2016 #6

Let K be a Galois extension of a field F with $[K : F] = 2015$. Prove that K is an extension by radicals of the field F .

3.18 Fall 2015 #5

Let $u = \sqrt{2 + \sqrt{2}}$, $v = \sqrt{2 - \sqrt{2}}$, and $E = \mathbb{Q}(u)$.

- Find (with justification) the minimal polynomial $f(x)$ of u over \mathbb{Q} .
- Show $v \in E$, and show that E is a splitting field of $f(x)$ over \mathbb{Q} .
- Determine the Galois group of E over \mathbb{Q} and determine all of the intermediate fields F such that $\mathbb{Q} \subset F \subset E$.

3.19 Fall 2015 #6

- a. Let G be a finite group. Show that there exists a field extension K/F with $\text{Gal}(K/F) = G$.

You may assume that for any natural number n there is a field extension with Galois group S_n .

- b. Let K be a Galois extension of F with $|\text{Gal}(K/F)| = 12$. Prove that there exists an intermediate field E of K/F with $[E : F] = 3$.
- c. With K/F as in (b), does an intermediate field L necessarily exist satisfying $[L : F] = 2$? Give a proof or counterexample.

3.20 Spring 2015 #2

Let \mathbb{F} be a finite field.

- a. Give (with proof) the decomposition of the additive group $(\mathbb{F}, +)$ into a direct sum of cyclic groups.
- b. The *exponent* of a finite group is the least common multiple of the orders of its elements. Prove that a finite abelian group has an element of order equal to its exponent.
- c. Prove that the multiplicative group $(\mathbb{F}^\times, \cdot)$ is cyclic.

3.21 Spring 2015 #5

Let $f(x) = x^4 - 5 \in \mathbb{Q}[x]$.

- a. Compute the Galois group of f over \mathbb{Q} .
- b. Compute the Galois group of f over $\mathbb{Q}(\sqrt{5})$.

3.22 Fall 2014 #1

Let $f \in \mathbb{Q}[x]$ be an irreducible polynomial and L a finite Galois extension of \mathbb{Q} . Let $f(x) = g_1(x)g_2(x) \cdots g_r(x)$ be a factorization of f into irreducibles in $L[x]$.

- a. Prove that each of the factors $g_i(x)$ has the same degree.
- b. Give an example showing that if L is not Galois over \mathbb{Q} , the conclusion of part (a) need not hold.

3.23 Fall 2014 #3

Consider the polynomial $f(x) = x^4 - 7 \in \mathbb{Q}[x]$ and let E/\mathbb{Q} be the splitting field of f .

- a. What is the structure of the Galois group of E/\mathbb{Q} ?
- b. Give an explicit description of all of the intermediate subfields $\mathbb{Q} \subset K \subset E$ in the form $K = \mathbb{Q}(\alpha), \mathbb{Q}(\alpha, \beta), \dots$ where α, β , etc are complex numbers. Describe the corresponding subgroups of the Galois group.

3.24 Spring 2014 #3

Let $F \subset C$ be a field extension with C algebraically closed.

- Prove that the intermediate field $C_{\text{alg}} \subset C$ consisting of elements algebraic over F is algebraically closed.
- Prove that if $F \rightarrow E$ is an algebraic extension, there exists a homomorphism $E \rightarrow C$ that is the identity on F .

3.25 Spring 2014 #4

Let $E \subset \mathbb{C}$ denote the splitting field over \mathbb{Q} of the polynomial $x^3 - 11$.

- Prove that if n is a squarefree positive integer, then $\sqrt{n} \notin E$.

Hint: you can describe all quadratic extensions of \mathbb{Q} contained in E .

- Find the Galois group of $(x^3 - 11)(x^2 - 2)$ over \mathbb{Q} .
- Prove that the minimal polynomial of $11^{1/3} + 2^{1/2}$ over \mathbb{Q} has degree 6.

3.26 Fall 2013 #5

Let L/K be a finite extension of fields.

- Define what it means for L/K to be *separable*.
- Show that if K is a finite field, then L/K is always separable.
- Give an example of a finite extension L/K that is not separable.

3.27 Fall 2013 #6

Let K be the splitting field of $x^4 - 2$ over \mathbb{Q} and set $G = \text{Gal}(K/\mathbb{Q})$.

