

# Complex Analysis Problems

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## 1 Topology and Functions of One Variable (8155a)

## 2 Several Variables (8155h)

## 3 Conformal Maps (8155c)

Notation:  $\mathbb{D}$  is the open unit disc,  $\mathbb{H}$  is the open upper half-plane.

### 3.1 1

Find a conformal map from  $\mathbb{D}$  to  $\mathbb{H}$ .

### 3.2 2

Find a conformal map from the strip  $\{z \in \mathbb{C} \mid 0 < \Im(z) < 1\}$  to  $\mathbb{H}$ .

### 3.3 3

Find a fractional linear transformation  $T$  which maps  $\mathbb{H}$  to  $\mathbb{D}$ , and explicitly describe the image of the first quadrant under  $T$ .

### 3.4 4

Find a conformal map from  $\{z \in \mathbb{C} \mid |z - i| > 1, \Re(z) > 0\}$  to  $\mathbb{H}$ .

### 3.5 5

Find a conformal map from  $\{z \in \mathbb{C} \mid |z| < 1, \left|z - \frac{1}{2}\right| > \frac{1}{2}\}$  to  $\mathbb{D}$ .

### 3.6 6

Find a conformal map from  $\{|z - 1| < 2\} \cap \{|z + 1| < 2\}$  to  $\mathbb{H}$ .

### 3.7 7

Let  $\Omega$  be the region inside the unit circle  $|z| = 1$  and outside the circle  $\left|z - \frac{1}{4}\right| = \frac{1}{4}$ .

Find an injective conformal map from  $\Omega$  onto some annulus  $\{r < |z| < 1\}$  for an appropriate constant  $r$ .

### 3.8 8

Let  $D$  be the region obtained by deleting the real interval  $[0, 1)$  from  $\mathbb{D}$ ; find a conformal map from  $D$  to  $\mathbb{D}$ .

**3.9 9**

Find a conformal map from  $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$  to  $\mathbb{D}$ .

**3.10 10**

Find a conformal map from  $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \geq 1\}$  to  $\mathbb{D}$ .

**3.11 11**

Find a bijective conformal map from  $G$  to  $\mathbb{H}$ , where

$$G := \{z \in \mathbb{C} \mid |z - 1| < \sqrt{2}, |z + 1| < \sqrt{2}\} \setminus [0, i).$$

**3.12 12**

Prove that TFAE for a Möbius transformation  $T$  given by  $T(z) = \frac{az + b}{cz + d}$ :

**4 Integrals and Cauchy's Theorem (8155d)**

Some interesting problems: 3, 4, 9, 10.

**4.1 1**

Suppose  $f, g : [0, 1] \rightarrow \mathbb{R}$  where  $f$  is Riemann integrable and for  $x, y \in [0, 1]$ ,

$$|g(x) - g(y)| \leq |f(x) - f(y)|.$$

Prove that  $g$  is Riemann integrable.

**4.2 2**

State and prove Green's Theorem for rectangles.

Then use it to prove Cauchy's Theorem for functions that are analytic in a rectangle.

**4.3 3**

Suppose  $\{f_n\}_{n \in \mathbb{N}}$  is a sequence of analytic functions on  $\mathbb{D}^\circ := \{z \in \mathbb{C} \mid |z| < 1\}$ .

Show that if  $f_n \rightarrow g$  for some  $g : \mathbb{D}^\circ \rightarrow \mathbb{C}$  uniformly on every compact  $K \subset \mathbb{D}^\circ$ , then  $g$  is analytic on  $\mathbb{D}^\circ$ .



**4.4 4**

Suppose  $\{f_n\}_{n \in \mathbb{N}}$  is a sequence of entire functions where

- $f_n \rightarrow g$  pointwise for some  $g : \mathbb{C} \rightarrow \mathbb{C}$ .
- On every line segment in  $\mathbb{C}$ ,  $f_n \rightarrow g$  uniformly.

Show that

- $g$  is entire, and
- $f_n \rightarrow g$  uniformly on every compact subset of  $\mathbb{C}$ .

**4.5 5**

Prove that there is no sequence of polynomials that uniformly converge to  $f(z) = \frac{1}{z}$  on  $S^1$ .

