

# Real Analysis Qualifying Exam Questions

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## 1 Undergraduate Analysis: Uniform Convergence

### 1.1 Fall 2018 # 1

Let  $f(x) = \frac{1}{x}$ . Show that  $f$  is uniformly continuous on  $(1, \infty)$  but not on  $(0, \infty)$ .

**1.2 Fall 2017 # 1**

Let

$$f(x) = s \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which  $f$  does and does not converge uniformly.

**1.3 Fall 2014 # 1**

Let  $\{f_n\}$  be a sequence of continuous functions such that  $\sum f_n$  converges uniformly.

Prove that  $\sum f_n$  is also continuous.

**1.4 Spring 2017 # 4**

Let  $f(x, y)$  on  $[-1, 1]^2$  be defined by

$$f(x, y) = \begin{cases} \frac{xy}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Determine if  $f$  is integrable.

**1.5 Spring 2015 # 1**

Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces,  $f : X \rightarrow Y$ , and  $x_0 \in X$ .

Prove that the following statements are equivalent:

1. For every  $\varepsilon > 0$   $\exists \delta > 0$  such that  $\rho(f(x), f(x_0)) < \varepsilon$  whenever  $d(x, x_0) < \delta$ .
2. The sequence  $\{f(x_n)\}_{n=1}^{\infty} \rightarrow f(x_0)$  for every sequence  $\{x_n\} \rightarrow x_0$  in  $X$ .

**1.6 Fall 2014 # 2**

Let  $I$  be an index set and  $\alpha : I \rightarrow (0, \infty)$ .

1. Show that

$$\sum_{i \in I} a(i) := \sup_{\substack{J \subset I \\ J \text{ finite}}} \sum_{i \in J} a(i) < \infty \implies I \text{ is countable.}$$

2. Suppose  $I = \mathbb{Q}$  and  $\sum_{q \in \mathbb{Q}} a(q) < \infty$ . Define

$$f(x) := \sum_{\substack{q \in \mathbb{Q} \\ q \leq x}} a(q).$$

Show that  $f$  is continuous at  $x \iff x \notin \mathbb{Q}$ .

## 1.7 Spring 2014 # 2

Let  $\{a_n\}$  be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that  $\sum a_n^2 < \infty$ .

Note: Assume  $a_n, b_n$  are all non-negative.

## 2 General Analysis

## 2.1 Spring 2020 # 1

Prove that if  $f : [0, 1] \rightarrow \mathbb{R}$  is continuous then

$$\lim_{k \rightarrow \infty} \int_0^1 kx^{k-1} f(x) dx = f(1).$$

*Solution.*

Concepts used:

- DCT
- Weierstrass Approximation Theorem

**Solution:**

- Suppose  $p$  is a polynomial, then

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_0^1 kx^{k-1} p(x) dx &= \lim_{k \rightarrow \infty} \int_0^1 \left( \frac{\partial}{\partial x} x^k \right) p(x) dx \\ &= \lim_{k \rightarrow \infty} \left[ x^k p(x) \Big|_0^1 - \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx \right] \quad \text{integrating by parts} \\ &= p(1) - \lim_{k \rightarrow \infty} \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx, \end{aligned}$$

- Thus it suffices to show that

$$\lim_{k \rightarrow \infty} \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx = 0.$$

- Integrating by parts a second time yields

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx &= \lim_{k \rightarrow \infty} \frac{x^{k+1}}{k+1} p'(x) \Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1} \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx \\ &= - \lim_{k \rightarrow \infty} \int_0^1 \frac{x^{k+1}}{k+1} \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx \\ &= - \int_0^1 \lim_{k \rightarrow \infty} \frac{x^{k+1}}{k+1} \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx \quad \text{by DCT} \\ &= - \int_0^1 0 \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx \\ &= 0. \end{aligned}$$

- The DCT can be applied here because  $f''$  is continuous and  $[0, 1]$  is compact, so  $f''$  is bounded on  $[0, 1]$  by a constant  $M$  and

$$\int_0^1 |x^k f''(x)| \leq \int_0^1 1 \cdot M = M < \infty.$$

- Now use the Weierstrass approximation theorem:
  - If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then for every  $\varepsilon > 0$  there exists a polynomial  $p_\varepsilon(x)$  such that  $\|f - p_\varepsilon\|_\infty < \varepsilon$ .
- Thus

$$\begin{aligned} \left| \int_0^1 kx^{k-1} p_\varepsilon(x) dx - \int_0^1 kx^{k-1} f(x) dx \right| &= \left| \int_0^1 kx^{k-1} (p_\varepsilon(x) - f(x)) dx \right| \\ &\leq \left| \int_0^1 kx^{k-1} \|p_\varepsilon - f\|_\infty dx \right| \\ &= \|p_\varepsilon - f\|_\infty \cdot \left| \int_0^1 kx^{k-1} dx \right| \\ &= \|p_\varepsilon - f\|_\infty \cdot x^k \Big|_0^1 \\ &= \|p_\varepsilon - f\|_\infty \xrightarrow{\varepsilon \rightarrow 0} 0 \end{aligned}$$

and the integrals are equal.

- By the first argument,

$$\int_0^1 kx^{k-1} p_\varepsilon(x) dx = p_\varepsilon(1) \text{ for each } \varepsilon$$

- Since uniform convergence implies pointwise convergence,  $p_\varepsilon(1) \xrightarrow{\varepsilon \rightarrow 0} f(1)$ .

## 2.2 Fall 2019 # 1.

Let  $\{a_n\}_{n=1}^\infty$  be a sequence of real numbers.

- a. Prove that if  $\lim_{n \rightarrow \infty} a_n = 0$ , then

$$\lim_{n \rightarrow \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

- b. Prove that if  $\sum_{n=1}^\infty \frac{a_n}{n}$  converges, then

$$\lim_{n \rightarrow \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

*Solution.*

Cesaro mean/summation. Break series apart into pieces that can be handled separately.

**2.2.1 a**

Prove a stronger result:

$$a_k \rightarrow S \implies S_N := \frac{1}{N} \sum_{k=1}^N a_k \rightarrow S.$$

Idea: once  $N$  is large enough,  $a_k \approx S$ , and all smaller terms will die off as  $N \rightarrow \infty$ .  
See this MSE answer.

- Use convergence  $a_k \rightarrow S$ : choose  $M$  large enough such that

$$k \geq M + 1 \implies |a_k - S| < \varepsilon.$$

Then

$$\begin{aligned} \left| \left( \frac{1}{N} \sum_{k=1}^N a_k \right) - S \right| &= \frac{1}{N} \left| \left( \sum_{k=1}^N a_k \right) - NS \right| \\ &= \frac{1}{N} \left| \left( \sum_{k=1}^N a_k \right) - \sum_{k=1}^N S \right| \\ &= \frac{1}{N} \left| \sum_{k=1}^N (a_k - S) \right| \\ &\leq \frac{1}{N} \sum_{k=1}^N |a_k - S| \\ &= \frac{1}{N} \sum_{k=1}^M |a_k - S| + \sum_{k=M+1}^N |a_k - S| \\ &\leq \frac{1}{N} \sum_{k=1}^M |a_k - S| + \sum_{k=M+1}^N \frac{\varepsilon}{2} \\ &= \frac{1}{N} \sum_{k=1}^M |a_k - S| + (N - M) \frac{\varepsilon}{2} \\ &\xrightarrow{\varepsilon \rightarrow 0} \frac{1}{N} \sum_{k=1}^M |a_k - S| + 0 \\ &\xrightarrow{N \rightarrow \infty} 0 + 0. \end{aligned}$$

Note:  $M$  is fixed, so the last sum is some constant  $c$ , and  $c/N \rightarrow 0$  as  $N \rightarrow \infty$  for any constant. To be more careful, choose  $M$  first to get  $\varepsilon/2$  for the tail, then choose  $N(M) > M$  for the remaining truncated part of the sum.

**2.2.2 b**

- Define

$$\Gamma_n := \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

- $\Gamma_1 = \sum_{k=1}^n \frac{a_k}{k}$  is the original series and each  $\Gamma_n$  is a tail of  $\Gamma_1$ , so by assumption  $\Gamma_n \xrightarrow{n \rightarrow \infty} 0$ .
- Compute

$$\frac{1}{n} \sum_{k=1}^n a_k = \frac{1}{n} (\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \Gamma_{n+1})$$

- This comes from consider the following summation:

$\Gamma_1 :$	$a_1$	$+\frac{a_2}{2}$	$+\frac{a_3}{3}$	$+\cdots$	
$\Gamma_2 :$		$\frac{a_2}{2}$	$+\frac{a_3}{3}$	$+\cdots$	
$\Gamma_3 :$			$\frac{a_3}{3}$	$+\cdots$	
$\sum_{i=1}^n \Gamma_i :$	$a_1$	$+a_2$	$+a_3$	$+\cdots$	$a_n + \frac{a_{n+1}}{n+1} + \cdots$

- Use part (a): since  $\Gamma_n \xrightarrow{n \rightarrow \infty} 0$ , we have  $\frac{1}{n} \sum_{k=1}^n \Gamma_k \xrightarrow{n \rightarrow \infty} 0$ .
- Also a minor check:  $\Gamma_n \rightarrow 0 \implies \frac{1}{n} \Gamma_n \rightarrow 0$ .
- Then

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n a_k &= \frac{1}{n} (\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \Gamma_{n+1}) \\ &= \left( \frac{1}{n} \sum_{k=1}^n \Gamma_k \right) - \left( \frac{1}{n} \Gamma_{n+1} \right) \\ &\xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

### 2.3 Fall 2018 # 4

Let  $f \in L^1([0, 1])$ . Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) |\sin nx| \, dx = \frac{2}{\pi} \int_0^1 f(x) \, dx$$

Hint: Begin with the case that  $f$  is the characteristic function of an interval.



