Complex Analysis Qualifying Exam Solutions

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1 Preface

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- Jingzhi Tie, for supplying many additional problems and solutions.

2 Topology and Functions of One Variable (8155a)

2.1 1

Let $x_0 = a, x_1 = b$, and set

$$x_n \coloneqq \frac{x_{n-1} + x_{n-2}}{2} \qquad n \geq 2.$$

Show that $\{x_n\}$ is a Cauchy sequence and find its limit in terms of a and b.

2.2 2

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim\limits_{x \to \pm\infty} f(x) = 0$. Prove that f is uniformly continuous.

2.3 3

Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is everywhere differentiable but f' is not continuous at 0.

2.4 4

Suppose $\{g_n\}$ is a uniformly convergent sequence of functions from $\mathbb{R}$ to $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous. Prove that the sequence $\{f \circ g_n\}$ is uniformly convergent.

2.5 5

Let f be differentiable on $[a,b]$. Say that f is *uniformly differentiable* iff

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \quad |x-y| < \delta \implies \left| \frac{f(x)-f(y)}{x-y} - f'(y) \right| < \varepsilon.$$

Prove that f is uniformly differentiable on $[a,b] \iff f'$ is continuous on $[a,b]$.

2.6 6

Suppose $A, B \subseteq \mathbb{R}^n$ are disjoint and compact. Prove that there exist $a \in A, b \in B$ such that

$$\|a-b\| = \inf \left\{ \|x-y\| \;\middle|\; x \in A,\, y \in B \right\}.$$

2.7 7

Suppose $A, B \subseteq \mathbb{R}^n$ are connected and not disjoint. Prove that $A \cup B$ is also connected.

2.8 8

Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ such that

$$f_n(x) \geq f_{n+1}(x) \geq 0 \quad \forall n \in \mathbb{N}, \forall x \in [0,1].$$

Prove that if $\{f_n\}$ converges pointwise to 0 on $[0,1]$ then it converges to 0 uniformly on $[0,1]$.

2.9 9

Show that if $E \subset [0,1]$ is uncountable, then there is some $t \in \mathbb{R}$ such that $E \cap (-\infty, t)$ and $E \cap (t, \infty)$ are also uncountable.

3 | Several Variables (8155h)

3.1 1

Is the following function continuous, differentiable, continuously differentiable?

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & \text{else.} \end{cases}$$

3.2 2

3.2.1 a

Complete this definition: "$f : \mathbb{R}^n \to \mathbb{R}^m$ is real-differentiable a point $p \in \mathbb{R}^n$ iff there exists a linear transformation…"

3.2.2 b

Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ whose first-order partial derivatives exist everywhere but f is not differentiable at $(0,0)$.

3.2.3 c

Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is real-differentiable everywhere but nowhere complex-differentiable.

3.3 3

Let $f : \mathbb{R}^2 \to \mathbb{R}$.

3.3.1 a

Define in terms of linear transformations what it means for f to be differentiable at a point $(a, b) \in \mathbb{R}^2$.

3.3.2 b

State a version of the inverse function theorem in this setting.

3.3.3 c

Identify $\mathbb{R}^2$ with $\mathbb{C}$ and give a necessary and sufficient condition for a real-differentiable function at (a, b) to be complex differentiable at the point $a + ib$.

3.4 4

Let $f = u + iv$ be complex-differentiable with continuous partial derivatives at a point $z = re^{i\theta}$ with $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

3.5 5

Let $P = (1, 3) \in \mathbb{R}^2$ and define

$$f(s, t) := ps^3 - 6st + t^2.$$

3.5.1 a

State the conclusion of the implicit function theorem concerning $f(s,t) = 0$ when f is considered a function $\mathbb{R}^2 \to \mathbb{R}$.

3.5.2 b

State the above conclusion when f is considered a function $\mathbb{C}^2 \to \mathbb{C}$.

3.5.3 c

Use the implicit function theorem for a function $\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ to prove (b).

There are various approaches: using the definition of the complex derivative, the Cauchy-Riemann equations, considering total derivatives, etc.

3.6 6

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable with $F(0,0) = 0$ and $\|\nabla F(0,0)\| < 1$.

Prove that there is some real number $r > 0$ such that $|F(x,y)| < r$ whenever $\|(x,y)\| < r$.

3.7 7

State the most general version of the implicit function theorem for real functions and outline how it can be proved using the inverse function theorem.

4 Several Variables: Extra Questions

4.1 ?

Let $f = u + iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$.

Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

4.2 ?

Give an example of a

Show that $f(z) = z^2$ is uniformly continuous in any open disk $|z| < R$, where $R > 0$ is fixed, but it is not uniformly continuous on $\mathbb{C}$.

4.2.1 1

Show that the function $u = u(x, y)$ given by

$$u(x, y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx \quad \text{for } n \in \mathbf{N}$$

is the solution on $D = \{(x, y) \mid x^2 + y^2 < 1\}$ of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = \frac{\sin nx}{n}.$$

4.2.2 2

Show that there exist points $(x, y) \in D$ such that $\limsup_{n\to\infty} |u(x, y)| = \infty$.

5 | Conformal Maps (8155c)

Notation: $\mathbb{D}$ is the open unit disc, $\mathbb{H}$ is the open upper half-plane.

5.1 1

Find a conformal map from $\mathbb{D}$ to $\mathbb{H}$.

5.2 2

Find a conformal map from the strip $\left\{ z \in \mathbb{C} \;\middle|\; 0 < \Im(z) < 1 \right\}$ to $\mathbb{H}$.

5.3 3

Find a fractional linear transformation T which maps $\mathbb{H}$ to $\mathbb{D}$, and explicitly describe the image of the first quadrant under T.

5.4 4

Find a conformal map from $\left\{ z \in \mathbb{C} \;\middle|\; |z - i| > 1, \Re(z) > 0 \right\}$ to $\mathbb{H}$.

5.5 5

Find a conformal map from $\left\{ z \in \mathbb{C} \;\middle|\; |z| < 1, \left| z - \frac{1}{2} \right| > \frac{1}{2} \right\}$ to $\mathbb{D}$.

5.6 6

Find a conformal map from $\{|z-1|<2\} \cap \{|z+1|<2\}$ to $\mathbb{H}$.

5.7 7

Let Ω be the region inside the unit circle $|z|=1$ and outside the circle $\left|z-\frac{1}{4}\right|=\frac{1}{4}$.

Find an injective conformal map from Ω onto some annulus $\{r<|z|<1\}$ for gonstant r.

5.8 8

Let D be the region obtained by deleting the real interval $[0,1)$ from $\mathbb{D}$; find a conformal map from D to $\mathbb{D}$.

5.9 9

Find a conformal map from $\mathbb{C} \setminus \left\{x \in \mathbb{R} \;\middle|\; x \leq 0\right\}$ to $\mathbb{D}$.

5.10 10

Find a conformal map from $\mathbb{C} \setminus \left\{x \in \mathbb{R} \;\middle|\; x \geq 1\right\}$ to $\mathbb{D}$.

5.11 11

Find a bijective conformal map from G to $\mathbb{H}$, where

$$G := \left\{z \in \mathbb{C} \;\middle|\; |z-1|<\sqrt{2},\ |z+1|<\sqrt{2}\right\} \setminus [0,i).$$

5.12 12

Prove that TFAE for a Möbius transformation T given by $T(z) = \dfrac{az+b}{cz+d}$:

a. T maps $\mathbb{R} \cup \{\infty\}$ to itself.
b. It is possible to choose a, b, c, d to be real numbers.
c. $\overline{T(z)} = T(\bar{z})$ for every $z \in \mathbb{CP}^1$.
d. There exist $\alpha \in \mathbb{R}, \beta \in \mathbb{C} \setminus \mathbb{R}$ such that $T(\alpha) = \alpha$ and $T(\overline{\beta}) = \overline{T(\beta)}$.

6 Extra Questions

6.1 12.

Find a conformal map from $D = \{z : \ |z| < 1, \ |z - 1/2| > 1/2\}$ to the unit disk $\Delta = \{z : \ |z| < 1\}$.

7 Integrals and Cauchy's Theorem (8155d)

Some interesting problems: 3, 4, 9, 10.

7.1 1

Suppose $f, g : [0,1] \to \mathbb{R}$ where f is Riemann integrable and for $x, y \in [0,1]$,

$$|g(x) - g(y)| \leq |f(x) - f(y)|.$$

Prove that g is Riemann integrable.

7.2 2

State and prove Green's Theorem for rectangles.

Then use it to prove Cauchy's Theory for functions that are analytic in a rectangle.

7.3 3 🚩

Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of analytic functions on $\mathbb{D} := \left\{z \in \mathbb{C} \;\middle|\; |z| < 1\right\}$.

Show that if $f_n \to g$ for some $g : \mathbb{D} \to \mathbb{C}$ uniformly on every compact $K \subset \mathbb{D}$, then g is analytic on $\mathbb{D}$.

7.4 4 🚩

Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of entire functions where

- $f_n \to g$ pointwise for some $g : \mathbb{C} \to \mathbb{C}$.
- On every line segment in $\mathbb{C}$, $f_n \to g$ uniformly.

Show that

- g is entire, and
- $f_n \to g$ uniformly on every compact subset of $\mathbb{C}$.

7.5 5 ✨

Prove that there is no sequence of polynomials that uniformly converge to $f(z) = \dfrac{1}{z}$ on S^1.

Solution:

Concepts Used:

- By Cauchy's integral formula, $\displaystyle\int_{S^1} f = 2\pi i$
- If p_j is any polynomial, then p_j is holomorphic in $\mathbb{D}$, so $\displaystyle\int_{S^1} p_j = 0$.
- Contradiction: compact sets in $\mathbb{C}$ are bounded, so

$$\left|\int f - \int p_j\right| \leq \int |p_j - f| \leq \int \|p_j - f\|_\infty = \|p_j - f\|_\infty \int_{S^1} 1\, dz = \|p_j - f\|_\infty \cdot 2\pi \to 0$$

which forces $\int f = \int p_j = 0.$

7.6 6 🚩

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that vanishes outside of some finite interval. For each $z \in \mathbb{C}$, define

$$g(z) = \int_{-\infty}^{\infty} f(t)e^{-izt}\, dt.$$

Show that g is entire.

7.7 7 🚩

Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire and

$$|f(z)| \leq |z|^{\frac{1}{2}} \quad \text{when } |z| > 10.$$

Prove that f is constant.

7.8 8 🚩

Let γ be a smooth curve joining two distinct points $a, b \in \mathbb{C}$.

Prove that the function

$$f(z) := \int_\gamma \frac{g(w)}{w - z}\, dw$$

is analytic in $\mathbb{C} \setminus \gamma$.

7.9 9 ✨

Suppose that $f : \mathbb{C} \to \mathbb{C}$ is continuous everywhere and analytic on $\mathbb{C} \setminus \mathbb{R}$ and prove that f is entire.

Something missing?

Solution:

Concepts Used:

- Note f is continuous on $\mathbb{C}$ since analytic implies continuous (f equals its power series, where the partials sums uniformly converge to it, and uniform limit of continuous is continuous).
- Strategy: take D a disc centered at a point $x \in \mathbb{R}$, show f is holomorphic in D by Morera's theorem.
- Let $\Delta \subset D$ be a triangle in D.
- Case 1: If $\Delta \cap \mathbb{R} = 0$, then f is holomorphic on Δ and $\int_\Delta f = 0$.
- Case 2: one side or vertex of Δ intersects $\mathbb{R}$, and wlog the rest of Δ is in $\mathbb{H}^+$.
  - Then let Δ_ε be the perturbation $\Delta + i\varepsilon = \left\{ z + i\varepsilon \;\middle|\; z \in \Delta \right\}$; then $\Delta_\varepsilon \cap \mathbb{R} = 0$ and $\int_{\Delta_\varepsilon} f = 0$.
  - Now let $\varepsilon \to 0$ and conclude by continuity of f (???)
    - ◇ We want
    $$\int_{\Delta_\varepsilon} f = \int_a^b f(\gamma_\varepsilon(t))\gamma_\varepsilon'(t)\,dt \overset{\varepsilon \to 0}{\rightarrow} \int_a^b f(\gamma(t))\gamma_\varepsilon'(t)\,dt = \int_\Delta f$$
    where $\gamma_\varepsilon, \gamma$ are curves parametrizing $\Delta_\varepsilon, \Delta$ respectively.
    - ◇ Since $\gamma, \gamma_\varepsilon$ are closed and bounded in $\mathbb{C}$, they are compact subsets. Thus it suffices to show that $f(\gamma_\varepsilon(t))\gamma_\varepsilon'(t)$ converges uniformly to $f(\gamma(t))\gamma'(t)$.
    - ◇ ??
- Case 3: Δ intersects both $\mathbb{H}^+$ and $\mathbb{H}^-$.
  - Break into smaller triangles, each of which falls into one of the previous two cases.

