

# Real Analysis Qualifying Exam Solutions

D. Zack Garza

Friday 19<sup>th</sup> June, 2020

## Contents

<b>1</b>	<b>Spring 2020</b>	<b>3</b>
1.1	1 . . . . .	3
1.2	2 . . . . .	4
1.2.1	a . . . . .	4
1.2.2	b . . . . .	5
1.3	3 . . . . .	5
1.3.1	a . . . . .	5
1.3.2	b . . . . .	6
1.3.3	c . . . . .	7
1.4	4 . . . . .	7
1.5	5 . . . . .	8
1.6	6 . . . . .	8
<b>2</b>	<b>Spring 2019</b>	<b>9</b>
2.1	1 . . . . .	9
2.1.1	a . . . . .	9
2.1.2	b . . . . .	9
2.2	2 . . . . .	9
2.2.1	a . . . . .	9
2.2.2	b . . . . .	10
2.3	3 . . . . .	10
2.4	4 . . . . .	11
2.4.1	a . . . . .	11
2.4.2	b . . . . .	11
2.5	5 . . . . .	12
2.5.1	a . . . . .	12
2.5.2	b . . . . .	12
<b>3</b>	<b>Fall 2019</b>	<b>14</b>
3.1	1 . . . . .	14
3.2	a . . . . .	14
3.3	b . . . . .	15
3.4	2 . . . . .	16

3.5	3	16
3.5.1	a	16
3.5.2	b	17
3.5.3	c	17
3.6	4	17
3.6.1	a	18
3.6.2	b	18
3.7	5	19
3.8	a	19
3.9	b	19
<b>4</b>	<b>Spring 2018</b>	<b>20</b>
4.1	1	20
4.2	2	21
4.2.1	a	21
4.2.2	b	21
4.3	3	22
4.4	4	22
4.5	5	23
<b>5</b>	<b>Fall 2018</b>	<b>24</b>
5.1	1	24
5.2	2	24
5.3	3	25
5.4	4	25
5.5	5	25
<b>6</b>	<b>Spring 2017</b>	<b>26</b>
6.1	1	26
6.2	2	27
6.2.1	a	27
6.2.2	b	27
6.3	3	28
6.3.1	a	28
6.3.2	b	28
6.4	4	28
6.5	5	28
<b>7</b>	<b>Fall 2017</b>	<b>29</b>
7.1	1	29
7.2	2	29
7.2.1	a	29
7.2.2	b	30
7.3	3	30
7.4	4	31
7.4.1	a	31
7.4.2	b	31

---

7.5	5	31
7.5.1	a	31
7.5.2	b	33
7.6	6	34
<b>8</b>	<b>Spring 2016</b>	<b>34</b>
8.1	1	34
<b>9</b>	<b>Fall 2016</b>	<b>34</b>
9.1	1	34
<b>10</b>	<b>Spring 2014</b>	<b>34</b>
10.1	1	34

## 1 Spring 2020

### 1.1 1

Concepts used:

- DCT
- Weierstrass Approximation Theorem

Suppose  $p$  is a polynomial, then

$$\begin{aligned}
 \lim_{k \rightarrow \infty} \int_0^1 k x^{k-1} p(x) dx &= \lim_{k \rightarrow \infty} \int_0^1 \left( \frac{\partial}{\partial x} x^k \right) p(x) dx \\
 &= \lim_{k \rightarrow \infty} \left[ x^k p(x) \Big|_0^1 - \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx \right] \quad \text{integrating by parts} \\
 &= p(1) - \lim_{k \rightarrow \infty} \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx,
 \end{aligned}$$

and thus it suffices to show that

$$\lim_{k \rightarrow \infty} \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx = 0.$$

Integrating by parts a second time yields

$$\begin{aligned}
 \lim_{k \rightarrow \infty} \int_0^1 x^k \left( \frac{\partial}{\partial x} p(x) \right) dx &= \lim_{k \rightarrow \infty} \frac{x^{k+1}}{k+1} p'(x) \Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1} \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx \\
 &= - \lim_{k \rightarrow \infty} \int_0^1 \frac{x^{k+1}}{k+1} \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx \\
 &= - \int_0^1 \lim_{k \rightarrow \infty} \frac{x^{k+1}}{k+1} \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx \quad \text{by DCT} \\
 &= - \int_0^1 0 \left( \frac{\partial^2}{\partial x^2} p(x) \right) dx \\
 &= 0.
 \end{aligned}$$

The DCT can be applied here because  $f''$  is continuous and  $[0, 1]$  is compact, so  $f''$  is bounded on  $[0, 1]$  by a constant  $M$  and  $\int_0^1 |x^k f''(x)| \leq \int_0^1 1 \cdot M = M < \infty$ .

We now use the Weierstrass approximation theorem: if  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then for every  $\varepsilon > 0$  there exists a polynomial  $p_\varepsilon(x)$  such that  $\|f - p_\varepsilon\|_\infty < \varepsilon$ . Thus

$$\begin{aligned} \left| \int_0^1 kx^{k-1} p_\varepsilon(x) dx - \int_0^1 kx^{k-1} f(x) dx \right| &= \left| \int_0^1 kx^{k-1} (p_\varepsilon(x) - f(x)) dx \right| \\ &\leq \left| \int_0^1 kx^{k-1} \|p_\varepsilon - f\|_\infty dx \right| \\ &= \|p_\varepsilon - f\|_\infty \cdot \left| \int_0^1 kx^{k-1} dx \right| \\ &= \|p_\varepsilon - f\|_\infty \cdot x^k \Big|_0^1 \\ &= \|p_\varepsilon - f\|_\infty \xrightarrow{\varepsilon \rightarrow 0} 0 \end{aligned}$$

and the integrals are equal. Finally, the first integral is equal to  $p_\varepsilon(1)$  for each  $\varepsilon$ , which converges to  $f(1)$  since uniform convergence implies pointwise convergence.

## 1.2 2

Concepts used:

- Definition of outer measure:  $m_*(E) = \inf_{\{Q_j\} \Rightarrow E} \sum |Q_j|$  where  $\{Q_j\}$  is a countable collection of closed cubes.
- Break  $\mathbb{R}$  into  $\coprod_{n \in \mathbb{Z}} [n, n+1)$ , each with finite measure.

### 1.2.1 a

Suppose  $m_*(E) = N < \infty$ .

Since  $m_*(E)$  is an infimum, by definition, for every  $\varepsilon > 0$  there exists a covering by closed cubes  $\{Q_i(\varepsilon)\}_{i=1}^\infty \Rightarrow E$  such that

$$\sum_{i=1}^\infty |Q_i(\varepsilon)| < N + \varepsilon.$$

Set  $\varepsilon_n = \frac{1}{n}$  to produce such a collection  $\{Q_i(\varepsilon_n)\}$  and set  $B_n := \bigcup_{i=1}^\infty Q_i(\varepsilon_n)$ . Then (theorem) the outer measure of cubes is *equal* to the sum of their volumes, so

$$m_*(B_n) = \sum_{i=1}^\infty |Q_i(\varepsilon_n)| < N + \varepsilon_n = N + \frac{1}{n}.$$

Now set  $B := \bigcap_{n=1}^\infty B_n$ .

- Since  $E \subseteq B_n$  for every  $n$ ,  $E \subseteq B$
- Since  $B$  is a countable intersection of countable unions of closed sets,  $B$  is Borel.
- Since  $B_n \subseteq B$  for every  $n$ , we can apply subadditivity to obtain the inequality

$$E \subseteq B \subseteq B_n \implies N \leq m_*(B) \leq m_*(B_n) < N + \frac{1}{n} \quad \text{for all } n \in \mathbb{Z}^{\geq 1}.$$

This forces  $m_*(E) = m_*(B)$ .

If  $m_*(E) = \infty$ , then take  $B = \mathbb{R}^n$  since  $m(\mathbb{R}^n) = \infty$ .

### 1.2.2 b

Suppose  $m_*(E) < \infty$ .

- By (a), find a Borel set  $B \supseteq E$  such that  $m_*(B) = m_*(E)$
- Note that  $E \subseteq B \implies B \cap E = E$  and  $B \cap E^c = B \setminus E$ .
- By assumption,

$$\begin{aligned} m_*(B) &= m_*(B \cap E) + m_*(B \cap E^c) \\ m_*(E) &= m_*(E) + m_*(B \setminus E) \\ m_*(E) - m_*(E) &= m_*(B \setminus E) \quad \text{since } m_*(E) < \infty \\ \implies m_*(B \setminus E) &= 0. \end{aligned}$$

- So take  $N = B \setminus E$ ; this shows  $m_*(N) = 0$  and  $E = B \setminus (B \setminus E) = B \setminus N$ .
- If  $m_*(E) = \infty$ 
  - Apply result to  $E_R := E \cap [R, R+1)^n \subset \mathbb{R}^n$  for  $R \in \mathbb{Z}$ , so  $E = \coprod_R E_R$
  - Obtain  $B_R, N_R$  such that  $E_R = B_R \setminus N_R$ ,  $m_*(E_R) = m_*(B_R)$ , and  $m_*(N_R) = 0$ .
  - Note that
    - \*  $B := \bigcup_R B_R$ , which is a union of Borel sets and thus still Borel
    - \*  $E = \bigcup_R E_R$
    - \*  $N := B \setminus E$
    - \*  $N' := \bigcup_R N_R$  which is a union of null sets and thus still null
  - Since  $E_R \subset B_R$  for every  $R$ , we have  $E \subset B$
  - We can compute

$$N = B \setminus E = \left( \bigcup_R B_R \right) \setminus \left( \bigcup_R E_R \right) \subseteq \bigcup_R (B_R \setminus E_R) = N'$$

where  $m_*(N) = 0$  and thus subadditivity forces  $m_*(N) = 0$ .

