Complex Analysis Problems

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1 Integrals and Cauchy's Theorem

1.1 1

Suppose $f, g : [0,1] \longrightarrow \mathbb{R}$ where f is Riemann integrable and for $x, y \in [0,1]$,

$$|g(x) - g(y)| \leq |f(x) - f(y)|.$$

Prove that g is Riemann integrable.

1.2 2

State and prove Green's Theorem for rectangles.

Then use it to prove Cauchy's Theory for functions that are analytic in a rectangle.

1.3 3

Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of analytic functions on $\mathbb{D}^\circ \coloneqq \left\{ z \in \mathbb{C} \;\middle|\; |z| < 1 \right\}$.

Show that if $f_n \longrightarrow g$ for some $g : \mathbb{D}^\circ \longrightarrow \mathbb{C}$ uniformly on every compact $K \subset \mathbb{D}^\circ$, then g is analytic on $\mathbb{D}^\circ$.

1.4 4

Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of entire functions where

- $f_n \longrightarrow g$ pointwise for some $g : \mathbb{C} \longrightarrow \mathbb{C}$.
- On every line segment in $\mathbb{C}$, $f_n \longrightarrow g$ uniformly.

Show that

- g is entire, and
- $f_n \longrightarrow g$ uniformly on every compact subset of $\mathbb{C}$.

1.5 5

Prove that there is no sequence of polynomials that uniformly converge to $f(z) = \dfrac{1}{z}$ on S^1.

1.6 6

Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function that vanishes outside of some finite interval. For each $z \in \mathbb{C}$, define

$$g(z) = \int_{-\infty}^{\infty} f(t) e^{-izt}\, dt.$$

Show that g is entire.

1.7 7

Suppose $f : \mathbb{C} \longrightarrow \mathbb{C}$ is entire and

$$|f(z)| \leq |z|^{\frac{1}{2}} \quad \text{when } |z| > 10.$$

Prove that f is constant.

1.8 8

Let γ be a smooth curve joining two distinct points $a, b \in \mathbb{C}$.

Prove that the function

$$f(z) \coloneqq \int_\gamma \frac{g(w)}{w - z}\, dw$$

is analytic in $\mathbb{C} \setminus \gamma$.

1.9 9

Suppose that $f : \mathbb{C} \longrightarrow \mathbb{C}$ is continuous everywhere and analytic on $\mathbb{C} \setminus \mathbb{R}$ and prove that f is entire.

1.10 10

Prove Liouville's theorem: suppose $f : \mathbb{C} \longrightarrow \mathbb{C}$ is entire and bounded. Use Cauchy's formula to prove that $f' \equiv 0$ and hence f is constant.

2 Liouville's Theorem, Power Series

2.1 1

Suppose f is analytic on a region Ω such that $\mathbb{D} \subseteq \Omega \subseteq \mathbb{C}$ and $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ is a power series with radius of convergence exactly 1.

a. Give an example of such an f that converges at every point of S^1.

b. Given an example of such an f which is analytic at 1 but $\sum_{n=0}^{\infty} a_n$ diverges.

c. Prove that f can not be analytic at *every* point of S^1.

2.2 2

Suppose f is entire and has Taylor series $\sum a_n z^n$ about 0.

a. Express a_n as a contour integral along the circle $|z| = R$.

b. Apply (a) to show that the above Taylor series converges uniformly on every bounded subset of $\mathbb{C}$.

c. Determine those functions f for which the above Taylor series converges uniformly on all of $\mathbb{C}$.

2.3 3

Suppose D is a domain and f, g are analytic on D.

Prove that if $fg = 0$ on D, then either $f \equiv 0$ or $g \equiv 0$ on D.

2.4 4

Suppose f is analytic on $\mathbb{D}^\circ$. Determine with proof which of the following are possible:

a. $f\left(\frac{1}{n}\right) = (-1)^n$ for each $n > 1$.

b. $f\left(\frac{1}{n}\right) = e^{-n}$ for each even integer $n > 1$ while $f\left(\frac{1}{n}\right) = 0$ for each odd integer $n > 1$.

c. $f\left(\frac{1}{n^2}\right) = \frac{1}{n}$ for each integer $n > 1$.

d. $f\left(\frac{1}{n}\right) = \frac{n-2}{n-1}$ for each integer $n > 1$.

2.5 5

Prove the Fundamental Theorem of Algebra (using complex analysis).

2.6 6

Find all entire functions that satisfy

$$|f(z)| \geq |z| \quad \forall z \in \mathbb{C}.$$

Prove this list is complete.

2.7 7

Suppose $\sum_{n=0}^{\infty} a_n z^n$ converges for some $z_0 \neq 0$.

a. Prove that the series converges absolutely for each z with $|z| < |z|_0$.

b. Suppose $0 < r < |z_0|$ and show that the series converges uniformly on $|z| \leq r$.

2.8 8

Suppose f is entire and suppose that for some integer $n \geq 1$,

$$\lim_{z \longrightarrow \infty} \frac{f(z)}{z^n} = 0.$$

Prove that f is a polynomial of degree at most $n-1$.

2.9 9

Find all entire functions satisfying

$$|f(z)| \leq |z|^{\frac{1}{2}} \quad \text{for } |z| > 10.$$

2.10 10

Prove that the following series converges uniformly on the set $\left\{ z \;\middle|\; \Im(z) < \ln 2 \right\}$:

$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{2^n}.$$

3 Spring 2020 Homework 1

4 Spring 2020 Homework 2

Note on notation: I sometimes use $f_x \coloneqq \frac{\partial f}{\partial x}$ to denote partial derivatives, and $\partial_z^n f$ as $f^{(n)}(z)$.

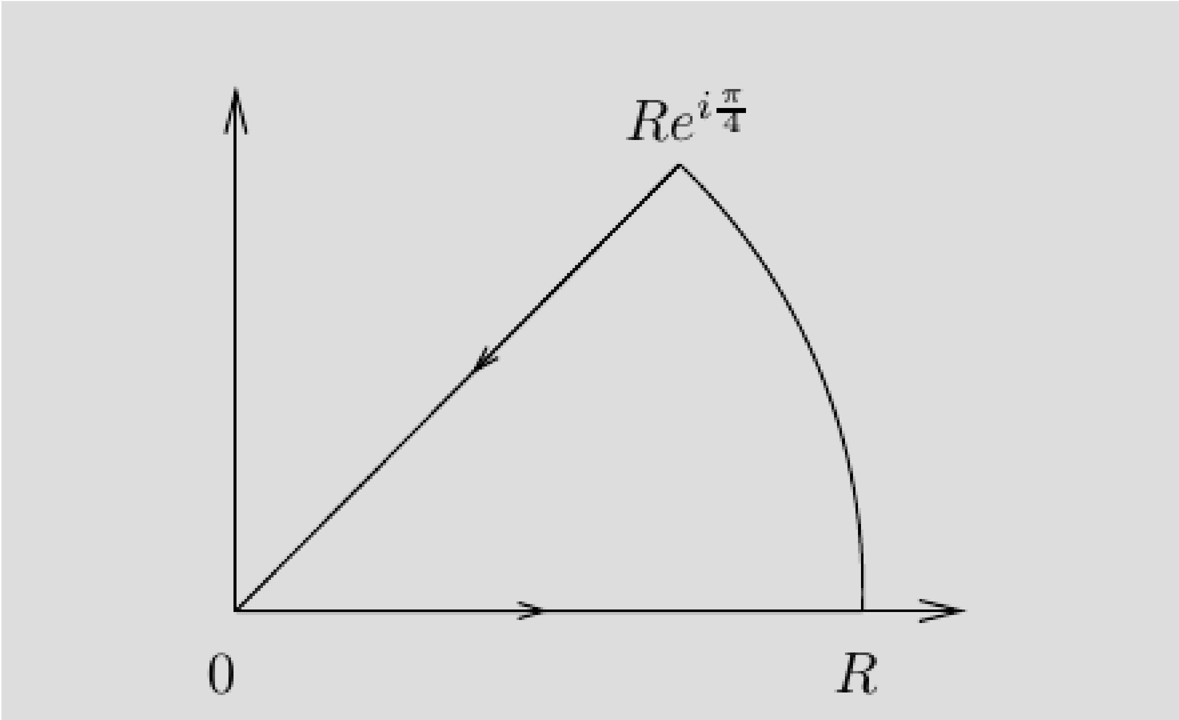
4.1 Stein And Shakarchi

4.1.1 2.6.1

Show that

$$\int_0^{\infty} \sin\left(x^2\right) dx = \int_0^{\infty} \cos\left(x^2\right) dx = \frac{\sqrt{2\pi}}{4}.$$

Hint: integrate e^{-x^2} over the following contour, using the fact that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$:



4.1.2 2.6.2

Show that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Hint: use the fact that this integral equals $\dfrac{1}{2i}\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}-1}{x} dx$, and integrate around an indented semicircle.

4.1.3 2.6.5

Suppose $f \in C^1_{\mathbb{C}}(\Omega)$ and $T \subset \Omega$ is a triangle with $T^\circ \subset \Omega$. Apply Green's theorem to show that $\displaystyle\int_T f(z)\ dz = 0.$

Assume that f' is continuous and prove Goursat's theorem.

Hint: Green's theorem states

$$\int_T F dx + G dy = \int_{T^\circ} \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dxdy.$$

4.1.4 2.6.6

Suppose that f is holomorphic on a punctured open set $\Omega \setminus \{w_0\}$ and let $T \subset \Omega$ be a triangle containing w_0. Prove that if f is bounded near w_0, then $\int_T f(z)\, dz = 0$.

4.1.5 2.6.7

Suppose $f : \mathbb{D} \longrightarrow \mathbb{C}$ is holomorphic and let $d \coloneqq \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$ be the diameter of the image of f. Show that $2\left|f'(0)\right| \leq d$, and that equality holds iff f is linear, so $f(z) = a_1 z + a_2$.

> Hint: $2f'(0) = \dfrac{1}{2\pi i}\displaystyle\int_{|\xi|=r} \frac{f(\xi) - f(-\xi)}{\xi^2}\, d\xi$ whenever $0 < r < 1$.

4.1.6 2.6.8

Suppose that f is holomorphic on the strip $S = \left\{x + iy \;\middle|\; x \in \mathbb{R},\ -1 < y < 1\right\}$ with $|f(z)| \leq A(1 + |z|)^\nu$ for ν some fixed real number. Show that for all $z \in S$, for each integer $n \geq 0$ there exists an $A_n \geq 0$ such that $\left|f^{(n)}(x)\right| \leq A_n(1 + |x|)^\nu$ for all $x \in \mathbb{R}$.

> Hint: Use the Cauchy inequalities.

4.1.7 2.6.9

Let $\Omega \subset \mathbb{C}$ be open and bounded and $\varphi : \Omega \longrightarrow \Omega$ holomorphic. Prove that if there exists a point $z_0 \in \Omega$ such that $\varphi(z_0) = z_0$ and $\varphi'(z_0) = 1$, then φ is linear.

> Hint: assume $z_0 = 0$ (explain why this can be done) and write $\varphi(z) = z + a_n z^n + O(z^{n+1})$ near 0. Let $\varphi_k = \varphi \circ \varphi \circ \cdots \circ \varphi$ and prove that $\varphi_k(z) = z + k a_n z^n + O(z^{n+1})$. Apply Cauchy's inequalities and let $k \longrightarrow \infty$ to conclude.

