Title

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Contents

1 Measure Theory

1.1 Useful Techniques

- $s = \inf\{x \in X\} \implies$ for every ε there is an $x \in X$ such that $x \leq s + \varepsilon$.
- Always consider bounded sets, and if E is unbounded write $E = \bigcup_n B_n(0) \bigcap E$ and use countable subadditivity or continuity of measure.

1.2 Definitions

Definition (Outer Measure) The outer measure of a set is given by

$$m_*(E) \coloneqq \inf_{\substack{\{Q_i\} \rightrightarrows E \\ \text{closed cubes}}} \sum |Q_i|.$$

Definition (Limsup and Liminf of Sets)

$$\begin{aligned}
\limsup_n A_n &\coloneqq \bigcap_n \bigcup_{j \geq n} A_j = \left\{ x \;\middle|\; x \in A_n \text{ for inf. many } n \right\} \\
\liminf_n A_n &\coloneqq \bigcup_n \bigcap_{j \geq n} A_j = \left\{ x \;\middle|\; x \in A_n \text{ for all except fin. many } n \right\}
\end{aligned}$$

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Definition (Lebesgue Measurable Set) A subset $E \subseteq \mathbb{R}^n$ is *Lebesgue measurable* iff for every $\varepsilon > 0$ there exists an open set $O \supseteq E$ such that $m_*(O \setminus E) < \varepsilon$. In this case, we define $m(E) \coloneqq m_*(E)$.

1.3 Theorems

Lemma Every open subset of $\mathbb{R}$ (resp $\mathbb{R}^n$) can be written as a unique countable union of disjoint (resp. almost disjoint) intervals (resp. cubes).

Lemma (Properties of Outer Measure)

- Montonicity: $E \subseteq F \implies m_*(E) \leq m_*(F)$.
- Countable Subadditivity: $m_*(\bigcup E_i) \leq \sum m_*(E_i)$.
- Approximation: For all E there exists a $G \supseteq E$ such that $m_*(G) \leq m_*(E) + \varepsilon$.
- Disjoint[1] Additivity: $m_*(A \coprod B) = m_*(A) + m_*(B)$.

Lemma (Subtraction of Measure)

$$m(A) = m(B) + m(C) \quad \text{and} \quad m(C) < \infty \implies m(A) - m(C) = m(B).$$

Lemma (Continuity of Measure)

$$\begin{aligned} E_i \nearrow E &\implies m(E_i) \longrightarrow m(E) \\ m(E_1) < \infty \text{ and } E_i \searrow E &\implies m(E_i) \longrightarrow m(E). \end{aligned}$$

Proof 1. Break into disjoint annuli $A_2 = E_2 \setminus E_1$, etc then apply countable disjoint additivity to $E = \coprod A_i$.

2. Use $E_1 = (\coprod E_j \setminus E_{j+1}) \coprod (\bigcap E_j)$, taking measures yields a telescoping sum,and use countable disjoint additivity.

Theorem Suppose E is measurable; then for every $\varepsilon > 0$,

1. There exists an open $O \supset E$ with $m(O \setminus E) < \varepsilon$
2. There exists a closed $F \subset E$ with $m(E \setminus F) < \varepsilon$
3. There exists a compact $K \subset E$ with $m(E \setminus K) < \varepsilon$.

Proof

- (1): Take $\{Q_i\} \rightrightarrows E$ and set $O = \bigcup Q_i$.
- (2): Since E^c is measurable, produce $O \supset E^c$ with $m(O \setminus E^c) < \varepsilon$.
  - Set $F = O^c$, so F is closed.
  - Then $F \subset E$ by taking complements of $O \supset E^c$
  - $E \setminus F = O \setminus E^c$ and taking measures yields $m(E \setminus F) < \varepsilon$
- (3): Pick $F \subset E$ with $m(E \setminus F) < \varepsilon/2$.
  - Set $K_n = F \bigcap \mathbb{D}_n$, a ball of radius n about 0.
  - Then $E \setminus K_n \searrow E \setminus F$
  - Since $m(E) < \infty$, there is an N such that $n \geq N \implies m(E \setminus K_n) < \varepsilon$.

Lemma Lebesgue measure is translation and dilation invariant.

Proof Obvious for cubes; if $Q_i \rightrightarrows E$ then $Q_i + k \rightrightarrows E + k$, etc.

Flesh out this proof.

Theorem (Non-Measurable Sets) There is a non-measurable set.

[1] This holds for outer measure **iff** $\operatorname{dist}(A, B) > 0$.

Proof

- Use AOC to choose one representative from every coset of $\mathbb{R}/\mathbb{Q}$ on $[0,1)$, which is countable, and assemble them into a set N
- Enumerate the rationals in $[0,1]$ as q_j, and define $N_j = N + q_j$. These intersect trivially.
- Define $M := \coprod N_j$, then $[0,1) \subseteq M \subseteq [-1,2)$, so the measure must be between 1 and 3. By translation invariance, $m(N_j) = m(N)$, and disjoint additivity forces $m(M) = 0$, a contradiction.

Proposition (Borel Characterization of Measurable Sets) If E is Lebesgue measurable, then $E = H \coprod N$ where $H \in F_\sigma$ and N is null.

> **Useful technique:** F_σ sets are Borel, so establish something for Borel sets and use this to extend it to Lebesgue.

Proof For every $\frac{1}{n}$ there exists a closed set $K_n \subset E$ such that $m(E \setminus K_n) \leq \frac{1}{n}$. Take $K = \bigcup K_n$, wlog $K_n \nearrow K$ so $m(K) = \lim m(K_n) = m(E)$. Take $N := E \setminus K$, then $m(N) = 0$.

Lemma If A_n are all measurable, $\limsup A_n$ and $\liminf A_n$ are measurable.

Proof Measurable sets form a sigma algebra, and these are expressed as countable unions/intersections of measurable sets.

Theorem (Borel-Cantelli) Let $\{E_k\}$ be a countable collection of measurable sets. Then

$$\sum_k m(E_k) < \infty \implies \text{almost every } x \in \mathbb{R} \text{ is in at most finitely many } E_k.$$

Proof

- If $E = \limsup_j E_j$ with $\sum m(E_j) < \infty$ then $m(E) = 0$.
- If E_j are measurable, then $\limsup_j E_j$ is measurable.
- If $\sum_j m(E_j) < \infty$, then $\sum_{j=N}^{\infty} m(E_j) \overset{N\longrightarrow\infty}{\longrightarrow} 0$ as the tail of a convergent sequence.
- $E = \limsup_j E_j = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j \implies E \subseteq \bigcup_{j=k}^{\infty}$ for all k
- $E \subset \bigcup_{j=k}^{\infty} \implies m(E) \leq \sum_{j=k}^{\infty} m(E_j) \overset{k\longrightarrow\infty}{\longrightarrow} 0$.

Lemma

- Characteristic functions are measurable
- If f_n are measurable, so are $|f_n|, \limsup f_n, \liminf f_n, \lim f_n$,
- Sums and differences of measurable functions are measurable,
- Cones $F(x,y) = f(x)$ are measurable,
- Compositions $f \circ T$ for T a linear transformation are measurable,
- "Convolution-ish" transformations $(x,y) \mapsto f(x-y)$ are measurable

Proof (Convolution) Take the cone on f to get $F(x,y) = f(x)$, then compose F with the linear transformation $T = [1,-1;1,0]$.