Title

D. Zack Garza

Sunday $16^{\rm th}$ August, 2020

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1 Undergraduate Analysis: Uniform Convergence

1.1 Fall 2018 # 1 💝

Let $f(x) = \frac{1}{x}$. Show that f is uniformly continuous on $(1, \infty)$ but not on $(0, \infty)$.

Solution:

Concepts Used:

• Uniform continuity:

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$$
 such that $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$.

• Negating uniform continuity: $\exists \varepsilon > 0$ such that $\forall \delta(\varepsilon)$ there exist x, y such that $|x-y| < \delta$ and $|f(x) - f(y)| > \varepsilon$.

Claim: $f(x) = \frac{1}{x}$ is uniformly continuous on (c, ∞) for any c > 0.

• Note that

$$|x|, |y| > c > 0 \implies |xy| = |x||y| > c^2 \implies \frac{1}{|xy|} < \frac{1}{c^2}.$$

• Letting ε be arbitrary, choose $\delta < \varepsilon c^2$.

- Note that δ does not depend on x, y.

• Then

$$|f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right|$$

$$= \frac{|x - y|}{xy}$$

$$\leq \frac{\delta}{xy}$$

$$< \frac{\delta}{c^2}$$

$$< \varepsilon,$$

which shows uniform continuity.

Claim: f is not uniformly continuous when c = 0.

• Toward a contradiction, let $\varepsilon < 1$.

• Let $x_n = \frac{1}{n}$ for $n \ge 1$.

• Choose n large enough such that $|x_n - x_{n+1}| = \frac{1}{n} - \frac{1}{n+1} < \delta$.

- Why this can be done: by the archimedean property of \mathbb{R} , choose n such that $\frac{1}{n} < \varepsilon$.

- Then

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \le \frac{1}{n} < \varepsilon \quad \text{since } n+1 > 1.$$

• Note $f(x_n) = n$ and thus

$$|f(x_{n+1}) - f(x_n)| = (n+1) - n = 1 > \varepsilon,$$

a contradiction.

1.2 Fall 2017 # 1 🔭

Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which f does and does not converge uniformly.

Solution:

Concepts Used:

•
$$f_N \longrightarrow f$$
 uniformly $\iff \|f_N - f\|_{\infty} \longrightarrow 0$.
• $\sum_{n=0}^{\infty} c_n x^n := \lim_{N \longrightarrow \infty} \sum_{n=0}^{N} c_n x^n$

I.e. an infinite sum is defined as the pointwise limit of its partial sums.

• If $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on a set A, then $\sup_{x \in A} |f_n(x)| \longrightarrow 0$.

• Set
$$f_N(x) = \sum_{n=1}^N \frac{x^n}{n!}$$
.

- Then by definition, $f_N(x) \longrightarrow f(x)$ pointwise on \mathbb{R} .

• For any compact interval [-M, M], we have

$$\begin{split} \|f_N(x) - f(x)\|_{\infty} &= \sup_{-M \le x \le M} \ \left| \sum_{n=N+1}^{\infty} \frac{x^n}{n!} \right| \\ &\le \sup_{-M \le x \le M} \ \sum_{n=N+1}^{\infty} \left| \frac{x^n}{n!} \right| \\ &\le \sum_{n=N+1}^{\infty} \frac{M^n}{n!} \\ &\le \sum_{n=0}^{\infty} \frac{M^n}{n!} \quad \text{since all additional terms are positive} \\ &= e^M \\ &< \infty, \end{split}$$

so $f_N \longrightarrow f$ uniformly on [-M, M] by the M-test.

Here we've used that e^x is equal to its power series expansion.

Thus f converges on any bounded interval, since any bounded interval is contained in

some larger compact interval.

Claim: f does not converge on \mathbb{R} .

- If $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on a set A, then $\sup_{x \in A} |f_n(x)| \longrightarrow 0$.
- But taking $A = \mathbb{R}$ and $g_n(x) = \frac{x^n}{n!}$, we have

$$\sup_{x \in \mathbb{R}} |g_n(x)| = \sup_{x \in \mathbb{R}} \frac{x^n}{n!} = \infty.$$

1.3 Fall 2014 # 1 🔆

Let $\{f_n\}$ be a sequence of continuous functions such that $\sum f_n$ converges uniformly.

