Title

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Contents

1 Modules

1.1 General Questions

1.1.1 Fall 2019 Final #2

Consider the $\mathbb{Z}$-submodule N of $\mathbb{Z}^3$ spanned by $f_1 = [-1, 0, 1], f_2 = [2, -3, 1], f_3 = [0, 3, 1], f_4 = [3, 1, 5]$. Find a basis for N and describe $\mathbb{Z}^3/N$.

1.1.2 Spring 2018 #6.

Let

$$M=\{(w,x,y,z)\in\mathbb{Z}^4 \,\middle|\, w+x+y+z\in 2\mathbb{Z}\},$$

and

$$N=\{(w,x,y,z)\in\mathbb{Z}^4 \,\middle|\, 4 \,\middle|\, (w-x),\ 4 \,\middle|\, (x-y),\ 4 \,\middle|\, (y-z)\}.$$

a. Show that N is a $\mathbb{Z}$-submodule of M .

b. Find vectors $u_1,u_2,u_3,u_4\in\mathbb{Z}^4$ and integers d_1,d_2,d_3,d_4 such that

$$\{u_1,u_2,u_3,u_4\}$$

is a free basis for M, and

$$\{d_1u_1,\ d_2u_2,\ d_3u_3,\ d_4u_4\}$$

is a free basis for N .

c. Use the previous part to describe M/N as a direct sum of cyclic $\mathbb{Z}$-modules.

1.1.3 Fall 2018 #6 ⋈

Let R be a commutative ring, and let M be an R-module. An R-submodule N of M is maximal if there is no R-module P with $N\subsetneq P\subsetneq M$.

a. Show that an R-submodule N of M is maximal $\iff$ M/N is a simple R-module: i.e., M/N is nonzero and has no proper, nonzero R-submodules.

b. Let M be a $\mathbb{Z}$-module. Show that a $\mathbb{Z}$-submodule N of M is maximal $\iff$ $\#M/N$ is a prime number.

c. Let M be the $\mathbb{Z}$-module of all roots of unity in $\mathbb{C}$ under multiplication. Show that there is no maximal $\mathbb{Z}$-submodule of M.

Solution.
a
By the correspondence theorem, submodules of M/N biject with submodules A of M containing N.
So

- M is maximal:
- $\iff$ no such (proper, nontrivial) submodule A exists
- $\iff$ there are no (proper, nontrivial) submodules of M/N
- $\iff$ M/N is simple.

b
Identify $\mathbb{Z}$-modules with abelian groups, then by (a), N is maximal $\iff$ M/N is simple $\iff$ M/N has no nontrivial proper subgroups.

By Cauchy's theorem, if $|M/N| = ab$ is a composite number, then $a \mid ab \implies$ there is an element (and thus a subgroup) of order a. In this case, M/N contains a nontrivial proper cyclic subgroup, so M/N is not simple. So $|M/N|$ can not be composite, and therefore must be prime.
c
Let $G = \left\{ x \in \mathbb{C} \mathrel{\Big|} x^n = 1 \text{ for some } n \in \mathbb{N} \right\}$, and suppose $H < G$ is a proper subgroup.
Then there must be a prime p such that the $\zeta_{p^k} \notin H$ for all k greater than some constant m – otherwise, we can use the fact that if $\zeta_{p^k} \in H$ then $\zeta_{p^\ell} \in H$ for all $\ell \leq k$, and if $\zeta_{p^k} \in H$ for all p and all k then $H = G$.
But this means there are infinitely many elements in $G \setminus H$, and so $\infty = [G : H] = |G/H|$ is not a prime. Thus by (b), H can not be maximal, a contradiction.

