Complex Analysis Qualifying Exam Review

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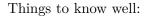
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A great deal of content borrowed from the following: https://web.stanford.edu/~chriseur/notes_pdf/Eur_ComplexAnalysis_Notes.pdf

1 | General Info / Tips / Techniques

1.1 Greatest Hits



- Estimates for derivatives, mean value theorem
- ??CauchyTheorem|Cauchy's Theorem
- ??CauchyIntegral|Cauchy's Integral Formula
- ??CauchyInequality|Cauchy's Inequality
- ??Morera]Morera's Theorem
- ??Liouville|Liouville's Theorem
- ??MaximumModulus]Maximum Modulus Principle
- ??Rouche]Rouché's Theorem
- ??SchwarzReflection|The Schwarz Reflection Principle
- ??SchwarzLemma]The Schwarz Lemma
- ??Casorati]Casorati-Weierstrass Theorem
- Properties of linear fractional transformations
- Automorphisms of \mathbb{D} , \mathbb{C} , \mathbb{CP}^1 .

1.2 Common Tricks

• Virtually any time: consider 1/f(z) and f(1/z).

Remark 1.2.1 (Showing a function is constant): If you want to show that a function f is constant, try one of the following:

- Write f = u + iv and use Cauchy-Riemann to show $u_x, u_y = 0$, etc.
- Show that f is entire and bounded.
 - If you additionally want to show f is zero, show $\lim_{z\to\infty} f(z) = 0$.

Fact 1.2.2

To show a function is holomorphic,

- Use Morera's theorem
- Find a primitive (sufficient but not necessary)

Fact 1.2.3

To count zeros:

- Rouche's theorem
- The argument principle
- Setting $w = e^z$ is useful.

1.3 Basic but Useful Facts

1.3.1 Arithmetic

Fact 1.3.1 (Some useful facts about basic complex algebra)

$$z\bar{z} = |z|^2$$
 $\operatorname{Arg}(z/w) = \operatorname{Arg}(z) - \operatorname{Arg}(w)$ $\Re(z) = \frac{z + \bar{z}}{2}$ $\Im(z) = \frac{z - \bar{z}}{2i}$.

Exponential forms of cosine and sine, where it's sometimes useful to set $w \coloneqq e^{iz}$:

$$\cos(z) = \frac{1}{2} \left(e^{iz} + e^{-iz} \right) = \frac{1}{2} (w + w^{-1})$$
$$\sin(z) = \frac{1}{2i} \left(e^{iz} - e^{-iz} \right) = \frac{1}{2i} (w - w^{-1}).$$

1.3 Basic but Useful Facts 5

Exponential forms of hyperbolic cosine and sin:

$$\begin{aligned} \cosh(z) &= \cos(iz) = \frac{1}{2} \left(e^z + e^{-z} \right) \\ \sinh(z) &= -i \sin(iz) = \frac{1}{2} \left(e^z - e^{-z} \right). \end{aligned}$$

Some other useful facts about the hyperbolic exponentials:

- They are periodic with period $2\pi i$.
- They are periodic with period $2\pi i$. $\frac{\partial}{\partial z}\cosh(z) = \sinh(z)$ and $\frac{\partial}{\partial z}\sinh(z) = \cosh(z)$. \sinh is odd and \cosh is even. $\cosh(z+i\pi) = -\cosh(z)$ and $\sinh(z+i\pi) = -\sinh(z)$.

- cosh has zeros at $\left\{i\pi\left(\frac{2k+1}{2}\right)\right\} = \left\{i\left(\pi/2 + k\pi\right)\right\}$, i.e. $\cdots, -\pi/2, \pi/2, 3\pi/2, \cdots$, the half-
- sinh has zeros at $\{i\pi k\}$, i.e. the integers.

Fact 1.3.2

Some computations that come up frequently:

$$|z \pm w|^2 = |z|^2 + |w|^z + 2\Re(\overline{w}z)$$

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)$$

$$\frac{1}{|a+b|} \le \frac{1}{|a|-|b|}$$

$$|e^z| = e^{\Re(z)}, \quad \arg(e^z) = \Im(z).$$

1.3.2 Calculus

Fact 1.3.3

Various differentials:

$$dz = dx + i \ dy$$
$$d\bar{z} = dx - i \ dy$$

$$f_z = f_x = f_y/i.$$

Integral of a complex exponential:

$$\int_0^{2\pi} e^{i\ell x} dx = \begin{cases} 2\pi & \ell = 0\\ 0 & \text{else} \end{cases}.$$

1.3 Basic but Useful Facts

1.4 Series

Fact 1.4.1 (Generalized Binomial Theorem)

Define $(n)_k$ to be the falling factorial

$$\prod_{j=0}^{k-1} (n-k) = n(n-1)\cdots(n-k+1)$$

and set $\binom{n}{k} := (n)_k/k!$, then

$$(x+y)^n = \sum_{k\geq 0} \binom{n}{k} x^k y^{n-k}.$$

Fact 1.4.2 (Some useful series)

1.4 Series

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{0 \le k \le N} z^k = \frac{1-z^{N+1}}{1-z}$$

$$\frac{1}{1-z} = \sum_{k \ge 0} z^k$$

$$e^z = \sum_{k \ge 0} \frac{z^k}{k!}$$

$$\sin(z) = \sum_{k \ge 1} (-1)^{\frac{k+1}{2}} \frac{z^k}{k!}$$

$$= z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 + \cdots$$

$$\cos(z) = \sum_{k \ge 0} (-1)^{\frac{k}{2}} \frac{z^k}{k!}$$

$$= 1 - \frac{1}{2!} z^2 + \frac{1}{4!} z^4 + \cdots$$

$$\cosh(z) = \sum_{k \ge 0} \frac{z^{2k}}{(2k)!}$$

$$\sinh(z) = \sum_{k \ge 0} \frac{z^{2k+1}}{(2k+1)!}$$

$$\log(1-x) = \sum_{k \ge 0} \frac{z^k}{k!} |z| < 1$$

$$\frac{\partial}{\partial z} \sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} a_{k+1} z^k$$

$$\sqrt{1+x} = (1+x)^{1/2} = 1 + (1/2)x + \frac{(1/2)(-1/2)}{2!} x^2 + \frac{(1/2)(-1/2)(-3/2)}{3!} x^3 + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{16}x^3 - \cdots$$

Fact 1.4.3

1.4 Series

Useful trick for expanding square roots:

$$\sqrt{z} = \sqrt{z_0 + z - z_0} = \sqrt{z_0 \left(1 + \frac{z - z_0}{z}\right)} = \sqrt{z_0} \sqrt{1 + u}, \quad u := \frac{z - z_0}{z}$$

$$\implies \sqrt{z} = \sqrt{z_0} \sum_{k > 0} \binom{1/2}{k} \left(\frac{z - z_0}{z}\right)^k.$$

2 | Calculus Preliminaries

2.1 Definitions

Definition 2.1.1 (Locally uniform convergence)

A sequence of functions f_n is said to converge **locally uniformly** on $\Omega \subseteq \mathbb{C}$ iff $f_n \to f$ uniformly on every compact subset $K \subseteq \Omega$.

Definition 2.1.2 (Equicontinuous Family)

A family of functions f_n is **equicontinuous** iff for every ε there exists a $\delta = \delta(\varepsilon)$ (not depending on n or f_n) such that $|x - y| < \varepsilon \implies |f_n(x) - f_n(y)| < \varepsilon$ for all n.

Remark 2.1.3: Recall Arzelà-Ascoli, an analog of Heine-Borel: for X compact Hausdorff, consider the the Banach space $C(X;\mathbb{R})$ equipped with the uniform norm $\|f\|_{\infty,X} := \sup_{x \in X} |f(x)|$. Then a subset $A \subseteq X$ is compact iff A is closed, uniformly bounded, and equicontinuous. As a consequence, if A is a sequence, it contains a subsequence converging uniformly to a continuous function. The proof is an $\varepsilon/3$ argument.

Definition 2.1.4 (Normal Family)

Remark 2.1.5: A continuous function on a compact set is uniformly continuous.

Definition 2.1.6 (Univalent functions)

A function $f \in \text{Hol}(U; \mathbb{C})$ is called **univalent** if f is injective.

Remark 2.1.7: If $f: \Omega \to \Omega'$ is a univalent surjection, f is invertible on Ω and f^{-1} is holomorphic. Compare to real functions: $f(x) = x^3$ is injective on (-c, c) for any c but f'(0) = 0 and $f^{-1}(x) := x^{1/3}$ is not differentiable at zero.

2.2 Theorems



Theorem 2.2.1 (Implicit Function Theorem).

Theorem 2.2.2 (Inverse Function Theorem).

For $f \in C^1(\mathbb{R}; \mathbb{R})$ with $f'(a) \neq 0$, then f is invertible in a neighborhood $U \ni a, g := f^{-1} \in C^1(U; \mathbb{R})$, and at b := f(a) the derivative of g is given by

$$g'(b) = \frac{1}{f'(a)}.$$

For $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with D_f invertible in a neighborhood of a, so $\det(J_f) \neq 0$, then setting b := F(a),

$$J_{F^{-1}}(q) = (J_F(p))^{-1}$$
.

The version for holomorphic functions: if $f \in \operatorname{Hol}(\mathbb{C};\mathbb{C})$ with $f'(p) \neq 0$ then there is a neighborhood $V \ni p$ with that $f \in \operatorname{BiHol}(V, f(V))$.

Theorem 2.2.3 (Green's Theorem).

If $\Omega \subseteq \mathbb{C}$ is bounded with $\partial \Omega$ piecewise smooth and $f, g \in C^1(\overline{\Omega})$, then

$$\int_{\partial\Omega} f \, dx + g \, dy = \iint_{\Omega} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dA.$$

In vector form,

$$\int_{\gamma} F \cdot dr = \iint_{R} \operatorname{curl} F \, dA.$$

2.3 Convergence

Remark 2.3.1: Recall that absolutely convergent implies convergent, but not conversely: $\sum k^{-1} = \infty$ but $\sum (-1)^k k^{-1} < \infty$. This converges because the even (odd) partial sums are monotone increasing/decreasing respectively and in (0,1), so they converge to a finite number. Their difference converges to 0, and their common limit is the limit of the sum.

Proposition 2.3.2 (Uniform Convergence of Series).

A series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly iff

$$\lim_{n \to \infty} \left\| \sum_{k \ge n} f_k \right\|_{\infty} = 0.$$

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Theorem 2.3.3 (Weierstrass M-Test).

If $\{f_n\}$ with $f_n: \Omega \to \mathbb{C}$ and there exists a sequence $\{M_n\}$ with $\|f_n\|_{\infty} \leq M_n$ and $\sum_{n \in \mathbb{N}} M_n < \infty$, then $f(x) := \sum_{n \in \mathbb{N}} f_n(x)$ converges absolutely and uniformly on Ω . Moreover, if the f_n are continuous, by the uniform limit theorem, f is again continuous.

2.4 Integrals



- grad $f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$.
 - If F = grad f for some f, F is a vector field.
- Given f(x,y) and $\gamma(t)$, the chain rule yields $\frac{\partial}{\partial t} (f \circ \gamma)(t) = \langle \operatorname{grad} f \circ \gamma)(t), \ \gamma'(t) \rangle$. For F(x,y) = [M(x,y), N(x,y)], $\operatorname{curl} F = \frac{\partial N}{\partial x} \frac{\partial M}{\partial y}$ and $\operatorname{Div} F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$.
- $\int_{\Omega} F \cdot dr = \int_{\Omega}^{b} F(\gamma(t)) \cdot \gamma'(t) dt$.

2.5 Series and Sequences

Remark 2.5.1: Note that if a power series converges uniformly, then summing commutes with integrating or differentiating.

Proposition 2.5.2 (Ratio Test).

Consider $\sum c_k z^k$, set $R = \lim \left| \frac{c_{k+1}}{c_k} \right|$, and recall the **ratio test**:

- $R \in (0,1) \implies$ convergence.
- $R \in (1, \infty] \implies$ divergence.
- R = 1 yields no information.

2.4 Integrals

Theorem 2.2 (Root Test). Suppose $\sum a_n(z-z_0)^n$ is a formal power series. Let

$$R = \liminf_{n \to \infty} |a_n|^{-\frac{1}{n}} = \frac{1}{\limsup_{n \to \infty} |a_n|^{\frac{1}{n}}} \in [0, +\infty].$$

Then $\sum_{n=0}^{\infty} a_n (z-z_0)^n$

- (a) converges absolutely in $\{z : |z z_0| < R\}$,
- (b) converges uniformly in $\{z : |z z_0| \le r\}$ for all r < R, and
- (c) diverges in $\{z: |z-z_0| > R\}$.

Figure 1: image_2021-05-27-15-40-58

Proposition 2.5.3 (Root Test).

Proposition 2.5.4(Radius of Convergence by the Root Test).

For
$$f(z) = \sum_{k \in \mathbb{N}} c_k z^k$$
, defining

$$\frac{1}{R} := \limsup_{k} |a_k|^{\frac{1}{k}},$$

then f converges absolutely and uniformly for $D_R := |z| < R$ and diverges for |z| > R. Moreover f is holomorphic in D_R , can be differentiated term-by-term, and $f' = \sum_{k \in \mathbb{N}} nc_k z^k$.

Fact 2.5.5

Recall the *p*-test:

$$\sum n^{-p} < \infty \iff p \in (1, \infty).$$

Fact 2.5.6

The product of two sequences is given by the Cauchy product

$$\sum a_k z^k \cdot \sum b_k z^k = \sum c_k z^k, \quad c_k := \sum_{j \le k} a_k b_{k-j}.$$

Fact 2.5.7

Recall how to carry out polynomial long division:

Polynomial long division

Fact 2.5.8 (Partial Fraction Decomposition)

- For every root r_i of multiplicity 1, include a term $A/(x-r_i)$.
- For any factors g(x) of multiplicity k, include terms A₁/g(x), A₂/g(x)², ···, A_k/g(x)^k.
 For irreducible quadratic factors h_i(x), include terms of the form Ax + B/h_i(x).

2.6 Exercises



Find the radius of convergences for the power series expansion of \sqrt{z} about $z_0 = 4 + 3i$.

Preliminaries

Definition 3.0.1 (Toy contour)

A closed Jordan curve that separates C into an exterior and interior region is referred to as a toy contour.

Fact 3.0.2 (Complex roots of a number)

The complex nth roots of $z := re^{i\theta}$ are given by

$$\left\{ \omega_k := r^{1/n} e^{i\left(\frac{\theta + 2k\pi}{n}\right)} \mid 0 \le k \le n - 1 \right\}.$$

Note that one root is $r^{1/n} \in \mathbb{R}$, and the rest are separated by angles of $2\pi/n$. Mnemonic:

$$z = re^{i\theta} = re^{i(\theta + 2k\pi)} \implies z^{1/n} = \cdots$$

3.1 Complex Log

Fact 3.1.1 (Complex Log)

For $z = re^{i\theta} \neq 0$, θ is of the form $\Theta + 2k\pi$ where $\Theta = \text{Arg } z$ We define

$$\log(z) = \ln(|z|) + i\operatorname{Arg}(z)$$

and $z^c := e^{c \log(z)}$. Thus

$$\log(re^{i\theta}) = \ln|r| + i\theta.$$

2.6 Exercises 13

Fact 3.1.2

Common trick:

$$f^{1/n} = e^{\frac{1}{n}\log(f)},$$

taking (say) a principal branch of log given by $\mathbb{C} \setminus (-\infty, 0] \times 0$.