7.10 10 ✨

Prove Liouville's theorem: suppose $f : \mathbb{C} \to \mathbb{C}$ is entire and bounded. Use Cauchy's formula to prove that $f' \equiv 0$ and hence f is constant.

Solution:

Concepts Used:

- Suffices to prove $f' = 0$ because $\mathbb{C}$ is connected (see Stein Ch 1, 3.4)

  - Idea: Fix w_0, show $f(w) = f(w_0)$ for any $w \neq w_0$
  - Connected = Path connected in $\mathbb{C}$, so take γ joining w to w_0.
  - f is a primitive for f', and $\int_\gamma f' = f(w) - f(w_0)$, but $f' = 0$.

- Fix $z_0 \in \mathbb{C}$, let B be the bound for f, so $|f(z)| \leq B$ for all z.
- Apply Cauchy inequalities: if f is holomorphic on $U \supset \overline{D}_R(z_0)$ then setting $\|f\|_C := \sup_{z \in C} |f(z)|$,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!\|f\|_C}{R^n}.$$

  - Yields $|f'(z_0)| \leq B/R$

- Take $R \to \infty$, QED.

8 | Extra

8.1 ?

Assume f is continuous in the region: $0 < |z-a| \leq R$, $0 \leq \arg(z-a) \leq \beta_0$ $(0 < \beta_0 \leq 2\pi)$ and the limit $\lim_{z \to a}(z-a)f(z) = A$ exists. Show that

$$\lim_{r \to 0} \int_{\gamma_r} f(z)dz = iA\beta_0 \ ,$$

where $\gamma_r := \{z \mid z = a + re^{it},\ 0 \leq t \leq \beta_0\}$.

9 | Liouville's Theorem, Power Series (8155e)

9.1 1 ✨

Suppose f is analytic on a region Ω such that $\mathbb{D} \subseteq \Omega \subseteq \mathbb{C}$ and $f(z) = \sum_{n=0}^\infty a_n z^n$ is a power series with radius of convergence exactly 1.

9.1.1 a ✨

Give an example of such an f that converges at every point of S^1.

9.1.2 b 🚩

Give an example of such an f which is analytic at 1 but $\sum_{n=0}^{\infty} a_n$ diverges.

9.1.3 c 🚩

Prove that f can not be analytic at *every* point of S^1.

Missing part (c)

Solution:

Concepts Used:

9.1.4 a ✨

- Take $\sum \frac{z^n}{n^2}$
- Then

$$|z| \leq 1 \implies \left|\frac{z^n}{n^2}\right| \leq \frac{1}{n^2}$$

  which is summable
- So the series converges for $|z| \leq 1$.

9.1.5 b 🚩

- Take $\sum \frac{z^n}{n}$;
- Then $z = 1$ yields the harmonic series, which diverges.
- For $z \in S^1 \setminus \{1\}$, we have $z = e^{2\pi i t}$ for $0 < t < 2\pi$.
- So fix t.
- Toward applying the Dirichlet test, set $a_n = 1/n, b_n = z^n$.

- Then for all N,

$$\left|\sum_{n=1}^{N} b_n\right| = \left|\sum_{n=1}^{N} b_n\right| = \left|\sum_{n=1}^{N} z^n\right| = \left|\frac{z - z^{N+1}}{|1-z|}\right| \le \frac{2}{1-z} < \infty.$$

- Thus $\sum a_n b_n < \infty$ and $\sum z^n/n$ converges.

c. ?

9.2 2

Suppose f is entire and has Taylor series $\sum a_n z^n$ about 0.

9.2.1 a

Express a_n as a contour integral along the circle $|z| = R$.

9.2.2 b

Apply (a) to show that the above Taylor series converges uniformly on every bounded subset of $\mathbb{C}$.

9.2.3 c

Determine those functions f for which the above Taylor series converges uniformly on all of $\mathbb{C}$.

9.3 3

Suppose D is a domain and f, g are analytic on D.

Prove that if $fg = 0$ on D, then either $f \equiv 0$ or $g \equiv 0$ on D.

9.4 4 🚩

Suppose f is analytic on $\mathbb{D}^\circ$. Determine with proof which of the following are possible:

a. $f\left(\frac{1}{n}\right) = (-1)^n$ for each $n > 1$.

b. $f\left(\frac{1}{n}\right) = e^{-n}$ for each even integer $n > 1$ while $f\left(\frac{1}{n}\right) = 0$ for each odd integer $n > 1$.

c. $f\left(\frac{1}{n^2}\right) = \frac{1}{n}$ for each integer $n > 1$.

d. $f\left(\frac{1}{n}\right) = \frac{n-2}{n-1}$ for each integer $n > 1$.

9.5 5 ✨

Prove the Fundamental Theorem of Algebra (using complex analysis).

Solution:

Concepts Used:

- Strategy: By contradiction with Liouville's Theorem
- Suppose p is non-constant and has no roots.
- Claim: $1/p(z)$ is a bounded holomorphic function on $\mathbb{C}$.
  - Holomorphic: clear? Since p has no roots.
  - Bounded: for $z \neq 0$, write
  $$\frac{P(z)}{z^n} = a_n + \left(\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right).$$
  - The term in parentheses goes to 0 as $|z| \to \infty$
  - Thus there exists an $R > 0$ such that
  $$|z| > R \implies \left|\frac{P(z)}{z^n}\right| \geq c := \frac{|a_n|}{2}.$$
  - So p is bounded below when $|z| > R$
  - Since p is continuous and has no roots in $|z| \leq R$, it is bounded below when $|z| \leq R$.
  - Thus p is bounded below on $\mathbb{C}$ and thus $1/p$ is bounded above on $\mathbb{C}$.
- By Liouville's theorem, $1/p$ is constant and thus p is constant, a contradiction.

9.6 6 ✨

Find all entire functions that satisfy

$$|f(z)| \geq |z| \quad \forall z \in \mathbb{C}.$$

Prove this list is complete.

Solution:

Concepts Used:

- Suppose f is entire and define $g(z) := \dfrac{z}{f(z)}$.
- By the inequality, $|g(z)| \leq 1$, so g is bounded.
- g potentially has singularities at the zeros $Z_f := f^{-1}(0)$, but since f is entire, g is holomorphic on $\mathbb{C} \setminus Z_f$.
- Claim: $Z_f = \{0\}$.
  - If $f(z) = 0$, then $|z| \leq |f(z)| = 0$ which forces $z = 0$.
- We can now apply Riemann's removable singularity theorem:
  - Check g is bounded on some open subset $D\setminus\{0\}$, clear since it's bounded everywhere
  - Check g is holomorphic on $D \setminus \{0\}$, clear since the only singularity of g is $z = 0$.
- By Riemann's removable singularity theorem, the singularity $z = 0$ is removable and g has an extension to an entire function $\tilde{g}$.
- By continuity, we have $|\tilde{g}(z)| \leq 1$ on all of $\mathbb{C}$
  - If not, then $|\tilde{g}(0)| = 1 + \varepsilon > 1$, but then there would be a domain $\Omega \subseteq \mathbb{C} \setminus \{0\}$ such that $1 < |\tilde{g}(z)| \leq 1 + \varepsilon$ on Ω, a contradiction.
- By Liouville, $\tilde{g}$ is constant, so $\tilde{g}(z) = c_0$ with $|c_0| \leq 1$
- Thus $f(z) = c_0^{-1} z := cz$ where $|c| \geq 1$

Thus all such functions are of the form $f(z) = cz$ for some $c \in \mathbb{C}$ with $|c| \geq 1$.

9.7 7 🚩

Suppose $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ converges for some $z_0 \neq 0$.

9.7.1 a

Prove that the series converges absolutely for each z with $|z| < |z|_0$.

9.7.2 b

Suppose $0 < r < |z_0|$ and show that the series converges uniformly on $|z| \leq r$.

9.8 8

Suppose f is entire and suppose that for some integer $n \geq 1$,

$$\lim_{z \to \infty} \frac{f(z)}{z^n} = 0.$$

Prove that f is a polynomial of degree at most $n - 1$.

9.9 9

Find all entire functions satisfying

$$|f(z)| \leq |z|^{\frac{1}{2}} \quad \text{for } |z| > 10.$$

9.10 10

Prove that the following series converges uniformly on the set $\left\{z \;\middle|\; \Im(z) < \ln 2\right\}$:

$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{2^n}.$$

10 Extra

10.1 ?

Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω. Show without using Picard's theorems that $f(z)$ is a constant.

Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω.

Show without using Picard's theorems that $f(z)$ is a constant.

10.2 ?

10.2.1 1

Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$.

Show that for $r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

10.2.2 2

Deduce Liouville's theorem from (1).

11 Laurent Expansions and Singularities (8155f)

11.1 1 ✨

Find the Laurent expansion of

$$f(z) = \frac{z+1}{z(z-1)}$$

Solution:

Concepts Used:

Let $f(z) = \dfrac{z+1}{z(z-1)}$.
About $z = 0$:

$$\begin{aligned}
f(z) &= (z+1)\left(-\frac{1}{z} + \frac{1}{z-1}\right) \\
&= -(z+1)\left(\frac{1}{z} + \sum_{n=0}^{\infty} z^n\right) \\
&= -(z+1)\sum_{n=-1}^{\infty} z^n \\
&= \frac{1}{z} + 2\sum_{n=0}^{\infty} z^n \\
&= -\frac{1}{z} - 2 - 2z - 2z^2 - \cdots .
\end{aligned}$$

About $z = 1$:

$$\begin{aligned}
f(z) &= \left(\frac{(1-z)-2}{1-z}\right)\left(\frac{1}{1-(1-z)}\right) \\
&= \left(1-\frac{2}{1-z}\right)\sum_{n=0}^{\infty}(1-z)^n \\
&= \sum_{n=0}^{\infty}(1-z)^n - 2\sum_{n=-1}^{\infty}(1-z)^n \\
&= -\frac{2}{1-z} - \sum_{n=0}^{\infty}(1-z)^n \\
&= \frac{2}{z-1} + \sum_{n=0}^{\infty}(-1)^{n+1}(z-1)^n \\
&= \frac{2}{z-1} - 1 + (z-1) - (z-1)^2 + \cdots .
\end{aligned}$$

about $z = 0$ and $z = 1$ respectively.

11.2 2 ✨

Find the Laurent expansions about $z = 0$ of the following functions:

$$\exp\frac{1}{z} \qquad\qquad \cos\left(\frac{1}{z}\right).$$

Solution:

Concepts Used:

Let $f(z) = \dfrac{z+1}{z(z-1)}$.

About $z = 0$:

$$
\begin{aligned}
f(z) &= (z+1)\left(-\frac{1}{z} + \frac{1}{z-1}\right) \\
&= -(z+1)\left(\frac{1}{z} + \sum_{n=0}^{\infty} z^n\right) \\
&= -(z+1)\sum_{n=-1}^{\infty} z^n \\
&= \frac{1}{z} + 2\sum_{n=0}^{\infty} z^n \\
&= -\frac{1}{z} - 2 - 2z - 2z^2 - \cdots .
\end{aligned}
$$

About $z = 1$:

$$
\begin{aligned}
f(z) &= \left(\frac{(1-z)-2}{1-z}\right)\left(\frac{1}{1-(1-z)}\right) \\
&= \left(1 - \frac{2}{1-z}\right)\sum_{n=0}^{\infty}(1-z)^n \\
&= \sum_{n=0}^{\infty}(1-z)^n - 2\sum_{n=-1}^{\infty}(1-z)^n \\
&= -\frac{2}{1-z} - \sum_{n=0}^{\infty}(1-z)^n \\
&= \frac{2}{z-1} + \sum_{n=0}^{\infty}(-1)^{n+1}(z-1)^n \\
&= \frac{2}{z-1} - 1 + (z-1) - (z-1)^2 + \cdots .
\end{aligned}
$$

11.3 3

Find the Laurent expansion of

$$f(z) = \frac{z+1}{z(z-1)^2}$$

about $z = 0$ and $z = 1$ respectively.

