

# Solutions to Mike's Compendium of Problems

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## 1 Preface

Note: usually (1/3) of the qual problems relate to point set (around 8 problems).

### 1.1 Problems to Revisit

- 4
- 6 (Without Heine-Borel)
- 8
- 10
- 11
- 14

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## 2 1: Point-Set

### 2.1 1 (Todo)

### 2.2 2

See Munkres p.164, especially for (ii).

- i. See definition section of review doc.
- ii. Direct proof:
  - Let  $\{U_i \mid j \in J\} \Rightarrow X$ ; then  $0 \in U_j$  for some  $j \in J$ .
  - In the subspace topology,  $U_i$  is given by some  $V \in \tau(\mathbb{R})$  such that  $V \cap X = U_i$ 
    - A basis for the subspace topology on  $\mathbb{R}$  is open intervals, so write  $V$  as a union of open intervals  $V = \bigcup_{k \in K} I_k$ .
    - Since  $0 \in U_j$ ,  $0 \in I_k$  for some  $k$ .
  - Since  $I_k$  is an interval, it contains infinitely many points of the form  $x_n = \frac{1}{n} \in X$
  - Then  $I_k \cap X \subset U_j$  contains infinitely many such points.
  - So there are only *finitely* many points in  $X \setminus U_j$ , each of which is in  $U_{j(n)}$  for some  $j(n) \in J$  depending on  $n$ .
  - So  $U_j$  and the *finitely* many  $U_{j(n)}$  form a finite subcover of  $X$ . ■
- iii. Todo: Need direct proof.

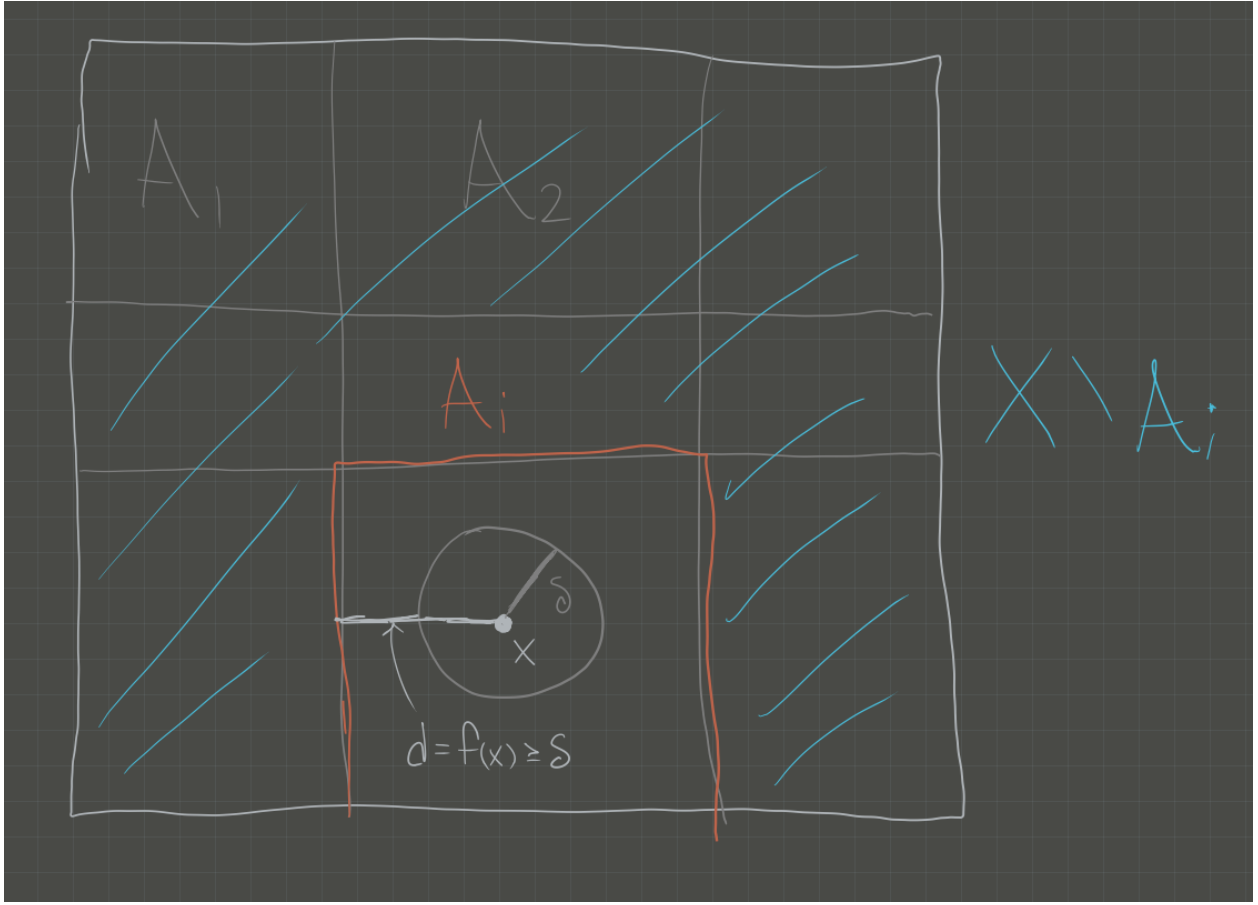
### 2.3 3 (Todo)

### 2.4 4

Statement: show that the *Lebesgue number* is well-defined for compact metric spaces.

Note: this is a question about the *Lebesgue Number*. See Wikipedia for detailed proof.

- Write  $U = \{U_i \mid i \in I\}$ , then  $X \subseteq \bigcup_{i \in I} U_i$ . Need to construct a  $\delta > 0$ .
- By compactness of  $X$ , choose a finite subcover  $U_1, \dots, U_n$ .
- Define the distance between a point  $x$  and a set  $Y \subset X$ :  $d(x, Y) = \inf_{y \in Y} d(x, y)$ .
  - **Claim:** the function  $d(\cdot, Y) : X \rightarrow \mathbb{R}$  is continuous for a fixed set.
  - Proof: Todo, not obvious.



- Define a function

$$f : X \longrightarrow \mathbb{R}$$

$$x \mapsto \frac{1}{n} \sum_{i=1}^n d(x, X \setminus U_i).$$

- Note this is a sum of continuous functions and thus continuous.

- **Claim:**

$$\delta := \inf_{x \in X} f(x) = \min_{x \in X} f(x) = f(x_{\min}) > 0$$

suffices.

