Real Analysis Qualifying Exam Review

D. Zack Garza

Table of Contents

Contents

Exercises from Folland:

- Chapter 1: Exercises 3, 7, 10, 12, 14 (with the sets in 3(a) being non-empty) Exercises 15, 17, 18, 19, 22(a), 24, 28 Exercises 26, 30 (also check out 31)

- Chapter 2: Exercises 2, 3, 7, 9 (in 9(c) you can use Exercise 1.29 without proof Exercises 10, 12, 13, 14, 16, 19 Exercises 24, 25, 28(a,b), 33, 34, 35, 38, 41 (note that 24 shows that upper sums are not needed in the definition of integrals, and the extra hypotheses also show that they are not desired either) Exercises 40, 44, 47, 49, 50, 51, 52, 54, 56, 58, 59

- Chapter 3: Exercises 3(b,c), 5, 6, 9, 12, 13, 14, 16, 20, 21, 22

1 | Basics

Notation	Definition
$\prod f \prod_\infty \coloneqq \sup_{x \in \operatorname{dom} f} \bigcup f x \bigcup$	The Sup norm
$\prod f \prod_{L^\infty} \coloneqq \inf \ M \geq 0 \bigcup \bigcup f x \bigcup \leq M \text{ for a.e. } x$	The L^∞ norm
$f_n \overset{n\to\infty}{\to} f$	Convergence of a sequence
$f x \overset{\bigcup x \bigcup \to\infty}{\to} 0$	Vanishing at infinity
$\mathcal{R}_{\bigcup x \bigcup \geq N} f \overset{N\to\infty}{\to} 0$	Having small tails

Notation	Definition
$H, \mathcal{H}$	A Hilbert space
X	A topological space

1.1 Useful Techniques

- General advice: try swapping the orders of limits, sums, integrals, etc.
- Limits:
  - Take the $\limsup$ or $\liminf$, which always exist, and aim for an inequality like
  $$c \leq \liminf a_n \leq \limsup a_n \leq c.$$
  - $\lim f_n = \limsup f_n = \liminf f_n$ iff the limit exists, so to show some g is a limit, show
  $$\limsup f_n \leq g \leq \liminf f_n \qquad \implies g = \lim f.$$
  - A limit does *not* exist if $\liminf a_n > \limsup a_n$.
- Sequences and Series
  - If f_n has a global maximum (computed using f_n' and the first derivative test) $M_n \to 0$, then $f_n \to 0$ uniformly.
  - For a fixed x, if $f = \mathcal{P} f_n$ converges *uniformly* on some $B_r x$ and each f_n is continuous at x, then f is also continuous at x .
- Equalities
  - Split into upper and lower bounds:
  $$a = b \iff a \leq b \text{ and } a \geq b.$$
  - Use an epsilon of room:
  $$\forall \epsilon, \ a < b + \varepsilon \implies a \leq b.$$

- Showing something is zero:

$$\forall\epsilon, \prod a \prod < \varepsilon \implies a = 0.$$

- Continuity / differentiability: show it holds on $\left(-M, M\right]$ for all M to get it to hold on $\mathbb{R}$.

- Simplifications:

  - To show something for a measurable set, show it for bounded/compact/elementary sets/
  - To show something for a function, show it for continuous, bounded, compactly supported, simple, indicator functions, L^1, etc
  - Replace a continuous sequence ($\varepsilon \to 0$) with an arbitrary countable sequence ($x_n \to 0$)
  - Intersect with a ball $B_r\mathbf{0} \subset \mathbb{R}^n$.

- Integrals

  - Calculus techniques: Taylor series, IVT, MVT, etc.
  - Break up $\mathbb{R}^n = \bigcup x \bigcup \leq 1 \mathcal{N} \bigcup x \bigcup > 1$.
    - ◇ Or break integration region into disjoint annuli.
  - Break up into $f > g \mathcal{N} f = g \mathcal{N} f < g$.
  - Tail estimates!
  - Most of what works for integrals will work for sums.

- Measure theory:

  - Always consider bounded sets, and if E is unbounded write $E = \cup_n B_n 0 \cap E$ and use countable subadditivity or continuity of measure.

  - F_σ sets are Borel, so establish something for Borel sets and use this to extend it to Lebesgue.

  - $s = \inf x \in X \implies$ for every ε there is an $x \in X$ such that $x \leq s + \varepsilon$.

- Approximate by dense subsets of functions

- Useful facts about compactly supported ($C_c\mathbb{R}$) continuous functions:

  - Uniformly continuous
  - Bounded almost everywhere

1.2 Definitions

Definition 1.2.1 (Uniform Continuity)
f is uniformly continuous iff

$$\begin{aligned}
&\forall\varepsilon \quad \exists\delta\varepsilon \bigcup \quad \forall x, y, \quad \bigcup x - y \bigcup < \delta \implies \bigcup f x - f y \bigcup < \varepsilon \\
\iff &\forall\varepsilon \quad \exists\delta\varepsilon \bigcup \quad \forall x, y, \quad \bigcup y \bigcup < \delta \implies \bigcup f x - y - f y \bigcup < \varepsilon.
\end{aligned}$$

Definition 1.2.2 (Nowhere Dense Sets)
A set S is **nowhere dense** iff the closure of S has empty interior iff every interval contains a subinterval that does not intersect S.

Definition 1.2.3 (Meager Sets)
A set is **meager** if it is a *countable* union of nowhere dense sets.

Definition 1.2.4 (F_σ and G_δ Sets)
An F_σ set is a union of closed sets, and a G_δ set is an intersection of opens. [a]

[a]Mnemonic: "F" stands for *ferme*, which is "closed" in French, and σ corresponds to a "sum", i.e. a union.

Definition 1.2.5 (Limsup/Liminf)

$$\begin{aligned}
\limsup_n a_n &= \lim_{n\to\infty} \sup_{j\geq n} a_j = \inf_{n\geq 0} \sup_{j\geq n} a_j \\
\liminf_n a_n &= \lim_{n\to\infty} \inf_{j\geq n} a_j = \sup_{n\geq 0} \inf_{j\geq n} a_j.
\end{aligned}$$

Definition 1.2.6 (Topological Notions)
Let X be a metric space and A a subset. Let A' denote the limit points of A, and $\overline{A} \coloneqq A \cup A'$ to be its closure.

- A **neighborhood** of p is an open set U_p containing p.
- An ε-**neighborhood** of p is an open ball $B_r p \coloneqq q \bigcup dp, q < r$ for some $r > 0$.
- A point $p \in X$ is an **accumulation point** of A iff every neighborhood U_p of p contains a point $q \in Q$
- A point $p \in X$ is a **limit point** of A iff every *punctured* neighborhood $U_p \setminus p$ contains a point $q \in A$.
- If $p \in A$ and p is not a limit point of A, then p is an **isolated point** of A.
- A is **closed** iff $A' \subset A$, so A contains all of its limit points.
- A point $p \in A$ is **interior** iff there is a neighborhood $U_p \subset A$ that is strictly contained in A.

- A is **open** iff every point of A is interior.
- A is **perfect** iff A is closed and $A \subset A'$, so every point of A is a limit point of A.
- A is **bounded** iff there is a real number M and a point $q \in X$ such that $dp, q < M$ for all $p \in A$.
- A is **dense** in X iff every point $x \in X$ is either a point of A, so $x \in A$, or a limit point of A, so $x \in A'$. I.e., $X \subset A \cup A'$.
  - Alternatively, $\overline{A} = X$, so the closure of A is X.

Definition 1.2.7 (Uniform Convergence)

$$\forall \varepsilon > 0 \ \exists n_0 = n_0 \varepsilon \ \forall x \in S \ \forall n > n_0 \ \bigcup f_n x - f x \bigcup < \varepsilon \ .$$

Negated:[a]

$$\exists \varepsilon > 0 \ \forall n_0 = n_0 \varepsilon \ \exists x = x n_0 \in S \ \exists n > n_0 \ \bigcup f_n x - f x \bigcup \geq \varepsilon \ .$$

[a] Slogan: to negate, find a bad x depending on n_0 that are larger than some ε.

Definition 1.2.8 (Pointwise Convergence)
A sequence of functions f_j is said to **converge pointwise** to f if and only if

$$\forall \varepsilon > 0 \forall x \in S \ \exists n_0 = n_0 x, \varepsilon \ \forall n > n_0 \ \bigcup f_n x - f x \bigcup < \varepsilon \ .$$

Definition 1.2.9 (Outer Measure)
The **outer measure** of a set is given by

$$m_* E \coloneqq \inf_{\substack{Q_i \rightrightarrows E \\ \text{closed cubes}}} \mathcal{P} \bigcup Q_i \bigcup.$$

Definition 1.2.10 (Limsup and Liminf of Sets)

$$\begin{aligned} \limsup_n A_n &\coloneqq \cap_n \cup_{j \geq n} A_j = x \bigcup x \in A_n \text{ for inf. many } n \\ \liminf_n A_n &\coloneqq \cup_n \cap_{j \geq n} A_j = x \bigcup x \in A_n \text{ for all except fin. many } n \end{aligned}$$

.

Definition 1.2.11 (Lebesgue Measurable Sets)
A subset $E \subseteq \mathbb{R}^n$ is **Lebesgue measurable** iff for every $\varepsilon > 0$ there exists an open set $O \supseteq E$ such that $m_* O \setminus E < \varepsilon$. In this case, we define $mE \coloneqq m_* E$.

Definition 1.2.12 (L^+: Measurable non-negative functions.)
$f \in L^+$ iff f is measurable and non-negative.

Definition 1.2.13 (Integrability)
A measurable function is **integrable** iff $\prod f \prod_1 < \infty$.

Definition 1.2.14 (The Infinity Norm)

$$\prod f \prod_\infty \coloneqq \inf_{\alpha \geq 0} \alpha \bigcup m \bigcup f \bigcup \geq \alpha = 0 \; .$$

Definition 1.2.15 (Essentially Bounded Functions)
A function $f : X \to \mathbb{C}$ is **essentially bounded** iff there exists a real number c such that $\mu \bigcup f \bigcup > x = 0$, i.e. $\prod f \prod_\infty < \infty$.

Definition 1.2.16 (L^∞)

$$L^\infty X \coloneqq f : X \to \mathbb{C} \bigcup f \text{ is essentially bounded } \coloneqq f : X \to \mathbb{C} \bigcup \prod f \prod_\infty < \infty \; .$$

Definition 1.2.17 (Dual Norm)
For X a normed vector space and $\Lambda \in X^\vee$,

$$\prod \Lambda \prod_{X^\vee} \coloneqq \sup_{x \in X \bigcup \prod x \prod_X \leq 1} \bigcup f x \bigcup .$$

Definition 1.2.18 (Convolution)

$$f * g x = \mathcal{R} \, f x - y g y dy.$$

Definition 1.2.19 (Fourier Transform)

$$f \xi = \mathcal{R} \, f x \; e^{2\pi i x \cdot \xi} \; dx.$$

Definition 1.2.20 (Dilation)

$$\varphi_t x = t^{-n} \varphi \; t^{-1} x \; .$$

Definition 1.2.21 (Approximations to the identity)
For $\varphi \in L^1$, the dilations satisfy $\mathcal{R} \, \varphi_t = \mathcal{R} \, \varphi$, and if $\mathcal{R} \, \varphi = 1$ then φ is an **approximate identity**.

Definition 1.2.22 (Baire Space)
A space X is a **Baire space** if and only if every countable intersections of open, dense sets is still dense.

1.2.1 Functional Analysis

Definition 1.2.23 (Orthonormal sequence)
A countable collection of elements u_i is **orthonormal** if and only if

1. $\coprod u_i,\ u_j \frown = 0$ for all $j \neq k$ and
2. $\prod u_j \prod^2 \coloneqq \coprod u_j,\ u_j \frown = 1$ for all j.