## 1.3 3

### 1.3.1 a

Stated integral equality:

- Let  $\varepsilon > 0$
- $C_c(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$  is dense so choose  $\{f_n\} \rightrightarrows f$  so  $\|f_n - f\|_1 \rightarrow 0$ .
- Choose  $n \gg 1$  so that  $\|f_n - f\| < \varepsilon$ . Fix this  $n$ .
- Since  $\{f_n\}$  are compactly supported, choose  $N_0 \gg 1$  such that  $f_n$  is zero outside of  $B_N(\mathbf{0})$ .
- Then

$$N \geq N_0 \implies \int_{|x|>N} |f| = \int_{|x|>N} |f - f_n + f_n| \leq \int_{|x|>N} |f - f_n| + \int_{|x|>N} |f_n| = \varepsilon + 0.$$

- Now take  $n \rightarrow \infty$ .

To see that this doesn't force  $f(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ :

- Take  $f(x)$  to be a train of boxes of height 1 and area  $1/2^j$  centered on even integers.
- Then the  $\int_{|x|>N} |f| = \sum_{j=N}^{\infty} 1/2^j$  which tends to zero as  $N \rightarrow \infty$ .
- However  $f(x) = 1$  for any even integer  $x > N$ , so  $f(x) \not\rightarrow 0$ .

### 1.3.2 b

- Since  $f$  is decreasing on  $[1, \infty)$ , for any  $t \in [x - n, x]$  we have

$$x - n \leq t \leq x \implies f(x) \leq f(t) \leq f(x - n).$$

- Integrate over  $[x, 2x]$ , using monotonicity of the integral:

$$\int_x^{2x} f(x) dt \leq \int_x^{2x} f(t) dt \leq \int_x^{2x} f(x - n) dt \implies xf(x) \leq \int_x^{2x} f(t) dt \leq xf(x - n).$$

- By (a),  $\lim_{x \rightarrow \infty} \int_x^{2x} f(t) dt = 0$  (?)
- So the LHS term  $\lim_{x \rightarrow \infty} xf(x) = 0$ .
- Since  $x > 1$ ,  $|f(x)| \leq |xf(x)|$
- Thus  $f(x) \rightarrow 0$  as well.

Alternatively showing  $f(x) \xrightarrow{x \rightarrow \infty} 0$ :

- Toward a contradiction, suppose not.
- If  $f(x) \rightarrow -\infty$ , then  $f \notin L^1(\mathbb{R})$ : choose  $x \gg 1$  so that  $|f(x)| > 1$ , then

$$\int_{\mathbb{R}} |f| \geq \int_x^{\infty} f(t) dt \geq \int_x^{\infty} 1 = \infty.$$

- WLOG replace  $f$  with  $-f$  to make  $f$  increasing (since  $\|f\|_1 = \|f\|_2$ ).
- Otherwise  $f(x) \rightarrow L$  with  $L < \infty$ . Fix  $\varepsilon > 0$ .

- Choose  $x \gg 1$  so that  $t \geq x \implies L - \varepsilon \leq f(t) \leq L$
- Then  $\int_x^\infty f \geq \int_x^\infty (L - \varepsilon) = \infty$ .

### 1.3.3 c

- No: take  $f(x) = \frac{1}{x \ln x}$
- Then  $\int f = \ln(\ln(x)) \rightarrow \infty$  is unbounded, so  $f \notin L^1([1, \infty))$ .
- But  $xf(x) = 1/\ln(x) \rightarrow 0$

## 1.4 4

Relevant concepts:

- Tonelli: non-negative and measurable yields measurability of slices and equality of iterated integrals
- Fubini:  $f(x, y) \in L^1$  yields *integrable* slices and equality of iterated integrals
- F/T: apply Tonelli to  $|f|$ ; if finite,  $f \in L^1$  and apply Fubini to  $f$

$$\begin{aligned}\|(f * g)(x)\|_1 &= \left\| \int_{\mathbb{R}} H(x, y) dy \right\|_1 \\ &:= \left\| \int_{\mathbb{R}} f(y)g(x - y) dy \right\|_1 \\ &\leq \int_{\mathbb{R}} \|f(y)g(x - y)\|_1 dy \\ &\leq \int_{\mathbb{R}} |f(y)| \cdot \|g(x - y)\|_1 dy \\ &\leq \int_{\mathbb{R}} |f(y)| \cdot \|g\|_1 dy \\ &\leq \|g\|_1 \int_{\mathbb{R}} |f(y)| dy \\ &\leq \|g\|_1 \|f\|_1 \\ &< \infty \quad \text{by assumption} \quad .\end{aligned}$$

Todo:

- Change of variables for  $x - y$ ?
- How to justify F/T?

**1.5 5**

Note that

$$\begin{aligned}\lim_n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} &= \frac{1}{\lim_n \left(1 + \frac{x^2}{n}\right)^1 \left(1 + \frac{x^2}{n}\right)^n} \\ &= \frac{1}{1 \cdot e^{x^2}} \\ &= e^{-x^2}.\end{aligned}$$

If passing the limit through the integral is justified, we will have  $\int_0^\infty e^{-x^2} = \frac{\sqrt{\pi}}{2}$ .

Computing the integral:

$$\begin{aligned}\left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2 &= \left(\int_{\mathbb{R}} e^{-x^2} dx\right) \left(\int_{\mathbb{R}} e^{-y^2} dy\right) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(x+y)^2} dx dy \\ &= \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta \quad u = r^2 \\ &= \frac{1}{2} \int_0^{2\pi} \int_0^\infty e^{-u} du d\theta \\ &= \frac{1}{2} \int_0^{2\pi} 1 d\theta \\ &= \pi,\end{aligned}$$

and now use the fact that the function is even.

Todo:

- Justify, MCT? Possible have

$$f_n(x) := \frac{1}{\left(1 + \frac{x^2}{n}\right)^{n+1}} \nearrow e^{-x^2}.$$

**1.6 6**

Concepts used:

- For  $e_n(x) := 2^{2\pi i n x}$ , the set  $\{e_n\}$  is an orthonormal basis for  $L^2([0, 1])$ .
- For any orthonormal set in a Hilbert space, Bessel's inequality:

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

- When a basis, the above is an inequality (Parseval)
- Arguing uniform convergence: since  $\{\widehat{f}(n)\} \in \ell^1(\mathbb{Z})$ , we should be able to apply the  $M$  test.



---

Want to show

$$\int_0^1 |f(x)|^2 dx \leq \int_0^1 |f(x)| dx$$
$$\sum_{n=1}^{\infty} |c_n|^2 \leq \sum_{n=1}^{\infty} |c_n|.$$

- First inequality: ?
- Second inequality: break into 2 pieces?

## 2 Spring 2019

### 2.1 1

#### 2.1.1 a

Let  $\{f_k\}$  be a Cauchy sequence in  $C(I)$ . For each fixed  $x \in [0, 1]$ , the sequence of real numbers  $\{f_k(x)\}$  is Cauchy in  $\mathbb{R}$ , which is complete, since

$$x_0 \in I \implies |f_k(x_0) - f_j(x_0)| \leq \sup_{x \in I} |f_k(x) - f_j(x)| = \|f_k - f_j\|_{\infty} \longrightarrow 0,$$

so we can define  $f(x) := \lim_k f_k(x)$ .

We also have

$$\|f_k - f\|_{\infty} = \left\| f_k - \lim_{j \rightarrow \infty} f_j \right\|_{\infty} = \lim_{j \rightarrow \infty} \|f_k - f_j\|_{\infty} \longrightarrow 0.$$

Finally,  $f$  is the uniform limit of continuous functions and thus continuous.

■

#### 2.1.2 b

It suffices to produce a Cauchy sequence that does not converge to a continuous function. Take

$$f_k(x) = \begin{cases} (x + \frac{1}{2})^k & x \in [0, \frac{1}{2}) \\ 1 & x \in [\frac{1}{2}, 1] \end{cases} \xrightarrow{k \rightarrow \infty} f(x) = \begin{cases} 0 & x \in [0, \frac{1}{2}) \\ 1 & x \in [\frac{1}{2}, 1] \end{cases},$$

which is Cauchy, but there is no  $g \in L^1$  that is continuous such that  $\|f - g\|_1 = 0$ .