**2.4 Fall 2017 # 4**

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

1. Show that  $f_n \rightarrow 0$  pointwise but not uniformly on  $[0, 1]$ .

Hint: Consider the maximum of  $f_n$ .

2.

$$\lim_{n \rightarrow \infty} \int_0^1 n(1-x)^n \sin x \, dx = 0$$

**2.5 Spring 2017 # 3**

Let

$$f_n(x) = ae^{-nax} - be^{-nbx} \quad \text{where } 0 < a < b.$$

Show that

- a.  $\sum_{n=1}^{\infty} |f_n|$  is not in  $L^1([0, \infty), m)$

Hint:  $f_n(x)$  has a root  $x_n$ .

b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0, \infty), m) \quad \text{and} \quad \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x) \, dm = \ln \frac{b}{a}$$

**2.6 Fall 2016 # 1**

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that  $f$  converges to a differentiable function on  $(1, \infty)$  and that

$$f'(x) = \sum_{n=1}^{\infty} \left( \frac{1}{n^x} \right)'.$$

Hint:

$$\left( \frac{1}{n^x} \right)' = -\frac{1}{n^x} \ln n$$

**2.7 Fall 2016 # 5**

Let  $\varphi \in L^\infty(\mathbb{R})$ . Show that the following limit exists and satisfies the equality

$$\lim_{n \rightarrow \infty} \left( \int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} dx \right)^{\frac{1}{n}} = \|\varphi\|_\infty.$$

**2.8 Fall 2016 # 6**

Let  $f, g \in L^2(\mathbb{R})$ . Show that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x)g(x+n) dx = 0$$

**2.9 Spring 2016 # 1**

For  $n \in \mathbb{N}$ , define

$$e_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad E_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

Show that  $e_n < E_n$ , and prove Bernoulli's inequality:

$$(1+x)^n \geq 1+nx \text{ for } -1 < x < \infty \text{ and } n \in \mathbb{N}$$

Use this to show the following:

1. The sequence  $e_n$  is increasing.
2. The sequence  $E_n$  is decreasing.
3.  $2 < e_n < E_n < 4$ .
4.  $\lim_{n \rightarrow \infty} e_n = \lim_{n \rightarrow \infty} E_n$ .

**2.10 Fall 2015 # 1**

Define

$$f(x) = c_0 + c_1x^1 + c_2x^2 + \dots + c_nx^n \text{ with } n \text{ even and } c_n > 0.$$

Show that there is a number  $x_m$  such that  $f(x_m) \leq f(x)$  for all  $x \in \mathbb{R}$ .

**3 Measure Theory: Sets****3.1 Spring 2020 # 2**

Let  $m_*$  denote the Lebesgue outer measure on  $\mathbb{R}$ .

**3.1.1 a.**

Prove that for every  $E \subseteq \mathbb{R}$  there exists a Borel set  $B$  containing  $E$  such that

$$m_*(B) = m_*(E).$$

**3.1.2 b.**

Prove that if  $E \subseteq \mathbb{R}$  has the property that

$$m_*(A) = m_*(A \cap E) + m_*(A \cap E^c)$$

for every set  $A \subseteq \mathbb{R}$ , then there exists a Borel set  $B \subseteq \mathbb{R}$  such that  $E = B \setminus N$  with  $m_*(N) = 0$ .

Be sure to address the case when  $m_*(E) = \infty$ .

*Solution.*

Concepts used:

- Definition of outer measure:  $m_*(E) = \inf_{\{Q_j\} \Rightarrow E} \sum |Q_j|$  where  $\{Q_j\}$  is a countable collection of closed cubes.
- Break  $\mathbb{R}$  into  $\prod_{n \in \mathbb{Z}} [n, n+1)$ , each with finite measure.
- Theorem:  $m_*(Q) = |Q|$  for  $Q$  a closed cube (i.e. the outer measure equals the volume).

**Proof (of Theorem)** Statement: if  $Q$  is a closed cube, then  $m_*(Q) = |Q|$ , the usual volume.

- $m_*(Q) \leq |Q|$ :
  - Since  $Q \subseteq Q$ ,  $Q \Rightarrow Q$  and  $m_*(Q) \leq |Q|$  since  $m_*$  is an infimum over such coverings.
- $|Q| \leq m_*(Q)$ :
  - Fix  $\varepsilon > 0$ .
  - Let  $\{Q_i\}_{i=1}^\infty \Rightarrow Q$  be arbitrary, it suffices to show that

$$|Q| \leq \left( \sum_{i=1}^\infty |Q_i| \right) + \varepsilon.$$

- Pick open cubes  $S_i$  such that  $Q_i \subseteq S_i$  and  $|Q_i| \leq |S_i| \leq (1 + \varepsilon)|Q_i|$ .
- Then  $\{S_i\} \Rightarrow Q$ , so by compactness of  $Q$  pick a finite subcover with  $N$  elements.
- Note

$$Q \subseteq \bigcup_{i=1}^N S_i \implies |Q| \leq \sum_{i=1}^N |S_i| \leq \sum_{i=1}^N (1 + \varepsilon)|Q_i| \leq (1 + \varepsilon) \sum_{i=1}^\infty |Q_i|.$$

- Taking an infimum over coverings on the RHS preserves the inequality, so

$$|Q| \leq (1 + \varepsilon)m_*(Q)$$

- Take  $\varepsilon \rightarrow 0$  to obtain final inequality.

**3.1.3 a**

- If  $m_*(E) = \infty$ , then take  $B = \mathbb{R}^n$  since  $m(\mathbb{R}^n) = \infty$ .
- Suppose  $N := m_*(E) < \infty$ .
- Since  $m_*(E)$  is an infimum, by definition, for every  $\varepsilon > 0$  there exists a covering by closed cubes  $\{Q_i(\varepsilon)\}_{i=1}^\infty \Rightarrow E$  depending on  $\varepsilon$  such that

$$\sum_{i=1}^\infty |Q_i(\varepsilon)| < N + \varepsilon.$$

- For each fixed  $n$ , set  $\varepsilon_n = \frac{1}{n}$  to produce such a covering  $\{Q_i(\varepsilon_n)\}_{i=1}^\infty$  and set  $B_n := \bigcup_{i=1}^\infty Q_i(\varepsilon_n)$ .
- The outer measure of cubes is *equal* to the sum of their volumes, so

$$m_*(B_n) = \sum_{i=1}^\infty |Q_i(\varepsilon_n)| < N + \varepsilon_n = N + \frac{1}{n}.$$

- Now set  $B := \bigcap_{n=1}^\infty B_n$ .
  - Since  $E \subseteq B_n$  for every  $n$ ,  $E \subseteq B$
  - Since  $B$  is a countable intersection of countable unions of closed sets,  $B$  is Borel.
  - Since  $B_n \subseteq B$  for every  $n$ , we can apply subadditivity to obtain the inequality

$$E \subseteq B \subseteq B_n \implies N \leq m_*(B) \leq m_*(B_n) < N + \frac{1}{n} \quad \text{for all } n \in \mathbb{Z}^{\geq 1}.$$

- This forces  $m_*(E) = m_*(B)$ .

### 3.1.4 b

Suppose  $m_*(E) < \infty$ .

- By (a), find a Borel set  $B \supseteq E$  such that  $m_*(B) = m_*(E)$
- Note that  $E \subseteq B \implies B \cap E = E$  and  $B \cap E^c = B \setminus E$ .
- By assumption,

$$\begin{aligned} m_*(B) &= m_*(B \cap E) + m_*(B \cap E^c) \\ m_*(E) &= m_*(E) + m_*(B \setminus E) \\ m_*(E) - m_*(E) &= m_*(B \setminus E) \quad \text{since } m_*(E) < \infty \\ \implies m_*(B \setminus E) &= 0. \end{aligned}$$

- So take  $N = B \setminus E$ ; this shows  $m_*(N) = 0$  and  $E = B \setminus (B \setminus E) = B \setminus N$ .

If  $m_*(E) = \infty$ :

- Apply result to  $E_R := E \cap [R, R+1)^n \subset \mathbb{R}^n$  for  $R \in \mathbb{Z}$ , so  $E = \bigsqcup_R E_R$
- Obtain  $B_R, N_R$  such that  $E_R = B_R \setminus N_R$ ,  $m_*(E_R) = m_*(B_R)$ , and  $m_*(N_R) = 0$ .
- Note that
  - $B := \bigcup_R B_R$  is a union of Borel sets and thus still Borel
  - $E = \bigcup_R E_R$
  - $N := B \setminus E$
  - $N' := \bigcup_R N_R$  is a union of null sets and thus still null
- Since  $E_R \subset B_R$  for every  $R$ , we have  $E \subset B$
- We can compute

$$N = B \setminus E = \left( \bigcup_R B_R \right) \setminus \left( \bigcup_R E_R \right) \subseteq \bigcup_R (B_R \setminus E_R) = \bigcup_R N_R := N'$$

where  $m_*(N') = 0$  since  $N'$  is null, and thus subadditivity forces  $m_*(N) = 0$ .

