

# Topology Qualifying Exam Notes

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## 0.1 Conventions

- $\pi_0(X)$  is the set of path components of  $X$ , and I write  $\pi_0(X) = \mathbb{Z}$  if  $X$  is path-connected (although it is not a group). Similarly,  $H_0(X)$  is a free abelian group on the set of path components of  $X$ .
- Lists start at entry 1, since all spaces are connected here and thus  $\pi_0 = H_0 = \mathbb{Z}$ . That is,
  - $\pi_*(X) = [\pi_1(X), \pi_2(X), \pi_3(X), \dots]$
  - $H_*(X) = [H_1(X), H_2(X), H_3(X), \dots]$
- For a finite index set  $I$ , it is the case that  $\prod_I G = \bigoplus_I G$  in **Grp**, i.e. the finite direct product and finite direct sum coincide. Otherwise, if  $I$  is infinite, the direct sum requires cofinitely many zero entries (i.e. finitely many nonzero entries), so here we always use  $\prod$ .

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In other words, there is an injective map

$$\bigoplus_I G \hookrightarrow \prod_I G$$

which is an isomorphism when  $|I| < \infty$

- $\mathbb{Z}^n := \prod_{i=1}^n \mathbb{Z} = \mathbb{Z} \times \mathbb{Z} \times \dots \mathbb{Z}$  is the free abelian group of rank  $n$ .
  - $x \in \mathbb{Z}^n = \langle a_1, \dots, a_n \rangle \implies x = \sum_{i=1}^n c_i a_i$  for some  $c_i \in \mathbb{Z}$ , i.e.  $a_i$  form a basis.
  - Example:  $x = 2a_1 + 4a_2 + a_1 - a_2 = 3a_1 + 3a_2$ .
- $\mathbb{Z}^{*n} := \mathbb{Z} * \mathbb{Z} * \dots \mathbb{Z}$  is the free product of  $n$  free abelian groups, i.e. a free (nonabelian) group on  $n$  generators.
  - $x \in \mathbb{Z}^{*n} = \langle a_1, \dots, a_n \rangle$  implies that  $x$  is a finite word in the noncommuting symbols  $a_i^k$  for  $k \in \mathbb{Z}$ .
  - Example:  $x = a_1^2 a_2^4 a_1 a_2^{-2}$
- $K(G, n)$  is an Eilenberg-MacLane space, the homotopy-unique space satisfying

$$\pi_k(K(G, n)) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

- $K(\mathbb{Z}, 1) = S^1$
- $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$
- $K(\mathbb{Z}_2, 1) = \mathbb{RP}^\infty$
- $M(G, n)$  is a Moore space, the homotopy-unique space satisfying
 
$$H_k(M(G, n); G) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$
  - $M(\mathbb{Z}, n) = S^n$
  - $M(\mathbb{Z}_2, 1) = \mathbb{RP}^2$
  - $M(\mathbb{Z}_p, n)$  is made by attaching  $e^{n+1}$  to  $S^n$  via a degree  $p$  map.

- $T^n = \prod_n S^1$  is the  $n$ -torus
- $D(k, X)$  is the space  $X$  with  $k \in \mathbb{N}$  distinct points deleted, i.e. the punctured space  $X - \{x_1, x_2, \dots, x_k\}$  where each  $x_i \in X$ .
- $\mathbb{RP}^n = S^n / S^0 = S^n / \mathbb{Z}_2$
- $\mathbb{CP}^n = S^{2n+1} / S^1$
- $B^n = \{\mathbf{v} \in \mathbb{R}^n \mid \|\mathbf{v}\| \leq 1\} \subset \mathbb{R}^n$
- $S^{n-1} = \partial B^n = \{\mathbf{v} \in \mathbb{R}^n \mid \|\mathbf{v}\| = 1\} \subset \mathbb{R}^n$

