Topology Qualifying Exam Notes

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Contents

1 Definitions

- Topology: Closed under arbitrary unions and finite intersections.
- Basis: A subset $\{B_i\}$ is a basis iff
  - $x \in X \implies x \in B_i$ for some i.
  - $x \in B_i \bigcap B_j \implies x \in B_k \subset B_i \bigcap B_k$.
  - Topology generated by this basis: $x \in N_x \implies x \in B_i \subset N_x$ for some i.
- Dense: A subset $Q \subset X$ is dense iff $y \in N_y \subset X \implies N_y \bigcap Q \neq \emptyset$ iff $\overline{Q} = X$.
- Neighborhood: A neighborhood of a point x is any open set containing x.
- Hausdorff
- Second Countable: admits a countable basis.
- Closed (several characterizations)
- Closure in a subspace: $Y \subset X \implies \mathrm{cl}_Y(A) \coloneqq \mathrm{cl}_X(A) \bigcap Y$.
- Bounded
- Compact: A topological space (X, τ) is **compact** if every open cover has a *finite* subcover.

That is, if $\left\{U_j \;\middle|\; j \in J\right\} \subset \tau$ is a collection of open sets such that $X \subseteq \bigcup_{j \in J} U_j$, then there exists a *finite* subset $J' \subset J$ such that $X \subseteq \bigcup_{j \in J'} U_j$.

- Locally compact For every $x \in X$, there exists a $K_x \ni x$ such that K_x is compact.
- Connected: There does not exist a disconnecting set $X = A \coprod B$ such that $\emptyset \neq A, B \subsetneq$, i.e. X is the union of two proper disjoint nonempty sets.

  Equivalently, X contains no proper nonempty clopen sets.

  – Additional condition for a subspace $Y \subset X$: $\mathrm{cl}_Y(A) \bigcap V = A \bigcap \mathrm{cl}_Y(B) = \emptyset$.
- Locally connected: A space is locally connected at a point x iff $\forall N_x \ni x$, there exists a $U \subset N_x$ containing x that is connected.
- Retract: A subspace $A \subset X$ is a *retract* of X iff there exists a continuous map $f : X \longrightarrow A$ such that $f \Big|_A = \mathrm{id}_A$. Equivalently it is a *left* inverse to the inclusion.
- Uniform Continuity: For $f : (X, d_x) \longrightarrow (Y, d_Y)$ metric spaces,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \quad d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

- Lebesgue number: For (X, d) a compact metric space and $\{U_\alpha\} \rightrightarrows X$, there exist $\delta_L > 0$ such that

$$A \subset X,\ \mathrm{diam}(A) < \delta_L \implies A \subseteq U_\alpha \text{ for some } \alpha.$$

- Paracompact
- Components: Set $x \sim y$ iff there exists a connected set $U \ni x, y$ and take equivalence classes.
- Path Components: Set $x \sim y$ iff there exists a path-connected set $U \ni x, y$ and take equivalence classes.
- Separable: Contains a countable dense subset.
- Limit Point: For $A \subset X$, x is a limit point of A if every punctured neighborhood P_x of x satisfies $P_x \bigcap A \neq \emptyset$, i.e. every neighborhood of x intersects A in some point other than x itself.

  Equivalently, x is a limit point of A iff $x \in \mathrm{cl}_X(A \setminus \{x\})$.