- That the infimum is a minimum:  $f$  is a continuous function on a compact set, apply the extreme value theorem: it attains its minimum.
- That  $\delta > 0$ : otherwise,  $\delta = 0 \implies \exists x_0$  such that  $d(x_0, X \setminus U_i) = 0$  for all  $i$ .
  - \* Forces  $x_0 \in X \setminus U_i$  for all  $i$ , but  $X \setminus \bigcup U_i = \emptyset$  since the  $U_i$  cover  $X$ .
- That it satisfies the Lebesgue condition:

$$\forall x \in X, \exists i \text{ such that } B_\delta(x) \subset U_i$$

- \* Let  $B_\delta(x) \ni x$ ; then by minimality  $f(x) \geq \delta$ .
- \* Thus it can *not* be the case that  $d(x, X \setminus U_i) < \delta$  for *every*  $i$ , otherwise

$$f(x) \leq \frac{1}{n}(\delta + \dots + \delta) = \frac{n\delta}{n} = \delta$$

- \* So there is some particular  $i$  such that  $d(x, X \setminus U_i) \geq \delta$ .
- \* But then  $B_\delta \subseteq U_i$  as desired.

## 2.5 5 (Todo)

## 2.6 6

Facts used:

- Cantor's Intersection Theorem
- Bases for standard topology on  $\mathbb{R}$ .
- Definition of compactness

- Toward a contradiction, let  $\{U_\alpha\} \rightrightarrows [0, 1]$  be an open cover with no finite subcover.
- Then either  $[0, \frac{1}{2}]$  or  $[\frac{1}{2}, 1]$  has no finite subcover; WLOG assume it is  $[0, \frac{1}{2}]$ .
- Then either  $[0, \frac{1}{4}]$  or  $[\frac{1}{4}, \frac{1}{2}]$  has no finite subcover
- Inductively defining  $[a_n, b_n]$  this way yields a sequence of closed, bounded, nested intervals (each with no finite subcover) with  $\text{diam}([a_n, b_n]) \leq \frac{1}{2^n} \rightarrow 0$ , so Cantor's Nested Interval theorem applies and the intersection contains exactly one point  $p \in [0, 1]$ .
- Since  $p \in [0, 1]$ ,  $p \in U_\alpha$  for some  $\alpha$ .
- Since a basis for  $\tau(\mathbb{R})$  is given by open intervals, we can find an  $\varepsilon > 0$  such that  $(p - \varepsilon, p + \varepsilon) \subseteq U_\alpha$
- Then if  $\frac{1}{2^N} < \varepsilon$ , for  $n \geq N$  we have

$$[a_n, b_n] \subseteq (p - \varepsilon, p + \varepsilon) \subseteq U_\alpha.$$

- But then  $U_\alpha \rightrightarrows [a_n, b_n]$ , yielding a finite subcover of  $[a_n, b_n]$ , a contradiction.

## 2.7 7 (Todo)

## 2.8 8

Topic: proof of the tube lemma.

Statement: show  $X, Y \in \text{Top}_{\text{compact}} \iff X \times Y \in \text{Top}_{\text{compact}}$

### 2.8.1 Proof 1

$\Leftarrow :$

- By universal properties, the product  $X \times Y$  is equipped with continuous projections
- The continuous image of a compact set is compact, and  $\pi_1(X \times Y) = X, \pi_2(X \times Y) = Y$
- So  $X, Y$  are compact.

$\Rightarrow :$

Proof of Tube Lemma:

- Let  $\{U_j \times V_j \mid j \in J\} \rightrightarrows X \times Y$ .
- Fix a point  $x_0 \in X$ , then  $\{x_0\} \times Y \subset N$  for some open set  $N$ .
- By the tube lemma, there is a  $U^x \subset X$  such that the tube  $U^x \times Y \subset N$ .

- Since  $\{x_0\} \times Y \cong Y$  which is compact, there is a finite subcover  $\{U_j \times V_j \mid j \leq n\} \Rightarrow \{x_0\} \times Y$ .
- “Integrate the  $X$ ”: write

$$W = \bigcap_{j=1}^n U_j,$$

then  $x_0 \in W$  and  $W$  is a finite intersection of open sets and thus open.

- Claim:  $\{U_j \times V_j \mid j \leq n\} \Rightarrow W \times Y$ 
  - Let  $(x, y) \in W \times Y$ ; want to show  $(x, y) \in U_j \times V_j$  for some  $j \leq n$ .
  - Then  $(x_0, y) \in \{x_0\} \times Y$  is on the same horizontal line
  - $(x_0, y) \in U_j \times V_j$  for some  $j$  by construction
  - So  $y \in V_j$  for this  $j$
  - Since  $x \in W$ ,  $x \in U_j$  for *every*  $j$ , thus  $x \in U_j$ .
  - So  $(x, y) \in U_j \times V_j$

Actual Proof:

- Let  $\{U_j \mid j \in J\} \Rightarrow X \times Y$ .
- Fix  $x_0 \in X$ , the slice  $\{x_0\} \times Y$  is compact and can be covered by finitely many elements  $\{U_j \mid j \leq m\} \Rightarrow \{x_0\} \times Y$ .
  - Sum: write  $N = \bigcup_{j=1}^m U_j$ ; then  $\{x_0\} \times Y \subset N$ .
  - Apply the tube lemma to  $N$ : produce  $\{x_0\} \times Y \in W \times Y \subset N$ ; then  $\{U_j \mid j \leq m\} \Rightarrow W \times Y$ .
- Now let  $x \in X$  vary: for each  $x \in X$ , produce  $W_x \times Y$  as above, then  $\{W_x \times Y \mid x \in X\} \Rightarrow X$ .
  - By above argument, every tube  $W_x \times Y$  can be covered by *finitely* many  $U_j$ .
- Since  $\{W_x \mid x \in X\} \Rightarrow X$  and  $X$  is compact, produce a finite subset  $\{W_k \mid k \leq m'\} \Rightarrow X$ .
- Then  $\{W_k \times Y \mid k \leq m'\} \Rightarrow X \times Y$ ; the claim is that it is a finite cover.
  - Finitely many  $k$
  - For each  $k$ , the tube  $W_k \times Y$  is covered by finitely by  $U_j$
  - And finite  $\times$  finite = finite. ■

Shorter mnemonic:

**19.U** It is sufficient to consider a cover consisting of elementary sets. Since  $Y$  is compact, each fiber  $x \times Y$  has a finite subcovering  $\{U_i^x \times V_i^x\}$ . Put  $W^x = \cap U_i^x$ . Since  $X$  is compact, the cover  $\{W^x\}_{x \in X}$  has a finite subcovering  $W^{x_j}$ . Then  $\{U_i^{x_j} \times V_i^{x_j}\}$  is the required finite subcovering.