Prove that $\sum f_n$ is also continuous.

Solution:

Claim: If $F_N \longrightarrow F$ uniformly with each F_N continuous, then F is continuous.

• Follows from an $\varepsilon/3$ argument:

$$|F(x) - F(y)| \le |F(x) - F_N(x)| + |F_N(x) - F_N(y)| + |F_N(y) - F(y)| \le \varepsilon \longrightarrow 0.$$

- The first and last $\varepsilon/3$ come from uniform convergence of $F_N \longrightarrow F$.
- The middle $\varepsilon/3$ comes from continuity of each F_N .
- Now setting $F_N := \sum_{n=1}^{N} f_n$ yields a finite sum of continuous functions, which is continuous.
- Each F_N is continuous and $F_N \longrightarrow F$ uniformly, so applying the claim yields the desired result.

1.4 Spring 2017 # 4 🦙

Let f(x,y) on $[-1,1]^2$ be defined by

$$f(x,y) = \begin{cases} \frac{xy}{(x^2 + y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Determine if f is integrable.

Solution:

Concepts Used:

• Just Calculus.

Switching to polar coordinates and integrating over one quarter of the unit disc $D \subseteq I^2$, we

have

$$\int_{I^2} f \, dA \ge \int_D f \, dA$$

$$\ge \int_0^{\pi/2} \int_0^1 \frac{\cos(\theta) \sin(\theta)}{r^4} r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \cos(\theta) \sin(\theta) \int_0^1 \frac{1}{r^3} \, dr \, d\theta$$

$$= \left(\int_0^1 \frac{1}{r^3} \, dr \right) \left(\int_0^{\pi/2} \cos(\theta) \sin(\theta) \, d\theta \right)$$

$$= \left(\int_0^1 \frac{1}{r^3} \, dr \right) \left(-\frac{1}{2} \cos^2(\theta) \Big|_0^{\pi/2} \right)$$

$$= -\frac{1}{2r^2} \Big|_0^1 \left(\frac{1}{2} \right)$$

$$= \left(\frac{1}{4} \right) \left(-1 + \lim_{r \to 0} \frac{1}{r^2} \right)$$

$$= \infty,$$

so f is not integrable.

1.5 Spring 2015 # 1 💝

Let (X, d) and (Y, ρ) be metric spaces, $f: X \longrightarrow Y$, and $x_0 \in X$.

Prove that the following statements are equivalent:

- 1. For every $\varepsilon > 0$ $\exists \delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$.
- 2. The sequence $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow f(x_0)$ for every sequence $\{x_n\} \longrightarrow x_0$ in X.

1.6 Fall 2014 # 2

Let I be an index set and $\alpha: I \longrightarrow (0, \infty)$.

1. Show that

$$\sum_{i \in I} a(i) := \sup_{\substack{J \subset I \\ J \text{ finite}}} \sum_{i \in J} a(i) < \infty \implies I \text{ is countable.}$$

2. Suppose $I = \mathbb{Q}$ and $\sum_{q \in \mathbb{Q}} a(q) < \infty$. Define

$$f(x) := \sum_{\substack{q \in \mathbb{Q} \\ q \le x}} a(q).$$

Show that f is continuous at $x \iff x \notin \mathbb{Q}$.

1.7 Spring 2014 # 2

Let $\{a_n\}$ be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that $\sum a_n^2 < \infty$.

Note: Assume a_n, b_n are all non-negative.