- Show that K/\mathbb{Q} contains both $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt[4]{2})$ and has degree 8 over \mathbb{Q} .
- Let $N = \text{Gal}(K/\mathbb{Q}(i))$ and $H = \text{Gal}(K/\mathbb{Q}(\sqrt[4]{2}))$. Show that N is normal in G and $NH = G$.

Hint: what field is fixed by NH ?

- Show that $\text{Gal}(K/\mathbb{Q})$ is generated by elements σ, τ , of orders 4 and 2 respectively, with $\tau\sigma\tau^{-1} = \sigma^{-1}$.

Equivalently, show it is the dihedral group of order 8.

- How many distinct quartic subfields of K are there? Justify your answer.

3.28 ★ Fall 2013 #7

Let $F = \mathbb{F}_2$ and let \bar{F} denote its algebraic closure.

- Show that \bar{F} is not a finite extension of F .
- Suppose that $\alpha \in \bar{F}$ satisfies $\alpha^{17} = 1$ and $\alpha \neq 1$. Show that $F(\alpha)/F$ has degree 8.

3.29 Spring 2013 #7

Let $f(x) = g(x)h(x) \in \mathbb{Q}[x]$ and $E, B, C/\mathbb{Q}$ be the splitting fields of f, g, h respectively.

- Prove that $\text{Gal}(E/B)$ and $\text{Gal}(E/C)$ are normal subgroups of $\text{Gal}(E/\mathbb{Q})$.
- Prove that $\text{Gal}(E/B) \cap \text{Gal}(E/C) = \{1\}$.
- If $B \cap C = \mathbb{Q}$, show that $\text{Gal}(E/B)\text{Gal}(E/C) = \text{Gal}(E/\mathbb{Q})$.
- Under the hypothesis of (c), show that $\text{Gal}(E/\mathbb{Q}) \cong \text{Gal}(E/B) \times \text{Gal}(E/C)$.
- Use (d) to describe $\text{Gal}(\mathbb{Q}[\alpha]/\mathbb{Q})$ where $\alpha = \sqrt{2} + \sqrt{3}$.

3.30 Spring 2013 #8

Let F be the field with 2 elements and K a splitting field of $f(x) = x^6 + x^3 + 1$ over F . You may assume that f is irreducible over F .

- Show that if r is a root of f in K , then $r^9 = 1$ but $r^3 \neq 1$.
- Find $\text{Gal}(K/F)$ and express each intermediate field between F and K as $F(\beta)$ for an appropriate $\beta \in K$.

3.31 Fall 2012 #3

Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 5. Assume that f has all but two roots in \mathbb{R} . Compute the Galois group of $f(x)$ over \mathbb{Q} and justify your answer.

3.32 Fall 2012 #4

Let $f(x) \in \mathbb{Q}[x]$ be a polynomial and K be a splitting field of f over \mathbb{Q} . Assume that $[K : \mathbb{Q}] = 1225$ and show that $f(x)$ is solvable by radicals.

3.33 Spring 2012 #1

Suppose that $F \subset E$ are fields such that E/F is Galois and $|\text{Gal}(E/F)| = 14$.

- Show that there exists a unique intermediate field K with $F \subset K \subset E$ such that $[K : F] = 2$.
- Assume that there are at least two distinct intermediate subfields $F \subset L_1, L_2 \subset E$ with $[L_i : F] = 7$. Prove that $\text{Gal}(E/F)$ is nonabelian.

3.34 Spring 2012 #4

Let $f(x) = x^7 - 3 \in \mathbb{Q}[x]$ and E/\mathbb{Q} be a splitting field of f with $\alpha \in E$ a root of f .

- Show that E contains a primitive 7th root of unity.
- Show that $E \neq \mathbb{Q}(\alpha)$.

3.35 Fall 2019 Midterm #6

Compute the Galois group of $f(x) = x^3 - 3x - 3 \in \mathbb{Q}[x]/\mathbb{Q}$.

3.36 Fall 2019 Midterm #7

Show that a field k of characteristic $p \neq 0$ is perfect \iff for every $x \in k$ there exists a $y \in k$ such that $y^p = x$.

3.37 Fall 2019 Midterm #8

Let k be a field of characteristic $p \neq 0$ and $f \in k[x]$ irreducible. Show that $f(x) = g(x^{p^d})$ where $g(x) \in k[x]$ is irreducible and separable. Conclude that every root of f has the same multiplicity p^d in the splitting field of f over k .