**4.6 6**

Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function that vanishes outside of some finite interval. For each  $z \in \mathbb{C}$ , define

$$g(z) = \int_{-\infty}^{\infty} f(t)e^{-izt} dt.$$

Show that  $g$  is entire.

**4.7 7**

Suppose  $f : \mathbb{C} \rightarrow \mathbb{C}$  is entire and

$$|f(z)| \leq |z|^{\frac{1}{2}} \quad \text{when } |z| > 10.$$

Prove that  $f$  is constant.

**4.8 8**

Let  $\gamma$  be a smooth curve joining two distinct points  $a, b \in \mathbb{C}$ .

Prove that the function

$$f(z) := \int_{\gamma} \frac{g(w)}{w - z} dw$$

is analytic in  $\mathbb{C} \setminus \gamma$ .

**4.9 9**

Suppose that  $f : \mathbb{C} \rightarrow \mathbb{C}$  is continuous everywhere and analytic on  $\mathbb{C} \setminus \mathbb{R}$  and prove that  $f$  is entire.

**4.10 10**

Prove Liouville's theorem: suppose  $f : \mathbb{C} \rightarrow \mathbb{C}$  is entire and bounded. Use Cauchy's formula to prove that  $f' \equiv 0$  and hence  $f$  is constant.

**5 Liouville's Theorem, Power Series (8155e)****5.1 1**

Suppose  $f$  is analytic on a region  $\Omega$  such that  $\mathbb{D} \subseteq \Omega \subseteq \mathbb{C}$  and  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  is a power series with radius of convergence exactly 1.

- Give an example of such an  $f$  that converges at every point of  $S^1$ .
- Given an example of such an  $f$  which is analytic at 1 but  $\sum_{n=0}^{\infty} a_n$  diverges.
- Prove that  $f$  can not be analytic at *every* point of  $S^1$ .

**5.2 2**

Suppose  $f$  is entire and has Taylor series  $\sum a_n z^n$  about 0.

- Express  $a_n$  as a contour integral along the circle  $|z| = R$ .
- Apply (a) to show that the above Taylor series converges uniformly on every bounded subset of  $\mathbb{C}$ .
- Determine those functions  $f$  for which the above Taylor series converges uniformly on all of  $\mathbb{C}$ .

**5.3 3**

Suppose  $D$  is a domain and  $f, g$  are analytic on  $D$ .

Prove that if  $fg = 0$  on  $D$ , then either  $f \equiv 0$  or  $g \equiv 0$  on  $D$ .

**5.4 4**

Suppose  $f$  is analytic on  $\mathbb{D}^\circ$ . Determine with proof which of the following are possible:

- $f\left(\frac{1}{n}\right) = (-1)^n$  for each  $n > 1$ .
- $f\left(\frac{1}{n}\right) = e^{-n}$  for each even integer  $n > 1$  while  $f\left(\frac{1}{n}\right) = 0$  for each odd integer  $n > 1$ .
- $f\left(\frac{1}{n^2}\right) = \frac{1}{n}$  for each integer  $n > 1$ .
- $f\left(\frac{1}{n}\right) = \frac{n-2}{n-1}$  for each integer  $n > 1$ .

**5.5 5**

Prove the Fundamental Theorem of Algebra (using complex analysis).

**5.6 6**

Find all entire functions that satisfy

$$|f(z)| \geq |z| \quad \forall z \in \mathbb{C}.$$

Prove this list is complete.

**5.7 7**

Suppose  $\sum_{n=0}^{\infty} a_n z^n$  converges for some  $z_0 \neq 0$ .

- Prove that the series converges absolutely for each  $z$  with  $|z| < |z_0|$ .
- Suppose  $0 < r < |z_0|$  and show that the series converges uniformly on  $|z| \leq r$ .

**5.8 8**

Suppose  $f$  is entire and suppose that for some integer  $n \geq 1$ ,

$$\lim_{z \rightarrow \infty} \frac{f(z)}{z^n} = 0.$$

Prove that  $f$  is a polynomial of degree at most  $n - 1$ .