Proposition 3.1.3 (Existence of complex log).

Suppose Ω is a simply-connected region such that $1 \in \Omega, 0 \notin \Omega$. Then there exists a branch of F(z) := Log(z) such that

- F is holomorphic on Ω ,
- $e^{F(z)} = z$ for all $z \in \Omega$
- $F(x) = \log(x)$ for $x \in \mathbb{R}$ in a neighborhood of 1.

Definition 3.1.4 (Principal branch and exponential)

Take \mathbb{C} and delete $\mathbb{R}^{\leq 0}$ to obtain the **principal branch** of the logarithm. Equivalently, this is define for all $z = re^{i\theta}$ where $\theta \in (-\pi, \pi)$.

Here the log is defined as

$$Log(z) := log(r) + i\theta$$
 $|\theta| < \pi$.

Similarly define

$$z^{\alpha} \coloneqq e^{\alpha \operatorname{Log}(z)}.$$

⚠ Warning 3.1.5

It's tempting to define

$$z^{\frac{1}{n}} := (re^{i\theta})^{\frac{1}{n}} = r^{\frac{1}{n}}e^{\frac{i\theta}{n}},$$

but this requires a branch cut to ensure continuity.

Remark 3.1.6: Note the problem: for $z := x + i0 \in \mathbb{R}^{\leq 0}$, just above the axis consider $z_+ := x + i\varepsilon$ and $z_- := x - i\varepsilon$. Then

- $\log(z_+) = \log|x| + i\pi$, and
- $\log(z_{-}) = \log|x| i\pi$.

So log can't even be made continuous if one crosses the branch. The issue is the **branch point** or **branch singularity** at z = 0.

Theorem 3.1.7 (Existence of log of a function).

If f is holomorphic and nonvanishing on a simply-connected region Ω , then there exists a holomorphic G on Ω such that

$$f(z) = e^{G(z)}.$$

3.1 Complex Log

3.2 Complex Calculus

Remark 3.2.1: When parameterizing integrals $\int_{\gamma} f(z) dz$, parameterize γ by θ and write $z = re^{i\theta}$ so $dz = ire^{i\theta} d\theta$.

⚠ Warning 3.2.2

 $f(z) = \sin(z), \cos(z)$ are unbounded on \mathbb{C} ! An easy way to see this: they are nonconstant and entire, thus unbounded by Liouville.

Example 3.2.3(?): You can show $f(z) = \sqrt{z}$ is not holomorphic by showing its integral over S^1 is nonzero. This is a direct computation:

$$\int_{S^1} z^{1/2} dz = \int_0^{2\pi} (e^{i\theta})^{1/2} i e^{i\theta} d\theta$$

$$= i \int_0^{2\pi} e^{\frac{i3\theta}{2}} d\theta$$

$$= i \left(\frac{2}{3i}\right) e^{\frac{i3\theta}{2}} \Big|_0^{2\pi}$$

$$= \frac{2}{3} \left(e^{3\pi i - 1}\right)$$

$$= -\frac{4}{3}.$$

Note an issue: a different parameterization yields a different (still nonzero) number

$$\cdots = \int_{-\pi}^{\pi} (e^{i\theta})^{1/2} i e^{i\theta} d\theta$$

$$= \frac{2}{3} \left(e^{\frac{3\pi i}{2}} - e^{\frac{-3\pi i}{2}} \right)$$

$$= -\frac{4i}{3}.$$

This is these are paths that don't lift to closed loops on the Riemann surface defined by $z \mapsto z^2$.

3.2.1 Holomorphy and Cauchy-Riemann

Definition 3.2.4 (Analytic)

A function $f: \Omega \to \mathbb{C}$ is analytic at $z_0 \in \Omega$ iff there exists a power series $g(z) = \sum a_n (z - z_0)^n$ with radius of convergence R > 0 and a neighborhood $U \ni z_0$ such that f(z) = g(z) on U.

Definition 3.2.5 (Complex differentiable / holomorphic /entire)

A function $f: \mathbb{C} \to \mathbb{C}$ is **complex differentiable** or **holomorphic** at z_0 iff the following limit

exists:

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(h)}{h}.$$

A function that is holomorphic on \mathbb{C} is said to be **entire**.

Equivalently, there exists an $\alpha \in \mathbb{C}$ such that

$$f(z_0 + h) - f(z_0) = \alpha h + R(h) \qquad R(h) \xrightarrow{h \to 0} 0.$$

In this case, $\alpha = f'(z_0)$.

Example 3.2.6 (Holomorphic vs non-holomorphic):

- f(z) := |z| is not holomorphic.
- $f(z) := \arg z$ is not holomorphic.
- $f(z) := \Re z$ is not holomorphic.
- $f(z) := \Im z$ is not holomorphic.
- $f(z) = \frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ but not holomorphic on \mathbb{C}
- $f(z) = \overline{z}$ is *not* holomorphic, but is real differentiable:

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{\overline{z_0}+\overline{h}-\overline{z_0}}{h} = \frac{\overline{h}}{h} = \frac{re^{-i\theta}}{re^{i\theta}} = e^{-2i\theta} \xrightarrow{h\to 0} e^{-2i\theta},$$

which is a complex number that depends on θ and is thus not a single value.

Definition 3.2.7 (Real (multivariate) differentiable)

A function $F: \mathbb{R}^n \to \mathbb{R}^m$ is **real-differentiable** at **p** iff there exists a linear transformation A such that

$$\frac{\|F(\mathbf{p} + \mathbf{h}) - F(\mathbf{p}) - A(\mathbf{h})\|}{\|\mathbf{h}\|} \stackrel{\|\mathbf{h}\| \to 0}{\to} 0.$$

Rewriting,

$$||F(\mathbf{p} + \mathbf{h}) - F(\mathbf{p}) - A(\mathbf{h})|| = ||\mathbf{h}|| ||R(\mathbf{h})|| \qquad ||R(\mathbf{h})|| \stackrel{||\mathbf{h}|| \to 0}{\longrightarrow} 0.$$

Equivalently,

$$F(\mathbf{p} + \mathbf{h}) - F(\mathbf{p}) = A(\mathbf{h}) + \|\mathbf{h}\| R(\mathbf{h}) \qquad \|R(\mathbf{h})\| \stackrel{\|\mathbf{h}\| \to 0}{\longrightarrow} 0.$$

Or in a slightly more useful form,

$$F(\mathbf{p} + \mathbf{h}) = F(\mathbf{p}) + A(\mathbf{h}) + R(\mathbf{h}) \qquad \qquad R \in o(\|\mathbf{h}\|), \text{ i.e. } \frac{\|R(\mathbf{h})\|}{\|\mathbf{h}\|} \xrightarrow{\mathbf{h} \to 0} 0.$$

Proposition 3.2.8 (Complex differentiable implies Cauchy-Riemann).

If f is differentiable at z_0 , then the limit defining $f'(z_0)$ must exist when approaching from any direction. Identify f(z) = f(x, y) and write $z_0 = x + iy$, then first consider $h \in RR$, so

Preliminaries Preliminaries

 $h = h_1 + ih_2$ with $h_2 = 0$. Then

$$f'(z_0) = \lim_{h_1 \to 0} \frac{f(x + h_1, y) - f(x, y)}{h_1} := \frac{\partial f}{\partial x}(x, y).$$

Taking $h \in i\mathbb{R}$ purely imaginary, so $h = ih_2$,

$$f'(z_0) = \lim_{ih_2 \to 0} \frac{f(x, y + h_2) - f(x, y)}{ih_2} := \frac{1}{i} \frac{\partial f}{\partial y}(x, y).$$

Equating,

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y},$$

and writing f = u + iv and 1/i = -i yields

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \frac{1}{i} \frac{\partial f}{\partial y} = \frac{1}{i} \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

Thus

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \,.$$

Proposition 3.2.9 (Polar Cauchy-Riemann equations).

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$
 and $\frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$.

Proof.

Setting

$$z = re^{i\theta} = r(\cos(\theta) + i\sin(\theta)) = x + iy$$

yields $x = r\cos(\theta), y = r\sin(\theta)$, one can identify

$$x_r = \cos(\theta), x_\theta = -r\sin(\theta)$$

$$y_r = \sin(\theta), y_\theta = r\cos(\theta).$$

Now apply the chain rule:

$$u_r = u_x x_r + u_y y_r$$

$$= v_y x_r - v_x y_r$$

$$= v_y \cos(\theta) - v_x \sin(\theta)$$

$$= \frac{1}{r} (v_y r \cos(\theta) - v_x r \sin(\theta))$$

$$= \frac{1}{r} (v_y y_\theta + v_x x_\theta)$$

$$= \frac{1}{r} v_\theta.$$
CR

Similarly,

$$\begin{aligned} v_r &= v_x x_r + v_y y_r \\ &= v_x \cos(\theta) + v_y \sin(\theta) \\ &= -u_y \cos(\theta) + u_x \sin(\theta) \end{aligned} \qquad \text{CR}$$

$$= \frac{1}{r} \left(-u_y r \cos(\theta) + u_x r \sin(\theta) \right) \\ &= \frac{1}{r} \left(-u_y y_\theta - u_x x_0 \right) \\ &= -\frac{1}{r} u_\theta. \end{aligned}$$

Thus

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$
 and $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$

Proposition $3.2.10 (Holomorphic\ functions\ are\ continuous.).$

f is holomorphic at z_0 iff there exists an $a \in \mathbb{C}$ such that

$$f(z_0 + h) - f(z_0) - ah = h\psi(h), \quad \psi(h) \stackrel{h \to 0}{\to} 0.$$

In this case, $a = f'(z_0)$.

Prove

3.2.2 Delbar and the Laplacian

Definition 3.2.11 (del and delbar operators)

$$\partial := \partial_z := \frac{1}{2} (\partial_x - i \partial_y)$$
 and $\bar{\partial} := \partial_{\bar{z}} = \frac{1}{2} (\partial_x + i \partial_y)$.

Moreover, $f' = \partial f + \bar{\partial} f$.

Proposition 3.2.12 (Holomorphic iff delbar vanishes).

f is holomorphic at z_0 iff $\bar{\partial} f(z_0) = 0$:

$$2\bar{\partial}f := (\partial_x + i\partial_y)(u + iv)$$

$$= u_x + iv_x + iu_y - v_y$$

$$= (u_x - v_y) + i(u_y + v_x)$$

$$= 0$$

by Cauchy-Riemann.

3.2.3 Harmonic Functions and the Laplacian

Definition 3.2.13 (Laplacian and Harmonic Functions)

A real function of two variables u(x,y) is **harmonic** iff it is in the kernel of the Laplacian operator:

$$\Delta u := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0.$$

Proposition 3.2.14 (Cauchy-Riemann implies holomorphic).

If f = u + iv with $u, v \in C^1(\mathbb{R})$ satisfying the Cauchy-Riemann equations on Ω , then f is holomorphic on Ω and

$$f'(z) = \partial f = \frac{1}{2} (u_x + iv_x).$$

Proposition 3.2.15 (Holomorphic functions have harmonic components).

If f(z) = u(x, y) + iv(x, y) is holomorphic, then u, v are harmonic.

Proof (?).

• By CR,

$$u_x = v_y u_y = -v_x.$$

• Differentiate with respect to x:

$$u_{xx} = v_{yx} u_{yx} = -v_{xx}.$$

• Differentiate with respect to y:

$$u_{xy} = v_{yy} u_{yy} = -v_{xy}.$$

• Clairaut's theorem: partials are equal, so

$$u_{xx} - v_{yx} = 0 \implies u_{xx} + u_{yy} = 0$$

$$v_{xx} + u_{yx} = 0 \implies v_{xx} + v_{yy} = 0$$

.

3.2.4 Exercises

Proposition 3.2.16 (Injectivity Relates to Derivatives).

If z_0 is a zero of f' of order n, then f is (n+1)-to-one in a neighborhood of z_0 .

 $\frac{Proof}{?}$

proof

Exercise 3.2.17 (Zero derivative implies constant)

Show that if f'=0 on a domain Ω , then f is constant on Ω

Solution:

Write f = u + iv, then $0 = 2f' = u_x + iv_x = u_y - iu_y$, so grad u = grad v = 0. Show f is constant along every straight line segment L by computing the directional derivative grad $u \cdot \mathbf{v} = 0$ along L connecting p, q. Then u(p) = u(q) = a some constant, and v(p) = v(q) = b, so f(z) = a + bi everywhere.

Exercise 3.2.18 (f and fbar holomorphic implies constant)

Show that if f and \bar{f} are both holomorphic on a domain Ω , then f is constant on Ω .

Solution:

- Strategy: show f' = 0.
- Write f = u + iv. Since f is analytic, it satisfies CR, so

$$u_x = v_y u_y = -v_x.$$

• Similarly write $\bar{f} = U + iV$ where U = u and V = -v. Since \bar{f} is analytic, it also satisfies CR, so

$$U_x = V_y U_y = -V_x$$

$$\implies u_x = -v_y$$
 $u_y = v_x$

- Add the LHS of these two equations to get $2u_x = 0 \implies u_x = 0$. Subtract the right-hand side to get $-2v_x = 0 \implies v_x = 0$
- Since f is analytic, it is holomorphic, so f' exists and satisfies $f' = u_x + iv_x$. But by above, this is zero.
- By the previous exercise, $f' = 0 \implies f$ is constant.

Exercise 3.2.19 (SS 1.13: Constant real/imaginary/magnitude implies constant) If f is holomorphic on Ω and any of the following hold, then f is constant:

- 1. $\Re(f)$ is constant.
- 2. $\Im(f)$ is constant.
- 3. |f| is constant.

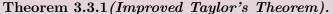
Solution:

Part 3:

- Write $|f| = c \in \mathbb{R}$.
- If c = 0, done, so suppose c > 0.
 Use f\(\bar{f} = |f|^2 = c^2\) to write \(\bar{f} = c^2/f\).
- Since $|f(z)| = 0 \iff f(z) = 0$, we have $f \neq 0$ on Ω , so \overline{f} is analytic.
- Similarly f is analytic, and f, \bar{f} analytic implies f' = 0 implies f is constant.

Finish

3.3 Power Series



If f is holomorphic on a region Ω with $D_R(z_0) \subseteq \Omega$, and for every $z \in D_r(z_0)$, f has a power series expansion of the following form:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$
 where $a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta$.

Proposition 3.3.2(Power Series are Smooth).

