

# Real Analysis Qualifying Exam Solutions

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## 1 Spring 2020

### 1.1 1

Concepts used:

- DCT
- Weierstrass Approximation Theorem

**Solution:**

- Suppose  $p$  is a polynomial, then

$$\begin{aligned}
 \lim_{k \rightarrow \infty} \int_0^1 k x^{k-1} p(x) dx &= \lim_{k \rightarrow \infty} \int_0^1 \left( \frac{\partial}{\partial x} x^k \right) p(x) dx \\
 &= \lim_{k \rightarrow \infty} \left[ x^k p(x) \Big|_0^1 - \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx \right] \quad \text{integrating by parts} \\
 &= p(1) - \lim_{k \rightarrow \infty} \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx,
 \end{aligned}$$

- Thus it suffices to show that

$$\lim_{k \rightarrow \infty} \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx = 0.$$

- Integrating by parts a second time yields

$$\begin{aligned}
\lim_{k \rightarrow \infty} \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx &= \lim_{k \rightarrow \infty} \frac{x^{k+1}}{k+1} p'(x) \Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1} \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx \\
&= - \lim_{k \rightarrow \infty} \int_0^1 \frac{x^{k+1}}{k+1} \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx \\
&= - \int_0^1 \lim_{k \rightarrow \infty} \frac{x^{k+1}}{k+1} \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx \quad \text{by DCT} \\
&= - \int_0^1 0 \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx \\
&= 0.
\end{aligned}$$

- The DCT can be applied here because  $f''$  is continuous and  $[0, 1]$  is compact, so  $f''$  is bounded on  $[0, 1]$  by a constant  $M$  and

$$\int_0^1 |x^k f''(x)| \leq \int_0^1 1 \cdot M = M < \infty.$$

- Now use the Weierstrass approximation theorem:
  - If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then for every  $\varepsilon > 0$  there exists a polynomial  $p_\varepsilon(x)$  such that  $\|f - p_\varepsilon\|_\infty < \varepsilon$ .
- Thus

$$\begin{aligned}
\left| \int_0^1 kx^{k-1} p_\varepsilon(x) dx - \int_0^1 kx^{k-1} f(x) dx \right| &= \left| \int_0^1 kx^{k-1} (p_\varepsilon(x) - f(x)) dx \right| \\
&\leq \left| \int_0^1 kx^{k-1} \|p_\varepsilon - f\|_\infty dx \right| \\
&= \|p_\varepsilon - f\|_\infty \cdot \left| \int_0^1 kx^{k-1} dx \right| \\
&= \|p_\varepsilon - f\|_\infty \cdot x^k \Big|_0^1 \\
&= \|p_\varepsilon - f\|_\infty \xrightarrow{\varepsilon \rightarrow 0} 0
\end{aligned}$$

and the integrals are equal.

- By the first argument,

$$\int_0^1 kx^{k-1} p_\varepsilon(x) dx = p_\varepsilon(1) \text{ for each } \varepsilon$$

- Since uniform convergence implies pointwise convergence,  $p_\varepsilon(1) \xrightarrow{\varepsilon \rightarrow 0} f(1)$ .

## 1.2 2

Concepts used:

- Definition of outer measure:  $m_*(E) = \inf_{\{Q_j\} \Rightarrow E} \sum |Q_j|$  where  $\{Q_j\}$  is a countable collection of closed cubes.
- Break  $\mathbb{R}$  into  $\prod_{n \in \mathbb{Z}} [n, n+1)$ , each with finite measure.
- Theorem:  $m_*(Q) = |Q|$  for  $Q$  a closed cube (i.e. the outer measure equals the volume).

*Proof (of Theorem).*

Statement: if  $Q$  is a closed cube, then  $m_*(Q) = |Q|$ , the usual volume.

- $m_*(Q) \leq |Q|$ :
  - Since  $Q \subseteq Q$ ,  $Q \Rightarrow Q$  and  $m_*(Q) \leq |Q|$  since  $m_*$  is an infimum over such coverings.
- $|Q| \leq m_*(Q)$ :
  - Fix  $\varepsilon > 0$ .
  - Let  $\{Q_i\}_{i=1}^\infty \Rightarrow Q$  be arbitrary, it suffices to show that

$$|Q| \leq \left( \sum_{i=1}^\infty |Q_i| \right) + \varepsilon.$$

- Pick open cubes  $S_i$  such that  $Q_i \subseteq S_i$  and  $|Q_i| \leq |S_i| \leq (1 + \varepsilon)|Q_i|$ .
- Then  $\{S_i\} \Rightarrow Q$ , so by compactness of  $Q$  pick a finite subcover with  $N$  elements.
- Note

$$Q \subseteq \bigcup_{i=1}^N S_i \implies |Q| \leq \sum_{i=1}^N |S_i| \leq \sum_{i=1}^N (1 + \varepsilon)|Q_i| \leq (1 + \varepsilon) \sum_{i=1}^\infty |Q_i|.$$

- Taking an infimum over coverings on the RHS preserves the inequality, so

$$|Q| \leq (1 + \varepsilon)m_*(Q)$$

- Take  $\varepsilon \rightarrow 0$  to obtain final inequality. ■

### 1.2.1 a

- If  $m_*(E) = \infty$ , then take  $B = \mathbb{R}^n$  since  $m(\mathbb{R}^n) = \infty$ .
- Suppose  $N := m_*(E) < \infty$ .
- Since  $m_*(E)$  is an infimum, by definition, for every  $\varepsilon > 0$  there exists a covering by closed cubes  $\{Q_i(\varepsilon)\}_{i=1}^\infty \Rightarrow E$  depending on  $\varepsilon$  such that

$$\sum_{i=1}^\infty |Q_i(\varepsilon)| < N + \varepsilon.$$

- For each fixed  $n$ , set  $\varepsilon_n = \frac{1}{n}$  to produce such a covering  $\{Q_i(\varepsilon_n)\}_{i=1}^\infty$  and set  $B_n := \bigcup_{i=1}^\infty Q_i(\varepsilon_n)$ .
- The outer measure of cubes is *equal* to the sum of their volumes, so

$$m_*(B_n) = \sum_{i=1}^\infty |Q_i(\varepsilon_n)| < N + \varepsilon_n = N + \frac{1}{n}.$$

- Now set  $B := \bigcap_{n=1}^{\infty} B_n$ .
  - Since  $E \subseteq B_n$  for every  $n$ ,  $E \subseteq B$
  - Since  $B$  is a countable intersection of countable unions of closed sets,  $B$  is Borel.
  - Since  $B_n \subseteq B$  for every  $n$ , we can apply subadditivity to obtain the inequality

$$E \subseteq B \subseteq B_n \implies N \leq m_*(B) \leq m_*(B_n) < N + \frac{1}{n} \quad \text{for all } n \in \mathbb{Z}^{\geq 1}.$$

- This forces  $m_*(E) = m_*(B)$ .

### 1.2.2 b

Suppose  $m_*(E) < \infty$ .

- By (a), find a Borel set  $B \supseteq E$  such that  $m_*(B) = m_*(E)$
- Note that  $E \subseteq B \implies B \cap E = E$  and  $B \cap E^c = B \setminus E$ .
- By assumption,

$$\begin{aligned} m_*(B) &= m_*(B \cap E) + m_*(B \cap E^c) \\ m_*(E) &= m_*(E) + m_*(B \setminus E) \\ m_*(E) - m_*(E) &= m_*(B \setminus E) \quad \text{since } m_*(E) < \infty \\ \implies m_*(B \setminus E) &= 0. \end{aligned}$$

- So take  $N = B \setminus E$ ; this shows  $m_*(N) = 0$  and  $E = B \setminus (B \setminus E) = B \setminus N$ .

If  $m_*(E) = \infty$ :

- Apply result to  $E_R := E \cap [R, R+1)^n \subset \mathbb{R}^n$  for  $R \in \mathbb{Z}$ , so  $E = \bigsqcup_R E_R$
- Obtain  $B_R, N_R$  such that  $E_R = B_R \setminus N_R$ ,  $m_*(E_R) = m_*(B_R)$ , and  $m_*(N_R) = 0$ .
- Note that
  - $B := \bigcup_R B_R$  is a union of Borel sets and thus still Borel
  - $E = \bigcup_R E_R$
  - $N := B \setminus E$
  - $N' := \bigcup_R N_R$  is a union of null sets and thus still null
- Since  $E_R \subset B_R$  for every  $R$ , we have  $E \subset B$
- We can compute

$$N = B \setminus E = \left( \bigcup_R B_R \right) \setminus \left( \bigcup_R E_R \right) \subseteq \bigcup_R (B_R \setminus E_R) = \bigcup_R N_R := N'$$

where  $m_*(N') = 0$  since  $N'$  is null, and thus subadditivity forces  $m_*(N) = 0$ .

### 1.3 3

Concepts used:

- Limits

- Cauchy Criterion for Integrals:  $\int_a^\infty f(x) dx$  converges iff for every  $\varepsilon > 0$  there exists an  $M_0$  such that  $A, B \geq M_0$  implies  $\left| \int_A^B f \right| < \varepsilon$ , i.e.  $\left| \int_A^B f \right| \xrightarrow{A \rightarrow \infty} 0$ .
- Integrals of  $L^1$  functions have vanishing tails:  $\int_N^\infty |f| \xrightarrow{N \rightarrow \infty} 0$ .
- Mean Value Theorem for Integrals:  $\int_a^b f(t) dt = (b-a)f(c)$  for some  $c \in [a, b]$ .

### 1.3.1 a

Stated integral equality:

- Let  $\varepsilon > 0$
- $C_c(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$  is dense so choose  $\{f_n\} \rightarrow f$  with  $\|f_n - f\|_1 \rightarrow 0$ .
- Since  $\{f_n\}$  are compactly supported, choose  $N_0 \gg 1$  such that  $f_n$  is zero outside of  $B_{N_0}(\mathbf{0})$ .
- Then

$$\begin{aligned}
 N \geq N_0 &\implies \int_{|x|>N} |f| = \int_{|x|>N} |f - f_n + f_n| \\
 &\leq \int_{|x|>N} |f - f_n| + \int_{|x|>N} |f_n| \\
 &= \int_{|x|>N} |f - f_n| \\
 &\leq \int_{|x|>N} \|f - f_n\|_1 \\
 &= \|f_n - f\|_1 \left( \int_{|x|>N} 1 \right) \\
 &\xrightarrow{n \rightarrow \infty} 0 \left( \int_{|x|>N} 1 \right) \\
 &= 0 \\
 &\xrightarrow{N \rightarrow \infty} 0.
 \end{aligned}$$

To see that this doesn't force  $f(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ :

- Take  $f(x)$  to be a train of rectangles of height 1 and area  $1/2^j$  centered on even integers.
- Then

$$\int_{|x|>N} |f| = \sum_{j=N}^{\infty} 1/2^j \xrightarrow{N \rightarrow \infty} 0$$

as the tail of a convergent sum.

- However  $f(x) = 1$  for infinitely many even integers  $x > N$ , so  $f(x) \not\rightarrow 0$  as  $|x| \rightarrow \infty$ .

### 1.3.2 b

#### Solution 1 ("Trick")

- Since  $f$  is decreasing on  $[1, \infty)$ , for any  $t \in [x - n, x]$  we have

$$x - n \leq t \leq x \implies f(x) \leq f(t) \leq f(x - n).$$

- Integrate over  $[x, 2x]$ , using monotonicity of the integral:

$$\begin{aligned} \int_x^{2x} f(x) dt &\leq \int_x^{2x} f(t) dt \leq \int_x^{2x} f(x - n) dt \\ \implies f(x) \int_x^{2x} dt &\leq \int_x^{2x} f(t) dt \leq f(x - n) \int_x^{2x} dt \\ \implies xf(x) &\leq \int_x^{2x} f(t) dt \leq xf(x - n). \end{aligned}$$

- By the Cauchy Criterion for integrals,  $\lim_{x \rightarrow \infty} \int_x^{2x} f(t) dt = 0$ .
- So the LHS term  $xf(x) \xrightarrow{x \rightarrow \infty} 0$ .
- Since  $x > 1$ ,  $|f(x)| \leq |xf(x)|$
- Thus  $f(x) \xrightarrow{x \rightarrow \infty} 0$  as well.

### Solution 2 (Variation on the Trick)

- Use mean value theorem for integrals:

$$\int_x^{2x} f(t) dt = xf(c_x) \quad \text{for some } c_x \in [x, 2x] \text{ depending on } x.$$

- Since  $f$  is decreasing,

$$\begin{aligned} x \leq c_x \leq 2x &\implies f(2x) \leq f(c_x) \leq f(x) \\ \implies 2xf(2x) &\leq 2xf(c_x) \leq 2xf(x) \\ \implies 2xf(2x) &\leq 2x \int_x^{2x} f(t) dt \leq 2xf(x) \end{aligned}$$

- By Cauchy Criterion,  $\int_x^{2x} f \rightarrow 0$ .
- So  $2xf(2x) \rightarrow 0$ , which by a change of variables gives  $uf(u) \rightarrow 0$ .
- Since  $u \geq 1$ ,  $f(u) \leq uf(u)$  so  $f(u) \rightarrow 0$  as well.

