# Real Analysis Review Notes

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Sunday 16th August, 2020

## Contents

# 1 Basics

## 1.1 Useful Techniques

- General advice: try swapping the orders of limits, sums, integrals, etc.
- Limits:
  - Take the lim sup or lim inf, which always exist, and aim for an inequality like

$$c \leq \liminf a_n \leq \limsup a_n \leq c.$$

- $\lim f_n = \limsup f_n = \liminf f_n$ iff the limit exists, so to show some $g$ is a limit, show
$$\limsup f_n \leq g \leq \liminf f_n \qquad (\implies g = \lim f).$$
- A limit does *not* exist if $\liminf a_n > \limsup a_n$.

- Sequences and Series
  - If $f_n$ has a global maximum (computed using $f_n'$ and the first derivative test) $M_n \longrightarrow 0$, then $f_n \longrightarrow 0$ uniformly.
  - For a fixed $x$, if $f = \sum f_n$ converges *uniformly* on some $B_r(x)$ and each $f_n$ is continuous at $x$, then $f$ is also continuous at $x$ .
- Equalities
  - Split into upper and lower bounds:
$$a = b \iff a \leq b \text{ and } a \geq b.$$
  - Use an epsilon of room:
$$a < b + \varepsilon \,\forall \varepsilon \implies a \leq b.$$
  - Showing something is zero:
$$|a| \leq \varepsilon \,\forall \varepsilon \implies a = 0.$$
- Simplifications:
  - To show something for a measurable set, show it for bounded/compact/elementary sets/
  - To show something for a function, show it for continuous, bounded, compactly supported, simple, indicator functions, $L^1$, etc
  - Replace a continuous sequence ($\varepsilon \longrightarrow 0$) with an arbitrary countable sequence ($x_n \longrightarrow 0$)
  - Intersect with a ball $B_r(\mathbf{0}) \subset \mathbb{R}^n$.
- Integrals
  - Break up $\mathbb{R}^n = \{|x| \leq 1\} \coprod \{|x| > 1\}$.

## 1.2 Definitions

**Definition 1.0.1** (Uniform Continuity)**.**
$f$ is uniformly continuous iff

$$\begin{aligned} &\forall \varepsilon \quad \exists \delta(\varepsilon) \Big| \quad \forall x, y, \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \\ \iff &\forall \varepsilon \quad \exists \delta(\varepsilon) \Big| \quad \forall x, y, \quad |y| < \delta \implies |f(x - y) - f(y)| < \varepsilon. \end{aligned}$$

**Definition (Nowhere Dense Sets)** A set $S$ is **nowhere dense** iff the closure of $S$ has empty interior iff every interval contains a subinterval that does not intersect $S$.

**Definition (Meager Sets)** A set is **meager** if it is a *countable* union of nowhere dense sets.

**Definition 1.0.2** ($F_\sigma$ and $G_\delta$ Sets)**.**
An $F_\sigma$ set is a union of closed sets, and a $G_\delta$ set is an intersection of opens.

Mnemonic: "F" stands for *ferme*, which is "closed" in French, and $\sigma$ corresponds to a "sum", i.e. a union.

**Theorem (Heine-Cantor)** Every continuous function on a compact space is uniformly continuous.

**Definition 1.0.3** (Limsup/Liminf)**.**

$$\limsup_n a_n = \lim_{n \longrightarrow \infty} \sup_{j \geq n} a_j = \inf_{n \geq 0} \sup_{j \geq n} a_j$$
$$\liminf_n a_n = \lim_{n \longrightarrow \infty} \inf_{j \geq n} a_j = \sup_{n \geq 0} \inf_{j \geq n} a_j.$$

## 1.3 Theorems

### 1.3.1 Topology / Sets

**Lemma** Metric spaces are compact iff they are sequentially compact, (i.e. every sequence has a convergent subsequence).

**Proposition** The unit ball in $C([0,1])$ with the sup norm is not compact.

**Proof** Take $f_k(x) = x^n$, which converges to a dirac delta at 1. The limit is not continuous, so no subsequence can converge.

**Proposition** A *finite* union of nowhere dense is again nowhere dense.

**Lemma (Convergent Sums Have Small Tails)**

$$\sum a_n < \infty \implies a_n \longrightarrow 0 \quad \text{and} \quad \sum_{k=N}^{\infty} a_n \stackrel{N \longrightarrow \infty}{\Longrightarrow} 0$$

**Theorem (Heine-Borel)** $X \subseteq \mathbb{R}^n$ is compact $\iff$ $X$ is closed and bounded.

**Lemma (Geometric Series)**

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \iff |x| < 1.$$

*Corollary:* $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k} = 1.$

**Lemma** The Cantor set is closed with empty interior.

**Proof** Its complement is a union of open intervals, and can't contain an interval since intervals have positive measure and $m(C_n)$ tends to zero.

**Corollary** The Cantor set is nowhere dense.

**Lemma** Singleton sets in $\mathbb{R}$ are closed, and thus $\mathbb{Q}$ is an $F_\sigma$ set.

**Theorem (Baire)** $\mathbb{R}$ is a **Baire space** (countable intersections of open, dense sets are still dense). Thus $\mathbb{R}$ can not be written as a countable union of nowhere dense sets.

**Lemma** Any nonempty set which is bounded from above (resp. below) has a well-defined supremum (resp. infimum).

### 1.3.2 Functions

**Proposition (Existence of Smooth Compactly Supported Functions)** There exist smooth compactly supported functions, e.g. take

$$f(x) = e^{-\frac{1}{x^2}} \chi_{(0,\infty)}(x).$$

**Lemma** There is a function discontinuous precisely on $\mathbb{Q}$.

**Proof** $f(x) = \dfrac{1}{n}$ if $x = r_n \in \mathbb{Q}$ is an enumeration of the rationals, and zero otherwise. The limit at every point is 0.

**Lemma** There *do not* exist functions that are discontinuous precisely on $\mathbb{R} \setminus \mathbb{Q}$.

**Proof** $D_f$ is always an $F_\sigma$ set, which follows by considering the oscillation $\omega_f$. $\omega_f(x) = 0 \iff f$ is continuous at $x$, and $D_f = \bigcup_n A_{\frac{1}{n}}$ where $A_\varepsilon = \{\omega_f \geq \varepsilon\}$ is closed.

**Proposition** A function $f : (a,b) \longrightarrow \mathbb{R}$ is Lipschitz $\iff$ $f$ is differentiable and $f'$ is bounded. In this case, $|f'(x)| \leq C$, the Lipschitz constant.

## 1.4 Uniform Convergence

**Theorem (Weierstrass Approximation)** If $[a,b] \subset \mathbb{R}$ is a closed interval and $f$ is continuous, then for every $\varepsilon > 0$ there exists a polynomial $p_\varepsilon$ such that $\|f - p_\varepsilon\|_{L^\infty([a,b])} \overset{\varepsilon \longrightarrow 0}{\longrightarrow} 0$.

**Theorem (Egorov)** Let $E \subseteq \mathbb{R}^n$ be measurable with $m(E) > 0$ and $\{f_k : E \longrightarrow \mathbb{R}\}$ be measurable functions such that

$$f(x) \coloneqq \lim_{k \longrightarrow \infty} f_k(x) < \infty$$

exists almost everywhere.

Then $f_k \longrightarrow f$ *almost uniformly*, i.e.

$$\forall \varepsilon > 0,\ \exists F \subseteq E \text{ closed such that } m(E \setminus F) < \varepsilon \text{ and } f_k \overset{u}{\to} f \text{ on } F.$$

**Proposition** The space $X = C([0,1])$, continuous functions $f : [0,1] \longrightarrow \mathbb{R}$, equipped with the norm $\|f\| = \sup_{x \in [0,1]} |f(x)|$, is a **complete** metric space.

**Proof**

1. Let $\{f_k\}$ be Cauchy in $X$.
2. Define a candidate limit using pointwise convergence:

   Fix an $x$; since

   $$|f_k(x) - f_j(x)| \leq \|f_k - f_k\| \longrightarrow 0$$

   the sequence $\{f_k(x)\}$ is Cauchy in $\mathbb{R}$. So define $f(x) \coloneqq \lim_k f_k(x)$.
3. Show that $\|f_k - f\| \longrightarrow 0$:

   $$|f_k(x) - f_j(x)| < \varepsilon \ \forall x \implies \lim_j |f_k(x) - f_j(x)| < \varepsilon \ \forall x$$

   Alternatively, $\|f_k - f\| \leq \|f_k - f_N\| + \|f_N - f_j\|$, where $N, j$ can be chosen large enough to bound each term by $\varepsilon/2$.

4. Show that $f \in X$:

   The uniform limit of continuous functions is continuous.

   > Note: in other cases, you may need to show the limit is bounded, or has bounded derivative, or whatever other conditions define $X$.

Theorem (Uniform Limits of Continuous Functions are Continuous)

**Theorem 1.1** ***(Uniform Limit Theorem).***
If $f_n \longrightarrow f$ pointwise and uniformly with each $f_n$ continuous, then $f$ is continuous.

> Slogan: "A uniform limit of continuous functions is continuous."

*Proof* .
- Follows from an $\varepsilon/3$ argument:

$$|F(x) - F(y| \leq |F(x) - F_N(x)| + |F_N(x) - F_N(y)| + |F_N(y) - F(y)| \leq \varepsilon \longrightarrow 0.$$

  - The first and last $\varepsilon/3$ come from uniform convergence of $F_N \longrightarrow F$.
  - The middle $\varepsilon/3$ comes from continuity of each $F_N$.
- So just need to choose $N$ large enough and $\delta$ small enough to make all 3 $\varepsilon$ bounds hold.

