Title

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Contents

1 Basics

Notation:

- $\|f\|_\infty \coloneqq \sup_{x\in \mathrm{dom}(f)} |f(x)|$
- $\|f\|_{L^\infty} \coloneqq \inf \left\{ M \geq 0 \;\middle|\; |f(x)| \leq M \text{for a.e. } x \right\}$.

1.1 Useful Techniques

- General advice: try swapping the orders of limits, sums, integrals, etc.
- Limits:
  - Take the $\limsup$ or $\liminf$, which always exist, and aim for an inequality like
  $$c \leq \liminf a_n \leq \limsup a_n \leq c.$$
  - $\lim f_n = \limsup f_n = \liminf f_n$ iff the limit exists, so to show some g is a limit, show
  $$\limsup f_n \leq g \leq \liminf f_n \qquad (\implies g = \lim f).$$
  - A limit does *not* exist if $\liminf a_n > \limsup a_n$.
- Sequences and Series
  - If f_n has a global maximum (computed using f_n' and the first derivative test) $M_n \to 0$, then $f_n \to 0$ uniformly.
  - For a fixed x, if $f = \sum f_n$ converges *uniformly* on some $B_r(x)$ and each f_n is continuous at x, then f is also continuous at x .
- Equalities
  - Split into upper and lower bounds:
  $$a = b \iff a \leq b \text{ and } a \geq b.$$
  - Use an epsilon of room:
  $$a < b + \varepsilon \,\forall \varepsilon \implies a \leq b.$$
  - Showing something is zero:
  $$|a| \leq \varepsilon \,\forall \varepsilon \implies a = 0.$$
- Simplifications:
  - To show something for a measurable set, show it for bounded/compact/elementary sets/

  - To show something for a function, show it for continuous, bounded, compactly supported, simple, indicator functions, L^1, etc
  - Replace a continuous sequence ($\varepsilon \to 0$) with an arbitrary countable sequence ($x_n \to 0$)
  - Intersect with a ball $B_r(\mathbf{0}) \subset \mathbb{R}^n$.

- Integrals

  - Break up $\mathbb{R}^n = \{|x| \leq 1\} \coprod \{|x| > 1\}$.
  - Break up into $\{f > g\} \coprod \{f = g\} \coprod \{f < g\}$.
  - Tail estimates!

- Continuity / differentiability: show it holds on $[-M, M]$ for all M to get it to hold on $\mathbb{R}$.

1.2 Definitions

Definition 1.2.1 (Uniform Continuity)
f is uniformly continuous iff

$$\begin{aligned} &\forall \varepsilon \quad \exists \delta(\varepsilon) \Bigm| \quad \forall x, y, \quad |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon \\ \iff &\forall \varepsilon \quad \exists \delta(\varepsilon) \Bigm| \quad \forall x, y, \quad |y| < \delta \implies |f(x-y) - f(y)| < \varepsilon. \end{aligned}$$

Definition 1.2.2 (Nowhere Dense Sets)
A set S is **nowhere dense** iff the closure of S has empty interior iff every interval contains a subinterval that does not intersect S.

Proposition 1.2.1 ***(Meager Sets).***
A set is **meager** if it is a *countable* union of nowhere dense sets.

Definition 1.2.3 (F_σ and G_δ Sets)
An F_σ set is a union of closed sets, and a G_δ set is an intersection of opens. [a]

[a] Mnemonic: "F" stands for *ferme*, which is "closed" in French, and σ corresponds to a "sum", i.e. a union.

Theorem 1.2.1 ***(Heine-Cantor).***
Every continuous function on a compact space is uniformly continuous.

Definition 1.2.4 (Limsup/Liminf)

$$\limsup_n a_n = \lim_{n\to\infty} \sup_{j\geq n} a_j = \inf_{n\geq 0} \sup_{j\geq n} a_j$$
$$\liminf_n a_n = \lim_{n\to\infty} \inf_{j\geq n} a_j = \sup_{n\geq 0} \inf_{j\geq n} a_j.$$

Definition 1.2.5 (Topological Notions)
Let X be a metric space and A a subset. Let A' denote the limit points of A, and $\overline{A} \coloneqq A \cup A'$ to be its closure.