### Solution 3 (Contradiction)

Just showing  $f(x) \xrightarrow{x \rightarrow \infty} 0$ :

- Toward a contradiction, suppose not.
- Since  $f$  is decreasing, it can not diverge to  $+\infty$



- If  $f(x) \rightarrow -\infty$ , then  $f \notin L^1(\mathbb{R})$ : choose  $x_0 \gg 1$  so that  $t \geq x_0 \implies f(t) < -1$ , then
- Then  $t \geq x_0 \implies |f(t)| \geq 1$ , so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f(t)| dt \geq \int_{x_0}^\infty 1 = \infty.$$

- Otherwise  $f(x) \rightarrow L \neq 0$ , some finite limit.
- If  $L > 0$ :
  - Fix  $\varepsilon > 0$ , choose  $x_0 \gg 1$  such that  $t \geq x_0 \implies L - \varepsilon \leq f(t) \leq L$
  - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L - \varepsilon) dt = \infty$$

- If  $L < 0$ :
  - Fix  $\varepsilon > 0$ , choose  $x_0 \gg 1$  such that  $t \geq x_0 \implies L \leq f(t) \leq L + \varepsilon$ .
  - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L) dt = \infty$$

Showing  $xf(x) \xrightarrow{x \rightarrow \infty} 0$ .

- Toward a contradiction, suppose not.
- (How to show that  $xf(x) \not\rightarrow +\infty$ ?)
- If  $xf(x) \rightarrow -\infty$ 
  - Choose a sequence  $\Gamma = \{\hat{x}_i\}$  such that  $x_i \rightarrow \infty$  and  $x_i f(x_i) \rightarrow -\infty$ .
  - Choose a subsequence  $\Gamma' = \{x_i\}$  such that  $x_i f(x_i) \leq -1$  for all  $i$  and  $x_i \leq x_{i+1}$ .
  - Choose a further subsequence  $S = \{x_i \in \Gamma' \mid 2x_i < x_{i+1}\}$ .
  - Then since  $f$  is always decreasing, for  $t \geq x_0$ ,  $|f|$  is increasing, and  $|f(x_i)| \leq |f(2x_i)|$ , so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f| \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(t)| dt \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(x_i)| = \sum_{x_i \in S} x_i f(x_i) \rightarrow \infty.$$

- If  $xf(x) \rightarrow L \neq 0$  for  $0 < L < \infty$ :
  - Fix  $\varepsilon > 0$ , choose an infinite sequence  $\{x_i\}$  such that  $L - \varepsilon \leq x_i f(x_i) \leq L$  for all  $i$ .

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)| dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i) dt = \sum_S x_i f(x_i) \geq \sum_S (L - \varepsilon) \rightarrow \infty.$$

- If  $xf(x) \rightarrow L \neq 0$  for  $-\infty < L < 0$ :
  - Fix  $\varepsilon > 0$ , choose an infinite sequence  $\{x_i\}$  such that  $L \leq x_i f(x_i) \leq L + \varepsilon$  for all  $i$ .

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)| dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i) dt = \sum_S x_i f(x_i) \geq \sum_S (L) \rightarrow \infty.$$

**Solution 4 (Akos's Suggestion)** For  $x \geq 1$ ,

$$|xf(x)| = \left| \int_x^{2x} f(x) dt \right| \leq \int_x^{2x} |f(x)| dt \leq \int_x^{2x} |f(t)| dt \leq \int_x^{\infty} |f(t)| dt \xrightarrow{x \rightarrow \infty} 0$$

where we've used

- Since  $f$  is decreasing and  $\lim_{x \rightarrow \infty} f(x) = 0$  from part (a),  $f$  is non-negative.
- Since  $f$  is positive and decreasing, for every  $t \in [a, b]$  we have  $|f(a)| \leq |f(t)|$ .
- By part (a), the last integral goes to zero.

**Solution 5 (Peter's)**

- Toward a contradiction, produce a sequence  $x_i \rightarrow \infty$  with  $x_i f(x_i) \rightarrow \infty$  and  $x_i f(x_i) > \varepsilon > 0$ , then

$$\begin{aligned} \int f(x) dx &\geq \sum_{i=1}^{\infty} \int_{x_i}^{x_{i+1}} f(x) dx \\ &\geq \sum_{i=1}^{\infty} \int_{x_i}^{x_{i+1}} f(x_{i+1}) dx \\ &= \sum_{i=1}^{\infty} f(x_{i+1}) \int_{x_i}^{x_{i+1}} dx \\ &\geq \sum_{i=1}^{\infty} (x_{i+1} - x_i) f(x_{i+1}) \\ &\geq \sum_{i=1}^{\infty} (x_{i+1} - x_i) \frac{\varepsilon}{x_{i+1}} \\ &= \varepsilon \sum_{i=1}^{\infty} \left(1 - \frac{x_{i-1}}{x_i}\right) \rightarrow \infty \end{aligned}$$

which can be ensured by passing to a subsequence where  $\sum \frac{x_{i-1}}{x_i} < \infty$ .

### 1.3.3 c

- No: take  $f(x) = \frac{1}{x \ln x}$
- Then by a  $u$ -substitution,

$$\int_0^x f = \ln(\ln(x)) \xrightarrow{x \rightarrow \infty} \infty$$

is unbounded, so  $f \notin L^1([1, \infty))$ .

- But

$$xf(x) = \frac{1}{\ln(x)} \xrightarrow{x \rightarrow \infty} 0.$$

## 1.4 4

Relevant concepts:

- Tonelli: non-negative and measurable yields measurability of slices and equality of iterated integrals
- Fubini:  $f(x, y) \in L^1$  yields *integrable* slices and equality of iterated integrals
- F/T: apply Tonelli to  $|f|$ ; if finite,  $f \in L^1$  and apply Fubini to  $f$

$$\begin{aligned}
 \|H(x)\|_1 &= \int_{\mathbb{R}} |H(x, y)| \, dx \\
 &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y)g(x-y) \, dy \right| \, dx \\
 &\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)g(x-y)| \, dy \right) \, dx \\
 &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)g(x-y)| \, dx \right) \, dy \quad \text{by Tonelli} \\
 &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)g(t)| \, dt \right) \, dy \quad \text{setting } t = x - y, \, dt = -dx \\
 &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)| \cdot |g(t)| \, dt \right) \, dy \\
 &= \int_{\mathbb{R}} |f(y)| \cdot \left( \int_{\mathbb{R}} |g(t)| \, dt \right) \, dy \\
 &:= \int_{\mathbb{R}} |f(y)| \cdot \|g\|_1 \, dy \\
 &= \|g\|_1 \int_{\mathbb{R}} |f(y)| \, dy \\
 &:= \|g\|_1 \|f\|_1 \\
 &< \infty \quad \text{by assumption} \quad .
 \end{aligned}$$

- $H$  is measurable on  $\mathbb{R}^2$ :
  - If we can show  $\tilde{f}(x, y) := f(y)$  and  $\tilde{g}(x, y) := g(x - y)$  are both measurable on  $\mathbb{R}^2$ , then  $H = \tilde{f} \cdot \tilde{g}$  is a product of measurable functions and thus measurable.
  - $f \in L^1$ , and  $L^1$  functions are measurable by definition.
  - The function  $(x, y) \mapsto g(x - y)$  is measurable on  $\mathbb{R}^2$ :
    - \* Let  $g$  be measurable on  $\mathbb{R}$ , then the cylinder function  $G(x, y) = g(x)$  on  $\mathbb{R}^2$  is always measurable
    - \* Define a linear transformation  $T := [1, -1; 0, 1]$  which sends  $(x, y) \rightarrow (x - y, y)$ , then  $T \in \text{GL}(2, \mathbb{R})$  is linear and thus measurable.
    - \* Then  $(G \circ T)(x, y) = G(x - y, y) = \tilde{g}(x - y)$ , so  $\tilde{g}$  is a composition of measurable functions and thus measurable.
- Apply **Tonelli** to  $|H|$ 
  - $H$  measurable implies  $|H|$  is measurable
  - $|H|$  is non-negative
  - So the iterated integrals are equal in the extended sense
  - The calculation shows the iterated integral is finite, to  $\int |H|$  is finite and  $H$  is thus integrable on  $\mathbb{R}^2$ .

Note: Fubini is not needed, since we're not calculating the actual integral, just showing  $H$  is integrable.

## 1.5 5

Concepts used:

- DCT
- Passing limits through products and quotients

Note that

$$\begin{aligned}\lim_n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} &= \frac{1}{\lim_n \left(1 + \frac{x^2}{n}\right)^1 \left(1 + \frac{x^2}{n}\right)^n} \\ &= \frac{1}{1 \cdot e^{x^2}} \\ &= e^{-x^2}.\end{aligned}$$

If passing the limit through the integral is justified, we will have

$$\begin{aligned}\lim_{n \rightarrow \infty} \int_0^n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \chi_{[0,n]} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \\ &= \int_{\mathbb{R}} \lim_{n \rightarrow \infty} \chi_{[0,n]} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \quad \text{by the DCT} \\ &= \int_{\mathbb{R}} \lim_{n \rightarrow \infty} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \\ &= \int_0^\infty e^{-x^2} dx \\ &= \frac{\sqrt{\pi}}{2}.\end{aligned}$$

Computing the last integral:

$$\begin{aligned}\left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2 &= \left(\int_{\mathbb{R}} e^{-x^2} dx\right) \left(\int_{\mathbb{R}} e^{-y^2} dy\right) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(x+y)^2} dx dy \\ &= \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta \quad u = r^2 \\ &= \frac{1}{2} \int_0^{2\pi} \int_0^\infty e^{-u} du d\theta \\ &= \frac{1}{2} \int_0^{2\pi} 1 d\theta \\ &= \pi,\end{aligned}$$

and now use the fact that the function is even so  $\int_0^\infty f = \frac{1}{2} \int_{\mathbb{R}} f$ .

Justifying the DCT:

- Apply Bernoulli's inequality:

$$1 + \frac{x^{2n+1}}{n} \geq 1 + \frac{x^2}{n} (1 + x^2) \geq 1 + x^2,$$

where the last inequality follows from the fact that  $1 + \frac{x^2}{n} \geq 1$

Flesh out

## 1.6 6

Concepts used:

- For  $e_n(x) := e^{2\pi i n x}$ , the set  $\{e_n\}$  is an orthonormal basis for  $L^2([0, 1])$ .
- For any orthonormal sequence in a Hilbert space, we have Bessel's inequality:

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

- When  $\{e_n\}$  is a basis, the above is an *equality* (Parseval)
- Arguing uniform convergence: since  $\{\hat{f}(n)\} \in \ell^1(\mathbb{Z})$ , we should be able to apply the  $M$  test.

### 1.6.1 a

Claim:  $\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$ .

- Set  $\mathbf{c} = \{c_k \mid k \in \mathbb{Z}\} \in \ell^1(\mathbb{Z})$ .
- It suffices to show that if  $\sum_{k \in \mathbb{Z}} |c_k| < \infty$  then  $\sum_{k \in \mathbb{Z}} |c_k|^2 < \infty$ .
- Let  $S = \{c_k \mid |c_k| \leq 1\}$ , then  $c_k \in S \implies |c_k|^2 \leq |c_k|$
- Claim:  $S^c$  can only contain finitely many elements, all of which are finite.
  - If not, either  $S^c := \{c_j\}_{j=1}^\infty$  is infinite with every  $|c_j| > 1$ , which forces

$$\sum_{c_k \in S^c} |c_k| = \sum_{j=1}^{\infty} |c_j| > \sum_{j=1}^{\infty} 1 = \infty.$$

– If any  $c_j = \infty$ , then  $\sum_{k \in \mathbb{Z}} |c_k| \geq c_j = \infty$ .

- So  $S^c$  is a finite set of finite integers, let  $N = \max \{|c_j|^2 \mid c_j \in S^c\} < \infty$ .

- Rewrite the sum

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} |c_k|^2 &= \sum_{c_k \in S} |c_k|^2 + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \sum_{c_k \in S} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \sum_{k \in \mathbb{Z}} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \quad \text{since the } |c_k| \text{ are all positive} \\
&= \|\mathbf{c}\|_{\ell^1} + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \|\mathbf{c}\|_{\ell^1} + |S^c| \cdot N \\
&< \infty.
\end{aligned}$$

Claim:  $L^2([0, 1]) \subseteq L^1([0, 1])$ .

- It suffices to show that  $\int |f|^2 < \infty \implies \int |f| < \infty$ .
- Define  $S = \{x \in [0, 1] \mid |f(x)| \leq 1\}$ , then  $x \in S^c \implies |f(x)|^2 \geq |f(x)|$ .
- Break up the integral:

$$\begin{aligned}
\int_{\mathbb{R}} |f| &= \int_S |f| + \int_{S^c} |f| \\
&\leq \int_S |f| + \int_{S^c} |f|^2 \\
&\leq \int_S |f| + \|f\|_2 \\
&\leq \sup_{x \in S} \{|f(x)|\} \cdot \mu(S) + \|f\|_2 \\
&= 1 \cdot \mu(S) + \|f\|_2 \quad \text{by definition of } S \\
&\leq 1 \cdot \mu([0, 1]) + \|f\|_2 \quad \text{since } S \subseteq [0, 1] \\
&= 1 + \|f\|_2 \\
&< \infty.
\end{aligned}$$

Note: this proof shows  $L^2(X) \subseteq L^1(X)$  whenever  $\mu(X) < \infty$ .

## 2 Fall 2019

### 2.1 1

Cesaro mean/summation. Break series apart into pieces that can be handled separately.

### 2.2 a

Prove a stronger result:

$$a_k \longrightarrow S \implies S_N := \frac{1}{N} \sum_{k=1}^N a_k \longrightarrow S.$$

Idea: once  $N$  is large enough,  $a_k \approx S$ , and all smaller terms will die off as  $N \rightarrow \infty$ .  
See this MSE answer.

