Title

D. Zack Garza

Monday 10th August, 2020

Contents

1	Mod	Modules		
	1.1	Genera	Questions	
			Fall 2019 Final #2 \dots 1	
			Spring 2018 #6	
		1.1.3	Fall $2018 \# 6 \bowtie \dots $ $2018 \# 6 \bowtie \dots$	
		1.1.4	Spring 2018 #7	

1 Modules

1.1 General Questions

1.1.1 Fall 2019 Final #2

Consider the \mathbb{Z} -submodule N of \mathbb{Z}^3 spanned by $f_1 = [-1, 0, 1], f_2 = [2, -3, 1], f_3 = [0, 3, 1], f_4 = [3, 1, 5]$. Find a basis for N and describe \mathbb{Z}^3/N .

1.1.2 Spring 2018 #6.

Let

$$M = \{(w, x, y, z) \in \mathbb{Z}^4 \mid w + x + y + z \in 2\mathbb{Z}\},\$$

and

$$N = \{(w, x, y, z) \in \mathbb{Z}^4 \mid 4 \mid (w - x), 4 \mid (x - y), 4 \mid (y - z)\}.$$

- a. Show that N is a \mathbb{Z} -submodule of M .
- b. Find vectors $u_1, u_2, u_3, u_4 \in \mathbb{Z}^4$ and integers d_1, d_2, d_3, d_4 such that

$$\{u_1, u_2, u_3, u_4\}$$

is a free basis for M, and

$$\{d_1u_1, d_2u_2, d_3u_3, d_4u_4\}$$

is a free basis for N .

c. Use the previous part to describe M/N as a direct sum of cyclic \mathbb{Z} -modules.

1.1.3 Fall 2018 #6 ⋈

Let R be a commutative ring, and let M be an R-module. An R-submodule N of M is maximal if there is no R-module P with $N \subseteq P \subseteq M$.

- a. Show that an R-submodule N of M is maximal $\iff M/N$ is a simple R-module: i.e., M/N is nonzero and has no proper, nonzero R-submodules.
- b. Let M be a \mathbb{Z} -module. Show that a \mathbb{Z} -submodule N of M is maximal $\iff \#M/N$ is a prime number.
- c. Let M be the \mathbb{Z} -module of all roots of unity in \mathbb{C} under multiplication. Show that there is no maximal \mathbb{Z} -submodule of M.

Solution.

a

By the correspondence theorem, submodules of M/N biject with submodules A of M containing N.

So

- M is maximal:
- \iff no such (proper, nontrivial) submodule A exists
- \iff there are no (proper, nontrivial) submodules of M/N
- $\iff M/N$ is simple.

b

Identify \mathbb{Z} -modules with abelian groups, then by (a), N is maximal $\iff M/N$ is simple \iff M/N has no nontrivial proper subgroups.

By Cauchy's theorem, if |M/N| = ab is a composite number, then $a \mid ab \implies$ there is an element (and thus a subgroup) of order a. In this case, M/N contains a nontrivial proper cyclic subgroup, so M/N is not simple. So |M/N| can not be composite, and therefore must be prime.

c Let $G = \{x \in \mathbb{C} \mid x^n = 1 \text{ for some } n \in \mathbb{N} \}$, and suppose H < G is a proper subgroup.

Then there must be a prime p such that the $\zeta_{p^k} \notin H$ for all k greater than some constant m – otherwise, we can use the fact that if $\zeta_{p^k} \in H$ then $\zeta_{p^\ell} \in H$ for all $\ell \leq k$, and if $\zeta_{p^k} \in H$ for all p and all p then p and all p then p is p and p and p and p and p and p are p in p and p in p in p and p and p in p in p and p in p in

But this means there are infinitely many elements in $G \setminus H$, and so $\infty = [G:H] = |G/H|$ is not a prime. Thus by (b), H can not be maximal, a contradiction.

1.1.4 Spring 2018 #7.

Let R be a PID and M be an R-module. Let p be a prime element of R. The module M is called $\langle p \rangle$ -primary if for every $m \in M$ there exists k > 0 such that $p^k m = 0$.

a. Suppose M is $\langle p \rangle$ -primary. Show that if $m \in M$ and $t \in R$, $t \notin \langle p \rangle$, then there exists $a \in R$ such that atm = m.

b. A submodule S of M is said to be *pure* if $S \cap rM = rS$ for all $r \in R$. Show that if M is $\langle p \rangle$ -primary, then S is pure if and only if $S \cap p^k M = p^k S$ for all $k \geq 0$.