

Topology Qualifying Exam Notes

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Tuesday 9th June, 2020

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1 Definitions

- Topology: Closed under arbitrary unions and finite intersections.
- Basis: A subset $\{B_i\}$ is a basis iff
 - $x \in X \implies x \in B_i$ for some i .
 - $x \in B_i \cap B_j \implies x \in B_k \subset B_i \cap B_j$.
 - Topology generated by this basis: $x \in N_x \implies x \in B_i \subset N_x$ for some i .
- Dense: A subset $Q \subset X$ is dense iff $y \in N_y \subset X \implies N_y \cap Q \neq \emptyset$ iff $\bar{Q} = X$.
- Neighborhood: A neighborhood of a point x is any open set containing x .
- Hausdorff
- Second Countable: admits a countable basis.
- Closed (several characterizations)
- Closure in a subspace: $Y \subset X \implies \text{cl}_Y(A) := \text{cl}_X(A) \cap Y$.
- Bounded
- Compact: A topological space (X, τ) is **compact** if every open cover has a *finite* subcover.
 That is, if $\{U_j \mid j \in J\} \subset \tau$ is a collection of open sets such that $X \subseteq \bigcup_{j \in J} U_j$, then there exists a *finite* subset $J' \subset J$ such that $X \subseteq \bigcup_{j \in J'} U_j$.
- Locally compact For every $x \in X$, there exists a $K_x \ni x$ such that K_x is compact.
- Connected: There does not exist a disconnecting set $X = A \coprod B$ such that $\emptyset \neq A, B \subsetneq X$, i.e. X is the union of two proper disjoint nonempty sets.
 Equivalently, X contains no proper nonempty clopen sets.
 - Additional condition for a subspace $Y \subset X$: $\text{cl}_Y(A) \cap V = A \cap \text{cl}_Y(B) = \emptyset$.
- Locally connected: A space is locally connected at a point x iff $\forall N_x \ni x$, there exists a $U \subset N_x$ containing x that is connected.
- Retract: A subspace $A \subset X$ is a *retract* of X iff there exists a continuous map $f : X \longrightarrow A$ such that $f|_A = \text{id}_A$. Equivalently it is a *left* inverse to the inclusion.

- Uniform Continuity: For $f : (X, d_X) \rightarrow (Y, d_Y)$ metric spaces,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

- Lebesgue number: For (X, d) a compact metric space and $\{U_\alpha\} \Rightarrow X$, there exist $\delta_L > 0$ such that

$$A \subset X, \text{diam}(A) < \delta_L \implies A \subseteq U_\alpha \text{ for some } \alpha.$$

- Paracompact
- Components: Set $x \sim y$ iff there exists a connected set $U \ni x, y$ and take equivalence classes.
- Path Components: Set $x \sim y$ iff there exists a path-connected set $U \ni x, y$ and take equivalence classes.
- Separable: Contains a countable dense subset.
- Limit Point: For $A \subset X$, x is a limit point of A if every punctured neighborhood P_x of x satisfies $P_x \cap A \neq \emptyset$, i.e. every neighborhood of x intersects A in some point other than x itself.

Equivalently, x is a limit point of A iff $x \in \text{cl}_X(A \setminus \{x\})$.

1.1 Algebraic

1.1.1 Homotopy

Todo: Merge the two van Kampen theorems.

Theorem 1.1 (Van Kampen).

The pushout is the northwest colimit of the following diagram

$$\begin{array}{ccc} A \amalg_Z B & \longleftarrow & A \\ \uparrow & & \uparrow \iota_A \\ B & \xleftarrow{\iota_B} & Z \end{array}$$

For groups, the pushout is given by the amalgamated free product: if $A = \langle G_A \mid R_A \rangle$, $B = \langle G_B \mid R_B \rangle$, then

$$A *_Z B = \langle G_A, G_B \mid R_A, R_B, T \rangle$$

where T is a set of relations given by

$$T = \{ \iota_A(z) \iota_B(z)^{-1} \mid z \in Z \}.$$

Suppose $X = U_1 \cup U_2$ such that $U_1 \cap U_2 \neq \emptyset$ is **path connected** (necessary condition). Then taking $x_0 \in U := U_1 \cap U_2$ yields a pushout of fundamental groups

$$\pi_1(X; x_0) = \pi_1(U_1; x_0) *_{\pi_1(U; x_0)} \pi_1(U_2; x_0).$$

Theorem 1.2 (Van Kampen).

If $X = U \bigcup V$ where $U, V, U \cap V$ are all path-connected then

$$\pi_1(X) = \pi_1 U *_{\pi_1(U \cap V)} \pi_1 V,$$

where the amalgamated product can be computed as follows: If we have presentations

$$\begin{aligned}\pi_1(U, w) &= \langle u_1, \dots, u_k \mid \alpha_1, \dots, \alpha_l \rangle \\ \pi_1(V, w) &= \langle v_1, \dots, v_m \mid \beta_1, \dots, \beta_n \rangle \\ \pi_1(U \cap V, w) &= \langle w_1, \dots, w_p \mid \gamma_1, \dots, \gamma_q \rangle\end{aligned}$$

then

$$\begin{aligned}\pi_1(X, w) &= \langle u_1, \dots, u_k, v_1, \dots, v_m \rangle \\ &\quad \text{mod } \langle \alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_n, I(w_1)J(w_1)^{-1}, \dots, I(w_p)J(w_p)^{-1} \rangle \\ &= \frac{\pi_1(U) * \pi_1(V)}{\langle \{I(w_i)J(w_i)^{-1} \mid 1 \leq i \leq p\} \rangle}\end{aligned}$$

where

$$\begin{aligned}I &: \pi_1(U \cap V, w) \rightarrow \pi_1(U, w) \\ J &: \pi_1(U \cap V, w) \rightarrow \pi_1(V, w).\end{aligned}$$

Theorem 1.3 (Seifert-van Kampen Theorem).