- Use convergence  $a_k \rightarrow S$ : choose  $M$  large enough such that

$$k \geq M + 1 \implies |a_k - S| < \varepsilon.$$

Then

$$\begin{aligned} \left| \left( \frac{1}{N} \sum_{k=1}^N a_k \right) - S \right| &= \frac{1}{N} \left| \left( \sum_{k=1}^N a_k \right) - NS \right| \\ &= \frac{1}{N} \left| \left( \sum_{k=1}^N a_k \right) - \sum_{k=1}^N S \right| \\ &= \frac{1}{N} \left| \sum_{k=1}^N (a_k - S) \right| \\ &\leq \frac{1}{N} \sum_{k=1}^N |a_k - S| \\ &= \frac{1}{N} \sum_{k=1}^M |a_k - S| + \sum_{k=M+1}^N |a_k - S| \\ &\leq \frac{1}{N} \sum_{k=1}^M |a_k - S| + \sum_{k=M+1}^N \frac{\varepsilon}{2} \\ &= \frac{1}{N} \sum_{k=1}^M |a_k - S| + (N - M) \frac{\varepsilon}{2} \\ &\xrightarrow{\varepsilon \rightarrow 0} \frac{1}{N} \sum_{k=1}^M |a_k - S| + 0 \\ &\xrightarrow{N \rightarrow \infty} 0 + 0. \end{aligned}$$

Note:  $M$  is fixed, so the last sum is some constant  $c$ , and  $c/N \rightarrow 0$  as  $N \rightarrow \infty$  for any constant. To be more careful, choose  $M$  first to get  $\varepsilon/2$  for the tail, then choose  $N(M) > M$  for the remaining truncated part of the sum.

## 2.3 b

- Define

$$\Gamma_n := \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

- $\Gamma_1 = \sum_{k=1}^{\infty} \frac{a_k}{k}$  is the original series and each  $\Gamma_n$  is a tail of  $\Gamma_1$ , so by assumption  $\Gamma_n \xrightarrow{n \rightarrow \infty} 0$ .

- Compute

$$\frac{1}{n} \sum_{k=1}^n a_k = \frac{1}{n} (\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \Gamma_{n+1})$$

- This comes from consider the following summation:

$\Gamma_1 :$	$a_1$	$+$	$\frac{a_2}{2}$	$+$	$\frac{a_3}{3}$	$+$	$\cdots$
$\Gamma_2 :$			$\frac{a_2}{2}$	$+$	$\frac{a_3}{3}$	$+$	$\cdots$
$\Gamma_3 :$				$+$	$\frac{a_3}{3}$	$+$	$\cdots$
$\sum_{i=1}^n \Gamma_i :$	$a_1$	$+$	$a_2$	$+$	$a_3$	$+$	$\cdots$
	$a_n$	$+$	$\frac{a_{n+1}}{n+1}$	$+$	$\cdots$		

- Use part (a): since  $\Gamma_n \xrightarrow{n \rightarrow \infty} 0$ , we have  $\frac{1}{n} \sum_{k=1}^n \Gamma_k \xrightarrow{n \rightarrow \infty} 0$ .
- Also a minor check:  $\Gamma_n \rightarrow 0 \implies \frac{1}{n} \Gamma_n \rightarrow 0$ .
- Then

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n a_k &= \frac{1}{n} (\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \Gamma_{n+1}) \\ &= \left( \frac{1}{n} \sum_{k=0}^n \Gamma_k \right) - \left( \frac{1}{n} \Gamma_{n+1} \right) \\ &\xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

■

## 2.4 2

DCT, and bounding in the right place. Don't evaluate the actual integral!

- By induction on the number of limits we can pass through the integral.



- For  $n = 1$  we first pass one derivative into the integral: let  $x_n \rightarrow x$  be any sequence converging to  $x$ , then

$$\begin{aligned}
\frac{\partial}{\partial x} \frac{\sin(x)}{x} &= \frac{\partial}{\partial x} \int_0^1 \cos(tx) dt \\
&= \lim_{x_n \rightarrow x} \frac{1}{x_n - x} \left( \int_0^1 \cos(tx_n) dt - \int_0^1 \cos(tx) dt \right) \\
&= \lim_{x_n \rightarrow x} \left( \int_0^1 \frac{\cos(tx_n) - \cos(tx)}{x_n - x} dt \right) \\
&= \lim_{x_n \rightarrow x} \left( \int_0^1 \left( t \sin(tx) \Big|_{x=\xi_n} \right) dt \right) \quad \text{where } \xi_n \in [x_n, x] \text{ by MVT, } \xi_n \rightarrow x \\
&= \lim_{\xi_n \rightarrow x} \left( \int_0^1 t \sin(t\xi_n) dt \right) \\
&\stackrel{\text{DCT}}{=} \int_0^1 \lim_{\xi_n \rightarrow x} t \sin(t\xi_n) dt \\
&= \int_0^1 t \sin(tx) dt
\end{aligned}$$

- Taking absolute values we obtain an upper bound

$$\begin{aligned}
\left| \frac{\partial}{\partial x} \frac{\sin(x)}{x} \right| &= \left| \int_0^1 t \sin(tx) dt \right| \\
&\leq \int_0^1 |t \sin(tx)| dt \\
&\leq \int_0^1 1 dt = 1,
\end{aligned}$$

since  $t \in [0, 1] \implies |t| < 1$ , and  $|\sin(xt)| \leq 1$  for any  $x$  and  $t$ .

- Note that this bound also justifies the DCT, since the functions  $f_n(t) = t \sin(t\xi_n)$  are uniformly dominated by  $g(t) = 1$  on  $L^1([0, 1])$ .

Note: integrating by parts here yields the actual formula:

$$\begin{aligned}
\int_0^1 t \sin(tx) dt &\stackrel{\text{IBP}}{=} \left( \frac{-t \cos(tx)}{x} \right) \Big|_{t=0}^{t=1} - \int_0^1 \frac{\cos(tx)}{x} dt \\
&= \frac{-\cos(x)}{x} - \frac{\sin(x)}{x^2} \\
&= \frac{x \cos(x) - \sin(x)}{x^2}.
\end{aligned}$$

- For the inductive step, we assume that we can pass  $n - 1$  limits through the integral and show

we can pass the  $n$ th through as well.

$$\begin{aligned}\frac{\partial^n}{\partial x^n} \frac{\sin(x)}{x} &= \frac{\partial^n}{\partial x^n} \int_0^1 \cos(tx) dt \\ &= \frac{\partial}{\partial x} \int_0^1 \frac{\partial^{n-1}}{\partial x^{n-1}} \cos(tx) dt \\ &= \frac{\partial}{\partial x} \int_0^1 t^{n-1} f_{n-1}(x, t) dt\end{aligned}$$

- Note that  $f_n(x, t) = \pm \sin(tx)$  when  $n$  is odd and  $f_n(x, t) = \pm \cos(tx)$  when  $n$  is even, and a constant factor of  $t$  is multiplied when each derivative is taken.

- We continue as in the base case:

$$\begin{aligned}\frac{\partial}{\partial x} \int_0^1 t^{n-1} f_{n-1}(x, t) dt &= \lim_{x_k \rightarrow x} \int_0^1 t^{n-1} \left( \frac{f_{n-1}(x_k, t) - f_{n-1}(x, t)}{x_k - x} \right) dt \\ &=_{\text{IVT}} \lim_{x_k \rightarrow x} \int_0^1 t^{n-1} \frac{\partial f_{n-1}}{\partial x}(\xi_k, t) dt \quad \text{where } \xi_k \in [x_k, x], \xi_k \rightarrow x \\ &=_{\text{DCT}} \int_0^1 \lim_{x_k \rightarrow x} t^{n-1} \frac{\partial f_{n-1}}{\partial x}(\xi_k, t) dt \\ &:= \int_0^1 \lim_{x_k \rightarrow x} t^n f_n(\xi_k, t) dt \\ &:= \int_0^1 t^n f_n(x, t) dt.\end{aligned}$$

- We've used the fact that  $f_0(x) = \cos(tx)$  is smooth as a function of  $x$ , and in particular continuous
- The DCT is justified because the functions  $h_{n,k}(x, t) = t^n f_n(\xi_k, t)$  are again uniformly (in  $k$ ) bounded by 1 since  $t \leq 1 \implies t^n \leq 1$  and each  $f_n$  is a sin or cosine.
- Now take absolute values

$$\begin{aligned}\left| \frac{\partial^n}{\partial x^n} \frac{\sin(x)}{x} \right| &= \left| \int_0^1 -t^n f_n(x, t) dt \right| \\ &\leq \int_0^1 |t^n f_n(x, t)| dt \\ &\leq \int_0^1 |t^n| |f_n(x, t)| dt \\ &\leq \int_0^1 |t^n| \cdot 1 dt \\ &\leq \int_0^1 t^n dt \quad \text{since } t \text{ is positive} \\ &= \frac{1}{n+1} \\ &< \frac{1}{n}.\end{aligned}$$

- We've again used the fact that  $f_n(x, t)$  is of the form  $\pm \cos(tx)$  or  $\pm \sin(tx)$ , both of which are bounded by 1.

■

**2.5 3**

Concepts used: - Borel-Cantelli: for a sequence of sets  $X_n$ ,

$$\begin{aligned}\limsup_n X_n &= \left\{ x \mid x \in X_n \text{ for infinitely many } n \right\} &= \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} X_n \\ \liminf_n X_n &= \left\{ x \mid x \in X_n \text{ for all but finitely many } n \right\} &= \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} X_n.\end{aligned}$$

- Properties of logs and exponentials:

$$\prod_n e^{x_n} = e^{\sum_n x_n} \quad \text{and} \quad \sum_n \log(x_n) = \log\left(\prod_n x_n\right).$$

- Tails of convergent sums vanish.
- Continuity of measure:  $B_n \searrow B$  and  $\mu(B_0) < \infty$  implies  $\lim_n \mu(B_n) = \mu(B)$ , and  $B_n \nearrow B \implies \lim_n \mu(B_n) = \mu(B)$ .

**2.5.1 a**

- The Borel  $\sigma$ -algebra is closed under countable unions/intersections/complements,
- $B = \limsup_n B_n$  is an intersection of unions of measurable sets.

**2.5.2 b**

- Tails of convergent sums go to zero, so  $\sum_{n \geq M} \mu(B_n) \xrightarrow{M \rightarrow \infty} 0$ ,
- $B_M := \bigcap_{m=1}^M \bigcup_{n \geq m} B_n \searrow B$ .

$$\begin{aligned}\mu(B_M) &= \mu\left(\bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} B_n\right) \\ &\leq \mu\left(\bigcup_{n \geq m} B_n\right) \quad \text{for all } m \in \mathbb{N} \text{ by countable subadditivity} \\ &\longrightarrow 0,\end{aligned}$$

- The result follows by continuity of measure.

**2.5.3 c**

- To show  $\mu(B) = 1$ , we'll show  $\mu(B^c) = 0$ .

- Let  $B_k = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^K B_n$ . Then

$$\begin{aligned}
 \mu(B_K^c) &= \mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^K B_n^c\right) \\
 &\leq \sum_{m=1}^{\infty} \mu\left(\bigcap_{n=m}^K B_n^c\right) \quad \text{by subadditivity} \\
 &= \sum_{m=1}^{\infty} \prod_{n=m}^K (1 - \mu(B_n)) \quad \text{by assumption} \\
 &\leq \sum_{m=1}^{\infty} \prod_{n=m}^K e^{-\mu(B_n^c)} \quad \text{by hint} \\
 &= \sum_{m=1}^{\infty} \exp\left(-\sum_{n=m}^K \mu(B_n^c)\right) \\
 &\stackrel{K \rightarrow \infty}{\rightarrow} 0
 \end{aligned}$$

since  $\sum_{n=m}^K \mu(B_n^c) \stackrel{K \rightarrow \infty}{\rightarrow} \infty$  by assumption

- We can apply continuity of measure since  $B_K^c \xrightarrow{K \rightarrow \infty} B^c$ .

Proving the hint: ?

■

## 2.6 4

Concepts used:

- Bessel's Inequality
- Pythagoras
- Surjectivity of the Riesz map
- Parseval's Identity
- Trick – remember to write out finite sum  $S_N$ , and consider  $\|x - S_N\|$ .

### 2.6.1 a

**Claim:**

$$\begin{aligned}
 0 \leq \left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\
 &\implies \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2.
 \end{aligned}$$

*Proof:* Let  $S_N = \sum_{n=1}^N \langle x, u_n \rangle u_n$ . Then

$$\begin{aligned} 0 &\leq \|x - S_N\|^2 \\ &= \langle x - S_N, x - S_N \rangle \\ &= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\ &\xrightarrow{N \rightarrow \infty} \|x\|^2 - \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2. \end{aligned}$$

### 2.6.2 b

1. Fix  $\{a_n\} \in \ell^2$ , then note that  $\sum |a_n|^2 < \infty \implies$  the tails vanish.
2. Define

$$x := \lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \sum_{k=1}^N a_k u_k$$

3.  $\{S_N\}$  Cauchy (by 1) and  $H$  complete  $\implies x \in H$ .
- 4.

$$\langle x, u_n \rangle = \left\langle \sum_k a_k u_k, u_n \right\rangle = \sum_k a_k \langle u_k, u_n \rangle = a_n \quad \forall n \in \mathbb{N}$$

since the  $u_k$  are all orthogonal.

- 5.

$$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$$

by Pythagoras since the  $u_k$  are normal.

Bonus: We didn't use completeness here, so the Fourier series may not actually converge to  $x$ . If  $\{u_n\}$  is **complete** (so  $x = 0 \iff \langle x, u_n \rangle = 0 \forall n$ ) then the Fourier series *does* converge to  $x$  and  $\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 = \|x\|^2$  for all  $x \in H$ .