Hint: recall that power series can be differentiated.

11.4 4

For the following functions, find the Laurent series about 0 and classify their singularities there:

$$\frac{\sin^2(z)}{z}$$
$$z \exp \frac{1}{z^2}$$
$$\frac{1}{z(4-z)}.$$

11.5 5

Find all entire functions with have poles at ∞.

11.6 6

Find all functions on the Riemann sphere that have a simple pole at $z = 2$ and a double pole at $z = \infty$, but are analytic elsewhere.

11.7 7

Let f be entire, and discuss (with proofs and examples) the types of singularities f might have (removable, pole, or essential) at $z = \infty$ in the following cases:

1. f has at most finitely many zeros in $\mathbb{C}$.
2. f has infinitely many zeros in $\mathbb{C}$.

11.8 8

Define

$$f(z) = \frac{\pi^2}{\sin^2(\pi z)}$$
$$g(z) = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}.$$

a. Show that f and g have the same singularities in $\mathbb{C}$.
b. Show that f and g have the same singular parts at each of their singularities.
c. Show that f, g each have period one and approach zero uniformly on $0 \leq x \leq 1$ as $|y| \to \infty$.
d. Conclude that $f = g$.

Not finished.

Solution:

Concepts Used:

Idea: show their $f - g$ is analytic by taking away all of the negative powers, and bounded by (c).

12 | Residues (8155g)

12.1 1

Calculate

$$\int_0^\infty \frac{1}{(1+z)^2(z+9x^2)}\,dx.$$

12.2 2

Let $a > 0$ and calculate

$$\int_0^\infty \frac{x\sin(x)}{x^2+a^2}\,dx.$$

12.3 3

Calculate

$$\int_0^\infty \frac{\sqrt{x}}{(x+1)^2}\,dx.$$

12.4 4

Calculate

$$\int_0^\infty \frac{\cos(x)-\cos(4x)}{x^2}\,dx.$$

12.5 5

Let $a > 0$ and calculate

$$\int_0^\infty \frac{x^2}{(x^2+a^2)^2}\,dx.$$

12.6 6

Calculate

$$\int_0^\infty \frac{\sin(x)}{x}\,dx.$$

12.7 7

Calculate

$$\int_0^\infty \frac{\sin(x)}{x(x^2+1)}\,dx.$$

12.8 8

Calculate

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx.$$

12.9 9

Calculate

$$\int_{-\infty}^\infty \frac{1+x^2}{1+x^4}\,dx.$$

12.10 10

Let $a > 0$ and calculate

$$\int_0^\infty \frac{\cos(x)}{(x^2+a^2)^2}\,dx.$$

12.11 11

Calculate

$$\int_0^\infty \frac{\sin^3(x)}{x^3}\,dx.$$

12.12 12

Let $n \in \mathbb{Z}^{\geq 1}$ and $0 < \theta < \pi$ and show that

$$\frac{1}{2\pi i}\int_{|z|=2} \frac{z^n}{1-3z\cos(\theta)+z^2}\,dz = \frac{\sin(n\theta)}{\sin(\theta)}.$$

12.13 13

Suppose $a > b > 0$ and calculate

$$\int_0^{2\pi} \frac{1}{(a + b\cos(\theta))^2}\, d\theta.$$

13 | Residue Theorem: Extra Questions

13.1 ?

Suppose that f is an analytic function in the region D which contains the point a. Let

$$F(z) = z - a - qf(z), \quad \text{where} \quad q \text{ is a complex parameter.}$$

13.1.1 1

Let $K \subset D$ be a circle with the center at point a and also we assume that $f(z) \neq 0$ for $z \in K$. Prove that the function F has one and only one zero $z = w$ on the closed disc $\overline{K}$ whose boundary is the circle K if $|q| < \min_{z \in K} \dfrac{|z-a|}{|f(z)|}$.

13.1.2 2

Let $G(z)$ be an analytic function on the disk $\overline{K}$. Apply the residue theorem to prove that

$$\frac{G(w)}{F'(w)} = \frac{1}{2\pi i} \int_K \frac{G(z)}{F(z)} dz,$$

where w is the zero from (1).

13.2 ?

Evaluate

$$\int_0^{\infty} \frac{x \sin x}{x^2 + a^2}\, dx.$$

13.3 ?

Show that

$$\int_0^\infty \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$$

using complex analysis, $0 < a < n$. Here n is a positive integer.

14 Rouche's Theorem (8155h)

14.1 1 ✨

Prove that for every $n \in \mathbb{Z}^{\geq 0}$ the following polynomial has no roots in the open unit disc:

$$f_n(x) := \sum_{k=0}^{n} \frac{z^k}{k!}.$$

Hint: check $n = 1, 2$ directly.

Solution:

Concepts Used:

Note

- $f_1(z) = 1 + z$, which has the single root $z = -1$ which is not inside $|z| < 1$.
- $f_2(z) = 1 + z + \frac{1}{2}z^2 = (z - (1+i))(z - (1-i))$, and $|1 \pm i| = \sqrt{2} > 1$.
- Note that $p_n(z) \overset{n\to\infty}{\longrightarrow} e^z$ uniformly on any compact set.
- Let r be arbitrary and fix $N := \mathbb{D}_r(0)$, then $p_n(z) \to e^z$ uniformly on $\overline{N}$.
- Set $g_n(z) := p_n(z)/e^z$, then $g_n \to 1$ uniformly on $\overline{N}$.
- Choose $n \gg 0$ so that $|f(z) - 1| < \varepsilon < 1$ for all $z \in \overline{N}$.
- So take $h(z) = 1$, then on ∂N,?

14.2 2 ✨

Assume that $|b| < 1$ and show that the following polynomial has exactly two roots (counting multiplicity) in $|z| < 1$:

$$f(z) := z^3 + 3z^2 + bz + b^2.$$

Solution:

Concepts Used:

Multiple versions of Rouches theorem!

- Set $h(z) = 3z^2$ and $g(z) = z^3 + bz + b^2$.
- Then on $|z| = 1$,

$$|g(z)| \leq 1 + b + b^2 < 3 = 3|z|^2 = \left|3z^2\right| = |h|,$$

  so g, h have the same number of roots in $|z| \leq_? 1$.
- But h evidently has two roots in this region.

14.3 3 🚩

Let $c \in \mathbb{C}$ with $|c| < \frac{1}{3}$. Show that on the open set $\left\{z \in \mathbb{C} \;\middle|\; \Re(z) < 1\right\}$, the function $f(z) := ce^z$ has exactly one fixed point.

14.4 4 ✨

How many roots does the following polynomial have in the open disc $|z| < 1$?

$$f(z) = z^7 - 4z^3 - 1.$$

Solution:

Concepts Used:

- Set $h(z) = -4z^3$ and $g(z) = z^7 - 1$, then on $|z| = 1$,

$$|g(z)| = \left|z^7 - 1\right| \leq 1 + 1 = 2 < 4 = \left|-4z^3\right| = |h(z)|.$$

- So h and $h + g$ have the same number of roots, but h has three roots here.

14.5 5 ✨

Let $n \in \mathbb{Z}^{\geq 0}$ and show that the equation

$$e^z = az^n$$

has n solutions in the open unit disc if $|a| > e$, and no solutions if $|a| < \frac{1}{e}$.

14.6 6 🚩

Let f be analytic in a domain D and fix $z_0 \in D$ with $w_0 := f(z_0)$. Suppose z_0 is a zero of $f(z) - w_0$ with finite multiplicity m. Show that there exists $\delta > 0$ and $\varepsilon > 0$ such that for each w such that $0 < |w - w_0| < \varepsilon$, the equation $f(z) - w = 0$ has exactly m *distinct* solutions inside the disc $|z - z_0| < \delta$.

14.7 7 🚩

For $k = 1, 2, \cdots, n$, suppose $|a_k| < 1$ and

$$f(z) := \left(\frac{z - a_1}{1 - \bar{a}_q z}\right)\left(\frac{z - a_2}{1 - \bar{a}_2 z}\right)\cdots\left(\frac{z - a_n}{1 - \bar{a}_n z}\right).$$

Show that $f(z) = b$ has n solutions in $|z| < 1$.

14.8 8

For each $n \in \mathbb{Z}^{\geq 1}$, let

$$P_n(z) = 1 + z + \frac{1}{2!}z^2 + \cdots + \frac{1}{n!}z^n.$$

Show that for sufficiently large n, the polynomial P_n has no zeros in $|z| < 10$, while the polynomial $P_n(z) - 1$ has precisely 3 zeros there.

14.9 9

Prove that

$$\max_{|z|=1} \left| a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n \right| \geq 1.$$

Hint: the first part of the problem asks for a statement of Rouche's theorem.

14.10 10

Use Rouche's theorem to prove the Fundamental Theorem of Algebra.

15 | Extras

15.1 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra:

If

$$P_n(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree n, then it has n zeros in $\mathbb{C}$.

15.2 ?

Suppose f is entire and there exist $A, R > 0$ and natural number N such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that (i) f is a polynomial and (ii) the degree of f is at least N.

16 Schwarz Lemma and Reflection Principle (8155i)

16.1 1

Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic and admits a continuous extension $\tilde{f} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that $|z| = 1 \implies |f(z)| = 1$.

16.1.1 a

Prove that f is a rational function.

16.1.2 b

Suppose that $z = 0$ is the unique zero of f. Show that

$$\exists n \in \mathbb{N}, \lambda \in S^1 \quad \text{suchthat} \quad f(z) = \lambda z^n.$$

16.1.3 c

Suppose that $a_1, \cdots, a_n \in \mathbb{D}$ are the zeros of f and prove that

$$\exists \lambda \in S^1 \quad \text{suchthat} \quad f(z) = \lambda \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}.$$

16.2 2

Let $\overline{B}(a, r)$ denote the closed disc of radius r about $a \in \mathbb{C}$. Let f be holomorphic on an open set containing $\overline{B}(a, r)$ and let

$$M := \sup_{z \in \overline{B}(a,r)} |f(z)|.$$

Prove that

$$z \in \overline{B}\left(a, \frac{r}{2}\right), \, z \neq a, \qquad \frac{|f(z) - f(a)|}{|z - a|} \leq \frac{2M}{r}.$$

16.3 3

Define

$$G := \left\{ z \in \mathbb{C} \;\middle|\; \Re(z) > 0, \, |z - 1| > 1 \right\}.$$

Find all of the injective conformal maps $G \to \mathbb{D}$. These may be expressed as compositions of maps, but explain why this list is complete.

16.4 4

Suppose $f : \mathbb{H} \cup \mathbb{R} \to \mathbb{C}$ satisfies the following:

- $f(i) = i$
- f is continuous
- f is analytic on $\mathbb{H}$
- $f(z) \in \mathbb{R} \iff z \in \mathbb{R}$.

Show that $f(\mathbb{H})$ is a dense subset of $\mathbb{H}$.

16.5 5

Suppose $f : \mathbb{D} \to \mathbb{H}$ is analytic and satisfies $f(0) = 2$. Find a sharp upper bound for $|f'(0)|$, and prove it is sharp by example.

16.6 6

Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic, has a single zero of order k at $z = 0$, and satsifies $\lim_{|z| \to 1} |f(z)| = 1$. Give with proof a formula for $f(z)$.

16.7 7

16.7.1 a

State the standard Schwarz reflection principle involving reflection across the real axis.

