

# Real Analysis Qualifying Exam Questions

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## 1 Undergraduate Analysis: Uniform Convergence

### 1.1 Fall 2018 # 1

Let  $f(x) = \frac{1}{x}$ . Show that  $f$  is uniformly continuous on  $(1, \infty)$  but not on  $(0, \infty)$ .

### 1.2 Fall 2017 # 1

Let

$$f(x) = s \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which  $f$  does and does not converge uniformly.

### 1.3 Fall 2014 # 1

Let  $\{f_n\}$  be a sequence of continuous functions such that  $\sum f_n$  converges uniformly.

Prove that  $\sum f_n$  is also continuous.

### 1.4 Spring 2017 # 4

Let  $f(x, y)$  on  $[-1, 1]^2$  be defined by

$$f(x, y) = \begin{cases} \frac{xy}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Determine if  $f$  is integrable.

**1.5 Spring 2015 # 1**

Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces,  $f : X \rightarrow Y$ , and  $x_0 \in X$ .

Prove that the following statements are equivalent:

1. For every  $\varepsilon > 0$   $\exists \delta > 0$  such that  $\rho(f(x), f(x_0)) < \varepsilon$  whenever  $d(x, x_0) < \delta$ .
2. The sequence  $\{f(x_n)\}_{n=1}^{\infty} \rightarrow f(x_0)$  for every sequence  $\{x_n\} \rightarrow x_0$  in  $X$ .

**1.6 Fall 2014 # 2**

Let  $I$  be an index set and  $\alpha : I \rightarrow (0, \infty)$ .

1. Show that

$$\sum_{i \in I} a(i) := \sup_{\substack{J \subset I \\ J \text{ finite}}} \sum_{i \in J} a(i) < \infty \implies I \text{ is countable.}$$

2. Suppose  $I = \mathbb{Q}$  and  $\sum_{q \in \mathbb{Q}} a(q) < \infty$ . Define

$$f(x) := \sum_{\substack{q \in \mathbb{Q} \\ q \leq x}} a(q).$$

Show that  $f$  is continuous at  $x \iff x \notin \mathbb{Q}$ .

**1.7 Spring 2014 # 2**

Let  $\{a_n\}$  be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that  $\sum a_n^2 < \infty$ .

Note: Assume  $a_n, b_n$  are all non-negative.

**2 General Analysis****2.1 Spring 2020 # 1**

Prove that if  $f : [0, 1] \rightarrow \mathbb{R}$  is continuous then

$$\lim_{k \rightarrow \infty} \int_0^1 kx^{k-1} f(x) dx = f(1).$$

**2.2 Fall 2019 # 1.**

Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of real numbers.

- a. Prove that if  $\lim_{n \rightarrow \infty} a_n = 0$ , then

$$\lim_{n \rightarrow \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

- b. Prove that if  $\sum_{n=1}^{\infty} \frac{a_n}{n}$  converges, then

$$\lim_{n \rightarrow \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

### 2.3 Fall 2018 # 4

Let  $f \in L^1([0, 1])$ . Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) |\sin nx| \, dx = \frac{2}{\pi} \int_0^1 f(x) \, dx$$

Hint: Begin with the case that  $f$  is the characteristic function of an interval.

### 2.4 Fall 2017 # 4

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

1. Show that  $f_n \rightarrow 0$  pointwise but not uniformly on  $[0, 1]$ .

Hint: Consider the maximum of  $f_n$ .

- 2.

$$\lim_{n \rightarrow \infty} \int_0^1 n(1-x)^n \sin x \, dx = 0$$

### 2.5 Spring 2017 # 3

Let

$$f_n(x) = ae^{-nax} - be^{-nbx} \quad \text{where } 0 < a < b.$$

Show that

- a.  $\sum_{n=1}^{\infty} |f_n|$  is not in  $L^1([0, \infty), m)$

Hint:  $f_n(x)$  has a root  $x_n$ .