### 2.2 2

#### 2.2.1 a

Lemma 1:  $\mu(\coprod_{k=1}^{\infty} E_k) = \lim_{N \rightarrow \infty} \sum_{k=1}^N \mu(E_k)$ .

Lemma 2:  $A = A \setminus B \coprod A \cap B$ .

Let  $A_k = F_k \setminus F_{k+1}$ , so the  $A_k$  are disjoint, and let  $A = \coprod_k A_k$ .

Let  $F = \bigcap_k F_k$ . Then  $F_1 = F \coprod A$  by lemma 2, so

$$\begin{aligned}
 \mu(F_1) &= \mu(F) + \mu(A) \\
 &= \mu(F) + \lim_{N \rightarrow \infty} \sum_k^N \mu(A_k) \quad \text{by Lemma 1} \\
 &= \mu(F) + \lim_{N \rightarrow \infty} \sum_k^N \mu(F_k) - \mu(F_{k+1}) \\
 &= \mu(F) + \lim_{N \rightarrow \infty} (\mu(F_1) - \mu(F_N)) \quad (\text{Telescoping}) \\
 &= \mu(F) + \mu(F_1) - \lim_{N \rightarrow \infty} \mu(F_N),
 \end{aligned}$$

and since the measure is finite,  $\mu(F_1) < \infty$  and can be subtracted, yielding

$$\begin{aligned}
 \mu(F_1) &= \mu(F) + \mu(F_1) - \lim_{N \rightarrow \infty} \mu(F_N) \\
 \implies \mu(F) &= \lim_{N \rightarrow \infty} \mu(F_N).
 \end{aligned}$$

### 2.2.2 b

Suppose toward a contradiction that there is some  $\varepsilon > 0$  for which no such  $\delta$  exists.

This means that we can take any sequence  $\delta_n \rightarrow 0$  and produce sets  $A_n$  such  $m(A) < \delta_n$  but  $\mu(A) > \varepsilon$ .

So choose the sequence  $\delta_n = \frac{1}{2^n}$  and define  $A_n$  accordingly, and let

$$A = \limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

Since

$$\mu\left(\bigcup_{k=n}^{\infty} A_k\right) \leq \sum_{k=n}^{\infty} \mu(A_k) \approx \frac{1}{2^n} \rightarrow 0,$$

by part (a) we have  $m(A) = 0$ . Now by assumption, we should thus have  $\mu(A) = 0$  as well.

However, again by part (a), we have

$$\mu(A) = \lim_n \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \geq \lim_n \mu(A_n) = \lim_n \varepsilon = \varepsilon > 0.$$

### 2.3 3

Since  $f_k \rightarrow f$  almost everywhere, we have  $\liminf_k f_k(x) = f(x)$  and since  $|f|^2 \in L^+$  we can apply Fatou:

$$\begin{aligned}
\|f\|_2^2 &= \int |f(x)|^2 \\
&= \int \liminf_k |f_k(x)|^2 \\
&\leq \liminf_k \int |f_k(x)|^2 \\
&= M^2,
\end{aligned}$$

so  $\|f\| \leq M < \infty$  and  $f \in L^2$ .

Let  $I = [0, 1]$ . Applying Egorov's theorem to produce sets  $F_\varepsilon$  such that  $f_k \xrightarrow{u} f$  on  $F_\varepsilon$  and taking  $F = \bigcap F_\varepsilon$ , we have

$$\int_I f_k = \int_{F_\varepsilon} f_k + \int_{F_\varepsilon^c} f_k \xrightarrow{\varepsilon \rightarrow 0} \int_F f_k + 0 \xrightarrow{k \rightarrow \infty} \int_F f,$$

using that fact that uniform converges allows commuting limits and integrals.

## 2.4 4

### 2.4.1 a

$\Rightarrow :$

Idea:  $\mathcal{A} = \{f(x) - t \geq 0\} \cap \{t \geq 0\}$ .

Define  $F(x, t) = f(x)$ ,  $G(x, t) = t$ , and  $H(x, y) = F(x, t) - G(x, t)$ , which are all measurable functions.

Then  $\mathcal{A} = \{H \geq 0\} \cap \{G \geq 0\}$  which is an intersection of measurable sets.

$\Leftarrow :$

By F.T., for almost every  $x \in \mathbb{R}^n$ , the  $x$ -slices are measurable, so

$$\mathcal{A}_x := \{t \in \mathbb{R} \mid (x, t) \in \mathcal{A}\} = [0, f(x)] \implies m(\mathcal{A}_x) = f(x)$$

But  $x \mapsto m(\mathcal{A}_x)$  is a measurable function, and is exactly to  $x \mapsto f(x)$ , so  $f$  is measurable.

### 2.4.2 b

We first note

$$\begin{aligned}
\mathcal{A} &= \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid 0 \leq t \leq f(x)\} \\
\mathcal{A}_t &= \{x \in \mathbb{R}^n \mid t \leq f(x)\}.
\end{aligned}$$

Then,

$$\begin{aligned}
 \int_{\mathbb{R}^n} f(x) \, dx &= \int_{\mathbb{R}^n} \int_0^{f(x)} 1 \, dt \, dx \\
 &= \int_{\mathbb{R}^n} \int_{\mathbb{R}} \chi_{\mathcal{A}} \, dt \, dx \\
 &\stackrel{F.T.}{=} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_{\mathcal{A}} \, dx \, dt \\
 &= \int_0^\infty \int_{\mathbb{R}^n} \chi_{\mathcal{A}} \, dx \, dt \\
 &= \int_0^\infty m(\mathcal{A}_t) \, dt,
 \end{aligned}$$

where we just note that  $\int \chi_{\mathcal{A}} = m(\mathcal{A})$ , and by F.T., all of these integrals are equal.

## 2.5 5

### 2.5.1 a

By Holder's inequality with  $p = q = 2$ , we have

$$\|f\|_1 = \|f \cdot 1\|_1 \leq \|f\|_2 \|1\|_2 = \|f\|_2 m(X)^{\frac{1}{2}} = \|f\|_2,$$

since  $X = [0, 1] \implies m(X) = 1$ .

So  $L^2(X) \subseteq L^1(X)$ , and since simple functions are dense in both spaces,  $L^2$  is dense in  $L^1$ .

### 2.5.2 b

**Step 1** Let  $\Lambda \in L^1(X)^\vee$ ; we'll show that in fact  $\Lambda \in L^2(X)^\vee$ , and by Riesz Representation for  $L^2$  there will be a  $g \in L^2$  such that  $\Lambda(f) = \langle f, g \rangle$ .

**Lemma:**  $m(X) < \infty \implies L^p(X) \subset L^2(X)$ .

*Proof:* Write Holder's inequality as  $\|fg\|_1 \leq \|f\|_a \|g\|_b$  where  $\frac{1}{a} + \frac{1}{b} = 1$ , then

$$\|f\|_p^p = \| |f|^p \|_1 \leq \| |f|^p \|_a \|1\|_b.$$

Now take  $a = \frac{2}{p}$  and this reduces to

$$\begin{aligned}
 \|f\|_p^p &\leq \|f\|_2^p m(X)^{\frac{1}{b}} \\
 \implies \|f\|_p &\leq \|f\|_2 \cdot O(m(X)) < \infty.
 \end{aligned}$$

Let  $f \in L^2$  be arbitrary – by the lemma,  $\|f\|_1 \leq C\|f\|_2$  for some constant  $C = O(m(X))$ .

Since  $\|\Lambda\|_{1^\vee} := \sup_{\|f\|_1=1} |\Lambda(f)|$ , given an arbitrary  $f \in L^1$ , we can define  $\hat{f} = f/\|f\|_1$ , so  $\|\hat{f}\|_1 = 1$ , and obtain

$$|\Lambda(\hat{f})| \leq \|\Lambda\|_{1^\vee},$$

since  $\|\Lambda\|_{1^\vee}$  is the *least* such bound over all  $f \in L^1$ , and thus

$$\begin{aligned} \frac{|\Lambda(f)|}{\|f\|_1} &= |\Lambda(\hat{f})| \leq \|\Lambda\|_{1^\vee} \\ \implies |\Lambda(f)| &\leq \|\Lambda\|_{1^\vee} \cdot \|f\|_1 \\ &\leq \|\Lambda\|_{1^\vee} \cdot C\|f\|_2, \end{aligned}$$

which is finite by assumption. So  $\Lambda \in (L^2)^\vee$  since it is bounded and thus continuous.