■

**Proposition 1.2** ***(Testing Uniform Convergence: The Sup Norm).***
$f_n \longrightarrow f$ uniformly iff there exists an $M_n$ such that $\|f_n - f\|_\infty \leq M_n \longrightarrow 0$.

> **Negating**: find an $x$ which depends on $n$ for which the norm is bounded *below*.

**Proposition 1.3** ***(Testing Uniform Convergence: The Weierstrass $M$-Test).***
If $\sup_{x \in A} |f_n(x)| \leq M_n$ for each $n$ where $\sum M_n < \infty$, then $\sum_{n=1}^\infty f_n(x)$ converges uniformly and absolutely on $A$.

Conversely, if $\sum f_n$ converges uniformly on $A$ then $\sup_{x \in A} |f_n(x)| \longrightarrow 0$.

**Lemma (Uniform Limits Commute with Integrals)** If $f_n \longrightarrow f$ uniformly, then $\int f_n = \int f$.

**Lemma (Uniform Convergence and Derivatives)** If $f_n' \longrightarrow g$ uniformly for some $g$ and $f_n \longrightarrow f$ pointwise (or at least at one point), then $g = f'$.

### 1.4.1 Series

**Lemma (Pointwise Convergence for a Series of Functions)** If $f_n(x) \leq M_n$ **for a fixed** $x$ where $\sum M_n < \infty$, then the series $f(x) = \sum f_n(x)$ converges pointwise.

**Lemma (Small Tails for Series of Functions)** If $\sum f_n$ converges then $f_n \longrightarrow 0$ uniformly.

**Lemma (M-test for Series)** If $|f_n(x)| \leq M_n$ which does not depend on $x$, then $\sum f_n$ converges uniformly.

**Lemma (p-tests)** Let $n$ be a fixed dimension and set $B = \left\{ x \in \mathbb{R}^n \;\middle|\; \|x\| \leq 1 \right\}$.

$$
\begin{aligned}
\sum \frac{1}{n^p} < \infty &\iff p > 1 \\
\int_\varepsilon^\infty \frac{1}{x^p} < \infty &\iff p > 1 \\
\int_0^1 \frac{1}{x^p} < \infty &\iff p < 1 \\
\int_B \frac{1}{|x|^p} < \infty &\iff p < n \\
\int_{B^c} \frac{1}{|x|^p} < \infty &\iff p > n
\end{aligned}
$$

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# 2 Measure Theory

## 2.1 Useful Techniques

- $s = \inf\{x \in X\} \implies$ for every $\varepsilon$ there is an $x \in X$ such that $x \leq s + \varepsilon$.
- Always consider bounded sets, and if $E$ is unbounded write $E = \bigcup_n B_n(0) \bigcap E$ and use countable subadditivity or continuity of measure.

## 2.2 Definitions

**Definition (Outer Measure)** The outer measure of a set is given by

$$m_*(E) \coloneqq \inf_{\substack{\{Q_i\} \rightrightarrows E \\ \text{closed cubes}}} \sum |Q_i|.$$

**Definition (Limsup and Liminf of Sets)**

$$
\begin{aligned}
\limsup_n A_n &\coloneqq \bigcap_n \bigcup_{j \geq n} A_j = \left\{ x \;\middle|\; x \in A_n \text{ for inf. many } n \right\} \\
\liminf_n A_n &\coloneqq \bigcup_n \bigcap_{j \geq n} A_j = \left\{ x \;\middle|\; x \in A_n \text{ for all except fin. many } n \right\}
\end{aligned}
$$

.

**Definition (Lebesgue Measurable Set)** A subset $E \subseteq \mathbb{R}^n$ is *Lebesgue measurable* iff for every $\varepsilon > 0$ there exists an open set $O \supseteq E$ such that $m_*(O \setminus E) < \varepsilon$. In this case, we define $m(E) \coloneqq m_*(E)$.

## 2.3 Theorems

**Lemma** Every open subset of $\mathbb{R}$ (resp $\mathbb{R}^n$) can be written as a unique countable union of disjoint (resp. almost disjoint) intervals (resp. cubes).

**Lemma (Properties of Outer Measure)**

- Montonicity: $E \subseteq F \implies m_*(E) \leq m_*(F)$.
- Countable Subadditivity: $m_*(\bigcup E_i) \leq \sum m_*(E_i)$.
- Approximation: For all $E$ there exists a $G \supseteq E$ such that $m_*(G) \leq m_*(E) + \varepsilon$.
- Disjoint[1] Additivity: $m_*(A \coprod B) = m_*(A) + m_*(B)$.

**Lemma (Subtraction of Measure)**

$$m(A) = m(B) + m(C) \quad \text{and} \quad m(C) < \infty \implies m(A) - m(C) = m(B).$$

**Lemma (Continuity of Measure)**

$$E_i \nearrow E \implies m(E_i) \longrightarrow m(E)$$
$$m(E_1) < \infty \text{ and } E_i \searrow E \implies m(E_i) \longrightarrow m(E).$$

**Proof** 1. Break into disjoint annuli $A_2 = E_2 \setminus E_1$, etc then apply countable disjoint additivity to $E = \coprod A_i$.

2. Use $E_1 = (\coprod E_j \setminus E_{j+1}) \coprod (\bigcap E_j)$, taking measures yields a telescoping sum,and use countable disjoint additivity.

**Theorem** Suppose $E$ is measurable; then for every $\varepsilon > 0$,

1. There exists an open $O \supset E$ with $m(O \setminus E) < \varepsilon$
2. There exists a closed $F \subset E$ with $m(E \setminus F) < \varepsilon$
3. There exists a compact $K \subset E$ with $m(E \setminus K) < \varepsilon$.

**Proof**

- (1): Take $\{Q_i\} \rightrightarrows E$ and set $O = \bigcup Q_i$.
- (2): Since $E^c$ is measurable, produce $O \supset E^c$ with $m(O \setminus E^c) < \varepsilon$.
  - Set $F = O^c$, so $F$ is closed.
  - Then $F \subset E$ by taking complements of $O \supset E^c$
  - $E \setminus F = O \setminus E^c$ and taking measures yields $m(E \setminus F) < \varepsilon$
- (3): Pick $F \subset E$ with $m(E \setminus F) < \varepsilon/2$.
  - Set $K_n = F \bigcap \mathbb{D}_n$, a ball of radius $n$ about 0.
  - Then $E \setminus K_n \searrow E \setminus F$
  - Since $m(E) < \infty$, there is an $N$ such that $n \geq N \implies m(E \setminus K_n) < \varepsilon$.

**Lemma** Lebesgue measure is translation and dilation invariant.

**Proof** Obvious for cubes; if $Q_i \rightrightarrows E$ then $Q_i + k \rightrightarrows E + k$, etc.

Flesh out this proof.

**Theorem (Non-Measurable Sets)** There is a non-measurable set.

[1] This holds for outer measure **iff** $\text{dist}(A, B) > 0$.

**Proof**

- Use AOC to choose one representative from every coset of $\mathbb{R}/\mathbb{Q}$ on $[0,1)$, which is countable, and assemble them into a set $N$
- Enumerate the rationals in $[0,1]$ as $q_j$, and define $N_j = N + q_j$. These intersect trivially.
- Define $M := \coprod N_j$, then $[0,1) \subseteq M \subseteq [-1,2)$, so the measure must be between 1 and 3. By translation invariance, $m(N_j) = m(N)$, and disjoint additivity forces $m(M) = 0$, a contradiction.

**Proposition (Borel Characterization of Measurable Sets)** If $E$ is Lebesgue measurable, then $E = H \coprod N$ where $H \in F_\sigma$ and $N$ is null.

> **Useful technique:** $F_\sigma$ sets are Borel, so establish something for Borel sets and use this to extend it to Lebesgue.

**Proof** For every $\frac{1}{n}$ there exists a closed set $K_n \subset E$ such that $m(E \setminus K_n) \leq \frac{1}{n}$. Take $K = \bigcup K_n$, wlog $K_n \nearrow K$ so $m(K) = \lim m(K_n) = m(E)$. Take $N := E \setminus K$, then $m(N) = 0$.

**Lemma** If $A_n$ are all measurable, $\limsup A_n$ and $\liminf A_n$ are measurable.

**Proof** Measurable sets form a sigma algebra, and these are expressed as countable unions/intersections of measurable sets.

**Theorem (Borel-Cantelli)** Let $\{E_k\}$ be a countable collection of measurable sets. Then

$$\sum_k m(E_k) < \infty \implies \text{almost every } x \in \mathbb{R} \text{ is in at most finitely many } E_k.$$

**Proof**

- If $E = \limsup_j E_j$ with $\sum m(E_j) < \infty$ then $m(E) = 0$.
- If $E_j$ are measurable, then $\limsup_j E_j$ is measurable.
- If $\sum_j m(E_j) < \infty$, then $\sum_{j=N}^{\infty} m(E_j) \overset{N\to\infty}{\longrightarrow} 0$ as the tail of a convergent sequence.
- $E = \limsup_j E_j = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j \implies E \subseteq \bigcup_{j=k}^{\infty}$ for all $k$
- $E \subset \bigcup_{j=k}^{\infty} \implies m(E) \leq \sum_{j=k}^{\infty} m(E_j) \overset{k\to\infty}{\longrightarrow} 0.$

**Lemma**

- Characteristic functions are measurable
- If $f_n$ are measurable, so are $|f_n|, \limsup f_n, \liminf f_n, \lim f_n$,
- Sums and differences of measurable functions are measurable,
- Cones $F(x,y) = f(x)$ are measurable,
- Compositions $f \circ T$ for $T$ a linear transformation are measurable,
- "Convolution-ish" transformations $(x,y) \mapsto f(x-y)$ are measurable

**Proof (Convolution)** Take the cone on $f$ to get $F(x,y) = f(x)$, then compose $F$ with the linear transformation $T = [1,-1;1,0]$.