16.7.2 b

Give a linear fractional transformation T mapping $\mathbb{D}$ to $\mathbb{H}$. Let $g(z) = \bar{z}$, and show

$$(T^{-1} \circ g \circ T)(z) = 1/\bar{z}.$$

16.7.3 c

Suppose that f is holomorphic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, and real on S^1. Show that f must be constant.

16.8 8

Suppose $f, g : \mathbb{D} \to \Omega$ are holomorphic with f injective and $f(0) = g(0)$.

Show that

$$\forall\, 0 < r < 1, \qquad g\left(\{|z| < r\}\right) \subseteq f\left(\{|z| < r\}\right).$$

The first part of this problem asks for a statement of the Schwarz lemma.

16.9 9

Let $S \coloneqq \left\{ z \in \mathbb{D} \;\middle|\; \Im(z) \geq 0 \right\}$. Suppose $f : S \to \mathbb{C}$ is continuous on S, real on $S \cap \mathbb{R}$, and holomorphic on S°.

Prove that f is the restriction of a holomorphic function on $\mathbb{D}$.

16.10 10

Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic. Prove that

$$\forall a \in \mathbb{D}, \qquad \frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2}.$$

17 Unsorted/Unknown

17.1 ?

17.2 ?

Let f be a continuous function in the region

$$D = \{z \,||z| > R, 0 \leq \arg Z \leq \theta\} \quad \text{where} \quad 0 \leq \theta \leq 2\pi.$$

If there exists k such that $\lim\limits_{z \to \infty} z f(z) = k$ for z in the region D.

Show that

$$\lim_{R' \to \infty} \int_L f(z) dz = i\theta k,$$

where L is the part of the circle $|z| = R'$ which lies in the region D.

17.2.1 3

If $z \in K$, prove that the function $\dfrac{1}{F(z)}$ can be represented as a convergent series with respect to q:

$$\frac{1}{F(z)} = \sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z-a)^{n+1}}.$$

17.3 ?

Show that

$$\int_0^\infty \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$$

using complex analysis, $0 < a < n$.

Here n is a positive integer.

17.4 11.

Let g be analytic for $|z| \le 1$ and $|g(z)| < 1$ for $|z| = 1$.

17.4.1 a

Show that g has a unique fixed point in $|z| < 1$.

17.4.2 b

What happens if we replace $|g(z)| < 1$ with $|g(z)| \le 1$ for $|z| = 1$?

Give an example if (a) is not true or give an proof if (a) is still true.

17.4.3 c

What happens if we simply assume that f is analytic for $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z) \not\equiv z$.

Can f have more than one fixed point in $|z| < 1$?

Hint: The map

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$$

may be useful.

17.5 ?

Let $f(z)$ be entire and assume that $f(z) \leq M|z|^2$ outside some disk for some constant M.

Show that $f(z)$ is a polynomial in z of degree ≤ 2.

17.6 ?

Let $a_n(z)$ be an analytic sequence in a domain D such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D.

Show that $\sum_{n=0}^{\infty} |a_n'(z)|$ converges uniformly on bounded and closed sub-regions of D.

17.7 16

Let $f(z)$ be analytic in an open set Ω except possibly at a point z_0 inside Ω.

Show that if $f(z)$ is bounded in near z_0, then $\int_\Delta f(z)dz = 0$ for all triangles Δ in Ω.

18 Riemann Mapping and Casorati-Weierstrass

18.1 10.

Let $f : \mathbb{C} \to \mathbb{C}$ be an injective analytic (also called univalent) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

19 | Spring 2020 Homework 1

19.1 1

Geometrically describe the following subsets of $\mathbb{C}$:

a. $|z-1| = 1$
b. $|z-1| = 2|z-2|$
c. $1/z = \bar{z}$
d. $\Re(z) = 3$
e. $\Im(z) = a$ with $a \in \mathbb{R}$.
f. $\Re(z) > a$ with $a \in \mathbb{R}$.
g. $|z-1| < 2|z-2|$

19.2 2

Prove the following inequality, and explain when equality holds:

$$|z+w| \geq ||z| - |w||.$$

19.3 3

Prove that the following polynomial has its roots outside of the unit circle:

$$p(z) = z^3 + 2z + 4.$$

Hint: What is the maximum value of the modulus of the first two terms if $|z| \leq 1$?

19.4 4

a. Prove that if $c > 0$,

$$|w_1| = c|w_2| \implies \left|w_1 - c^2 w_2\right| = c|w_1 - w_2|.$$

b. Prove that if $c > 0$ and $c \neq 1$, with $z_1 \neq z_2$, then the following equation represents a circle:

$$\left|\frac{z - z_1}{z - z_2}\right| = c.$$

Find its center and radius.

Hint: use part (a)

19.5 5

a. Let $z, w \in \mathbb{C}$ with $\bar{z}w \neq 1$. Prove that

$$\left|\frac{w - z}{1 - \overline{w}z}\right| < 1 \quad \text{ if } |z| < 1,\ |w| < 1$$

with equality when $|z| = 1$ or $|w| = 1$.

b. Prove that for a fixed $w \in \mathbb{D}$, the mapping $F : z \mapsto \dfrac{w - z}{1 - \overline{w}z}$ satisfies

- F maps $\mathbb{D}$ to itself and is holomorphic.
- $F(0) = w$ and $F(w) = 0$.
- $|z| = 1$ implies $|F(z)| = 1$.

19.6 6

Use nth roots of unity to show that

$$2^{n-1} \sin\left(\frac{\pi}{n}\right) \sin\left(\frac{2\pi}{n}\right) \cdots \sin\left(\frac{(n-1)\pi}{n}\right) = n.$$

Hint:

$$\begin{aligned} 1 - \cos(2\theta) &= 2\sin^2(\theta) \\ 2\sin(2\theta) &= 2\sin(\theta)\cos(\theta). \end{aligned}$$

19.7 7

Prove that $f(z) = |z|^2$ has a derivative at $z = 0$ and nowhere else.

19.8 8

Let $f(z)$ be analytic in a domain, and prove that f is constant if it satisfies any of the following conditions:

a. $|f(z)|$ is constant.
b. $\Re(f(z))$ is constant.
c. $\arg(f(z))$ is constant.
d. $\overline{f(z)}$ is analytic.

How do you generalize (a) and (b)?

19.9 9

Prove that if $z \mapsto f(z)$ is analytic, then $z \mapsto \overline{f(\bar{z})}$ is analytic.

19.10 10

a. Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

b. Use (a) to show that the logarithm function, defined as

$$\log z = \log r + i\theta \text{ where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi.$$

is holomorphic on the region $r > 0, -\pi < \theta < \pi$.

Also show that this function is not continuous in $r > 0$.

19.11 11

Prove that the distinct complex numbers z_1, z_2, z_3 are the vertices of an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

20 Spring 2020 Homework 2

Note on notation: I sometimes use $f_x \coloneqq \dfrac{\partial f}{\partial x}$ to denote partial derivatives, and $\partial_z^n f$ as $f^{(n)}(z)$.

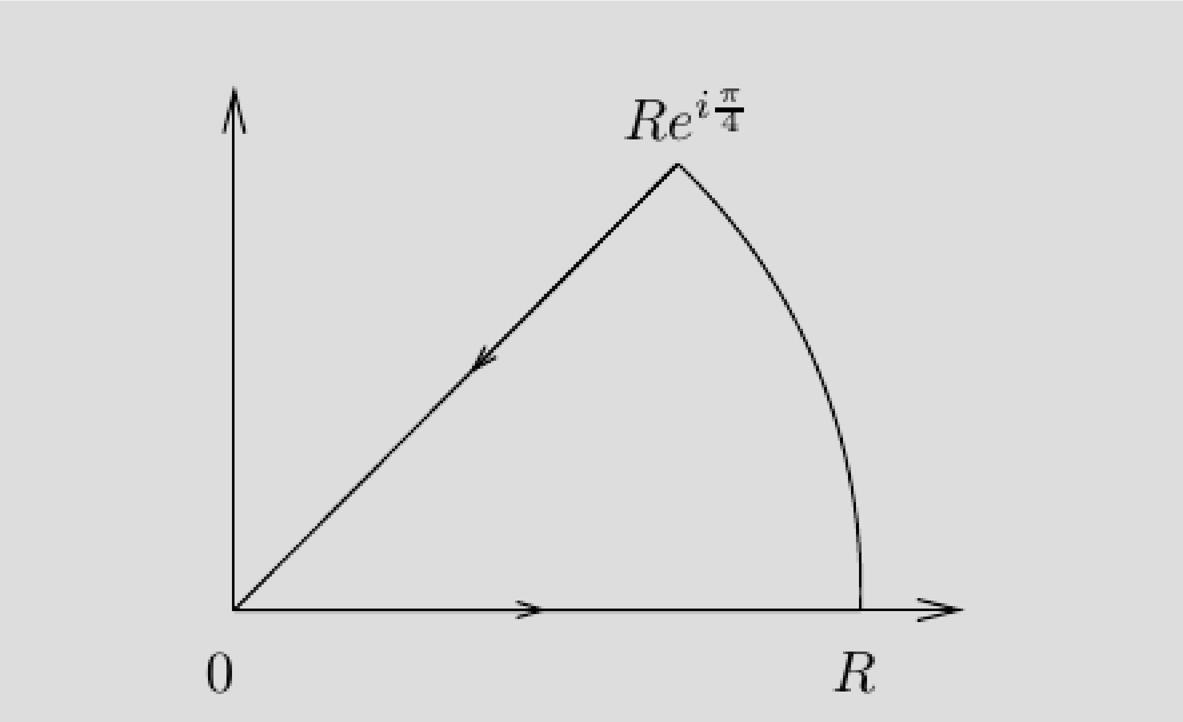
20.1 Stein And Shakarchi

20.1.1 2.6.1

Show that

$$\int_0^\infty \sin\left(x^2\right) dx = \int_0^\infty \cos\left(x^2\right) dx = \frac{\sqrt{2\pi}}{4}.$$

Hint: integrate e^{-x^2} over the following contour, using the fact that $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$:



20.1.2 2.6.2

Show that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Hint: use the fact that this integral equals $\frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{ix}-1}{x} dx$, and integrate around an indented semicircle.

20.1.3 2.6.5

Suppose $f \in C^1_{\mathbb{C}}(\Omega)$ and $T \subset \Omega$ is a triangle with $T^\circ \subset \Omega$. Apply Green's theorem to show that $\int_T f(z)\, dz = 0$.

Assume that f' is continuous and prove Goursat's theorem.

Hint: Green's theorem states

$$\int_T F dx + G dy = \int_{T^\circ} \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx dy.$$

20.1.4 2.6.6

Suppose that f is holomorphic on a punctured open set $\Omega \setminus \{w_0\}$ and let $T \subset \Omega$ be a triangle containing w_0. Prove that if f is bounded near w_0, then $\int_T f(z)\, dz = 0$.

20.1.5 2.6.7

Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and let $d \coloneqq \sup_{z,w \in \mathbb{D}} |f(z) - f(w)|$ be the diameter of the image of f. Show that $2|f'(0)| \leq d$, and that equality holds iff f is linear, so $f(z) = a_1 z + a_2$.

> *Hint:* $2f'(0) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi) - f(-\xi)}{\xi^2}\, d\xi$ *whenever* $0 < r < 1$.

20.1.6 2.6.8

Suppose that f is holomorphic on the strip $S = \left\{ x + iy \;\middle|\; x \in \mathbb{R},\ -1 < y < 1 \right\}$ with $|f(z)| \leq A(1 + |z|)^\nu$ for ν some fixed real number. Show that for all $z \in S$, for each integer $n \geq 0$ there exists an $A_n \geq 0$ such that $\left| f^{(n)}(x) \right| \leq A_n (1 + |x|)^\nu$ for all $x \in \mathbb{R}$.

> *Hint: Use the Cauchy inequalities.*

20.1.7 2.6.9

Let $\Omega \subset \mathbb{C}$ be open and bounded and $\varphi : \Omega \to \Omega$ holomorphic. Prove that if there exists a point $z_0 \in \Omega$ such that $\varphi(z_0) = z_0$ and $\varphi'(z_0) = 1$, then φ is linear.

