Algebra Qualifying Exam Review

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1 Preface

Some fun resources:

- The Line with Two Origins

1.1 Notation

- $A \times B, \prod X_j$ are direct products.
- $A \oplus B, \bigoplus_j X_j$ are direct sums, the subset of $A \times B$ where only finitely many terms are nonzero.
  - $\mathbb{Z}^n$ denotes the direct sum of n copies of the group $\mathbb{Z}$.
  - Note that $A \oplus B \hookrightarrow A \times B$.
- $A * B, *_j X_J$ are free products, $F_n \coloneqq \mathbb{Z}^{*n}$ is the free group on n generators.
  - Note that the abelianization yields $(*_j X_j) = \bigoplus_j X_j$.

2 Summary and Topics: Point-Set Topology

- Connectedness
- Compactness
- Hausdorff Spaces
- Path-Connectedness

3 Definitions: Point-Set Topology

3.1 Basics

Definition (Topology) Closed under arbitrary unions and finite intersections.

Definition (Neighborhood) A neighborhood of a point x is *any* open set containing x.

Definition (Limit Point) For $A \subset X$, x is a limit point of A if every punctured neighborhood P_x of x satisfies $P_x \cap A \neq \emptyset$, i.e. every neighborhood of x intersects A in some point other than x itself.

Equivalently, x is a limit point of A iff $x \in \mathrm{cl}_X(A \setminus \{x\})$.

Definition (Closed) There are several characterizations of a closed set:

- Closure of a subspace: $Y \subset X \implies \mathrm{cl}_Y(A) \coloneqq \mathrm{cl}_X(A) \cap Y$.
- A set is closed iff it contains all of its limit points.

Definition (Basis) For X a space and $\mathcal{B}$ a collection of subsets, $\mathcal{B}$ is a *basis* for (X, τ_X) iff every open $U \in \tau_X$ is a union of elements in $\mathcal{B}$.

Definition (Topology Generated by a Basis) For X an arbitrary set, a collection of subsets $\mathcal{B}$ is a *basis for* X iff $\mathcal{B}$ is closed under intersections, and every intersection of basis elements contains another basis element. The set of unions of elements in B is a topology and is denoted *the topology generated by* $\mathcal{B}$.

Definition (Neighborhood Basis) If $p \in X$, a *neighborhood basis* at p is a collection $\mathcal{B}_p$ of neighborhoods of p such that if N_p is a neighborhood of p, then $N_p \supseteq B$ for at least one $B \in \mathcal{B}_p$.

Definition (Cover) A collection of subsets $\{U_\alpha\}$ of X is said to *cover* X iff $X = \cup_\alpha U_\alpha$. If $A \subseteq X$ is a subspace, then this collection *covers* A iff $A \subseteq \cup_\alpha U_\alpha$

Definition (Refinement) A cover $\mathcal{V} \rightrightarrows X$ is a *refinement* of $\mathcal{U} \rightrightarrows X$ iff for each $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $V \subseteq U$.

3.2 Analysis

Definition (Dense) A subset $Q \subset X$ is dense iff $y \in N_y \subset X \implies N_y \cap Q \neq \emptyset$ iff $\overline{Q} = X$.

Definition (Bounded) A set S in a metric space (X, d) is *bounded* iff there exists an $m \in \mathbb{R}$ such that $d(x, y) < m$ for every $x, y \in S$.

Definition (Uniform Continuity) For $f : (X, d_x) \to (Y, d_Y)$ metric spaces,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \quad d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

Definition (Lebesgue number) For (X, d) a compact metric space and $\{U_\alpha\} \rightrightarrows X$, there exist $\delta_L > 0$ such that

$$A \subset X,\ \mathrm{diam}(A) < \delta_L \implies A \subseteq U_\alpha \text{ for some } \alpha.$$

3.3 Connectedness

Definition (Connected) There does not exist a disconnecting set $X = A \coprod B$ such that $\varnothing \neq A, B \subsetneq$, i.e. X is the union of two proper disjoint nonempty sets.

Equivalently, X contains no proper nonempty clopen sets.

Additional condition for a subspace $Y \subset X$: $\mathrm{cl}_Y(A) \cap V = A \cap \mathrm{cl}_Y(B) = \varnothing$.

Definition (Locally Connected) A space is *locally connected* at a point x iff $\forall N_x \ni x$, there exists a $U \subset N_x$ containing x that is connected.

Definition (Locally Path-Connected) A space is *locally path-connected* if it admits a basis of path-connected open subsets.

Definition (Components) Set $x \sim y$ iff there exists a connected set $U \ni x, y$ and take equivalence classes.

Definition (Path Components) Set $x \sim y$ iff there exists a path-connected set $U \ni x, y$ and take equivalence classes.

3.4 Compactness

Definition (Compact) A topological space (X, τ) is **compact** if every open cover has a *finite* subcover.

That is, if $\left\{U_j \;\middle|\; j \in J\right\} \subset \tau$ is a collection of open sets such that $X = \cup_{j \in J} U_j$, then there exists a *finite* subset $J' \subset J$ such that $X \subseteq \cup_{j \in J'} U_j$.

Definition (Locally Compact) A space X is *locally compact* iff every $x \in X$ has a neighborhood contained in a compact subset of X.

Definition (Paracompact) A topological space X is *paracompact* iff every open cover of X admits an open locally finite refinement.

Definition (Precompact) A subset $A \subseteq X$ is *precompact* iff $\mathrm{cl}_X(A)$ is compact.

3.5 Separability

Definition (Locally Finite) A collection of subsets $\mathcal{S}$ of X is *locally finite* iff each point of M has a neighborhood that intersects at most finitely many elements of $\mathcal{S}$.

Definition (Separable) A space X is *separable* iff X contains a countable dense subset.

Definition (Hausdorff) A topo.ogical space X is *Hausdorff* iff for every $p \neq q \in X$ there exist disjoint open sets $U \ni p$ and $V \ni q$.

Definition (First Countable) A space is *first-countable* iff every point admits a countable neighborhood basis.

Definition (Second Countable) A space is *second-countable* iff it admits a countable basis.

Definition (Regular)

Todo

Definition (Normal)

Todo

3.6 Misc

Definition (Normal)

Todo

3.7 Todo

- Saturated
- Quotient Map
- The subspace topology
- The quotient topology
- The product topology
- Topological Embedding
- Continuous map
- Open and Closed maps

4 Examples

4.1 Common Spaces and Operations

4.1.1 Point-Set

- Finite discrete sets with the discrete topology
- Subspaces of $\mathbb{R}$: $(a,b),(a,b],(a,\infty)$, etc.
  - $\{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{Z}^{\geq 1} \right\}$
- $\mathbb{Q}$
- The topologist's sine curve
- One-point compactifications
- $\mathbb{R}^\omega$
- Hawaiian earring
- Cantor set

Non-Hausdorff spaces:

- The cofinite topology on any infinite set.
- $\mathbb{R}/\mathbb{Q}$
- The line with two origins.

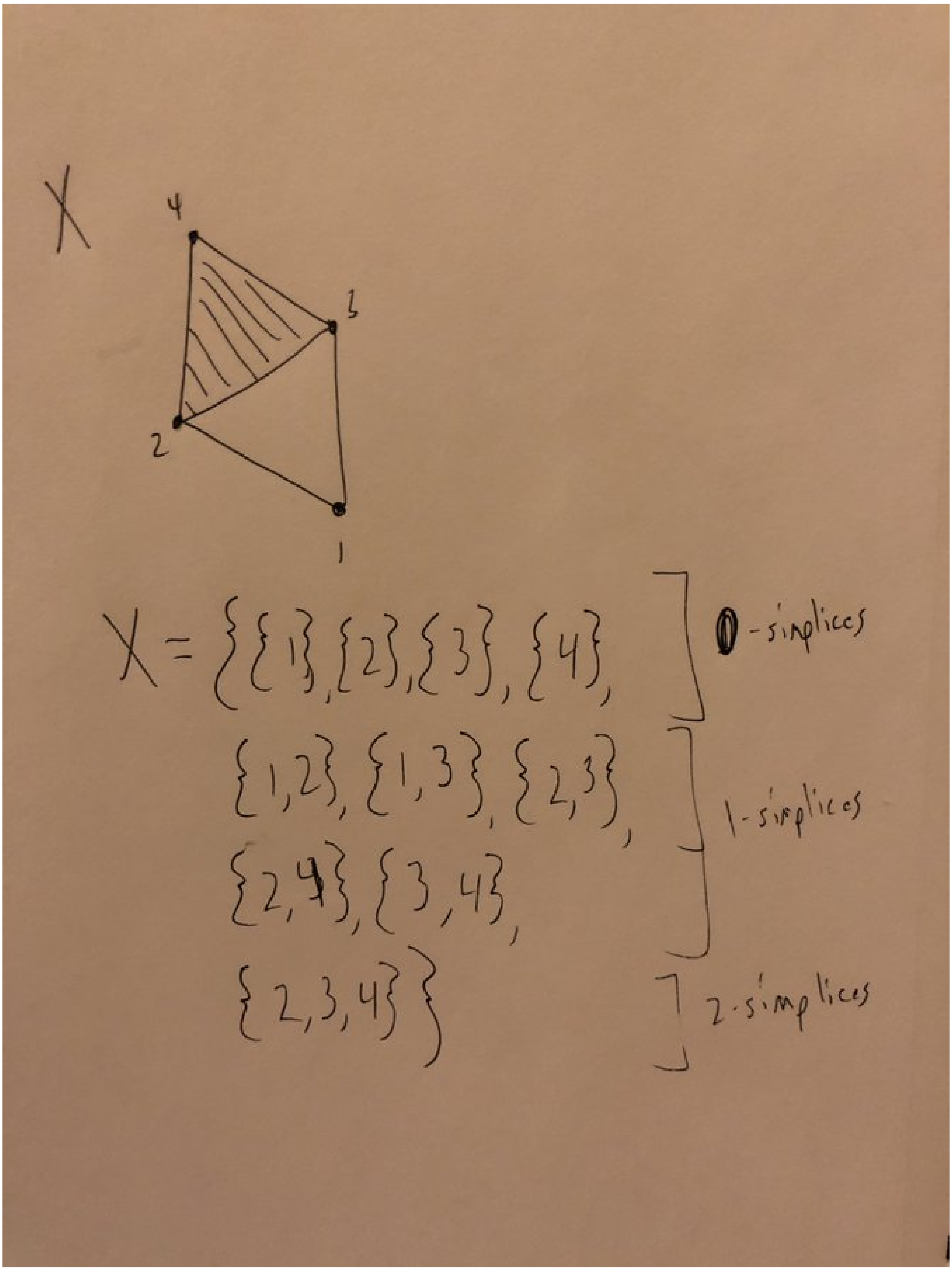
General Spaces:

$$S^n, \mathbb{D}^n, T^n, \mathbb{RP}^n, \mathbb{CP}^n, \mathbb{M}, \mathbb{K}, \Sigma_g, \mathbb{RP}^\infty, \mathbb{CP}^\infty.$$

"Constructed" Spaces

- Knot complements in S^3
- Covering spaces (hyperbolic geometry)
- Lens spaces
- Matrix groups
- Prism spaces
- Pair of pants
- Seifert surfaces
- Surgery
- Simplicial Complexes

– Nice minimal example:



Exotic/Pathological Spaces

- $\mathbb{HP}^n$
- Dunce Cap
- Horned sphere

Operations

- Cartesian product $A \times B$
- Wedge product $A \vee B$
- Connect Sum $A \# B$
- Quotienting A/B
- Puncturing $A \setminus \{a_i\}$
- Smash product
- Join
- Cones
- Suspension
- Loop space
- Identifying a finite number of points

4.2 Alternative Topologies

- Discrete
- Cofinite
- Discrete and Indiscrete
- Uniform

The cofinite topology:

- Non-Hausdorff
- Compact

The discrete topology:

- Discrete iff points are open
- Always Hausdorff
- Compact iff finite
- Totally disconnected
- If the domain, every map is continuous

The indiscrete topology:

- Only open sets are $\varnothing, X$
- Non-Hausdorff
- If the codomain, every map is continuous
- Compact

5 Theorems

Properties preserved and not preserved by continuous functions: Link

- Properties pushed forward through continuous maps:
  - Compactness?
  - Connectedness (when surjective)
  - Separability
  - Density **only when** f is surjective
  - **Not** openness
  - **Not** closedness

5.1 Metric Spaces and Analysis

Theorem (Cantor's Intersection Theorem) A bounded collection of nested closed sets $C_1 \supset C_2 \supset \cdots$ in a metric space X is nonempty $\iff$ X is complete.

Theorem (Cantor's Nested Intervals Theorem) If $\left\{[a_n, b_n] \;\middle|\; n \in \mathbb{Z}^{\geq 0}\right\}$ is a nested sequence of **closed and bounded** intervals, then their intersection is nonempty. If $\operatorname{diam}([a_n, b_n]) \overset{n\to\infty}{0}$, then the intersection contains exactly one point.

Proposition A continuous function on a compact set is uniformly continuous.

Proof Take $\left\{B_{\frac{\varepsilon}{2}}(y) \;\middle|\; y \in Y\right\} \rightrightarrows Y$, pull back to an open cover of X, has Lebesgue number $\delta_L > 0$, then $x' \in B_{\delta_L}(x) \implies f(x), f(x') \in B_{\frac{\varepsilon}{2}}(y)$ for some y.

Corollary Lipschitz continuity implies uniform continuity (take $\delta = \varepsilon/C$)

Counterexample to the converse: $f(x) = \sqrt{x}$ on $[0,1]$ has unbounded derivative.

Theorem (Extreme Value Theorem) For $f : X \to Y$ continuous with X compact and Y ordered in the order topology, there exist points $c, d \in X$ such that $f(x) \in [f(c), f(d)]$ for every x.

Theorem A metric space X is sequentially compact iff it is complete and totally bounded.

Theorem A metric space is totally bounded iff every sequence has a Cauchy subsequence.

Theorem A metric space is compact iff it is complete and totally bounded.

Theorem (Baire) If X is a complete metric space, then the intersection of countably many dense open sets is dense in X.

5.2 Connectedness

Theorem (Tube Lemma)

Todo

5.3 Compactness

Theorem $U \subset X$ a Hausdorff spaces is closed $\iff$ it is compact.

Theorem A closed subset A of a compact set B is compact.

Proof

- Let $\{A_i\} \rightrightarrows A$ be a covering of A by sets open in A.
- Each $A_i = B_i \cap A$ for some B_i open in B (definition of subspace topology)
- Define $V = \{B_i\}$, then $V \rightrightarrows A$ is an open cover.
- Since A is closed, $W \coloneqq B \setminus A$ is open
- Then $V \cup W$ is an open cover of B, and has a finite subcover $\{V_i\}$
- Then $\{V_i \cap A\}$ is a finite open cover of A.

Theorem The continuous image of a compact set is compact.

Theorem A closed subset of a Hausdorff space is compact.

Theorem A continuous bijection $f : X \to Y$ where X is compact and Y is Hausdorff is an open map and hence a homeomorphism.

5.4 Separability

Proposition A retract of a Hausdorff/connected/compact space is closed/connected/compact respectively.
Theorem Points are closed in T_1 spaces.

5.5 Maps and Homeomorphism

Theorem A continuous bijective open map is a homeomorphism.

Theorem (Munkres 18.1) For $f : X \to Y$, TFAE:

- f is continuous
- $A \subset X \implies f(\mathrm{cl}_X(A)) \subset \mathrm{cl}_X(f(A))$
- B closed in $Y \implies f^{-1}(B)$ closed in X.

- For each $x \in X$ and each neighborhood $V \ni f(x)$, there is a neighborhood $U \ni x$ such that $f(U) \subset V$.

Proof

Todo, see Munkres page 104

Theorem (Lee A.52) If $f : X \to Y$ is continuous where X is compact and Y is Hausdorff, then

- f is a closed map.
- If f is surjective, f is a quotient map.
- If f is injective, f is a topological embedding.
- If f is bijective, it is a homeomorphism.

6 Topics

- Algebraic topology topics:
  - Classification of compact surfaces
  - Euler characteristic
  - Connect sum
  - Homology and cohomology groups
  - Fundamental group
  - Singular/cellular/simplicial homology
  - Mayer-Vietoris long exact sequences for homology and cohomology
  - Diagram chasing
  - Degree of maps from $S^n \to S^n$
  - Orientability, compactness
  - Top-level homology and cohomology
  - Reduced homology and cohomology
  - Relative homology
  - Homotopy and homotopy invariance
  - Deformation retract
  - Retract
  - Excision
  - Kunneth formula
  - Factoring maps
  - Fundamental theorem of algebra
- Algebraic topology theorems:
  - Brouwer fixed point theorem
  - Poincare lemma
  - Poincare duality
  - de Rham theorem
  - Seifert-van Kampen theorem

- Covering space theory topics:
  - Covering maps
  - Free actions
  - Properly discontinuous action
  - Universal cover
  - Correspondence between covering spaces and subgroups of the fundamental group of the base.
  - Lifting paths
  - Homotopy lifting property
  - Deck transformations
  - The action of the fundamental group
  - Normal/regular cover

7 AT Summary

7.1 Different Types of Product/Sum Structures

- Cartesian Product $X \times Y, \prod_i X_i$
- Direct Sum $X \oplus Y, \bigoplus_i X_i$
- Direct Product $X * Y, *_i X_i$
  - Element-wise multiplication, allows infinitely many entries
  - $*_i X_i = \oplus_i X_i$ for $i < \infty$
- Tensor Product $X \otimes Y, \bigotimes_i X_i, X^{\otimes_i}$
- For a finite index set I, $\prod_I G = \bigoplus_I G$ in **Grp**, i.e. the finite direct product and finite direct sum coincide.

  Otherwise, if I is infinite, the direct sum requires cofinitely many zero entries (i.e. finitely many nonzero entries), so here we always use $\prod$.

  In other words, there is an injective map

  $$\bigoplus_I G \hookrightarrow \prod_I G$$

  which is an isomorphism when $|I| < \infty$

- The free abelian group of rank n:

$$\mathbb{Z}^n := \prod_{i=1}^{n} \mathbb{Z} = \mathbb{Z} \times \mathbb{Z} \times \ldots \mathbb{Z}.$$

  - $x \in \mathbb{Z}^n = \langle a_1, \cdots, a_n \rangle \implies x = \sum_n c_i a_i$ for some $c_i \in \mathbb{Z}$, i.e. a_i form a basis.
  - Example: $x = 2a_1 + 4a_2 + a_1 - a_2 = 3a_1 + 3a_2$.

- The **free product** of n free abelian groups:

$$\mathbb{Z}^{*n} := \mathop{\ast}_{i=1}^{n} \mathbb{Z} = \mathbb{Z} * \mathbb{Z} * \ldots \mathbb{Z}$$

  This is a free *nonabelian* group on n generators.

  - $x \in \mathbb{Z}^{*n} = \langle a_1, \ldots, a_n \rangle$ implies that x is a finite word in the noncommuting symbols a_i^k for $k \in \mathbb{Z}$.
  - Example: $x = a_1^2 a_2^4 a_1 a_2^{-2}$

Proposition There are no nontrivial homomorphisms from finite groups into free groups.

In particular, any homomorphism $\mathbb{Z}_n \to \mathbb{Z}$ is trivial.

Proof Homomorphisms preserve torsion; the former has n-torsion while the latter does not.

This is especially useful if you have some $f : A \to B$ and you look at the induced homomorphism $f_* : \pi_1(A) \to \pi_1(B)$. If the former is finite and the latter contains a copy of $\mathbb{Z}$, then f_* has to be the trivial map $f_*([\alpha]) = e \in \pi_1(B)$ for every $[\alpha] \in \pi_1(A)$.

7.2 Conventions

- Generally assume spaces are connected.
- $\pi_0(X)$ is the set of path components of X, and I write $\pi_0(X) = \mathbb{Z}$ if X is path-connected (although it is not a group). Similarly, $H_0(X)$ is a free abelian group on the set of path components of X.
- Lists start at entry 1, since all spaces are connected here and thus $\pi_0 = H_0 = \mathbb{Z}$. That is,
  - $\pi_*(X) = [\pi_1(X), \pi_2(X), \pi_3(X), \cdots]$
  - $H_*(X) = [H_1(X), H_2(X), H_3(X), \cdots]$

8 | Definitions: Algebraic Topology

- Acyclic
- Alexander duality
- Basis
  - For an R-module M, a basis B is a linearly independent generating set.
- Boundary
- Boundary of a manifold
  - Points $x \in M^n$ defined by

$$\partial M = \{x \in M : H_n(M, M - \{x\}; \mathbb{Z}) = 0\}$$

- Cap Product
  - Denoting $\Delta^p \xrightarrow{\sigma} X \in C_p(X; G)$, a map that sends pairs (p-chains, q-cochains) to $(p-q)$-chains $\Delta^{p-q} \to X$ by

$$\begin{aligned} H_p(X;R) \times H^q(X;R) &\xrightarrow{\frown} H_{p-q}(X;R) \\ \sigma \frown \psi &= \psi(F_0^q(\sigma)) F_q^p(\sigma) \end{aligned}$$

    where F_i^j is the face operator, which acts on a simplicial map σ by restriction to the face spanned by $[v_i \dots v_j]$, i.e. $F_i^j(\sigma) = \sigma|_{[v_i \dots v_j]}$.