# 3 Integration

Notation:

- "$f$ vanishes at infinity" means $f(x) \overset{|x|\longrightarrow\infty}{\longrightarrow} 0$.
- "$f$ has small tails" means $\displaystyle\int_{|x|\geq N} f \overset{N\longrightarrow\infty}{\longrightarrow} 0$.

## 3.1 Useful Techniques

- Break integration domain up into disjoint annuli.
- Break integrals or sums into $x < 1$ and $x \geq 1$.
- Calculus techniques: Taylor series, IVT, ...
- Approximate by dense subsets of functions
- Useful facts about compactly supported continuous functions:
  - Uniformly continuous
  - Bounded

## 3.2 Definitions

**Definition ($L^+$)** $f \in L^+$ iff $f$ is measurable and non-negative.

**Definition (Integrable)** A measurable function is integrable iff $\|f\|_1 < \infty$.

**Definition (The Infinity Norm)**

$$\|f\|_\infty \coloneqq \inf_{\alpha\geq 0}\left\{\alpha \;\middle|\; m\left\{|f| \geq \alpha\right\} = 0\right\}.$$

**Definition (Essentially Bounded Functions)** A function $f : X \longrightarrow \mathbb{C}$ is *essentially bounded* iff there exists a real number $c$ such that $\mu(\{|f| > x\}) = 0$, i.e. $\|f\|_\infty < \infty$.

If $f \in L^\infty(X)$, then $f$ is equal to some bounded function $g$ almost everywhere.

**Definition (L infty)**

$$L^\infty(X) \coloneqq \left\{f : X \longrightarrow \mathbb{C} \;\middle|\; f \text{ is essentially bounded }\right\} \coloneqq \left\{f : X \longrightarrow \mathbb{C} \;\middle|\; \|f\|_\infty < \infty\right\},$$

Example:

- $f(x) = x\chi_{\mathbb{Q}}(x)$ is essentially bounded but not bounded.

## 3.3 Theorems

Useful facts about $C_c$ functions:

- Bounded almost everywhere
- Uniformly continuous

**Theorem (p-Test for Integrals in $\RR$)**

$$\int_0^1 \frac{1}{x^p} < \infty \iff p < 1$$
$$\int_1^\infty \frac{1}{x^p} < \infty \iff p > 1.$$

Slogan: big powers of $x$ help us in neighborhoods of infinity and hurt around zero

Some integrable functions:

- $\int \frac{1}{1+x^2} = \arctan(x) \overset{x \longrightarrow \infty}{\pi}/2 < \infty$
- Any bounded function (or continuous on a compact set, by EVT)
- $\int_0^1 \frac{1}{\sqrt{x}} < \infty$
- $\int_0^1 \frac{1}{x^{1-\varepsilon}} < \infty$
- $\int_1^\infty \frac{1}{x^{1+\varepsilon}} < \infty$

Some non-integrable functions:

- $\int_0^1 \frac{1}{x} = \infty.$
- $\int_1^\infty \frac{1}{x} = \infty.$
- $\int_1^\infty \frac{1}{\sqrt{x}} = \infty$
- $\int_1^\infty \frac{1}{x^{1-\varepsilon}} = \infty$
- $\int_0^1 \frac{1}{x^{1+\varepsilon}} = \infty$

### 3.3.1 Convergence Theorems

**Theorem 3.1** ***(Monotone Convergence).***
If $f_n \in L^+$ and $f_n \nearrow f$ a.e., then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \int f_n \longrightarrow \int f.$$

Needs to be positive and increasing.

**Theorem 3.2** ***(Dominated Convergence).***
If $f_n \in L^1$ and $f_n \longrightarrow f$ a.e. with $|f_n| \leq g$ for some $g \in L^1$, then $f \in L^1$ and

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \int f_n \longrightarrow \int f < \infty,$$

and more generally,

$$\int |f_n - f| \longrightarrow 0.$$

Positivity *not* needed.

**Theorem (Generalized DCT)** If

- $f_n \in L^1$ with $f_n \longrightarrow f$ a.e.,
- There exist $g_n \in L^1$ with $|f_n| \leq g_n$, $g_n \geq 0$.
- $g_n \longrightarrow g$ a.e. with $g \in L^1$, and
- $\lim \int g_n = \int g$,

then $f \in L^1$ and $\lim \int f_n = \int f < \infty$.

Note that this is the DCT with $|f_n| < |g|$ relaxed to $|f_n| < g_n \longrightarrow g \in L^1$.

**Proof (Generalized DCT)** Proceed by showing $\limsup \int f_n \leq \int f \leq \liminf \int f_n$:

- $\int f \geq \limsup \int f_n$:

$$\begin{aligned}
\int g - \int f &= \int (g - f) \\
&\leq \liminf \int (g_n - f_n) \quad \text{Fatou} \\
&= \lim \int g_n + \liminf \int (-f_n) \\
&= \lim \int g_n - \limsup \int f_n \\
&= \int g - \limsup \int f_n
\end{aligned}$$

$$\implies \int f \geq \limsup \int f_n.$$

  - Here we use $g_n - f_n \overset{n \longrightarrow \infty}{g} - f$ with $0 \leq |f_n| - f_n \leq g_n - f_n$, so $g_n - f_n$ are nonnegative (and measurable) and Fatou's lemma applies.

- $\int f \le \liminf \int f_n$:

$$\begin{aligned}\int g + \int f = \int (g+f) \\ &\le \liminf \int (g_n + f_n) \\ &= \lim \int g_n + \liminf \int f_n \\ &= \int g + \liminf f_n\end{aligned}$$

$$\int f \le \liminf \int f_n.$$

  - Here we use that $g_n + f_n \longrightarrow g + f$ with $0 \le |f_n| + f_n \le g_n + f_n$ so Fatou's lemma again applies.

**Lemma (Converges in \$L^1\$ implies convergence of \$L^1\$ norms)** If $f \in L^1$, then

$$\int |f_n - f| \longrightarrow 0 \iff \int |f_n| \longrightarrow \int |f|.$$

**Proof** Let $g_n = |f_n| - |f_n - f|$, then $g_n \longrightarrow |f|$ and

$$|g_n| = ||f_n| - |f_n - f|| \le |f_n - (f_n - f)| = |f| \in L^1,$$

so the DCT applies to $g_n$ and

$$\begin{aligned}\|f_n - f\|_1 = \int |f_n - f| + |f_n| - |f_n| = \int |f_n| - g_n \\ \longrightarrow_{DCT} \lim \int |f_n| - \int |f|.\end{aligned}$$

**Theorem (Fatou's Lemma)** If $f_n$ is a sequence of nonnegative measurable functions, then

$$\int \liminf_n f_n \le \liminf_n \int f_n$$
$$\limsup_n \int f_n \le \int \limsup_n f_n.$$

**Theorem (Tonelli)** For $f(x, y)$ **non-negative and measurable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is a **measurable** function
- $F(x) = \int f(x, y)\, dy$ is a **measurable** function,
- For $E$ measurable, the slices $E_x \coloneqq \left\{ y \,\middle|\, (x, y) \in E \right\}$ are measurable.
- $\int f = \int \int F$, i.e. any iterated integral is equal to the original.

**Theorem (Fubini)** For $f(x, y)$ **integrable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is an **integrable** function

- $F(x) \coloneqq \int f(x,y)\ dy$ is an **integrable** function,
- For $E$ measurable, the slices $E_x \coloneqq \left\{ y \;\middle|\; (x,y) \in E \right\}$ are measurable.
- $\int f = \int \int f(x,y)$, i.e. any iterated integral is equal to the original

**Theorem (Fubini/Tonelli)** If any iterated integral is **absolutely integrable**, i.e. $\int \int |f(x,y)| < \infty$, then $f$ is integrable and $\int f$ equals any iterated integral.

**Corollary (Measurable Slices)** Let $E$ be a measurable subset of $\mathbb{R}^n$. Then

- For almost every $x \in \mathbb{R}^{n_1}$, the slice $E_x \coloneqq \left\{ y \in \mathbb{R}^{n_2} \;\middle|\; (x,y) \in E \right\}$ is measurable in $\mathbb{R}^{n_2}$.
- The function

$$\begin{aligned} F : \mathbb{R}^{n_1} &\longrightarrow \mathbb{R} \\ x &\mapsto m(E_x) = \int_{\mathbb{R}^{n_2}} \chi_{E_x}\ dy \end{aligned}$$

is measurable and

$$m(E) = \int_{\mathbb{R}^{n_1}} m(E_x)\ dx = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{E_x}\ dy\ dx$$

**Proof (Measurable Slices)**

$\implies$:

- Let $f$ be measurable on $\mathbb{R}^n$.
- Then the cylinders $F(x,y) = f(x)$ and $G(x,y) = f(y)$ are both measurable on $\mathbb{R}^{n+1}$.
- Write $\mathcal{A} = \{G \leq F\} \bigcap \{G \geq 0\}$; both are measurable.

$\impliedby$:

- Let $A$ be measurable in $\mathbb{R}^{n+1}$.
- Define $A_x = \left\{ y \in \mathbb{R} \;\middle|\; (x,y) \in \mathcal{A} \right\}$, then $m(A_x) = f(x)$.
- By the corollary, $A_x$ is measurable set, $x \mapsto A_x$ is a measurable function, and $m(A) = \int f(x)\ dx$.
- Then explicitly, $f(x) = \chi_A$, which makes $f$ a measurable function.