- b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0, \infty), m) \quad \text{and} \quad \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x) \, dm = \ln \frac{b}{a}$$

**2.6 Fall 2016 # 1**

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that  $f$  converges to a differentiable function on  $(1, \infty)$  and that

$$f'(x) = \sum_{n=1}^{\infty} \left( \frac{1}{n^x} \right)'.$$

Hint:

$$\left( \frac{1}{n^x} \right)' = -\frac{1}{n^x} \ln n$$

**2.7 Fall 2016 # 5**

Let  $\varphi \in L^\infty(\mathbb{R})$ . Show that the following limit exists and satisfies the equality

$$\lim_{n \rightarrow \infty} \left( \int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2} dx \right)^{\frac{1}{n}} = \|\varphi\|_\infty.$$

**2.8 Fall 2016 # 6**

Let  $f, g \in L^2(\mathbb{R})$ . Show that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x)g(x+n) dx = 0$$

**2.9 Spring 2016 # 1**

For  $n \in \mathbb{N}$ , define

$$e_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad E_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

Show that  $e_n < E_n$ , and prove Bernoulli's inequality:

$$(1+x)^n \geq 1+nx \quad \text{for } -1 < x < \infty \text{ and } n \in \mathbb{N}$$

Use this to show the following:

1. The sequence  $e_n$  is increasing.
2. The sequence  $E_n$  is decreasing.
3.  $2 < e_n < E_n < 4$ .
4.  $\lim_{n \rightarrow \infty} e_n = \lim_{n \rightarrow \infty} E_n$ .

**2.10 Fall 2015 # 1**

Define

$$f(x) = c_0 + c_1x^1 + c_2x^2 + \dots + c_nx^n \text{ with } n \text{ even and } c_n > 0.$$

Show that there is a number  $x_m$  such that  $f(x_m) \leq f(x)$  for all  $x \in \mathbb{R}$ .

**3 Measure Theory: Sets****3.1 Spring 2020 # 2**

Let  $m_*$  denote the Lebesgue outer measure on  $\mathbb{R}$ .

- a. Prove that for every  $E \subseteq \mathbb{R}$  there exists a Borel set  $B$  containing  $E$  such that

$$m_*(B) = m_*(E).$$

- b. Prove that if  $E \subseteq \mathbb{R}$  has the property that

$$m_*(A) = m_*(A \cap E) + m_*(A \cap E^c)$$

for every set  $A \subseteq \mathbb{R}$ , then there exists a Borel set  $B \subseteq \mathbb{R}$  such that  $E = B \setminus N$  with  $m_*(N) = 0$ .

Be sure to address the case when  $m_*(E) = \infty$ .

**3.2 Fall 2019 # 3.**

Let  $(X, \mathcal{B}, \mu)$  be a measure space with  $\mu(X) = 1$  and  $\{B_n\}_{n=1}^\infty$  be a sequence of  $\mathcal{B}$ -measurable subsets of  $X$ , and

$$B := \left\{ x \in X \mid x \in B_n \text{ for infinitely many } n \right\}.$$

- a. Argue that  $B$  is also a  $\mathcal{B}$ -measurable subset of  $X$ .
- b. Prove that if  $\sum_{n=1}^\infty \mu(B_n) < \infty$  then  $\mu(B) = 0$ .
- c. Prove that if  $\sum_{n=1}^\infty \mu(B_n) = \infty$  **and** the sequence of set complements  $\{B_n^c\}_{n=1}^\infty$  satisfies

$$\mu\left(\bigcap_{n=k}^K B_n^c\right) = \prod_{n=k}^K (1 - \mu(B_n))$$

for all positive integers  $k$  and  $K$  with  $k < K$ , then  $\mu(B) = 1$ .

Hint: Use the fact that  $1 - x \leq e^{-x}$  for all  $x$ .

