# UGA Complex Analysis Qualifying Exam Questions and Solutions

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# Saturday 15<sup>th</sup> August, 2020

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#### **Preface**

I'd like to thank the following individuals for their contributions to this document:

- Edward Azoff, for supplying a problem sheet broken out by topic.
- Mentzelos Melistas, for explaining and documenting many solutions to these questions.
- Jingzhi Tie, for supplying many additional problems and solutions.

Add references.

### 1 Topology and Functions of One Variable (8155a)

# 1.1 1

Let  $x_0 = a, x_1 = b$ , and set

$$x_n \coloneqq \frac{x_{n-1} + x_{n-2}}{2} \quad n \ge 2.$$

Show that  $\{x_n\}$  is a Cauchy sequence and find its limit in terms of a and b.

### 1.2 2

Suppose  $f: \mathbb{R} \longrightarrow \mathbb{R}$  is continuous and  $\lim_{x \longrightarrow \pm \infty} f(x) = 0$ . Prove that f is uniformly continuous.

# 1.3 3

Give an example of a function  $f: \mathbb{R} \longrightarrow \mathbb{R}$  that is everywhere differentiable but f' is not continuous at 0.

# 1.4 4

Suppose  $\{g_n\}$  is a uniformly convergent sequence of functions from  $\mathbb{R}$  to  $\mathbb{R}$  and  $f: \mathbb{R} \longrightarrow \mathbb{R}$  is uniformly continuous. Prove that the sequence  $\{f \circ g_n\}$  is uniformly convergent.

# 1.5 5

Let f be differentiable on [a, b]. Say that f is uniformly differentiable iff

$$\forall \varepsilon > 0, \ \exists \delta > 0 \text{ such that } |x - y| < \delta \implies \left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \varepsilon.$$

1.6 6

Prove that f is uniformly differentiable on  $[a, b] \iff f'$  is continuous on [a, b].

1.6 6

Suppose  $A, B \subseteq \mathbb{R}^n$  are disjoint and compact. Prove that there exist  $a \in A, b \in B$  such that

$$||a - b|| = \inf \{ ||x - y|| \mid x \in A, y \in B \}.$$

1.7 7

Suppose  $A, B \subseteq \mathbb{R}^n$  are connected and not disjoint. Prove that  $A[\ ]B$  is also connected.

1.8 8

Suppose  $\{f_n\}_{n\in\mathbb{N}}$  is a sequence of continuous functions  $f_n:[0,1]\longrightarrow\mathbb{R}$  such that

$$f_n(x) \ge f_{n+1}(x) \ge 0 \quad \forall n \in \mathbb{N}, \, \forall x \in [0, 1].$$

Prove that if  $\{f_n\}$  converges pointwise to 0 on [0,1] then it converges to 0 uniformly on [0,1].

1.9 9

Show that if  $E \subset [0,1]$  is uncountable, then there is some  $t \in \mathbb{R}$  such that  $E \cap (-\infty,t)$  and  $E \cap (t,\infty)$  are also uncountable.

### 2 Several Variables (8155h)

2.1 1

Is the following function continuous, differentiable, continuously differentiable?

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & (x,y) \neq (0,0) \\ 0 & \text{else.} \end{cases}$$

2.2 2

2.2.1 a

Complete this definition: " $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$  is real-differentiable a point  $p \in \mathbb{R}^n$  iff there exists a linear transformation..."

2.2.2 b

Give an example of a function  $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$  whose first-order partial derivatives exist everywhere but f is not differentiable at (0,0).

2.3 3

2.2.3 c

Give an example of a function  $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$  which is real-differentiable everywhere but nowhere complex-differentiable.

2.3 3

Let  $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ .

2.3.1 a

Define in terms of linear transformations what it means for f to be differentiable at a point  $(a,b) \in \mathbb{R}^2$ .

2.3.2 b

State a version of the inverse function theorem in this setting.

2.3.3 c

Identify  $\mathbb{R}^2$  with  $\mathbb{C}$  and give a necessary and sufficient condition for a real-differentiable function at (a, b) to be complex differentiable at the point a + ib.

2.4 4

Let f = u + iv be complex-differentiable with continuous partial derivatives at a point  $z = re^{i\theta}$  with  $r \neq 0$ . Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta} \,.$$

2.5 5

Let  $P = (1,3) \in \mathbb{R}^2$  and define

$$f(s,t) := ps^3 - 6st + t^2.$$

2.5.1 a

State the conclusion of the implicit function theorem concerning f(s,t) = 0 when f is considered a function  $\mathbb{R}^2 \longrightarrow \mathbb{R}$ .

2.5.2 b

State the above conclusion when f is considered a function  $\mathbb{C}^2 \longrightarrow \mathbb{C}$ .

### 2.5.3 c

Use the implicit function theorem for a function  $\mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$  to prove (b).

There are various approaches: using the definition of the complex derivative, the Cauchy-Riemann equations, considering total derivatives, etc.

# 2.6 6

Let  $F: \mathbb{R}^2 \longrightarrow \mathbb{R}$  be continuously differentiable with F(0,0) = 0 and  $\|\nabla F(0,0)\| < 1$ .

Prove that there is some real number r > 0 such that |F(x,y)| < r whenever ||(x,y)|| < r.

# 2.7 7

State the most general version of the implicit function theorem for real functions and outline how it can be proved using the inverse function theorem.

### 3 Conformal Maps (8155c)

Notation:  $\mathbb{D}$  is the open unit disc,  $\mathbb{H}$  is the open upper half-plane.

# 3.1 1

Find a conformal map from  $\mathbb D$  to  $\mathbb H.$ 

# 3.2 2

Find a conformal map from the strip  $\{z \in \mathbb{C} \mid 0 < \Im(z) < 1\}$  to  $\mathbb{H}$ .

# 3.3 3

Find a fractional linear transformation T which maps  $\mathbb H$  to  $\mathbb D$ , and explicitly describe the image of the first quadrant under T.

# 3.4 4

Find a conformal map from  $\{z \in \mathbb{C} \mid |z-i| > 1, \Re(z) > 0\}$  to  $\mathbb{H}$ .

# 3.5 5

Find a conformal map from  $\left\{z\in\mathbb{C}\ \Big|\ |z|<1,\ \left|z-\frac{1}{2}\right|>\frac{1}{2}\right\}$  to  $\mathbb{D}.$ 

3.6 6

Find a conformal map from  $\{|z-1|<2\}\bigcap\{|z+1|<2\}$  to  $\mathbb{H}$ .

3.7 7

Let  $\Omega$  be the region inside the unit circle |z|=1 and outside the circle  $\left|z-\frac{1}{4}\right|=\frac{1}{4}$ .

Find an injective conformal map from  $\Omega$  onto some annulus  $\{r < |z| < 1\}$  for gonstant r.

3.8 8

Let D be the region obtained by deleting the real interval [0,1) from  $\mathbb{D}$ ; find a conformal map from D to  $\mathbb{D}$ .

3.9 9

Find a conformal map from  $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$  to  $\mathbb{D}$ .

3.10 10

Find a conformal map from  $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \geq 1\}$  to  $\mathbb{D}$ .

3.11 11

Find a bijective conformal map from G to  $\mathbb{H}$ , where

$$G \coloneqq \left\{ z \in \mathbb{C} \mid |z - 1| < \sqrt{2}, |z + 1| < \sqrt{2} \right\} \setminus [0, i).$$

3.12 12

Prove that TFAE for a Möbius transformation T given by  $T(z) = \frac{az+b}{cz+d}$ :

- a. T maps  $\mathbb{R} \bigcup \{\infty\}$  to itself.
- b. It is possible to choose a, b, c, d to be real numbers.
- c.  $\overline{T(z)} = T(\overline{z})$  for every  $z \in \mathbb{CP}^1$ .
- d. There exist  $\alpha \in \mathbb{R}, \beta \in \mathbb{C} \setminus \mathbb{R}$  such that  $T(\alpha) = \alpha$  and  $T(\overline{\beta}) = \overline{T(\beta)}$ .

4 Integrals and Cauchy's Theorem (8155d)

Some interesting problems: 3, 4, 9, 10.

# 4.1 1

Suppose  $f, g: [0,1] \longrightarrow \mathbb{R}$  where f is Riemann integrable and for  $x, y \in [0,1]$ ,

$$|g(x) - g(y)| \le |f(x) - f(y)|.$$

Prove that q is Riemann integrable.

# 4.2 2

State and prove Green's Theorem for rectangles.

Then use it to prove Cauchy's Theory for functions that are analytic in a rectangle.

# 4.3 3

Suppose  $\{f_n\}_{n\in\mathbb{N}}$  is a sequence of analytic functions on  $\mathbb{D}:=\{z\in\mathbb{C}\ \big|\ |z|<1\}$ .

Show that if  $f_n \longrightarrow g$  for some  $g: \mathbb{D} \longrightarrow \mathbb{C}$  uniformly on every compact  $K \subset \mathbb{D}$ , then g is analytic on  $\mathbb{D}$ .

# 4.4 4

Suppose  $\{f_n\}_{n\in\mathbb{N}}$  is a sequence of entire functions where

- f<sub>n</sub> → g pointwise for some g: C → C.
  On every line segment in C, f<sub>n</sub> → g uniformly.

Show that

- g is entire, and
- $f_n \longrightarrow g$  uniformly on every compact subset of  $\mathbb{C}$ .

### 4.5 5

Prove that there is no sequence of polynomials that uniformly converge to  $f(z) = \frac{1}{z}$  on  $S^1$ .

### **Solution:**

#### Concepts Used:

- By Cauchy's integral formula,  $\int_{S^1} f = 2\pi i$
- If  $p_j$  is any polynomial, then  $p_j$  is holomorphic in  $\mathbb{D}$ , so  $\int_{S^1} p_j = 0$ .
- Contradiction: compact sets in  $\mathbb C$  are bounded, so

$$\left| \int f - \int p_j \right| \le \int |p_j - f| \le \int \|p_j - f\|_{\infty} = \|p_j - f\|_{\infty} \int_{S^1} 1 \, dz = \|p_j - f\|_{\infty} \cdot 2\pi \longrightarrow 0$$
 which forces 
$$\int f = \int p_j = 0.$$

### 4.6 6

Suppose that  $f: \mathbb{R} \longrightarrow \mathbb{R}$  is a continuous function that vanishes outside of some finite interval. For each  $z \in \mathbb{C}$ , define

$$g(z) = \int_{-\infty}^{\infty} f(t)e^{-izt} dt.$$

Show that q is entire.

### 4.7 7

Suppose  $f: \mathbb{C} \longrightarrow \mathbb{C}$  is entire and

$$|f(z)| \le |z|^{\frac{1}{2}}$$
 when  $|z| > 10$ .

Prove that f is constant.

# 4.8 8

Let  $\gamma$  be a smooth curve joining two distinct points  $a, b \in \mathbb{C}$ .

Prove that the function

$$f(z) := \int_{\gamma} \frac{g(w)}{w - z} \, dw$$

is analytic in  $\mathbb{C} \setminus \gamma$ .

### 4.9 9 🐪

Suppose that  $f: \mathbb{C} \longrightarrow \mathbb{C}$  is continuous everywhere and analytic on  $\mathbb{C} \setminus \mathbb{R}$  and prove that f is entire.

### Something missing?

#### Solution:

### Concepts Used:

- Note f is continuous on  $\mathbb{C}$  since analytic implies continuous (f equals its power series, where the partials sums uniformly converge to it, and uniform limit of continuous is continuous).
- Strategy: take D a disc centered at a point  $x \in \mathbb{R}$ , show f is holomorphic in D by Morera's theorem.
- Let  $\Delta \subset D$  be a triangle in D.
- Case 1: If  $\Delta \cap \mathbb{R} = 0$ , then f is holomorphic on  $\Delta$  and  $\int_{\Delta} f = 0$ .
- Case 2: one side or vertex of  $\Delta$  intersects  $\mathbb{R}$ , and wlog the rest of  $\Delta$  is in  $\mathbb{H}^+$ .
  - Then let  $\Delta_{\varepsilon}$  be the perturbation  $\Delta + i\varepsilon = \{z + i\varepsilon \mid z \in \Delta\}$ ; then  $\Delta_{\varepsilon} \cap \mathbb{R} = 0$  and

$$\int_{\Delta_{\varepsilon}} f = 0.$$
- Now let  $\varepsilon \longrightarrow 0$  and conclude by continuity of  $f$  (???)

$$\int_{\Delta_{\varepsilon}} f = \int_{a}^{b} f(\gamma_{\varepsilon}(t)) \gamma_{\varepsilon}'(t) dt \xrightarrow{\varepsilon \longrightarrow 0} \int_{a}^{b} f(\gamma(t)) \gamma_{\varepsilon}'(t) dt = \int_{\Delta} f$$

where  $\gamma_{\varepsilon}, \gamma$  are curves parametrizing  $\Delta_{\varepsilon}, \Delta$  respectively.