**Proposition (Differentiating Under an Integral)** If $\left| \frac{\partial}{\partial t} f(x,t) \right| \leq g(x) \in L^1$, then letting $F(t) = \int f(x,t)\ dt$,

$$\begin{aligned} \frac{\partial}{\partial t} F(t) &\coloneqq \lim_{h \to 0} \int \frac{f(x,t+h) - f(x,t)}{h} dx \\ &\overset{DCT}{=} \int \frac{\partial}{\partial t} f(x,t)\ dx. \end{aligned}$$

To justify passing the limit, let $h_k \longrightarrow 0$ be any sequence and define

$$f_k(x,t) = \frac{f(x,t+h_k) - f(x,t)}{h_k},$$

so $f_k \overset{\text{pointwise}}{\longrightarrow} \frac{\partial}{\partial t} f$.

Apply the MVT to $f_k$ to get $f_k(x,t) = f_k(\xi, t)$ for some $\xi \in [0, h_k]$, and show that $f_k(\xi, t) \in L_1$.

**Proposition (Swapping Sum and Integral)** If $f_n$ are non-negative and $\sum \int |f|_n < \infty$, then $\sum \int f_n = \int \sum f_n$.

**Proof** MCT. Let $F_N = \sum^N f_n$ be a finite partial sum; then there are simple functions $\varphi_n \nearrow f_n$ and so $\sum^N \varphi_n \nearrow F_N$, so apply MCT.

**Lemma** If $f_k \in L^1$ and $\sum \|f_k\|_1 < \infty$ then $\sum f_k$ converges almost everywhere and in $L^1$.

**Proof** Define $F_N = \sum^N f_k$ and $F = \lim_N F_N$, then $\|F_N\|_1 \leq \sum^N \|f_k\| < \infty$ so $F \in L^1$ and $\|F_N - F\|_1 \longrightarrow 0$ so the sum converges in $L^1$. Almost everywhere convergence: ?

## 3.4 $L^1$ Facts

**Lemma (Translation Invariance)** The Lebesgue integral is translation invariant, i.e. $\int f(x)\, dx = \int f(x+h)\, dx$ for any $h$.

**Proof**

- For characteristic functions, $\int_E f(x+h) = \int_{E+h} f(x) = m(E+h) = m(E) = \int_E f$ by translation invariance of measure.
- So this also holds for simple functions by linearity
- For $f \in L^+$, choose $\varphi_n \nearrow f$ so $\int \varphi_n \longrightarrow \int f$.
- Similarly, $\tau_h \varphi_n \nearrow \tau_h f$ so $\int \tau_h f \longrightarrow \int f$
- Finally $\left\{ \int \tau_h \varphi \right\} = \left\{ \int \varphi \right\}$ by step 1, and the suprema are equal by uniqueness of limits.

**Lemma (Integrals Distribute Over Disjoint Sets)** If $X \subseteq A \bigcup B$, then $\int_X f \leq \int_A f + \int_{A^c} f$ with equality iff $X = A \coprod B$.

**Lemma (Unif. Cts. L1 Functions Vanish at Infinity)** If $f \in L^1$ and $f$ is uniformly continuous, then $f(x) \overset{|x| \longrightarrow \infty}{\longrightarrow} 0$.

> Doesn't hold for general $L^1$ functions, take any train of triangles with height 1 and summable areas.

**Lemma (L1 Functions Have Small Tails)** If $f \in L^1$, then for every $\varepsilon$ there exists a radius $R$ such that if $A = B_R(0)^c$, then $\int_A |f| < \varepsilon$.

**Proof** Approximate with compactly supported functions. Take $g \xrightarrow{L_1} f$ with $g \in C_c$, then choose $N$ large enough so that $g = 0$ on $E := B_N(0)^c$, then $\int_E |f| \leq \int_E |f-g| + \int_E |g|$.

**Lemma ($L^1$ Functions Have Absolutely Continuity)** $m(E) \longrightarrow 0 \implies \int_E f \longrightarrow 0$.

**Proof** Approximate with compactly supported functions. Take $g \xrightarrow{L_1} f$, then $g \leq M$ so $\int_E f \leq \int_E f - g + \int_E g \longrightarrow 0 + M \cdot m(E) \longrightarrow 0$.

**Lemma ($L^1$ Functions Are Finite Almost Everywhere)** If $f \in L^1$, then $m(\{f(x) = \infty\}) = 0$.

**Proof** Idea: Split up domain Let $A = \{f(x) = \infty\}$, then $\infty > \int f = \int_A f + \int_{A^c} f = \infty \cdot m(A) + \int_{A^c} f \implies m(X) = 0$.

**Proposition (Continuity in $L^1$)**

$$\|\tau_h f - f\|_1 \xrightarrow{h \longrightarrow 0} 0$$

**Proof** Approximate with compactly supported functions. Take $g \xrightarrow{L_1} f$ with $g \in C_c$.

$$\begin{aligned}
\int f(x+h) - f(x) &\leq \int f(x+h) - g(x+h) + \int g(x+h) - g(x) + \int g(x) - f(x) \\
&\xrightarrow{? \longrightarrow ?} 2\varepsilon + \int g(x+h) - g(x) \\
&= \int_K g(x+h) - g(x) + \int_{K^c} g(x+h) - g(x) \\
&\xrightarrow{??} 0,
\end{aligned}$$

which follows because we can enlarge the support of $g$ to $K$ where the integrand is zero on $K^c$, then apply uniform continuity on $K$.

**Proposition (Integration by Parts, Special Case)**

$$\begin{aligned}
F(x) := \int_0^x f(y)dy \quad &\text{and} \quad G(x) := \int_0^x g(y)dy \\
\implies \int_0^1 F(x)g(x)dx &= F(1)G(1) - \int_0^1 f(x)G(x)dx.
\end{aligned}$$

**Proof** Fubini-Tonelli, and sketch region to change integration bounds.

**Theorem (Lebesgue Density)**

$$A_h(f)(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(y)dy \implies \|A_h(f) - f\| \xrightarrow{h \longrightarrow 0} 0.$$

**Proof** Fubini-Tonelli, and sketch region to change integration bounds, and continuity in $L^1$.

**Theorem 3.3** ***(Commuting Sums with Integrals).***

If $\{f_n\}$ integrable with either $\sum \int |f_n| < \infty$ or $\int \sum |f_n| < \infty$, then

$$\int \sum f_n = \sum \int f_n.$$

*Proof .*

- By Tonelli, if $f_n(x) \geq 0$ for all $n$, taking the counting measure allows interchanging the order of "integration".
- By Fubini on $|f_n|$, if either "iterated integral" is finite then the result follows.

■

## 3.5 $L^p$ Spaces

**Lemma** The following are dense subspaces of $L^2([0,1])$:

- Simple functions
- Step functions
- $C_0([0,1])$
- Smoothly differentiable functions $C_0^\infty([0,1])$
- Smooth compactly supported functions $C_c^\infty$ Theorem :

$$m(X) < \infty \implies \lim_{p \longrightarrow \infty} \|f\|_p = \|f\|_\infty.$$

**Proof**

- Let $M = \|f\|_\infty$.
- For any $L < M$, let $S = \{|f| \geq L\}$.
- Then $m(S) > 0$ and

$$\begin{aligned}
\|f\|_p &= \left( \int_X |f|^p \right)^{\frac{1}{p}} \\
&\geq \left( \int_S |f|^p \right)^{\frac{1}{p}} \\
&\geq L\ m(S)^{\frac{1}{p}} \overset{p \longrightarrow \infty}{\longrightarrow} L \\
&\implies \liminf_p \|f\|_p \geq M.
\end{aligned}$$

We also have

$$\begin{aligned}
\|f\|_p &= \left( \int_X |f|^p \right)^{\frac{1}{p}} \\
&\leq \left( \int_X M^p \right)^{\frac{1}{p}} \\
&= M\ m(X)^{\frac{1}{p}} \xrightarrow{p \longrightarrow \infty} M \\
&\implies \limsup_p \|f\|_p \leq M \blacksquare.
\end{aligned}$$

**Theorem (Dual Lp Spaces)** For $p \neq \infty$, $(L^p)^\vee \cong L^q$.

**Proof (p=1)** ?

**Proof (p=2)** Use Riesz Representation for Hilbert spaces.

*Proof .*
$L^1 \subset (L^\infty)^\vee$, since the isometric mapping is always injective, but *never* surjective. ■

# 4 Fourier Transform and Convolution

## 4.1 The Fourier Transform

**Definition (Convolution)**

$$f * g(x) = \int f(x-y)g(y)dy.$$

**Definition (The Fourier Transform)**

$$\widehat{f}(\xi) = \int f(x)\ e^{2\pi i x \cdot \xi}\ dx.$$

**Lemma** If $\widehat{f} = \widehat{g}$ then $f = g$ almost everywhere.

**Lemma (Riemann-Lebesgue: Fourier transforms have small tails)**

$$f \in L^1 \implies \widehat{f}(\xi) \to 0 \text{ as } |\xi| \to \infty,$$

if $f \in L^1$, then $\widehat{f}$ is continuous and bounded.

**Proof**

- Boundedness:

$$\left|\widehat{f}(\xi)\right| \leq \int |f| \cdot \left|e^{2\pi i x \cdot \xi}\right| = \|f\|_1.$$

- Continuity:
  - $\left|\widehat{f}(\xi_n) - \widehat{f}(\xi)\right|$
  - Apply DCT to show $a \overset{n\longrightarrow\infty}{\longrightarrow} 0$.

**Theorem (Fourier Inversion)**

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(x) e^{2\pi i x \cdot \xi} d\xi.$$

**Proof** Idea: Fubini-Tonelli doesn't work directly, so introduce a convergence factor, take limits, and use uniqueness of limits.