1.1.4 Spring 2018 #7.

Let R be a PID and M be an R-module. Let p be a prime element of R. The module M is called $\langle p \rangle$ *-primary* if for every $m \in M$ there exists $k > 0$ such that $p^k m = 0$.

a. Suppose M is $\langle p \rangle$-primary. Show that if $m \in M$ and $t \in R$, $t \notin \langle p \rangle$, then there exists $a \in R$ such that $atm = m$.

b. A submodule S of M is said to be *pure* if $S \cap rM = rS$ for all $r \in R$. Show that if M is $\langle p \rangle$-primary, then S is pure if and only if $S \cap p^k M = p^k S$ for all $k \geq 0$.

1.1.5 Fall 2016 #6

Let R be a ring and $f : M \longrightarrow N$ and $g : N \longrightarrow M$ be R-module homomorphisms such that $g \circ f = \mathrm{id}_M$. Show that $N \cong \operatorname{im} f \oplus \ker g$.

1.1.6 Spring 2016 #4

Let R be a ring with the following commutative diagram of R-modules, where each row represents a short exact sequence of R-modules:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

Prove that if α and γ are isomorphisms then β is an isomorphism.

1.1.7 Spring 2015 #8

Let R be a PID and M a finitely generated R-module.

a. Prove that there are R-submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that for all $0 \leq i \leq n-1$, the module M_{i+1}/M_i is cyclic.

b. Is the integer n in part (a) uniquely determined by M? Prove your answer.

1.1.8 Fall 2012 #6

Let R be a ring and M an R-module. Recall that M is *Noetherian* iff any strictly increasing chain of submodule $M_1 \subsetneq M_2 \subsetneq \cdots$ is finite. Call a proper submodule $M' \subsetneq M$ *intersection-decomposable* if it can not be written as the intersection of two proper submodules $M' = M_1 \bigcap M_2$ with $M_i \subsetneq M$.

Prove that for every Noetherian module M, any proper submodule $N \subsetneq M$ can be written as a finite intersection $N = N_1 \bigcap \cdots \bigcap N_k$ of intersection-indecomposable modules.

1.1.9 Fall 2019 Final #1

Let A be an abelian group, and show A is a $\mathbb{Z}$-module in a unique way.

1.2 Torsion and the Structure Theorem

1.2.1 $\star$ Fall 2019 #5 $\bowtie$

Let R be a ring and M an R-module.

Recall that the set of torsion elements in M is defined by

$$\operatorname{Tor}(m) = \{m \in M \mid \exists r \in R,\ r \neq 0,\ rm = 0\}.$$

a. Prove that if R is an integral domain, then $\operatorname{Tor}(M)$ is a submodule of M .

b. Give an example where $\operatorname{Tor}(M)$ is not a submodule of M.

c. If R has zero-divisors, prove that every non-zero R-module has non-zero torsion elements.

Solution.

One-step submodule test.

a It suffices to show that

$$r \in R,\ t_1, t_2 \in \operatorname{Tor}(M) \implies rt_1 + t_2 \in \operatorname{Tor}(M).$$

We have

$$\begin{aligned} t_1 \in \operatorname{Tor}(M) &\implies \exists s_1 \neq 0 \text{ such that } s_1 t_1 = 0 \\ t_2 \in \operatorname{Tor}(M) &\implies \exists s_2 \neq 0 \text{ such that } s_2 t_2 = 0. \end{aligned}$$

Since R is an integral domain, $s_1 s_2 \neq 0$. Then

$$\begin{aligned} s_1 s_2 (rt_1 + t_2) &= s_1 s_2 r t_1 + s_1 s_2 t_2 \\ &= s_2 r (s_1 t_1) + s_1 (s_2 t_2) \quad \text{since } R \text{ is commutative} \\ &= s_2 r(0) + s_1(0) \\ &= 0. \end{aligned}$$

b Let $R = \mathbb{Z}/6\mathbb{Z}$ as a $\mathbb{Z}/6\mathbb{Z}$-module, which is not an integral domain as a ring.
Then $[3]_6 \curvearrowright [2]_6 = [0]_6$ and $[2]_6 \curvearrowright [3]_6 = [0]_6$, but $[2]_6 + [3]_6 = [5]_6$, where 5 is coprime to 6, and thus $[n]_6 \curvearrowright [5]_6 = [0] \implies [n]_6 = [0]_6$. So $[5]_6$ is *not* a torsion element.
So the set of torsion elements are not closed under addition, and thus not a submodule.

c Suppose R has zero divisors $a, b \neq 0$ where $ab = 0$. Then for any $m \in M$, we have $b \curvearrowright m := bm \in M$ as well, but then

$$a \curvearrowright bm = (ab) \curvearrowright m = 0 \curvearrowright m = 0_M,$$

so m is a torsion element for any m.