By Riesz Representation for  $L^2$ , there is a  $g \in L^2$  such that for all  $f \in L^2$ ,  $\Lambda(f) = \langle f, g \rangle$

**Step 2** By Holder, we already have

$$\begin{aligned} \|\Lambda\|_{1^\vee} &= \sup_{\|f\|_1=1} |\Lambda(f)| \\ &= \sup_{\|f\|_1=1} \left| \int_X fg \right| \\ &\leq \sup_{\|f\|_1=1} \|fg\|_1 \\ &\leq \sup_{\|f\|_1=1} \|f\|_1 \|g\|_\infty \\ &= \|g\|_\infty, \end{aligned}$$

so it just remains to show that  $\|g\|_\infty \leq \|\Lambda\|_{1^\vee}$ .

Suppose otherwise, so  $\|g\|_\infty > \|\Lambda\|_{1^\vee}$ .

Then there exists some  $E \subseteq X$  with  $m(E) > 0$  such that  $x \in E \implies |g(x)| > \|\Lambda\|_{1^\vee}$ .

Define

$$h = \frac{1}{m(E)} \frac{\bar{g}}{|g|} \chi_E.$$

---


$$\begin{aligned}
\Lambda(h) &= \int_X hg \\
&= \int_X \frac{1}{m(E)} \frac{g\bar{g}}{|g|} \chi_E \\
&= \frac{1}{m(E)} \int_E |g| \\
&\geq \frac{1}{m(E)} \|g\|_\infty m(E) \\
&= \|g\|_\infty \\
&> \|\Lambda\|_{1^\vee},
\end{aligned}$$

a contradiction. ■

### 3 Fall 2019

#### 3.1 1

Cesaro mean/summation. Break series apart into pieces that can be handled separately.

#### 3.2 a

Prove a stronger result:

$$a_n \longrightarrow A \implies \frac{1}{N} \sum_{k=1}^N a_k \longrightarrow A.$$

Idea: once  $N$  is large enough,  $a_k \approx A$ , and all smaller terms will die off as  $N \rightarrow \infty$ .  
See this MSE answer.

Suppose  $S_k \rightarrow S$ . Choose  $\ell$  large enough such that

$$k \geq \ell \implies |S_k - S| < \varepsilon.$$

With  $\ell$  fixed, choose  $N$  large enough such that

$$k \leq \ell \implies \frac{|S_k - S|}{N} < \varepsilon.$$

Then

$$\begin{aligned}
 \left| \left( \frac{1}{N} \sum_{k=1}^N S_k \right) - S \right| &= \frac{1}{N} \left| \sum_{k=1}^N (S_k - S) \right| \\
 &\leq \frac{1}{N} \sum_{k=1}^N |S_k - S| \\
 &= \sum_{k=1}^{\ell} \frac{|S_k - S|}{N} + \sum_{k=\ell+1}^N \frac{|S_k - S|}{N} \\
 &\longrightarrow 0.
 \end{aligned}$$

### 3.3 b

Define

$$\Gamma_n := \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

Then  $\Gamma_1 = \sum_k \frac{a_k}{k}$  and each  $\Gamma_n$  is a tail of this series, so by assumption  $\Gamma_n \longrightarrow 0$ .

Then

$$\begin{aligned}
 \frac{1}{n} \sum_{k=1}^n a_k &= \frac{1}{n} (\Gamma_0 + \Gamma_1 + \cdots + \Gamma_n - \Gamma_{n+1}) \\
 &\longrightarrow 0.
 \end{aligned}$$

This comes from consider the following summation:

$\Gamma_1 :$	$a_1$	$+\frac{a_2}{2}$	$+\frac{a_3}{3}$	$+\cdots$	
$\Gamma_2 :$		$\frac{a_2}{2}$	$+\frac{a_3}{3}$	$+\cdots$	
$\Gamma_3 :$			$\frac{a_3}{3}$	$+\cdots$	
$\sum_{i=1}^n \Gamma_i :$	$a_1$	$+a_2$	$+a_3$	$+\cdots$	$a_n + \frac{a_{n+1}}{n+1} + \cdots$

■

## 3.4 2

DCT, and bounding in the right place. Don't evaluate the actual integral!

Use the fact that  $\int_0^1 \cos(tx) dt = \sin(x)/x$ , then

$$\begin{aligned}
 \left| \frac{\partial^n}{\partial x} \sin(x)/x \right| &= \left| \frac{\partial^n}{\partial x} \int_0^1 \cos(tx) dt \right| \\
 &= ? \left| \int_0^1 \frac{\partial^n}{\partial x} \cos(tx) dt \right| \\
 &= \left| \int_0^1 -t^n \sin(tx) dt \right| \quad \text{for } n \text{ odd} \\
 &\leq \int_0^1 |t^n \sin(tx)| dt \\
 &\leq \int_0^1 t^n dt \\
 &= \frac{1}{n+1} \\
 &< \frac{1}{n}.
 \end{aligned}$$

Where the DCT is justified by noting that  $f(t) = \cos(tx)$  is dominated by  $g(t) = 1$  on  $[0, 1]$ , which integrates to 1. ■

## 3.5 3

Borel-Cantelli.

Use the following observation: for a sequence of sets  $X_n$ ,

$$\begin{aligned}
 \limsup_n X_n &= \left\{ x \mid x \in X_n \text{ for infinitely many } n \right\} &= \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} X_n \\
 \liminf_n X_n &= \left\{ x \mid x \in X_n \text{ for all but finitely many } n \right\} &= \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} X_n.
 \end{aligned}$$

And recall

$$\prod_n e^{x_n} = e^{\sum_n x_n} \quad \text{and} \quad \sum_n \log(x_n) = \log \left( \prod_n x_n \right).$$

## 3.5.1 a

The Borel  $\sigma$ -algebra is closed under countable unions/intersections/complements, and  $B = \limsup_n B_n$  is an intersection of unions of measurable sets.



**3.5.2 b**

We'll use the fact that tails of convergent sums go to zero, so  $\sum_{n \geq M} \mu(B_n) \xrightarrow{M \rightarrow \infty} 0$ , and  $B_M :=$

$$\bigcap_{m=1}^M \bigcup_{n \geq m} B_n \searrow B.$$

$$\begin{aligned} \mu(B_M) &= \mu \left( \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} B_n \right) \\ &\leq \mu \left( \bigcup_{n \geq m} B_n \right) \quad \text{for all } m \in \mathbb{N} \\ &\longrightarrow 0, \end{aligned}$$

and the result follows by continuity of measure.

**3.5.3 c**

To show  $\mu(B) = 1$ , we'll show  $\mu(B^c) = 0$ .

Let  $B_k = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^K B_n$ . Then

$$\begin{aligned} \mu(B_K^c) &= \mu \left( \bigcup_{m=1}^{\infty} \bigcap_{n=m}^K B_n^c \right) \\ &\leq \sum_{m=1}^{\infty} \mu \left( \bigcap_{n=m}^K B_n^c \right) \quad \text{by subadditivity} \\ &= \sum_{m=1}^{\infty} \prod_{n=m}^K (1 - \mu(B_n)) \\ &\leq \sum_{m=1}^{\infty} \prod_{n=m}^K e^{-\mu(B_n)} \quad \text{by hint} \\ &= \sum_{m=1}^{\infty} e^{-\sum_{n=m}^K \mu(B_n)} \\ &\longrightarrow 0 \end{aligned}$$

since  $\sum_{n=m}^K \mu(B_n) \longrightarrow \infty$ , and we can apply continuity of measure since  $B_K^c \xrightarrow{K \rightarrow \infty} B^c$ . ■

**3.6 4**

Bessel's Inequality, surjectivity of Riesz map, and Parseval's Identity.  
Trick – remember to write out finite sum  $S_N$ , and consider  $\|x - S_N\|$ .

## 3.6.1 a

Claim:

$$\begin{aligned} 0 \leq \left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\ &\implies \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2. \end{aligned}$$

*Proof:* Let  $S_N = \sum_{n=1}^N \langle x, u_n \rangle u_n$ . Then

$$\begin{aligned} 0 &\leq \|x - S_N\|^2 \\ &= \langle x - S_N, x - S_N \rangle \\ &= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\ &\xrightarrow{N \rightarrow \infty} \|x\|^2 - \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2. \end{aligned}$$

## 3.6.2 b

1. Fix  $\{a_n\} \in \ell^2$ , then note that  $\sum |a_n|^2 < \infty \implies$  the tails vanish.
2. Define

$$x := \lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \sum_{k=1}^N a_k u_k$$

3.  $\{S_N\}$  Cauchy (by 1) and  $H$  complete  $\implies x \in H$ .
- 4.

$$\langle x, u_n \rangle = \left\langle \sum_k a_k u_k, u_n \right\rangle = \sum_k a_k \langle u_k, u_n \rangle = a_n \quad \forall n \in \mathbb{N}$$

since the  $u_k$  are all orthogonal.

- 5.

$$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$$

by Pythagoras since the  $u_k$  are normal.