## 2.8.2 Proof 2

Let  $\pi_X, \pi_Y$  denote the canonical projections, which we can note are continuous and preserve open sets.

$\Rightarrow$  : Suppose  $X \times Y$  is compact, and let  $\{U_\alpha\}, \{V_\beta\}$  be open covers of  $X$  and  $Y$  respectively.

Let  $T_{\alpha\beta} = U_\alpha \times V_\beta$ ; then  $\{T_{\alpha\beta}\}$  is an open cover of  $X \times Y$ . So there is a finite subcover  $\{T_{ij}\}$ ,  $\{\pi_X(T_{ij})\}$  is an open cover of  $X$ , and similarly for  $Y$ . So both  $X, Y$  are compact.

$\Leftarrow$  : Suppose  $X$  and  $Y$  are compact, and let  $U_\alpha \Rightarrow X \times Y$  be an open cover. Let  $\pi_Y : X \times Y \rightarrow Y$  be the canonical projection; then  $\{\pi_Y(U_\alpha)\} \Rightarrow Y$  and by compactness of  $Y$  there is a finite subcover of the form  $\{\pi_Y(U_i) \mid 1 \leq i \leq n\}$ . Then  $\{V_{x,i} := \{x\} \times U_i\}$  is an open cover of  $\{x\} \times Y$  for any fixed  $x$ .

So if we fix an  $x \in X$ , we can let  $V_{x,i} \Rightarrow \{x\} \times Y$  be any finite subcollection covering this slice. By the Tube Lemma, there is an open set  $W_x$  such that  $\{x\} \times Y \subset W_x \times Y \subset \bigcup V_{x,i} = \{x\} \times Y$ .

Then  $\{W_x\} \Rightarrow X$  as  $x$  varies is an open cover of  $X$ , and by compactness of  $X$ , there are finitely many  $x_j \in X$  such that  $W_{x_j} \Rightarrow X$ . But then  $X \times Y = \bigcup_j W_{x_j} \times Y = \bigcup_j \bigcup_i W_{x_j} \times V_{x_j,i} \subset \bigcup_\alpha U_\alpha$  is a finite cover.

### 2.8.3 Proof of Tube Lemma

Todo: Prove tube lemma.

## 2.9 9

### 2.10 10

#### 2.10.1 Proof 1

$X$  is connected:

- Write  $X = L \coprod G$  where  $L = \{0\} \times [-1, 1]$  and  $G = \{\Gamma(\sin(x)) \mid x \in (0, 1]\}$  is the graph of  $\sin(x)$ .
- $L \cong [0, 1]$  which is connected
  - Claim: Every interval is connected (todo)
- Claim:  $G$  is connected (i.e. as the graph of a continuous function on a connected set)
  - The function

$$\begin{aligned} f : (0, 1] &\longrightarrow [-1, 1] \\ x &\mapsto \sin(x) \end{aligned}$$

is continuous (how to prove?)

- Products of continuous functions are continuous iff all of the components are continuous.
- Claim: The diagonal map  $\Delta : Y \rightarrow Y \times Y$  where  $\Delta(t) = (t, t)$  is continuous for any  $Y$  since  $\Delta = (\text{id}, \text{id})$ 
  - \* Product of identity functions, which are continuous.
- The composition of continuous function is continuous, therefore

$$\begin{aligned} F : (0, 1] &\xrightarrow{\Delta} (0, 1]^2 \xrightarrow{(\text{id}, f)} (0, 1] \times [-1, 1] \\ t &\mapsto (t, t) \mapsto (t, f(t)) \end{aligned}$$

- Then  $G = F((0, 1])$  is the continuous image of a connected set and thus connected.
- Claim:  $X$  is connected

- Suppose there is a disconnecting cover  $X = A \coprod B$  such that  $\bar{A} \cap B = A \cap \bar{B} = \emptyset$  and  $A, B \neq \emptyset$ .
- WLOG let  $(x, \sin(x)) \in B$  for  $x > 0$  (otherwise just relabeling  $A, B$ )
- Claim:  $B = G$ 
  - \* It can't be the case that  $A$  intersects  $G$ : otherwise

$$X = A \coprod B \implies G = (A \cap G) \coprod (B \cap G)$$

disconnects  $G$ . So  $A \cap G = \emptyset$ , forcing  $A \subseteq L$

- \* Similarly  $L$  can not be disconnected, so  $B \cap L = \emptyset$  forcing  $B \subset G$
- \* So  $A \subset L$  and  $B \subset G$ , and since  $X = A \coprod B$ , this forces  $A = L$  and  $B = G$ .
- But any open set  $U$  in the subspace topology  $L \subset \mathbb{R}^2$  (generated by open balls) containing  $(0, 0) \in L$  is the restriction of a ball  $V \subset \mathbb{R}^2$  of radius  $r > 0$ , i.e.  $U = V \cap X$ .
- \* But any such ball contains points of  $G$ :

$$n \gg 0 \implies \frac{1}{n\pi} < r \implies \exists g \in G \text{ s.t. } g \in U.$$

- \* So  $U \cap L \cap G \neq \emptyset$ , contradicting  $L \cap G = \emptyset$ .
- Claim:  $X$  is *not* path-connected.
  - Todo: “can't get from  $L$  to  $G$  in finite time”.
  - Toward a contradiction, choose a continuous function  $f : I \longrightarrow X$  with  $f(0) \in G$  and  $f(1) \in L$ .
    - \* Since  $L \cong [0, 1]$ , use path-connectedness to create a path  $f(1) \longrightarrow (0, 1)$
    - \* Concatenate paths and reparameterize to obtain  $f(1) = (0, 1) \in L \subset \mathbb{R}^2$ .
  - Let  $\varepsilon = \frac{1}{2}$ ; by continuity there exists a  $\delta \in I$  such that

$$t \in B_\delta(1) \subset I \implies f(t) \in B_\varepsilon(\mathbf{0}) \in X$$

- Using the fact that  $[1 - \delta, 1]$  is connected,  $f([1 - \delta, 1]) \subset X$  is connected.
- Let  $f(1 - \delta) = \mathbf{x}_0 = (x_0, y_0) \in X \subset \mathbb{R}^2$ .
- Define a composite map

$$F : [0, 1] \longrightarrow \mathbb{R}F \qquad \qquad \qquad := \mathfrak{p}_{x\text{-axis}} \circ f.$$

- \*  $F$  is continuous as a composition of continuous functions.
- Then  $F([1 - \delta, 1]) \subset \mathbb{R}$  is connected and thus must be an interval  $(a, b)$
- Since  $f(1) = \mathbf{0}$  which has  $x$ -component zero,  $[0, b] \subset (a, b)$ .
- Since  $f(1 - \delta) = \mathbf{x}$ ,  $F(\mathbf{x}) = x_0$  and this  $[0, x_0] \subset (a, b)$ .
- Thus for all  $x \in (0, x_0]$  there exists a  $t \in [1 - \delta, 1]$  such that  $f(t) = (x, \sin\left(\frac{1}{x}\right))$ .
- Now toward the contradiction, choose  $x = \frac{1}{2n\pi - \pi/2} \in \mathbb{R}$  with  $n$  large enough such that  $x \in (0, x_0]$ .
  - \* Note that  $\sin\left(\frac{1}{x}\right) = -1$  by construction.
  - \* Apply the previous statement: there exists a  $t$  such that  $f(t) = (x, \sin\left(\frac{1}{x}\right)) = (x, -1)$ .