- \* Since  $\gamma, \gamma_{\varepsilon}$  are closed and bounded in  $\mathbb{C}$ , they are compact subsets. Thus it suffices to show that  $f(\gamma_{\varepsilon}(t))\gamma'_{\varepsilon}(t)$  converges uniformly to  $f(\gamma(t))\gamma'(t)$ .
- \* ??
- Case 3:  $\Delta$  intersects both  $\mathbb{H}^+$  and  $\mathbb{H}^-$ .
  - Break into smaller triangles, each of which falls into one of the previous two cases.

### 4.10 10

Prove Liouville's theorem: suppose  $f:\mathbb{C}\longrightarrow\mathbb{C}$  is entire and bounded. Use Cauchy's formula to prove that  $f'\equiv 0$  and hence f is constant.

### Solution:

### Concepts Used:

- Suffices to prove f' = 0 because  $\mathbb{C}$  is connected (see Stein Ch 1, 3.4)
  - Idea: Fix  $w_0$ , show  $f(w) = f(w_0)$  for any  $w \neq w_0$
  - Connected = Path connected in  $\mathbb{C}$ , so take  $\gamma$  joining w to  $w_0$ .
  - f is a primitive for f', and  $\int_{\gamma} f' = f(w) f(w_0)$ , but f' = 0.
- Fix  $z_0 \in \mathbb{C}$ , let B be the bound for f, so  $|f(z)| \leq B$  for all z.
- Apply Cauchy inequalities: if f is holomorphic on  $U \supset \overline{D}_R(z_0)$  then setting  $||f||_C := \sup_{z \in C} |f(z)|$ ,

$$\left| f^{(n)}(z_0) \right| \le \frac{n! \|f\|_C}{R^n}.$$

- Yields  $|f'(z_0)| \leq B/R$
- Take  $R \longrightarrow \infty$ , QED.

### 5 Liouville's Theorem, Power Series (8155e)

### 5.1 1 🙀

Suppose f is analytic on a region  $\Omega$  such that  $\mathbb{D} \subseteq \Omega \subseteq \mathbb{C}$  and  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  is a power series with radius of convergence exactly 1.

5.1.1 a

Give an example of such an f that converges at every point of  $S^1$ .

5.1.2 b

Give an example of such an f which is analytic at 1 but  $\sum_{n=1}^{\infty} a_n$  diverges.

5.1.3 c

Prove that f can not be analytic at every point of  $S^1$ .

Missing part (c)

Solution:

Concepts Used:

5.1.4 a

Take  $\sum \frac{z^n}{n^2}$ ; then  $|z| \le 1 \implies \left|\frac{z^n}{n^2}\right| \le \frac{1}{n^2}$  which is summable, so the series converges for

5.1.5 b

Take  $\sum \frac{z^n}{n}$ ; then z = 1 yields the harmonic series, which diverges. • For  $z \in S^1 \setminus \{1\}$ , we have  $z = e^{2\pi i t}$  for  $0 < t < 2\pi$ .

- Toward applying the Dirichlet test, set  $a_n = 1/n, b_n = z^n$ .
- Then for all N,

$$\left| \sum_{n=1}^{N} b_n \right| = \left| \sum_{n=1}^{N} b_n \right| = \left| \sum_{n=1}^{N} z^n \right| = \left| \frac{z - z^{N+1}}{|1 - z|} \right| \le \frac{2}{1 - z} < \infty.$$

• Thus  $\sum a_n b_n < \infty$  and  $\sum z^n/n$  converges.

5.2 2

Suppose f is entire and has Taylor series  $\sum a_n z^n$  about 0.

5.2.1 a

Express  $a_n$  as a contour integral along the circle |z| = R.

### 5.2.2 b

Apply (a) to show that the above Taylor series converges uniformly on every bounded subset of  $\mathbb{C}$ .

### 5.2.3 c

Determine those functions f for which the above Taylor series converges uniformly on all of  $\mathbb{C}$ .

# 5.3 3

Suppose D is a domain and f, g are analytic on D.

Prove that if fg = 0 on D, then either  $f \equiv 0$  or  $g \equiv 0$  on D.

### 5.4 4

Suppose f is analytic on  $\mathbb{D}^{\circ}$ . Determine with proof which of the following are possible:

a. 
$$f\left(\frac{1}{n}\right) = (-1)^n$$
 for each  $n > 1$ .

b. 
$$f\left(\frac{1}{n}\right) = e^{-n}$$
 for each even integer  $n > 1$  while  $f\left(\frac{1}{n}\right) = 0$  for each odd integer  $n > 1$ .

c. 
$$f\left(\frac{1}{n^2}\right) = \frac{1}{n}$$
 for each integer  $n > 1$ .

d. 
$$f\left(\frac{1}{n}\right) = \frac{n-2}{n-1}$$
 for each integer  $n > 1$ .

# 5.5 5 🐪

Prove the Fundamental Theorem of Algebra (using complex analysis).

### Solution:

#### Concepts Used:

- Strategy: By contradiction with Liouville's Theorem
- Suppose *p* is non-constant and has no roots.
- Claim: 1/p(z) is a bounded holomorphic function on  $\mathbb{C}$ .
  - Holomorphic: clear? Since p has no roots.
  - Bounded: for  $z \neq 0$ , write

$$\frac{P(z)}{z^n} = a_n + \left(\frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n}\right).$$

- The term in parentheses goes to 0 as  $|z| \longrightarrow \infty$
- Thus there exists an R > 0 such that

$$|z| > R \implies \left| \frac{P(z)}{z^n} \right| \ge c \coloneqq \frac{|a_n|}{2}.$$

- So p is bounded below when |z| > R
- Since p is continuous and has no roots in  $|z| \leq R$ , it is bounded below when  $|z| \leq R$ .
- Thus p is bounded below on  $\mathbb{C}$  and thus 1/p is bounded above on  $\mathbb{C}$ .
- By Liouville's theorem, 1/p is constant and thus p is constant, a contradiction.

### 5.6 6

Find all entire functions that satisfy

$$|f(z)| \ge |z| \quad \forall z \in \mathbb{C}.$$

Prove this list is complete.

#### Solution:

#### Concepts Used:

- Suppose f is entire and define  $g(z) := \frac{z}{f(z)}$ .
- By the inequality,  $|g(z)| \le 1$ , so g is bounded.
- g potentially has singularities at the zeros  $Z_f := f^{-1}(0)$ , but since f is entire, g is holomorphic on  $\mathbb{C} \setminus Z_f$ .
- Claim:  $Z_f = \{0\}.$ 
  - If f(z) = 0, then  $|z| \le |f(z)| = 0$  which forces z = 0.
- We can now apply Riemann's removable singularity theorem:
  - Check g is bounded on some open subset  $D \setminus \{0\}$ , clear since it's bounded everywhere
  - Check g is holomorphic on  $D \setminus \{0\}$ , clear since the only singularity of g is z = 0.
- By Riemann's removable singularity theorem, the singularity z = 0 is removable and g has an extension to an entire function  $\tilde{q}$ .
- By continuity, we have  $|\tilde{g}(z)| \leq 1$  on all of  $\mathbb{C}$ 
  - If not, then  $|\tilde{g}(0)| = 1 + \varepsilon > 1$ , but then there would be a domain  $\Omega \subseteq \mathbb{C} \setminus \{0\}$  such that  $1 < |\tilde{g}(z)| \le 1 + \varepsilon$  on  $\Omega$ , a contradiction.
- By Liouville,  $\tilde{g}$  is constant, so  $\tilde{g}(z) = c_0$  with  $|c_0| \leq 1$
- Thus  $f(z) = c_0^{-1}z := cz$  where  $|c| \ge 1$

Thus all such functions are of the form f(z) = cz for some  $c \in \mathbb{C}$  with  $|c| \ge 1$ .

### 5.7 7

Suppose  $\sum_{n=0}^{\infty} a_n z^n$  converges for some  $z_0 \neq 0$ .

### 5.7.1 a

Prove that the series converges absolutely for each z with  $|z| < |z|_0$ .

5.7.2 b

Suppose  $0 < r < |z_0|$  and show that the series converges uniformly on  $|z| \le r$ .

5.8 8

Suppose f is entire and suppose that for some integer  $n \geq 1$ ,

$$\lim_{z \longrightarrow \infty} \frac{f(z)}{z^n} = 0.$$

Prove that f is a polynomial of degree at most n-1.

5.9 9

Find all entire functions satisfying

$$|f(z)| \le |z|^{\frac{1}{2}}$$
 for  $|z| > 10$ .

5.10 10

Prove that the following series converges uniformly on the set  $\{z \mid \Im(z) < \ln 2\}$ :

$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{2^n}.$$

# 6 Laurent Expansions and Singularities (8155f)

6.1 1 🐪

Find the Laurent expansion of

$$f(z) = \frac{z+1}{z(z-1)}$$

Solution:

Concepts Used:

Let 
$$f(z) = \frac{z+1}{z(z-1)}$$
.

$$f(z) = (z+1)\left(-\frac{1}{z} + \frac{1}{z-1}\right)$$

$$= -(z+1)\left(\frac{1}{z} + \sum_{n=0}^{\infty} z^n\right)$$

$$= -(z+1)\sum_{n=-1}^{\infty} z^n$$

$$= \frac{1}{z} + 2\sum_{n=0}^{\infty} z^n$$

$$= -\frac{1}{z} - 2 - 2z - 2z^2 - \cdots$$

About z = 1:

$$f(z) = \left(\frac{(1-z)-2}{1-z}\right) \left(\frac{1}{1-(1-z)}\right)$$

$$= \left(1 - \frac{2}{1-z}\right) \sum_{n=0}^{\infty} (1-z)^n$$

$$= \sum_{n=0}^{\infty} (1-z)^n - 2 \sum_{n=-1}^{\infty} (1-z)^n$$

$$= -\frac{2}{1-z} - \sum_{n=0}^{\infty} (1-z)^n$$

$$= \frac{2}{z-1} + \sum_{n=0}^{\infty} (-1)^{n+1} (z-1)^n$$

$$= \frac{2}{z-1} - 1 + (z-1) - (z-1)^2 + \cdots$$

about z = 0 and z = 1 respectively.

# 6.2 2 🙀

Find the Laurent expansions about z=0 of the following functions:

$$\exp\left(\frac{1}{z}\right) \qquad \qquad \cos\left(\frac{1}{z}\right).$$

#### Solution:

Concepts Used:

Let 
$$f(z) = \frac{z+1}{z(z-1)}$$
.  
About  $z = 0$ :

$$f(z) = (z+1)\left(-\frac{1}{z} + \frac{1}{z-1}\right)$$

$$= -(z+1)\left(\frac{1}{z} + \sum_{n=0}^{\infty} z^n\right)$$

$$= -(z+1)\sum_{n=-1}^{\infty} z^n$$

$$= \frac{1}{z} + 2\sum_{n=0}^{\infty} z^n$$

$$= -\frac{1}{z} - 2 - 2z - 2z^2 - \cdots$$

About z = 1:

$$f(z) = \left(\frac{(1-z)-2}{1-z}\right) \left(\frac{1}{1-(1-z)}\right)$$

$$= \left(1 - \frac{2}{1-z}\right) \sum_{n=0}^{\infty} (1-z)^n$$

$$= \sum_{n=0}^{\infty} (1-z)^n - 2 \sum_{n=-1}^{\infty} (1-z)^n$$

$$= -\frac{2}{1-z} - \sum_{n=0}^{\infty} (1-z)^n$$

$$= \frac{2}{z-1} + \sum_{n=0}^{\infty} (-1)^{n+1} (z-1)^n$$

$$= \frac{2}{z-1} - 1 + (z-1) - (z-1)^2 + \cdots$$

# 6.3 3

Find the Laurent expansion of

$$f(z) = \frac{z+1}{z(z-1)^2}$$

about z = 0 and z = 1 respectively.