**3.3 Spring 2019 # 2**

Let  $\mathcal{B}$  denote the set of all Borel subsets of  $\mathbb{R}$  and  $\mu : \mathcal{B} \rightarrow [0, \infty)$  denote a finite Borel measure on  $\mathbb{R}$ .

- a. Prove that if  $\{F_k\}$  is a sequence of Borel sets for which  $F_k \supseteq F_{k+1}$  for all  $k$ , then

$$\lim_{k \rightarrow \infty} \mu(F_k) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right)$$

- b. Suppose  $\mu$  has the property that  $\mu(E) = 0$  for every  $E \in \mathcal{B}$  with Lebesgue measure  $m(E) = 0$ . Prove that for every  $\varepsilon > 0$  there exists  $\delta > 0$  so that if  $E \in \mathcal{B}$  with  $m(E) < \delta$ , then  $\mu(E) < \varepsilon$ .

**3.4 Fall 2018 # 2**

Let  $E \subset \mathbb{R}$  be a Lebesgue measurable set. Show that there is a Borel set  $B \subset E$  such that  $m(E \setminus B) = 0$ .

**3.5 Spring 2018 # 1**

Define

$$E := \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N} \right\}.$$

Prove that  $m(E) = 0$ .

**3.6 Fall 2017 # 2**

Let  $f(x) = x^2$  and  $E \subset [0, \infty) := \mathbb{R}^+$ .

1. Show that

$$m^*(E) = 0 \iff m^*(f(E)) = 0.$$

2. Deduce that the map

$$\begin{aligned} \varphi : \mathcal{L}(\mathbb{R}^+) &\longrightarrow \mathcal{L}(\mathbb{R}^+) \\ E &\mapsto f(E) \end{aligned}$$

is a bijection from the class of Lebesgue measurable sets of  $[0, \infty)$  to itself.

**3.7 Spring 2017 # 2**

- a. Let  $\mu$  be a measure on a measurable space  $(X, \mathcal{M})$  and  $f$  a positive measurable function.

Define a measure  $\lambda$  by

$$\lambda(E) := \int_E f \, d\mu, \quad E \in \mathcal{M}$$



Show that for  $g$  any positive measurable function,

$$\int_X g \, d\lambda = \int_X fg \, d\mu$$

b. Let  $E \subset \mathbb{R}$  be a measurable set such that

$$\int_E x^2 \, dm = 0.$$

Show that  $m(E) = 0$ .

### 3.8 Fall 2016 # 4

Let  $(X, \mathcal{M}, \mu)$  be a measure space and suppose  $\{E_n\} \subset \mathcal{M}$  satisfies

$$\lim_{n \rightarrow \infty} \mu(X \setminus E_n) = 0.$$

Define

$$G := \left\{ x \in X \mid x \in E_n \text{ for only finitely many } n \right\}.$$

Show that  $G \in \mathcal{M}$  and  $\mu(G) = 0$ .

### 3.9 Spring 2016 # 3

Let  $f$  be Lebesgue measurable on  $\mathbb{R}$  and  $E \subset \mathbb{R}$  be measurable such that

$$0 < A = \int_E f(x) dx < \infty.$$

Show that for every  $0 < t < 1$ , there exists a measurable set  $E_t \subset E$  such that

$$\int_{E_t} f(x) dx = tA.$$

### 3.10 Spring 2016 # 5

Let  $(X, \mathcal{M}, \mu)$  be a measure space. For  $f \in L^1(\mu)$  and  $\lambda > 0$ , define

$$\varphi(\lambda) = \mu(\{x \in X \mid f(x) > \lambda\}) \quad \text{and} \quad \psi(\lambda) = \mu(\{x \in X \mid f(x) < -\lambda\})$$

Show that  $\varphi, \psi$  are Borel measurable and

$$\int_X |f| \, d\mu = \int_0^\infty [\varphi(\lambda) + \psi(\lambda)] \, d\lambda$$

### 3.11 Fall 2015 # 2

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be Lebesgue measurable.