4.1.8 2.6.10

Can every continuous function on $\overline{\mathbb{D}}$ be uniformly approximated by polynomials in the variable z?

> Hint: compare to Weierstrass for the real interval.

4.1.9 2.6.13

Suppose f is analytic, defined on all of $\mathbb{C}$, and for each $z_0 \in \mathbb{C}$ there is at least one coefficient in the expansion $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ is zero. Prove that f is a polynomial.

> Hint: use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.

4.1.10 2.6.14

Suppose that f is holomorphic in an open set containing $\mathbb{D}$ except for a pole $z_0 \in \partial\mathbb{D}$. Let $\sum_{n=0}^{\infty} a_n z^n$ be the power series expansion of f in $\mathbb{D}$, and show that $\lim \frac{a_n}{a_{n+1}} = z_0$.

4.1.11 2.6.15

Suppose f is continuous and nonvanishing on $\overline{\mathbb{D}}$, and holomorphic in $\mathbb{D}$. Prove that if $|z| = 1 \implies |f(z)| = 1$, then f is constant.

> Hint: Extend f to all of $\mathbb{C}$ by $f(z) = 1/\overline{f(1/\bar{z})}$ for any $|z| > 1$, and argue as in the Schwarz reflection principle.

4.2 Additional Problems

4.2.1 1

Let $a_n \neq 0$ and show that

$$\lim_{n \longrightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L \implies \lim_{n \longrightarrow \infty} |a_n|^{\frac{1}{n}} = L.$$

In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

4.2.2 2

Let f be a power series centered at the origin. Prove that f has a power series expansion about any point in its disc of convergence.

4.2.3 3

Prove the following:

a. $\sum_n n z^n$ does not converge at any point of S^1

b. $\sum_n \frac{z^n}{n^2}$ converges at every point of S^1.

c. $\sum_n \frac{z^n}{n}$ converges at every point of S^1 except $z = 1$.

4.2.4 4

Without using Cauchy's integral formula, show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{dz}{(z - \alpha)(z - \beta)} = \frac{2\pi i}{\alpha - \beta}$$

where γ denotes the circle centered at the origin of radius r with positive orientation.

4.2.5 5

Assume f is continuous in the region $\left\{x+iy \;\middle|\; x \geq x_0,\ 0 \leq y \leq b\right\}$, and the following limit exists independent of y:

$$\lim_{x \longrightarrow +\infty} f(x+iy) = A.$$

Show that if $\gamma_x := \left\{z = x+it \;\middle|\; 0 \leq t \leq b\right\}$, then

$$\lim_{x \longrightarrow +\infty} \int_{\gamma_x} f(z)\, dz = iAb.$$

4.2.6 6

Show by example that there exists a function $f(z)$ that is holomorphic on $\left\{z \in \mathbb{C} \;\middle|\; 0 < |z| < 1\right\}$ and for all $r < 1$,

$$\int_{|z|=r} f(z)\, dz = 0,$$

but f is not holomorphic at $z = 0$.

4.2.7 7

Let f be analytic on a region R and suppose $f'(z_0) \neq 0$ for some $z_0 \in R$. Show that if C is a circle of sufficiently small radius centered at z_0, then

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{dz}{f(z) - f(z_0)}.$$

Hint: use the inverse function theorem.

4.2.8 8

Assume two functions $u, b : \mathbb{R}^2 \longrightarrow \mathbb{R}$ have continuous partial derivatives at (x_0, y_0). Show that $f := u + iv$ has derivative $f'(z_0)$ at $z_0 == x_0 + iy_0$ if and only if

$$\lim_{r \to 0} \frac{1}{\pi r^2} \int_{|z-z_0|=r} f(z) dz = 0.$$

4.2.9 9 (Cauchy's Formula for Exterior Regions)

Let γ be a piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2. Assume f' exists in an open set containing γ and Ω_2 with $\lim_{z \longrightarrow \infty} f(z) = A$. Show that

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1 \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}.$$

4.2.10 10

Let $f(z)$ be bounded and analytic in $\mathbb{C}$. Let $a \neq b$ be any fixed complex numbers. Show that the following limit exists:

$$\lim_{R \longrightarrow \infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}\, dz.$$

Use this to show that $f(z)$ must be constant.

4.2.11 11

Suppose $f(z)$ is entire and

$$\lim_{z \longrightarrow \infty} \frac{f(z)}{z} = 0.$$

Show that $f(z)$ is a constant.

4.2.12 12

Let f be analytic in a domain D and γ be a closed curve in D. For any $z_0 \in D$ not on γ, show that

$$\int_\gamma \frac{f'(z)}{(z-z_0)} dz = \int_\gamma \frac{f(z)}{(z-z_0)^2} dz.$$

Give a generalization of this result.

4.2.13 13

Compute

$$\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z}$$

and use it to show that

$$\in_0^{2\pi} \cos^{2n}(\theta)\, d\theta = 2\pi \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right).$$

5 Spring 2020 Homework 3

5.1 Problems From Tie

5.1.1 1

Prove that if f has two Laurent series expansions,

$$f(z) = \sum c_n (z-a)^n \quad \text{and} \quad f(z) = \sum c_n' (z-a)^n$$

then $c_n = c_n'$.

Solution By taking the difference of two such expansions, it suffices to show that if f is identically zero and $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$ about some point a, then $c_n = 0$ for all n.

Under this assumption, let $D_\varepsilon(a)$ be a disc about a and γ be any contoured contained in its interior. Then for each n, we can apply the formula

$$\begin{aligned} c_n &= \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - a)^{n+1}} \, d\xi \\ &= \frac{1}{2\pi i} \int_\gamma \frac{0}{(\xi - a)^{n+1}} \, d\xi \quad \text{by assumption} \\ &= 0, \end{aligned}$$

which shows that $c_n = 0$ for all n.

■

5.1.2 2

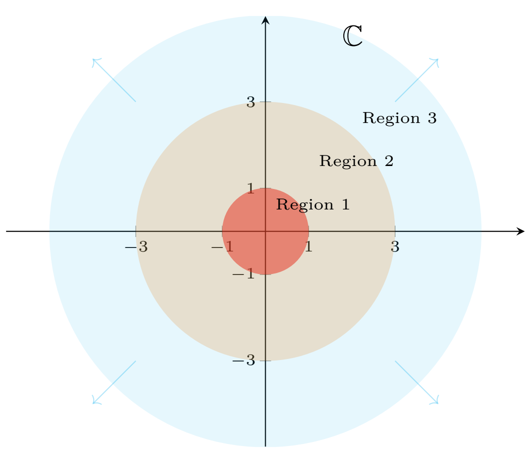
Find Laurent series expansions of

$$\frac{1}{1-z^2} + \frac{1}{3-z}$$

How many such expansions are there? In what domains are each valid?

Solution Note that f has poles at $z = -1, 1, 3$, all with multiplicity 1, and so there are 3 regions to consider:

1. $|z| < 1$
2. $1 < |z| < 3$
3. $3 < |z|$.



Region 1: Take the following expansion:

$$
\begin{aligned}
f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\
&= \sum_{n\geq 0} z^{2n} + \frac{1}{3}\left(\frac{1}{1-\frac{3}{z}}\right) \\
&= \sum_{n\geq 0} z^{2n} + \frac{1}{3}\sum_{n\geq 0}\left(\frac{1}{3}\right)^n z^n \\
&= \sum_{n\geq 0} z^{2n} + \sum_{n\geq 0}\left(\frac{1}{3}\right)^{n+1} z^n
\end{aligned}
$$

.

Noting $\left|z^2\right| < 1$ implies then $|z| < 1$, and that the first term converges for $\left|z^2\right| < 1$ and the second for $\left|\frac{z}{3}\right| < 1 \iff |z| < 3$, this expansion converges to f on the region $|z| < 1$.

Region 2: Take the following expansion:

$$
\begin{aligned}
f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\
&= -\frac{1}{z^2}\left(\frac{1}{1-\frac{1}{z^2}}\right) - \frac{1}{3}\left(\frac{1}{1-\frac{z}{3}}\right) \\
&= -\frac{1}{z^2}\sum_{n\geq 0} z^{-2n} + \sum_{n\geq 0}\left(\frac{1}{3}\right)^{n+1} z^n \\
&= -\sum_{n\geq 2}\frac{1}{z^{2n}} + \sum_{n\geq 0}\left(\frac{1}{3}\right)^{n+1} z^n
\end{aligned}
$$

.

By construction, the first term converges for $\left|\frac{1}{z^2}\right| < 1 \iff |z| > 1$ and the second for $|z| < 3$.

Region 3: Take the following expansion:

$$
\begin{aligned}
f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} \\
&= -\frac{1}{z^2}\left(\frac{1}{1-\frac{1}{z^2}}\right) - \frac{1}{z}\left(\frac{1}{1-\frac{3}{z}}\right) \\
&= -\frac{1}{z^2}\sum_{n\geq 0}\frac{1}{z^{2n}} - \frac{1}{z}\sum_{n\geq 0} 3^n\frac{1}{z^n} \\
&= -\sum_{n\geq 2}\frac{1}{z^{2n}} - \sum_{n\geq 1}\left(\frac{1}{3}\right)^{n-1}\frac{1}{z^n}.
\end{aligned}
$$

Note: in principle, terms could be collected here.

By construction, this converges on $\left\{|z|^2 > 1\right\} \bigcap \{|z| > 3\} = \{|z| > 3\}$.

■

5.1.3 3

Let P, Q be polynomials with no common zeros. Assume a is a root of Q. Find the principal part of P/Q at $z = a$ in terms of P and Q if a is (1) a simple root, and (2) a double root.

Solution We'll use the following definition: if $f : \mathbb{C} \longrightarrow \mathbb{C}$ is analytic with Laurent expansion $f(z) = \displaystyle\sum_{k=-\infty}^{\infty} c_k (z-a)^k$ at the point $a \in \mathbb{C}$, then the **principal part** of f at a is given by

$$\sum_{k=-1}^{-\infty} c_k (z-a)^k = c_{-1}(z-a)^{-1} + c_{-2}(z-a)^{-2} + \cdots.$$

Without loss of generality (by performing polynomial long division if necessary), assume that $\deg P < \deg Q$. By the method used in the theorem that proves meromorphic functions are rational, if we let $a_1, \cdots, a_n$ be the finitely many zeros of $Q(z)$, these are the finitely many poles of $P(z)/Q(z)$, and we can write

$$\frac{P(z)}{Q(z)} := f(z) = P_\infty(z) + \sum_{i=1}^{n} P_{a_i}(z)$$

where $P_w(z)$ denotes the principal part of f at the point w.

Note that if w is a pole of order ℓ, we can explicitly write

$$P_w(z) = \frac{\alpha_1}{z-w} + \frac{\alpha_2}{(z-w)^2} + \cdots + \frac{\alpha_\ell}{(z-w)^\ell}$$

for some constants $\alpha_i \in \mathbb{C}$, and thus the first equation expresses f in terms of its partial fraction decomposition.

Thus if a is a simple root of $Q(z)$, it is a simple pole of f, and thus we have $P_a(z) = \dfrac{\alpha_1}{z-a}$, which consists of a single term. Since we can write $f(z) = P_\infty(z) + P_a(z) + \cdots$ where none of the remaining terms involve a, it follows by definition that $\alpha_1 = \operatorname{Res}(f, a)$ and so

$$P_a(z) = \frac{\operatorname{Res}(f(z), a)}{z-a},$$

where we can use a known formula to express $\operatorname{Res}(f(z), a) = \dfrac{P(a)}{Q'(a)}$.