2 General Analysis

2.1 Spring 2020 # 1 🔆

Prove that if $f:[0,1]\longrightarrow \mathbb{R}$ is continuous then

$$\lim_{k \to \infty} \int_0^1 kx^{k-1} f(x) \, dx = f(1).$$

Solution:

Concepts Used:

- DCT
- Weierstrass Approximation Theorem
- Suppose p is a polynomial, then

$$\begin{split} \lim_{k \longrightarrow \infty} \int_0^1 k x^{k-1} p(x) \, dx &= \lim_{k \longrightarrow \infty} \int_0^1 \left(\frac{\partial}{\partial x} \, x^k \right) \! p(x) \, dx \\ &= \lim_{k \longrightarrow \infty} \left[x^k p(x) \Big|_0^1 - \int_0^1 x^k \left(\frac{\partial}{\partial x} \, p(x) \right) dx \right] \quad \text{integrating by parts} \\ &= p(1) - \lim_{k \longrightarrow \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} \, p(x) \right) dx, \end{split}$$

• Thus it suffices to show that

$$\lim_{k \to \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx = 0.$$

• Integrating by parts a second time yields

$$\lim_{k \to \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx = \lim_{k \to \infty} \frac{x^{k+1}}{k+1} p'(x) \Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx$$

$$= -\lim_{k \to \infty} \int_0^1 \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx$$

$$= -\int_0^1 \lim_{k \to \infty} \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \quad \text{by DCT}$$

$$= -\int_0^1 0 \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx$$

$$= 0.$$

– The DCT can be applied here because f'' is continuous and [0,1] is compact, so f'' is bounded on [0,1] by a constant M and

$$\int_0^1 \left| x^k f''(x) \right| \le \int_0^1 1 \cdot M = M < \infty.$$

- Now use the Weierstrass approximation theorem:
 - If $f:[a,b] \longrightarrow \mathbb{R}$ is continuous, then for every $\varepsilon > 0$ there exists a polynomial $p_{\varepsilon}(x)$ such that $||f p_{\varepsilon}||_{\infty} < \varepsilon$.
- Thus

$$\left| \int_0^1 kx^{k-1} p_{\varepsilon}(x) \, dx - \int_0^1 kx^{k-1} f(x) \, dx \right| = \left| \int_0^1 kx^{k-1} (p_{\varepsilon}(x) - f(x)) \, dx \right|$$

$$\leq \left| \int_0^1 kx^{k-1} || p_{\varepsilon} - f ||_{\infty} \, dx \right|$$

$$= || p_{\varepsilon} - f ||_{\infty} \cdot \left| \int_0^1 kx^{k-1} \, dx \right|$$

$$= || p_{\varepsilon} - f ||_{\infty} \cdot x^k \Big|_0^1$$

$$= || p_{\varepsilon} - f ||_{\infty} \xrightarrow{\varepsilon \longrightarrow 0} 0$$

and the integrals are equal.

• By the first argument,

$$\int_0^1 kx^{k-1} p_{\varepsilon}(x) dx = p_{\varepsilon}(1) \text{ for each } \varepsilon$$

• Since uniform convergence implies pointwise convergence, $p_{\varepsilon}(1) \stackrel{\varepsilon \longrightarrow 0}{\longrightarrow} f(1)$.

2.2 Fall 2019 # 1 🦙

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

2.2.1 a

Prove that if $\lim_{n \to \infty} a_n = 0$, then

$$\lim_{n\to\infty}\frac{a_1+\cdots+a_n}{n}=0$$

2.2.2 b

Prove that if $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges, then

$$\lim_{n\to\infty}\frac{a_1+\cdots+a_n}{n}=0$$

Solution:

Concepts Used:

- Cesaro mean/summation.
- Break series apart into pieces that can be handled separately.

2.2.3 a

Prove a stronger result:

$$a_k \longrightarrow S \implies S_N := \frac{1}{N} \sum_{k=1}^N a_k \longrightarrow S.$$

Idea: once N is large enough, $a_k \approx S$, and all smaller terms will die off as $N \longrightarrow \infty$.

• Use convergence $a_k \longrightarrow S$: choose M large enough such that

$$k \ge M + 1 \implies |a_k - S| < \varepsilon.$$

Then

$$\left| \left(\frac{1}{N} \sum_{k=1}^{N} a_k \right) - S \right| = \frac{1}{N} \left| \left(\sum_{k=1}^{N} a_k \right) - NS \right|$$

$$= \frac{1}{N} \left| \left(\sum_{k=1}^{N} a_k \right) - \sum_{k=1}^{N} S \right|$$