1. Show that there is a sequence of simple functions  $s_n(x)$  such that  $s_n(x) \rightarrow f(x)$  for all  $x \in \mathbb{R}$ .
2. Show that there is a Borel measurable function  $g$  such that  $g = f$  almost everywhere.

**3.12 Spring 2015 # 3**

Let  $\mu$  be a finite Borel measure on  $\mathbb{R}$  and  $E \subset \mathbb{R}$  Borel. Prove that the following statements are equivalent:

1.  $\forall \varepsilon > 0$  there exists  $G$  open and  $F$  closed such that

$$F \subseteq E \subseteq G \quad \text{and} \quad \mu(G \setminus F) < \varepsilon.$$

2. There exists a  $V \in G_\delta$  and  $H \in F_\sigma$  such that

$$H \subseteq E \subseteq V \quad \text{and} \quad \mu(V \setminus H) = 0$$

**3.13 Spring 2014 # 3**

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and suppose

$$\forall x \in \mathbb{R}, \quad f(x) \geq \limsup_{y \rightarrow x} f(y)$$

Prove that  $f$  is Borel measurable.

**3.14 Spring 2014 # 4**

Let  $(X, \mathcal{M}, \mu)$  be a measure space and suppose  $f$  is a measurable function on  $X$ . Show that

$$\lim_{n \rightarrow \infty} \int_X f^n d\mu = \begin{cases} \infty \\ \mu(f^{-1}(1)) \end{cases} \quad \text{or}$$

and characterize the collection of functions of each type.

**3.15 Spring 2017 # 1**

Let  $K$  be the set of numbers in  $[0, 1]$  whose decimal expansions do not use the digit 4.

We use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 with  $399\cdots$ . For example,  $0.8754 = 0.8753999\cdots$ .

Show that  $K$  is a compact, nowhere dense set without isolated points, and find the Lebesgue measure  $m(K)$ .

**3.16 Spring 2016 # 2**

Let  $0 < \lambda < 1$  and construct a Cantor set  $C_\lambda$  by successively removing middle intervals of length  $\lambda$ .

Prove that  $m(C_\lambda) = 0$ .

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## 4 Measure Theory: Functions

### 4.1 Fall 2016 # 2

Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be measurable with

$$\int_a^b f(x) \, dx = \int_a^b g(x) \, dx.$$

Show that either

1.  $f(x) = g(x)$  almost everywhere, or
2. There exists a measurable set  $E \subset [a, b]$  such that

$$\int_E f(x) \, dx > \int_E g(x) \, dx$$

### 4.2 Spring 2016 # 4

Let  $E \subset \mathbb{R}$  be measurable with  $m(E) < \infty$ . Define

$$f(x) = m(E \cap (E + x)).$$

Show that

1.  $f \in L^1(\mathbb{R})$ .
2.  $f$  is uniformly continuous.
3.  $\lim_{|x| \rightarrow \infty} f(x) = 0$ .

Hint:

$$\chi_{E \cap (E+x)}(y) = \chi_E(y) \chi_E(y-x)$$

## 5 Integrals: Convergence

### 5.1 Fall 2019 # 2.

Prove that

$$\left| \frac{d^n}{dx^n} \frac{\sin x}{x} \right| \leq \frac{1}{n}$$

for all  $x \neq 0$  and positive integers  $n$ .

Hint: Consider  $\int_0^1 \cos(tx) \, dt$

### 5.2 Spring 2020 # 5

Compute the following limit and justify your calculations:

$$\lim_{n \rightarrow \infty} \int_0^n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx.$$

**5.3 Spring 2019 # 3**

Let  $\{f_k\}$  be any sequence of functions in  $L^2([0, 1])$  satisfying  $\|f_k\|_2 \leq M$  for all  $k \in \mathbb{N}$ .

