

# Topology Qualifying Exam Solutions

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## Contents

<b>1</b>	<b>Definitions</b>	<b>1</b>
<b>2</b>	<b>Theorems</b>	<b>2</b>
<b>3</b>	<b>General Topology</b>	<b>2</b>
3.1	2 . . . . .	2
3.1.1	Alternate Proof of (ii) . . . . .	3
3.2	4 . . . . .	3
3.3	6 . . . . .	5
3.3.1	Proof 1 (DZG) . . . . .	5
3.4	8 . . . . .	5
3.4.1	Proof 1 (DZG) . . . . .	5
<b>4</b>	<b>10</b>	<b>6</b>
<b>5</b>	<b>12</b>	<b>7</b>
<b>6</b>	<b>14</b>	<b>7</b>
<b>7</b>	<b>16</b>	<b>7</b>
7.1	18 . . . . .	9

## 1 Definitions

- Closed (several characterizations)
- Closure in a subspace:  $Y \subset X \implies \text{cl}_Y(A) := \text{cl}_X(A) \cap Y$ .
- Bounded
- Compact
- Connectedness: There does not exist a disconnecting set  $X = A \amalg B$  such that  $\emptyset \neq A, B \subsetneq X$ , i.e.  $X$  is the union of two proper disjoint nonempty sets. Equivalently,  $X$  contains no proper nonempty clopen sets.
- Subspace topology
- Retract: A subspace  $A \subset X$  is a *retract* of  $X$  iff there exists a continuous map  $f : X \longrightarrow A$  such that  $f|_A = \text{id}_A$ . Equivalently it is a *left* inverse to the inclusion.

- 
- Uniform Continuity: For  $f : (X, d_X) \longrightarrow (Y, d_Y)$  metric spaces,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

- Lebesgue number: For  $(X, d)$  a compact metric space and  $\{U_\alpha\} \rightrightarrows X$ , there exist  $\delta_L > 0$  such that  $A \subset X \implies \text{diam}(A) < \delta_L$  and  $A \subseteq U_\alpha$  for some  $\alpha$ .

## 2 Theorems

- Closed subsets of Hausdorff spaces are compact? (check)
- Cantor's intersection theorem?
- Tube lemma
- Properties pushed forward through continuous maps:
  - Compactness?
  - Connectedness (when surjective)
  - Separability
  - Density **only when**  $f$  is surjective
  - **Not** openness
  - **Not** closedness
- Results that only work for metric spaces
  - ?
- A retract of a Hausdorff/connected/compact space is closed/connected/compact respectively.
- A continuous function on a compact set is uniformly continuous.
  - Proof: take  $\{B_{\frac{\varepsilon}{2}}(y) \mid y \in Y\} \rightrightarrows Y$ , pull back to an open cover of  $X$ , has Lebesgue number  $\delta_L > 0$ , then  $x' \in B_{\delta_L}(x) \implies f(x), f(x') \in B_{\frac{\varepsilon}{2}}(y)$  for some  $y$ .

## 3 General Topology

### 3.1 2

Statement: state the definition of compactness, determine if the sets  $\{0\} \cup \left\{\frac{1}{n}\right\}, (0, 1]$  are compact.

- A topological space  $(X, \tau)$  is **compact** if every open cover has a *finite* subcover. That is, if  $\{U_j \mid j \in J\} \subset \tau$  is a collection of open sets such that  $X \subseteq \bigcup_{j \in J} U_j$ , then there exists a *finite* subset  $J' \subset J$  such that  $X \subseteq \bigcup_{j \in J'} U_j$ .
- Use Heine-Borel theorem: a set  $U \subset \mathbb{R}^n$  is compact  $\iff U$  is *closed* and *bounded*.
  - $X$  is closed in  $\mathbb{R}$ , since we can write its complement as an arbitrary union of open intervals:

$$X^c = (-\infty, 0) \cup \left( \bigcup_{n \in \mathbb{Z}^+} \left( \frac{1}{n}, \frac{1}{n+1} \right) \right) \cup (1, \infty)$$

- $X$  is *bounded*, since we can pick  $r = 1$ , then  $x, y \in X \implies d(x, y) \leq r = 1$ .

- iii. Use Heine-Borel again:  $X$  is not closed because it does not contain all of its limit points, e.g. the sequence  $\left\{x_n := \frac{1}{n} \mid n \in \mathbb{Z}^{\geq 1}\right\} \subset X$  but  $x_n \xrightarrow{n \rightarrow \infty} 0 \in X^c$ . Thus  $X$  is **not** compact.

