Topology Qualifying Exam Solutions

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Friday 22nd May, 2020

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1 General Topology

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Statement: state the definition of compactness, determine if the sets $\{0\} \bigcup \left\{ \frac{1}{n} \right\}$, $(0, 1]$ are compact.

i. A topological space (X, τ) is **compact** if every open cover has a *finite* subcover. That is, if $\left\{ U_j \;\middle|\; j \in J \right\} \subset \tau$ is a collection of open sets such that $X \subseteq \bigcup_{j \in J} U_j$, then there exists a *finite* subset $J' \subset J$ such that $X \subseteq \bigcup_{j \in J'} U_j$.

ii. Use Heine-Borel theorem: a set $U \subset \mathbb{R}^n$ is compact $\iff$ U is *closed* and *bounded*.

- X is closed in $\mathbb{R}$, since we can write its complement as an arbitrary union of open intervals:

$$X^c = (-\infty, 0) \bigcup \left(\bigcup_{n \in \mathbb{Z}^+} \left(\frac{1}{n}, \frac{1}{n+1} \right) \right) \bigcup (1, \infty)$$

- X is *bounded*, since we can pick $r = 1$, then $x, y \in X \implies d(x, y) \leq r = 1$.

iii. Use Heine-Borel again: X is not closed because it does not contain all of its limit points, e.g. the sequence $\left\{ x_n := \frac{1}{n} \;\middle|\; n \in \mathbb{Z}^{\geq 1} \right\} \subset X$ but $x_n \overset{n \to \infty}{\longrightarrow} 0 \in X^c$. Thus is is **not** compact.

1.1.1 Alternate Proof of (ii)

See Munkres p.164

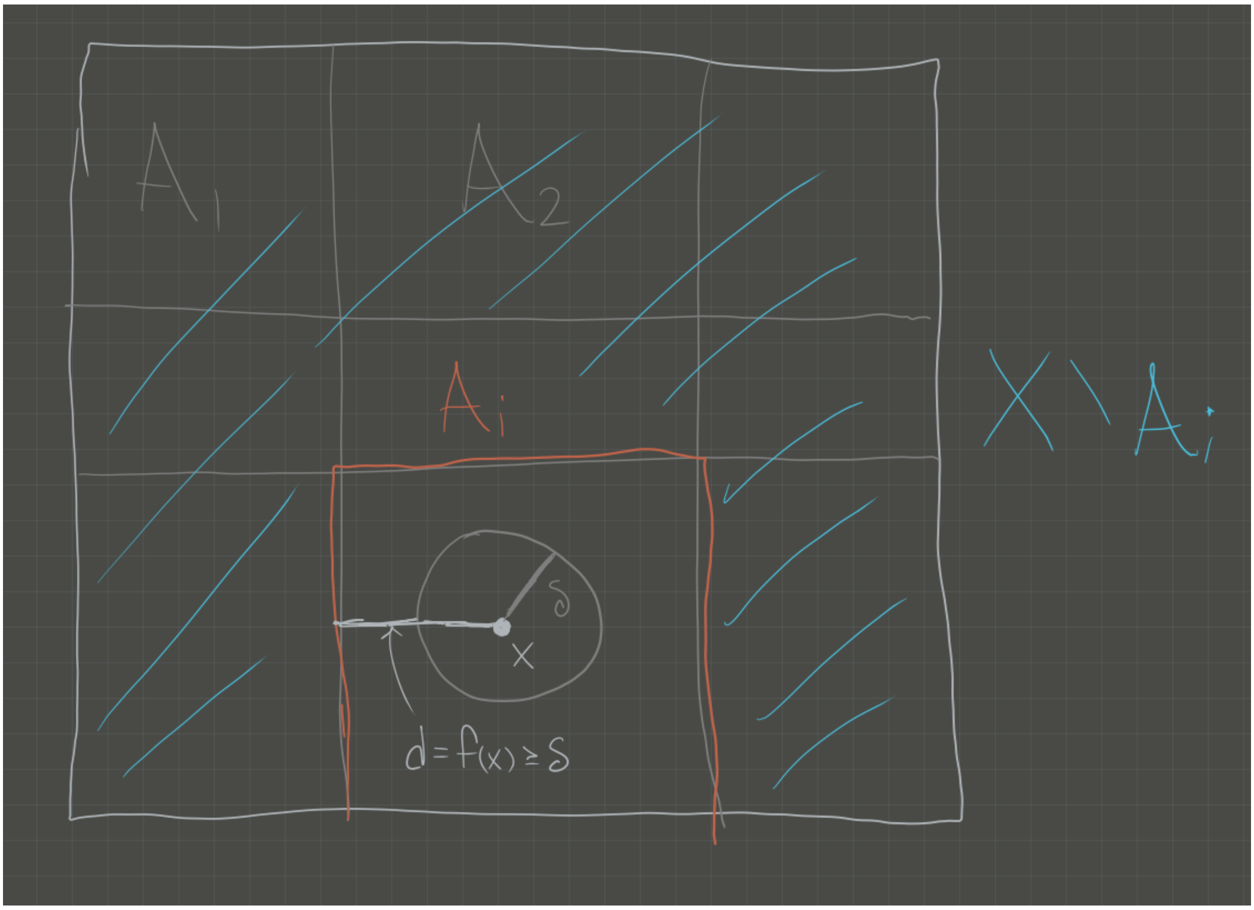
- Let $\left\{U_i \mid j \in J\right\} \rightrightarrows X$; then $0 \in U_j$ for some $j \in J$.
- In the subspace topology, U_i is given by some $V \in \tau(\mathbb{R})$ such that $V \bigcap X = U_i$
  - A basis for the subspace topology on $\mathbb{R}$ is open intervals, so write V as a union of open intervals $V = \bigcup_{k \in K} I_k$.
  - Since $0 \in U_j$, $0 \in I_k$ for some k.
- Since I_k is an interval, it contains infinitely many points of the form $x_n = \frac{1}{n} \in X$
- Then $I_k \bigcap X \subset U_j$ contains infinitely many such points.
- So there are only *finitely* many points in $X \setminus U_j$, each of which is in $U_{j(n)}$ for some $j(n) \in J$ depending on n.
- So U_j and the *finitely* many $U_{j(n)}$ form a finite subcover of X. $\blacksquare$

