# Complex Analysis Qualifying Exam Solutions

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## Contents



# 1 Week 1

## 1.1 Integrals and Cauchy's Theorem

### 1.1.1 5

Show that there is no sequence of polynomials converging uniformly to $f(z) = 1/z$ on $S^1$.

Solution

- By Cauchy's integral formula, $\int_{S^1} f = 2\pi i$
- If $p_j$ is any polynomial, then $p_j$ is holomorphic in $\mathbb{D}$, so $\int_{S^1} p_j = 0$.
- Contradiction: compact sets in $\mathbb{C}$ are bounded, so

$$\left| \int f - \int p_j \right| \leq \int |p_j - f| \leq \int \|p_j - f\|_\infty = \|p_j - f\|_\infty \int_{S^1} 1 \, dz = \|p_j - f\|_\infty \cdot 2\pi \longrightarrow 0$$

which forces $\int f = \int p_j = 0$.

#### 1.1.2 9

#### 1.1.3 10

Suppose $f : \mathbb{C} \longrightarrow \mathbb{C}$ is entire and bounded, and use Cauchy's theorem to prove that $f' \equiv 0$ and thus $f$ is constant.

Solution

- Suffices to prove $f' = 0$ because $\mathbb{C}$ is connected (see Stein Ch 1, 3.4)
  - Idea: Fix $w_0$, show $f(w) = f(w_0)$ for any $w \neq w_0$
  - Connected = Path connected in $\mathbb{C}$, so take $\gamma$ joining $w$ to $w_0$.
  - $f$ is a primitive for $f'$, and $\int_\gamma f' = f(w) - f(w_0)$, but $f' = 0$.
- Fix $z_0 \in \mathbb{C}$, let $B$ be the bound for $f$, so $|f(z)| \leq B$ for all $z$.
- Apply Cauchy inequalities: if $f$ is holomorphic on $U \supset \overline{D}_R(z_0)$ then setting $\|f\|_C \coloneqq \sup_{z \in C} |f(z)|$,

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! \|f\|_C}{R^n}.$$

  - Yields $\left| f'(z_0) \right| \leq B/R$
- Take $R \longrightarrow \infty$, QED.

### 1.2 Liouville. The Fundamental Theorem of Algebra, Power Series

#### 1.2.1 1

Suppose $f$ is analytic on $\Omega \supseteq \mathbb{D}$ whose power series $\sum a_n z^n$ has radius of convergence 1.

a. Give an example of an $f$ which converges at every point on $S^1$.
b. Give an example of an $f$ which is analytic at $z = 1$ but $\sum a_n$ diverges.
c. Prove that $f$ can not be analytic at every point of $S^1$.

Solution:

a. Take $\sum \frac{z^n}{n^2}$; then $|z| \leq 1 \implies \left| \frac{z^n}{n^2} \right| \leq \frac{1}{n^2}$ which is summable, so the series converges for $|z| \leq 1$.

b. Take $\sum \frac{z^n}{n}$; then $z = 1$ yields the harmonic series, which diverges.

   - For $z \in S^1 \setminus \{1\}$, we have $z = e^{2\pi i t}$ for $0 < t < 2\pi$.
   - So fix $t$.
   - Toward applying the Dirichlet test, set $a_n = 1/n, b_n = z^n$.
   - Then for all $N$,

$$\left| \sum_{n=1}^N b_n \right| = \left| \sum_{n=1}^N b_n \right| = \left| \sum_{n=1}^N z^n \right| = \left| \frac{z - z^{N+1}}{|1 - z|} \right| \leq \frac{2}{1 - z} < \infty.$$

   - Thus $\sum a_n b_n < \infty$ and $\sum z^n / n$ converges.

c. ?

### 1.2.2 5

Prove the Fundamental Theorem of Algebra: every non-constant polynomial $p(z) = a_n z^n + \cdots + a_0 \in \mathbb{C}[x]$ has a root in $\mathbb{C}$.

Solution:

- Strategy: By contradiction with Liouville's Theorem
- Suppose $p$ is non-constant and has no roots.
- Claim: $1/p(z)$ is a bounded holomorphic function on $\mathbb{C}$.
  - Holomorphic: clear? Since $p$ has no roots.
  - Bounded: for $z \neq 0$, write
$$\frac{P(z)}{z^n} = a_n + \left(\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right).$$
  - The term in parentheses goes to 0 as $|z| \longrightarrow \infty$
  - Thus there exists an $R > 0$ such that
$$|z| > R \implies \left|\frac{P(z)}{z^n}\right| \geq c := \frac{|a_n|}{2}.$$
  - So $p$ is bounded below when $|z| > R$
  - Since $p$ is continuous and has no roots in $|z| \leq R$, it is bounded below when $|z| \leq R$.
  - Thus $p$ is bounded below on $\mathbb{C}$ and thus $1/p$ is bounded above on $\mathbb{C}$.
- By Liouville's theorem, $1/p$ is constant and thus $p$ is constant, a contradiction.