Any power series is smooth (and thus holomorphic) on its disc of convergence, and its derivatives can be obtained using term-by-term differentiation:

$$\frac{\partial}{\partial z} f(z) = \frac{\partial}{\partial z} \sum_{k \ge 0} c_k (z - z_0)^k = \sum_{k \ge 1} k c_k (z - z_0)^k.$$

Moreover, the coefficients are given by

$$c_k = \frac{f^{(n)}(z_0)}{n!}.$$

Remark 3.3.3: By an application of the Cauchy integral formula (see S&S 7.1) if f is holomorphic on $D_R(z_0)$ there is a formula for all $k \ge 0$ and all 0 < r < R:

$$c_k = \frac{1}{2\pi r^k} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta.$$

Proposition 3.3.4 (Exponential is uniformly convergent in discs).

 $f(z) = e^z$ is uniformly convergent in any disc in \mathbb{C} .

Proof.

Apply the estimate

$$|e^z| \le \sum \frac{|z|^n}{n!} = e^{|z|}.$$

Now by the M-test,

$$|z| \le R < \infty \implies \left| \sum \frac{z^n}{n!} \right| \le e^R < \infty.$$

Lemma 3.3.5 (Dirichlet's Test).

Given two sequences of real numbers $\{a_k\}, \{b_k\}$ which satisfy

- 1. The sequence of partial sums $\{A_n\}$ is bounded,
- $b_k \searrow 0.$

then

$$\sum_{k\geq 1} a_k b_k < \infty.$$

Proof (?).

See http://www.math.uwaterloo.ca/~krdavids/Comp/Abel.pdf

Use summation by parts. For a fixed $\sum a_k b_k$, write

$$\sum_{n=1}^{m} x_n Y_n + \sum_{n=1}^{m} X_n y_{n+1} = X_m Y_{m+1}.$$

Set $x_n := a_n, y_N := b_n - b_{n-1}$, so $X_n = A_n$ and $Y_n = b_n$ as a telescoping sum. Importantly, all y_n are negative, so $|y_n| = |b_n - b_{n-1}| = b_{n-1} - b_n$, and moreover $a_n b_n = x_n Y_n$ for all n. We have

$$\begin{split} \sum_{n\geq 1} a_n b_n &= \lim_{N\to\infty} \sum_{n\leq N} x_n Y_n \\ &= \lim_{N\to\infty} \sum_{n\leq N} X_N Y_N - \sum_{n\leq N} X_n y_{n+1} \\ &= -\sum_{n\geq 1} X_n y_{n+1}, \end{split}$$

where in the last step we've used that

$$|X_N| = |A_N| \le M \implies |X_N Y_N| = |X_N| |b_{n+1}| \le M b_{n+1} \to 0.$$

So it suffices to bound the latter sum:

$$\begin{split} \sum_{k \geq n} |X_k y_{k+1}| &\leq M \sum_{k \geq 1} |y_{k+1}| \\ &\leq M \sum_{k \geq 1} b_k - b_{k+1} \\ &\leq 2M (b_1 - b_{n+1}) \\ &< 2M b_1. \end{split}$$

Theorem 3.3.6 (Abel's Theorem).

If $\sum_{k=1}^{\infty} c_k z^j$ converges on |z| < 1 then

$$\lim_{z \to 1^{-}} \sum_{k \in \mathbb{N}} c_k z^k = \sum_{k \in \mathbb{N}} c_k.$$

Lemma 3.3.7 (Abel's Test).

If $f(z) := \sum_{k} c_k z^k$ is a power series with $c_k \in \mathbb{R}^{\geq 0}$ and $c_k \searrow 0$, then f converges on S^1 except possibly at z = 1.

Example 3.3.8 (application of Abel's theorem): What is the value of the alternating harmonic series? Integrate a geometric series to obtain

$$\sum_{n} \frac{(-1)^k z^k}{n} = \log(z+1)$$
 |z| < 1.

Since $c_k := (-1)^k/k \searrow 0$, this converges at z = 1, and by Abel's theorem $f(1) = \log(2)$.

Remark 3.3.9: The converse to Abel's theorem is false: take $f(z) = \sum (-z)^n = 1/(1+z)$. Then $f(1) = 1 - 1 + 1 - \cdots$ diverges at 1, but 1/1 + 1 = 1/2. So the limit $s := \lim_{x \to 1^-} f(x)1/2$, but $\sum a_n$ doesn't converge to s.

Proposition 3.3.10 (Summation by Parts).

Setting $A_n := \sum_{k=1}^n b_k$ and $B_0 := 0$,

$$\sum_{k=m}^{n} a_k b_k = A_n b_n - A_{m-1} b_m - \sum_{k=m}^{n-1} A_k (b_{k+1} - b_k).$$

Compare this to integrating by parts:

$$\int_a^b fg = F(b)g(b) - F(a)g(a) - \int_a^b Fg'.$$

Note there is a useful form for taking the product of sums:

$$A_n B_n = \sum_{k=1}^n A_k b_k + \sum_{k=1}^n a_k B_{k-1}.$$

Proof (?).

An inelegant proof: define $A_n := \sum_{k \le n} a_k$, use that $a_k = A_k - A_{k-1}$, reindex, and peel a top/bottom term off of each sum to pattern-match.

Behold:

$$\begin{split} \sum_{m \leq k \leq n} a_k b_k &= \sum_{m \leq k \leq n} (A_k - A_{k-1}) b_k \\ &= \sum_{m \leq k \leq n} A_k b_k - \sum_{m \leq k \leq n} A_{k-1} b_k \\ &= \sum_{m \leq k \leq n} A_k b_k - \sum_{m-1 \leq k \leq n-1} A_k b_{k+1} \\ &= A_n b_n + \sum_{m \leq k \leq n-1} A_k b_k - \sum_{m-1 \leq k \leq n-1} A_k b_{k+1} \\ &= A_n b_n - A_{m-1} b_m + \sum_{m \leq k \leq n-1} A_k b_k - \sum_{m \leq k \leq n-1} A_k b_{k+1} \\ &= A_n b_n - A_{m-1} b_m + \sum_{m \leq k \leq n-1} A_k (b_k - b_{k+1}) \\ &= A_n b_n - A_{m-1} b_m - \sum_{m \leq k \leq n-1} A_k (b_{k+1} - b_k). \end{split}$$

Proposition 3.3.11(?).

If f is non-constant, then f' is analytic and the zeros of f' are isolated. If f, g are analytic with f' = g', then f - g is constant.

3.3.1 Exercises: Series

Exercise 3.3.12 (Application of summation by parts)

Use summation by parts to show that $\sin(n)/n$ converges.

Exercise 3.3.13 (1.20: Series convergence on the circle)

Show that

- 1. $\sum kz^k$ diverges on S^1 . 2. $\sum k^{-2}z^k$ converges on S^1 . 3. $\sum k^{-1}z^k$ converges on $S^1\setminus\{1\}$ and diverges at 1.

plution: 1. Use that $|z^k| = 1$ and $\sum c_k z^k < \infty \implies |c_k| \to 0$, but $|kz^k| = |k| \to \infty$ here. 2. Use that absolutely convergent implies convergent, and $\sum |k^{-2}z^k| = \sum |k^{-2}|$ converges

- 3. If z=1, this is the harmonic series. Otherwise take $a_k=1/k, b_k=e^{ik\theta}$ where $\theta\in(0,2\pi)$ is some constant, and apply Dirichlet's test. It suffices to bound the partial sums of the b_k . Recalling that $\sum_{k\leq N} r^k = (1-r^{N+1})/(1-r)$,

$$\left\| \sum_{k \le m} e^{ik\theta} \right\| = \left\| \frac{1 - e^{i(m+1)\theta}}{1 - e^{i\theta}} \right\| \le \frac{2}{\|1 - e^{i\theta}\|} := M,$$

which is a constant. Here we've used that two points on S^1 are at most distance 2 from each other.

Exercise 3.3.14 (Laurent expansions inside and outside of a disc)

Expand $f(z) = \frac{1}{z(z-1)}$ in both

Solution:

$$\frac{1}{z(z-1)} = -\frac{1}{z}\frac{1}{1-z} = -\frac{1}{z}\sum z^k.$$

4

and

$$\frac{1}{z(z-1)} = \frac{1}{z^2(1-\frac{1}{z})} = \frac{1}{z^2} \sum \left(\frac{1}{z}\right)^k.$$

Exercise 3.3.15 (Laurent expansions about different points)

Find the Laurent expansion about z = 0 and z = 1 respectively of the following function:

$$f(z) \coloneqq \frac{z+1}{z(z-1)}.$$

Solution:

Note: once you see that everything is in terms of powers of $(z - z_0)$, you're essentially done. For z = 0:

$$\begin{split} \frac{z+1}{z(z-1)} &= \frac{1}{z} \frac{z+1}{z-1} \\ &= -\frac{z+1}{z} \frac{1}{1-z} \\ &= -\left(1 + \frac{1}{z}\right) \sum_{k \geq 0} z^k. \end{split}$$

For z = 1:

$$\begin{split} \frac{z+1}{z(z-1)} &= \frac{1}{z-1} \left(1 + \frac{1}{z} \right) \\ &= \frac{1}{z-1} \left(1 + \frac{1}{1-(1-z)} \right) \\ &= \frac{1}{z-1} \left(1 + \sum_{k \ge 0} (1-z)^k \right) \\ &= \frac{1}{z-1} \left(1 + \sum_{k \ge 0} (-1)^k (z-1)^k \right). \end{split}$$

Exercise 3.3.16 (?)

Show that a real-valued holomorphic function must be constant.

4 | Cauchy's Theorem

4.1 Complex Integrals



Definition 4.1.1 (Complex Integral)

$$\int_{\gamma} f dz := \int_{I} f(\gamma(t)) \gamma'(t) dt = \int_{\gamma} (u + iv) dx \wedge (-v + iu) dy.$$

Theorem 4.1.2 (Cauchy-Goursat Theorem).

If f is holomorphic on a region Ω with $\pi_1\Omega=1$, then for any closed path $\gamma\subseteq\Omega$,

$$\int_{\gamma} f(z) \, dz = 0.$$

Slogan 4.1.3

Closed path integrals of holomorphic functions vanish.

4.2 Applications of Cauchy's Theorem

4.2.1 Integral Formulas and Estimates

See reference

Theorem 4.2.1 (Cauchy Integral Formula).

Suppose f is holomorphic on Ω , then for any $z_0 \in \Omega$ and any open disc $\overline{D_R(z_0)}$ such that $\gamma := \partial \overline{D_R(z_0)} \subseteq \Omega$,

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z_0} d\xi$$

and

$$\frac{\partial^n f}{\partial z^n}(z_0) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi.$$

Proof. It follows from a consequence of Cauchy's theorem (see above) that if $C(z_0, r)$ denotes the circle of radius r around z_0 for a sufficiently small r > 0 then

$$\left| \frac{1}{2\pi i} \int_{C} \frac{f(z)}{z - z_{0}} dz - f(z_{0}) \right| = \left| \frac{1}{2\pi i} \int_{C(z_{0}, r)} \frac{f(z) - f(z_{0})}{z - z_{0}} dz \right|
= \left| \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{f(z_{0} + re^{i\theta}) - f(z_{0})}{re^{i\theta}} ire^{i\theta} d\theta \right|
\leq \frac{1}{2\pi} 2\pi \times \sup_{\theta \in [0, 2\pi)} |f(z_{0} + re^{i\theta}) - f(z_{0})|
\text{(by } ML \text{ inequality)}.$$

As f is continuous it follows that the righthand side goes to zero as r tends to zero. This completes the proof.

Figure 2: image_2021-05-27-16-54-06

Proof (?).

Proof (?).

Proof. (*) Using Cauchy's integral formula we can write that

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \to 0} \frac{1}{2\pi i h} \int_C (\frac{f(z)}{z - z_0 - h} - \frac{f(z)}{z - z_0}) dz$$

$$(C \text{ is so chosen that the point } z_0 + h \text{ is enclosed by } C)$$

$$= \lim_{h \to 0} \frac{1}{2\pi i h} \int_C \frac{f(z)h}{(z - z_0 - h)(z - z_0)} dz.$$

So we need to prove that

$$\left| \int_{C} \frac{f(z)}{(z - z_0 - h)(z - z_0)} dz - \int_{C} \frac{f(z)}{(z - z_0)^2} dz \right|$$

$$= \left| \int_{C} \frac{f(z)h}{(z - z_0 - h)(z - z_0)^2} dz \right| \to 0, \text{ as } h \to 0.$$

We will basically use ML inequality to prove this. Note that, as f is continuous it is bounded on C by M (say). Let $\alpha = \min\{|z - z_0| : z \in C\}$. Then $|z - z_0|^2 \ge \alpha^2$ and $\alpha \le |z - z_0| = |z - z_0 - h + h| \le |z - z_0 - h| + |h|$ and hence for $|h| \le \frac{\alpha}{2}$ (after all h is going to be small) we get $|z - z_0 - h| \ge \alpha - |h| \ge \frac{\alpha}{2}$. Therefore

$$\left| \int_C \frac{f(z)h}{(z-z_0-h)(z-z_0)^2} dz \right| \le \frac{M|h|l}{\frac{\alpha}{2}\alpha^2} = \frac{2M|h|l}{\alpha^3} \to 0,$$

as $h \to 0$. By repeating exactly the same technique we get $f^2(z_0) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^3} dz$ and so on.

Theorem 4.2.2 (Cauchy's Inequality / Cauchy's Estimate).

For $z_0 \in D_R(z_0) \subset \Omega$, setting $M := \sup_{z \in \gamma} |f(z)|$ so $|f(z)| \leq M$ on γ

$$\left| f^{(n)}(z_0) \right| \le \frac{n!}{2\pi} \int_0^{2\pi} \frac{M}{R^{n+1}} R \, d\theta = \frac{Mn!}{R^n}.$$

Proof (of Cauchy's inequality).

- Given $z_0 \in \Omega$, pick the largest disc $D_R(z_0) \subset \Omega$ and let $C = \partial D_R$.
- Then apply the integral formula.

$$\left| f^{(n)}(z_0) \right| = \left| \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right|
= \left| \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f\left(z_0 + re^{i\theta}\right) rie^{i\theta}}{(re^{i\theta})^{n+1}} d\theta \right|
\leq \frac{n!}{2\pi} \int_0^{2\pi} \left| \frac{f\left(z_0 + re^{i\theta}\right) rie^{i\theta}}{(re^{i\theta})^{n+1}} \right| d\theta
= \frac{n!}{2\pi} \int_0^{2\pi} \frac{\left| f\left(z_0 + re^{i\theta}\right) \right|}{r^n} d\theta
\leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{M}{r^n} d\theta
= \frac{Mn!}{r^n}.$$

Slogan 4.2.3

The *n*th Taylor coefficient of an analytic function is at most $\sup_{|z|=R} |f|/R^n$.

Theorem 4.2.4 (Mean Value Property for Holomorphic Functions).

If f is holomorphic on $D_r(z_0)$

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta = \frac{1}{\pi r^2} \iint_{D_r(z_0)} f(z) dA.$$

Taking the real part of both sides, one can replace f = u + iv with u.

4.2.2 Liouville

Theorem 4.2.5 (Liouville's Theorem).