Similarly, if now a is a root of multiplicity 2 of $Q(z)$, a is a pole of order 2 of f and $P_a(z) = \dfrac{\alpha_1}{z-a} + \dfrac{\alpha_2}{(z-a)^2}$ with precisely two terms. Thus as before, $\alpha_1 = \operatorname{Res}(f(z), a)$, and now $\alpha_2 =$

$\operatorname{Res}((z-a)f(z),a)$, and we have

$$P_a(z) = \frac{\operatorname{Res}(f(z),a)}{z-a} + \frac{\operatorname{Res}((z-a)f(z),a)}{(z-a)^2}.$$

$\blacksquare$

5.1.4 4

Let f be non-constant, analytic in $|z| > 0$, where $f(z_n) = 0$ for infinitely many points z_n with $\lim_{n \longrightarrow \infty} z_n = 0$.
Show that $z = 0$ is an essential singularity for f.

Example: $f(z) = \sin(1/z)$.

Solution We first note that $z = 0$ is in fact a singularity of f, since the zeros of analytic functions are isolated.

The point $z = 0$ can not be a pole because (by definition) this would force $\lim_{z \longrightarrow 0} |f(z)| = \infty$. Explicitly, this would mean that for every $R > 0$, there would exist a $\delta > 0$ such that $z \in D_\delta(0) \implies |f(z)| > R$.

However, since $z_n \longrightarrow 0$ and $f(z_n) = 0 < R$ for every n, every $D_\delta(0)$ contains a point z_N that violates this condition.

Similarly, $z = 0$ can not be removable, since the function

$$g(z) = \begin{cases} 0 & z = 0 \\ f(z) & \text{otherwise} \end{cases}$$

defines an analytic continuation of f. However, it is a theorem that the zeros of an analytic function are isolated, whereas every neighborhood of $z = 0$ (which is a zero of g) contains infinitely many distinct zeros of the form z_n, a contradiction.

■

5.1.5 5

Show that if f is entire and $\lim_{z \longrightarrow \infty} f(z) = \infty$, then f is a polynomial.

Solution Since f is entire, it is analytic on $\mathbb{C}$, so there is an expansion $f(z) = \sum_{k=0}^{\infty} c_k z^k$ that converges to f everywhere. Let $F(z) = f(1/z)$; then $\lim_{z \longrightarrow 0} F(z) = \infty$ by assumption.

This also implies that since $z = \infty$ is a pole of f, the point $z = 0$ is a pole of F, say of order N.

However, we can expand $F(z) = \sum_{k=0}^{\infty} c_k \frac{1}{z^k}$. Since this is a Laurent expansion for F about $z = 0$, which is a pole of order N, we must in fact have $F(z) = \sum_{k=0}^{N} c_k \frac{1}{z_k}$, i.e. there are only N terms in this expansion.

This implies that $f(z) = \sum_{k=0}^{N} c_k z^k$, which has finitely many terms and is thus a polynomial.

■

5.1.6 6

a. Show that

$$\int_0^{2\pi} \log \left|1 - e^{i\theta}\right| \, d\theta = 0$$

b. Show that this identity is equivalent to SS 3.8.9:

$$\int_0^1 \log(\sin(\pi x)) \, dx = -\log 2.$$

Solution Part (a) Let I be the integral in question, then substituting $z = e^{i\theta}$ and $\dfrac{dz}{iz} = d\theta$ yields

$$I = \int_{S^1} \frac{\log|1-z|}{iz} \, dz := \Re\left(\int_{S^1} f(z) \, dz\right),$$

where

$$f(z) := \frac{\log(1-z)}{iz},$$

S^1 denotes the unit circle in $\mathbb{C}$, and since by definition

$$\log_{\mathbb{C}}(z) = \log_{\mathbb{R}}(|z|) + i\arg(z)$$

where the subscripts denote the complex and real logarithms respectively, we have

$$\log_{\mathbb{C}} |1-z| = \Re(\log_{\mathbb{C}}(1-z)).$$

So it suffices to show that $\displaystyle\int_{S^1} f(z) \, dz = 0$.

The claim is that $z = 0$ is a removable singularity and thus f is holomorphic in the unit disc. The singularity is removable because we have

$$\begin{aligned}
\lim_{z \longrightarrow 0} f(z) &= \lim_{z \longrightarrow 0} \frac{\log(1-z)}{iz} \\
&= \lim_{z \longrightarrow 0} \frac{\frac{1}{1-z}}{i} \qquad \text{by L'Hopital's} \\
&= -i,
\end{aligned}$$

so the modified function

$$F(z) = \begin{cases} -i & z = 0 \\ f(z) & \text{otherwise} \end{cases}$$

is holomorphic, making $z = 0$ removable.

Since f is also analytic, the Cauchy-Goursat theorem applies and $\displaystyle\int_{S^1} f = 0$.

Solution Part (b) No clue how to relate these two!

5.1.7 7

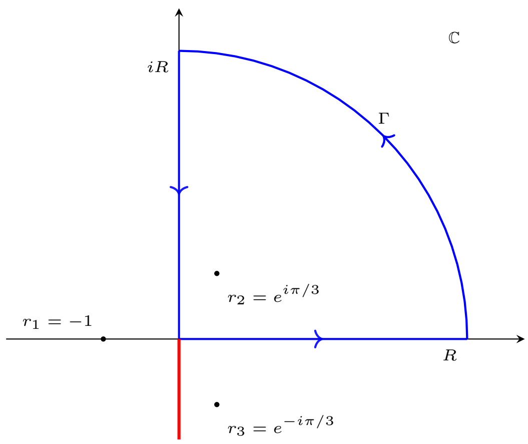
Let $0 < a < 4$ and evaluate

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^3}\,dx$$

Solution Let I denote the integral in question. We will compute this using a closed contour and the residue theorem, so first note that

$$z^3+1 = (z+1)(z-e^{i\pi 3})(z-e^{-i\pi 3}) := (z-r_1)(z-r_2)(z-r_3).$$

Defining $z^\alpha = e^{\alpha \log z}$ for $\alpha \in \mathbb{R}$, we'll take the following contour Γ shown in blue along with a branch cut for the logarithm function indicated in red:



Letting

$$f(z) := \frac{z^{\alpha-1}}{z^3+1} := \frac{P(z)}{Q(z)},$$

we find that only $z = r_2$ will contribute a term to $\displaystyle\int_\Gamma f$. Noting that each pole is simple of order 1, we have

$$\text{Res}(f(z), z = r_i) = \frac{P(r_i)}{Q'(r_i)} = \frac{r_i^{\alpha-1}}{3r_i^2} = \frac{r_i^{\alpha-3}}{3}$$

We thus have

$$\begin{aligned}\text{Res}(f(z), z = r_2) &= \frac{1}{3}e^{\frac{i\pi(\alpha-3)}{3}} \\ \implies \int_\Gamma f(z)\,dz &= \frac{2\pi i}{3}e^{\frac{i\pi(\alpha-3)}{3}}.\end{aligned}$$

We can now compute the contributions to the integral along the semicircular arc and the portion along the imaginary axis.

Along the arc, Jordan's lemma applies since $\dfrac{1}{R^3+1} \overset{R\longrightarrow\infty}{\longrightarrow} 0$, and thus this contribution vanishes.

Along the imaginary axis, we can make the following change of variables:

$$\begin{aligned}
\int_R^0 f(iy)\, dy &= -\int_0^R \frac{(iy)^{\alpha-1}}{(iy)^3+1}\, dy \\
&= -\frac{1}{i}\int_0^R \frac{t^{\alpha-1}}{t^3+1}\, dt \qquad\qquad (t = iz,\ dt = idz) \\
&= iI,
\end{aligned}$$

which is i times the original integral.

We thus have

$$\begin{aligned}
\operatorname{Res}(f(z), z = r_2) &= \int_\Gamma f \\
&= \int_0^R f + \int_{C_R} f + \int_{iR}^0 f \\
&\overset{R\longrightarrow\infty}{\longrightarrow} I + 0 + iI \\
&= (1+i)I,
\end{aligned}$$

and so

$$I = \frac{\operatorname{Res}(f(z), z = r_2)}{1+i} = \frac{2\pi i}{3(1+i)} e^{\frac{i\pi(\alpha-3)}{3}}.$$

■

Note: this seems to be wrong, because plugging in $a = 1, 2, 3$ doesn't result in a real value.

5.1.8 8

Prove the fundamental theorem of Algebra using

a. Rouche's Theorem.
b. The maximum modulus principle.

Solution (Rouche) We want to show that every $f \in \mathbb{C}[x]$ has precisely n roots, and we'll use the follow formulation of Rouche's theorem:

Theorem 5.1 ***(Rouche).***
If f, g are holomorphic on $D(z_0)$ with $f, g \neq 0$ and $|f - g| < |f| + |g|$ on $\partial D(z_0)$, then f and g has the same number of zeros within D.

We'll also use without proof the fact that the function $h(z) = z^n$ has precisely n zeros (counted with multiplicity).

Suppose $f(z) = a_n z^n + \cdots + a_1 z + a_0$ where $a_n \neq 0$ and define

$$g(z) \coloneqq a_n z^n.$$

Noting that polynomials are entire, f, g are nonzero by assumption, and fixing $|z| = R > 1$, we have

$$\begin{aligned}
|f - g| &= \left|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right| \\
&= \left|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 + a_n z^n - a_n z^n\right| \\
&\leq \left|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 + a_n z^n\right| + \left|-a_n z^n\right| \quad \text{by the triangle inequality} \\
&= |f| + |g|
\end{aligned}$$

the conditions of Rouche's theorem apply and f, g have the same number of roots. Since g has precisely n roots, f does as well.

This is much simpler than other proofs out there, so I suspect something is slightly wrong but I couldn't sort out what it was.

■

Solution (Maximum Modulus Principle) Toward a contradiction, suppose f is non-constant and has *no* zeros. Then $g(z) \coloneqq 1/f(z)$ is non-constant and holomorphic on $\mathbb{C}$.

Using the fact that $\lim_{z \longrightarrow \infty} f(z) = \infty$ for any polynomial f, pick R large enough such that

$$|z| \geq R \implies |f(z)| > |f(0)|,$$

which inverted yields,

$$|z| \geq R \implies |g(z)| < |g(0)|.$$

Noting that $S_R \coloneqq \{|z| \geq R\}$ is closed (as the complement of the open set $\{|z| < R\}$), bounded (by the argument above), and thus compact by Heine-Borel, g attains a maximum on S_R.

But by the maximum modulus principle, this forces g to be constant, and since $g = \dfrac{1}{f}$, it must also be true that f is constant.

■

5.1.9 9

Let f be analytic in a region D and γ a rectifiable curve in D with interior in D. Prove that if $f(z)$ is real for all $z \in \gamma$, then f is constant.

Solution Without loss of generality, assume $0 \in D$ (by considering the translate $f(z) - w$ if necessary) and γ is not entirely contained in $\mathbb{R}$ (by taking a homotopic curve).

Since f is analytic in D, we can write its Laurent series expansion about $z = 0$:

$$f(z) = c_0 + c_1 z + \cdots \qquad \text{for } z \in D.$$

For $z \in \gamma$ we can write $z = x + iy$ where $y \neq 0$.