**3.2 Fall 2019 # 3.**

Let  $(X, \mathcal{B}, \mu)$  be a measure space with  $\mu(X) = 1$  and  $\{B_n\}_{n=1}^\infty$  be a sequence of  $\mathcal{B}$ -measurable subsets of  $X$ , and

$$B := \left\{x \in X \mid x \in B_n \text{ for infinitely many } n\right\}.$$

- Argue that  $B$  is also a  $\mathcal{B}$ -measurable subset of  $X$ .
- Prove that if  $\sum_{n=1}^\infty \mu(B_n) < \infty$  then  $\mu(B) = 0$ .
- Prove that if  $\sum_{n=1}^\infty \mu(B_n) = \infty$  **and** the sequence of set complements  $\{B_n^c\}_{n=1}^\infty$  satisfies

$$\mu\left(\bigcap_{n=k}^K B_n^c\right) = \prod_{n=k}^K (1 - \mu(B_n))$$

for all positive integers  $k$  and  $K$  with  $k < K$ , then  $\mu(B) = 1$ .

Hint: Use the fact that  $1 - x \leq e^{-x}$  for all  $x$ .

*Solution.*

Concepts used:

- Borel-Cantelli: for a sequence of sets  $X_n$ ,

$$\begin{aligned} \limsup_n X_n &= \left\{x \mid x \in X_n \text{ for infinitely many } n\right\} &= \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} X_n \\ \liminf_n X_n &= \left\{x \mid x \in X_n \text{ for all but finitely many } n\right\} &= \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} X_n. \end{aligned}$$

- Properties of logs and exponentials:

$$\prod_n e^{x_n} = e^{\sum_n x_n} \quad \text{and} \quad \sum_n \log(x_n) = \log\left(\prod_n x_n\right).$$

- Tails of convergent sums vanish.
- Continuity of measure:  $B_n \searrow B$  and  $\mu(B_0) < \infty$  implies  $\lim_n \mu(B_n) = \mu(B)$ , and  $B_n \nearrow B \implies \lim_n \mu(B_n) = \mu(B)$ .

**3.2.1 a**

- The Borel  $\sigma$ -algebra is closed under countable unions/intersections/complements,
- $B = \limsup_n B_n$  is an intersection of unions of measurable sets.

**3.2.2 b**

- Tails of convergent sums go to zero, so  $\sum_{n \geq M} \mu(B_n) \xrightarrow{M \rightarrow \infty} 0$ ,
- $B_M := \bigcap_{m=1}^M \bigcup_{n \geq m} B_n \searrow B$ .

$$\begin{aligned}
\mu(B_M) &= \mu\left(\bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} B_n\right) \\
&\leq \mu\left(\bigcup_{n \geq m} B_n\right) \quad \text{for all } m \in \mathbb{N} \text{ by countable subadditivity} \\
&\longrightarrow 0,
\end{aligned}$$

- The result follows by continuity of measure.

### 3.2.3 c

- To show  $\mu(B) = 1$ , we'll show  $\mu(B^c) = 0$ .
- Let  $B_K = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^K B_n$ . Then

$$\begin{aligned}
\mu(B_K^c) &= \mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^K B_n^c\right) \\
&\leq \sum_{m=1}^{\infty} \mu\left(\bigcap_{n=m}^K B_n^c\right) \quad \text{by subadditivity} \\
&= \sum_{m=1}^{\infty} \prod_{n=m}^K (1 - \mu(B_n)) \quad \text{by assumption} \\
&\leq \sum_{m=1}^{\infty} \prod_{n=m}^K e^{-\mu(B_n^c)} \quad \text{by hint} \\
&= \sum_{m=1}^{\infty} \exp\left(-\sum_{n=m}^K \mu(B_n^c)\right) \\
&\xrightarrow{K \rightarrow \infty} 0
\end{aligned}$$

since  $\sum_{n=m}^K \mu(B_n^c) \xrightarrow{K \rightarrow \infty} \infty$  by assumption

- We can apply continuity of measure since  $B_K^c \xrightarrow{K \rightarrow \infty} B^c$ .

Proving the hint: ?

### 3.3 Spring 2019 # 2

Let  $\mathcal{B}$  denote the set of all Borel subsets of  $\mathbb{R}$  and  $\mu : \mathcal{B} \rightarrow [0, \infty)$  denote a finite Borel measure on  $\mathbb{R}$ .

- a. Prove that if  $\{F_k\}$  is a sequence of Borel sets for which  $F_k \supseteq F_{k+1}$  for all  $k$ , then

$$\lim_{k \rightarrow \infty} \mu(F_k) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right)$$

- b. Suppose  $\mu$  has the property that  $\mu(E) = 0$  for every  $E \in \mathcal{B}$  with Lebesgue measure  $m(E) = 0$ . Prove that for every  $\varepsilon > 0$  there exists  $\delta > 0$  so that if  $E \in \mathcal{B}$  with  $m(E) < \delta$ , then  $\mu(E) < \varepsilon$ .

### 3.4 Fall 2018 # 2

Let  $E \subset \mathbb{R}$  be a Lebesgue measurable set. Show that there is a Borel set  $B \subset E$  such that  $m(E \setminus B) = 0$ .

### 3.5 Spring 2018 # 1

Define

$$E := \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N} \right\}.$$

Prove that  $m(E) = 0$ .

### 3.6 Fall 2017 # 2

Let  $f(x) = x^2$  and  $E \subset [0, \infty) := \mathbb{R}^+$ .

1. Show that

$$m^*(E) = 0 \iff m^*(f(E)) = 0.$$

2. Deduce that the map

$$\begin{aligned} \varphi : \mathcal{L}(\mathbb{R}^+) &\longrightarrow \mathcal{L}(\mathbb{R}^+) \\ E &\mapsto f(E) \end{aligned}$$

is a bijection from the class of Lebesgue measurable sets of  $[0, \infty)$  to itself.

### 3.7 Spring 2017 # 2

- a. Let  $\mu$  be a measure on a measurable space  $(X, \mathcal{M})$  and  $f$  a positive measurable function.

Define a measure  $\lambda$  by

$$\lambda(E) := \int_E f \, d\mu, \quad E \in \mathcal{M}$$

Show that for  $g$  any positive measurable function,

$$\int_X g \, d\lambda = \int_X fg \, d\mu$$

- b. Let  $E \subset \mathbb{R}$  be a measurable set such that

$$\int_E x^2 \, dm = 0.$$

Show that  $m(E) = 0$ .

**3.8 Fall 2016 # 4**

Let  $(X, \mathcal{M}, \mu)$  be a measure space and suppose  $\{E_n\} \subset \mathcal{M}$  satisfies

$$\lim_{n \rightarrow \infty} \mu(X \setminus E_n) = 0.$$

Define

$$G := \left\{ x \in X \mid x \in E_n \text{ for only finitely many } n \right\}.$$

Show that  $G \in \mathcal{M}$  and  $\mu(G) = 0$ .

**3.9 Spring 2016 # 3**

Let  $f$  be Lebesgue measurable on  $\mathbb{R}$  and  $E \subset \mathbb{R}$  be measurable such that

$$0 < A = \int_E f(x) dx < \infty.$$

Show that for every  $0 < t < 1$ , there exists a measurable set  $E_t \subset E$  such that

$$\int_{E_t} f(x) dx = tA.$$

**3.10 Spring 2016 # 5**

Let  $(X, \mathcal{M}, \mu)$  be a measure space. For  $f \in L^1(\mu)$  and  $\lambda > 0$ , define

$$\varphi(\lambda) = \mu(\{x \in X \mid f(x) > \lambda\}) \quad \text{and} \quad \psi(\lambda) = \mu(\{x \in X \mid f(x) < -\lambda\})$$

Show that  $\varphi, \psi$  are Borel measurable and

$$\int_X |f| d\mu = \int_0^\infty [\varphi(\lambda) + \psi(\lambda)] d\lambda$$

**3.11 Fall 2015 # 2**

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be Lebesgue measurable.

1. Show that there is a sequence of simple functions  $s_n(x)$  such that  $s_n(x) \rightarrow f(x)$  for all  $x \in \mathbb{R}$ .
2. Show that there is a Borel measurable function  $g$  such that  $g = f$  almost everywhere.

**3.12 Spring 2015 # 3**

Let  $\mu$  be a finite Borel measure on  $\mathbb{R}$  and  $E \subset \mathbb{R}$  Borel. Prove that the following statements are equivalent:

1.  $\forall \varepsilon > 0$  there exists  $G$  open and  $F$  closed such that

$$F \subseteq E \subseteq G \quad \text{and} \quad \mu(G \setminus F) < \varepsilon.$$

2. There exists a  $V \in G_\delta$  and  $H \in F_\sigma$  such that

$$H \subseteq E \subseteq V \quad \text{and} \quad \mu(V \setminus H) = 0$$



**3.13 Spring 2014 # 3**

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and suppose

$$\forall x \in \mathbb{R}, \quad f(x) \geq \limsup_{y \rightarrow x} f(y)$$

Prove that  $f$  is Borel measurable.