Bonus: We didn't use completeness here, so the Fourier series may not actually converge to  $x$ . If  $\{u_n\}$  is **complete** (so  $x = 0 \iff \langle x, u_n \rangle = 0 \forall n$ ) then the Fourier series *does* converge to  $x$  and  $\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 = \|x\|^2$  for all  $x \in H$ .

■

**3.7 5**

Continuity in  $L^1$  (recall that DCT won't work! Notes 19.4, prove it for a dense subset first).  
Lebesgue differentiation in 1-dimensional case. See HW 5.6.

**3.8 a**

Choose  $g \in C_c^0$  such that  $\|f - g\|_1 \rightarrow 0$ .

By translation invariance,  $\|\tau_h f - \tau_h g\|_1 \rightarrow 0$ .

Write

$$\begin{aligned} \|\tau f - f\|_1 &= \|\tau_h f - g + g - \tau_h g + \tau_h g - f\|_1 \\ &\leq \|\tau_h f - \tau_h g\| + \|g - f\| + \|\tau_h g - g\| \\ &\rightarrow \|\tau_h g - g\|, \end{aligned}$$

so it suffices to show that  $\|\tau_h g - g\| \rightarrow 0$  for  $g \in C_c^0$ .

Fix  $\varepsilon > 0$ . Enlarge the support of  $g$  to  $K$  such that

$$|h| \leq 1 \text{ and } x \in K^c \implies |g(x-h) - g(x)| = 0.$$

By uniform continuity of  $g$ , pick  $\delta \leq 1$  small enough such that

$$x \in K, |h| \leq \delta \implies |g(x-h) - g(x)| < \varepsilon,$$

then

$$\int_K |g(x-h) - g(x)| \leq \int_K \varepsilon = \varepsilon \cdot m(K) \rightarrow 0.$$

**3.9 b**

We have

$$\begin{aligned} \int_{\mathbb{R}} |A_h(f)(x)| \, dx &= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y) \, dy \right| \, dx \\ &\leq \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)| \, dy \, dx \\ &=_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)| \, \mathbf{dx} \, \mathbf{dy} \\ &= \int_{\mathbb{R}} |f(y)| \, dy \\ &= \|f\|_1. \end{aligned}$$

and (rough sketch)

---


$$\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x) - f(x)| \, dx &= \int_{\mathbb{R}} \left| \left( \frac{1}{2h} \int_{B(h,x)} f(y) \, dy \right) - f(x) \right| \, dx \\
&= \int_{\mathbb{R}} \left| \left( \frac{1}{2h} \int_{B(h,x)} f(y) \, dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x) \, dy \right| \, dx \\
&\leq_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{B(h,x)} |f(y-x) - f(x)| \, \mathbf{d}\mathbf{x} \, dy \\
&\leq \frac{1}{2h} \int_{\mathbb{R}} \|\tau_x f - f\|_1 \, dy \\
&\longrightarrow 0 \quad \text{by (a).}
\end{aligned}$$

■

## 4 Spring 2018

### 4.1 1

We'll show that  $m(E) \cap [n, n+1] = 0$  for all  $n \in \mathbb{Z}$ ; then the result will follow from that fact that

$$m(E) = m\left(\bigcup_{n \in \mathbb{Z}} E \cap [n, n+1]\right) \leq \sum m(E \cap [n, n+1]) = 0$$

By translation invariance of measure, it suffices to show  $m(E \cap [0, 1]) = 0$ .

Define

$$E_j := \left\{ x \in [0, 1] \mid \exists p \in \mathbb{Z}^{\geq 0} \text{ s.t. } \left| x - \frac{p}{j} \right| < \frac{1}{j^3} \right\}.$$

Note that we can write  $E_j$  is a union of intervals

$$\begin{aligned}
E_j &= (1, \frac{1}{j^3}) \\
&\quad \amalg B_{\frac{1}{j^3}}\left(\frac{1}{j}\right) \amalg B_{\frac{1}{j^3}}\left(\frac{2}{j}\right) \amalg \cdots \amalg B_{\frac{1}{j^3}}\left(\frac{j-1}{j}\right) \\
&\quad \amalg (1 - \frac{1}{j^3}, 1),
\end{aligned}$$

from which we can conclude that  $E_j$  is Borel and thus Lebesgue measurable, and that for each  $j$ , there are exactly  $j+1$  possible choices for a numerator (corresponding to the  $j+1$  sets appearing above.)

The first and last intervals are length  $\frac{1}{j^3}$  and the remaining  $(j+1) - 2 = j-1$  intervals are length  $\frac{2}{j^3}$ , so we find that

$$m(E_j) = 2\left(\frac{1}{j^3}\right) + (j-1)\left(\frac{2}{j^3}\right) = \frac{2}{j^2}$$

We can then note that

$$\sum_{j \in \mathbb{N}} m(E_j) \leq 2 \sum_{j \in \mathbb{N}} \frac{1}{j^2} < \infty,$$

which converges by the  $p$ -test for sums.

Since  $\{E_j\}$  is a countable collection of measurable sets such that  $\sum m(E_j) < \infty$ , Borel-Cantelli applies and  $m(\limsup_j E_j) = 0$ , where we can just note that  $\limsup_j E_j = E \cap [0, 1]$ . ■

## 4.2 2

### 4.2.1 a

Since  $x < 1 \implies x^n \longrightarrow 0$  and  $x > 1 \implies x^n \longrightarrow \infty$ , we have

$$f_n(x) = \frac{x}{1+x^n} \xrightarrow{n \rightarrow \infty} f(x) = \begin{cases} 0, & x = 0 \\ x, & x < 1 \\ \frac{1}{2}, & x = 1 \\ 0, & x > 1 \end{cases}.$$

If  $f_n \longrightarrow f$  uniformly on  $[0, \infty)$ , it would converge uniformly on every subset.

But  $f_n(x)$  is clearly continuous on  $(0, \infty)$ , and if the convergence was uniform then  $f$  would be continuous. However  $f$  has a clear discontinuity at  $x = 1$ .

### 4.2.2 b

If the DCT applies, we can interchange the limit and integral, and the value would be the area under the graph of  $f$  which is  $\int_0^1 x \, dx = \frac{1}{2}$ .

To justify the DCT, write

$$\int_0^\infty f_n(x) \, dx = \int_0^1 f_n(x) \, dx + \int_1^\infty f_n(x) \, dx.$$

Then

$$x \in [0, 1] \implies \frac{x}{1+x^n} < \frac{1}{1+x^n} < 1$$

and  $\int_0^1 1 \, dx = 1 < \infty$ .

On the other hand,

$$x \in (1, \infty) \implies \frac{x}{1+x^n} \approx O\left(\frac{1}{x^{n-1}}\right),$$

and so for  $n > 2$  the integral will converge by the  $p$ -test.

**4.3 3**

Since  $|f(x)| \leq \|f\|_\infty$  almost everywhere, we have

$$\|f\|_p^p = \int_X |f(x)|^p dx \leq \int_X \|f\|_\infty^p dx = \|f\|_\infty^p \cdot m(X) = \|f\|_\infty^p,$$

so  $\|f\|_p \leq \|f\|_\infty$  for all  $p$  and taking  $\lim_{p \rightarrow \infty}$  preserves this inequality.

Conversely, let  $\varepsilon > 0$ . Define

$$S_\varepsilon := \left\{ x \in \mathbb{R} \mid |f(x)| \geq \|f\|_\infty - \varepsilon \right\}.$$

Then

$$\begin{aligned} \|f\|_p^p &= \int_X |f(x)|^p dx \\ &\geq \int_{S_\varepsilon} |f(x)|^p dx \\ &\geq \int_{S_\varepsilon} (\|f\|_\infty - \varepsilon)^p dx \\ &= (\|f\|_\infty - \varepsilon)^p \cdot m(S_\varepsilon) \\ \implies \|f\|_p &\geq (\|f\|_\infty - \varepsilon) \cdot m(S_\varepsilon)^{\frac{1}{p}} \\ &\xrightarrow{p \rightarrow \infty} (\|f\|_\infty - \varepsilon) \\ &\xrightarrow{\varepsilon \rightarrow 0} \|f\|_\infty. \end{aligned}$$

So  $\|f\|_p \geq \|f\|_\infty$ .