■

## 2.7 5

Continuity in  $L^1$  (recall that DCT won't work! Notes 19.4, prove it for a dense subset first).  
Lebesgue differentiation in 1-dimensional case. See HW 5.6.

**2.8 a**

Choose  $g \in C_c^0$  such that  $\|f - g\|_1 \rightarrow 0$ .

By translation invariance,  $\|\tau_h f - \tau_h g\|_1 \rightarrow 0$ .

Write

$$\begin{aligned} \|\tau f - f\|_1 &= \|\tau_h f - g + g - \tau_h g + \tau_h g - f\|_1 \\ &\leq \|\tau_h f - \tau_h g\| + \|g - f\| + \|\tau_h g - g\| \\ &\rightarrow \|\tau_h g - g\|, \end{aligned}$$

so it suffices to show that  $\|\tau_h g - g\| \rightarrow 0$  for  $g \in C_c^0$ .

Fix  $\varepsilon > 0$ . Enlarge the support of  $g$  to  $K$  such that

$$|h| \leq 1 \text{ and } x \in K^c \implies |g(x - h) - g(x)| = 0.$$

By uniform continuity of  $g$ , pick  $\delta \leq 1$  small enough such that

$$x \in K, |h| \leq \delta \implies |g(x - h) - g(x)| < \varepsilon,$$

then

$$\int_K |g(x - h) - g(x)| \leq \int_K \varepsilon = \varepsilon \cdot m(K) \rightarrow 0.$$

**2.9 b**

We have

$$\begin{aligned} \int_{\mathbb{R}} |A_h(f)(x)| \, dx &= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y) \, dy \right| \, dx \\ &\leq \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)| \, dy \, dx \\ &=_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)| \, dx \, dy \\ &= \int_{\mathbb{R}} |f(y)| \, dy \\ &= \|f\|_1. \end{aligned}$$

and (rough sketch)

---


$$\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x) - f(x)| \, dx &= \int_{\mathbb{R}} \left| \left( \frac{1}{2h} \int_{B(h,x)} f(y) \, dy \right) - f(x) \right| \, dx \\
&= \int_{\mathbb{R}} \left| \left( \frac{1}{2h} \int_{B(h,x)} f(y) \, dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x) \, dy \right| \, dx \\
&\leq_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{B(h,x)} |f(y-x) - f(x)| \, \mathbf{d}\mathbf{x} \, dy \\
&\leq \frac{1}{2h} \int_{\mathbb{R}} \|\tau_x f - f\|_1 \, dy \\
&\longrightarrow 0 \quad \text{by (a).}
\end{aligned}$$

■

### 3 Spring 2019

#### 3.1 1

##### 3.1.1 a

- Let  $\{f_n\}$  be a Cauchy sequence in  $C(I, \|\cdot\|_{\infty})$ , so  $\lim_n \lim_m \|f_m - f_n\|_{\infty} = 0$ , we will show it converges to some  $f$  in this space.
- For each fixed  $x_0 \in [0, 1]$ , the sequence of real numbers  $\{f_n(x_0)\}$  is Cauchy in  $\mathbb{R}$  since

$$x_0 \in I \implies |f_m(x_0) - f_n(x_0)| \leq \sup_{x \in I} |f_m(x) - f_n(x)| := \|f_m - f_n\|_{\infty} \xrightarrow{m > n \rightarrow \infty} 0,$$

- Since  $\mathbb{R}$  is complete, this sequence converges and we can define  $f(x) := \lim_{k \rightarrow \infty} f_n(x)$ .
- Thus  $f_n \rightarrow f$  pointwise by construction
- Claim:  $\|f - f_n\| \xrightarrow{n \rightarrow \infty} 0$ , so  $f_n$  converges to  $f$  in  $C([0, 1], \|\cdot\|_{\infty})$ .

– Proof:

- \* Fix  $\varepsilon > 0$ ; we will show there exists an  $N$  such that  $n \geq N \implies \|f_n - f\| < \varepsilon$
- \* Fix an  $x_0 \in I$ . Since  $f_n \rightarrow f$  pointwise, choose  $N_1$  large enough so that

$$n \geq N_1 \implies |f_n(x_0) - f(x_0)| < \varepsilon/2.$$

- \* Since  $\|f_n - f_m\|_{\infty} \rightarrow 0$ , choose and  $N_2$  large enough so that

$$n, m \geq N_2 \implies \|f_n - f_m\|_{\infty} < \varepsilon/2.$$

\* Then for  $n, m \geq \max(N_1, N_2)$ , we have

$$\begin{aligned}
 |f_n(x_0) - f(x_0)| &= |f_n(x_0) - f(x_0) + f_m(x_0) - f_m(x_0)| \\
 &= |f_n(x_0) - f_m(x_0) + f_m(x_0) - f(x_0)| \\
 &\leq |f_n(x_0) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \\
 &< |f_n(x_0) - f_m(x_0)| + \frac{\varepsilon}{2} \\
 &\leq \sup_{x \in I} |f_n(x) - f_m(x)| + \frac{\varepsilon}{2} \\
 &< \|f_n - f_m\|_\infty + \frac{\varepsilon}{2} \\
 &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
 \implies |f_n(x_0) - f(x_0)| &< \varepsilon \\
 \implies \sup_{x \in I} |f_n(x_0) - f(x_0)| &\leq \sup_{x \in I} \varepsilon \quad \text{by order limit laws} \\
 \implies \|f_n - f\| &\leq \varepsilon
 \end{aligned}$$

- $f$  is the uniform limit of continuous functions and thus continuous, so  $f \in C([0, 1])$ .

### 3.1.2 b

- It suffices to produce a Cauchy sequence that does not converge to a continuous function.
- Take the following sequence of functions:
  - $f_1$  increases linearly from 0 to 1 on  $[0, 1/2]$  and is 1 on  $[1/2, 1]$
  - $f_2$  is 0 on  $[0, 1/4]$  increases linearly from 0 to 1 on  $[1/4, 1/2]$  and is 1 on  $[1/2, 1]$
  - $f_3$  is 0 on  $[0, 3/8]$  increases linearly from 0 to 1 on  $[3/8, 1/2]$  and is 1 on  $[1/2, 1]$
  - $f_3$  is 0 on  $[0, (1/2 - 3/8)/2]$  increases linearly from 0 to 1 on  $[(1/2 - 3/8)/2, 1/2]$  and is 1 on  $[1/2, 1]$

Idea: take sequence starting points for the triangles:  $0, 0 + \frac{1}{4}, 0 + \frac{1}{4} + \frac{1}{8}, \dots$  which converges to  $1/2$  since  $\sum_{k=1}^{\infty} \frac{1}{2^k} = -\frac{1}{2} + \sum_{k=0}^{\infty} \frac{1}{2^k}$ .





- Then each  $f_n$  is clearly integrable, since its graph is contained in the unit square.
- $\{f_n\}$  is Cauchy: geometrically subtracting areas yields a single triangle whose area tends to 0.
- But  $f_n$  converges to  $\chi_{[\frac{1}{2},1]}$  which is discontinuous.

Todo: show that  $\int_0^1 |f_n(x) - f_m(x)| dx \rightarrow 0$  rigorously, show that no  $g \in L^1([0,1])$  can converge to this indicator function.

## 3.2 2

### 3.2.1 a

See Folland p.26

- Lemma 1:  $\mu(\coprod_{k=1}^{\infty} E_k) = \lim_{N \rightarrow \infty} \sum_{k=1}^N \mu(E_k)$ .
- Suppose  $F_0 \supseteq F_1 \supseteq \dots$ .
- Let  $A_k = F_k \setminus F_{k+1}$ , since the  $F_k$  are nested the  $A_k$  are disjoint
- Set  $A := \coprod_{k=1}^{\infty} A_k$  and  $F := \bigcap_{k=1}^{\infty} F_k$ .
- Note  $X = X \setminus Y \coprod X \cap Y$  for any two sets (just write  $X \setminus Y := X \cap Y^c$ )
- Note that  $A$  contains anything that was removed from  $F_0$  when passing from any  $F_j$  to  $F_{j+1}$ , while  $F$  contains everything that is never removed at any stage, and these are disjoint possibilities.

- Thus  $F_0 = F \coprod A$ , so

$$\begin{aligned}
\mu(F_0) &= \mu(F) + \mu(A) \\
&= \mu(F) + \mu\left(\coprod_{k=1}^{\infty} A_k\right) \\
&= \mu(F) + \lim_{n \rightarrow \infty} \sum_{k=0}^n \mu(A_k) \quad \text{by countable additivity} \\
&= \mu(F) + \lim_{n \rightarrow \infty} \sum_{k=0}^n \mu(F_k) - \mu(F_{k+1}) \\
&= \mu(F) + \lim_{n \rightarrow \infty} (\mu(F_1) - \mu(F_n)) \quad (\text{Telescoping}) \\
&= \mu(F) + \mu(F_1) - \lim_{n \rightarrow \infty} \mu(F_n),
\end{aligned}$$

- Since  $\mu$  is a finite measure,  $\mu(F_1) < \infty$  and can be subtracted, yielding

$$\begin{aligned}
\mu(F_1) &= \mu(F) + \mu(F_1) - \lim_{n \rightarrow \infty} \mu(F_n) \\
\implies \mu(F) &= \lim_{n \rightarrow \infty} \mu(F_n) \\
\implies \mu\left(\bigcap_{k=1}^{\infty} F_k\right) &= \lim_{n \rightarrow \infty} \mu(F_n).
\end{aligned}$$

### 3.2.2 b

- Toward a contradiction, negate the implication: suppose there exists an  $\varepsilon > 0$  such that for all  $\delta$ , we have  $m(E) < \delta$  but  $\mu(E) > \varepsilon$ .
- The sequence  $\left\{\delta_n := \frac{1}{2^n}\right\}_{n \in \mathbb{N}}$  and produce sets  $A_n \in \mathcal{B}$  such  $m(A_n) < \frac{1}{2^n}$  but  $\mu(A_n) > \varepsilon$ .
- Define

$$\begin{aligned}
F_n &:= \bigcup_{j \geq n} A_j \\
C_m &:= \bigcap_{k=1}^m F_k \\
A &:= C_{\infty} := \bigcap_{k=1}^{\infty} F_k.
\end{aligned}$$

- Note that  $F_1 \supseteq F_2 \supseteq \dots$ , since each increase in index unions fewer sets.
- By continuity for the Lebesgue measure,

$$m(A) = m\left(\bigcap_{k=1}^{\infty} F_k\right) = \lim_{k \rightarrow \infty} m(F_k) = \lim_{k \rightarrow \infty} m\left(\bigcup_{j \geq k} A_j\right) \leq \lim_{k \rightarrow \infty} \sum_{j \geq k} m(A_j) = \lim_{k \rightarrow \infty} \sum_{j \geq k} \frac{1}{2^j} = 0,$$

which follows because this is the tail of a convergent sum

- Thus  $m(A) = 0$  and by assumption, this implies  $\mu(A) = 0$ .

- However, by part (a),

$$\mu(A) = \lim_n \mu \left( \bigcup_{k=n}^{\infty} A_k \right) \geq \lim_n \mu(A_n) = \lim_n \varepsilon = \varepsilon > 0.$$

All messed up.

### 3.3 3

Concepts used:

- Definition of  $L^+$ : space of measurable function  $X \rightarrow [0, \infty]$ .
- Fatou: For any sequence of  $L^+$  functions,  $\int \liminf f_n \leq \liminf \int f_n$ .
- Egorov's Theorem: If  $E \subseteq \mathbb{R}^n$  is measurable,  $m(E) > 0$ ,  $f_k : E \rightarrow \mathbb{R}$  a sequence of measurable functions where  $\lim_{n \rightarrow \infty} f_n(x)$  exists and is finite a.e., then  $f_n \rightarrow f$  *almost uniformly*: for every  $\varepsilon > 0$  there exists a closed subset  $F_\varepsilon \subseteq E$  with  $m(E \setminus F_\varepsilon) < \varepsilon$  and  $f_n \rightarrow f$  uniformly on  $F_\varepsilon$ .

$L^2$  bound:

- Since  $f_k \rightarrow f$  almost everywhere,  $\liminf_n f_n(x) = f(x)$  a.e.
- $\|f_n\|_2 < \infty$  implies each  $f_n$  is measurable and thus  $|f_n|^2 \in L^+$ , so we can apply Fatou:

$$\begin{aligned} \|f\|_2^2 &= \int |f(x)|^2 \\ &= \int \liminf_n |f_n(x)|^2 \\ &\leq \liminf_n \int |f_n(x)|^2 \\ &\leq \liminf_n M \\ &= M. \end{aligned}$$

- Thus  $\|f\|_2 \leq \sqrt{M} < \infty$  implying  $f \in L^2$ .

Equality of Integrals:

What is the "right" proof here that uses the first part?

- Take the sequence  $\varepsilon_n = \frac{1}{n}$
- Apply Egorov's theorem: obtain a set  $F_\varepsilon$  such that  $f_n \rightarrow f$  uniformly on  $F_\varepsilon$  and  $m(I \setminus F_\varepsilon) < \varepsilon$ .

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \int_0^1 f_n - f \right| &\leq \lim_{n \rightarrow \infty} \int_0^1 |f_n - f| \\ &= \lim_{n \rightarrow \infty} \left( \int_{F_\varepsilon} |f_n - f| + \int_{I \setminus F_\varepsilon} |f_n - f| \right) \\ &= \int_{F_\varepsilon} \lim_{n \rightarrow \infty} |f_n - f| + \lim_{n \rightarrow \infty} \int_{I \setminus F_\varepsilon} |f_n - f| \quad \text{by uniform convergence} \\ &= 0 + \lim_{n \rightarrow \infty} \int_{I \setminus F_\varepsilon} |f_n - f|, \end{aligned}$$

so it suffices to show  $\int_{I \setminus F_\varepsilon} |f_n - f| \xrightarrow{n \rightarrow \infty} 0$ .