**3.14 Spring 2014 # 4**

Let  $(X, \mathcal{M}, \mu)$  be a measure space and suppose  $f$  is a measurable function on  $X$ . Show that

$$\lim_{n \rightarrow \infty} \int_X f^n d\mu = \begin{cases} \infty \\ \mu(f^{-1}(1)) \end{cases} \quad \text{or}$$

and characterize the collection of functions of each type.

**3.15 Spring 2017 # 1**

Let  $K$  be the set of numbers in  $[0, 1]$  whose decimal expansions do not use the digit 4.

We use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 with  $399\ldots$ . For example,  $0.8754 = 0.8753999\ldots$ .

Show that  $K$  is a compact, nowhere dense set without isolated points, and find the Lebesgue measure  $m(K)$ .

**3.16 Spring 2016 # 2**

Let  $0 < \lambda < 1$  and construct a Cantor set  $C_\lambda$  by successively removing middle intervals of length  $\lambda$ .

Prove that  $m(C_\lambda) = 0$ .

**4 Measure Theory: Functions****4.1 Fall 2016 # 2**

Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be measurable with

$$\int_a^b f(x) dx = \int_a^b g(x) dx.$$

Show that either

1.  $f(x) = g(x)$  almost everywhere, or
2. There exists a measurable set  $E \subset [a, b]$  such that

$$\int_E f(x) dx > \int_E g(x) dx$$

**4.2 Spring 2016 # 4**

Let  $E \subset \mathbb{R}$  be measurable with  $m(E) < \infty$ . Define

$$f(x) = m(E \cap (E + x)).$$

Show that

1.  $f \in L^1(\mathbb{R})$ .
2.  $f$  is uniformly continuous.
3.  $\lim_{|x| \rightarrow \infty} f(x) = 0$ .

Hint:

$$\chi_{E \cap (E+x)}(y) = \chi_E(y) \chi_E(y-x)$$

**5 Integrals: Convergence****5.1 Fall 2019 # 2.**

Prove that

$$\left| \frac{d^n}{dx^n} \frac{\sin x}{x} \right| \leq \frac{1}{n}$$

for all  $x \neq 0$  and positive integers  $n$ .

Hint: Consider  $\int_0^1 \cos(tx) dt$

*Solution.*

Concepts used:

- DCT
- Bounding in the right place. Don't evaluate the actual integral!

**Solution:**

- By induction on the number of limits we can pass through the integral.
- For  $n = 1$  we first pass one derivative into the integral: let  $x_n \rightarrow x$  be any sequence

converging to  $x$ , then

$$\begin{aligned}
 \frac{\partial}{\partial x} \frac{\sin(x)}{x} &= \frac{\partial}{\partial x} \int_0^1 \cos(tx) dt \\
 &= \lim_{x_n \rightarrow x} \frac{1}{x_n - x} \left( \int_0^1 \cos(tx_n) dt - \int_0^1 \cos(tx) dt \right) \\
 &= \lim_{x_n \rightarrow x} \left( \int_0^1 \frac{\cos(tx_n) - \cos(tx)}{x_n - x} dt \right) \\
 &= \lim_{x_n \rightarrow x} \left( \int_0^1 \left( t \sin(tx) \Big|_{x=\xi_n} \right) dt \right) \quad \text{where } \xi_n \in [x_n, x] \text{ by MVT, } \xi_n \rightarrow x \\
 &= \lim_{\xi_n \rightarrow x} \left( \int_0^1 t \sin(t\xi_n) dt \right) \\
 &\stackrel{\text{DCT}}{=} \int_0^1 \lim_{\xi_n \rightarrow x} t \sin(t\xi_n) dt \\
 &= \int_0^1 t \sin(tx) dt
 \end{aligned}$$

- Taking absolute values we obtain an upper bound

$$\begin{aligned}
 \left| \frac{\partial}{\partial x} \frac{\sin(x)}{x} \right| &= \left| \int_0^1 t \sin(tx) dt \right| \\
 &\leq \int_0^1 |t \sin(tx)| dt \\
 &\leq \int_0^1 1 dt = 1,
 \end{aligned}$$

since  $t \in [0, 1] \implies |t| < 1$ , and  $|\sin(xt)| \leq 1$  for any  $x$  and  $t$ .

- Note that this bound also justifies the DCT, since the functions  $f_n(t) = t \sin(t\xi_n)$  are uniformly dominated by  $g(t) = 1$  on  $L^1([0, 1])$ .

Note: integrating by parts here yields the actual formula:

$$\begin{aligned}
 \int_0^1 t \sin(tx) dt &\stackrel{\text{IBP}}{=} \left( \frac{-t \cos(tx)}{x} \right) \Big|_{t=0}^{t=1} - \int_0^1 \frac{\cos(tx)}{x} dt \\
 &= \frac{-\cos(x)}{x} - \frac{\sin(x)}{x^2} \\
 &= \frac{x \cos(x) - \sin(x)}{x^2}.
 \end{aligned}$$

- For the inductive step, we assume that we can pass  $n - 1$  limits through the integral and show we can pass the  $n$ th through as well.

$$\begin{aligned}
 \frac{\partial^n}{\partial x^n} \frac{\sin(x)}{x} &= \frac{\partial^n}{\partial x^n} \int_0^1 \cos(tx) dt \\
 &= \frac{\partial}{\partial x} \int_0^1 \frac{\partial^{n-1}}{\partial x^{n-1}} \cos(tx) dt \\
 &= \frac{\partial}{\partial x} \int_0^1 t^{n-1} f_{n-1}(x, t) dt
 \end{aligned}$$

- Note that  $f_n(x, t) = \pm \sin(tx)$  when  $n$  is odd and  $f_n(x, t) = \pm \cos(tx)$  when  $n$  is even, and a constant factor of  $t$  is multiplied when each derivative is taken.
- We continue as in the base case:

$$\begin{aligned}
 \frac{\partial}{\partial x} \int_0^1 t^{n-1} f_{n-1}(x, t) dt &= \lim_{x_k \rightarrow x} \int_0^1 t^{n-1} \left( \frac{f_{n-1}(x_k, t) - f_{n-1}(x, t)}{x_k - x} \right) dt \\
 &=_{\text{IVT}} \lim_{x_k \rightarrow x} \int_0^1 t^{n-1} \frac{\partial f_{n-1}}{\partial x}(\xi_k, t) dt \quad \text{where } \xi_k \in [x_k, x], \xi_k \rightarrow x \\
 &=_{\text{DCT}} \int_0^1 \lim_{x_k \rightarrow x} t^{n-1} \frac{\partial f_{n-1}}{\partial x}(\xi_k, t) dt \\
 &:= \int_0^1 \lim_{x_k \rightarrow x} t^n f_n(\xi_k, t) dt \\
 &:= \int_0^1 t^n f_n(x, t) dt.
 \end{aligned}$$

- We've used the fact that  $f_0(x) = \cos(tx)$  is smooth as a function of  $x$ , and in particular continuous
- The DCT is justified because the functions  $h_{n,k}(x, t) = t^n f_n(\xi_k, t)$  are again uniformly (in  $k$ ) bounded by 1 since  $t \leq 1 \implies t^n \leq 1$  and each  $f_n$  is a sin or cosine.
- Now take absolute values

$$\begin{aligned}
 \left| \frac{\partial^n}{\partial x^n} \frac{\sin(x)}{x} \right| &= \left| \int_0^1 -t^n f_n(x, t) dt \right| \\
 &\leq \int_0^1 |t^n f_n(x, t)| dt \\
 &\leq \int_0^1 |t^n| |f_n(x, t)| dt \\
 &\leq \int_0^1 |t^n| \cdot 1 dt \\
 &\leq \int_0^1 t^n dt \quad \text{since } t \text{ is positive} \\
 &= \frac{1}{n+1} \\
 &< \frac{1}{n}.
 \end{aligned}$$

- We've again used the fact that  $f_n(x, t)$  is of the form  $\pm \cos(tx)$  or  $\pm \sin(tx)$ , both of which are bounded by 1.

## 5.2 Spring 2020 # 5

Compute the following limit and justify your calculations:

$$\lim_{n \rightarrow \infty} \int_0^n \left( 1 + \frac{x^2}{n} \right)^{-(n+1)} dx.$$

Not finished, flesh out.

*Solution.*

Concepts used:

- DCT
- Passing limits through products and quotients

Note that

$$\begin{aligned}\lim_n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} &= \frac{1}{\lim_n \left(1 + \frac{x^2}{n}\right)^1 \left(1 + \frac{x^2}{n}\right)^n} \\ &= \frac{1}{1 \cdot e^{x^2}} \\ &= e^{-x^2}.\end{aligned}$$

If passing the limit through the integral is justified, we will have

$$\begin{aligned}\lim_{n \rightarrow \infty} \int_0^n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \chi_{[0,n]} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \\ &= \int_{\mathbb{R}} \lim_{n \rightarrow \infty} \chi_{[0,n]} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \quad \text{by the DCT} \\ &= \int_{\mathbb{R}} \lim_{n \rightarrow \infty} \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx \\ &= \int_0^\infty e^{-x^2} \\ &= \frac{\sqrt{\pi}}{2}.\end{aligned}$$

Computing the last integral:

$$\begin{aligned}\left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2 &= \left(\int_{\mathbb{R}} e^{-x^2} dx\right) \left(\int_{\mathbb{R}} e^{-y^2} dx\right) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(x+y)^2} dx \\ &= \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta \quad u = r^2 \\ &= \frac{1}{2} \int_0^{2\pi} \int_0^\infty e^{-u} du d\theta \\ &= \frac{1}{2} \int_0^{2\pi} 1 \\ &= \pi,\end{aligned}$$

and now use the fact that the function is even so  $\int_0^\infty f = \frac{1}{2} \int_{\mathbb{R}} f$ .