$$= \frac{1}{N} \left| \sum_{k=1}^{N} (a_k - S) \right|$$

$$\leq \frac{1}{N} \sum_{k=1}^{N} |a_k - S|$$

$$= \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} |a_k - S|$$

$$\leq \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} \frac{\varepsilon}{2}$$

$$= \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + (N - M) \frac{\varepsilon}{2}$$

$$\stackrel{\varepsilon}{\Longrightarrow} \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + 0$$

$$\stackrel{N \longrightarrow \infty}{\Longrightarrow} 0 + 0.$$

Note: M is fixed, so the last sum is some constant c, and $c/N \longrightarrow 0$ as $N \longrightarrow \infty$ for any constant. To be more careful, choose M first to get $\varepsilon/2$ for the tail, then choose N(M) > M for the remaining truncated part of the sum.

2.2.4 b

• Define

$$\Gamma_n := \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

- $\Gamma_1 = \sum_{k=1}^n \frac{a_k}{k}$ is the original series and each Γ_n is a tail of Γ_1 , so by assumption $\Gamma_n \stackrel{n \longrightarrow \infty}{\longrightarrow} 0$.
- Compute

$$\frac{1}{n}\sum_{k=1}^{n}a_k=\frac{1}{n}(\Gamma_1+\Gamma_2+\cdots+\Gamma_n-\mathbf{\Gamma_{n+1}})$$

• This comes from consider the following summation:

$$\Gamma_1:$$
 $a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots$
 $\Gamma_2:$ $\frac{a_2}{2} + \frac{a_3}{3} + \cdots$
 $\Gamma_3:$ $\frac{a_3}{3} + \cdots$
 $\sum_{i=1}^n \Gamma_i:$ $a_1 + a_2 + a_3 + \cdots + a_n + \frac{a_{n+1}}{n+1} + \cdots$

- Use part (a): since $\Gamma_n \xrightarrow{n \to \infty} 0$, we have $\frac{1}{n} \sum_{k=1}^n \Gamma_k \xrightarrow{n \to \infty} 0$.
- Also a minor check: $\Gamma_n \longrightarrow 0 \implies \frac{1}{n}\Gamma_n \longrightarrow 0$.

$$\frac{1}{n} \sum_{k=1}^{n} a_k = \frac{1}{n} (\Gamma_1 + \Gamma_2 + \dots + \Gamma_n - \mathbf{\Gamma_{n+1}})$$
$$= \left(\frac{1}{n} \sum_{k=0}^{n} \Gamma_k\right) - \left(\frac{1}{n} \Gamma_{n+1}\right)$$
$$\stackrel{n \to \infty}{\longrightarrow} 0.$$

2.3 Fall 2018 # 4 처

Let $f \in L^1([0,1])$. Prove that

$$\lim_{n \to \infty} \int_0^1 f(x) |\sin nx| \ dx = \frac{2}{\pi} \int_0^1 f(x) \ dx$$

> Hint: Begin with the case that f is the characteristic function of an interval.

Add concepts.

Solution:

Concepts Used:

• ?

Case of characteristic function

- First suppose $f(x) = \chi_{[0,1]}(x)$.
- Note that $\sin(nx)$ has a period of $2\pi/n$, and thus $\left|\frac{n}{2\pi}\right|$ full periods in [0,1].
- Taking the absolute value yields a new function with half the period, so a period of π/n

and $|\pi/n|$ full periods in [0,1].

• We can compute the integral over one full period (which is independent of which period is chosen), and since $\sin(x)$ is positive and agrees with $|\sin(nx)|$ on the first period, we have

$$\int_{\text{One Period}} |\sin(nx)| \, dx = \int_0^{\pi/n} \sin(nx) \, dx$$

$$= \frac{1}{n} \int_0^{\pi} \sin(u) \, du \quad u = nx$$

$$= \frac{1}{n} - \cos(u) \Big|_0^{\pi}$$

$$= \frac{2}{n}.$$

• Then break the integral up into integrals over periods P_1, P_2, \dots, P_N where $N := \lfloor n/\pi \rfloor$:

$$\int_{0}^{1} |\sin(nx)| dx = \left(\sum_{j=1}^{N} \int_{P_{j}} |\sin(nx)| dx\right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| dx$$

$$= \left(\sum_{j=1}^{N} \frac{2}{n}\right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| dx$$

$$= N\left(\frac{2}{n}\right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| dx$$

$$\coloneqq \left\lfloor \frac{n}{\pi} \right\rfloor \frac{2}{n} + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| dx$$

$$= \frac{2}{\pi} + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)| dx$$

$$\coloneqq \frac{2}{\pi} + R(n)$$

so it suffices to show that $R(n) \stackrel{n \longrightarrow \infty}{\longrightarrow} 0$. Need to justify removing floor function and cancellation.