- Cellular Homology
- CW Cell
  - An n-cell of X, say e^n, is the image of a map $\Phi : B^n \to X$. That is, $e^n = \Phi(B^n)$. Attaching an n-cell to X is equivalent to forming the space $B^n \coprod_f X$ where $f : \partial B^n \to X$.
    - ◇ A 0-cell is a point.
    - ◇ A 1-cell is an interval $[-1,1] = B^1 \subset \mathbb{R}^1$. Attaching requires a map from $S^0 = \{-1,+1\} \to X$
    - ◇ A 2-cell is a solid disk $B^2 \subset \mathbb{R}^2$ in the plane. Attaching requires a map $S^1 \to X$.
    - ◇ A 3-cell is a solid ball $B^3 \subset \mathbb{R}^3$. Attaching requires a map from the sphere $S^2 \to X$.
- Cellular Map
  - A map $X \xrightarrow{f} Y$ is said to be cellular if $f(X^{(n)}) \subseteq Y^{(n)}$ where $X^{(n)}$ denotes the n- skeleton.

- Chain
  - An element $c \in C_p(X;R)$ can be represented as the singular p simplex $\Delta^p \to X$.
- Chain Homotopy
  - Given two maps between chain complexes $(C_*, \partial_C) \xrightarrow{f,\ g} (D_*, \partial_D)$, a chain homotopy is a family $h_i : C_i \to B_{i+1}$ satisfying

$$f_i - g_i = \partial_{B,i-1} \circ h_n + h_{i+1} \circ \partial_{A,i}$$

    .
- Chain Map
  - A map between chain complexes $(C_*, \partial_C) \xrightarrow{f} (D_*, \partial_D)$ is a chain map iff each component $C_i \xrightarrow{f_i} D_i$ satisfies

$$f_{i-1} \circ \partial_{C,i} = \partial_{D,i} \circ f_i$$

    (i.e this forms a commuting ladder)
- Closed manifold
  - A manifold that is compact, with or without boundary.
- Coboundary
- Cochain
  - An cochain $c \in C^p(X;R)$ is a map $c \in \hom(C_p(X;R), R)$ on chains.
- Cocycle

Definition 8.0.1 (Constant Map)
A *constant map* $f : X \to Y$ iff $f(X) = y_0$ for some $y_0 \in Y$, i.e. for every $x \in X$ the output value $f(x) = y_0$ is the same.

Definition 8.0.2 (Colimit)
For a directed system $(X_i, f_{ij}$, the *colimit* is an object X with a sequence of projections $\pi_i : X \to X_i$ such that for any Y mapping into the system, the following diagram commutes:

$$\begin{tikzcd}
 & & Y \arrow[d, dashed, "\exists!"] \arrow[ddl, "\psi_j"'] \arrow[ddr, "\psi_i"] & & \\
 & & X \arrow[dl, "\pi_j"] \arrow[dr, "\pi_i"'] & & \\
\cdots \arrow[r] & X_j \arrow[rr, "f_{ij}"] & & X_i \arrow[r] & \cdots
\end{tikzcd}$$

Example 8.0.3:
- Products
- Pullbacks
- Inverse/Projective limits
- The p-adic integers $\mathbb{Z}_p$.

- Compact
  - A space X is compact iff every open cover of X has a finite subcover.
- Cone
  - For a space X, defined as
$$CX = \frac{X \times I}{X \times \{0\}}.$$
    Example: The cone on the circle CS^1

  Note that the cone embeds X in a contractible space CX.
- Contractible
  - A space is contractible if its identity map is nullhomotopic.
- Contractible
- Coproduct
- Covering Space
- Cup Product
  - A map taking pairs (p-cocycles, q-cocycles) to $(p+q)$-cocyles by
$$\begin{aligned} H^p(X;R) \times H^q(X;R) &\xrightarrow{\smile} H^{p+q}(X;R) \\ (a \cup b)(\sigma) &= a(\sigma \circ I_0^p)\ b(\sigma \circ I_p^{p+q}) \end{aligned}$$

where $\Delta^{p+q} \xrightarrow{\sigma} X$ is a singular $p+q$ simplex and

$$I_i^j : [i, \cdots, j] \hookrightarrow \Delta^{p+q}$$

is an embedding of the $(j-i)$-simplex into a $(p+q)$-simplex. On a manifold, the cup product is Poincare dual to the intersection of submanifolds.

- Applications
  - ◇ $T^2 \not\simeq S^2 \vee S^1 \vee S^1$.

Proof

- CW Complex
- Cycle
- Deck Transformation
- Deformation
- Deformation Retract
  - A map r in $A \underset{r}{\overset{\iota}{\rightleftarrows}} X$ that is a retraction (so $r \circ \iota = \mathrm{id}_A$) **that also satisfies** $\iota \circ r \simeq \mathrm{id}_X$.

Note that this is equality in one direction, but only homotopy equivalence in the other.

  - Equivalently, a map $F : I \times X \to X$ such that
    - ◇ $F_0(x) = \mathrm{id}_X$
    - ◇ $F_t(x)\Big|_A = \mathrm{id}_A$
    - ◇ $F_1(X) = A$
- Degree of a Map
- Derived Functor
  - For a functor T and an R-module A, a *left derived functor* $(L_n T)$ is defined as $h_n(TP_A)$, where P_A is a projective resolution of A.
- Dimension of a manifold
  - For $x \in M$, the only nonvanishing homology group $H_i(M, M - \{x\}; \mathbb{Z})$
- Direct Limit
- Direct Product
- Direct Sum

- Eilenberg-MacLane Space
- Euler Characteristic
- Exact Functor
  - A functor T is *right exact* if a short exact sequence
    $$0 \to A \to B \to C \to 0$$
    yields an exact sequence
    $$\ldots TA \to TB \to TC \to 0,$$
    and is *left exact* if it yields
    $$0 \to TA \to TB \to TC \to \ldots$$
    Thus a functor is exact iff it is both left and right exact, yielding
    $$0 \to TA \to TB \to TC \to 0$$
  - Examples:
    - ◊ $\cdot \otimes_R \cdot$ is a right exact bifunctor.
- Exact Sequence
- Excision
- Ext Group
- Flat
  - An R-module is flat if $A \otimes_R \cdot$ is an exact functor.
- Free and Properly Discontinuous
- Free module
  - A -module M with a basis $S = \{s_i\}$ of generating elements. Every such module is the image of a unique map $\mathcal{F}(S) = R^S \twoheadrightarrow M$, and if $M = \left\langle S \mid \mathcal{R} \right\rangle$ for some set of relations $\mathcal{R}$, then $M \cong R^S/\mathcal{R}$.
- Free Product
- Free product with amalgamation
- Fundamental Class

  - For a connected, closed, orientable manifold, $[M]$ is a generator of $H_n(M;\mathbb{Z}) = \mathbb{Z}$.

- Fundamental classes
- Fundamental Group
- Generating Set
  - $S = \{s_i\}$ is a generating set for an R- module M iff

$$x \in M \implies x = \sum r_i s_i$$

    for some coefficients $r_i \in R$ (where this sum may be infinite).

- Gluing Along a Map
- Group Ring
- Homologous
- Homotopic
- Homotopy
- Homotopy Class
- Homotopy Equivalence
- Homotopy Extension Property
- Homotopy Groups
- Homotopy Lifting Property
- Injection
  - A map ι with a **left** inverse f satisfying $f \circ \iota = \mathrm{id}$
- Intersection Pairing For a manifold M, a map on homology defined by

$$\begin{aligned} H_{\widehat{i}}M \otimes H_{\widehat{j}}M &\to H_{\widehat{i+j}}X \\ \alpha \otimes \beta &\mapsto \langle \alpha, \beta \rangle \end{aligned}$$

obtained by conjugating the cup product with Poincare Duality, i.e.

$$\langle \alpha, \beta \rangle = [M] \frown ([\alpha]^\vee \smile [\beta]^\vee)$$

Then, if $[A], [B]$ are transversely intersecting submanifolds representing α, β, then

$$\langle \alpha, \beta \rangle = [A \cap B]$$

.

If $\widehat{i} = j$ then $\langle \alpha, \beta \rangle \in H_0 M = \mathbb{Z}$ is the signed number of intersection points.

- Inverse Limit
- Intersection Pairing
  - The pairing obtained from dualizing Poincare Duality to obtain

    $$\mathrm{F}(H_i M) \otimes \mathrm{F}(H_{n-i} M) \to \mathbb{Z}$$

    Computed as an oriented intersection number between two homology classes (perturbed to be transverse).
- Intersection Form
  - The nondegenerate bilinear form cohomology induced by the Kronecker Pairing:

    $$I : H^k(M_n) \times H^{n-k}(M^n) \to \mathbb{Z}$$

    where $n = 2k$.
    - When k is odd, I is skew-symmetric and thus a *symplectic form.*
    - When k is even (and thus $n \equiv 0 \pmod 4$) this is a symmetric form.
    - Satisfies $I(x, y) = (-1)^{k(n-k)} I(y, x)$
- Kronecker Pairing
  - A map pairing a chain with a cochain, given by

    $$\begin{aligned} H^n(X; R) \times H_n(X; R) &\to R \\ ([\psi, \alpha]) &\mapsto \psi(\alpha) \end{aligned}$$

    which is a nondegenerate bilinear form.
- Kronecker Product
- Lefschetz duality
- Lefschetz Number
- Lens Space
- Local Degree

- At a point $x \in V \subset M$, a generator of $H_n(V, V - \{x\})$. The degree of a map $S^n \to S^n$ is the sum of its local degrees.

- Local Orientation
- Limit
- Linear Independence
  - A generating S for a module M is linearly independent if $\sum r_i s_i = 0_M \implies \forall i,\ r_i = 0$ where $s_i \in S, r_i \in R$.
- Local homology
  - $H_n(X, X - A; \mathbb{Z})$ is the local homology at A, also denoted $H_n(X \mid A)$
- Local Homology
- Local orientation of a manifold
  - At a point $x \in M^n$, a choice of a generator μ_x of $H_n(M, M - \{x\}) = \mathbb{Z}$.
- Long exact sequence
- Loop Space
- Manifold
  - An n-manifold is a Hausdorff space in which each neighborhood has an open neighborhood homeomorphic to $\mathbb{R}^n$.
- Manifold with boundary
  - A manifold in which open neighborhoods may be isomorphic to either $\mathbb{R}^n$ or a half-space $\left\{\mathbf{x} \in \mathbb{R}^n \mid x_i > 0\right\}$.
- Mapping Cone
- Mapping Cylinder
- Mapping Path Space
- Mayer-Vietoris Sequence
- Monodromy

- Moore Space
- N-cell
- N-connected

Definition 8.0.4 (Nullhomotopic)
A map $X \xrightarrow{f} Y$ is *nullhomotopic* if it is homotopic to a constant map $X \xrightarrow{g} \{y_0\}$; that is, there exists a homotopy

$$\begin{aligned} &F : X \times I \to Y \\ &F\big|_{X\times\{0\}} = f \quad F(x,0) = f(x) \\ &F\big|_{X\times\{1\}} = g \quad F(x,1) = g(x) = y_0 \end{aligned}$$

.

- Orientable manifold
  - A manifold for which an orientation exists, see "Orientation of a Manifold".
- Orientation Cover
  - For any manifold M, a two sheeted orientable covering space $\tilde{M}_o$. M is orientable iff $\tilde{M}$ is disconnected. Constructed as

$$\tilde{M} = \coprod_{x\in M} \left\{ \mu_x \;\middle|\; \mu_x \text{ is a local orientation} \right\}$$

- Orientation of a manifold
  - A family of $\{\mu_x\}_{x\in M}$ with local consistency: if $x, y \in U$ then μ_x, μ_y are related via a propagation.
    - ◇ Formally, a function

$$\begin{aligned} M^n \to \coprod_{x\in M} H(X \mid \{x\}) \\ x \mapsto \mu_x \end{aligned}$$

      such that $\forall x \exists N_x$ in which $\forall y \in N_x$, the preimage of each μ_y under the map $H_n(M \mid N_x) \twoheadrightarrow H_n(M \mid y)$ is a single generator μ_{N_x}.
  - TFAE:
    - ◇ M is orientable.
    - ◇ The map $W : (M, x) \to \mathbb{Z}_2$ is trivial.
    - ◇ $\tilde{M}_o = M \coprod \mathbb{Z}_2$ (two sheets).
    - ◇ $\tilde{M}_o$ is disconnected
    - ◇ The projection $\tilde{M}_o \to M$ admits a section.

- Oriented manifold
- Path
- Path Lifting Property
- Perfect Pairing
  - A pairing alone is an R-bilinear module map, or equivalently a map out of a tensor product since $p : M \otimes_R N \to L$ can be partially applied to yield $\varphi : M \to L^N = \hom_R(N, L)$. A pairing is **perfect** when φ is an isomorphism.
    - ◇ Example: $\det_M : k^2 \times k^2 \to k$
- Poincare Duality
  - For a closed, orientable n-manifold, following map $[M] \frown \cdot$ is an isomorphism:

$$\begin{aligned} D : H^k(M; R) \to H_{n-k}(M; R) \\ D(\alpha) = [M] \frown \alpha \end{aligned}$$

- Projective Resolution
- Properly Discontinuous
- Pullback
- Pushout
- Quasi-isomorphism
- R-orientability
- Relative boundaries
- Relative cycles
- Relative homotopy groups
- Retraction
  - A map r in $A \underset{r}{\overset{\iota}{\leftrightarrows}} X$ satisfying

$$r \circ \iota = \mathrm{id}_A .$$

    Equivalently $X \twoheadrightarrow_r A$ and $r|_A = \mathrm{id}_A$. If X retracts onto A, then i_* is injective.

- Short exact sequence

- Simplicial Complex
- Simplicial Map
  - For a map

$$K \xrightarrow{f} L$$

    between simplicial complexes, f is a simplicial map if for any set of vertices $\{v_i\}$ spanning a simplex in K, the set $\{f(v_i)\}$ are the vertices of a simplex in L.

- Simply Connected
- Singular Chain

$$x \in C_n(x) \implies X = \sum_i n_i \sigma_i = \sum_i n_i (\Delta^n \xrightarrow{\sigma_i} X)$$

- Singular Cochain

$$x \in C^n(x) \implies X = \sum_i n_i \psi_i = \sum_i n_i (\sigma_i \xrightarrow{\psi_i} X)$$

- Singular Homology
- Smash Product
- Surjection
  - A map π with a **right** inverse f satisfying

$$\pi \circ f = \mathrm{id}$$

- Suspension Compact represented as $\Sigma X = CX \coprod_{\mathrm{id}_X} CX$, two cones on X glued along X. Explicitly given by

$$\Sigma X = \frac{X \times I}{(X \times \{0\}) \cup (X \times \{1\}) \cup (\{x_0\} \times I)}$$

- Tor Group
  - For an R-module

$$\mathrm{Tor}_R^n(\,\cdot\,, B) = L_n(\,\cdot\, \otimes_R B)$$

    where L_n denotes the nth left derived functor.

- Universal Cover
- Universal Coefficient Theorem for Cohomology
- Universal Coefficient Theorem for Change of Coefficient Ring
- Weak Homotopy Equivalence
- Weak Topology
- Wedge Product

9 Examples: Algebraic Topology

9.1 Standard Spaces and Modifications

- $K(G,n)$ is an Eilenberg-MacLane space, the homotopy-unique space satisfying

$$\pi_k(K(G,n)) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

  - $K(\mathbb{Z},1) = S^1$
  - $K(\mathbb{Z},2) = \mathbb{CP}^\infty$
  - $K(\mathbb{Z}/2\mathbb{Z},1) = \mathbb{RP}^\infty$

- $M(G,n)$ is a Moore space, the homotopy-unique space satisfying

$$H_k(M(G,n);G) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

  - $M(\mathbb{Z},n) = S^n$
  - $M(\mathbb{Z}/2\mathbb{Z},1) = \mathbb{RP}^2$
  - $M(\mathbb{Z}/p\mathbb{Z},n)$ is made by attaching e^{n+1} to S^n via a degree p map.

- $\mathbb{RP}^n = S^n/S^0 = S^n/\mathbb{Z}/2\mathbb{Z}$
- $\mathbb{CP}^n = S^{2n+1}/S^1$
- $T^n = \prod_n S^1$ is the n-torus
- $D(k,X)$ is the space X with $k \in \mathbb{N}$ distinct points deleted, i.e. the punctured space $X - \{x_1, x_2, \dots x_k\}$ where each $x_i \in X$.

- $B^n = \left\{ \mathbf{v} \in \mathbb{R}^n \,\middle|\, \|\mathbf{v}\| \leq 1 \right\} \subset \mathbb{R}^n$
- $S^{n-1} = \partial B^n = \left\{ \mathbf{v} \in \mathbb{R}^n \,\middle|\, \|\mathbf{v}\| = 1 \right\} \subset \mathbb{R}^n$

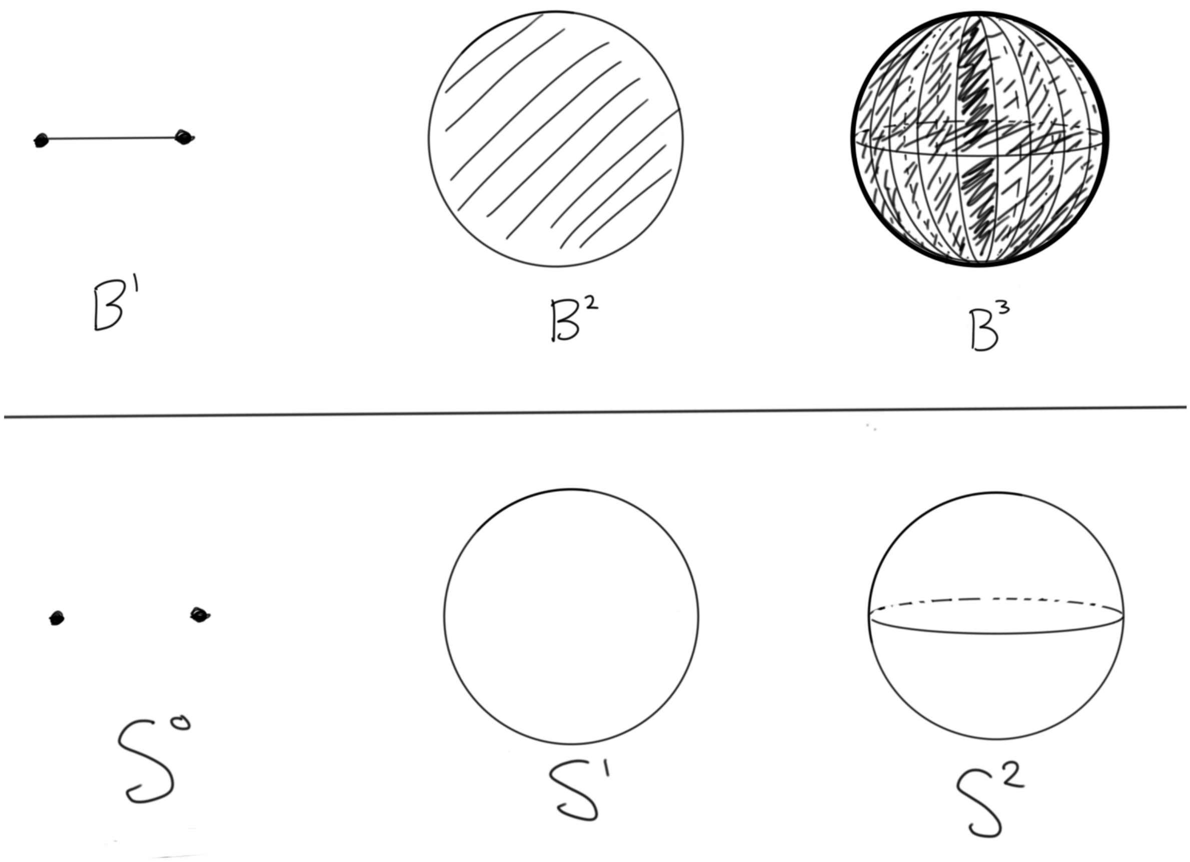


Figure 1: Low-Dimensional Spheres/Discs/Balls

- The "generalized uniform bouquet"? $\mathcal{B}^n(m) = \bigvee_{i=1}^{n} S^m$
- The real Grassmannian, $Gr(n, k, \mathbb{R})$, i.e. the set of k dimensional subspaces of $\mathbb{R}^n$
- The Stiefel manifold $V_n(k)$
- Possible modifications to a space X:
  - Remove k points by taking $D(k, X)$
  - Remove a line segment
  - Remove an entire line/axis
  - Remove a hole

    - Quotient by a group action (e.g. antipodal map, or rotation)
    - Remove a knot
    - Take complement in ambient space

- Lie Groups

    - The real general linear group, $\mathrm{GL}_n(\mathbb{R})$
        - The real special linear group $SL_n(\mathbb{R})$
        - The real orthogonal group, $O_n(\mathbb{R})$
            - The real special orthogonal group, $SO_n(\mathbb{R})$
        - The real unitary group, $U_n(\mathbb{R})$
            - The real special unitary group, $SU_n(\mathbb{R})$
        - The real symplectic group $Sp(n)$

- "Geometric" Stuff

    - Affine n-space over a field $\mathbb{A}^n(k) = k^n \rtimes GL_n(k)$
    - The projective space $\mathbb{P}^n(k)$
        - The projective linear group over a ring R, $PGL_n(R)$
        - The projective special linear group over a ring R, $PSL_n(R)$
        - The modular groups $PSL_n(\mathbb{Z})$
            - Specifically $PSL_2(\mathbb{Z})$

9.2 Facts About Low Dimensional and/or Standard Spaces

- $S^{2n+1} \subset \mathbb{C}^{n+1}$
- $\mathbb{RP}^1 \cong S^1$
- $\mathbb{RP}^n \cong S^n/S^0 \cong S^n/\mathbb{Z}/2\mathbb{Z}$.
- $\mathbb{CP}^1 \cong S^2$
- $\mathcal{M} \simeq S^1$
- $\mathbb{CP}^n = \mathbb{C}^n \coprod \mathbb{CP}^{n-1} = \coprod_{i=0}^{n} \mathbb{C}^i$
- $\mathbb{CP}^n = S^{2n+1}/S^n$
- $S^n/S^k \cong S^n \vee \Sigma S^k$.