## 1 Table of Homotopy and Homology Structures

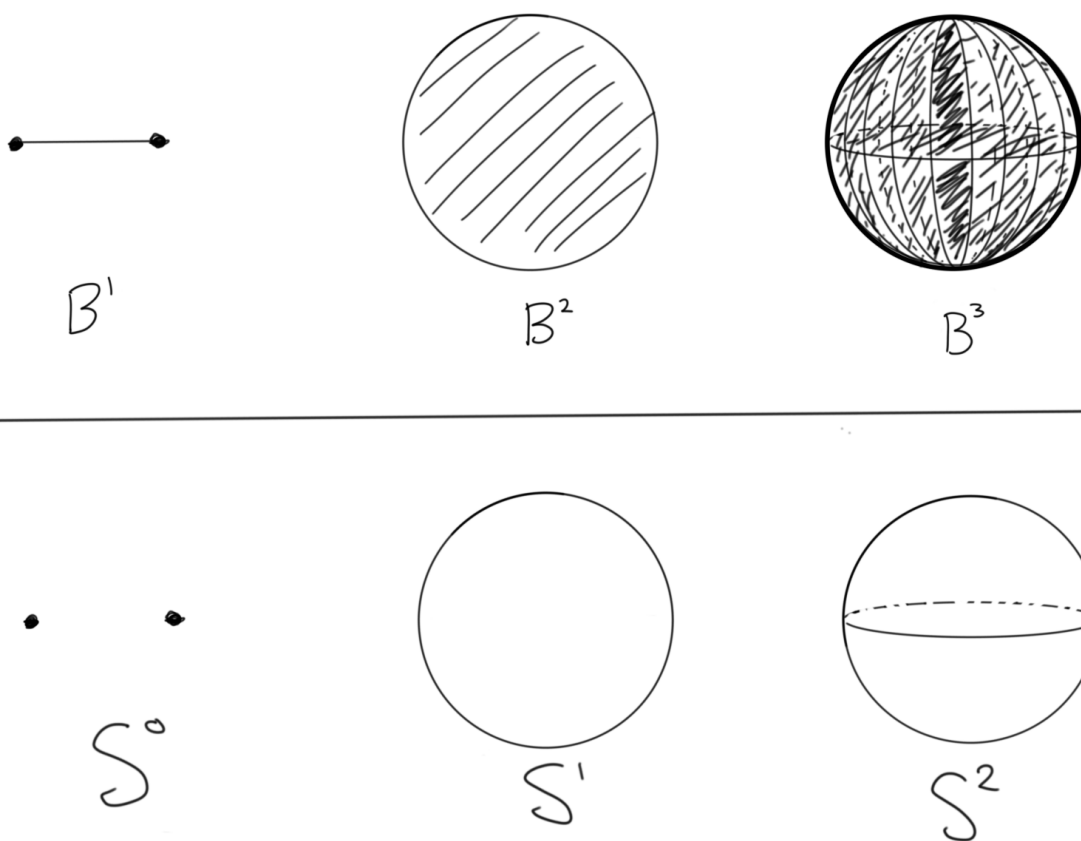


Figure 1: sphere ball correct

$X$	$\pi_*(X)$	$H_*(X)$	CW Structure	$H^*(X)$
$\mathbb{R}^1$	0	0	$\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot x$	0
$\mathbb{R}^n$	0	0	$(\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot x)^n$	0
$D(k, \mathbb{R}^n)$	$\pi_* \bigvee^k S^1$	$\bigoplus^k H_* M(\mathbb{Z}, 1)$	$1 + kx$	?
$B^n$	$\pi_*(\mathbb{R}^n)$	$H_*(\mathbb{R}^n)$	$1 + x^n + x^{n+1}$	0
$S^n$	$[0 \dots, \mathbb{Z}, ? \dots]$	$H_* M(\mathbb{Z}, n)$	$1 + x^n$	$\mathbb{Z}[x]/(x^2)$
$D(k, S^n)$	$\pi_* \bigvee^{k-1} S^1$	$\bigoplus^{k-1} H_* M(\mathbb{Z}, 1)$	$1 + (k-1)x^1$	?
$T^2$	$\pi_* S^1 \times \pi_* S^1$	$(H_* M(\mathbb{Z}, 1))^2 \times H_* M(\mathbb{Z}, 2)$	$1 + 2x + x^2$	$\Lambda(1x_1, 1x_2)$
$T^n$	$\prod \pi_* S^1$	$\prod_{i=1}^n (H_* M(\mathbb{Z}, i))^{\binom{n}{i}}$	$(1+x)^n$	$\Lambda(1x_1, 1x_2, \dots, 1x_n)$
$D(k, T^n)$	$[0, 0, 0, 0, \dots]?$	$[0, 0, 0, 0, \dots]?$	$1 + x$	?
$S^1 \vee S^1$	$\pi_* S^1 * \pi_* S^1$	$(H_* M(\mathbb{Z}, 1))^2$	$1 + 2x$	?
$\bigvee_n S^1$	$*^n \pi_* S^1$	$\prod H_* M(\mathbb{Z}, 1)$	$1 + x$	?
$\mathbb{RP}^1$	$\pi_* S^1$	$H_* M(\mathbb{Z}, 1)$	$1 + x$	${}_0\mathbb{Z} \times {}_1\mathbb{Z}$
$\mathbb{RP}^2$	$\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) + \pi_* S^2$	$H_* M(\mathbb{Z}_2, 1)$	$1 + x + x^2$	${}_0\mathbb{Z} \times {}_2\mathbb{Z}/2\mathbb{Z}$
$\mathbb{RP}^3$	$\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) + \pi_* S^3$	$H_* M(\mathbb{Z}_2, 1) + H_* M(\mathbb{Z}, 3)$	$1 + x + x^2 + x^3$	${}_0\mathbb{Z} \times {}_2\mathbb{Z}/2\mathbb{Z} \times {}_3\mathbb{Z}$
$\mathbb{RP}^4$	$\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) + \pi_* S^4$	$H_* M(\mathbb{Z}_2, 1) + H_* M(\mathbb{Z}_2, 3)$	$1 + x + x^2 + x^3 + x^4$	${}_0\mathbb{Z} \times ({}_2\mathbb{Z}/2\mathbb{Z})^2$
$\mathbb{RP}^n, n \geq 4$ even	$\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) + \pi_* S^n$	$\prod_{\text{odd } i < n} H_* M(\mathbb{Z}_2, i)$	$\sum_{i=1}^n x^i$	${}_0\mathbb{Z} \times \prod_{i=1}^{n/2} {}_2\mathbb{Z}/2\mathbb{Z}$