\* But then

$$\|f(t) - f(x)\| = \|(x, -1) - (0, 1)\| = \|(x, 2)\| > \frac{1}{2},$$

contradicting continuity of  $f$ .

### 2.10.2 Proof 2?

Let  $X = A \cup B$  with  $A = \{(0, y) \mid y \in [-1, 1]\}$  and  $B = \{(x, \sin(1/x)) \mid x \in (0, 1]\}$ . Since  $B$  is the graph of a continuous function, which is always connected. Moreover,  $X = \bar{A}$ , and the closure of a connected set is still connected.

Alternative direct argument: the subspace  $X' = B \cup \{\mathbf{0}\}$  is not connected. If it were, write  $X' = U \coprod V$ , where wlog  $\mathbf{0} \in U$ . Then there is an open such that  $\mathbf{0} \in N_r(\mathbf{0}) \subset U$ . But any neighborhood about zero intersects  $B$ , so we must have  $V \subset B$  as a strict inclusion. But then  $U \cap B$  and  $V$  disconnects  $B$ , a connected set, which is a contradiction.

To see that  $X$  is not path-connected, suppose toward a contradiction that there is a continuous function  $f : I \rightarrow X \subset \mathbb{R}^2$ . In particular,  $f$  is continuous at  $\mathbf{0}$ , and so

$$\forall \varepsilon \quad \exists \delta \quad \|\mathbf{x}\| < \delta \implies \|f(\mathbf{x})\| < \varepsilon.$$

where the norm is the standard Euclidean norm.

However, we can pick  $\varepsilon < 1$ , say, and consider points of the form  $\mathbf{x}_n = (\frac{1}{2n\pi}, 0)$ . In particular, we can pick  $n$  large enough such that  $\|\mathbf{x}_n\|$  is as small as we like, whereas  $\|f(\mathbf{x}_n)\| = 1 > \varepsilon$  for all  $n$ , a contradiction.

## 2.11 11 (Todo)

Consider the (continuous) projection  $\pi : \mathbb{R}^2 \rightarrow \mathbb{RP}^1$  given by  $(x, y) \mapsto [y/x, 1]$  in homogeneous coordinates. (I.e. this sends points to lines through the origin with rational slope).

Note that the image of  $\pi$  is  $\mathbb{RP}^1 \setminus \{\infty\}$ , which is homeomorphic to  $\mathbb{R}$ .

If we now define  $f = \pi|_X$ , we have  $f(X) \rightarrow \mathbb{Q} \subset \mathbb{R}$ . If  $X$  were connected, then  $f(X)$  would also be connected, but  $\mathbb{Q} \subset \mathbb{R}$  is disconnected, a contradiction.

## 2.12 12 (Todo)

- Using the fact that  $[0, \infty) \subset \mathbb{R}$  is Hausdorff, any retract must be closed, so any closed interval  $[\varepsilon, N]$  for  $0 \leq \varepsilon \leq N \leq \infty$ .
  - Note that  $\varepsilon = N$  yields all one point sets  $\{x_0\}$  for  $x_0 \geq 0$ .
- No finite discrete sets occur, since the retract of a connected set is connected.
- ?

**2.13 13 (Todo)****2.14 14**

- Take two connected sets  $X, Y$ ; then there exists  $p \in X \cap Y$ .
- Toward a contradiction: write  $X \cup Y = A \coprod B$  with both  $A, B \subset A \coprod B$  open.
- Since  $p \in X \cup Y = A \coprod B$ , WLOG  $p \in A$ . We will show  $B$  must be empty.
- Claim:  $A \cap X$  is clopen in  $X$ .
  - $A \cap X$  is open in  $X$ : ?
  - $A \cap X$  is closed in  $X$ : ?
- The only clopen sets of a connected set are empty or the entire thing, and since  $p \in A$ , we must have  $A \cap X = X$ .
- By the same argument,  $A \cap Y = Y$ .
- So  $A \cap (X \cup Y) = (A \cap X) \cup (A \cap Y) = X \cup Y$
- Since  $A \subset X \cup Y$ ,  $A \cap (X \cup Y) = A$
- Thus  $A = X \cup Y$ , forcing  $B = \emptyset$ .

**2.14.1 Proof 2?**

Let  $X := \bigcup_{\alpha} X_{\alpha}$ , and let  $p \in \bigcap X_{\alpha}$ . Suppose toward a contradiction that  $X = A \coprod B$  with  $A, B$  nonempty, disjoint, and relatively open as subspaces of  $X$ . Wlog, suppose  $p \in A$ , so let  $q \in B$  be arbitrary.

Then  $q \in X_{\alpha}$  for some  $\alpha$ , so  $q \in B \cap X_{\alpha}$ . We also have  $p \in A \cap X_{\alpha}$ .

But then these two sets disconnect  $X_{\alpha}$ , which was assumed to be connected – a contradiction.

**2.15 15 (Todo)****2.16 16****2.16.1 Proof 1**

Topic: closure and connectedness in the subspace topology. See Munkres p.148

- $S \subset X$  is **not** connected if  $S$  with the subspace topology is not connected.
  - I.e. there exist  $A, B \subset S$  such that
    - \*  $A, B \neq \emptyset$ ,
    - \*  $A \cap B = \emptyset$ ,
    - \*  $A \coprod B = S$ .
- Or equivalently, there exists a nontrivial  $A \subset S$  that is clopen in  $S$ .

Show stronger statement: this is an iff.

$\implies$  :

- Suppose  $S$  is not connected; we then have sets  $A \cup B = S$  from above and it suffices to show  $\text{cl}_Y(A) \cap B = A \cap \text{cl}_X(B) = \emptyset$ .
- $A$  is open by assumption and  $Y \setminus A = B$  is closed in  $Y$ , so  $A$  is clopen.