• Showing this: ???????????

No clue how to show this.

General case

Not sure. Approximate f by simple functions...?

2.4 Fall 2017 # 4 🔆

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

1. Show that $f_n \longrightarrow 0$ pointwise but not uniformly on [0,1].

Hint: Consider the maximum of f_n .

2.

$$\lim_{n \to \infty} \int_0^1 n(1-x)^n \sin x \, dx = 0$$

Add concepts.

Walk through

Solution:

Concepts Used:

• ?

2.4.1 a

Let $G(x) = \sum_{n=1}^{\infty} nx(1-x)^n$. Applying the ratio test, we have

$$\left| \frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n} \right| = \frac{n+1}{n} |1-x| \stackrel{n \longrightarrow \infty}{\longrightarrow} |1-x| < 1 \iff 0 \le x \le 2,$$

and in particular, this series converges on [0,2]. Thus its terms go to zero, and $nx(1-x)^n \longrightarrow 0$ on $[0,1] \subset [0,2]$.

To see that the convergence is not uniform, let $x_n = \frac{1}{n}$ and $\varepsilon > \frac{1}{e}$, then

$$\sup_{x \in [0,1]} |nx(1-x)^n - 0| \ge |nx_n(1-x_n)^n| = \left| \left(1 - \frac{1}{n}\right)^n \right| \stackrel{n \to \infty}{\longrightarrow} e^{-1} > \varepsilon.$$

2.4.2 b

Note: could use the first part with $\sin(x) \le x$, but then integral ends up more complicated. Noting that $\sin(x) \le 1$, we have We have

$$\left| \int_0^1 n(1-x)^n \sin(x) \right| \le \int_0^1 |n(1-x)^n \sin(x)|$$

$$\le \int_0^1 |n(1-x)^n|$$

$$= n \int_0^1 (1-x)^n$$

$$= -\frac{n(1-x)^{n+1}}{n+1}$$

$$\stackrel{n \to \infty}{\longrightarrow} 0.$$

2.5 Spring 2017 # 3 🦙

Let

$$f_n(x) = ae^{-nax} - be^{-nbx}$$
 where $0 < a < b$.

Show that

a.
$$\sum_{n=1}^{\infty} |f_n| \text{ is not in } L^1([0,\infty),m)$$

Hint: $f_n(x)$ has a root x_n .

b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0,\infty),m) \text{ and } \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x) dm = \ln \frac{b}{a}$$

Not complete

Add concepts

Walk through

Solution:

Concepts Used:

• ?

2.5.1 a

Letting $x_n := \frac{1}{n}$, we have

$$\sum_{k=1}^{\infty} |f_k(x)| \ge |f_n(x_n)| = |ae^{-ax} - be^{-bx}| := M.$$

In particular, $\sup_{x} |f_n(x)| \not\longrightarrow 0$, so the terms do not go to zero and the sum can not converge.

2.5.2 b

?

2.6 Fall 2016 # 1 🔭

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that f converges to a differentiable function on $(1, \infty)$ and that

$$f'(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n^x}\right)'.$$

Hint:

$$\left(\frac{1}{n^x}\right)' = -\frac{1}{n^x} \ln n$$

Add concepts

Solution:

Concepts Used:

. 7

• Set
$$f_N(x) := \sum_{n=1}^N n^{-x}$$
, so $f(x) = \lim_{N \to \infty} f_N(x)$.

