Topology Qualifying Exam Solutions

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Contents

1 Definitions

- Hausdorff
- Second Countable: admits a countable basis.
- Closed (several characterizations)
- Closure in a subspace: $Y \subset X \implies \mathrm{cl}_Y(A) := \mathrm{cl}_X(A) \bigcap Y$.
- Bounded

- Compact
- Locally compact For every $x \in X$, there exists a $K_x \ni x$ such that K_x is compact.
- Connected: There does not exist a disconnecting set $X = A\coprod B$ such that $\emptyset \neq A, B \subsetneq$, i.e. X is the union of two proper disjoint nonempty sets. Equivalently, X contains no proper nonempty clopen sets.
  - Additional condition for a subspace $Y \subset X$: $\mathrm{cl}_Y(A) \bigcap V = A \bigcap \mathrm{cl}_Y(B) = \emptyset$.
- Locally connected: A space is locally connected at a point x iff $\forall N_x \ni x$, there exists a $U \subset N_x$ containing x that is connected.
- Retract: A subspace $A \subset X$ is a *retract* of X iff there exists a continuous map $f : X \longrightarrow A$ such that $f \Big|_A = \mathrm{id}_A$. Equivalently it is a *left* inverse to the inclusion.
- Uniform Continuity: For $f : (X, d_x) \longrightarrow (Y, d_Y)$ metric spaces,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \quad d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

- Lebesgue number: For (X, d) a compact metric space and $\{U_\alpha\} \rightrightarrows X$, there exist $\delta_L > 0$ such that

$$A \subset X,\ \mathrm{diam}(A) < \delta_L \implies A \subseteq U_\alpha \text{ for some } \alpha.$$

- Paracompact
- Components: Set $x \sim y$ iff there exists a connected set $U \ni x, y$ and take equivalence classes
- Path Components: Set $x \sim y$ iff there exists a path-connected set $U \ni x, y$ and take equivalence classes
- Separable: countable dense subset.

2 Theorems

- Closed subsets of Hausdorff spaces are compact? (check)
- Cantor's intersection theorem?
- Tube lemma
- Properties pushed forward through continuous maps:
  - Compactness?
  - Connectedness (when surjective)
  - Separability
  - Density **only when** f is surjective
  - **Not** openness
  - **Not** closedness
- Results that only work for metric spaces
  - ?
- A retract of a Hausdroff/connected/compact space is closed/connected/compact respectively.
- A continuous function on a compact set is uniformly continuous.

- Proof: take $\left\{ B_{\frac{\varepsilon}{2}}(y) \;\middle|\; y \in Y \right\} \rightrightarrows Y$, pull back to an open cover of X, has Lebesgue number $\delta_L > 0$, then $x' \in B_{\delta_L}(x) \implies f(x), f(x') \in B_{\frac{\varepsilon}{2}}(y)$ for some y.
- Lipschitz continuity implies uniform continuity (take $\delta = \varepsilon/C$)
  - Counterexample to converse: $f(x) = \sqrt{x}$ on $[0,1]$ has unbounded derivative.
- Extreme Value Theorem: for $f : X \longrightarrow Y$ continuous with X compact and Y ordered in the order topology, there exist points $c, d \in X$ such that $f(x) \in [f(c), f(d)]$ for every x.

3 Sandbox of Spaces

- Finite discrete sets with the discrete topology
- Subspaces of $\mathbb{R}$: $(a,b), (a,b], (a,\infty)$, etc.
  - $\{0\} \bigcup \left\{ \frac{1}{n \;\middle|\; n \in \mathbb{Z}^{\geq 1}} \right\}$
- $\mathbb{Q}$
- The topologist's sine curve
- One-point compactifications
- $\mathbb{R}^\omega$

Alternative topologies to consider:

- Cofinite
- Discrete and Indiscrete
- Uniform

4 General Topology

4.1 2

Statement: state the definition of compactness, determine if the sets $\{0\} \bigcup \left\{ \frac{1}{n} \right\}, (0,1]$ are compact.

i. A topological space (X, τ) is **compact** if every open cover has a *finite* subcover. That is, if $\left\{ U_j \;\middle|\; j \in J \right\} \subset \tau$ is a collection of open sets such that $X \subseteq \bigcup_{j \in J} U_j$, then there exists a *finite* subset $J' \subset J$ such that $X \subseteq \bigcup_{j \in J'} U_j$.

ii. Use Heine-Borel theorem: a set $U \subset \mathbb{R}^n$ is compact $\iff$ U is *closed* and *bounded*.