2 Examples

2.1 Common Spaces and Operations

Point-Set:

- Finite discrete sets with the discrete topology
- Subspaces of $\mathbb{R}$: $(a, b), (a, b], (a, \infty)$, etc.
  – $\{0\} \bigcup \left\{\frac{1}{n} \;\middle|\; n \in \mathbb{Z}^{\geq 1}\right\}$

- $\mathbb{Q}$
- The topologist's sine curve
- One-point compactifications
- $\mathbb{R}^\omega$
- Hawaiian earring
- Cantor set

Non-Hausdorff spaces:

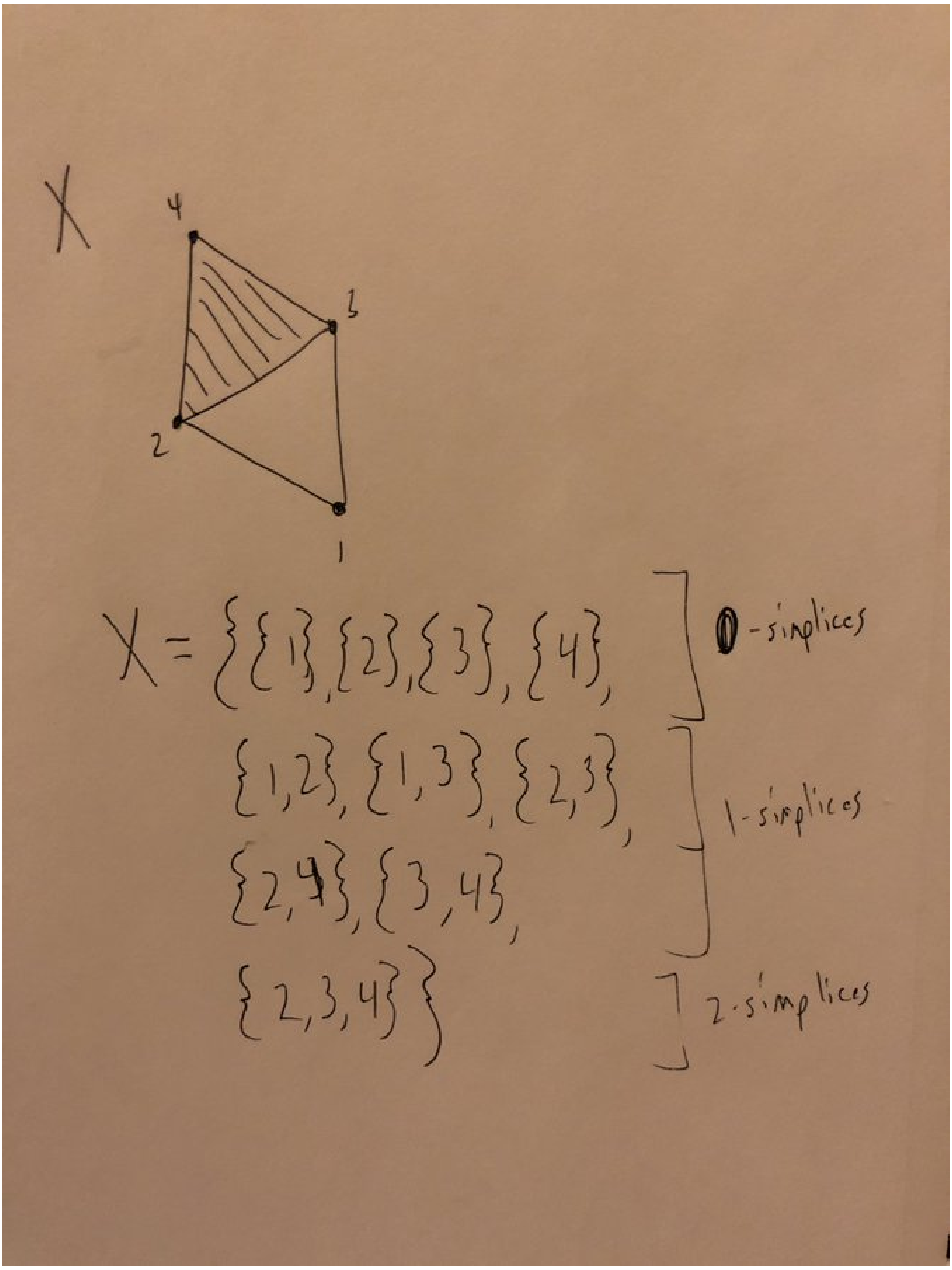
- The cofinite topology on any infinite set.
- $\mathbb{R}/\mathbb{Q}$
- The line with two origins.

