# Complex Analysis Qualifying Exam Notes

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# 1 Preface

References

- Simon

# 2 Theorems

**Theorem 2.1** ***(Summation by Parts).***
Define the forward difference operator $\Delta f_k = f_{k+1} - f_k$, then

$$\sum_{k=m}^{n} f_k \Delta g_k + \sum_{k=m}^{n-1} g_{k+1} \Delta f_k = f_n g_{n+1} - f_m g_m.$$

Note: compare to $\int_a^b f\,dg + \int_a^b g\,df = f(b)g(b) - f(a)g(a)$.

**Theorem 2.2** ***(Morera's Theorem).***
If $f$ is continuous on a domain $\Omega$ and $\int_T f = 0$ for every triangle $T \subset \Omega$, then $f$ is holomorphic.

**Theorem 2.3** ***(Cauchy Integral Formula).***
Suppose $f$ is holomorphic on $\Omega$, then

$$f(z) = \frac{1}{2\pi i} \oint_{\partial\Omega}$$

and

$$\frac{\partial^n f}{\partial z^n}(z) - \frac{n!}{2\pi i} \oint_{\partial\Omega} \frac{f\xi}{(\xi - z)^{n+1}}\, d\xi.$$

**Theorem 2.4** ***(Cauchy's Inequality).***
For $z_o \in D_R(z_0) \subset \Omega$, we have

$$\left|f^{(n)}(z_0)\right| \le \frac{n!}{2\pi}\int_0^{2\pi} \frac{\|f\|_{C_R}}{R^{n+1}} R d\theta = \frac{n!\|f\|_{C_R}}{R^n}.$$

**Theorem 2.5** ***(Liouville).***
If $f$ is entire and bounded, $f$ is constant.

**Theorem 2.6** ***(Argument Principle).***
?

**Theorem 2.7** ***(Green's).***
If $\Omega \subseteq \mathbb{C}$ is bounded with $\partial\Omega$ piecewise smooth and $f, g \in C^1(\bar{\Omega})$, then

$$\int_{\partial\Omega} f\,dx + g\,dy = \iint_\Omega \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.$$

**Theorem 2.8** ***(Rouche).***
If $f, g$ are analytic on a domain $\Omega$ with finitely many zeros in $\Omega$ and $\gamma \subset \Omega$ is a closed curve surrounding each point exactly once, where $|g| < |f|$ on $\gamma$, then $f$ and $f + g$ have the same number of zeros.

**Example 2.1.**

- Take $P(z) = z^4 + 6z + 3$.
- On $|z| < 2$:
  - Set $f(z) = z^4$ and $g(z) = 6z + 3$, then $|g(z)| \le 6|z| + 3 = 15 < 16 = |f(z)|$.
  - So $P$ has 4 zeros here.
- On $|z| < 1$:
  - Set $f(z) = 6z$ and $g(z) = z^4 + 3$.
  - Check $|g(z)| \le |z|^4 + 3 = 4 < 6 = |f(z)|$.
  - So $P$ has 1 zero here.

**Example 2.2.**

- Claim: the equation $\alpha z e^z = 1$ where $|\alpha| > e$ has exactly one solution in $\mathbb{D}$.
- Set $f(z) = \alpha z$ and $g(z) = e^{-z}$.
- Estimate at $|z| = 1$ we have $|g| = \left|e^{-z}\right| = e^{-\Re(z)} \le e^1 < |\alpha| = |f(z)|$
- $f$ has one zero at $z_0 = 0$, thus so does $f + g$.

**Theorem 2.9** ***(Open Mapping).***
Any holomorphic non-constant map is an open map.

**Theorem 2.10** ***(Maximum Modulus).***
If $f$ is holomorphic and nonconstant on an open region $\Omega$, then $|f|$ can not attain a maximum on $\Omega$. If $\Omega$ is bounded and $f$ is continuous on $\overline{\Omega}$, then $\max_{\overline{\Omega}} |f|$ occurs on $\partial\Omega$.

Conversely, if $f$ attains a local maximum at $z_0 \in \Omega$, then $f$ is constant on $\Omega$.

**Theorem 2.11** ***(Casorati-Weierstrass).***
If $f$ is holomorphic on $\Omega \setminus \{z_0\}$ where $z_0$ is an essential singularity, then for every $V \subset \Omega \setminus \{z_0\}$, $f(V)$ is dense in $\mathbb{C}$.