Suppose $X = U_1 \bigcup U_2$ where $U := U_1 \cap U_2 \neq \emptyset$ is path-connected, and let $\{\text{pt}\} \in U$. Then the maps $i_1 : U_1 \rightarrow X$ and $i_2 : U_2 \rightarrow X$ induce the following group homomorphisms:

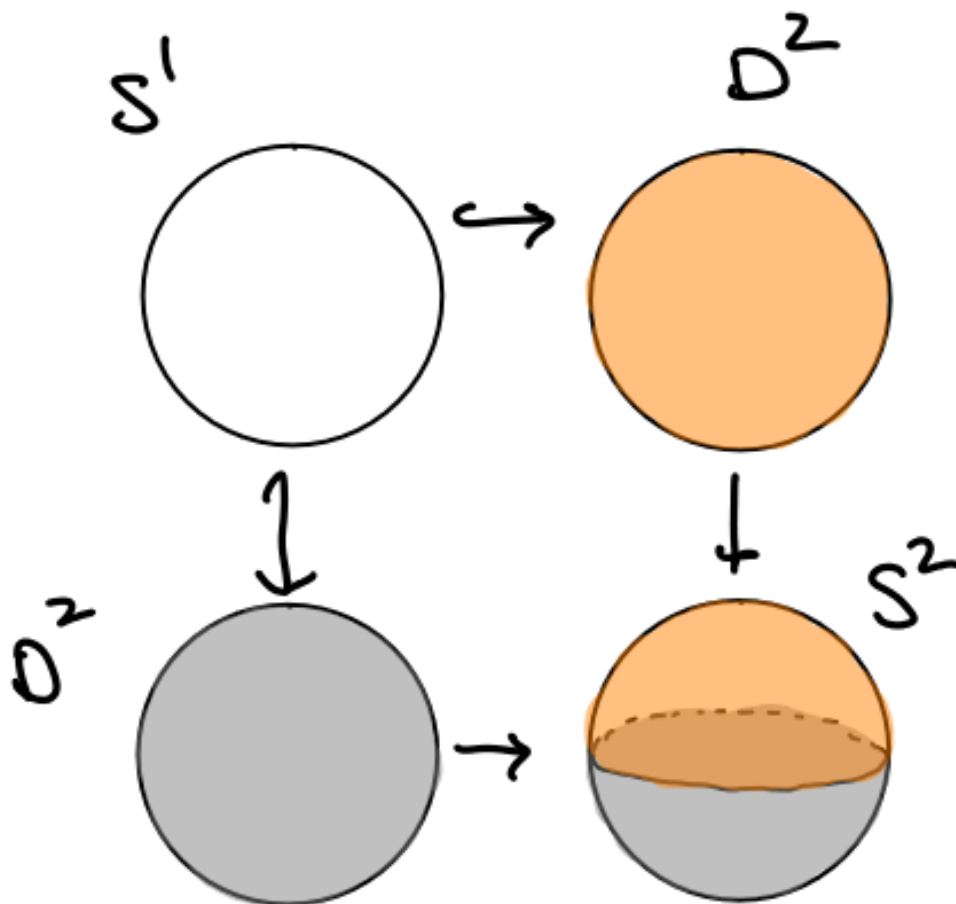
$$\begin{aligned}i_1^* &: \pi_1(U_1, \{\text{pt}\}) \rightarrow \pi_1(X, \{\text{pt}\}) \\ i_2^* &: \pi_1(U_2, \{\text{pt}\}) \rightarrow \pi_1(X, \{\text{pt}\})\end{aligned}$$

and letting $P = \pi_1(U, \{\text{pt}\})$, there is a natural isomorphism

$$\pi_1(X, \{\text{pt}\}) \cong \pi_1(U_1, \{\text{pt}\}) *_P \pi_1(U_2, \{\text{pt}\})$$

where $*_P$ is the amalgamated free product over P .

(Todo: formulate in terms of pushouts)



Examples

Example 1.1.

$A = \mathbb{Z}/4\mathbb{Z} = \langle x \mid x^4 \rangle$, $B = \mathbb{Z}/6\mathbb{Z} = \langle y \mid y^6 \rangle$, $Z = \mathbb{Z}/2\mathbb{Z} = \langle z \mid z^2 \rangle$. Then we can identify Z as a subgroup of A, B using $\iota_A(z) = x^2$ and $\iota_B(z) = y^3$. So

$$A *_Z B = \langle x, y \mid x^4, y^6, x^2 y^{-3} \rangle$$

.

- Computing $\pi_1(S^1 \vee S^1)$
- Computing $\pi_1(S^1 \times S^1)$
- Counterexample when $U \cap V$ isn't path-connected: S^1 with U, V neighborhoods of the poles.

1.1.2 Homology

Useful fact: since \mathbb{Z} is free, any exact sequence of the form $0 \longrightarrow \mathbb{Z}^n \longrightarrow A \longrightarrow \mathbb{Z}^m \longrightarrow 0$ splits and $A \cong \mathbb{Z}^n \times \mathbb{Z}^m$.

Useful fact: $\tilde{H}_*(A \vee B) \cong H_*(A) \times H_*(B)$.

Theorem 1.4 (Mayer Vietoris).

Let $X = A^\circ \cup B^\circ$; then there is a SES of chain complexes

$$0 \longrightarrow C_n(A \cap B) \xrightarrow{x \mapsto (x, -x)} C_n(A) \oplus C_n(B) \xrightarrow{(x, y) \mapsto x + y} C_n(A + B) \longrightarrow 0$$

where $C_n(A + B)$ denotes the chains that are sums of chains in A and chains in B . This yields a LES in homology:

$$\cdots \longrightarrow H_n(A \cap B) \xrightarrow{x \mapsto (x, -x)} H_n(A) \oplus H_n(B) \xrightarrow{(x, y) \mapsto x + y} H_n(X) \longrightarrow \cdots$$

2 Theorems

2.1 Point-Set

Theorem 2.1.

$U \subset X$ a Hausdorff spaces is closed \iff it is compact.

Theorem 2.2 (Cantor's Intersection Theorem).

A bounded collection of nested closed sets $C_1 \supset C_2 \supset \cdots$ in a metric space X is nonempty $\iff X$ is complete.

- Tube lemma
- Properties pushed forward through continuous maps:
 - Compactness?
 - Connectedness (when surjective)
 - Separability
 - Density **only when** f is surjective
 - **Not** openness
 - **Not** closedness
- A retract of a Hausdorff/connected/compact space is closed/connected/compact respectively.

Proposition 2.3.

A continuous function on a compact set is uniformly continuous.

Proof.

Take $\{B_{\frac{\varepsilon}{2}}(y) \mid y \in Y\} \rightrightarrows Y$, pull back to an open cover of X , has Lebesgue number $\delta_L > 0$, then $x' \in B_{\delta_L}(x) \implies f(x), f(x') \in B_{\frac{\varepsilon}{2}}(y)$ for some y . ■

Corollary 2.4.

Lipschitz continuity implies uniform continuity (take $\delta = \varepsilon/C$)

Counterexample to converse: $f(x) = \sqrt{x}$ on $[0, 1]$ has unbounded derivative.

Theorem 2.5 (Extreme Value Theorem).

For $f : X \rightarrow Y$ continuous with X compact and Y ordered in the order topology, there exist points $c, d \in X$ such that $f(x) \in [f(c), f(d)]$ for every x .

Theorem 2.6.

Points are closed in T_1 spaces.

Theorem 2.7.

A metric space X is sequentially compact iff it is complete and totally bounded.

Theorem 2.8.

A metric space is totally bounded iff every sequence has a Cauchy subsequence.

Theorem 2.9.

A metric space is compact iff it is complete and totally bounded.

Theorem 2.10 (Baire).