General Spaces:

$$S^n, \mathbb{D}^n, T^n, \mathbb{RP}^n, \mathbb{CP}^n, \mathbb{M}, \mathbb{K}, \Sigma_g, \mathbb{RP}^\infty, \mathbb{CP}^\infty.$$

"Constructed" Spaces

- Knot complements in S^3
- Covering spaces (hyperbolic geometry)
- Lens spaces
- Matrix groups
- Prism spaces
- Pair of pants
- Seifert surfaces
- Surgery
- Simplicial Complexes
  - Nice minimal example:



Exotic/Pathological Spaces

- $\mathbb{HP}^n$
- Dunce Cap

- Horned sphere

Operations

- Cartesian product $A \times B$
- Wedge product $A \vee B$
- Connect Sum $A \# B$
- Quotienting A/B
- Puncturing $A \setminus \{a_i\}$
- Smash product
- Join
- Cones
- Suspension
- Loop space
- Identifying a finite number of points

2.2 Alternative Topologies

- Discrete
- Cofinite
- Discrete and Indiscrete
- Uniform

The cofinite topology:

- Non-Hausdorff
- Compact

The discrete topology:

- Discrete iff points are open
- Always Hausdorff
- Compact iff finite
- Totally disconnected
- If the domain, every map is continuous

The indiscrete topology:

- Only open sets are $\emptyset, X$
- Non-Hausdorff
- If the codomain, every map is continuous
- Compact

3 Theorems

3.1 Point-Set

- Closed subsets of Hausdorff spaces are compact? (check)

- Cantor's intersection theorem?

- Tube lemma

- Properties pushed forward through continuous maps:
  - Compactness?
  - Connectedness (when surjective)
  - Separability
  - Density **only when** f is surjective
  - **Not** openness
  - **Not** closedness
- A retract of a Hausdorff/connected/compact space is closed/connected/compact respectively.

Proposition 3.1.
A continuous function on a compact set is uniformly continuous.

Proof .
Take $\left\{ B_{\frac{\varepsilon}{2}}(y) \,\middle|\, y \in Y \right\} \rightrightarrows Y$, pull back to an open cover of X, has Lebesgue number $\delta_L > 0$, then $x' \in B_{\delta_L}(x) \implies f(x), f(x') \in B_{\frac{\varepsilon}{2}}(y)$ for some y. ∎

- Lipschitz continuity implies uniform continuity (take $\delta = \varepsilon / C$)
  - Counterexample to converse: $f(x) = \sqrt{x}$ on $[0,1]$ has unbounded derivative.
- Extreme Value Theorem: for $f : X \longrightarrow Y$ continuous with X compact and Y ordered in the order topology, there exist points $c, d \in X$ such that $f(x) \in [f(c), f(d)]$ for every x.

Theorem 3.2.
Points are closed in T_1 spaces.

Theorem 3.3.
A metric space X is sequentially compact iff it is complete and totally bounded.

Theorem 3.4.
A metric space is totally bounded iff every sequence has a Cauchy subsequence.

Theorem 3.5.
A metric space is compact iff it is complete and totally bounded.

Theorem 3.6 ***(Baire)*****.**
If X is a complete metric space, then the intersection of countably many dense open sets is dense in X.

Theorem 3.7.
A continuous bijective open map is a homeomorphism.

Theorem 3.8.
A closed subset A of a compact set B is compact.

Proof .

- Let $\{A_i\} \rightrightarrows A$ be a covering of A by sets open in A.
- Each $A_i = B_i \bigcap A$ for some B_i open in B (definition of subspace topology)
- Define $V = \{B_i\}$, then $V \rightrightarrows A$ is an open cover.
- Since A is closed, $W := B \setminus A$ is open
- Then $V \bigcup W$ is an open cover of B, and has a finite subcover $\{V_i\}$
- Then $\left\{V_i \bigcap A\right\}$ is a finite open cover of A.