9.3 Table of Homotopy and Homology Structures

10 Low Dimensional Homology Examples

$$\begin{array}{rcllllllll}
S^1 & = & [& \mathbb{Z}, & \mathbb{Z}, & 0, & 0, & 0, & 0 \to &] \\
\mathcal{M} & = & [& \mathbb{Z}, & \mathbb{Z}, & 0, & 0, & 0, & 0 \to &] \\
\mathbb{RP}^1 & = & [& \mathbb{Z}, & \mathbb{Z}, & 0, & 0, & 0, & 0 \to &] \\
\mathbb{RP}^2 & = & [& \mathbb{Z}, & \mathbb{Z}_2, & 0, & 0, & 0, & 0 \to &] \\
\mathbb{RP}^3 & = & [& \mathbb{Z}, & \mathbb{Z}_2, & 0, & \mathbb{Z}, & 0, & 0 \to &] \\
\mathbb{RP}^4 & = & [& \mathbb{Z}, & \mathbb{Z}_2, & 0, & \mathbb{Z}_2, & 0, & 0 \to &] . \\
S^2 & = & [& \mathbb{Z}, & 0, & \mathbb{Z}, & 0, & 0, & 0 \to &] \\
\mathbb{T}^2 & = & [& \mathbb{Z}, & \mathbb{Z}^2, & \mathbb{Z}, & 0, & 0, & 0 \to &] \\
\mathbb{K} & = & [& \mathbb{Z}, & \mathbb{Z} \oplus \mathbb{Z}_2, & 0, & 0, & 0, & 0 \to &] \\
\mathbb{CP}^1 & = & [& \mathbb{Z}, & 0, & \mathbb{Z}, & 0, & 0, & 0 \to &] \\
\mathbb{CP}^2 & = & [& \mathbb{Z}, & 0, & \mathbb{Z}, & 0, & \mathbb{Z}, & 0 \to &]
\end{array}$$

X	$\pi_*(X)$	$H_*(X)$	CW Structure	$H^*(X)$
$\mathbb{R}^1$	0	0	$\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot x$	0
$\mathbb{R}^n$	0	0	$(\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot x)^n$	0
$D(k, \mathbb{R}^n)$	$\pi_* \bigvee^{k} S^1$	$\bigoplus_k H_* M(\mathbb{Z}, 1)$	$1 + kx$	?
B^n	$\pi_*(\mathbb{R}^n)$	$H_*(\mathbb{R}^n)$	$1 + x^n + x^{n+1}$	0
S^n	$[0 \ldots, \mathbb{Z}, ? \ldots]$	$H_* M(\mathbb{Z}, n)$	$1 + x^n$ or $\sum_{i=0}^{n} 2x^i$	$\mathbb{Z}[{}_n x]/(x^2)$
$D(k, S^n)$	$\pi_* \bigvee^{k-1} S^1$	$\bigoplus_{k-1} H_* M(\mathbb{Z}, 1)$	$1 + (k-1)x^1$	?
T^2	$\pi_* S^1 \times \pi_* S^1$	$(H_* M(\mathbb{Z}, 1))^2 \times H_* M(\mathbb{Z}, 2)$	$1 + 2x + x^2$	$\Lambda({}_1x_1, {}_1x_2)$
T^n	$\prod^{n} \pi_* S^1$	$\prod_{i=1}^{n} (H_* M(\mathbb{Z}, i))^{\binom{n}{i}}$	$(1 + x)^n$	$\Lambda({}_1x_1, {}_1x_2, \ldots {}_1x_n)$
$D(k, T^n)$	$[0, 0, 0, 0, \ldots]$?	$[0, 0, 0, 0, \ldots]$?	$1 + x$	?
$S^1 \vee S^1$	$\pi_* S^1 * \pi_* S^1$	$(H_* M(\mathbb{Z}, 1))^2$	$1 + 2x$	?
$\bigvee^{n} S^1$	$*^n \pi_* S^1$	$\prod H_* M(\mathbb{Z}, 1)$	$1 + x$	?
$\mathbb{RP}^1$	$\pi_* S^1$	$H_* M(\mathbb{Z}, 1)$	$1 + x$	${}_0\mathbb{Z} \times {}_1\mathbb{Z}$
$\mathbb{RP}^2$	$\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) + \pi_* S^2$	$H_* M(\mathbb{Z}/2\mathbb{Z}, 1)$	$1 + x + x^2$	${}_0\mathbb{Z} \times {}_2\mathbb{Z}/2\mathbb{Z}$
$\mathbb{RP}^3$	$\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) + \pi_* S^3$	$H_* M(\mathbb{Z}/2\mathbb{Z}, 1) + H_* M(\mathbb{Z}, 3)$	$1 + x + x^2 + x^3$	${}_0\mathbb{Z} \times {}_2\mathbb{Z}/2\mathbb{Z} \times {}_3\mathbb{Z}$
$\mathbb{RP}^4$	$\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) + \pi_* S^4$	$H_* M(\mathbb{Z}/2\mathbb{Z}, 1) + H_* M(\mathbb{Z}/2\mathbb{Z}, 3)$	$1 + x + x^2 + x^3 + x^4$	${}_0\mathbb{Z} \times ({}_2\mathbb{Z}/2\mathbb{Z})^2$
$\mathbb{RP}^n, n \geq 4$ even	$\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) +$ $\pi_* S^n$	$\prod_{\text{odd } i<n} H_* M(\mathbb{Z}/2\mathbb{Z}, i)$	$\sum_{i=1}^{n} x^i$	${}_0\mathbb{Z} \times \prod_{i=1}^{n/2} {}_2\mathbb{Z}/2\mathbb{Z}$
$\mathbb{RP}^n, n \geq 4$ odd	$\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) +$ $\pi_* S^n$	$\prod_{\text{odd } i \leq n-2} H_* M(\mathbb{Z}/2\mathbb{Z}, i) \times H_* S^n$	$\sum_{i=1}^{n} x^i$	$H^*(\mathbb{RP}^{n-1}) \times {}_n\mathbb{Z}$
$\mathbb{CP}^1$	$\pi_* K(\mathbb{Z}, 2) + \pi_* S^3$	$H_* S^2$	$x^0 + x^2$	$\mathbb{Z}[{}_2x]/({}_2x^2)$
$\mathbb{CP}^2$	$\pi_* K(\mathbb{Z}, 2) + \pi_* S^5$	$H_* S^2 \times H_* S^4$	$x^0 + x^2 + x^4$	$\mathbb{Z}[{}_2x]/({}_2x^3)$
$\mathbb{CP}^n, n \geq 2$	$\pi_* K(\mathbb{Z}, 2) + \pi_* S^{2n+1}$	$\prod_{i=1}^{n} H_* S^{2i}$	$\sum_{i=1}^{n} x^{2i}$	$\mathbb{Z}[{}_2x]/({}_2x^{n+1})$
Mobius Band	$\pi_* S^1$	$H_* S^1$	$1 + x$	?
Klein Bottle	$K(\mathbb{Z} \rtimes_{-1} \mathbb{Z}, 1)$	$H_* S^1 \times H_* \mathbb{RP}^\infty$	$1 + 2x + x^2$	?

Facts used to compute the above table:

- $\mathbb{R}^n$ is a contractible space, and so $[S^m, \mathbb{R}^n] = 0$ for all n, m which makes its homotopy groups all zero.

- $D(k, \mathbb{R}^n) = \mathbb{R}^n - \{x_1 \dots x_k\} \simeq \bigvee_{i=1}^{k} S^1$ by a deformation retract.

- $S^n \cong B^n / \partial B^n$ and employs an attaching map

$$\begin{aligned} \varphi : (D^n, \partial D^n) &\to S^n \\ (D^n, \partial D^n) &\mapsto (e^n, e^0). \end{aligned}$$

- $B^n \simeq \mathbb{R}^n$ by normalizing vectors.

- Use the inclusion $S^n \hookrightarrow B^{n+1}$ as the attaching map.

- $\mathbb{CP}^1 \cong S^2$.

- $\mathbb{RP}^1 \cong S^1$.

- Use $\left[\pi_1, \prod\right] = 0$ and the universal cover $\mathbb{R}^1 \twoheadrightarrow S^1$ to yield the cover $\mathbb{R}^n \twoheadrightarrow T^n$.

- Take the universal double cover $S^n \twoheadrightarrow^{\times 2} \mathbb{RP}^n$ to get equality in $\pi_{i \geq 2}$.

- Use $\mathbb{CP}^n = S^{2n+1}/S^1$

- Alternatively, the fundamental group is $\mathbb{Z} * \mathbb{Z}/bab^{-1}a$. Use the fact the $\tilde{K} = \mathbb{R}^2$.

- $M \simeq S^1$ by deformation-retracting onto the center circle.

- $D(1, S^n) \cong \mathbb{R}^n$ and thus $D(k, S^n) \cong D(k-1, \mathbb{R}^n) \cong \bigvee^{k-1} S^1$

11 | Theorems: Algebraic Topology

11.1 Fundamental Group

Conjugacy in π_1:

- See Hatcher 1.19, p.28

- See Hatcher's proof that π_1 is a group
- See change of basepoint map
- For a graph G, we always have $\pi_1(G) \cong \mathbb{Z}^n$ where $n = |E(G - T)|$, the complement of the set of edges in any maximal tree. Equivalently, $n = 1 - \chi(G)$. Moreover, $X \simeq \bigvee^n S^1$ in this case.

To calculate $\pi_1(X)$: construct a universal cover $\tilde{X}$, then find a group $G \curvearrowright \tilde{X}$ such that $\tilde{X}/G = X$; then $\pi_1(X) = G$ by uniqueness of universal covers.

Proposition 11.1.1 *(?).*
$\pi_0(X) = \mathbb{Z}$ iff X is simply connected.

- H_1 is the abelianization of π_1.
- Homotopy commutes with products: $\pi_k \prod X_i = \prod \pi_k X_i$.
- Homotopy splits wedge products: $\pi_1 \bigvee X_i = *\pi_1 X_i$.

11.2 Homotopy

Merge Van
pen theorem

Theorem (Van Kampen) The pushout is the northwest colimit of the following diagram

$$\begin{tikzcd}
A \coprod_Z B & A \arrow[l] \\
B \arrow[u] & Z \arrow[u, "\iota_A", leftrightarrow] \arrow[l, "\iota_B", leftrightarrow]
\end{tikzcd}$$

For groups, the pushout is given by the amalgamated free product: if $A = \left\langle G_A \mid R_A \right\rangle, B = \left\langle G_B \mid R_B \right\rangle$, then

$$A *_Z B = \left\langle G_A, G_B \mid R_A, R_B, T \right\rangle$$

where T is a set of relations given by

$$T = \left\{ \iota_A(z)\iota_B(z)^{-1} \mid z \in Z \right\}.$$

Suppose $X = U_1 \cup U_2$ such that $U_1 \cap U_2 \neq \varnothing$ is **path connected** (necessary condition). Then taking $x_0 \in U \coloneqq U_1 \cap U_2$ yields a pushout of fundamental groups

$$\pi_1(X; x_0) = \pi_1(U_1; x_0) *_{\pi_1(U; x_0)} \pi_1(U_2; x_0).$$

Theorem (Van Kampen) If $X = U \cup V$ where $U, V, U \cap V$ are all path-connected then

$$\pi_1(X) = \pi_1 U *_{\pi_1(U \cap V)} \pi_1 V,$$

where the amalgamated product can be computed as follows: If we have presentations

$$\begin{aligned}
\pi_1(U, w) &= \left\langle u_1, \cdots, u_k \;\middle|\; \alpha_1, \cdots, \alpha_l \right\rangle \\
\pi_1(V, w) &= \left\langle v_1, \cdots, v_m \;\middle|\; \beta_1, \cdots, \beta_n \right\rangle \\
\pi_1(U \cap V, w) &= \left\langle w_1, \cdots, w_p \;\middle|\; \gamma_1, \cdots, \gamma_q \right\rangle
\end{aligned}$$

then

$$\begin{aligned}
\pi_1(X, w) = &\langle u_1, \cdots, u_k, v_1, \cdots, v_m \rangle \\
&\pmod{\left\langle \alpha_1, \cdots, \alpha_l, \beta_1, \cdots, \beta_n, I(w_1) J(w_1)^{-1}, \cdots, I(w_p) J(w_p)^{-1} \right\rangle} \\
= &\frac{\pi_1(U) * \pi_1(B)}{\left\langle \left\{ I(w_i) J(w_i)^{-1} \;\middle|\; 1 \le i \le p \right\} \right\rangle}
\end{aligned}$$

where

$$\begin{aligned}
I &: \pi_1(U \cap V, w) \to \pi_1(U, w) \\
J &: \pi_1(U \cap V, w) \to \pi_1(V, w).
\end{aligned}$$

Theorem (Seifert-van Kampen Theorem) Suppose $X = U_1 \cup U_2$ where $U \coloneqq U_1 \cap U_2 \neq \varnothing$ is path-connected, and let $\{\mathrm{pt}\} \in U$. Then the maps $i_1 : U_1 \to X$ and $i_2 : U_2 \to X$ induce the following group homomorphisms:

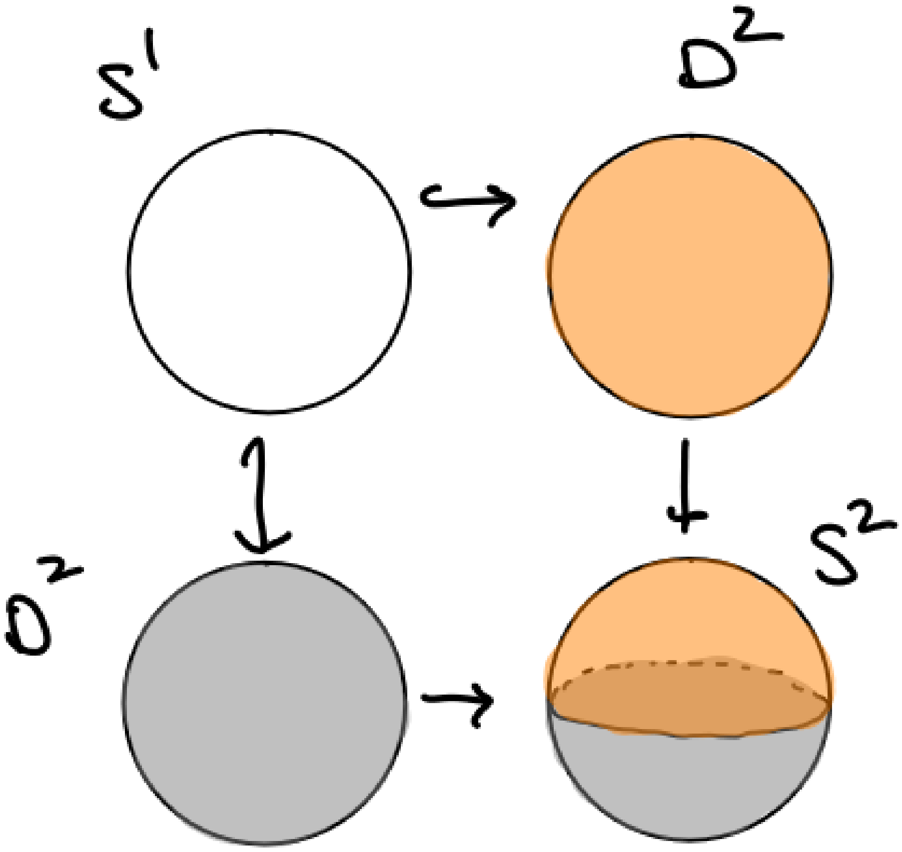
$$\begin{aligned}
i_1^* &: \pi_1(U_1, \{\mathrm{pt}\}) \to \pi_1(X, \{\mathrm{pt}\}) \\
i_2^* &: \pi_1(U_2, \{\mathrm{pt}\}) \to \pi_1(X, \{\mathrm{pt}\})
\end{aligned}$$

and letting $P = \pi_1(U), \{\mathrm{pt}\}$, there is a natural isomorphism

$$\pi_1(X, \{\mathrm{pt}\}) \cong \pi_1(U_1, \{\mathrm{pt}\}) \ *_P \ \pi_1(U_2, \{\mathrm{pt}\})$$

where $*_P$ is the amalgamated free product over P.

Formulate in terms of pushouts.



Note that the hypothesis that $U \cap V$ is path-connected is necessary: take S^1 with U, V neighborhoods of the poles, whose intersection is two disjoint components.

Example (of pushing out with Van Kampen) $A = \mathbb{Z}/4\mathbb{Z} = \left\langle x \;\middle|\; x^4 \right\rangle, B = \mathbb{Z}/6\mathbb{Z} = \left\langle y \;\middle|\; x^6 \right\rangle, Z = \mathbb{Z}/2\mathbb{Z} = \left\langle z \;\middle|\; z^2 \right\rangle$.

Then we can identify Z as a subgroup of A, B using $\iota_A(z) = x^2$ and $\iota_B(z) = y^3$.

So

$$A *_Z B = \left\langle x, y \;\middle|\; x^4, y^6, x^2 y^{-3} \right\rangle$$

.

Theorem 11.2.1 ***(Whitehead's Theorem).***
A map $X \xrightarrow{f} Y$ on CW complexes that is a weak homotopy equivalence (inducing isomorphisms in homotopy) is in fact a homotopy equivalence.

Warning 11.2.2
Individual maps may not work: take $S^2 \times \mathbb{RP}^3$ and $S^3 \times \mathbb{RP}^2$ which have isomorphic homotopy but not homology.

Theorem 11.2.3 ***(Hurewicz).***
The Hurewicz map on an $n-1$-connected space X is an isomorphism $\pi_{k\leq n}X \to H_{k\leq n}X$.

I.e. for the minimal $i \geq 2$ for which $\pi_i X \neq 0$ but $\pi_{\leq i-1}X = 0$, $\pi_i X \cong H_i X$.

Theorem 11.2.4 ***(Cellular Approximation).***
Any continuous map between CW complexes is homotopy equivalent to a cellular map.

Applications:

- $\pi_{k\leq n} S^n = 0$
- $\pi_n(X) \cong \pi_n(X^{(n)})$

Theorem 11.2.5 ***(Freudenthal Suspension).***

Todo

- $\pi_{i\geq 2}(X)$ is always abelian.
- The ranks of π_0 and H_0 are the number of path components, and $\pi_0(X) = \mathbb{Z}$ iff X is simply connected.
  - X simply connected $\implies \pi_k(X) \cong H_k(X)$ up to and including the first nonvanishing H_k
  - $H_1(X) = \mathrm{Ab}(\pi_1 X)$, the abelianization.
- $\pi_k \bigvee X \neq \prod \pi_k X$ (counterexample: $S^1 \vee S^2$)
  - Nice case: $\pi_1 \bigvee X = *\pi_1 X$ by Van Kampen.
- $\pi_i(\widehat{X}) \cong \pi_i(X)$ for $i \geq 2$ whenever $\widehat{X} \twoheadrightarrow X$ is a universal cover.
- $\pi_i(S^n) = 0$ for $i < n$, $\pi_n(S^n) = \mathbb{Z}$
  - Not necessarily true that $\pi_i(S^n) = 0$ when $i > n$!!!
    - E.g. $\pi_3(S^2) = \mathbb{Z}$ by Hopf fibration

- $S^n/S^k \simeq S^n \vee \Sigma S^k$
  - $\Sigma S^n = S^{n+1}$
- General mantra: homotopy plays nicely with products, homology with wedge products.[1]
- $\pi_k \prod X = \prod \pi_k X$ by LES.[2]

In general, homotopy groups behave nicely under homotopy pull-backs (e.g., fibrations and products), but not homotopy push-outs (e.g., cofibrations and wedges). Homology is the opposite.

Constructing a $K(\pi, 1)$: since $\pi = \left\langle S \mid R \right\rangle = F(S)/R$, take $\bigvee^{|S|} S^1 \cup_{|R|} e^2$. In English, wedge a circle for each generator and attach spheres for relations.

Proposition 11.2.6 *(Contracting Spaces in Products).*

$$X \times \mathbb{R}^n \simeq X \times \{\mathrm{pt}\} \cong X.$$

---

12 | The Fundamental Group (Unsorted)

12.1 Lemma: The fundamental group of a CW-complex only depends on the 1-skeleton, and $H^k(X)$ only depends on the k-skeleton.

12.2 Definition: Homotopy

Let X, Y be topological spaces and $f, g : X \to Y$ continuous maps. Then a *homotopy* from f to g is a continuous function

$F : X \times I \to Y$

[1] More generally, in **Top**, we can look at $A \leftarrow \{\mathrm{pt}\} \to B$ – then $A \times B$ is the pullback and $A \vee B$ is the pushout. In this case, homology $h : \mathbf{Top} \to \mathbf{Grp}$ takes pushouts to pullbacks but doesn't behave well with pullbacks. Similarly, while π takes pullbacks to pullbacks, it doesn't behave nicely with pushouts.