- Take the modified integral:

$$
\begin{aligned}
I_t(x) &= \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}\ e^{-\pi t^2|\xi|^2} \\
&= \int \widehat{f}(\xi)\varphi(\xi) \\
&= \int f(\xi)\widehat{\varphi}(\xi) \\
&= \int f(\xi)\widehat{\widehat{g}}(\xi - x) \\
&= \int f(\xi)g_t(x-\xi)\ d\xi \\
&= \int f(y-x)g_t(y)\ dy \quad (\xi = y - x) \\
&= (f * g_t) \\
&\longrightarrow f \text{ in } L^1 \text{ as } t \longrightarrow 0.
\end{aligned}
$$

- We also have

$$
\begin{aligned}
\lim_{t\longrightarrow 0} I_t(x) &= \lim_{t\longrightarrow 0} \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}\ e^{-\pi t^2|\xi|^2} \\
&= \lim_{t\longrightarrow 0} \int \widehat{f}(\xi)\varphi(\xi) \\
&=_{DCT} \int \widehat{f}(\xi) \lim_{t\longrightarrow 0} \varphi(\xi) \\
&= \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}
\end{aligned}
$$

.

- So

$$
I_t(x) \longrightarrow \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi} \quad \text{pointwise and} \quad \|I_t(x) - f(x)\|_1 \longrightarrow 0.
$$

- So there is a subsequence $I_{t_n}$ such that $I_{t_n}(x) \longrightarrow f(x)$ almost everywhere
- Thus $f(x) = \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}$ almost everywhere by uniqueness of limits.

**Proposition (Eigenfunction of the Fourier Transform)**

$$
g(x) := e^{-\pi|t|^2} \implies \widehat{g}(\xi) = g(\xi) \quad \text{and} \quad \widehat{g}_t(x) = g(tx) = e^{-\pi t^2|x|^2}.
$$

**Proposition (Properties of the Fourier Transform)**

?????.

## 4.2 Approximate Identities

**Definition (Dilation)**

$$\varphi_t(x) = t^{-n}\varphi\left(t^{-1}x\right).$$

**Definition (Approximation to the Identity)** For $\varphi \in L^1$, the dilations satisfy $\int \varphi_t = \int \varphi$, and if $\int \varphi = 1$ then $\varphi$ is an *approximate identity.*

Example: $\varphi(x) = e^{-\pi x^2}$

**Theorem (Convolution Against Approximate Identities Converge in $L^1$)**

$$\|f * \varphi_t - f\|_1 \overset{t\longrightarrow 0}{\longrightarrow} 0.$$

**Proof**

$$\begin{aligned}
\|f - f * \varphi_t\|_1 &= \int f(x) - \int f(x-y)\varphi_t(y)\ dydx \\
&= \int f(x) \int \varphi_t(y)\ dy - \int f(x-y)\varphi_t(y)\ dydx \\
&= \int \int \varphi_t(y)[f(x) - f(x-y)]\ dydx \\
&=_{FT} \int \int \varphi_t(y)[f(x) - f(x-y)]\ dxdy \\
&= \int \varphi_t(y) \int f(x) - f(x-y)\ dxdy \\
&= \int \varphi_t(y)\|f - \tau_y f\|_1 dy \\
&= \int_{y<\delta} \varphi_t(y)\|f - \tau_y f\|_1 dy + \int_{y\geq\delta} \varphi_t(y)\|f - \tau_y f\|_1 dy \\
&\leq \int_{y<\delta} \varphi_t(y)\varepsilon + \int_{y\geq\delta} \varphi_t(y)\left(\|f\|_1 + \|\tau_y f\|_1\right) dy \quad \text{by continuity in } L^1 \\
&\leq \varepsilon + 2\|f\|_1 \int_{y\geq\delta} \varphi_t(y) dy \\
&\leq \varepsilon + 2\|f\|_1 \cdot \varepsilon \quad \text{since } \varphi_t \text{ has small tails} \\
&\overset{\varepsilon\longrightarrow 0}{\longrightarrow} 0.
\end{aligned}$$

**Theorem (Convolutions Vanish at Infinity)**

$$f, g \in L^1 \text{ and bounded} \implies \lim_{|x|\to\infty} (f * g)(x) = 0.$$

**Proof**

- Choose $M \geq f, g$.
- By small tails, choose $N$ such that $\int_{B_N^c} |f|, \int_{B_n^c} |g| < \varepsilon$

- Note

$$|f * g| \leq \int |f(x-y)|\ |g(y)|\ dy := I.$$

- Use $|x| \leq |x-y| + |y|$, take $|x| \geq 2N$ so either

$$|x-y| \geq N \implies I \leq \int_{\{x-y \geq N\}} |f(x-y)| M\ dy \leq \varepsilon M \longrightarrow 0$$

then

$$|y| \geq N \implies I \leq \int_{\{y \geq N\}} M|g(y)|\ dy \leq M\varepsilon \longrightarrow 0.$$

Proposition (Young's Inequality?) :

$$\frac{1}{r} := \frac{1}{p} + \frac{1}{q} - 1 \implies \|f * g\|_r \leq \|f\|_p \|g\| q.$$

**Corollary** Take $q = 1$ to obtain

$$\|f * g\|_p \leq \|f\| p \|g\| 1.$$

**Corollary** If $f, g \in L^1$ then $f * g \in L^1$.

# 5 Functional Analysis

## 5.1 Definitions

Notation: $H$ denotes a Hilbert space.

**Definition (Orthonormal Sequence)** ?

**Definition (Basis)** A set $\{u_n\}$ is a *basis* for a Hilbert space $\mathcal{H}$ iff it is dense in $\mathcal{H}$.

**Definition (Complete)** A collection of vectors $\{u_n\} \subset H$ is *complete* iff $\langle x,\ u_n \rangle = 0$ for all $n \iff x = 0$ in $H$.

**Definition (Dual Space)**

$$X^\vee := \left\{ L : X \longrightarrow \mathbb{C} \;\middle|\; L \text{ is continuous} \right\}.$$

**Definition** A map $L : X \longrightarrow \mathbb{C}$ is a linear functional iff

$$L(\alpha \mathbf{x} + \mathbf{y}) = \alpha L(\mathbf{x}) + L(\mathbf{y})..$$

**Definition (Operator Norm)**

$$\|L\|_{X^\vee} := \sup_{\substack{x \in X \\ \|x\| = 1}} |L(x)|.$$

**Definition (Banach Space)** A complete normed vector space.

**Definition (Hilbert Space)** An inner product space which is a Banach space under the induced norm.

## 5.2 Theorems

**Theorem (Bessel's Inequality)** For any orthonormal set $\{u_n\} \subseteq \mathcal{H}$ a Hilbert space (not necessarily a basis),

$$\left\| x - \sum_{n=1}^{N} \langle x, u_n \rangle \, u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n \rangle|^2$$

and thus

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2.$$

**Proof**

- Let $S_N = \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n$

$$\begin{aligned}
\|x - S_N\|^2 &= \langle x - S_n,\ x - S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\langle x,\ S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\left\langle x,\ \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n \right\rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^{N} \langle x,\ \langle x,\ u_n \rangle u_n \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^{N} \overline{\langle x,\ u_n \rangle} \langle x,\ u_n \rangle \\
&= \|x\|^2 + \left\| \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n \right\|^2 - 2 \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 - 2 \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
&= \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2.
\end{aligned}$$

- By continuity of the norm and inner product, we have

$$\begin{aligned}
\lim_{N \longrightarrow \infty} \|x - S_N\|^2 &= \lim_{N \longrightarrow \infty} \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
\implies \left\| x - \lim_{N \longrightarrow \infty} S_N \right\|^2 &= \|x\|^2 - \lim_{N \longrightarrow \infty} \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
\implies \left\| x - \sum_{n=1}^{\infty} \langle x,\ u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2.
\end{aligned}$$

- Then noting that $0 \leq \|x - S_N\|^2$,

$$\begin{aligned} 0 &\leq \|x\|^2 - \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2 \\ \implies \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2 &\leq \|x\|^2 \blacksquare. \end{aligned}$$

**Theorem (Riesz Representation for Hilbert Spaces)** If $\Lambda$ is a continuous linear functional on a Hilbert space $H$, then there exists a unique $y \in H$ such that

$$\forall x \in H, \quad \Lambda(x) = \langle x,\ y \rangle..$$

**Proof**

- Define $M \coloneqq \ker \Lambda$.
- Then $M$ is a closed subspace and so $H = M \oplus M^\perp$
- There is some $z \in M^\perp$ such that $\|z\| = 1$.
- Set $u \coloneqq \Lambda(x)z - \Lambda(z)x$
- Check

$$\Lambda(u) = \Lambda(\Lambda(x)z - \Lambda(z)x) = \Lambda(x)\Lambda(z) - \Lambda(z)\Lambda(x) = 0 \implies u \in M$$

- Compute

$$\begin{aligned} 0 &= \langle u,\ z \rangle \\ &= \langle \Lambda(x)z - \Lambda(z)x,\ z \rangle \\ &= \langle \Lambda(x)z,\ z \rangle - \langle \Lambda(z)x,\ z \rangle \\ &= \Lambda(x)\langle z,\ z \rangle - \Lambda(z)\langle x,\ z \rangle \\ &= \Lambda(x)\|z\|^2 - \Lambda(z)\langle x,\ z \rangle \\ &= \Lambda(x) - \Lambda(z)\langle x,\ z \rangle \\ &= \Lambda(x) - \left\langle x,\ \overline{\Lambda(z)}z \right\rangle, \end{aligned}$$