- We can obtain a bound using Holder's inequality with  $p = q = 2$ :

$$\begin{aligned}
\int_{I \setminus F_\varepsilon} |f_n - f| &\leq \left( \int_{I \setminus F_\varepsilon} |f_n - f|^2 \right)^{1/2} \left( \int_{I \setminus F_\varepsilon} 1^2 \right)^{1/2} \\
&= \left( \int_{I \setminus F_\varepsilon} |f_n - f|^2 \right)^{1/2} \mu(F_\varepsilon) \\
&\leq \|f_n - f\|_2 \mu(F_\varepsilon) \\
&\leq (\|f_n\|_2 + \|f\|_2) \mu(F_\varepsilon) \\
&\leq 2M \cdot \mu(F_\varepsilon)
\end{aligned}$$

where  $M$  is now a constant not depending on  $\varepsilon$  or  $n$ .

- Now take a nested sequence of sets  $F_\varepsilon$  with  $\mu(F_\varepsilon) \rightarrow 0$  and applying continuity of measure yields the desired statement.

### 3.4 4

See S&S p.82.

#### 3.4.1 a

$\Rightarrow$  :

- Suppose  $f$  is a measurable function.
- Note that  $\mathcal{A} = \{f(x) - t \geq 0\} \cap \{t \geq 0\}$ .
- Define  $F(x, t) = f(x)$ ,  $G(x, t) = t$ , which are cylinders on measurable functions and thus measurable.
- Define  $H(x, y) = F(x, t) - G(x, t)$ , which are linear combinations of measurable functions and thus measurable.
- Then  $\mathcal{A} = \{H \geq 0\} \cap \{G \geq 0\}$  as a countable intersection of measurable sets, which is again measurable.

$\Leftarrow$  :

- Suppose  $\mathcal{A}$  is a measurable set.
- Then FT on  $\chi_{\mathcal{A}}$  implies that for almost every  $x \in \mathbb{R}^n$ , the  $x$ -slices  $\mathcal{A}_x$  are measurable and

$$\mathcal{A}_x := \{t \in \mathbb{R} \mid (x, t) \in \mathcal{A}\} = [0, f(x)] \implies m(\mathcal{A}_x) = f(x) - 0 = f(x)$$

- But  $x \mapsto m(\mathcal{A}_x)$  is a measurable function, and is exactly the function  $x \mapsto f(x)$ , so  $f$  is measurable.

#### 3.4.2 b

- Note

$$\begin{aligned}
\mathcal{A} &= \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid 0 \leq t \leq f(x)\} \\
\mathcal{A}_t &= \{x \in \mathbb{R}^n \mid t \leq f(x)\}.
\end{aligned}$$

- Then

$$\begin{aligned}
 \int_{\mathbb{R}^n} f(x) \, dx &= \int_{\mathbb{R}^n} \int_0^{f(x)} 1 \, dt \, dx \\
 &= \int_{\mathbb{R}^n} \int_0^\infty \chi_{\mathcal{A}} \, dt \, dx \\
 &\stackrel{F.T.}{=} \int_0^\infty \int_{\mathbb{R}^n} \chi_{\mathcal{A}} \, dx \, dt \\
 &= \int_0^\infty m(\mathcal{A}_t) \, dt,
 \end{aligned}$$

where we just use that  $\int \chi_{\mathcal{A}} = m(\mathcal{A})$

- By F.T., all of these integrals are equal.

Why is FT justified.

### 3.5 5

Concepts used:

- Holders' inequality:  $\|fg\|_1 \leq \|f\|_p \|g\|_q$
- Riesz Representation for  $L^2$ : If  $\Lambda \in (L^2)^\vee$  then there exists a unique  $g \in L^2$  such that  $\Lambda(f) = \int fg$ .
- $\|f\|_{L^\infty(X)} := \inf \{t \geq 0 \mid |f(x)| \leq t \text{ almost everywhere}\}.$
- **Lemma:**  $m(X) < \infty \implies L^p(X) \subset L^2(X).$

*Proof:* Write Holder's inequality as  $\|fg\|_1 \leq \|f\|_a \|g\|_b$  where  $\frac{1}{a} + \frac{1}{b} = 1$ , then

$$\|f\|_p^p = \| |f|^p \|_1 \leq \| |f|^p \|_a \|1\|_b.$$

Now take  $a = \frac{2}{p}$  and this reduces to

$$\begin{aligned}
 \|f\|_p^p &\leq \|f\|_2^p m(X)^{\frac{1}{p}} \\
 \implies \|f\|_p &\leq \|f\|_2 \cdot O(m(X)) < \infty.
 \end{aligned}$$

#### 3.5.1 a

- Note  $X = [0, 1] \implies m(X) = 1.$
- By Holder's inequality with  $p = q = 2$ ,

$$\|f\|_1 = \|f \cdot 1\|_1 \leq \|f\|_2 \cdot \|1\|_2 = \|f\|_2 \cdot m(X)^{\frac{1}{2}} = \|f\|_2,$$

- Thus  $L^2(X) \subseteq L^1(X)$
- Since they share a common dense subset (simple functions)  $L^2$  is dense in  $L^1$

What theorem is this using?

## 3.5.2 b

Let  $\Lambda \in L^1(X)^\vee$  be arbitrary.

**(i): Existence of  $g$  Representing  $\Lambda$ .**

- Let  $f \in L^2 \subseteq L^1$  be arbitrary
- Claim:  $\Lambda \in L^1(X)^\vee \implies \Lambda \in L^2(X)^\vee$ .
  - Suffices to show that  $\|\Gamma\|_{L^2(X)^\vee} := \sup_{\|f\|_2=1} |\Gamma(f)| < \infty$ , since bounded implies continuous.
  - By the lemma,  $\|f\|_1 \leq C\|f\|_2$  for some constant  $C \approx m(X)$ .
  - Note

$$\|\Lambda\|_{L^1(X)^\vee} := \sup_{\|f\|_1=1} |\Lambda(f)|$$

- Define  $\hat{f} = \frac{f}{\|f\|_1}$  so  $\|\hat{f}\|_1 = 1$
- Since  $\|\Lambda\|_{1^\vee}$  is a supremum over *all*  $f \in L^1(X)$  with  $\|f\|_1 = 1$ ,

$$|\Lambda(\hat{f})| \leq \|\Lambda\|_{(L^1(X))^\vee},$$

- Then

$$\begin{aligned} \frac{|\Lambda(f)|}{\|f\|_1} &= |\Lambda(\hat{f})| \leq \|\Lambda\|_{L^1(X)^\vee} \\ \implies |\Lambda(f)| &\leq \|\Lambda\|_{1^\vee} \cdot \|f\|_1 \\ &\leq \|\Lambda\|_{1^\vee} \cdot C\|f\|_2 < \infty \quad \text{by assumption,} \end{aligned}$$

- So  $\Lambda \in (L^2)^\vee$ .
- Now apply Riesz Representation for  $L^2$ : there is a  $g \in L^2$  such that

$$f \in L^2 \implies \Lambda(f) = \langle f, g \rangle := \int_0^1 f(x) \overline{g(x)} dx.$$

**(ii):  $g$  is in  $L^\infty$** 

- It suffices to show  $\|g\|_{L^\infty(X)} < \infty$ .
- Since we're assuming  $\|\Gamma\|_{L^1(X)^\vee} < \infty$ , it suffices to show the stated equality.
- Claim:  $\|\Lambda\|_{L^1(X)^\vee} = \|g\|_{L^\infty(X)}$

- The result follows because  $\Lambda$  was assumed to be in  $L^1(X)^\vee$ , so  $\|\Lambda\|_{L^1(X)^\vee} < \infty$ .

Is this assumed..?  
Or did we show  
it..?

---

–  $\leq$ :

$$\begin{aligned}
\|\Lambda\|_{L^1(X)^\vee} &= \sup_{\|f\|_1=1} |\Lambda(f)| \\
&= \sup_{\|f\|_1=1} \left| \int_X f \bar{g} \right| \quad \text{by (i)} \\
&= \sup_{\|f\|_1=1} \int_X |f \bar{g}| \\
&:= \sup_{\|f\|_1=1} \|fg\|_1 \\
&\leq \sup_{\|f\|_1=1} \|f\|_1 \|g\|_\infty \quad \text{by Holder with } p=1, q=\infty \\
&= \|g\|_\infty,
\end{aligned}$$

–  $\geq$ :

- \* Suppose toward a contradiction that  $\|g\|_\infty > \|\Lambda\|_{L^1(X)^\vee}$ .
- \* Then there exists some  $E \subseteq X$  with  $m(E) > 0$  such that

$$x \in E \implies |g(x)| > \|\Lambda\|_{L^1(X)^\vee}.$$

- \* Define

$$h = \frac{1}{m(E)} \frac{\bar{g}}{|g|} \chi_E.$$

- \* Note  $\|h\|_{L^1(X)} = 1$ .
- \* Then

$$\begin{aligned}
\Lambda(h) &= \int_X hg \\
&:= \int_X \frac{1}{m(E)} \frac{g\bar{g}}{|g|} \chi_E \\
&= \frac{1}{m(E)} \int_E |g| \\
&\geq \frac{1}{m(E)} \|g\|_\infty m(E) \\
&= \|g\|_\infty \\
&> \|\Lambda\|_{L^1(X)^\vee},
\end{aligned}$$

a contradiction since  $\|\Lambda\|_{L^1(X)^\vee}$  is the supremum over all  $h_\alpha$  with  $\|h_\alpha\|_{L^1(X)} = 1$ .

## 4 Fall 2018

### 4.1 1

Concepts used:

- Uniform continuity.

Show a stronger statement:  $f(x) = \frac{1}{x}$  is uniformly continuous on any interval of the form  $(c, \infty)$  where  $c > 0$ .

- Note that

$$|x|, |y| > c > 0 \implies |xy| = |x||y| > c^2 \implies \frac{1}{|xy|} < \frac{1}{c^2}.$$

- Letting  $\varepsilon$  be arbitrary, choose  $\delta < \varepsilon c^2$ .
- Note that  $\delta$  does not depend on  $x, y$ .
- Then

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{x} - \frac{1}{y} \right| \\ &= \frac{|x - y|}{xy} \\ &\leq \frac{\delta}{xy} \\ &< \frac{\delta}{c^2} \\ &< \varepsilon, \end{aligned}$$

which shows uniform continuity.

To see that  $f$  is not uniformly continuous when  $c = 0$ :

Note: negating uniform continuity says  $\exists \varepsilon > 0$  such that  $\forall \delta(\varepsilon)$  there exist  $x, y$  such that  $|x - y| < \delta$  and  $|f(x) - f(y)| > \varepsilon$ .

- Let  $\varepsilon < 1$ .
- Let  $x_n = \frac{1}{n}$  for  $n \geq 1$ .
- Choose  $n$  large enough such that  $|x_n - x_{n+1}| = \frac{1}{n} - \frac{1}{n+1} < \delta$ .
  - Why this can be done: by the archimedean property of  $\mathbb{R}$ , choose  $n$  such that  $\frac{1}{n} < \varepsilon$ .
  - Then

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \leq \frac{1}{n} < \varepsilon \quad \text{since } n+1 > 1.$$

- Note  $f(x_n) = n$  and thus

$$|f(x_n) - f(x_{n+1})| = n - (n+1) = 1 > \varepsilon.$$

■

## 4.2 2

Concepts used:



- Definition of measurability: there exists an open  $O \supset E$  such that  $m_*(O \setminus E) < \varepsilon$  for all  $\varepsilon > 0$ .
- Theorem:  $E$  is Lebesgue measurable iff there exists a closed set  $F \subseteq E$  such that  $m_*(E \setminus F) < \varepsilon$  for all  $\varepsilon > 0$ .
- Every  $F_\sigma, G_\delta$  is Borel.
- Claim:  $E$  is measurable  $\iff$  for every  $\varepsilon$  there exist  $F_\varepsilon \subset E \subset G_\varepsilon$  with  $F_\varepsilon$  closed and  $G_\varepsilon$  open and  $m(G_\varepsilon \setminus E) < \varepsilon$  and  $m(E \setminus F_\varepsilon) < \varepsilon$ .
  - Proof: existence of  $G_\varepsilon$  is the definition of measurability.
  - Existence of  $F_\varepsilon$ : ?
- Claim:  $E$  is measurable  $\implies$  there exists an open  $O \supseteq E$  such that  $m(O \setminus E) = 0$ .
  - Since  $E$  is measurable, for each  $n \in \mathbb{N}$  choose  $G_n \supseteq E$  such that  $m_*(G_n \setminus E) < \frac{1}{n}$ .
  - Set  $O_N := \bigcap_{n=1}^N G_n$  and  $O := \bigcap_{n=1}^\infty G_n$ .
  - Suppose  $E$  is bounded.
    - \* Note  $O_N \searrow O$  and  $m_*(O_1) < \infty$  if  $E$  is bounded, since in this case

$$m_*(G_n \setminus E) = m_*(G_1) - m_*(E) < 1 \iff m_*(G_1) < m_*(E) + \frac{1}{n} < \infty.$$

- \* Note  $O_N \setminus E \searrow O \setminus E$  since  $O_N \setminus E := O_N \cap E^c \supseteq O_{N+1} \cap E^c$  for all  $N$ , and again  $m_*(O_1 \setminus E) < \infty$ .
- \* So it's valid to apply continuity of measure from above:

$$\begin{aligned} m_*(O \setminus E) &= \lim_{N \rightarrow \infty} m_*(O_N \setminus E) \\ &\leq \lim_{N \rightarrow \infty} m_*(G_N \setminus E) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} = 0, \end{aligned}$$

where the inequality uses subadditivity on  $\bigcap_{n=1}^N G_n \subseteq G_N$

- Suppose  $E$  is unbounded.
  - \* Write  $E^k = E \cap [k, k+1]^d \subset \mathbb{R}^d$  as the intersection of  $E$  with an annulus, and note that  $E = \coprod_{k \in \mathbb{N}} E_k$ .
  - \* Each  $E_k$  is bounded, so apply the previous case to obtain  $O_k \supseteq E_k$  with  $m(O_k \setminus E_k) = 0$ .
  - \* So write  $O_k = E_k \coprod N_k$  where  $N_k := O_k \setminus E_k$  is a null set.
  - \* Define  $O = \bigcup_{k \in \mathbb{N}} O_k$ , note that  $E \subseteq O$ .
  - \* Now note

$$\begin{aligned} O \setminus E &= \left( \coprod_k O_k \right) \setminus \left( \coprod_k E_k \right) \\ &\subseteq \coprod_k (O_k \setminus E_k) \\ \implies m_*(O \setminus E) &\leq m_*\left( \coprod_k (O_k \setminus E_k) \right) = 0, \end{aligned}$$

since any countable union of null sets is again null.