### 3.1.1 Alternate Proof of (ii)

See Munkres p.164

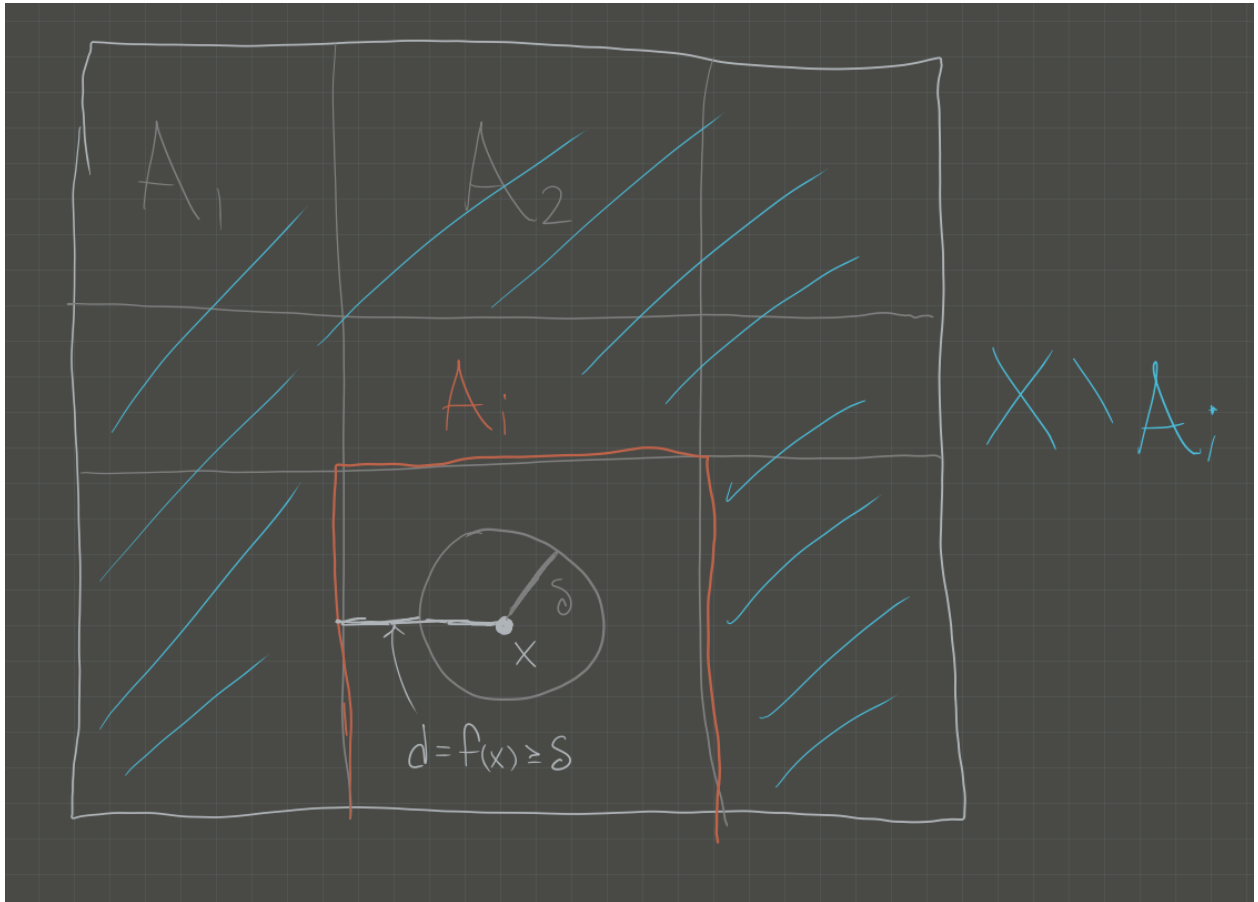
- Let  $\{U_i \mid i \in J\} \Rightarrow X$ ; then  $0 \in U_j$  for some  $j \in J$ .
- In the subspace topology,  $U_i$  is given by some  $V \in \tau(\mathbb{R})$  such that  $V \cap X = U_i$ 
  - A basis for the subspace topology on  $\mathbb{R}$  is open intervals, so write  $V$  as a union of open intervals  $V = \bigcup_{k \in K} I_k$ .
  - Since  $0 \in U_j$ ,  $0 \in I_k$  for some  $k$ .
- Since  $I_k$  is an interval, it contains infinitely many points of the form  $x_n = \frac{1}{n} \in X$ .
- Then  $I_k \cap X \subset U_j$  contains infinitely many such points.
- So there are only *finitely* many points in  $X \setminus U_j$ , each of which is in  $U_{j(n)}$  for some  $j(n) \in J$  depending on  $n$ .
- So  $U_j$  and the *finitely* many  $U_{j(n)}$  form a finite subcover of  $X$ . ■