3.38 Fall 2019 Midterm #9

Let $n \geq 3$ and ζ_n be a primitive n th root of unity. Show that $[\mathbb{Q}(\zeta_n + \zeta_n^{-1}) : \mathbb{Q}] = \varphi(n)/2$ for φ the totient function. 10. Let L/K be a finite normal extension - Show that if L/K is cyclic and E/K is normal with $L/E/K$ then L/E and E/K are cyclic. - Show that if L/K is cyclic then there exists exactly one extension E/K of degree n with $L/E/K$ for each divisor n of $[L : K]$.

4 Modules**4.0.1 Fall 2019 Final #2**

Consider the \mathbb{Z} -submodule N of \mathbb{Z}^3 spanned by $f_1 = [-1, 0, 1], f_2 = [2, -3, 1], f_3 = [0, 3, 1], f_4 = [3, 1, 5]$. Find a basis for N and describe \mathbb{Z}^3/N .

4.0.2 Spring 2018 #6.

Let

$$M = \{(w, x, y, z) \in \mathbb{Z}^4 \mid w + x + y + z \in 2\mathbb{Z}\},$$

and

$$N = \{(w, x, y, z) \in \mathbb{Z}^4 \mid 4 \mid (w - x), 4 \mid (x - y), 4 \mid (y - z)\}.$$

(a) Show that N is a \mathbb{Z} -submodule of M .

(b) Find vectors $u_1, u_2, u_3, u_4 \in \mathbb{Z}^4$ and integers d_1, d_2, d_3, d_4 such that

$$\{u_1, u_2, u_3, u_4\}$$

is a free basis for M , and

$$\{d_1 u_1, d_2 u_2, d_3 u_3, d_4 u_4\}$$

is a free basis for N .

-
- (c) Use the previous part to describe M/N as a direct sum of cyclic \mathbb{Z} -modules.

4.0.3 Fall 2018 #6.

Let R be a commutative ring, and let M be an R -module. An R -submodule N of M is maximal if there is no R -module P with $N \subsetneq P \subsetneq M$.

- Show that an R -submodule N of M is maximal $\iff M/N$ is a simple R -module: i.e., M/N is nonzero and has no proper, nonzero R -submodules.
- Let M be a \mathbb{Z} -module. Show that a \mathbb{Z} -submodule N of M is maximal $\iff \#M/N$ is a prime number.
- Let M be the \mathbb{Z} -module of all roots of unity in \mathbb{C} under multiplication. Show that there is no maximal \mathbb{Z} -submodule of M .

4.0.4 Spring 2018 #7.

Let R be a PID and M be an R -module. Let p be a prime element of R . The module M is called $\langle p \rangle$ -primary if for every $m \in M$ there exists $k > 0$ such that $p^k m = 0$.

- Suppose M is $\langle p \rangle$ -primary. Show that if $m \in M$ and $t \in R$, $t \notin \langle p \rangle$, then there exists $a \in R$ such that $atm = m$.
- A submodule S of M is said to be *pure* if $S \cap rM = rS$ for all $r \in R$. Show that if M is $\langle p \rangle$ -primary, then S is pure if and only if $S \cap p^k M = p^k S$ for all $k \geq 0$.

4.0.5 Fall 2016 #6

Let R be a ring and $f : M \rightarrow N$ and $g : N \rightarrow M$ be R -module homomorphisms such that $g \circ f = \text{id}_M$. Show that $N \cong \text{im } f \oplus \ker g$.

4.0.6 Spring 2016 #4

Let R be a ring with the following commutative diagram of R -modules, where each row represents a short exact sequence of R -modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

Prove that if α and γ are isomorphisms then β is an isomorphism.

4.0.7 Spring 2015 #8

Let R be a PID and M a finitely generated R -module.

- Prove that there are R -submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that for all $0 \leq i \leq n-1$, the module M_{i+1}/M_i is cyclic.

- b. Is the integer n in part (a) uniquely determined by M ? Prove your answer.

4.0.8 Fall 2012 #6

Let R be a ring and M an R -module. Recall that M is *Noetherian* iff any strictly increasing chain of submodule $M_1 \subsetneq M_2 \subsetneq \cdots$ is finite. Call a proper submodule $M' \subsetneq M$ *intersection-decomposable* if it can not be written as the intersection of two proper submodules $M' = M_1 \cap M_2$ with $M_i \subsetneq M$.

