Title

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Contents

0.1 Series of Groups

Definition 0.0.1 (Normal Series)**.**
A **normal series** of a group G is a sequence $G \longrightarrow G^1 \longrightarrow G^2 \longrightarrow \cdots$ such that $G^{i+1} \trianglelefteq G_i$ for every i.

Definition 0.0.2 (Central Series)**.**
A **central series** for a group G is a terminating normal series $G \longrightarrow G^1 \longrightarrow \cdots \longrightarrow \{e\}$ such that each quotient is **central**, i.e. $[G, G^i] \leq G^{i-1}$ for all i.

Definition 0.0.3 (Composition Series)**.**
A **composition series** of a group G is a finite normal series such that G^{i+1} is a *maximal proper* normal subgroup of G^i.

Theorem 0.1 ***(Jordan-Holder).***
Any two composition series of a group have the same length and isomorphic composition factors (up to permutation).

Definition 0.1.1 (Simple Groups)**.**
A group G is **simple** iff $H \trianglelefteq G \implies H = \{e\}, G$, i.e. it has no non-trivial proper subgroups.

Proposition 0.2.
If G is *not* simple, then G is an extension of any of its normal subgroups. I.e. for any $N \trianglelefteq G$, $G \cong E$ for some extension of the form $N \longrightarrow E \longrightarrow G/N$.

Definition 0.2.1 (Lower Central Series)**.**
Set $G^0 = G$ and $G^{i+1} = [G, G^i]$, then $G^0 \geq G^1 \geq \cdots$ is the *lower central series* of G.

Mnemonic: “lower” because the chain is descending. Iterate the adjoint map $[\cdot, G]$, if this terminates then the map is nilpotent, so call G nilpotent!

Definition 0.2.2 (Upper Central Series)**.**
Set $Z_0 = 1$, $Z_1 = Z(G)$, and $Z_{i+1} \leq G$ to be the subgroup satisfying $Z_{i+1}/Z_i = Z(G/Z_i)$. Then $Z_0 \leq Z_1 \leq \cdots$ is the *upper central series* of G.
Equivalently, since $Z_i \trianglelefteq G$, there is a quotient map $\pi : G \longrightarrow G/Z_i$, so define $Z_{i+1} \coloneqq \pi^{-1}(Z(G/Z_i))$ (?).

Mnemonic: "upper" because the chain is ascending. "Take higher centers".

Definition 0.2.3 (Derived Series)**.**
Set $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$, then $G^{(0)} \geq G^{(1)} \geq \cdots$ is the *derived series* of G.

Definition 0.2.4 (Solvable)**.**
A group G is **solvable** iff G has a terminating normal series with abelian composition factors, i.e.

$$G \longrightarrow G^1 \longrightarrow \cdots \longrightarrow \{e\} \text{ with } G^i/G^{i+1} \text{ abelian for all } i.$$

Theorem 0.3 ***(Characterization of Solvable).***
A group G is solvable iff its derived series terminates.

Theorem 0.4 ***(S_n is Almost Always Solvable).***
If $n \geq 4$ then S_n is solvable.

Lemmas:

- G is solvable iff G has a terminating *derived series*.
- Solvable groups satisfy the 2 out of 3 property
- Abelian $\implies$ solvable
- Every group of order less than 60 is solvable.

Definition 0.4.1 (Nilpotent)**.**
A group G is **nilpotent** iff G has a terminating upper central series.

Moral: the adjoint map is nilpotent.

Theorem 0.5 ***(Nilpotents Have All Sylows Normal).***
A group G is nilpotent iff all of its Sylow p-subgroups are normal for every p dividing $|G|$.

Theorem 0.6 ***(Nilpotent Implies Maximal Normals).***
A group G is nilpotent iff every maximal subgroup is normal.

Theorem 0.7 ***(Characterization of Nilpotent Groups).***
G is nilpotent iff G has an upper central series terminating at G.

Theorem 0.8 ***(Characterization of Nilpotent Groups).***
G is nilpotent iff G has a lower central series terminating at 1.

Proposition 0.9.
For G a finite group, TFAE:

- G is nilpotent
- Normalizers grow (i.e.$H < N_G(H)$ whenever H is proper)
- Every Sylow-p subgroup is normal
- G is the direct product of its Sylow p-subgroups
- Every maximal subgroup is normal
- G has a terminating *Lower* Central Series
- G has a terminating *Upper* Central Series

Lemmas:

- Nilpotent groups satisfy the 2 out of 3 property. Todo. Specify.
- G has normal subgroups of order d for *every* d dividing $|G|$