- So  $O \supseteq E$  with  $m(O \setminus E) = 0$ .
- Theorem: since  $E$  is measurable,  $E^c$  is measurable

- Proof: It suffices to write  $E^c$  as the union of two measurable sets,  $E^c = S \cup (E^c - S)$ , where  $S$  is to be determined.
- We'll produce an  $S$  such that  $m_*(E^c - S) = 0$  and use the fact that any subset of a null set is measurable.
- Since  $E$  is measurable, for every  $\varepsilon > 0$  there exists an open  $\mathcal{O}_\varepsilon \supseteq E$  such that  $m_*(\mathcal{O}_\varepsilon \setminus E) < \varepsilon$ .
- Take the sequence  $\left\{ \varepsilon_n := \frac{1}{n} \right\}$  to produce a sequence of sets  $\mathcal{O}_n$ .
- Note that each  $\mathcal{O}_n^c$  is closed and

$$\mathcal{O}_n \supseteq E \iff \mathcal{O}_n^c \subseteq E^c.$$

- Set  $S := \bigcup_n \mathcal{O}_n^c$ , which is a union of closed sets, thus an  $F_\sigma$  set, thus Borel, thus measurable.
- Note that  $S \subseteq E^c$  since each  $\mathcal{O}_n \subseteq E^c$ .
- Note that

$$\begin{aligned} E^c \setminus S &:= E^c \setminus \left( \bigcup_{n=1}^{\infty} \mathcal{O}_n^c \right) \\ &:= E^c \cap \left( \bigcup_{n=1}^{\infty} \mathcal{O}_n^c \right)^c \quad \text{definition of set minus} \\ &= E^c \cap \left( \bigcap_{n=1}^{\infty} \mathcal{O}_n \right)^c \quad \text{De Morgan's law} \\ &= E^c \cup \left( \bigcap_{n=1}^{\infty} \mathcal{O}_n \right) \\ &:= \left( \bigcap_{n=1}^{\infty} \mathcal{O}_n \right) \setminus E \\ &\subseteq \mathcal{O}_N \setminus E \quad \text{for every } N \in \mathbb{N}. \end{aligned}$$

- Then by subadditivity,

$$m_*(E^c \setminus S) \leq m_*(\mathcal{O}_N \setminus E) \leq \frac{1}{N} \quad \forall N \implies m_*(E^c \setminus S) = 0.$$

- Thus  $E^c \setminus S$  is measurable.

#### 4.2.1 Indirect Proof

- Since  $E$  is measurable,  $E^c$  is measurable.
- Since  $E^c$  is measurable exists an open  $\mathcal{O} \supseteq E^c$  such that  $m(\mathcal{O} \setminus E^c) = 0$ .
- Set  $B := \mathcal{O}^c$ , then  $\mathcal{O} \supseteq E^c \iff \mathcal{O}^c \subseteq E \iff B \subseteq E$ .
- Computing measures yields

$$E \setminus B := E \setminus \mathcal{O}^c := E \cap (\mathcal{O}^c)^c = E \cap \mathcal{O} = \mathcal{O} \cap (E^c)^c := \mathcal{O} \setminus E^c,$$

thus  $m(E \setminus B) = m(\mathcal{O} \setminus E^c) = 0$ .

- Since  $\mathcal{O}$  is open,  $B$  is closed and thus Borel.

## 4.2.2 Direct Proof

?

Try to construct the set.

## 4.3 3

Concepts used:

- Mean Value Theorem
- DCT

$$\begin{aligned}\frac{\partial}{\partial t} F(t) &= \frac{\partial}{\partial t} \int_{\mathbb{R}} f(x) \cos(xt) \, dx \\ &\stackrel{DCT}{=} \int_{\mathbb{R}} f(x) \frac{\partial}{\partial t} \cos(xt) \, dx \\ &= \int_{\mathbb{R}} x f(x) \cos(xt) \, dx,\end{aligned}$$

so it only remains to justify the DCT.

- Fix  $t$ , then let  $t_n \rightarrow t$  be arbitrary.
- Define

$$h_n(x, t) = f(x) \left( \frac{\cos(tx) - \cos(t_n x)}{t_n - t} \right) \xrightarrow{n \rightarrow \infty} \frac{\partial}{\partial t} (f(x) \cos(xt))$$

since  $\cos(tx)$  is differentiable in  $t$  and this is the limit definition of differentiability.

- Note that

$$\begin{aligned}\frac{\partial}{\partial t} \cos(tx) &:= \lim_{t_n \rightarrow t} \frac{\cos(tx) - \cos(t_n x)}{t_n - t} \\ &\stackrel{MVT}{=} \frac{\partial}{\partial t} \cos(tx) \Big|_{t=\xi_n} \quad \text{for some } \xi_n \in [t, t_n] \text{ or } [t_n, t] \\ &= x \sin(\xi_n x)\end{aligned}$$

where  $\xi_n \xrightarrow{n \rightarrow \infty} t$  since wlog  $t_n \leq \xi_n \leq t$  and  $t_n \nearrow t$ .

- We then have

$$|h_n(x)| = |f(x)x \sin(\xi_n x)| \leq |xf(x)| \quad \text{since } |\sin(\xi_n x)| \leq 1$$

for every  $x$  and every  $n$ .

- Since  $xf(x) \in L^1(\mathbb{R})$  by assumption, the DCT applies.

## 4.4 4

Case of characteristic function

- First suppose  $f(x) = \chi_{[0,1]}(x)$ .

- Note that  $\sin(nx)$  has a period of  $2\pi/n$ , and thus  $\left\lfloor \frac{n}{2\pi} \right\rfloor$  full periods in  $[0, 1]$ .
- Taking the absolute value yields a new function with half the period, so a period of  $\pi/n$  and  $\lfloor \pi/n \rfloor$  full periods in  $[0, 1]$ .
- We can compute the integral over one full period (which is independent of *which* period is chosen), and since  $\sin(x)$  is positive and agrees with  $|\sin(nx)|$  on the first period, we have

$$\begin{aligned} \int_{\text{One Period}} |\sin(nx)| dx &= \int_0^{\pi/n} \sin(nx) dx \\ &= \frac{1}{n} \int_0^\pi \sin(u) du \quad u = nx \\ &= \frac{1}{n} - \cos(u) \Big|_0^\pi \\ &= \frac{2}{n}. \end{aligned}$$

- Then break the integral up into integrals over periods  $P_1, P_2, \dots, P_N$  where  $N := \lfloor n/\pi \rfloor$ :

$$\begin{aligned} \int_0^1 |\sin(nx)| dx &= \left( \sum_{j=1}^N \int_{P_j} |\sin(nx)| dx \right) + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)| dx \\ &= \left( \sum_{j=1}^N \frac{2}{n} \right) + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)| dx \\ &= N \left( \frac{2}{n} \right) + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)| dx \\ &:= \left\lfloor \frac{n}{\pi} \right\rfloor \frac{2}{n} + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)| dx \\ &= \frac{2}{\pi} + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)| dx \\ &:= \frac{2}{\pi} + R(n) \end{aligned}$$

so it suffices to show that  $R(n) \xrightarrow{n \rightarrow \infty} 0$ .

- Showing this: ???????????

General case

## 4.5 5

Concepts used:

- Claim: If  $E \subseteq \mathbb{R}^a \times \mathbb{R}^b$  is a measurable set, then for almost every  $y \in \mathbb{R}^b$ , the slice  $E^y$  is measurable and

$$m(E) = \int_{\mathbb{R}^b} m(E^y) dy.$$

- Set  $g = \chi_E$ , which is non-negative and measurable, so apply Tonelli.

Need to justify removing floor function and cancellation.

No clue how to show this.

Not sure. Approximate  $f$  by simple functions...?

- Conclude that  $g^y = \chi_{E^y}$  is measurable, the function  $y \mapsto \int g^y(x) dx$  is measurable, and  $\int \int g^y(x) dx dy = \int g$ .
- But  $\int g = m(E)$  and  $\int \int g^y(x) dx dy = \int m(E^y) dy$ .

**Solution**

Note:  $f$  is a function  $\mathbb{R} \rightarrow \mathbb{R}$  in the original problem, but here I've assumed  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ .

- Since  $f \geq 0$ , set

$$E := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) > t\} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid 0 \leq t < f(x)\}.$$

- Claim: since  $f$  is measurable,  $E$  is measurable and thus  $m(E)$  makes sense.
  - Since  $f$  is measurable,  $F(x, t) := t - f(x)$  is measurable on  $\mathbb{R}^n \times \mathbb{R}$ .
  - Then write  $E = \{F < 0\} \cap \{t \geq 0\}$  as an intersection of measurable sets.
- We have slices

$$E^t := \{x \in \mathbb{R}^n \mid (x, t) \in E\} = \{x \in \mathbb{R}^n \mid 0 \leq t < f(x)\}$$

$$E^x := \{t \in \mathbb{R} \mid (x, t) \in E\} = \{t \in \mathbb{R} \mid 0 \leq t \leq f(x)\} = [0, f(x)].$$

- $E_t$  is precisely the set that appears in the original RHS integrand.
- $m(E^x) = f(x)$ .
- Claim:  $\chi_E$  satisfies the conditions of Tonelli, and thus  $m(E) = \int \chi_E$  is equal to any iterated integral.
  - Non-negative: clear since  $0 \leq \chi_E \leq 1$
  - Measurable: characteristic functions of measurable sets are measurable.
- Conclude:
  1. For almost every  $x$ ,  $E^x$  is a measurable set,  $x \mapsto m(E^x)$  is a measurable function, and  $m(E) = \int_{\mathbb{R}^n} m(E^x) dx$
  2. For almost every  $t$ ,  $E^t$  is a measurable set,  $t \mapsto m(E^t)$  is a measurable function, and  $m(E) = \int_{\mathbb{R}} m(E^t) dt$
- On one hand,

$$\begin{aligned}
 m(E) &= \int_{\mathbb{R}^{n+1}} \chi_E(x, t) \\
 &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_E(x, t) dt dx \quad \text{by Tonelli} \\
 &= \int_{\mathbb{R}^n} m(E^x) dx \quad \text{first conclusion} \\
 &= \int_{\mathbb{R}^n} f(x) dx.
 \end{aligned}$$

- On the other hand,

$$\begin{aligned}
 m(E) &= \int_{\mathbb{R}^{n+1}} \chi_E(x, t) \\
 &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_E(x, t) dx dt \quad \text{by Tonelli} \\
 &= \int_{\mathbb{R}} m(E^t) dt \quad \text{second conclusion.}
 \end{aligned}$$

- Thus

$$\int_{\mathbb{R}^n} f dx = m(E) = \int_{\mathbb{R}} m(E^t) dt = \int_{\mathbb{R}} m(\{x \mid f(x) > t\}) dt.$$

■

## 4.6 6

- Note that  $x^{\frac{1}{n}} \xrightarrow{n \rightarrow \infty} 1$  for any  $0 < x < \infty$ .
- Thus the integrand converges to  $\frac{1}{e^x}$ , which is integrable on  $(0, \infty)$  and integrates to 1.
- Break the integrand up:

$$\int_0^\infty \frac{1}{(1 + \frac{x}{n})^n x^{\frac{1}{n}}} dx = \int_0^1 \frac{1}{(1 + \frac{x}{n})^n x^{\frac{1}{n}}} dx = \int_1^\infty \frac{1}{(1 + \frac{x}{n})^n x^{\frac{1}{n}}} dx.$$

## 5 Spring 2018

### 5.1 1

- We'll show that  $m(E) \cap [n, n+1] = 0$  for all  $n \in \mathbb{Z}$ ; then the result follows from

$$m(E) = m\left(\bigcup_{n \in \mathbb{Z}} E \cap [n, n+1]\right) \leq \sum_{n=1}^\infty m(E \cap [n, n+1]) = 0.$$

- By translation invariance of measure, it suffices to show  $m(E \cap [0, 1]) = 0$ .
- Define

$$E_j := \left\{ x \in [0, 1] \mid \exists p \in \mathbb{Z}^{\geq 0} \text{ s.t. } \left| x - \frac{p}{j} \right| < \frac{1}{j^3} \right\}.$$

- Note that  $E_j \subseteq \coprod_{p \in \mathbb{Z}^{\geq 0}} B_{j^{-3}}\left(\frac{p}{j}\right)$ , i.e. a union over integers  $p$  of intervals of radius  $1/j^3$  around the points  $p/j$ . Since  $1/j^3 < 1/j$ , this union is in fact disjoint.