If X is a complete metric space, then the intersection of countably many dense open sets is dense in X .

Theorem 2.11.

A continuous bijective open map is a homeomorphism.

Theorem 2.12.

A closed subset A of a compact set B is compact.

Proof .

- Let $\{A_i\} \rightrightarrows A$ be a covering of A by sets open in A .
- Each $A_i = B_i \cap A$ for some B_i open in B (definition of subspace topology)
- Define $V = \{B_i\}$, then $V \rightrightarrows A$ is an open cover.
- Since A is closed, $W := B \setminus A$ is open
- Then $V \cup W$ is an open cover of B , and has a finite subcover $\{V_i\}$
- Then $\{V_i \cap A\}$ is a finite open cover of A . ■

Theorem 2.13.

The continuous image of a compact set is compact.

Theorem 2.14.

A closed subset of a Hausdorff space is compact.

Theorem 2.15.

A continuous bijection $f : X \rightarrow Y$ where X is compact and Y is Hausdorff is an open map and hence a homeomorphism.

3 Examples

3.1 Common Spaces and Operations

Point-Set:

- Finite discrete sets with the discrete topology
- Subspaces of \mathbb{R} : (a, b) , $(a, b]$, (a, ∞) , etc.
 - $\{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{Z}^{\geq 1} \right\}$
- \mathbb{Q}
- The topologist's sine curve
- One-point compactifications
- \mathbb{R}^ω
- Hawaiian earring
- Cantor set

Non-Hausdorff spaces:

- The cofinite topology on any infinite set.
- \mathbb{R}/\mathbb{Q}
- The line with two origins.

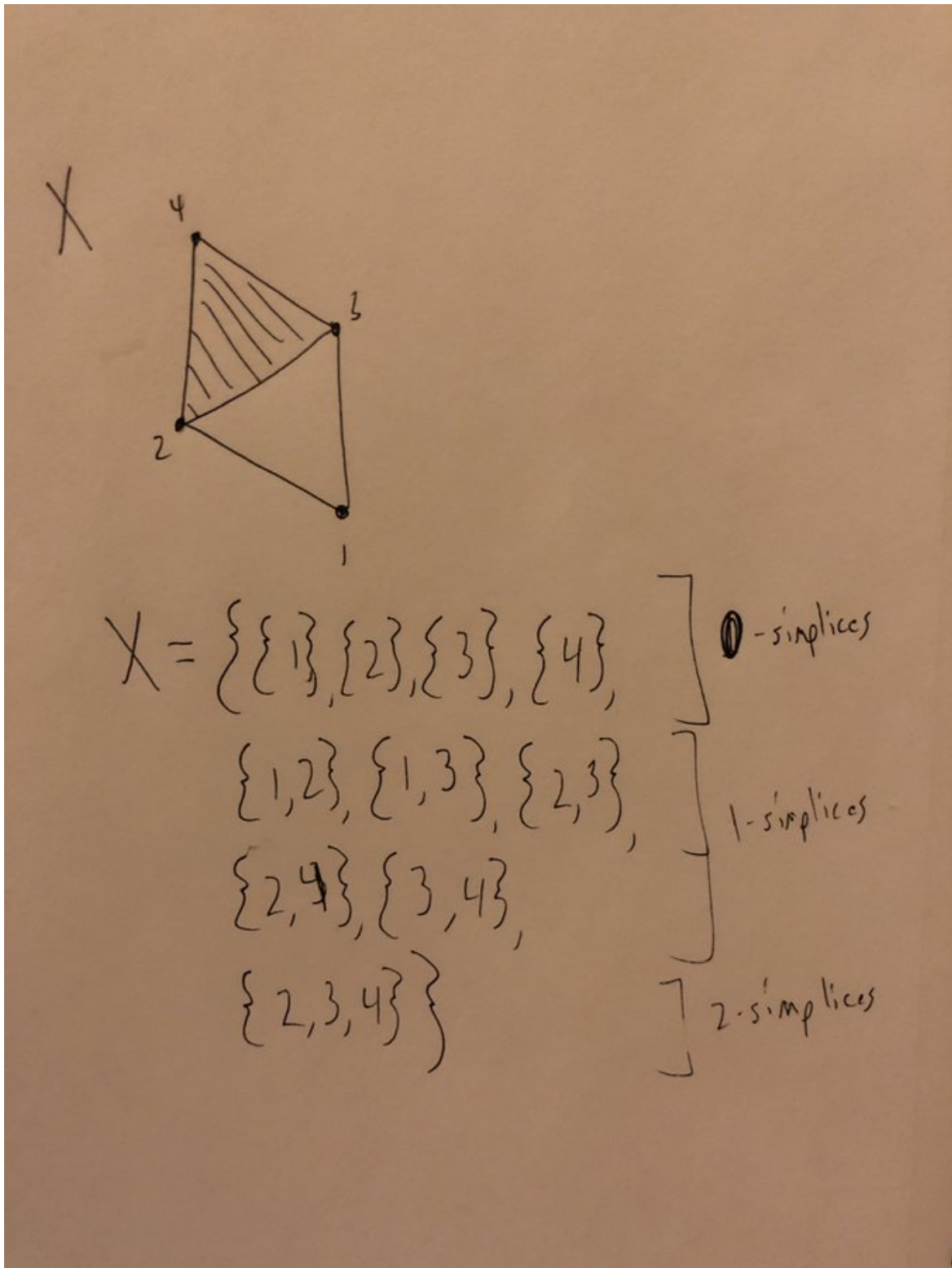
General Spaces:

$$S^n, \mathbb{D}^n, T^n, \mathbb{RP}^n, \mathbb{CP}^n, \mathbb{M}, \mathbb{K}, \Sigma_g, \mathbb{RP}^\infty, \mathbb{CP}^\infty.$$

“Constructed” Spaces

- Knot complements in S^3
- Covering spaces (hyperbolic geometry)
- Lens spaces
- Matrix groups
- Prism spaces
- Pair of pants
- Seifert surfaces
- Surgery
- Simplicial Complexes

– Nice minimal example:



Exotic/Pathological Spaces

- $\mathbb{H}\mathbb{P}^n$

- Dunce Cap
- Horned sphere

Operations

- Cartesian product $A \times B$
- Wedge product $A \vee B$
- Connect Sum $A \# B$
- Quotienting A/B
- Puncturing $A \setminus \{a_i\}$
- Smash product
- Join
- Cones
- Suspension
- Loop space
- Identifying a finite number of points

3.2 Alternative Topologies

- Discrete
- Cofinite
- Discrete and Indiscrete
- Uniform

The cofinite topology:

- Non-Hausdorff
- Compact

The discrete topology:

- Discrete iff points are open
- Always Hausdorff
- Compact iff finite
- Totally disconnected
- If the domain, every map is continuous

The indiscrete topology:

- Only open sets are \emptyset, X
- Non-Hausdorff
- If the codomain, every map is continuous
- Compact

4 AT Summary

4.1 Conventions

- $\pi_0(X)$ is the set of path components of X , and I write $\pi_0(X) = \mathbb{Z}$ if X is path-connected (although it is not a group). Similarly, $H_0(X)$ is a free abelian group on the set of path components of X .