**5.9 9**

Find all entire functions satisfying

$$|f(z)| \leq |z|^{\frac{1}{2}} \quad \text{for } |z| > 10.$$

**5.10 10**

Prove that the following series converges uniformly on the set  $\{z \mid \Im(z) < \ln 2\}$ :

$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{2^n}.$$

**6 Laurent Expansions and Singularities (8155f)****6.1 1**

Find the Laurent expansion of

$$f(z) = \frac{z+1}{z(z-1)}$$

about  $z = 0$  and  $z = 1$  respectively.

**6.2 2**

Find the Laurent expansions about  $z = 0$  of the following functions:

$$\exp\left(\frac{1}{z}\right) \qquad \cos\left(\frac{1}{z}\right).$$

**6.3 3**

Find the Laurent expansion of

$$f(z) = \frac{z+1}{z(z-1)^2}$$

about  $z = 0$  and  $z = 1$  respectively.

Hint: recall that power series can be differentiated.

**6.4 4**

For the following functions, find the Laurent series about 0 and classify their singularities there:

$$\frac{\sin^2(z)}{z} \\ z \exp\left(\frac{1}{z^2}\right) \\ \frac{1}{z(4-z)}.$$

**6.5 5**

Find all entire functions which have poles at  $\infty$ .

**6.6 6**

Find all functions on the Riemann sphere that have a simple pole at  $z = 2$  and a double pole at  $z = \infty$ , but are analytic elsewhere.

**6.7 7**

Let  $f$  be entire, and discuss (with proofs and examples) the types of singularities  $f$  might have (removable, pole, or essential) at  $z = \infty$  in the following cases:

1.  $f$  has at most finitely many zeros in  $\mathbb{C}$ .
2.  $f$  has infinitely many zeros in  $\mathbb{C}$ .

**6.8 8**

Define

$$f(z) = \frac{\pi^2}{\sin^2(\pi z)}$$
$$g(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z - n)^2}.$$

- Show that  $f$  and  $g$  have the same singularities in  $\mathbb{C}$ .
- Show that  $f$  and  $g$  have the same singular parts at each of their singularities.
- Show that  $f, g$  each have period one and approach zero uniformly on  $0 \leq x \leq 1$  as  $|y| \rightarrow \infty$ .
- Conclude that  $f = g$ .

**7 Residues (8155g)****7.1 1**

Calculate

$$\int_0^\infty \frac{1}{(1+x)^2(x+9x^2)} dx.$$

**7.2 2**

Let  $a > 0$  and calculate

$$\int_0^\infty \frac{x \sin(x)}{x^2 + a^2} dx.$$

**7.3 3**

Calculate

$$\int_0^\infty \frac{\sqrt{x}}{(x+1)^2} dx.$$

**7.4 4**

Calculate

$$\int_0^\infty \frac{\cos(x) - \cos(4x)}{x^2} dx.$$

**7.5 5**

Let  $a > 0$  and calculate

$$\int_0^\infty \frac{x^2}{(x^2 + a^2)^2} dx.$$

**7.6 6**

Calculate

$$\int_0^\infty \frac{\sin(x)}{x} dx.$$

**7.7 7**

Calculate

$$\int_0^\infty \frac{\sin(x)}{x(x^2 + 1)} dx.$$

**7.8 8**

Calculate

$$\int_0^\infty \frac{\sqrt{x}}{1 + x^2} dx.$$

**7.9 9**

Calculate

$$\int_{-\infty}^\infty \frac{1 + x^2}{1 + x^4} dx.$$

**7.10 10**Let  $a > 0$  and calculate

$$\int_0^\infty \frac{\cos(x)}{(x^2 + a^2)^2} dx.$$

**7.11 11**

Calculate

$$\int_0^\infty \frac{\sin^3(x)}{x^3} dx.$$

**7.12 12**Let  $n \in \mathbb{Z}^{\geq 1}$  and  $0 < \theta < \pi$  and show that

$$\frac{1}{2\pi i} \int_{|z|=2} \frac{z^n}{1 - 3z \cos(\theta) + z^2} dz = \frac{\sin(n\theta)}{\sin(\theta)}.$$

**7.13 13**

Suppose  $a > b > 0$  and calculate

$$\int_0^{2\pi} \frac{1}{(a + b \cos(\theta))^2} d\theta.$$

**8 Rouché's Theorem (8155h)****8.1 1**

Prove that for every  $n \in \mathbb{Z}^{\geq 0}$  the following polynomial has no roots in the open unit disc:

$$f_n(x) := \sum_{k=0}^n \frac{z^k}{k!}.$$

Hint: check  $n = 1, 2$  directly.