[2] This follows because $X \times Y \twoheadrightarrow X$ is a fiber bundle, so use LES in homotopy and the fact that $\pi_{i \geq 2} \in \mathbf{Ab}$.

such that

$F(x,0) = f(x)$ and $F(x,1) = g(x)$

for all $x \in X$. If such a homotopy exists, we write $f \simeq g$. This is an equivalence relation on $\mathrm{Hom}(X,Y)$, and the set of such classes is denoted $[X,Y] \coloneqq \hom(X,Y)/\simeq$.

12.3 Definition: Nullhomotopic

If f is homotopic to a constant map, say $f : x \mapsto y_0$ for some fixed $y_0 \in Y$, then f is said to be *nullhomotopic*. In other words, if $f : X \to Y$ is nullhomotopic, then there exists a homotopy $H : X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = y_0$.

Note that constant maps (or anything homotopic) induce zero homomorphisms.

13 Theorem: Any two continuous functions into a convex set are homotopic.

Proof: The linear homotopy. Supposing X is convex, for any two points $x, y \in X$, the line $tx+(1-t)y$ is contained in X for every $t \in [0,1]$. So let $f, g : Z \to X$ be any continuous functions into X. Then define $H : Z \times I \to X$ by $H(z,t) = tf(z)+(1-t)g(z)$, the linear homotopy between f, g. By convexity, the image is contained in X for every t, z, so this is a homotopy between f, g.

13.1 Definition: Homotopy Equivalence

Let $f : X \to Y$ be a continuous map, then f is said to be a *homotopy equivalence* if there exists a continuous map $g : X \to Y$ such that

$f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Such a map g is called a homotopy inverse of f, the pair of maps is a homotopy equivalence.

If such an f exists, we write $X \simeq Y$ and say X and Y have the same homotopy type, or that they are homotopy equivalent.

Note that homotopy equivalence is strictly weaker than homeomorphic equivalence, i.e., $X \cong Y$ implies $X \simeq Y$ but not necessarily the converse.

13.2 Definition: Contractible

A topological space X is *contractible* if X is homotopy equivalent to a point, i.e. $X \simeq \{x_0\}$. This means that there exists a pair of homotopy inverses $f : X \to \{x_0\}$ and $g : \{x_0\} \to X$ such that $f \circ g = \mathrm{id}_{\{x_0\}}$ and $g \circ f = \mathrm{id}_X$.

This is a useful property, because it supplies you with a homotopy.

14 Definition: Deformation Retract

Let X be a topological space and $A \subset X$ be a subspace, then a *retraction* of X onto A is a map $r : X \to X$ such that the image of X is A and r restricted to A is the identity map on A.

Note that this definition isn't very useful, as every space has at least one retraction - for example, the constant map $r : X \to \{x_0\}$ for any $x\_0 \in X$.

A *deformation retract* is a homotopy $H : X \times I \to X$ from the identity on X to the identity on A that fixes A at all times. That is,

$$\begin{aligned} H : X \times I &\to X \\ H(x,0) &= \mathrm{id}_X \\ H(x,1) &= \mathrm{id}_A \\ x \in A \implies H(x,t) &\in A \quad \forall t \end{aligned}$$

Equivalently, this requires that $H|_A = \mathrm{id}_A$

A deformation retract between a space and a subspace is a homotopy equivalence, and further $X \simeq Y$ iff there is a Z such that both X and Y are deformation retracts of Z. Moreover, if A and B both have deformation retracts onto a common space X, then $A \simeq B$.

14.1 Definition: The Fundamental Group / 1st Homotopy Group

Given a pointed space (X, x_0), we define the fundamental group $\pi_1(X)$ as follows:

- Take the set $L = \left\{ \alpha : S^1 \to X \,\middle|\, \alpha(0) = \alpha(1) = x_0 \right\}$.
- Define an equivalence relation $\alpha \sim \beta$ iff there exists a homotopy $H : S^1 \times I \to X$ such that $H(s,0) = \alpha(s)$ and $H(s,1) = \beta(s)$, i.e. if $f \simeq g$ in X.

  - Symmetric:
  - Reflexive:
  - Transitive:

- Define $L/\sim$, which contains elements like $[\alpha]$ and $[\mathrm{id}_{x_0}]$, the equivalence classes of loops after quotienting by this relation.
- Define a product structure: for $[\alpha], [\beta] \in L/\sim$, define $[\alpha][\beta] = [\alpha \cdot \beta]$, where we just need to define a product structure on bona fide loops. Just do this by reparameterizing: $(f \cdot g)(s) = \nVdash[s \in \left[0, \frac{1}{2}\right]] f(2s) + \nVdash[s \in \left[\frac{1}{2}, 1\right]] g(2s-1)$
  - Continuous: by the pasting lemma and assumed continuity of f, g
  - Well-defined:
- Check that this is actually a group
  - Identity element:
  - Closure:
  - Associativity:
  - Inverses:
- Summary:
  - Elements of the fundamental group are *homotopy classes of loops*.
  - Continuous maps between spaces induce *some* homomorphism on fundamental groups.
    - ◊ Inclusions

15 | Theorem: X is simply connected iff it has trivial fundamental group.

By definition, X is simply connected iff X is path connected and every loop contracts to a point.

$\Rightarrow$: Suppose X is simply connected. Then every loop in X contracts to a point, so if α is a loop in X, $[\alpha] = [\mathrm{id}_{x_0}]$, the identity element of $\pi_1(X)$. But then there is only one element in in this group.

$\Leftarrow$: Suppose $\pi_1(X) = 0$. Then there is just one element in the fundamental group, the identity element, so if α is a loop in X then $[\alpha] = [\mathrm{id}_{x_0}]$. So there is a homotopy taking α to the constant map, which is a contraction of α to a point.

16 | Covering Spaces

When covering spaces are involved in any way, try computing Euler characteristics - this sometimes yields nice numerical constraints.

Picture to keep in mind

Path lifting:

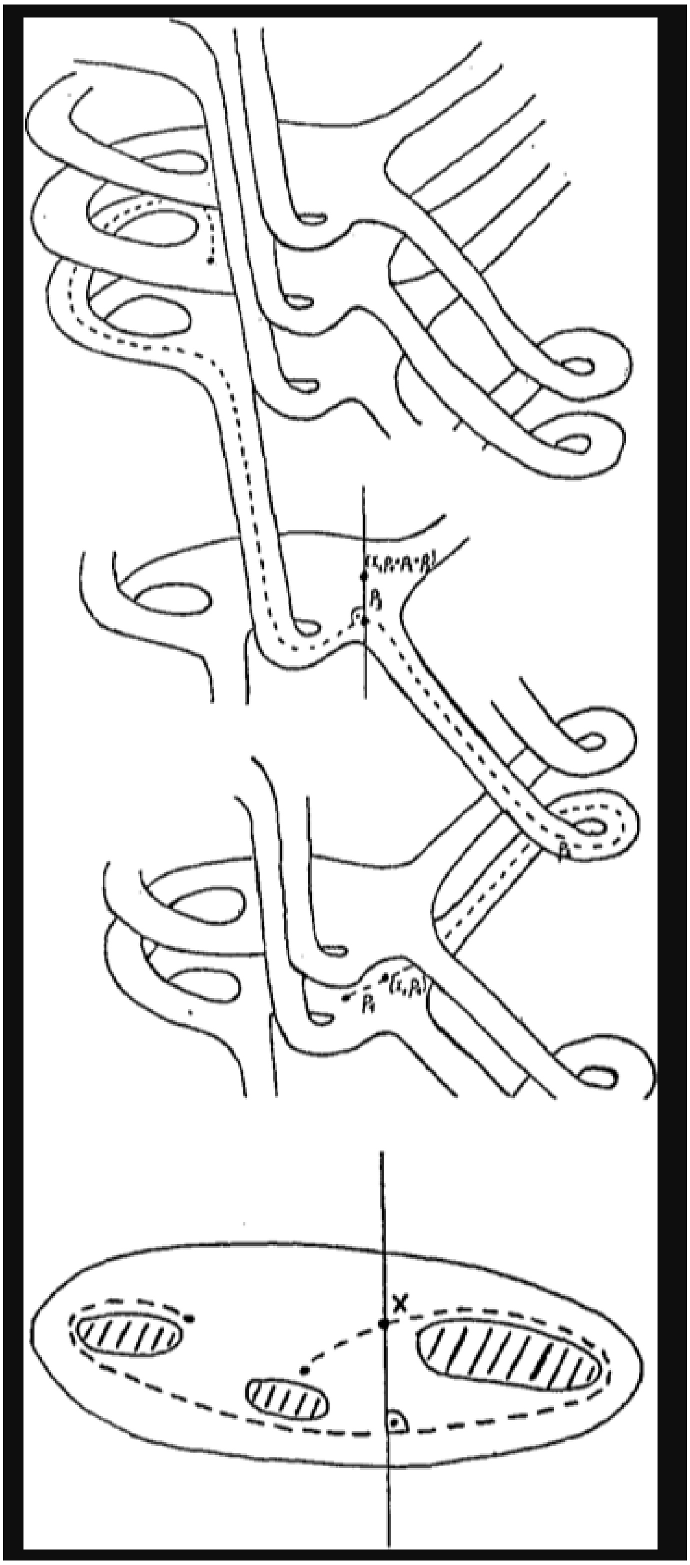


Figure 2: image_2021-01-09-00-19-03

16.1 Useful Covering Spaces

- $\mathbb{R} \xrightarrow{\pi} S^1 \leftarrow \mathbb{Z}$
- $\mathbb{R}^n \xrightarrow{\pi} T^n \leftarrow \mathbb{Z}^n$
- $\mathbb{RP}^n \xrightarrow{\pi} S^n \leftarrow \mathbb{Z}_2$
- $\vee_n S^1 \xrightarrow{\pi} C^n \leftarrow \mathbb{Z}^{*n}$ where C^n is the n-valent Cayley Graph
- $M \xrightarrow{\pi} \tilde{M} \leftarrow \mathbb{Z}_2$, the orientation double cover
- $T^2 \xrightarrow{\times 2} \mathbb{K}$
- $L_{p/q} \xrightarrow{\pi} S^3 \leftarrow \mathbb{Z}_q$
- $\mathbb{C}^* \xrightarrow{z^n} \mathbb{C} \leftarrow \mathbb{Z}_n$
- For $A \xrightarrow{\pi(\times d)} B$, we have $\chi(A) = d\chi(B)$
- Covering spaces of orientable manifolds are orientable.

16.2 Theorems

Theorem 16.2.1 ***(Lifts to Universal Cover (H. 1.33)).***
If $f : Y \to X$ with Y path-connected and locally path-connected, then there is a unique lift $\widehat{f} : Y \to \widehat{X} \iff f_*(\pi_1 Y) \subset \pi_*(\pi_1 \widehat{X})$.

16.3 Useful Facts

- Covering maps inject fundamental groups.
  - If $\tilde{X} \twoheadrightarrow_p X$ is a covering space, then $\pi_1(\tilde{X}) \hookrightarrow \pi_1(X)$ as a subgroup.
- The preimage of a boundary point under a covering map must also be a boundary point
- An n-sheeted covering space $\tilde{X} \twoheadrightarrow X$ satisfies $\chi(\tilde{X}) = n\chi(X)$ when $\tilde{X}$ is compact.
- For surfaces, covering spaces satisfy $\Sigma_{ij+1} \twoheadrightarrow \Sigma_{i+1}$ for some i, j.
- $\operatorname{Deck}(\tilde{X}) \coloneqq \left\{ \varphi \in \hom_{\mathbf{Top}}(\tilde{X}, \tilde{X}) : p \circ \varphi = p \right\} \cong \pi_1(X)$
- $\tilde{X} \twoheadrightarrow_{\times k} X \implies [\pi_1(\tilde{X}) : \pi_1(X)] = k$ where $k = |p^{-1}(x_0)|$

- Normal subgroups correspond to regular coverings (where automorphisms act freely/transitively, so highly symmetric)

16.4 Definition: Covering Maps

A covering map of a space is a map $p : \tilde{X} \to X$ such that each open set $U \in X$ pulls back to a disjoint union of open sets (called sheets) in $\tilde{X}$ (referred to as the covering space). That is, $p^{-1}(U) = \coprod_i V_i \subseteq \tilde{X}$.

If $\tilde{X}$is simply connected, it is the universal covering space - that is, for any other covering space Y of X, $\tilde{X}$ is also a cover of Y. We also have $\mathrm{Aut}(\tilde{X}) \cong \pi_1(X)$ for universal covers - for other covers, $\mathrm{Aut}(\tilde{X}) \cong N(\Gamma)/\Gamma$ where $N(\cdot)$ is the normalizer and Γ is the set of homotopy classes of loops in $\tilde{X}$ that are lifted from loops in X.

Covering spaces of X are in (contravariant) galois correspondence with subgroups of $\pi_1(X)$, i.e. the covering map induces an injective map on fundamental groups.

The number of sheets of a covering space is equal to $[p^*(\pi_1(\tilde{X})) : \pi_1(X)]$.

16.4.1 Example: Covering spaces

Identify $S^1 \subset \mathbb{C}$, then every map $p_n : S^1 \to S^1$ given by $z \mapsto z^n$ a yields a covering space $\tilde{X}_n$. Note the induced map $p_n^* : \pi_1(S^1) \to \pi_1(S^1)$ is given by $[\omega_1] \mapsto [\omega_n] = n[\omega_1]$ and so $p_n^*(\pi_1(S^1)) = \mathbb{Z}_n = \mathrm{Aut}(\tilde{X}_n)$. (This can also be seen the other way, by looking at deck transformations which are rotations of the circle by $2\pi/n$)

The universal cover of S^1 is $\mathbb{R}$; this is an infinitely sheeted cover. The fiber above x_0 is equal to $\mathbb{Z}$. $A \coloneqq B$

The universal cover of $\mathbb{RP}^n$ is S^n; this is a two-sheeted cover. The fiber above x_0 contains the two antipodal points.

The universal cover of $T = S^1 \times S^1$ is $\tilde{X} = \mathbb{R} \times \mathbb{R}$. The fiber above the base point contains every point on the integer lattice $\mathbb{Z} \times \mathbb{Z} = \pi_1(T) = \mathrm{Aut}(\tilde{X})$

16.5 Theorem: Homotopy Lifting

The setup: given $p : \tilde{X} \twoheadrightarrow X$ a covering space of X, a map $f : Y \to X$, and a homotopy $H : Y \times I \to X$ such that $f_0 \coloneqq H(y, 0)$ has a lift $\tilde{f}_0 : Y \to \tilde{X}$.

Then there is a unique homotopy $\tilde{H} : Y \times I \to \tilde{X}$ satisfying $p \circ \tilde{H} = H$ In other words, if the $t = 0$ portion of a homotopy can be lifted to a cover, the entire homotopy can.

16.6 Theorem: Lifting Criterion

Let $p : \tilde{X} \twoheadrightarrow X$ be a covering of X, and let $f : Y \to X$ be a map. Then there is an induced homomorphism $f^* : \pi_1(Y) \to \pi_1(X)$. There is also an induced map $p^* : \pi_1(\tilde{X}) \to \pi_1(X)$. We then have the following condition:

There exists a lift $\tilde{f} : Y \to \tilde{X}$ satisfying $p \circ \tilde{f} = f$ iff $f^*(\pi_1(Y)) \subseteq p^*(\pi_1(\tilde{X}))$, i.e. when the fundamental group of Y injects into the projected fundamental group of the cover.

Note that if Y is simply connected, then $\pi_1(Y) = 0$ and this holds automatically!

Moreover, lifts are *unique* if they agree at a single point.

(Technically you need the base space to be connected and "locally pathwise connected")

16.7 Theorem: Fundamental theorem of covering spaces

For every subgroup $G \leq \pi_1(X)$, there is a corresponding covering space $X_G \twoheadrightarrow X$ such that $\pi_1(X_G) = G$. The universal cover is obtained by taking G to be the trivial group.

Alternative phrasing: there is a contravariant, inclusion-reversing map from subgroups of $\pi_1(X)$ to covering spaces of X.

16.8 Theorem: If Y is contractible, every map $f : X \to Y$ is nullhomotopic.

If Y is contractible, then Y has the homotopy type of a point. So there is a homotopy $H : Y \times I \to Y$ between id_Y and a constant map $c : y \mapsto y_0$. So construct $H' : X \times I \to Y$ as $H'(x,t) = H(f(x),t)$; then $H'(x,0) = H(f(x),0) = (\mathrm{id}_x \circ f)(x) = f(x)$ and $H'(x,1) = H(f(x),1) = (c \circ f)(x) = c(y) = y_0$ for some y. So H' is a homotopy between f and a constant map, and f is nullhomotopic.

16.9 Theorem: Any map that factors through a contractible space is nullhomotopic.

Suppose we have the following commutative diagram:

Then $f = p \circ \tilde{f}$. Every map into a contractible space is nullhomotopic, so if Z is contractible, then there is a homotopy $\tilde{H} : X \times I \to Z$ from $\tilde{f}$ to a constant map c. But then $p \circ \tilde{H} : X \times I \to Y$ is also a homotopy from f to the constant map $p \circ c$.

16.10 Application: Showing when there is no covering map $f : X \to Y$

This can be done by lifting f to $\tilde{f} : X \to \tilde{Y}$, the universal cover. If the covering space happens to be contractible, you get that $\tilde{f}$ is nullhomotopic. So there is a homotopy $\tilde{H} : X \times I \to \tilde{Y}$ - but then $p \circ \tilde{H} : X \to Y$ descends to a homotopy of f. If you leave f arbitrary, this would force $\pi_1(Y) = 0$.

17 Definition: Monodromy Action

Given X connected and locally connected, $p : \tilde{X} \to X$ a covering, and α a loop at $x \in X$, let $\tilde{\alpha}$ be its lift and $\tilde{x} \in p^{-1}(x)$ be the lifted point in the fiber above x. Then α acts on $\tilde{x}$ from the right, by the rule $\tilde{x} \backsim \alpha = \tilde{\alpha}(1)$.

Then $\mathrm{stab}(\tilde{x}) = p_*(\pi_1(\tilde{X}, \tilde{x})) \subseteq \pi_1(X, x)$, and this induces a homomorphism $\pi_1(X, x) \to \mathrm{Aut}(p^{-1}(x))$ which is a permutation of elements in the fiber above x.

17.1 Definition: Freely and Properly Discontinuous Group Actions

Todo

17.2 Theorem: If G induces a free and properly discontinuous group action on X, then $p : X \to X/G$ is a covering space

Here X/G denotes $X/\sim$ where $\forall x, y \in X, x \sim y \iff \exists g \in G \mid g.x = y$, i.e. all elements in a single orbit are identified.

17.2.1 Proof:

Construct a map $\varphi : G \to \pi_1(X/G, G.x_0)$ by $g \mapsto [p \circ \gamma_g]$

where $\gamma_g(0) = x_0$ and $\gamma_g(1) = G.x_0$.

- This is homomorphism:
- This is well-defined:

17.3 Application: Fundamental group of the circle

17.4 Application: Fundamental group of the real projective plane

17.5 Constructing Covering Spaces

For a wedge product $X = \bigvee_i^n \tilde{X}_i$, the covering space $\tilde{X}$ is constructed as a tree in which each $\tilde{X}_i$ is a vertex with one of i colors denoting which space it covers. The neighborhood of each colored vertex has edges corresponding to $\pi_1(X_i)$.

If X and Y are two reasonable spaces with universal covers $\tilde{X}$ and $\tilde{Y}$, there is a nice picture of the universal cover $\widetilde{X \vee Y}$ which has the combinatorial pattern of an infinite tree. The tree is bipartite with vertices labeled by the symbols X and Y. The edges from an X vertex are bijective correspondence with the fundamental group $\pi_1(X)$, and likewise for Y vertices and $\pi_1(Y)$. To make $\widetilde{X \vee Y}$, replace each X vertex by $\tilde{X}$ and each Y vertex by $\tilde{Y}$. The base point of X lifts to $|\pi_1(X)|$ points in $\tilde{X}$, and likewise for Y. In $\widetilde{X \vee Y}$, copies of $\tilde{X}$ are attached to copies of $\tilde{Y}$ at lifts of base points.