## 3.2 4

Statement: show that the *Lebesgue number* is well-defined for compact metric spaces.

Note: this is a question about the *Lebesgue Number*. See Wikipedia for detailed proof.

- Write  $U = \{U_i \mid i \in I\}$ , then  $X \subseteq \bigcup_{i \in I} U_i$ . Need to construct a  $\delta > 0$ .
- By compactness of  $X$ , choose a finite subcover  $U_1, \dots, U_n$ .
- Define the distance between a point  $x$  and a set  $Y \subset X$ :  $d(x, Y) = \inf_{y \in Y} d(x, y)$ .
  - **Claim:** the function  $d(\cdot, Y) : X \rightarrow \mathbb{R}$  is continuous for a fixed set.
  - Proof: Todo, not obvious.



- Define a function

$$f : X \longrightarrow \mathbb{R}$$

$$x \mapsto \frac{1}{n} \sum_{i=1}^n d(x, X \setminus U_i).$$

- Note this is a sum of continuous functions and thus continuous.

- **Claim:**

$$\delta := \inf_{x \in X} f(x) = \min_{x \in X} f(x) = f(x_{\min}) > 0$$

suffices.

- That the infimum is a minimum:  $f$  is a continuous function on a compact set, apply the extreme value theorem: it attains its minimum.
- That  $\delta > 0$ : otherwise,  $\delta = 0 \implies \exists x_0$  such that  $d(x_0, X \setminus U_i) = 0$  for all  $i$ .
  - \* Forces  $x_0 \in X \setminus U_i$  for all  $i$ , but  $X \setminus \bigcup U_i = \emptyset$  since the  $U_i$  cover  $X$ .
- That it satisfies the Lebesgue condition:

$$\forall x \in X, \exists i \text{ such that } B_\delta(x) \subset U_i$$

- \* Let  $B_\delta(x) \ni x$ ; then by minimality  $f(x) \geq \delta$ .
- \* Thus it can *not* be the case that  $d(x, X \setminus U_i) < \delta$  for *every*  $i$ , otherwise

$$f(x) \leq \frac{1}{n}(\delta + \dots + \delta) = \frac{n\delta}{n} = \delta$$

- \* So there is some particular  $i$  such that  $d(x, X \setminus U_i) \geq \delta$ .
- \* But then  $B_\delta \subseteq U_i$  as desired.

### 3.3 6

Statement: prove that  $[0, 1] \subset \mathbb{R}$  is compact.

#### 3.3.1 Proof 1 (DZG)

Todo: find a direct proof.

### 3.4 8

Topic: proof of the tube lemma.

Statement: show  $X, Y \in \text{Top}_{\text{compact}} \iff X \times Y \in \text{Top}_{\text{compact}}$

#### 3.4.1 Proof 1 (DZG)

$\Leftarrow :$

- By universal properties, the product  $X \times Y$  is equipped with continuous projections
- The continuous image of a compact set is compact, and  $\pi_1(X \times Y) = X, \pi_2(X \times Y) = Y$
- So  $X, Y$  are compact.

$\Rightarrow :$

Proof of Tube Lemma:

- Let  $\{U_j \times V_j \mid j \in J\} \Rightarrow X \times Y$ .
- Fix a point  $x_0 \in X$ , then  $\{x_0\} \times Y \subset N$  for some open set  $N$ .
- By the tube lemma, there is a  $U^x \subset X$  such that the tube  $U^x \times Y \subset N$ .
- Since  $\{x_0\} \times Y \cong Y$  which is compact, there is a finite subcover  $\{U_j \times V_j \mid j \leq n\} \Rightarrow \{x_0\} \times Y$ .
- “Integrate the  $X$ ”: write

$$W = \bigcap_{j=1}^n U_j,$$

then  $x_0 \in W$  and  $W$  is a finite intersection of open sets and thus open.