- Write  $\text{cl}_Y(A) := \text{cl}_X(A) \cap Y$ .
- Since  $A$  is closed in  $Y$ ,  $A = \text{cl}_Y(A)$  by definition, so  $A = \text{cl}_Y(A) = \text{cl}_X(A) \cap Y$ .
- Since  $A \cap B = \emptyset$ , we then have  $\text{cl}_Y(A) \cap B = \emptyset$ .
- The same argument applies to  $B$ , so  $\text{cl}_Y(B) \cap A = \emptyset$ .

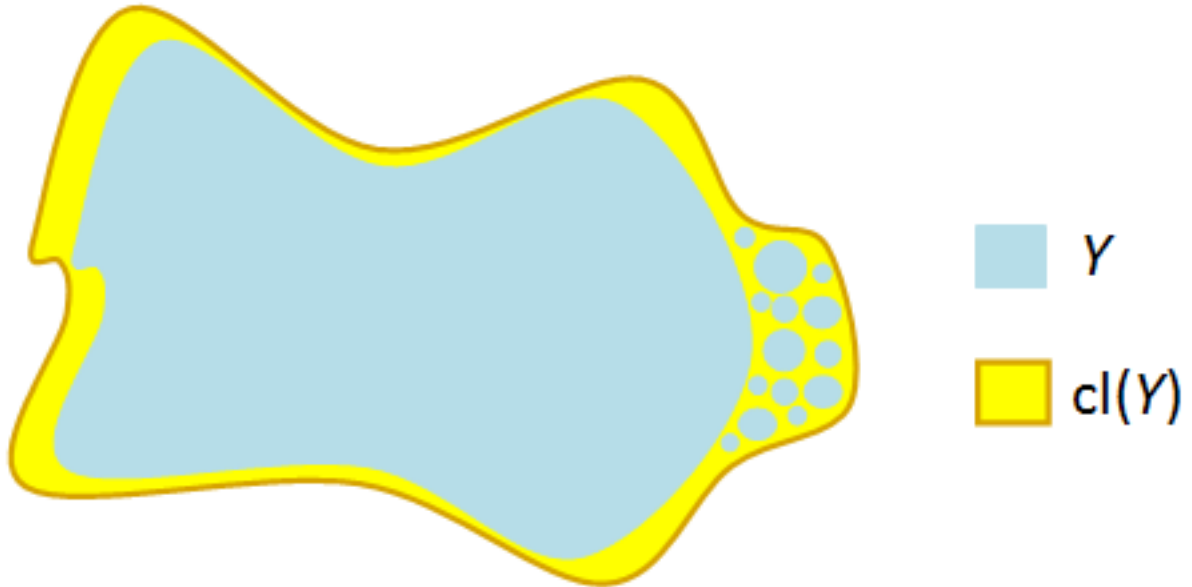
$\Leftarrow$  :

- Suppose displayed condition holds; given such  $A, B$  we will show they are clopen in  $Y$ .
- Since  $\text{cl}_Y(A) \cap B = \emptyset$ , (claim) we have  $\text{cl}_Y(A) = A$  and thus  $A$  is closed in  $Y$ .
  - Why?

$$\begin{aligned}
 \text{cl}_Y(A) &:= \text{cl}_X(A) \cap Y \\
 &= \text{cl}_X(A) \cap (A \amalg B) \\
 &= (\text{cl}_X(A) \cap A) \amalg (\text{cl}_X(A) \cap B) \\
 &= A \amalg (\text{cl}_X(A) \cap B) \quad \text{since } A \subset \text{cl}_Y(A) \\
 &= A \amalg (\text{cl}_Y(A) \cap B) \quad \text{since } B \subset Y \\
 &= A \amalg \emptyset \quad \text{using the assumption} \\
 &= A.
 \end{aligned}$$

- But  $A = Y \setminus B$  where  $B$  is closed, so  $A$  is open and thus a nontrivial clopen subset.

■



### 2.16.2 Proof 2

Lemma:  $X$  is connected iff the only subsets of  $X$  that are closed and open are  $\emptyset, X$ .

If  $S \subset X$  is not connected, then there exists a subset  $A \subset S$  that is both open and closed in the subspace topology, where  $A \neq \emptyset, S$ .

Suppose  $S$  is not connected, then choose  $A$  as above. Then  $B = S \setminus A$  yields a pair  $A, B$  that disconnects  $S$ . Since  $A$  is closed in  $S$ ,  $\bar{A} = A$  and thus  $\bar{A} \cap B = A \cap B = \emptyset$ . Similarly, since  $A$  is open,  $B$  is closed, and  $\bar{B} = B \implies \bar{B} \cap A = B \cap A = \emptyset$ .

## 2.17 17 (Todo)

### 2.18 18

- Define a new function

$$g : X \longrightarrow \mathbb{R}$$

$$x \mapsto d_X(x, f(x)).$$

- Attempt to minimize. Claim:  $g$  is a continuous function.
- Given claim, a continuous function on a compact space attains its infimum, so set

$$m := \inf_{x \in X} g(x)$$

and produce  $x_0 \in X$  such that  $g(x_0) = m$ .

- Then

$$m > 0 \iff d(x_0, f(x_0)) > 0 \iff x_0 \neq f(x_0).$$

- Now apply  $f$  and use the assumption that  $f$  is a contraction to contradict minimality of  $m$ :

$$\begin{aligned} d(f(f(x_0)), f(x_0)) &\leq C \cdot d(f(x_0), x_0) \\ &< d(f(x_0), x_0) \quad \text{since } C < 1 \\ &\leq m \end{aligned}$$

- Proof that  $g$  is continuous: use the definition of  $g$ , the triangle inequality, and that  $f$  is a contraction:

$$\begin{aligned} d(x, f(x)) &\leq d(x, y) + d(y, f(y)) + d(f(x), f(y)) \\ \implies d(x, f(x)) - d(y, f(y)) &\leq d(x, y) + d(f(x), f(y)) \\ \implies g(x) - g(y) &\leq d(x, y) + C \cdot d(x, y) = (C + 1) \cdot d(x, y) \end{aligned}$$

- This shows that  $g$  is Lipschitz continuous with constant  $C + 1$  (implies uniformly continuous, but not used).

**2.19 19 (Todo)**

Statement: prove that the product of two connected spaces is connected.

**Solution:**

Use the fact that a union of spaces containing a common point is still connected. Fix a point  $(a, b) \in X \times Y$ . Since the horizontal slice  $X_b := X \times \{b\}$  is homeomorphic to  $X$  which is connected, as are all of the vertical slices  $Y_x := \{x\} \times Y \cong Y$  (for any  $x$ ), the “T-shaped” space  $T_x := X_b \cup Y_x$  is connected for each  $x$ .