- X is closed in $\mathbb{R}$, since we can write its complement as an arbitrary union of open intervals:

$$X^c = (-\infty, 0) \bigcup \left(\bigcup_{n \in \mathbb{Z}^+} \left(\frac{1}{n}, \frac{1}{n+1} \right) \right) \bigcup (1, \infty)$$

- X is *bounded*, since we can pick $r = 1$, then $x, y \in X \implies d(x,y) \leq r = 1$.

iii. Use Heine-Borel again: X is not closed because it does not contain all of its limit points, e.g. the sequence $\left\{ x_n := \frac{1}{n} \;\middle|\; n \in \mathbb{Z}^{\geq 1} \right\} \subset X$ but $x_n \overset{n \longrightarrow \infty}{\longrightarrow} 0 \in X^c$. Thus is is **not** compact.

4.1.1 Alternate Proof of (ii)

See Munkres p.164

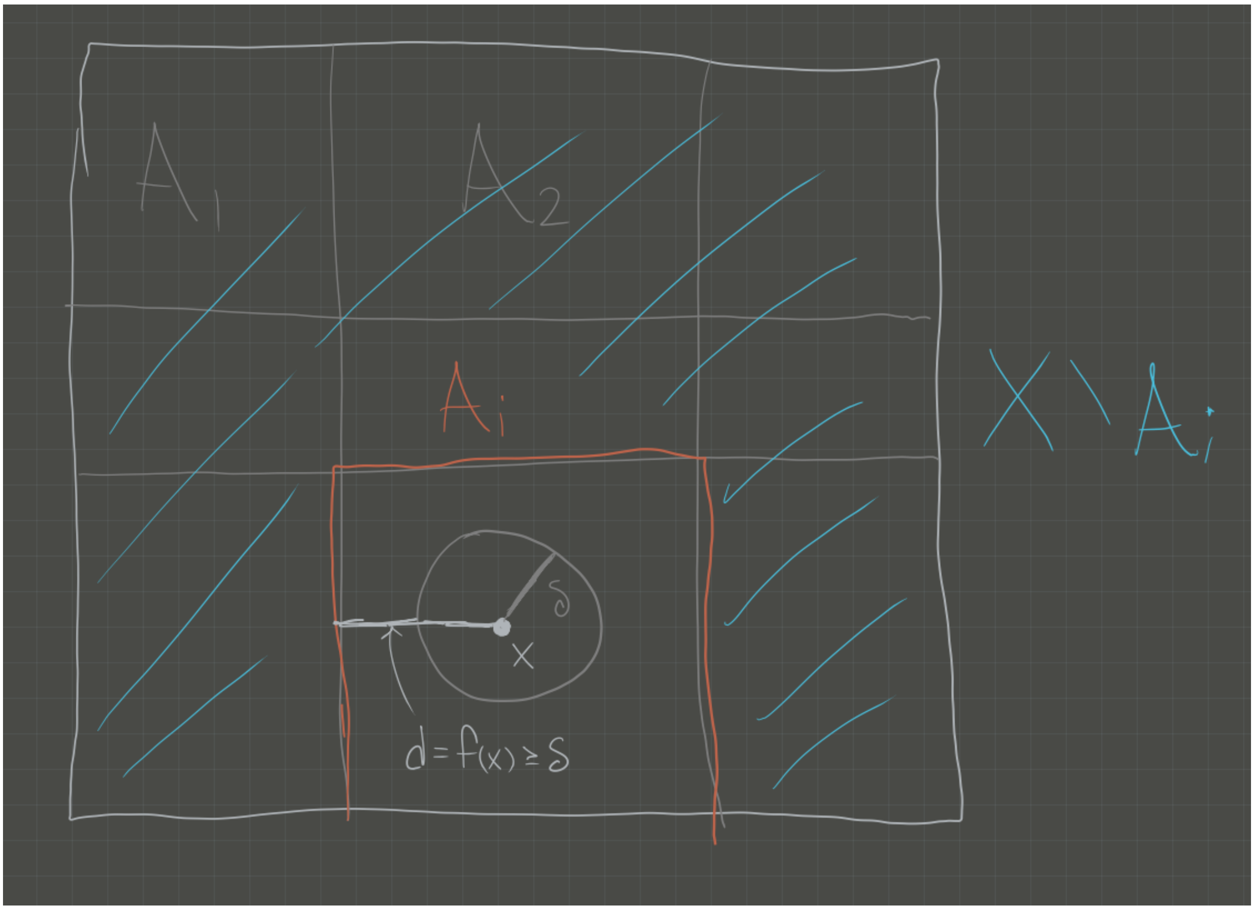
- Let $\left\{U_i \;\middle|\; j \in J\right\} \rightrightarrows X$; then $0 \in U_j$ for some $j \in J$.
- In the subspace topology, U_i is given by some $V \in \tau(\mathbb{R})$ such that $V \bigcap X = U_i$
  - A basis for the subspace topology on $\mathbb{R}$ is open intervals, so write V as a union of open intervals $V = \bigcup_{k \in K} I_k$.
  - Since $0 \in U_j$, $0 \in I_k$ for some k.
- Since I_k is an interval, it contains infinitely many points of the form $x_n = \frac{1}{n} \in X$
- Then $I_k \bigcap X \subset U_j$ contains infinitely many such points.
- So there are only *finitely* many points in $X \setminus U_j$, each of which is in $U_{j(n)}$ for some $j(n) \in J$ depending on n.
- So U_j and the *finitely* many $U_{j(n)}$ form a finite subcover of X. ■