- Claim:  $\{U_j \times V_j \mid j \leq n\} \Rightarrow W \times Y$ 
  - Let  $(x, y) \in W \times Y$ ; want to show  $(x, y) \in U_j \times V_j$  for some  $j \leq n$ .
  - Then  $(x_0, y) \in \{x_0\} \times Y$  is on the same horizontal line
  - $(x_0, y) \in U_j \times V_j$  for some  $j$  by construction
  - So  $y \in V_j$  for this  $j$
  - Since  $x \in W$ ,  $x \in U_j$  for every  $j$ , thus  $x \in U_j$ .
  - So  $(x, y) \in U_j \times V_j$

Actual Proof:

- Let  $\{U_j \mid j \in J\} \Rightarrow X \times Y$ .

- 
- Fix  $x_0 \in X$ , the slice  $\{x_0\} \times Y$  is compact and can be covered by finitely many elements  $\{U_j \mid j \leq m\} \Rightarrow \{x_0\} \times Y$ .
    - Sum: write  $N = \bigcup_{j=1}^m U_j$ ; then  $\{x_0\} \times Y \subset N$ .
    - Apply the tube lemma to  $N$ : produce  $\{x_0\} \times Y \in W \times Y \subset N$ ; then  $\{U_j \mid j \leq m\} \Rightarrow W \times Y$ .
  - Now let  $x \in X$  vary: for each  $x \in X$ , produce  $W_x \times Y$  as above, then  $\{W_x \times Y \mid x \in X\} \Rightarrow X$ .
    - By above argument, every tube  $W_x \times Y$  can be covered by *finitely* many  $U_j$ .
  - Since  $\{W_x \mid x \in X\} \Rightarrow X$  and  $X$  is compact, produce a finite subset  $\{W_k \mid k \leq m'\} \Rightarrow X$ .
  - Then  $\{W_k \times Y \mid k \leq m'\} \Rightarrow X \times Y$ ; the claim is that it is a finite cover.
    - Finitely many  $k$
    - For each  $k$ , the tube  $W_k \times Y$  is covered by finitely by  $U_j$
    - And finite  $\times$  finite = finite. ■

Shorter mnemonic:

**19.U** It is sufficient to consider a cover consisting of elementary sets. Since  $Y$  is compact, each fiber  $x \times Y$  has a finite subcovering  $\{U_i^x \times V_i^x\}$ . Put  $W^x = \bigcap U_i^x$ . Since  $X$  is compact, the cover  $\{W^x\}_{x \in X}$  has a finite subcovering  $W^{x_j}$ . Then  $\{U_i^{x_j} \times V_i^{x_j}\}$  is the required finite subcovering.

## 4 10

$X$  is connected:

- Write  $X = L \amalg G$  where  $L = \{0\} \times [-1, 1]$  and  $G = \{\Gamma(\sin(x)) \mid x \in (0, 1]\}$  is the graph of  $\sin(x)$ .
- $L \cong [0, 1]$  which is connected
  - Claim: Every interval is connected (todo)
- Claim:  $G$  is connected
  - The function

$$f : (0, 1] \longrightarrow [-1, 1]$$

$$x \mapsto \sin(x)$$

is continuous (how to prove?)

- Claim: The diagonal map  $\Delta : Y \longrightarrow Y \times Y$  where  $\Delta(t) = (t, t)$  is continuous for any  $Y$  since  $\Delta = (\text{id}, \text{id})$
- The composition of continuous function is continuous
- So the composition is continuous:

$$F : (0, 1] \xrightarrow{\Delta} (0, 1]^2 \xrightarrow{(\text{id}, f)} (0, 1] \times [-1, 1]$$

$$t \mapsto (t, t) \mapsto (t, f(t))$$

- Then  $G = F((0, 1])$  is the continuous image of a connected set and thus connected.