$\mathbb{RP}^n, n \geq 4$ odd	$\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) + \pi_* S^n$	$\prod_{\text{odd } i \leq n-2} H_* M(\mathbb{Z}_2, i) \times H_* S^n$	$\sum_{i=1}^n x^i$	$H^*(\mathbb{RP}^{n-1}) \times {}_n\mathbb{Z}$
$\mathbb{CP}^1$	$\pi_* K(\mathbb{Z}, 2) + \pi_* S^3$	$H_* S^2$	$x^0 + x^2$	$\mathbb{Z}[2x]/(2x^2)$
$\mathbb{CP}^2$	$\pi_* K(\mathbb{Z}, 2) + \pi_* S^5$	$H_* S^2 \times H_* S^4$	$x^0 + x^2 + x^4$	$\mathbb{Z}[2x]/(2x^3)$
$\mathbb{CP}^n, n \geq 2$	$\pi_* K(\mathbb{Z}, 2) + \pi_* S^{2n+1}$	$\prod_{i=1}^n H_* S^{2i}$	$\sum_{i=1}^n x^{2i}$	$\mathbb{Z}[2x]/(2x^{n+1})$
Mobius Band	$\pi_* S^1$	$H_* S^1$	$1 + x$	?
Klein Bottle	$K(\mathbb{Z} \rtimes_{-1} \mathbb{Z}, 1)$	$H_* S^1 \times H_* \mathbb{RP}^\infty$	$1 + 2x + x^2$	?

Facts used to compute the above table:

- $\mathbb{R}^n$  is a contractible space, and so  $[S^m, \mathbb{R}^n] = 0$  for all  $n, m$  which makes its homotopy groups all zero.
- $D(k, \mathbb{R}^n) = \mathbb{R}^n - \{x_1 \dots x_k\} \simeq \bigvee_{i=1}^k S^1$  by a deformation retract.
- $S^n \cong B^n / \partial B^n$  and employs an attaching map

$$\begin{aligned} \phi : (D^n, \partial D^n) &\longrightarrow S^n \\ (D^n, \partial D^n) &\mapsto (e^n, e^0). \end{aligned}$$

- $B^n \simeq \mathbb{R}^n$  by normalizing vectors.
- Use the inclusion  $S^n \hookrightarrow B^{n+1}$  as the attaching map.
- $\mathbb{CP}^1 \cong S^2$ .
- $\mathbb{RP}^1 \cong S^1$ .
- Use  $\pi_1 \prod = \prod \pi_1$  and the universal cover  $\mathbb{R}^1 \twoheadrightarrow S^1$  to yield the cover  $\mathbb{R}^n \twoheadrightarrow T^n$ .

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Use the fact that  $D(1, S^n) \cong \mathbb{R}^n$  and thus  $D(k, S^n) \cong D(k-1, \mathbb{R}^n) \cong \bigvee^{k-1} S^1$  # Euler Characteristics  
- Only surfaces with positive  $\chi$ : -  $\chi S^2 = 2$  -  $\chi \mathbb{RP}^2 = 1$  -  $\chi B^2 = 1$  - Manifolds with zero  $\chi$  -  $T^2, K, M, S^1 \times I$  - Manifolds with negative  $\chi$  -  $\Sigma_{g \geq 2}$  by  $\chi(X) = 2 - 2g$ .