- Choose $y \coloneqq \overline{\Lambda(z)}z$.
- Check uniqueness:

$$\begin{aligned} \langle x,\ y \rangle = \langle x,\ y' \rangle &\quad \forall x \\ \implies \langle x,\ y - y' \rangle = 0 &\quad \forall x \\ \implies \langle y - y',\ y - y' \rangle &= 0 \\ \implies \|y - y'\| &= 0 \\ \implies y - y' = \mathbf{0} &\implies y = y'. \end{aligned}$$

**Theorem (Continuous iff Bounded)** Let $L : X \longrightarrow \mathbb{C}$ be a linear functional, then the following are equivalent:

1. $L$ is continuous
2. $L$ is continuous at zero
3. $L$ is bounded, i.e. $\exists c \geq 0 \;\Big|\; |L(x)| \leq c\|x\|$ for all $x \in H$

**Proof**

$2 \implies 3$: Choose $\delta < 1$ such that

$$\|x\| \leq \delta \implies |L(x)| < 1.$$

Then

$$\begin{aligned} |L(x)| &= \left| L\left( \frac{\|x\|}{\delta} \frac{\delta}{\|x\|} x \right) \right| \\ &= \frac{\|x\|}{\delta} \left| L\left( \delta \frac{x}{\|x\|} \right) \right| \\ &\leq \frac{\|x\|}{\delta} 1, \end{aligned}$$

so we can take $c = \dfrac{1}{\delta}$. ■

$3 \implies 1$:

We have $|L(x-y)| \leq c\|x-y\|$, so given $\varepsilon \geq 0$ simply choose $\delta = \dfrac{\varepsilon}{c}$.

**Theorem (Operator Norm is a Norm)** If $H$ is a Hilbert space, then $(H^\vee, \|\cdot\|_{\text{op}})$ is a normed space.

**Proof** The only nontrivial property is the triangle inequality, but

$$\|L_1 + L_2\|_{\text{op}} = \sup |L_1(x) + L_2(x)| \leq \sup |L_1(x)| + |\sup L_2(x)| = \|L_1\|_{\text{op}} + \|L_2\|_{\text{op}}.$$

**Theorem (Completeness in Operator Norm)** If $X$ is a normed vector space, then $(X^\vee, \|\cdot\|_{\text{op}})$ is a Banach space.

**Proof**

- Let $\{L_n\}$ be Cauchy in $X^\vee$.
- Then for all $x \in C$, $\{L_n(x)\} \subset \mathbb{C}$ is Cauchy and converges to something denoted $L(x)$.
- Need to show $L$ is continuous and $\|L_n - L\| \longrightarrow 0$.
- Since $\{L_n\}$ is Cauchy in $X^\vee$, choose $N$ large enough so that

$$n, m \geq N \implies \|L_n - L_m\| < \varepsilon \implies |L_m(x) - L_n(x)| < \varepsilon \quad \forall x \;\Big|\; \|x\| = 1.$$

- Take $n \longrightarrow \infty$ to obtain

$$\begin{aligned} m \geq N &\implies |L_m(x) - L(x)| < \varepsilon \quad \forall x \;\Big|\; \|x\| = 1 \\ &\implies \|L_m - L\| < \varepsilon \longrightarrow 0. \end{aligned}$$

- Continuity:

$$\begin{aligned}|L(x)| &= |L(x) - L_n(x) + L_n(x)| \\ &\leq |L(x) - L_n(x)| + |L_n(x)| \\ &\leq \varepsilon\|x\| + c\|x\| \\ &= (\varepsilon + c)\|x\| \blacksquare.\end{aligned}$$

**Theorem (Riesz-Fischer)** Let $U = \{u_n\}_{n=1}^\infty$ be an orthonormal set (not necessarily a basis), then

1. There is an isometric surjection

$$\begin{aligned}\mathcal{H} &\longrightarrow \ell^2(\mathbb{N}) \\ \mathbf{x} &\mapsto \{\langle \mathbf{x},\ \mathbf{u}_n \rangle\}_{n=1}^\infty\end{aligned}$$

i.e. if $\{a_n\} \in \ell^2(\mathbb{N})$, so $\sum |a_n|^2 < \infty$, then there exists a $\mathbf{x} \in \mathcal{H}$ such that

$$a_n = \langle \mathbf{x},\ \mathbf{u}_n \rangle \quad \forall n.$$

2. $\mathbf{x}$ can be chosen such that

$$\|\mathbf{x}\|^2 = \sum |a_n|^2$$

> Note: the choice of $\mathbf{x}$ is unique $\iff$ $\{u_n\}$ is **complete**, i.e. $\langle \mathbf{x},\ \mathbf{u}_n \rangle = 0$ for all $n$ implies $\mathbf{x} = \mathbf{0}$.

**Proof**

- Given $\{a_n\}$, define $S_N = \sum^N a_n \mathbf{u}_n$.
- $S_N$ is Cauchy in $\mathcal{H}$ and so $S_N \longrightarrow \mathbf{x}$ for some $\mathbf{x} \in \mathcal{H}$.
- $\langle x,\ u_n \rangle = \langle x - S_N,\ u_n \rangle + \langle S_N,\ u_n \rangle \longrightarrow a_n$
- By construction, $\|x - S_N\|^2 = \|x\|^2 - \sum^N |a_n|^2 \longrightarrow 0$, so $\|x\|^2 = \sum^\infty |a_n|^2$.

# 6 Extra Problems

## 6.1 Greatest Hits

- $\star$: Show that for $E \subseteq \mathbb{R}^n$, TFAE:
  1. $E$ is measurable
  2. $E = H \bigcup Z$ here $H$ is $F_\sigma$ and $Z$ is null
  3. $E = V \setminus Z'$ where $V \in G_\delta$ and $Z'$ is null.
- $\star$: Show that if $E \subseteq \mathbb{R}^n$ is measurable then $m(E) = \sup\left\{ m(K) \,\middle|\, K \subset E \text{ compact} \right\}$ iff for all $\varepsilon > 0$ there exists a compact $K \subseteq E$ such that $m(K) \geq m(E) - \varepsilon$.
- $\star$: Show that cylinder functions are measurable, i.e. if $f$ is measurable on $\mathbb{R}^s$, then $F(x, y) := f(x)$ is measurable on $\mathbb{R}^s \times \mathbb{R}^t$ for any $t$.

- $\star$: Prove that the Lebesgue integral is translation invariant, i.e. if $\tau_h(x) = x + h$ then $\displaystyle\int \tau_h f = \int f$.
- $\star$: Prove that the Lebesgue integral is dilation invariant, i.e. if $f_\delta(x) = \dfrac{f(\frac{x}{\delta})}{\delta^n}$ then $\displaystyle\int f_\delta = \int f$.
- $\star$: Prove continuity in $L^1$, i.e.

$$f \in L^1 \Longrightarrow \lim_{h \to 0} \int |f(x+h) - f(x)| = 0.$$

- $\star$: Show that

$$f, g \in L^1 \implies f * g \in L^1 \quad \text{and} \quad \|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

- $\star$: Show that if $X \subseteq \mathbb{R}$ with $\mu(X) < \infty$ then

$$\|f\|_p \overset{p \longrightarrow \infty}{\longrightarrow} \|f\|_\infty.$$

## 6.2 By Topic

### 6.2.1 Topology

- Show that every compact set is closed and bounded.
- Show that if a subset of a metric space is complete and totally bounded, then it is compact.
- Show that if $K$ is compact and $F$ is closed with $K, F$ disjoint then $\text{dist}(K, F) > 0$.

### 6.2.2 Continuity

- Show that a continuous function on a compact set is uniformly continuous.

### 6.2.3 Differentiation

- Show that if $f \in C^1(\mathbb{R})$ and both $\lim_{x \longrightarrow \infty} f(x)$ and $\lim_{x \longrightarrow \infty} f'(x)$ exist, then $\lim_{x \longrightarrow \infty} f'(x)$ must be zero.

### 6.2.4 Advanced Limitology

- If $f$ is continuous, is it necessarily the case that $f'$ is continuous?
- If $f_n \longrightarrow f$, is it necessarily the case that $f_n'$ converges to $f'$ (or at all)?
- Is it true that the sum of differentiable functions is differentiable?
- Is it true that the limit of integrals equals the integral of the limit?
- Is it true that a limit of continuous functions is continuous?
- Show that a subset of a metric space is closed iff it is complete.
- Show that if $m(E) < \infty$ and $f_n \longrightarrow f$ uniformly, then $\displaystyle\lim \int_E f_n = \int_E f$.

Uniform Convergence

- Show that a uniform limit of bounded functions is bounded.

- Show that a uniform limit of continuous function is continuous.
  - I.e. if $f_n \longrightarrow f$ uniformly with each $f_n$ continuous then $f$ is continuous.
- Show that if $f_n \longrightarrow f$ pointwise, $f_n' \longrightarrow g$ uniformly for some $f, g$, then $f$ is differentiable and $g = f'$.
- Prove that uniform convergence implies pointwise convergence implies a.e. convergence, but none of the implications may be reversed.
- Show that $\sum \frac{x^n}{n!}$ converges uniformly on any compact subset of $\mathbb{R}$.

Measure Theory

- Show that continuity of measure from above/below holds for outer measures.
- Show that a countable union of null sets is null.

Measurability

- Show that $f = 0$ a.e. iff $\int_E f = 0$ for every measurable set $E$.

Integrability

- Show that if $f$ is a measurable function, then $f = 0$ a.e. iff $\int f = 0$.
- Show that a bounded function is Lebesgue integrable iff it is measurable.
- Show that simple functions are dense in $L^1$.
- Show that step functions are dense in $L^1$.
- Show that smooth compactly supported functions are dense in $L^1$.