4.2 4

Statement: show that the *Lebesgue number* is well-defined for compact metric spaces.

Note: this is a question about the *Lebesgue Number*. See Wikipedia for detailed proof.

- Write $U = \left\{U_i \;\middle|\; i \in I\right\}$, then $X \subseteq \bigcup_{i \in I} U_i$. Need to construct a $\delta > 0$.
- By compactness of X, choose a finite subcover $U_1, \cdots, U_n$.
- Define the distance between a point x and a set $Y \subset X$: $d(x, Y) = \inf_{y \in Y} d(x, y)$.
  - **Claim**: the function $d(\,\cdot\,, Y) : X \longrightarrow \mathbb{R}$ is continuous for a fixed set.
  - Proof: Todo, not obvious.



- Define a function

$$\begin{aligned} f : X &\longrightarrow \mathbb{R} \\ x &\mapsto \frac{1}{n}\sum_{i=1}^{n} d(x, X \setminus U_i). \end{aligned}$$

  - Note this is a sum of continuous functions and thus continuous.
- **Claim**:

$$\delta := \inf_{x\in X} f(x) = \min_{x\in X} f(x) = f(x_{\text{min}}) > 0$$

  suffices.
  - That the infimum is a minimum: f is a continuous function on a compact set, apply the extreme value theorem: it attains its minimum.
  - That $\delta > 0$: otherwise, $\delta = 0 \implies \exists x_0$ such that $d(x_0, X \setminus U_i) = 0$ for all i.
    * Forces $x_0 \in X \setminus U_i$ for all i, but $X \setminus \bigcup U_i = \emptyset$ since the U_i cover X.
  - That it satisfies the Lebesgue condition:

$$\forall x \in X, \exists i \quad \text{such that} \quad B_\delta(x) \subset U_i$$

    * Let $B_\delta(x) \ni x$; then by minimality $f(x) \geq \delta$.
    * Thus it can *not* be the case that $d(x, X \setminus U_i) < \delta$ for *every* i, otherwise

$$f(x) \leq \frac{1}{n}(\delta + \cdots + \delta) = \frac{n\delta}{n} = \delta$$

    - $*$ So there is some particular i such that $d(x, X \setminus U_i) \geq \delta$.
    - $*$ But then $B_\delta \subseteq U_i$ as desired.

4.3 6

Statement: prove that $[0,1] \subset \mathbb{R}$ is compact.

4.3.1 Proof 1 (DZG)

Todo: find a direct proof.

4.4 8

Topic: proof of the tube lemma.

Statement: show $X, Y \in \text{Top}_{\text{compact}} \iff X \times Y \in \text{Top}_{\text{compact}}$

4.4.1 Proof 1 (DZG)

$\Longleftarrow$:

- By universal properties, the product $X \times Y$ is equipped with continuous projections
- The continuous image of a compact set is compact, and $\pi_1(X \times Y) = X, p_2(X \times Y) = Y$
- So X, Y are compact.