Example: $S^1 \vee S^1 \to \mathbb{Z} * \mathbb{Z}$

Example: $\mathbb{RP}^2 \vee \mathbb{RP}^2 \to \mathbb{Z}_2 * \mathbb{Z}_2$

Example: $\mathbb{RP}^2 \vee T^2 \to \mathbb{Z}_2 * \mathbb{Z}$

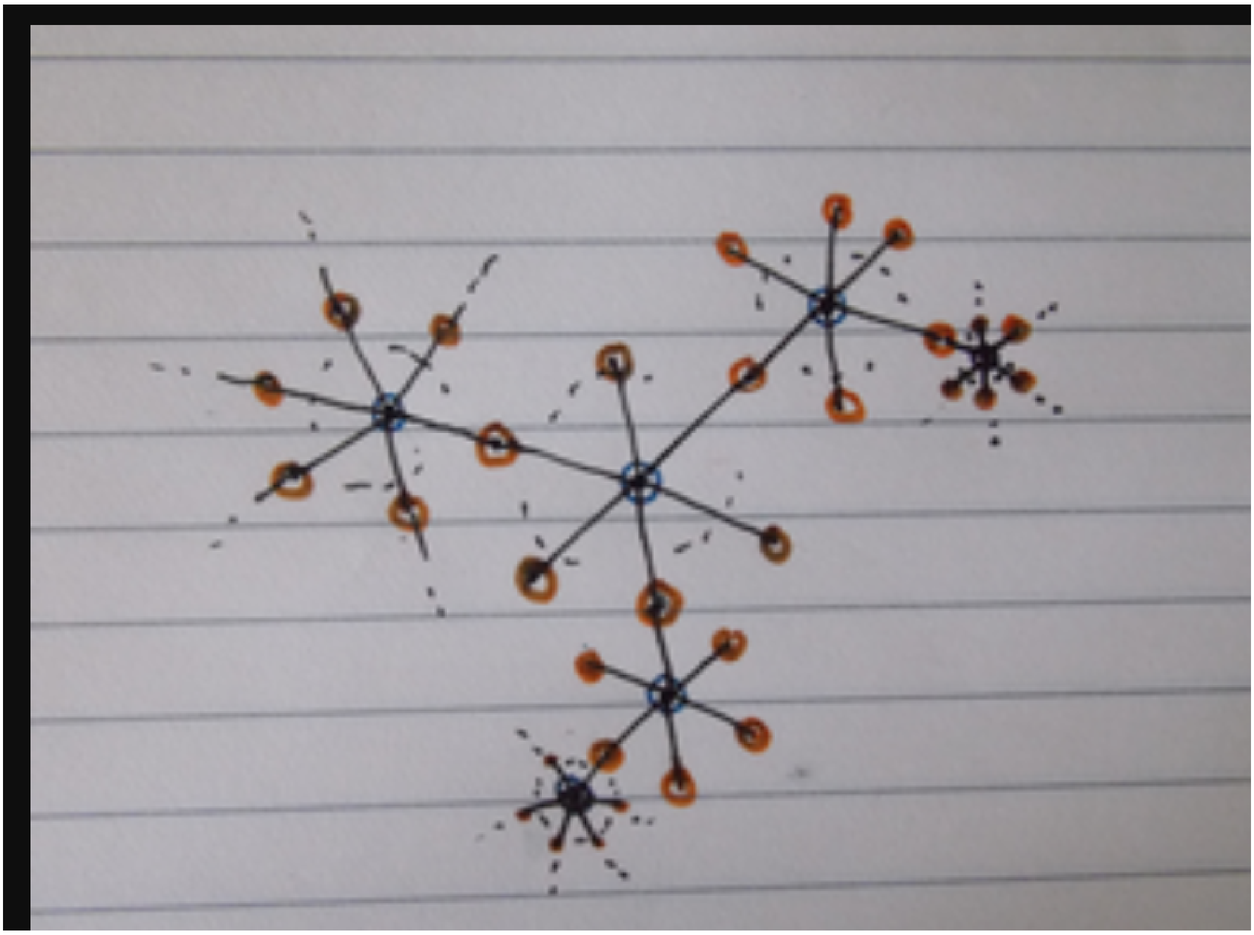


Figure 3: Image

17.6 Application: Every subgroup of a free group is free

Idea for a particular case: use the fact that $\pi_1(\bigvee^k S^1) = \mathbb{Z}^{*k}$, so if $G \leq \mathbb{Z}^{*k}$ then there is a covering space $X \twoheadrightarrow \bigvee^k S^1$ such that $\pi_1(X) = G$. Since X can be explicitly constructed as a graph, i.e. a CW complex with only a 1-skeleton, $\pi_1(X)$ is free on its maximal tree. ■

18 | CW and Simplicial Complexes

18.1 Useful Facts

- To build a Moore space $M(n, \mathbb{Z}_p)$, take $X = S^n$ and attach e^{n+1} via a map $\Phi : S^n = \partial B^{n+1} \to X^{(n)} = S^n$ of degree p.
  - To obtain $M(n, \prod G_i)$ take the corresponding $\bigvee X_i$
  - Can also use Mayer Veitoris to conclude $H_{n+1}(\Sigma X) = H_n(X)$, and just suspend spaces with known homology.

18.2 Theorem: Van Kampen's Theorem

Claim: If $X = U \cup V$ and $U \cap V$ is nonempty and "nice", then $\pi_1(X) = \pi_1(U) \ *_{\pi_1(U \cap V)} \ \pi_1(V)$.

18.2.1 Proof

- Construct a map going backwards
- Show it is surjective
  - "There and back" paths
- Show it is injective
  - Divide $I \times I$ into a grid

18.3 Definition: CW Complex

18.3.1 Examples

- $\mathbb{RP}^n = e^1 \cup e^2 \cup \cdots \cup e^n$
- $\mathbb{CP}^n = e^2 \cup e^4 \cup \cdots e^{2n}$
- $S^\infty = \varinjlim S^n$

19 | Definition: The Degree of Map $S^n \to S^n$

Given any $f : S^n \to S^n$, there are induced maps on homotopy and homology groups. Taking $f^* : H^n(S^n) \to H^n(S^n)$ and identifying $H^n(S^n) \cong \mathbb{Z}$, we have $f^* : \mathbb{Z} \to \mathbb{Z}$. But homomorphisms of this type are entirely determined by their action on generators. So if $f^*(1) = n$, define n to be the degree of f.

Properties and examples:

- $\deg\ \mathrm{id}_{S^n} = 1$
- $\deg(f \circ g) = \deg f \cdot \deg g$
- $\deg r = -1$ where r is any rotation about a hyperplane, i.e. $r([x_1 \cdots x_i \cdots x_n]) = [x_1 \cdots - x_i \cdots x_n]$.
- The antipodal map on $S^n \subset \mathbb{R}^{n+1}$ is the composition of $n+1$ reflections, so $\deg \alpha = (-1)^{n+1}$.

20 | Definition: Simplicial Complex

Given a simplex $\sigma = [v_1 \cdots v_n]$, define the face map $\partial_i : \Delta^n \to \Delta^{n-1}$, where $\partial_i \sigma = [v_1 \cdots \widehat{v_i} \cdots v_n]$.

A simplicial complex is a set K satisfying

1. $\sigma \in K \implies \partial_i \sigma \in K$
2. $\sigma, \tau \in K \implies \sigma \cap \tau = \varnothing,\ \partial_i \sigma,\ \text{or}\ \partial_i \tau$

    1. This amounts to saying that any collection of $(n-1)$-simplices uniquely determines an n-simplex (or its lack thereof), or that that map $\Delta^k \to X$ is a continuous injection from the standard simplex in $\mathbb{R}^n$.

3. $|K \cap B_\varepsilon(\sigma)| < \infty$ for every $\sigma \in K$, identifying $\sigma \subseteq \mathbb{R}^n$.

To write down a simplicial complex, label the vertices with increasing integers. Then each n-cell will correspond to a set of $n+1$ of these integers - throw them in a list.

20.1 Examples of Simplicial Complexes

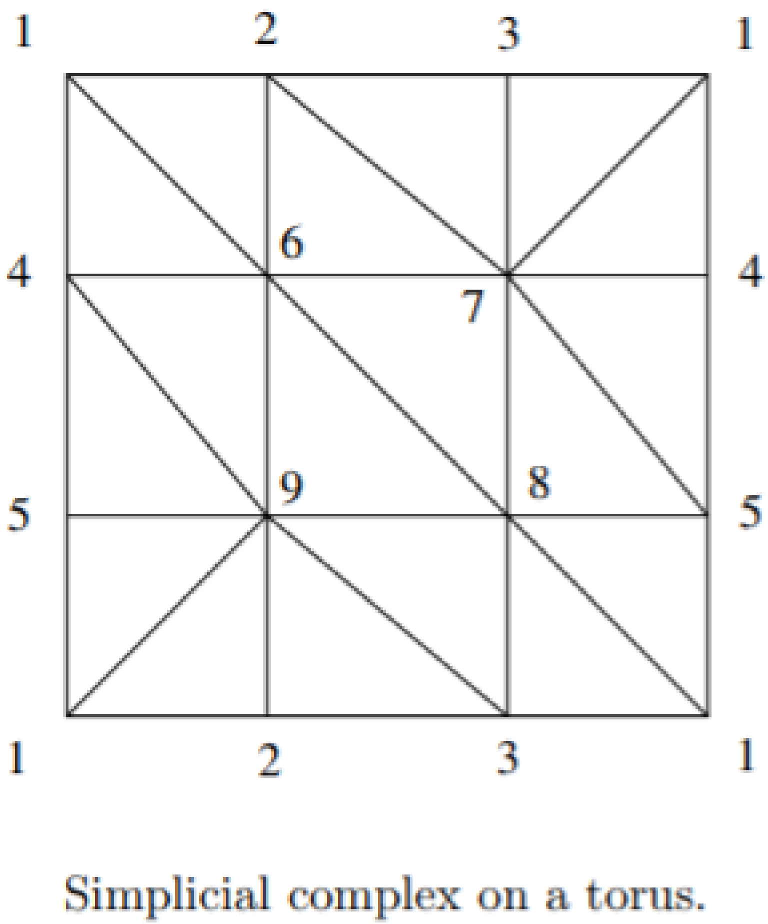


Figure 4: Torus

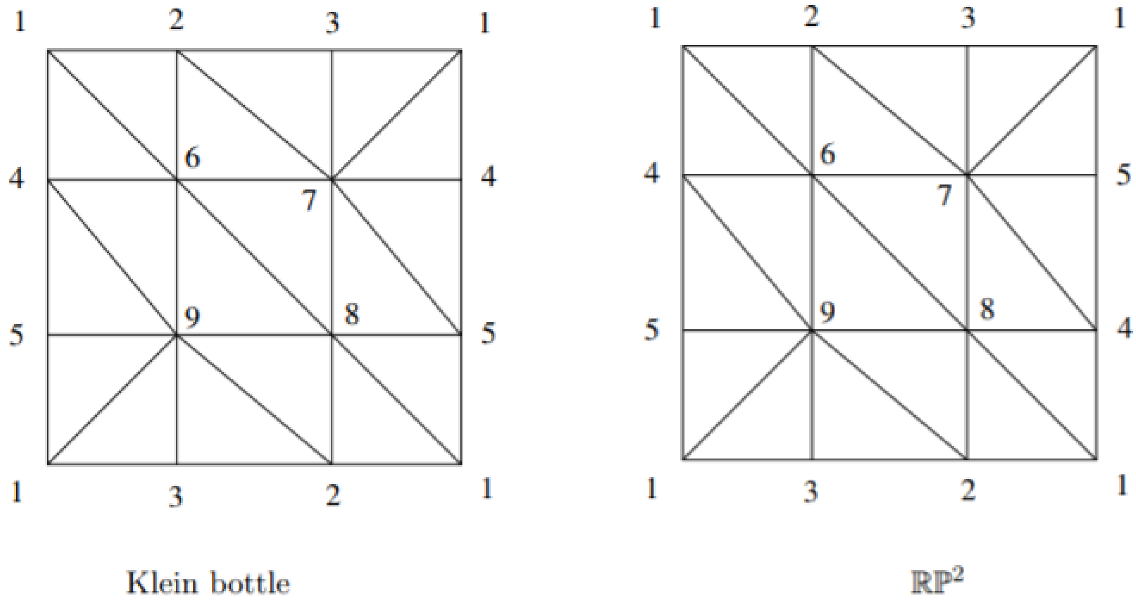


Figure 5: Klein Bottle and $\mathbb{RP}^2$

For counterexamples, note that this fails to be a triangulation of T:

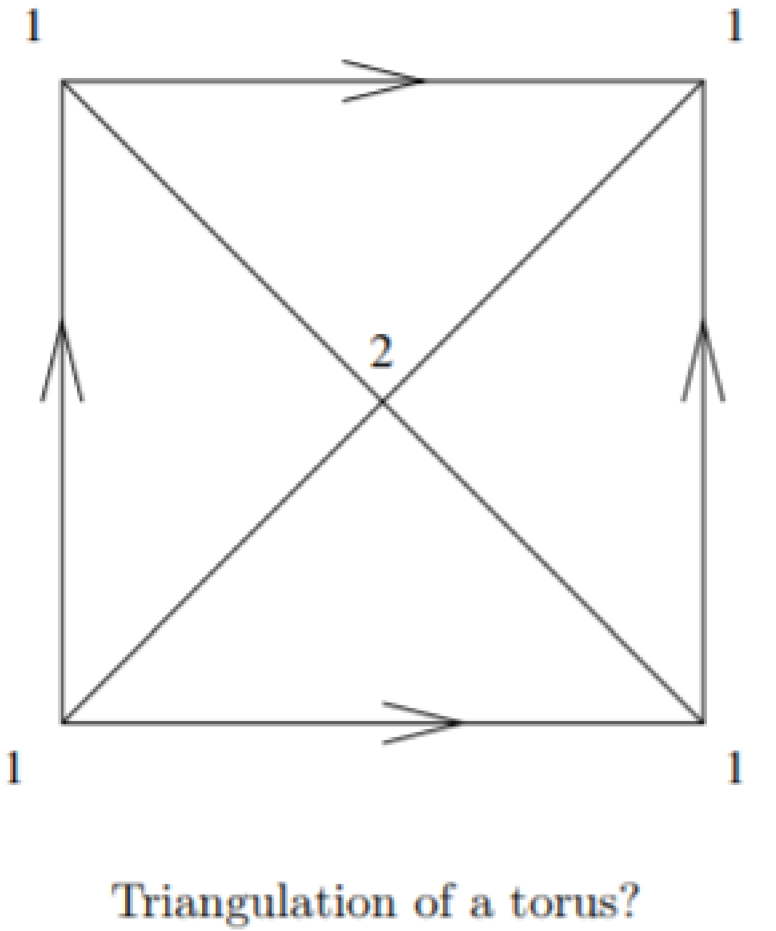


Figure 6: Not a Torus

This fails - for example, the simplex $[1, 2, 1]$ does not uniquely determine a triangle in the above picture.

20.2 Templates for Triangulation

You can always triangulate a space by triangulating something homeomorphic, so for common spaces you can work with these fundamental domains:

Examples 4.17. The common surfaces $\mathbb{S}^2$, $\mathbb{T}^2$, K and $\mathbb{P}^2$ all have presentations:

(1) The sphere: $\langle a \mid aa^{-1}\rangle$ or $\langle a, b \mid abb^{-1}a^{-1}\rangle$
(2) The torus: $\langle a, b \mid aba^{-1}b^{-1}\rangle$
(3) The projective plane: $\langle a \mid aa\rangle$ or $\langle a, b \mid abab\rangle$
(4) The Klein Bottle: $\langle a, b \mid abab^{-1}\rangle$

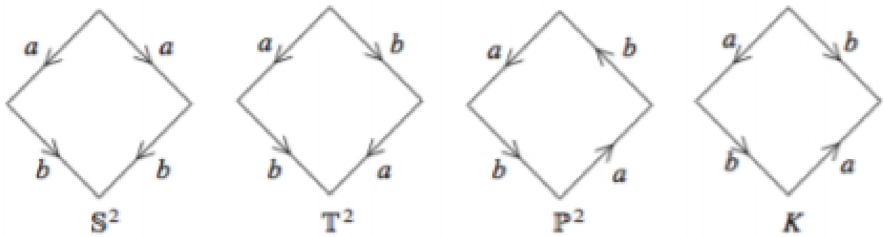


FIGURE 14. Polygonal presentation of $\mathbb{S}^2$, $\mathbb{T}^2$, $\mathbb{P}^2$, and K.

Figure 7: 1513064067523

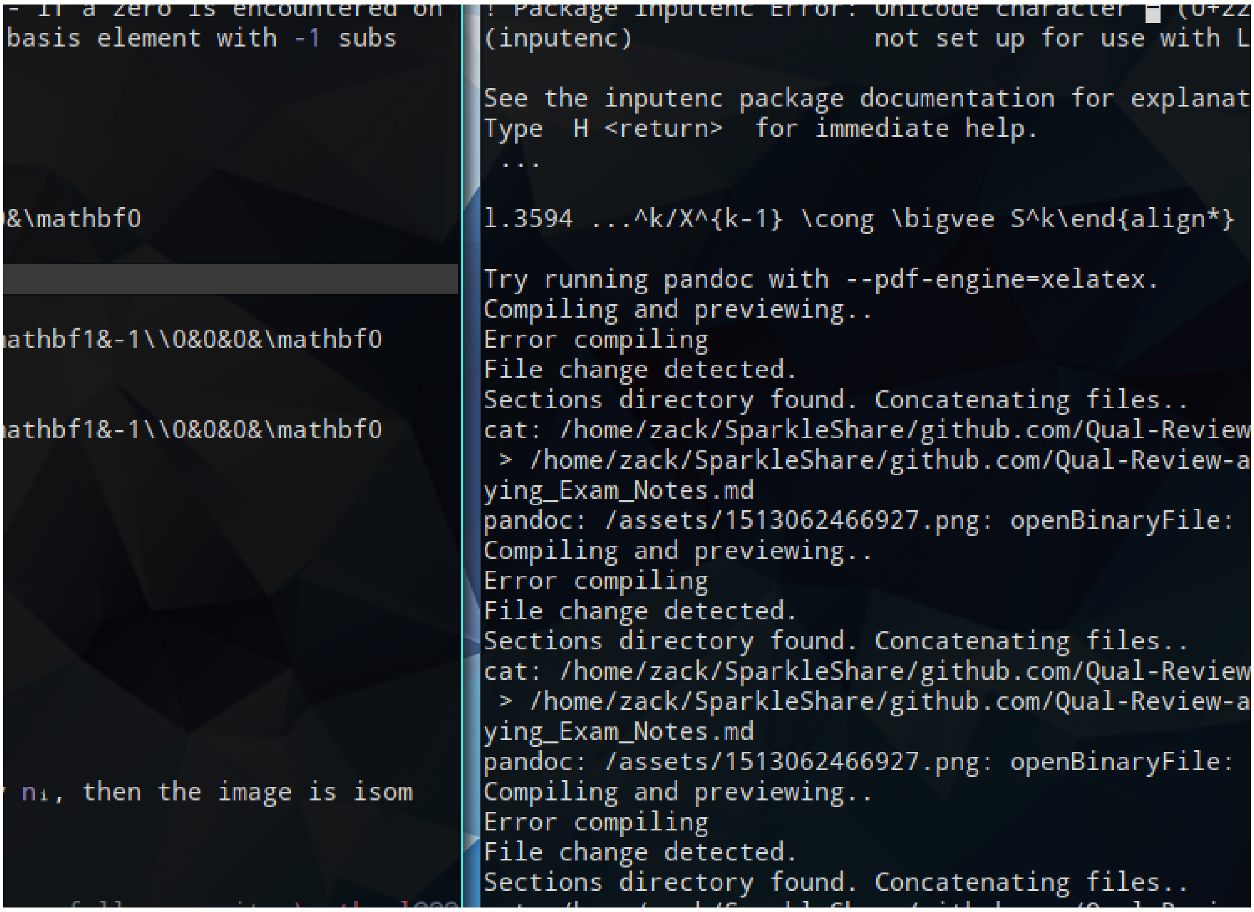


Figure 8: Image

21 Homology

21.1 Unsorted

- $H_n(X/A) \cong \tilde{H}_n(X, A)$ when $A \subset X$ has a neighborhood that deformation retracts onto it.
- $H_n(\bigvee_\alpha X_\alpha) = \bigoplus_\alpha H_n X_\alpha$
- Useful fact: since $\mathbb{Z}$ is free, any exact sequence of the form $0 \to \mathbb{Z}^n \to A \to \mathbb{Z}^m \to 0$ splits and $A \cong \mathbb{Z}^n \times \mathbb{Z}^m$.
- Useful fact: $\tilde{H}_*(A \vee B) \cong H_*(A) \times H_*(B)$.