**Theorem 2.12** ***(Cayley Transform).***
The fractional linear transformation given by $F(z) = \dfrac{i-z}{i+z}$ maps $\mathbb{D} \longrightarrow \mathbb{H}$ with inverse $G(w) = i\dfrac{1-w}{1+w}$.

**Theorem 2.13** ***(Continuation Principle).***
If $f$ is holomorphic on a bounded connected domain $\Omega$ and there exists a sequence $\{z_i\}$ with a limit point in $\Omega$ such that $f(z_i) = 0$, then $f \equiv 0$ on $\Omega$.

**Theorem 2.14** ***(Schwarz Reflection).***
If $f$ is continuous and holomorphic on $\mathbb{H}^+$ and real-valued on $\mathbb{R}$, then the extension defined by $F(z) = \overline{f(\overline{z})}$ for $z \in \mathbb{H}^-$ is a well-defined holomorphic function on $\mathbb{C}$.

Note: $\mathbb{H}^+, \mathbb{H}^-$ can be replaced with any region symmetric about a line segment $L \subseteq \mathbb{R}$.

**Theorem 2.15** ***(Schwarz Lemma).***
If $f : \mathbb{D} \longrightarrow \mathbb{D}$ is holomorphic with $f(0) = 0$, then

1. $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.
2. If there exists a $0 \neq z_0 \in \mathbb{D}$ with $f(z) = z$, then $f$ is a rotation.
3. $|f'(0)| \leq 1$ with equality $\iff$ $f$ is a rotation.

# 3 Stuff

### 3.0.1 Fundamental Theorem of Algebra: Argument Principle

- Let $P(z) = a_n z^n + \cdots + a_0$ and $g(z) = P'(z)/P(z)$, note $P$ is holomorphic
- Since $\lim_{|z| \longrightarrow \infty} P(z) = \infty$, there exist an $R > 0$ such that $P$ has no roots in $\{|z| \geq R\}$.
- Apply the argument principle:

$$N(0) = \frac{1}{2\pi i} \oint_{|\xi|=R} g(\xi)\, d\xi.$$

- Check that $\lim_{|z \longrightarrow \infty|} z g(z) = n$, so $g$ has a simple pole at $\infty$
- Then $g$ has a Laurent series $\dfrac{n}{z} + \dfrac{c_2}{z^2} + \cdots$
- Integrate term-by-term to get $N(0) = n$.

### 3.0.2 Fundamental Theorem of Algebra: Rouche's Theorem

- Let $P(z) = a_n z^n + \cdots + a_0$
- Set $f(z) = a_n z^n$ and $g(z) = P(z) - f(z) = a_{n-1} z^{n-1} + \cdots + a_0$, so $f + g = P$.
- Choose $R > \max\left(\frac{|a_{n-1}| + \cdots + |a_0|}{|a_n|}, 1\right)$, then

$$
\begin{aligned}
|g(z)| &\coloneqq |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| \\
&\leq |a_{n-1} z^{n-1}| + \cdots + |a_1 z| + |a_0| \quad \text{by the triangle inequality} \\
&= |a_{n-1}| \cdot |z^{n-1}| + \cdots + |a_1| \cdot |z| + |a_0| \\
&= |a_{n-1}| \cdot R^{n-1} + \cdots + |a_1| R + |a_0| \\
&\leq |a_{n-1}| \cdot R^{n-1} + |a_{n-2}| \cdot R^{n-1} + \cdots + |a_1| \cdot R^{n-1} + |a_0| \cdot R^{n-1} \quad \text{since } R > 1 \implies R^{a+b} \geq R^a \\
&= R^{n-1} \left(|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0|\right) \\
&\leq R^{n-1} \left(|a_n| \cdot R\right) \quad \text{by choice of } R \\
&= R^n |a_n| \\
&= |a_n z^n| \\
&\coloneqq |f(z)|
\end{aligned}
$$

- Then $a_n z^n$ has $n$ zeros in $|z| < R$, so $f + g$ also has $n$ zeros.