• If an interchange of limits is justified, we have

$$\begin{split} \frac{\partial}{\partial x} & \lim_{N \to \infty} \sum_{n=1}^{N} n^{-x} = \lim_{h \to 0} \lim_{N \to \infty} \frac{1}{h} \left[\left(\sum_{n=1}^{N} n^{-x} \right) - \left(\sum_{n=1}^{N} n^{-(x+h)} \right) \right] \\ & \stackrel{?}{=} \lim_{N \to \infty} \lim_{h \to 0} \frac{1}{h} \left[\left(\sum_{n=1}^{N} n^{-x} \right) - \left(\sum_{n=1}^{N} n^{-(x+h)} \right) \right] \\ & = \lim_{N \to \infty} \lim_{h \to 0} \frac{1}{h} \left[\sum_{n=1}^{N} n^{-x} - n^{-(x+h)} \right] \quad (1) \\ & = \lim_{N \to \infty} \sum_{n=1}^{N} \lim_{h \to 0} \frac{1}{h} \left[n^{-x} - n^{-(x+h)} \right] \quad \text{since this is a finite sum} \\ & \coloneqq \lim_{N \to \infty} \sum_{n=1}^{N} \frac{\partial}{\partial x} \left(\frac{1}{n^x} \right) \\ & = \lim_{N \to \infty} \sum_{n=1}^{N} -\frac{\ln(n)}{n^x}, \end{split}$$

where the combining of sums in (1) is valid because $\sum n^{-x}$ is absolutely convergent for x > 1 by the *p*-test.

- Thus it suffices to justify the interchange of limits and show that the last sum converges on $(1, \infty)$.
- Claim: $\sum n^{-x} \ln(n)$ converges.
 - Use the fact that for any fixed $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{\ln(n)}{n^{\varepsilon}} \stackrel{\text{L.H.}}{=} \lim_{n \to \infty} \frac{1/n}{\varepsilon n^{\varepsilon - 1}} = \lim_{n \to \infty} \frac{1}{\varepsilon n^{\varepsilon}} = 0,$$

– This implies that for a fixed $\varepsilon > 0$ and for any constant c > 0 there exists an N large enough such that $n \geq N$ implies $\ln(n)/n^{\varepsilon} < c$, i.e. $\ln(n) < cn^{\varepsilon}$.

- Taking c = 1, we have $n \ge N \implies \ln(n) < n^{\varepsilon}$
- We thus break up the sum:

$$\sum_{n \in \mathbb{N}} \frac{\ln(n)}{n^x} = \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{\ln(n)}{n^x}$$

$$\leq \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{n^{\varepsilon}}{n^x}$$

$$\coloneqq C_{\varepsilon} + \sum_{n=N}^{\infty} \frac{n^{\varepsilon}}{n^x} \quad \text{with } C_{\varepsilon} < \infty \text{ a constant}$$

$$= C_{\varepsilon} + \sum_{n=N}^{\infty} \frac{1}{n^{x-\varepsilon}},$$

where the last term converges by the *p*-test if $x - \varepsilon > 1$.

- But ε can depend on x, and if $x \in (1, \infty)$ is fixed we can choose $\varepsilon < |x 1|$ to ensure this.
- Claim: the interchange of limits is justified.

2.7 Fall 2016 # 5 💝

Let $\varphi \in L^{\infty}(\mathbb{R})$. Show that the following limit exists and satisfies the equality

$$\lim_{n \to \infty} \left(\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} \, dx \right)^{\frac{1}{n}} = \|\varphi\|_{\infty}.$$

Walk through

Add concepts

Solution:

Concepts Used:

• ?

Let L be the LHS and R be the RHS.

Claim: $L \leq R.$ - Since $|\varphi| \leq \|\varphi\|_{\infty}$ a.e., we can write

$$L^{\frac{1}{n}} := \int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1 + x^2}$$

$$\leq \int_{\mathbb{R}} \frac{\|\varphi\|_{\infty}^n}{1 + x^2}$$

$$= \|\varphi\|_{\infty}^n \int_{\mathbb{R}} \frac{1}{1 + x^2}$$

$$= \|\varphi\|_{\infty}^n \arctan(x)\Big|_{-\infty}^{\infty}$$

$$= \|\varphi\|_{\infty}^n \left(\frac{\pi}{2} - \frac{-\pi}{2}\right)$$

$$= \pi \|\varphi\|_{\infty}^n$$

$$\implies L^{\frac{1}{n}} \leq \sqrt[n]{\pi \|\varphi\|_{\infty}^{n}}$$

$$\implies L \leq \pi^{\frac{1}{n}} \|\varphi\|_{\infty}$$

$$\stackrel{n \to \infty}{\longrightarrow} \|\varphi\|_{\infty},$$

where we've used the fact that $c^{\frac{1}{n}} \stackrel{n \longrightarrow \infty}{\longrightarrow} 1$ for any constant c.