## 2 Useful Facts and Techniques

- Fundamental group:
  - Van Kampen
- Homotopy Groups
  - Hurewicz map
- Homology
  - Mayer-Vietoris
    - \*  $(X = A \bigcup B) \mapsto (\bigcap, \oplus, \bigcup)$  in homology
  - LES of a pair
    - \*  $(A \hookrightarrow X) \mapsto (A, X, X/A)$
  - Excision
- $\pi_{i \geq 2}(X)$  is always abelian.
- The ranks of  $\pi_0$  and  $H_0$  are the number of path components, and  $\pi_0(X) = \mathbb{Z}$  iff  $X$  is simply connected.
  - $X$  simply connected implies  $\pi_k(X) \cong H_k(X)$  up to and including the first nonvanishing  $H_k$
  - $H_1(X) = \pi_1 X / [\pi_1 X, \pi_1 X]$ , the abelianization.
- General mantra: homotopy plays nicely with products, homology with wedge products.<sup>1</sup>

In general, homotopy groups behave nicely under homotopy pull-backs (e.g., fibrations and products), but not homotopy push-outs (e.g., cofibrations and wedges). Homology is the opposite.

- $\pi_k \prod X = \prod \pi_k X$  by LES.<sup>2</sup>
- $H_k \prod X \neq \prod H_k X$  due to torsion.

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<sup>1</sup>More generally, in **Top**, we can look at  $A \leftarrow \{\text{pt}\} \rightarrow B$  – then  $A \times B$  is the pullback and  $A \vee B$  is the pushout. In this case, homology  $h : \mathbf{Top} \rightarrow \mathbf{Grp}$  takes pushouts to pullbacks but doesn't behave well with pullbacks. Similarly, while  $\pi$  takes pullbacks to pullbacks, it doesn't behave nicely with pushouts.

<sup>2</sup>This follows because  $X \times Y \rightarrow X$  is a fiber bundle, so use LES in homotopy and the fact that  $\pi_{i \geq 2} \in \mathbf{Ab}$ .

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- Nice case:  $H_k(A \times B) = \prod_{i+j=k} H_i A \otimes H_j B$  by Kunneth when all groups are torsion-free.<sup>3</sup>
  - $H_k \bigvee X = \prod H_k X$  by Mayer-Vietoris.<sup>4</sup>
  - $\pi_k \bigvee X \neq \prod \pi_k X$  (counterexample:  $S^1 \vee S^2$ )
    - Nice case:  $\pi_1 \bigvee X = * \pi_1 X$  by Van Kampen.
  - $\pi_i(\widehat{X}) \cong \pi_i(X)$  for  $i \geq 2$  whenever  $\widehat{X} \rightarrow X$  is a universal cover.
  - Groups and Group Actions
    - $\pi_0(G) = G$  for  $G$  a discrete topological group.
    - $\pi_k(G/H) = \pi_k(G)$  if  $\pi_k(H) = \pi_{k-1}(H) = 0$ .
    - $\pi_1(X/G) = \pi_0(G)$  when  $G$  acts freely/transitively on  $X$ .
  - Manifolds
    - $H^n(M^n) = \mathbb{Z}$  if  $M^n$  is orientable and zero if  $M^n$  is nonorientable.
    - Poincare Duality:  $H_i M^n \cong H^{n-i} M^n$  iff  $M^n$  is closed and orientable.

### 3 Other Interesting Things To Consider

- The “generalized uniform bouquet”?  $\mathcal{B}^n(m) = \bigvee_{i=1}^n S^m$
- Lie Groups
  - The real general linear group,  $GL_n(\mathbb{R})$ 
    - \* The real special linear group  $SL_n(\mathbb{R})$
    - \* The real orthogonal group,  $O_n(\mathbb{R})$ 
      - The real special orthogonal group,  $SO_n(\mathbb{R})$
    - \* The real unitary group,  $U_n(\mathbb{R})$ 
      - The real special unitary group,  $SU_n(\mathbb{R})$
    - \* The real symplectic group  $Sp(n)$
- “Geometric” Stuff
  - Affine  $n$ -space over a field  $\mathbb{A}^n(k) = k^n \rtimes GL_n(k)$
  - The projective space  $\mathbb{P}^n(k)$ 
    - \* The projective linear group over a ring  $R$ ,  $PGL_n(R)$
    - \* The projective special linear group over a ring  $R$ ,  $PSL_n(R)$

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<sup>3</sup>The generalization of Kunneth is as follows: write  $\mathcal{P}(n, k)$  be the set of partitions of  $n$  into  $k$  parts, i.e.  $\curvearrowright \in \mathcal{P}(n, k) \implies \curvearrowright = (x_1, x_2, \dots, x_k)$  where  $\sum x_i = n$ . Then

$$H_n\left(\prod_{j=1}^k X_j\right) = \bigoplus_{\curvearrowright \in \mathcal{P}(n, k)} \bigotimes_{i=1}^k H_{x_i}(X_i).$$