Prove that for every Noetherian module M , any proper submodule $N \subsetneq M$ can be written as a finite intersection $N = N_1 \cap \cdots \cap N_k$ of intersection-indecomposable modules.

4.0.9 Fall 2019 Final #1

Let A be an abelian group, and show A is a \mathbb{Z} -module in a unique way.

4.1 Torsion and the Structure Theorem

4.1.1 Spring 2012 #5

Let M be a finitely generated module over a PID R .

- M_t be the set of torsion elements of M , and show that M_t is a submodule of M .
- Show that M/M_t is torsion free.
- Prove that $M \cong M_t \oplus F$ where F is a free module.

4.1.2 Spring 2017 #5

Let R be an integral domain and let M be a nonzero torsion R -module.

- Prove that if M is finitely generated then the annihilator in R of M is nonzero.
- Give an example of a non-finitely generated torsion R -module whose annihilator is (0) , and justify your answer.

4.1.3 ★ Fall 2019 #5

Let R be a ring and M an R -module.

Recall that the set of torsion elements in M is defined by

$$\text{Tor}(M) = \{m \in M \mid \exists r \in R, r \neq 0, rm = 0\}.$$

- Prove that if R is an integral domain, then $\text{Tor}(M)$ is a submodule of M .
- Give an example where $\text{Tor}(M)$ is not a submodule of M .
- If R has zero-divisors, prove that every non-zero R -module has non-zero torsion elements.

4.1.4 ★ Spring 2019 #5.

Let R be an integral domain. Recall that if M is an R -module, the *rank* of M is defined to be the maximum number of R -linearly independent elements of M .

- (a) Prove that for any R -module M , the rank of $\text{Tor}(M)$ is 0.
- (b) Prove that the rank of M is equal to the rank of $M/\text{Tor}(M)$.
- (c) Suppose that M is a non-principal ideal of R .

Prove that M is torsion-free of rank 1 but not free.

4.1.5 ★ Spring 2020 #6

Let R be a ring with unity.

- a. Give a definition for a free module over R .
- b. Define what it means for an R -module to be torsion free.
- c. Prove that if F is a free module, then any short exact sequence of R -modules of the following form splits:

$$0 \longrightarrow N \longrightarrow M \longrightarrow F \longrightarrow 0.$$

- d. Let R be a PID. Show that any finitely generated R -module M can be expressed as a direct sum of a torsion module and a free module. You may assume that a finitely generated torsionfree module over a PID is free.

4.1.6 Fall 2019 Final #3

Let $R = k[x]$ for k a field and let M be the R -module given by

$$M = \frac{k[x]}{(x-1)^3} \oplus \frac{k[x]}{(x^2+1)^2} \oplus \frac{k[x]}{(x-1)(x^2+1)^4} \oplus \frac{k[x]}{(x+2)(x^2+1)^2}.$$

Describe the elementary divisors and invariant factors of M .

4.1.7 Fall 2019 Final #4

Let $I = (2, x)$ be an ideal in $R = \mathbb{Z}[x]$, and show that I is not a direct sum of nontrivial cyclic R -modules.

4.1.8 Fall 2019 Final #5

Let R be a PID.

- Classify irreducible R -modules up to isomorphism.
- Classify indecomposable R -modules up to isomorphism.

4.1.9 Fall 2019 Final #6

Let V be a finite-dimensional k -vector space and $T : V \rightarrow V$ a non-invertible k -linear map. Show that there exists a k -linear map $S : V \rightarrow V$ with $T \circ S = 0$ but $S \circ T \neq 0$.

4.1.10 Fall 2019 Final #7

Let $A \in M_n(\mathbb{C})$ with $A^2 = A$. Show that A is similar to a diagonal matrix, and exhibit an explicit diagonal matrix similar to A .

4.1.11 Fall 2019 Final #8

Exhibit the rational canonical form for - $A \in M_6(\mathbb{Q})$ with minimal polynomial $(x-1)(x^2+1)^2$. - $A \in M_{10}(\mathbb{Q})$ with minimal polynomial $(x^2+1)^2(x^3+1)$.

4.1.12 Fall 2019 Final #9

Exhibit the rational and Jordan canonical forms for the following matrix $A \in M_4(\mathbb{C})$:

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{pmatrix}.$$

4.1.13 Fall 2019 Final #10

Show that the eigenvalues of a Hermitian matrix A are real and that $A = PDP^{-1}$ where P is an invertible matrix with orthogonal columns.