- Intersecting with  $[0, 1]$ , we can write  $E_j$  as a union of intervals:

$$E_j = \left(0, \frac{1}{j^3}\right) \amalg B_{\frac{1}{j^3}}\left(\frac{1}{j}\right) \amalg B_{\frac{1}{j^3}}\left(\frac{2}{j}\right) \amalg \cdots \amalg B_{\frac{1}{j^3}}\left(\frac{j-1}{j}\right) \amalg \left(1 - \frac{1}{j^3}, 1\right),$$

from which we can conclude that  $E_j$  is Borel and thus Lebesgue measurable, and that for each  $j$ , there are exactly  $j + 1$  possible choices for a numerator (corresponding to the  $j + 1$  sets appearing above.)

The first and last intervals are length  $\frac{1}{j^3}$  and the remaining  $(j + 1) - 2 = j - 1$  intervals are length  $\frac{2}{j^3}$ , so we find that

$$m(E_j) = 2\left(\frac{1}{j^3}\right) + (j - 1)\left(\frac{2}{j^3}\right) = \frac{2}{j^2}$$

We can then note that

$$\sum_{j \in \mathbb{N}} m(E_j) \leq 2 \sum_{j \in \mathbb{N}} \frac{1}{j^2} < \infty,$$

which converges by the  $p$ -test for sums.

Since  $\{E_j\}$  is a countable collection of measurable sets such that  $\sum m(E_j) < \infty$ , Borel-Cantelli applies and  $m(\limsup_j E_j) = 0$ , where we can just note that  $\limsup_j E_j = E \cap [0, 1]$ . ■

## 5.2 2

### 5.2.1 a

Since  $x < 1 \implies x^n \longrightarrow 0$  and  $x > 1 \implies x^n \longrightarrow \infty$ , we have

$$f_n(x) = \frac{x}{1 + x^n} \xrightarrow{n \rightarrow \infty} f(x) = \begin{cases} 0, & x = 0 \\ x, & x < 1 \\ \frac{1}{2}, & x = 1 \\ 0, & x > 1 \end{cases}.$$

If  $f_n \longrightarrow f$  uniformly on  $[0, \infty)$ , it would converge uniformly on every subset.

But  $f_n(x)$  is clearly continuous on  $(0, \infty)$ , and if the convergence was uniform then  $f$  would be continuous. However  $f$  has a clear discontinuity at  $x = 1$ .

### 5.2.2 b

If the DCT applies, we can interchange the limit and integral, and the value would be the area under the graph of  $f$  which is  $\int_0^1 x \, dx = \frac{1}{2}$ .

To justify the DCT, write

$$\int_0^\infty f_n(x) \, dx = \int_0^1 f_n(x) \, dx + \int_1^\infty f_n(x) \, dx.$$

Then

$$x \in [0, 1] \implies \frac{x}{1+x^n} < \frac{1}{1+x^n} < 1$$

and  $\int_0^1 1 \, dx = 1 < \infty$ .

On the other hand,

$$x \in (1, \infty) \implies \frac{x}{1+x^n} \approx O\left(\frac{1}{x^{n-1}}\right),$$

and so for  $n > 2$  the integral will converge by the  $p$ -test.

### 5.3 3

Since  $|f(x)| \leq \|f\|_\infty$  almost everywhere, we have

$$\|f\|_p^p = \int_X |f(x)|^p \, dx \leq \int_X \|f\|_\infty^p \, dx = \|f\|_\infty^p \cdot m(X) = \|f\|_\infty^p,$$

so  $\|f\|_p \leq \|f\|_\infty$  for all  $p$  and taking  $\lim_{p \rightarrow \infty}$  preserves this inequality.

Conversely, let  $\varepsilon > 0$ . Define

$$S_\varepsilon := \left\{ x \in \mathbb{R} \mid |f(x)| \geq \|f\|_\infty - \varepsilon \right\}.$$

Then

$$\begin{aligned} \|f\|_p^p &= \int_X |f(x)|^p \, dx \\ &\geq \int_{S_\varepsilon} |f(x)|^p \, dx \\ &\geq \int_{S_\varepsilon} (\|f\|_\infty - \varepsilon)^p \, dx \\ &= (\|f\|_\infty - \varepsilon)^p \cdot m(S_\varepsilon) \\ \implies \|f\|_p &\geq (\|f\|_\infty - \varepsilon) \cdot m(S_\varepsilon)^{\frac{1}{p}} \\ &\xrightarrow{p \rightarrow \infty} (\|f\|_\infty - \varepsilon) \\ &\xrightarrow{\varepsilon \rightarrow 0} \|f\|_\infty. \end{aligned}$$

So  $\|f\|_p \geq \|f\|_\infty$ .

■



**5.4 4**

Fix  $k \in \mathbb{Z}$ . Since  $e^{2\pi i k x}$  is continuous on the compact interval  $[0, 1]$ , it is uniformly continuous, and is thus there is a sequence of polynomials  $P_\ell$  such that

$$P_{\ell,k} \xrightarrow{\ell \rightarrow \infty} e^{2\pi i k x} \text{ uniformly on } [0, 1].$$

Note that by linearity,

$$\int f(x)x^n = 0 \quad \forall n \implies \int f(x)P_{\ell,k}(x) = 0 \quad \forall \ell \in \mathbb{N}$$

But then the  $k$ th Fourier coefficient of  $f$  is given by

$$\begin{aligned} \langle f, e_k \rangle &= \int_0^1 f(x)e^{-2\pi i k x} dx \\ &= \int_0^1 f(x) \lim_{\ell \rightarrow \infty} P_\ell(x) \\ &= \lim_{\ell \rightarrow \infty} \int_0^1 f(x)P_\ell(x) \quad \text{by uniform convergence} \\ &= \lim_{\ell \rightarrow \infty} 0 \\ &= 0 \quad \forall k \in \mathbb{Z}, \end{aligned}$$

so  $\hat{f}$  is the zero function, and  $\hat{f} = 0 \iff f = 0$  almost everywhere. ■

**5.5 5**

$$\text{Moral: } \int |f_n - f| \longrightarrow \iff \int f_n = \int f.$$

Since if  $\int |f_n| \longrightarrow \int |f|$  then we can define

$$\begin{aligned} h_n &= |f_n - f| && \longrightarrow 0 \text{ a.e.} \\ g_n &= |f_n| + |f| && \longrightarrow 2|f| \text{ a.e.} \end{aligned}$$

$$\begin{aligned} \int 2|f| &= \int \liminf (g_n - h_n) \\ &= \int \liminf g_n - \int \liminf h_n \\ &= \int 2|f| - \int \liminf h_n \\ &\stackrel{\text{Fatou}}{\leq} \int 2|f| + \limsup \int h_n, \end{aligned}$$

---

which forces  $\int h_n = \int |f_n - f| \rightarrow 0$ .

But then

$$\left| \int f_n - \int f \right| = \left| \int f_n - f \right| \leq \int |f_n - f| \rightarrow 0,$$

so  $\int f_n \rightarrow \int f$ .

■

## 6 Fall 2017

### 6.1 1

Note that  $f(x) = e^x$  is entire and thus equal to its power series. So  $f(x) = \sum_{j=0}^{\infty} \frac{1}{j!} x^j$ .

Letting  $f_N(x) = \sum_{j=1}^N \frac{1}{j!} x^j$ , we have  $f_N(x) \rightarrow f(x)$  pointwise on  $(-\infty, \infty)$ .

For any compact interval  $[-M, M]$ , we have

$$\begin{aligned} \|f_N(x) - f(x)\|_{\infty} &= \sup_{-M \leq x \leq M} \left| \sum_{j=N+1}^{\infty} \frac{1}{j!} x^j \right| \\ &\leq \sup_{-M \leq x \leq M} \sum_{j=N+1}^{\infty} \frac{1}{j!} |x|^j \\ &\leq \sum_{j=N+1}^{\infty} \frac{1}{j!} M^j \\ &\leq \sum_{j=0}^{\infty} \frac{1}{j!} M^j \\ &= e^M \\ &< \infty, \end{aligned}$$

so  $f_N \rightarrow f$  uniformly on  $[-M, M]$  by the M-test. Thus it converges on any bounded interval.

It does not converge on  $\mathbb{R}$ , since  $x^N$  is unbounded.

### 6.2 2

#### 6.2.1 a

It suffices to consider the bounded case, i.e.  $E \subseteq B_M(0)$  for some  $M$ . Then write  $E_n = B_n(0) \cap E$  and apply the theorem to  $E_n$ , and by subadditivity,  $m^*(E) = m^*\left(\bigcup_n E_n\right) \leq \sum_n m^*(E_n) = 0$ .

**Lemma:**  $f(x) = x^2$ ,  $f^{-1}(x) = \sqrt{x}$  are Lipschitz on any compact subset of  $[0, \infty)$ .

*Proof:* Let  $g = f$  or  $f^{-1}$ . Then  $g \in C^1([0, M])$  for any  $M$ , so  $g$  is differentiable and  $g'$  is continuous. Since  $g'$  is continuous on a compact interval, it is bounded, so  $|g'(x)| \leq L$  for all  $x$ . Applying the MVT,

$$|f(x) - f(y)| = f'(c)|x - y| \leq L|x - y|.$$

**Lemma:** If  $g$  is Lipschitz on  $\mathbb{R}^n$ , then  $m(E) = 0 \implies m(g(E)) = 0$ .

*Proof:* If  $g$  is Lipschitz, then

$$g(B_r(x)) \subseteq B_{Lr}(x),$$

which is a dilated ball/cube, and so

$$m^*(B_{Lr}(x)) \leq L^n \cdot m^*(B_r(x)).$$

Now choose  $\{Q_j\} \rightrightarrows E$ ; then  $\{g(Q_j)\} \rightrightarrows g(E)$ .

By the above observation,

$$|g(Q_j)| \leq L^n |Q_j|,$$

and so

$$m^*(g(E)) \leq \sum_j |g(Q_j)| \leq \sum_j L^n |Q_j| = L^n \sum_j |Q_j| \longrightarrow 0.$$

Now just take  $g(x) = x^2$  for one direction, and  $g(x) = f^{-1}(x) = \sqrt{x}$  for the other. ■

### 6.2.2 b

Lemma:  $E$  is measurable iff  $E = K \coprod N$  for some  $K$  compact,  $N$  null.

Write  $E = K \coprod N$  where  $K$  is compact and  $N$  is null.

Then  $\varphi^{-1}(E) = \varphi^{-1}(K \coprod N) = \varphi^{-1}(K) \coprod \varphi^{-1}(N)$ .

Since  $\varphi^{-1}(N)$  is null by part (a) and  $\varphi^{-1}(K)$  is the preimage of a compact set under a continuous map and thus compact,  $\varphi^{-1}(E) = K' \coprod N'$  where  $K'$  is compact and  $N'$  is null, so  $\varphi^{-1}(E)$  is measurable.

So  $\varphi$  is a measurable function, and thus yields a well-defined map  $\mathcal{L}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathbb{R})$  since it preserves measurable sets. Restricting to  $[0, \infty)$ ,  $f$  is bijection, and thus so is  $\varphi$ . ■

## 6.3 3

From homework:  $E$  is Lebesgue measurable iff there exists a finite union of closed cubes  $A$  such that  $m(E \Delta A) < \varepsilon$ .

It suffices to show that  $S$  is dense in simple functions, and since simple functions are *finite* linear combinations of characteristic functions, it suffices to show this for  $\chi_A$  for  $A$  a measurable set.

Let  $s = \chi_A$ . By regularity of the Lebesgue measure, choose an open set  $O \supseteq A$  such that  $m(O \setminus A) < \varepsilon$ .

$O$  is an open subset of  $\mathbb{R}$ , and thus  $O = \coprod_{j \in \mathbb{N}} I_j$  is a disjoint union of countably many open intervals.

Now choose  $N$  large enough such that  $m(O \Delta I_{N,n}) < \varepsilon = \frac{1}{n}$  where we define  $I_{N,n} := \coprod_{j=1}^N I_j$ .

Now define  $f_n = \chi_{I_{N,n}}$ , then

$$\|s - f_n\|_1 = \int |\chi_A - \chi_{I_{N,n}}| = m(A \Delta I_{N,n}) \xrightarrow{n \rightarrow \infty} 0.$$

Since any simple function is a finite linear combination of  $\chi_{A_i}$ , we can do this for each  $i$  to extend this result to all simple functions. But simple functions are dense in  $L^1$ , so  $S$  is dense in  $L^1$ .

## 6.4 4

## 6.4.1 a

Let  $G(x) = \sum_{n=1}^{\infty} nx(1-x)^n$ . Applying the ratio test, we have

$$\left| \frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n} \right| = \frac{n+1}{n} |1-x| \xrightarrow{n \rightarrow \infty} |1-x| < 1 \iff 0 \leq x \leq 2,$$

and in particular, this series converges on  $[0, 2]$ . Thus its terms go to zero, and  $nx(1-x)^n \rightarrow 0$  on  $[0, 1] \subset [0, 2]$ .

To see that the convergence is not uniform, let  $x_n = \frac{1}{n}$  and  $\varepsilon > \frac{1}{e}$ , then

$$\sup_{x \in [0,1]} |nx(1-x)^n - 0| \geq |nx_n(1-x_n)^n| = \left| \left(1 - \frac{1}{n}\right)^n \right| \xrightarrow{n \rightarrow \infty} e^{-1} > \varepsilon.$$

## 6.4.2 b

Note: could use the first part with  $\sin(x) \leq x$ , but then integral ends up more complicated.