> *Hint: assume* $z_0 = 0$ *(explain why this can be done) and write* $\varphi(z) = z + a_n z^n + O(z^{n+1})$ *near* 0*. Let* $\varphi_k = \varphi \circ \varphi \circ \cdots \circ \varphi$ *and prove that* $\varphi_k(z) = z + k a_n z^n + O(z^{n+1})$*. Apply Cauchy's inequalities and let* $k \to \infty$ *to conclude.*

20.1.8 2.6.10

Can every continuous function on $\overline{\mathbb{D}}$ be uniformly approximated by polynomials in the variable z?

Hint: compare to Weierstrass for the real interval.

20.1.9 2.6.13

Suppose f is analytic, defined on all of $\mathbb{C}$, and for each $z_0 \in \mathbb{C}$ there is at least one coefficient in the expansion $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ is zero. Prove that f is a polynomial.

Hint: use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.

20.1.10 2.6.14

Suppose that f is holomorphic in an open set containing $\mathbb{D}$ except for a pole $z_0 \in \partial\mathbb{D}$. Let $\sum_{n=0}^{\infty} a_n z^n$ be the power series expansion of f in $\mathbb{D}$, and show that $\lim \frac{a_n}{a_{n+1}} = z_0$.

20.1.11 2.6.15

Suppose f is continuous and nonvanishing on $\overline{\mathbb{D}}$, and holomorphic in $\mathbb{D}$. Prove that if $|z| = 1 \implies |f(z)| = 1$, then f is constant.

Hint: Extend f to all of $\mathbb{C}$ by $f(z) = 1/\overline{f(1/\overline{z})}$ for any $|z| > 1$, and argue as in the Schwarz reflection principle.

20.2 Additional Problems

20.2.1 1

Let $a_n \neq 0$ and show that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L \implies \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L.$$

In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

20.2.2 2

Let f be a power series centered at the origin. Prove that f has a power series expansion about any point in its disc of convergence.

20.2.3 3

Prove the following:

a. $\sum_n nz^n$ does not converge at any point of S^1

b. $\sum_n \frac{z^n}{n^2}$ converges at every point of S^1.

c. $\sum_n \frac{z^n}{n}$ converges at every point of S^1 except $z = 1$.

20.2.4 4

Without using Cauchy's integral formula, show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{dz}{(z-\alpha)(z-\beta)} = \frac{2\pi i}{\alpha - \beta}$$

where γ denotes the circle centered at the origin of radius r with positive orientation.

20.2.5 5

Assume f is continuous in the region $\left\{ x + iy \;\middle|\; x \geq x_0,\ 0 \leq y \leq b \right\}$, and the following limit exists independent of y:

$$\lim_{x \to +\infty} f(x + iy) = A.$$

Show that if $\gamma_x := \left\{ z = x + it \;\middle|\; 0 \leq t \leq b \right\}$, then

$$\lim_{x \to +\infty} \int_{\gamma_x} f(z)\, dz = iAb.$$

20.2.6 6

Show by example that there exists a function $f(z)$ that is holomorphic on $\left\{z \in \mathbb{C} \,\middle|\, 0<|z|<1\right\}$ and for all $r<1$,

$$\int_{|z|=r} f(z)\, dz = 0,$$

but f is not holomorphic at $z=0$.

20.2.7 7

Let f be analytic on a region R and suppose $f'(z_0) \neq 0$ for some $z_0 \in R$. Show that if C is a circle of sufficiently small radius centered at z_0, then

$$\frac{2\pi i}{f'\left(z_0\right)} = \int_C \frac{dz}{f(z)-f\left(z_0\right)}.$$

Hint: use the inverse function theorem.

20.2.8 8

Assume two functions $u, b : \mathbb{R}^2 \to \mathbb{R}$ have continuous partial derivatives at (x_0, y_0). Show that $f := u + iv$ has derivative $f'(z_0)$ at $z_0 == x_0 + iy_0$ if and only if

$$\lim_{r \to 0} \frac{1}{\pi r^2} \int_{|z-z_0|=r} f(z) dz = 0.$$

20.2.9 9 (Cauchy's Formula for Exterior Regions)

Let γ be a piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2. Assume f' exists in an open set containing γ and Ω_2 with $\lim_{z \to \infty} f(z) = A$. Show that

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1 \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}.$$

20.2.10 10

Let $f(z)$ be bounded and analytic in $\mathbb{C}$. Let $a \neq b$ be any fixed complex numbers. Show that the following limit exists:

$$\lim_{R \to \infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}\, dz.$$

Use this to show that $f(z)$ must be constant.

20.2.11 11

Suppose $f(z)$ is entire and

$$\lim_{z\to\infty} \frac{f(z)}{z} = 0.$$

Show that $f(z)$ is a constant.

20.2.12 12

Let f be analytic in a domain D and γ be a closed curve in D. For any $z_0 \in D$ not on γ, show that

$$\int_\gamma \frac{f'(z)}{(z-z_0)} dz = \int_\gamma \frac{f(z)}{(z-z_0)^2} dz.$$

Give a generalization of this result.

20.2.13 13

Compute

$$\int_{|z|=1} \left(z+\frac{1}{z}\right)^{2n} \frac{dz}{z}$$

and use it to show that

$$\int_0^{2\pi} \cos^{2n}(\theta)\, d\theta = 2\pi \left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right).$$

21 | Spring 2020 Homework 3

21.1 Stein and Shakarchi

21.1.1 3.8.1

Use the following formula to show that the complex zeros of $\sin(\pi z)$ are exactly the integers, and they are each of order 1:

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}.$$

Calculate the residue of $\dfrac{1}{\sin(\pi z)}$ at $z = n \in \mathbb{Z}$.

21.1.2 3.8.2

Evaluate the integral

$$\int_{\mathbb{R}} \frac{dx}{1+x^4}.$$

What are the poles of $\dfrac{1}{1+z^4}$?

21.1.3 3.8.4

Show that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}, \qquad \text{for all } a > 0.$$

21.1.4 3.8.5

Show that if $\xi \in \mathbb{R}$, then

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{(1+x^2)^2} dx = \frac{\pi}{2}(1 + 2\pi|\xi|)e^{-2\pi|\xi|}.$$

21.1.5 3.8.6

Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \pi.$$

21.1.6 3.8.7

Show that

$$\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{3/2}}, \qquad \text{whenever } a > 1.$$

21.1.7 3.8.8

Show that if $a, b \in \mathbb{R}$ with $a > |b|$, then

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}.$$

21.1.8 3.8.9

Show that

$$\int_0^1 \log(\sin \pi x)dx = -\log 2.$$

Hint: use the following contour.

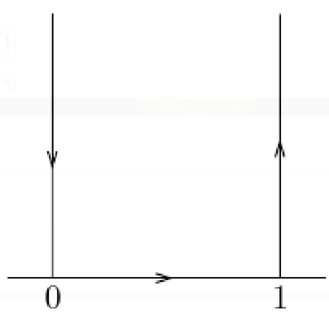


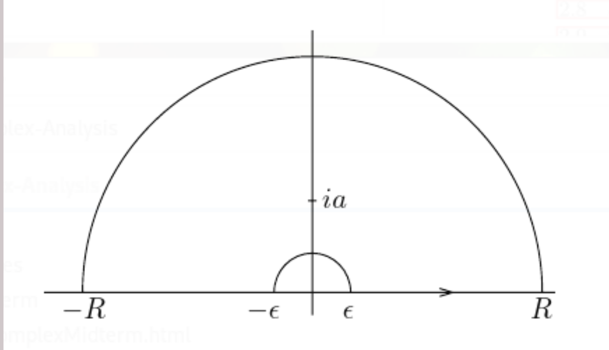
Figure 9. Contour in Exercise 9

21.1.9 3.8.10

Show that if $a > 0$, then

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx = \frac{\pi}{2a} \log a.$$

Hint: use the following contour.



21.1.10 3.8.14

Prove that all entire functions that are injective are of the form $f(z) = az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$.

Hint: Apply the Casorati-Weierstrass theorem to $f(1/z)$.

21.1.11 3.8.15

Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

a. Prove that if f is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \leq AR^k + B$$

for all $R > 0$, some integer $k \geq 0$, and some constants $A, B > 0$, then f is a polynomial of degree $\leq k$.

b. Show that if f is holomorphic in the unit disc, is bounded, and converges uniformly to zero in the sector $\theta < \arg(z) < \varphi$ as $|z| \to 0$, then $f \equiv 0$.

c. Let $w_1, \cdots w_n$ be points on $S^1 \subset \mathbb{C}$. Prove that there exists a point $z \in S^1$ such that the product of the distances from z to the points w_j is at least 1.

Conclude that there exists a point $w \in S^1$ such that the product of the above distances is *exactly* 1.

d. Show that if the real part of an entire function is bounded, then f is constant.

21.1.12 3.8.17

Let f be non-constant and holomorphic in an open set containing the closed unit disc.

a. Show that if $|f(z)| = 1$ whenever $|z| = 1$, then the image of f contains the unit disc.

> *Hint: Show that $f(z) = w_0$ has a root for every $w_0 \in \mathbb{D}$, for which it suffices to show that $f(z) = 0$ has a root. Conclude using the maximum modulus principle.*

b. If $|f(z)| \geq 1$ whenever $|z| = 1$ and there exists a $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$, then the image of f contains the unit disc.

21.1.13 3.8.19

Prove that maximum principle for harmonic functions, i.e.

a. If u is a non-constant real-valued harmonic function in a region Ω, then u can not attain a maximum or a minimum in Ω.

b. Suppose Ω is a region with compact closure $\overline{\Omega}$. If u is harmonic in Ω and continuous in $\overline{\Omega}$, then

$$\sup_{z \in \Omega} |u(z)| \leq \sup_{z \in \overline{\Omega} - \Omega} |u(z)|.$$

> *Hint: to prove (a), assume u attains a local maximum at z_0. Let f be holomorphic near z_0 with $\Re(f) = u$, and show that f is not an open map. Then (a) implies (b).*

21.2 Extra Problems

21.2.1 1

Problem Prove that if f has two Laurent series expansions,

$$f(z) = \sum c_n (z-a)^n \quad \text{and} \quad f(z) = \sum c'_n (z-a)^n$$

then $c_n = c'_n$.

21.2.2 2

Problem Find Laurent series expansions of

$$\frac{1}{1-z^2} + \frac{1}{3-z}$$

How many such expansions are there? In what domains are each valid?

21.2.3 3

Problem Let P, Q be polynomials with no common zeros. Assume a is a root of Q. Find the principal part of P/Q at $z = a$ in terms of P and Q if a is (1) a simple root, and (2) a double root.

21.2.4 4

Problem Let f be non-constant, analytic in $|z| > 0$, where $f(z_n) = 0$ for infinitely many points z_n with $\lim_{n\to\infty} z_n = 0$.

Show that $z = 0$ is an essential singularity for f.

Example: $f(z) = \sin(1/z)$.

21.2.5 5

Problem Show that if f is entire and $\lim_{z\to\infty} f(z) = \infty$, then f is a polynomial.

21.2.6 6

Problem

a. Show (without using 3.8.9 in the S&S) that

$$\int_0^{2\pi} \log \left|1 - e^{i\theta}\right| \, d\theta = 0$$

b. Show that this identity is equivalent to S&S 3.8.9:

$$\int_0^1 \log(\sin(\pi x)) \, dx = -\log 2.$$

21.2.7 7

Problem Let $0 < a < 4$ and evaluate

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^3} \, dx$$

21.2.8 8

Problem Prove the fundamental theorem of Algebra using

a. Rouche's Theorem.
b. The maximum modulus principle.

21.2.9 9

Problem Let f be analytic in a region D and γ a rectifiable curve in D with interior in D.

Prove that if $f(z)$ is real for all $z \in \gamma$, then f is constant.

21.2.10 10

Problem For $a > 0$, evaluate

$$\int_0^{\pi/2} \frac{d\theta}{a + \sin^2 \theta}$$

21.2.11 11

Problem Find the number of roots of $p(z) = 4z^4 - 6z + 3$ in $|z| < 1$ and $1 < |z| < 2$ respectively.

21.2.12 12

Problem Prove that $z^4 + 2z^3 - 2z + 10$ has exactly one root in each open quadrant.