Hint: recall that power series can be differentiated.

# 6.4 4

For the following functions, find the Laurent series about 0 and classify their singularities there:

$$\frac{\sin^2(z)}{z}$$

$$z \exp\left(\frac{1}{z^2}\right)$$

$$\frac{1}{z(4-z)}.$$

# 6.5 5

Find all entire functions with have poles at  $\infty$ .

# 6.6 6

Find all functions on the Riemann sphere that have a simple pole at z=2 and a double pole at  $z=\infty$ , but are analytic elsewhere.

# 6.7 7

Let f be entire, and discuss (with proofs and examples) the types of singularities f might have (removable, pole, or essential) at  $z = \infty$  in the following cases:

- 1. f has at most finitely many zeros in  $\mathbb{C}$ .
- 2. f has infinitely many zeros in  $\mathbb{C}$ .

# 6.8 8

Define

$$f(z) = \frac{\pi^2}{\sin^2(\pi z)}$$
$$g(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}.$$

- a. Show that f and g have the same singularities in  $\mathbb{C}$ .
- b. Show that f and g have the same singular parts at each of their singularities.
- c. Show that f, g each have period one and approach zero uniformly on  $0 \le x \le 1$  as  $|y| \longrightarrow \infty$ .
- d. Conclude that f = g.

#### Not finished

### Solution:

### Concepts Used:

Idea: show their f - g is analytic by taking away all of the negative powers, and bounded by (c).

# **7 Residues (8155g)**

7.1 1

Calculate

$$\int_0^\infty \frac{1}{(1+z)^2(z+9x^2)} \, dx.$$

7.2 2

Let a > 0 and calculate

$$\int_0^\infty \frac{x \sin(x)}{x^2 + a^2} \, dx.$$

7.3 3

Calculate

$$\int_0^\infty \frac{\sqrt{x}}{(x+1)^2} \, dx.$$

7.4 4

Calculate

$$\int_0^\infty \frac{\cos(x) - \cos(4x)}{x^2} \, dx.$$

7.5 5

Let a > 0 and calculate

$$\int_0^\infty \frac{x^2}{(x^2 + a^2)^2} \, dx.$$

7.6 6

Calculate

$$\int_0^\infty \frac{\sin(x)}{x} \, dx.$$

# 7.7 7

Calculate

$$\int_0^\infty \frac{\sin(x)}{x(x^2+1)} \, dx.$$

# 7.8 8

Calculate

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2} \, dx.$$

# 7.9 9

Calculate

$$\int_{-\infty}^{\infty} \frac{1+x^2}{1+x^4} \, dx.$$

# 7.10 10

Let a > 0 and calculate

$$\int_0^\infty \frac{\cos(x)}{(x^2 + a^2)^2} \, dx.$$

# 7.11 11

Calculate

$$\int_0^\infty \frac{\sin^3(x)}{x^3} \, dx.$$

# 7.12 12

Let  $n \in \mathbb{Z}^{\geq 1}$  and  $0 < \theta < \pi$  and show that

$$\frac{1}{2\pi i} \int_{|z|=2} \frac{z^n}{1 - 3z\cos(\theta) + z^2} dz = \frac{\sin(n\theta)}{\sin(\theta)}.$$

# 7.13 13

Suppose a > b > 0 and calculate

$$\int_0^{2\pi} \frac{1}{(a+b\cos(\theta))^2} \, d\theta.$$

### 8 Rouche's Theorem (8155h)

### 8.1 1 🙀

Prove that for every  $n \in \mathbb{Z}^{\geq 0}$  the following polynomial has no roots in the open unit disc:

$$f_n(x) \coloneqq \sum_{k=0}^n \frac{z^k}{k!}.$$

Hint: check n = 1, 2 directly.

#### Solution:

### Concepts Used:

Note

- $f_1(z) = 1 + z$ , which has the single root z = -1 which is not inside |z| < 1.
- $f_2(z) = 1 + z + \frac{1}{2}z^2 = (z (1+i))(z (1-i))$ , and  $|1 \pm i| = \sqrt{2} > 1$ .
- Note that  $p_n(z) \stackrel{\sim}{n \to \infty}^z$  uniformly on any compact set.
- Let r be arbitrary and fix  $N := \mathbb{D}_r(0)$ , then  $p_n(z) \longrightarrow e^z$  uniformly on  $\overline{N}$ .
- Set  $g_n(z) := p_n(z)/e^z$ , then  $g_n \longrightarrow 1$  uniformly on  $\overline{N}$ .
- Choose  $n \gg 0$  so that  $|f(z) 1| < \varepsilon < 1$  for all  $z \in \overline{N}$ .
- So take h(z) = 1, then on  $\partial N$ ,?

### 8.2 2

Assume that |b| < 1 and show that the following polynomial has exactly two roots (counting multiplicity) in |z| < 1:

$$f(z) \coloneqq z^3 + 3z^2 + bz + b^2.$$

#### Solution:

### Concepts Used:

Multiple versions of Rouches theorem!

- Set  $h(z) = 3z^2$  and  $g(z) = z^3 + bz + b^2$ .
- Then on |z|=1,

$$|g(z)| \le 1 + b + b^2 < 3 = 3|z|^2 = |3z^2| = |h|,$$

so g, h have the same number of roots in  $|z| \leq_? 1$ .

• But h evidently has two roots in this region.

# 8.3 3

Let  $c \in \mathbb{C}$  with  $|c| < \frac{1}{3}$ . Show that on the open set  $\{z \in \mathbb{C} \mid \Re(z) < 1\}$ , the function  $f(z) := ce^z$  has exactly one fixed point.

### 8.4 4 🐪

How many roots does the following polynomial have in the open disc |z| < 1?

$$f(z) = z^7 - 4z^3 - 1.$$

### Solution:

### Concepts Used:

• Set  $h(z) = -4z^3$  and  $g(z) = z^7 - 1$ , then on |z| = 1,

$$|g(z)| = |z^7 - 1| \le 1 + 1 = 2 < 4 = |-4z^3| = |h(z)|.$$

• So h and h+g have the same number of roots, but h has three roots here.

### 8.5 5 🐪

Let  $n \in \mathbb{Z}^{\geq 0}$  and show that the equation

$$e^z = az^n$$

has n solutions in the open unit disc if |a| > e, and no solutions if  $|a| < \frac{1}{e}$ .

# 8.6 6

Let f be analytic in a domain D and fix  $z_0 \in D$  with  $w_0 := f(z_0)$ . Suppose  $z_0$  is a zero of  $f(z) - w_0$  with finite multiplicity m. Show that there exists  $\delta > 0$  and  $\varepsilon > 0$  such that for each w such that  $0 < |w - w_0| < \varepsilon$ , the equation f(z) - w = 0 has exactly m distinct solutions inside the disc  $|z - z_0| < \delta$ .

# 8.7 7

For  $k = 1, 2, \dots, n$ , suppose  $|a_k| < 1$  and

$$f(z) := \left(\frac{z - a_1}{1 - \overline{a}_q z}\right) \left(\frac{z - a_2}{1 - \overline{a}_2 z}\right) \cdots \left(\frac{z - a_n}{1 - \overline{a}_n z}\right).$$

Show that f(z) = b has n solutions in |z| < 1.

# 8.8 8

For each  $n \in \mathbb{Z}^{\geq 1}$ , let

$$P_n(z) = 1 + z + \frac{1}{2!}z^2 + \dots + \frac{1}{n!}z^n.$$

Show that for sufficiently large n, the polynomial  $P_n$  has no zeros in |z| < 10, while the polynomial  $P_n(z) - 1$  has precisely 3 zeros there.

# 8.9 9

Prove that

$$\max_{|z|=1} \left| a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + z^n \right| \ge 1.$$

Hint: the first part of the problem asks for a statement of Rouche's theorem.

# 8.10 10

Use Rouche's theorem to prove the Fundamental Theorem of Algebra.

### 9 Schwarz Lemma and Reflection Principle (8155i)

# 9.1 1

Suppose  $f: \mathbb{D} \longrightarrow \mathbb{D}$  is analytic and admits a continuous extension  $\tilde{f}: \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$  such that  $|z| = 1 \Longrightarrow |f(z)| = 1$ .

# 9.1.1 a

Prove that f is a rational function.

# 9.1.2 b

Suppose that z = 0 is the unique zero of f. Show that

$$\exists n \in \mathbb{N}, \lambda \in S^1$$
 such that  $f(z) = \lambda z^n$ .

# 9.1.3 c

Suppose that  $a_1, \dots, a_n \in \mathbb{D}$  are the zeros of f and prove that

$$\exists \lambda \in S^1$$
 such that  $f(z) = \lambda \prod_{j=1}^n \frac{z - a_j}{1 - \overline{a_j} z}$ .

# 9.2 2

Let  $\overline{B}(a,r)$  denote the closed disc of radius r about  $a \in \mathbb{C}$ . Let f be holomorphic on an open set containing  $\overline{B}(a,r)$  and let

$$M \coloneqq \sup_{z \in \overline{B}(a,r)} |f(z)|.$$

Prove that

$$z \in \overline{B}\left(a, \frac{r}{2}\right), \ z \neq a, \qquad \frac{|f(z) - f(a)|}{|z - a|} \leq \frac{2M}{r}.$$

# 9.3 3

Define

$$G := \left\{ z \in \mathbb{C} \mid \Re(z) > 0, \, |z - 1| > 1 \right\}.$$

Find all of the injective conformal maps  $G \longrightarrow \mathbb{D}$ . These may be expressed as compositions of maps, but explain why this list is complete.

# 9.4 4

Suppose  $f: \mathbb{H} \bigcup \mathbb{R} \longrightarrow \mathbb{C}$  satisfies the following:

- f(i) = i
- f is continuous
- f is analytic on  $\mathbb{H}$
- $f(z) \in \mathbb{R} \iff z \in \mathbb{R}$ .

Show that  $f(\mathbb{H})$  is a dense subset of  $\mathbb{H}$ .

### 9.5 5

Suppose  $f: \mathbb{D} \longrightarrow \mathbb{H}$  is analytic and satisfies f(0) = 2. Find a sharp upper bound for |f'(0)|, and prove it is sharp by example.

# 9.6 6

Suppose  $f: \mathbb{D} \longrightarrow \mathbb{D}$  is analytic, has a single zero of order k at z=0, and satsifies  $\lim_{|z| \longrightarrow 1} |f(z)| = 1$ . Give with proof a formula for f(z).

### 9.7 7

### 9.7.1 a

State the standard Schwarz reflection principle involving reflection across the real axis.

### 9.7.2 b

Give a linear fractional transformation T mapping  $\mathbb{D}$  to  $\mathbb{H}$ . Let  $g(z) = \bar{z}$ , and show

$$(T^{-1} \circ g \circ T)(z) = 1/\bar{z}.$$

# 9.7.3 c

Suppose that f is holomorphic on  $\mathbb{D}$ , continuous on  $\overline{\mathbb{D}}$ , and real on  $S^1$ . Show that f must be constant.

# 9.8 8

Suppose  $f, g : \mathbb{D} \longrightarrow \Omega$  are holomorphic with f injective and f(0) = g(0).

Show that

$$\forall 0 < r < 1, \qquad g(\{|z| < r\}) \subseteq f(\{|z| < r\}).$$

The first part of this problem asks for a statement of the Schwarz lemma.

# 9.9 9

Let  $S := \{z \in \mathbb{D} \mid \Im(z) \geq 0\}$ . Suppose  $f: S \longrightarrow \mathbb{C}$  is continuous on S, real on  $S \cap \mathbb{R}$ , and holomorphic on  $S^{\circ}$ .

Prove that f is the restriction of a holomorphic function on  $\mathbb{D}$ .