5 Linear Algebra: Canonical Forms

5.1 Spring 2020 #7

Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 6 & 1 \\ -16 & -16 & -2 \end{bmatrix} \in M_3(\mathbb{C}).$$

- Find the Jordan canonical form J of A .
- Find an invertible matrix P such that $P^{-1}AP = J$. You should not need to compute P^{-1} .
- Write down the minimal polynomial of A .

5.2 Spring 2020 #8

Let $T : V \rightarrow V$ be a linear transformation where V is a finite-dimensional vector space over \mathbb{C} . Prove the Cayley-Hamilton theorem: if $p(x)$ is the characteristic polynomial of T , then $p(T) = 0$. You may use canonical forms.

5.3 Spring 2012 #8

Let V be a finite-dimensional vector space over a field k and $T : V \rightarrow V$ a linear transformation.

- Provide a definition for the *minimal polynomial* in $k[x]$ for T .
- Define the *characteristic polynomial* for T .
- Prove the Cayley-Hamilton theorem: the linear transformation T satisfies its characteristic polynomial.

5.4 Spring 2019 #7

Let p be a prime number. Let A be a $p \times p$ matrix over a field F with 1 in all entries except 0 on the main diagonal.

Determine the Jordan canonical form (JCF) of A

- When $F = \mathbb{Q}$,
- When $F = \mathbb{F}_p$.

Hint: In both cases, all eigenvalues lie in the ground field. In each case find a matrix P such that $P^{-1}AP$ is in JCF.

5.5 Spring 2018 #4

Let

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 1 & -3 \\ 1 & 2 & -4 \end{bmatrix} \in M_3(\mathbb{C})$$

- Find the Jordan canonical form J of A .
- Find an invertible matrix P such that $P^{-1}AP = J$.

You should not need to compute P^{-1} .

5.6 Spring 2017 #6

Let A be an $n \times n$ matrix with all entries equal to 0 except for the $n - 1$ entries just above the diagonal being equal to 2.

- What is the Jordan canonical form of A , viewed as a matrix in $M_n(\mathbb{C})$?
- Find a nonzero matrix $P \in M_n(\mathbb{C})$ such that $P^{-1}AP$ is in Jordan canonical form.

5.7 Spring 2016 #1

Let

$$A = \begin{pmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{pmatrix} \in M_3(\mathbb{C}).$$

- Find the Jordan canonical form J of A .
- Find an invertible matrix P such that $P^{-1}AP = J$. You do not need to compute P^{-1} .

5.8 Spring 2016 #7

Let $D = \mathbb{Q}[x]$ and let M be a $\mathbb{Q}[x]$ -module such that

$$M \cong \frac{\mathbb{Q}[x]}{(x-1)^3} \oplus \frac{\mathbb{Q}[x]}{(x^2+1)^3} \oplus \frac{\mathbb{Q}[x]}{(x-1)(x^2+1)^5} \oplus \frac{\mathbb{Q}[x]}{(x+2)(x^2+1)^2}.$$

Determine the elementary divisors and invariant factors of M .

5.9 Spring 2015 #6

Let F be a field and n a positive integer, and consider

$$A = \begin{bmatrix} 1 & \dots & 1 \\ & \ddots & \\ 1 & \dots & 1 \end{bmatrix} \in M_n(F).$$

Show that A has a Jordan normal form over F and find it.

Hint: treat the cases $n \cdot 1 \neq 0$ in F and $n \cdot 1 = 0$ in F separately.

5.10 Fall 2014 #5

Let T be a 5×5 complex matrix with characteristic polynomial $\chi(x) = (x-3)^5$ and minimal polynomial $m(x) = (x-3)^2$. Determine all possible Jordan forms of T .

5.11 Spring 2013 #5

Let $T : V \rightarrow V$ be a linear map from a 5-dimensional \mathbb{C} -vector space to itself and suppose $f(T) = 0$ where $f(x) = x^2 + 2x + 1$.

- Show that there does not exist any vector $v \in V$ such that $Tv = v$, but there *does* exist a vector $w \in V$ such that $T^2w = w$.
- Give all of the possible Jordan canonical forms of T .