Prove that if  $f_k \rightarrow f$  almost everywhere, then  $f \in L^2([0, 1])$  with  $\|f\|_2 \leq M$  and

$$\lim_{k \rightarrow \infty} \int_0^1 f_k(x) dx = \int_0^1 f(x) dx$$

Hint: Try using Fatou's Lemma to show that  $\|f\|_2 \leq M$  and then try applying Egorov's Theorem.

**5.4 Fall 2018 # 6**

Compute the following limit and justify your calculations:

$$\lim_{n \rightarrow \infty} \int_1^n \frac{dx}{\left(1 + \frac{x}{n}\right)^n \sqrt[n]{x}}$$

**5.5 Fall 2018 # 3**

Suppose  $f(x)$  and  $xf(x)$  are integrable on  $\mathbb{R}$ . Define  $F$  by

$$F(t) := \int_{-\infty}^{\infty} f(x) \cos(xt) dx$$

Show that

$$F'(t) = - \int_{-\infty}^{\infty} xf(x) \sin(xt) dx.$$

**5.6 Spring 2018 # 5**

Suppose that

- $f_n, f \in L^1$ ,
- $f_n \rightarrow f$  almost everywhere, and
- $\int |f_n| \rightarrow \int |f|$ .

Show that  $\int f_n \rightarrow \int f$

**5.7 Spring 2018 # 2**

Let

$$f_n(x) := \frac{x}{1 + x^n}, \quad x \geq 0.$$

- a. Show that this sequence converges pointwise and find its limit. Is the convergence uniform on  $[0, \infty)$ ?
- b. Compute

$$\lim_{n \rightarrow \infty} \int_0^{\infty} f_n(x) dx$$

**5.8 Fall 2016 # 3**

Let  $f \in L^1(\mathbb{R})$ . Show that

$$\lim_{x \rightarrow 0} \int_{\mathbb{R}} |f(y-x) - f(y)| dy = 0$$

**5.9 Fall 2015 # 3**

Compute the following limit:

$$\lim_{n \rightarrow \infty} \int_1^n \frac{ne^{-x}}{1+nx^2} \sin\left(\frac{x}{n}\right) dx$$

**5.10 Fall 2015 # 4**

Let  $f : [1, \infty) \rightarrow \mathbb{R}$  such that  $f(1) = 1$  and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Show that the following limit exists and satisfies the equality

$$\lim_{x \rightarrow \infty} f(x) \leq 1 + \frac{\pi}{4}$$

**6 Integrals: Approximation****6.1 Spring 2018 # 3**

Let  $f$  be a non-negative measurable function on  $[0, 1]$ .

Show that

$$\lim_{p \rightarrow \infty} \left( \int_{[0,1]} f(x)^p dx \right)^{\frac{1}{p}} = \|f\|_{\infty}.$$

**6.2 Spring 2018 # 4**

Let  $f \in L^2([0, 1])$  and suppose

$$\int_{[0,1]} f(x)x^n dx = 0 \text{ for all integers } n \geq 0.$$

Show that  $f = 0$  almost everywhere.

**6.3 Spring 2015 # 2**

Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be continuous with period 1. Prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(n\alpha) = \int_0^1 f(t) dt \quad \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

Hint: show this first for the functions  $f(t) = e^{2\pi i k t}$  for  $k \in \mathbb{Z}$ .

**6.4 Fall 2014 # 4**

Let  $g \in L^\infty([0, 1])$  Prove that

$$\int_{[0,1]} f(x)g(x) dx = 0 \quad \text{for all continuous } f : [0, 1] \longrightarrow \mathbb{R} \implies g(x) = 0 \text{ almost everywhere.}$$

**7  $L^1$** **7.1 Spring 2020 # 3**

a. Prove that if  $g \in L^1(\mathbb{R})$  then

$$\lim_{N \rightarrow \infty} \int_{|x| \geq N} |f(x)| dx = 0,$$

and demonstrate that it is not necessarily the case that  $f(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ .

b. Prove that if  $f \in L^1([1, \infty))$  and is decreasing, then  $\lim_{x \rightarrow \infty} f(x) = 0$  and in fact  $\lim_{x \rightarrow \infty} xf(x) = 0$ .

c. If  $f : [1, \infty) \rightarrow [0, \infty)$  is decreasing with  $\lim_{x \rightarrow \infty} xf(x) = 0$ , does this ensure that  $f \in L^1([1, \infty))$ ?