1.2.2 ⋆ Spring 2019 #5 ⋈

Let R be an integral domain. Recall that if M is an R-module, the *rank* of M is defined to be the maximum number of R-linearly independent elements of M .

a. Prove that for any R-module M, the rank of $\operatorname{Tor}(M)$ is 0.

b. Prove that the rank of M is equal to the rank of of $M/\operatorname{Tor}(M)$.

c. Suppose that M is a non-principal ideal of R.

   Prove that M is torsion-free of rank 1 but not free.

Solution.
Part a

- Suppose toward a contradiction $\operatorname{Tor}(M)$ has rank $n \geq 1$.
- Then $\operatorname{Tor}(M)$ has a linearly independent generating set $B = \{\mathbf{r}_1, \cdots, \mathbf{r}_n\}$, so in particular

$$\sum_{i=1}^{n} s_i \mathbf{r}_i = 0 \implies s_i = 0_R \, \forall i.$$

- Let $\mathbf{r}$ be any of of these generating elements.
- Since $\mathbf{r} \in \operatorname{Tor}(M)$, there exists an $s \in R \setminus 0_R$ such that $s\mathbf{r} = 0_M$.
- Then $s\mathbf{r} = 0$ with $s \neq 0$, so $\{\mathbf{r}\} \subseteq B$ is *not* a linearly independent set, a contradiction.

Part b

- Let $n = \operatorname{rank} M$, and let $\mathcal{B} = \{\mathbf{r}_i\}_{i=1}^{n} \subseteq R$ be a generating set.
- Let $\tilde{M} := M/\operatorname{Tor}(M)$ and $\pi : M \longrightarrow M'$ be the canonical quotient map.

Claim:

$$\tilde{\mathcal{B}} := \pi(\mathcal{B}) = \{\mathbf{r}_i + \operatorname{Tor}(M)\}$$

is a basis for $\tilde{M}$.

- **Linearly Independent:**
  - Suppose that

$$\sum_{i=1}^{n} s_i(\mathbf{r}_i + \operatorname{Tor}(M)) = \mathbf{0}_{\tilde{M}}.$$

- Then using the definition of coset addition/multiplication, we can write this as

$$\sum_{i=1}^{n}(s_i\mathbf{r}_i + \mathrm{Tor}(M)) = \left(\sum_{i=1}^{n} s_i\mathbf{r}_i\right) + \mathrm{Tor}(M) = 0_{\tilde{M}}.$$

- Since $\tilde{\mathbf{x}} = 0 \in \tilde{M} \iff \tilde{\mathbf{x}} = \mathbf{x} + \mathrm{Tor}(M)$ where $\mathbf{x} \in \mathrm{Tor}(M)$, this forces $\sum s_i\mathbf{r}_i \in \mathrm{Tor}(M)$.
- Then there exists a scalar $\alpha \in R^{\bullet}$ such that $\alpha \sum s_i\mathbf{r}_i = 0_M$.
- Since R is an integral domain and $\alpha \neq 0$, we must have $\sum s_i\mathbf{r}_i = 0_M$.
- Since $\{\mathbf{r}_i\}$ was linearly independent in M, we must have $s_i = 0_R$ for all i.