# 9.10 10

Suppose  $f: \mathbb{D} \longrightarrow \mathbb{D}$  is analytic. Prove that

$$\forall a \in \mathbb{D}, \qquad \frac{|f'(a)|}{1 - |f(a)|^2} \le \frac{1}{1 - |a|^2}.$$

# 10 Spring 2020 Homework 1

#### 10.1 1

Geometrically describe the following subsets of  $\mathbb{C}$ :

- a. |z 1| = 1
- b. |z-1| = 2|z-2|
- c.  $1/z = \bar{z}$
- d.  $\Re(z) = 3$
- e.  $\Im(z) = a$  with  $a \in \mathbb{R}$ .
- f.  $\Re(z) > a$  with  $a \in \mathbb{R}$ .
- g. |z-1| < 2|z-2|

#### 10.2 2

Prove the following inequality, and explain when equality holds:

$$|z+w| > ||z| - |w||$$
.

### 10.3 3

Prove that the following polynomial has its roots outside of the unit circle:

$$p(z) = z^3 + 2z + 4.$$

Hint: What is the maximum value of the modulus of the first two terms if  $|z| \leq 1$ ?

### 10.4 4

a. Prove that if c > 0,

$$|w_1| = c|w_2| \implies |w_1 - c^2w_2| = c|w_1 - w_2|.$$

b. Prove that if c > 0 and  $c \neq 1$ , with  $z_1 \neq z_2$ , then the following equation represents a circle:

$$\left| \frac{z - z_1}{z - z_2} \right| = c.$$

Find its center and radius.

Hint: use part (a)

### 10.5 5

a. Let  $z, w \in \mathbb{C}$  with  $\bar{z}w \neq 1$ . Prove that

$$\left| \frac{w - z}{1 - \overline{w}z} \right| < 1 \quad \text{if } |z| < 1, \ |w| < 1$$

with equality when |z| = 1 or |w| = 1.

b. Prove that for a fixed  $w \in \mathbb{D}$ , the mapping  $F: z \mapsto \frac{w-z}{1-\overline{w}z}$  satisfies

- F maps  $\mathbb D$  to itself and is holomorphic.
- F(0) = w and F(w) = 0.
- |z| = 1 implies |F(z)| = 1.

### 10.6 6

Use nth roots of unity to show that

$$2^{n-1}\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\cdots\sin\left(\frac{(n-1)\pi}{n}\right)=n.$$

Hint:

$$1 - \cos(2\theta) = 2\sin^2(\theta)$$
$$2\sin(2\theta) = 2\sin(\theta)\cos(\theta).$$

### 10.7 7

Prove that  $f(z) = |z|^2$  has a derivative at z = 0 and nowhere else.

### 10.8 8

Let f(z) be analytic in a domain, and prove that f is constant if it satisfies any of the following conditions:

- a. |f(z)| is constant.
- b.  $\Re(f(z))$  is constant.
- c. arg(f(z)) is constant.
- d. f(z) is analytic.

How do you generalize (a) and (b)?

### 10.9 9

Prove that if  $z\mapsto f(z)$  is analytic, then  $z\mapsto \overline{f(\bar{z})}$  is analytic.

### 10.10 10

a. Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$
 and  $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$ .

b. Use (a) to show that the logarithm function, defined as

$$\log z = \log r + i\theta$$
 where  $z = re^{i\theta}$  with  $-\pi < \theta < \pi$ .

is holomorphic on the region  $r > 0, -\pi < \theta < \pi$ .

Also show that this function is not continuous in r > 0.

### 10.11 11

Prove that the distinct complex numbers  $z_1, z_2, z_3$  are the vertices of an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

### 11 Spring 2020 Homework 2

Note on notation: I sometimes use  $f_x := \frac{\partial f}{\partial x}$  to denote partial derivatives, and  $\partial_z^n f$  as  $f^{(n)}(z)$ .

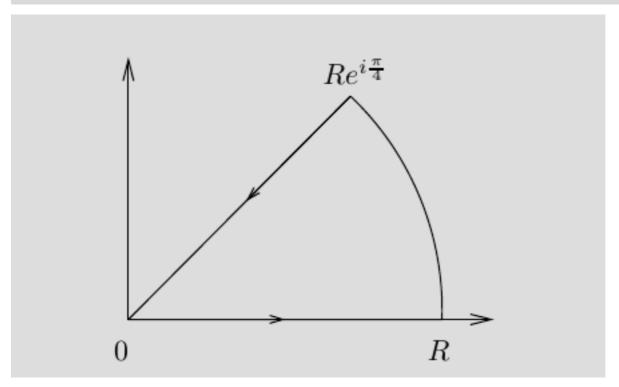
### 11.1 Stein And Shakarchi

### 11.1.1 2.6.1

Show that

$$\int_0^\infty \sin\left(x^2\right) dx = \int_0^\infty \cos\left(x^2\right) dx = \frac{\sqrt{2\pi}}{4}.$$

Hint: integrate  $e^{-x^2}$  over the following contour, using the fact that  $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ :



### 11.1.2 2.6.2

Show that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Hint: use the fact that this integral equals  $\frac{1}{2i} \int_{-\infty}^{\infty} \frac{e^{ix} - 1}{x} dx$ , and integrate around an indented semicircle.

#### 11.1.3 2.6.5

Suppose  $f \in C^1_{\mathbb{C}}(\Omega)$  and  $T \subset \Omega$  is a triangle with  $T^{\circ} \subset \Omega$ . Apply Green's theorem to show that  $\int_T f(z) \ dz = 0$ .

Assume that f' is continuous and prove Goursat's theorem.

Hint: Green's theorem states

$$\int_{T} F dx + G dy = \int_{T^{\circ}} \left( \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx dy.$$

#### 11.1.4 2.6.6

Suppose that f is holomorphic on a punctured open set  $\Omega \setminus \{w_0\}$  and let  $T \subset \Omega$  be a triangle containing  $w_0$ . Prove that if f is bounded near  $w_0$ , then  $\int_T f(z) dz = 0$ .

#### 11.1.5 2.6.7

Suppose  $f: \mathbb{D} \longrightarrow \mathbb{C}$  is holomorphic and let  $d := \sup_{z,w \in \mathbb{D}} |f(z) - f(w)|$  be the diameter of the image of f. Show that  $2|f'(0)| \leq d$ , and that equality holds iff f is linear, so  $f(z) = a_1z + a_2$ .

Hint: 
$$2f'(0) = \frac{1}{2\pi i} \int_{|\xi| = r} \frac{f(\xi) - f(-\xi)}{\xi^2} d\xi$$
 whenever  $0 < r < 1$ .

#### 11.1.6 2.6.8

Suppose that f is holomorphic on the strip  $S = \{x + iy \mid x \in \mathbb{R}, -1 < y < 1\}$  with  $|f(z)| \le A(1+|z|)^{\nu}$  for  $\nu$  some fixed real number. Show that for all  $z \in S$ , for each integer  $n \ge 0$  there exists an  $A_n \ge 0$  such that  $|f^{(n)}(x)| \le A_n(1+|x|)^{\nu}$  for all  $x \in \mathbb{R}$ .

Hint: Use the Cauchy inequalities.

#### 11.1.7 2.6.9

Let  $\Omega \subset \mathbb{C}$  be open and bounded and  $\varphi : \Omega \longrightarrow \Omega$  holomorphic. Prove that if there exists a point  $z_0 \in \Omega$  such that  $\varphi(z_0) = z_0$  and  $\varphi'(z_0) = 1$ , then  $\varphi$  is linear.

Hint: assume  $z_0 = 0$  (explain why this can be done) and write  $\varphi(z) = z + a_n z^n + O(z^{n+1})$  near 0. Let  $\varphi_k = \varphi \circ \varphi \circ \cdots \circ \varphi$  and prove that  $\varphi_k(z) = z + k a_n z^n + O(z^{n+1})$ . Apply Cauchy's inequalities and let  $k \longrightarrow \infty$  to conclude.

#### 11.1.8 2.6.10

Can every continuous function on  $\overline{\mathbb{D}}$  be uniformly approximated by polynomials in the variable z?

Hint: compare to Weierstrass for the real interval.

#### 11.1.9 2.6.13

Suppose f is analytic, defined on all of  $\mathbb{C}$ , and for each  $z_0 \in \mathbb{C}$  there is at least one coefficient in the expansion  $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$  is zero. Prove that f is a polynomial.

Hint: use the fact that  $c_n n! = f^{(n)}(z_0)$  and use a countability argument.

#### 11.1.10 2.6.14

Suppose that f is holomorphic in an open set containing  $\mathbb{D}$  except for a pole  $z_0 \in \partial \mathbb{D}$ . Let  $\sum_{n=0}^{\infty} a_n z^n$  be the power series expansion of f in  $\mathbb{D}$ , and show that  $\lim \frac{a_n}{a_{n+1}} = z_0$ .

#### 11.1.11 2.6.15

Suppose f is continuous and nonvanishing on  $\overline{\mathbb{D}}$ , and holomorphic in  $\mathbb{D}$ . Prove that if  $|z| = 1 \implies |f(z)| = 1$ , then f is constant.

Hint: Extend f to all of  $\mathbb{C}$  by  $f(z) = 1/\overline{f(1/\overline{z})}$  for any |z| > 1, and argue as in the Schwarz reflection principle.

#### 11.2 Additional Problems

#### 11.2.1 1

Let  $a_n \neq 0$  and show that

$$\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = L \implies \lim_{n \to \infty} |a_n|^{\frac{1}{n}} = L.$$

In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

#### 11.2.2 2

Let f be a power series centered at the origin. Prove that f has a power series expansion about any point in its disc of convergence.

#### 11.2.3 3

Prove the following:

- a.  $\sum_{n} nz^{n}$  does not converge at any point of  $S^{1}$
- b.  $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$  converges at every point of  $S^1$ .

c.  $\sum_{n} \frac{z^n}{n}$  converges at every point of  $S^1$  except z = 1.

#### 11.2.4 4

Without using Cauchy's integral formula, show that if |a| < r < |b|, then

$$\int_{\gamma} \frac{dz}{(z-\alpha)(z-\beta)} = \frac{2\pi i}{\alpha - \beta}$$

where  $\gamma$  denotes the circle centered at the origin of radius r with positive orientation.

#### 11.2.5 5

Assume f is continuous in the region  $\{x+iy \mid x \geq x_0, \ 0 \leq y \leq b\}$ , and the following limit exists independent of y:

$$\lim_{x \to +\infty} f(x + iy) = A.$$

Show that if  $\gamma_x := \{z = x + it \mid 0 \le t \le b\}$ , then

$$\lim_{x \longrightarrow +\infty} \int_{\gamma_x} f(z) \, dz = iAb.$$

#### 11.2.6 6

Show by example that there exists a function f(z) that is holomorphic on  $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$  and for all r < 1,

$$\int_{|z|=r} f(z) \, dz = 0,$$

but f is not holomorphic at z = 0.

#### 11.2.7 7

Let f be analytic on a region R and suppose  $f'(z_0) \neq 0$  for some  $z_0 \in R$ . Show that if C is a circle of sufficiently small radius centered at  $z_0$ , then

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{dz}{f(z) - f(z_0)}.$$

Hint: use the inverse function theorem.

### 11.2.8 8

Assume two functions  $u, b : \mathbb{R}^2 \longrightarrow \mathbb{R}$  have continuous partial derivatives at  $(x_0, y_0)$ . Show that f := u + iv has derivative  $f'(z_0)$  at  $z_0 := x_0 + iy_0$  if and only if

$$\lim_{r \to 0} \frac{1}{\pi r^2} \int_{|z-z_0|=r} f(z) dz = 0.$$

### 11.2.9 9 (Cauchy's Formula for Exterior Regions)

Let  $\gamma$  be a piecewise smooth simple closed curve with interior  $\Omega_1$  and exterior  $\Omega_2$ . Assume f' exists in an open set containing  $\gamma$  and  $\Omega_2$  with  $\lim_{z \to \infty} f(z) = A$ . Show that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1 \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}.$$

### 11.2.10 10

Let f(z) be bounded and analytic in  $\mathbb{C}$ . Let  $a \neq b$  be any fixed complex numbers. Show that the following limit exists:

$$\lim_{R \to \infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

Use this to show that f(z) must be constant.