- 
- Claim:  $X$  is connected
    - Suppose there is a disconnecting cover  $X = A \amalg B$  such that  $\bar{A} \cap B = A \cap \bar{B} = \emptyset$  and  $A, B \neq \emptyset$ .
    - WLOG suppose  $(x, \sin(x)) \in B$  for  $x > 0$ .
    - Claim:  $B = G$ 
      - \* It can't be the case that  $A$  intersects  $G$ : otherwise  $X = A \amalg B \implies G = (A \cap G) \amalg (B \cap V)$  disconnects  $G$ . So  $A \cap G = \emptyset$ , forcing  $A \subseteq L$
      - \* Similarly  $L$  can not be disconnected, so  $B \cap L = \emptyset$  forcing  $B \subset G$
      - \* So  $A \subset L$  and  $B \subset G$ , and since  $X = A \amalg B$ , this forces  $A = L$  and  $B = G$ .
    - But any open set  $U$  in the subspace topology  $L \subset \mathbb{R}^2$  (generated by open balls) containing  $(0, 0) \in L$  is the restriction of a ball  $V \subset \mathbb{R}^2$  of positive radius  $r > 0$ , i.e.  $U = V \cap X$ .
      - \* But any such ball contains points of  $G$ : namely take  $n$  large enough such that  $\frac{1}{n\pi} < r$ .
      - \* So  $U \cap L \cap G \neq \emptyset$ , contradicting  $L \cap G = \emptyset$ .

## 5 12

- Using the fact that  $[0, \infty) \subset \mathbb{R}$  is Hausdorff, any retract must be closed, so any closed interval  $[\varepsilon, N]$  for  $0 \leq \varepsilon \leq N \leq \infty$ .
  - Note that  $\varepsilon = N$  yields all one point sets  $\{x_0\}$  for  $x_0 \geq 0$ .
- No finite discrete sets occur, since the retract of a connected set is connected.
- ?

## 6 14

- Take two connected sets  $X, Y$ ; then there exists  $p \in X \cap Y$ .
- Write  $X \cup Y = A \amalg B$  with both  $A, B \subset A \amalg B$  open.
- Since  $p \in X \cup Y = A \amalg B$ , WLOG  $p \in A$ . We will show  $B$  must be empty.
- Claim:  $A \cap X$  is clopen in  $X$ .
  - $A \cap X$  is open in  $X$ : ?
  - $A \cap X$  is closed in  $X$ : ?
- The only clopen sets of a connected set are empty or the entire thing, and since  $p \in A$ , we must have  $A \cap X = X$ .
- By the same argument,  $A \cap Y = Y$ .
- So  $A \cap (X \cup Y) = (A \cap X) \cup (A \cap Y) = X \cup Y$
- Since  $A \subset X \cup Y$ ,  $A \cap (X \cup Y) = A$
- Thus  $A = X \cup Y$ , forcing  $B = \emptyset$ .

## 7 16

Topic: closure and connectedness in the subspace topology. See Munkres p.148

- 
- $S \subset X$  is **not** connected if  $S$  with the subspace topology is not connected.
    - I.e. there exist  $A, B \subset S$  such that
      - \*  $A, B \neq \emptyset$ ,
      - \*  $A \cap B = \emptyset$ ,
      - \*  $A \amalg B = S$ .
  - Or equivalently, there exists a nontrivial  $A \subset S$  that is clopen in  $S$ .

Show stronger statement: this is an iff.

$\implies$  :

- Suppose  $S$  is not connected; we then have sets  $A \cup B = S$  from above and it suffices to show  $\text{cl}_Y(A) \cap B = A \cap \text{cl}_X(B) = \emptyset$ .
- $A$  is open by assumption and  $Y \setminus A = B$  is closed in  $Y$ , so  $A$  is clopen.
- Write  $\text{cl}_Y(A) := \text{cl}_X(A) \cap Y$ .
- Since  $A$  is closed in  $Y$ ,  $A = \text{cl}_Y(A)$  by definition, so  $A = \text{cl}_Y(A) = \text{cl}_X(A) \cap Y$ .
- Since  $A \cap B = \emptyset$ , we then have  $\text{cl}_Y(A) \cap B = \emptyset$ .
- The same argument applies to  $B$ , so  $\text{cl}_Y(B) \cap A = \emptyset$ .