- Lists start at entry 1, since all spaces are connected here and thus $\pi_0 = H_0 = \mathbb{Z}$. That is,
 - $\pi_*(X) = [\pi_1(X), \pi_2(X), \pi_3(X), \dots]$
 - $H_*(X) = [H_1(X), H_2(X), H_3(X), \dots]$
- For a finite index set I , $\prod_I G = \bigoplus_I G$ in **Grp**, i.e. the finite direct product and finite direct sum coincide.

Otherwise, if I is infinite, the direct sum requires cofinitely many zero entries (i.e. finitely many nonzero entries), so here we always use \prod .

In other words, there is an injective map

$$\bigoplus_I G \hookrightarrow \prod_I G$$

which is an isomorphism when $|I| < \infty$

- The free abelian group of rank n :

$$\mathbb{Z}^n := \prod_{i=1}^n \mathbb{Z} = \mathbb{Z} \times \mathbb{Z} \times \dots \mathbb{Z}.$$

- $x \in \mathbb{Z}^n = \langle a_1, \dots, a_n \rangle \implies x = \sum_{i=1}^n c_i a_i$ for some $c_i \in \mathbb{Z}$, i.e. a_i form a basis.
- Example: $x = 2a_1 + 4a_2 + a_1 - a_2 = 3a_1 + 3a_2$.

- The **free product** of n free abelian groups:

$$\mathbb{Z}^{*n} := \bigstar_{i=1}^n \mathbb{Z} = \mathbb{Z} * \mathbb{Z} * \dots \mathbb{Z}$$

This is a free *nonabelian* group on n generators.

- $x \in \mathbb{Z}^{*n} = \langle a_1, \dots, a_n \rangle$ implies that x is a finite word in the noncommuting symbols a_i^k for $k \in \mathbb{Z}$.
- Example: $x = a_1^2 a_2^4 a_1 a_2^{-2}$

- $K(G, n)$ is an Eilenberg-MacLane space, the homotopy-unique space satisfying

$$\pi_k(K(G, n)) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

- $K(\mathbb{Z}, 1) = S^1$
- $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$
- $K(\mathbb{Z}/2\mathbb{Z}, 1) = \mathbb{RP}^\infty$

- $M(G, n)$ is a Moore space, the homotopy-unique space satisfying

$$H_k(M(G, n); G) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

- $M(\mathbb{Z}, n) = S^n$
- $M(\mathbb{Z}/2\mathbb{Z}, 1) = \mathbb{RP}^2$

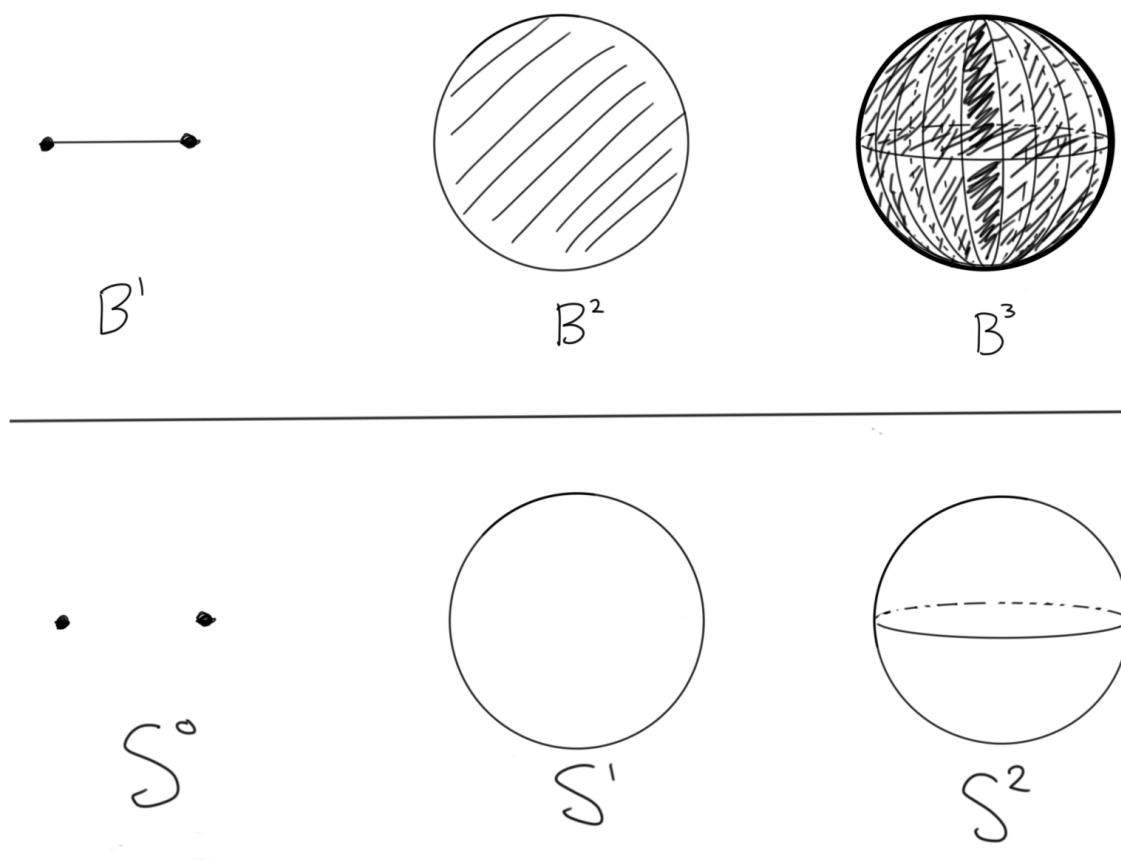


Figure 1: Low-Dimensional Spheres/Discs/Balls

- $M(\mathbb{Z}/p\mathbb{Z}, n)$ is made by attaching e^{n+1} to S^n via a degree p map.
- $B^n = \{\mathbf{v} \in \mathbb{R}^n \mid \|\mathbf{v}\| \leq 1\} \subset \mathbb{R}^n$
- $S^{n-1} = \partial B^n = \{\mathbf{v} \in \mathbb{R}^n \mid \|\mathbf{v}\| = 1\} \subset \mathbb{R}^n$
- $\mathbb{RP}^n = S^n/S^0 = S^n/\mathbb{Z}/2\mathbb{Z}$
- $\mathbb{CP}^n = S^{2n+1}/S^1$
- $T^n = \prod_n S^1$ is the n -torus
- $D(k, X)$ is the space X with $k \in \mathbb{N}$ distinct points deleted, i.e. the punctured space $X - \{x_1, x_2, \dots, x_k\}$ where each $x_i \in X$.

