Title

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Contents

0.1 Ring Theory

Basic Structure

- Show that if an ideal $I \trianglelefteq R$ contains a unit then $I = R$.
- Show that $R^\times$ need not be closed under addition.

Ideals

Problem.
Every $a \in R$ for a finite ring is either a unit or a zero divisor.

Solution:
- Let $a \in R$ and define $\varphi(x) = ax$.
- If φ is injective, then it is surjective, so $1 = ax$ for some $x \implies x^{-1} = a$.
- Otherwise, $ax_1 = ax_2$ with $x_1 \neq x_2 \implies a(x_1 - x_2) = 0$ and $x_1 - x_2 \neq 0$
- So a is a zero divisor.

Problem.
Maximal $\implies$ prime, but generally not the converse.

Solution:
- Suppose $\mathfrak{m}$ is maximal, $ab \in \mathfrak{m}$, and $b \notin \mathfrak{m}$.
- Then there is a containment of ideals $\mathfrak{m} \subsetneq \mathfrak{m} + (b) \implies \mathfrak{m} + (b) = R$.
- So

$$1 = m + rb \implies a = am + r(ab),$$

  but $am \in \mathfrak{m}$ and $ab \in \mathfrak{m} \implies a \in \mathfrak{m}$.

Counterexample: $(0) \in \mathbb{Z}$ is prime since $\mathbb{Z}$ is a domain, but not maximal since it is properly contained in any other ideal.

- Show that every proper ideal is contained in a maximal ideal
- Show that if $x \in R$ a PID, then x is irreducible $\iff \langle x \rangle \trianglelefteq R$ is maximal.

- Show that intersections, products, and sums of ideals are ideals.
- Show that the union of two ideals need not be an ideal.
- Show that every ring has a proper maximal ideal.
- Show that $I \trianglelefteq R$ is maximal iff R/I is a field.
- Show that $I \trianglelefteq R$ is prime iff R/I is an integral domain.
- Show that $\displaystyle\bigcup_{\mathfrak{m} \in \operatorname{maxSpec}(R)} = R \setminus R^\times$.
- Show that $\operatorname{maxSpec}(R) \subsetneq \operatorname{Spec}(R)$ but the containment is strict.
- $\star$ Show that if x is not a unit, then x is contained in some maximal ideal.
- Show that every prime ideal is radical.
- Show that the nilradical is given by $\mathfrak{N}(R) = \operatorname{rad}(0)$.
- Show that $\operatorname{rad}(IJ) = \operatorname{rad}(I) \bigcap \operatorname{rad}(J)$
- Show that if $\operatorname{Spec}(R) \subseteq \operatorname{maxSpec}(R)$ then R is a UFD.
- Show that if R is Noetherian then every ideal is finitely generated.

Characterizing Certain Ideals

- Show that the nilradical of a ring is the intersection of all prime ideals $I \trianglelefteq R$.
- Show that for an ideal $I \trianglelefteq R$, its radical is the intersection of all prime ideals containing I.
- Show that $\operatorname{rad}(I)$ is the intersection of all prime ideals containing I.

Problem.
The nilradical is contained in the Jacobson radical, i.e.

$$\mathfrak{N}(R) \subseteq \mathfrak{J}(R).$$

Solution: Maximal $\implies$ prime, and so if x is in every prime ideal, it is necessarily in every maximal ideal as well.

Problem.
$R/\mathfrak{N}(R)$ has no nonzero nilpotent elements.

Solution:

$$\begin{aligned} a + \mathfrak{N}(R) \text{ nilpotent} &\implies (a + \mathfrak{N}(R))^n := a^n + \mathfrak{N}(R) = \mathfrak{N}(R) \\ &\implies a^n \in \mathfrak{N}(R) \\ &\implies \exists \ell \text{ such that } (a^n)^\ell = 0 \\ &\implies a \in \mathfrak{N}(R). \end{aligned}$$

Problem.
The nilradical is the intersection of all prime ideals, i.e.

$$\mathfrak{N}(R) = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(R)} \mathfrak{p}$$

Solution:

- $\mathfrak{N} \subseteq \bigcap \mathfrak{p}$:
- $x \in \mathfrak{N} \implies x^n = 0 \in \mathfrak{p} \implies x \in \mathfrak{p}$ or $x^{n-1} \in \mathfrak{p}$.
- $\mathfrak{N}^c \subseteq \bigcup \mathfrak{p}^c$:
- Define $S = \left\{ I \trianglelefteq R \;\middle|\; a^n \notin I \text{ for any } n \right\}$.
- Then apply Zorn's lemma to get a maximal ideal $\mathfrak{m}$, and maximal $\implies$ prime.

Misc

- Show that localizing a ring at a prime ideal produces a local ring.
- Show that R is a local ring iff for every $x \in R$, either x or $1 - x$ is a unit.
- Show that if R is a local ring then $R \setminus R^\times$ is a proper ideal that is contained in $\mathfrak{J}(R)$.
- Show that if $R \neq 0$ is a ring in which every non-unit is nilpotent then R is local.
- Show that every prime ideal is primary.
- Show that every prime ideal is irreducible.
- Show that