<sup>4</sup> $\bigvee$  is the coproduct in the category  $\mathbf{Top}_0$  of pointed topological spaces, and alternatively,  $X \vee Y$  is the pushout in  $\mathbf{Top}$  of  $X \leftarrow \{\text{pt}\} \rightarrow Y$

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- \* The modular groups  $PSL_n(\mathbb{Z})$ 
    - Specifically  $PSL_2(\mathbb{Z})$
  - The real Grassmannian,  $Gr(n, k, \mathbb{R})$ , i.e. the set of  $k$  dimensional subspaces of  $\mathbb{R}^n$
  - The Stiefel manifold  $V_n(k)$
  - Possible modifications to a space  $X$ :
    - Remove  $k$  points by taking  $D(k, X)$
    - Remove a line segment
    - Remove an entire line/axis
    - Remove a hole
    - Quotient by a group action (e.g. antipodal map, or rotation)
    - Remove a knot
    - Take complement in ambient space
  - Assorted info about other Lie Groups:
  - $O_n, U_n, SO_n, SU_n, Sp_n$
  - $\pi_k(U_n) = \mathbb{Z} \cdot \mathbb{1} [k \text{ odd}]$ 
    - $\pi_1(U_n) = 1$
  - $\pi_k(SU_n) = \mathbb{Z} \cdot \mathbb{1} [k \text{ odd}]$ 
    - $\pi_1(SU_n) = 0$
  - $\pi_k(U_n) = \mathbb{Z}_2 \cdot \mathbb{1} [k = 0, 1 \pmod 8] + \mathbb{Z} \cdot \mathbb{1} [k = 3, 7 \pmod 8]$
  - $\pi_k(Sp_n) = \mathbb{Z}_2 \cdot \mathbb{1} [k = 4, 5 \pmod 8] + \mathbb{Z} \cdot \mathbb{1} [k = 3, 7 \pmod 8]$

## 4 Spheres

- $\pi_i(S^n) = 0$  for  $i < n$ ,  $\pi_n(S^n) = \mathbb{Z}$ 
  - Not necessarily true that  $\pi_i(S^n) = 0$  when  $i > n$ !!!
    - \* E.g.  $\pi_3(S^2) = \mathbb{Z}$  by Hopf fibration
- $H_i(S^n) = \mathbb{1} [i \in \{0, n\}]$
- $H_n(\bigvee_i X_i) \cong \prod_i H_n(X_i)$  for “good pairs”
  - Corollary:  $H_n(\bigvee_k S^n) = \mathbb{Z}^k$
- $S^n/S^k \simeq S^n \vee \Sigma S^k$ 
  - $\Sigma S^n = S^{n+1}$
- $S^n$  has the CW complex structure of  $2$   $k$ -cells for each  $0 \leq k \leq n$ .

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## 5 Definitions

- Topology: Closed under arbitrary unions and finite intersections.
- Basis: A subset  $\{B_i\}$  is a basis iff
  - $x \in X \implies x \in B_i$  for some  $i$ .
  - $x \in B_i \cap B_j \implies x \in B_k \subset B_i \cap B_j$ .
  - Topology generated by this basis:  $x \in N_x \implies x \in B_i \subset N_x$  for some  $i$ .
- Dense: A subset  $Q \subset X$  is dense iff  $y \in N_y \subset X \implies N_y \cap Q \neq \emptyset$  iff  $\bar{Q} = X$ .
- Neighborhood: A neighborhood of a point  $x$  is any open set containing  $x$ .
- Hausdorff
- Second Countable: admits a countable basis.
- Closed (several characterizations)
- Closure in a subspace:  $Y \subset X \implies \text{cl}_Y(A) := \text{cl}_X(A) \cap Y$ .
- Bounded
- Compact: A topological space  $(X, \tau)$  is **compact** if every open cover has a *finite* subcover.  
 That is, if  $\{U_j \mid j \in J\} \subset \tau$  is a collection of open sets such that  $X \subseteq \bigcup_{j \in J} U_j$ , then there exists a *finite* subset  $J' \subset J$  such that  $X \subseteq \bigcup_{j \in J'} U_j$ .
- Locally compact For every  $x \in X$ , there exists a  $K_x \ni x$  such that  $K_x$  is compact.
- Connected: There does not exist a disconnecting set  $X = A \amalg B$  such that  $\emptyset \neq A, B \subsetneq X$ , i.e.  $X$  is the union of two proper disjoint nonempty sets.  
 Equivalently,  $X$  contains no proper nonempty clopen sets.
  - Additional condition for a subspace  $Y \subset X$ :  $\text{cl}_Y(A) \cap V = A \cap \text{cl}_Y(B) = \emptyset$ .
- Locally connected: A space is locally connected at a point  $x$  iff  $\forall N_x \ni x$ , there exists a  $U \subset N_x$  containing  $x$  that is connected.
- Retract: A subspace  $A \subset X$  is a *retract* of  $X$  iff there exists a continuous map  $f : X \longrightarrow A$  such that  $f|_A = \text{id}_A$ . Equivalently it is a *left* inverse to the inclusion.
- Uniform Continuity: For  $f : (X, d_x) \longrightarrow (Y, d_Y)$  metric spaces,
 