**7.2 Fall 2019 # 5.**

a. Show that if  $f$  is continuous with compact support on  $\mathbb{R}$ , then

$$\lim_{y \rightarrow 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dx = 0$$

b. Let  $f \in L^1(\mathbb{R})$  and for each  $h > 0$  let

$$\mathcal{A}_h f(x) := \frac{1}{2h} \int_{|y| \leq h} f(x-y) dy$$

c. Prove that  $\|\mathcal{A}_h f\|_1 \leq \|f\|_1$  for all  $h > 0$ .

ii. Prove that  $\mathcal{A}_h f \rightarrow f$  in  $L^1(\mathbb{R})$  as  $h \rightarrow 0^+$ .

**7.3 Fall 2017 # 3**

Let

$$S = \text{span}_{\mathbb{C}} \left\{ \chi_{(a,b)} \mid a, b \in \mathbb{R} \right\},$$

the complex linear span of characteristic functions of intervals of the form  $(a, b)$ .

Show that for every  $f \in L^1(\mathbb{R})$ , there exists a sequence of functions  $\{f_n\} \subset S$  such that

$$\lim_{n \rightarrow \infty} \|f_n - f\|_1 = 0$$

**7.4 Spring 2015 # 4**

Define

$$f(x, y) := \begin{cases} \frac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Carefully show that  $f \in L^1(\mathbb{R}^2)$ .

**7.5 Fall 2014 # 3**

Let  $f \in L^1(\mathbb{R})$ . Show that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } m(E) < \delta \implies \int_E |f(x)| dx < \varepsilon$$

**7.6 Spring 2014 # 1**

1. Give an example of a continuous  $f \in L^1(\mathbb{R})$  such that  $f(x) \not\rightarrow 0$  as  $|x| \rightarrow \infty$ .
2. Show that if  $f$  is *uniformly* continuous, then

$$\lim_{|x| \rightarrow \infty} f(x) = 0.$$

**8 Fubini-Tonelli****8.1 Spring 2020 # 4**

Let  $f, g \in L^1(\mathbb{R})$ . Argue that  $H(x, y) := f(y)g(x-y)$  defines a function in  $L^1(\mathbb{R}^2)$  and deduce from this fact that

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x-y) dy$$

defines a function in  $L^1(\mathbb{R})$  that satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

**8.2 Spring 2019 # 4**

Let  $f$  be a non-negative function on  $\mathbb{R}^n$  and  $\mathcal{A} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq f(x)\}$ .

Prove the validity of the following two statements:

- a.  $f$  is a Lebesgue measurable function on  $\mathbb{R}^n \iff \mathcal{A}$  is a Lebesgue measurable subset of  $\mathbb{R}^{n+1}$
- b. If  $f$  is a Lebesgue measurable function on  $\mathbb{R}^n$ , then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x) dx = \int_0^\infty m(\{x \in \mathbb{R}^n : f(x) \geq t\}) dt$$

**8.3 Fall 2018 # 5**

Let  $f \geq 0$  be a measurable function on  $\mathbb{R}$ . Show that

$$\int_{\mathbb{R}} f = \int_0^{\infty} m(\{x : f(x) > t\}) dt$$

**8.4 Fall 2015 # 5**

Let  $f, g \in L^1(\mathbb{R})$  be Borel measurable.

1. Show that
  - The function

$$F(x, y) := f(x - y)g(y)$$

is Borel measurable on  $\mathbb{R}^2$ , and

- For almost every  $y \in \mathbb{R}$ ,

$$F_y(x) := f(x - y)g(y)$$

is integrable with respect to  $y$ .