- **Spanning:**
  - Write $\pi(\mathcal{B}) = \{\mathbf{r}_i + \mathrm{Tor}(M)\}_{i=1}^{n}$ as a set of cosets.
  - Letting $\mathbf{x} \in M'$ be arbitrary, we can write $\mathbf{x} = \mathbf{m} + \mathrm{Tor}(M)$ for some $\mathbf{m} \in M$ where $\pi(\mathbf{m}) = \mathbf{x}$ by surjectivity of π.
  - Since $\mathcal{B}$ is a basis for M, we have $\mathbf{m} = \sum_{i=1}^{n} s_i\mathbf{r}_i$, and so

$$\begin{aligned} \mathbf{x} &= \pi(\mathbf{m}) \\ &\coloneqq \pi\left(\sum_{i=1}^{n} s_i\mathbf{r}_i\right) \\ &= \sum_{i=1}^{n} s_i\pi(\mathbf{r}_i) \quad \text{since } \pi \text{ is an } R\text{-module morphism} \\ &\coloneqq \sum_{i=1}^{n} s_i(\mathbf{r}_i + \mathrm{Tor}(M)), \end{aligned}$$

  which expresses $\mathbf{x}$ as a linear combination of elements in $\mathcal{B}'$.

Part c

Notation: Let 0_R denote $0 \in R$ regarded as a ring element, and $\mathbf{0} \in R$ denoted 0_R regarded as a module element (where R is regarded as an R-module over itself)

M **is not free:**

- **Claim**: If $I \subseteq R$ is an ideal *and* a free R-module, then I is principal .
  - Suppose I is free and let $I = \langle B \rangle$ for some basis, we will show $|B| = 1$>
  - Toward a contradiction, suppose $|B| \geq 2$ and let $m_1, m_2 \in B$.
  - Then since R is commutative, $m_2m_1 - m_1m_2 = 0$ and this yields a linear dependence
  - So B has only one element m.
  - But then $I = \langle m \rangle = R_m$ is cyclic as an R- module and thus principal as an ideal of R.
  - Now since M was assumed to *not* be principal, M is not free (using the contrapositive of the claim).

M **is rank 1**:

- For any module, we can take an element $\mathbf{m} \in M^{\bullet}$ and consider the cyclic submodule $R\mathbf{m}$.
- Since M is not principle, it is not the zero ideal, and contains at least two elements. So we can consider an element $\mathbf{m} \in M$.
- We have $\mathrm{rank}_R(M) \geq 1$, since $R\mathbf{m} \leq M$ and $\{m\}$ is a subset of some spanning set.

- $R\mathbf{m}$ can not be linearly dependent, since R is an integral domain and $M \subseteq R$, so $\alpha\mathbf{m} = \mathbf{0} \implies \alpha = 0_R$.
- Claim: since R is commutative, $\operatorname{rank}_R(M) \leq 1$.
  - If we take two elements $\mathbf{m}, \mathbf{n} \in M^\bullet$, then since $m, n \in R$ as well, we have $nm = mn$ and so

$$(n)\mathbf{m} + (-m)\mathbf{n} = 0_R = \mathbf{0}$$

    is a linear dependence.

M **is torsion-free**:

- Let $\mathbf{x} \in \operatorname{Tor}M$, then there exists some $r \neq 0 \in R$ such that $r\mathbf{x} = \mathbf{0}$.
- But $\mathbf{x} \in R$ as well and R is an integral domain, so $\mathbf{x} = 0_R$, and thus $\operatorname{Tor}(M) = \{0_R\}$.

1.2.3 ⋆ Spring 2020 #6 ⋈

Let R be a ring with unity.

a. Give a definition for a free module over R.

b. Define what it means for an R-module to be torsion free.

c. Prove that if F is a free module, then any short exact sequence of R-modules of the following form splits:

$$0 \longrightarrow N \longrightarrow M \longrightarrow F \longrightarrow 0.$$

d. Let R be a PID. Show that any finitely generated R-module M can be expressed as a direct sum of a torsion module and a free module.

> You may assume that a finitely generated torsionfree module over a PID is free.