$\Longrightarrow$:

Proof of Tube Lemma:

- Let $\left\{ U_j \times V_j \;\middle|\; j \in J \right\} \rightrightarrows X \times Y$.
- Fix a point $x_0 \in X$, then $\{x_0\} \times Y \subset N$ for some open set N.
- By the tube lemma, there is a $U^x \subset X$ such that the tube $U^x \times Y \subset N$.
- Since $\{x_0\} \times Y \cong Y$ which is compact, there is a finite subcover $\left\{ U_j \times V_j \;\middle|\; j \leq n \right\} \rightrightarrows \{x_0\} \times Y$.
- "Integrate the X": write

$$W = \bigcap_{j=1}^{n} U_j,$$

  then $x_0 \in W$ and W is a finite intersection of open sets and thus open.
- Claim: $\left\{ U_j \times V_j \;\middle|\; j \leq n \right\} \rightrightarrows W \times Y$
    - Let $(x, y) \in W \times Y$; want to show $(x, y) \in U_j \times V_j$ for some $j \leq n$.
    - Then $(x_0, y) \in \{x_0\} \times Y$ is on the same horizontal line
    - $(x_0, y) \in U_j \times V_j$ for some j by construction
    - So $y \in V_j$ for this j
    - Since $x \in W$, $x \in U_j$ for *every* j, thus $x \in U_j$.
    - So $(x, y) \in U_j \times V_j$

Actual Proof:

- Let $\left\{ U_j \;\middle|\; j \in J \right\} \rightrightarrows X \times Y$.

- Fix $x_0 \in X$, the slice $\{x_0\} \times Y$ is compact and can be covered by finitely many elements $\left\{U_j \;\middle|\; j \leq m\right\} \rightrightarrows \{x_0\} \times Y$.
  - Sum: write $N = \bigcup_{j=1}^{m} U_j$; then $\{x_0\} \times Y \subset N$.
  - Apply the tube lemma to N: produce $\{x_0\} \times Y \in W \times Y \subset N$; then $\left\{U_j \;\middle|\; j \leq m\right\} \rightrightarrows W \times Y$.
- Now let $x \in X$ vary: for each $x \in X$, produce $W_x \times Y$ as above, then $\left\{W_x \times Y \;\middle|\; x \in X\right\} \rightrightarrows X$.
  - By above argument, every tube $W_x \times Y$ can be covered by *finitely* many U_j.
- Since $\left\{W_x \;\middle|\; x \in X\right\} \rightrightarrows X$ and X is compact, produce a finite subset $\left\{W_k \;\middle|\; k \leq m'\right\} \rightrightarrows X$.
- Then $\left\{W_k \times Y \;\middle|\; k \leq m'\right\} \rightrightarrows X \times Y$; the claim is that it is a finite cover.
  - Finitely many k
  - For each k, the tube $W_k \times Y$ is covered by finitely by U_j
  - And finite $\times$ finite $=$ finite. ∎

Shorter mnemonic:

19.U It is sufficient to consider a cover consisting of elementary sets. Since Y is compact, each fiber $x \times Y$ has a finite subcovering $\{U_i^x \times V_i^x\}$. Put $W^x = \cap U_i^x$. Since X is compact, the cover $\{W^x\}_{x \in X}$ has a finite subcovering W^{x_j}. Then $\{U_i^{x_j} \times V_i^{x_j}\}$ is the required finite subcovering.

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X is connected:

- Write $X = L \coprod G$ where $L = \{0\} \times [-1, 1]$ and $G = \left\{\Gamma(\sin(x)) \;\middle|\; x \in (0, 1]\right\}$ is the graph of $\sin(x)$.
- $L \cong [0, 1]$ which is connected
  - Claim: Every interval is connected (todo)
- Claim: G is connected
  - The function

$$\begin{aligned} f : (0,1] &\longrightarrow [-1,1] \\ x &\mapsto \sin(x) \end{aligned}$$

    is continuous (how to prove?)
  - Claim: The diagonal map $\Delta : Y \longrightarrow Y \times Y$ where $\Delta(t) = (t, t)$ is continuous for any Y since $\Delta = (\mathrm{id}, \mathrm{id})$
  - The composition of continuous function is continuous
  - So the composition is continuous:

$$\begin{aligned} F : (0,1] \xrightarrow{\Delta} (0,1]^2 &\xrightarrow{(\mathrm{id}, f)} (0,1] \times [-1,1] \\ t \mapsto (t,t) &\mapsto (t, f(t)) \end{aligned}$$

  - Then $G = F((0, 1])$ is the continuous image of a connected set and thus connected.