- $H_n(\bigvee_\alpha X_\alpha) = \bigoplus_\alpha H_n X_\alpha$
- $H_n(X, A) \cong H_n(X/A)$
- $H_n(X) = 0 \iff X$ has no n-cells.
- $C^0 X = \{\mathrm{pt}\} \implies d_1 : C^1 \to C^0$ is the zero map.
- $H^*(X; \mathbb{F}) = \hom(H_*(X; \mathbb{F}), \mathbb{F})$ for a field.
- Useful tools:
  - Mayer-Vietoris
    - $(X = A \cup B) \mapsto (\cap, \oplus, \cup)$ in homology
  - LES of a pair
    - $(A \hookrightarrow X) \mapsto (A, X, X/A)$
  - Excision
- $H_k \prod X \neq \prod H_k X$ due to torsion.
  - Nice case: $H_k(A \times B) = \prod_{i+j=k} H_i A \otimes H_j B$ by Kunneth when all groups are torsion-free.[3]
- $H_k \bigvee X = \prod H_k X$ by Mayer-Vietoris.[4]
- $H_i(S^n) = \mathbb{1}[i \in \{0, n\}]$
- $H_n(\bigvee_i X_i) \cong \prod_i H_n(X_i)$ for "good pairs"
  - Corollary: $H_n(\bigvee_k S^n) = \mathbb{Z}^k$

$$X = A \cup B \implies A \cap B \to A \oplus B \to A \cup B \xrightarrow{\delta} \cdots (X, A) \implies \qquad A \to X \to X, A \xrightarrow{\delta} \cdots$$

$$A \to B \to C \implies \mathrm{Tor}(A, G) \to \mathrm{Tor}(B, G) \to \mathrm{Tor}(C, G) \xrightarrow{\delta_\downarrow} \cdots$$

$$A \to B \to C \implies \mathrm{Ext}(A, G) \to \mathrm{Ext}(B, G) \to \mathrm{Ext}(C, G) \xrightarrow{\delta_\uparrow} \cdots$$

[3] The generalization of Kunneth is as follows: write $\mathcal{P}(n, k)$ be the set of partitions of n into k parts, i.e. $\mathbf{x} \in \mathcal{P}(n, k) \implies \mathbf{x} = (x_1, x_2, \ldots, x_k)$ where $\sum x_i = n$. Then

$$H_n\left(\prod_{j=1}^k X_j\right) = \bigoplus_{\mathbf{x} \in \mathcal{P}(n,k)} \bigotimes_{i=1}^k H_{x_i}(X_i).$$

[4] $\bigvee$ is the coproduct in the category $\mathbf{Top}_0$ of pointed topological spaces, and alternatively, $X \vee Y$ is the pushout in $\mathbf{Top}$ of $X \leftarrow \{\mathrm{pt}\} \to Y$

21.2 Constructing a CW Complex with Prescribed Homology

- Given $G = \bigoplus G_i$, and want a space such that $H_i X = G$? Construct $X = \bigvee X_i$ and then $H_i(\bigvee X_i) = \bigoplus H_i X_i$. Reduces problem to: given a group H, find a space Y such that $H_n(Y) = G$.
  - Attach an e^n to a point to get $H_n = \mathbb{Z}$
  - Then attach an e^{n+1} with attaching map of degree d to get $H_n = \mathbb{Z}_d$

21.3 Mayer-Vietoris

Theorem (Mayer Vietoris) Let $X = A^\circ \cup B^\circ$; then there is a SES of chain complexes

$$0 \to C_n(A \cap B) \xrightarrow{x \mapsto (x,-x)} C_n(A) \oplus C_n(B) \xrightarrow{(x,y) \mapsto x+y} C_n(A+B) \to 0$$

where $C_n(A+B)$ denotes the chains that are sums of chains in A and chains in B. This yields a LES in homology:

$$\cdots \to H_n(A \cap B) \xrightarrow{x \mapsto (x,-x)} H_n(A) \oplus H_n(B) \xrightarrow{(x,y) \mapsto x+y} H_n(X) \to \cdots.$$

Given $A, B \subset X$ such that $A^\circ \cup B^\circ = X$, there is a long exact sequence in homology:

$$\begin{tikzcd}
 & & \cdots \arrow[dll, "\delta_3"'] \\
H_2(A \cap B) \arrow[r, "(i^*,-j^*)_2"] & H_2 A \oplus H_2 B \arrow[r, "(l^*-r^*)_2"] & H_2(A \cup B) \arrow[dll, "\delta_2"'] \\
H_1(A \cap B) \arrow[r, "(i^*,-j^*)_1"] & H_1 A \oplus H_1 B \arrow[r, "(l^*-r^*)_1"] & H_1(A \cup B) \arrow[dll, "\delta_1"'] \\
H_0(A \cap B) \arrow[r, "(i^*,-j^*)_0"] & H_0 A \oplus H_0 B \arrow[r, "(l^*-r^*)_0"] & H_0(A \cup B) \arrow[dll, "\delta_0"'] \\
0 & &
\end{tikzcd}$$

This is sometimes written in the following compact form:

$$\cdots H_n(A \cap B) \xrightarrow{(i^*,\ j^*)} H_n(A) \oplus H_n(B) \xrightarrow{l^* - r^*} H_n(X) \xrightarrow{\delta} H_{n-1}(A \cap B) \cdots$$

Where

- $i : A \cap B \hookrightarrow A$ induces $i^* : H_*(A \cap B) \to H_*(A)$
- $j : A \cap B \hookrightarrow B$ induces $j^* : H_*(A \cap B) \to H_*(B)$
- $l : A \hookrightarrow A \cup B$ induces $l^* : H_*(A) \to H_*(X)$
- $r : B \hookrightarrow A \cup B$ induces $r^* : H_*(B) \to H_*(X)$

The connecting homomorphisms $\delta_n : H_n(X) \to H_{n-1}(X)$ are defined by taking a class $[\alpha] \in H_n(X)$, writing it as an n-cycle z, then decomposing $z = \sum c_i$ where each c_i is an $x + y$ chain. Then $\partial(c_i) = \partial(x + y) = 0$, since the boundary of a cycle is zero, so $\partial(x) = -\partial(y)$. So then just define $\delta([\alpha]) = [\partial x] = [-\partial y]$.

Handy mnemonic diagram:

$$\begin{array}{ccccc} & & A \cap B & & \\ & \swarrow & & \searrow & \\ A \cup B & & \longleftarrow & & A \oplus B \end{array}.$$

21.3.1 Application: Isomorphisms in the homology of spheres.

Proposition 21.3.1 *(?).*

$$H^i(S^n) \cong H^{i-1}(S^{n-1}).$$

Proof .
Write $X = A \cup B$, the northern and southern hemispheres, so that $A \cap B = S^{n-1}$, the equator. In the LES, we have:

$$H^{i+1}(S^n) \to H^i(S^{n-1}) \to H^i A \oplus H^i B \to H^i S^n \to H^{i-1}(S^{n-1}) \to H^{i-1} A \oplus H^{i-1} B.$$

But A, B are contractible, so $H^i A = H^i B = 0$, so we have

$$H^{i+1}(S^n) \to H^i(S^{n-1}) \to 0 \oplus 0 \to H^i(S^n) \to H^{i-1}(S^{n-1}) \to 0.$$

In particular, we have the shape $0 \to A \to B \to 0$ in an exact sequence, which is always an isomorphism.

■

Theorem 21.3.2 ***(Eilenber-Zilber).***
Given two spaces X, Y, there are chain maps

$$F : C_*(X \times Y; R) \to C_*(X; R) \otimes_R C_*(Y; R)$$
$$G : C_*(X; R) \otimes_R C_*(Y; R) \to C_*(X \times Y; R)$$

such that $FG = \mathrm{id}$ and $GF \simeq \mathrm{id}$. In particular,

$$H_*(X \times Y; R) \cong H_*(X; R) \otimes_R H_*(Y; R).$$

Theorem 21.3.3 ***(Kunneth).***
There exists a short exact sequence

$$0 \to \bigoplus_{i+j=k} H_j(X; R) \otimes_R H_i(Y; R) \to H_k(X \times Y; R) \to \bigoplus_{i+j=k-1} \mathrm{Tor}^1_R(H_i(X; R), H_j(Y; R))$$

It has a non-canonical splitting given by

$$H_k(X \times Y) = \left(\bigoplus_{i+j=k} H_i X \oplus H_j Y \right) \oplus \bigoplus_{i+j=k-1} \mathrm{Tor}(H_i X, H_j Y)$$

Theorem 21.3.4 ***(UCT for Change of Group).***
For changing coefficients from $\mathbb{Z}$ to G an arbitrary group, there are short exact sequences

$$0 \to H_i X \otimes G \to H_i(X; G) \to \mathrm{Tor}(H_{i-1} X, G) \to 0$$

$$0 \to \mathrm{Ext}(H_{i-1} X, G) \to H^i(X; G) \to \hom(H_i X, G) \to 0$$

which split unnaturally:

$$H_i(X; G) = (H_i X \otimes G) \oplus \mathrm{Tor}(H_{i-1} X; G)$$

$$H^i(X; G) = \hom(H_i X, G) \oplus \mathrm{Ext}(H_{i-1} X; G)$$

When $H_i X$ are all finitely generated, write $H_i(X; \mathbb{Z}) = \mathbb{Z}^{\beta_i} \oplus T_i$. Then

$$H^i(X; \mathbb{Z}) = \mathbb{Z}^{\beta_i} \oplus T_{i-1}.$$

21.3.2 Useful Long Exact Sequences

Mayer Vietoris

$$\cdots \to H^i(X) \to H^i(U) \oplus H^i(V) \to H^i(U \cap V) \xrightarrow{\delta} H^{i+1}(X) \to \cdots$$

LES of a Pair

$$\cdots \to H_i(A) \to H_i(X) \to H_i(X,A) \xrightarrow{\delta} H_{i-1}(A) \to \cdots$$

21.3.3 Useful Short Exact Sequences

Note that $\mathrm{Ext}^0_R = \hom_R$ *and* $\mathrm{Tor}^0_R = \otimes_R$

Homology to Cohomology

$$0 \to \mathrm{Tor}^0_{\mathbb{Z}}(H_i(X;\mathbb{Z}), A) \to H_i(X;A) \to \mathrm{Tor}^1_{\mathbb{Z}}(H_{i-1}(X;\mathbb{Z}), A) \to 0.$$

Cohomology to Dual of Homology

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}(X;\mathbb{Z}), A) \to H^i(X;A) \to \mathrm{Ext}^0_{\mathbb{Z}}(H_i(X;\mathbb{Z}), A) \to 0.$$

Product of Spaces to Tensor Product in Homology

$$0 \to \bigoplus_{i+j=k} H_i(X;R) \otimes_R H_j(Y;R) \to H_k(X \times Y;R) \to \bigoplus_{i+j=k-1} \mathrm{Tor}^R_1(H_i(X;R), H_j(Y;R)) \to 0$$

21.3.4 Useful Shortcuts

- Cohomology: If A is a field, then

$$H^i(X;A) \cong \hom(H_i(X;A), A)$$

- Kunneth: If R is a freely generated free R-module (or a PID or a field), then

$$H_k(X \times Y) \cong \bigoplus_{i+j=k} H_i(X) \otimes H_j(Y) \bigoplus_{i+j=k-1} \mathrm{Tor}(H_i(X), H_j(X))$$

- Universal Coefficients Theorem: If X is a finite CW complex then

$$H^i(X;\mathbb{Z}) = F(H_i(X;\mathbb{Z})) \times T(H_{i-1}(X;\mathbb{Z}))$$
$$H_i(X;\mathbb{Z}) = F(H^i(X;\mathbb{Z})) \times T(H^{i+1}(X;\mathbb{Z}))$$

21.4 Cellular Homology

- S^n has the CW complex structure of 2 k-cells for each $0 \le k \le n$.

How to compute:

1. Write cellular complex

$$0 \to C^n \to C^{n-1} \to \cdots C^2 \to C^1 \to C^0 \to 0$$

2. Compute differentials $\partial_i : C^i \to C^{i-1}$

   3. *Note: if C^0 is a point, ∂_1 is the zero map.*
   4. *Note: $H_n X = 0 \iff C^n = \varnothing$.*
   5. Compute degrees: Use $\partial_n(e_i^n) = \sum_i d_i e_i^{n-1}$ where

   $$d_i = \deg(\text{Attach } e_i^n \to \text{Collapse } X^{n-1}\text{-skeleton}),$$

   which is a map $S^{n-1} \to S^{n-1}$.
      1. Alternatively, choose orientations for both spheres. Then pick a point in the target, and look at points in the fiber. Sum them up with a weight of +1 if the orientations match and -1 otherwise.
   6. Note that $\mathbb{Z}^m \xrightarrow{f} \mathbb{Z}^n$ has an $n \times m$ matrix
   7. Row reduce, image is span of rows with pivots. Kernel can be easily found by taking RREF, padding with zeros so matrix is square and has all diagonals, then reading down diagonal - if a zero is encountered on nth element, take that column vector as a basis element with -1 substituted in for the nth entry. e.g.

$$\begin{array}{cccc} \mathbf{1} & 2 & 0 & 2 \\ 0 & 0 & \mathbf{1} & -1 \\ 0 & 0 & 0 & \mathbf{0} \end{array} \to \begin{array}{cccccccc} \mathbf{1} & 2 & 0 & 2 & \mathbf{1} & 2 & 0 & 2 \\ 0 & \mathbf{0} & 0 & 0 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & \mathbf{1} & -1 & 0 & 0 & \mathbf{1} & -1 \\ 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & \mathbf{0} \end{array}$$

$$\ker = \begin{array}{cc} 2 & 3 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{array}$$

$$\operatorname{im} = \langle a + 2b + 2d, c - d \rangle .$$

   6. Or look at elementary divisors, say n_i, then the image is isomorphic to $\bigoplus n_i \mathbb{Z}$

22 | Homology

22.1 Useful Facts

- $H_*(A\#B)$: Use the fact that $A\#B = A \cup_{S^n} B$ to apply Mayer-Vietoris.
- $H_n(X,A) \cong_? H_n(X/A, \{\text{pt}\})$
- For CW complexes $X = \left\{X^{(i)}\right\}$, we have

$$H_n(X^{(k)}, X^{(k-1)}) \cong \begin{cases} \mathbb{Z}[\{e^n\}] & k = n, \\ 0 & \text{otherwise} \end{cases} \qquad \text{since } X^k/X^{k-1} \cong \bigvee S^k$$

23 | Fixed Points and Degree Theory

Theorem (Lefschetz Fixed Point) If $\Lambda_f \neq 0$ then f has a fixed point, where $X \circlearrowleft_f$ and $\Lambda_f = \sum_{k \geq 0} (-1)^k \operatorname{Tr}(H_k(X;\mathbb{Q}) \circlearrowleft_{f_*})$.

Theorem: (Brouwer Fixed Point) Every $B^n \circlearrowleft_f$ has a fixed point.

Theorem (Hairy Ball) There is no non-vanishing tangent vector field on even dimensional spheres.

Theorem (Borsuk-Ulam) For every $S^n \xrightarrow{f} \mathbb{R}^n \exists x \in S^n$ such that $f(x) = f(-x)$.

Theorem (Ham Sandwich)

Todo

Review and lect notes fr Hatcher.

24 | Surfaces and Manifolds

24.1 Classification of Surfaces

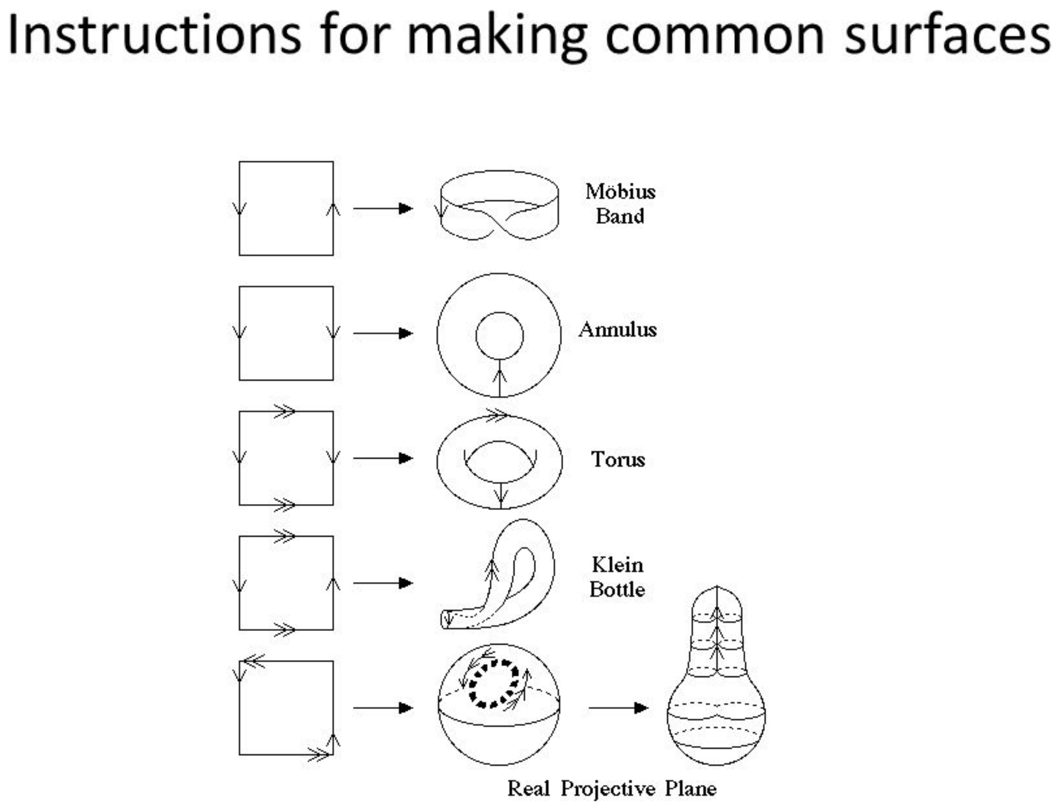


Figure 9: Pasting Diagrams for Surfaces

The most common spaces appearing in this theory:

- $\mathbb{M}$ the Möbius Strip
- $\mathbb{S}^2$,
- $\mathbb{T}^2 \coloneqq S^1 \times S^1$,
- $\mathbb{RP}^2$
- $\mathbb{K}$ the Klein bottle
- $\Sigma_n \coloneqq \#_{i=1}^n \mathbb{T}^2$.

Theorem 24.1.1 ***(Classification of Surfaces).***
The set of surfaces under connect sum forms a monoid with the presentation

$$\left\langle \mathbb{S}^2, \mathbb{RP}^2, \mathbb{T} \;\middle|\; \mathbb{S}^2 = 0, 3\mathbb{RP}^2 = \mathbb{RP}^2 + \mathbb{T}^2 \right\rangle.$$

Surfaces are classified up to homeomorphism by orientability and χ, or equivalently "genus"

- In orientable case, actual genus, g equals the number of copies of $\mathbb{T}^2$.
- In nonorientable case, k equals the number of copies of $\mathbb{RP}^2$.

In each case, there is a formula

$$\chi(X) = \begin{cases} 2-2g-b & \text{orientable} \\ 2-k & \text{non-orientable.} \end{cases}$$

Orientable?	-4	-3	-2	-1	0	1	2
Yes	Σ_3	$\varnothing$	Σ_2	$\varnothing$	$\mathbb{T}^2, S^1 \times I$	$\mathbb{D}^2$	$\mathbb{S}^2$
No	?	?	?	?	$\mathbb{K}, \mathbb{M}$	$\mathbb{RP}^2$	$\varnothing$

Proposition 24.1.2 ***(Inclusion-Exclusion).***

$$X = U \cup V \implies \chi(X) = \chi(U) + \chi(V) - \chi(U \cap V).$$

Proof .
Todo ∎

Proof.

Corollary 24.1.3 ***(Euler for Connect Sums).***

$$\chi(A \# B) = \chi(A) + \chi(B) - 2.$$

Proof .
Set $U = A, B = V$, then by definition of the connect sum, $A \cap B = \mathbb{S}^2$ where $\chi(\mathbb{S}^2) = 2$ ∎

Proposition 24.1.4 ***(Decomposing $\mathbb{RP}^2$).***

$$\mathbb{RP}^2 = \mathbb{M} \coprod_{\mathrm{id}_{\partial\mathbb{M}}} \mathbb{M}.$$

Proposition 24.1.5 ***(Decomposing a Klein Bottle).***

$$\mathbb{K} \cong \mathbb{RP}^2 \# \mathbb{RP}^2.$$

Proof .
Todo ∎

Proof.

Proposition 24.1.6 ***(Rewriting a Klein Bottle).***

$$\mathbb{RP}^2 \# \mathbb{K} \cong \mathbb{RP}^2 \# \mathbb{T}^2.$$

Proof .
Todo ■

Proof.

24.2 Manifolds

To show something is not a manifold, try looking at local homology. Can use point-set style techniques like removing points, i.e. $H_1(X, X - \{\mathrm{pt}\})$; this should essentially always yield $\mathbb{Z}$ by excision arguments.

- M^n closed/connected $\implies H_n = \mathbb{Z}$ and $\mathrm{Tor}(H_{n-1}) = 0$
- 3-manifolds:
  - Orientable: $H_* = (\mathbb{Z}, \mathbb{Z}^r, \mathbb{Z}^r, \mathbb{Z})$
  - Nonorientable: $H_* = (\mathbb{Z}, \mathbb{Z}^r, \mathbb{Z}^{r-1} \oplus \mathbb{Z}_2, \mathbb{Z})$
- $H^n(M^n) = \mathbb{Z}$ if M^n is orientable and zero if M^n is nonorientable.
- Poincaré Duality: $H_i M^n =\cong H^{n-i} M^n$ iff M^n is closed and orientable.

On the complements of spaces in $\mathbb{R}^3$:

My personal crutch is to just think about complements in S^3, which are usually easier since knot complements in S^3 are always $K(\pi, 1)$s. Now if K is a knot and X is its complement in S^3, then you can prove that its complement in $\mathbb{R}^3$ is homotopy equivalent to $S^2 \vee X$

If M is a closed 3-manifold and K is a nullhomologous knot in M, then $H_1(X - n(K)) \cong H_1(X) \times \mathbb{Z}$ where $n(K)$ is a tubular neighborhood.

Proposition 24.2.1 ***(Homology of Sphere minus a knot).***
For $M = S^3 \setminus K$, $H_*(M) = [\mathbb{Z}, \mathbb{Z}, 0, 0, \cdots]$.

Proof .
Apply Mayer-Vietoris, taking $S^3 = n(K) \cup (S^3 - K)$, where $n(K) \simeq S^1$ and $S^3 - K \cap n(K) \simeq T^2$. Use the fact that $S^3 - K$ is a connected, open 3-manifold, so $H^3(S^3 - K) = 0$.

■

- Every $\mathbb{C}$-manifold is canonically orientable.