Justifying the DCT:

- Apply Bernoulli's inequality:

$$1 + \frac{x^{2^{n+1}}}{n} \geq 1 + \frac{x^2}{n} (1 + x^2) \geq 1 + x^2,$$

where the last inequality follows from the fact that  $1 + \frac{x^2}{n} \geq 1$

### 5.3 Spring 2019 # 3

Let  $\{f_k\}$  be any sequence of functions in  $L^2([0, 1])$  satisfying  $\|f_k\|_2 \leq M$  for all  $k \in \mathbb{N}$ .

Prove that if  $f_k \rightarrow f$  almost everywhere, then  $f \in L^2([0, 1])$  with  $\|f\|_2 \leq M$  and

$$\lim_{k \rightarrow \infty} \int_0^1 f_k(x) dx = \int_0^1 f(x) dx$$

Hint: Try using Fatou's Lemma to show that  $\|f\|_2 \leq M$  and then try applying Egorov's Theorem.

### 5.4 Fall 2018 # 6

Compute the following limit and justify your calculations:

$$\lim_{n \rightarrow \infty} \int_1^n \frac{dx}{\left(1 + \frac{x}{n}\right)^n \sqrt[n]{x}}$$

### 5.5 Fall 2018 # 3

Suppose  $f(x)$  and  $xf(x)$  are integrable on  $\mathbb{R}$ . Define  $F$  by

$$F(t) := \int_{-\infty}^{\infty} f(x) \cos(xt) dx$$

Show that

$$F'(t) = - \int_{-\infty}^{\infty} xf(x) \sin(xt) dx.$$

### 5.6 Spring 2018 # 5

Suppose that

- $f_n, f \in L^1$ ,
- $f_n \rightarrow f$  almost everywhere, and
- $\int |f_n| \rightarrow \int |f|$ .

Show that  $\int f_n \rightarrow \int f$

**5.7 Spring 2018 # 2**

Let

$$f_n(x) := \frac{x}{1+x^n}, \quad x \geq 0.$$

- Show that this sequence converges pointwise and find its limit. Is the convergence uniform on  $[0, \infty)$ ?
- Compute

$$\lim_{n \rightarrow \infty} \int_0^\infty f_n(x) dx$$

**5.8 Fall 2016 # 3**

Let  $f \in L^1(\mathbb{R})$ . Show that

$$\lim_{x \rightarrow 0} \int_{\mathbb{R}} |f(y-x) - f(y)| dy = 0$$

**5.9 Fall 2015 # 3**

Compute the following limit:

$$\lim_{n \rightarrow \infty} \int_1^n \frac{ne^{-x}}{1+nx^2} \sin\left(\frac{x}{n}\right) dx$$

**5.10 Fall 2015 # 4**

Let  $f : [1, \infty) \rightarrow \mathbb{R}$  such that  $f(1) = 1$  and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Show that the following limit exists and satisfies the equality

$$\lim_{x \rightarrow \infty} f(x) \leq 1 + \frac{\pi}{4}$$

**6 Integrals: Approximation****6.1 Spring 2018 # 3**

Let  $f$  be a non-negative measurable function on  $[0, 1]$ .

Show that

$$\lim_{p \rightarrow \infty} \left( \int_{[0,1]} f(x)^p dx \right)^{\frac{1}{p}} = \|f\|_\infty.$$

**6.2 Spring 2018 # 4**

Let  $f \in L^2([0, 1])$  and suppose

$$\int_{[0,1]} f(x)x^n dx = 0 \text{ for all integers } n \geq 0.$$

Show that  $f = 0$  almost everywhere.

**6.3 Spring 2015 # 2**

Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be continuous with period 1. Prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(n\alpha) = \int_0^1 f(t) dt \quad \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

Hint: show this first for the functions  $f(t) = e^{2\pi i k t}$  for  $k \in \mathbb{Z}$ .

**6.4 Fall 2014 # 4**

Let  $g \in L^\infty([0, 1])$  Prove that

$$\int_{[0,1]} f(x)g(x) dx = 0 \quad \text{for all continuous } f : [0, 1] \rightarrow \mathbb{R} \implies g(x) = 0 \text{ almost everywhere.}$$

**7  $L^1$** **7.1 Spring 2020 # 3**

a. Prove that if  $g \in L^1(\mathbb{R})$  then

$$\lim_{N \rightarrow \infty} \int_{|x| \geq N} |f(x)| dx = 0,$$

and demonstrate that it is not necessarily the case that  $f(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ .

b. Prove that if  $f \in L^1([1, \infty))$  and is decreasing, then  $\lim_{x \rightarrow \infty} f(x) = 0$  and in fact  $\lim_{x \rightarrow \infty} x f(x) = 0$ .

c. If  $f : [1, \infty) \rightarrow [0, \infty)$  is decreasing with  $\lim_{x \rightarrow \infty} x f(x) = 0$ , does this ensure that  $f \in L^1([1, \infty))$ ?

*Solution.*

Concepts used:

- Limits
- Cauchy Criterion for Integrals:  $\int_a^\infty f(x) dx$  converges iff for every  $\varepsilon > 0$  there exists an  $M_0$  such that  $A, B \geq M_0$  implies  $\left| \int_A^B f \right| < \varepsilon$ , i.e.  $\left| \int_A^B f \right| \xrightarrow{A \rightarrow \infty} 0$ .
- Integrals of  $L^1$  functions have vanishing tails:  $\int_N^\infty |f| \xrightarrow{N \rightarrow \infty} 0$ .



- Mean Value Theorem for Integrals:  $\int_a^b f(t) dt = (b-a)f(c)$  for some  $c \in [a, b]$ .

**7.1.1 a**

Stated integral equality:

- Let  $\varepsilon > 0$
- $C_c(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$  is dense so choose  $\{f_n\} \rightarrow f$  with  $\|f_n - f\|_1 \rightarrow 0$ .
- Since  $\{f_n\}$  are compactly supported, choose  $N_0 \gg 1$  such that  $f_n$  is zero outside of  $B_{N_0}(\mathbf{0})$ .
- Then

$$\begin{aligned}
 N \geq N_0 &\implies \int_{|x|>N} |f| = \int_{|x|>N} |f - f_n + f_n| \\
 &\leq \int_{|x|>N} |f - f_n| + \int_{|x|>N} |f_n| \\
 &= \int_{|x|>N} |f - f_n| \\
 &\leq \int_{|x|>N} \|f - f_n\|_1 \\
 &= \|f_n - f\|_1 \left( \int_{|x|>N} 1 \right) \\
 &\xrightarrow[n \rightarrow \infty]{} 0 \left( \int_{|x|>N} 1 \right) \\
 &= 0 \\
 &\xrightarrow[N \rightarrow \infty]{} 0.
 \end{aligned}$$

To see that this doesn't force  $f(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ :

- Take  $f(x)$  to be a train of rectangles of height 1 and area  $1/2^j$  centered on even integers.
- Then

$$\int_{|x|>N} |f| = \sum_{j=N}^{\infty} 1/2^j \xrightarrow[N \rightarrow \infty]{} 0$$

as the tail of a convergent sum.

- However  $f(x) = 1$  for infinitely many even integers  $x > N$ , so  $f(x) \not\rightarrow 0$  as  $|x| \rightarrow \infty$ .

**7.1.2 b****Solution 1 ("Trick")**

- Since  $f$  is decreasing on  $[1, \infty)$ , for any  $t \in [x-n, x]$  we have

$$x-n \leq t \leq x \implies f(x) \leq f(t) \leq f(x-n).$$

- Integrate over  $[x, 2x]$ , using monotonicity of the integral:

$$\begin{aligned} \int_x^{2x} f(x) dt &\leq \int_x^{2x} f(t) dt \leq \int_x^{2x} f(x-n) dt \\ \implies f(x) \int_x^{2x} dt &\leq \int_x^{2x} f(t) dt \leq f(x-n) \int_x^{2x} dt \\ \implies xf(x) &\leq \int_x^{2x} f(t) dt \leq xf(x-n). \end{aligned}$$

- By the Cauchy Criterion for integrals,  $\lim_{x \rightarrow \infty} \int_x^{2x} f(t) dt = 0$ .
- So the LHS term  $xf(x) \xrightarrow{x \rightarrow \infty} 0$ .
- Since  $x > 1$ ,  $|f(x)| \leq |xf(x)|$
- Thus  $f(x) \xrightarrow{x \rightarrow \infty} 0$  as well.