1.2 4

Statement: show that the *Lebesgue number* is well-defined for compact metric spaces.

Note: this is a question about the *Lebesgue Number*. See Wikipedia for detailed proof.

- Write $U = \left\{U_i \mid i \in I\right\}$, then $X \subseteq \bigcup_{i \in I} U_i$. Need to construct a $\delta > 0$.
- By compactness of X, choose a finite subcover $U_1, \cdots, U_n$.
- Define the distance between a point x and a set $Y \subset X$: $d(x, Y) = \inf_{y \in Y} d(x, y)$.
  - **Claim**: the function $d(\,\cdot\,, Y) : X \longrightarrow \mathbb{R}$ is continuous for a fixed set.
  - Proof: Todo, not obvious.



- Define a function

$$f : X \longrightarrow \mathbb{R}$$
$$x \mapsto \frac{1}{n}\sum_{i=1}^{n} d(x, X \setminus U_i).$$

  - Note this is a sum of continuous functions and thus continuous.
- **Claim**:

$$\delta := \inf_{x \in X} f(x) = \min_{x \in X} f(x) = f(x_{\min}) > 0$$

  suffices.
  - That the infimum is a minimum: f is a continuous function on a compact set, apply the extreme value theorem: it attains its minimum.
  - That $\delta > 0$: otherwise, $\delta = 0 \implies \exists x_0$ such that $d(x_0, X \setminus U_i) = 0$ for all i.
    * Forces $x_0 \in X \setminus U_i$ for all i, but $X \setminus \bigcup U_i = \emptyset$ since the U_i cover X.
  - That it satisfies the Lebesgue condition:

$$\forall x \in X, \exists i \quad \text{such that} \quad B_\delta(x) \subset U_i$$

    * Let $B_\delta(x) \ni x$; then by minimality $f(x) \geq \delta$.
    * Thus it can *not* be the case that $d(x, X \setminus U_i) < \delta$ for *every* i, otherwise

$$f(x) \leq \frac{1}{n}(\delta + \cdots + \delta) = \frac{n\delta}{n} = \delta$$

  * So there is some particular i such that $d(x, X \setminus U_i) \geq \delta$.
  * But then $B_\delta \subseteq U_i$ as desired.

1.3 6

Statement: prove that $[0,1] \subset \mathbb{R}$ is compact.

1.3.1 Proof 1 (DZG)

- Use the Heine-Borel property? Compact sets are closed and bounded.
  - Bounded: take $r = 1$, then $x, y \in [0,1] \implies d(x,y) \leq r = 1$.
  - Closed: Write its complements as $(-\infty, 0) \bigcup (1, \infty)$, an arbitrary union of open intervals.

Todo: find a direct proof.

1.4 8

Topic: proof of the tube lemma.

Statement: show $X, Y \in \text{Top}_{\text{compact}} \iff X \times Y \in \text{Top}_{\text{compact}}$

1.4.1 Proof 1 (DZG)

$\impliedby$:

- By universal properties, the product $X \times Y$ is equipped with continuous projections
- The continuous image of a compact set is compact, and $\pi_1(X \times Y) = X, p_2(X \times Y) = Y$
- So X, Y are compact.