2. Show that  $f * g \in L^1(\mathbb{R})$  and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

**8.5 Spring 2014 # 5**

Let  $f, g \in L^1([0, 1])$  and for all  $x \in [0, 1]$  define

$$F(x) := \int_0^x f(y) dy \quad \text{and} \quad G(x) := \int_0^x g(y) dy.$$

Prove that

$$\int_0^1 F(x)g(x) dx = F(1)G(1) - \int_0^1 f(x)G(x) dx$$

**9  $L^2$  and Fourier Analysis****9.1 Spring 2020 # 6****9.1.1 a**

Show that

$$L^2([0, 1]) \subseteq L^1([0, 1]) \quad \text{and} \quad \ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z}).$$



**9.1.2 b**

For  $f \in L^1([0, 1])$  define

$$\widehat{f}(n) := \int_0^1 f(x) e^{-2\pi i n x} dx.$$

Prove that if  $f \in L^1([0, 1])$  and  $\{\widehat{f}(n)\} \in \ell^1(\mathbb{Z})$  then

$$S_N f(x) := \sum_{|n| \leq N} \widehat{f}(n) e^{2\pi i n x}.$$

converges uniformly on  $[0, 1]$  to a continuous function  $g$  such that  $g = f$  almost everywhere.

Hint: One approach is to argue that if  $f \in L^1([0, 1])$  with  $\{\widehat{f}(n)\} \in \ell^1(\mathbb{Z})$  then  $f \in L^2([0, 1])$ .

**9.2 Fall 2017 # 5**

Let  $\varphi$  be a compactly supported smooth function that vanishes outside of an interval  $[-N, N]$  such that  $\int_{\mathbb{R}} \varphi(x) dx = 1$ .

For  $f \in L^1(\mathbb{R})$ , define

$$K_j(x) := j\varphi(jx), \quad f * K_j(x) := \int_{\mathbb{R}} f(x-y) K_j(y) dy$$

and prove the following:

1. Each  $f * K_j$  is smooth and compactly supported.
- 2.

$$\lim_{j \rightarrow \infty} \|f * K_j - f\|_1 = 0$$

Hint:

$$\lim_{y \rightarrow 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dy = 0$$

**9.3 Spring 2017 # 5**

Let  $f, g \in L^2(\mathbb{R})$ . Prove that the formula

$$h(x) := \int_{-\infty}^{\infty} f(t) g(x-t) dt$$

defines a uniformly continuous function  $h$  on  $\mathbb{R}$ .

**9.4 Spring 2015 # 6**

Let  $f \in L^1(\mathbb{R})$  and  $g$  be a bounded measurable function on  $\mathbb{R}$ .

1. Show that the convolution  $f * g$  is well-defined, bounded, and uniformly continuous on  $\mathbb{R}$ .
2. Prove that one further assumes that  $g \in C^1(\mathbb{R})$  with bounded derivative, then  $f * g \in C^1(\mathbb{R})$  and

$$\frac{d}{dx}(f * g) = f * \left(\frac{d}{dx}g\right)$$

**9.5 Fall 2014 # 5**

1. Let  $f \in C_c^0(\mathbb{R}^n)$ , and show

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^n} |f(x+t) - f(x)| dx = 0.$$

2. Extend the above result to  $f \in L^1(\mathbb{R}^n)$  and show that

$$f \in L^1(\mathbb{R}^n), \quad g \in L^\infty(\mathbb{R}^n) \quad \implies \quad f * g \text{ is bounded and uniformly continuous.}$$

**10 Functional Analysis: General****10.1 Fall 2019 # 4.**

Let  $\{u_n\}_{n=1}^\infty$  be an orthonormal sequence in a Hilbert space  $\mathcal{H}$ .