Convergence

- Prove Fatou's lemma using the Monotone Convergence Theorem.
- Show that if $\{f_n\}$ is in $L^1$ and $\sum \int |f_n| < \infty$ then $\sum f_n$ converges to an $L^1$ function and

$$\int \sum f_n = \sum \int f_n.$$

Convolution

- Show that if $f \in L^1$ and $g$ is bounded, then $f * g$ is bounded and uniformly continuous.
- If $f, g$ are compactly supported, is it necessarily the case that $f * g$ is compactly supported?
- Show that under any of the following assumptions, $f * g$ vanishes at infinity:
  - $f, g \in L^1$ are both bounded.
  - $f, g \in L^1$ with just $g$ bounded.
  - $f, g$ smooth and compactly supported (and in fact $f * g$ is smooth)
  - $f \in L^1$ and $g$ smooth and compactly supported (and in fact $f * g$ is smooth)
- Show that if $f \in L^1$ and $g'$ exists with $\frac{\partial g}{\partial x_i}$ all bounded, then

$$\frac{\partial}{\partial x_i}(f * g) = f * \frac{\partial g}{\partial x_i}$$

Fourier Analysis

- Show that if $f \in L^1$ then $\widehat{f}$ is bounded and uniformly continuous.
- Is it the case that $f \in L^1$ implies $\widehat{f} \in L^1$?

- Show that if $f, \widehat{f} \in L^1$ then $f$ is bounded, uniformly continuous, and vanishes at infinity.
  - Show that this is not true for arbitrary $L^1$ functions.
- Show that if $f \in L^1$ and $\widehat{f} = 0$ almost everywhere then $f = 0$ almost everywhere.
  - Prove that $\widehat{f} = \widehat{g}$ implies that $f = g$ a.e.
- Show that if $f, g \in L^1$ then

$$\int \widehat{f} g = \int f \widehat{g}.$$

  - Give an example showing that this fails if $g$ is not bounded.
- Show that if $f \in C^1$ then $f$ is equal to its Fourier *series*.

Approximate Identities

- Show that if $\varphi$ is an approximate identity, then

$$\|f * \varphi_t - f\|_1 \overset{t \longrightarrow 0}{\longrightarrow} 0.$$

  - Show that if additionally $|\varphi(x)| \leq c(1 + |x|)^{-n-\varepsilon}$ for some $c, \varepsilon > 0$, then this converges is almost everywhere.
- Show that is $f$ is bounded and uniformly continuous and $\varphi_t$ is an approximation to the identity, then $f * \varphi_t$ uniformly converges to $f$.

$L^p$ Spaces

- Show that if $E \subseteq \mathbb{R}^n$ is measurable with $\mu(E) < \infty$ and $f \in L^p(X)$ then

$$\|f\|_{L^p(X)} \overset{p \longrightarrow \infty}{\longrightarrow} \|f\|_\infty.$$

- Is it true that the converse to the DCT holds? I.e. if $\displaystyle\int f_n \longrightarrow \int f$, is there a $g \in L^p$ such that $f_n < g$ a.e. for every $n$?
- Prove continuity in $L^p$: If $f$ is uniformly continuous then for all $p$,

$$\|\tau_h f - f\|_p \overset{h \longrightarrow 0}{\longrightarrow} 0.$$

- Prove the following inclusions of $L^p$ spaces for $m(X) < \infty$:

$$L^\infty(X) \subset L^2(X) \subset L^1(X)$$
$$\ell^2(\mathbb{Z}) \subset \ell^1(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z}).$$

# 7 Practice Exam (November 2014)

## 7.1 1: Fubini-Tonelli

### 7.1.1 a

Carefully state Tonelli's theorem for a nonnegative function $F(x, t)$ on $\mathbb{R}^n \times \mathbb{R}$.

#### 7.1.2 b

Let $f : \mathbb{R}^n \longrightarrow [0, \infty]$ and define

$$\mathcal{A} \coloneqq \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; 0 \le t \le f(x) \right\}.$$

Prove the validity of the following two statements:

1. $f$ is Lebesgue measurable on $\mathbb{R}^n \iff \mathcal{A}$ is a Lebesgue measurable subset of $\mathbb{R}^{n+1}$.
2. If $f$ is Lebesgue measurable on $\mathbb{R}^n$ then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)dx = \int_0^\infty m\left( \left\{ x \in \mathbb{R}^n \;\middle|\; f(x) \ge t \right\} \right) dt.$$

### 7.2 2: Convolutions and the Fourier Transform

#### 7.2.1 a

Let $f, g \in L^1(\mathbb{R}^n)$ and give a definition of $f * g$.

#### 7.2.2 b

Prove that if $f, g$ are integrable and bounded, then

$$(f * g)(x) \overset{|x| \longrightarrow \infty}{\longrightarrow} 0.$$

#### 7.2.3 c

1. Define the *Fourier transform* of an integrable function $f$ on $\mathbb{R}^n$.
2. Give an outline of the proof of the Fourier inversion formula.
3. Give an example of a function $f \in L^1(\mathbb{R}^n)$ such that $\widehat{f}$ is not in $L^1(\mathbb{R}^n)$.

### 7.3 3: Hilbert Spaces

Let $\{u_n\}_{n=1}^\infty$ be an orthonormal sequence in a Hilbert space $H$.

#### 7.3.1 a

Let $x \in H$ and verify that

$$\left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|_H^2 = \|x\|_H^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 .$$

for any $N \in \mathbb{N}$ and deduce that

$$\sum_{n=1}^\infty |\langle x, u_n \rangle|^2 \le \|x\|_H^2.$$

### 7.3.2 b

Let $\{a_n\}_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ and prove that there exists an $x \in H$ such that $a_n = \langle x,\ u_n\rangle$ for all $n \in \mathbb{N}$, and moreover $x$ may be chosen such that

$$\|x\|_H = \left(\sum_{n\in\mathbb{N}} |a_n|^2\right)^{\frac{1}{2}}.$$

**Proof**

- Take $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
- Define $x \coloneqq \lim_{N\longrightarrow\infty} S_N$ where $S_N = \sum_{k=1}^{N} a_k u_k$
- $\{S_N\}$ is Cauchy and $H$ is complete, so $x \in H$.
- By construction,

  $$\langle x,\ u_n\rangle = \left\langle \sum_k a_k u_k,\ u_n \right\rangle = \sum_k a_k \langle u_k,\ u_n\rangle = a_n$$

  since the $u_k$ are all orthogonal.
- By Pythagoras since the $u_k$ are normal,

  $$\|x\|^2 = \left\|\sum_k a_k u_k\right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2.$$

### 7.3.3 c

Prove that if $\{u_n\}$ is *complete*, Bessel's inequality becomes an equality.

**Proof** Let $x$ and $u_n$ be arbitrary.

$$\begin{aligned}
\left\langle x - \sum_{k=1}^{\infty} \langle x,\ u_k\rangle u_k,\ u_n \right\rangle &= \langle x,\ u_n\rangle - \left\langle \sum_{k=1}^{\infty} \langle x,\ u_k\rangle u_k,\ u_n \right\rangle \\
&= \langle x,\ u_n\rangle - \sum_{k=1}^{\infty} \langle \langle x,\ u_k\rangle u_k,\ u_n\rangle \\
&= \langle x,\ u_n\rangle - \sum_{k=1}^{\infty} \langle x,\ u_k\rangle \langle u_k,\ u_n\rangle \\
&= \langle x,\ u_n\rangle - \langle x,\ u_n\rangle = 0 \\
\implies x - \sum_{k=1}^{\infty} \langle x,\ u_k\rangle u_k = 0 \quad &\text{by completeness.}
\end{aligned}$$

So

$$x = \sum_{k=1}^{\infty} \langle x,\ u_k\rangle u_k \implies \|x\|^2 = \sum_{k=1}^{\infty} |\langle x,\ u_k\rangle|^2. \blacksquare.$$

## 7.4 4: $L^p$ Spaces

### 7.4.1 a

Prove Holder's inequality: let $f \in L^p, g \in L^q$ with $p, q$ conjugate, and show that

$$\|fg\|_p \leq \|f\|_p \cdot \|g\|_q.$$

### 7.4.2 b

Prove Minkowski's Inequality:

$$1 \leq p < \infty \implies \|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Conclude that if $f, g \in L^p(\mathbb{R}^n)$ then so is $f+g$.

### 7.4.3 c

Let $X = [0,1] \subset \mathbb{R}$.

1. Give a definition of the Banach space $L^\infty(X)$ of essentially bounded functions of $X$.
2. Let $f$ be non-negative and measurable on $X$, prove that

$$\int_X f(x)^p \, dx \overset{p \longrightarrow \infty}{\longrightarrow} \begin{cases} \infty & \text{or} \\ m\left(\left\{f^{-1}(1)\right\}\right) \end{cases},$$

and characterize the functions of each type

**Proof**

$$\begin{aligned} \int f^p &= \int_{x<1} f^p + \int_{x=1} f^p + \int_{x>1} f^p \\ &= \int_{x<1} f^p + \int_{x=1} 1 + \int_{x>1} f^p \\ &= \int_{x<1} f^p + m(\{f=1\}) + \int_{x>1} f^p \\ &\overset{p \longrightarrow \infty}{\longrightarrow} 0 + m(\{f=1\}) + \begin{cases} 0 & m(\{x \geq 1\}) = 0 \\ \infty & m(\{x \geq 1\}) > 0. \end{cases} \end{aligned}$$

Justify passing limit into integrals.

## 7.5 5: Dual Spaces

Let $X$ be a normed vector space.

### 7.5.1 a

Give the definition of what it means for a map $L : X \longrightarrow \mathbb{C}$ to be a *linear functional*.

