Title

D. Zack Garza

Monday 10th August, 2020

Contents

1 Fields and Galois Theory

1.1 ⋆ Fall 2016 #5

How many monic irreducible polynomials over $\mathbb{F}_p$ of prime degree ℓ are there? Justify your answer.

1.2 ⋆ Fall 2013 #7

Let $F = \mathbb{F}_2$ and let $\overline{F}$ denote its algebraic closure.

a. Show that $\overline{F}$ is not a finite extension of F.

b. Suppose that $\alpha \in \overline{F}$ satisfies $\alpha^{17} = 1$ and $\alpha \neq 1$. Show that $F(\alpha)/F$ has degree 8.

1.3 Spring 2020 #3

Let E be an extension field of F and $\alpha \in E$ be algebraic of odd degree over F.

a. Show that $F(\alpha) = F(\alpha^2)$.

b. Prove that α^{2020} is algebraic of odd degree over F.

1.4 Spring 2020 #4

Let $f(x) = x^4 - 2 \in \mathbb{Q}[x]$.

a. Define what it means for a finite extension field E of a field F to be a Galois extension.

b. Determine the Galois group $\mathrm{Gal}(E/\mathbb{Q})$ for the polynomial $f(x)$, and justify your answer carefully.

c. Exhibit a subfield K in (b) such that $\mathbb{Q} \leq K \leq E$ with K not a Galois extension over $\mathbb{Q}$. Explain.

1.5 Fall 2019 #4 $\bowtie$

Let F be a finite field with q elements.

Let n be a positive integer relatively prime to q and let ω be a primitive nth root of unity in an extension field of F.

Let $E = F[\omega]$ and let $k = [E : F]$.

(a) Prove that n divides $q^k - 1$.

(b) Let m be the order of q in $\mathbb{Z}/n\mathbb{Z}^\times$. Prove that m divides k.

(c) Prove that $m = k$.