4.2 Table of Homotopy and Homology Structures

X	$\pi_*(X)$	$H_*(X)$	CW Structure	$H^*(X)$
\mathbb{R}^1	0	0	$\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot x$	0

4.2 Table of Homotopy and Homology Structures

X	$\pi_*(X)$	$H_*(X)$	CW Structure	$H^*(X)$
\mathbb{R}^n	0	0	$(\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot x)^n$	0
$D(k, \mathbb{R}^n)$	$\pi_* \bigvee^k S^1$	$\bigoplus^k H_* M(\mathbb{Z}, 1)$	$1 + kx$?
B^n	$\pi_*(\mathbb{R}^n)$	$H_*(\mathbb{R}^n)$	$1 + x^n + x^{n+1}$	0
S^n	$[0, \dots, \mathbb{Z}, ? \dots]$	$H_* M(\mathbb{Z}, n)$	$1 + x^n$ or $\sum_{i=0}^n 2x^i$	$\mathbb{Z}[x]/(x^2)$
$D(k, S^n)$	$\pi_* \bigvee^{k-1} S^1$	$\bigoplus^{k-1} H_* M(\mathbb{Z}, 1)$	$1 + (k-1)x^1$?
T^2	$\pi_* S^1 \times \pi_* S^1$	$(H_* M(\mathbb{Z}, 1))^2 \times H_* M(\mathbb{Z}, 2)$	$1 + 2x + x^2$	$\Lambda(1x_1, 1x_2)$
T^n	$\prod_n \pi_* S^1$	$\prod_{i=1}^n (H_* M(\mathbb{Z}, i))^{\binom{n}{i}}$	$(1+x)^n$	$\Lambda(1x_1, 1x_2, \dots, 1x_n)$
$D(k, T^n)$	$[0, 0, 0, 0, \dots]?$	$[0, 0, 0, 0, \dots]?$	$1 + x$?
$S^1 \vee S^1$	$\pi_* S^1 * \pi_* S^1$	$(H_* M(\mathbb{Z}, 1))^2$	$1 + 2x$?
$\bigvee_n S^1$	$*^n \pi_* S^1$	$\prod H_* M(\mathbb{Z}, 1)$	$1 + x$?
\mathbb{RP}^1	$\pi_* S^1$	$H_* M(\mathbb{Z}, 1)$	$1 + x$	${}_0\mathbb{Z} \times {}_1\mathbb{Z}$
\mathbb{RP}^2	$\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) + \pi_* S^2$	$H_* M(\mathbb{Z}/2\mathbb{Z}, 1)$	$1 + x + x^2$	${}_0\mathbb{Z} \times {}_2\mathbb{Z}/2\mathbb{Z}$
\mathbb{RP}^3	$\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) + \pi_* S^3$	$H_* M(\mathbb{Z}/2\mathbb{Z}, 1) + H_* M(\mathbb{Z}, 3)$	$1 + x + x^2 + x^3$	${}_0\mathbb{Z} \times {}_2\mathbb{Z}/2\mathbb{Z} \times {}_3\mathbb{Z}$
\mathbb{RP}^4	$\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) + \pi_* S^4$	$H_* M(\mathbb{Z}/2\mathbb{Z}, 1) + H_* M(\mathbb{Z}/2\mathbb{Z}, 3)$	$1 + x + x^2 + x^3 + x^4$	${}_0\mathbb{Z} \times ({}_2\mathbb{Z}/2\mathbb{Z})^2$
$\mathbb{RP}^n, n \geq 4$ even	$\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) + \pi_* S^n$	$\prod_{\text{odd } i < n} H_* M(\mathbb{Z}/2\mathbb{Z}, i)$	$\sum_{i=1}^n x^i$	${}_0\mathbb{Z} \times \prod_{i=1}^{n/2} {}_2\mathbb{Z}/2\mathbb{Z}$
$\mathbb{RP}^n, n \geq 4$ odd	$\pi_* K(\mathbb{Z}/2\mathbb{Z}, 1) + \pi_* S^n$	$\prod_{\text{odd } i \leq n-2} H_* M(\mathbb{Z}/2\mathbb{Z}, i) \times H_* S^n$	$\sum_{i=1}^n x^i$	$H^*(\mathbb{RP}^{n-1}) \times {}_n\mathbb{Z}$
\mathbb{CP}^1	$\pi_* K(\mathbb{Z}, 2) + \pi_* S^3$	$H_* S^2$	$x^0 + x^2$	$\mathbb{Z}[2x]/(2x^2)$
\mathbb{CP}^2	$\pi_* K(\mathbb{Z}, 2) + \pi_* S^5$	$H_* S^2 \times H_* S^4$	$x^0 + x^2 + x^4$	$\mathbb{Z}[2x]/(2x^3)$
$\mathbb{CP}^n, n \geq 2$	$\pi_* K(\mathbb{Z}, 2) + \pi_* S^{2n+1}$	$\prod_{i=1}^n H_* S^{2i}$	$\sum_{i=1}^n x^{2i}$	$\mathbb{Z}[2x]/(2x^{n+1})$
Mobius Band	$\pi_* S^1$	$H_* S^1$	$1 + x$?
Klein Bottle	$K(\mathbb{Z} \rtimes_{-1} \mathbb{Z}, 1)$	$H_* S^1 \times H_* \mathbb{RP}^\infty$	$1 + 2x + x^2$?

Facts used to compute the above table:

- \mathbb{R}^n is a contractible space, and so $[S^m, \mathbb{R}^n] = 0$ for all n, m which makes its homotopy groups all zero.
- $D(k, \mathbb{R}^n) = \mathbb{R}^n - \{x_1 \dots x_k\} \simeq \bigvee_{i=1}^k S^1$ by a deformation retract.
- $S^n \cong B^n / \partial B^n$ and employs an attaching map

$$\begin{aligned} \varphi : (D^n, \partial D^n) &\longrightarrow S^n \\ (D^n, \partial D^n) &\mapsto (e^n, e^0). \end{aligned}$$

- $B^n \simeq \mathbb{R}^n$ by normalizing vectors.
- Use the inclusion $S^n \hookrightarrow B^{n+1}$ as the attaching map.
- $\mathbb{CP}^1 \cong S^2$.