Actually true? Need conditions?

Claim: $R \leq L$.

- We will show that $R \leq L + \varepsilon$ for every $\varepsilon > 0$.
- Set

$$S_{\varepsilon} := \left\{ x \in \mathbb{R}^n \mid |\varphi(x)| \ge ||\varphi||_{\infty} - \varepsilon \right\}.$$

• Then we have

$$\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} dx \ge \int_{S_{\varepsilon}} \frac{|\varphi(x)|^n}{1+x^2} dx \quad S_{\varepsilon} \subset \mathbb{R}$$

$$\ge \int_{S_{\varepsilon}} \frac{(\|\varphi\|_{\infty} - \varepsilon)^n}{1+x^2} dx \quad \text{by definition of } S_{\varepsilon}$$

$$= (\|\varphi\|_{\infty} - \varepsilon)^n \int_{S_{\varepsilon}} \frac{1}{1+x^2} dx$$

$$= (\|\varphi\|_{\infty} - \varepsilon)^n C_{\varepsilon} \quad \text{where } C_{\varepsilon} \text{ is some constant}$$

$$\implies \left(\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} \, dx \right)^{\frac{1}{n}} \ge (\|\varphi\|_{\infty} - \varepsilon) C_{\varepsilon}^{\frac{1}{n}}$$

$$\stackrel{n \longrightarrow \infty}{\longrightarrow} (\|\varphi\|_{\infty} - \varepsilon) \cdot 1$$

$$\stackrel{\varepsilon \longrightarrow 0}{\longrightarrow} \|\varphi\|_{\infty},$$

where we've again used the fact that $c^{\frac{1}{n}} \longrightarrow 1$ for any constant.

2.8 Fall 2016 # 6 🐪

Let $f, g \in L^2(\mathbb{R})$. Show that

$$\lim_{n \to \infty} \int_{\mathbb{R}} f(x)g(x+n) \, dx = 0$$

Add concepts

Solution:

Concepts Used:

- '
- Use the fact that L^p has small tails: if $h \in L^2(\mathbb{R})$, then for any $\varepsilon > 0$,

$$\forall \varepsilon, \exists N \in \mathbb{N} \text{ such that } \int_{|x| \ge N} |h(x)|^2 dx < \varepsilon.$$

How to prove small tails in L^p ?

- So choose n large enough so the tails of both f and g are smaller than ε .
- Apply Cauchy-Schwarz:

$$\left| \int_{\mathbb{R}} f(x)g(x+n) \, dx \right| \le \int_{\mathbb{R}} |f(x)g(x+n)| \, dx$$
$$\le \int_{\mathbb{R}} .$$

2.9 Spring 2016 # 1

For $n \in \mathbb{N}$, define

$$e_n = \left(1 + \frac{1}{n}\right)^n$$
 and $E_n = \left(1 + \frac{1}{n}\right)^{n+1}$

Show that $e_n < E_n$, and prove Bernoulli's inequality:

$$(1+x)^n > 1 + nx$$
 for $-1 < x < \infty$ and $n \in \mathbb{N}$

Use this to show the following:

- 1. The sequence e_n is increasing.
- 2. The sequence E_n is decreasing.
- 3. $2 < e_n < E_n < 4$.
- $4. \lim_{n \to \infty} e_n = \lim_{n \to \infty} E_n.$

2.10 Fall 2015 # 1

Define

$$f(x) = c_0 + c_1 x^1 + c_2 x^2 + \ldots + c_n x^n$$
 with n even and $c_n > 0$.

Show that there is a number x_m such that $f(x_m) \leq f(x)$ for all $x \in \mathbb{R}$.