- A **neighborhood** of p is an open set U_p containing p.
- An ε-**neighborhood** of p is an open ball $B_r(p) \coloneqq \left\{ q \,\middle|\, d(p,q) < r \right\}$ for some $r > 0$.
- A point $p \in X$ is an **accumulation point** of A iff every neighborhood U_p of p contains a point $q \in Q$
- A point $p \in X$ is a **limit point** of A iff every *punctured* neighborhood $U_p \setminus \{p\}$ contains a point $q \in A$.
- If $p \in A$ and p is not a limit point of A, then p is an **isolated point** of A.
- A is **closed** iff $A' \subset A$, so A contains all of its limit points.
- A point $p \in A$ is **interior** iff there is a neighborhood $U_p \subset A$ that is strictly contained in A.
- A is **open** iff every point of A is interior.
- A is **perfect** iff A is closed and $A \subset A'$, so every point of A is a limit point of A.
- A is **bounded** iff there is a real number M and a point $q \in X$ such that $d(p,q) < M$ for all $p \in A$.
- A is **dense** in X iff every point $x \in X$ is either a point of A, so $x \in A$, or a limit point of A, so $x \in A'$. I.e., $X \subset A \cup A'$.
  - Alternatively, $\overline{A} = X$, so the closure of A is X.

1.3 Theorems

1.3.1 Topology / Sets

Lemma 1.1 ***(?).***
Metric spaces are compact iff they are sequentially compact, (i.e. every sequence has a convergent subsequence).

Proposition 1.3.1 ***(?).***
The unit ball in $C([0,1])$ with the sup norm is not compact.

Proof (?).
Take $f_k(x) = x^n$, which converges to $\chi(x=1)$. The limit is not continuous, so no subsequence can converge. ■

Proposition 1.3.2 ***(?).***
A *finite* union of nowhere dense is again nowhere dense.

Proposition 1.3.3 ***(Convergent Sums Have Small Tails).***

$$\sum a_n < \infty \implies a_n \to 0 \quad \text{and} \quad \sum_{k=N}^{\infty} a_n \overset{N\to\infty}{\to} 0$$

Theorem 1.3.1 ***(Heine-Borel).***
$X \subseteq \mathbb{R}^n$ is compact $\iff$ X is closed and bounded.

Proposition 1.3.4 ***(Geometric Series).***

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \iff |x| < 1.$$

Corollary 1.3.1 ***(?).***

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = 1.$$

Lemma 1.2 *(?)*.
The Cantor set is closed with empty interior.

Proof (?).
Its complement is a union of open intervals, and can't contain an interval since intervals have positive measure and $m(C_n)$ tends to zero. ∎

Corollary 1.3.2 *(?)*.
The Cantor set is nowhere dense.

Lemma 1.3 *(?)*.
Singleton sets in $\mathbb{R}$ are closed, and thus $\mathbb{Q}$ is an F_σ set.

Theorem 1.3.2 *(Baire)*.
$\mathbb{R}$ is a **Baire space** (countable intersections of open, dense sets are still dense). Thus $\mathbb{R}$ can not be written as a countable union of nowhere dense sets.

Lemma 1.4 *(?)*.
Any nonempty set which is bounded from above (resp. below) has a well-defined supremum (resp. infimum).

1.3.2 Functions

Proposition 1.3.5 *(Existence of Smooth Compactly Supported Functions)*.
There exist smooth compactly supported functions, e.g. take

$$f(x) = e^{-\frac{1}{x^2}} \chi_{(0,\infty)}(x).$$

Lemma 1.5 *(?)*.
There is a function discontinuous precisely on $\mathbb{Q}$.

Proof (?).
$f(x) = \dfrac{1}{n}$ if $x = r_n \in \mathbb{Q}$ is an enumeration of the rationals, and zero otherwise. The limit at every point is 0. ∎

Proposition 1.3.6 ***(?).***
There *do not* exist functions that are discontinuous precisely on $\mathbb{R} \setminus \mathbb{Q}$.

Proof (?).
D_f is always an F_σ set, which follows by considering the oscillation ω_f. $\omega_f(x) = 0 \iff f$ is continuous at x, and $D_f = \cup_n A_{\frac{1}{n}}$ where $A_\varepsilon = \{\omega_f \geq \varepsilon\}$ is closed. ■

Proposition 1.3.7 ***(?).***
A function $f : (a, b) \to \mathbb{R}$ is Lipschitz $\iff$ f is differentiable and f' is bounded. In this case, $|f'(x)| \leq C$, the Lipschitz constant.