$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$
- Lebesgue number: For  $(X, d)$  a compact metric space and  $\{U_\alpha\} \rightrightarrows X$ , there exist  $\delta_L > 0$  such that
 
$$A \subset X, \text{diam}(A) < \delta_L \implies A \subseteq U_\alpha \text{ for some } \alpha.$$
- Paracompact



- 
- Components: Set  $x \sim y$  iff there exists a connected set  $U \ni x, y$  and take equivalence classes.
  - Path Components: Set  $x \sim y$  iff there exists a path-connected set  $U \ni x, y$  and take equivalence classes.
  - Separable: Contains a countable dense subset.
  - Limit Point: For  $A \subset X$ ,  $x$  is a limit point of  $A$  if every punctured neighborhood  $P_x$  of  $x$  satisfies  $P_x \cap A \neq \emptyset$ , i.e. every neighborhood of  $x$  intersects  $A$  in some point other than  $x$  itself.

Equivalently,  $x$  is a limit point of  $A$  iff  $x \in \text{cl}_X(A \setminus \{x\})$ .

## 6 Examples

### 6.1 Common Spaces and Operations

Point-Set:

- Finite discrete sets with the discrete topology
- Subspaces of  $\mathbb{R}$ :  $(a, b)$ ,  $(a, b]$ ,  $(a, \infty)$ , etc.  
 $-\ \{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{Z}^{\geq 1} \right\}$
- $\mathbb{Q}$
- The topologist's sine curve
- One-point compactifications
- $\mathbb{R}^\omega$
- Hawaiian earring
- Cantor set

Non-Hausdorff spaces:

- The cofinite topology on any infinite set.
- $\mathbb{R}/\mathbb{Q}$
- The line with two origins.

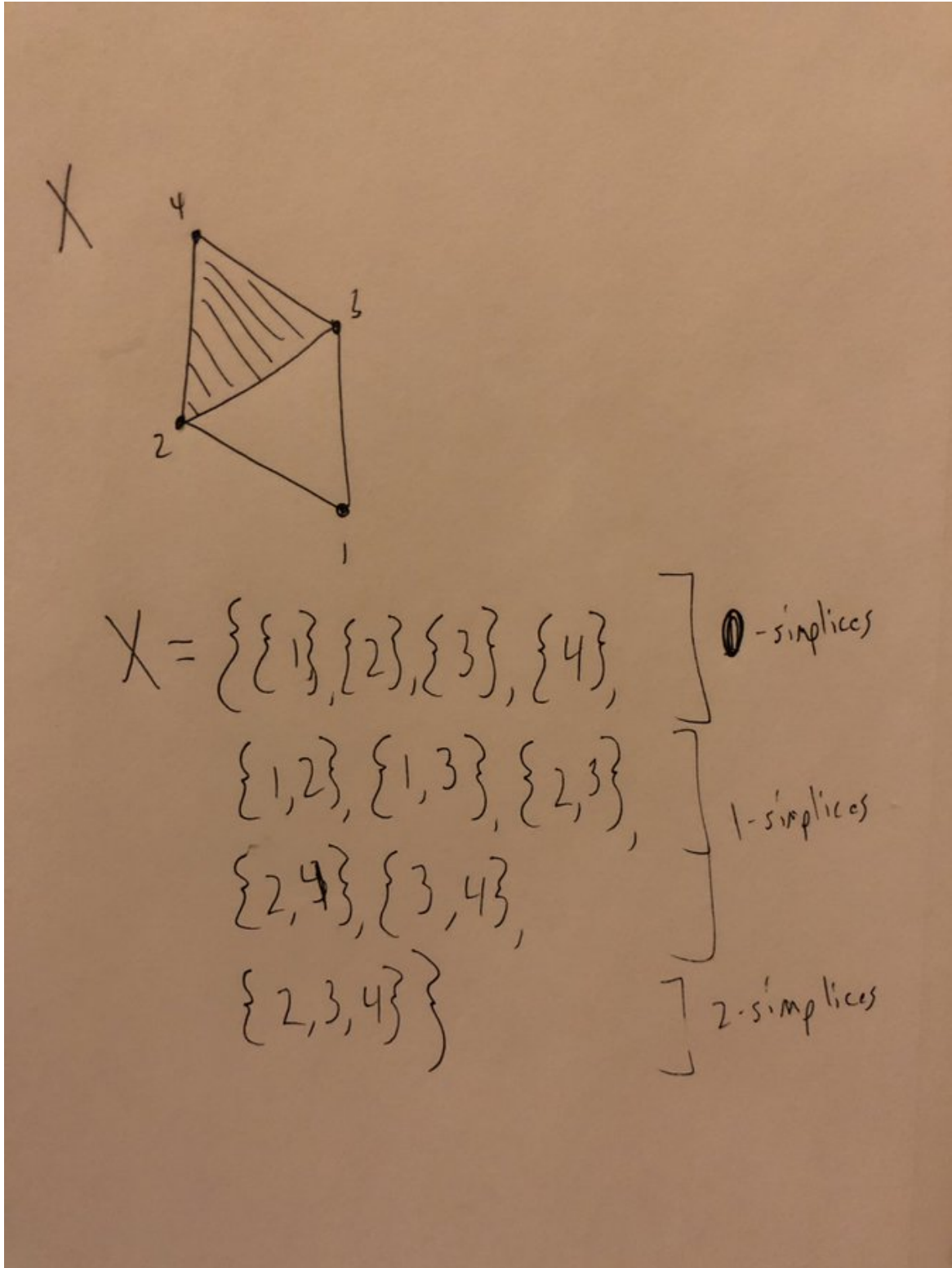
General Spaces:

$$S^n, \mathbb{D}^n, T^n, \mathbb{RP}^n, \mathbb{CP}^n, \mathbb{M}, \mathbb{K}, \Sigma_g, \mathbb{RP}^\infty, \mathbb{CP}^\infty.$$

“Constructed” Spaces

- Knot complements in  $S^3$
- Covering spaces (hyperbolic geometry)
- Lens spaces
- Matrix groups
- Prism spaces
- Pair of pants
- Seifert surfaces

- Surgery
- Simplicial Complexes
  - Nice minimal example:



### Exotic/Pathological Spaces

- $\mathbb{H}\mathbb{P}^n$
- Dunce Cap
- Horned sphere

### Operations

- Cartesian product  $A \times B$
- Wedge product  $A \vee B$
- Connect Sum  $A \# B$
- Quotienting  $A/B$
- Puncturing  $A \setminus \{a_i\}$
- Smash product
- Join
- Cones
- Suspension
- Loop space
- Identifying a finite number of points

## 6.2 Alternative Topologies

- Discrete
- Cofinite
- Discrete and Indiscrete
- Uniform

The cofinite topology:

- Non-Hausdorff
- Compact

The discrete topology:

- Discrete iff points are open
- Always Hausdorff
- Compact iff finite
- Totally disconnected
- If the domain, every map is continuous

The indiscrete topology:

- Only open sets are  $\emptyset, X$
- Non-Hausdorff
- If the codomain, every map is continuous
- Compact

## 7 Theorems

### 7.1 Point-Set

- Closed subsets of Hausdorff spaces are compact? (check)

- Cantor's intersection theorem?
- Tube lemma
- Properties pushed forward through continuous maps:
  - Compactness?
  - Connectedness (when surjective)
  - Separability
  - Density **only when**  $f$  is surjective
  - **Not** openness
  - **Not** closedness
- A retract of a Hausdorff/connected/compact space is closed/connected/compact respectively.

**Proposition 7.1.**

A continuous function on a compact set is uniformly continuous.

*Proof.*

Take  $\{B_{\frac{\varepsilon}{2}}(y) \mid y \in Y\} \Rightarrow Y$ , pull back to an open cover of  $X$ , has Lebesgue number  $\delta_L > 0$ , then  $x' \in B_{\delta_L}(x) \Rightarrow f(x), f(x') \in B_{\frac{\varepsilon}{2}}(y)$  for some  $y$ . ■

- Lipschitz continuity implies uniform continuity (take  $\delta = \varepsilon/C$ )
  - Counterexample to converse:  $f(x) = \sqrt{x}$  on  $[0, 1]$  has unbounded derivative.
- Extreme Value Theorem: for  $f : X \rightarrow Y$  continuous with  $X$  compact and  $Y$  ordered in the order topology, there exist points  $c, d \in X$  such that  $f(x) \in [f(c), f(d)]$  for every  $x$ .