### 3.0.3 Fundamental Theorem of Algebra: Liouville's Theorem

- Suppose $p$ is nonconstant and has no roots, then $\frac{1}{p}$ is entire
- Write $g(z) \coloneqq \frac{p(z)}{z^n} = a_n \left(\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right)$
- Outside a disc:
  - Note $\lim_{z \longrightarrow \infty} = 0$ for the parenthesized terms, so there exists an $R$ large enough such that $|g(z)| \geq \frac{1}{2}|a_n|$
  - Then $|p(z)| \geq \frac{R^n}{2}|a_n|$ implies $\frac{1}{p}$ is bounded in $|z| > R$
- Inside a disc:
  - $p$ is continuous with no roots so $p$ is bounded below on $|z| < R$.
  - $p$ is continuous on a compact set and thus achieves a min $A$
  - Set $B = \min(A, \frac{R^n}{2}|a_n|)$, then $p \geq B$ on $|z| < R$.
- Thus $p$ is bounded below everywhere and thus $\frac{1}{p}$ is bounded above everywhere, thus bounded.
- Thus $\frac{1}{p}$ is constant, forcing $p$ to be constant.

### 3.0.4 Fundamental Theorem of Algebra: Open Mapping Theorem

- $p$ induces a continuous map $\mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$
- The continuous image of compact space is compact;
- Since the codomain is Hausdorff space, the image is closed.

- $p$ is holomorphic and non-constant, so by the Open Mapping Theorem, the image is open.
- Thus the image is clopen in $\mathbb{CP}^1$.
- The image is nonempty, since $p(1) = \sum a_i \in \mathbb{C}$
- $\mathbb{CP}^1$ is connected
- But the only nonempty clopen subset of a connected space is the entire space.
- So $p$ is surjective, and $p^{-1}(0)$ is nonempty.
- So $p$ has a root.

# 4 Appendix

$$\begin{aligned} dz &= dx + i\ dy \\ d\bar{z} &= dx - i\ dy \\ f_z &= f_x = i^{-1} f_y \\ \int_0^{2\pi} e^{i\ell x} dx &= \begin{cases} 2\pi & (\ell = 0) \\ 0 & (\ell \neq 0) \end{cases} . \end{aligned}$$

- Holomorphic: once complex differentiable in neighborhoods of every point.
- Analytic: equal to its Taylor series expansion

Collection of facts used on problem sets

## 4.1 Things to know well:

- Cauchy Integral Formula
- Estimates for derivatives, mean value theorem
- Rouché's theorem
- Casorati-Weierstrass
- The 8 types of conformal maps

## 4.2 Theorems

### 4.2.1 The Argument Principle

**Theorem 4.1** ***(Statement 1).***
For $f$ meromorphic in $\gamma^\circ$,

$$\Delta_\gamma \arg f(z) = 2\pi(Z_f - P_f).$$

### 4.2.2 Rouche

**Theorem 4.2** ***(Statement 1).***
Suppose $f = g + h$ with $g \neq 0, \infty$ on $\gamma$ with $|g| > |h|$ on $\gamma$. Then

$$\Delta_\gamma \arg(f) = \Delta_\gamma \arg(h) \quad \text{and } Z_f - P_f = Z_g - P_g.$$

## 4.3 Misc Prereq

**Standard forms of conic sections:**

- Circle: $x^2 + y^2 = r^2$
- Ellipse: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$
- Hyperbola: $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$
  - Rectangular Hyperbola: $xy = \frac{c^2}{2}$.
- Parabola: $-4ax + y^2 = 0$.

Mnemonic: Write $f(x, y) = Ax^2 + Bxy + Cy^2 + \cdots$, then consider the discriminant $\Delta = B^2 - 4AC$:

- $\Delta < 0 \iff$ ellipse
  - $\Delta < 0$ and $A = C, B = 0 \iff$ circle
- $\Delta = 0 \iff$ parabola
- $\Delta > 0 \iff$ hyperbola

**Completing the square:**

$$
\begin{aligned}
x^2 - bx &= (x - s)^2 - s^2 \quad \text{where} s = \frac{b}{2} \\
x^2 + bx &= (x + s)^2 - s^2 \quad \text{where} s = \frac{b}{2}.
\end{aligned}
$$

**Useful Properties**

- $\Re(z) = \frac{1}{2}(z + \bar{z})$ and $\Im(z) = \frac{1}{2i}(z - \bar{z})$.
- $z\bar{z} = |z|^2$
- $\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$
- $\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$.

**Useful Series**

$$
\begin{aligned}
\sum_{k=1}^{n} k &= \frac{n(n+1)}{2} \\
\sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6} \\
\sum_{k=1}^{n} k^3 &= \frac{n^2(n+1)^2}{4} \\
\log(z) &= \sum_{j=0}^{\infty} (-1)^j \frac{(z-a)^j}{j}
\end{aligned}
$$

**Cauchy-Riemann Equations**

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$
.