### 11.2.11 11

Suppose f(z) is entire and

$$\lim_{z \longrightarrow \infty} \frac{f(z)}{z} = 0.$$

Show that f(z) is a constant.

### 11.2.12 12

Let f be analytic in a domain D and  $\gamma$  be a closed curve in D. For any  $z_0 \in D$  not on  $\gamma$ , show that

$$\int_{\gamma} \frac{f'(z)}{(z - z_0)} dz = \int_{\gamma} \frac{f(z)}{(z - z_0)^2} dz.$$

Give a generalization of this result.

### 11.2.13 13

Compute

$$\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z}$$

and use it to show that

$$\int_0^{2\pi} \cos^{2n}(\theta) d\theta = 2\pi \left( \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \right).$$

# 12 Spring 2020 Homework 3

# 12.1 Stein and Shakarchi

### 12.1.1 3.8.1

Use the following formula to show that the complex zeros of  $\sin(\pi z)$  are exactly the integers, and they are each of order 1:

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}.$$

Calculate the residue of  $\frac{1}{\sin(\pi z)}$  at  $z = n \in \mathbb{Z}$ .

# 12.1.2 3.8.2

Evaluate the integral

$$\int_{\mathbb{R}} \frac{dx}{1+x^4}.$$

What are the poles of  $\frac{1}{1+z^4}$ ?

### 12.1.3 3.8.4

Show that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}, \quad \text{for all } a > 0.$$

### 12.1.4 3.8.5

Show that if  $\xi \in \mathbb{R}$ , then

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{(1+x^2)^2} dx = \frac{\pi}{2} (1+2\pi |\xi|) e^{-2\pi |\xi|}.$$

### 12.1.5 3.8.6

Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \pi.$$

### 12.1.6 3.8.7

Show that

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos \theta)^2} = \frac{2\pi a}{(a^2 - 1)^{3/2}}, \quad \text{whenever } a > 1.$$

# 12.1.7 3.8.8

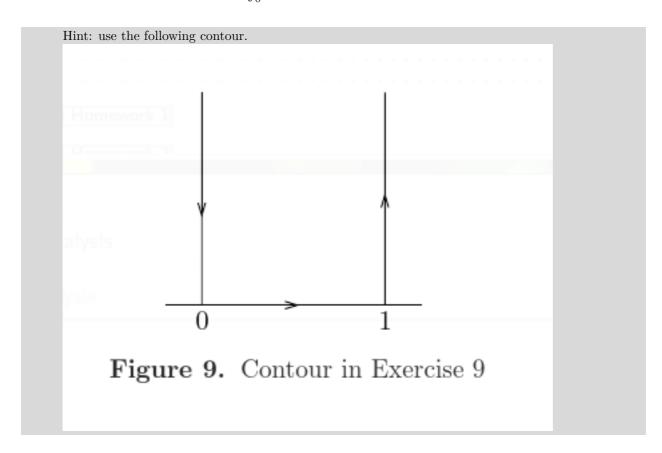
Show that if  $a, b \in \mathbb{R}$  with a > |b|, then

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

# 12.1.8 3.8.9

Show that

$$\int_0^1 \log(\sin \pi x) dx = -\log 2.$$



# 12.1.9 3.8.10

Show that if a > 0, then

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx = \frac{\pi}{2a} \log a.$$



### 12.1.10 3.8.14

Prove that all entire functions that are injective are of the form f(z) = az + b with  $a, b \in \mathbb{C}$  and  $a \neq 0$ .

Hint: Apply the Casorati-Weierstrass theorem to f(1/z).

# 12.1.11 3.8.15

Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

a. Prove that if f is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \le AR^k + B$$

for all R > 0, some integer  $k \ge 0$ , and some constants A, B > 0, then f is a polynomial of degree  $\le k$ .

- b. Show that if f is holomorphic in the unit disc, is bounded, and converges uniformly to zero in the sector  $\theta < \arg(z) < \varphi$  as  $|z| \longrightarrow 0$ , then  $f \equiv 0$ .
- c. Let  $w_1, \dots w_n$  be points on  $S^1 \subset \mathbb{C}$ . Prove that there exists a point  $z \in S^1$  such that the product of the distances from z to the points  $w_i$  is at least 1.

Conclude that there exists a point  $w \in S^1$  such that the product of the above distances is exactly 1.

d. Show that if the real part of an entire function is bounded, then f is constant.

#### 12.1.12 3.8.17

Let f be non-constant and holomorphic in an open set containing the closed unit disc.

a. Show that if |f(z)| = 1 whenever |z| = 1, then the image of f contains the unit disc.

Hint: Show that  $f(z) = w_0$  has a root for every  $w_0 \in \mathbb{D}$ , for which it suffices to show that f(z) = 0 has a root. Conclude using the maximum modulus principle.

b. If  $|f(z)| \ge 1$  whenever |z| = 1 and there exists a  $z_0 \in \mathbb{D}$  such that  $|f(z_0)| < 1$ , then the image of f contains the unit disc.

# 12.1.13 3.8.19

Prove that maximum principle for harmonic functions, i.e.

- a. If u is a non-constant real-valued harmonic function in a region  $\Omega$ , then u can not attain a maximum or a minimum in  $\Omega$ .
- b. Suppose  $\Omega$  is a region with compact closure  $\overline{\Omega}$ . If u is harmonic in  $\Omega$  and continuous in  $\overline{\Omega}$ , then

$$\sup_{z\in\Omega}|u(z)|\leq \sup_{z\in\overline{\Omega}-\Omega}|u(z)|.$$

Hint: to prove (a), assume u attains a local maximum at  $z_0$ . Let f be holomorphic near  $z_0$  with  $\Re(f) = u$ , and show that f is not an open map. Then (a) implies (b).

# 12.2 Extra Problems

# 12.2.1 1

**Problem** Prove that if f has two Laurent series expansions,

$$f(z) = \sum c_n(z-a)^n$$
 and  $f(z) = \sum c'_n(z-a)^n$ 

then  $c_n = c'_n$ .

#### 12.2.2 2

**Problem** Find Laurent series expansions of

$$\frac{1}{1 - z^2} + \frac{1}{3 - z}$$

How many such expansions are there? In what domains are each valid?

# 12.2.3 3

**Problem** Let P, Q be polynomials with no common zeros. Assume a is a root of Q. Find the principal part of P/Q at z=a in terms of P and Q if a is (1) a simple root, and (2) a double root.

### 12.2.4 4

**Problem** Let f be non-constant, analytic in |z| > 0, where  $f(z_n) = 0$  for infinitely many points  $z_n$  with  $\lim_{n \to \infty} z_n = 0$ .

Show that z = 0 is an essential singularity for f.

Example: 
$$f(z) = \sin(1/z)$$
.

# 12.2.5 5

**Problem** Show that if f is entire and  $\lim_{z \to \infty} f(z) = \infty$ , then f is a polynomial.

# 12.2.6 6

### **Problem**

a. Show (without using 3.8.9 in the S&S) that

$$\int_0^{2\pi} \log \left| 1 - e^{i\theta} \right| \, d\theta = 0$$

b. Show that this identity is equivalent to S&S 3.8.9:

$$\int_0^1 \log(\sin(\pi x)) \ dx = -\log 2.$$

# 12.2.7 7

**Problem** Let 0 < a < 4 and evaluate

$$\int_0^\infty \frac{x^{\alpha - 1}}{1 + x^3} \ dx$$

### 12.2.8 8

**Problem** Prove the fundamental theorem of Algebra using

- a. Rouche's Theorem.
- b. The maximum modulus principle.

### 12.2.9 9

**Problem** Let f be analytic in a region D and  $\gamma$  a rectifiable curve in D with interior in D.

Prove that if f(z) is real for all  $z \in \gamma$ , then f is constant.

### 12.2.10 10

**Problem** For a > 0, evaluate

$$\int_0^{\pi/2} \frac{d\theta}{a + \sin^2 \theta}$$

### 12.2.11 11

**Problem** Find the number of roots of  $p(z) = 4z^4 - 6z + 3$  in |z| < 1 and 1 < |z| < 2 respectively.

### 12.2.12 12

**Problem** Prove that  $z^4 + 2z^3 - 2z + 10$  has exactly one root in each open quadrant.

# 12.2.13 13

**Problem** Prove that for a > 0,  $z \tan z - a$  has only real roots.

# 12.2.14 14

**Problem** Let f be nonzero, analytic on a bounded region  $\Omega$  and continuous on its closure  $\overline{\Omega}$ .

Show that if  $|f(z)| \equiv M$  is constant for  $z \in \partial \Omega$ , then  $f(z) \equiv Me^{i\theta}$  for some real constant  $\theta$ .

# 13 Extra Questions from Jingzhi Tie

# 13.1 Fall 2009

# 13.1.1 ?

(1) Assume  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  converges in |z| < R. Show that for r < R,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} .$$

(2) Deduce Liouville's theorem from (1).

### 13.1.2 ?

Let f be a continuous function in the region

$$D = \{z \mid |z| > R, 0 \le \arg z \le \theta\}$$
 where  $1 \le \theta \le 2\pi$ .

If there exists k such that  $\lim_{z \to \infty} z f(z) = k$  for z in the region D. Show that

$$\lim_{R' \longrightarrow \infty} \int_{L} f(z) dz = i\theta k,$$

where L is the part of the circle |z| = R' which lies in the region D.

#### 13.1.3 ?

Suppose that f is an analytic function in the region D which contains the point a. Let

$$F(z) = z - a - qf(z)$$
, where q is a complex parameter.

- (1) Let  $K \subset D$  be a circle with the center at point a and also we assume that  $f(z) \neq 0$  for  $z \in K$ . Prove that the function F has one and only one zero z = w on the closed disc  $\overline{K}$  whose boundary is the circle K if  $|q| < \min_{z \in K} \frac{|z - a|}{|f(z)|}$ .
- (2) Let G(z) be an analytic function on the disk  $\overline{K}$ . Apply the residue theorem to prove that  $\frac{G(w)}{F'(w)} = \frac{1}{2\pi i} \int_K \frac{G(z)}{F(z)} dz$ , where w is the zero from (1).
- (3) If  $z \in K$ , prove that the function  $\frac{1}{F(z)}$  can be represented as a convergent series with respect to q:  $\frac{1}{F(z)} = \sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z-a)^{n+1}}$ .

#### 13.1.4 ?

Evaluate

$$\int_0^\infty \frac{x \sin x}{x^2 + a^2} \, dx.$$

### 13.1.5 ?

Let f = u + iv be differentiable (i.e. f'(z) exists) with continuous partial derivatives at a point  $z = re^{i\theta}$ ,  $r \neq 0$ . Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

### 13.1.6 ?

Show that  $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$  using complex analysis, 0 < a < n. Here n is a positive integer.

#### 13.1.7 ?

For s > 0, the **gamma function** is defined by  $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$ .

- 1. Show that the gamma function is analytic in the half-plane  $\Re(s) > 0$ , and is still given there by the integral formula above.
- 2. Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need  $\Gamma(1-s) = t \int_0^\infty e^{-vt} (vt)^{-s} dv$  for t > 0.

#### 13.1.8 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree n, then it has n zeros in  $\mathbb{C}$ .

#### 13.1.9 ?

Suppose f is entire and there exist A, R > 0 and natural number N such that

$$|f(z)| \ge A|z|^N$$
 for  $|z| \ge R$ .

Show that (i) f is a polynomial and (ii) the degree of f is at least N.

### 13.1.10 ?

Let  $f: \mathbb{C} \to \mathbb{C}$  be an injective analytic (also called *univalent*) function. Show that there exist complex numbers  $a \neq 0$  and b such that f(z) = az + b.

### 13.1.11 ?

Let g be analytic for  $|z| \le 1$  and |g(z)| < 1 for |z| = 1.

- 1. Show that g has a unique fixed point in |z| < 1.
- 2. What happens if we replace |g(z)| < 1 with  $|g(z)| \le 1$  for |z| = 1? Give an example if (a) is not true or give an proof if (a) is still true.
- 3. What happens if we simply assume that f is analytic for |z| < 1 and |f(z)| < 1 for |z| < 1? Suppose that  $f(z) \not\equiv z$ . Can f have more than one fixed point in |z| < 1?

Hint: The map  $\psi_{\alpha}(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$  may be useful.