### Solution 2 (Variation on the Trick)

- Use mean value theorem for integrals:

$$\int_x^{2x} f(t) dt = xf(c_x) \quad \text{for some } c_x \in [x, 2x] \text{ depending on } x.$$

- Since  $f$  is decreasing,

$$\begin{aligned} x \leq c_x \leq 2x &\implies f(2x) \leq f(c_x) \leq f(x) \\ \implies 2xf(2x) &\leq 2xf(c_x) \leq 2xf(x) \\ \implies 2xf(2x) &\leq 2x \int_x^{2x} f(t) dt \leq 2xf(x) \end{aligned}$$

- By Cauchy Criterion,  $\int_x^{2x} f \rightarrow 0$ .
- So  $2xf(2x) \rightarrow 0$ , which by a change of variables gives  $uf(u) \rightarrow 0$ .
- Since  $u \geq 1$ ,  $f(u) \leq uf(u)$  so  $f(u) \rightarrow 0$  as well.

### Solution 3 (Contradiction)

Just showing  $f(x) \xrightarrow{x \rightarrow \infty} 0$ :

- Toward a contradiction, suppose not.
- Since  $f$  is decreasing, it can not diverge to  $+\infty$
- If  $f(x) \rightarrow -\infty$ , then  $f \notin L^1(\mathbb{R})$ : choose  $x_0 \gg 1$  so that  $t \geq x_0 \implies f(t) < -1$ , then
- Then  $t \geq x_0 \implies |f(t)| \geq 1$ , so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f(t)| dt \geq \int_{x_0}^\infty 1 = \infty.$$

- Otherwise  $f(x) \rightarrow L \neq 0$ , some finite limit.
- If  $L > 0$ :
  - Fix  $\varepsilon > 0$ , choose  $x_0 \gg 1$  such that  $t \geq x_0 \implies L - \varepsilon \leq f(t) \leq L$
  - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L - \varepsilon) dt = \infty$$

- If  $L < 0$ :
  - Fix  $\varepsilon > 0$ , choose  $x_0 \gg 1$  such that  $t \geq x_0 \implies L \leq f(t) \leq L + \varepsilon$ .
  - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L) dt = \infty$$

Showing  $xf(x) \xrightarrow{x \rightarrow \infty} 0$ .

- Toward a contradiction, suppose not.
- (How to show that  $xf(x) \not\xrightarrow{x \rightarrow \infty} +\infty$ ?)
- If  $xf(x) \rightarrow -\infty$ 
  - Choose a sequence  $\Gamma = \{\hat{x}_i\}$  such that  $x_i \rightarrow \infty$  and  $x_i f(x_i) \rightarrow -\infty$ .
  - Choose a subsequence  $\Gamma' = \{x_i\}$  such that  $x_i f(x_i) \leq -1$  for all  $i$  and  $x_i \leq x_{i+1}$ .
  - Choose a further subsequence  $S = \{x_i \in \Gamma' \mid 2x_i < x_{i+1}\}$ .
  - Then since  $f$  is always decreasing, for  $t \geq x_0$ ,  $|f|$  is increasing, and  $|f(x_i)| \leq |f(2x_i)|$ , so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f| \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(t)| dt \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(x_i)| = \sum_{x_i \in S} x_i f(x_i) \rightarrow \infty.$$

- If  $xf(x) \rightarrow L \neq 0$  for  $0 < L < \infty$ :
  - Fix  $\varepsilon > 0$ , choose an infinite sequence  $\{x_i\}$  such that  $L - \varepsilon \leq x_i f(x_i) \leq L$  for all  $i$ .

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)| dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i) dt = \sum_S x_i f(x_i) \geq \sum_S (L - \varepsilon) \rightarrow \infty.$$

- If  $xf(x) \rightarrow L \neq 0$  for  $-\infty < L < 0$ :
  - Fix  $\varepsilon > 0$ , choose an infinite sequence  $\{x_i\}$  such that  $L \leq x_i f(x_i) \leq L + \varepsilon$  for all  $i$ .

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)| dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i) dt = \sum_S x_i f(x_i) \geq \sum_S (L) \rightarrow \infty.$$

**Solution 4 (Akos's Suggestion)** For  $x \geq 1$ ,

$$|xf(x)| = \left| \int_x^{2x} f(x) dt \right| \leq \int_x^{2x} |f(x)| dt \leq \int_x^{2x} |f(t)| dt \leq \int_x^\infty |f(t)| dt \xrightarrow{x \rightarrow \infty} 0$$

where we've used

- Since  $f$  is decreasing and  $\lim_{x \rightarrow \infty} f(x) = 0$  from part (a),  $f$  is non-negative.
- Since  $f$  is positive and decreasing, for every  $t \in [a, b]$  we have  $|f(a)| \leq |f(t)|$ .
- By part (a), the last integral goes to zero.

**Solution 5 (Peter's)**

- Toward a contradiction, produce a sequence  $x_i \rightarrow \infty$  with  $x_i f(x_i) \rightarrow \infty$  and  $x_i f(x_i) >$

$\varepsilon > 0$ , then

$$\begin{aligned}
 \int f(x) dx &\geq \sum_{i=1}^{\infty} \int_{x_i}^{x_{i+1}} f(x) dx \\
 &\geq \sum_{i=1}^{\infty} \int_{x_i}^{x_{i+1}} f(x_{i+1}) dx \\
 &= \sum_{i=1}^{\infty} f(x_{i+1}) \int_{x_i}^{x_{i+1}} dx \\
 &\geq \sum_{i=1}^{\infty} (x_{i+1} - x_i) f(x_{i+1}) \\
 &\geq \sum_{i=1}^{\infty} (x_{i+1} - x_i) \frac{\varepsilon}{x_{i+1}} \\
 &= \varepsilon \sum_{i=1}^{\infty} \left(1 - \frac{x_{i-1}}{x_i}\right) \rightarrow \infty
 \end{aligned}$$

which can be ensured by passing to a subsequence where  $\sum \frac{x_{i-1}}{x_i} < \infty$ .

### 7.1.3 c

- No: take  $f(x) = \frac{1}{x \ln x}$
- Then by a  $u$ -substitution,

$$\int_0^x f = \ln(\ln(x)) \xrightarrow{x \rightarrow \infty} \infty$$

is unbounded, so  $f \notin L^1([1, \infty))$ .

- But

$$xf(x) = \frac{1}{\ln(x)} \xrightarrow{x \rightarrow \infty} 0.$$

## 7.2 Fall 2019 # 5.

### 7.2.1 a

Show that if  $f$  is continuous with compact support on  $\mathbb{R}$ , then

$$\lim_{y \rightarrow 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dx = 0$$

### 7.2.2 b

Let  $f \in L^1(\mathbb{R})$  and for each  $h > 0$  let

$$\mathcal{A}_h f(x) := \frac{1}{2h} \int_{|y| \leq h} f(x-y) dy$$

- i. Prove that  $\|\mathcal{A}_h f\|_1 \leq \|f\|_1$  for all  $h > 0$ .
- ii. Prove that  $\mathcal{A}_h f \rightarrow f$  in  $L^1(\mathbb{R})$  as  $h \rightarrow 0^+$ .

*Solution.*

Continuity in  $L^1$  (recall that DCT won't work! Notes 19.4, prove it for a dense subset first).  
 Lebesgue differentiation in 1-dimensional case. See HW 5.6.

### 7.3 a

Choose  $g \in C_c^0$  such that  $\|f - g\|_1 \rightarrow 0$ .

By translation invariance,  $\|\tau_h f - \tau_h g\|_1 \rightarrow 0$ .

Write

$$\begin{aligned} \|\tau f - f\|_1 &= \|\tau_h f - g + g - \tau_h g + \tau_h g - f\|_1 \\ &\leq \|\tau_h f - \tau_h g\| + \|g - f\| + \|\tau_h g - g\| \\ &\rightarrow \|\tau_h g - g\|, \end{aligned}$$

so it suffices to show that  $\|\tau_h g - g\| \rightarrow 0$  for  $g \in C_c^0$ .

Fix  $\varepsilon > 0$ . Enlarge the support of  $g$  to  $K$  such that

$$|h| \leq 1 \text{ and } x \in K^c \implies |g(x-h) - g(x)| = 0.$$

By uniform continuity of  $g$ , pick  $\delta \leq 1$  small enough such that

$$x \in K, |h| \leq \delta \implies |g(x-h) - g(x)| < \varepsilon,$$

then

$$\int_K |g(x-h) - g(x)| \leq \int_K \varepsilon = \varepsilon \cdot m(K) \rightarrow 0.$$

### 7.4 b

We have

$$\begin{aligned} \int_{\mathbb{R}} |A_h(f)(x)| \, dx &= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y) \, dy \right| \, dx \\ &\leq \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)| \, dy \, dx \\ &=_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)| \, dx \, dy \\ &= \int_{\mathbb{R}} |f(y)| \, dy \\ &= \|f\|_1. \end{aligned}$$

and (rough sketch)

$$\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x) - f(x)| \, dx &= \int_{\mathbb{R}} \left| \left( \frac{1}{2h} \int_{B(h,x)} f(y) \, dy \right) - f(x) \right| \, dx \\
&= \int_{\mathbb{R}} \left| \left( \frac{1}{2h} \int_{B(h,x)} f(y) \, dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x) \, dy \right| \, dx \\
&\leq_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{B(h,x)} |f(y-x) - f(x)| \, \mathbf{d}\mathbf{x} \, dy \\
&\leq \frac{1}{2h} \int_{\mathbb{R}} \|\tau_x f - f\|_1 \, dy \\
&\rightarrow 0 \quad \text{by (a).}
\end{aligned}$$

### 7.5 Fall 2017 # 3

Let

$$S = \text{span}_{\mathbb{C}} \left\{ \chi_{(a,b)} \mid a, b \in \mathbb{R} \right\},$$

the complex linear span of characteristic functions of intervals of the form  $(a, b)$ .