Solution.
Let R be a ring with 1.

a An R-module M is **free** if any of the following conditions hold:

- M admits an R-linearly independent spanning set $\{\mathbf{b}_\alpha\}$, so

$$m \in M \implies m = \sum_\alpha r_\alpha \mathbf{b}_\alpha$$

  and

$$\sum_\alpha r_\alpha \mathbf{b}_\alpha = 0_M \implies r_\alpha = 0_R$$

  for all α.
- $M \cong \bigoplus_\alpha R$ are isomorphic as R-modules.
- There is a nonempty set X and an inclusion $X \hookrightarrow M$ such that for every R-modules N, every map $X \longrightarrow N$ lifts to a unique map $M \longrightarrow N$, so the following diagram commutes:

$$
\begin{tikzcd}
M \arrow[dr, dotted, "\exists ! \tilde f"] & \\
X \arrow[u, hook] \arrow[r, "f"] & N
\end{tikzcd}
$$

b M is **torsionfree** iff $M_t := \left\{ m \in M \,\middle|\, \mathrm{Ann}(m) \neq 0 \right\} \leq M$ is the trivial submodule, where $\mathrm{Ann}(m) := \left\{ r \in R \,\middle|\, r \cdot m = 0_M \right\} \trianglelefteq R$.

c

- Let the following be an SES where F is a free R-module:

$$0 \longrightarrow N \longrightarrow M \xrightarrow{\pi} F \longrightarrow 0.$$

- Since F is free, there is a generating set $X = \{x_\alpha\}$ and a map $\iota : X \hookrightarrow M$ satisfying the 3rd property from (a).
- If we construct a map $f : X \longrightarrow M$, then the universal property of free modules will give a lift $\tilde f : F \longrightarrow M$
- Note $\{\iota(x_\alpha)\} \subseteq F$ and π is surjective, so choose fibers $\{y_\alpha\} \subseteq M$ such that

$$\pi(y_\alpha) = \iota(x_\alpha).$$

- Define a map

$$
\begin{aligned}
f : X &\longrightarrow M \\
x_\alpha &\mapsto y_\alpha.
\end{aligned}
$$

- By the universal property, this yields a map $h : F \longrightarrow M$, commutativity forces $(h \circ \iota)(x_\alpha) = y_\alpha$, i.e. we have a diagram

$$
\begin{tikzcd}
 & & & X = \{x_\alpha\} \arrow[dl, "f"'] \arrow[d, hook, "\iota"] & \\
0 \arrow[r] & N \arrow[r] & M \arrow[r, "\pi"'] & F \arrow[r] \arrow[l, dotted, bend right, "\exists ! h"'] & 0
\end{tikzcd}
$$

- It remains to check that it's a section:

$$
\begin{aligned}
f \in F \implies f &= \sum_\alpha r_\alpha \iota(x_\alpha) \\
\implies (\pi \circ h)(f) &= \pi\left(h\left(\sum_\alpha r_\alpha \iota(x_\alpha) \right) \right) \\
&= \pi\left(\sum_\alpha r_\alpha h(\iota(x_\alpha)) \right) \\
&= \pi\left(\sum_\alpha r_\alpha y_\alpha \right) \\
&= \sum_\alpha r_\alpha \pi(y_\alpha) \\
&= \sum_\alpha r_\alpha \iota(x_\alpha) \\
&:= f
\end{aligned}
$$

- Checking $(h \circ \pi)(m) = m$: seems to be hard!
- Both $\pi \circ h$ and id_F are two maps that agree on the spanning set $\{\iota(x_\alpha)\}$, so in fact they are *equal*.