■

**4.4 4**

Fix  $k \in \mathbb{Z}$ . Since  $e^{2\pi i k x}$  is continuous on the compact interval  $[0, 1]$ , it is uniformly continuous, and is thus there is a sequence of polynomials  $P_\ell$  such that

$$P_{\ell,k} \xrightarrow{\ell \rightarrow \infty} e^{2\pi i k x} \text{ uniformly on } [0, 1].$$

Note that by linearity,

$$\int f(x) x^n = 0 \quad \forall n \implies \int f(x) P_{\ell,k}(x) = 0 \quad \forall \ell \in \mathbb{N}$$

But then the  $k$ th Fourier coefficient of  $f$  is given by

$$\begin{aligned}
 \langle f, e_k \rangle &= \int_0^1 f(x) e^{-2\pi i k x} dx \\
 &= \int_0^1 f(x) \lim_{\ell \rightarrow \infty} P_\ell(x) \\
 &= \lim_{\ell \rightarrow \infty} \int_0^1 f(x) P_\ell(x) \quad \text{by uniform convergence} \\
 &= \lim_{\ell \rightarrow \infty} 0 \\
 &= 0 \quad \forall k \in \mathbb{Z},
 \end{aligned}$$

so  $\hat{f}$  is the zero function, and  $\hat{f} = 0 \iff f = 0$  almost everywhere. ■

## 4.5 5

$$\text{Moral: } \int |f_n - f| \rightarrow 0 \iff \int f_n = \int f.$$

Since if  $\int |f_n| \rightarrow \int |f|$  then we can define

$$\begin{aligned}
 h_n &= |f_n - f| && \rightarrow 0 \text{ a.e.} \\
 g_n &= |f_n| + |f| && \rightarrow 2|f| \text{ a.e.}
 \end{aligned}$$

$$\begin{aligned}
 \int 2|f| &= \int \liminf (g_n - h_n) \\
 &= \int \liminf g_n - \int \liminf h_n \\
 &= \int 2|f| - \int \liminf h_n \\
 &\stackrel{\text{Fatou}}{\leq} \int 2|f| + \limsup \int h_n,
 \end{aligned}$$

which forces  $\int h_n = \int |f_n - f| \rightarrow 0$ .

But then

$$\left| \int f_n - \int f \right| = \left| \int f_n - f \right| \leq \int |f_n - f| \rightarrow 0,$$

so  $\int f_n \rightarrow \int f$ . ■

---

## 5 Fall 2018

Note: this is considered...not the most useful or representative exam of all time.

### 5.1 1

We'll show a stronger statement:  $f(x) = \frac{1}{x}$  is uniformly continuous on any interval of the form  $(c, \infty)$  where  $c > 0$ .

We can use that fact that  $x, y > c \implies xy > c^2 \implies \frac{1}{xy} < \frac{1}{c^2}$ .

Letting  $\varepsilon$  be arbitrary, choose  $\delta < \varepsilon c^2$ . Then

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{x} - \frac{1}{y} \right| \\ &= \frac{|x - y|}{xy} \\ &\leq \frac{\delta}{xy} \\ &< \frac{\delta}{c^2} \\ &< \varepsilon, \end{aligned}$$

which shows uniform continuity since  $\delta$  does not depend on  $x$  or  $y$ .

To see that  $f$  is not uniformly continuous when  $c = 0$ , let  $\varepsilon < 1$  be arbitrary.

Let  $x_n = \frac{1}{n}$ . Then choose  $n$  large enough such that  $|x_n - x_{n+1}| = \frac{1}{n} - \frac{1}{n+1} < \delta$ . Then just note that  $f(x_n) = n$  and thus  $|f(x_n) - f(x_{n+1})| = n - (n+1) = 1 > \varepsilon$ .

### 5.2 2

First consider the bounded case where  $m(E) < \infty$ .

$E$  is measurable  $\iff$  for every  $\varepsilon$  there exist  $F_\varepsilon \subset E \subset G_\varepsilon$  with  $F_\varepsilon$  closed and  $G_\varepsilon$  open and  $m(G_\varepsilon \setminus E) < \varepsilon$  and  $m(E \setminus F_\varepsilon) < \varepsilon$ .

So take the sequence  $\varepsilon_n = \frac{1}{n} \rightarrow 0$  to produce a sequence of closed sets  $F_n$  such that  $m(E \setminus F_n) < \frac{1}{n}$  for all  $n$ , and let  $F := \bigcup_n F_n$ , which is clearly an  $F_\sigma$  and thus Borel set.

Since  $F_n \subseteq F_{n+1}$ , we have  $F_n \nearrow F$  and so by continuity of measure,

$$m(F) = \lim_n m(F_n) < \lim_n \left( \frac{1}{n} \right) \rightarrow 0.$$

If  $E$  is not bounded, let  $E_N = B_N(0) \cap E$  which is bounded. Then  $E_N \nearrow E$ , and for each  $N$  we can find an  $F_N$  by the previous case such that  $m(E_N \setminus F_N) = 0$ .



So take  $F := \bigcup_N F_N$  so  $F_N \nearrow F$ . Then

$$E_N \setminus F_N \nearrow E \setminus F \implies m(E \setminus F) = \lim_N m(E_N \setminus F_N) = 0.$$

### 5.3 3

$$\begin{aligned} \frac{\partial}{\partial t} F(t) &= \frac{\partial}{\partial t} \int_{\mathbb{R}} f(x) \cos(xt) \, dx \\ &\stackrel{DCT}{=} \int_{\mathbb{R}} f(x) \frac{\partial}{\partial t} \cos(xt) \, dx \\ &= \int_{\mathbb{R}} x f(x) \cos(xt) \, dx, \end{aligned}$$

so it only remains to justify the DCT.

Fix  $t$ , then let  $t_n \rightarrow t$  be any sequence. Then

$$\begin{aligned} \frac{\partial}{\partial t} \cos(tx) &:= \lim_{t_n \rightarrow t} \frac{\cos(tx) - \cos(t_n x)}{t_n - t} \\ &\stackrel{MVT}{=} \frac{\partial}{\partial t} \cos(tx) \Big|_{t=\xi_n} \quad \text{for some } \xi_n \in [t, t_n] \text{ or } [t_n, t] \\ &= x \sin(\xi_n x). \end{aligned}$$

So we can define

$$h_n(x, t) = f(x) \left( \frac{\cos(tx) - \cos(t_n x)}{t_n - t} \right)$$

and note that  $h_n \rightarrow \frac{\partial}{\partial t} [f(x) \cos(xt)]$  pointwise.

We then have  $|h_n| = |f(x)x \sin(\xi_n x)| \leq |xf(x)|$  for every  $n$  by the above argument, and since  $g(x) := xf(x) \in L^1(\mathbb{R})$  by assumption, the DCT can be applied.

### 5.4 4

???

Apparently “easy” part: let  $f(x) = \chi_{[0, \pi]}$ , then  $\int_{\mathbb{R}} f(x) |\sin(nx)| \, dx = \int_0^\pi |\sin(nx)| \, dx = 2$ , and so  $\int_0^1 |\sin(nx)| \, dx = \frac{2}{\pi}$ , none of which depend on  $n$ .

Now approximate  $f$  by step functions.

### 5.5 5

???

---

## 6 Spring 2017

### 6.1 1

$A$  is nowhere dense  $\iff$  every interval  $I$  contains a subinterval  $S \subseteq A^c$ .

**$K$  is compact:**

It suffices to show that  $K^c := [0, 1] \setminus K$  is open; then  $K$  will be a closed and bounded subset of  $\mathbb{R}$  and thus compact by Heine-Borel.

We can identify  $K^c$  as the set of real numbers in  $[0, 1]$  whose decimal expansion **does** use a 4. Let  $x \in K^c$ , and suppose a 4 occurs as the  $k$ th digit and write

$$x = 0.d_1d_2\cdots d_{k-1}4d_{k+1}\cdots = \sum_{j=1}^k d_j 10^{-j} + 4 \cdot 10^{-k} + \sum_{j=k+1}^{\infty} d_j 10^{-j}.$$

Then if we set  $r < 10^{-k}$  and pick any  $y \in [0, 1]$  such that  $y \in B_r(x)$ , then  $|x - y| < 10^{-k}$ . If we write  $y = \sum_{j=1}^{\infty} c_j 10^{-j}$ , this means that for all  $j \leq k$  we have  $d_j = c_j$ , and in particular  $d_k = 4 = c_k$ , so  $y$  has a 4 in its decimal expansion.

But then  $K^c = \bigcup_x B_r(x)$  is a union of open sets and thus open.

**$K$  is nowhere dense and  $m(K) = 0$ :**

Since  $K$  is closed, we'll show that  $K$  can not properly contain any interval, so  $(\overline{K})^\circ = \emptyset$ .

As in the construction of the Cantor set, let

- $K_1$  denote  $[0, 1]$  with 1 interval  $[0.4, 0.5]$  of length  $\frac{1}{10}$  deleted
- $K_2$  denote  $K_1$  with 9 intervals  $[0.04, 0.05], [0.14, 0.15], \dots [0.94, 0.95]$  length  $\frac{1}{100}$  deleted
- $K_n$  denote  $K_{n-1}$  with  $9^{n-1}$  such intervals of length  $10^{-n}$  deleted.

Then  $K = \bigcap K_n$ , and

$$m(K) = 1 - m(K^c) = 1 - \sum_{j=0}^{\infty} \frac{9^j}{10^{j+1}} = 1 - \frac{1}{10} \left( \frac{1}{1 - \frac{9}{10}} \right) = 0,$$

and since any interval has strictly positive measure,  $K$  can not contain any interval.