Note that  $(a, b) \in T_x$  for every  $x$ , so  $\bigcup_{x \in X} T_x = X \times Y$  is connected.

Image

**2.20 20**

Space	Connected	Locally Connected
$\mathbb{R}$	✓	✓
$[0, 1] \cup [2, 3]$		✓
Sine Curve	✓	
$\mathbb{Q}$		

- See definitions in intro.
- Claim: the Topologist’s sine curve  $X$  suffices.

Proof:

- Claim 1:  $X$  is connected.
  - Intervals and graphs of cts functions are connected, so the only problem point is 0.
- Claim 2:  $X$  is **not** locally path connected.
  - Take any  $B_\varepsilon(0) \in \mathbb{R}^2$ ; then  $\pi_X B_\varepsilon(0)$  yields infinitely many arcs, each intersecting the graph at two points on  $\partial B_\varepsilon(0)$ .
  - These are homeomorphic to a collection of disjoint embedded open intervals, and any disjoint union of intervals is clearly not connected. ■

Todo: what’s the picture?

**2.21 Problem 23**

Note: this is precisely the cofinite topology.

1.  $\mathbb{R} \in \tau$  since  $\mathbb{R} \setminus \mathbb{R} = \emptyset$  is trivially a finite set, and  $\emptyset \in \tau$  by definition.
2. If  $U_i \in \tau$  then  $(\bigcup_{i=1}^n U_i)^c = \bigcap_{i=1}^n U_i^c$  is an intersection of finite sets and thus finite, so  $\bigcup_{i=1}^n U_i \in \tau$ .
3. If  $U_i \in \tau$ , then  $(\bigcap_{i=1}^n U_i)^c = \bigcup_{i=1}^n U_i^c$  is a finite union of finite sets and thus finite, so  $\bigcap_{i=1}^n U_i \in \tau$ .

So  $\tau$  forms a topology.

To see that  $(\mathbb{R}, \tau)$  is compact, let  $\{U_i\} \rightrightarrows \mathbb{R}$  be an open cover by elements in  $\tau$ .

Fix any  $U_\alpha$ , then  $U_\alpha^c = \{p_1, \dots, p_n\}$  is finite, say of size  $n$ . So pick  $U_1 \ni p_1, \dots, U_n \ni p_n$ ; then  $\mathbb{R} \subset U_\alpha \bigcup_{i=1}^n U_i$  is a finite cover.

## 2.22 Problem 27

Notes: use diagonal trick to construct the Cauchy sequence.

### 2.22.1 a

$\implies :$

If  $X$  is totally bounded, let  $\varepsilon = \frac{1}{n}$  for each  $n$ , and let  $\{x_i\}$  be an arbitrary sequence. For  $n = 1$ , pick a finite open cover  $\{U_i\}_n$  such that  $\text{diam} U_i < \frac{1}{n}$  for every  $i$ .

Choose  $V_1$  such that there are infinitely many  $x_i \in V_1$ . (Why?) Note that  $\text{diam} V_i < 1$ . Now choose  $x_i \in V_1$  arbitrarily and define it to be  $y_1$ .

Then since  $V_1$  is totally bounded, repeat this process to obtain  $V_2 \subseteq V_1$  with  $\text{diam}(V_2) < \frac{1}{2}$ , and choose  $x_i \in V_2$  arbitrarily and define it to be  $y_2$ .

This yields a nested family of sets  $V_1 \supseteq V_2 \supseteq \dots$  and a sequence  $\{y_i\}$  such that  $d(y_i, y_j) < \max(\frac{1}{i}, \frac{1}{j}) \rightarrow 0$ , so  $\{y_i\}$  is a Cauchy subsequence.

$\longleftarrow :$

Then fix  $\varepsilon > 0$  and pick  $x_1$  arbitrarily and define  $S_1 = B(\varepsilon, x_1)$ . Then pick  $x_2 \in S_1^c$  and define  $S_2 = S_1 \bigcup B(\varepsilon, x_2)$ , and so on. Continue by picking  $x_{n+1} \in S_n^c$  (Since  $X$  is not totally bounded, this can always be done) and defining  $S_{n+1} = S_n \bigcup B(\varepsilon, x_{n+1})$ .

Then  $\{x_n\}$  is not Cauchy, because  $d(x_i, x_j) > \varepsilon$  for every  $i \neq j$ .

### 2.22.2 b

Take  $X = C^0([0, 1])$  with the sup-norm, then  $f_n(x) = x^n$  are all bounded by 1, but  $\|f_i - f_j\| = 1$  for every  $i, j$ , so no subsequence can be Cauchy, so  $X$  can not be totally bounded.

Moreover,  $\{f_n\}$  is closed. (Why?)

## 2.23 Problem 30

Let  $A \subset X$  be compact, and pick a fixed  $x \in X \setminus A$ . Since  $X$  is Hausdorff, for arbitrary  $a \in A$ , there exists opens  $U_a \ni a$  and  $U_{x,a} \ni x$  such that  $V_a \cap U_{x,a} = \emptyset$ . Then  $\{U_a \mid a \in A\} \rightrightarrows A$ , so by compactness there is a finite subcover  $\{U_{a_i}\} \rightrightarrows A$ .

Now take  $U = \bigcup_i U_{a_i}$  and  $V_x = \bigcap_i V_{a_i, x}$ , so  $U \cap V = \emptyset$ . Note that both  $U$  and  $V_x$  are open.

But then defining  $V := \bigcup_{x \in X \setminus A} V_x$ , we have  $X \setminus A \subset V$  and  $V \cap A = \emptyset$ , so  $V = X \setminus A$ , which is open and thus  $A$  is closed.

## 2.24 Problem 31

### 2.24.1 a

Theorems used:

- Continuous bijection + open map (or closed map)  $\implies$  homeomorphism.
- **Closed** subsets of compact sets are compact.
- The continuous image of a compact set is compact.
- Closed subsets of Hausdorff spaces are compact.

So we'll show that  $f$  is a closed map.

Let  $U \in X$  be closed.

- Since  $X$  is compact,  $U$  is compact
- Since  $f$  is continuous,  $f(U)$  is compact
- Since  $Y$  is Hausdorff,  $f(U)$  is closed.

### 2.24.2 b

Note that any finite space is clearly compact.

Take  $f : ([2], \tau_1) \longrightarrow ([2], \tau_2)$  to be the identity map, where  $\tau_1$  is the discrete topology and  $\tau_2$  is the indiscrete topology. Any map into an indiscrete topology is continuous, and  $f$  is clearly a bijection.