If f is entire and bounded, f is constant.

Proof (of Liouville).

- Since f is bounded, $f(z) \leq M$ uniformly on \mathbb{C} .
- Apply Cauchy's estimate for the 1st derivative:

$$|f'(z)| \le \frac{1! ||f||_{C_R}}{R} \le \frac{M}{R} \stackrel{R \to \infty}{\longrightarrow} 0,$$

so f'(z) = 0 for all z.

Exercise 2.E. [SSh03, 2.15] Suppose f is continuous and non-zero on $\overline{\mathbb{D}}$ and holomorphic on \mathbb{D} such that |f(z)| = 1 for all |z| = 1. Show that f is then constant.

Figure 3: image_2021-05-17-11-54-14

Exercise 4.2.6 (?)

4.2.3 Continuation Principle

Theorem 4.2.7 (Continuation Principle / Identity Theorem).

If f is holomorphic on a bounded connected domain Ω and there exists a sequence $\{z_i\}$ with a limit point in Ω such that $f(z_i) = 0$, then $f \equiv 0$ on Ω .

Slogan 4.2.8

Two functions agreeing on a set with a limit point are equal on a domain.

Proof (?).

Apply Improved Taylor Theorem?

todo

Exercise 2.D. [SSh03, 2.13] If f is holomorphic on a region Ω and for each $z_0 \in \Omega$ at least one coefficient in the power series expansion $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$ is zero. Then show that f is a polynomial.

Figure 4: image_2021-05-17-11-53-33

Exercise 4.2.9 (?)

4.3 Exercises

Exercise 4.3.1 (Primitives imply vanishing integral)

Show that if f has a primitive F on Ω then $\int_{\gamma} f = 0$ for every closed curve $\gamma \subseteq \Omega$.

Exercise 4.3.2 (?)

Prove the uniform limit theorem for holomorphic functions: if $f_n \to f$ locally uniformly and each f_n is holomorphic then f is holomorphic.

Solution:

This is S&S Theorem 5.2. Statement: if $f_n \to f$ uniformly locally uniformly on Ω then f is holomorphic on Ω .

• Let $D \subset \Omega$ with $\overline{\mathbb{D}} \subset \Omega$ and $\Delta \subset D$ be a triangle.

4.3 Exercises 31

- Apply Goursat: ∫_Δ f_n = 0.
 f_n → f uniformly on Δ since it is closed and bounded and thus compact by Heine-Borel, so f is continuous and

$$\lim_{n} \int_{\Delta} f_n = \int_{\Delta} \lim_{n} f_n := \int_{\Delta} f.$$

• Apply Morera's theorem: $\int_{\Lambda} f$ vanishes on every triangle in Ω , so f is holomorphic on

Exercise 4.3.3 (?)

Prove that if $f_n \to f$ locally uniformly with f_n holomorphic, then $f'_n \to f'$ locally uniformly and f' is holomorphic.

Solution:

- Simplifying step: for some reason, it suffices to assume $f_n \to f$ uniformly on all of Ω ?
- Take Ω_R to be Ω with a buffer of R, so $d(z,\partial\Omega) > R$ for every $z \in \overline{\Omega_R}$.
- It suffices to show the following bound for F any holomorphic function on Ω :

$$\sup_{z \in \Omega_R} |F'(z)| \le \frac{1}{R} \sup_{\zeta \in \Omega} |F(\zeta)| \qquad \forall R,$$

where on the right we take the sup over all Ω .

- Then take $F := f_n f$ and $R \to 0$ to conclude, since the right-hand side is a constant not depending on Ω_R .
- For any $z \in \Omega_R$, we have $\overline{D_R(z)} \subseteq \Omega_R$, so Cauchy's integral formula can be applied:

$$\begin{split} |F'(z)| &= \left| \frac{1}{2\pi i} \int_{\partial D_R(z)} \frac{F(\xi)}{(\xi - z)^2} d\xi \right| \\ &\leq \frac{1}{2\pi} \int_{\partial D_R(z)} \frac{|F(\xi)|}{|\xi - z|^2} d\xi \\ &\leq \frac{1}{2\pi} \int_{\partial D_R(z)} \frac{\sup_{\zeta \in \Omega} |F(\zeta)|}{|\xi - z|^2} d\xi \\ &= \frac{1}{2\pi} \sup_{\zeta \in \Omega} |F(\zeta)| \int_{\partial D_R(z)} \frac{1}{R^2} d\xi \\ &= \frac{1}{2\pi} \sup_{\zeta \in \Omega} |F(\zeta)| \frac{1}{R^2} \int_{\partial D_R(z)} d\xi \\ &= \frac{1}{2\pi} \sup_{\zeta \in \Omega} |F(\zeta)| \frac{1}{R^2} 2\pi R \\ &\leq \frac{1}{2\pi} \sup_{\zeta \in \Omega} |F(\zeta)| \frac{1}{R^2} (2\pi R) \\ &= \frac{1}{R} \sup_{\zeta \in \Omega} |F(\zeta)|. \end{split}$$

Now

$$||f'_n - f'||_{\infty,\Omega_R} \le \frac{1}{R} ||f_n - f||_{\infty,\Omega},$$

where if R is fixed then by uniform convergence of $f_n \to f$, for n large enough $||f_n - f|| < \varepsilon/R$.

4.4 Morera's Theorem

Theorem 4.4.1 (Morera's Theorem).

If f is continuous on a domain Ω and $\int_T f = 0$ for every triangle $T \subset \Omega$, then f is holomorphic.

Slogan 4.4.2

If every integral along a triangle vanishes, implies holomorphic.

Corollary 4.4.3 (Sufficient condition for a sequence to converge to a holomorphic function).

If $\{f_n\}_{n\in\mathbb{N}}$ is a holomorphic sequence on a region Ω which uniformly converges to f on every compact subset $K\subseteq\Omega$, then f is holomorphic, and $f'_n\to f'$ uniformly on every such compact subset K.

4.4 Morera's Theorem 33

Proof (?).

Commute limit with integral and apply Morera's theorem.

Remark 4.4.4: This can be applied to series of the form $\sum_{k} f_k(z)$.

4.4.1 Symmetric Regions

In this section, take Ω to be a region symmetric about the real axis, so $z \in \Omega \iff \bar{z} \in \Omega$. Partition this set as $\Omega^+ \subseteq \mathbb{H}, I \subseteq \mathbb{R}, \Omega^- \subseteq \overline{\mathbb{H}}$.

Theorem 4.4.5 (Symmetry Principle).

Suppose that f^+ is holomorphic on Ω^+ and f^- is holomorphic on Ω^- , and f extends continuously to I with $f^+(x) = f^-(x)$ for $x \in I$. Then the following piecewise-defined function is holomorphic on Ω :

$$f(z) := \begin{cases} f^{+}(z) & z \in \Omega^{+} \\ f^{-}(z) & z \in \Omega^{-} \\ f^{+}(z) = f^{-}(z) & z \in I. \end{cases}$$

Proof (?).

Apply Morera?

Theorem 4.4.6 (Schwarz Reflection).

If f is continuous and holomorphic on \mathbb{H}^+ and real-valued on \mathbb{R} , then the extension defined by $F^-(z) = \overline{f(\overline{z})}$ for $z \in \mathbb{H}^-$ is a well-defined holomorphic function on \mathbb{C} .

Proof (?).

Apply the symmetry principle.

Remark 4.4.7: \mathbb{H}^+ , \mathbb{H}^- can be replaced with any region symmetric about a line segment $L \subseteq \mathbb{R}$.

5 Zeros and Singularities

Definition 5.0.1 (Singularity)

A point z_0 is an **isolated singularity** if $f(z_0)$ is undefined but f(z) is defined in a punctured neighborhood $D(z_0) \setminus \{z_0\}$ of z_0 .

There are three types of isolated singularities:

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- Removable singularities
- Poles
- Essential singularities

Definition 5.0.2 (Removable Singularities)

If z_0 is a singularity of f, then z_0 is a **removable singularity** iff there exists a holomorphic function g such that f(z) = g(z) in a punctured neighborhood of z_0 . Equivalently,

$$\lim_{z \to z_0} (z - z_0) f(z) = 0.$$

Equivalently, f is bounded on a neighborhood of z_0 .

Remark 5.0.3: Singularities can be classified by Laurent expansions $f(z) = \sum_{k \in \mathbb{Z}} c_k z^k$:

- Essential singularity: infinitely many negative terms.
- Pole of order N: truncated at k = -N, so $c_{N-\ell} = 0$ for all ℓ .
- Removable singularity: truncated at k = 0, so $c_{\leq -1} = 0$.

Example 5.0.4 (Removable singularities):

- $f(z) := \sin(z)/z$ has a removable singularity at z = 0, and one can redefine f(0) := 1.
- If f(z) = p(z)/q(z) with $q(z_0) = 0$ and $p(z_0) = 0$, then z_0 is removable with $f(z_0) := p'(z_0)/q'(z_0)$.

Example 5.0.5 (Essential singularities): $f(z) := e^{1/z}$ has an essential singularity at z = 0, since we can expand and pick up infinitely many negative terms:

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

In fact there exists a neighborhood of zero such that $f(U) = \mathbb{C} \setminus \{0\}$. Similarly $g(z) := \sin\left(\frac{1}{z}\right)$ has an essential singularity at z = 0, and there is a neighborhood V of zero such that $g(V) = \mathbb{C}$.

Example 5.0.6(?): The singularities of a rational function are always isolated, since there are finitely many zeros of any polynomial. The function F(z) := Log(z) has a singularity at z = 0 that is **not** isolated, since every neighborhood intersects the branch cut $(-\infty, 0) \times \{0\}$, where F is not even defined. The function $G(z) := 1/\sin(\pi/z)$ has a non-isolated singularity at 0 and isolated singularities at 1/n for all n.

⚠ Warning 5.0.7

 $f(z) := z^{\frac{1}{2}}$ has a singularity at zero that does not fall under this classification -z = 0 is a **branch** singularity and admits no Laurent expansion around z = 0.

A similar example: $(z(z-1))^{\frac{1}{2}}$ has two branch singularities at z=0,1.

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Theorem 5.0.8 (Extension over removable singularities).

If f is holomorphic on $\Omega \setminus \{z_0\}$ where z_0 is a removable singularity, then there is a unique holomorphic extension of f to all of Ω .

Proof (?).

Take γ to be a circle centered at z_0 and use

$$f(z) := \int_{\gamma} \frac{f(\xi)}{\xi - z} dx.$$

This is valid for $z \neq z_0$, but the right-hand side is analytic. (?)

Revisit

Theorem 5.0.9 (Improved Taylor Remainder Theorem).

If f is analytic on a region Ω containing z_0 , then f can be written as

$$f(z) = \left(\sum_{k=0}^{n-1} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k\right) + R_n(z) (z - z_0)^n,$$

where R_n is analytic.

Definition 5.0.10 (Zeros)

If f is analytic and not identically zero on Ω with $f(z_0) = 0$, then there exists a nonvanishing holomorphic function g such that

$$f(z) = (z - z_0)^n q(z).$$

We refer to z_0 as a **zero of order** n.

$\textbf{Definition 5.0.11} \ (\textbf{Poles (and associated terminology)})$

A pole z_0 of a function f(z) is a zero of $g(z) := \frac{1}{f(z)}$. Equivalently, $\lim_{z \to z_0} f(z) = \infty$. In this case there exists a minimal n and a holomorphic h such that

$$f(z) = (z - z_0)^{-n} h(z).$$

Such an n is the order of the pole. A pole of order 1 is said to be a simple pole.

Definition 5.0.12 (Principal Part and Residue)

If f has a pole of order n at z_0 , then there exist a holomorphic G in a neighborhood of z_0 such that

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \dots + \frac{a_{-1}}{z - z_0} + G(z) := P(z) + G(z).$$

The term P(z) is referred to as the *principal part of* f at z_0 consists of terms with negative degree, and the *residue* of f at z_0 is the coefficient a_{-1} .

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Definition 5.0.13 (Essential Singularity)

A singularity z_0 is essential iff it is neither removable nor a pole. Equivalently, a Laurent series expansion about z_0 has a principal part with infinitely many terms.

Theorem 5.0.14 (Casorati-Weierstrass).

If f is holomorphic on $\Omega \setminus \{z_0\}$ where z_0 is an essential singularity, then for every $V \subset \Omega \setminus \{z_0\}$, f(V) is dense in \mathbb{C} .

Slogan 5.0.15

The image of a punctured disc at an essential singularity is dense in \mathbb{C} .

Proof (of Casorati-Weierstrass).

Pick $w \in \mathbb{C}$ and suppose toward a contradiction that $D_R(w) \cap f(V)$ is empty. Consider

$$g(z) := \frac{1}{f(z) - w},$$

and use that it's bounded to conclude that z_0 is either removable or a pole for f.

Definition 5.0.16 (Singularities at infinity)

For any f holomorphic on an unbounded region, we say $z = \infty$ is a singularity (of any of the above types) of f if g(z) := f(1/z) has a corresponding singularity at z = 0.

Definition 5.0.17 (Meromorphic)

A function $f: \Omega \to \mathbb{C}$ is meromorphic iff there exists a sequence $\{z_n\}$ such that

- $\{z_n\}$ has no limit points in Ω .
- f is holomorphic in $\Omega \setminus \{z_n\}$.
- f has poles at the points $\{z_n\}$.

Equivalently, f is holomorphic on Ω with a discrete set of points delete which are all poles of f.

Theorem 5.0.18 (Meromorphic implies rational).

Meromorphic functions on \mathbb{C} are rational functions.

Proof(?).

Consider f(z) - P(z), subtracting off the principal part at each pole z_0 , to get a bounded entire function and apply Liouville.

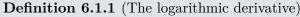
Theorem 5.0.19 (Riemann Extension Theorem).

A singularity of a holomorphic function is removable if and only if the function is bounded in some punctured neighborhood of the singular point.

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6 Counting Zeros and Poles

6.1 Argument Principle



The logarithmic derivative is defined as

$$\partial_{\log} f := \frac{f'}{f}.$$

It converts all poles and zeros of f into simple poles of $\partial_{\log f}$.

Exercise 6.1.2 (?)

Show that $\partial_{\log}(fg) = \partial_{\log}f + \partial_{\log}g$, i.e.

$$\frac{(fg)'}{fq} = \frac{f'}{f} + \frac{g'}{q}.$$

Definition 6.1.3 (Winding Number)

For $\gamma \subseteq \Omega$ a closed curve not passing through a point z_0 , the winding number of γ about z_0 (or the index) is defined as

$$\operatorname{Ind}_{z=z_0}(\gamma) \coloneqq \frac{1}{2\pi i} \int_{\gamma} \frac{1}{\xi - z_0} d\xi.$$

Theorem 6.1.4(Argument Principle, Zeros/Poles Version).

For f meromorphic in Ω with multisets of zeros $Z_f := \{z_j\}$ and poles $P_f := \{p_k\}$ (so repeated with multiplicity) for $\gamma := \partial \Omega$ not intersect

$$\frac{1}{2\pi i} \int_{\gamma} \partial_{\log} f(z) \, dz = \# Z_f - \# P_f,$$

where $\#Z_f$ and $\#P_f$ are the number of zeros and poles respectively, counted with multiplicity.