- $\mathbb{RP}^1 \cong S^1$.
- Use $[\pi_1, \coprod] = 0$ and the universal cover $\mathbb{R}^1 \rightarrow S^1$ to yield the cover $\mathbb{R}^n \rightarrow T^n$.
- Take the universal double cover $S^n \twoheadrightarrow^{\times 2} \mathbb{RP}^n$ to get equality in $\pi_{i \geq 2}$.
- Use $\mathbb{CP}^n = S^{2n+1}/S^1$
- Alternatively, the fundamental group is $\mathbb{Z} * \mathbb{Z}/bab^{-1}a$. Use the fact the $\tilde{K} = \mathbb{R}^2$.
- $M \simeq S^1$ by deformation-retracting onto the center circle.
- $D(1, S^n) \cong \mathbb{R}^n$ and thus $D(k, S^n) \cong D(k-1, \mathbb{R}^n) \cong \bigvee^{k-1} S^1$

4.3 Euler Characteristics

- Only surfaces with positive χ :
 - $\chi S^2 = 2$
 - $\chi \mathbb{RP}^2 = 1$
 - $\chi B^2 = 1$
- Manifolds with zero χ
 - $T^2, K, M, S^1 \times I$
- Manifolds with negative χ
 - $\Sigma_{g \geq 2}$ by $\chi(X) = 2 - 2g$.

4.4 Useful Facts and Techniques

- Homotopy Groups
 - Hurewicz map
- Homology
 - Mayer-Vietoris
 - * $(X = A \cup B) \mapsto (\bigcap, \oplus, \bigcup)$ in homology
 - LES of a pair
 - * $(A \hookrightarrow X) \mapsto (A, X, X/A)$
 - Excision
- $\pi_{i \geq 2}(X)$ is always abelian.
- The ranks of π_0 and H_0 are the number of path components, and $\pi_0(X) = \mathbb{Z}$ iff X is simply connected.
 - X simply connected $\implies \pi_k(X) \cong H_k(X)$ up to and including the first nonvanishing H_k
 - $H_1(X) = \text{Ab}(\pi_1 X)$, the abelianization.
- General mantra: homotopy plays nicely with products, homology with wedge products.¹

¹More generally, in **Top**, we can look at $A \leftarrow \{\text{pt}\} \rightarrow B$ – then $A \times B$ is the pullback and $A \vee B$ is the pushout. In this case, homology $h : \mathbf{Top} \rightarrow \mathbf{Grp}$ takes pushouts to pullbacks but doesn't behave well with pullbacks. Similarly, while π takes pullbacks to pullbacks, it doesn't behave nicely with pushouts.

In general, homotopy groups behave nicely under homotopy pull-backs (e.g., fibrations and products), but not homotopy push-outs (e.g., cofibrations and wedges). Homology is the opposite.

- $\pi_k \prod X = \prod \pi_k X$ by LES.²
- $H_k \prod X \neq \prod H_k X$ due to torsion.
 - Nice case: $H_k(A \times B) = \prod_{i+j=k} H_i A \otimes H_j B$ by Kunneth when all groups are torsion-free.³
- $H_k \bigvee X = \prod H_k X$ by Mayer-Vietoris.⁴
- $\pi_k \bigvee X \neq \prod \pi_k X$ (counterexample: $S^1 \vee S^2$)
 - Nice case: $\pi_1 \bigvee X = * \pi_1 X$ by Van Kampen.
- $\pi_i(\widehat{X}) \cong \pi_i(X)$ for $i \geq 2$ whenever $\widehat{X} \rightarrow X$ is a universal cover.
- Groups and Group Actions
 - $\pi_0(G) = G$ for G a discrete topological group.
 - $\pi_k(G/H) = \pi_k(G)$ if $\pi_k(H) = \pi_{k-1}(H) = 0$.
 - $\pi_1(X/G) = \pi_0(G)$ when G acts freely/transitively on X .
- Manifolds
 - $H^n(M^n) = \mathbb{Z}$ if M^n is orientable and zero if M^n is nonorientable.
 - Poincare Duality: $H_i M^n \cong H^{n-i} M^n$ iff M^n is closed and orientable.

4.5 Other Interesting Things To Consider

- The “generalized uniform bouquet”? $\mathcal{B}^n(m) = \bigvee_{i=1}^n S^m$
- Lie Groups
 - The real general linear group, $GL_n(\mathbb{R})$
 - * The real special linear group $SL_n(\mathbb{R})$
 - * The real orthogonal group, $O_n(\mathbb{R})$
 - The real special orthogonal group, $SO_n(\mathbb{R})$
 - * The real unitary group, $U_n(\mathbb{R})$
 - The real special unitary group, $SU_n(\mathbb{R})$

²This follows because $X \times Y \rightarrow X$ is a fiber bundle, so use LES in homotopy and the fact that $\pi_{i \geq 2} \in \mathbf{Ab}$.

³The generalization of Kunneth is as follows: write $\mathcal{P}(n, k)$ be the set of partitions of n into k parts, i.e. $\mathbf{x} \in \mathcal{P}(n, k) \implies \mathbf{x} = (x_1, x_2, \dots, x_k)$ where $\sum x_i = n$. Then

$$H_n \left(\prod_{j=1}^k X_j \right) = \bigoplus_{\mathbf{x} \in \mathcal{P}(n, k)} \bigotimes_{i=1}^k H_{x_i}(X_i).$$

⁴ \bigvee is the coproduct in the category \mathbf{Top}_0 of pointed topological spaces, and alternatively, $X \vee Y$ is the pushout in \mathbf{Top} of $X \leftarrow \{\text{pt}\} \rightarrow Y$

- * The real symplectic group $Sp(n)$
- “Geometric” Stuff
 - Affine n -space over a field $\mathbb{A}^n(k) = k^n \rtimes GL_n(k)$
 - The projective space $\mathbb{P}^n(k)$
 - * The projective linear group over a ring R , $PGL_n(R)$
 - * The projective special linear group over a ring R , $PSL_n(R)$
 - * The modular groups $PSL_n(\mathbb{Z})$
 - Specifically $PSL_2(\mathbb{Z})$
- The real Grassmannian, $Gr(n, k, \mathbb{R})$, i.e. the set of k dimensional subspaces of \mathbb{R}^n
- The Stiefel manifold $V_n(k)$
- Possible modifications to a space X :
 - Remove k points by taking $D(k, X)$
 - Remove a line segment
 - Remove an entire line/axis
 - Remove a hole
 - Quotient by a group action (e.g. antipodal map, or rotation)
 - Remove a knot
 - Take complement in ambient space
- Assorted info about other Lie Groups:
- $O_n, U_n, SO_n, SU_n, Sp_n$
- $\pi_k(U_n) = \mathbb{Z} \cdot \mathbb{1} [k \text{ odd}]$
 - $\pi_1(U_n) = 1$
- $\pi_k(SU_n) = \mathbb{Z} \cdot \mathbb{1} [k \text{ odd}]$
 - $\pi_1(SU_n) = 0$
- $\pi_k(U_n) = \mathbb{Z}/2\mathbb{Z} \cdot \mathbb{1} [k = 0, 1 \pmod 8] + \mathbb{Z} \cdot \mathbb{1} [k = 3, 7 \pmod 8]$
- $\pi_k(Sp_n) = \mathbb{Z}/2\mathbb{Z} \cdot \mathbb{1} [k = 4, 5 \pmod 8] + \mathbb{Z} \cdot \mathbb{1} [k = 3, 7 \pmod 8]$