Let  $g$  be the inverse map; then note that  $1 \in \tau_1$  but  $g^{-1}(1) = 1$  is not in  $\tau_2$ , so  $g$  is not continuous. ■

## 2.25 Problem 32

$\implies :$

- Let  $p \in \Delta^c$ .
- Then  $p$  is of the form  $(x, y)$  where  $x \neq y$  and  $x, y \in X$ .
- Since  $X$  is Hausdorff, pick  $N_x, N_y$  in  $X$  such that  $N_x \cap N_y = \emptyset$ .
- Then  $N_p := N_x \times N_y$  is an open set in  $X^2$  containing  $p$ .
- Claim:  $N_p \cap \Delta = \emptyset$ .
  - If  $q \in N_p \cap \Delta$ , then  $q = (z, z)$  where  $z \in X$ , and  $q \in N_p \implies q \in N_x \cap N_y = \emptyset$ .
- Then  $\Delta^c = \bigcup_p N_p$  is open.

$\longleftarrow :$

- Let  $x \neq y \in X$ .

- Consider  $(x, y) \in \Delta^c \subset X^2$ , which is open.
- Thus  $(x, y) \in B$  for some box in the product topology.
- $B = U \times V$  where  $U, V$  are open in  $X$ , and  $B \subset \Delta^c$ .
- So  $x \in U, y \in V$ .
- Claim:  $U \bigcap V = \emptyset$ .
  - Otherwise,  $z \in U \bigcap V \implies (z, z) \in B$ , but  $B \bigcap \Delta^c = \emptyset$ .

## 2.26 Problem 38

$\mathbb{R}$  is clearly Hausdorff, and  $\mathbb{R}/\mathbb{Q}$  has the indiscrete topology, and is thus non-Hausdorff. So take the quotient map  $\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Q}$ .

Direct proof that  $\mathbb{R}/\mathbb{Q}$  isn't Hausdorff:

- Pick  $[x] \subset U \neq [y] \subset V \in \mathbb{R}/\mathbb{Q}$  and suppose  $U \cap V = \emptyset$ .
- Pull back  $U \longrightarrow A, V \longrightarrow B$  open disjoint sets in  $\mathbb{R}$
- Both  $A, B$  contain intervals, so they contain rationals  $p \in A, q \in B$
- Then  $[p] = [q] \in U \bigcap V$ .

## 2.27 Problem 42

Proof that  $\mathbb{R}/\mathbb{Q}$  has the indiscrete topology:

- Let  $U \subset \mathbb{R}/\mathbb{Q}$  be open and nonempty, show  $U = \mathbb{R}/\mathbb{Q}$ .
- Let  $[x] \in U$ , then  $x \in \pi^{-1}(U) := V \subset \mathbb{R}$  is open.
- Then  $V$  contains an interval  $(a, b)$
- Every  $y \in V$  satisfies  $y + q \in V$  for all  $q \in \mathbb{Q}$ , since  $y + q - y \in \mathbb{Q} \implies [y + q] = [y]$ .
- So  $(a - q, b + q) \in V$  for all  $q \in \mathbb{Q}$ .
- So  $\bigcup_{q \in \mathbb{Q}} (a - q, b + q) \in V \implies \mathbb{R} \subset V$ .
- So  $\pi(V) = \mathbb{R}/\mathbb{Q} = U$ , and thus the only open sets are the entire space and the empty set.

## 2.28 Problem 44

### 2.28.1 a

- Suppose  $X$  has a countable basis  $B = \{B_i\}$ .
- Choose an arbitrary  $x_i \in B_i$  for each  $i$ . Define  $Q = \{x_i\}$ .
- Let  $y \in N_y \subset X$ .
- By definition of a basis, there exists some  $B_i$  such that  $y \in B_i \subset N_y$ .
- Since  $x_i \in B_i$ ,  $Q \bigcap N_y \neq \emptyset$ .
- Thus  $Q$  is dense in  $X$ .

### 2.28.2 b

- Let  $\{q_i\}$  be a countable dense subset.
- Define  $B_{i,j} = B_{\frac{1}{j}}(q_i)$ , which is still countable.
- Property 1: Every  $x \in B_{i,j}$ 
  - Take  $x \in N_{\frac{1}{2}}(x) \ni q_j$  by density.



- 
- Then  $x \in B_{\frac{1}{2},j}$ .
  - Property 2:  $x \in B_{i_1,j_1} \cap B_{i_2,j_2} \implies x \in B_{i_3,j_3} \subset B_{i_1,j_1} \cap B_{i_2,j_2}$ :
    - Take  $i < \min(i_1, i_2)$ , then  $N_i(x) \ni q_j$  for some  $j$ .
    - Thus  $x \in B_{i,j}$ .

### 3 Spring 2019

#### 3.1 Problem 1

Complete and **totally** bounded  $\implies$  compact. - Definition: A space  $X$  is *totally bounded* if for every  $\varepsilon > 0$ , there is a finite cover  $X \subseteq \bigcup B_\alpha(\varepsilon)$  such that the radius of each ball is less than  $\varepsilon$ .

- Definition: A subset of a space  $S \subset X$  is *bounded* if there exists a  $B(r)$  such that  $r < \infty$  and  $S \subseteq B(r)$  - Totally bounded  $\implies$  bounded - Counterexample to converse:  $\mathbb{N}$  with the discrete metric. - Equivalent for Euclidean metric - Compact  $\implies$  totally bounded.

Counterexample for problem: the unit ball in any Hilbert (or Banach) space of infinite dimension is closed, bounded, and not compact.

Proof: Inductively, let  $\mathbf{x}_1 \in B(1, \mathbf{0})$  and  $A_1 = \text{span}(\mathbf{x}_1)$ , then choose  $s = \mathbf{x} + A_1 \in B(1, \mathbf{0})/A_1$  such that  $\|s\| = \frac{1}{2}$  and then a representative  $\mathbf{x}_2$  such that  $\|\mathbf{x}_2\| \leq 1$ . Then  $\|\mathbf{x}_2 - \mathbf{x}_1\| \geq \frac{1}{2}$ . Then, let  $A_2 = \text{span}(\mathbf{x}_1, \mathbf{x}_2)$ , (which is closed) and repeat this for  $s = \mathbf{x} + A_2 \in B(1, \mathbf{0})/A_2$  to get an  $\mathbf{x}_3$  such that  $\|\mathbf{x}_3 - \mathbf{x}_{\leq 2}\| \geq \frac{1}{2}$ . This produces a non-convergent sequence in the closed ball, so it can not be compact.