### 13.1.12 ?

Find a conformal map from  $D = \{z: |z| < 1, |z - 1/2| > 1/2\}$  to the unit disk  $\Delta = \{z: |z| < 1\}$ .

# 13.1.13 ?

Let f(z) be entire and assume values of f(z) lie outside a bounded open set  $\Omega$ . Show without using Picard's theorems that f(z) is a constant.

# 13.1.14 ?

(1) Assume  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  converges in |z| < R. Show that for r < R,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} .$$

(2) Deduce Liouville's theorem from (1).

### 13.1.15 ?

Let f(z) be entire and assume that  $f(z) \leq M|z|^2$  outside some disk for some constant M. Show that f(z) is a polynomial in z of degree  $\leq 2$ .

### 13.1.16 ?

Let  $a_n(z)$  be an analytic sequence in a domain D such that  $\sum_{n=0}^{\infty} |a_n(z)|$  converges uniformly on bounded and closed sub-regions of D. Show that  $\sum_{n=0}^{\infty} |a'_n(z)|$  converges uniformly on bounded and closed sub-regions of D.

# 13.1.17 ?

Let f(z) be analytic in an open set  $\Omega$  except possibly at a point  $z_0$  inside  $\Omega$ . Show that if f(z) is bounded in near  $z_0$ , then  $\int_{\Omega} f(z)dz = 0$  for all triangles  $\Delta$  in  $\Omega$ .

#### 13.1.18 ?

Assume f is continuous in the region:  $0 < |z - a| \le R$ ,  $0 \le \arg(z - a) \le \beta_0$   $(0 < \beta_0 \le 2\pi)$  and the limit  $\lim_{z \to a} (z - a) f(z) = A$  exists. Show that

$$\lim_{r\to 0} \int_{\gamma_-} f(z)dz = iA\beta_0 ,$$

where  $\gamma_r := \{ z \mid z = a + re^{it}, \ 0 \le t \le \beta_0 \}.$ 

# 13.1.19 ?

Show that  $f(z) = z^2$  is uniformly continuous in any open disk |z| < R, where R > 0 is fixed, but it is not uniformly continuous on  $\mathbb{C}$ .

#### 13.1.20 ?

(1) Show that the function u = u(x, y) given by

$$u(x,y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx$$
 for  $n \in \mathbb{N}$ 

is the solution on  $D = \{(x,y) | x^2 + y^2 < 1\}$  of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,0) = \frac{\sin nx}{n}.$$

(2) Show that there exist points  $(x,y) \in D$  such that  $\limsup_{n \to \infty} |u(x,y)| = \infty$ .

#### 13.2 Fall 2011

### 13.2.1 ?

(1) Assume  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  converges in |z| < R. Show that for r < R,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} .$$

(2) Deduce Liouville's theorem from (1).

#### 13.2.2 ?

Let f be a continuous function in the region

$$D = \{z \mid |z| > R, 0 \le \arg Z \le \theta\} \quad \text{where} \quad 0 \le \theta \le 2\pi.$$

If there exists k such that  $\lim_{z \to \infty} z f(z) = k$  for z in the region D. Show that

$$\lim_{R' \to \infty} \int_L f(z) dz = i\theta k,$$

where L is the part of the circle |z| = R' which lies in the region D.

### 13.2.3 ?

Suppose that f is an analytic function in the region D which contains the point a. Let

$$F(z) = z - a - qf(z)$$
, where q is a complex parameter.

- (1) Let  $K \subset D$  be a circle with the center at point a and also we assume that  $f(z) \neq 0$  for  $z \in K$ . Prove that the function F has one and only one zero z = w on the closed disc  $\overline{K}$  whose boundary is the circle K if  $|q| < \min_{z \in K} \frac{|z - a|}{|f(z)|}$ .
- (2) Let G(z) be an analytic function on the disk  $\overline{K}$ . Apply the residue theorem to prove that  $\frac{G(w)}{F'(w)} = \frac{1}{2\pi i} \int_K \frac{G(z)}{F(z)} dz$ , where w is the zero from (1).
- (3) If  $z \in K$ , prove that the function  $\frac{1}{F(z)}$  can be represented as a convergent series with respect to q:  $\frac{1}{F(z)} = \sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z-a)^{n+1}}$ .

### 13.2.4 ?

Evaluate  $\int_0^\infty \frac{x \sin x}{x^2 + a^2} dx$ .

# 13.2.5 ?

Let f = u + iv be differentiable (i.e. f'(z) exists) with continuous partial derivatives at a point  $z = re^{i\theta}$ ,  $r \neq 0$ . Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

### 13.2.6 ?

Show that  $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$  using complex analysis, 0 < a < n. Here n is a positive integer.

### 13.2.7 ?

For s > 0, the **gamma function** is defined by  $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$ .

- 1. Show that the gamma function is analytic in the half-plane  $\Re(s) > 0$ , and is still given there by the integral formula above.
- 2. Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need 
$$\Gamma(1-s) = t \int_0^\infty e^{-vt} (vt)^{-s} dv$$
 for  $t > 0$ .

### 13.2.8 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree n, then it has n zeros in  $\mathbb{C}$ .

### 13.2.9 ?

Suppose f is entire and there exist A, R > 0 and natural number N such that

$$|f(z)| \ge A|z|^N$$
 for  $|z| \ge R$ .

Show that (i) f is a polynomial and (ii) the degree of f is at least N.

#### 13.2.10 ?

Let  $f: \mathbb{C} \to \mathbb{C}$  be an injective analytic (also called univalent) function. Show that there exist complex numbers  $a \neq 0$  and b such that f(z) = az + b.

#### 13.2.11 ?

Let g be analytic for  $|z| \le 1$  and |g(z)| < 1 for |z| = 1.

- Show that g has a unique fixed point in |z| < 1.
- What happens if we replace |g(z)| < 1 with  $|g(z)| \le 1$  for |z| = 1? Give an example if (a) is not true or give an proof if (a) is still true.
- What happens if we simply assume that f is analytic for |z| < 1 and |f(z)| < 1 for |z| < 1? Suppose that  $f(z) \not\equiv z$ . Can f have more than one fixed point in |z| < 1?

Hint: The map 
$$\psi_{\alpha}(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$$
 may be useful.

# 13.2.12 ?

Find a conformal map from  $D = \{z : |z| < 1, |z - 1/2| > 1/2\}$  to the unit disk  $\Delta = \{z : |z| < 1\}$ .

### 13.2.13 ?

Let f(z) be entire and assume values of f(z) lie outside a bounded open set  $\Omega$ . Show without using Picard's theorems that f(z) is a constant.

### 13.2.14 ?

Let f(z) be entire and assume values of f(z) lie outside a bounded open set  $\Omega$ . Show without using Picard's theorems that f(z) is a constant.

### 13.2.15 ?

(1) Assume  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  converges in |z| < R. Show that for r < R,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} .$$

(2) Deduce Liouville's theorem from (1).

### 13.2.16 ?

Let f(z) be entire and assume that  $f(z) \leq M|z|^2$  outside some disk for some constant M. Show that f(z) is a polynomial in z of degree  $\leq 2$ .

#### 13.2.17 ?

Let  $a_n(z)$  be an analytic sequence in a domain D such that  $\sum_{n=0}^{\infty} |a_n(z)|$  converges uniformly on bounded and closed sub-regions of D. Show that  $\sum_{n=0}^{\infty} |a'_n(z)|$  converges uniformly on bounded and closed sub-regions of D.

### 13.2.18 ?

Let f(z) be analytic in an open set  $\Omega$  except possibly at a point  $z_0$  inside  $\Omega$ . Show that if f(z) is bounded in near  $z_0$ , then  $\int_{\Delta} f(z)dz = 0$  for all triangles  $\Delta$  in  $\Omega$ .

### 13.2.19 ?

Assume f is continuous in the region:  $0 < |z - a| \le R$ ,  $0 \le \arg(z - a) \le \beta_0$   $(0 < \beta_0 \le 2\pi)$  and the limit  $\lim_{z \to a} (z - a) f(z) = A$  exists. Show that

$$\lim_{r \to 0} \int_{\gamma_r} f(z) dz = iA\beta_0 ,$$

where  $\gamma_r := \{ z \mid z = a + re^{it}, \ 0 \le t \le \beta_0 \}.$ 

#### 13.2.20 ?

Show that  $f(z) = z^2$  is uniformly continuous in any open disk |z| < R, where R > 0 is fixed, but it is not uniformly continuous on  $\mathbb{C}$ .

(1) Show that the function u = u(x, y) given by

$$u(x,y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx$$
 for  $n \in \mathbb{N}$ 

is the solution on  $D = \{(x,y) | x^2 + y^2 < 1\}$  of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,0) = \frac{\sin nx}{n}.$$

(2) Show that there exist points  $(x,y) \in D$  such that  $\limsup_{n \to \infty} |u(x,y)| = \infty$ .

### 13.3 Spring 2014

### 13.3.1 ?

The question provides some insight into Cauchy's theorem. Solve the problem without using the Cauchy theorem.

1. Evaluate the integral  $\int_{\gamma} z^n dz$  for all integers n. Here  $\gamma$  is any circle centered at the origin with the positive (counterclockwise) orientation.

- 2. Same question as (a), but with  $\gamma$  any circle not containing the origin.
- 3. Show that if |a| < r < |b|, then  $\int_{\gamma} \frac{dz}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$ . Here  $\gamma$  denotes the circle centered at the origin, of radius r, with the positive orientation.

#### 13.3.2 ?

(1) Assume the infinite series  $\sum_{n=0}^{\infty} c_n z^n$  converges in |z| < R and let f(z) be the limit. Show that for r < R,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} .$$

(2) Deduce Liouville's theorem from (1). Liouville's theorem: If f(z) is entire and bounded, then f is constant.

### 13.3.3 ?

Let f be a continuous function in the region

$$D = \{z \mid |z| > R, 0 \le \arg Z \le \theta\}$$
 where  $0 \le \theta \le 2\pi$ .

If there exists k such that  $\lim_{z \to \infty} z f(z) = k$  for z in the region D. Show that

$$\lim_{R' \to \infty} \int_L f(z) dz = i\theta k,$$

where L is the part of the circle |z| = R' which lies in the region D.

# 13.3.4 ?

Evaluate  $\int_0^\infty \frac{x \sin x}{x^2 + a^2} dx$ .

# 13.3.5 ?

Let f = u + iv be differentiable (i.e. f'(z) exists) with continuous partial derivatives at a point  $z = re^{i\theta}$ ,  $r \neq 0$ . Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

# 13.3.6 ?

Show that  $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$  using complex analysis, 0 < a < n. Here n is a positive integer.

#### 13.3.7 ?

For s > 0, the **gamma function** is defined by  $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$ .

- Show that the gamma function is analytic in the half-plane  $\Re(s) > 0$ , and is still given there by the integral formula above.
- Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need 
$$\Gamma(1-s) = t \int_0^\infty e^{-vt} (vt)^{-s} dv$$
 for  $t > 0$ .

### 13.3.8 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree n, then it has n zeros in C.

#### 13.3.9 ?

Suppose f is entire and there exist A, R > 0 and natural number N such that

$$|f(z)| \ge A|z|^N$$
 for  $|z| \ge R$ .

Show that (i) f is a polynomial and (ii) the degree of f is at least N.

### 13.3.10 ?

Let  $f: \mathbb{C} \to \mathbb{C}$  be an injective analytic (also called univalent) function. Show that there exist complex numbers  $a \neq 0$  and b such that f(z) = az + b.

### 13.3.11 ?

Let g be analytic for  $|z| \le 1$  and |g(z)| < 1 for |z| = 1.

- Show that g has a unique fixed point in |z| < 1.
- What happens if we replace |g(z)| < 1 with  $|g(z)| \le 1$  for |z| = 1? Give an example if (a) is not true or give an proof if (a) is still true.
- What happens if we simply assume that f is analytic for |z| < 1 and |f(z)| < 1 for |z| < 1? Suppose that  $f(z) \not\equiv z$ . Can f have more than one fixed point in |z| < 1?

Hint: The map 
$$\psi_{\alpha}(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$$
 may be useful.