Definition 1.2.24 (Basis of a Hilbert space)
A set u_n is a **basis** for a Hilbert space $\mathcal{H}$ iff it is dense in $\mathcal{H}$.

Definition 1.2.25 (Completeness of a Hilbert space)
A collection of vectors $u_n \subset H$ is **complete** iff $\coprod x,\ u_n \frown = 0$ for all $n \iff x = 0$ in H.

Definition 1.2.26 (Dual of a Hilbert space)
The **dual** of a Hilbert space H is defined as

$$H^\vee \coloneqq L : H \to \mathbb{C} \bigcup L \text{ is continuous } .$$

Definition 1.2.27 (Linear functionals)
A map $L : X \to \mathbb{C}$ is a **linear functional** iff

$$L\alpha\mathbf{x} + \mathbf{y} = \alpha L\mathbf{x} + L\mathbf{y}..$$

Definition 1.2.28 (Operator norm)
The **operator norm** of an operator L is defined as

$$\prod L \prod_{X^\vee} \coloneqq \sup_{\substack{x \in X \\ \prod x \prod = 1}} \bigcup Lx \bigcup .$$

Definition 1.2.29 (Banach Space)
A space is a **Banach space** if and only if it is a complete normed vector space.

Definition 1.2.30 (Hilbert Space)
A **Hilbert space** is an inner product space which is a Banach space under the induced norm.

1.3 Theorems

1.3.1 Topology / Sets

Theorem 1.3.1 *(Heine-Cantor).*
Every continuous function on a compact space is uniformly continuous.

Proposition 1.3.2 *(Compact if and only if sequentially compact for metric spaces).*
Metric spaces are compact iff they are sequentially compact, (i.e. every sequence has a convergent subsequence).

Proposition 1.3.3 *(A unit ball that is not compact).*
The unit ball in $C\left(\left[0,1\right]\right)$ with the sup norm is not compact.

Proof (?).
Take $f_k(x) = x^n$, which converges to $\chi(x) = 1$. The limit is not continuous, so no subsequence can converge. ∎

Proposition 1.3.4 *(?).*
A *finite* union of nowhere dense is again nowhere dense.

Proposition 1.3.5 *(Convergent Sums Have Small Tails).*

$$\sum a_n < \infty \implies a_n \to 0 \quad \text{and} \quad \sum_{k=N}^{\infty} a_n \overset{N\to\infty}{\to} 0$$

Theorem 1.3.6 *(Heine-Borel).*
$X \subseteq \mathbb{R}^n$ is compact $\iff$ X is closed and bounded.

Proposition 1.3.7 *(Geometric Series).*

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \iff |x| < 1.$$

Corollary 1.3.8 *(?).*

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = 1.$$

Proposition 1.3.9 *(?).*
The Cantor set is closed with empty interior.

Proof (?).
Its complement is a union of open intervals, and can't contain an interval since intervals have positive measure and mC_n tends to zero.

■

Corollary 1.3.10 ***(?).***
The Cantor set is nowhere dense.

Proposition 1.3.11 ***(?).***
Singleton sets in $\mathbb{R}$ are closed, and thus $\mathbb{Q}$ is an F_σ set.

Theorem 1.3.12 ***(Baire).***
$\mathbb{R}$ is a **Baire space** Thus $\mathbb{R}$ can not be written as a countable union of nowhere dense sets.

Lemma 1.3.13 ***(?).***
Any nonempty set which is bounded from above (resp. below) has a well-defined supremum (resp. infimum).

1.3.2 Functions

Proposition 1.3.14 ***(Existence of Smooth Compactly Supported Functions).***
There exist smooth compactly supported functions, e.g. take

$$f x = e^{-\frac{1}{x^2}} \chi_{0,\infty} x.$$

Lemma 1.3.15 ***(Function discontinuous on the rationals).***
There is a function discontinuous precisely on $\mathbb{Q}$.

Proof (?).
$f x = \dfrac{1}{n}$ if $x = r_n \in \mathbb{Q}$ is an enumeration of the rationals, and zero otherwise. The limit at every point is 0.

■

Proposition 1.3.16 ***(No functions discontinuous on the irrationals).***
There *do not* exist functions that are discontinuous precisely on $\mathbb{R} \setminus \mathbb{Q}$.

Proof (?).
D_f is always an F_σ set, which follows by considering the oscillation ω_f. $\omega_f x = 0 \iff f$ is continuous at x, and $D_f = \cup_n A_{\frac{1}{n}}$ where $A_\varepsilon = \omega_f \geq \varepsilon$ is closed.

■

Proposition 1.3.17 ***(Lipschitz $\iff$ differentiable with bounded derivative.).***
A function $f : a, b \to \mathbb{R}$ is Lipschitz $\iff$ f is differentiable and f' is bounded. In this case, $\bigcup f' x \bigcup \leq C$, the Lipschitz constant.

1.4 Uniform Convergence

Proposition 1.4.1 ***(Testing Uniform Convergence: The Sup Norm Test).***
$f_n \to f$ uniformly iff there exists an M_n such that $\prod f_n - f \prod_\infty \leq M_n \to 0$.

Remark 1.4.2 ***(Negating the Sup Norm test):*** **Negating**: find an x which depends on n for which $\prod f_n \prod_\infty > \varepsilon$ (negating small tails) or $\prod f_n - f_m \prod > \varepsilon$ (negating the Cauchy criterion).

1.4.1 Example: Completeness of a Normed Function Space

:::{.proposition title=" CI is complete"} The space $X = C\left(0, 1\right]$, continuous functions $f : \left(0, 1\right] \to \mathbb{R}$, equipped with the norm

$$\prod f \prod_\infty \coloneqq \sup_{x \in \left(0,1\right]} \bigcup f x \bigcup$$

is a **complete** metric space. :::

Proof .

1. Let f_k be Cauchy in X.
2. Define a candidate limit using pointwise convergence:

   Fix an x; since

   $$\bigcup f_k x - f_j x \bigcup \leq \prod f_k - f_k \prod \to 0$$

   the sequence $f_k x$ is Cauchy in $\mathbb{R}$. So define $f x \coloneqq \lim_k f_k x$.
3. Show that $\prod f_k - f \prod \to 0$:

   $$\bigcup f_k x - f_j x \bigcup < \varepsilon \ \forall x \implies \lim_j \bigcup f_k x - f_j x \bigcup < \varepsilon \ \forall x$$

   Alternatively, $\prod f_k - f \prod \leq \prod f_k - f_N \prod + \prod f_N - f_j \prod$, where N, j can be chosen large enough to bound each term by $\varepsilon \Uparrow 2$.
4. Show that $f \in X$:

   The uniform limit of continuous functions is continuous.

■

Remark 1.4.3: In other cases, you may need to show the limit is bounded, or has bounded derivative, or whatever other conditions define X.

Theorem 1.4.4 ***(Uniform Limit Theorem).***
If $f_n \to f$ pointwise and uniformly with each f_n continuous, then f is continuous. [a]

[a] Slogan: a uniform limit of continuous functions is continuous.

Proof .

- Follows from an $\varepsilon_{\Uparrow}3$ argument:

$$\bigcup Fx - Fy \bigcup \leq \bigcup Fx - F_N x \bigcup + \bigcup F_N x - F_N y \bigcup + \bigcup F_N y - Fy \bigcup \leq \varepsilon \to 0.$$

  - The first and last $\varepsilon_{\Uparrow}3$ come from uniform convergence of $F_N \to F$.
  - The middle $\varepsilon_{\Uparrow}3$ comes from continuity of each F_N.

- So just need to choose N large enough and δ small enough to make all 3 ε bounds hold.

∎

Proposition 1.4.5 ***(Uniform Limits Commute with Integrals).***
If $f_n \to f$ uniformly, then $\mathcal{R} f_n = \mathcal{R} f$.

1.4.2 Series

Proposition 1.4.6 ***(p-tests).***
Let n be a fixed dimension and set $B = x \in \mathbb{R}^n \bigcup \prod x \prod \leq 1$.

$$\begin{aligned}
\mathcal{P} \frac{1}{n^p} < \infty &\iff p > 1 \\
\mathcal{R}_{\varepsilon}^{\infty} \frac{1}{x^p} < \infty &\iff p > 1 \\
\mathcal{R}_0^1 \frac{1}{x^p} < \infty &\iff p < 1 \\
\mathcal{R}_B \frac{1}{\bigcup x \bigcup^p} < \infty &\iff p < n \\
\mathcal{R}_{B^c} \frac{1}{\bigcup x \bigcup^p} < \infty &\iff p > n
\end{aligned}$$

.

Proposition 1.4.7 ***(Comparison Test).***
If $0 \leq a_n \leq b_n$, then

- $\mathcal{P} b_n < \infty \implies \mathcal{P} a_n < \infty$, and
- $\mathcal{P} a_n = \infty \implies \mathcal{P} b_n = \infty$.

Proposition 1.4.8 *(Small Tails for Series of Functions).*
If $\mathcal{P} f_n$ converges then $f_n \to 0$ uniformly.

Corollary 1.4.9 *(Term by Term Continuity Theorem).*
If f_n are continuous and $\mathcal{P} f_n \to f$ converges uniformly, then f is continuous.

Proposition 1.4.10 *(Weak M-Test).*
If $f_n x \leq M_n$ **for a fixed** x where $\mathcal{P} M_n < \infty$, then the series $f x = \mathcal{P} f_n x$ converges.[a]

[a] Note that this is only pointwise convergence of f, whereas the full M-test gives uniform convergence.

Proposition 1.4.11 *(The Weierstrass M-Test).*
If $\sup_{x \in A} \bigcup f_n x \bigcup \leq M_n$ for each n where $\mathcal{P} M_n < \infty$, then $\mathcal{P}_{n=1}^{\infty} f_n x$ converges uniformly and absolutely on A. [a] Conversely, if $\mathcal{P} f_n$ converges uniformly on A then $\sup_{x \in A} \bigcup f_n x \bigcup \to 0$.

[a] It suffices to show $\bigcup f_n x \bigcup \leq M_n$ for some M_n not depending on x.

Proposition 1.4.12 *(Cauchy criterion for sums).*
f_n are uniformly Cauchy (so $\prod f_n - f_m \prod_\infty < \varepsilon$) iff f_n is uniformly convergent.

Derivatives

Theorem 1.4.13 *(Term by Term Differentiability Theorem).*
If f_n are differentiable, $\mathcal{P} f_n' \to g$ uniformly, and there exists one point[a] x_0 such that $\mathcal{P} f_n x$ converges, then there exist an f such that $\mathcal{P} f_n \to f$ uniformly and $f' = g$.[b]

[a] So this implicitly holds if f is the pointwise limit of f_n.
[b] See Abbott theorem 6.4.3, pp 168.