### 1.2.3 6

Find all entire functions $f$ which satisfy the following inequality, and prove the list is complete:
$$|f(z)| \geq |z|.$$

Solution:

- Suppose $f$ is entire and define $g(z) := \dfrac{z}{f(z)}$.
- By the inequality, $|g(z)| \leq 1$, so $g$ is bounded.
- $g$ potentially has singularities at the zeros $Z_f := f^{-1}(0)$, but since $f$ is entire, $g$ is holomorphic on $\mathbb{C} \setminus Z_f$.
- Claim: $Z_f = \{0\}$.
  - If $f(z) = 0$, then $|z| \leq |f(z)| = 0$ which forces $z = 0$.
- We can now apply Riemann's removable singularity theorem:
  - Check $g$ is bounded on some open subset $D \setminus \{0\}$, clear since it's bounded everywhere
  - Check $g$ is holomorphic on $D \setminus \{0\}$, clear since the only singularity of $g$ is $z = 0$.
- By Riemann's removable singularity theorem, the singularity $z = 0$ is removable and $g$ has an extension to an entire function $\tilde{g}$.
- By continuity, we have $|\tilde{g}(z)| \leq 1$ on all of $\mathbb{C}$
  - If not, then $|\tilde{g}(0)| = 1 + \varepsilon > 1$, but then there would be a domain $\Omega \subseteq \mathbb{C} \setminus \{0\}$ such that $1 < |\tilde{g}(z)| \leq 1 + \varepsilon$ on $\Omega$, a contradiction.
- By Liouville, $\tilde{g}$ is constant, so $\tilde{g}(z) = c_0$ with $|c_0| \leq 1$
- Thus $f(z) = c_0^{-1} z := cz$ where $|c| \geq 1$

Thus all such functions are of the form $f(z) = cz$ for some $c \in \mathbb{C}$ with $|c| \geq 1$.

## 2 Integrals and Cauchy's Theorem

### 2.1 9

- Note $f$ is continuous on $\mathbb{C}$ since analytic implies continuous ($f$ equals its power series, where the partials sums uniformly converge to it, and uniform limit of continuous is continuous).
- Strategy: take $D$ a disc centered at a point $x \in \mathbb{R}$, show $f$ is holomorphic in $D$ by Morera's theorem.
- Let $\Delta \subset D$ be a triangle in $D$.
- Case 1: If $\Delta \bigcap \mathbb{R} = 0$, then $f$ is holomorphic on $\Delta$ and $\displaystyle\int_\Delta f = 0$.
- Case 2: one side or vertex of $\Delta$ intersects $\mathbb{R}$, and wlog the rest of $\Delta$ is in $\mathbb{H}^+$.
  - Then let $\Delta_\varepsilon$ be the perturbation $\Delta + i\varepsilon = \left\{ z + i\varepsilon \;\middle|\; z \in \Delta \right\}$; then $\Delta_\varepsilon \bigcap \mathbb{R} = 0$ and $\displaystyle\int_{\Delta_\varepsilon} f = 0$.
  - Now let $\varepsilon \longrightarrow 0$ and conclude by continuity of $f$ (???)
    - We want
      $$\int_{\Delta_\varepsilon} f = \int_a^b f(\gamma_\varepsilon(t))\gamma_\varepsilon'(t)\,dt \overset{\varepsilon \longrightarrow 0}{\longrightarrow} \int_a^b f(\gamma(t))\gamma_\varepsilon'(t)\,dt = \int_\Delta f$$
      where $\gamma_\varepsilon, \gamma$ are curves parametrizing $\Delta_\varepsilon, \Delta$ respectively.
    - Since $\gamma, \gamma_\varepsilon$ are closed and bounded in $\mathbb{C}$, they are compact subsets. Thus it suffices to show that $f(\gamma_\varepsilon(t))\gamma_\varepsilon'(t)$ converges uniformly to $f(\gamma(t))\gamma'(t)$.
    - ??
- Case 3: $\Delta$ intersects both $\mathbb{H}^+$ and $\mathbb{H}^-$.
  - Break into smaller triangles, each of which falls into one of the previous two cases.

## 3 Laurent Polynomials

### 3.1 1

Let $f(z) = \dfrac{z+1}{z(z-1)}$.

About $z = 0$:

$$\begin{aligned}
f(z) &= (z+1)\left(-\frac{1}{z} + \frac{1}{z-1}\right) \\
&= -(z+1)\left(\frac{1}{z} + \sum_{n=0}^\infty z^n\right) \\
&= -(z+1)\sum_{n=-1}^\infty z^n \\
&= \frac{1}{z} + 2\sum_{n=0}^\infty z^n \\
&= -\frac{1}{z} - 2 - 2z - 2z^2 - \cdots .
\end{aligned}$$

About $z = 1$:

$$
\begin{aligned}
f(z) &= \left(\frac{(1-z)-2}{1-z}\right)\left(\frac{1}{1-(1-z)}\right) \\
&= \left(1 - \frac{2}{1-z}\right)\sum_{n=0}^{\infty}(1-z)^n \\
&= \sum_{n=0}^{\infty}(1-z)^n - 2\sum_{n=-1}^{\infty}(1-z)^n \\
&= -\frac{2}{1-z} - \sum_{n=0}^{\infty}(1-z)^n \\
&= \frac{2}{z-1} + \sum_{n=0}^{\infty}(-1)^{n+1}(z-1)^n \\
&= \frac{2}{z-1} - 1 + (z-1) - (z-1)^2 + \cdots .
\end{aligned}
$$

## 3.2 2

$$
e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{z}\right)^n = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots .
$$

$$
\begin{aligned}
\cos\left(\frac{1}{z}\right) &= \frac{1}{2}\left(e^{\frac{i}{z}} + e^{-\frac{i}{z}}\right) \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\left(\frac{i}{z}\right)^n + \left(\frac{-i}{z}\right)^n\right) \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{z}\right)^{2n}.
\end{aligned}
$$