Then

$$\begin{aligned} f(z) &= f(x+iy) \\ &= c_0 + c_1(x+iy) + c_2(x+iy)^2 + \cdots \\ &= c_0 + \left(c_1 x + c_2 x^2 + \cdots\right) + i\left(c_1 y + c_2 y^2 + c_2 xy + \cdots\right) \\ &\subset \mathbb{R} \quad \text{by assumption,} \end{aligned}$$

where the second parenthesized term must vanish for all $x + iy \in \gamma$,

But since there is at least one $z \in \gamma$ with $y \neq 0$, this forces $c_1 = c_2 = \cdots = 0$, and thus $f(z) = c_0$ must be constant.

■

5.1.10 10

For $a > 0$, evaluate

$$\int_0^{\pi/2} \frac{d\theta}{a + \sin^2\theta}$$

Solution We have

$$
\begin{aligned}
I &:= \int_0^{\pi/2} \frac{1}{1+\sin^2(\theta)}\, d\theta \\
&= \int_{\gamma_1} \frac{1}{a + \left(\frac{z-z^{-1}}{2i}\right)^2} \frac{-i\, dz}{z} \qquad \text{where } \gamma_1 \text{ is } \frac{1}{4} \text{ of the unit circle } S^1 \\
&= -i \int_{\gamma_1} \frac{1}{z} \left(\frac{1}{a + \left(-\frac{1}{4}\right)(z^2 - 2 + z^{-2})} \right) dz \\
&= 4i \int_{\gamma_1} \frac{1}{z} \left(\frac{1}{z^2 - (2+4a) + z^{-2}} \right) dz \\
&= 4i \int_{\gamma_1} \frac{z}{z^4 - (2+4a)z^2 + 1}\, dz \\
&= i \oint_{S^1} \frac{z}{z^4 - (2+4a)z^2 + 1}\, dz \\
&= \frac{i}{2} \oint_{2\cdot S^1} \frac{1}{u^2 - (2+4a)u + 1}\, du \qquad \text{using } u = z^2, \frac{1}{2}\, du = z\, dz \\
&:= \frac{i}{2} \oint_{2\cdot S^1} \frac{1}{f_a(u)}\, du \\
&= \frac{i}{2} \cdot 2\pi i \cdot \sum \operatorname{Res}\left(\frac{1}{f_a(u)}, u = r_i \right),
\end{aligned}
$$

where $2 \cdot S^1$ denotes the contour wrapping around the unit circle twice and r_i denote the poles contained in the region bounded by S^1. We can now compute the last integral by the residue theorem.

Factor the denominator as

$$
f_a(u) = u^2 - (2+4a)u + 1 = (u - r_1)(u - r_2),
$$

where the r_i are given by $(1+2a) \pm 4\sqrt{a^2+a}$ using the quadratic formula. We can then write a partial fraction decomposition

$$
\begin{aligned}
\frac{1}{f_a(u)} &:= \frac{1}{u^2 - (2+4a)u + 1} \\
&= \frac{1}{(u-r_1)(u-r_2)} \\
&= \frac{A}{u-r_1} + \frac{B}{u-r_2} \\
&= \frac{\operatorname{Res}_{u=r_1} 1/f(u)}{u-r_1} + \frac{\operatorname{Res}_{u=r_2} 1/f(u)}{u-r_2} \\
&= \frac{1/f'(r_1)}{u-r_1} + \frac{1/f'(r_2)}{u-r_2} \\
&= -\frac{1}{8\sqrt{a^2+a}(u-r_1)} + \frac{1}{8\sqrt{a^2+a}(u-r_2)}.
\end{aligned}
$$

Since $|r_2| = \left|(1+2a) + 4\sqrt{a^2+a}\right| > 1$, we find that the only relevant pole inside of S^1 is r_1. Reading off the residue from the above decomposition, we thus have

$$\begin{aligned} I &= \frac{i}{2} \cdot 2\pi i \cdot \sum \mathrm{Res}_{u=r_i} \frac{1}{f_a(u)} \\ &= -\pi \cdot \mathrm{Res}_{u=r_1} \frac{1}{f_a(u)} \\ &= \frac{\pi}{8\sqrt{a^2+a}}. \end{aligned}$$

■

Note: I know I'm off by a constant here at least, since $a = 1$ should reduce to $\pi/2\sqrt{2}$.

5.1.11 11

Find the number of roots of $p(z) = 4z^4 - 6z + 3$ in $|z| < 1$ and $1 < |z| < 2$ respectively.

Solution For $|z| < 1$, take $f(z) = -6z$ and $g(z) = z^4 + 3$, noting that $f + g = p$. Using the maximum modulus principal, we know that the max/mins of f, g occur on $|z| = 1$, on which we have

$$|g(z)| = 4 < 6 = |f(z)|,$$

so Rouche's theorem applies and both p and f have the same number of zeros. Since f clearly has **one** zero, p has one zero in this region.

Now consider $|z| < 2$ and set $f(z) = z^4$ and $g(z) = -6z + 3$. By a similar argument, we have

$$|g(z)| = 15 < 16 = |f|$$

on $|z| = 2$, and thus f and p have the same number of zeros in this region. Since f has **four** zeros here, so does p.

Thus p has $4 - 1 =$ **3 zeros** on $1 \leq |z| \leq 2$.

■

5.1.12 12

Prove that $z^4 + 2z^3 - 2z + 10$ has exactly one root in each open quadrant.

Solution Let $f(z) = z^4 + 2z^3 - 2z + 10$, and consider the following contour:

Image

By the argument principle, we have

$$\Delta_\Gamma \arg f(z) = 2\pi(Z - P),$$

where Z is the number of zeros of f in the region Ω enclosed by Γ and P is the number of poles in Ω.

Since polynomials are holomorphic on $\mathbb{C}$, by the argument principle it suffices to show that

- f does not have any roots on the real or imaginary axes
- f does not vanish on Γ, and
- $\Delta_\Gamma \arg f(z) = 1$, where Δ_Γ denotes the total change in the argument of f over Γ.

It will follow by symmetry that f has exactly one root in each quadrant.

Claim 5.2.

- f has no roots on the coordinate axes.
- $\Delta_{\gamma_1} \arg f(z) = 0$
- $\Delta_{\gamma_2} \arg f(z) = 2\pi$
- $\Delta_{\gamma_3} \arg f(z) = 0$

Given the claim, we would have

$$\Delta_\Gamma \arg f(z) = 2\pi = 2\pi(Z - 0) \implies Z = 1,$$

which is what we wanted to show.

Proof of Claim:

γ_2: For $R \gg 0$, we have $f(z) \sim z^4$. Along γ_2, the argument of z ranges from 0 to $\dfrac{\pi}{2}$, and thus the argument of z^4 ranges from 0 to $4 \cdot \dfrac{\pi}{2} = 2\pi$.

γ_1: By cases, for $z \in \mathbb{R}$,

- If $|z| > 1$, then $z^3 > z$ and so

$$\begin{aligned} f(z) &= (z^4 + 10) + (2z^3 - 2z) \\ &> (z^4 + 10) + (2z - 2z) \\ &= z^4 + 10 \\ &> 0, \end{aligned}$$

  so f is strictly positive and does not change argument on $(\pm 1, \pm\infty)$ or $i \cdot (\pm 1, \pm\infty)$.

- If $|z| \leq 1$,

$$\begin{aligned} \left| -z^4 - 2z^3 + 2z \right| &\leq |z|^4 + 2|z|^3 + 2|z| \\ &\leq 1 + 2 + 2 \\ &= 5 \\ &< 10 \\ \implies f(z) &= 10 - (-z^4 - 2z^3 + 2z) > 0, \end{aligned}$$

  so f is strictly positive and does not change argument $(0, \pm 1)$ or $i \cdot (0, \pm 1)$.

■

5.1.13 13

Prove that for $a > 0$, $z \tan z - a$ has only real roots.

Solution We can extend Rouche's theorem in the following way: if $f = g + h$ with $|g| > |h|$ on γ then $Z_f - P_f = Z_g - P_g$, where Z, P denote the number of zeros and poles respectively.

So we proceed by explicitly counting the number of real roots Z_f of $f(z) = z\tan(z) - a$ on a certain arbitrary real interval, then extend that interval to a rectangle in $\mathbb{C}$ and apply Rouche to show that there are still Z_f zeros within the rectangle. This will imply that the only roots on that region are real, and in the limit as the length of the interval goes to infinity, this will remain true (since any potential root must fall within such a bounded rectangle).

Fix some parameter $N \in \mathbb{Q}$ (to be determined) and consider the interval $[-N\pi - \varepsilon, N\pi + \varepsilon]$ for some $0 < \varepsilon \ll 1$. In this interval, we can compute $\sin(x) = 0 \iff x = 2k\pi$, yielding $2N + 1$ zeros (including $x = 0$), and thus $x\tan(x)$ has exactly $2N + 1$ zeros here.

We can also compute $\cos(x) = 0 \iff x = (2k + 1)\pi$, yielding $2N$ zeros and thus $2N$ poles of $x\tan(x)$.

Thus letting $\tilde{Z}_f, \tilde{P}_f$ denote the number of real zeros/poles of f, we have

$$\tilde{Z}_f - \tilde{P}_f = (2N + 1) - 2N = 1.$$

I got stuck here. It's not clear that $g(z) = z\tan(z), h(z) = a$ works at all. Trying to analyze the related function $\tan(z) - \dfrac{a}{z}$ didn't seem to help either. General idea: decompose $f = g + h$, try to bound $|h| < |g|$ on edges of the rectangle $R = [-N\pi, N\pi] \times i[-N\pi, N\pi]$, use Rouche to get $Z_f - P_f = Z_g - P_g \implies Z_f = P_f + Z_g - P_g$ and use the fact that $\cos(z)$ has only real zeros to count P_f in R.

5.1.14 14

Let f be nonzero, analytic on a bounded region Ω and continuous on its closure $\overline{\Omega}$.
Show that if $|f(z)| \equiv M$ is constant for $z \in \partial\Omega$, then $f(z) \equiv Me^{i\theta}$ for some real constant θ.

Solution By the maximum modulus principle applied to f in $\overline{\Omega}$, we know that $\max|f| = M$. Similarly, the maximum modulus principle applied to $\dfrac{1}{f}$ in $\overline{\Omega^c}$ since f is nonzero in Ω, and we can conclude that $\min|f| = M$ as well. Thus $|f| = M$ is constant on $\overline{\Omega}$.

So consider the function $g(z) = |f(z)|$; from the above observation, we find that $g(\overline{\Omega}) = \{M\}$. Letting S_M be the circle of radius M, this implies that $f(\Omega) \subseteq S_M$. In particular, $S_M \subset \mathbb{C}$ is a closed set.

However, by the open mapping theorem, $f(\Omega) \subset \mathbb{C}$ must be an open set. A basis for the topology on $\mathbb{C}$ is given by open discs, so in particular, the open sets of $\mathbb{C}$ have real dimension either zero or two.

Since S_M has real dimension 1, $f(\Omega)$ must have dimension zero and is thus a collection of points. Since f is continuous, the image can only be one point, i.e. $f(\Omega) = \{\text{pt}\} \in S_M$. So f is constant.

