

Real Analysis Qualifying Exam Notes

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1 Practice Exam 2 (November 2014)

1.1 1: Fubini-Tonelli

1.1.1 a

Carefully state Tonelli's theorem for a nonnegative function $F(x, t)$ on $\mathbb{R}^n \times \mathbb{R}$.

1.1.2 b

Let $f : \mathbb{R}^n \rightarrow [0, \infty]$ and define

$$\mathcal{A} := \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid 0 \leq t \leq f(x) \right\}.$$

Prove the validity of the following two statements:

1. f is Lebesgue measurable on $\mathbb{R}^n \iff \mathcal{A}$ is a Lebesgue measurable subset of \mathbb{R}^{n+1} .
2. If f is Lebesgue measurable on \mathbb{R}^n then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x) dx = \int_0^\infty m\left(\left\{x \in \mathbb{R}^n \mid f(x) \geq t\right\}\right) dt.$$

1.2 2: Convolutions and the Fourier Transform**1.2.1 a**

Let $f, g \in L^1(\mathbb{R}^n)$ and give a definition of $f * g$.

1.2.2 b

Prove that if f, g are integrable and bounded, then

$$(f * g)(x) \xrightarrow{|x| \rightarrow \infty} 0.$$

1.2.3 c

1. Define the *Fourier transform* of an integrable function f on \mathbb{R}^n .
2. Give an outline of the proof of the Fourier inversion formula.
3. Give an example of a function $f \in L^1(\mathbb{R}^n)$ such that \hat{f} is not in $L^1(\mathbb{R}^n)$.

1.3 3: Hilbert Spaces

Let $\{u_n\}_{n=1}^\infty$ be an orthonormal sequence in a Hilbert space H .

1.3.1 a

Let $x \in H$ and verify that

$$\left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|_H^2 = \|x\|_H^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2.$$

for any $N \in \mathbb{N}$ and deduce that

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|_H^2.$$

1.3.2 b

Let $\{a_n\}_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$ and prove that there exists an $x \in H$ such that $a_n = \langle x, u_n \rangle$ for all $n \in \mathbb{N}$, and moreover x may be chosen such that

$$\|x\|_H = \left(\sum_{n \in \mathbb{N}} |a_n|^2 \right)^{\frac{1}{2}}.$$

Proof .

- Take $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
- Define $x := \lim_{N \rightarrow \infty} S_N$ where $S_N = \sum_{k=1}^N a_k u_k$
- $\{S_N\}$ is Cauchy and H is complete, so $x \in H$.
- By construction,

$$\langle x, u_n \rangle = \left\langle \sum_k a_k u_k, u_n \right\rangle = \sum_k a_k \langle u_k, u_n \rangle = a_n$$

since the u_k are all orthogonal.

- By Pythagoras since the u_k are normal,

$$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2.$$

■

1.3.3 c

Prove that if $\{u_n\}$ is *complete*, Bessel's inequality becomes an equality.

Proof .

Let x and u_n be arbitrary.

$$\begin{aligned} \left\langle x - \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k, u_n \right\rangle &= \langle x, u_n \rangle - \left\langle \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k, u_n \right\rangle \\ &= \langle x, u_n \rangle - \sum_{k=1}^{\infty} \langle \langle x, u_k \rangle u_k, u_n \rangle \\ &= \langle x, u_n \rangle - \sum_{k=1}^{\infty} \langle x, u_k \rangle \langle u_k, u_n \rangle \\ &= \langle x, u_n \rangle - \langle x, u_n \rangle = 0 \\ \implies x - \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k &= 0 \quad \text{by completeness.} \end{aligned}$$

So

$$x = \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k \implies \|x\|^2 = \sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2. \blacksquare$$

■

1.4 4: Lp Spaces

1.4.1 a

?

1.4.2 c

Definition (Infinity Norm):

$$L^\infty(X) = \left\{ f : X \longrightarrow \mathbb{C} \mid \|f\|_\infty < \infty \right\}$$

where

$$\|f\|_\infty = \inf_{\alpha \geq 0} \left\{ \alpha \mid m\{|f| \geq \alpha\} = 0 \right\}.$$

Theorem:

$$m(X) < \infty \implies \lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty.$$

Proof: Let $M = \|f\|_\infty$. For any $L < M$, let $S = \{|f| \geq L\}$. Then $m(S) > 0$ and

$$\begin{aligned} \|f\|_p &= \left(\int_X |f|^p \right)^{\frac{1}{p}} \\ &\geq \left(\int_S |f|^p \right)^{\frac{1}{p}} \\ &\geq L m(S)^{\frac{1}{p}} \xrightarrow{p \rightarrow \infty} L \\ &\implies \liminf_p \|f\|_p \geq M. \end{aligned}$$

We also have

$$\begin{aligned} \|f\|_p &= \left(\int_X |f|^p \right)^{\frac{1}{p}} \\ &\leq \left(\int_X M^p \right)^{\frac{1}{p}} \\ &= M m(X)^{\frac{1}{p}} \xrightarrow{p \rightarrow \infty} M \\ &\implies \limsup_p \|f\|_p \leq M. \blacksquare \end{aligned}$$

Note: this doesn't help with this problem at all.