1.4 Uniform Convergence

Definition 1.4.1 (Uniform Convergence)

$$(\forall \varepsilon > 0)\, (\exists n_0 = n_0(\varepsilon))\, (\forall x \in S)\, (\forall n > n_0)\, (|f_n(x) - f(x)| < \varepsilon)\,.$$

Negated:[a]

$$(\exists \varepsilon > 0)\, (\forall n_0 = n_0(\varepsilon))\, (\exists x = x(n_0) \in S)\, (\exists n > n_0)\, (|f_n(x) - f(x)| \geq \varepsilon)\,.$$

[a] Slogan: to negate, find a bad x depending on n_0 that are larger than some ε.

Compare this to the definition of pointwise convergence:

Definition 1.4.2 (Pointwise Convergence)

$$(\forall \varepsilon > 0)(\forall x \in S)\, (\exists n_0 = n_0(x, \varepsilon))\, (\forall n > n_0)\, (|f_n(x) - f(x)| < \varepsilon)\,.$$

Proposition 1.4.1 ***(Testing Uniform Convergence: The Sup Norm).***
$f_n \to f$ uniformly iff there exists an M_n such that $\|f_n - f\|_\infty \leq M_n \to 0$.

Negating: find an x which depends on n for which $\|f_n\|_\infty > \varepsilon$ (negating small tails) or $\|f_n - f_m\| > \varepsilon$ (negating the Cauchy criterion).

Proposition 1.4.2 *(?).*
The space $X = C([0,1])$, continuous functions $f : [0,1] \to \mathbb{R}$, equipped with the norm

$$\|f\|_\infty \coloneqq \sup_{x\in[0,1]} |f(x)|$$

is a **complete** metric space.

Proof .

1. Let $\{f_k\}$ be Cauchy in X.
2. Define a candidate limit using pointwise convergence:

   Fix an x; since

   $$|f_k(x) - f_j(x)| \leq \|f_k - f_k\| \to 0$$

   the sequence $\{f_k(x)\}$ is Cauchy in $\mathbb{R}$. So define $f(x) \coloneqq \lim_k f_k(x)$.
3. Show that $\|f_k - f\| \to 0$:

   $$|f_k(x) - f_j(x)| < \varepsilon \ \forall x \implies \lim_j |f_k(x) - f_j(x)| < \varepsilon \ \forall x$$

   Alternatively, $\|f_k - f\| \leq \|f_k - f_N\| + \|f_N - f_j\|$, where N, j can be chosen large enough to bound each term by $\varepsilon/2$.
4. Show that $f \in X$:

   The uniform limit of continuous functions is continuous.

■

Note: in other cases, you may need to show the limit is bounded, or has bounded derivative, or whatever other conditions define X.

Theorem 1.4.1 *(Uniform Limit Theorem).*
If $f_n \to f$ pointwise and uniformly with each f_n continuous, then f is continuous. [a]

[a] Slogan: a uniform limit of continuous functions is continuous.

Proof .

- Follows from an $\varepsilon/3$ argument:

  $$|F(x) - F(y| \leq |F(x) - F_N(x)| + |F_N(x) - F_N(y)| + |F_N(y) - F(y)| \leq \varepsilon \to 0.$$

  – The first and last $\varepsilon/3$ come from uniform convergence of $F_N \to F$.
  – The middle $\varepsilon/3$ comes from continuity of each F_N.
- So just need to choose N large enough and δ small enough to make all 3 ε bounds hold.

■

Proposition 1.4.3 *(Uniform Limits Commute with Integrals).*
If $f_n \to f$ uniformly, then $\int f_n = \int f$.

1.4.1 Series

Proposition 1.4.4 *(p-tests).*
Let n be a fixed dimension and set $B = \left\{ x \in \mathbb{R}^n \;\middle|\; \|x\| \leq 1 \right\}$.

$$\begin{aligned}
\sum \frac{1}{n^p} < \infty &\iff p > 1 \\
\int_\varepsilon^\infty \frac{1}{x^p} < \infty &\iff p > 1 \\
\int_0^1 \frac{1}{x^p} < \infty &\iff p < 1 \\
\int_B \frac{1}{|x|^p} < \infty &\iff p < n \\
\int_{B^c} \frac{1}{|x|^p} < \infty &\iff p > n
\end{aligned}$$

.