21.2.13 13

Problem Prove that for $a > 0$, $z \tan z - a$ has only real roots.

21.2.14 14

Problem Let f be nonzero, analytic on a bounded region Ω and continuous on its closure $\overline{\Omega}$.

Show that if $|f(z)| \equiv M$ is constant for $z \in \partial\Omega$, then $f(z) \equiv Me^{i\theta}$ for some real constant θ.

22 Extra Questions from Jingzhi Tie

22.1 Fall 2009

22.1.1 ?

(1) Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} .$$

(2) Deduce Liouville's theorem from (1).

22.1.2 ?

Let f be a continuous function in the region

$$D = \{z \mid |z| > R, 0 \le \arg z \le \theta\} \quad \text{where} \quad 1 \le \theta \le 2\pi.$$

If there exists k such that $\lim_{z\to\infty} zf(z) = k$ for z in the region D. Show that

$$\lim_{R'\to\infty} \int_L f(z)dz = i\theta k,$$

where L is the part of the circle $|z| = R'$ which lies in the region D.

22.1.3 ?

Suppose that f is an analytic function in the region D which contains the point a. Let

$$F(z) = z - a - qf(z), \quad \text{where } q \text{ is a complex parameter.}$$

(1) Let $K \subset D$ be a circle with the center at point a and also we assume that $f(z) \neq 0$ for $z \in K$. Prove that the function F has one and only one zero $z = w$ on the closed disc $\overline{K}$ whose boundary is the circle K if $|q| < \min_{z\in K} \dfrac{|z-a|}{|f(z)|}$.

(2) Let $G(z)$ be an analytic function on the disk $\overline{K}$. Apply the residue theorem to prove that $\dfrac{G(w)}{F'(w)} = \dfrac{1}{2\pi i}\displaystyle\int_K \frac{G(z)}{F(z)}dz$, where w is the zero from (1).

(3) If $z \in K$, prove that the function $\dfrac{1}{F(z)}$ can be represented as a convergent series with respect to q: $\dfrac{1}{F(z)} = \displaystyle\sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z-a)^{n+1}}$.

22.1.4 ?

Evaluate

$$\int_0^\infty \frac{x \sin x}{x^2 + a^2}\, dx.$$

22.1.5 ?

Let $f = u + iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

22.1.6 ?

Show that $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^n}dx = \frac{\pi}{n\sin\frac{a\pi}{n}}$ using complex analysis, $0 < a < n$. Here n is a positive integer.

22.1.7 ?

For $s > 0$, the **gamma function** is defined by $\Gamma(s) = \displaystyle\int_0^\infty e^{-t}t^{s-1}dt$.

1. Show that the gamma function is analytic in the half-plane $\Re(s) > 0$, and is still given there by the integral formula above.

2. Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need $\Gamma(1-s) = t\displaystyle\int_0^\infty e^{-vt}(vt)^{-s}dv$ *for* $t > 0$.

22.1.8 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree n, then it has n zeros in $\mathbb{C}$.

22.1.9 ?

Suppose f is entire and there exist $A, R > 0$ and natural number N such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that (i) f is a polynomial and (ii) the degree of f is at least N.

22.1.10 ?

Let $f : \mathbb{C} \to \mathbb{C}$ be an injective analytic (also called *univalent*) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

22.1.11 ?

Let g be analytic for $|z| \leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

1. Show that g has a unique fixed point in $|z| < 1$.
2. What happens if we replace $|g(z)| < 1$ with $|g(z)| \leq 1$ for $|z| = 1$? Give an example if (a) is not true or give an proof if (a) is still true.
3. What happens if we simply assume that f is analytic for $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z) \not\equiv z$. Can f have more than one fixed point in $|z| < 1$?

Hint: The map $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ *may be useful.*

22.1.12 ?

Find a conformal map from $D = \{z : \ |z| < 1, \ |z - 1/2| > 1/2\}$ to the unit disk $\Delta = \{z : \ |z| < 1\}$.

22.1.13 ?

Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω. Show without using Picard's theorems that $f(z)$ is a constant.

22.1.14 ?

(1) Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \ .$$

(2) Deduce Liouville's theorem from (1).

22.1.15 ?

Let $f(z)$ be entire and assume that $f(z) \leq M|z|^2$ outside some disk for some constant M. Show that $f(z)$ is a polynomial in z of degree ≤ 2.

22.1.16 ?

Let $a_n(z)$ be an analytic sequence in a domain D such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D. Show that $\sum_{n=0}^{\infty} |a_n'(z)|$ converges uniformly on bounded and closed sub-regions of D.

22.1.17 ?

Let $f(z)$ be analytic in an open set Ω except possibly at a point z_0 inside Ω. Show that if $f(z)$ is bounded in near z_0, then $\int_\Delta f(z)dz = 0$ for all triangles Δ in Ω.

22.1.18 ?

Assume f is continuous in the region: $0 < |z-a| \leq R,\ 0 \leq \arg(z-a) \leq \beta_0$ $(0 < \beta_0 \leq 2\pi)$ and the limit $\lim_{z \to a}(z-a)f(z) = A$ exists. Show that

$$\lim_{r \to 0} \int_{\gamma_r} f(z)dz = iA\beta_0 \ ,$$

where $\gamma_r := \{z \mid z = a + re^{it},\ 0 \leq t \leq \beta_0\}$.

22.1.19 ?

Show that $f(z) = z^2$ is uniformly continuous in any open disk $|z| < R$, where $R > 0$ is fixed, but it is not uniformly continuous on $\mathbb{C}$.

22.1.20 ?

(1) Show that the function $u = u(x, y)$ given by

$$u(x, y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx \quad \text{for } n \in \mathbf{N}$$

is the solution on $D = \{(x, y) \; |x^2 + y^2 < 1\}$ of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = \frac{\sin nx}{n}.$$

(2) Show that there exist points $(x, y) \in D$ such that $\limsup_{n\to\infty} |u(x, y)| = \infty$.

22.2 Fall 2011

22.2.1 ?

(1) Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \; .$$

(2) Deduce Liouville's theorem from (1).

22.2.2 ?

Let f be a continuous function in the region

$$D = \{z \; ||z| > R, 0 \leq \arg Z \leq \theta\} \quad \text{where} \quad 0 \leq \theta \leq 2\pi.$$

If there exists k such that $\lim_{z\to\infty} zf(z) = k$ for z in the region D. Show that

$$\lim_{R'\to\infty} \int_L f(z)dz = i\theta k,$$

where L is the part of the circle $|z| = R'$ which lies in the region D.

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Suppose that f is an analytic function in the region D which contains the point a. Let

$$F(z) = z - a - qf(z), \quad \text{where} \quad q \text{ is a complex parameter.}$$

(1) Let $K \subset D$ be a circle with the center at point a and also we assume that $f(z) \neq 0$ for $z \in K$. Prove that the function F has one and only one zero $z = w$ on the closed disc $\overline{K}$ whose boundary is the circle K if $|q| < \min_{z\in K} \frac{|z-a|}{|f(z)|}$.

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(3) If $z \in K$, prove that the function $\dfrac{1}{F(z)}$ can be represented as a convergent series with respect to q: $\dfrac{1}{F(z)} = \displaystyle\sum_{n=0}^{\infty} \dfrac{(qf(z))^n}{(z-a)^{n+1}}$.

22.2.4 ?

Evaluate $\displaystyle\int_0^\infty \frac{x\sin x}{x^2+a^2}\,dx$.

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Let $f = u + iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

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Show that $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^n}dx = \frac{\pi}{n\sin\frac{a\pi}{n}}$ using complex analysis, $0 < a < n$. Here n is a positive integer.

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For $s > 0$, the **gamma function** is defined by $\Gamma(s) = \displaystyle\int_0^\infty e^{-t}t^{s-1}dt$.

1. Show that the gamma function is analytic in the half-plane $\Re(s) > 0$, and is still given there by the integral formula above.

2. Apply the formula in the previous question to show that

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Hint: You may need $\Gamma(1-s) = t\displaystyle\int_0^\infty e^{-vt}(vt)^{-s}dv$ *for* $t > 0$.

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Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree n, then it has n zeros in $\mathbb{C}$.

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Suppose f is entire and there exist $A, R > 0$ and natural number N such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that (i) f is a polynomial and (ii) the degree of f is at least N.

22.2.10 ?

Let $f : \mathbb{C} \to \mathbb{C}$ be an injective analytic (also called univalent) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

22.2.11 ?

Let g be analytic for $|z| \leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

- Show that g has a unique fixed point in $|z| < 1$.
- What happens if we replace $|g(z)| < 1$ with $|g(z)| \leq 1$ for $|z| = 1$? Give an example if (a) is not true or give an proof if (a) is still true.
- What happens if we simply assume that f is analytic for $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z) \not\equiv z$. Can f have more than one fixed point in $|z| < 1$?

Hint: The map $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ *may be useful.*

22.2.12 ?

Find a conformal map from $D = \{z : \ |z| < 1, \ |z - 1/2| > 1/2\}$ to the unit disk $\Delta = \{z : \ |z| < 1\}$.

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Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω. Show without using Picard's theorems that $f(z)$ is a constant.

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Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω. Show without using Picard's theorems that $f(z)$ is a constant.

22.2.15 ?

(1) Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} .$$

(2) Deduce Liouville's theorem from (1).

22.2.16 ?

Let $f(z)$ be entire and assume that $f(z) \leq M|z|^2$ outside some disk for some constant M. Show that $f(z)$ is a polynomial in z of degree ≤ 2.

22.2.17 ?

Let $a_n(z)$ be an analytic sequence in a domain D such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D. Show that $\sum_{n=0}^{\infty} |a_n'(z)|$ converges uniformly on bounded and closed sub-regions of D.

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Let $f(z)$ be analytic in an open set Ω except possibly at a point z_0 inside Ω. Show that if $f(z)$ is bounded in near z_0, then $\int_\Delta f(z)dz = 0$ for all triangles Δ in Ω.

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Assume f is continuous in the region: $0 < |z-a| \leq R,\ 0 \leq \arg(z-a) \leq \beta_0\ (0 < \beta_0 \leq 2\pi)$ and the limit $\lim_{z \to a}(z-a)f(z) = A$ exists. Show that

$$\lim_{r \to 0} \int_{\gamma_r} f(z)dz = iA\beta_0 \ ,$$

where $\gamma_r := \{z \mid z = a + re^{it},\ 0 \leq t \leq \beta_0\}$.

22.2.20 ?

Show that $f(z) = z^2$ is uniformly continuous in any open disk $|z| < R$, where $R > 0$ is fixed, but it is not uniformly continuous on $\mathbb{C}$.

(1) Show that the function $u = u(x,y)$ given by

$$u(x,y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx \quad \text{for } n \in \mathbf{N}$$

is the solution on $D = \{(x,y) \mid x^2 + y^2 < 1\}$ of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,0) = \frac{\sin nx}{n}.$$

(2) Show that there exist points $(x,y) \in D$ such that $\limsup_{n \to \infty} |u(x,y)| = \infty$.

22.3 Spring 2014

22.3.1 ?

The question provides some insight into Cauchy's theorem. Solve the problem without using the Cauchy theorem.

1. Evaluate the integral $\int_\gamma z^n dz$ for all integers n. Here γ is any circle centered at the origin with the positive (counterclockwise) orientation.

2. Same question as (a), but with γ any circle not containing the origin.

3. Show that if $|a| < r < |b|$, then $\int_\gamma \frac{dz}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$. Here γ denotes the circle centered at the origin, of radius r, with the positive orientation.

22.3.2 ?

(1) Assume the infinite series $\sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$ and let $f(z)$ be the limit. Show that for $r < R$,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} .$$

(2) Deduce Liouville's theorem from (1). Liouville's theorem: If $f(z)$ is entire and bounded, then f is constant.

22.3.3 ?

Let f be a continuous function in the region

$$D = \{z\, ||z| > R, 0 \leq \arg Z \leq \theta\} \quad \text{where} \quad 0 \leq \theta \leq 2\pi.$$

If there exists k such that $\lim_{z\to\infty} zf(z) = k$ for z in the region D. Show that

$$\lim_{R'\to\infty}\int_L f(z)dz = i\theta k,$$

where L is the part of the circle $|z| = R'$ which lies in the region D.

