Title

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1 Spring 2017

1.1 1

Concepts used:

- Definition: A is *nowhere dense* $\iff$ every interval I contains a subinterval $S \subseteq A^c$.

Solution

- Claim: K **is compact**.
  - It suffices to show that $K^c := [0,1] \setminus K$ is open; Then K will be a closed and bounded subset of $\mathbb{R}$ and thus compact by Heine-Borel.
  - Strategy: write K^c as the union of open balls (since these form a basis for the Euclidean topology on $\mathbb{R}$).
    - Do this by showing every point $x \in K^c$ is an interior point, i.e. x admits a neighborhood N_x such that $N_x \subseteq K^c$.
  - Identify K^c as the set of real numbers in $[0,1]$ whose decimal expansion **does** contain a 4.
    - We will show that there exists a neighborhood small enough such that all points in it contain a 4 in their decimal expansions.

- Let $x \in K^c$, suppose a 4 occurs as the kth digit, and write

$$x = 0.d_1 d_2 \cdots d_{k-1}\ 4\ d_{k+1} \cdots = \left(\sum_{j=1}^{k} d_j 10^{-j}\right) + \left(4 \cdot 10^{-k}\right) + \left(\sum_{j=k+1}^{\infty} d_j 10^{-j}\right).$$

- Set $r_x < 10^{-k}$ and let $y \in [0,1] \bigcap B_{r_x}(x)$ be arbitrary.
- Thus $|x-y| < r_x < 10^{-k}$, and the first k digits of x and y must agree:

$$x - y = \sum_{i=1}^{\infty} d_j 10^j - \sum_{i=1}^{\infty} c_j 10^j = \sum_{i=1}^{\infty} (d_j - c_j) 10^j$$

$$\text{Thus } |x-y| \leq \sum_{j=1}^{\infty} |d_j - c_j| 10^j < 10^{-k} \iff |d_j - c_j| = 0 \quad \forall j \leq k.$$

- Write $y = \sum_{j=1}^{\infty} c_j 10^{-j}$, this means that for all $j \leq k$ we have $d_j = c_j$, and in particular $d_k = 4 = c_k$, so y has a 4 in its decimal expansion.
- But then $K^c = \bigcup_x B_r(x)$ is a union of open sets and thus open.

* Claim: K **is nowhere dense and** $m(K) = 0$**:**

Since K is closed, we'll show that K can not properly contain any interval, so $(\overline{K})^\circ = \emptyset$.

As in the construction of the Cantor set, let

* K_1 denote $[0,1]$ with 1 interval $[0.4, 0.5]$ of length $\frac{1}{10}$ deleted
* K_2 denote K_1 with 9 intervals $[0.04, 0.05],\ [0.14, 0.15], \cdots [0.94, 0.95]$ length $\frac{1}{100}$ deleted
* K_n denote K_{n-1} with 9^{n-1} such intervals of length 10^{-n} deleted.

Then $K = \bigcap K_n$, and

$$m(K) = 1 - m(K^c) = 1 - \sum_{j=0}^{\infty} \frac{9^n}{10^{n+1}} = 1 - \frac{1}{10}\left(\frac{1}{1 - \frac{9}{10}}\right) = 0,$$

and since any interval has strictly positive measure, K can not contain any interval.

* Claim: K **has no isolated points**:

A point $x \in K$ is isolated iff there there is an open ball $B_r(x)$ containing x such that $B_r(x) \bigcap K = \emptyset$, so every point in this ball has a 4 in its decimal expansion.

Note that $m(K_n) = \left(\frac{9}{10}\right)^n \longrightarrow 0$ and that the endpoints of intervals are never removed and are thus elements of K. Then for every ε, we can choose n such that $\left(\frac{9}{10}\right)^n < \varepsilon$; then there is an endpoint of a removed interval e_n satisfying $|x - e_n| \leq \left(\frac{9}{10}\right)^n < \varepsilon$.

So every ball containing x contains some endpoint of a removed interval, and thus an element of K.

$\blacksquare$

1.2 2

$\lambda \ll \mu \iff E \in \mathcal{M}, \mu(E) = 0 \implies \lambda(E) = 0.$

1.2.1 a

By Radon-Nikodym, if $\lambda \ll \mu$ then $d\lambda = f d\mu$, which would yield

$$\int g \; d\lambda = \int gf \; d\mu.$$

So let E be measurable and suppose $\mu(E) = 0$. Then

$$\lambda(E) := \int_E f \; d\mu = \lim_n \left\{ \varphi_n := \sum_j c_j \mu(E_j) \right\},$$

where we take a sequence of simple functions increasing to f.

But since each $E_j \subseteq E$, we must have $\mu(E_j) = 0$ for any such E_j, so every such φ_n must be zero and thus $\lambda(E) = 0$.

1.2.2 b

By Radon-Nikodym, there exists a positive f such that

$$\int g \; dm = \int gf \; d\mu,$$

where we can take $g(x) = x^2$, then the LHS is zero by assumption and thus so is the RHS.

Note that gf is positive.

Define $A_k = \left\{ x \in X \;\middle|\; gf\chi_E > \frac{1}{k} \right\}$, then by Chebyshev

$$\mu(A_k) \leq k \int_E gf \; d\mu = 0,$$

which holds for every k.

Then noting that $A_k \searrow A := \left\{ x \in E \;\middle|\; x^2 > 0 \right\}$, and gf is positive, we have

$$x \in E \iff gf\chi_E(x) > 0 \iff x \in A,$$

so $E = A$ and $\mu(E) = \mu(A)$.

But since $m \ll \mu$ by construction, we can conclude that $m(E) = 0$.

$\blacksquare$

1.3 3

1.3.1 a

Letting $x_n := \frac{1}{n}$, we have

$$\sum_{k=1}^{\infty} |f_k(x)| \geq |f_n(x_n)| = \left| ae^{-ax} - be^{-bx} \right| := M.$$

In particular, $\sup_x |f_n(x)| \not\longrightarrow 0$, so the terms do not go to zero and the sum can not converge.

1.3.2 b

?

1.4 4

Switching to polar coordinates and integrating over a half-circle contained in I^2, we have

$$\int_{I^2} f \geq \int_0^\pi \int_0^1 \frac{\cos(\theta)\sin(\theta)}{r^2}\, dr\, d\theta = \infty,$$

so f is not integrable.

1.5 5

> See https://math.stackexchange.com/questions/507263/prove-that-c1a-b-with-the-c1-norm-is-a-banach-space

This is clearly a norm, which we'll write $\| \cdot \|_u$

Let f_n be a Cauchy sequence and define a candidate limit $f(x) = \lim_n f_n(x)$.

Then noting that $\|f_n\|_\infty, \|f_n'\|\infty \leq \|f_n\|_u < \infty$, both f_n, f_n are Cauchy sequences in $C^0([a,b], \| \cdot \|_\infty)$, which is a Banach space.

So $f_n \longrightarrow f$ uniformly, and $f_n' \longrightarrow g$ uniformly for some g, and moreover $f, g \in C^0([a,b])$.

We thus have

$$\begin{aligned} f_n(x) - f_n(a) &\xrightarrow{u} f(x) - f(a) \\ \int_a^x f_n' &\xrightarrow{u} \int_a^x g, \end{aligned}$$

and by the FTC, the left-hand sides are equal, and by uniqueness of limits so are the right-hand sides, so $f' = g$.

Since $f, f' \in C^0([a,b])$, they are bounded, and so $\|f\|_u < \infty$. This means that $\|f_n - f\|_u \longrightarrow 0$, so f_n converges to f, which is in the same space.

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