- If M^n is **closed and connected**, then $H_{\geq n}(X) = 0$ and M^n is orientable iff $H_n(X) = \mathbb{Z}$.
- If M^n is a **closed orientable manifold without boundary**, then $H^k(M^n; F) \cong H_{n-k}(M^n; F)$ for a field F.
- This is a strict implication, so failure of the RHS implies missing conditions on the LHS.
- The intersection pairing is nondegenerate modulo torsion.
- If M^n is a **closed orientable manifold with boundary** then $H_k(M^n; \mathbb{Z}) \cong H^{n-k}(M^n, \partial M^n; \mathbb{Z})$
- M^n closed, connected, and orientable $\implies H_n = \mathbb{Z}$ and $\mathrm{Tor}(H_{n-1}) = 0$
- M^n closed and n odd implies $\chi(M^n) = 0$.
- Any map $X \to Y$ with X factors through the orientation cover $\tilde{Y}_o$.
  - If Y is non-orientable, this is a double cover.
- If n is odd, $\chi(M^n) = 0$ by Poincaré Duality.

Theorem 24.2.2 *(Poincare Duality).*
Todo

Theorem 24.2.3 *(Lefschetz Duality).*
Todo

25 Extra Problems: Algebraic Topology

25.1 Homotopy 101

- Show that if $X \xrightarrow{f} X^n$ is not surjective, then f is nullhomotopic.

25.2 π_1

- Compute $\pi_1(S^1 \vee S^1)$
- Compute $\pi_1(S^1 \times S^1)$

25.3 Surfaces

- Show that if $M^{\text{orientable}} \xrightarrow{\pi_k} M^{\text{non-orientable}}$ is a k-fold cover, then k is even or ∞.
- Show that M is orientable if $\pi_1(M)$ has no subgroup of index 2.

26 | Fall 2014

26.1 1

Let $X = \mathbb{R}^3 - \Delta^{(1)}$, the complement of the skeleton of regular tetrahedron, and compute $\pi_1(X)$ and $H_*(X)$.

Lay the graph out flat in the plane, then take a maximal tree - these leaves 3 edges, and so $\pi_1(X) = \mathbb{Z}^{*3}$.

Moreover $X \simeq S^1 \vee S^1 \vee S^1$ which has only a 1-skeleton, thus $H_*(X) = [\mathbb{Z}, \mathbb{Z}^3, 0 \to]$.

26.2 2

Let $X = S^1 \times B^2 - L$ where L is two linked solid torii inside a larger solid torus. Compute $H_*(X)$. ?

26.3 3

Let L be a 3-manifold with homology $[\mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z}, \ldots]$ and let $X = L \times \Sigma L$. Compute $H_*(X), H^*(X)$.

Useful facts:

- $H_k(X \times Y) \cong \bigoplus_{i+j=k} H_i(X) \otimes H_j(Y) \bigoplus_{i+j=k-1} \mathrm{Tor}(H_i(X), H_j(Y))$
- $\tilde{H}_i(\Sigma X) = \tilde{H}_{i-1}(X)$

We will use the fact that $H_*(\Sigma L) = [\mathbb{Z}, \mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z}]$.

Represent $H_*(L)$ by $p(x,y) = 1 + yx + x^3$ and $H_*(\Sigma L)$ by $q(x,y) = 1 + x + yx^2 + x^4$, we can extract the free part of $H_*(X)$ by multiplying

$$p(x,y)q(x,y) = 1 + (1+y)x + 2yx^2 + (y^2+1)x^3 + 2x^4 + 2yx^5 + x^7$$

where multiplication corresponds to the tensor product, addition to the direct sum/product.

So the free portion is

$$\begin{aligned} H_*(X) &= [\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}_3 \otimes \mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z}_3 \otimes \mathbb{Z}_3, \mathbb{Z}^2, \mathbb{Z}_3^2, 0, \mathbb{Z}] \\ &= [\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}^2, \mathbb{Z}_3^2, 0, \mathbb{Z}] \end{aligned}$$

We can add in the correction from torsion by noting that only terms of the form $\mathrm{Tor}(\mathbb{Z}_3, \mathbb{Z}_3) = \mathbb{Z}_3$ survive. These come from the terms $i = 1, j = 2$, so $i + j = k - 1 \implies k = 1 + 2 + 1 = 4$ and there is thus an additional torsion term appearing in dimension 4. So we have

$$\begin{aligned} H_*(X) &= [\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z} \times \mathbb{Z}_3, \mathbb{Z}^2 \times \mathbb{Z}_3, \mathbb{Z}_3^2, 0, \mathbb{Z}] \\ &= [\mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}^2, 0, 0, \mathbb{Z}] \times [0, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3^2, 0, 0] \end{aligned}$$

and

$$\begin{aligned} H^*(X) &= [\mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}^2, 0, 0, \mathbb{Z}] \times [0, 0, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3^2, 0] \\ &= [\mathbb{Z}, \mathbb{Z}, \mathbb{Z}_3, \mathbb{Z} \times \mathbb{Z}_3, \mathbb{Z}^2 \times \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3^2, \mathbb{Z}]. \end{aligned}$$

■

26.4 4

Let M be a closed, connected, oriented 4-manifold such that $H_2(M; \mathbb{Z})$ has rank 1. Show that there is not a free $\mathbb{Z}_2$ action on M.

Useful facts:

- $X \twoheadrightarrow_{\times p} Y$ induces $\chi(X) = p\chi(Y)$
- Moral: always try a simple Euler characteristic argument first!

We know that $H_*(M) = [\mathbb{Z}, A, \mathbb{Z} \times G, A, \mathbb{Z}]$ for some group A and some torsion group G. Letting $n = \mathrm{rank}(A)$ and taking the Euler characteristic, we have $\chi(M) = (1)1 + (-1)n + (1)1 + (-1)n + (1)1 = 3 - 2n$. Note that this is odd for any n.

However, a free action of $\mathbb{Z}_2 \curvearrowright M$ would produce a double covering $M \twoheadrightarrow_{\times 2} M/\mathbb{Z}_2$, and multiplicativity of Euler characteristics would force $\chi(M) = 2\chi(M/\mathbb{Z}_2)$ and thus $3 - 2n = 2k$ for some integer k. This would require $3 - 2n$ to be even, so we have a contradiction. ■

26.5 5

Let X be T^2 with a 2-cell attached to the interior along a longitude. Compute $\pi_2(X)$.

Useful facts:

- $T^2 = e^0 + e^1_1 + e^1_2 + e^2$ as a CW complex.
- $S^2/(x_0 \sim x_1) \simeq S^2 \wedge S^1$ when x_0, x_1 are two distinct points. (Picture: sphere with a string handle connecting north/south poles.)
- $\pi_{\geq 2}(\tilde{X}) \cong \pi_{\geq 2}(X)$ for $\tilde{X} \twoheadrightarrow X$ the universal cover.

Write $T^2 = e^0 + e^1_1 + e^1_2 + e^2$, where the first and second 1-cells denote the longitude and meridian respectively. By symmetry, we could have equivalently attached a disk to the meridian instead of the longitude, filling the center hole in the torus. Contract this disk to a point, then pull it vertically in both directions to obtain S^2 with two points identified, which is homotopy-equivalent to $S^2 \vee S_1$.

Take the universal cover, which is $\mathbb{R}^1 \cup_{\mathbb{Z}} S^2$ and has the same π_2. This is homotopy-equivalent to $\bigvee_{i\in\mathbb{Z}} S^2$ and so $\pi_2(X) = \prod_{i\in\mathbb{Z}} \mathbb{Z}$ generated by each distinct copy of S^2. (Alternatively written as $\mathbb{Z}[t, t^{-1}]$).

27 | Summer 2003

27.1 1

Describe all possible covering maps between S^2, T^2, K

Useful facts:

1. $\tilde{X} \twoheadrightarrow X$ induces $\pi_1(\tilde{X}) \hookrightarrow \pi_1(X)$
2. $\chi(\tilde{X}) = n\chi(X)$
3. $\pi_n(X) = [S^n, X]$
4. $Y \to X$ with $\pi_1(Y) = 0$ and $\tilde{X} \simeq \{\text{pt}\} \implies$ every $Y \xrightarrow{f} X$ is nullhomotopic.
5. $\pi_*(T^2) = [\mathbb{Z} * \mathbb{Z}, 0 \to]$
6. $\pi_*(K) = [\mathbb{Z} \rtimes_{\mathbb{Z}_2} \mathbb{Z}, 0 \to]$
7. Universal covers are homeomorphic.
8. $\pi_{\geq 2}(\tilde{X}) \cong \pi_{\geq 2}(X)$

Spaces

- $S^2 \twoheadrightarrow T^2$
- $S^2 \twoheadrightarrow K$
- $K \twoheadrightarrow S^2$
- $T^2 \twoheadrightarrow S^2$
  - All covered by the fact that
$$\mathbb{Z} = \pi_2(S^2) \neq \pi_2(X) = 0$$
    for $X = T^2, K$.
- $K \twoheadrightarrow T^2$
  - Doesn't cover, would induce $\pi_1(K) \hookrightarrow \pi_1(T^2) \implies \mathbb{Z} \rtimes \mathbb{Z} \hookrightarrow \mathbb{Z}^2$ but this would be a non-abelian subgroup of an abelian group.
- $T^2 \twoheadrightarrow K$
  - ?

■

27.2 2

Show that $\mathbb{Z}^{*2}$ has subgroups isomorphic to $\mathbb{Z}^{*n}$ for every n.

Facts Used 1. $\pi_1(\bigvee^k S^1) = \mathbb{Z}^{*k}$ 2. $\tilde{X} \twoheadrightarrow X \implies \pi_1(\tilde{X}) \hookrightarrow \pi_1(X)$ 3. Every subgroup $G \leq \pi_1(X)$ corresponds to a covering space $X_G \twoheadrightarrow X$ 4. $A \subseteq B \implies F(A) \leq F(B)$ for free groups.

It is easier to prove the stronger claim that $\mathbb{Z}^{\mathbb{N}} \leq \mathbb{Z}^{*2}$ (i.e. the free group on countably many generators) and use fact 4 above.

Just take the covering space $\tilde{X} \twoheadrightarrow S^1 \vee S^1$ defined via the gluing map $\mathbb{R} \cup_{\mathbb{Z}} S^1$ which attaches a circle to each integer point, taking 0 as the base point. Then let a denote a translation and b denote traversing a circle, so we have $\pi_1(\tilde{X}) = \langle \cup_{n \in \mathbb{Z}} a^n b a^{-n} \rangle$ which is a free group on countably many generators. Since $\tilde{X}$ is a covering space, $\pi_1(\tilde{X}) \hookrightarrow \pi_1(S^1 \vee S^1) = \mathbb{Z}^{*2}$. By 4, we can restrict this to n generators for any n to get a subgroup, and $A \leq B \leq C \implies A \leq C$ as groups.

■

27.3 3

Construct a space having $H_*(X) = [\mathbb{Z}, 0, 0, 0, 0, \mathbb{Z}_4, 0 \to]$.

Facts used: - Construction of Moore Spaces - $\tilde{H}_n(\Sigma X) = \tilde{H}_{n-1}(X)$, using $\Sigma X = C_X \cup_X C_X$ and Mayer-Vietoris.

Take $X = e^0 \cup_{\Phi_1} e^5 \cup_{\Phi_2} e^6$, where

$$\Phi_1 : \partial B^5 = S^4 \xrightarrow{z \mapsto z^0} e^0$$
$$\Phi_2 : \partial B^6 = S^5 \xrightarrow{z \mapsto z^4} e^5.$$

where $\deg \Phi_2 = 4$. ■

27.4 4

Compute the complement of a knotted solid torus in S^3.

Facts used:

- $H_*(T^2) = [\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}, 0 \to]$
- $N^{(1)} \simeq S^1$, so $H_{\geq 2}(N) = 0$.
- A SES $0 \to A \to B \to F \to 0$ with F free splits.
- $0 \to A \to B \xrightarrow{\cong} C \to D \to 0$ implies $A = D = 0$.

Let N be the knotted solid torus, so that $\partial N = T^2$, and let $X = S^3 - N$. Then

- $S^3 = N \cup_{T^2} X$
- $N \cap X = T^2$

and we apply Mayer-Vietoris to S^3:

$$\begin{array}{ll}
4 & H_4(T^2) \to H_4(N) \times H_4(X) \to H_4(S^3) \\
3 & H_3(T^2) \to H_3(N) \times H_3(X) \to H_3(S^3) \\
2 & H_2(T^2) \to H_2(N) \times H_2(X) \to H_2(S^3) \\
1 & H_1(T^2) \to H_1(N) \times H_1(X) \to H_1(S^3) \\
0 & H_0(T^2) \to H_0(N) \times H_0(X) \to H_0(S^3)
\end{array}.$$

where we can plug in known information and deduce some maps:

$$
\begin{array}{llrl}
4 & 0 \to & 0 & \to 0 \xrightarrow{\partial_4} \\
3 & 0 \to & H_3(X) & \to \mathbb{Z} \xrightarrow{\partial_3} \\
2 & \mathbb{Z} \to & H_2(X) & \to 0 \xrightarrow{\partial_2} \\
1 & \mathbb{Z}^2 \cong & \mathbb{Z} \times H_1(X) & \to 0 \xrightarrow{\partial_1} \\
0 & \mathbb{Z} \to & \mathbb{Z} \times H_0(X) & \to \mathbb{Z} \to 0
\end{array}
$$

(1) (2) (3) (4) (5) (6)

We then deduce: - $H_0(X) = \mathbb{Z}$ by the splitting of the line 0 SES

$$0 \to \mathbb{Z} \to \mathbb{Z} \times H_0(X) \to \mathbb{Z} \to 0$$

yielding $Z \times H_0(X) \cong \mathbb{Z} \times \mathbb{Z}$. - $H_1(X) = \mathbb{Z}$ by the line 1 SES

$$0 \to \mathbb{Z}^2 \to \mathbb{Z} \times H_1(X) \to 0$$

which yields an isomorphism. - $H_2(X) = H_3(X) = 0$ by examining the SES spanning lines 3 and 2:

$$0 \hookrightarrow H_3(X) \hookrightarrow \mathbb{Z} \xrightarrow{\cong \partial_3} \mathbb{Z} \twoheadrightarrow H_2(X) \twoheadrightarrow 0$$

Since ∂_3 must be an isomorphism, this forces the edge terms to be zero.

■

27.5 5

Compute the homology and cohomology of a closed, connected, oriented 3-manifold M with $\pi_1(M) = \mathbb{Z}^{*2}$.

Facts used: - M closed, connected, oriented $\implies H_i(M) \cong H^{n-i}(M)$ - $H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$ - For orientable manifolds $H_n(M^n) = \mathbb{Z}$

Homology

- Since M is connected, $H_0 = \mathbb{Z}$
- Since $\pi_1(M) = \mathbb{Z}^{*2}$, H_1 is the abelianization and $H_1(X) = \mathbb{Z}^2$
- Since M is closed/connected/oriented, Poincare Duality holds and $H_2 = H^{3-2} = H^1 = \mathbf{F}H_1 + \mathbf{T}H_0$ by UCT. Since $H_0 = \mathbb{Z}$ is torsion-free, we have $H_2(M) = H_1(M) = \mathbb{Z}^2$.
- Since M is an orientable manifold, $H_3(M) = \mathbb{Z}$
- So $H_*(M) = [\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^2, \mathbb{Z}, 0 \to]$

Cohomology

- By Poincare Duality, $H^*(M) = \widehat{H_*(M)} = [\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^2, \mathbb{Z}, 0 \to]$. (Where the hat denotes reversing the list.)

■

27.6 6

Compute $\text{Ext}(\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z} \times \mathbb{Z}_4 \times \mathbb{Z}_5)$

Facts Used:

1. $\text{Ext}(\mathbb{Z}, \mathbb{Z}_m) = \mathbb{Z}_m$
2. $\text{Ext}(\mathbb{Z}_m, \mathbb{Z}) = 0$
3. $\text{Ext}(\prod_i A_i, \prod_j B_j) = \prod_i \prod_j \text{Ext}(A_i, B_j)$

Break it up into a bigraded complex, take Ext of the pieces, and sum over the complex: $\text{Ext}(\downarrow, \to)$ | $\mathbb{Z}$ | $\mathbb{Z}_4$ | $\mathbb{Z}_5$ ——————————|———|———|——— $\mathbb{Z}$ | 0 | 0 | 0 $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 $\mathbb{Z}_3$ | $\mathbb{Z}_3$ | 0 | 0

So the answer is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 = \mathbb{Z}_{12}$.

■

27.7 7

Show there is no homeomorphism $\mathbb{CP}^2 \circlearrowleft_f$ such that $f(\mathbb{CP}^1)$ is disjoint from $\mathbb{CP}_1 \subset \mathbb{CP}_2$.

Facts used:

1. Every homeomorphism induces isomorphisms on homotopy/homology/cohomology.
2. $H^*(\mathbb{CP}^2) = \mathbb{Z}[\alpha]/(\alpha^2)$ where $\deg \alpha = 2$.
3. $[f(X)] = f_*([X])$
4. $a \cdot b = 0 \implies a = 0$ or $b = 0$ (nondegeneracy).

Supposing such a homeomorphism exists, we would have $[\mathbb{CP}^1] \cdot [f(\mathbb{CP}^1)] = 0$ by the definition of these submanifolds being disjoint.

But $[\mathbb{CP}^1] \cdot [f(\mathbb{CP}^1)] = [\mathbb{CP}^1] \cdot f_*([\mathbb{CP}^1])$, where

$$f_* : H^*(\mathbb{CP}^2) \to H^*(\mathbb{CP}^2)$$

is the induced map on cohomology.

Since the intersection pairing is nondegenerate, either $[\mathbb{CP}^1] = 0$ or $f_*([\mathbb{CP}^1]) = 0$.

We know that $H^*(\mathbb{CP}^2) = \mathbb{Z}[\alpha]/\alpha^2$ where $\alpha = [\mathbb{CP}^1]$, however, so this forces $f_*([\mathbb{CP}^1]) = 0$. But since this was a generator of H^*, we have $f_*(H^*(\mathbb{CP}^2)) = 0$, so f is not an isomorphism on cohomology. ■

27.8 8

Describe the universal cover of $X = (S^1 \times S^1) \vee S^2$ and compute $\pi_2(X)$.

Facts used: - $\pi_{\geq 2}(\tilde{X}) \cong \pi_{\geq 2}(X)$ - Structure of the universal cover of a wedge product - $\mathbb{R}^2 \twoheadrightarrow_p T^2 = S^1 \times S^1$

$\tilde{X} = \mathbb{R}^2 \cup_{\mathbb{Z}^2} S^2$, so $\pi_2(X) \cong \pi_2(\tilde{X}) = \prod_{i,j \in \mathbb{Z}^2} \mathbb{Z} = \mathbb{Z}^{\mathbb{Z}^2} = \mathbb{Z}^{\aleph_0}$. ■

27.9 9

Let $S^3 \to E \to S^5$ be a fiber bundle and compute $H_3(E)$.

Facts used: - Homotopy LES - Hurewicz - $0 \to A \to B \to 0$ exact iff $A \cong B$

From the LES in homotopy we have

$$
\begin{aligned}
&4 \quad \pi_4(S^3) \to \pi_4(E) \to \pi_4(S^5) && (7)\\
&3 \quad \pi_3(S^3) \to \pi_3(E) \to \pi_3(S^5) && (8)\\
&2 \quad \pi_2(S^3) \to \pi_2(E) \to \pi_2(S^5) && (9)\\
&1 \quad \pi_1(S^3) \to \pi_1(E) \to \pi_1(S^5) && (10)\\
&0 \quad \pi_0(S^3) \to \pi_0(E) \to \pi_0(S^5) && (11)\\
& && (12)
\end{aligned}
$$

and plugging in known information yields

$$
\begin{array}{llll}
4 & \pi_4(S^3) \to & \pi_4(E) & \to 0 \\
3 & \mathbb{Z} \to & \pi_3(E) & \to 0 \\
2 & 0 \to & \pi_2(E) & \to 0 \\
1 & 0 \to & \pi_1(E) & \to 0 \\
0 & \mathbb{Z} \to & \pi_0(E) & \to \mathbb{Z}
\end{array}
\quad
\begin{array}{r}
(13)\\(14)\\(15)\\(16)\\(17)\\(18)
\end{array}
$$

where rows 3 and 4 force $\pi_3(E) \cong \mathbb{Z}$, rows 0 and 1 force $\pi_0(E) = \mathbb{Z}$, and the remaining rows force $\pi_1(E) = \pi_2(E) = 0$.

By Hurewicz, we thus have $H_3(E) = \pi_3(E) = \mathbb{Z}$.

■

28 | Fall 2017 Final

28.1 1

Let X be the subspace of the unit cube I^3 consisting of the union of the 6 faces and the 4 internal diagonals. Compute $\pi_1(X)$.

Solution:

28.2 2

Let X be an arbitrary topological space, and compute $\pi_1(\Sigma X)$.

Solution:

Write $\Sigma X = U \cup V$ where $U = \Sigma X - (X \times [0, 1/2])$ and $U = \Sigma X - X \times [1/2, 1])$. Then $U \cap V = X \times \{1/2\} \cong X$, so $\pi_1(U \cap V) = \pi_1(X)$.

But both U and V can be identified by the cone on X, given by $CX = \dfrac{X \times I}{X \times 1}$, by just rescaling the interval with the maps:

$i_U : U \to CX$ where $(x, s) \mapsto (x, 2s - 1)$ (The second component just maps $[1/2, 1] \to [0, 1]$.)

$i_V : V \to CX$ where $(x,s) \mapsto (x,2s)$. (The second component just maps $[0,1/2] \to [0,1]$)

But CX is contractible by the homotopy $H : CX \times I \to CX$ where $H((c,s),t) = (c, s(1-t))$.

So $\pi_1(U) = \pi_1(V) = 0$.

By Van Kampen, we have $\pi_1(X) = 0 *_{\pi_1(X)} 0 = 0$.