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Theorem 3.9.
The continuous image of a compact set is compact.

Theorem 3.10.
A closed subset of a Hausdorff space is compact.

3.2 Algebraic

Todo: Merge the two van Kampen theorems.

Theorem 3.11 ***(Van Kampen).***
The pushout is the northwest colimit of the following diagram

$$\begin{tikzcd} A\coprod_Z B & A \arrow[l] \\ B & Z \arrow[u, leftrightarrow, "\iota_A"'] \arrow[l, leftrightarrow, "\iota_B"] \end{tikzcd}$$

For groups, the pushout is given by the amalgamated free product: if $A = \left\langle G_A \;\middle|\; R_A \right\rangle, B = \left\langle G_B \;\middle|\; R_B \right\rangle$, then $A *_Z B = \left\langle G_A, G_B \;\middle|\; R_A, R_B, T \right\rangle$ where T is a set of relations given by $T = \left\{\iota_A(z)\iota_B(z)^{-1} \;\middle|\; z \in Z\right\}$.
Example: $A = \mathbb{Z}/4\mathbb{Z} = \left\langle x \;\middle|\; x^4 \right\rangle, B = \mathbb{Z}/6\mathbb{Z} = \left\langle y \;\middle|\; x^6 \right\rangle, Z = \mathbb{Z}/2\mathbb{Z} = \left\langle z \;\middle|\; z^2 \right\rangle$. Then we can identify Z as a subgroup of A, B using $\iota_A(z) = x^2$ and $\iota_B(z) = y^3$. So

$$A *_Z B = \left\langle x, y \;\middle|\; x^4, y^6, x^2 y^{-3} \right\rangle$$

.
Suppose $X = U_1 \bigcup U_2$ such that $U_1 \bigcap U_2 \neq \emptyset$ is path connected. Then taking $x_0 \in U := U_1 \bigcap U_2$ yields a pushout of fundamental groups

$$\pi_1(X; x_0) = \pi_1(U_1; x_0) *_{\pi_1(U; x_0)} \pi_1(U_2; x_0).$$

Theorem 3.12 ***(Van Kampen).***
If $X = U \bigcup V$ where $U, V, U \bigcap V$ are all path-connected then

$$\pi_1(X) = \pi_1 U *_{\pi_1(U \bigcap V)} \pi_1 V,$$

where the amalgamated product can be computed as follows: If we have presentations

$$\pi_1(U, w) = \left\langle u_1, \cdots, u_k \,\middle|\, \alpha_1, \cdots, \alpha_l \right\rangle$$
$$\pi_1(V, w) = \left\langle v_1, \cdots, v_m \,\middle|\, \beta_1, \cdots, \beta_n \right\rangle$$
$$\pi_1(U \cap V, w) = \left\langle w_1, \cdots, w_p \,\middle|\, \gamma_1, \cdots, \gamma_q \right\rangle$$

then

$$\begin{aligned}\pi_1(X, w) &= \langle u_1, \cdots, u_k, v_1, \cdots, v_m \rangle \\ &\qquad \text{mod } \left\langle \alpha_1, \cdots, \alpha_l, \beta_1, \cdots, \beta_n, I\left(w_1\right) J\left(w_1\right)^{-1}, \cdots, I\left(w_p\right) J\left(w_p\right)^{-1} \right\rangle \\ &= \frac{\pi_1(U) * \pi_1(B)}{\left\langle \left\{ I(w_i) J(w_i)^{-1} \,\middle|\, 1 \leq i \leq p \right\} \right\rangle}\end{aligned}$$

where

$$I : \pi_1(U \cap V, w) \to \pi_1(U, w)$$
$$J : \pi_1(U \cap V, w) \to \pi_1(V, w).$$