$\impliedby$  :

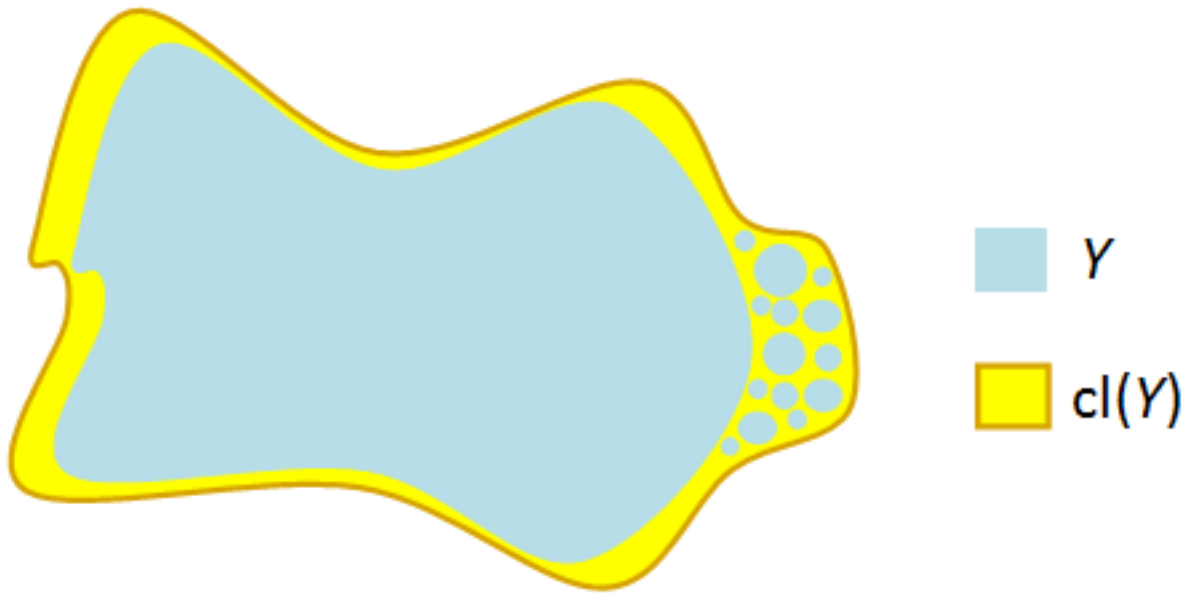
- Suppose displayed condition holds; given such  $A, B$  we will show they are clopen in  $Y$ .
- Since  $\text{cl}_Y(A) \cap B = \emptyset$ , (claim) we have  $\text{cl}_Y(A) = A$  and thus  $A$  is closed in  $Y$ .
  - Why?

$$\begin{aligned}
 \text{cl}_Y(A) &:= \text{cl}_X(A) \cap Y \\
 &= \text{cl}_X(A) \cap (A \amalg B) \\
 &= (\text{cl}_X(A) \cap A) \amalg (\text{cl}_X(A) \cap B) \\
 &= A \amalg (\text{cl}_X(A) \cap B) \quad \text{since } A \subset \text{cl}_Y(A) \\
 &= A \amalg (\text{cl}_Y(A) \cap B) \quad \text{since } B \subset Y \\
 &= A \amalg \emptyset \quad \text{using the assumption} \\
 &= A.
 \end{aligned}$$

- But  $A = Y \setminus B$  where  $B$  is closed, so  $A$  is open and thus a nontrivial clopen subset.

■





## 7.1 18

- Define a new function

$$g : X \longrightarrow \mathbb{R}$$

$$x \mapsto d_X(x, f(x)).$$

- Attempt to minimize. Claim:  $g$  is a continuous function.
- Given claim, a continuous function on a compact space attains its infimum, so set

$$m := \inf_{x \in X} g(x)$$

and produce  $x_0 \in X$  such that  $g(x_0) = m$ .

- Then

$$m > 0 \iff d(x_0, f(x_0)) > 0 \iff x_0 \neq f(x_0).$$

- Now apply  $f$  and use the assumption that  $f$  is a contraction to contradict minimality of  $m$ :

$$\begin{aligned} d(f(f(x_0)), f(x_0)) &\leq C \cdot d(f(x_0), x_0) \\ &< d(f(x_0), x_0) \quad \text{since } C < 1 \\ &\leq m \end{aligned}$$

- Proof that  $g$  is continuous: use the definition of  $g$ , the triangle inequality, and that  $f$  is a

contraction:

$$\begin{aligned}d(x, f(x)) &\leq d(x, y) + d(y, f(y)) + d(f(x), f(y)) \\ \implies d(x, f(x)) - d(y, f(y)) &\leq d(x, y) + d(f(x), f(y)) \\ \implies g(x) - g(y) &\leq d(x, y) + C \cdot d(x, y) = (C + 1) \cdot d(x, y)\end{aligned}$$

- This shows that  $g$  is Lipschitz continuous with constant  $C + 1$  (implies uniformly continuous, but not used).