4.6 Spheres

- $\pi_i(S^n) = 0$ for $i < n$, $\pi_n(S^n) = \mathbb{Z}$
 - Not necessarily true that $\pi_i(S^n) = 0$ when $i > n$!!!
 - * E.g. $\pi_3(S^2) = \mathbb{Z}$ by Hopf fibration
- $H_i(S^n) = \mathbb{1} [i \in \{0, n\}]$
- $H_n(\bigvee_i X_i) \cong \prod_i H_n(X_i)$ for “good pairs”
 - Corollary: $H_n(\bigvee_k S^n) = \mathbb{Z}^k$
- $S^n/S^k \simeq S^n \vee \Sigma S^k$

$$- \Sigma S^n = S^{n+1}$$

- S^n has the CW complex structure of 2 k -cells for each $0 \leq k \leq n$.

5 Fall 2014

5.1 1. Let $X = \mathbb{R}^3 - \Delta^{(1)}$, the complement of the skeleton of regular tetrahedron, and compute $\pi_1(X)$ and $H_*(X)$.

Lay the graph out flat in the plane, then take a maximal tree - these leaves 3 edges, and so $\pi_1(X) = \mathbb{Z}^{*3}$.

Moreover $X \simeq S^1 \vee S^1 \vee S^1$ which has only a 1-skeleton, thus $H_*(X) = [\mathbb{Z}, \mathbb{Z}^3, 0 \rightarrow]$.

5.2 2. Let $X = S^1 \times B^2 - L$ where L is two linked solid torii inside a larger solid torus. Compute $H_*(X)$.

?

5.3 3. Let L be a 3-manifold with homology $[\mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z}, \dots]$ and let $X = L \times \Sigma L$. Compute $H_*(X), H^*(X)$.

Useful facts:

- $H_k(X \times Y) \cong \bigoplus_{i+j=k} H_i(X) \otimes H_j(Y) \oplus \bigoplus_{i+j=k-1} \text{Tor}(H_i(X), H_j(Y))$
- $\tilde{H}_i(\Sigma X) = \tilde{H}_{i-1}(X)$

We will use the fact that $H_*(\Sigma L) = [\mathbb{Z}, \mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z}]$.

Represent $H_*(L)$ by $p(x, y) = 1 + yx + x^3$ and $H_*(\Sigma L)$ by $q(x, y) = 1 + x + yx^2 + x^4$, we can extract the free part of $H_*(X)$ by multiplying

$$p(x, y)q(x, y) = 1 + (1 + y)x + 2yx^2 + (y^2 + 1)x^3 + 2x^4 + 2yx^5 + x^7$$

where multiplication corresponds to the tensor product, addition to the direct sum/product.

So the free portion is

$$\begin{aligned} H_*(X) &= [\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}_3 \otimes \mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z}_3 \otimes \mathbb{Z}_3, \mathbb{Z}^2, \mathbb{Z}_3^2, 0, \mathbb{Z}] \\ &= [\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}^2, \mathbb{Z}_3^2, 0, \mathbb{Z}] \end{aligned}$$

We can add in the correction from torsion by noting that only terms of the form $\text{Tor}(\mathbb{Z}_3, \mathbb{Z}_3) = \mathbb{Z}_3$ survive. These come from the terms $i = 1, j = 2$, so $i + j = k - 1 \implies k = 1 + 2 + 1 = 4$ and there is thus an additional torsion term appearing in dimension 4. So we have

$$\begin{aligned} H_*(X) &= [\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}^2 \times \mathbb{Z}_3, \mathbb{Z}_3^2, 0, \mathbb{Z}] \\ &= [\mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}^2, 0, 0, \mathbb{Z}] \times [0, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3^2, 0, \mathbb{Z}] \end{aligned}$$

5.4 4. Let M be a closed, connected, oriented 4-manifold such that $H_2(M; \mathbb{Z})$ has rank 1. Show that there is not a free \mathbb{Z}_2 action on M .

and

$$\begin{aligned} H^*(X) &= [\mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}^2, 0, 0, \mathbb{Z}] \times [0, 0, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3^2, 0] \\ &= [\mathbb{Z}, \mathbb{Z}, \mathbb{Z}_3, \mathbb{Z} \times \mathbb{Z}_3, \mathbb{Z}^2 \times \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3^2, \mathbb{Z}]. \end{aligned}$$

■

5.4 4. Let M be a closed, connected, oriented 4-manifold such that $H_2(M; \mathbb{Z})$ has rank 1. Show that there is not a free \mathbb{Z}_2 action on M .

Useful facts:

- $X \rightarrow_{\times p} Y$ induces $\chi(X) = p\chi(Y)$
- Moral: always try a simple Euler characteristic argument first!

We know that $H_*(M) = [\mathbb{Z}, A, \mathbb{Z} \times G, A, \mathbb{Z}]$ for some group A and some torsion group G . Letting $n = \text{rank}(A)$ and taking the Euler characteristic, we have $\chi(M) = (1)1 + (-1)n + (1)1 + (-1)n + (1)1 = 3 - 2n$. Note that this is odd for any n .

However, a free action of $\mathbb{Z}_2 \curvearrowright M$ would produce a double covering $M \rightarrow_{\times 2} M/\mathbb{Z}_2$, and multiplicativity of Euler characteristics would force $\chi(M) = 2\chi(M/\mathbb{Z}_2)$ and thus $3 - 2n = 2k$ for some integer k . This would require $3 - 2n$ to be even, so we have a contradiction. ■

5.5 5. Let X be T^2 with a 2-cell attached to the interior along a longitude. Compute $\pi_2(X)$.

Useful facts:

- $T^2 = e^0 + e_1^1 + e_2^1 + e^2$ as a CW complex.
- $S^2/(x_0 \sim x_1) \simeq S^2 \wedge S^1$ when x_0, x_1 are two distinct points. (Picture: sphere with a string handle connecting north/south poles.)
- $\pi_{\geq 2}(\tilde{X}) \cong \pi_{\geq 2}(X)$ for $\tilde{X} \rightarrow X$ the universal cover.

Write $T^2 = e^0 + e_1^1 + e_2^1 + e^2$, where the first and second 1-cells denote the longitude and meridian respectively. By symmetry, we could have equivalently attached a disk to the meridian instead of the longitude, filling the center hole in the torus. Contract this disk to a point, then pull it vertically in both directions to obtain S^2 with two points identified, which is homotopy-equivalent to $S^2 \vee S_1$.