■

6 Extra Questions from Jingzhi Tie

6.1 Fall 2009

6.1.1 ?

```
  (1) Assume $\displaystyle f(z) = \sum_{n=0}^\infty c_n z^n$
converges in $|z| < R$. Show that for $r <R$,
$$\frac{1}{2 \pi} \int_0^{2 \pi} |f(r e^{i \theta})|^2 d \theta =
\sum_{n=0}^\infty |c_n|^2 r^{2n} \; .$$

(2) Deduce Liouville's theorem from (1).
```

6.1.2 ?

```
  Let $f$ be a continuous function in the region
$$D=\{z \suchthat  \abs{z}>R, 0\leq \arg z\leq \theta\}\quad\text{where}\quad
1\leq \theta \leq 2\pi.$$ If there exists $k$ such that
$\displaystyle{\lim_{z\to\infty} zf(z)=k}$ for $z$ in the region
$D$.
Show that $$\lim_{R'\to\infty} \int_{L} f(z) dz=i\theta k,$$
where $L$ is the part of the circle $|z|=R'$ which lies in the
region $D$.
```

6.1.3 ?

```
  Suppose that $f$ is an analytic function in the region $D$ which
contains the point $a$. Let
$$F(z)= z-a-qf(z),\quad \text{where}~ q \ \text{is a complex
parameter}.$$

(1) Let $K\subset D$ be a circle with the center at
point $a$ and also we assume that $f(z)\not =0$ for $z\in K$. Prove
that the function $F$ has one and only one zero $z=w$ on the closed
disc $\bar{K}$ whose boundary is the circle $K$ if $\displaystyle{
|q|<\min_{z\in K} \frac{|z-a|}{|f(z)|}.}$\

(2) Let $G(z)$ be an analytic function on the disk $\bar{K}$. Apply
the residue theorem to prove that
$\displaystyle{ \frac{G(w)}{F'(w)}=\frac{1}{2\pi
i}\int_K \frac{G(z)}{F(z)} dz,}$ where $w$ is the zero from (1).\

(3) If $z\in K$, prove that the function
```

$\displaystyle{\frac{1}{F(z)}}$ can be represented as a convergent series with respect to q: $\displaystyle{ \frac{1}{F(z)}=\sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z-a)^{n+1}}.}$

6.1.4 ?

Evaluate $$\displaystyle{ \int_{0}^{\infty}\frac{x\sin x}{x^2+a^2} \, dx }.$$

6.1.5 ?

Let $f=u+iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z=re^{i\theta}$, $r\not= 0$. Show that
$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\quad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

6.1.6 ?

Show that $\displaystyle \int_0^\infty \frac{x^{a-1}}{1+x^n} dx=\frac{\pi}{n\sin \frac{a\pi}{n}}$ using complex analysis, $0< a < n$. Here n is a positive integer.

6.1.7 ?

For $s>0$, the **gamma function** is defined by $\displaystyle{\Gamma(s)=\int_0^{\infty} e^{-t}t^{s-1} dt}$.

1. Show that the gamma function is analytic in the half-plane $\Re (s)>0$, and is still given there by the integral formula above.

2. Apply the formula in the previous question to show that $$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin \pi s}.$$

> Hint: You may need $\displaystyle{\Gamma(1-s)=t \int_0^{\infty}e^{-vt}(vt)^{-s} dv}$ for $t>0$.

6.1.8 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If
$$P_n(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n\quad (a_n \neq 0)$$
is a polynomial of degree n, then it has n zeros in $\mathbb C$.

6.1.9 ?

Suppose f is entire and there exist $A, R >0$ and natural number N such that $$|f(z)| \geq A |z|^N\ \text{for}\ |z| \geq R.$$ Show

that
(i) f is a polynomial and
(ii) the degree of f is at least N.

6.1.10 ?

Let $f: {\mathbb C} \rightarrow {\mathbb C}$ be an injective
analytic (also called *univalent*) function. Show that there exist
complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

6.1.11 ?

Let g be analytic for $|z|\leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

1. Show that g has a unique fixed point in $|z| < 1$.

2. What happens if we replace $|g(z)| < 1$ with $|g(z)|\leq 1$ for
   $|z|=1$? Give an example if (a) is not true or give an proof
   if (a) is still true.

3. What happens if we simply assume that f is analytic for
   $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z)
   \not\equiv z$. Can f have more than one fixed point in
   $|z| < 1$?

> Hint: The map $\displaystyle{\psi_{\alpha}(z)=\frac{\alpha-z}{1-\bar{\alpha}z}}$ may be useful.

6.1.12 ?

Find a conformal map from $D = \{z :\ |z| < 1,\ |z - 1/2| > 1/2\}$
to the unit disk $\Delta=\{z: \ |z|<1\}$.

6.1.13 ?

Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded*

open set Ω. Show without using Picard's theorems that $f(z)$ is a constant.

6.1.14 ?

(1) Assume $\displaystyle f(z) = \sum_{n=0}^\infty c_n z^n$ converges
in $|z| < R$. Show that for $r <R$,
$$\frac{1}{2 \pi} \int_0^{2 \pi} |f(r e^{i \theta})|^2 d \theta
= \sum_{n=0}^\infty |c_n|^2 r^{2n} \; .$$

(2) Deduce Liouville's theorem from (1).

6.1.15 ?

Let $f(z)$ be entire and assume that $f(z) \leq M |z|^2$ outside some

disk for some constant M. Show that $f(z)$ is a polynomial in z of degree ≤ 2.

6.1.16 ?

Let $a_n(z)$ be an analytic sequence in a domain D such that

$\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D. Show that $\sum_{n=0}^{\infty} |a_n'(z)|$ converges uniformly on bounded and closed sub-regions of D.

6.1.17 ?

Let $f(z)$ be analytic in an open set Ω except possibly at a
point z_0 inside Ω. Show that if $f(z)$ is bounded in near
z_0, then $\displaystyle \int_\Delta f(z) dz = 0$ for all triangles
Δ in Ω.

6.1.18 ?

Assume f is continuous in the region:

$0 < |z-a| \leq R,\ 0 \leq \arg(z-a) \leq \beta_0\ (0 < \beta_0 \leq 2\pi)$ and the limit $\lim_{z\to a}(z-a)f(z) = A$ exists. Show that

$$\lim_{r\to 0}\int_{\gamma_r} f(z)dz = iA\beta_0\ ,$$

where $\gamma_r := \{z \mid z = a + re^{it},\ 0 \leq t \leq \beta_0\}$.

6.1.19 ?

Show that $f(z) = z^2$ is uniformly continuous in any open disk

$|z| < R$, where $R > 0$ is fixed, but it is not uniformly continuous on $\mathbb{C}$.

6.1.20 ?

(1) Show that the function $u=u(x,y)$ given by
$$u(x,y)=\frac{e^{ny}-e^{-ny}}{2n^2}\sin nx\quad \text{for}\ n\in {\mathbf N}$$
is the solution on $D=\{(x,y)\ | x^2+y^2<1\}$ of the Cauchy problem for
the Laplace equation
$$\frac{\partial ^2u}{\partial x^2}+\frac{\partial ^2u}{\partial y^2}=0,\quad
u(x,0)=0,\quad \frac{\partial u}{\partial y}(x,0)=\frac{\sin nx}{n}.$$
(2) Show that there exist points $(x,y)\in D$ such that
$\displaystyle{\limsup_{n\to\infty} |u(x,y)|=\infty}$.

6.2 Fall 2011

6.2.1 ?

(1) Assume $\displaystyle f(z) = \sum_{n=0}^\infty c_n z^n$
converges in $|z| < R$. Show that for $r <R$,
$$\frac{1}{2 \pi} \int_0^{2 \pi} |f(r e^{i \theta})|^2 d \theta =
\sum_{n=0}^\infty |c_n|^2 r^{2n} \; .$$

(2) Deduce Liouville's theorem from (1).

6.2.2 ?

Let f be a continuous function in the region
$$D=\{z\ | |z|>R, 0\leq \arg Z\leq \theta\}\quad\text{where}\quad
0\leq \theta \leq 2\pi.$$ If there exists k such that
$\displaystyle{\lim_{z\to\infty} zf(z)=k}$ for z in the region
D. Show that $$\lim_{R'\to\infty} \int_{L} f(z) dz=i\theta k,$$
where L is the part of the circle $|z|=R'$ which lies in the
region D.

6.2.3 ?

Suppose that f is an analytic function in the region D which
contains the point a. Let
$$F(z)= z-a-qf(z),\quad \text{where}\quad q \ \text{is a complex
parameter}.$$

(1) Let $K\subset D$ be a circle with the center at
point a and also we assume that $f(z)\not =0$ for $z\in K$. Prove
that the function F has one and only one zero $z=w$ on the closed
disc $\bar{K}$ whose boundary is the circle K if $\displaystyle{
|q|<\min_{z\in K} \frac{|z-a|}{|f(z)|}.}$\

(2) Let $G(z)$ be an analytic function on the disk $\bar{K}$. Apply
the residue theorem to prove that
$\displaystyle{ \frac{G(w)}{F'(w)}=\frac{1}{2\pi
i}\int_K \frac{G(z)}{F(z)} dz,}$ where w is the zero from (1).\

(3) If $z\in K$, prove that the function
$\displaystyle{\frac{1}{F(z)}}$ can be represented as a convergent
series with respect to q: $\displaystyle{
\frac{1}{F(z)}=\sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z-a)^{n+1}}.}$

6.2.4 ?

Evaluate $\displaystyle{ \int_{0}^{\infty}\frac{x\sin x}{x^2+a^2} \,
dx }$.

6.2.5 ?

Let $f=u+iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z=re^{i\theta}$, $r\not= 0$. Show that
$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\quad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

6.2.6 ?

Show that $\displaystyle \int_0^\infty \frac{x^{a-1}}{1+x^n} dx=\frac{\pi}{n\sin \frac{a\pi}{n}}$ using complex analysis, $0< a < n$. Here n is a positive integer.

6.2.7 ?

For $s>0$, the **gamma function** is defined by $\displaystyle{\Gamma(s)=\int_0^{\infty} e^{-t}t^{s-1} dt}$.

1. Show that the gamma function is analytic in the half-plane $\Re (s)>0$, and is still given there by the integral formula above.

2. Apply the formula in the previous question to show that $$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin \pi s}.$$

> Hint: You may need $\displaystyle{\Gamma(1-s)=t \int_0^{\infty}e^{-vt}(vt)^{-s} dv}$ for $t>0$.

6.2.8 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If
$$P_n(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n\quad (a_n \neq 0)$$
is a polynomial of degree n, then it has n zeros in $\mathbb C$.

6.2.9 ?

Suppose f is entire and there exist $A, R >0$ and natural number N such that $$|f(z)| \geq A |z|^N\ \text{for}\ |z| \geq R.$$ Show that (i) f is a polynomial and (ii) the degree of f is at least N.

6.2.10 ?

Let $f: {\mathbb C} \rightarrow {\mathbb C}$ be an injective analytic (also called univalent) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

6.2.11 ?

Let g be analytic for $|z|\leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

- Show that g has a unique fixed point in $|z| < 1$.

- What happens if we replace $|g(z)| < 1$ with $|g(z)|\leq 1$ for $|z|=1$? Give an example if (a) is not true or give an proof if (a) is still true.

- What happens if we simply assume that f is analytic for $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z) \not\equiv z$. Can f have more than one fixed point in $|z| < 1$?

> Hint: The map
$\displaystyle{\psi_{\alpha}(z)=\frac{\alpha-z}{1-\bar{\alpha}z}}$
> may be useful.