1.5 Commuting Limiting Operations

Proposition 1.5.1 *(Limits of bounded functions need not be bounded).*

$$\lim_{n \to \infty} \sup_{x \in X} \bigcup f_n x \bigcup \neq \sup_{x \in X} \bigcup \lim_{n \to \infty} f_n x \bigcup.$$

Proposition 1.5.2 *(Limits of continuous functions need not be continuous).*

$$\lim_{k \to \infty} \lim_{n \to \infty} f_n x_k \neq \lim_{n \to \infty} \lim_{k \to \infty} f_n x_k.$$

Proposition 1.5.3 ***(Limits of differentiable functions need not be differentiable).***

$$\lim_{n\to\infty} \frac{\partial}{\partial x} f_n \neq \frac{\partial}{\partial n} \lim_{n\to\infty} f_n \,.$$

Proposition 1.5.4 ***(?).***

$$\lim_{n\to\infty} \mathcal{R}_a^b\, f_n x\, dx \neq \mathcal{R}_a^b \lim_{n\to\infty} f_n x \;\, dx.$$

1.6 Slightly Advanced Stuff

Theorem 1.6.1 ***(Weierstrass Approximation).***
If $\left(a,b\right] \subset \mathbb{R}$ is a closed interval and f is continuous, then for every $\varepsilon > 0$ there exists a polynomial p_ε such that $\prod f - p_\varepsilon \prod_{L^\infty \left(a,b\right]} \overset{\varepsilon\to 0}{\to} 0$.
Equivalently, polynomials are dense in the Banach space $C\left(0,1\right], \prod \cdot \prod_\infty$.

Theorem 1.6.2 ***(Egorov).***
Let $E \subseteq \mathbb{R}^n$ be measurable with $mE > 0$ and $f_k : E \to \mathbb{R}$ be measurable functions such that

$$f x \coloneqq \lim_{k\to\infty} f_k x < \infty$$

exists almost everywhere.
Then $f_k \to f$ *almost uniformly*, i.e.

$$\forall \varepsilon > 0,\ \exists F \subseteq E \text{ closed such that } mE \setminus F < \varepsilon \quad \text{and} \quad f_k \to f \text{ uniformly on } F.$$

1.7 Examples

Example 1.7.1 ***(?):*** A series of continuous functions that does *not* converge uniformly but is still continuous:

$$g x \coloneqq \mathcal{P} \frac{1}{1 + n^2 x}.$$

Take $x = 1_{\Uparrow} n^2$.

2 Measure Theory

2.1 Theorems

Proposition 2.1.1 ***(Opens are unions of almost disjoint intervals.).***
Every open subset of $\mathbb{R}$ (resp $\mathbb{R}^n$) can be written as a unique countable union of disjoint (resp. almost disjoint) intervals (resp. cubes).

Proposition 2.1.2 ***(Properties of Outer Measure).***

1. Monotonicity: $E \subseteq F \implies m_*E \leq m_*F$.
2. Countable Subadditivity: $m_* \cup E_i \leq \mathcal{P}\, m_* E_i$.
3. Approximation: For all E there exists a $G \supseteq E$ such that $m_*G \leq m_*E + \varepsilon$.
4. Disjoint[a] Additivity: $m_* A \mathcal{N} B = m_* A + m_* B$.

[a] This holds for outer measure **iff** $\operatorname{dist} A, B > 0$.

Proposition 2.1.3 ***(Subtraction of Measures).***

$$mA = mB + mC \quad \text{and} \quad mC < \infty \implies mA - mC = mB.$$

Proposition 2.1.4 ***(Continuity of Measure).***

$$E_i \nearrow E \implies mE_i \to mE$$
$$mE_1 < \infty \text{ and } E_i \searrow E \implies mE_i \to mE.$$

Proof (of continuity of measure).

1. Break into disjoint annuli $A_2 = E_2 \setminus E_1$, etc then apply countable disjoint additivity to $E = \mathcal{N} A_i$.
2. Use $E_1 = \mathcal{N} E_j \setminus E_{j+1} \mathcal{N} \cap E_j$, taking measures yields a telescoping sum,and use countable disjoint additivity.

■

Theorem 2.1.5 ***(Measurable sets can be approximated by open/closed/compact sets.).***
Suppose E is measurable; then for every $\varepsilon > 0$,

1. There exists an open $O \supset E$ with $mO \setminus E < \varepsilon$
2. There exists a closed $F \subset E$ with $mE \setminus F < \varepsilon$

3. There exists a compact $K \subset E$ with $mE \setminus K < \varepsilon$.

Proof (that measurable sets can be approximated).

- (1): Take $Q_i \rightrightarrows E$ and set $O = \cup Q_i$.
- (2): Since E^c is measurable, produce $O \supset E^c$ with $mO \setminus E^c < \varepsilon$.
  - Set $F = O^c$, so F is closed.
  - Then $F \subset E$ by taking complements of $O \supset E^c$
  - $E \setminus F = O \setminus E^c$ and taking measures yields $mE \setminus F < \varepsilon$
- (3): Pick $F \subset E$ with $mE \setminus F < \varepsilon_{\Uparrow} 2$.
  - Set $K_n = F \cap \mathbb{D}_n$, a ball of radius n about 0.
  - Then $E \setminus K_n \searrow E \setminus F$
  - Since $mE < \infty$, there is an N such that $n \geq N \implies mE \setminus K_n < \varepsilon$.

■

Proposition 2.1.6 ***(Translation and Dilation Invariance).***
Lebesgue measure is translation and dilation invariant.

Proof ((Todo) of translation/dilation invariance).
Obvious for cubes; if $Q_i \rightrightarrows E$ then $Q_i + k \rightrightarrows E + k$, etc.

■

Theorem 2.1.7 ***(Non-measurable sets exist).***
There is a non-measurable set.

Proof (Constructing a non-measurable set).

- Use AOC to choose one representative from every coset of $\mathbb{R}_{\Uparrow}\mathbb{Q}$ on $[0, 1]$, which is countable, and assemble them into a set N
- Enumerate the rationals in $[0, 1]$ as q_j, and define $N_j = N + q_j$. These intersect trivially.
- Define $M \coloneqq \mathcal{N} N_j$, then $[0, 1] \subseteq M \subseteq [-1, 2]$, so the measure must be between 1 and 3.
- By translation invariance, $mN_j = mN$, and disjoint additivity forces $mM = 0$, a contradiction.

■

Proposition 2.1.8 ***(Borel Characterization of Measurable Sets).***
If E is Lebesgue measurable, then $E = H \mathcal{N} N$ where $H \in F_\sigma$ and N is null.

Proof (of Borel characterization).
For every $\frac{1}{n}$ there exists a closed set $K_n \subset E$ such that $mE \setminus K_n \leq \frac{1}{n}$. Take $K = \cup K_n$, wlog $K_n \nearrow K$ so $mK = \lim mK_n = mE$. Take $N \coloneqq E \setminus K$, then $mN = 0$.

■

Proposition 2.1.9 ***(Limsups/infs of measurable sets are measurable.).***
If A_n are all measurable, $\limsup A_n$ and $\liminf A_n$ are measurable.

Proof (That limsups/infs are measurable).
Measurable sets form a sigma algebra, and these are expressed as countable unions/intersections of measurable sets.

■

Theorem 2.1.10 ***(Borel-Cantelli).***
Let E_k be a countable collection of measurable sets. Then

$$\sum_k mE_k < \infty \implies \text{almost every } x \in \mathbb{R} \text{ is in at most finitely many } E_k.$$

Proof (of Borel-Cantelli).

- If $E = \limsup_j E_j$ with $\sum mE_j < \infty$ then $mE = 0$.
- If E_j are measurable, then $\limsup_j E_j$ is measurable.
- If $\sum_j mE_j < \infty$, then $\sum_{j=N}^{\infty} mE_j \overset{N\to\infty}{\to} 0$ as the tail of a convergent sequence.
- $E = \limsup_j E_j = \cap_{k=1}^{\infty} \cup_{j=k}^{\infty} E_j \implies E \subseteq \cup_{j=k}^{\infty}$ for all k
- $E \subset \cup_{j=k}^{\infty} \implies mE \leq \sum_{j=k}^{\infty} mE_j \overset{k\to\infty}{\to} 0.$

■

Proposition 2.1.11 ***(Extending the class of measurable functions.).*** •
Characteristic functions are measurable
- If f_n are measurable, so are $\bigcup f_n \bigcup, \limsup f_n, \liminf f_n, \lim f_n$,
- Sums and differences of measurable functions are measurable,
- Cones $Fx, y = fx$ are measurable,
- Compositions $f \circ T$ for T a linear transformation are measurable,
- "Convolution-ish" transformations $x, y \mapsto fx - y$ are measurable

Proof (Convolution).
Take the cone on f to get $Fx, y = fx$, then compose F with the linear transformation $T = \left(1, -1; 1, 0 \right]$.

■

3 Integration

3.1 Theorems

Remark 3.1.1: If $f \in L^\infty X$, then f is equal to some bounded function g almost everywhere.

Example 3.1.2 *(?)*: $fx = x\chi_{\mathbb{Q}}x$ is essentially bounded but not bounded.

Theorem 3.1.3 *(p-Test for Integrals)*.

$$\mathcal{R}_0^1 \frac{1}{x^p} < \infty \iff p < 1$$
$$\mathcal{R}_1^\infty \frac{1}{x^p} < \infty \iff p > 1.$$

Slogan 3.1.4
Large powers of x help us in neighborhoods of infinity and hurt around zero

3.1.1 Convergence Theorems

Theorem 3.1.5 *(Monotone Convergence)*.
If $f_n \in L^+$ and $f_n \nearrow f$ almost everywhere, then

$$\lim \mathcal{R} f_n = \mathcal{R} \lim f_n = \mathcal{R} f \quad \text{i.e. } \mathcal{R} f_n \to \mathcal{R} f.$$

Needs to be positive and increasing.

Theorem 3.1.6 *(Dominated Convergence)*.
If $f_n \in L^1$ and $f_n \to f$ almost everywhere with $\bigcup f_n \bigcup \leq g$ for some $g \in L^1$, then $f \in L^1$ and

$$\lim \mathcal{R} f_n = \mathcal{R} \lim f_n = \mathcal{R} f \quad \text{i.e. } \mathcal{R} f_n \to \mathcal{R} f < \infty,$$

and more generally,

$$\mathcal{R} \bigcup f_n - f \bigcup \to 0.$$

Positivity not *needed.*

Theorem 3.1.7 *(Generalized DCT)*.
If

- $f_n \in L^1$ with $f_n \to f$ almost everywhere,
- There exist $g_n \in L^1$ with $\bigcup f_n \bigcup \le g_n$, $g_n \ge 0$.
- $g_n \to g$ almost everywhere with $g \in L^1$, and
- $\lim \mathcal{R}\, g_n = \mathcal{R}\, g$,

then $f \in L^1$ and $\lim \mathcal{R}\, f_n = \mathcal{R}\, f < \infty$.

Note that this is the DCT with $\bigcup f_n \bigcup < \bigcup g \bigcup$ relaxed to $\bigcup f_n \bigcup < g_n \to g \in L^1$.

Proof .
Proceed by showing $\limsup \mathcal{R}\, f_n \le \mathcal{R}\, f \le \liminf \mathcal{R}\, f_n$:

- $\mathcal{R}\, f \ge \limsup \mathcal{R}\, f_n$:

$$
\begin{aligned}
\mathcal{R}\, g - \mathcal{R}\, f &= \mathcal{R}\, g - f \\
&\le \liminf \mathcal{R}\, g_n - f_n \qquad \text{Fatou} \\
&= \lim \mathcal{R}\, g_n + \liminf \mathcal{R} - f_n \\
&= \lim \mathcal{R}\, g_n - \limsup \mathcal{R}\, f_n \\
&= \mathcal{R}\, g - \limsup \mathcal{R}\, f_n
\end{aligned}
$$

$$
\implies \mathcal{R}\, f \ge \limsup \mathcal{R}\, f_n.
$$

  – Here we use $g_n - f_n {}^{n\to\infty} g - f$ with $0 \le \bigcup f_n \bigcup - f_n \le g_n - f_n$, so $g_n - f_n$ are nonnegative (and measurable) and Fatou's lemma applies.