Show that for every  $f \in L^1(\mathbb{R})$ , there exists a sequence of functions  $\{f_n\} \subset S$  such that

$$\lim_{n \rightarrow \infty} \|f_n - f\|_1 = 0$$

### 7.6 Spring 2015 # 4

Define

$$f(x, y) := \begin{cases} \frac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Carefully show that  $f \in L^1(\mathbb{R}^2)$ .

### 7.7 Fall 2014 # 3

Let  $f \in L^1(\mathbb{R})$ . Show that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } m(E) < \delta \implies \int_E |f(x)| \, dx < \varepsilon$$

### 7.8 Spring 2014 # 1

1. Give an example of a continuous  $f \in L^1(\mathbb{R})$  such that  $f(x) \not\rightarrow 0$  as  $|x| \rightarrow \infty$ .
2. Show that if  $f$  is *uniformly* continuous, then

$$\lim_{|x| \rightarrow \infty} f(x) = 0.$$

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## 8 Fubini-Tonelli

### 8.1 Spring 2020 # 4

Let  $f, g \in L^1(\mathbb{R})$ . Argue that  $H(x, y) := f(y)g(x - y)$  defines a function in  $L^1(\mathbb{R}^2)$  and deduce from this fact that

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x - y) dy$$

defines a function in  $L^1(\mathbb{R})$  that satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

*Solution.*

Relevant concepts:

- Tonelli: non-negative and measurable yields measurability of slices and equality of iterated integrals
- Fubini:  $f(x, y) \in L^1$  yields *integrable* slices and equality of iterated integrals
- F/T: apply Tonelli to  $|f|$ ; if finite,  $f \in L^1$  and apply Fubini to  $f$

$$\begin{aligned} \|H(x)\|_1 &= \int_{\mathbb{R}} |H(x, y)| dx \\ &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y)g(x - y) dy \right| dx \\ &\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)g(x - y)| dy \right) dx \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)g(x - y)| dx \right) dy \quad \text{by Tonelli} \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)g(t)| dt \right) dy \quad \text{setting } t = x - y, dt = -dx \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y)| \cdot |g(t)| dt \right) dy \\ &= \int_{\mathbb{R}} |f(y)| \cdot \left( \int_{\mathbb{R}} |g(t)| dt \right) dy \\ &:= \int_{\mathbb{R}} |f(y)| \cdot \|g\|_1 dy \\ &= \|g\|_1 \int_{\mathbb{R}} |f(y)| dy \\ &:= \|g\|_1 \|f\|_1 \\ &< \infty \quad \text{by assumption} \quad . \end{aligned}$$

- $H$  is measurable on  $\mathbb{R}^2$ :
  - If we can show  $\tilde{f}(x, y) := f(y)$  and  $\tilde{g}(x, y) := g(x - y)$  are both measurable on  $\mathbb{R}^2$ , then  $H = \tilde{f} \cdot \tilde{g}$  is a product of measurable functions and thus measurable.
  - $f \in L^1$ , and  $L^1$  functions are measurable by definition.
  - The function  $(x, y) \mapsto g(x - y)$  is measurable on  $\mathbb{R}^2$ :
    - \* Let  $g$  be measurable on  $\mathbb{R}$ , then the cylinder function  $G(x, y) = g(x)$  on  $\mathbb{R}^2$  is always measurable

- \* Define a linear transformation  $T := [1, -1; 0, 1]$  which sends  $(x, y) \rightarrow (x - y, y)$ , then  $T \in \text{GL}(2, \mathbb{R})$  is linear and thus measurable.
- \* Then  $(G \circ T)(x, y) = G(x - y, y) = \tilde{g}(x - y)$ , so  $\tilde{g}$  is a composition of measurable functions and thus measurable.
- Apply **Tonelli** to  $|H|$ 
  - $H$  measurable implies  $|H|$  is measurable
  - $|H|$  is non-negative
  - So the iterated integrals are equal in the extended sense
  - The calculation shows the iterated integral is finite, to  $\int |H|$  is finite and  $H$  is thus integrable on  $\mathbb{R}^2$ .

Note: Fubini is not needed, since we're not calculating the actual integral, just showing  $H$  is integrable.

## 8.2 Spring 2019 # 4

Let  $f$  be a non-negative function on  $\mathbb{R}^n$  and  $\mathcal{A} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq f(x)\}$ .

Prove the validity of the following two statements:

- a.  $f$  is a Lebesgue measurable function on  $\mathbb{R}^n \iff \mathcal{A}$  is a Lebesgue measurable subset of  $\mathbb{R}^{n+1}$
- b. If  $f$  is a Lebesgue measurable function on  $\mathbb{R}^n$ , then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x) dx = \int_0^\infty m(\{x \in \mathbb{R}^n : f(x) \geq t\}) dt$$

## 8.3 Fall 2018 # 5

Let  $f \geq 0$  be a measurable function on  $\mathbb{R}$ . Show that

$$\int_{\mathbb{R}} f = \int_0^\infty m(\{x : f(x) > t\}) dt$$

## 8.4 Fall 2015 # 5

Let  $f, g \in L^1(\mathbb{R})$  be Borel measurable.

1. Show that
  - The function

$$F(x, y) := f(x - y)g(y)$$

is Borel measurable on  $\mathbb{R}^2$ , and

- For almost every  $y \in \mathbb{R}$ ,

$$F_y(x) := f(x - y)g(y)$$

is integrable with respect to  $y$ .

2. Show that  $f * g \in L^1(\mathbb{R})$  and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$



**8.5 Spring 2014 # 5**

Let  $f, g \in L^1([0, 1])$  and for all  $x \in [0, 1]$  define

$$F(x) := \int_0^x f(y) dy \quad \text{and} \quad G(x) := \int_0^x g(y) dy.$$

Prove that

$$\int_0^1 F(x)g(x) dx = F(1)G(1) - \int_0^1 f(x)G(x) dx$$

**9  $L^2$  and Fourier Analysis****9.1 Spring 2020 # 6****9.1.1 a**

Show that

$$L^2([0, 1]) \subseteq L^1([0, 1]) \quad \text{and} \quad \ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z}).$$

**9.1.2 b**

For  $f \in L^1([0, 1])$  define

$$\hat{f}(n) := \int_0^1 f(x)e^{-2\pi i n x} dx.$$

Prove that if  $f \in L^1([0, 1])$  and  $\{\hat{f}(n)\} \in \ell^1(\mathbb{Z})$  then

$$S_N f(x) := \sum_{|n| \leq N} \hat{f}(n)e^{2\pi i n x}.$$

converges uniformly on  $[0, 1]$  to a continuous function  $g$  such that  $g = f$  almost everywhere.

Hint: One approach is to argue that if  $f \in L^1([0, 1])$  with  $\{\hat{f}(n)\} \in \ell^1(\mathbb{Z})$  then  $f \in L^2([0, 1])$ .

*Solution.*

Concepts used:

- For  $e_n(x) := e^{2\pi i n x}$ , the set  $\{e_n\}$  is an orthonormal basis for  $L^2([0, 1])$ .
- For any orthonormal sequence in a Hilbert space, we have Bessel's inequality:

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

- When  $\{e_n\}$  is a basis, the above is an *equality* (Parseval)
- Arguing uniform convergence: since  $\{\hat{f}(n)\} \in \ell^1(\mathbb{Z})$ , we should be able to apply the  $M$  test.

**9.1.3 a**

Claim:  $\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$ .

- Set  $\mathbf{c} = \{c_k \mid k \in \mathbb{Z}\} \in \ell^1(\mathbb{Z})$ .
- It suffices to show that if  $\sum_{k \in \mathbb{Z}} |c_k| < \infty$  then  $\sum_{k \in \mathbb{Z}} |c_k|^2 < \infty$ .
- Let  $S = \{c_k \mid |c_k| \leq 1\}$ , then  $c_k \in S \implies |c_k|^2 \leq |c_k|$
- Claim:  $S^c$  can only contain finitely many elements, all of which are finite.
  - If not, either  $S^c := \{c_j\}_{j=1}^\infty$  is infinite with every  $|c_j| > 1$ , which forces

$$\sum_{c_k \in S^c} |c_k| = \sum_{j=1}^\infty |c_j| > \sum_{j=1}^\infty 1 = \infty.$$

- If any  $c_j = \infty$ , then  $\sum_{k \in \mathbb{Z}} |c_k| \geq c_j = \infty$ .

- So  $S^c$  is a finite set of finite integers, let  $N = \max\{|c_j|^2 \mid c_j \in S^c\} < \infty$ .
- Rewrite the sum

$$\begin{aligned} \sum_{k \in \mathbb{Z}} |c_k|^2 &= \sum_{c_k \in S} |c_k|^2 + \sum_{c_k \in S^c} |c_k|^2 \\ &\leq \sum_{c_k \in S} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \\ &\leq \sum_{k \in \mathbb{Z}} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \quad \text{since the } |c_k| \text{ are all positive} \\ &= \|\mathbf{c}\|_{\ell^1} + \sum_{c_k \in S^c} |c_k|^2 \\ &\leq \|\mathbf{c}\|_{\ell^1} + |S^c| \cdot N \\ &< \infty. \end{aligned}$$

Claim:  $L^2([0, 1]) \subseteq L^1([0, 1])$ .