6.2.12 ?

Find a conformal map from $D = \{z :\ |z| < 1,\ |z - 1/2| > 1/2\}$ to the unit disk $\Delta=\{z: \ |z|<1\}$.

6.2.13 ?

Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω. Show without using Picard's theorems that $f(z)$ is a constant.

6.2.14 ?

Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω. Show without using Picard's theorems that $f(z)$ is a constant.

6.2.15 ?

(1) Assume $\displaystyle f(z) = \sum_{n=0}^\infty c_n z^n$ converges in $|z| < R$. Show that for $r <R$,
$$\frac{1}{2 \pi} \int_0^{2 \pi} |f(r e^{i \theta})|^2 d \theta = \sum_{n=0}^\infty |c_n|^2 r^{2n} \; .$$

(2) Deduce Liouville's theorem from (1).

6.2.16 ?

Let $f(z)$ be entire and assume that $f(z) \leq M |z|^2$ outside some

disk for some constant M. Show that $f(z)$ is a polynomial in z of
degree ≤ 2.

6.2.17 ?

Let $a_n(z)$ be an analytic sequence in a domain D such that
$\displaystyle \sum_{n=0}^\infty |a_n(z)|$ converges uniformly on
bounded and closed sub-regions of D. Show that
$\displaystyle \sum_{n=0}^\infty |a'_n(z)|$ converges uniformly on
bounded and closed sub-regions of D.

6.2.18 ?

Let $f(z)$ be analytic in an open set Ω except possibly at a
point z_0 inside Ω. Show that if $f(z)$ is bounded in near
z_0, then $\displaystyle \int_\Delta f(z) dz = 0$ for all triangles
Δ in Ω.

6.2.19 ?

Assume f is continuous in the region:
$0< |z-a| \leq R, \; 0 \leq \arg(z-a) \leq \beta_0$
($0 < \beta_0 \leq 2 \pi$) and the limit
$\displaystyle \lim_{z \rightarrow a} (z-a) f(z) = A$ exists. Show that
$$\lim_{r \rightarrow 0} \int_{\gamma_r} f(z) dz = i A \beta_0 \; , \; \;$$
where
$\gamma_r : = \{ z \; | \; z = a + r e^{it}, \; 0 \leq t \leq \beta_0 \}.$

6.2.20 ?

Show that $f(z) = z^2$ is uniformly continuous in any open disk
$|z| < R$, where $R>0$ is fixed, but it is not uniformly continuous on
$\mathbb C$.

(1) Show that the function $u=u(x,y)$ given by
$$u(x,y)=\frac{e^{ny}-e^{-ny}}{2n^2}\sin nx\quad \text{for}\ n\in {\mathbf N}$$
is the solution on $D=\{(x,y)\ | x^2+y^2<1\}$ of the Cauchy problem for
the Laplace equation
$$\frac{\partial ^2u}{\partial x^2}+\frac{\partial ^2u}{\partial y^2}=0,\quad
u(x,0)=0,\quad \frac{\partial u}{\partial y}(x,0)=\frac{\sin nx}{n}.$$
(2) Show that there exist points $(x,y)\in D$ such that
$\displaystyle{\limsup_{n\to\infty} |u(x,y)|=\infty}$.

6.3 Spring 2014

6.3.1 ?

The question provides some insight into Cauchy's theorem. Solve the problem without using the Cauchy theorem.

1. Evaluate the integral $\displaystyle{\int_{\gamma} z^n dz}$ for all integers n. Here γ is any circle centered at the origin with the positive (counterclockwise) orientation.

2. Same question as (a), but with γ any circle not containing the origin.

3. Show that if $|a|<r<|b|$, then $\displaystyle{\int_{\gamma}\frac{dz}{(z-a)(z-b)} dz=\frac{2\pi i}{a-b}}$. Here γ denotes the circle centered at the origin, of radius r, with the positive orientation.

6.3.2 ?

(1) Assume the infinite series
$\displaystyle \sum_{n=0}^\infty c_n z^n$ converges in $|z| < R$
and let $f(z)$ be the limit. Show that for $r <R$,
$$\frac{1}{2 \pi} \int_0^{2 \pi} |f(r e^{i \theta})|^2 d \theta =
\sum_{n=0}^\infty |c_n|^2 r^{2n} \; .$$

(2) Deduce Liouville's theorem from (1). Liouville's theorem: If
$f(z)$ is entire and bounded, then f is constant.

6.3.3 ?

Let f be a continuous function in the region
$$D=\{z\ | |z|>R, 0\leq \arg Z\leq \theta\}\quad\text{where}\quad
0\leq \theta \leq 2\pi.$$ If there exists k such that
$\displaystyle{\lim_{z\to\infty} zf(z)=k}$ for z in the region
D. Show that $$\lim_{R'\to\infty} \int_{L} f(z) dz=i\theta k,$$
where L is the part of the circle $|z|=R'$ which lies in the
region D.

6.3.4 ?

Evaluate $\displaystyle{ \int_{0}^{\infty}\frac{x\sin x}{x^2+a^2} \,
dx }$.

6.3.5 ?

Let $f=u+iv$ be differentiable (i.e. $f'(z)$ exists) with continuous
partial derivatives at a point $z=re^{i\theta}$, $r\not= 0$. Show
that
$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\quad
\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

6.3.6 ?

Show that $\displaystyle \int_0^\infty \frac{x^{a-1}}{1+x^n}
dx=\frac{\pi}{n\sin \frac{a\pi}{n}}$ using complex analysis, $0< a <
n$. Here n is a positive integer.

6.3.7 ?

For $s>0$, the **gamma function** is defined by
$\displaystyle{\Gamma(s)=\int_0^{\infty} e^{-t}t^{s-1} dt}$.

- Show that the gamma function is analytic in the half-plane
  $\Re (s)>0$, and is still given there by the integral formula
  above.

- Apply the formula in the previous question to show that
  $$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin \pi s}.$$

> Hint: You may need $\displaystyle{\Gamma(1-s)=t \int_0^{\infty}e^{-vt}(vt)^{-s} dv}$ for $t>0$.

6.3.8 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If
$$P_n(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n\quad (a_n \neq 0)$$
is a polynomial of degree n, then it has n zeros in $\mathbf C$.

6.3.9 ?

Suppose f is entire and there exist $A, R >0$ and natural number
N such that $$|f(z)| \geq A |z|^N\ \text{for}\ |z| \geq R.$$ Show
that (i) f is a polynomial and (ii) the degree of f is at least
N.

6.3.10 ?

Let $f: {\mathbb C} \rightarrow {\mathbb C}$ be an injective
analytic (also called univalent) function. Show that there exist
complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

6.3.11 ?

Let g be analytic for $|z|\leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

- Show that g has a unique fixed point in $|z| < 1$.

- What happens if we replace $|g(z)| < 1$ with $|g(z)|\leq 1$ for
  $|z|=1$? Give an example if (a) is not true or give an proof

if (a) is still true.

- What happens if we simply assume that f is analytic for
  $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z)
  \not\equiv z$. Can f have more than one fixed point in
  $|z| < 1$?

> Hint: The map
$\displaystyle{\psi_{\alpha}(z)=\frac{\alpha-z}{1-\bar{\alpha}z}}$
> may be useful.

6.3.12 ?

Find a conformal map from $D = \{z :\ |z| < 1,\ |z - 1/2| > 1/2\}$
to the unit disk $\Delta=\{z: \ |z|<1\}$.

6.4 Fall 2015

6.4.1 ?

Let $a_n \neq 0$ and assume that $\displaystyle
\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L$. Show that
$\displaystyle
\lim_{n \rightarrow \infty}
\sqrt[n]{|a_n|} = L.
%p_n^{\frac{1}{n}} = L.$ In particular, this shows that when
applicable, the ratio test can be used to calculate the radius of
convergence of a power series.

6.4.2 ?

(a) Let z, w be complex numbers, such that $\bar{z} w \neq 1$.
Prove that
$$\abs{\frac{w - z}{1 - \bar{w} z}} < 1 \; \; \; \mbox{if} \; |z| < 1 \; \mbox{and}\; |w| < 1,$$
and also that
$$\abs{\frac{w - z}{1 - \bar{w} z}} = 1 \; \; \; \mbox{if} \; |z| = 1 \; \mbox{or}\; |w| = 1.$$

(b) Prove that for fixed w in the unit disk $\mathbb D$, the
mapping $$F: z \mapsto \frac{w - z}{1 - \bar{w} z}$$ satisfies the
following conditions:

(i) F maps $\mathbb D$ to itself and is holomorphic.

(ii) F interchanges 0 and w, namely, $F(0) = w$ and
$F(w) = 0$.

(iii) $|F(z)| = 1$ if $|z| = 1$.

(iv) $F: {\mathbb D} \mapsto {\mathbb D}$ is bijective.

> Hint: Calculate $F \circ F$.

6.4.3 ?

Use n-th roots of unity (i.e. solutions of $z^n - 1 =0$) to show
that
$$2^{n-1} \sin\frac{\pi}{n} \sin\frac{2\pi}{n} \cdots \sin\frac{(n-1)\pi}{n}
= n
\; .$$

> Hint: $1 - \cos 2 \theta = 2 \sin^2 \theta,\; \sin 2 \theta = 2 \sin \theta \cos \theta$.

(a) Show that in polar coordinates, the Cauchy-Riemann
equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}
\; \; \; \text{and} \; \; \;
\frac{\partial v}{\partial r} = - \frac{1}{r} \frac{\partial u}{\partial \theta}$$

(b) Use these equations to show that the logarithm function
defined by $$\log z = \log r + i \theta \; \;
\mbox{where} \; z = r e^{i \theta } \; \mbox{with} \; - \pi < \theta < \pi$$
is a holomorphic function in the region
$r>0, \; - \pi < \theta < \pi$. Also show that $\log z$ defined
above is not continuous in $r>0$.

6.4.4 ?

Assume f is continuous in the region:
$x \geq x_0, \; 0 \leq y \leq b$ and the limit
$$\displaystyle \lim_{x \rightarrow + \infty} f(x + iy) = A$$ exists
uniformly with respect to y (independent of y). Show that
$$\lim_{x \rightarrow + \infty} \int_{\gamma_x} f(z) dz = iA b \; , \; \;$$
where $\gamma_x : = \{ z \; | \; z = x + it, \; 0 \leq t \leq b\}.$

6.4.5 ?

(Cauchy's formula for "exterior" region) Let γ be piecewise
smooth simple closed curve with interior Ω_1 and exterior
Ω_2. Assume $f'(z)$ exists in an open set containing γ
and Ω_2 and $\lim_{z \rightarrow \infty } f(z) = A$. Show
that
$$\frac{1}{2 \pi i} \int_\gamma \frac{f(\xi)}{\xi - z} \, d \xi =
\begin{cases}
A,   &   \text{if\ $z \in \Omega_1$}, \\

-f (z) + A, & \text{if\ $z \in \Omega_2$}
\end{cases}$$

6.4.6 ?

Let $f(z)$ be bounded and analytic in $\mathbb C$. Let $a \neq b$ be any fixed complex numbers. Show that the following limit exists
$$\lim_{R \rightarrow \infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$
Use this to show that $f(z)$ must be a constant (Liouville's theorem).

6.4.7 ?