- $\mathcal{R}\, f \le \liminf \mathcal{R}\, f_n$:

$$
\begin{aligned}
\mathcal{R}\, g + \mathcal{R}\, f &= \mathcal{R} g + f \\
&\le \liminf \mathcal{R}\, g_n + f_n \\
&= \lim \mathcal{R}\, g_n + \liminf \mathcal{R}\, f_n \\
&= \mathcal{R}\, g + \liminf f_n
\end{aligned}
$$

$$
\mathcal{R}\, f \le \liminf \mathcal{R}\, f_n.
$$

  – Here we use that $g_n + f_n \to g + f$ with $0 \le \bigcup f_n \bigcup + f_n \le g_n + f_n$ so Fatou's lemma again applies.

■

Proposition 3.1.8 ***(Convergence in L^1 implies convergence of L^1 norm).***
If $f \in L^1$, then

$$
\mathcal{R} \bigcup f_n - f \bigcup \to 0 \iff \mathcal{R} \bigcup f_n \bigcup \to \mathcal{R} \bigcup f \bigcup.
$$

Proof .
Let $g_n = |f_n| - |f_n - f|$, then $g_n \to |f|$ and

$$|g_n| = ||f_n| - |f_n - f|| \leq |f_n - f_n - f| = |f| \in L^1,$$

so the DCT applies to g_n and

$$\begin{aligned}\|f_n - f\|_1 &= \int |f_n - f| + |f_n| - |f_n| = \int |f_n| - g_n \\ &\to_{DCT} \lim \int |f_n| - \int |f|.\end{aligned}$$

■

Theorem 3.1.9 ***(Fatou).***
If f_n is a sequence of nonnegative measurable functions, then

$$\int \liminf_n f_n \leq \liminf_n \int f_n$$
$$\limsup_n \int f_n \leq \int \limsup_n f_n.$$

Theorem 3.1.10 ***(Tonelli (Non-Negative, Measurable)).***
For $f(x, y)$ **non-negative and measurable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is a **measurable** function
- $F(x) = \int f(x, y)\, dy$ is a **measurable** function,
- For E measurable, the slices $E_x \coloneqq \{y \mid (x, y) \in E\}$ are measurable.
- $\int f = \int \int F$, i.e. any iterated integral is equal to the original.

Theorem 3.1.11 ***(Fubini (Integrable)).***
For $f(x, y)$ **integrable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is an **integrable** function
- $F(x) \coloneqq \int f(x, y)\, dy$ is an **integrable** function,
- For E measurable, the slices $E_x \coloneqq \{y \mid (x, y) \in E\}$ are measurable.
- $\int f = \int \int f(x, y)$, i.e. any iterated integral is equal to the original

Theorem 3.1.12 ***(Fubini-Tonelli).***
If any iterated integral is **absolutely integrable**, i.e. $\int \int |f(x, y)| < \infty$, then f is integrable and $\int f$ equals any iterated integral.

Proposition 3.1.13 ***(Measurable Slices).***
Let E be a measurable subset of $\mathbb{R}^n$. Then

- For almost every $x \in \mathbb{R}^{n_1}$, the slice $E_x \coloneqq \{y \in \mathbb{R}^{n_2} \mid (x, y) \in E\}$ is measurable in $\mathbb{R}^{n_2}$.
- The function

$$\begin{aligned} F : \mathbb{R}^{n_1} &\to \mathbb{R} \\ x &\mapsto mE_x = \mathcal{R}_{\mathbb{R}^{n_2}} \chi_{E_x}\, dy \end{aligned}$$

is measurable and

$$mE = \mathcal{R}_{\mathbb{R}^{n_1}} mE_x\, dx = \mathcal{R}_{\mathbb{R}^{n_1}} \mathcal{R}_{\mathbb{R}^{n_2}} \chi_{E_x}\, dy\, dx.$$

Proof .
$\implies$:

- Let f be measurable on $\mathbb{R}^n$.
- Then the cylinders $Fx, y = fx$ and $Gx, y = fy$ are both measurable on $\mathbb{R}^{n+1}$.
- Write $\mathcal{A} = G \leq F \cap G \geq 0$; both are measurable.

$\impliedby$:

- Let A be measurable in $\mathbb{R}^{n+1}$.
- Define $A_x = y \in \mathbb{R} \bigcup x, y \in \mathcal{A}$, then $mA_x = fx$.
- By the corollary, A_x is measurable set, $x \mapsto A_x$ is a measurable function, and $mA = \mathcal{R} fx\, dx$.
- Then explicitly, $fx = \chi_A$, which makes f a measurable function.

■

Proposition 3.1.14 ***(Differentiating Under an Integral).***
If $\bigcup \frac{\partial}{\partial t} fx, t \bigcup \leq gx \in L^1$, then letting $Ft = \mathcal{R} fx, t\, dt$,

$$\begin{aligned} \frac{\partial}{\partial t} Ft &\coloneqq \lim_{h \to 0} \mathcal{R} \frac{fx, t + h - fx, t}{h} dx \\ &\overset{\text{DCT}}{=} \mathcal{R} \frac{\partial}{\partial t} fx, t\, dx. \end{aligned}$$

To justify passing the limit, let $h_k \to 0$ be any sequence and define

$$f_k x, t = \frac{fx, t + h_k - fx, t}{h_k},$$

so $f_k \overset{\text{pointwise}}{\to} \frac{\partial}{\partial t} f$.
Apply the MVT to f_k to get $f_k x, t = f_k \xi, t$ for some $\xi \in \left(0, h_k \right]$, and show that $f_k \xi, t \in L_1$.

Proposition 3.1.15 ***(Commuting Sums with Integrals (non-negative)).***
If f_n are non-negative and $\mathcal{P} \mathcal{R} \bigcup f \bigcup_n < \infty$, then $\mathcal{P} \mathcal{R} f_n = \mathcal{R} \mathcal{P} f_n$.

Proof .
- Idea: MCT.
- Let $F_N = \sum^N f_n$ be a finite partial sum;
- Then there are simple functions $\varphi_n \nearrow f_n$
- So $\sum^N \varphi_n \nearrow F_N$ and MCT applies

■

Theorem 3.1.16 ***(Commuting Sums with Integrals (integrable)).***
If f_n integrable with either $\sum \int |f_n| < \infty$ or $\int \sum |f_n| < \infty$, then

$$\int \sum f_n = \sum \int f_n.$$

Proof .
- By Tonelli, if $f_n x \geq 0$ for all n, taking the counting measure allows interchanging the order of "integration".
- By Fubini on $|f_n|$, if either "iterated integral" is finite then the result follows.

■

Proposition 3.1.17 ***(?).***
If $f_k \in L^1$ and $\sum \|f_k\|_1 < \infty$ then $\sum f_k$ converges almost everywhere and in L^1.

Proof (?).
Define $F_N = \sum^N f_k$ and $F = \lim_N F_N$, then $\|F_N\|_1 \leq \sum^N \|f_k\| < \infty$ so $F \in L^1$ and $\|F_N - F\|_1 \to 0$ so the sum converges in L^1. Almost everywhere convergence: ?

■

3.2 Examples of (Non)Integrable Functions

Example 3.2.1 ***(Examples of integrable functions):***

- $\int \frac{1}{1+x^2} = \arctan x \overset{x\to\infty}{\to} \pi/2 < \infty$
- Any bounded function (or continuous on a compact set, by EVT)
- $\int_0^1 \frac{1}{x} < \infty$
- $\int_0^1 \frac{1}{x^{1-\varepsilon}} < \infty$
- $\int_1^\infty \frac{1}{x^{1+\varepsilon}} < \infty$

Example 3.2.2 ***(Examples of non-integrable functions):***

- $\mathcal{R}_0^1 \frac{1}{x} = \infty$.
- $\mathcal{R}_1^\infty \frac{1}{x} = \infty$.
- $\mathcal{R}_1^\infty \frac{1}{x} = \infty$
- $\mathcal{R}_1^\infty \frac{1}{x^{1-\varepsilon}} = \infty$
- $\mathcal{R}_0^1 \frac{1}{x^{1+\varepsilon}} = \infty$

3.3 L^1 Facts

Proposition 3.3.1 ***(Zero in L^1 iff zero almost everywhere).***
For $f \in L^+$,

$$\mathcal{R} f = 0 \qquad \Longleftrightarrow \qquad f \equiv 0 \text{ almost everywhere.}$$

Proof .

- Obvious for simple functions:
  - If $fx = \mathcal{P}_{j=1}^{n} c_j \chi_{E_j}$, then $\mathcal{R} f = 0$ iff for each j, either $c_j = 0$ or $mE_j = 0$.
  - Since nonzero c_j correspond to sets where $f \neq 0$, this says $m\ \ f \neq 0\ \ = 0$.
- $\Longleftarrow$:
  - If $f = 0$ almost everywhere and $\varphi \nearrow f$, then $\varphi = 0$ almost everywhere since $\varphi x \leq fx$ -Then

$$\mathcal{R} f = \sup_{\varphi \leq f} \mathcal{R} \varphi = \sup_{\varphi \leq f} 0 = 0.$$

- $\Longrightarrow$:
  - Instead show negating "$f = 0$ almost everywhere" implies $\mathcal{R} f \neq 0$.
  - Write $f \neq 0 = \cup_{n \in \mathbb{N}} S_n$ where $S_n \coloneqq x \bigcup fx \geq \frac{1}{n}$.
  - Since "not $f = 0$ almost everywhere", there exists an n such that $mS_n > 0$.
  - Then

$$0 < \frac{1}{n} \chi_{E_n} \leq f \implies 0 < \mathcal{R} \frac{1}{n} \chi_{E_n} \leq \mathcal{R} f.$$

■

Proposition 3.3.2 ***(Translation Invariance).***
The Lebesgue integral is translation invariant, i.e.

$$\mathcal{R} fx\ dx = \mathcal{R} fx + h\ dx \qquad \text{for any} h.$$

Proof .

- Let $E \subseteq X$; for characteristic functions,

$$\mathcal{R}_X\, \chi_E x + h = \mathcal{R}_X\, \chi_{E+h} x = mE + h = mE = \mathcal{R}_X\, \chi_E x$$

  by translation invariance of measure.
- So this also holds for simple functions by linearity.
- For $f \in L^+$, choose $\varphi_n \nearrow f$ so $\mathcal{R}\, \varphi_n \to \mathcal{R}\, f$.
- Similarly, $\tau_h \varphi_n \nearrow \tau_h f$ so $\mathcal{R}\, \tau_h f \to \mathcal{R}\, f$
- Finally $\mathcal{R}\, \tau_h \varphi = \mathcal{R}\, \varphi$ by step 1, and the suprema are equal by uniqueness of limits.

■

Proposition 3.3.3 ***(Integrals distribute over disjoint sets).***
If $X \subseteq A \cup B$, then $\mathcal{R}_X\, f \leq \mathcal{R}_A\, f + \mathcal{R}_{A^c}\, f$ with equality iff $X = A \mathcal{N} B$.

Proposition 3.3.4 ***(Uniformly continuous L^1 functions vanish at infinity.).***
If $f \in L^1$ and f is uniformly continuous, then $f x \overset{\bigcup x \bigcup \to \infty}{\to} 0$.

Warning 3.3.5
This doesn't hold for general L^1 functions, take any train of triangles with height 1 and summable areas.

Theorem 3.3.6 ***(Small Tails in L^1).***
If $f \in L^1$, then for every ε there exists a radius R such that if $A = B_R 0^c$, then $\mathcal{R}_A \bigcup f \bigcup < \varepsilon$.

Proof .