**$K$  has no isolated points:**

A point  $x \in K$  is isolated iff there is an open ball  $B_r(x)$  containing  $x$  such that  $B_r(x) \cap K = \{x\}$ , so every point in this ball has a 4 in its decimal expansion.

Note that  $m(K_n) = \left(\frac{9}{10}\right)^n \rightarrow 0$  and that the endpoints of intervals are never removed and are thus elements of  $K$ . Then for every  $\varepsilon$ , we can choose  $n$  such that  $\left(\frac{9}{10}\right)^n < \varepsilon$ ; then there is an endpoint of a removed interval  $e_n$  satisfying  $|x - e_n| \leq \left(\frac{9}{10}\right)^n < \varepsilon$ .

So every ball containing  $x$  contains some endpoint of a removed interval, and thus an element of  $K$ . ■

## 6.2 2

$$\lambda \ll \mu \iff E \in \mathcal{M}, \mu(E) = 0 \implies \lambda(E) = 0.$$

### 6.2.1 a

By Radon-Nikodym, if  $\lambda \ll \mu$  then  $d\lambda = f d\mu$ , which would yield

$$\int g d\lambda = \int gf d\mu.$$

So let  $E$  be measurable and suppose  $\mu(E) = 0$ . Then

$$\lambda(E) := \int_E f d\mu = \lim_n \left\{ \varphi_n := \sum_j c_j \mu(E_j) \right\},$$

where we take a sequence of simple functions increasing to  $f$ .

But since each  $E_j \subseteq E$ , we must have  $\mu(E_j) = 0$  for any such  $E_j$ , so every such  $\varphi_n$  must be zero and thus  $\lambda(E) = 0$ .

### 6.2.2 b

By Radon-Nikodym, there exists a positive  $f$  such that

$$\int g dm = \int gf d\mu,$$

where we can take  $g(x) = x^2$ , then the LHS is zero by assumption and thus so is the RHS.

Note that  $gf$  is positive.

Define  $A_k = \left\{ x \in X \mid gf\chi_E > \frac{1}{k} \right\}$ , then by Chebyshev

$$\mu(A_k) \leq k \int_E gf d\mu = 0,$$

which holds for every  $k$ .

Then noting that  $A_k \searrow A := \left\{ x \in E \mid x^2 > 0 \right\}$ , and  $gf$  is positive, we have

$$x \in E \iff gf\chi_E(x) > 0 \iff x \in A,$$

so  $E = A$  and  $\mu(E) = \mu(A)$ .

But since  $m \ll \mu$  by construction, we can conclude that  $m(E) = 0$ . ■

**6.3 3****6.3.1 a**

Letting  $x_n := \frac{1}{n}$ , we have

$$\sum_{k=1}^{\infty} |f_k(x)| \geq |f_n(x_n)| = |ae^{-ax} - be^{-bx}| := M.$$

In particular,  $\sup_x |f_n(x)| \not\rightarrow 0$ , so the terms do not go to zero and the sum can not converge.

**6.3.2 b**

?

**6.4 4**

Switching to polar coordinates and integrating over a half-circle contained in  $I^2$ , we have

$$\int_{I^2} f \geq \int_0^\pi \int_0^1 \frac{\cos(\theta) \sin(\theta)}{r^2} dr d\theta = \infty,$$

so  $f$  is not integrable.

**6.5 5**

See <https://math.stackexchange.com/questions/507263/prove-that-c1a-b-with-the-c1-norm-is-a-banach-space>

This is clearly a norm, which we'll write  $\|\cdot\|_u$

Let  $f_n$  be a Cauchy sequence and define a candidate limit  $f(x) = \lim_n f_n(x)$ .

Then noting that  $\|f_n\|_\infty, \|f'_n\|_\infty \leq \|f_n\|_u < \infty$ , both  $f_n, f'_n$  are Cauchy sequences in  $C^0([a, b], \|\cdot\|_\infty)$ , which is a Banach space.

So  $f_n \rightarrow f$  uniformly, and  $f'_n \rightarrow g$  uniformly for some  $g$ , and moreover  $f, g \in C^0([a, b])$ .

We thus have

$$\begin{aligned} f_n(x) - f_n(a) &\xrightarrow{u} f(x) - f(a) \\ \int_a^x f'_n &\xrightarrow{u} \int_a^x g, \end{aligned}$$

and by the FTC, the left-hand sides are equal, and by uniqueness of limits so are the right-hand sides, so  $f' = g$ .

Since  $f, f' \in C^0([a, b])$ , they are bounded, and so  $\|f\|_u < \infty$ . This means that  $\|f_n - f\|_u \rightarrow 0$ , so  $f_n$  converges to  $f$ , which is in the same space.

■

---

## 7 Fall 2017

### 7.1 1

Note that  $f(x) = e^x$  is entire and thus equal to its power series. So  $f(x) = \sum_{j=0}^{\infty} \frac{1}{j!} x^j$ .

Letting  $f_N(x) = \sum_{j=1}^N \frac{1}{j!} x^j$ , we have  $f_N(x) \rightarrow f(x)$  pointwise on  $(-\infty, \infty)$ .

For any compact interval  $[-M, M]$ , we have

$$\begin{aligned} \|f_N(x) - f(x)\|_{\infty} &= \sup_{-M \leq x \leq M} \left| \sum_{j=N+1}^{\infty} \frac{1}{j!} x^j \right| \\ &\leq \sup_{-M \leq x \leq M} \sum_{j=N+1}^{\infty} \frac{1}{j!} |x|^j \\ &\leq \sum_{j=N+1}^{\infty} \frac{1}{j!} M^j \\ &\leq \sum_{j=0}^{\infty} \frac{1}{j!} M^j \\ &= e^M \\ &< \infty, \end{aligned}$$

so  $f_N \rightarrow f$  uniformly on  $[-M, M]$  by the M-test. Thus it converges on any bounded interval.

It does not converge on  $\mathbb{R}$ , since  $x^N$  is unbounded.

### 7.2 2

#### 7.2.1 a

It suffices to consider the bounded case, i.e.  $E \subseteq B_M(0)$  for some  $M$ . Then write  $E_n = B_n(0) \cap E$  and apply the theorem to  $E_n$ , and by subadditivity,  $m^*(E) = m^*\left(\bigcup_n E_n\right) \leq \sum_n m^*(E_n) = 0$ .

**Lemma:**  $f(x) = x^2, f^{-1}(x) = \sqrt{x}$  are Lipschitz on any compact subset of  $[0, \infty)$ .

*Proof:* Let  $g = f$  or  $f^{-1}$ . Then  $g \in C^1([0, M])$  for any  $M$ , so  $g$  is differentiable and  $g'$  is continuous. Since  $g'$  is continuous on a compact interval, it is bounded, so  $|g'(x)| \leq L$  for all  $x$ . Applying the MVT,

$$|f(x) - f(y)| = |f'(c)| |x - y| \leq L |x - y|.$$

**Lemma:** If  $g$  is Lipschitz on  $\mathbb{R}^n$ , then  $m(E) = 0 \implies m(g(E)) = 0$ .

*Proof:* If  $g$  is Lipschitz, then

$$g(B_r(x)) \subseteq B_{Lr}(g(x)),$$

which is a dilated ball/cube, and so

$$m^*(B_{Lr}(x)) \leq L^n \cdot m^*(B_r(x)).$$

Now choose  $\{Q_j\} \rightrightarrows E$ ; then  $\{g(Q_j)\} \rightrightarrows g(E)$ .

By the above observation,

$$|g(Q_j)| \leq L^n |Q_j|,$$

and so

$$m^*(g(E)) \leq \sum_j |g(Q_j)| \leq \sum_j L^n |Q_j| = L^n \sum_j |Q_j| \rightarrow 0.$$

Now just take  $g(x) = x^2$  for one direction, and  $g(x) = f^{-1}(x) = \sqrt{x}$  for the other. ■

### 7.2.2 b

Lemma:  $E$  is measurable iff  $E = K \coprod N$  for some  $K$  compact,  $N$  null.

Write  $E = K \coprod N$  where  $K$  is compact and  $N$  is null.

Then  $\varphi^{-1}(E) = \varphi^{-1}(K \coprod N) = \varphi^{-1}(K) \coprod \varphi^{-1}(N)$ .

Since  $\varphi^{-1}(N)$  is null by part (a) and  $\varphi^{-1}(K)$  is the preimage of a compact set under a continuous map and thus compact,  $\varphi^{-1}(E) = K' \coprod N'$  where  $K'$  is compact and  $N'$  is null, so  $\varphi^{-1}(E)$  is measurable.