Proof(?).

- If z_0 is a zero of f of order m, write $f(z) = (z z_0)^m g(z)$ with g(z) holomorphic and nonzero on some neighborhood of z_0 .
- Compute

$$\partial_{\log} f(z) = \frac{m (z - z_0)^{m-1} g(z) + (z - z_0)^m g'(z)}{(z - z_0)^m g(z)}$$
$$= \frac{m}{z - z_0} + \partial_{\log} g(z),$$

so z_0 is a simple pole of $\partial_{\log} f$ and $\underset{z=z_0}{\operatorname{res}} \partial_{\log} f = m$.

• If z_0 is a pole of f of order m, write $f(z) = (z - z_0)^{-m} g(z)$, then

$$\partial_{\log} f = \frac{-m}{z - z_0} + \partial_{\log} g,$$

so z_0 is a simple pole and $\underset{z=z_0}{\text{Res }} \partial_{\log f} = -m$.

• Now apply the residue theorem, and group residues according to sign:

$$\begin{split} \frac{1}{2\pi i} \int_{\gamma} \partial_{\log} f(z) \, dz &= \sum_{z_i \in P_{\partial_{\log} f}} \mathop{\mathrm{Res}}_{z=z_i} \partial_{\log} f(z) \\ &= \sum_{z_k \in Z_f} \mathop{\mathrm{Res}}_{z=z_k} f(z) - \sum_{z_j \in P_f} \mathop{\mathrm{Res}}_{z=z_j} f(z). \end{split}$$

Theorem 6.1.5 (Argument Principle, Index Version).

With the same setup as above,

$$\frac{1}{2\pi i} \int_{\gamma} \partial_{\log} f(z) \, dz = \operatorname{Ind}_{w=0} (f \circ \gamma)(w).$$

Proof (?).

Make the change of variables w = f(z), then $z = \gamma(t) \mapsto w = (f \circ \gamma)(t)$ and dw = f'(z) dz, so

$$\frac{1}{2\pi i} \int_{\gamma} \partial_{\log} f(z) \, dz = \frac{1}{2\pi i} \int_{f \circ \gamma} \frac{1}{w} \, dw := \operatorname{Ind}_{w=0}(f \circ \gamma)(w).$$

Example 6.1.6 (Using the index version of the argument principle): Let $f(z) = z^2 + z = z(z+1)$.

• $\gamma_1 \coloneqq \{|z| = 2\}$ contains 2 zeros and 0 poles, so $f \circ \gamma$ winds twice around zero counterclockwise.

• $\gamma_2 := \left\{ |z| = \frac{1}{2} \right\}$ contains 1 zero and 0 poles, so $f \circ \gamma$ winds once.

6.2 Rouché

Theorem 6.2.1 (Rouch'e's Theorem). If

- f, g are meromorphic on Ω
- $\gamma \subset \Omega$ is a toy contour winding about each zero/pole of f, g exactly once,

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•
$$|g| < |f|$$
 on γ

then

$$\operatorname{Ind}_{z=0}(f\circ\gamma)(z)=\operatorname{Ind}_{z=0}((f+g)\circ\gamma)(z)\implies Z_f-P_f=Z_{f+g}-P_{f+g}.$$

In particular, if f, g are holomorphic, they have the same number of zeros in Ω .

Slogan 6.2.2

The number of zeros/poles are determined by a dominating function.

Prove

Exercise 6.2.3 (?)

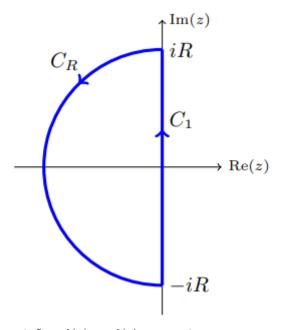
Show that $h(z) = z^5 + 3z + 1$ has 5 zeros in $|z| \le 2$.

Exercise 6.2.4 (?)

Show that $h(z) = z + 3 + 2e^z$ has one root in $\{\Re(z) \le 0\}$.

Solution:

Use the following contour:



Take $g(z) \coloneqq 2e^z < f(z) \coloneqq f(z) \coloneqq z + 3$.

Corollary 6.2.5 (Open Mapping).

Any holomorphic non-constant map is an open map.

Prove

6.2 Rouché 40

Corollary 6.2.6 (Maximum Modulus).

If f is holomorphic and nonconstant on an open connected region Ω , then |f| can not attain a maximum on Ω . If Ω is bounded and f is continuous on $\overline{\Omega}$, then $\max_{\overline{\Omega}} |f|$ occurs on $\partial\Omega$.

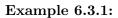
Conversely, if f attains a local supremum at $z_0 \in \Omega$, then f is constant on Ω .

Prove

Corollary 6.2.7(?).

If f is nonzero on Ω , then f attains a minimum on $\partial\Omega$. This follows from applying the MMP to 1/f.

6.3 Counting Zeros



- Take $P(z) = z^4 + 6z + 3$.
- On |z| < 2:
 - Set $f(z) = z^4$ and g(z) = 6z + 3, then $|g(z)| \le 6|z| + 3 = 15 < 16 = |f(z)|$.
 - So P has 4 zeros here.
- On |z| < 1:
 - Set f(z) = 6z and $g(z) = z^4 + 3$.
 - Check $|g(z)| \le |z|^4 + 3 = 4 < 6 = |f(z)|$.
 - − So P has 1 zero here.

Example 6.3.2:

- Claim: the equation $\alpha z e^z = 1$ where $|\alpha| > e$ has exactly one solution in \mathbb{D} .
- Set $f(z) = \alpha z$ and $g(z) = e^{-z}$.
- Estimate at |z|=1 we have $|g|=\left|e^{-z}\right|=e^{-\Re(z)}\leq e^1<|\alpha|=|f(z)|$
- f has one zero at $z_0 = 0$, thus so does f + g.

7 Residues

7.1 Basics

Remark 7.1.1: Check: do you need residues at all?? You may be able to just compute an integral!

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• Directly by parameterization:

$$\int_{\gamma} f dz = \int_{a}^{b} f(z(t)) z'(t) dt \qquad \text{for } z(t) \text{ a parameterization of } \gamma,$$

• Finding a primitive F, then

$$\int_{\gamma} f = F(b) - F(a).$$

- Note: you can parameterize a circle around z_0 using

$$z = z_0 + re^{i\theta}.$$

Fact 7.1.2 (Integrating z^k around S^1 powers residues) The major fact that reduces integrals to residues:

$$\int_{\gamma} z^k dz = \int_0^{2\pi} e^{ik\theta} i e^{i\theta} d\theta = \int_0^{2\pi} e^{i(k+1)\theta} d\theta = \begin{cases} 2\pi i & k = -1 \\ 0 & \text{else.} \end{cases}.$$

Thus

$$\int \sum_{k>-M} c_k z^k = \sum_{k>-M} \int c_k z^k = 2\pi i c_{-1},$$

i.e. the integral picks out the c_{-1} coefficient in a Laurent series expansion.

Example 7.1.3(?): Consider

$$f(z) := \frac{e^{iz}}{1 + z^2}$$

where $z \neq \pm i$, and attempt to integrate

$$\int_{\mathbb{R}} f(z) \, dz.$$

Use a semicircular contour γ_R where $z = Re^{it}$ and check

$$\begin{split} \sup_{z \in \gamma_R} |f(z)| &= \max_{t \in [0,\pi} \frac{1}{1 + (Re^{it})^2} \\ &= \max_{t \in [0,\pi} \frac{1}{1 + R^2 e^{2it}} \\ &= \frac{1}{R^2 - 1}. \end{split}$$

7.2 Estimates

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Proposition 7.2.1(Length bound / ML Estimate).

$$\left| \int_{\gamma} f \right| \leq ML \coloneqq \sup_{z \in \gamma} |f| \cdot \operatorname{length}(\gamma).$$

Proof (?).

$$\left| \int_{\gamma} f(z) dz \right| \leq \sup_{t \in [a,b]} |f(z(t))| \int_{a}^{b} |z'(t)| dt \leq \sup_{z \in \gamma} |f(z)| \cdot \operatorname{length}(\gamma).$$

Proposition 7.2.2 (Jordan's Lemma).

Suppose that $f(z) = e^{iaz}g(z)$ for some g, and let $C_R := \{z = Re^{it} \mid t \in [0, \pi]\}$. Then

$$\left| \int_{C_R} f(z) \, dz \right| \le \frac{\pi M_R}{a}$$

where $M_R := \sup_{t \in [0,\pi]} \left| g(Re^{it}) \right|$.

Proof (?).

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$$\begin{split} \left| \int_{C_R} f(z) \, dz \right| &= \left| \int_{C_R} e^{iaz} g(z) \, dz \right| \\ &= \left| \int_{[0,\pi]} e^{ia \left(Re^{it}\right)} g(Re^{it}) i Re^{it} \, dt \right| \\ &\leq \int_{[0,\pi]} \left| e^{ia \left(Re^{it}\right)} g(Re^{it}) i Re^{it} \right| \, dt \\ &= R \int_{[0,\pi]} \left| e^{ia \left(Re^{it}\right)} g(Re^{it}) \right| \, dt \\ &\leq R M_R \int_{[0,\pi]} \left| e^{ia \left(Re^{it}\right)} \right| \, dt \\ &= R M_R \int_{[0,\pi]} e^{\Re(iaRe^{it})} \, dt \\ &= R M_R \int_{[0,\pi]} e^{\Re(iaR(\cos(t)+i\sin(t)))} \, dt \\ &= R M_R \int_{[0,\pi]} e^{-aR\sin(t)} \, dt \\ &= 2R M_R \int_{[0,\pi/2]} e^{-aR\sin(t)} \, dt \\ &\leq 2R M_R \int_{[0,\pi/2]} e^{-aR\left(\frac{2t}{\pi}\right)} \, dt \\ &= 2R M_R \left(\frac{\pi}{2aR}\right) \left(1 - e^{-aR}\right) \\ &= \frac{\pi M_R}{a}. \end{split}$$

where we've used that on $[0, \pi/2]$, there is an inequality $2t/\pi \le \sin(t)$. This is obvious from a picture, since $\sin(t)$ is a height on S^1 and $2t/\pi$ is a height on a diagonal line:

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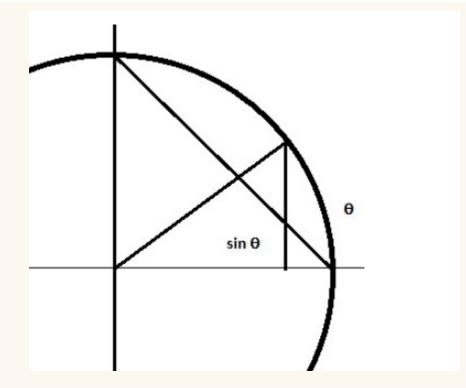


Figure 5: $image_2021-06-09-01-29-22$

7.3 Residue Formulas

Theorem 7.3.1 (The Residue Theorem).

Let f be meromorphic on a region Ω with poles $\{z_1, z_2, \dots, z_N\}$. Then for any $\gamma \in \Omega \setminus \{z_1, z_2, \dots, z_N\}$,

$$\frac{1}{2\pi i} \int_{\gamma} f(z) dz = \sum_{j=1}^{N} n_{\gamma}(z_j) \operatorname{Res}_{z=z_j} f.$$

If γ is a toy contour, then

$$\frac{1}{2\pi i} \int_{\gamma} f \, dz = \sum_{j=1}^{N} \operatorname{Res}_{z=z_{j}} f.$$

Proposition 7.3.2 (Residue formula for higher order poles).

If f has a pole z_0 of order n, then

$$\operatorname{Res}_{z=z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{\partial}{\partial z} \right)^{n-1} (z - z_0)^n f(z).$$

Proposition 7.3.3 (Residue formula for simple poles).

As a special case, if z_0 is a simple pole of f, then

$$\operatorname{Res}_{z=z_0} f = \lim_{z \to z_0} (z - z_0) f(z).$$

Corollary 7.3.4(Better derivative formula that sometimes works for simple poles). If additionally f = g/h where $h(z_0) = 0$ and $h'(z_0) \neq 0$,

$$\operatorname{Res}_{z=z_0} \frac{g(z)}{h(z)} = \frac{g(z_0)}{h'(z_0)}.$$

Proof (?).

Apply L'Hopital:

$$(z - z_0) \frac{g(z)}{h(z)} = \frac{(z - z_0)g(z)}{h(z)} \stackrel{LH}{=} \frac{g(z) + (z - z_0)g'(z)}{h'(z)} \stackrel{z \to z_0}{\longrightarrow} \frac{g(z_0)}{h'(z_0)}.$$

Example 7.3.5 (Residue of a simple pole (order 1)): Let $f(z) = \frac{1}{1+z^2}$, then $g(z) = 1, h(z) = 1+z^2$, and h'(z) = 2z so that $h'(i) = 2i \neq 0$. Thus

$$\operatorname{Res}_{z=i} \frac{1}{1+z^2} = \frac{1}{2i}.$$

Proposition 7.3.6 (Residue at infinity).

$$\mathop{\rm Res}_{z=\infty} f(z) = \mathop{\rm Res}_{z=0} g(z) \qquad \qquad g(z) \coloneqq -\frac{1}{z^2} f\left(\frac{1}{z}\right).$$

7.3.1 Exercises

Some good computations here.

Exercise 7.3.7

Show that the complex zeros of $f(z) := \sin(\pi z)$ are exactly \mathbb{Z} , and each is order 1. Calculate the residue of $1/\sin(\pi x)$ at $z = n \in \mathbb{Z}$.

Exerci

Exe

Exercise 3.A. [SSh03, 3.1] Show that the complex zeros of $\sin \pi z$ are exactly at the integers, and are each of order 1. Calculate the residue of $1/\sin \pi x$ are $z=n\in\mathbb{Z}$.

Exercise 3.C. [SSh03, 3.8] Prove that

 $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$

Exercise 7.3.8 (?)

$$\int_{\mathbb{R}} \frac{1}{(1+x^2)^2} \, dx.$$

Solution:

- Factor $(1+z^2)^2 = ((z-i)(z+i))^2$, so f has poles at $\pm i$ of order 2.
- Take a semicircular contour $\gamma := I_R \cup D_R$, then $f(z) \approx 1/z^4 \to 0$ for large R and $\int_{D_R} f \to 0$.
- $$\begin{split} &\int_{D_R} f \to 0. \\ \bullet & \text{ Note } \int_{I_R} f \to \int_{\mathbb{R}} f, \text{ so } \int_{\gamma} f \to \int_{\mathbb{R}} f. \end{split}$$
- $\int_{\gamma} f = 2\pi i \sum_{z=0}^{\infty} \lim_{z=z_0} f$, and $z_0 = i$ is the only pole in this region.
- Compute

$$\operatorname{Res}_{z=i} f = \lim_{z \to i} \frac{1}{(2-1)!} \frac{\partial}{\partial z} (z-i)^2 f(z)$$

$$= \lim_{z \to i} \frac{\partial}{\partial z} \frac{1}{(z+i)^2}$$

$$= \lim_{z \to i} \frac{-2}{(z+i)^3}$$

$$= -\frac{2}{(2i)^3}$$

$$= \frac{1}{4i}$$

$$\implies \int_{\gamma} f = \frac{2\pi i}{4i} = \pi/2,$$

Exercise 7.3.9 (?)