- Claim: X is connected
  - Suppose there is a disconnecting cover $X = A \coprod B$ such that $\overline{A} \bigcap B = A \bigcap \overline{B} = \emptyset$ and $A, B \neq \emptyset$.
  - WLOG suppose $(x, \sin(x)) \in B$ for $x > 0$.
  - Claim: $B = G$
    * It can't be the case that A intersects G: otherwise $X = A \coprod B \implies G = (A \bigcap G) \coprod (B \bigcap V)$ disconnects G. So $A \bigcap G = \emptyset$, forcing $A \subseteq L$
    * Similarly L can not be disconnected, so $B \bigcap L = \emptyset$ forcing $B \subset G$
    * So $A \subset L$ and $B \subset G$, and since $X = A \coprod B$, this forces $A = L$ and $B = G$.
  - But any open set U in the subspace topology $L \subset \mathbb{R}^2$ (generated by open balls) containing $(0,0) \in L$ is the restriction of a ball $V \subset \mathbb{R}^2$ of positive radius $r > 0$, i.e. $U = V \bigcap X$.
    * But any such ball contains points of G: namely take n large enough such that $\frac{1}{n\pi} < r$.
    * So $U \bigcap L \bigcap G \neq \emptyset$, contradicting $L \bigcap G = \emptyset$.

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- Using the fact that $[0, \infty) \subset \mathbb{R}$ is Hausdorff, any retract must be closed, so any closed interval $[\varepsilon, N]$ for $0 \leq \varepsilon \leq N \leq \infty$.
  - Note that $\varepsilon = N$ yields all one point sets $\{x_0\}$ for $x_0 \geq 0$.
- No finite discrete sets occur, since the retract of a connected set is connected.
- ?

7 14

- Take two connected sets X, Y; then there exists $p \in X \bigcap Y$.
- Write $X \bigcup Y = A \coprod B$ with both $A, B \subset A \coprod B$ open.
- Since $p \in X \bigcup Y = A \coprod B$, WLOG $p \in A$. We will show B must be empty.
- Claim: $A \bigcap X$ is clopen in X.
  - $A \bigcap X$ is open in X: ?
  - $A \bigcap X$ is closed in X: ?
- The only clopen sets of a connected set are empty or the entire thing, and since $p \in A$, we must have $A \bigcap X = X$.
- By the same argument, $A \bigcap Y = Y$.
- So $A \bigcap \left(X \bigcup Y \right) = \left(A \bigcap X \right) \bigcup \left(A \bigcap Y \right) = X \bigcup Y$
- Since $A \subset X \bigcup Y$, $A \bigcap \left(X \bigcup Y \right) = A$
- Thus $A = X \bigcup Y$, forcing $B = \emptyset$.

8 16

Topic: closure and connectedness in the subspace topology. See Munkres p.148

- $S \subset X$ is **not** connected if S with the subspace topology is not connected.
  - I.e. there exist $A, B \subset S$ such that
    - $A, B \neq \emptyset$,
    - $A \bigcap B = \emptyset$,
    - $A \coprod B = S$.
- Or equivalently, there exists a nontrivial $A \subset S$ that is clopen in S.

Show stronger statement: this is an iff.

$\implies$:

- Suppose S is not connected; we then have sets $A \bigcup B = S$ from above and it suffices to show $\mathrm{cl}_Y(A) \bigcap B = A \bigcap \mathrm{cl}_X(B) = \emptyset$.
- A is open by assumption and $Y \setminus A = B$ is closed in Y, so A is clopen.
- Write $\mathrm{cl}_Y(A) \coloneqq \mathrm{cl}_X(A) \bigcap Y$.
- Since A is closed in Y, $A = \mathrm{cl}_Y(A)$ by definition, so $A = \mathrm{cl}_Y(A) = \mathrm{cl}_X(A) \bigcap Y$.
- Since $A \bigcap B = \emptyset$, we then have $\mathrm{cl}_Y(A) \bigcap B = \emptyset$.
- The same argument applies to B, so $\mathrm{cl}_Y(B) \bigcap A = \emptyset$.