- Approximate with compactly supported functions.
- Take $g \overset{L_1}{\to} f$ with $g \in C_c$
- Then choose N large enough so that $g = 0$ on $E \coloneqq B_N 0$
- Then

$$\mathcal{R}_E \bigcup f \bigcup \leq \mathcal{R}_E \bigcup f - g \bigcup + \mathcal{R}_E \bigcup g \bigcup.$$

■

Proposition 3.3.7 ***(L^1 functions are absolutely continuous.).***
$mE \to 0 \implies \mathcal{R}_E\, f \to 0$.

Proof (?).
Approximate with compactly supported functions. Take $g \overset{L_1}{\to} f$, then $g \leq M$ so $\mathcal{R}_E\, f \leq \mathcal{R}_E\, f - g + \mathcal{R}_E\, g \to 0 + M \cdot mE \to 0$.

■

Proposition 3.3.8 *(L^1 functions are finite almost everywhere.).*
If $f \in L^1$, then $m f x = \infty = 0$.

Proof (?).
Idea: Split up domain Let $A = f x = \infty$, then $\infty > \mathcal{R} f = \mathcal{R}_A f + \mathcal{R}_{A^c} f = \infty \cdot mA + \mathcal{R}_{A^c} f \implies mX = 0$.

■

Theorem 3.3.9 *(Continuity in L^1).*

$$\prod \tau_h f - f \prod_1 \overset{h \to 0}{\to} 0$$

Proof .
Approximate with compactly supported functions. Take $g \overset{L_1}{\to} f$ with $g \in C_c$.

$$\begin{aligned}
\mathcal{R} f x + h - f x &\leq \mathcal{R} f x + h - g x + h + \mathcal{R} g x + h - g x + \mathcal{R} g x - f x \\
&\overset{? \to ?}{\to} 2\varepsilon + \mathcal{R} g x + h - g x \\
&= \mathcal{R}_K g x + h - g x + \mathcal{R}_{K^c} g x + h - g x \\
&\overset{??}{\to} 0,
\end{aligned}$$

which follows because we can enlarge the support of g to K where the integrand is zero on K^c, then apply uniform continuity on K.

■

Proposition 3.3.10 *(Integration by parts, special case).*

$$\begin{aligned}
F x := \mathcal{R}_0^x f y dy \quad &\text{and} \quad G x := \mathcal{R}_0^x g y dy \\
\implies \mathcal{R}_0^1 F x g x dx &= F 1 G 1 - \mathcal{R}_0^1 f x G x dx.
\end{aligned}$$

Proof (?).
Fubini-Tonelli, and sketch region to change integration bounds.

■

Theorem 3.3.11 *(Lebesgue Density).*

$$A_h f x := \frac{1}{2h} \mathcal{R}_{x-h}^{x+h} f y dy \implies \prod A_h f - f \prod \overset{h \to 0}{\to} 0.$$

Proof (?).
Fubini-Tonelli, and sketch region to change integration bounds, and continuity in L^1.

■

3.4 Lp Facts

Proposition 3.4.1 *(Dense subspaces of $L^2 I$).*
The following are dense subspaces of $L^2 \left(0, 1\right]$:

- Simple functions
- Step functions
- $C_0 \left(0, 1\right]$
- Smoothly differentiable functions $C_0^\infty \left(0, 1\right]$
- Smooth compactly supported functions C_c^∞

Theorem 3.4.2 *(?).*

$$mX < \infty \implies \lim_{p \to \infty} \left\lVert f \right\rVert_p = \left\lVert f \right\rVert_\infty .$$

Proof (?).
Let $M = \left\lVert f \right\rVert_\infty$.

- For any $L < M$, let $S = \left| f \right| \geq L$.
- Then $mS > 0$ and

$$\begin{aligned} \left\lVert f \right\rVert_p &= \mathcal{R}_X \left| f \right|^p \frac{1}{p} \\ &\geq \mathcal{R}_S \left| f \right|^p \frac{1}{p} \\ &\geq L \; mS^{\frac{1}{p}} \overset{p \to \infty}{\to} L \\ &\implies \liminf_p \left\lVert f \right\rVert_p \geq M. \end{aligned}$$

We also have

$$\begin{aligned} \left\lVert f \right\rVert_p &= \mathcal{R}_X \left| f \right|^p \frac{1}{p} \\ &\leq \mathcal{R}_X \; M^p \frac{1}{p} \\ &= M \; mX^{\frac{1}{p}} \xrightarrow{p \to \infty} M \\ &\implies \limsup_p \left\lVert f \right\rVert_p \leq M \blacksquare . \end{aligned}$$

■

Theorem 3.4.3 *(Duals for L^p spaces).*
For $1 \leq p < \infty$, $L^{p\vee} \cong L^q$.

Proof ($p = 1$ case).
?

■

todo

Proof ($p = 2$ case).
Use Riesz Representation for Hilbert spaces.

■

Proposition 3.4.4 ***(L^1 is not quite the dual of L^∞.).***
$L^1 \subset L^{\infty\vee}$, since the isometric mapping is always injective, but *never* surjective.

4 | Fourier Transform and Convolution

4.1 The Fourier Transform

Proposition 4.1.1 ***(?).***
If $f = g$ then $f = g$ almost everywhere.

Proposition 4.1.2 ***(Riemann-Lebesgue: Fourier transforms have small tails.).***

$$f \in L^1 \implies f\xi \to 0 \text{ as } \bigcup^{\xi}\bigcup \to \infty,$$

if $f \in L^1$, then f is continuous and bounded.

Proof (?).

- Boundedness:

$$\bigcup f\xi \bigcup \leq \mathcal{R} \bigcup f \bigcup \cdot \bigcup e^{2\pi i x \cdot \xi} \bigcup = \prod f \prod_1 .$$

- Continuity:
- $\bigcup f\xi_n - f\xi \bigcup$
- Apply DCT to show $a \overset{n\to\infty}{\to} 0$.

■

Theorem 4.1.3 ***(Fourier Inversion).***

$$fx = \mathcal{R}_{\mathbb{R}^n} fxe^{2\pi ix\cdot\xi}d\xi.$$

⚠ **Warning 4.1.4**
Fubini-Tonelli does not work here!

Proof (?).
Idea: Fubini-Tonelli doesn't work directly, so introduce a convergence factor, take limits, and use uniqueness of limits.

- Take the modified integral:

$$\begin{aligned}
I_t x &= \mathcal{R}\, f\xi\ e^{2\pi ix\cdot\xi}\ e^{-\pi t^2 \bigcup \xi \bigcup^2} \\
&= \mathcal{R}\, f\xi\varphi\xi \\
&= \mathcal{R}\, f\xi\varphi\xi \\
&= \mathcal{R}\, f\xi g\xi - x \\
&= \mathcal{R}\, f\xi g_t x - \xi\ d\xi \\
&= \mathcal{R}\, fy - xg_t y\ dy \quad \xi = y - x \\
&= f * g_t \\
&\to f \text{ in } L^1 \text{ as } t \to 0.
\end{aligned}$$

- We also have

$$\begin{aligned}
\lim_{t\to 0} I_t x &= \lim_{t\to 0} \mathcal{R}\, f\xi\ e^{2\pi ix\cdot\xi}\ e^{-\pi t^2 \bigcup \xi \bigcup^2} \\
&= \lim_{t\to 0} \mathcal{R}\, f\xi\varphi\xi \\
&=_{DCT} \mathcal{R}\, f\xi \lim_{t\to 0} \varphi\xi \\
&= \mathcal{R}\, f\xi\ e^{2\pi ix\cdot\xi}
\end{aligned}$$

.

- So

$$I_t x \to \mathcal{R}\, f\xi\ e^{2\pi ix\cdot\xi} \text{ pointwise and } \coprod I_t x - fx \coprod_1 \to 0.$$

- So there is a subsequence I_{t_n} such that $I_{t_n}x \to fx$ almost everywhere
- Thus $fx = \mathcal{R}\, f\xi\ e^{2\pi ix\cdot\xi}$ almost everywhere by uniqueness of limits.

■

Proposition 4.1.5 *(Eigenfunction of the Fourier transform).*

$$gx \coloneqq e^{-\pi \bigcup t \bigcup^2} \implies g\xi = g\xi \quad \text{and} \quad g_t x = gtx = e^{-\pi t^2 \bigcup x \bigcup^2}.$$

4.2 Approximate Identities

Example 4.2.1 *(of an approximation to the identity.):*

$$\varphi x \coloneqq e^{-\pi x^2}.$$

Theorem 4.2.2 *(Convolving against an approximate identity converges in L^1.).*

$$\prod f * \varphi_t - f \prod_1 \overset{t\to 0}{\to} 0.$$

Proof (?).

$$\begin{aligned}
\prod f - f * \varphi_t \prod_1 &= \mathcal{R}\, fx - \mathcal{R}\, fx - y\varphi_t y\ dydx \\
&= \mathcal{R}\, fx\, \mathcal{R}\, \varphi_t y\ dy - \mathcal{R}\, fx - y\varphi_t y\ dydx \\
&= \mathcal{R}\,\mathcal{R}\, \varphi_t y \left(fx - fx - y \right) dydx \\
&=_{FT} \mathcal{R}\,\mathcal{R}\, \varphi_t y \left(fx - fx - y \right) dxdy \\
&= \mathcal{R}\, \varphi_t y\, \mathcal{R}\, fx - fx - y\ dxdy \\
&= \mathcal{R}\, \varphi_t y \prod f - \tau_y f \prod_1 dy \\
&= \mathcal{R}_{y<\delta}\, \varphi_t y \prod f - \tau_y f \prod_1 dy + \mathcal{R}_{y\geq\delta}\, \varphi_t y \prod f - \tau_y f \prod_1 dy \\
&\leq \mathcal{R}_{y<\delta}\, \varphi_t y \varepsilon + \mathcal{R}_{y\geq\delta}\, \varphi_t y \prod f \prod_1 + \prod \tau_y f \prod_1\ dy \quad \text{by continuity in } L^1 \\
&\leq \varepsilon + 2\prod f \prod_1 \mathcal{R}_{y\geq\delta}\, \varphi_t y dy \\
&\leq \varepsilon + 2\prod f \prod_1 \cdot \varepsilon \quad \text{since } \varphi_t \text{ has small tails} \\
&\overset{\varepsilon\to 0}{\to} 0.
\end{aligned}$$

■

Theorem 4.2.3 *(Convolutions vanish at infinity).*

$$f, g \in L^1 \text{ and bounded} \implies \lim_{\bigcup x \bigcup \to\infty} f * gx = 0.$$

Proof (?).
- Choose $M \geq f, g$.
- By small tails, choose N such that $\int_{B_N^c} |f|, \int_{B_n^c} |g| < \varepsilon$
- Note

$$|f * g| \leq \int |f x - y| \, |g y| \, dy \coloneqq I.$$

- Use $|x| \leq |x - y| + |y|$, take $|x| \geq 2N$ so either

$$|x - y| \geq N \implies I \leq \int_{x-y \geq N} |f x - y| M \, dy \leq \varepsilon M \to 0$$

then

$$|y| \geq N \implies I \leq \int_{y \geq N} M |g y| \, dy \leq M\varepsilon \to 0.$$

■

Proposition 4.2.4 ***(Corollary of Young's inequality).***
Take $q = 1$ in Young's inequality to obtain

$$\|f * g\|_p \leq \|f\|_p \|g\|_1.$$

Proposition 4.2.5 ***(L^1 is closed under convolution.).***
If $f, g \in L^1$ then $f * g \in L^1$.

5 | Functional Analysis

5.1 Theorems

Theorem 5.1.1 ***(Bessel's Inequality).***
For any orthonormal set $u_n \subseteq \mathcal{H}$ a Hilbert space (not necessarily a basis),

$$\left\| x - \sum_{n=1}^{N} \langle x, u_n \rangle u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n \rangle|^2$$

and thus

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2.$$

Proof (of Bessel's inequality).