22.3.4 ?

Evaluate $\displaystyle\int_0^{\infty} \frac{x\sin x}{x^2+a^2}\,dx.$

22.3.5 ?

Let $f = u + iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

22.3.6 ?

Show that $\displaystyle\int_0^{\infty} \frac{x^{a-1}}{1+x^n}dx = \frac{\pi}{n\sin\frac{a\pi}{n}}$ using complex analysis, $0 < a < n$. Here n is a positive integer.

22.3.7 ?

For $s > 0$, the **gamma function** is defined by $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$.

- Show that the gamma function is analytic in the half-plane $\Re(s) > 0$, and is still given there by the integral formula above.
- Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need $\Gamma(1-s) = t \int_0^\infty e^{-vt}(vt)^{-s} dv$ *for* $t > 0$.

22.3.8 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree n, then it has n zeros in **C**.

22.3.9 ?

Suppose f is entire and there exist $A, R > 0$ and natural number N such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that (i) f is a polynomial and (ii) the degree of f is at least N.

22.3.10 ?

Let $f : \mathbb{C} \to \mathbb{C}$ be an injective analytic (also called univalent) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

22.3.11 ?

Let g be analytic for $|z| \leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

- Show that g has a unique fixed point in $|z| < 1$.
- What happens if we replace $|g(z)| < 1$ with $|g(z)| \leq 1$ for $|z| = 1$? Give an example if (a) is not true or give an proof if (a) is still true.
- What happens if we simply assume that f is analytic for $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z) \not\equiv z$. Can f have more than one fixed point in $|z| < 1$?

Hint: The map $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ *may be useful.*

22.3.12 ?

Find a conformal map from $D = \{z : \ |z| < 1, \ |z - 1/2| > 1/2\}$ to the unit disk $\Delta = \{z : \ |z| < 1\}$.

22.4 Fall 2015

22.4.1 ?

Let $a_n \neq 0$ and assume that $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$. Show that $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$. In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

22.4.2 ?

(a) Let z, w be complex numbers, such that $\bar{z}w \neq 1$. Prove that

$$\left| \frac{w - z}{1 - \overline{w} z} \right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left| \frac{w - z}{1 - \overline{w} z} \right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

(b) Prove that for fixed w in the unit disk $\mathbb{D}$, the mapping

$$F : z \mapsto \frac{w - z}{1 - \overline{w} z}$$

satisfies the following conditions:

(c) F maps $\mathbb{D}$ to itself and is holomorphic.

(ii) F interchanges 0 and w, namely, $F(0) = w$ and $F(w) = 0$.

(iii) $|F(z)| = 1$ if $|z| = 1$.

(iv) $F : \mathbb{D} \mapsto \mathbb{D}$ is bijective.

Hint: Calculate $F \circ F$.

22.4.3 ?

Use n-th roots of unity (i.e. solutions of $z^n - 1 = 0$) to show that

$$2^{n-1} \sin\frac{\pi}{n} \sin\frac{2\pi}{n} \cdots \sin\frac{(n-1)\pi}{n} = n\ .$$

Hint: $1 - \cos 2\theta = 2\sin^2\theta,\ \sin 2\theta = 2\sin\theta\cos\theta$.

(a) Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

(b) Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \ \text{ where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

is a holomorphic function in the region $r > 0,\ -\pi < \theta < \pi$. Also show that $\log z$ defined above is not continuous in $r > 0$.

22.4.4 ?

Assume f is continuous in the region: $x \geq x_0,\ 0 \leq y \leq b$ and the limit

$$\lim_{x\to+\infty} f(x+iy) = A$$

exists uniformly with respect to y (independent of y). Show that

$$\lim_{x\to+\infty} \int_{\gamma_x} f(z)dz = iAb\ ,$$

where $\gamma_x := \{z \mid z = x + it,\ 0 \leq t \leq b\}$.

22.4.5 ?

(Cauchy's formula for "exterior" region) Let γ be piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2. Assume $f'(z)$ exists in an open set containing γ and Ω_2 and $\lim_{z\to\infty} f(z) = A$. Show that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\,d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

22.4.6 ?

Let $f(z)$ be bounded and analytic in $\mathbb{C}$. Let $a \neq b$ be any fixed complex numbers. Show that the following limit exists

$$\lim_{R\to\infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

Use this to show that $f(z)$ must be a constant (Liouville's theorem).

22.4.7 ?

Prove by *justifying all steps* that for all $\xi \in \mathbb{C}$ we have $e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\xi} dx$.

> *Hint: You may use that fact in Example 1 on p. 42 of the textbook without proof, i.e., you may assume the above is true for real values of ξ.*

22.4.8 ?

Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Let denote the the power series in the open disc. Show that (1) $c_n \neq 0$ for all large enough n's, and (2) $\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = z_0$.

22.4.9 ?

Let $f(z)$ be a non-constant analytic function in $|z| > 0$ such that $f(z_n) = 0$ for infinite many points z_n with $\lim_{n\to\infty} z_n = 0$. Show that $z = 0$ is an essential singularity for $f(z)$. (An example of such a function is $f(z) = \sin(1/z)$.)

22.4.10 ?

Let f be entire and suppose that $\lim_{z\to\infty} f(z) = \infty$. Show that f is a polynomial.

22.4.11 ?

Expand the following functions into Laurent series in the indicated regions:

(a) $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$, $2 < |z| < 3$, $3 < |z| < +\infty$.

(b) $f(z) = \sin \dfrac{z}{1-z}$, $0 < |z-1| < +\infty$

22.4.12 ?

Assume $f(z)$ is analytic in region D and Γ is a rectifiable curve in D with interior in D. Prove that if $f(z)$ is real for all $z \in \Gamma$, then $f(z)$ is a constant.

22.4.13 ?

Find the number of roots of $z^4 - 6z + 3 = 0$ in $|z| < 1$ and $1 < |z| < 2$ respectively.

22.4.14 ?

Prove that $z^4 + 2z^3 - 2z + 10 = 0$ has exactly one root in each open quadrant.

22.4.15 ?

(1) Let $f(z) \in H(\mathbb{D})$, $\mathrm{Re}(f(z)) > 0$, $f(0) = a > 0$. Show that

$$|\frac{f(z)-a}{f(z)+a}| \leq |z|, \quad |f'(0)| \leq 2a.$$

(2) Show that the above is still true if $\mathrm{Re}(f(z)) > 0$ is replaced with $\mathrm{Re}(f(z)) \geq 0$.

22.4.16 ?

Assume $f(z)$ is analytic in $\mathbb{D}$ and $f(0) = 0$ and is not a rotation (i.e. $f(z) \neq e^{i\theta}z$). Show that $\sum_{n=1}^{\infty} f^n(z)$ converges uniformly to an analytic function on compact subsets of $\mathbb{D}$, where $f^{n+1}(z) = f(f^n(z))$.

22.4.17 ?

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic and one-to-one in $|z| < 1$. For $0 < r < 1$, let D_r be the disk $|z| < r$. Show that the area of $f(D_r)$ is finite and is given by

$$S = \pi \sum_{n=1}^{\infty} n|c_n|^2 r^{2n}.$$

(Note that in general the area of $f(D_1)$ is infinite.)

22.4.18 ?

Let $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ be analytic and one-to-one in $r_0 < |z| < R_0$. For $r_0 < r < R < R_0$, let $D(r, R)$ be the annulus $r < |z| < R$. Show that the area of $f(D(r, R))$ is finite and is given by

$$S = \pi \sum_{n=-\infty}^{\infty} n|c_n|^2 (R^{2n} - r^{2n}).$$

22.5 Spring 2015

22.5.1 ?

Let $a_n(z)$ be an analytic sequence in a domain D such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D. Show that $\sum_{n=0}^{\infty} |a_n'(z)|$ converges uniformly on bounded and closed sub-regions of D.

22.5.2 ?

Let f_n, f be analytic functions on the unit disk $\mathbb{D}$. Show that the following are equivalent.

(i) $f_n(z)$ converges to $f(z)$ uniformly on compact subsets in $\mathbb{D}$.

(ii) $\displaystyle\int_{|z|=r} |f_n(z) - f(z)|\,|dz|$ converges to 0 if $0 < r < 1$.

22.5.3 ?

Let f and g be non-zero analytic functions on a region Ω. Assume $|f(z)| = |g(z)|$ for all z in Ω. Show that $f(z) = e^{i\theta} g(z)$ in Ω for some $0 \le \theta < 2\pi$.

22.5.4 ?

Suppose f is analytic in an open set containing the unit disc $\mathbb{D}$ and $|f(z)| = 1$ when $|z|$=1. Show that either $f(z) = e^{i\theta}$ for some $\theta \in \mathbb{R}$ or there are finite number of $z_k \in \mathbb{D}$, $k \le n$ and $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} \displaystyle\prod_{k=1}^{n} \frac{z - z_k}{1 - \bar{z}_k z}$.

Also cf. Stein et al, 1.4.7, 3.8.17

22.5.5 ?

(1) Let $p(z)$ be a polynomial, $R > 0$ any positive number, and $m \ge 1$ an integer. Let $M_R = \sup\{|z^m p(z) - 1| : |z| = R\}$. Show that $M_R > 1$.

(2) Let $m \ge 1$ be an integer and $K = \{z \in \mathbb{C} : r \le |z| \le R\}$ where $r < R$. Show (i) using (1) as well as, (ii) without using (1) that there exists a positive number $\varepsilon_0 > 0$ such that for each polynomial $p(z)$,

$$\sup\{|p(z) - z^{-m}| : z \in K\} \ge \varepsilon_0 .$$

22.5.6 ?

Let $f(z) = \dfrac{1}{z} + \dfrac{1}{z^2 - 1}$. Find all the Laurent series of f and describe the largest annuli in which these series are valid.

22.5.7 ?

Suppose f is entire and there exist $A, R > 0$ and natural number N such that $|f(z)| \le A|z|^N$ for $|z| \ge R$. Show that (i) f is a polynomial and (ii) the degree of f is at most N.

22.5.8 ?

Suppose f is entire and there exist $A, R > 0$ and natural number N such that $|f(z)| \geq A|z|^N$ for $|z| \geq R$. Show that (i) f is a polynomial and (ii) the degree of f is at least N.

22.5.9 ?

(1) Explicitly write down an example of a non-zero analytic function in $|z| < 1$ which has infinitely zeros in $|z| < 1$.

(2) Why does not the phenomenon in (1) contradict the uniqueness theorem?

22.5.10 ?

(1) Assume u is harmonic on open set O and z_n is a sequence in O such that $u(z_n) = 0$ and $\lim z_n \in O$. Prove or disprove that u is identically zero. What if O is a region?

(2) Assume u is harmonic on open set O and $u(z) = 0$ on a disc in O. Prove or disprove that u is identically zero. What if O is a region?

(3) Formulate and prove a Schwarz reflection principle for harmonic functions

cf. Theorem 5.6 on p.60 of Stein et al.

Hint: Verify the mean value property for your new function obtained by Schwarz reflection principle.

22.5.11 ?

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s < r$, there exists a constant $c > 0$ such that

$$||f||_{(\infty,s)} \leq c||f||_{(1,r)},$$

where $|f||_{(\infty,s)} = \sup_{z\in D_s(z_0)}|f(z)|$ and $||f||_{(1,r)} = \int_{D_r(z_0)} |f(z)|dxdy$.

Note: Exercise 3.8.20 on p.107 in Stein et al is a straightforward consequence of this stronger result using the integral form of the Cauchy-Schwarz inequality in real analysis.

22.5.12 ?

(1) Let f be analytic in $\Omega : 0 < |z-a| < r$ except at a sequence of poles $a_n \in \Omega$ with $\lim_{n\to\infty} a_n = a$. Show that for any $w \in \mathbb{C}$, there exists a sequence $z_n \in \Omega$ such that $\lim_{n\to\infty} f(z_n) = w$.

(2) Explain the similarity and difference between the above assertion and the Weierstrass-Casorati theorem.

22.5.13 ?

Compute the following integrals.