**Theorem 7.2.**

Points are closed in  $T_1$  spaces.

**Theorem 7.3.**

A metric space  $X$  is sequentially compact iff it is complete and totally bounded.

**Theorem 7.4.**

A metric space is totally bounded iff every sequence has a Cauchy subsequence.

**Theorem 7.5.**

A metric space is compact iff it is complete and totally bounded.

**Theorem 7.6 (Baire).**

If  $X$  is a complete metric space, then the intersection of countably many dense open sets is dense in  $X$ .

**Theorem 7.7.**

A continuous bijective open map is a homeomorphism.

**Theorem 7.8.**

A closed subset  $A$  of a compact set  $B$  is compact.

*Proof.*

- Let  $\{A_i\} \rightrightarrows A$  be a covering of  $A$  by sets open in  $A$ .
- Each  $A_i = B_i \cap A$  for some  $B_i$  open in  $B$  (definition of subspace topology)
- Define  $V = \{B_i\}$ , then  $V \rightrightarrows A$  is an open cover.
- Since  $A$  is closed,  $W := B \setminus A$  is open
- Then  $V \cup W$  is an open cover of  $B$ , and has a finite subcover  $\{V_i\}$
- Then  $\{V_i \cap A\}$  is a finite open cover of  $A$ .

■

**Theorem 7.9.**

The continuous image of a compact set is compact.

**Theorem 7.10.**

A closed subset of a Hausdorff space is compact.

**7.2 Algebraic**

Todo: Merge the two van Kampen theorems.

**Theorem 7.11 (Van Kampen).**

The pushout is the northwest colimit of the following diagram

$$\begin{array}{ccc} A \amalg_Z B & \longleftarrow & A \\ & & \uparrow \iota_A \\ & & B \\ & \longleftarrow \iota_B & Z \end{array}$$

For groups, the pushout is given by the amalgamated free product: if  $A = \langle G_A \mid R_A \rangle$ ,  $B = \langle G_B \mid R_B \rangle$ , then  $A *_Z B = \langle G_A, G_B \mid R_A, R_B, T \rangle$  where  $T$  is a set of relations given by  $T = \{ \iota_A(z) \iota_B(z)^{-1} \mid z \in Z \}$ .

Example:  $A = \mathbb{Z}/4\mathbb{Z} = \langle x \mid x^4 \rangle$ ,  $B = \mathbb{Z}/6\mathbb{Z} = \langle y \mid y^6 \rangle$ ,  $Z = \mathbb{Z}/2\mathbb{Z} = \langle z \mid z^2 \rangle$ . Then we can identify  $Z$  as a subgroup of  $A, B$  using  $\iota_A(z) = x^2$  and  $\iota_B(z) = y^3$ . So

$$A *_Z B = \langle x, y \mid x^4, y^6, x^2 y^{-3} \rangle$$

Suppose  $X = U_1 \cup U_2$  such that  $U_1 \cap U_2 \neq \emptyset$  is path connected. Then taking  $x_0 \in U := U_1 \cap U_2$  yields a pushout of fundamental groups

$$\pi_1(X; x_0) = \pi_1(U_1; x_0) *_{\pi_1(U; x_0)} \pi_1(U_2; x_0).$$

**Theorem 7.12 (Van Kampen).**

If  $X = U \bigcup V$  where  $U, V, U \cap V$  are all path-connected then

$$\pi_1(X) = \pi_1 U *_{\pi_1(U \cap V)} \pi_1 V,$$

where the amalgamated product can be computed as follows: If we have presentations

$$\begin{aligned}\pi_1(U, w) &= \langle u_1, \dots, u_k \mid \alpha_1, \dots, \alpha_l \rangle \\ \pi_1(V, w) &= \langle v_1, \dots, v_m \mid \beta_1, \dots, \beta_n \rangle \\ \pi_1(U \cap V, w) &= \langle w_1, \dots, w_p \mid \gamma_1, \dots, \gamma_q \rangle\end{aligned}$$

then

$$\begin{aligned}\pi_1(X, w) &= \langle u_1, \dots, u_k, v_1, \dots, v_m \rangle \\ &\quad \text{mod } \langle \alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_n, I(w_1)J(w_1)^{-1}, \dots, I(w_p)J(w_p)^{-1} \rangle \\ &= \frac{\pi_1(U) * \pi_1(V)}{\langle \{I(w_i)J(w_i)^{-1} \mid 1 \leq i \leq p\} \rangle}\end{aligned}$$

where

$$\begin{aligned}I &: \pi_1(U \cap V, w) \rightarrow \pi_1(U, w) \\ J &: \pi_1(U \cap V, w) \rightarrow \pi_1(V, w).\end{aligned}$$