Use a direct Laurent expansion to show

$$\operatorname{Res}_{z=0} \frac{1}{z - \sin(z)} = \frac{3!}{5 \cdot 4}.$$

Note the necessity: one doesn't know the order of the pole at zero, so it's unclear how many derivatives to take.

Solution:

Expand:

$$\frac{1}{z - \sin(z)} = z^{-1} \left(1 - z^{-1} \sin(z) \right)^{-1}$$

$$= z^{-1} \left(1 - z^{-1} \left(z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 - \cdots \right) \right)^{-1}$$

$$= z^{-1} \left(1 - \left(1 - \frac{1}{3!} z^2 + \frac{1}{5!} z^4 - \cdots \right) \right)^{-1}$$

$$= z^{-1} \left(\frac{1}{3!} z^2 - \frac{1}{5!} z^4 + \cdots \right)^{-1}$$

$$= z^{-1} \cdot 3! z^{-2} \left(1 - \frac{1}{5!/3!} z^2 + \cdots \right)^{-1}$$

$$= \frac{3!}{z^3} \left(\frac{1}{1 - \left(\frac{1}{5 \cdot 4} z^2 + \cdots \right)} \right)$$

$$= \frac{3!}{z^3} \left(1 + \left(\frac{1}{5 \cdot 4} z^2 \right) + \left(\frac{1}{5 \cdot 4} z^2 \right)^2 + \cdots \right)$$

$$= 3! z^{-3} + \frac{3!}{5 \cdot 4} z^{-1} + O(z)$$

Exercise 7.3.10 (?)

Compute

$$\operatorname{Res}_{z=0} \frac{1}{z^2 \sin(z)}.$$

Solution:

First expand $(\sin(z))^{-1}$:

$$\begin{split} \frac{1}{\sin(z)} &= \left(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots\right)^{-1} \\ &= z^{-1} \left(1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \cdots\right)^{-1} \\ &= z^{-1} \left(1 + \left(\frac{1}{3!}z^2 - \frac{1}{5!}z^4 + \cdots\right) + \left(\frac{1}{3!}z^2 - \cdots\right)^2 + \cdots\right) \\ &= z^{-1} \left(1 + \frac{1}{3!}z^2 \pm O(z^4)\right), \end{split}$$

using that $(1-x)^{-1} = 1 + x + x^2 + \cdots$

Thus

$$z^{-2} (\sin(z))^{-1} = z^{-2} \cdot z^{-1} \left(1 + \frac{1}{3!} z^2 \pm O(z^4) \right)$$
$$= z^{-3} + \frac{1}{3!} z^{-1} + O(z).$$

Exercise 7.3.11 (Keyhole contour and ML estimate)

Compute

$$\int_{[0,\infty]} \frac{\log(x)}{(1+x^2)^2} \, dx.$$

Solution:

Factor $(1+z^2)^2 = (z+i^2(z-i)^2)$. Take a keyhole contour similar to the following:

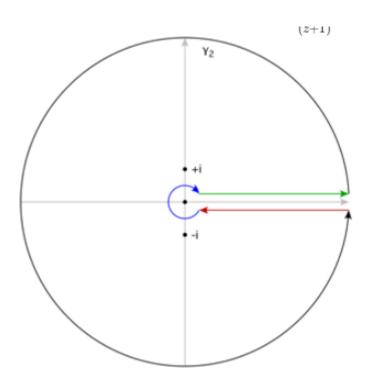


Figure 6: image_2021-06-09-02-11-59

Show that outer radius R and inner radius ρ circles contribute zero in the limit by the ML estimate? Compute the residues by just applying the formula and manually computing derivatives:

$$\begin{aligned} \operatorname*{Res}_{z=\pm i} f(z) &= \lim_{z \to \pm i} \frac{\partial}{\partial z} \frac{\log^2(z)}{(z \pm i)^2} \\ &= \lim_{z \to \pm i} \frac{2 \log(z) (z \pm i)^2 - 2 (z \pm i)^2 \log^2(z)}{((z \pm i)^2)^2} \\ &= \frac{2 \log(\pm i) (\pm 2i)^2 - 2 (\pm 2i)^2 \log^2(\pm i)}{(\pm 2i)^4} \\ &=_{?} \frac{\pi}{4} \pm \frac{i\pi^2}{16}. \end{aligned}$$

See p.4: http://www.math.toronto.edu/mnica/complex1.pdf

Exercise 7.3.12 (Sinc Function) Show

$$\int_{(0,\infty)} \frac{\sin(x)}{x} \, dx = \frac{\pi}{2}.$$

8 Integrals

Solution:

Take an indented semicircle. Let I be the original integral, then

$$I = \frac{1}{2i} \int_{\mathbb{R}} \frac{e^{iz} - 1}{z} \, dz.$$

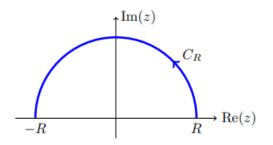
Exercise 3.E. [SSh03, 3.14] Prove that all entire functions that are also injective take the form f(z) = az + b with $a, b \in \mathbb{C}$ and $a \neq 0$.

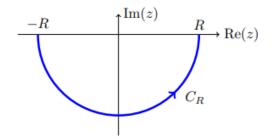
Figure 7: image_2021-05-17-13-33-55

8 | Integrals

See this very detailed note.

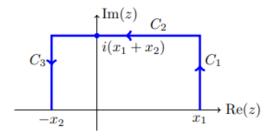
• For integrals that decay faster than $1/z^{\alpha}$, $\alpha > 1$: semicircular contours.

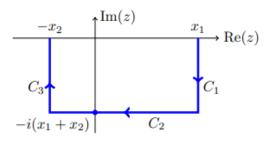




• For integrals that decay like 1/z: rectangular contours.

where $\psi_1 + \psi_2 + \psi_3$ is the recombiguing point shown selection on the reference.





Rectangular paths of height and width $x_1 + x_2$.

• If a trigonometric function is in the numerator, check if $I \approx \Re(\tilde{I})$ where \tilde{I} replaces cosines/sines with e^{iz} .

Integrals 51

8

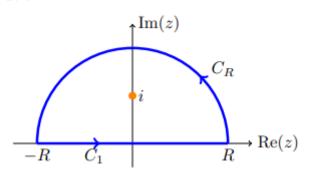
• For rational functions of cos, sin: set $2\cos(z)=z+z^{-1}, 2\sin(z)=z-z^{-1}, d\theta=\frac{dz}{iz}$ to reduce to a residue count in $|z|\leq 1$.

Exercise 8.0.1 (?)

$$\int_{\mathbb{R}} \frac{1}{(1+x)^2} = \frac{\pi}{2}.$$

Use that $f(z) \sim 1/z^4$.

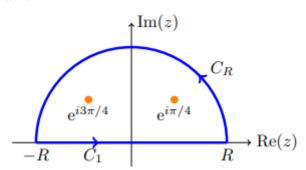
Solution:



Exercise 8.0.2 (?)

$$\int_{\mathbb{R}} \frac{1}{x^4 + 1} = \frac{\pi\sqrt{2}}{2}.$$

Solution:



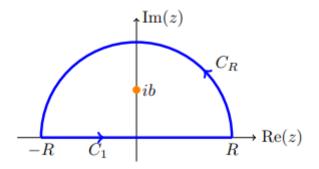
Exercise 8.0.3 (?)

$$\int_0^\infty \frac{\cos(x)}{x^2 + b^2} dx = \frac{\pi e^{-b}}{2b}..$$

8 Integrals

Solution:

Extend to $\int_{\mathbb{P}}$ using that f is even.



Exercise 8.0.4 (Trigonometric functions)

$$\int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos(\theta)} = \begin{cases} \frac{2\pi}{a^2 - 1} & \text{if } |a| > 1\\ \frac{2\pi}{1 - a^2} & \text{if } |a| < 1 \end{cases}.$$

Solution:

Write $2\cos(z) = z + z^{-1}$ on S^1 to get

$$=\int_{\left|z\right|=1}\frac{1}{i\left(\left(1+a^{2}\right)z-a\left(z^{2}+1\right)\right)}dz.$$

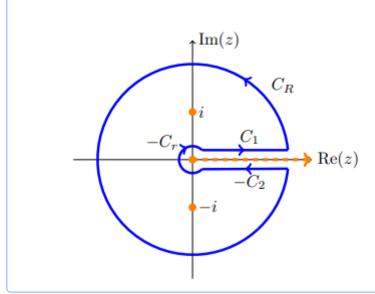
8.1 Branch Cuts

Exercise 8.1.1 (?)

$$\int_0^\infty \frac{x^{\frac{1}{3}}}{1+x^2} \, dx = \frac{\pi}{\sqrt{3}}.$$

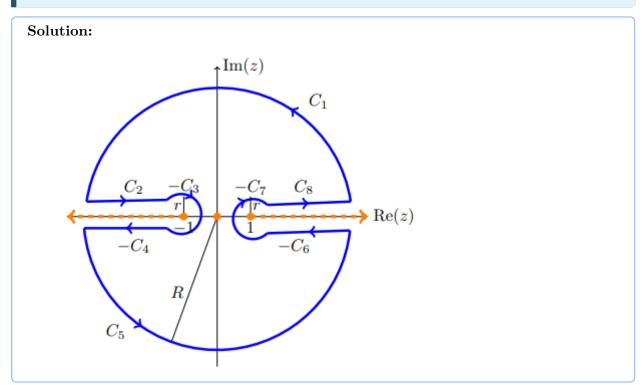
Solution:

9 Integrals



Exercise 8.1.2 (?)

$$\int_{1}^{\infty} \frac{dx}{x\sqrt{x^2 - 1}} = \frac{\pi}{2}.$$



9 Conformal Maps / Linear Fractional Transformations

Definition 9.0.1 (Conformal Map / Biholomorphism)

A map f is **conformal** on Ω iff f is complex-differentiable, $f'(z) \neq 0$ for $z \in \Omega$, and f preserves signed angles (so f is orientation-preserving). Conformal implies holomorphic, and a bijective conformal map has conformal inverse automatically.

A bijective conformal map $f:U\to V$ biholomorphism, and we say U and V are biholomorphic. Importantly, bijective holomorphic maps always have holomorphic inverses. Self-biholomorphisms of a domain Ω form a group $\operatorname{Aut}(\Omega)$.

Remark 9.0.2: There is an oft-used weaker condition that $f'(z) \neq 0$ for any point. Note that that this condition alone doesn't necessarily imply f is holomorphic, since anti-holomorphic maps may have nonzero derivatives. For example, take $f(z) = \bar{z}$, so f(x+iy) = x - iy – this does not satisfy the Cauchy-Riemann equations.

Remark 9.0.3: A bijective holomorphic map automatically has a holomorphic inverse. This can be weakened: an injective holomorphic map satisfies $f'(z) \neq 0$ and f^{-1} is well-defined on its range and holomorphic.

Definition 9.0.4 (Linear fractional transformation / Mobius transformation) A map of the following form is a *linear fractional transformation*:

$$T(z) = \frac{az+b}{cz+d},$$

where the denominator is assumed to not be a multiple of the numerator. These have inverses given by

$$T^{-1}(w) = \frac{dw - b}{-cw + a}.$$

Proposition 9.0.5(?).

Given any three points z_1, z_2, z_3 , the following Möbius transformation sends them to $1, 0, \infty$ respectively:

$$T(z) := \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$
$$z_1 \mapsto 0$$
$$z_2 \mapsto 1$$
$$z_3 \mapsto \infty.$$

Such a map is sometimes denoted $(z; z_1, z_2, z_3)$. One can use this to produce a map sending any three points to any other three points:

$$T(z) := (w; w_1, w_2, w_3)^{-1} \circ (z; z_1, z_2, z_3).$$

Example 9.0.6(?):

- $(z, i, 1, -1) : \mathbb{D} \to \mathbb{H}$
- $(z,0,-1,1): \mathbb{D} \cap \mathbb{H} \to Q_1$.

Theorem 9.0.7 (Cayley Transform).

The fractional linear transformation given by $F(z) = \frac{i-z}{i+z}$ maps $\mathbb{D} \to \mathbb{H}$ with inverse $G(w) = i\frac{1-w}{1+w}$.

Theorem 9.0.8 (Characterization of conformal maps).

Conformal maps $\mathbb{D} \to \mathbb{D}$ have the form

$$g(z) = \lambda \frac{1-a}{1-\bar{a}z}, \quad |a| < 1, \quad |\lambda| = 1.$$

Theorem 9.0.9 (Riemann Mapping).

If Ω is simply connected, nonempty, and not \mathbb{C} , then for every $z_0 \in \Omega$ there exists a unique conformal map $F: \Omega \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$.

Thus any two such sets Ω_1, Ω_2 are conformally equivalent.

9.1 By Type

Notation	Definition
$\mathbb{D} \coloneqq \left\{ z \mid z \le 1 \right\}$	The unit disc
$\mathbb{H} := \left\{ x + iy \mid y > 0 \right\}$	The upper half-plane
$X_{rac{1}{2}}$	A "half version of X ", see examples
$\mathbb{H}_{rac{1}{2}}$	The first quadrant
$\mathbb{H}_{rac{1}{2}}$ $\mathbb{D}_{rac{1}{2}}$	The portion of the first quadrant inside the unit disc
$L := \left\{ x + iy \mid x \in \mathbb{R}, \ 0 < y < \pi \right\}$	The horizontal strip

Remark 9.1.1 (Notation):

Theorem 9.1.2 (Classification of Conformal Maps).

There are 8 major types of conformal maps:

Type/Domains	Formula
Translation	$z \mapsto z + h$
Dilation	$z \mapsto cz$
Rotation	$z\mapsto e^{i heta}$

Sectors to sectors	$z\mapsto z^n$
$\mathbb{D}_{\frac{1}{2}} \to \mathbb{H}_{\frac{1}{2}}$, the first quadrant	$z \mapsto \frac{1+z}{1-z}$
$\mathbb{H} o S$	$z \mapsto \log(z)$
$\mathbb{D}_{rac{1}{2}} ightarrow L_{rac{1}{2}}$	$z \mapsto \log(z)$
$S_{rac{1}{2}}^{2} ightarrow \mathbb{D}_{rac{1}{2}}^{2}$	$z \mapsto e^{iz}$
$\mathbb{D}_{rac{1}{2}} o\mathbb{H}$	$z \mapsto \frac{1}{2} \left(z + \frac{1}{z} \right)$
$L_{rac{1}{2}} o \mathbb{H}$	$z\mapsto\sin(z)$

Pictures!