- Let $S_N = \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n$

$$
\begin{aligned}
\|x - S_N\|^2 &= \langle x - S_n,\ x - S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \langle x,\ S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \Big\langle x,\ \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n \Big\rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^{N} \langle x,\ \langle x,\ u_n \rangle u_n \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^{N} \overline{\langle x,\ u_n \rangle} \langle x,\ u_n \rangle \\
&= \|x\|^2 + \Big\| \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n \Big\|^2 - 2 \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 - 2 \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
&= \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2.
\end{aligned}
$$

- By continuity of the norm and inner product, we have

$$
\begin{aligned}
\lim_{N\to\infty} \|x - S_N\|^2 &= \lim_{N\to\infty} \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
\implies \Big\|x - \lim_{N\to\infty} S_N\Big\|^2 &= \|x\|^2 - \lim_{N\to\infty} \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
\implies \Big\|x - \sum_{n=1}^{\infty} \langle x,\ u_n \rangle u_n\Big\|^2 &= \|x\|^2 - \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2.
\end{aligned}
$$

- Then noting that $0 \le \|x - S_N\|^2$,

$$
\begin{aligned}
0 &\le \|x\|^2 - \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2 \\
\implies \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2 &\le \|x\|^2 \ \blacksquare\ .
\end{aligned}
$$

■

Theorem 5.1.2 ***(Riesz Representation for Hilbert Spaces).***
If Λ is a continuous linear functional on a Hilbert space H, then there exists a unique $y \in H$ such that

$$\forall x \in H, \quad \Lambda x = \langle x,\ y \rangle..$$

Proof (?).

- Define $M \coloneqq \ker \Lambda$.

- Then M is a closed subspace and so $H = M \oplus M^{\perp}$
- There is some $z \in M^{\perp}$ such that $\prod z \prod = 1$.
- Set $u \coloneqq \Lambda xz - \Lambda zx$
- Check

$$\Lambda u = \Lambda\Lambda xz - \Lambda zx = \Lambda x\Lambda z - \Lambda z\Lambda x = 0 \implies u \in M$$

- Compute

$$\begin{aligned} 0 &= \coprod u,\ z^{\sim} \\ &= \coprod \Lambda xz - \Lambda zx,\ z^{\sim} \\ &= \coprod \Lambda xz,\ z^{\sim} - \coprod \Lambda zx,\ z^{\sim} \\ &= \Lambda x \coprod z,\ z^{\sim} - \Lambda z \coprod x,\ z^{\sim} \\ &= \Lambda x \prod z \prod^{2} - \Lambda z \coprod x,\ z^{\sim} \\ &= \Lambda x - \Lambda z \coprod x,\ z^{\sim} \\ &= \Lambda x - \coprod x,\ \overline{\Lambda z} z^{\sim}, \end{aligned}$$

- Choose $y \coloneqq \overline{\Lambda z} z$.
- Check uniqueness:

$$\begin{aligned} \coprod x,\ y^{\sim} &= \coprod x,\ y'^{\sim} \quad \forall x \\ \implies \coprod x,\ y - y'^{\sim} &= 0 \quad \forall x \\ \implies \coprod y - y',\ y - y'^{\sim} &= 0 \\ \implies \prod y - y' \prod &= 0 \\ \implies y - y' &= \mathbf{0} \implies y = y'. \end{aligned}$$

∎

Theorem 5.1.3 ***(Functionals are continuous if and only if bounded).***
Let $L : X \to \mathbb{C}$ be a linear functional, then the following are equivalent:

1. L is continuous
2. L is continuous at zero
3. L is bounded, i.e. $\exists c \geq 0 \bigcup \bigcup Lx \bigcup \leq c \prod x \prod$ for all $x \in H$

Proof (?).

$2 \implies 3$: Choose $\delta < 1$ such that

$$\|x\| \leq \delta \implies |Lx| < 1.$$

Then

$$\begin{aligned}
|Lx| &= \left|L \frac{\|x\|}{\delta} \frac{\delta}{\|x\|} x\right| \\
&= \frac{\|x\|}{\delta} \left|L\, \delta \frac{x}{\|x\|}\right| \\
&\leq \frac{\|x\|}{\delta} 1,
\end{aligned}$$

so we can take $c = \frac{1}{\delta}$. ∎

$3 \implies 1$:

We have $|Lx - y| \leq c\|x - y\|$, so given $\varepsilon \geq 0$ simply choose $\delta = \frac{\varepsilon}{c}$. ∎

Theorem 5.1.4 ***(The operator norm is a norm).***
If H is a Hilbert space, then $H^\vee, \|\cdot\|_{\mathrm{op}}$ is a normed space.

Proof (?).
The only nontrivial property is the triangle inequality, but

$$\|L_1 + L_2\|_{\mathrm{op}} = \sup |L_1 x + L_2 x| \leq \sup |L_1 x| + |\sup L_2 x| = \|L_1\|_{\mathrm{op}} + \|L_2\|_{\mathrm{op}}.$$

∎

Theorem 5.1.5 ***(The operator norm on $X^\vee$ yields a Banach space).***
If X is a normed vector space, then $X^\vee, \|\cdot\|_{\mathrm{op}}$ is a Banach space.

Proof (?).

- Let L_n be Cauchy in $X^\vee$.
- Then for all $x \in C$, $L_n x \subset \mathbb{C}$ is Cauchy and converges to something denoted Lx.
- Need to show L is continuous and $\|L_n - L\| \to 0$.
- Since L_n is Cauchy in $X^\vee$, choose N large enough so that

$$n, m \geq N \implies \|L_n - L_m\| < \varepsilon \implies |L_m x - L_n x| < \varepsilon \quad \forall x \mid \|x\| = 1.$$

- Take $n \to \infty$ to obtain

$$\begin{aligned}
m \geq N &\implies |L_m x - Lx| < \varepsilon \quad \forall x \mid \|x\| = 1 \\
&\implies \|L_m - L\| < \varepsilon \to 0.
\end{aligned}$$

- Continuity:

$$\begin{aligned}\bigcup Lx \bigcup &= \bigcup Lx - L_n x + L_n x \bigcup \\ &\leq \bigcup Lx - L_n x \bigcup + \bigcup L_n x \bigcup \\ &\leq \varepsilon \prod x \prod + c \prod x \prod \\ &= \varepsilon + c \prod x \prod \blacksquare\,.\end{aligned}$$

■

Theorem 5.1.6 ***(Riesz-Fischer).***
Let $U = u_n{}_{n=1}^{\infty}$ be an orthonormal set (not necessarily a basis), then

1. There is an isometric surjection

$$\begin{aligned}\mathcal{H} &\to \ell^2\mathbb{N} \\ \mathbf{x} &\mapsto \coprod \mathbf{x},\ \mathbf{u}_n \smile_{n=1}^{\infty}\end{aligned}$$

i.e. if $a_n \in \ell^2\mathbb{N}$, so $\mathcal{P}\bigcup a_n \bigcup^2 < \infty$, then there exists a $\mathbf{x} \in \mathcal{H}$ such that

$$a_n = \coprod \mathbf{x},\ \mathbf{u}_n \smile \quad \forall n.$$

2. $\mathbf{x}$ can be chosen such that

$$\prod \mathbf{x} \prod^2 = \mathcal{P} \bigcup a_n \bigcup^2$$

Note: the choice of $\mathbf{x}$ *is unique* $\iff$ u_n *is* ***complete****, i.e.* $\coprod \mathbf{x},\ \mathbf{u}_n \smile = 0$ *for all* n *implies* $\mathbf{x} = \mathbf{0}$.

Proof (?).

- Given a_n, define $S_N = \overset{N}{\mathcal{P}} a_n \mathbf{u}_n$.
- S_N is Cauchy in $\mathcal{H}$ and so $S_N \to \mathbf{x}$ for some $\mathbf{x} \in \mathcal{H}$.
- $\coprod x,\ u_n \smile = \coprod x - S_N,\ u_n \smile + \coprod S_N,\ u_n \smile \to a_n$
- By construction, $\prod x - S_N \prod^2 = \prod x \prod^2 - \overset{N}{\mathcal{P}} \bigcup a_n \bigcup^2 \to 0$, so $\prod x \prod^2 = \overset{\infty}{\mathcal{P}} \bigcup a_n \bigcup^2$.

■

6 Extra Problems

6.1 Greatest Hits

- $\star$: Show that for $E \subseteq \mathbb{R}^n$, TFAE:

  1. E is measurable
  2. $E = H \cup Z$ here H is F_σ and Z is null
  3. $E = V \setminus Z'$ where $V \in G_\delta$ and Z' is null.

- $\star$: Show that if $E \subseteq \mathbb{R}^n$ is measurable then $mE = \sup mK \bigcup K \subset E \text{ compact}$ iff for all $\varepsilon > 0$ there exists a compact $K \subseteq E$ such that $mK \geq mE - \varepsilon$.

- $\star$: Show that cylinder functions are measurable, i.e. if f is measurable on $\mathbb{R}^s$, then $Fx, y \coloneqq fx$ is measurable on $\mathbb{R}^s \times \mathbb{R}^t$ for any t.

- $\star$: Prove that the Lebesgue integral is translation invariant, i.e. if $\tau_h x = x + h$ then $\mathcal{R}\,\tau_h f = \mathcal{R}\, f$.

- $\star$: Prove that the Lebesgue integral is dilation invariant, i.e. if $f_\delta x = \frac{f\frac{x}{\delta}}{\delta^n}$ then $\mathcal{R}\, f_\delta = \mathcal{R}\, f$.

- $\star$: Prove continuity in L^1, i.e.

$$f \in L^1 \implies \lim_{h \to 0} \mathcal{R} \bigcup f x + h - f x \bigcup = 0.$$

- $\star$: Show that

$$f, g \in L^1 \implies f * g \in L^1 \quad \text{and} \quad \prod f * g \prod_1 \leq \prod f \prod_1 \prod g \prod_1.$$

- $\star$: Show that if $X \subseteq \mathbb{R}$ with $\mu X < \infty$ then

$$\prod f \prod_p \overset{p \to \infty}{\to} \prod f \prod_\infty.$$

6.2 By Topic

6.2.1 Topology

- Show that every compact set is closed and bounded.
- Show that if a subset of a metric space is complete and totally bounded, then it is compact.
- Show that if K is compact and F is closed with K, F disjoint then $\mathrm{dist} K, F > 0$.

6.2.2 Continuity

- Show that a continuous function on a compact set is uniformly continuous.

6.2.3 Differentiation

- Show that if $f \in C^1\mathbb{R}$ and both $\lim_{x\to\infty} fx$ and $\lim_{x\to\infty} f'x$ exist, then $\lim_{x\to\infty} f'x$ must be zero.

6.2.4 Advanced Limitology

- If f is continuous, is it necessarily the case that f' is continuous?
- If $f_n \to f$, is it necessarily the case that f_n' converges to f' (or at all)?
- Is it true that the sum of differentiable functions is differentiable?
- Is it true that the limit of integrals equals the integral of the limit?
- Is it true that a limit of continuous functions is continuous?
- Show that a subset of a metric space is closed iff it is complete.
- Show that if $mE < \infty$ and $f_n \to f$ uniformly, then $\lim \mathcal{R}_E f_n = \mathcal{R}_E f$.

Uniform Convergence

- Show that a uniform limit of bounded functions is bounded.
- Show that a uniform limit of continuous function is continuous.
  - I.e. if $f_n \to f$ uniformly with each f_n continuous then f is continuous.
- Show that if $f_n \to f$ pointwise, $f_n' \to g$ uniformly for some f, g, then f is differentiable and $g = f'$.
- Prove that uniform convergence implies pointwise convergence implies a.e. convergence, but none of the implications may be reversed.
- Show that $\mathcal{P} \dfrac{x^n}{n!}$ converges uniformly on any compact subset of $\mathbb{R}$.