#### 13.3.12 ?

Find a conformal map from  $D=\{z:\ |z|<1,\ |z-1/2|>1/2\}$  to the unit disk  $\Delta=\{z:\ |z|<1\}.$ 

# 13.4 Fall 2015

# 13.4.1 ?

Let  $a_n \neq 0$  and assume that  $\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = L$ . Show that  $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L$ . In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

### 13.4.2 ?

(a) Let z, w be complex numbers, such that  $\bar{z}w \neq 1$ . Prove that

$$\left| \frac{w - z}{1 - \overline{w}z} \right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left| \frac{w-z}{1-\overline{w}z} \right| = 1$$
 if  $|z| = 1$  or  $|w| = 1$ .

(b) Prove that for fixed w in the unit disk  $\mathbb{D}$ , the mapping

$$F: z \mapsto \frac{w-z}{1-\overline{w}z}$$

satisfies the following conditions:

- (c) F maps  $\mathbb{D}$  to itself and is holomorphic.
- (ii) F interchanges 0 and w, namely, F(0) = w and F(w) = 0.
- (iii) |F(z)| = 1 if |z| = 1.
- (iv)  $F: \mathbb{D} \mapsto \mathbb{D}$  is bijective.

Hint: Calculate  $F \circ F$ .

### 13.4.3 ?

Use n-th roots of unity (i.e. solutions of  $z^n - 1 = 0$ ) to show that

$$2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n}=n.$$

Hint:  $1 - \cos 2\theta = 2\sin^2 \theta$ ,  $\sin 2\theta = 2\sin \theta \cos \theta$ .

(a) Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$
 and  $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$ 

(b) Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta$$
 where  $z = re^{i\theta}$  with  $-\pi < \theta < \pi$ 

is a holomorphic function in the region r > 0,  $-\pi < \theta < \pi$ . Also show that  $\log z$  defined above is not continuous in r > 0.

#### 13.4.4 ?

Assume f is continuous in the region:  $x \ge x_0$ ,  $0 \le y \le b$  and the limit

$$\lim_{x \to +\infty} f(x + iy) = A$$

exists uniformly with respect to y (independent of y). Show that

$$\lim_{x \to +\infty} \int_{\gamma_x} f(z) dz = iAb ,$$

where  $\gamma_x := \{ z \mid z = x + it, \ 0 \le t \le b \}.$ 

#### 13.4.5 ?

(Cauchy's formula for "exterior" region) Let  $\gamma$  be piecewise smooth simple closed curve with interior  $\Omega_1$  and exterior  $\Omega_2$ . Assume f'(z) exists in an open set containing  $\gamma$  and  $\Omega_2$  and  $\lim_{z\to\infty} f(z) = A$ . Show that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

# 13.4.6 ?

Let f(z) be bounded and analytic in  $\mathbb{C}$ . Let  $a \neq b$  be any fixed complex numbers. Show that the following limit exists

$$\lim_{R \to \infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

Use this to show that f(z) must be a constant (Liouville's theorem).

#### 13.4.7 ?

Prove by justifying all steps that for all  $\xi \in \mathbb{C}$  we have  $e^{-\pi \xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x \xi} dx$ .

Hint: You may use that fact in Example 1 on p. 42 of the textbook without proof, i.e., you may assume the above is true for real values of  $\xi$ .

#### 13.4.8 ?

Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at  $z_0$  on the unit circle. Let denote the power series in the open disc. Show that (1)  $c_n \neq 0$  for all large enough n's, and (2)  $\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = z_0$ .

### 13.4.9 ?

Let f(z) be a non-constant analytic function in |z| > 0 such that  $f(z_n) = 0$  for infinite many points  $z_n$  with  $\lim_{n \to \infty} z_n = 0$ . Show that z = 0 is an essential singularity for f(z). (An example of such a function is  $f(z) = \sin(1/z)$ .)

### 13.4.10 ?

Let f be entire and suppose that  $\lim_{z\to\infty} f(z) = \infty$ . Show that f is a polynomial.

# 13.4.11 ?

Expand the following functions into Laurent series in the indicated regions:

(a) 
$$f(z) = \frac{z^2 - 1}{(z+2)(z+3)}$$
,  $2 < |z| < 3$ ,  $3 < |z| < +\infty$ .

(b) 
$$f(z) = \sin \frac{z}{1-z}$$
,  $0 < |z-1| < +\infty$ 

# 13.4.12 ?

Assume f(z) is analytic in region D and  $\Gamma$  is a rectifiable curve in D with interior in D. Prove that if f(z) is real for all  $z \in \Gamma$ , then f(z) is a constant.

### 13.4.13 ?

Find the number of roots of  $z^4 - 6z + 3 = 0$  in |z| < 1 and 1 < |z| < 2 respectively.

# 13.4.14 ?

Prove that  $z^4 + 2z^3 - 2z + 10 = 0$  has exactly one root in each open quadrant.

# 13.4.15 ?

(1) Let  $f(z) \in H(\mathbb{D})$ , Re(f(z)) > 0, f(0) = a > 0. Show that

$$\left| \frac{f(z) - a}{f(z) + a} \right| \le |z|, \quad |f'(0)| \le 2a.$$

(2) Show that the above is still true if Re(f(z)) > 0 is replaced with  $Re(f(z)) \ge 0$ .

### 13.4.16 ?

Assume f(z) is analytic in  $\mathbb{D}$  and f(0) = 0 and is not a rotation (i.e.  $f(z) \neq e^{i\theta}z$ ). Show that  $\sum_{n=1}^{\infty} f^n(z)$  converges uniformly to an analytic function on compact subsets of  $\mathbb{D}$ , where  $f^{n+1}(z) = f(f^n(z))$ .

#### 13.4.17 ?

Let  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  be analytic and one-to-one in |z| < 1. For 0 < r < 1, let  $D_r$  be the disk |z| < r. Show that the area of  $f(D_r)$  is finite and is given by

$$S = \pi \sum_{n=1}^{\infty} n|c_n|^2 r^{2n}.$$

(Note that in general the area of  $f(D_1)$  is infinite.)

### 13.4.18 ?

Let  $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$  be analytic and one-to-one in  $r_0 < |z| < R_0$ . For  $r_0 < r < R < R_0$ , let D(r,R) be the annulus r < |z| < R. Show that the area of f(D(r,R)) is finite and is given by

$$S = \pi \sum_{n = -\infty}^{\infty} n |c_n|^2 (R^{2n} - r^{2n}).$$

### 13.5 Spring 2015

# 13.5.1 ?

Let  $a_n(z)$  be an analytic sequence in a domain D such that  $\sum_{n=0}^{\infty} |a_n(z)|$  converges uniformly on bounded and closed sub-regions of D. Show that  $\sum_{n=0}^{\infty} |a'_n(z)|$  converges uniformly on bounded and closed sub-regions of D.

#### 13.5.2 ?

Let  $f_n, f$  be analytic functions on the unit disk  $\mathbb{D}$ . Show that the following are equivalent.

- (i)  $f_n(z)$  converges to f(z) uniformly on compact subsets in  $\mathbb{D}$ .
- (ii)  $\int_{|z|=r} |f_n(z) f(z)| |dz|$  converges to 0 if 0 < r < 1.

### 13.5.3 ?

Let f and g be non-zero analytic functions on a region  $\Omega$ . Assume |f(z)| = |g(z)| for all z in  $\Omega$ . Show that  $f(z) = e^{i\theta}g(z)$  in  $\Omega$  for some  $0 \le \theta < 2\pi$ .

#### 13.5.4 ?

Suppose f is analytic in an open set containing the unit disc  $\mathbb{D}$  and |f(z)|=1 when |z|=1. Show that either  $f(z)=e^{i\theta}$  for some  $\theta\in\mathbb{R}$  or there are finite number of  $z_k\in\mathbb{D},\,k\leq n$  and  $\theta\in\mathbb{R}$  such that  $f(z)=e^{i\theta}\prod_{k=1}^n\frac{z-z_k}{1-\bar{z}_kz}$ .

Also cf. Stein et al, 1.4.7, 3.8.17

#### 13.5.5 ?

- (1) Let p(z) be a polynomial, R > 0 any positive number, and  $m \ge 1$  an integer. Let  $M_R = \sup\{|z^m p(z) 1| : |z| = R\}$ . Show that  $M_R > 1$ .
- (2) Let  $m \ge 1$  be an integer and  $K = \{z \in \mathbb{C} : r \le |z| \le R\}$  where r < R. Show (i) using (1) as well as, (ii) without using (1) that there exists a positive number  $\varepsilon_0 > 0$  such that for each polynomial p(z),

$$\sup\{|p(z) - z^{-m}| : z \in K\} \ge \varepsilon_0.$$

#### 13.5.6 ?

Let  $f(z) = \frac{1}{z} + \frac{1}{z^2 - 1}$ . Find all the Laurent series of f and describe the largest annuli in which these series are valid.

# 13.5.7 ?

Suppose f is entire and there exist A, R > 0 and natural number N such that  $|f(z)| \le A|z|^N$  for  $|z| \ge R$ . Show that (i) f is a polynomial and (ii) the degree of f is at most N.

### 13.5.8 ?

Suppose f is entire and there exist A, R > 0 and natural number N such that  $|f(z)| \ge A|z|^N$  for  $|z| \ge R$ . Show that (i) f is a polynomial and (ii) the degree of f is at least N.

#### 13.5.9 ?

- (1) Explicitly write down an example of a non-zero analytic function in |z| < 1 which has infinitely zeros in |z| < 1.
- (2) Why does not the phenomenon in (1) contradict the uniqueness theorem?

#### 13.5.10 ?

- (1) Assume u is harmonic on open set O and  $z_n$  is a sequence in O such that  $u(z_n) = 0$  and  $\lim z_n \in O$ . Prove or disprove that u is identically zero. What if O is a region?
- (2) Assume u is harmonic on open set O and u(z) = 0 on a disc in O. Prove or disprove that u is identically zero. What if O is a region?

(3) Formulate and prove a Schwarz reflection principle for harmonic functions

cf. Theorem 5.6 on p.60 of Stein et al.

Hint: Verify the mean value property for your new function obtained by Schwarz reflection principle.

#### 13.5.11 ?

Let f be holomorphic in a neighborhood of  $D_r(z_0)$ . Show that for any s < r, there exists a constant c > 0 such that

$$||f||_{(\infty,s)} \le c||f||_{(1,r)},$$

where 
$$|f||_{(\infty,s)} = \sup_{z \in D_s(z_0)} |f(z)|$$
 and  $||f||_{(1,r)} = \int_{D_r(z_0)} |f(z)| dx dy$ .

Note: Exercise 3.8.20 on p.107 in Stein et al is a straightforward consequence of this stronger result using the integral form of the Cauchy-Schwarz inequality in real analysis.

### 13.5.12 ?

- (1) Let f be analytic in  $\Omega: 0 < |z-a| < r$  except at a sequence of poles  $a_n \in \Omega$  with  $\lim_{n \to \infty} a_n = a$ . Show that for any  $w \in \mathbb{C}$ , there exists a sequence  $z_n \in \Omega$  such that  $\lim_{n \to \infty} f(z_n) = w$ .
- (2) Explain the similarity and difference between the above assertion and the Weierstrass-Casorati theorem.

#### 13.5.13 ?

Compute the following integrals.

$$i \int_{0}^{\infty} \frac{1}{(1+x^{n})^{2}} dx, \ n \ge 1 \ (ii) \int_{0}^{\infty} \frac{\cos x}{(x^{2}+a^{2})^{2}} dx, \ a \in \mathbb{R} \ (iii) \int_{0}^{\pi} \frac{1}{a+\sin\theta} d\theta, \ a > 1$$

$$iv \int_{0}^{\frac{\pi}{2}} \frac{d\theta}{a+\sin^{2}\theta}, \ a > 0. \ (v) \int_{|z|=2}^{\infty} \frac{1}{(z^{5}-1)(z-3)} dz \ (v) \int_{-\infty}^{\infty} \frac{\sin \pi a}{\cosh \pi x + \cos \pi a} e^{-ix\xi} dx, \ 0 < a < 1,$$

$$\xi \in \mathbb{R} \ (vi) \int_{|z|=1}^{\infty} \cot^{2} z \, dz.$$

### 13.5.14 ?

Compute the following integrals.

$$\begin{split} i & \int_0^\infty \frac{\sin x}{x} \, dx \text{ (ii) } \int_0^\infty (\frac{\sin x}{x})^2 \, dx \text{ (iii) } \int_0^\infty \frac{x^{a-1}}{(1+x)^2} \, dx, \, 0 < a < 2 \\ i & \int_0^\infty \frac{\cos ax - \cos bx}{x^2} \, dx, \, a, b > 0 \text{ (ii) } \int_0^\infty \frac{x^{a-1}}{1+x^n} \, dx, \, 0 < a < n \\ iii & \int_0^\infty \frac{\log x}{1+x^n} \, dx, \, n \ge 2 \text{ (iv) } \int_0^\infty \frac{\log x}{(1+x^2)^2} \, dx \text{ (v) } \int_0^\pi \log |1-a\sin\theta| d\theta, \, a \in \mathbb{C} \end{split}$$

#### 13.5.15 ?

Let 0 < r < 1. Show that polynomials  $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$  have no zeros in |z| < r for all sufficiently large n's.