- It suffices to show that  $\int |f|^2 < \infty \implies \int |f| < \infty$ .
- Define  $S = \{x \in [0, 1] \mid |f(x)| \leq 1\}$ , then  $x \in S^c \implies |f(x)|^2 \geq |f(x)|$ .
- Break up the integral:

$$\begin{aligned} \int_{\mathbb{R}} |f| &= \int_S |f| + \int_{S^c} |f| \\ &\leq \int_S |f| + \int_{S^c} |f|^2 \\ &\leq \int_S |f| + \|f\|_2 \\ &\leq \sup_{x \in S} \{|f(x)|\} \cdot \mu(S) + \|f\|_2 \\ &= 1 \cdot \mu(S) + \|f\|_2 \quad \text{by definition of } S \\ &\leq 1 \cdot \mu([0, 1]) + \|f\|_2 \quad \text{since } S \subseteq [0, 1] \\ &= 1 + \|f\|_2 \\ &< \infty. \end{aligned}$$

Note: this proof shows  $L^2(X) \subseteq L^1(X)$  whenever  $\mu(X) < \infty$ .

## 9.2 Fall 2017 # 5

Let  $\varphi$  be a compactly supported smooth function that vanishes outside of an interval  $[-N, N]$  such that  $\int_{\mathbb{R}} \varphi(x) dx = 1$ .

For  $f \in L^1(\mathbb{R})$ , define

$$K_j(x) := j\varphi(jx), \quad f * K_j(x) := \int_{\mathbb{R}} f(x-y)K_j(y) dy$$

and prove the following:

1. Each  $f * K_j$  is smooth and compactly supported.
- 2.

$$\lim_{j \rightarrow \infty} \|f * K_j - f\|_1 = 0$$

Hint:

$$\lim_{y \rightarrow 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dy = 0$$

## 9.3 Spring 2017 # 5

Let  $f, g \in L^2(\mathbb{R})$ . Prove that the formula

$$h(x) := \int_{-\infty}^{\infty} f(t)g(x-t) dt$$

defines a uniformly continuous function  $h$  on  $\mathbb{R}$ .

## 9.4 Spring 2015 # 6

Let  $f \in L^1(\mathbb{R})$  and  $g$  be a bounded measurable function on  $\mathbb{R}$ .

1. Show that the convolution  $f * g$  is well-defined, bounded, and uniformly continuous on  $\mathbb{R}$ .
2. Prove that one further assumes that  $g \in C^1(\mathbb{R})$  with bounded derivative, then  $f * g \in C^1(\mathbb{R})$  and

$$\frac{d}{dx}(f * g) = f * \left( \frac{d}{dx}g \right)$$

**9.5 Fall 2014 # 5**

1. Let  $f \in C_c^0(\mathbb{R}^n)$ , and show

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^n} |f(x+t) - f(x)| dx = 0.$$

2. Extend the above result to  $f \in L^1(\mathbb{R}^n)$  and show that

$$f \in L^1(\mathbb{R}^n), \quad g \in L^\infty(\mathbb{R}^n) \quad \implies \quad f * g \text{ is bounded and uniformly continuous.}$$

**10 Functional Analysis: General****10.1 Fall 2019 # 4.**

Let  $\{u_n\}_{n=1}^\infty$  be an orthonormal sequence in a Hilbert space  $\mathcal{H}$ .

**10.1.1 a**

Prove that for every  $x \in \mathcal{H}$  one has

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

**10.1.2 b**

Prove that for any sequence  $\{a_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$  there exists an element  $x \in \mathcal{H}$  such that

$$a_n = \langle x, u_n \rangle \text{ for all } n \in \mathbb{N}$$

and

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$$

*Solution.*

Concepts used:

- Bessel's Inequality
- Pythagoras
- Surjectivity of the Riesz map
- Parseval's Identity
- Trick – remember to write out finite sum  $S_N$ , and consider  $\|x - S_N\|$ .

**10.1.3 a****Claim:**

$$\begin{aligned}
0 \leq \left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\
&\implies \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2.
\end{aligned}$$

*Proof:* Let  $S_N = \sum_{n=1}^N \langle x, u_n \rangle u_n$ . Then

$$\begin{aligned}
0 &\leq \|x - S_N\|^2 \\
&= \langle x - S_N, x - S_N \rangle \\
&= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\
&\xrightarrow{N \rightarrow \infty} \|x\|^2 - \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2.
\end{aligned}$$

**10.1.4 b**

1. Fix  $\{a_n\} \in \ell^2$ , then note that  $\sum |a_n|^2 < \infty \implies$  the tails vanish.
2. Define

$$x := \lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \sum_{k=1}^N a_k u_k$$

3.  $\{S_N\}$  Cauchy (by 1) and  $H$  complete  $\implies x \in H$ .
- 4.

$$\langle x, u_n \rangle = \left\langle \sum_k a_k u_k, u_n \right\rangle = \sum_k a_k \langle u_k, u_n \rangle = a_n \quad \forall n \in \mathbb{N}$$

since the  $u_k$  are all orthogonal.

- 5.

$$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$$

by Pythagoras since the  $u_k$  are normal.

Bonus: We didn't use completeness here, so the Fourier series may not actually converge to  $x$ . If  $\{u_n\}$  is **complete** (so  $x = 0 \iff \langle x, u_n \rangle = 0 \forall n$ ) then the Fourier series *does* converge to  $x$  and  $\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 = \|x\|^2$  for all  $x \in H$ .

**10.2 Spring 2019 # 5****10.2.1 a**

Show that  $L^2([0, 1]) \subseteq L^1([0, 1])$  and argue that  $L^2([0, 1])$  in fact forms a dense subset of  $L^1([0, 1])$ .

**10.2.2 b**

Let  $\Lambda$  be a continuous linear functional on  $L^1([0, 1])$ .

Prove the Riesz Representation Theorem for  $L^1([0, 1])$  by following the steps below:

- i. Establish the existence of a function  $g \in L^2([0, 1])$  which represents  $\Lambda$  in the sense that

$$\Lambda(f) = \int_0^1 f(x)g(x)dx \text{ for all } f \in L^2([0, 1]).$$

Hint: You may use, without proof, the Riesz Representation Theorem for  $L^2([0, 1])$ .

- ii. Argue that the  $g$  obtained above must in fact belong to  $L^\infty([0, 1])$  and represent  $\Lambda$  in the sense that

$$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}dx \quad \text{for all } f \in L^1([0, 1])$$

with

$$\|g\|_{L^\infty([0,1])} = \|\Lambda\|_{L^1([0,1])^\vee}$$

**10.3 Spring 2016 # 6**

Without using the Riesz Representation Theorem, compute

$$\sup \left\{ \left| \int_0^1 f(x)e^x dx \right| \mid f \in L^2([0, 1], m), \|f\|_2 \leq 1 \right\}$$

**10.4 Spring 2015 # 5**

Let  $\mathcal{H}$  be a Hilbert space.

1. Let  $x \in \mathcal{H}$  and  $\{u_n\}_{n=1}^N$  be an orthonormal set. Prove that the best approximation to  $x$  in  $\mathcal{H}$  by an element in  $\text{span}_{\mathbb{C}} \{u_n\}$  is given by

$$\hat{x} := \sum_{n=1}^N \langle x, u_n \rangle u_n.$$

2. Conclude that finite dimensional subspaces of  $\mathcal{H}$  are always closed.

**10.5 Fall 2015 # 6**

Let  $f : [0, 1] \rightarrow \mathbb{R}$  be continuous. Show that

$$\sup \left\{ \|fg\|_1 \mid g \in L^1[0, 1], \|g\|_1 \leq 1 \right\} = \|f\|_\infty$$

**10.6 Fall 2014 # 6**

Let  $1 \leq p, q \leq \infty$  be conjugate exponents, and show that

$$f \in L^p(\mathbb{R}^n) \implies \|f\|_p = \sup_{\|g\|_q=1} \left| \int f(x)g(x)dx \right|$$

**11 Functional Analysis: Banach Spaces****11.1 Spring 2019 # 1**

Let  $C([0, 1])$  denote the space of all continuous real-valued functions on  $[0, 1]$ .

- Prove that  $C([0, 1])$  is complete under the uniform norm  $\|f\|_u := \sup_{x \in [0, 1]} |f(x)|$ .
- Prove that  $C([0, 1])$  is not complete under the  $L^1$ -norm  $\|f\|_1 = \int_0^1 |f(x)| dx$ .

**11.2 Spring 2017 # 5**

Show that the space  $C^1([a, b])$  is a Banach space when equipped with the norm

$$\|f\| := \sup_{x \in [a, b]} |f(x)| + \sup_{x \in [a, b]} |f'(x)|.$$

**11.3 Fall 2017 # 6**

Let  $X$  be a complete metric space and define a norm

$$\|f\| := \max\{|f(x)| : x \in X\}.$$

Show that  $(C^0(\mathbb{R}), \|\cdot\|)$  (the space of continuous functions  $f : X \rightarrow \mathbb{R}$ ) is complete.