Take the universal cover, which is $\mathbb{R}^1 \bigcup_{\mathbb{Z}} S^2$ and has the same π_2 . This is homotopy-equivalent to $\bigvee_{i \in \mathbb{Z}} S^2$ and so $\pi_2(X) = \prod_{i \in \mathbb{Z}} \mathbb{Z}$ generated by each distinct copy of S^2 . (Alternatively written as $\mathbb{Z}[t, t^{-1}]$).

6 1

Let X be the subspace of the unit cube I^3 consisting of the union of the 6 faces and the 4 internal diagonals. Compute $\pi_1(X)$.

Solution:

7 2

Let X be an arbitrary topological space, and compute $\pi_1(\Sigma X)$.

Solution:

Write $\Sigma X = U \bigcup V$ where $U = \Sigma X - (X \times [0, 1/2])$ and $V = \Sigma X - X \times [1/2, 1]$. Then $U \bigcap V = X \times \{1/2\} \cong X$, so $\pi_1(U \bigcap V) = \pi_1(X)$.

But both U and V can be identified by the cone on X , given by $CX = \frac{X \times I}{X \times 1}$, by just rescaling the interval with the maps:

$i_U : U \longrightarrow CX$ where $(x, s) \mapsto (x, 2s - 1)$ (The second component just maps $[1/2, 1] \longrightarrow [0, 1]$.)

$i_V : V \longrightarrow CX$ where $(x, s) \mapsto (x, 2s)$. (The second component just maps $[0, 1/2] \longrightarrow [0, 1]$)

But CX is contractible by the homotopy $H : CX \times I \longrightarrow CX$ where $H((c, s), t) = (c, s(1 - t))$.

So $\pi_1(U) = \pi_1(V) = 0$.

By Van Kampen, we have $\pi_1(X) = 0 *_{\pi_1(X)} 0 = 0$.

8 3

Let $X = S^1 \times S^1$ and $A \subset X$ be a subspace with $A \cong S^1 \vee S^1$. Show that there is no retraction from X to A .

Solution:

We have $\pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1)$ since S^1 is path-connected (by a lemma from the problem sets), and this equals $\mathbb{Z} \times \mathbb{Z}$.

We also have $\pi_1(S^1 \vee S^1) = \pi_1(S^1) *_{\{pt\}} \pi_1(S^1)$, which by Van-Kampen is $\mathbb{Z} * \mathbb{Z}$.

Suppose X retracts onto A , we can then look at the inclusion $\iota : A \hookrightarrow X$. The induced homomorphism $\iota_* : \pi_1(A) \hookrightarrow \pi_1(X)$ is then also injective, so we've produced an injection from $f : \mathbb{Z} * \mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Z}$.

This is a contradiction, because no such injection can exist. In particular, the commutator $[a, b]$ is nontrivial in the source. But $f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1}$ since f is a homomorphism, but since the target is a commutative group, this has to equal $f(a)f(a)^{-1}f(b)f(b)^{-1} = e$. So there is a non-trivial element in the kernel of f , and f can not be injective - a contradiction.

9 4

Show that for every map $f : S^2 \longrightarrow S^1$, there is a point $x \in S^2$ such that $f(x) = f(-x)$.

Solution:

Suppose towards a contradiction that f does not possess this property, so there is no $x \in S^2$ such that $f(x) = f(-x)$.

Then define $g : S^2 \longrightarrow S^1$ by $g(x) = f(x) - f(-x)$; by assumption, this is a nontrivial map, i.e. $g(x) \neq 0$ for any $x \in S^2$.

In particular, $-g(-x) = -(f(-x) - f(x)) = f(x) - f(-x) = g(x)$, so $-g(x) = g(-x)$ and thus g commutes with the antipodal map $\alpha : S^2 \rightarrow S^2$.

This means g is constant on the fibers of the quotient map $p : S^2 \rightarrow \mathbb{RP}^2$, and thus descends to a well defined map $\tilde{g} : \mathbb{RP}^2 \rightarrow S^1$, and since $S^1 \cong \mathbb{RP}^1$, we can identify this with a map $\tilde{g} : \mathbb{RP}^2 \rightarrow \mathbb{RP}^1$ which thus induces a homomorphism $\tilde{g}_* : \pi_1(\mathbb{RP}^2) \rightarrow \pi_1(\mathbb{RP}^1)$.

Since g was nontrivial, \tilde{g} is nontrivial, and by functoriality of π_1 , \tilde{g}_* is nontrivial.

But $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$ and $\pi_1(\mathbb{RP}^1) = \mathbb{Z}$, and $\tilde{g}_* : \mathbb{Z}_2 \rightarrow \mathbb{Z}$ can only be the trivial homomorphism - a contradiction.

Alternate Solution

Use covering space $\mathbb{R} \rightarrow S^1$?

10 5

How many path-connected 2-fold covering spaces does $S^1 \vee \mathbb{RP}^2$ have? What are the total spaces?

Solution:

First note that $\pi_1(X) = \pi_1(S^1) *_{\{\text{pt}\}} \pi_1(\mathbb{RP}^2)$ by Van-Kampen, and this is equal to $\mathbb{Z} * \mathbb{Z}_2$.

11 6

Let $G = \langle a, b \rangle$ and $H \leq G$ where $H = \langle aba^{-1}b^{-1}, a^2ba^{-2}b^{-1}, a^{-1}bab^{-1}, aba^{-2}b^{-1}a \rangle$. To what well-known group is H isomorphic?

Solution:

12 Extra Problems

1. Compute $\pi_1(X)$ where $X := S^2 / \sim$, where $x \sim -x$ only for x on the equator $S^1 \hookrightarrow S^2$.
 - Hint: try cellular homology. Should yield $[\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}, 0, \dots]$.
3. Show that a local homeomorphism between compact Hausdorff spaces is a covering space.
4. Describe all connected covering spaces of $\mathbb{RP}^2 \vee \mathbb{RP}^2$.
5. Compute the homology of the Klein bottle using the Mayer-Vietoris sequence and a decomposition $K = M \coprod_f M$.
6. Show that if $X = S^2 \coprod_{\text{id}} S^2$ is a pushout along the equators, then $H_n(X) = [\mathbb{Z}, 0, \mathbb{Z}^3, 0, \dots]$.
7. Use the Kunneth formula to compute $H^*(S^2 \times S^2; \mathbb{Z})$.
 - Known to be $[\mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z}, 0, 0, \dots]$.
9. Compute $H^*(S^2 \vee S^2 \vee S^4)$
 - Known to be $[\mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z}, 0, 0, \dots]$.
10. Show that $\chi(\Sigma_g + \Sigma_h) = \chi(\Sigma_g) + \chi(\Sigma_h) - 2$.

Suggested by Ernest

1. Let X be a compact space and let A be a closed subspace. Show that A is compact.
2. Let $f : X \rightarrow Y$ be a continuous function, with X compact. Show that $f(X)$ is compact.
3. Let A be a compact subspace of a Hausdorff space X . Show that A is closed.
4. Show that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.