#### 7.5.2 b

Define what it means for $L$ to be *bounded* and show $L$ is bounded $\iff$ $L$ is continuous.

#### 7.5.3 c

Prove that $(X^\vee, \|\cdot\|_{\text{op}})$ is a Banach space.

# 8 Midterm Exam 2 (November 2018)

## 8.1 1 (Integration by Parts)

Let $f, g \in L^1([0,1])$, define $F(x) = \displaystyle\int_0^x f$ and $G(x) = \displaystyle\int_0^x g$, and show

$$\int_0^1 F(x)g(x)\,dx = F(1)G(1) - \int_0^1 f(x)G(x)\,dx.$$

## 8.2 2

Let $\varphi \in L^1(\mathbb{R}^n)$ such that $\displaystyle\int \varphi = 1$ and define $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$.

Show that if $f$ is bounded and uniformly continuous then $f * \varphi_t \overset{t\longrightarrow 0}{f}$ uniformly.

## 8.3 3

Let $g \in L^\infty([0,1])$.

a. Prove

$$\|g\|_{L^p([0,1])} \overset{p\longrightarrow\infty}{\longrightarrow} \|g\|_{L^\infty([0,1])}.$$

b. Prove that the map

$$\begin{aligned} \Lambda_g : L^1([0,1]) &\longrightarrow \mathbb{C} \\ f &\mapsto \int_0^1 fg \end{aligned}$$

defines an element of $L^1([0,1])^\vee$ with $\|\Lambda_g\|_{L^1([0,1])^\vee} = \|g\|_{L^\infty([0,1])}$.

Note: 4 is a repeat.

# 9 Midterm Exam 2 (December 2014)

## 9.1 1

Note: (a) is a repeat.

- Let $\Lambda \in L^2(X)^\vee$.

- Show that $M := \left\{ f \in L^2(X) \;\middle|\; \Lambda(f) = 0 \right\} \subseteq L^2(X)$ is a closed subspace, and $L^2(X) = M \oplus M \perp$.
- Prove that there exists a unique $g \in L^2(X)$ such that $\Lambda(f) = \int_X g\bar{f}$.

### 9.2 2

a. In parts:

- Given a definition of $L^\infty(\mathbb{R}^n)$.
- Verify that $\|\cdot\|_\infty$ defines a norm on $L^\infty(\mathbb{R}^n)$.
- Carefully proved that $(L^\infty(\mathbb{R}^n), \|\cdot\|_\infty)$ is a Banach space.

b. Prove that for any measurable $f : \mathbb{R}^n \longrightarrow \mathbb{C}$,

$$L^1(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \quad \text{and} \quad \|f\|_2 \leq \|f\|_1^{\frac{1}{2}} \cdot \|f\|_\infty^{\frac{1}{2}}.$$

### 9.3 3

a. Prove that if $f, g : \mathbb{R}^n \longrightarrow \mathbb{C}$ is both measurable then $F(x, y) := f(x)$ and $h(x, y) := f(x-y)g(y)$ is measurable on $\mathbb{R}^n \times \mathbb{R}^n$.

b. Show that if $f \in L^1(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, then $f * g \in L^1(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$ is well defined, and carefully show that it satisfies the following properties:

$$\|f * g\|_\infty \leq \|g\|_1 \|f\|_\infty \|f * g\|_1 \qquad \leq \|g\|_1 \|f\|_1 \|f * g\|_2 \leq \|g\|_1 \|f\|_2.$$

Hint: first show $|f * g|^2 \leq \|g\|_1 \left(|f|^2 * |g|\right)$.

### 9.4 4 (Weierstrass Approximation Theorem)

Note: (a) is a repeat.

Let $f : [0, 1] \longrightarrow \mathbb{R}$ be continuous, and prove the Weierstrass approximation theorem: for any $\varepsilon > 0$ there exists a polynomial $P$ such that $\|f - P\|_\infty < \varepsilon$.

## 10 Inequalities and Equalities

**Proposition (Reverse Triangle Inequality)**

$$\left| \|x\| - \|y\| \right| \leq \|x - y\|.$$

**Proposition (Chebyshev's Inequality)**

$$\mu(\{x : |f(x)| > \alpha\}) \leq \left( \frac{\|f\|_p}{\alpha} \right)^p.$$

**Proposition (Holder's Inequality When Surjective)**

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \|fg\|_1 \leq \|f\|_p \|g\|_q.$$

*Application:* For finite measure spaces,

$$1 \leq p < q \leq \infty \implies L^q \subset L^p \quad (\text{ and } \ell^p \subset \ell^q).$$

**Proof (Holder's Inequality)** Fix $p, q$, let $r = \dfrac{q}{p}$ and $s = \dfrac{r}{r-1}$ so $r^{-1} + s^{-1} = 1$. Then let $h = |f|^p$:

$$\|f\|_p^p = \|h \cdot 1\|_1 \leq \|1\|_s \|h\|_r = \mu(X)^{\frac{1}{s}} \|f\|_q^{\frac{q}{r}} \implies \|f\|_p \leq \mu(X)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q.$$

Note: doesn't work for $\ell_p$ spaces, but just note that $\sum |x_n| < \infty \implies x_n < 1$ for large enough $n$, and thus $p < q \implies |x_n|^q \leq |x_n|^q$.

**Proof (Holder's Inequality)** It suffices to show this when $\|f\|_p = \|g\|_q = 1$, since

$$\|fg\|_1 \leq \|f\|_p \|f\|_q \iff \int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \leq 1.$$

Using $AB \leq \dfrac{1}{p} A^p + \dfrac{1}{q} B^q$, we have

$$\int |f||g| \leq \int \frac{|f|^p}{p} \frac{|g|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1.$$

**Proposition (Cauchy-Schwarz Inequality)**

$$|\langle f,\ g\rangle| = \|fg\|_1 \leq \|f\|_2 \|g\|_2 \quad \text{with equality} \quad \iff f = \lambda g.$$

Note: Relates inner product to norm, and only happens to relate norms in $L^1$.

**Proof** ?

**Proposition (Minkowski's Inequality:)**

$$1 \leq p < \infty \implies \|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Note: does not handle $p = \infty$ case. Use to prove $L^p$ is a normed space.

**Proof**

- We first note

$$|f+g|^p = |f+g||f+g|^{p-1} \leq (|f|+|g|)\,|f+g|^{p-1}.$$

- Note that if $p, q$ are conjugate exponents then

$$\begin{aligned}
\frac{1}{q} &= 1 - \frac{1}{p} = \frac{p-1}{p} \\
q &= \frac{p}{p-1}.
\end{aligned}$$

- Then taking integrals yields

$$\begin{aligned}
\|f+g\|_p^p &= \int |f+g|^p \\
&\leq \int (|f|+|g|)\,|f+g|^{p-1} \\
&= \int |f||f+g|^{p-1} + \int |g||f+g|^{p-1} \\
&= \left\|f(f+g)^{p-1}\right\|_1 + \left\|g(f+g)^{p-1}\right\|_1 \\
&\leq \|f\|_p \left\|(f+g)^{p-1})\right\|_q + \|g\|_p \left\|(f+g)^{p-1})\right\|_q \\
&= \left(\|f\|_p + \|g\|_p\right) \left\|(f+g)^{p-1})\right\|_q \\
&= \left(\|f\|_p + \|g\|_p\right) \left(\int |f+g|^{(p-1)q}\right)^{\frac{1}{q}} \\
&= \left(\|f\|_p + \|g\|_p\right) \left(\int |f+g|^{p}\right)^{1-\frac{1}{p}} \\
&= \left(\|f\|_p + \|g\|_p\right) \frac{\int |f+g|^p}{\left(\int |f+g|^p\right)^{\frac{1}{p}}} \\
&= \left(\|f\|_p + \|g\|_p\right) \frac{\|f+g\|_p^p}{\|f+g\|_p}.
\end{aligned}$$

- Cancelling common terms yields

$$\begin{aligned}
1 &\leq \left(\|f\|_p + \|g\|_p\right) \frac{1}{\|f+g\|_p} \\
&\implies \|f+g\|_p \leq \|f\|_p + \|g\|_p.
\end{aligned}$$

**Proposition (Young's Inequality*)**

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \implies \|f * g\|_r \leq \|f\|_p \|g\|_q$$

**Application**: Some useful specific cases:

$$\begin{aligned}
\|f * g\|_1 &\leq \|f\|_1 \|g\|_1 \\
\|f * g\|_p &\leq \|f\|_1 \|g\| p, \\
\|f * g\|_\infty &\leq \|f\|_2 \|g\|_2 \\
\|f * g\|_\infty &\leq \|f\|_p \|g\|_q.
\end{aligned}$$

**Proposition (Bessel's Inequality:)**

For $x \in H$ a Hilbert space and $\{e_k\}$ an orthonormal sequence,

$$\sum_{k=1}^{\infty} \|\langle x,\ e_k \rangle\|^2 \leq \|x\|^2.$$

Note: this does not need to be a basis.

**Proposition (Parseval's Identity:)** Equality in Bessel's inequality, attained when $\{e_k\}$ is a *basis*, i.e. it is complete, i.e. the span of its closure is all of $H$.

## 10.1 Less Explicitly Used Inequalities

**Proposition (AM-GM Inequality)**

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

**Proposition (Jensen's Inequality)**

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Proposition (???) :

$$AB \leq \frac{A^p}{p} + \frac{B^q}{q}.$$

**Proposition (? Inequality)**

$$(a+b)^p \leq 2^p(a^p + b^p).$$

**Proposition (Bernoulli's Inequality)**

$$(1+x)^n \geq 1 + nx \quad x \geq -1, \text{ or } n \in 2\mathbb{Z} \text{ and } \forall x.$$