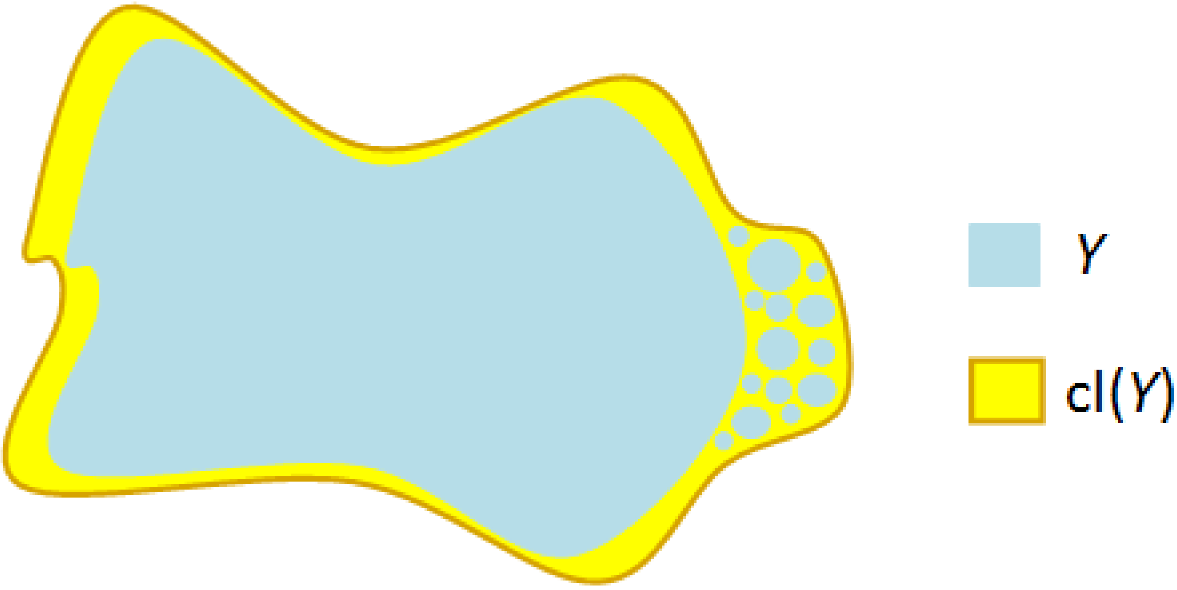
$\impliedby$:

- Suppose displayed condition holds; given such A, B we will show they are clopen in Y.
- Since $\mathrm{cl}_Y(A) \bigcap B = \emptyset$, (claim) we have $\mathrm{cl}_Y(A) = A$ and thus A is closed in Y.
  - Why?

$$
\begin{aligned}
\mathrm{cl}_Y(A) &\coloneqq \mathrm{cl}_X(A) \bigcap Y \\
&= \mathrm{cl}_X(A) \bigcap \left(A \coprod B\right) \\
&= \left(\mathrm{cl}_X(A) \bigcap A\right) \coprod \left(\mathrm{cl}_X(A) \bigcap B\right) \\
&= A \coprod \left(\mathrm{cl}_X(A) \bigcap B\right) \quad \text{since } A \subset \mathrm{cl}_Y(A) \\
&= A \coprod \left(\mathrm{cl}_Y(A) \bigcap B\right) \quad \text{since } B \subset Y \\
&= A \coprod \emptyset \quad \text{using the assumption} \\
&= A.
\end{aligned}
$$

- But $A = Y \setminus B$ where B is closed, so A is open and thus a nontrivial clopen subset.

$\blacksquare$



8.1 18

- Define a new function

$$\begin{aligned} g : X &\longrightarrow \mathbb{R} \\ x &\mapsto d_X(x, f(x)). \end{aligned}$$

- Attempt to minimize. Claim: g is a continuous function.
- Given claim, a continuous function on a compact space attains its infimum, so set

$$m := \inf_{x \in X} g(x)$$

  and produce $x_0 \in X$ such that $g(x) = m$.
- Then

$$m > 0 \iff d(x_0, f(x_0)) > 0 \iff x_0 \neq f(x_0).$$

- Now apply f and use the assumption that f is a contraction to contradict minimality of m:

$$\begin{aligned} d(f(f(x_0)), f(x_0)) &\leq C \cdot d(f(x_0), x_0) \\ &< d(f(x_0), x_0) \quad \text{since } C < 1 \\ &\leq m \end{aligned}$$

- Proof that g is continuous: use the definition of g, the triangle inequality, and that f is a contraction:

$$\begin{aligned} & d(x,f(x)) \le d(x,y) + d(y,f(y)) + d(f(x),f(y)) \\ \implies & d(x,f(x)) - d(y,f(y)) \le d(x,y) + d(f(x),f(y)) \\ \implies & g(x) - g(y) \le d(x,y) + C \cdot d(x,y) = (C+1) \cdot d(x,y) \end{aligned}$$

- This shows that g is Lipschitz continuous with constant $C + 1$ (implies uniformly continuous, but not used).

8.2 20

See definitions in intro.