$\implies$:

Proof of Tube Lemma:

- Let $\left\{ U_j \times V_j \;\middle|\; j \in J \right\} \rightrightarrows X \times Y$.
- Fix a point $x_0 \in X$, then $\{x_0\} \times Y \subset N$ for some open set N.
- By the tube lemma, there is a $U^x \subset X$ such that the tube $U^x \times Y \subset N$.
- Since $\{x_0\} \times Y \cong Y$ which is compact, there is a finite subcover $\left\{ U_j \times V_j \;\middle|\; j \leq n \right\} \rightrightarrows \{x_0\} \times Y$.
- "Integrate the X": write

$$W = \bigcap_{j=1}^{n} U_j,$$

  then $x_0 \in W$ and W is a finite intersection of open sets and thus open.
- Claim: $\left\{ U_j \times V_j \;\middle|\; j \leq n \right\} \rightrightarrows W \times Y$
  - Let $(x, y) \in W \times Y$; want to show $(x, y) \in U_j \times V_j$ for some $j \leq n$.
  - Then $(x_0, y) \in \{x_0\} \times Y$ is on the same horizontal line
  - $(x_0, y) \in U_j \times V_j$ for some j by construction
  - So $y \in V_j$ for this j
  - Since $x \in W$, $x \in U_j$ for *every* j, thus $x \in U_j$.

- So $(x, y) \in U_j \times V_j$

Actual Proof:

- Let $\left\{U_j \mid j \in J\right\} \rightrightarrows X \times Y$.
- Fix $x_0 \in X$, the slice $\{x_0\} \times Y$ is compact and can be covered by finitely many elements $\left\{U_j \mid j \leq m\right\} \rightrightarrows \{x_0\} \times Y$.
  - Sum: write $N = \bigcup_{j=1}^{m} U_j$; then $\{x_0\} \times Y \subset N$.
  - Apply the tube lemma to N: produce $\{x_0\} \times Y \in W \times Y \subset N$; then $\left\{U_j \mid j \leq m\right\} \rightrightarrows W \times Y$.
- Now let $x \in X$ vary: for each $x \in X$, produce $W_x \times Y$ as above, then $\left\{W_x \times Y \mid x \in X\right\} \rightrightarrows X$.
  - By above argument, every tube $W_x \times Y$ can be covered by *finitely* many U_j.
- Since $\left\{W_x \mid x \in X\right\} \rightrightarrows X$ and X is compact, produce a finite subset $\left\{W_k \mid k \leq m'\right\} \rightrightarrows X$.
- Then $\left\{W_k \times Y \mid k \leq m'\right\} \rightrightarrows X \times Y$; the claim is that it is a finite cover.
  - Finitely many k
  - For each k, the tube $W_k \times Y$ is covered by finitely by U_j
  - And finite $\times$ finite $=$ finite. ■

Shorter mnemonic:

> ***19.U*** It is sufficient to consider a cover consisting of elementary sets. Since Y is compact, each fiber $x \times Y$ has a finite subcovering $\{U_i^x \times V_i^x\}$. Put $W^x = \cap U_i^x$. Since X is compact, the cover $\{W^x\}_{x \in X}$ has a finite subcovering W^{x_j}. Then $\{U_i^{x_j} \times V_i^{x_j}\}$ is the required finite subcovering.

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X is connected:

- Write $X = V \coprod S$ where $V = \{0\} \times [-1, 1]$ and $S = \left\{\Gamma(\sin(x)) \mid x \in (0, 1]\right\}$ is the graph of $\sin(x)$.
- $V \cong [0, 1]$ which is connected
  - Claim: Every interval is connected (todo)
- Claim: S is connected
  - The function

$$
\begin{aligned}
f : (0, 1] &\longrightarrow [-1, 1] \\
x &\mapsto \sin(x)
\end{aligned}
$$

    is continuous (how to prove?)
  - Claim: The diagonal map $\Delta : X \longrightarrow X \times X$ where $\Delta(x) = (x, x)$ is continuous since $\Delta = (\mathrm{id}, \mathrm{id})$
  - The composition of continuous function is continuous

- So the composition is continuous:

$$F : (0,1] \xrightarrow{\Delta} (0,1]^2 \xrightarrow{(\mathrm{id}, f)} (0,1] \times [-1,1]$$
$$x \mapsto (x,x) \mapsto (x, f(x))$$

  - Then $V = F((0,1])$ is the continuous image of a connected set and thus connected.
- Suppose there is a disconnecting cover $X = A \coprod B$ such that $\overline{A} \bigcap B = A \bigcap \overline{B} = \emptyset$ and $A, B \neq \emptyset$.
- WLOG suppose $(x, \sin(x)) \in B$ for $x > 0$.
- Claim: $B = V$
  - V can not be disconnected, so it can't be the case that A intersects V: otherwise $X = A \coprod B \implies V = (A \bigcap V) \coprod B$ disconnecting V. So $A \bigcap V = \emptyset$
  - Similarly V can not be disconnected, so $B \bigcap S = \emptyset$
  - So $A \subset S$ and $B \subset V$, and since $X = A \coprod B$, this forces $A = V$ and $B = S$.
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