So  $\varphi$  is a measurable function, and thus yields a well-defined map  $\mathcal{L}(\mathbb{R}) \rightarrow \mathcal{L}(\mathbb{R})$  since it preserves measurable sets. Restricting to  $[0, \infty)$ ,  $f$  is bijection, and thus so is  $\varphi$ . ■

### 7.3 3

From homework:  $E$  is Lebesgue measurable iff there exists a finite union of closed cubes  $A$  such that  $m(E \Delta A) < \varepsilon$ .

It suffices to show that  $S$  is dense in simple functions, and since simple functions are *finite* linear combinations of characteristic functions, it suffices to show this for  $\chi_A$  for  $A$  a measurable set.

Let  $s = \chi_A$ . By regularity of the Lebesgue measure, choose an open set  $O \supseteq A$  such that  $m(O \setminus A) < \varepsilon$ .

$O$  is an open subset of  $\mathbb{R}$ , and thus  $O = \coprod_{j \in \mathbb{N}} I_j$  is a disjoint union of countably many open intervals.

Now choose  $N$  large enough such that  $m(O \Delta I_{N,n}) < \varepsilon = \frac{1}{n}$  where we define  $I_{N,n} := \coprod_{j=1}^N I_j$ .

Now define  $f_n = \chi_{I_{N,n}}$ , then

$$\|s - f_n\|_1 = \int |\chi_A - \chi_{I_{N,n}}| = m(A \Delta I_{N,n}) \xrightarrow{n \rightarrow \infty} 0.$$

Since any simple function is a finite linear combination of  $\chi_{A_i}$ , we can do this for each  $i$  to extend this result to all simple functions. But simple functions are dense in  $L^1$ , so  $S$  is dense in  $L^1$ .

## 7.4 4

### 7.4.1 a

Let  $G(x) = \sum_{n=1}^{\infty} nx(1-x)^n$ . Applying the ratio test, we have

$$\left| \frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n} \right| = \frac{n+1}{n} |1-x| \xrightarrow{n \rightarrow \infty} |1-x| < 1 \iff 0 \leq x \leq 2,$$

and in particular, this series converges on  $[0, 2]$ . Thus its terms go to zero, and  $nx(1-x)^n \rightarrow 0$  on  $[0, 1] \subset [0, 2]$ .

To see that the convergence is not uniform, let  $x_n = \frac{1}{n}$  and  $\varepsilon > \frac{1}{e}$ , then

$$\sup_{x \in [0,1]} |nx(1-x)^n - 0| \geq |nx_n(1-x_n)^n| = \left| \left(1 - \frac{1}{n}\right)^n \right| \xrightarrow{n \rightarrow \infty} e^{-1} > \varepsilon.$$

### 7.4.2 b

Note: could use the first part with  $\sin(x) \leq x$ , but then integral ends up more complicated.

Noting that  $\sin(x) \leq 1$ , we have We have

$$\begin{aligned} \left| \int_0^1 n(1-x)^n \sin(x) \right| &\leq \int_0^1 |n(1-x)^n \sin(x)| \\ &\leq \int_0^1 |n(1-x)^n| \\ &= n \int_0^1 (1-x)^n \\ &= -\frac{n(1-x)^{n+1}}{n+1} \\ &\xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

## 7.5 5

### 7.5.1 a

**Lemma:** If  $\varphi \in C_c^1$ , then  $(f * \varphi)' = f * \varphi'$  almost everywhere.

*Silly Proof:*

$$\begin{aligned}
\mathcal{F}((f * \varphi)') &= 2\pi i \xi \mathcal{F}(f * \varphi) \\
&= 2\pi i \xi \mathcal{F}(f) \mathcal{F}(\varphi) \\
&= \mathcal{F}(f) \cdot (2\pi i \xi \mathcal{F}(\varphi)) \\
&= \mathcal{F}(f) \cdot \mathcal{F}(\varphi') \\
&= \mathcal{F}(f * \varphi').
\end{aligned}$$

*Actual proof:*

$$\begin{aligned}
(f * \varphi)'(x) &= (\varphi * f)'(x) \\
&= \lim_{h \rightarrow 0} \frac{(\varphi * f)'(x+h) - (\varphi * f)'(x)}{h} \\
&= \lim_{h \rightarrow 0} \int \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \\
&\stackrel{DCT}{=} \int \lim_{h \rightarrow 0} \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \\
&= \int \varphi'(x-y) f(y) \\
&= (\varphi' * f)(x) \\
&= (f * \varphi')(x).
\end{aligned}$$

To see that the DCT is justified, we can apply the MVT on the interval  $[0, h]$  to  $f$  to obtain

$$\frac{\varphi(x+h-y) - \varphi(x-y)}{h} = \varphi'(c) \quad c \in [0, h],$$

and since  $\varphi'$  is continuous and compactly supported,  $\varphi'$  is bounded by some  $M < \infty$  by the extreme value theorem and thus

$$\begin{aligned}
\int \left| \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \right| &= \int |\varphi'(c) f(y)| \\
&\leq \int |M| |f| \\
&= |M| \int |f| < \infty,
\end{aligned}$$

since  $f \in L^1$  by assumption, so we can take  $g := |M||f|$  as the dominating function.

Applying this theorem infinitely many times shows that  $f * \varphi$  is smooth.

To see that  $f * \varphi$  is compactly supported, approximate  $f$  by a *continuous* compactly supported function  $h$ , so  $\|h - f\|_1 \xrightarrow{L^1} 0$ .

Now let  $g_x(y) = \varphi(x - y)$ , and note that  $\text{supp}(g) = x - \text{supp}(\varphi)$  which is still compact.



But since  $\text{supp}(h)$  is bounded, there is some  $N$  such that

$$|x| > N \implies A_x := \text{supp}(h) \cap \text{supp}(g_x) = \emptyset$$

and thus

$$\begin{aligned} (h * \varphi)(x) &= \int_{\mathbb{R}} \varphi(x-y)h(y) dy \\ &= \int_{A_x} g_x(y)h(y) dy \\ &= 0, \end{aligned}$$

so  $\{x \mid f * g(x) = 0\}$  is open, and its complement is closed and bounded and thus compact.

### 7.5.2 b

$$\begin{aligned} \|f * K_j - f\|_1 &= \int \left| \int f(x-y)K_j(y) dy - f(x) \right| dx \\ &= \int \left| \int f(x-y)K_j(y) dy - \int f(x)K_j(y) dy \right| dx \\ &= \int \left| \int (f(x-y) - f(x))K_j(y) dy \right| dx \\ &\leq \int \int |(f(x-y) - f(x))| \cdot |K_j(y)| dy dx \\ &\stackrel{FT}{=} \int \int |(f(x-y) - f(x))| \cdot |K_j(y)| \mathbf{dx} \mathbf{dy} \\ &= \int |K_j(y)| \left( \int |(f(x-y) - f(x))| dx \right) dy \\ &= \int |K_j(y)| \cdot \|f - \tau_y f\|_1 dy. \end{aligned}$$

We now split the integral up into pieces.

1. Chose  $\delta$  small enough such that  $|y| < \delta \implies \|f - \tau_y f\|_1 < \varepsilon$  by continuity of translation in  $L^1$ , and
2. Since  $\varphi$  is compactly supported, choose  $J$  large enough such that

$$j > J \implies \int_{|y| \geq \delta} |K_j(y)| dy = \int_{|y| \geq \delta} |j\varphi(jy)| = 0$$

Then

$$\begin{aligned} \|f * K_j - f\|_1 &\leq \int |K_j(y)| \cdot \|f - \tau_y f\|_1 dy \\ &= \int_{|y| < \delta} |K_j(y)| \cdot \|f - \tau_y f\|_1 dy + \int_{|y| \geq \delta} |K_j(y)| \cdot \|f - \tau_y f\|_1 dy \\ &= \varepsilon \int_{|y| < \delta} |K_j(y)| + 0 \\ &\leq \varepsilon(1) \longrightarrow 0. \end{aligned}$$

■

**7.6 6**

Should be supremum maybe..?

Let  $\{f_k\}$  be a Cauchy sequence, so  $\|f_k\| < \infty$  for all  $k$ . Then for a fixed  $x$ , the sequence  $f_k(x)$  is Cauchy in  $\mathbb{R}$  and thus converges to some  $f(x)$ , so define  $f$  by  $f(x) := \lim_{k \rightarrow \infty} f_k(x)$ .

Then  $\|f_k - f\| = \max_{x \in X} |f_k(x) - f(x)| \xrightarrow{k \rightarrow \infty} 0$ , and thus  $f_k \rightarrow f$  uniformly and thus  $f$  is continuous. It just remains to show that  $f$  has bounded norm.

Choose  $N$  large enough so that  $\|f - f_N\| < \varepsilon$ , and write  $\|f_N\| := M < \infty$

$$\|f\| \leq \|f - f_N\| + \|f_N\| < \varepsilon + M < \infty.$$

**8 Spring 2016****8.1 1****9 Fall 2016****9.1 1****10 Spring 2014****10.1 1**