- a. Prove that for every  $x \in \mathcal{H}$  one has

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

- b. Prove that for any sequence  $\{a_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$  there exists an element  $x \in \mathcal{H}$  such that

$$a_n = \langle x, u_n \rangle \text{ for all } n \in \mathbb{N}$$

and

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$$

**10.2 Spring 2019 # 5****10.2.1 a**

Show that  $L^2([0, 1]) \subseteq L^1([0, 1])$  and argue that  $L^2([0, 1])$  in fact forms a dense subset of  $L^1([0, 1])$ .

**10.2.2 b**

Let  $\Lambda$  be a continuous linear functional on  $L^1([0, 1])$ .

Prove the Riesz Representation Theorem for  $L^1([0, 1])$  by following the steps below:

- i. Establish the existence of a function  $g \in L^2([0, 1])$  which represents  $\Lambda$  in the sense that

$$\Lambda(f) = \int_0^1 f(x)g(x)dx \text{ for all } f \in L^2([0, 1]).$$

Hint: You may use, without proof, the Riesz Representation Theorem for  $L^2([0, 1])$ .

- ii. Argue that the  $g$  obtained above must in fact belong to  $L^\infty([0, 1])$  and represent  $\Lambda$  in the sense that

$$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}dx \quad \text{for all } f \in L^1([0, 1])$$

with

$$\|g\|_{L^\infty([0,1])} = \|\Lambda\|_{L^1([0,1])^\vee}$$

**10.3 Spring 2016 # 6**

Without using the Riesz Representation Theorem, compute

$$\sup \left\{ \left| \int_0^1 f(x)e^x dx \right| \mid f \in L^2([0, 1], m), \|f\|_2 \leq 1 \right\}$$

**10.4 Spring 2015 # 5**

Let  $\mathcal{H}$  be a Hilbert space.

1. Let  $x \in \mathcal{H}$  and  $\{u_n\}_{n=1}^N$  be an orthonormal set. Prove that the best approximation to  $x$  in  $\mathcal{H}$  by an element in  $\text{span}_{\mathbb{C}}\{u_n\}$  is given by

$$\hat{x} := \sum_{n=1}^N \langle x, u_n \rangle u_n.$$

2. Conclude that finite dimensional subspaces of  $\mathcal{H}$  are always closed.

**10.5 Fall 2015 # 6**

Let  $f : [0, 1] \rightarrow \mathbb{R}$  be continuous. Show that

$$\sup \left\{ \|fg\|_1 \mid g \in L^1[0, 1], \|g\|_1 \leq 1 \right\} = \|f\|_\infty$$

**10.6 Fall 2014 # 6**

Let  $1 \leq p, q \leq \infty$  be conjugate exponents, and show that

$$f \in L^p(\mathbb{R}^n) \implies \|f\|_p = \sup_{\|g\|_q=1} \left| \int f(x)g(x)dx \right|$$

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## 11 Functional Analysis: Banach Spaces

### 11.1 Spring 2019 # 1

Let  $C([0, 1])$  denote the space of all continuous real-valued functions on  $[0, 1]$ .

- Prove that  $C([0, 1])$  is complete under the uniform norm  $\|f\|_\infty := \sup_{x \in [0, 1]} |f(x)|$ .
- Prove that  $C([0, 1])$  is not complete under the  $L^1$ -norm  $\|f\|_1 = \int_0^1 |f(x)| \, dx$ .

### 11.2 Spring 2017 # 5

Show that the space  $C^1([a, b])$  is a Banach space when equipped with the norm

$$\|f\| := \sup_{x \in [a, b]} |f(x)| + \sup_{x \in [a, b]} |f'(x)|.$$

### 11.3 Fall 2017 # 6

Let  $X$  be a complete metric space and define a norm

$$\|f\| := \max\{|f(x)| : x \in X\}.$$

Show that  $(C^0(\mathbb{R}), \|\cdot\|)$  (the space of continuous functions  $f : X \rightarrow \mathbb{R}$ ) is complete.