Second counterexample:  $(\mathbb{R}, (x, y) \mapsto \frac{|x - y|}{1 + |x - y|})$ .

Best counterexample:  $X = \left( \mathbb{Z}, \rho(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \right)$ . This metric makes  $X$  complete for any  $X$ , then take  $\mathbb{N} \subset X$ . All sets are closed, and bounded, so we have a complete, closed, bounded set that is not compact – take that cover  $U_i = B(1, i)$ .

Useful tool:  $(X, d) \cong_{\text{Top}} (X, \min(d(x, y), 1))$  where the RHS is now a bounded space. This preserves all topological properties (e.g. compactness).

#### 3.2 Problem 2

Definition:  $(X, \tau)$  where  $\tau \subseteq \mathcal{P}(X)$  is a *topological space* iff

- $\emptyset, X \in \tau$
- $\{U_i\}_{i \in I} \subseteq \tau \implies \bigcup_{i \in I} U_i \in \tau$
- $\{U_i\}_{i \in \mathbb{N}} \subseteq \tau \implies \bigcap_{i \in \mathbb{N}} U_i \in \tau$

We can write  $\overline{(X, \tau)} = (X \coprod \{\text{pt}\}, \tau \cup \tau')$  where  $\tau' = \{U \coprod \{\text{pt}\} \mid X - U \text{ is compact}\}$ . We need to show that  $T := \tau \cup \tau'$  forms a topology.

- We have  $\emptyset, X \in \tau \implies \emptyset, X \in \tau \cup \tau'$ .

- We just need to check that  $\tau'$  is closed under arbitrary unions. Let  $\{U_i\} \subset \tau'$ , so  $X - U_i = K_i$  a compact set for each  $i$ . Then  $\bigcup_i U_i = \bigcup_i X - (X - U_i) = \bigcup_i X - K_i = X - \bigcap_i K_i$

### 3.3 7

Let  $f : S^1 \xrightarrow{\times k} S^1$ .

**Claim:** The inclusion  $S^1 \rightarrow C_\varphi$  induces an isomorphism  $\pi_1(C_\varphi) \cong \pi_1(S^1)/H$  where  $H = N_{\pi_1(S^1)}(\langle f^* \rangle)$  is the normal subgroup generated by the induced map  $f^* \pi_1(S^1) \rightarrow \pi_1(S^1)$ .

Since  $f$  is a  $k$ -fold cover, the induced map is multiplication by  $k$  on the generator  $\alpha \in \pi_1(S^1)$ , i.e.  $\alpha \mapsto \alpha^k$ . But then  $\pi_1(S^1) \cong \mathbb{Z}$  and  $H \cong k\mathbb{Z}$ , so  $\pi_1(C_\varphi) \cong \mathbb{Z}/m\mathbb{Z}$ . ■

## 4 2: Fundamental Group

### 4.1 1

**Proposition:**  $\gamma_1 \simeq \gamma_2 \iff \gamma_1, \gamma_2$  are conjugate in  $\pi_1(X, x_0)$ , i.e.  $\exists [\alpha] \in \pi_1$  such that  $[\gamma_1] = [\alpha][\gamma_2][\alpha]^{-1}$ .

**Proof:**

$\implies$  : Clear, since  $\gamma_1 \sim \gamma_2 \implies [\gamma_1] = [\gamma_2] \in \pi_1(X)$ , so take  $\alpha(t) = x_0$  the constant loop for all  $t$ .

$\impliedby$  : ? Forgot how these arguments go.

Image ■

Counterexample where homotopic loops are not equal in  $\pi_1$ , but just conjugate:

Image

## 5 Covering Spaces

### 5.1 1b

Homotopy lifting property:

$$\begin{array}{ccc}
 & & \tilde{X} \\
 & \nearrow \exists \tilde{H} & \downarrow \pi \\
 Y \times I & \xrightarrow{H} & X
 \end{array}$$

$\pi$  clearly induces a map  $p_*$  on  $\pi_1$  by functoriality, so we'll show that  $\ker p_*$  is trivial. Let  $\gamma : S^1 \rightarrow \tilde{X} \in \pi_1(\tilde{X})$  and suppose  $\alpha := p_*(\gamma) = [e] \in \pi_1(X)$ . We'll show  $\gamma \simeq [e]$  in  $\pi_1(\tilde{X})$ .

Since  $\alpha = [e]$ ,  $\alpha \simeq \text{const.}$  and thus there is a homotopy  $H : I \times S^1 \rightarrow X$  such that  $H_0 = \text{const.}(x_0)$  and  $H_1 = \gamma$ . By the HLP, this lifts to  $\tilde{H} : I \times S^1 \rightarrow \tilde{X}$ . Noting that  $\pi^{-1}(\text{const.}(x_0))$  is still a constant loop, this says that  $\gamma$  is homotopic to a constant loop and thus nullhomotopic.

## 5.2 1c

Since both spaces are path-connected, the degree of the covering map  $\pi$  is precisely the index of the included fundamental group. This forces  $\pi$  to be a degree 1 covering and hence a homeomorphism.

## 5.3 6

Note  $\pi_1 \mathbb{RP}^2 = \mathbb{Z}/2\mathbb{Z}$ , so  $\pi_1 X = (\mathbb{Z}/2\mathbb{Z})^2$ .

The pullback of any neighborhood of the basepoint needs to be locally homeomorphic to one of

- $S^2 \vee S^2$
- $\mathbb{RP}^2 \vee S^2$

And so *all* possibilities for regular covering spaces are given by

- $\bigvee^{2k} S^2$  “beads” wrapped into a necklace for any  $k \geq 1$
- $\mathbb{RP}^2 \vee (\bigvee^k S^2) \vee \mathbb{RP}^2$
- $\vee^\infty S^2$ , the universal cover

To get a threefold cover, we want the basepoint to lift to three preimages, so we can take

- $S^2 \vee S^2 \vee S^2$  wrapped
- $\mathbb{RP}^2 \vee S^2 \vee \mathbb{RP}^2$ .

## 5.4 7

- $\mathbb{RP}^3 \vee S^2 \vee \mathbb{RP}^3$ , which has  $\pi_2 = 0 * \mathbb{Z} * 0 = \mathbb{Z}$  since  $\pi_{i \geq 1} X = \pi_{i \geq 1} \tilde{X}$  and  $\mathbb{RP}^3 = S^3$ .
- $\mathbb{RP}^2 \vee S^3 \vee \mathbb{RP}^2$ , which has  $\pi_2 = \mathbb{Z} * 0 * \mathbb{Z} = \mathbb{Z} * \mathbb{Z} \neq \mathbb{Z}$

## 5.5 8

Yes,

Image