5.12 ★ Spring 2012 #7

Consider the following matrix as a linear transformation from $V := \mathbb{C}^5$ to itself:

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -4 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

- Find the invariant factors of A .
- Express V in terms of a direct sum of indecomposable $\mathbb{C}[x]$ -modules.
- Find the Jordan canonical form of A .

6 Linear Algebra: Diagonalizability**6.1 Spring 2015 #3**

Let F be a field and V a finite dimensional F -vector space, and let $A, B : V \rightarrow V$ be commuting F -linear maps. Suppose there is a basis \mathcal{B}_1 with respect to which A is diagonalizable and a basis \mathcal{B}_2 with respect to which B is diagonalizable.

Prove that there is a basis \mathcal{B}_3 with respect to which A and B are both diagonalizable.

6.2 Spring 2013 #6

Let V be a finite dimensional vector space over a field F and let $T : V \rightarrow V$ be a linear operator with characteristic polynomial $f(x) \in F[x]$.

- Show that $f(x)$ is irreducible in $F[x] \iff$ there are no proper nonzero subspaces $W < V$ with $T(W) \subseteq W$.
- If $f(x)$ is irreducible in $F[x]$ and the characteristic of F is 0, show that T is diagonalizable when we extend the field to its algebraic closure.

6.3 Spring 2019 #1

Let A be a square matrix over the complex numbers. Suppose that A is nonsingular and that A^{2019} is diagonalizable over \mathbb{C} .

Show that A is also diagonalizable over \mathbb{C} .

6.4 Fall 2017 #7

Let F be a field and let V and W be vector spaces over F .

Make V and W into $F[x]$ -modules via linear operators T on V and S on W by defining $X \cdot v = T(v)$ for all $v \in V$ and $X \cdot w = S(w)$ for all $w \in W$.

Denote the resulting $F[x]$ -modules by V_T and W_S respectively.

- (a) Show that an $F[x]$ -module homomorphism from V_T to W_S consists of an F -linear transformation $R : V \rightarrow W$ such that $RT = SR$.
- (b) Show that $VT \cong WS$ as $F[x]$ -modules \iff there is an F -linear isomorphism $P : V \rightarrow W$ such that $T = P^{-1}SP$.
- (c) Recall that a module M is *simple* if $M \neq 0$ and any proper submodule of M must be zero. Suppose that V has dimension 2. Give an example of F, T with V_T simple.
- (d) Assume F is algebraically closed. Prove that if V has dimension 2, then any V_T is not simple.

6.5 Fall 2016 #2

Let A, B be two $n \times n$ matrices with the property that $AB = BA$. Suppose that A and B are diagonalizable. Prove that A and B are *simultaneously* diagonalizable.

7 Linear Algebra: Misc

7.1 Fall 2012 #7

Let k be a field of characteristic zero and $A, B \in M_n(k)$ be two square $n \times n$ matrices over k such that $AB - BA = A$. Prove that $\det A = 0$.

Moreover, when the characteristic of k is 2, find a counterexample to this statement.

7.2 Fall 2012 #8

Prove that any nondegenerate matrix $X \in M_n(\mathbb{R})$ can be written as $X = UT$ where U is orthogonal and T is upper triangular.

7.3 Fall 2012 #5

Let U be an infinite-dimensional vector space over a field k , $f : U \rightarrow U$ a linear map, and $\{u_1, \dots, u_m\} \subset U$ vectors such that U is generated by $\{u_1, \dots, u_m, f^d(u_1), \dots, f^d(u_m)\}$ for some $d \in \mathbb{N}$.

Prove that U can be written as a direct sum $U \cong V \oplus W$ such that

1. V has a basis consisting of some vector $v_1, \dots, v_n, f^d(v_1), \dots, f^d(v_n)$ for some $d \in \mathbb{N}$, and
2. W is finite-dimensional.

Moreover, prove that for any other decomposition $U \cong V' \oplus W'$, one has $W' \cong W$.

7.4 Fall 2015 #7

- a. Show that two 3×3 matrices over \mathbb{C} are similar \iff their characteristic polynomials are equal and their minimal polynomials are equal.
- b. Does the conclusion in (a) hold for 4×4 matrices? Justify your answer with a proof or counterexample.

7.5 Fall 2018 #4

Let V be a finite dimensional vector space over a field (the field is not necessarily algebraically closed).

Let $\varphi : V \rightarrow V$ be a linear transformation. Prove that there exists a decomposition of V as $V = U \oplus W$, where U and W are φ -invariant subspaces of V , $\varphi|_U$ is nilpotent, and $\varphi|_W$ is nonsingular.