Measure Theory

- Show that continuity of measure from above/below holds for outer measures.

- Show that a countable union of null sets is null.

Measurability

- Show that $f = 0$ a.e. iff $\mathcal{R}_E f = 0$ for every measurable set E.

Integrability

- Show that if f is a measurable function, then $f = 0$ a.e. iff $\mathcal{R} f = 0$.
- Show that a bounded function is Lebesgue integrable iff it is measurable.
- Show that simple functions are dense in L^1.
- Show that step functions are dense in L^1.
- Show that smooth compactly supported functions are dense in L^1.

Convergence

- Prove Fatou's lemma using the Monotone Convergence Theorem.
- Show that if f_n is in L^1 and $\mathcal{P}\mathcal{R}\bigcup f_n \bigcup < \infty$ then $\mathcal{P} f_n$ converges to an L^1 function and

$$\mathcal{R}\mathcal{P} f_n = \mathcal{P}\mathcal{R} f_n.$$

Convolution

- Show that if $f \in L^1$ and g is bounded, then $f * g$ is bounded and uniformly continuous.
- If f, g are compactly supported, is it necessarily the case that $f * g$ is compactly supported?
- Show that under any of the following assumptions, $f * g$ vanishes at infinity:
  - $f, g \in L^1$ are both bounded.
  - $f, g \in L^1$ with just g bounded.
  - f, g smooth and compactly supported (and in fact $f * g$ is smooth)
  - $f \in L^1$ and g smooth and compactly supported (and in fact $f * g$ is smooth)
- Show that if $f \in L^1$ and g' exists with $\dfrac{\partial g}{\partial x_i}$ all bounded, then

$$\frac{\partial}{\partial x_i} f * g = f * \frac{\partial g}{\partial x_i}$$

Fourier Analysis

- Show that if $f \in L^1$ then f is bounded and uniformly continuous.
- Is it the case that $f \in L^1$ implies $f \in L^1$?
- Show that if $f, f \in L^1$ then f is bounded, uniformly continuous, and vanishes at infinity.
  - Show that this is not true for arbitrary L^1 functions.
- Show that if $f \in L^1$ and $f = 0$ almost everywhere then $f = 0$ almost everywhere.
  - Prove that $f = g$ implies that $f = g$ a.e.
- Show that if $f, g \in L^1$ then

$$\mathcal{R} fg = \mathcal{R} fg.$$

  - Give an example showing that this fails if g is not bounded.
- Show that if $f \in C^1$ then f is equal to its Fourier *series*.

Approximate Identities

- Show that if φ is an approximate identity, then

$$\prod f * \varphi_t - f \prod_1 \overset{t\to 0}{\to} 0.$$

  - Show that if additionally $\bigcup \varphi x \bigcup \leq c1 + \bigcup x \bigcup^{-n-\varepsilon}$ for some $c, \varepsilon > 0$, then this converges is almost everywhere.

- Show that is f is bounded and uniformly continuous and φ_t is an approximation to the identity, then $f * \varphi_t$ uniformly converges to f.

L^p Spaces

- Show that if $E \subseteq \mathbb{R}^n$ is measurable with $\mu E < \infty$ and $f \in L^p X$ then

$$\prod f \prod_{L^p X} \overset{p\to\infty}{\to} \prod f \prod_\infty .$$

- Is it true that the converse to the DCT holds? I.e. if $\mathcal{R} f_n \to \mathcal{R} f$, is there a $g \in L^p$ such that $f_n < g$ a.e. for every n?
- Prove continuity in L^p: If f is uniformly continuous then for all p,

$$\prod \tau_h f - f \prod_p \overset{h\to 0}{\to} 0.$$

- Prove the following inclusions of L^p spaces for $mX < \infty$:

$$L^\infty X \subset L^2 X \subset L^1 X$$
$$\ell^2 \mathbb{Z} \subset \ell^1 \mathbb{Z} \subset \ell^\infty \mathbb{Z}.$$

7 | Midterm Exam 2 (December 2014)

7.1 1

Note: (a) is a repeat.

- Let $\Lambda \in L^2 X^\vee$.
  - Show that $M \coloneqq f \in L^2 X \bigcup \Lambda f = 0 \subseteq L^2 X$ is a closed subspace, and $L^2 X = M \oplus M \perp$.
  - Prove that there exists a unique $g \in L^2 X$ such that $\Lambda f = \mathcal{R}_X \, g \overline{f}$.

7.2 2

a. In parts:

- Given a definition of $L^\infty\mathbb{R}^n$.
- Verify that $\prod\cdot\prod_\infty$ defines a norm on $L^\infty\mathbb{R}^n$.
- Carefully proved that $L^\infty\mathbb{R}^n, \prod\cdot\prod_\infty$ is a Banach space.

b. Prove that for any measurable $f : \mathbb{R}^n \to \mathbb{C}$,

$$L^1\mathbb{R}^n \cap L^\infty\mathbb{R}^n \subset L^2\mathbb{R}^n \quad \text{and} \quad \prod f\prod_2 \leq \prod f\prod_1^{\frac{1}{2}} \cdot \prod f\prod_\infty^{\frac{1}{2}}.$$

7.3 3

a. Prove that if $f, g : \mathbb{R}^n \to \mathbb{C}$ is both measurable then $Fx, y \coloneqq fx$ and $hx, y \coloneqq fx - ygy$ is measurable on $\mathbb{R}^n \times \mathbb{R}^n$.

b. Show that if $f \in L^1\mathbb{R}^n \cap L^\infty\mathbb{R}^n$ and $g \in L^1\mathbb{R}^n$, then $f * g \in L^1\mathbb{R}^n \cap L^\infty\mathbb{R}^n$ is well defined, and carefully show that it satisfies the following properties:

$$\prod f * g\prod_\infty \leq \prod g\prod_1 \prod f\prod_\infty \prod f * g\prod_1 \quad \leq \prod g\prod_1 \prod f\prod_1 \prod f * g\prod_2 \leq \prod g\prod_1 \prod f\prod_2.$$

Hint: first show $\bigcup f * g\bigcup^2 \leq \prod g\prod_1 \bigcup f\bigcup^2 * \bigcup g\bigcup$.

7.4 4 (Weierstrass Approximation Theorem)

Note: (a) is a repeat.

Let $f : \binom{0,1}{} \rfloor \to \mathbb{R}$ be continuous, and prove the Weierstrass approximation theorem: for any $\varepsilon > 0$ there exists a polynomial P such that $\prod f - P\prod_\infty < \varepsilon$.

8 | Midterm Exam 1 (October 2018)

8.1 Problem 1

Let $E \subseteq \mathbb{R}^n$ be bounded. Prove the following are equivalent:

1. For any $\epsilon > 0$ there exists and open set G and a closed set F such that

$$F \subseteq E \subseteq G \qquad\qquad mG \setminus F < \epsilon.$$

2. There exists a G_δ set V and an F_σ set H such that

$$mV \setminus H = 0.$$

8.2 Problem 2

Let $f_k{}_{k=1}^{\infty}$ be a sequence of extended real-valued Lebesgue measurable functions.

a. Prove that $\sup\limits_k f_k$ is a Lebesgue measurable function.

b. Prove that if $\lim\limits_{k\to\infty} f_k x$ exists for every $x \in \mathbb{R}^n$ then $\lim\limits_{k\to\infty} f_k$ is also a measurable function.

8.3 Problem 3

8.3.1 a

Prove that if $E \subseteq \mathbb{R}^n$ is a Lebesgue measurable set, then for any $h \in \mathbb{R}$ the set

$$E + h \coloneqq x + h \bigcup x \in E$$

is also Lebesgue measurable and satisfies $mE + h = mE$.

8.3.2 b

Prove that if f is a non-negative measurable function on $\mathbb{R}^n$ and $h \in \mathbb{R}^n$ then the function

$$\tau_h dx \coloneqq fx - h$$

is a non-negative measurable function and

$$\mathcal{R}\, fx\, dx = \mathcal{R}\, fx - h\, dx.$$

8.4 Problem 4

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function.

a. Prove that for all $\alpha > 0$,

$$A_\alpha \coloneqq x \in \mathbb{R}^n \bigcup \bigcup fx \bigcup > \alpha \implies mA_\alpha \leq \frac{1}{\alpha} \mathcal{R} \bigcup fx \bigcup \, dx.$$

b. Prove that

$$\mathcal{R} \bigcup fx \bigcup \, dx = 0 \iff f = 0 \text{ almost everywhere.}$$

8.5 Problem 5

Let $f_k{}_{k=1}^{\infty} \subseteq L^2 \big(0, 1 \big\rfloor$ be a sequence which *converges in* L^1 to a function f.

a. Prove that $f \in L^1 \big(0, 1 \big\rfloor$.

b. Give an example illustrating that f_k may not converge to f almost everywhere.

c. Prove that f_k must contain a subsequence that converges to f almost everywhere.

9 | Midterm Exam 2 (November 2018)

9.1 Problem 1

Let $f, g \in L^1 \big(0, 1 \big\rfloor$, define $Fx = \mathcal{R}_0^x \, fy \, dy$ and $Gx = \mathcal{R}_0^x \, gy \, dy$, and show

$$\mathcal{R}_0^1 \, Fxgx \, dx = F1G1 - \mathcal{R}_0^1 \, fxGx \, dx.$$

9.2 Problem 2

Let $\varphi \in L^1 \mathbb{R}^n$ such that $\mathcal{R} \, \varphi = 1$ and define $\varphi_t x = t^{-n} \varphi t^{-1} x$. Show that if f is bounded and uniformly continuous then $f * \varphi_t \overset{t \to 0}{\to} f$ uniformly.

9.3 Problem 3

Let $g \in L^\infty \left(0,1\right]$.

a. Prove

$$\prod g \prod_{L^p\left(0,1\right]} \overset{p\to\infty}{\to} \prod g \prod_{L^\infty\left(0,1\right]}.$$

b. Prove that the map

$$\begin{aligned}\Lambda_g : L^1\left(0,1\right] &\to \mathbb{C}\\ f &\mapsto \mathcal{R}_0^1 fg\end{aligned}$$

defines an element of $L^1\left(0,1\right]^\vee$ with $\prod \Lambda_g \prod_{L^1\left(0,1\right]^\vee} = \prod g \prod_{L^\infty\left(0,1\right]}$.

9.4 Problem 4

See section 10.3

10 Practice Exam (November 2014)

10.1 1: Fubini-Tonelli

10.1.1 a

Carefully state Tonelli's theorem for a nonnegative function Fx,t on $\mathbb{R}^n \times \mathbb{R}$.

10.1.2 b

Let $f : \mathbb{R}^n \to \left(0,\infty\right]$ and define

$$\mathcal{A} \coloneqq x,t \in \mathbb{R}^n \times \mathbb{R} \bigcup 0 \le t \le fx .$$

Prove the validity of the following two statements:

1. f is Lebesgue measurable on $\mathbb{R}^n \iff \mathcal{A}$ is a Lebesgue measurable subset of $\mathbb{R}^{n+1}$.
2. If f is Lebesgue measurable on $\mathbb{R}^n$ then

$$m\mathcal{A} = \mathcal{R}_{\mathbb{R}^n} fxdx = \mathcal{R}_0^\infty m\ x \in \mathbb{R}^n \bigcup fx \geq t\ dt.$$

10.2 2: Convolutions and the Fourier Transform

10.2.1 a

Let $f, g \in L^1\mathbb{R}^n$ and give a definition of $f * g$.