Prove by *justifying all steps* that for all $\xi \in {\mathbb C}$ we have $\displaystyle
e^{- \pi \xi^2} = \int_{- \infty}^\infty e^{- \pi x^2} e^{2 \pi i x \xi} dx \; .$

> Hint: You may use that fact in Example 1 on p. 42 of the textbook
without proof, i.e., you may assume the above is true for real
values of ξ.

6.4.8 ?

Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Let
$\displaystyle
%f(z) = \sum_{n = 1}^\infty a_n z^n
f(z) = \sum_{n = 1}^\infty c_n z^n$ denote the the power series in the open disc. Show that (1) $c_n \neq 0$ for all large enough n's, and (2)
$\displaystyle \lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}}= z_0$.

6.4.9 ?

Let $f(z)$ be a non-constant analytic function in $|z|>0$ such that $f(z_n) = 0$ for infinite many points z_n with $\lim_{n \rightarrow \infty} z_n =0$. Show that $z=0$ is an essential singularity for $f(z)$. (An example of such a function is $f(z) = \sin (1/z)$.)

6.4.10 ?

Let f be entire and suppose that $\lim_{z \rightarrow \infty} f(z) = \infty$. Show that f is a polynomial.

6.4.11 ?

Expand the following functions into Laurent series in the indicated regions:

(a)
$\displaystyle f(z) = \frac{z^2 - 1}{ (z+2)(z+3)}, \; \; 2 < |z| < 3$,
$3 < |z| < + \infty$.

(b)
$\displaystyle f(z) = \sin \frac{z}{1-z}, \; \; 0 < |z-1| < + \infty$

6.4.12 ?

Assume $f(z)$ is analytic in region D and Γ is a rectifiable curve in D with interior in D. Prove that if $f(z)$ is real for all $z \in \Gamma$, then $f(z)$ is a constant.

6.4.13 ?

Find the number of roots of $z^4 - 6z + 3 =0$ in $|z|<1$ and $1 < |z| < 2$ respectively.

6.4.14 ?

Prove that $z^4 + 2 z^3 - 2z + 10 =0$ has exactly one root in each open quadrant.

6.4.15 ?

(1) Let $f(z) \in H({\mathbb D})$, $\text{Re}(f(z)) >0$, $f(0)= a>0$. Show that $$|\frac{f(z)-a}{f(z)+a}| \leq |z|, \; \; \; |f'(0)| \leq 2a.$$

(2) Show that the above is still true if $\text{Re}(f(z)) >0$ is replaced with $\text{Re}(f(z)) \geq 0$.

6.4.16 ?

Assume $f(z)$ is analytic in ${\mathbb D}$ and $f(0)=0$ and is not a rotation (i.e. $f(z) \neq e^{i \theta} z$). Show that $\displaystyle \sum_{n=1}^\infty f^{n}(z)$ converges uniformly to an analytic function on compact subsets of ${\mathbb D}$, where $f^{n+1}(z) = f(f^{n}(z))$.

6.4.17 ?

Let $f(z) = \sum_{n=0}^\infty c_n z^n$ be analytic and one-to-one in $|z| < 1$. For $0<r<1$, let D_r be the disk $|z|<r$. Show that the area of $f(D_r)$ is finite and is given by
$$S = \pi \sum_{n=1}^\infty n |c_n|^2 r^{2n}.$$ (Note that in general the area of $f(D_1)$ is infinite.)

6.4.18 ?

Let $f(z) = \sum_{n= -\infty}^\infty c_n z^n$ be analytic and one-to-one in $r_0< |z| < R_0$. For $r_0<r<R<R_0$, let $D(r,R)$ be the annulus $r<|z|<R$. Show that the area of $f(D(r,R))$ is finite and is given by
$$S = \pi \sum_{n=- \infty}^\infty n |c_n|^2 (R^{2n} - r^{2n}).$$

6.5 Spring 2015

6.5.1 ?

Let $a_n(z)$ be an analytic sequence in a domain D such that $\displaystyle \sum_{n=0}^\infty |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D. Show that $\displaystyle \sum_{n=0}^\infty |a'_n(z)|$ converges uniformly on bounded and closed sub-regions of D.

6.5.2 ?

Let f_n, f be analytic functions on the unit disk ${\mathbb D}$. Show that the following are equivalent.

(i) $f_n(z)$ converges to $f(z)$ uniformly on compact subsets in $\mathbb D$.

(ii) $\int_{|z|= r} |f_n(z) - f(z)| \, |dz|$ converges to 0 if $0< r<1$.

6.5.3 ?

Let f and g be non-zero analytic functions on a region Ω. Assume $|f(z)| = |g(z)|$ for all z in Ω. Show that $f(z) = e^{i \theta} g(z)$ in Ω for some $0 \leq \theta < 2 \pi$.

6.5.4 ?

Suppose f is analytic in an open set containing the unit disc $\mathbb D$ and $|f(z)| =1$ when $|z|$=1. Show that either

$f(z) = e^{i \theta}$ for some $\theta \in \mathbb R$ or there are finite number of $z_k \in \mathbb D$, $k \leq n$ and $\theta \in \mathbb R$ such that $\displaystyle f(z) = e^{i\theta} \prod_{k=1}^n \frac{z-z_k}{1 - \bar{z}_k z } \, .$

> Also cf. Stein et al, 1.4.7, 3.8.17

6.5.5 ?

(1) Let $p(z)$ be a polynomial, $R>0$ any positive number, and $m \geq 1$ an integer. Let $M_R = \sup \{ |z^{m} p(z) - 1|: |z| = R \}$. Show that $M_R>1$.

(2) Let $m \geq 1$ be an integer and $K = \{z \in {\mathbb C}: r \leq |z| \leq R \}$ where $r<R$. Show (i) using (1) as well as, (ii) without using (1) that there exists a positive number $\varepsilon_0>0$ such that for each polynomial $p(z)$,
$$\sup \{|p(z) - z^{-m}|: z \in K \} \geq \varepsilon_0 \, .$$

6.5.6 ?

Let $\displaystyle f(z) = \frac{1}{z} + \frac{1}{z^2 -1}$. Find all the Laurent series of f and describe the largest annuli in which these series are valid.

6.5.7 ?

Suppose f is entire and there exist $A, R >0$ and natural number N such that $|f(z)| \leq A |z|^N$ for $|z| \geq R$. Show that (i) f is a polynomial and (ii) the degree of f is at most N.

6.5.8 ?

Suppose f is entire and there exist $A, R >0$ and natural number N such that $|f(z)| \geq A |z|^N$ for $|z| \geq R$. Show that (i) f is a polynomial and (ii) the degree of f is at least N.

6.5.9 ?

(1) Explicitly write down an example of a non-zero analytic function in $|z|<1$ which has infinitely zeros in $|z|<1$.

(2) Why does not the phenomenon in (1) contradict the uniqueness theorem?

6.5.10 ?

(1) Assume u is harmonic on open set O and z_n is a sequence in O such that $u(z_n) = 0$ and $\lim z_n \in O$. Prove or disprove that u is identically zero. What if O is a region?

(2) Assume u is harmonic on open set O and $u(z) = 0$ on a disc in O. Prove or disprove that u is identically zero. What if O is a region?

(3) Formulate and prove a Schwarz reflection principle for harmonic functions

> cf. Theorem 5.6 on p.60 of Stein et al.

> Hint: Verify the mean value property for your new function obtained by Schwarz reflection principle.

6.5.11 ?

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s<r$, there exists a constant $c>0$ such that $$||f||_{(\infty, s)} \leq c ||f||_{(1, r)},$$ where $\displaystyle |f||_{(\infty, s)} = \text{sup}_{z \in D_s(z_0)}|f(z)|$ and $\displaystyle ||f||_{(1, r)} = \int_{D_r(z_0)} |f(z)|dx dy$.

> Note: Exercise 3.8.20 on p.107 in Stein et al is a straightforward consequence of this stronger result using the integral form of the Cauchy-Schwarz inequality in real analysis.

6.5.12 ?

(1) Let f be analytic in $\Omega: 0<|z-a|<r$ except at a sequence of poles $a_n \in \Omega$ with $\lim_{n \rightarrow \infty} a_n = a$. Show that for any $w \in \mathbb C$, there exists a sequence $z_n \in \Omega$ such that $\lim_{n \rightarrow \infty} f(z_n) = w$.

(2) Explain the similarity and difference between the above assertion and the Weierstrass-Casorati theorem.

6.5.13 ?

Compute the following integrals.

\(i\) $\displaystyle \int_0^\infty \frac{1}{(1 + x^n)^2} \, dx$, $n \geq 1$ (ii) $\displaystyle \int_0^\infty \frac{\cos x}{(x^2 + a^2)^2} \, dx$,

$a \in \mathbb R$ (iii)
$\displaystyle \int_0^\pi \frac{1}{a + \sin \theta} \, d \theta$,
$a>1$

\(iv\) $\displaystyle \int_0^{\frac{\pi}{2}}
\frac{d \theta}{a+ \sin ^2 \theta},$ $a >0$. (v)
$\displaystyle \int_{|z|=2} \frac{1}{(z^{5} -1) (z-3)} \, dz$ (v)
$\displaystyle \int_{- \infty}^{\infty} \frac{\sin \pi a}{\cosh \pi x + \cos \pi a} e^{- i x \xi} \,
$0< a <1$, $\xi \in \mathbb R$ (vi)
$\displaystyle \int_{|z| = 1} \cot^2 z \, dz$.

6.5.14 ?

Compute the following integrals.

\(i\) $\displaystyle \int_0^\infty \frac{\sin x}{x} \, dx$ (ii)
$\displaystyle \int_0^\infty (\frac{\sin x}{x})^2 \, dx$ (iii)
$\displaystyle \int_0^\infty \frac{x^{a-1}}{(1 + x)^2} \, dx$,
$0< a < 2$

\(i\)
$\displaystyle \int_0^\infty \frac{\cos a x - \cos bx}{x^2} dx$,
$a, b >0$ (ii)
$\displaystyle \int_0^\infty \frac{x^{a-1}}{1 + x^n} \, dx$,
$0< a < n$

\(iii\) $\displaystyle \int_0^\infty \frac{\log x}{1 + x^n} \, dx$,
$n \geq 2$ (iv)
$\displaystyle \int_0^\infty \frac{\log x}{(1 + x^2)^2} dx$ (v)
$\displaystyle \int_0^{\pi} \log|1 - a \sin \theta| d \theta$,
$a \in \mathbb C$

6.5.15 ?

Let $0<r<1$. Show that polynomials
$P_n(z) = 1 + 2z + 3 z^2 + \cdots + n z^{n-1}$ have no zeros in
$|z|<r$ for all sufficiently large n's.

6.5.16 ?

Let f be an analytic function on a region Ω. Show that f
is a constant if there is a simple closed curve γ in Ω
such that its image $f(\gamma)$ is contained in the real axis.

6.5.17 ?

(1) Show that $\displaystyle \frac{\pi^2}{\sin^2 \pi z}$ and

$\displaystyle g(z) = \sum_{n = - \infty}^{ \infty} \frac{1}{(z-n)^2}$
have the same principal part at each integer point.

(2) Show that
$\displaystyle h(z) = \frac{\pi^2}{\sin^2 \pi z} - g(z)$ is bounded
on $\mathbb C$ and conclude that
$\displaystyle \frac{\pi^2}{\sin^2 \pi z} = \sum_{n = - \infty}^{ \infty} \frac{1}{(z-n)^2} \, .$

6.5.18 ?