7.6 Fall 2018 #5

Let A be an $n \times n$ matrix.

- (a) Suppose that v is a column vector such that the set $\{v, Av, \dots, A^{n-1}v\}$ is linearly independent. Show that any matrix B that commutes with A is a polynomial in A .
- (b) Show that there exists a column vector v such that the set $\{v, Av, \dots, A^{n-1}v\}$ is linearly independent \iff the characteristic polynomial of A equals the minimal polynomial of A .

7.7 Fall 2019 #8

Let $\{e_1, \dots, e_n\}$ be a basis of a real vector space V and let

$$\Lambda := \left\{ \sum r_i e_i \mid r_i \in \mathbb{Z} \right\}$$

Let \cdot be a non-degenerate ($v \cdot w = 0$ for all $w \in V \iff v = 0$) symmetric bilinear form on V such that the Gram matrix $M = (e_i \cdot e_j)$ has integer entries.

Define the dual of Λ to be

$$\Lambda^\vee := \{v \in V \mid v \cdot x \in \mathbb{Z} \text{ for all } x \in \Lambda\}.$$

- (a) Show that $\Lambda \subset \Lambda^\vee$.
- (b) Prove that $\det M \neq 0$ and that the rows of M^{-1} span Λ^\vee .
- (c) Prove that $\det M = |\Lambda^\vee / \Lambda|$.

7.8 Spring 2012 #6

Let k be a field and let the group $G = \text{GL}(m, k) \times \text{GL}(n, k)$ acts on the set of $m \times n$ matrices $M_{m,n}(k)$ as follows:

$$(A, B) \cdot X = AXB^{-1}$$

where $(A, B) \in G$ and $X \in M_{m,n}(k)$.

- a. State what it means for a group to act on a set. Prove that the above definition yields a group action.
- b. Exhibit with justification a subset S of $M_{m,n}(k)$ which contains precisely one element of each orbit under this action.

7.9 Spring 2014 #7

Let $G = \text{GL}(3, \mathbb{Q}[x])$ be the group of invertible 3×3 matrices over $\mathbb{Q}[x]$. For each $f \in \mathbb{Q}[x]$, let S_f be the set of 3×3 matrices A over $\mathbb{Q}[x]$ such that $\det(A) = cf(x)$ for some nonzero constant $c \in \mathbb{Q}$.

- a. Show that for $(P, Q) \in G \times G$ and $A \in S_f$, the formula

$$(P, Q) \cdot A := PAQ^{-1}$$

gives a well defined map $G \times G \times S_f \rightarrow S_f$ and show that this map gives a group action of $G \times G$ on S_f .

- b. For $f(x) = x^3(x^2 + 1)^2$, give one representative from each orbit of the group action in (a), and justify your assertion.

7.10 Fall 2014 #4

Let F be a field and T an $n \times n$ matrix with entries in F . Let I be the ideal consisting of all polynomials $f \in F[x]$ such that $f(T) = 0$.

Show that the following statements are equivalent about a polynomial $g \in I$:

- a. g is irreducible.
b. If $k \in F[x]$ is nonzero and of degree strictly less than g , then $k[T]$ is an invertible matrix.

7.11 Fall 2015 #8

Let V be a vector space over a field F and V^\vee its dual. A *symmetric bilinear form* (\cdot, \cdot) on V is a map $V \times V \rightarrow F$ satisfying

$$(av_1 + bv_2, w) = a(v_1, w) + b(v_2, w) \quad \text{and} \quad (v_1, v_2) = (v_2, v_1)$$

for all $a, b \in F$ and $v_1, v_2 \in V$. The form is *nondegenerate* if the only element $w \in V$ satisfying $(v, w) = 0$ for all $v \in V$ is $w = 0$.

Suppose (\cdot, \cdot) is a nondegenerate symmetric bilinear form on V . If W is a subspace of V , define

$$W^\perp := \{v \in V \mid (v, w) = 0 \text{ for all } w \in W\}.$$

- a. Show that if X, Y are subspaces of V with $Y \subset X$, then $X^\perp \subseteq Y^\perp$.
b. Define an injective linear map

$$\psi : Y^\perp / X^\perp \hookrightarrow (X/Y)^\vee$$

which is an isomorphism if V is finite dimensional.