Solution:

$$\begin{aligned}\int f^p &= \int_{x \leq 1} f^p + \int_{x=1} f^p + \int_{x \geq 1} f^p \\ &= \int_{x \leq 1} f^p + \int_{x=1} 1 + \int_{x \geq 1} f^p \\ &= \int_{x \leq 1} f^p + m(\{f = 1\}) + \int_{x \geq 1} f^p \\ &\xrightarrow{p \rightarrow \infty} 0 + m(\{f = 1\}) + \begin{cases} 0 & m(\{x \geq 1\}) = 0 \\ \infty & m(\{x \geq 1\}) > 0. \end{cases}\end{aligned}$$

1.5 5: Dual Spaces

Let X be a normed vector space.

1.5.1 a

Give the definition of what it means for a map $L : X \rightarrow \mathbb{C}$ to be a *linear functional*.

1.5.2 b

Prove Minkowski's Inequality:

$$1 \leq p < \infty \implies \|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Conclude that if $f, g \in L^p(\mathbb{R}^n)$ then so is $f + g$.

1.5.3 c

Let $X = [0, 1] \subset \mathbb{R}$.

1. Give a definition of the Banach space $L^\infty(X)$ of essentially bounded functions of X .
2. Let f be non-negative and measurable on X , prove that

$$\int_X f(x)^p dx \xrightarrow{p \rightarrow \infty} \infty \quad \text{or} \quad m(\{f^{-1}(1)\}) = 0,$$

and characterize the functions of each type

2 Qual: Fall 2019

2.1 1

See phone photo?

2.2 2

DCT?

2.3 3

“Follow your nose.”

2.4 4

See Problem Set 8.

Bessel’s Inequality: For any orthonormal set in a Hilbert space (not necessarily a basis), we have

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

Proof:

$$0 \leq \left\| x - \sum_{k=1}^n \langle x, e_k \rangle e_k \right\|^2$$

Corollary (Parseval’s Identity): If $\text{span}\{u_n\}$ is dense in \mathcal{H} , so it is a basis, then this is an equality.**Riesz-Fischer:** Let $U = \{u_n\}_{n=1}^{\infty}$ be an orthonormal set (not necessarily a basis), then

1. There is an isometric surjection

$$\begin{aligned} \mathcal{H} &\longrightarrow \ell^2(\mathbb{N}) \\ \mathbf{x} &\mapsto \{\langle \mathbf{x}, \mathbf{u}_n \rangle\}_{n=1}^{\infty} \end{aligned}$$

i.e. if $\{a_n\} \in \ell^2(\mathbb{N})$, so $\sum |a_n|^2 < \infty$, then there exists a $\mathbf{x} \in \mathcal{H}$ such that

$$a_n = \langle \mathbf{x}, \mathbf{u}_n \rangle \quad \forall n.$$

2. \mathbf{x} can be chosen such that

$$\|\mathbf{x}\|^2 = \sum |a_n|^2$$

Note: the choice of \mathbf{x} is unique $\iff \{u_n\}$ is **complete**, i.e. $\langle \mathbf{x}, \mathbf{u}_n \rangle = 0$ for all n implies $\mathbf{x} = \mathbf{0}$.

Proof:

- Given $\{a_n\}$, define $S_N = \sum_{n=1}^N a_n \mathbf{u}_n$.
- S_N is Cauchy in \mathcal{H} and so $S_N \longrightarrow \mathbf{x}$ for some $\mathbf{x} \in \mathcal{H}$.
- $\langle x, u_n \rangle = \langle x - S_N, u_n \rangle + \langle S_N, u_n \rangle \longrightarrow a_n$
- By construction, $\|x - S_N\|^2 = \|x\|^2 - \sum_{n=1}^N |a_n|^2 \longrightarrow 0$, so $\|x\|^2 = \sum_{n=1}^{\infty} |a_n|^2$.

2.5 5

See Problem Set 5.

Heine-Cantor theorem: Every continuous function on a compact set is uniformly continuous.

Uniform continuity:

$$\begin{aligned} & \forall \varepsilon \quad \exists \delta(\varepsilon) \mid \quad \forall x, y, \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \\ \iff & \forall \varepsilon \quad \exists \delta(\varepsilon) \mid \quad \forall x, y, \quad |y| < \delta \implies |f(x - y) - f(y)| < \varepsilon \end{aligned}$$

Fubini-Tonelli interchange of integrals, where the change of bounds becomes very important.

Continuity in L^1 :

$$\lim_{y \rightarrow 0} \|\tau_y f - f\|_1 = 0.$$