Noting that  $\sin(x) \leq 1$ , we have We have

$$\begin{aligned}
 \left| \int_0^1 n(1-x)^n \sin(x) \right| &\leq \int_0^1 |n(1-x)^n \sin(x)| \\
 &\leq \int_0^1 |n(1-x)^n| \\
 &= n \int_0^1 (1-x)^n \\
 &= -\frac{n(1-x)^{n+1}}{n+1} \\
 &\xrightarrow{n \rightarrow \infty} 0.
 \end{aligned}$$

## 6.5 5

### 6.5.1 a

**Lemma:** If  $\varphi \in C_c^1$ , then  $(f * \varphi)' = f * \varphi'$  almost everywhere.

*Silly Proof:*

$$\begin{aligned}
 \mathcal{F}((f * \varphi)') &= 2\pi i \xi \mathcal{F}(f * \varphi) \\
 &= 2\pi i \xi \mathcal{F}(f) \mathcal{F}(\varphi) \\
 &= \mathcal{F}(f) \cdot (2\pi i \xi \mathcal{F}(\varphi)) \\
 &= \mathcal{F}(f) \cdot \mathcal{F}(\varphi') \\
 &= \mathcal{F}(f * \varphi').
 \end{aligned}$$

*Actual proof:*

$$\begin{aligned}
 (f * \varphi)'(x) &= (\varphi * f)'(x) \\
 &= \lim_{h \rightarrow 0} \frac{(\varphi * f)'(x+h) - (\varphi * f)'(x)}{h} \\
 &= \lim_{h \rightarrow 0} \int \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \\
 &\stackrel{DCT}{=} \int \lim_{h \rightarrow 0} \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \\
 &= \int \varphi'(x-y) f(y) \\
 &= (\varphi' * f)(x) \\
 &= (f * \varphi')(x).
 \end{aligned}$$

To see that the DCT is justified, we can apply the MVT on the interval  $[0, h]$  to  $f$  to obtain

$$\frac{\varphi(x+h-y) - \varphi(x-y)}{h} = \varphi'(c) \quad c \in [0, h],$$

and since  $\varphi'$  is continuous and compactly supported,  $\varphi'$  is bounded by some  $M < \infty$  by the extreme value theorem and thus

$$\begin{aligned} \int \left| \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \right| &= \int |\varphi'(c)f(y)| \\ &\leq \int |M||f| \\ &= |M| \int |f| < \infty, \end{aligned}$$

since  $f \in L^1$  by assumption, so we can take  $g := |M||f|$  as the dominating function.

Applying this theorem infinitely many times shows that  $f * \varphi$  is smooth.

To see that  $f * \varphi$  is compactly supported, approximate  $f$  by a *continuous* compactly supported function  $h$ , so  $\|h - f\|_1 \xrightarrow{L^1} 0$ .

Now let  $g_x(y) = \varphi(x - y)$ , and note that  $\text{supp}(g) = x - \text{supp}(\varphi)$  which is still compact.

But since  $\text{supp}(h)$  is bounded, there is some  $N$  such that

$$|x| > N \implies A_x := \text{supp}(h) \cap \text{supp}(g_x) = \emptyset$$

and thus

$$\begin{aligned} (h * \varphi)(x) &= \int_{\mathbb{R}} \varphi(x - y)h(y) dy \\ &= \int_{A_x} g_x(y)h(y) dy \\ &= 0, \end{aligned}$$

so  $\{x \mid f * g(x) = 0\}$  is open, and its complement is closed and bounded and thus compact.

### 6.5.2 b

$$\begin{aligned} \|f * K_j - f\|_1 &= \int \left| \int f(x - y)K_j(y) dy - f(x) \right| dx \\ &= \int \left| \int f(x - y)K_j(y) dy - \int f(x)K_j(y) dy \right| dx \\ &= \int \left| \int (f(x - y) - f(x))K_j(y) dy \right| dx \\ &\leq \int \int |(f(x - y) - f(x))| \cdot |K_j(y)| dy dx \\ &\stackrel{FT}{=} \int \int |(f(x - y) - f(x))| \cdot |K_j(y)| \mathbf{dx} \mathbf{dy} \\ &= \int |K_j(y)| \left( \int |(f(x - y) - f(x))| dx \right) dy \\ &= \int |K_j(y)| \cdot \|f - \tau_y f\|_1 dy. \end{aligned}$$

We now split the integral up into pieces.

1. Chose  $\delta$  small enough such that  $|y| < \delta \implies \|f - \tau_y f\|_1 < \varepsilon$  by continuity of translation in  $L^1$ , and
2. Since  $\varphi$  is compactly supported, choose  $J$  large enough such that

$$j > J \implies \int_{|y| \geq \delta} |K_j(y)| dy = \int_{|y| \geq \delta} |j\varphi(jy)| dy = 0$$

Then

$$\begin{aligned} \|f * K_j - f\|_1 &\leq \int |K_j(y)| \cdot \|f - \tau_y f\|_1 dy \\ &= \int_{|y| < \delta} |K_j(y)| \cdot \|f - \tau_y f\|_1 dy + \int_{|y| \geq \delta} |K_j(y)| \cdot \|f - \tau_y f\|_1 dy \\ &= \varepsilon \int_{|y| \geq \delta} |K_j(y)| dy + 0 \\ &\leq \varepsilon(1) \longrightarrow 0. \end{aligned}$$

■

## 6.6 6

Should be supremum maybe..?

Let  $\{f_k\}$  be a Cauchy sequence, so  $\|f_k\| < \infty$  for all  $k$ . Then for a fixed  $x$ , the sequence  $f_k(x)$  is Cauchy in  $\mathbb{R}$  and thus converges to some  $f(x)$ , so define  $f$  by  $f(x) := \lim_{k \rightarrow \infty} f_k(x)$ .

Then  $\|f_k - f\| = \max_{x \in X} |f_k(x) - f(x)| \xrightarrow{k \rightarrow \infty} 0$ , and thus  $f_k \rightarrow f$  uniformly and thus  $f$  is continuous. It just remains to show that  $f$  has bounded norm.

Choose  $N$  large enough so that  $\|f - f_N\| < \varepsilon$ , and write  $\|f_N\| := M < \infty$

$$\|f\| \leq \|f - f_N\| + \|f_N\| < \varepsilon + M < \infty.$$

## 7 Spring 2017

### 7.1 1

$A$  is nowhere dense  $\iff$  every interval  $I$  contains a subinterval  $S \subseteq A^c$ .

**$K$  is compact:**

It suffices to show that  $K^c := [0, 1] \setminus K$  is open; then  $K$  will be a closed and bounded subset of  $\mathbb{R}$  and thus compact by Heine-Borel.

We can identify  $K^c$  as the set of real numbers in  $[0, 1]$  whose decimal expansion **does** use a 4. Let  $x \in K^c$ , and suppose a 4 occurs as the  $k$ th digit and write

$$x = 0.d_1 d_2 \cdots d_{k-1} 4 d_{k+1} \cdots = \sum_{j=1}^k d_j 10^{-j} + 4 \cdot 10^{-k} + \sum_{j=k+1}^{\infty} d_j 10^{-j}.$$

Then if we set  $r < 10^{-k}$  and pick any  $y \in [0, 1]$  such that  $y \in B_r(x)$ , then  $|x - y| < 10^{-k}$ . If we write  $y = \sum_{j=1}^{\infty} c_j 10^{-j}$ , this means that for all  $j \leq k$  we have  $d_j = c_j$ , and in particular  $d_k = 4 = c_k$ , so  $y$  has a 4 in its decimal expansion.

But then  $K^c = \bigcup_x B_r(x)$  is a union of open sets and thus open.

**$K$  is nowhere dense and  $m(K) = 0$ :**

Since  $K$  is closed, we'll show that  $K$  can not properly contain any interval, so  $(\overline{K})^\circ = \emptyset$ .

As in the construction of the Cantor set, let

- $K_1$  denote  $[0, 1]$  with 1 interval  $[0.4, 0.5]$  of length  $\frac{1}{10}$  deleted
- $K_2$  denote  $K_1$  with 9 intervals  $[0.04, 0.05], [0.14, 0.15], \dots [0.94, 0.95]$  length  $\frac{1}{100}$  deleted
- $K_n$  denote  $K_{n-1}$  with  $9^{n-1}$  such intervals of length  $10^{-n}$  deleted.

Then  $K = \bigcap K_n$ , and

$$m(K) = 1 - m(K^c) = 1 - \sum_{j=0}^{\infty} \frac{9^j}{10^{j+1}} = 1 - \frac{1}{10} \left( \frac{1}{1 - \frac{9}{10}} \right) = 0,$$

and since any interval has strictly positive measure,  $K$  can not contain any interval.

**$K$  has no isolated points:**

A point  $x \in K$  is isolated iff there is an open ball  $B_r(x)$  containing  $x$  such that  $B_r(x) \cap K = \{x\}$ , so every point in this ball has a 4 in its decimal expansion.

Note that  $m(K_n) = \left(\frac{9}{10}\right)^n \rightarrow 0$  and that the endpoints of intervals are never removed and are thus elements of  $K$ . Then for every  $\varepsilon$ , we can choose  $n$  such that  $\left(\frac{9}{10}\right)^n < \varepsilon$ ; then there is an endpoint of a removed interval  $e_n$  satisfying  $|x - e_n| \leq \left(\frac{9}{10}\right)^n < \varepsilon$ .

So every ball containing  $x$  contains some endpoint of a removed interval, and thus an element of  $K$ . ■

## 7.2 2

$$\lambda \ll \mu \iff E \in \mathcal{M}, \mu(E) = 0 \implies \lambda(E) = 0.$$

### 7.2.1 a

By Radon-Nikodym, if  $\lambda \ll \mu$  then  $d\lambda = f d\mu$ , which would yield

$$\int g d\lambda = \int g f d\mu.$$



So let  $E$  be measurable and suppose  $\mu(E) = 0$ . Then

$$\lambda(E) := \int_E f \, d\mu = \lim_n \left\{ \varphi_n := \sum_j c_j \mu(E_j) \right\},$$

where we take a sequence of simple functions increasing to  $f$ .

But since each  $E_j \subseteq E$ , we must have  $\mu(E_j) = 0$  for any such  $E_j$ , so every such  $\varphi_n$  must be zero and thus  $\lambda(E) = 0$ .

### 7.2.2 b

By Radon-Nikodym, there exists a positive  $f$  such that

$$\int g \, dm = \int gf \, d\mu,$$

where we can take  $g(x) = x^2$ , then the LHS is zero by assumption and thus so is the RHS.

Note that  $gf$  is positive.

Define  $A_k = \left\{ x \in X \mid gf\chi_E > \frac{1}{k} \right\}$ , then by Chebyshev

$$\mu(A_k) \leq k \int_E gf \, d\mu = 0,$$

which holds for every  $k$ .

Then noting that  $A_k \searrow A := \{x \in E \mid x^2 > 0\}$ , and  $gf$  is positive, we have

$$x \in E \iff gf\chi_E(x) > 0 \iff x \in A,$$

so  $E = A$  and  $\mu(E) = \mu(A)$ .

But since  $m \ll \mu$  by construction, we can conclude that  $m(E) = 0$ . ■

## 7.3 3

### 7.3.1 a

Letting  $x_n := \frac{1}{n}$ , we have

$$\sum_{k=1}^{\infty} |f_k(x)| \geq |f_n(x_n)| = \left| ae^{-ax} - be^{-bx} \right| := M.$$

In particular,  $\sup_x |f_n(x)| \not\rightarrow 0$ , so the terms do not go to zero and the sum can not converge.

**7.3.2 b**

?

**7.4 4**

Switching to polar coordinates and integrating over a half-circle contained in  $I^2$ , we have

$$\int_{I^2} f \geq \int_0^\pi \int_0^1 \frac{\cos(\theta) \sin(\theta)}{r^2} dr d\theta = \infty,$$

so  $f$  is not integrable.

**7.5 5**

See <https://math.stackexchange.com/questions/507263/prove-that-c1a-b-with-the-c1-norm-is-a-banach-space>

This is clearly a norm, which we'll write  $\|\cdot\|_u$

Let  $f_n$  be a Cauchy sequence and define a candidate limit  $f(x) = \lim_n f_n(x)$ .

Then noting that  $\|f_n\|_\infty, \|f'_n\|_\infty \leq \|f_n\|_u < \infty$ , both  $f_n, f'_n$  are Cauchy sequences in  $C^0([a, b], \|\cdot\|_\infty)$ , which is a Banach space.

So  $f_n \rightarrow f$  uniformly, and  $f'_n \rightarrow g$  uniformly for some  $g$ , and moreover  $f, g \in C^0([a, b])$ .

We thus have

$$\begin{aligned} f_n(x) - f_n(a) &\xrightarrow{u} f(x) - f(a) \\ \int_a^x f'_n &\xrightarrow{u} \int_a^x g, \end{aligned}$$

and by the FTC, the left-hand sides are equal, and by uniqueness of limits so are the right-hand sides, so  $f' = g$ .

Since  $f, f' \in C^0([a, b])$ , they are bounded, and so  $\|f\|_u < \infty$ . This means that  $\|f_n - f\|_u \rightarrow 0$ , so  $f_n$  converges to  $f$ , which is in the same space.

■

**8 Fall 2016****8.1 1****9 Spring 2016****9.1 1****10 Spring 2014****10.1 1**