#### 13.5.16 ?

Let f be an analytic function on a region  $\Omega$ . Show that f is a constant if there is a simple closed curve  $\gamma$  in  $\Omega$  such that its image  $f(\gamma)$  is contained in the real axis.

### 13.5.17 ?

- (1) Show that  $\frac{\pi^2}{\sin^2 \pi z}$  and  $g(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$  have the same principal part at each integer point.
- (2) Show that  $h(z) = \frac{\pi^2}{\sin^2 \pi z} g(z)$  is bounded on  $\mathbb{C}$  and conclude that  $\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ .

### 13.5.18 ?

Let f(z) be an analytic function on  $\mathbb{C}\setminus\{z_0\}$ , where  $z_0$  is a fixed point. Assume that f(z) is bijective from  $\mathbb{C}\setminus\{z_0\}$  onto its image, and that f(z) is bounded outside  $D_r(z_0)$ , where r is some fixed positive number. Show that there exist  $a, b, c, d \in \mathbb{C}$  with  $ad - bc \neq 0$ ,  $c \neq 0$  such that  $f(z) = \frac{az + b}{cz + d}$ .

### 13.5.19 ?

Assume f(z) is analytic in  $\mathbb{D}: |z| < 1$  and f(0) = 0 and is not a rotation (i.e.  $f(z) \neq e^{i\theta}z$ ). Show that  $\sum_{n=1}^{\infty} f^n(z)$  converges uniformly to an analytic function on compact subsets of  $\mathbb{D}$ , where  $f^{n+1}(z) = f(f^n(z))$ .

### 13.5.20 ?

Let f be a non-constant analytic function on  $\mathbb{D}$  with  $f(\mathbb{D}) \subseteq \mathbb{D}$ . Use  $\psi_a(f(z))$  (where a = f(0),  $\psi_a(z) = \frac{a-z}{1-\bar{a}z}$ ) to prove that  $\frac{|f(0)|-|z|}{1+|f(0)||z|} \le |f(z)| \le \frac{|f(0)|+|z|}{1-|f(0)||z|}$ .

#### 13.5.21 ?

Find a conformal map

- 1. from  $\{z: |z-1/2| > 1/2, \operatorname{Re}(z) > 0\}$  to  $\mathbb{H}$
- 2. from  $\{z: |z-1/2| > 1/2, |z| < 1\}$  to  $\mathbb{D}$
- 3. from the intersection of the disk  $|z+i| < \sqrt{2}$  with  $\mathbb H$  to  $\mathbb D$ .

4. from  $\mathbb{D}\setminus[a,1)$  to  $\mathbb{D}\setminus[0,1)$  (0 < a < 1).

Short solution possible using Blaschke factor

5. from  $\{z: |z| < 1, \text{Re}(z) > 0\} \setminus (0, 1/2]$  to  $\mathbb{H}$ .

### 13.5.22 ?

Let C and C' be two circles and let  $z_1 \in C$ ,  $z_2 \notin C$ ,  $z_1' \in C'$ ,  $z_2' \notin C'$ . Show that there is a unique fractional linear transformation f with f(C) = C' and  $f(z_1) = z_1'$ ,  $f(z_2) = z_2'$ .

### 13.5.23 ?

Assume  $f_n \in H(\Omega)$  is a sequence of holomorphic functions on the region  $\Omega$  that are uniformly bounded on compact subsets and  $f \in H(\Omega)$  is such that the set  $\{z \in \Omega : \lim_{n \to \infty} f_n(z) = f(z)\}$  has a limit point in  $\Omega$ . Show that  $f_n$  converges to f uniformly on compact subsets of  $\Omega$ .

# 13.5.24 ?

Let  $\psi_{\alpha}(z) = \frac{\alpha - z}{1 - \overline{\alpha}z}$  with  $|\alpha| < 1$  and  $\mathbb{D} = \{z : |z| < 1\}$ . Prove that

• 
$$\frac{1}{\pi} \iint_{\mathbb{D}} |\psi_{\alpha}'|^2 dx dy = 1.$$

• 
$$\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_{\alpha}| dx dy = \frac{1 - |\alpha|^2}{|\alpha|^2} \log \frac{1}{1 - |\alpha|^2}.$$

# 13.5.25 ?

Prove that  $f(z) = -\frac{1}{2}\left(z + \frac{1}{z}\right)$  is a conformal map from half disc  $\{z = x + iy : |z| < 1, y > 0\}$  to upper half plane  $\mathbb{H} = \{z = x + iy : y > 0\}$ .

#### 13.5.26 ?

Let  $\Omega$  be a simply connected open set and let  $\gamma$  be a simple closed contour in  $\Omega$  and enclosing a bounded region U anticlockwise. Let  $f: \Omega \longrightarrow \mathbb{C}$  be a holomorphic function and  $|f(z)| \leq M$  for all  $z \in \gamma$ . Prove that  $|f(z)| \leq M$  for all  $z \in U$ .

### 13.5.27 ?

Compute the following integrals. (i)  $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx$ , 0 < a < n (ii)  $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$ 

### 13.5.28 ?

Let 0 < r < 1. Show that polynomials  $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$  have no zeros in |z| < r for all sufficiently large n's.

### 13.5.29 ?

Let f be holomorphic in a neighborhood of  $D_r(z_0)$ . Show that for any s < r, there exists a constant c > 0 such that

$$||f||_{(\infty,s)} \le c||f||_{(1,r)},$$

where  $||f||_{(\infty,s)} = \sup_{z \in D_s(z_0)} |f(z)|$  and  $||f||_{(1,r)} = \int_{D_r(z_0)} |f(z)| dx dy$ .

# 13.5.30 ?

Let  $\psi_{\alpha}(z) = \frac{\alpha - z}{1 - \overline{\alpha}z}$  with  $|\alpha| < 1$  and  $\mathbb{D} = \{z : |z| < 1\}$ . Prove that

• 
$$\frac{1}{\pi} \iint_{\mathbb{D}} |\psi_{\alpha}'|^2 dx dy = 1.$$

• 
$$\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_{\alpha}| dx dy = \frac{1 - |\alpha|^2}{|\alpha|^2} \log \frac{1}{1 - |\alpha|^2}.$$

Prove that  $f(z) = -\frac{1}{2}\left(z + \frac{1}{z}\right)$  is a conformal map from half disc  $\{z = x + iy : |z| < 1, y > 0\}$  to upper half plane  $\mathbb{H} = \{z = x + iy : y > 0\}$ .

### 13.5.31 ?

Let  $\Omega$  be a simply connected open set and let  $\gamma$  be a simple closed contour in  $\Omega$  and enclosing a bounded region U anticlockwise. Let  $f: \Omega \longrightarrow \mathbb{C}$  be a holomorphic function and  $|f(z)| \leq M$  for all  $z \in \gamma$ . Prove that  $|f(z)| \leq M$  for all  $z \in U$ .

#### 13.5.32 ?

Compute the following integrals. (i)  $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx$ , 0 < a < n (ii)  $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$ 

# 13.5.33 ?

Let 0 < r < 1. Show that polynomials  $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$  have no zeros in |z| < r for all sufficiently large n's.

### 13.5.34 ?

Let f be holomorphic in a neighborhood of  $D_r(z_0)$ . Show that for any s < r, there exists a constant c > 0 such that

$$||f||_{(\infty,s)} \le c||f||_{(1,r)},$$

where  $||f||_{(\infty,s)} = \sup_{z \in D_s(z_0)} |f(z)|$  and  $||f||_{(1,r)} = \int_{D_r(z_0)} |f(z)| dx dy$ .

# 13.6 Fall 2016

### 13.6.1 ?

Let u(x,y) be harmonic and have continuous partial derivatives of order three in an open disc of radius R > 0.

(a) Let two points (a, b), (x, y) in this disk be given. Show that the following integral is independent of the path in this disk joining these points:

$$v(x,y) = \int_{a,b}^{x,y} \left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right).$$

(b)

- (i) Prove that u(x,y) + iv(x,y) is an analytic function in this disc.
- (ii) Prove that v(x, y) is harmonic in this disc.

### 13.6.2 ?

- (a) f(z) = u(x,y) + iv(x,y) be analytic in a domain  $D \subset \mathbb{C}$ . Let  $z_0 = (x_0,y_0)$  be a point in D which is in the intersection of the curves  $u(x,y) = c_1$  and  $v(x,y) = c_2$ , where  $c_1$  and  $c_2$  are constants. Suppose that  $f'(z_0) \neq 0$ . Prove that the lines tangent to these curves at  $z_0$  are perpendicular.
- (b) Let  $f(z) = z^2$  be defined in  $\mathbb{C}$ .
- (c) Describe the level curves of Re(f) and of Im(f).
- (ii) What are the angles of intersections between the level curves Re(f) = 0 and Im(f)? Is your answer in agreement with part a) of this question?

# 13.6.3 ?

(a)  $f: D \to \mathbb{C}$  be a continuous function, where  $D \subset \mathbb{C}$  is a domain.Let  $\alpha: [a, b] \to D$  be a smooth curve. Give a precise definition of the *complex line integral* 

$$\int_{\alpha} f$$
.

(b) Assume that there exists a constant M such that  $|f(\tau)| \leq M$  for all  $\tau \in \text{Image}(\alpha)$ . Prove that

$$\left| \int_{\alpha} f \right| \leq M \times \operatorname{length}(\alpha).$$

(c) Let  $C_R$  be the circle |z| = R, described in the counterclockwise direction, where R > 1. Provide an upper bound for  $|\int_{C_R} \frac{\log(z)}{z^2}|$ , which depends only on R and other constants.

# 13.6.4 ?

(a) Let Let  $f: \mathbb{C} \to \mathbb{C}$  be an entire function. Assume the existence of a non-negative integer m, and of positive constants L and R, such that for all z with |z| > R the inequality

$$|f(z)| \le L|z|^m$$

holds. Prove that f is a polynomial of degree  $\leq m$ .

(b) Let  $f:\mathbb{C}\to\mathbb{C}$  be an entire function. Suppose that there exists a real number M such that for all  $z\in\mathbb{C}$ 

$$\operatorname{Re}(f) \leq M$$
.

Prove that f must be a constant.

# 13.6.5 ?

Prove that all the roots of the complex polynomial

$$z^7 - 5z^3 + 12 = 0$$

lie between the circles |z| = 1 and |z| = 2.

### 13.6.6 ?

(a) Let F be an analytic function inside and on a simple closed curve C, except for a pole of order  $m \ge 1$  at z=a inside C. Prove that

$$\frac{1}{2\pi i} \oint_C F(\tau) d\tau = \lim_{\tau \to a} \frac{d^{m-1}}{d\tau^{m-1}} ((\tau - a)^m F(\tau)).$$

(b) Evaluate

$$\oint_C \frac{e^{\tau}}{(\tau^2 + \pi^2)^2} d\tau$$

where C is the circle |z| = 4.

### 13.6.7 ?

Find the conformal map that takes the upper half-plane comformally onto the half-strip  $\{w = x + iy : -\pi/2 < x < \pi/2 \ y > 0\}$ .

#### 13.6.8 ?

Compute the integral  $\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{\cosh \pi x} dx$  where  $\cosh z = \frac{e^z + e^{-z}}{2}$ .