Proposition 1.4.5 *(Comparison Test).*
If $0 \leq a_n \leq b_n$, then

- $\sum b_n < \infty \implies \sum a_n < \infty$, and
- $\sum a_n = \infty \implies \sum b_n = \infty$.

Proposition 1.4.6 *(Small Tails for Series of Functions).*
If $\sum f_n$ converges then $f_n \to 0$ uniformly.

Corollary 1.4.1 *(Term by Term Continuity Theorem).*
If f_n are continuous and $\sum f_n \to f$ converges uniformly, then f is continuous.

Proposition 1.4.7 *(Weak M-Test).*
If $f_n(x) \leq M_n$ **for a fixed** x where $\sum M_n < \infty$, then the series $f(x) = \sum f_n(x)$ converges.[a]

[a] Note that this is only pointwise convergence of f, whereas the full M-test gives uniform convergence.

Proposition 1.4.8 ***(The Weierstrass M-Test).***

If $\sup_{x\in A}|f_n(x)| \leq M_n$ for each n where $\sum M_n < \infty$, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly and absolutely on A. [a] Conversely, if $\sum f_n$ converges uniformly on A then $\sup_{x\in A}|f_n(x)| \to 0$.

[a] It suffices to show $|f_n(x)| \leq M_n$ for some M_n not depending on x.

Proposition 1.4.9 ***(Cauchy criterion for sums).***
f_n are uniformly Cauchy (so $\|f_n - f_m\|_\infty < \varepsilon$) iff f_n is uniformly convergent.

Derivatives

Theorem 1.4.2 ***(Term by Term Differentiability Theorem).***
If f_n are differentiable, $\sum f_n' \to g$ uniformly, and there exists one point[a] x_0 such that $\sum f_n(x)$ converges, then there exist an f such that $\sum f_n \to f$ uniformly and $f' = g$.[b]

[a] So this implicitly holds if f is the pointwise limit of f_n.
[b] See Abbott theorem 6.4.3, pp 168.

1.5 Commuting Limiting Operations

Proposition 1.5.1 ***(Limits of bounded functions need not be bounded).***

$$\lim_{n\to\infty}\sup_{x\in X}|f_n(x)| \neq \sup_{x\in X}\left|\lim_{n\to\infty} f_n(x)\right|.$$

Proposition 1.5.2 ***(Limits of continuous functions need not be continuous).***

$$\lim_{k\to\infty}\lim_{n\to\infty} f_n(x_k) \neq \lim_{n\to\infty}\lim_{k\to\infty} f_n(x_k).$$

Proposition 1.5.3 ***(Limits of differentiable functions need not be differentiable).***

$$\lim_{n\to\infty}\frac{\partial}{\partial x} f_n \neq \frac{\partial}{\partial n}\left(\lim_{n\to\infty} f_n\right).$$

Proposition 1.5.4 *(?).*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx \neq \int_a^b \lim_{n\to\infty} (f_n(x))\,dx.$$

1.6 Slightly Advanced Stuff

Theorem 1.6.1 ***(Weierstrass Approximation).***
If $[a,b] \subset \mathbb{R}$ is a closed interval and f is continuous, then for every $\varepsilon > 0$ there exists a polynomial p_ε such that $\|f - p_\varepsilon\|_{L^\infty([a,b])} \overset{\varepsilon\to 0}{\to} 0$.
Equivalently, polynomials are dense in the Banach space $C([0,1], \|\cdot\|_\infty)$.

Theorem 1.6.2 ***(Egorov).***
Let $E \subseteq \mathbb{R}^n$ be measurable with $m(E) > 0$ and $\{f_k : E \to \mathbb{R}\}$ be measurable functions such that

$$f(x) := \lim_{k\to\infty} f_k(x) < \infty$$

exists almost everywhere.
Then $f_k \to f$ *almost uniformly*, i.e.

$$\forall \varepsilon > 0,\ \exists F \subseteq E \text{ closed such that } m(E \setminus F) < \varepsilon \ \text{ and } \ f_k \to f \text{ uniformly on } F.$$

1.7 Examples

Example 1.7.1(?): A series of continuous functions that does *not* converge uniformly but is still continuous:

$$g(x) := \sum \frac{1}{1+n^2 x}.$$

Take $x = 1/n^2$.