Proposition 9.1.3 (Half-plane to Disc).

$$F: \mathbb{H}^{\circ} \rightleftharpoons \mathbb{D}^{\circ}$$

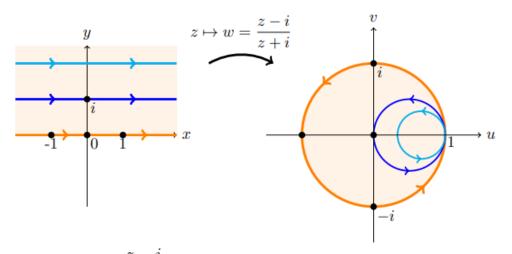
$$\left\{z \mid \Im(z) > 0\right\} \rightleftharpoons \left\{w \mid |w| < 1\right\}$$

$$z \mapsto \frac{i-z}{i+z}$$

$$i\left(\frac{1-w}{1+w}\right) \longleftrightarrow w.$$

Boundary behavior:

• This maps $\mathbb{R} \to \partial \mathbb{D}$, where $F(\infty) = -1$, and as $x \in \mathbb{R}$ ranges from $-\infty \to \infty$, F(x) travels from z = -1 counter-clockwise through S^1 (starting at z = -1 and moving through the lower half first).



So this extends to a map $\mathbb{H} \to \mathbb{D}$.

Mnemonic: every $z \in \mathbb{H}$ is closer to i than -i.

Remark 9.1.4: Some write a similar map:

$$\mathbb{H}^{\circ} \to \mathbb{D}^{\circ}$$
$$z \mapsto \frac{z-i}{z+i}.$$

This is just a composition of the above map with the flip $z\mapsto -z$:

$$-\frac{i-z}{i+z} = \frac{z-i}{i+z} = \frac{z-i}{z+i}.$$

Proposition 9.1.5 (Right half-plane to Disc).

$$\mathbb{H}_R \rightleftharpoons \mathbb{D}$$

$$\left\{ z \mid \Re(z) > 0 \right\} \rightleftharpoons \left\{ w \mid |w| < 1 \right\}$$

$$z \mapsto \frac{1 - z}{1 + z}$$

$$\frac{1 - w}{1 + w} \longleftrightarrow w.$$

Just map the *right* half-plane \mathbb{H}_R to the disc \mathbb{D} by precomposing with a rotation $e^{i\pi/2} = i$:

$$\mathbb{H}_R \to \mathbb{H} \to \mathbb{D}$$

$$z \mapsto iz \mapsto \frac{i - (iz)}{i + (iz)} = \frac{i(1-z)}{i(1+z)} = \frac{1-z}{1+z}.$$

This can easily be inverted:

$$w = \frac{1+z}{1+z}$$

$$\implies -(1-w) + z(w+1) = 0$$

$$\implies z = \frac{1-w}{1+w}.$$

Boundary behavior: Just a rotated version of $\mathbb{H} \to \mathbb{D}!$

Mnemonic: every $z \in \mathbb{H}_R$ is closed to 1 than -1.

Proposition 9.1.6 (Sector to sector).

For $0 < \alpha < 2$:

$$F_{\alpha}: S_{\frac{\pi}{\alpha}}^{\circ} \rightleftharpoons S_{\pi}^{\circ} = \mathbb{H}^{\circ}$$

$$\left\{z \mid 0 < \operatorname{Arg}(z) < \frac{\pi}{\alpha}\right\} \rightleftharpoons \left\{w \mid 0 < \operatorname{Arg}(w) < \pi\right\}$$

$$z \mapsto z^{\alpha}$$

$$w^{\frac{1}{\alpha}} \leftrightarrow w.$$

Note that if you look at the image of $\mathbb H$ under $z\mapsto z^{\alpha}$, you get

$$\Big\{z\ \Big|\ 0<\operatorname{Arg}(z)<\pi\Big\} \rightleftharpoons \{0<\operatorname{Arg}(w)<\alpha\pi\}$$

For the inverse, choose a branch cut of log deleting the negative real axis, or more generally fix $0 < \arg w < w^{\frac{1}{\alpha}}$.

Boundary behavior:

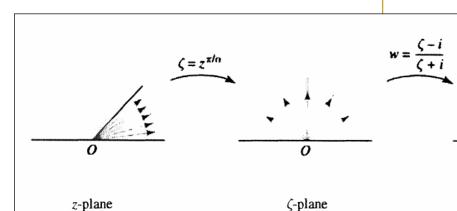
- As x travels from $-\infty \to 0$, $F_{\alpha}(x)$ travels away from infinity along the ray $\theta = \alpha \pi$, so $L = \{e^{t\alpha\pi} \mid t \in (0,\infty)\}$, from $\infty \to 0$.
- As x travels from $0 \to \infty$, $F_{\alpha}(x)$ travels from $0 \to \infty$ along \mathbb{R} .

Proposition 9.1.7 (Sector to Disc).

The unmotivated formula first:

$$F: S_{\alpha} \to \mathbb{D}$$

$$\left\{z \mid 0 < \operatorname{Arg}(z) < \alpha \right\} \rightleftharpoons \left\{w \mid |w| < 1 \right\}$$
$$z \rightleftharpoons \frac{z^{\frac{\pi}{\alpha}} - i}{z^{\frac{\pi}{\alpha}} + i}.$$



Idea: compose some known functions.

$$S_{\alpha} \to S_{\pi} = \mathbb{H} \to \mathbb{D}$$
$$z \mapsto z^{\frac{\pi}{\alpha}} \mapsto \frac{z - i}{z + i} \Big|_{z = z^{\frac{\pi}{\alpha}}}.$$

Proposition 9.1.8 (Upper half-disc to first quadrant).

$$\left\{z \mid |z| < 1, \, \Im(z) > 0\right\} \rightleftharpoons \left\{w \mid \Re(w) > 0, \, \Im(w) > 0\right\}$$
$$z \mapsto \frac{1+z}{1-z}$$
$$\frac{w-1}{w+1} \leftrightarrow w.$$

- Why this lands in the first quadrant:
 - Use that squares are non-negative and $z = x + iy \in \mathbb{D} \implies x^2 + y^2 < 1$:

$$f(z) = \frac{1 - (x^2 + y^2)}{(1 - x)^2 + y^2} + i \frac{2y}{(1 - x)^2 + y^2}.$$

- Why the inverse lands in the unit disc:
 - For w in Q1, the distance from w to 1 is smaller than from w to -1.
 - Check that if w = u + iv where u, v > 0, the imaginary part of the image is positive:

$$\frac{w-1}{w+1} = \frac{(w-1)\overline{(w+1)}}{|w+1|^2}$$

$$= \frac{(u-1+iv)(u+1-iv)}{(u+1)^2+v^2}$$

$$= \frac{u^2+v^2+1}{(u+1)^2+v^2} + i\left(\frac{2v}{(u+1)^2+v^2}\right).$$

Boundary behavior:

• On the upper half circle $\{e^{it} \mid t \in (0,\pi)\}$, write

$$f(z) = \frac{1 + e^{i\theta}}{1 - e^{i\theta}} = \frac{e^{-i\theta/2} + e^{i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}} = \frac{i}{\tan(\theta/2)},$$

so as t ranges $0 \to \pi$ we have f(z) ranging from $0 \to i\infty$ along the imaginary axis.

• As x ranges from $-1 \to 1$ in \mathbb{R} , f(z) ranges from $0 \to \infty$ with f(0) = 1.

Proposition 9.1.9 (Log: Upper half-plane to horizontal strip).

$$\mathbb{H} \rightleftharpoons \mathbb{R} \times (0, \pi)$$

$$\left\{ z \mid \Im(z) > 0 \right\} \rightleftharpoons \left\{ w \mid \Im(z) \in (0, \pi) \right\}$$

$$z \mapsto \log(z)$$

$$e^{w} \longleftrightarrow w.$$

• Why this lands in a strip: use that $\arg(z) \in (0,\pi)$ and $\log(z) = |z| + i \arg(z)$.

Boundary behavior:

- As x travels from $-\infty \to 0$, F(x) travels horizontally from $\infty + i\pi$ to $-\infty + i\pi$.
- As x travels from $o \to \infty$, F(x) travels from $-\infty \to \infty$ in \mathbb{R} .

Remark 9.1.10: This extends to a function $\mathbb{C} \setminus \mathbb{R}^{\leq 0} \to \mathbb{R} \times (-\pi, \pi)$. Circles of radius R are mapped to vertical line segments connecting $\ln(R) + i\pi$ to $\ln(R) - i\pi$, and rays are mapped to horizontal lines.

Remark 9.1.11: One can find other specific images of the logarithm:

$$\left\{ z \mid |z| < 1, \, \Im(z) > 0 \right\} \rightleftharpoons \mathbb{R}^{<0} \times (0, \pi)$$
$$\left\{ z \mid |z| > 1, \, \Im(z) > 0 \right\} \rightleftharpoons \mathbb{R}^{>0} \times (0, \pi)$$

For the upper half-disc to the negative horizontal half-strip: - As x travels $0 \to 1$ in \mathbb{R} , $\log(x)$ travels from $-\infty \to 0$. - As x travels from -1 to 1 along $S^1 \cap \mathbb{H}$, $\log(x)$ travels from $0 \to i\pi$ vertically. - As x travels from $-1 \to 0$, $\log(x)$ travels from $0 + i\pi \to i - \infty + i\pi$ along the top of the strip.

Proposition 9.1.12 (Half-discs to half strips).

$$F: (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^{>0} \to \mathbb{D} \cap \mathbb{H}$$
$$z \mapsto e^{iz}$$
$$\frac{\log(w)}{i}? \longleftrightarrow w.$$

This uses that $e^{iz} = e^{-\Im(z)}e^{i\Re(z)}$.

Boundary behavior:

Proposition 9.1.13 (Half-disc to upper half-plane).

$$F:? \rightleftharpoons ?$$

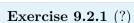
$$z \mapsto -\frac{1}{2} \left(z + z^{-1} \right)$$

Proposition 9.1.14 (Upper half-plane to vertical half-strip).

$$? \rightleftharpoons ?$$
$$z \mapsto \sin(z)$$

٠

9.2 Exercises



Find a conformal map from the upper half-disc to the upper half-plane.

Solution:

Solution: The map $T_0^{-1}(z)$ maps B to the second quadrant. Then multiplying by -i maps this to the first quadrant. Then squaring maps this to the upper half-plane. In the end we have

$$f(z) = \left(-i\left(\frac{iz+i}{-z+1}\right)\right)^2.$$

10 Schwarz Reflection

10.1 Schwarz

Theorem 10.1.1 (Schwarz Lemma).

If $f: \mathbb{D} \to \mathbb{D}$ is holomorphic with f(0) = 0, then

- 1. $|f(z)| \le |z|$ for all $z \in \mathbb{D}$ 2. $|f'(0)| \le 1$.

Moreover, if

- $|f(z_0)| = |z_0|$ for any $z_0 \in \mathbb{D}$, or |f'(0)| = 1,

then f is a rotation.

Proof (?).

Apply the maximum modulus principle to f(z)/z.

Exercise 10.1.2 (?)

Show that $\operatorname{Aut}(\mathbb{C}) = \{z \mapsto az + b \mid a \in \mathbb{C}^{\times}, b \in \mathbb{C}\}.$

Theorem 10.1.3 (Biholomorphisms of the disc).

9.2 Exercises 62

$$\operatorname{Aut}_{\mathbb{C}}(\mathbb{D}) = \left\{ z \mapsto e^{i\theta} \left(\frac{\alpha - z}{1 - \overline{\alpha}z} \right) \right\}.$$

Proof (?).

 $Schwarz\ lemma.$

Theorem 10.1.4(?).

$$\operatorname{Aut}_{\mathbb{C}}(\mathbb{H}) = \left\{ z \mapsto \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{C}, ad-bc = 1 \right\} \cong \operatorname{PSL}_2(\mathbb{R}).$$

?

11 | Schwarz Lemma

Montel's theorem

Normal families

Schwarz lemma

Equicontinuity

Schwarz Lemma 63

12 | Linear Fractional Transformations

13 | Montel's Theorem

14 Unsorted Theorems

Theorem 14.0.1 (Riemann's Removable Singularity Theorem).

If f is holomorphic on Ω except possibly at z_0 and f is bounded on $\Omega \setminus \{z_0\}$, then z_0 is a removable singularity.

Theorem 14.0.2 (Little Picard).

If $f: \mathbb{C} \to \mathbb{C}$ is entire and nonconstant, then $\operatorname{im}(f)$ is either \mathbb{C} or $\mathbb{C} \setminus \{z_0\}$ for some point z_0 .

Corollary 14.0.3.

The ring of holomorphic functions on a domain in $\mathbb C$ has no zero divisors.

Proof. ???

Find the proof!

Morera

Proposition 14.0.4(Bounded Complex Analytic Functions form a Banach Space). For $\Omega \subseteq \mathbb{C}$, show that $A(\mathbb{C}) := \{ f : \Omega \to \mathbb{C} \mid f \text{ is bounded} \}$ is a Banach space.

Proof.

Apply Morera's Theorem and Cauchy's Theorem

$15 \left| \begin{array}{c} \text{Proofs of the Fundamental Theorem of} \\ \text{Algebra} \end{array} \right|$

15.0.1 Argument Principle

Proof (using the argument principle).

- Let $P(z) = a_n z^n + \cdots + a_0$ and g(z) = P'(z)/P(z), note P is holomorphic
- Since $\lim_{|z|\to\infty} P(z) = \infty$, there exist an R > 0 such that P has no roots in $\{|z| \ge R\}$.
- Apply the argument principle:

$$N(0) = \frac{1}{2\pi i} \oint_{|\xi|=R} g(\xi) d\xi.$$

- Check that $\lim_{|z\to\infty|}zg(z)=n,$ so g has a simple pole at ∞
- Then g has a Laurent series $\frac{n}{z} + \frac{c_2}{z^2} + \cdots$
- Integrate term-by-term to get N(0) = n.

15.0.2 Rouche's Theorem

Proof (using Rouche's theorem).

Proof. Let

$$P(z) = z^{n} + a_{n-1}z^{n-1} + \ldots + a_0$$

be an *n*th order polynomial. Let $f(z) = z^n$ and h = P - f. Choose an R such that $R > \max(1, n|a_{n-1}|, \ldots, n|a_0|)$. Then on |z| = R we have

$$|h| \le |a_{n-1}|R^{n-1} + |a_{n-2}|R^{n-2} + \dots + |a_0| \le \frac{R}{n}R^{n-1} + \frac{R}{n}R^{n-2} + \dots + \frac{R}{n} < R^n.$$

On |z| = R we have $|f(z)| = R^n$, so we have shown |h| < |f| on the curve. Thus, the corollary to Rouchés theorem says f + h and f have the same number of zeros inside |z| = R. Since we know f has exactly n zeros inside the curve the same is true for the polynomial f + h. Now let R go to infinity, we've shown that f + h has exactly n zeros in the entire plane.

Note. The proof gives a simple bound on the size of the zeros: they are all have magnitude less than or equal to $\max(1, n|a_{n-1}|, \dots, n|a_0|)$.