Short proof:

- Free implies projective
- Universal property of projective modules: for every surjective $\pi : M \longrightarrow N$ and every $f : P \longrightarrow N$ there exists a unique lift $\tilde{f} : P \longrightarrow M$:

$$\begin{tikzcd} & P \arrow[dl, dotted, "\exists ! \tilde{f}"'] \arrow[d, "f"] \\ M \arrow[r, "\pi"] & N \end{tikzcd}$$

- Take the identity map:

$$\begin{tikzcd} & & & F \arrow[dl, "\exists ! h"'] \arrow[d, "\mathrm{id}_F"] & \\ 0 \arrow[r] & N \arrow[r] & M \arrow[r] & F \arrow[r] & 0 \end{tikzcd}$$

d

- Claim: if R is a PID and M is a finitely generated R-module, then $M \cong M_t \oplus M/M_t$ where $M_t \leq M$ is the torsion submodule.
- Claim: M/M_t is torsionfree, and since a f.g. torsion free module over a PID is free, M/M_t is free.
  - Let $m + M_t \in M/M_t$ and suppose it is torsion, we will show that is must be the zero coset.
  - Then there exists an $r \in R$ such that $r(m + M_t) = M_t$
  - Then $rm + M_t = M_t$, so $rm \in M_t$.
  - By definition of M_t, every element is torsion, so there exists some $s \in R$ such $s(rm) = 0_M$.
  - Then $(sr)m = 0_M$ which forces $m \in M_t$
  - Then $m + M_t = M_t$, so $m + M_t$ is the zero coset.
- There is a SES
$$0 \longrightarrow M_t \longrightarrow M \longrightarrow M/M_t \longrightarrow 0$$
and since M/M_t is free, by (c) this sequence splits and $M \cong M \oplus M/M_t$.

1.2.4 Spring 2012 #5

Let M be a finitely generated module over a PID R.

a. M_t be the set of torsion elements of M, and show that M_t is a submodule of M.

b. Show that M/M_t is torsion free.

c. Prove that $M \cong M_t \oplus F$ where F is a free module.

1.2.5 Spring 2017 #5

Let R be an integral domain and let M be a nonzero torsion R-module.

a. Prove that if M is finitely generated then the annihilator in R of M is nonzero.

b. Give an example of a non-finitely generated torsion R-module whose annihilator is (0), and justify your answer.

1.2.6 Fall 2019 Final #3

Let $R = k[x]$ for k a field and let M be the R-module given by

$$M = \frac{k[x]}{(x-1)^3} \oplus \frac{k[x]}{\left(x^2+1\right)^2} \oplus \frac{k[x]}{(x-1)\left(x^2+1\right)^4} \oplus \frac{k[x]}{(x+2)\left(x^2+1\right)^2}.$$

Describe the elementary divisors and invariant factors of M.

1.2.7 Fall 2019 Final #4

Let $I = (2, x)$ be an ideal in $R = \mathbb{Z}[x]$, and show that I is not a direct sum of nontrivial cyclic R-modules.

1.2.8 Fall 2019 Final #5

Let R be a PID.

- Classify irreducible R-modules up to isomorphism.
- Classify indecomposable R-modules up to isomorphism.

1.2.9 Fall 2019 Final #6

Let V be a finite-dimensional k-vector space and $T : V \longrightarrow V$ a non-invertible k-linear map. Show that there exists a k-linear map $S : V \longrightarrow V$ with $T \circ S = 0$ but $S \circ T \neq 0$.

1.2.10 Fall 2019 Final #7

Let $A \in M_n(\mathbb{C})$ with $A^2 = A$. Show that A is similar to a diagonal matrix, and exhibit an explicit diagonal matrix similar to A.

1.2.11 Fall 2019 Final #8

Exhibit the rational canonical form for - $A \in M_6(\mathbb{Q})$ with minimal polynomial $(x-1)(x^2+1)^2$. - $A \in M_{10}(\mathbb{Q})$ with minimal polynomial $(x^2+1)^2(x^3+1)$.

1.2.12 Fall 2019 Final #9

Exhibit the rational and Jordan canonical forms for the following matrix $A \in M_4(\mathbb{C})$:

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{pmatrix}.$$

1.2.13 Fall 2019 Final #10

Show that the eigenvalues of a Hermitian matrix A are real and that $A = PDP^{-1}$ where P is an invertible matrix with orthogonal columns.