10.2.2 b

Prove that if f, g are integrable and bounded, then

$$f * gx \overset{\bigcup x \bigcup \to \infty}{\to} 0.$$

10.2.3 c

1. Define the *Fourier transform* of an integrable function f on $\mathbb{R}^n$.
2. Give an outline of the proof of the Fourier inversion formula.
3. Give an example of a function $f \in L^1\mathbb{R}^n$ such that f is not in $L^1\mathbb{R}^n$.

10.3 3: Hilbert Spaces

Let $u_n{}_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space H.

10.3.1 a

Let $x \in H$ and verify that

$$\prod x - \sum_{n=1}^{N} \coprod x, u_n \frown u_n \prod_H^2 = \prod \prod x_H^2 - \sum_{n=1}^{N} \bigcup \coprod x, u_n \frown \bigcup^2.$$

for any $N \in \mathbb{N}$ and deduce that

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|_H^2.$$

10.3.2 b

Let $(a_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$ and prove that there exists an $x \in H$ such that $a_n = \langle x,\ u_n \rangle$ for all $n \in \mathbb{N}$, and moreover x may be chosen such that

$$\|x\|_H = \left(\sum_{n \in \mathbb{N}} |a_n|^2 \right)^{\frac{1}{2}}.$$

Proof

- Take $a_n \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
- Define $x \coloneqq \lim_{N \to \infty} S_N$ where $S_N = \sum_{k=1}^{N} a_k u_k$
- S_N is Cauchy and H is complete, so $x \in H$.
- By construction,

  $$\langle x,\ u_n \rangle = \left\langle \sum_k a_k u_k,\ u_n \right\rangle = \sum_k a_k \langle u_k,\ u_n \rangle = a_n$$

  since the u_k are all orthogonal.
- By Pythagoras since the u_k are normal,

  $$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2.$$

10.3.3 c

Prove that if $\{u_n\}$ is *complete*, Bessel's inequality becomes an equality.

Proof Let x and u_n be arbitrary.

$$
\begin{aligned}
\left\langle x - \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k,\ u_n \right\rangle &= \langle x,\ u_n \rangle - \left\langle \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k,\ u_n \right\rangle \\
&= \langle x,\ u_n \rangle - \sum_{k=1}^{\infty} \langle \langle x,\ u_k \rangle u_k,\ u_n \rangle \\
&= \langle x,\ u_n \rangle - \sum_{k=1}^{\infty} \langle x,\ u_k \rangle \langle u_k,\ u_n \rangle \\
&= \langle x,\ u_n \rangle - \langle x,\ u_n \rangle = 0 \\
\implies x - \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k = 0 \quad &\text{by completeness.}
\end{aligned}
$$

So

$$
x = \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k \implies \left\| x \right\|^2 = \sum_{k=1}^{\infty} \left| \langle x,\ u_k \rangle \right|^2 . \blacksquare .
$$

10.4 4: L^p Spaces

10.4.1 a

Prove Holder's inequality: let $f \in L^p, g \in L^q$ with p, q conjugate, and show that

$$
\left\| fg \right\|_p \leq \left\| f \right\|_p \cdot \left\| g \right\|_q .
$$

10.4.2 b

Prove Minkowski's Inequality:

$$
1 \leq p < \infty \implies \left\| f + g \right\|_p \leq \left\| f \right\|_p + \left\| g \right\|_p .
$$

Conclude that if $f, g \in L^p(\mathbb{R}^n)$ then so is $f + g$.

10.4.3 c

Let $X = (0, 1] \subset \mathbb{R}$.

1. Give a definition of the Banach space $L^\infty(X)$ of essentially bounded functions of X.
2. Let f be non-negative and measurable on X, prove that

$$
\int_X f(x)^p \, dx \overset{p \to \infty}{\to} \begin{cases} \infty & \text{or} \\ m\left(f^{-1}(1)\right) \end{cases},
$$

and characterize the functions of each type

Solution:

$$
\begin{aligned}
\mathcal{R} f^p &= \mathcal{R}_{x<1} f^p + \mathcal{R}_{x=1} f^p + \mathcal{R}_{x>1} f^p \\
&= \mathcal{R}_{x<1} f^p + \mathcal{R}_{x=1} 1 + \mathcal{R}_{x>1} f^p \\
&= \mathcal{R}_{x<1} f^p + m f = 1 + \mathcal{R}_{x>1} f^p \\
&\overset{p\to\infty}{\to} 0 + m f = 1 + \begin{cases} 0 & m x \geq 1 = 0 \\ \infty & m x \geq 1 > 0. \end{cases}
\end{aligned}
$$

10.5 5: Dual Spaces

Let X be a normed vector space.

10.5.1 a

Give the definition of what it means for a map $L : X \to \mathbb{C}$ to be a *linear functional.*

10.5.2 b

Define what it means for L to be *bounded* and show L is bounded $\iff$ L is continuous.

10.5.3 c

Prove that $X^\vee, \prod \cdot \prod_{\text{op}}$ is a Banach space.

11 Common Inequalities

Proposition 11.0.1 ***(Reverse Triangle Inequality).***

$$\bigcup \prod x \prod - \prod y \prod \bigcup \leq \prod x - y \prod.$$

Proposition 11.0.2 ***(Chebyshev's Inequality).***

$$\mu x : |fx| > \alpha \le \frac{\|f\|_p^{\,p}}{\alpha}.$$

Proposition 11.0.3 ***(Holder's Inequality).***

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \|fg\|_1 \le \|f\|_p \|g\|_q.$$

Proof (of Holder's inequality).
It suffices to show this when $\|f\|_p = \|g\|_q = 1$, since

$$\|fg\|^1 \le \|f\|^p \|f\|^q \iff \int \frac{|f|}{\|f\|^p} \frac{|g|}{\|g\|^q} \le 1.$$

Using $AB \le \frac{1}{p}A^p + \frac{1}{q}B^q$, we have

$$\int |f| |g| \le \int \frac{|f|^p}{p} \frac{|g|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1.$$

■

Example 11.0.4 ***(Application of Holder's inequality: containment of L^p spaces):*** For finite measure spaces,

$$1 \le p < q \le \infty \implies L^q \subset L^p \quad \text{and } \ell^p \subset \ell^q.$$

Proof (of containment of L^p spaces).
Fix p, q, let $r = \frac{q}{p}$ and $s = \frac{r}{r-1}$ so $r^{-1} + s^{-1} = 1$. Then let $h = |f|^p$:

$$\|f\|_p^p = \|h \cdot 1\|_1 \le \|1\|_s \|h\|_r = \mu X^{\frac{1}{s}} \|f\|_q^{\frac{q}{r}} \implies \|f\|_p \le \mu X^{\frac{1}{p} - \frac{1}{q}} \|f\|_q.$$

Note: doesn't work for ℓ_p spaces, but just note that $\sum |x_n| < \infty \implies x_n < 1$ for large enough n, and thus $p < q \implies |x_n|^q \le |x_n|^q$.

■

Proposition 11.0.5 ***(Cauchy-Schwarz Inequality).***

$$|\langle f, g \rangle| = \|fg\|_1 \le \|f\|_2 \|g\|_2 \qquad \text{with equality} \iff f = \lambda g.$$

Remark 11.0.6: In general, Cauchy-Schwarz relates inner product to norm, and only happens to relate norms in L^1.

Proposition 11.0.7 ***(Minkowski's Inequality).***

$$1 \leq p < \infty \implies \|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Remark 11.0.8: This does not handle $p = \infty$ case. Use to prove L^p is a normed space.

Proof (?).

- We first note

$$|f+g|^p = |f+g||f+g|^{p-1} \leq |f| + |g| \, |f+g|^{p-1}.$$

- Note that if p, q are conjugate exponents then

$$\begin{aligned} \frac{1}{q} &= 1 - \frac{1}{p} = \frac{p-1}{p} \\ q &= \frac{p}{p-1}. \end{aligned}$$

- Then taking integrals yields

$$\begin{aligned} \|f+g\|_p^p &= \int |f+g|^p \\ &\leq \int |f| + |g| \, |f+g|^{p-1} \\ &= \int |f||f+g|^{p-1} + \int |g||f+g|^{p-1} \\ &= \|f \, f+g^{p-1}\|_1 + \|g \, f+g^{p-1}\|_1 \\ &\leq \|f\|_p \, \|f+g^{p-1}\|_q + \|g\|_p \, \|f+g^{p-1}\|_q \\ &= \|f\|_p + \|g\|_p \, \|f+g^{p-1}\|_q \\ &= \|f\|_p + \|g\|_p \, \int |f+g|^{p-1q}{}^{\frac{1}{q}} \\ &= \|f\|_p + \|g\|_p \, \int |f+g|^{p}{}^{1-\frac{1}{p}} \\ &= \|f\|_p + \|g\|_p \, \frac{\int |f+g|^p}{\int |f+g|^p{}^{\frac{1}{p}}} \\ &= \|f\|_p + \|g\|_p \, \frac{\|f+g\|_p^p}{\|f+g\|_p}. \end{aligned}$$

- Cancelling common terms yields

$$\begin{aligned} 1 &\leq \|f\|_p + \|g\|_p \, \frac{1}{\|f+g\|_p} \\ &\implies \|f+g\|_p \leq \|f\|_p + \|g\|_p. \end{aligned}$$

■

Proposition 11.0.9 ***(Young's Inequality).***

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1 \implies \|f*g\|_r \le \|f\|_p\|g\|_q$$

Remark 11.0.10 ***(some useful special cases):***

$$\begin{aligned}
\|f*g\|_1 &\le \|f\|_1\|g\|_1\\
\|f*g\|_p &\le \|f\|_1\|g\|_p,\\
\|f*g\|_\infty &\le \|f\|_2\|g\|_2\\
\|f*g\|_\infty &\le \|f\|_p\|g\|_q.
\end{aligned}$$

Proposition 11.0.11 ***(Bessel's Inequality).***
For $x \in H$ a Hilbert space and e_k an orthonormal sequence,

$$\sum_{k=1}^{\infty} \left\| \langle x, e_k \rangle \right\|^2 \le \|x\|^2.$$

Note that this does not need to be a basis.

Proposition 11.0.12 ***(Parseval's Identity).***
Equality in Bessel's inequality, attained when e_k is a *basis*, i.e. it is complete, i.e. the span of its closure is all of H.

12 | Less Explicitly Used Inequalities

Proposition 12.0.1 ***(AM-GM Inequality).***

$$ab \le \frac{a+b}{2}.$$

Proposition 12.0.2 ***(Jensen's Inequality).***

$$ftx + 1 - ty \le tfx + 1 - tfy.$$

Proposition 12.0.3 ***(Young's Product Inequality).***

$$AB \leq \frac{A^p}{p} + \frac{B^q}{q}.$$

Proposition 12.0.4 ***(?).***

$$a + b^p \leq 2^{p-1} a^p + b^p.$$

Proposition 12.0.5 ***(Bernoulli's Inequality).***

$$1 + x^n \geq 1 + nx \quad x \geq -1, \text{ or } n \in 2\mathbb{Z} \text{ and } \forall x.$$

Proposition 12.0.6 ***(Exponential Inequality).***

$$\forall t \in \mathbb{R}, \quad 1 + t \leq e^t.$$

Proof .

- It's an equality when $t = 0$.
- $\frac{\partial}{\partial t} 1 + t < \frac{\partial t}{\partial e}^t \iff t < 0$

■

Proposition 12.0.7 ***(Young's Convolution Inequality).***

$$\frac{1}{r} \coloneqq \frac{1}{p} + \frac{1}{q} - 1 \implies \prod f * g \prod_r \leq \prod f \prod_p \prod g \prod q.$$