• Let $P(z) = a_n z^n + \dots + a_0$

• Set
$$f(z) = a_n z^n$$
 and $g(z) = P(z) - f(z) = a_{n-1} z^{n-1} + \dots + a_0$, so $f + g = P$.
• Choose $R > \max\left(\frac{|a_{n-1}| + \dots + |a_0|}{|a_n|}, 1\right)$, then

• Choose
$$R > \max\left(\frac{|a_{n-1}| + \dots + |a_0|}{|a_n|}, 1\right)$$
, then

$$\begin{split} |g(z)| &\coloneqq |a_{n-1}z^{n-1} + \dots + a_1z + a_0| \\ &\le |a_{n-1}z^{n-1}| + \dots + |a_1z| + |a_0| \quad \text{by the triangle inequality} \\ &= |a_{n-1}| \cdot |z^{n-1}| + \dots + |a_1| \cdot |z| + |a_0| \\ &= |a_{n-1}| \cdot R^{n-1} + \dots + |a_1|R + |a_0| \\ &\le |a_{n-1}| \cdot R^{n-1} + |a_{n-2}| \cdot R^{n-1} + \dots + |a_1| \cdot R^{n-1} + |a_0| \cdot R^{n-1} \quad \text{since } R > 1 \implies R^{a+b} \ge R^a \\ &= R^{n-1} \left(|a_{n-1}| + |a_{n-2}| + \dots + |a_1| + |a_0| \right) \\ &\le R^{n-1} \left(|a_n| \cdot R \right) \quad \text{by choice of } R \\ &= R^n |a_n| \\ &= |a_n z^n| \\ &\coloneqq |f(z)| \end{split}$$

• Then $a_n z^n$ has n zeros in |z| < R, so f + g also has n zeros.

15.0.3 Liouville's Theorem

Proofs of the Fundamental Theorem of Algebra

Proof (using Liouville's theorem).

- Suppose p is nonconstant and has no roots, then $\frac{1}{p}$ is entire. We will show it is also bounded and thus constant, a contradiction.
- Write $p(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n} \right)$
- Outside a disc:
 - Note that $p(z) \stackrel{z \to \infty}{\to} \infty$. so there exists an R large enough such that $|p(z)| \ge \frac{1}{A}$ for any fixed chosen constant A.
 - Then $|1/p(z)| \le A$ outside of |z| > R, i.e. 1/p(z) is bounded there.
- Inside a disc:
 - -p is continuous with no roots and thus must be bounded below on |z| < R.
 - p is entire and thus continuous, and since $\overline{D}_r(0)$ is a compact set, p achieves a min A there
 - Set $C := \min(A, B)$, then $|p(z)| \ge C$ on all of $\mathbb C$ and thus $|1/p(z)| \le C$ everywhere.
 - So 1/p(z) is bounded an entire and thus constant by Liouville's theorem but this forces p to be constant. \mathcal{E}

15.0.4 Open Mapping Theorem

Proof (using the Open Mapping theorem).

- p induces a continuous map $\mathbb{CP}^1 \to \mathbb{CP}^1$
- The continuous image of compact space is compact;
- Since the codomain is Hausdorff space, the image is closed.
- p is holomorphic and non-constant, so by the Open Mapping Theorem, the image is open.
- Thus the image is clopen in \mathbb{CP}^1 .
- The image is nonempty, since $p(1) = \sum a_i \in \mathbb{C}$
- \mathbb{CP}^1 is connected
- But the only nonempty clopen subset of a connected space is the entire space.
- So p is surjective, and $p^{-1}(0)$ is nonempty.
- So p has a root.

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15.0.5 Generalized Liouville

Theorem 15.0.1 (Generalized Liouville).

If X is a compact complex manifold, any holomorphic $f: X \to \mathbb{C}$ is constant.

Lemma 15.0.2(?).

If $f: X \to Y$ is a nonconstant holomorphic map between Riemann surfaces with X compact, then

- f must be surjective,
- Y must be compact,
- $f^{-1}(q)$ is finite for all $q \in Y$,
- The branch and ramification loci consist of finitely many points.

Proof (of FTA, using Generalized Liouville).

Given a nonconstant $p \in \mathbb{C}[x]$, regard it as a function $p : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ by extending so that $p(\infty) = \infty$. Since p is nonconstant, by the lemma p is surjective, so there exists some $x \neq \infty$ in $\mathbb{P}^1(\mathbb{C})$ with p(x) = 0.

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Definition 16.0.1 (Gamma function)

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

Remark 16.0.2: Some interesting properties of Γ : $\Gamma(z+1)=z\Gamma(z)$ and has simple poles at $z=0,-1,-2,\cdots$ with residues $\underset{z=-m}{\operatorname{Res}}\Gamma(z)=(-1)^m/m!$. There is also a factorization

$$\Gamma(z) = \frac{1}{ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{\frac{-z}{n}}}$$

where $\gamma := \lim_{N \to \infty} \sum_{n=1^N} \frac{1}{n} - \log(N)$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

which yields a product factorization for $\sin(\pi z)$.

$$\mathcal{L}(t^{z-1}, s = 1) = \Gamma(z) \text{ and } \mathcal{L}(t^n, s = 1) = \Gamma(n+1).$$

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Theorem 16.0.3 (Uniformization).

Every Riemann surface S is the quotient of a free proper holomorphic action of a group G on the universal cover \tilde{S} of S, so $S \cong \tilde{S}/G$ is a biholomorphism. Moreover, \tilde{S} is biholomorphic to

- \mathbb{CP}^1
- \mathbb{C}

16.1 Misc Basic Algebra



Fact 16.1.1 (Standard forms of conic sections)

- Circle: $x^2 + y^2 = r^2$ Ellipse: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ Hyperbola: $\left(\frac{x}{a}\right)^2 \left(\frac{y}{b}\right)^2 = 1$
 - Rectangular Hyperbola: $xy = \frac{c^2}{2}$.
- Parabola: $-4ax + y^2 = 0$.

Mnemonic: Write $f(x,y) = Ax^2 + Bxy + Cy^2 + \cdots$, then consider the discriminant $\Delta = B^2 - 4AC$:

• $\Delta < 0 \iff \text{ellipse}$

$$-\Delta < 0$$
 and $A = C, B = 0 \iff$ circle

- $\Delta = 0 \iff \text{parabola}$
- $\Delta > 0 \iff$ hyperbola

Fact 16.1.2 (Completing the square)

$$x^{2} - bx = (x - s)^{2} - s^{2}$$
 where $s = \frac{b}{2}$

$$x^{2} + bx = (x+s)^{2} - s^{2}$$
 where $s = \frac{b}{2}$.

Fact 16.1.3

The sum of the interior angles of an *n*-gon is $(n-2)\pi$, where each angle is $\frac{n-2}{n}\pi$.

Appendix

Definition 16.1.4 (The Dirichlet Problem)

Given a bounded piecewise continuous function $u: S^1 \to \mathbb{R}$, is there a unique extension to a continuous harmonic function $\tilde{u}: \mathbb{D} \to \mathbb{R}$?

Remark 16.1.5: More generally, this is a boundary value problem for a region where the *values* of the function on the boundary are given. Compare to prescribing conditions on the normal vector on the boundary, which would be a Neumann BVP. Why these show up: a harmonic function on a simply connected region has a harmonic conjugate, and solutions of BVPs are always analytic functions with harmonic real/imaginary parts.

Example 16.1.6 (Dirichlet problem on the strip): See section 27, example 1 in Brown and Churchill. On the strip $(x,y) \in (0,\pi) \times (0,\infty)$, set up the BVP for temperature on a thin plate with no sinks/sources:

$$\Delta T = 0$$
 $T(0, y) = 0, T(\pi, y) = 0 \ \forall y$

$$T(x,0) = \sin(x)$$

$$T(x,y) \stackrel{y \to \infty}{\longrightarrow} 0.$$

Then the following function is harmonic on \mathbb{R}^2 and satisfies that Dirichlet problem:

$$T(x,y) = e^{-y}\sin(x) = \Re(-ie^{iz}) = \Im(e^{iz}).$$

Exercise 16.1.7 (?)

Show that there is no continuous square root function defined on all of \mathbb{C} .

Suppose $f(z)^2 = z$. Then f is a section to the covering map

$$p: \mathbb{C}^{\times} \to \mathbb{C}^{\times}$$
$$z \mapsto z^2,$$

so $p \circ f = \mathrm{id}$. Using $\pi_1(\mathbb{C}^\times) = \mathbb{Z}$, the induced maps are $p_*(1) = 2$ and $f_*(1) = n$ for some $n \in \mathbb{Z}$. But then $p_* \circ f_*$ is multiplication by 2n, contradicting $p_* \circ f_* = \mathrm{id}$ by functoriality.

Definition 16.1.8 (Laplace Transform)

$$\mathcal{L}(f,\xi) := \int_{\mathbb{R}} e^{-\xi x} f(x) dx.$$

- $f(t) = c \rightleftharpoons F(s) = cs^{-1}$ $f(t) = e^{at} \rightleftharpoons F(s) = (s a)^{-1}$. $f(t) = t \rightleftharpoons F(s) = s^{-2}$. $f(t) = \cos(at) \rightleftharpoons F(s) = s(s^2 + a^2)^{-1}$.

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Remark 16.1.9:

$$\widehat{f}(\xi) \coloneqq \mathcal{L}(f, i\xi) \coloneqq \int_{\mathbb{R}} f(x) e^{-i\xi x} dx.$$

Fact 16.1.10 (Table of Laplace Transforms)

Properties and Rules

We assume that f(t) = 0 for t < 0.

<u>Function</u>	<u>Transform</u>	
f(t)	$F(s) = \int_0^\infty f(t)e^{-st} dt$	(Definition)
a f(t) + b g(t)	a F(s) + b G(s)	(Linearity)
$e^{at}f(t)$	F(s-a)	$(s ext{-shift})$
f'(t)	sF(s) - f(0)	
f''(t)	$s^2 F(s) - s f(0) - f'(0)$	
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}$	$^{-1)}(0)$
tf(t)	-F'(s)	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	
f(t-a)	$e^{-as}F(s)$	(t-translation or t -shift)
$\int_0^t f(\tau) d\tau$	$\frac{F(s)}{s}$	(integration rule)
$rac{f(t)}{t}$	$\int_{s}^{\infty} F(\sigma) d\sigma$	

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<u>Function</u>	<u>Transform</u>	Region of convergence
1	1/s	$\operatorname{Re}(s) > 0$
e^{at}	1/(s-a)	$\operatorname{Re}(s) > \operatorname{Re}(a)$
t	$1/s^2$	$\operatorname{Re}(s) > 0$
t^n	$n!/s^{n+1}$	$\operatorname{Re}(s) > 0$
$\cos(\omega t)$	$s/(s^2+\omega^2)$	$\operatorname{Re}(s) > 0$
$\sin(\omega t)$	$\omega/(s^2+\omega^2)$	$\operatorname{Re}(s) > 0$
$e^{at}\cos(\omega t)$	$(s-a)/((s-a)^2+\omega^2)$	$\operatorname{Re}(s) > \operatorname{Re}(a)$
$e^{at}\sin(\omega t)$	$\omega/((s-a)^2+\omega^2)$	$\operatorname{Re}(s) > \operatorname{Re}(a)$
$\delta(t)$	1	all s
$\delta(t-a)$	e^{-as}	all s
$\cosh(kt) = \frac{e^{kt} + e^{-kt}}{2}$	$s/(s^2 - k^2)$	$\operatorname{Re}(s) > k$
$\sinh(kt) = \frac{e^{kt} - e^{-kt}}{2}$	$k/(s^2 - k^2)$	Re(s) > k
$\frac{1}{2\omega^3}(\sin(\omega t) - \omega t \cos(\omega t))$	$\frac{1}{(s^2 + \omega^2)^2}$	$\operatorname{Re}(s) > 0$
$\frac{t}{2\omega}\sin(\omega t)$	$\frac{s}{(s^2 + \omega^2)^2}$	$\operatorname{Re}(s) > 0$
$\frac{1}{2\omega}(\sin(\omega t) + \omega t \cos(\omega t))$	$\frac{s^2}{(s^2 + \omega^2)^2}$	$\operatorname{Re}(s) > 0$
$t^n e^{at}$	$n!/(s-a)^{n+1}$	$\operatorname{Re}(s) > \operatorname{Re}(a)$
$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$	Re(s) > 0
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$	$\operatorname{Re}(s) > 0$

Basics

- Show that $\frac{1}{z}\sum_{k=1}^{\infty}\frac{z^k}{k}$ converges on $S^1\setminus\{1\}$ using summation by parts.
- Show that any power series is continuous on its domain of convergence.
- Show that a uniform limit of continuous functions is continuous.

??

• Show that if f is holomorphic on $\mathbb D$ then f has a power series expansion that converges uniformly on every compact $K \subset \mathbb D$.

- Show that any holomorphic function f can be uniformly approximated by polynomials.
- Show that if f is holomorphic on a connected region Ω and $f' \equiv 0$ on Ω , then f is constant on Ω .
- Show that if |f| = 0 on $\partial \Omega$ then either f is constant or f has a zero in Ω .
- Show that if $\{f_n\}$ is a sequence of holomorphic functions converging uniformly to a function f on every compact subset of Ω , then f is holomorphic on Ω and $\{f'_n\}$ converges uniformly to f' on every such compact subset.
- Show that if each f_n is holomorphic on Ω and $F := \sum f_n$ converges uniformly on every compact subset of Ω , then F is holomorphic.
- Show that if f is once complex differentiable at each point of Ω , then f is holomorphic.

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- Prove the triangle inequality
- Prove the reverse triangle inequality
- Show that $\sum z^{k-1}/k$ converges for all $z \in S^1$ except z = 1.
- What is an example of a noncontinuous limit of continuous functions?
- Show that the uniform limit of continuous functions is continuous.
- Show that f is holomorphic if and only if $\bar{\partial} f = 0$.
- Show $n^{\frac{1}{n}} \stackrel{n \to \infty}{\to} 1$.
- Show that if f is holomorphic with f' = 0 on Ω then f is constant.
- Show that holomorphic implies analytic.
- Use Cauchy's inequality to prove Liouville's theorem

Problem 17.0.1 (?)

What is a pair of conformal equivalences between \mathbb{H} and \mathbb{D} ?

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Solution:

$$F: HH \to \mathbb{D}$$

$$z \mapsto \frac{i-z}{i+z}$$

$$G: \mathbb{D} \to \mathbb{H}$$

$$w \mapsto i\frac{1-w}{1+w}.$$

Mnemonic: any point in $\mathbb H$ is closer to i than -i, so |F(z)| < 1.

• Maps $\mathbb{R} \to S^1 \setminus \{-1\}$.

Problem 17.0.2 (?)

What is conformal equivalence $\mathbb{H} \rightleftharpoons S := \{ w \in \mathbb{C} \mid 0 < \arg(w) < \alpha \pi \}$?

Solution:

$$f(z) = z^{\alpha}$$
.