$i \int_0^\infty \frac{1}{(1+x^n)^2}\, dx,\ n \geq 1$ (ii) $\int_0^\infty \frac{\cos x}{(x^2+a^2)^2}\, dx,\ a \in \mathbb{R}$ (iii) $\int_0^\pi \frac{1}{a+\sin\theta}\, d\theta,\ a > 1$

$iv \int_0^{\frac{\pi}{2}} \frac{d\theta}{a+\sin^2\theta},\ a > 0.$ (v) $\int_{|z|=2} \frac{1}{(z^5-1)(z-3)}\, dz$ (v) $\int_{-\infty}^\infty \frac{\sin \pi a}{\cosh \pi x + \cos \pi a} e^{-ix\xi}\, dx,\ 0 < a < 1,$ $\xi \in \mathbb{R}$ (vi) $\int_{|z|=1} \cot^2 z\, dz.$

22.5.14 ?

Compute the following integrals.

$i \int_0^\infty \frac{\sin x}{x}\, dx$ (ii) $\int_0^\infty (\frac{\sin x}{x})^2\, dx$ (iii) $\int_0^\infty \frac{x^{a-1}}{(1+x)^2}\, dx,\ 0 < a < 2$

$i \int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx,\ a, b > 0$ (ii) $\int_0^\infty \frac{x^{a-1}}{1+x^n}\, dx,\ 0 < a < n$

$iii \int_0^\infty \frac{\log x}{1+x^n}\, dx,\ n \geq 2$ (iv) $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$ (v) $\int_0^\pi \log|1 - a\sin\theta| d\theta,\ a \in \mathbb{C}$

22.5.15 ?

Let $0 < r < 1$. Show that polynomials $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ have no zeros in $|z| < r$ for all sufficiently large n's.

22.5.16 ?

Let f be an analytic function on a region Ω. Show that f is a constant if there is a simple closed curve γ in Ω such that its image $f(\gamma)$ is contained in the real axis.

22.5.17 ?

(1) Show that $\dfrac{\pi^2}{\sin^2 \pi z}$ and $g(z) = \displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ have the same principal part at each integer point.

(2) Show that $h(z) = \dfrac{\pi^2}{\sin^2 \pi z} - g(z)$ is bounded on $\mathbb{C}$ and conclude that $\dfrac{\pi^2}{\sin^2 \pi z} = \displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$.

22.5.18 ?

Let $f(z)$ be an analytic function on $\mathbb{C}\backslash\{z_0\}$, where z_0 is a fixed point. Assume that $f(z)$ is bijective from $\mathbb{C}\backslash\{z_0\}$ onto its image, and that $f(z)$ is bounded outside $D_r(z_0)$, where r is some fixed positive number. Show that there exist $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$, $c \neq 0$ such that $f(z) = \dfrac{az+b}{cz+d}$.

22.5.19 ?

Assume $f(z)$ is analytic in $\mathbb{D} : |z| < 1$ and $f(0) = 0$ and is not a rotation (i.e. $f(z) \neq e^{i\theta} z$). Show that $\displaystyle\sum_{n=1}^{\infty} f^n(z)$ converges uniformly to an analytic function on compact subsets of $\mathbb{D}$, where $f^{n+1}(z) = f(f^n(z))$.

22.5.20 ?

Let f be a non-constant analytic function on $\mathbb{D}$ with $f(\mathbb{D}) \subseteq \mathbb{D}$. Use $\psi_a(f(z))$ (where $a = f(0)$, $\psi_a(z) = \dfrac{a-z}{1-\bar{a}z}$) to prove that $\dfrac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \dfrac{|f(0)| + |z|}{1 - |f(0)||z|}$.

22.5.21 ?

Find a conformal map

1. from $\{z : |z - 1/2| > 1/2, \mathrm{Re}(z) > 0\}$ to $\mathbb{H}$

2. from $\{z : |z - 1/2| > 1/2, |z| < 1\}$ to $\mathbb{D}$
3. from the intersection of the disk $|z + i| < \sqrt{2}$ with $\mathbb{H}$ to $\mathbb{D}$.
4. from $\mathbb{D}\backslash[a, 1)$ to $\mathbb{D}\backslash[0, 1)$ $(0 < a < 1)$.

$$Shortsolutionpossibleusing Blaschkefactor$$

5. from $\{z : |z| < 1, \mathrm{Re}(z) > 0\}\backslash(0, 1/2]$ to $\mathbb{H}$.

22.5.22 ?

Let C and C' be two circles and let $z_1 \in C$, $z_2 \notin C$, $z_1' \in C'$, $z_2' \notin C'$. Show that there is a unique fractional linear transformation f with $f(C) = C'$ and $f(z_1) = z_1'$, $f(z_2) = z_2'$.

22.5.23 ?

Assume $f_n \in H(\Omega)$ is a sequence of holomorphic functions on the region Ω that are uniformly bounded on compact subsets and $f \in H(\Omega)$ is such that the set $\{z \in \Omega : \lim_{n\to\infty} f_n(z) = f(z)\}$ has a limit point in Ω. Show that f_n converges to f uniformly on compact subsets of Ω.

22.5.24 ?

Let $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha}z}$ with $|\alpha| < 1$ and $\mathbb{D} = \{z : \ |z| < 1\}$. Prove that

- $\dfrac{1}{\pi}\iint_{\mathbb{D}} |\psi_\alpha'|^2 dxdy = 1.$
- $\dfrac{1}{\pi}\iint_{\mathbb{D}} |\psi_\alpha'| dxdy = \dfrac{1 - |\alpha|^2}{|\alpha|^2} \log \dfrac{1}{1 - |\alpha|^2}.$

22.5.25 ?

Prove that $f(z) = -\dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$ is a conformal map from half disc $\{z = x + iy : \ |z| < 1, \ y > 0\}$ to upper half plane $\mathbb{H} = \{z = x + iy : \ y > 0\}$.

22.5.26 ?

Let Ω be a simply connected open set and let γ be a simple closed contour in Ω and enclosing a bounded region U anticlockwise. Let $f:\ \Omega \to \mathbb{C}$ be a holomorphic function and $|f(z)| \leq M$ for all $z \in \gamma$. Prove that $|f(z)| \leq M$ for all $z \in U$.

22.5.27 ?

Compute the following integrals. (i) $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^n}\,dx$, $0 < a < n$ (ii) $\displaystyle\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx$

22.5.28 ?

Let $0 < r < 1$. Show that polynomials $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ have no zeros in $|z| < r$ for all sufficiently large n's.

22.5.29 ?

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s < r$, there exists a constant $c > 0$ such that

$$\|f\|_{(\infty,s)} \leq c\|f\|_{(1,r)},$$

where $\|f\|_{(\infty,s)} = \sup_{z\in D_s(z_0)}|f(z)|$ and $\|f\|_{(1,r)} = \displaystyle\int_{D_r(z_0)} |f(z)|dxdy$.

22.5.30 ?

Let $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha}z}$ with $|\alpha| < 1$ and $\mathbb{D} = \{z:\ |z| < 1\}$. Prove that

- $\displaystyle\frac{1}{\pi}\iint_{\mathbb{D}} |\psi'_\alpha|^2 dxdy = 1.$
- $\displaystyle\frac{1}{\pi}\iint_{\mathbb{D}} |\psi'_\alpha| dxdy = \frac{1-|\alpha|^2}{|\alpha|^2}\log\frac{1}{1-|\alpha|^2}.$

Prove that $f(z) = -\dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$ is a conformal map from half disc $\{z = x + iy:\ |z| < 1,\ y > 0\}$ to upper half plane $\mathbb{H} = \{z = x + iy:\ y > 0\}$.

22.5.31 ?

Let Ω be a simply connected open set and let γ be a simple closed contour in Ω and enclosing a bounded region U anticlockwise. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function and $|f(z)| \leq M$ for all $z \in \gamma$. Prove that $|f(z)| \leq M$ for all $z \in U$.

22.5.32 ?

Compute the following integrals. (i) $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^n}\,dx$, $0 < a < n$ (ii) $\displaystyle\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx$

22.5.33 ?

Let $0 < r < 1$. Show that polynomials $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ have no zeros in $|z| < r$ for all sufficiently large n's.

22.5.34 ?

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s < r$, there exists a constant $c > 0$ such that

$$\|f\|_{(\infty,s)} \leq c\|f\|_{(1,r)},$$

where $\|f\|_{(\infty,s)} = \sup_{z\in D_s(z_0)}|f(z)|$ and $\|f\|_{(1,r)} = \displaystyle\int_{D_r(z_0)} |f(z)|dxdy$.

22.6 Fall 2016

22.6.1 ?

Let $u(x, y)$ be harmonic and have continuous partial derivatives of order three in an open disc of radius $R > 0$.

(a) Let two points $(a, b), (x, y)$ in this disk be given. Show that the following integral is independent of the path in this disk joining these points:

$$v(x, y) = \int_{a,b}^{x,y} \left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right).$$

(b) `{=tex}` `\hfill`

(i) Prove that $u(x,y) + iv(x,y)$ is an analytic function in this disc.

(ii) Prove that $v(x,y)$ is harmonic in this disc.

22.6.2 ?

(a) $f(z) = u(x,y) + iv(x,y)$ be analytic in a domain $D \subset \mathbb{C}$. Let $z_0 = (x_0, y_0)$ be a point in D which is in the intersection of the curves $u(x,y) = c_1$ and $v(x,y) = c_2$, where c_1 and c_2 are constants. Suppose that $f'(z_0) \neq 0$. Prove that the lines tangent to these curves at z_0 are perpendicular.

(b) Let $f(z) = z^2$ be defined in $\mathbb{C}$.

(c) Describe the level curves of $\mathrm{Re}(f)$ and of $\mathrm{Im}(f)$.

(ii) What are the angles of intersections between the level curves $\mathrm{Re}(f) = 0$ and $\mathrm{Im}(f)$? Is your answer in agreement with part a) of this question?

22.6.3 ?

(a) $f : D \to \mathbb{C}$ be a continuous function, where $D \subset \mathbb{C}$ is a domain.Let $\alpha : [a,b] \to D$ be a smooth curve. Give a precise definition of the *complex line integral*

$$\int_\alpha f.$$

(b) Assume that there exists a constant M such that $|f(\tau)| \leq M$ for all $\tau \in \mathrm{Image}(\alpha)$. Prove that

$$\left|\int_\alpha f\right| \leq M \times \mathrm{length}(\alpha).$$

(c) Let C_R be the circle $|z| = R$, described in the counterclockwise direction, where $R > 1$. Provide an upper bound for $\left|\int_{C_R} \frac{\log(z)}{z^2}\right|$, which depends only on R and other constants.

22.6.4 ?

(a) Let Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Assume the existence of a non-negative integer m, and of positive constants L and R, such that for all z with $|z| > R$ the inequality

$$|f(z)| \leq L|z|^m$$

holds. Prove that f is a polynomial of degree $\leq m$.

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose that there exists a real number M such that for all $z \in \mathbb{C}$

$$\mathrm{Re}(f) \leq M.$$

Prove that f must be a constant.

22.6.5 ?

Prove that all the roots of the complex polynomial

$$z^7 - 5z^3 + 12 = 0$$

lie between the circles $|z| = 1$ and $|z| = 2$.

22.6.6 ?

(a) Let F be an analytic function inside and on a simple closed curve C, except for a pole of order $m \geq 1$ at $z = a$ inside C. Prove that

$$\frac{1}{2\pi i} \oint_C F(\tau) d\tau = \lim_{\tau \to a} \frac{d^{m-1}}{d\tau^{m-1}} ((\tau - a)^m F(\tau))).$$

(b) Evaluate

$$\oint_C \frac{e^\tau}{(\tau^2 + \pi^2)^2} d\tau$$

where C is the circle $|z| = 4$.

22.6.7 ?

Find the conformal map that takes the upper half-plane comformally onto the half-strip $\{w = x + iy : \ -\pi/2 < x < \pi/2 \ y > 0\}$.

22.6.8 ?

Compute the integral $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{\cosh \pi x} dx$ where $\cosh z = \dfrac{e^z + e^{-z}}{2}$.