28.3 3

Let $X = S^1 \times S^1$ and $A \subset X$ be a subspace with $A \cong S^1 \vee S^1$. Show that there is no retraction from X to A.

Solution:

We have $\pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1)$ since S^1 is path-connected (by a lemma from the problem sets), and this equals $\mathbb{Z} \times \mathbb{Z}$.

We also have $\pi_1(S^1 \vee S^1) = \pi_1(S^1) *_{\{pt\}} \pi_1(S^1)$, which by Van-Kampen is $\mathbb{Z} * \mathbb{Z}$.

Suppose X retracts onto A, we can then look at the inclusion $\iota : A \hookrightarrow X$. The induced homomorphism $\iota_* : \pi_1(A) \hookrightarrow \pi_1(X)$ is then also injective, so we've produced an injection from $f : \mathbb{Z} * \mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Z}$.

This is a contradiction, because no such injection can exists. In particular, the commutator $[a,b]$ is nontrivial in the source. But $f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1}$ since f is a homomorphism, but since the target is a commutative group, this has to equal $f(a)f(a)^{-1}f(b)f(b)^{-1} = e$. So there is a non-trivial element in the kernel of f, and f can not be injective - a contradiction.

28.4 4

Show that for every map $f : S^2 \to S^1$, there is a point $x \in S^2$ such that $f(x) = f(-x)$.

Solution:

Suppose towards a contradiction that f does not possess this property, so there is no $x \in S^2$ such that $f(x) = f(-x)$.

Then define $g : S^2 \to S^1$ by $g(x) = f(x) - f(-x)$; by assumption, this is a nontrivial map, i.e. $g(x) \neq 0$ for *any* $x \in S^2$.

In particular, $-g(-x) = -(f(-x) - f(x)) = f(x) - f(-x) = g(x)$, so $-g(x) = g(-x)$ and thus g commutes with the antipodal map $\alpha : S^2 \to S^2$.

This means g is constant on the fibers of the quotient map $p : S^2 \to \mathbb{RP}2$, and thus descends to a well defined map $\tilde{g} : \mathbb{RP}2 \to S^1$, and since $S^1 \cong \mathbb{RP}1$, we can identify this with a map $\tilde{g} : \mathbb{RP}2 \to \mathbb{RP}1$ which thus induces a homomorphism $\tilde{g}_* : \pi_1(\mathbb{RP}2) \to \pi_1(\mathbb{RP}1)$.

Since g was nontrivial, $\tilde{g}$ is nontrivial, and by functoriality of π_1, $\tilde{g}_*$ is nontrivial.

But $\pi_1(\mathbb{RP}2) = \mathbb{Z}_2$ and $\pi_1(\mathbb{RP}1) = \mathbb{Z}$, and $\tilde{g}_* : \mathbb{Z}^2 \to \mathbb{Z}$ can only be the trivial homomorphism - a contradiction.

Alternate Solution

Use covering space $\mathbb{R} \twoheadrightarrow S^1$?

28.5 5

How many path-connected 2-fold covering spaces does $S^1 \vee \mathbb{RP}2$ have? What are the total spaces?

Solution:

First note that $\pi_1(X) = \pi_1(S^1) *_{\{pt\}} \pi_1(\mathbb{RP}2)$ by Van-Kampen, and this is equal to $\mathbb{Z} * \mathbb{Z}_2$.

28.6 6

Let $G =< a, b >$ and $H \leq G$ where $H =< aba^{-1}b^{-1},\ a^2ba^{-2}b^{-1},\ a^{-1}bab^{-1},\ aba^{-2}b^{-1}a >$. To what well-known group is H isomorphic?

Solution:

29 Appendix: Homological Algebra

29.1 Exact Sequences

The sequence $A \xrightarrow{f_1} B \xrightarrow{f_2} C$ is exact if and only if $\operatorname{im} f_i = \ker f_{i+1}$ and thus $f_2 \circ f_1 = 0$.

Some useful results:

- $0 \to A \hookrightarrow_f B$ is exact iff f is **injective**

- $B \twoheadrightarrow_f C \to 0$ is exact iff f is **surjective**
- $0 \to A \to B \to 0$ is exact iff $A \cong B$.
- $A \hookrightarrow B \to C \to D \twoheadrightarrow E$ iff $C = 0$
- $0 \to A \to B \xrightarrow{\cong} C \to D \to 0$ iff $A = D = 0$.
  - Todo: Proof
- $0 \to A \to B \to C \to 0$ splits iff C is free.

Can think of $C \cong \dfrac{B}{\operatorname{im} f_1}$.

The sequences *splits* when a morphism $f_2^{-1} : C \to B$ exists. In **Ab**, this means $B \cong A \oplus C$, in **Grp** it's $B \cong A \rtimes_\varphi C$.

Examples:

- $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \dfrac{\mathbb{Z}}{2\mathbb{Z}} \to 0$
- $1 \to N \xrightarrow{\iota} G \xrightarrow{p} \dfrac{G}{N} \to 1$
  - Groups and normal subgroups
- $1 \to \dfrac{\mathbb{Z}}{n\mathbb{Z}} \xrightarrow{\iota} D_{2n} \xrightarrow{?} \dfrac{\mathbb{Z}}{2\mathbb{Z}} \to 1$
  - Dihedral group and cyclic groups
- $0 \to I \cap J \xrightarrow{\Delta : x \mapsto (x,x)} I \oplus J \xrightarrow{f:(x,y)\mapsto x-y} I + J \to 0$
  - R-Modules
- $0 \to \dfrac{R}{I \cap J} \xrightarrow{\Delta : x \mapsto (x,x)} \dfrac{R}{I} \oplus \dfrac{R}{J} \xrightarrow{f:(x,y)\mapsto x-y} \dfrac{R}{I + J} \to 0$
- $0 \to \mathbb{H}_1 \xrightarrow{\nabla} \mathbb{H}_{\text{curl}} \xrightarrow{\nabla \times} \mathbb{H}_{\text{div}} \xrightarrow{\nabla \cdot} \mathbb{L}_2 \to 0$
  - Since $\nabla \times \nabla F = \nabla \cdot \nabla \times \bar{v} = 0$ in Hilbert spaces

Remark 29.1.1: Is $f_1 \circ f_2 = 0$ equivalent to exactness..? Answer: yes, every exact sequence is a chain complex with trivial homology. Therefore homology measures the failure of exactness.

Alternatively stated: Exact sequences are chain complexes with no cycles.

Any LES $A_1 \to \cdots \to A_6$ decomposes into a twisted collection of SES's; define $C_k = \ker(A_k \to A_{k+1}) \cong \operatorname{im}(A_{k-1} \to A_k)) \cong \operatorname{coker}(A_{k-2} \to A_{k-1})$, then all diagonals here are exact:

29.2 Five Lemma

If m, p are isomorphisms, l is an **surjection**, and q is an **injection**, then n is an **isomorphism**.

Proof: diagram chase two "four lemmas", one on each side. Full proof here.

29.3 Free Resolutions

The canonical example:

$$0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \xrightarrow{(\text{mod } m)} \mathbb{Z}_m \to 0$$

Or more generally for a finitely generated group $G = \langle g_1, g_2, \cdots, g_n \rangle$,

$$\cdots \to \ker(f) \to F[g_1, g_2, \cdots, g_n] \xrightarrow{f} G \to 0$$

where F denotes taking the free group.

Every abelian groups has a resolution of this form and length 2.

29.4 Computing Tor

$$\text{Tor}(A, B) = h[\cdots \to A_n \otimes B \to A_{n-1} \otimes B \to \cdots A_1 \otimes B \to 0]$$

where A_* is any free resolution of A.

Shorthand/mnemonic:

$$\text{Tor} : \mathcal{F}(A) \to (\cdot \otimes B) \to H_*$$

29.5 Computing Ext

$$\text{Ext}(A, B) = h[\cdots \hom(A, B_n) \to \hom(A, B_{n-1}) \to \cdots \to \hom(A, B_1) \to 0]$$

where B_* is a any free resolution of B.

Shorthand/mnemonic:

$$\text{Ext} : \mathcal{F}(B) \to \hom(A, \cdot) \to H_*$$

29.6 Properties of Tensor Products

- $A \otimes B \cong B \otimes A$
- $(\cdot) \otimes_R R^n = \mathrm{id}$
- $\bigoplus_i A_i \otimes \bigoplus_j B_j = \bigoplus_i \bigoplus_j (A_i \otimes B_j)$
- $\mathbb{Z}_m \otimes \mathbb{Z}_n = \mathbb{Z}_d$
- $\mathbb{Z}_n \otimes A = A/nA$

29.7 Properties of Hom

- $\hom_R(\bigoplus_i A_i, \prod B_j) = \bigoplus_i \prod_j \hom(A_i, B_j)$
- Contravariant in first slot, covariant in second
- Exact over vector spaces

29.8 Properties of Tor

- $\mathrm{Tor}^0_R(A, B) = A \otimes_R B$
- $\mathrm{Tor}(\bigoplus_i A_i, \bigoplus_j B) = \bigoplus_i \bigoplus_j \mathrm{Tor}(\mathbf{T}A_i, \mathbf{T}B_j)$ where $\mathbf{T}G$ is the torsion component of G.
- $\mathrm{Tor}(\mathbb{Z}_n, G) = \ker(g \mapsto ng) = \left\{ g \in G \;\middle|\; ng = 0 \right\}$
- $\mathrm{Tor}(A, B) = \mathrm{Tor}(B, A)$

29.9 Properties of Ext

- $\mathrm{Ext}^0_R(A, B) = \hom_R(A, B)$
- $\mathrm{Ext}(\bigoplus_i A_i, \prod_j B_j) = \bigoplus_i \prod_j \mathrm{Ext}(\mathbf{T}A_i, B_j)$
- $\mathrm{Ext}(F, G) = 0$ if F is free
- $\mathrm{Ext}(\mathbb{Z}_n, G) \cong G/nG$

29.10 Hom/Ext/Tor Tables

hom	$\mathbb{Z}_m$	$\mathbb{Z}$	$\mathbb{Q}$
$\mathbb{Z}_n$	$\mathbb{Z}_d$	0	0
$\mathbb{Z}$	$\mathbb{Z}_m$	$\mathbb{Z}$	$\mathbb{Q}$
$\mathbb{Q}$	0	0	$\mathbb{Q}$

Tor	$\mathbb{Z}_m$	$\mathbb{Z}$	$\mathbb{Q}$
$\mathbb{Z}_n$	$\mathbb{Z}_d$	0	0
$\mathbb{Z}$	0	0	0
$\mathbb{Q}$	0	0	0

Ext	$\mathbb{Z}_m$	$\mathbb{Z}$	$\mathbb{Q}$
$\mathbb{Z}_n$	$\mathbb{Z}_d$	$\mathbb{Z}_n$	0
$\mathbb{Z}$	0	0	0
$\mathbb{Q}$	0	$\mathcal{A}/\mathbb{Q}$	0

Where $d = \gcd(m, n)$ and $\mathbb{Z}_0 \coloneqq 0$.

Things that behave like "the zero functor":

- $\operatorname{Ext}(\mathbb{Z}, \cdot)$
- $\operatorname{Tor}(\cdot, \mathbb{Z}), \operatorname{Tor}(\mathbb{Z}, \cdot)$
- $\operatorname{Tor}(\cdot, \mathbb{Q}), \operatorname{Tor}(\mathbb{Q}, \cdot)$

Thins that behave like "the identity functor":

- $\hom(\mathbb{Z}, \cdot)$
- $\cdot \otimes_{\mathbb{Z}} \mathbb{Z}$ and $\mathbb{Z} \otimes_{\mathbb{Z}} \cdot$

For description of $\mathcal{A}$, see here. This is a certain ring of adeles.

30 | Appendix: ?

- Assorted info about other Lie Groups:
- $O_n, U_n, SO_n, SU_n, Sp_n$
- $\pi_k(U_n) = \mathbb{Z} \cdot \mathbb{1}[k \text{ odd}]$
  - $\pi_1(U_n) = 1$

- $\pi_k(SU_n) = \mathbb{Z} \cdot \mathbb{1}[k \text{ odd}]$
  - $\pi_1(SU_n) = 0$
- $\pi_k(U_n) = \mathbb{Z}/2\mathbb{Z} \cdot \mathbb{1}[k = 0, 1 \pmod 8] + \mathbb{Z} \cdot \mathbb{1}[k = 3, 7 \pmod 8]$
- $\pi_k(SP_n) = \mathbb{Z}/2\mathbb{Z} \cdot \mathbb{1}[k = 4, 5 \pmod 8] + \mathbb{Z} \cdot \mathbb{1}[k = 3, 7 \pmod 8]$
- Groups and Group Actions
  - $\pi_0(G) = G$ for G a discrete topological group.
  - $\pi_k(G/H) = \pi_k(G)$ if $\pi_k(H) = \pi_{k-1}(H) = 0$.
  - $\pi_1(X/G) = \pi_0(G)$ when G acts freely/transitively on X.

30.1 Cap and Cup Products

$$\cup : H^p \times H^q \to H^{p+q}; (a^p \cup b^q)(\sigma) = a^p(\sigma \circ F_p) b^q(\sigma \circ B_q)$$

where F_p, B_q is embedding into a $p+q$ simplex.

For f continuous, $f^*(a \cup b) = f^*a \cup f^*b$

It satisfies the Leibniz rule

$$\partial(a^p \cup b^q) = \partial a^p \cup b^q + (-1)^p(a^p \cup \partial b^q)$$

$$\cap : H_p \times H^q \to H_{p-q}; \sigma \cap \psi = \psi(F \circ \sigma)(B \circ \sigma)$$

where F, B are the front/back face maps.

Given $\psi \in C^q, \varphi \in C^p, \sigma : \Delta^{p+q} \to X$, we have

$$\psi(\sigma \cap \varphi) = (\varphi \cup \psi)(\sigma)$$
$$\langle \varphi \cup \psi,\ \sigma \rangle = \langle \psi,\ \sigma \cap \varphi \rangle$$

Let M^n be a closed oriented smooth manifold, and $A^{\widehat{i}}, B^{\widehat{j}} \subseteq X$ be submanifolds of codimension i and j respectively that intersect transversely (so $\forall p \in A \cap B$, the inclusion-induced map $T_pA \times T_pB \to T_pX$ is surjective.)

Then $A \cap B$ is a submanifold of codimension $i + j$ and there is a short exact sequence

$$0 \to T_p(A \cap B) \to T_pA \times T_pB \to T_pX \to 0$$

which determines an orientation on $A \cap B$.

Then the images under inclusion define homology classes

- $[A] \in H_{\widehat{i}}X$
- $[B] \in H_{\widehat{j}}X$
- $[A \cap B] \in H_{\widehat{i+j}}X$.

Denoting their Poincare duals by

- $[A]^\vee \in H^i X$
- $[B]^\vee \in H^j X$
- $[A \cap B]^\vee \in H^{i+j} X$

We then have

$$[A]^\vee \smile [B]^\vee = [A \cap B]^\vee \in H^{i+j} X$$

Example: in $\mathbb{CP}^n$, each even-dimensional cohomology $H^{2i}\mathbb{CP}^n$ has a generator α_i with is Poincare dual to an $\widehat{i}$ plane. A generic $\widehat{i}$ plane intersects a $\widehat{j}$ plane in a $\widehat{i+j}$ plane, yielding $\alpha_i \smile \alpha_j = \alpha_{i+j}$ for $i + j \leq n$.

Example: For T^2, we have - $H_1 T^2 = \mathbb{Z}^2$ generated by $[A], [B]$, the longitudinal and meridian circles. - $H_0 T^2 = \mathbb{Z}$ generated by $[p]$, the class of a point.

Then $A \cap B = \pm[p]$, and so

$$\begin{aligned} [A]^\vee \smile [B]^\vee &= [p]^\vee \\ [B]^\vee \smile [A]^\vee &= -[p]^\vee \end{aligned}$$

30.2 The Long Exact Sequence of a Pair

LES of pair $(A, B) \implies \cdots H_n(B) \to H_n(A) \to H_n(A, B) \to H_{n-1}(B) \cdots$

$$\begin{array}{ccccc} & & B & & \\ & \diagup & & \diagdown & \\ (A,B) & & \longleftarrow & & A \end{array}.$$

3.1.3 Example. The cases $n = 1, 2$ and part of the case $n = 3$ are shown in the figure below.

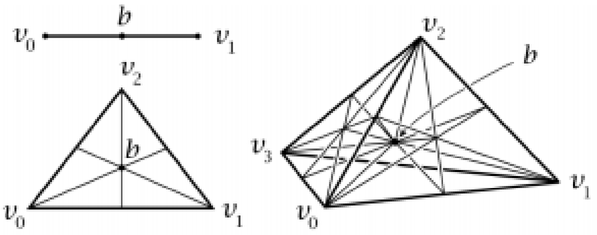


Figure 3.1: Barycentric subdivision [10].

Figure 10: Barycentric Subdivision

30.3 Tables

Homotopy groups of real projective spaces

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
RP^1	Z	0	0	0	0	0	0	0	0	0	0	0
RP^2	Z_2	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2 \times Z_2$
RP^3	Z_2	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2 \times Z_2$
RP^4	Z_2	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	$Z_2 \times Z_2$	$Z_2 \times Z_2$	$Z_{24} \times Z_3$	Z_{15}	Z_2

Figure 11: Higher homotopy groups of $\mathbb{RP}^n$

Homotopy groups of complex projective spaces												
	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
CP^1	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2 \times Z_2$
CP^2	0	Z	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}
CP^3	0	Z	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0
CP^4	0	Z	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}

Figure 12: Higher homotopy groups of $\mathbb{CP}^n$

Homotopy groups of spheres												
	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
S^1	Z	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\boxed{Z}$	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2 \times Z_2$
S^3	0	0	Z	$\boxed{Z_2}$	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2 \times Z_2$
S^4	0	0	0	Z	Z_2	$\boxed{Z_2}$	$Z \times Z_{12}$	$Z_2 \times Z_2$	$Z_2 \times Z_2$	$Z_{24} \times Z_3$	Z_{15}	Z_2
S^5	0	0	0	0	Z	Z_2	Z_2	$\boxed{Z_{24}}$	Z_2	Z_2	Z_2	Z_{30}
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	$\boxed{0}$	Z	Z_2
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	$\boxed{0}$
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0

Figure 13: Homotopy groups of spheres.

A1.1.3.4 Exceptional groups

Homotopy groups of exceptional groups												
	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
G_2	0	0	Z	0	0	Z_3	0	Z_2	Z_6	0	$Z \times Z_2$	0
F_4	0	0	Z	0	0	0	0	Z_2	Z_2	0	$Z \times Z_2$	0
E_6	0	0	Z	0	0	0	0	0	Z	0	Z	Z_{12}
E_7	0	0	Z	0	0	0	0	0	0	0	Z	Z_2
E_8	0	0	Z	0	0	0	0	0	0	0	0	0

Figure 14: Homotopy groups of exceptional groups

30.4 Homotopy Groups of Lie Groups

- $O(n)$: $\pi_k O_n = ?$
- $U(n) : \pi_k U_n$ is $\mathbb{Z}$ in odd degrees and $\pi_1 U_n = 1$

Check

- $SU(n) : \pi_k U_n$ is $\mathbb{Z}$ in odd degrees and $\pi_1 U_n = 0$.
- $U_n : \pi_k(U_n)$ is $\mathbb{Z}/2\mathbb{Z}$ in degrees?

30.5 Higher Homotopy

- $n \geq 2 \implies \pi_n(X) \in \mathbf{Ab}$
- $\Sigma S^n = S^{n+1}$
- $[\Sigma^n X, Y] \cong [X, \Omega^n Y]$
- $\pi * n(\Omega X) = \pi * n + 1(X)$
  - $\pi_n(X) \cong \pi_0(\Omega^n X)$
- $n \geq 2 \implies \pi_n(S^1) = 0$

- $k < n \implies \pi_k(S^n) = 0$
- $\pi_n(X)$ is the obstruction to $f : S^n \to X$ being lifted to $\widehat{f} : D^{n+1} \to X$
- $\pi_n(X) \cong H_n(X)$ for the first n such that $\pi_n(X) \neq 0$; $\forall k < n,\ H_k(X) = 0$.
- $k + 2 \leq 2n \implies \pi_k(S^n) \cong \pi_{k+1}(S^{n+1})$
- $\pi_k(S^n) = \pi_{k+1} S^{n+1} = \cdots = \pi_{k+i} S^{n+i}$
- $F \to E \to B$ a fibration yields $\cdots \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi * n - 1(F) \cdots$
- Freundenthal suspension, stable homotopy groups

30.6 Higher Homotopy Groups of the Sphere

- $\pi_n(S^n) = \mathbb{Z}$
- $\pi_{n+1} S^n = \mathbb{Z}_2$ for $n \geq 4$
- $\pi_{n+2}(S^n) \cong \mathbb{Z}_2$
- $\pi_{n+3} S^n = \mathbb{Z}_8$ for $n \geq 5$
- $\pi_5 S^2 = \mathbb{Z}_2$
- $\pi_6 S^3 = \mathbb{Z}_4$
- $\pi_7 S^4 = \mathbb{Z} \oplus \mathbb{Z}_4$
- $\pi_k S^2 \cong \pi_k S^3$
- $\pi_3 S^2 \cong \mathbb{Z}$
- $\pi_4 S^2 \cong \mathbb{Z}_2$

30.7 Misc

- $\Omega(\cdot)$ is an exact functor.