## 4.4 Useful Techniques

**Showing a function is constant:**

- Write $f = u + iv$ and use Cauchy-Riemann to show $u_x, u_y = 0$, etc.
- Show that $f$ is entire and bounded.

**Showing a function is zero**: Show $f$ is entire, bounded, and $\lim_{z \longrightarrow \infty} f(z) = 0$.

**Deriving Polar Cauchy-Riemann:** See walkthrough here. Take derivative along two paths, along a ray with constant angle $\theta_0$ and along a circular arc of constant radius $r_0$. Then equate real and imaginary parts. See problem set 1.

**Computing Arguments:** $\mathrm{Arg}(z/w) = \mathrm{Arg}(z) - \mathrm{Arg}(w)$.

The sum of the interior angles of an $n$-gon is $(n-2)\pi$, where each angle is $\frac{n-2}{n}\pi$.

## 4.5 Residues

If $p$ is a simple pole, $\mathrm{Res}(p, f) = \lim_{z \longrightarrow p} (z-p) f(z)$. Example: Let $f(z) = \frac{1}{1+z^2}$, then $\mathrm{Res}(i, f) = \frac{1}{2i}$.

Green's Theorem: Todo

$$\frac{\partial}{\partial z} \sum_{j=0}^{\infty} a_j z^j = \sum_{j=0}^{\infty} a_{j+1} z^j.$$

## 4.6 Pithy Statements

- Little Picard: $f$ misses at most one point and is a homeomorphism onto its image.
- Baire's Theorem: The intersection of open dense sets is open.
- Casorati-Weierstrass: The image of a disc punctured at an essential singularity is dense in $\mathbb{C}$.
- Open Mapping: Holomorphic functions preserve open sets.
- Argument Principle: The logarithmic derivative measures the difference of zeros and poles.
- Liouville: Bounded entire functions are constant.
- Maximum Modulus: Holomorphic functions take extrema only on boundaries.
- Cauchy Inequalities: The $n$th Taylor coefficient is at most $\sup_{|z|=R} |f|/R^n$.
- Cauchy's Theorem: Integrals of holomorphic functions vanish.
- Morera: Integrals vanishing along every rectangle implies holomorphic.
- Schwarz Reflection: ???
- Identity Theorem: Two functions agreeing on a set with a limit point are equal on a domain.

- The ring of holomorphic functions on a domain in $\mathbb{C}$ has no zero divisors (by the identity principle).

## 4.7 Precise Refinements

**Cauchy Inequality:** Given $z_0 \in \Omega$, pick the largest disc $D_R(z_0) \subset \Omega$ and let $C_R = \partial D_R$. Using the integral formula, defining $\|f\|_{C_R} = \max_{|z-z_0|=R} |f(z)|$

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{\|f\|_{C_R}}{R^{n+1}} R \, d\theta = \frac{n!\|f\|_{C_R}}{R^n}.$$

Basics

- Show that $\displaystyle\frac{1}{z}\sum_{k=1}^{\infty} \frac{z^k}{k}$ converges on $S^1 \setminus \{1\}$ using summation by parts.
- Show that any power series is continuous on its domain of convergence.
- Show that a uniform limit of continuous functions is continuous.

??

- Show that if $f$ is holomorphic on $\mathbb{D}$ then $f$ has a power series expansion that converges uniformly on every compact $K \subset \mathbb{D}$.
- Show that any holomorphic function $f$ can be uniformly approximated by polynomials.
- Show that if $f$ is holomorphic on a connected region $\Omega$ and $f' \equiv 0$ on $\Omega$, then $f$ is constant on $\Omega$.
- Show that if $|f| = 0$ on $\partial\Omega$ then either $f$ is constant or $f$ has a zero in $\Omega$.
- Show that if $\{f_n\}$ is a sequence of holomorphic functions converging uniformly to a function $f$ on every compact subset of $\Omega$, then $f$ is holomorphic on $\Omega$ and $\{f_n'\}$ converges uniformly to $f'$ on every such compact subset.
- Show that if each $f_n$ is holomorphic on $\Omega$ and $F \coloneqq \sum f_n$ converges uniformly on every compact subset of $\Omega$, then $F$ is holomorphic.
- Show that if $f$ is once complex differentiable at each point of $\Omega$, then $f$ is holomorphic.