Let $f(z)$ be an analytic function on
${\mathbb C} \backslash \{ z_0 \}$, where z_0 is a fixed point.
Assume that $f(z)$ is bijective from
${\mathbb C} \backslash \{ z_0 \}$ onto its image, and that $f(z)$
is bounded outside $D_r(z_0)$, where r is some fixed positive
number. Show that there exist $a, b, c, d \in \mathbb C$ with
$ad-bc \neq 0$, $c \neq 0$ such that
$\displaystyle f(z) = \frac{az + b}{cz + d}$.

6.5.19 ?

Assume $f(z)$ is analytic in ${\mathbb D}: |z|<1$ and $f(0)=0$ and
is not a rotation (i.e. $f(z) \neq e^{i \theta} z$). Show that
$\displaystyle \sum_{n=1}^\infty f^{n}(z)$ converges uniformly to an
analytic function on compact subsets of ${\mathbb D}$, where
$f^{n+1}(z) = f(f^{n}(z))$.

6.5.20 ?

Let f be a non-constant analytic function on $\mathbb D$ with
$f(\mathbb D) \subseteq \mathbb D$. Use $\psi_{a} (f(z))$ (where
$a=f(0)$, $\displaystyle \psi_a(z) = \frac{a - z}{1 - \bar{a}z}$) to
prove that $\displaystyle
\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq
\frac{|f(0)| + |z|}{1 - |f(0)||z|}$.

6.5.21 ?

Find a conformal map

1. from $\{ z: |z - 1/2| > 1/2, \text{Re}(z)>0 \}$ to $\mathbb H$

2. from $\{ z: |z - 1/2| > 1/2, |z| <1 \}$ to $\mathbb D$

3. from the intersection of the disk $|z + i| < \sqrt{2}$ with
   ${\mathbb H}$ to ${\mathbb D}$.

4. from ${\mathbb D} \backslash [a, 1)$ to
   ${\mathbb D} \backslash [0, 1)$ ($0<a<1)$. \[Short solution
   possible using Blaschke factor\]

5. from $\{ z: |z| < 1, \text{Re}(z) > 0 \} \backslash (0, 1/2]$ to
   $\mathbb H$.

6.5.22 ?

Let C and C' be two circles and let $z_1 \in C$, $z_2 \notin C$,
$z'_1 \in C'$, $z'_2 \notin C'$. Show that there is a unique
fractional linear transformation f with $f(C) = C'$ and
$f(z_1) = z'_1$, $f(z_2) = z'_2$.

6.5.23 ?

Assume $f_n \in H(\Omega)$ is a sequence of holomorphic functions on
the region Ω that are uniformly bounded on compact subsets
and $f \in H(\Omega)$ is such that the set
$\displaystyle \{z \in \Omega: \lim_{n \rightarrow \infty} f_n(z) = f(z) \}$
has a limit point in Ω. Show that f_n converges to f
uniformly on compact subsets of Ω.

6.5.24 ?

Let
$\displaystyle{\psi_{\alpha}(z)=\frac{\alpha-z}{1-\bar{\alpha}z}}$
with $|\alpha|<1$ and ${\mathbb D}=\{z:\ |z|<1\}$. Prove that

- $\displaystyle{\frac{1}{\pi}\iint_{{\mathbb D}} |\psi'_{\alpha}|^2 dx dy =1}$.

- $\displaystyle{\frac{1}{\pi}\iint_{{\mathbb D}} |\psi'_{\alpha}| dx dy =\frac{1-|\alpha|^2}{|\alph
  \log \frac{1}{1-|\alpha|^2}}$.

6.5.25 ?

Prove that
$\displaystyle{f(z)=-\frac{1}{2}\left(z+\frac{1}{z}\right)}$ is a
conformal map from half disc $\{z=x+iy:\ |z|<1,\ y>0\}$ to upper
half plane ${\mathbb H}=\{z=x+iy:\ y>0\}$.

6.5.26 ?

Let Ω be a simply connected open set and let γ be a
simple closed contour in Ω and enclosing a bounded region U
anticlockwise. Let $f: \ \Omega \to {\mathbb C}$ be a holomorphic
function and $|f(z)|\leq M$ for all $z\in \gamma$. Prove that

$|f(z)|\leq M$ for all $z\in U$.

6.5.27 ?

Compute the following integrals. (i)
$\displaystyle \int_0^\infty \frac{x^{a-1}}{1 + x^n} \, dx$,
$0< a < n$ (ii)
$\displaystyle \int_0^\infty \frac{\log x}{(1 + x^2)^2}\, dx$

6.5.28 ?

Let $0<r<1$. Show that polynomials
$P_n(z) = 1 + 2z + 3 z^2 + \cdots + n z^{n-1}$ have no zeros in
$|z|<r$ for all sufficiently large n's.

6.5.29 ?

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that
for any $s<r$, there exists a constant $c>0$ such that
$$\|f\|_{(\infty, s)} \leq c \|f\|_{(1, r)},$$ where
$\displaystyle \|f\|_{(\infty, s)} = \text{sup}_{z \in D_s(z_0)}|f(z)|$
and $\displaystyle \|f\|_{(1, r)} = \int_{D_r(z_0)} |f(z)|dx dy$.

6.5.30 ?

Let $\displaystyle{\psi_{\alpha}(z)=\frac{\alpha-z}{1-\bar{\alpha}z}}$
with $|\alpha|<1$ and ${\mathbb D}=\{z:\ |z|<1\}$. Prove that

- $\displaystyle{\frac{1}{\pi}\iint_{{\mathbb D}} |\psi'_{\alpha}|^2 dx dy =1}$.
- $\displaystyle{\frac{1}{\pi}\iint_{{\mathbb D}} |\psi'_{\alpha}| dx dy =\frac{1-|\alpha|^2}{|\alph
  \log \frac{1}{1-|\alpha|^2}}$.

Prove that $\displaystyle{f(z)=-\frac{1}{2}\left(z+\frac{1}{z}\right)}$
is a conformal map from half disc $\{z=x+iy:\ |z|<1,\ y>0\}$ to upper
half plane $\mathbb H=\{z=x+iy:\ y>0\}$.

6.5.31 ?

Let Ω be a simply connected open set and let γ be a simple
closed contour in Ω and enclosing a bounded region U
anticlockwise. Let $f: \ \Omega \to {\mathbb C}$ be a holomorphic
function and $|f(z)|\leq M$ for all $z\in \gamma$. Prove that
$|f(z)|\leq M$ for all $z\in U$.

6.5.32 ?

Compute the following integrals. (i)
$\displaystyle \int_0^\infty \frac{x^{a-1}}{1 + x^n} \, dx$, $0< a < n$
(ii) $\displaystyle \int_0^\infty \frac{\log x}{(1 + x^2)^2}\, dx$

6.5.33 ?

Let $0<r<1$. Show that polynomials
$P_n(z) = 1 + 2z + 3 z^2 + \cdots + n z^{n-1}$ have no zeros in $|z|<r$
for all sufficiently large n's.

6.5.34 ?

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for
any $s<r$, there exists a constant $c>0$ such that
$$\|f\|_{(\infty, s)} \leq c \|f\|_{(1, r)},$$ where
$\displaystyle \|f\|_{(\infty, s)} = \text{sup}_{z \in D_s(z_0)}|f(z)|$
and $\displaystyle \|f\|_{(1, r)} = \int_{D_r(z_0)} |f(z)|dx dy$.

6.6 Fall 2016

6.6.1 ?

Let $u(x,y)$ be harmonic and have continuous partial derivatives of
order three in an open disc of radius $R>0$.

(a) Let two points $(a,b), (x,y)$ in this disk be given. Show that
the following integral is independent of the path in this disk
joining these points:
$$v(x,y) = \int_{a,b}^{x,y} (-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy).$$

(b) \hfill

(i) Prove that $u(x,y)+i v(x,y)$ is an analytic function in this
disc.

(ii) Prove that $v(x,y)$ is harmonic in this disc.

6.6.2 ?

(a) $f(z)= u(x,y) +i v(x,y)$ be analytic in a domain
$D\subset {\mathbb C}$. Let $z_0=(x_0,y_0)$ be a point in D which
is in the intersection of the curves $u(x,y)= c_1$ and $v(x,y)=c_2$,
where c_1 and c_2 are constants. Suppose that $f'(z_0)\neq 0$.
Prove that the lines tangent to these curves at z_0 are
perpendicular.

(b) Let $f(z)=z^2$ be defined in ${\mathbb C}$.
  (i) Describe the
       level curves of $\mbox{\textrm Re}{(f)}$ and of $\mbox{Im}{(f)}$.

  (ii) What are the angles of intersections between the level curves
       $\mbox{\textrm Re}{(f)}=0$ and $\mbox{\textrm Im}{(f)}$? Is your answer in
       agreement with part a) of this question?

6.6.3 ?

(a) $f: D\rightarrow {\mathbb C}$ be a continuous function, where
$D\subset {\mathbb C}$ is a domain.Let $\alpha:[a,b]\rightarrow D$
be a smooth curve. Give a precise definition of the *complex line
integral* $$\int_{\alpha} f.$$

(b) Assume that there exists a constant M such that
$|f(\tau)|\leq M$ for all $\tau\in \mbox{\textrm Image}(\alpha$). Prove
that
$$\big | \int_{\alpha} f \big |\leq M \times \mbox{\textrm length}(\alpha).$$

(c) Let C_R be the circle $|z|=R$, described in the
counterclockwise direction, where $R>1$. Provide an upper bound for
$\big | \int_{C_R} \dfrac{\log{(z)} }{z^2} \big |,$ which depends
[only]{.underline} on R and other constants.

6.6.4 ?

(a) Let Let $f:{\mathbb C}\rightarrow {\mathbb C}$ be an entire
function. Assume the existence of a non-negative integer m, and of
positive constants L and R, such that for all z with $|z|>R$
the inequality $$|f(z)| \leq L |z|^m$$ holds. Prove that f is a
polynomial of degree $\leq m$.

(b) Let $f:{\mathbb C}\rightarrow {\mathbb C}$ be an entire
function. Suppose that there exists a real number M such that for
all $z\in {\mathbb C}$ $$\mbox{\textrm Re} (f) \leq M.$$ Prove that f
must be a constant.

6.6.5 ?

Prove that all the roots of the complex polynomial
$$z^7 - 5 z^3 +12 =0$$ lie between the circles $|z|=1$ and $|z|=2$.

6.6.6 ?

(a) Let F be an analytic function inside and on a simple closed
curve C, except for a pole of order $m\geq 1$ at $z=a$ inside C.

Prove that

$$\frac{1}{2 \pi i}\oint_{C} F(\tau) d\tau =
\lim_{\tau\rightarrow a} \frac{d^{m-1}}{d\tau^{m-1}}\big((\tau-a)^m F(\tau))\big).$$

(b) Evaluate $$\oint_{C}\frac{e^{\tau}}{(\tau^2+\pi^2)^2}d\tau$$
where C is the circle $|z|=4$.

6.6.7 ?

Find the conformal map that takes the upper half-plane comformally
onto the half-strip $\{
w=x+iy:\ -\pi/2<x<\pi/2\ y>0\}$.

6.6.8 ?

Compute the integral
$\displaystyle{\int_{-\infty}^{\infty} \frac{e^{-2\pi ix\xi}}{\cosh\pi x}dx}$
where $\displaystyle{\cosh z=\frac{e^{z}+e^{-z}}{2}}$.