**8.2 2**

Assume that  $|b| < 1$  and show that the following polynomial has exactly two roots (counting multiplicity) in  $|z| < 1$ :

$$f(z) := z^3 + 3z^2 + bz + b^2.$$

**8.3 3**

Let  $c \in \mathbb{C}$  with  $|c| < \frac{1}{3}$ . Show that on the open set  $\{z \in \mathbb{C} \mid \Re(z) < 1\}$ , the function  $f(z) := ce^z$  has exactly one fixed point.

**8.4 4**

How many roots does the following polynomial have in the open disc  $|z| < 1$ ?

$$f(z) = z^7 - 4z^3 - 1.$$

**8.5 5**

Let  $n \in \mathbb{Z}^{\geq 0}$  and show that the equation

$$e^z = az^n$$

has  $n$  solutions in the open unit disc if  $|a| > e$ , and no solutions if  $|a| < \frac{1}{e}$ .

**8.6 6**

Let  $f$  be analytic in a domain  $D$  and fix  $z_0 \in D$  with  $w_0 := f(z_0)$ . Suppose  $z_0$  is a zero of  $f(z) - w_0$  with finite multiplicity  $m$ . Show that there exists  $\delta > 0$  and  $\varepsilon > 0$  such that for each  $w$  such that  $0 < |w - w_0| < \varepsilon$ , the equation  $f(z) - w = 0$  has exactly  $m$  *distinct* solutions inside the disc  $|z - z_0| < \delta$ .

**8.7 7**

For  $k = 1, 2, \dots, n$ , suppose  $|a_k| < 1$  and

$$f(z) := \left( \frac{z - a_1}{1 - \bar{a}_1 z} \right) \left( \frac{z - a_2}{1 - \bar{a}_2 z} \right) \cdots \left( \frac{z - a_n}{1 - \bar{a}_n z} \right).$$

Show that  $f(z) = b$  has  $n$  solutions in  $|z| < 1$ .

**8.8 8**

For each  $n \in \mathbb{Z}^{\geq 1}$ , let

$$P_n(z) = 1 + z + \frac{1}{2!}z^2 + \cdots + \frac{1}{n!}z^n.$$

Show that for sufficiently large  $n$ , the polynomial  $P_n$  has no zeros in  $|z| < 10$ , while the polynomial  $P_n(z) - 1$  has precisely 3 zeros there.

**8.9 9**

Prove that

$$\max_{|z|=1} \left| a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n \right| \geq 1.$$

Hint: the first part of the problem asks for a statement of Rouché's theorem.

**8.10 10**

Use Rouché's theorem to prove the Fundamental Theorem of Algebra.

**9 Schwarz Lemma and Reflection Principle (8155i)****10 Spring 2020 Homework 1****10.1 1**

Geometrically describe the following subsets of  $\mathbb{C}$ :

- $|z - 1| = 1$
- $|z - 1| = 2|z - 2|$
- $1/z = \bar{z}$
- $\Re(z) = 3$
- $\Im(z) = a$  with  $a \in \mathbb{R}$ .
- $\Re(z) > a$  with  $a \in \mathbb{R}$ .
- $|z - 1| < 2|z - 2|$



**10.2 2**

Prove the following inequality, and explain when equality holds:

$$|z + w| \geq ||z| - |w||.$$

**10.3 3**

Prove that the following polynomial has its roots outside of the unit circle:

$$p(z) = z^3 + 2z + 4.$$

Hint: What is the maximum value of the modulus of the first two terms if  $|z| \leq 1$ ?

**10.4 4**

a. Prove that if  $c > 0$ ,

$$|w_1| = c|w_2| \implies |w_1 - c^2 w_2| = c|w_1 - w_2|.$$

b. Prove that if  $c > 0$  and  $c \neq 1$ , with  $z_1 \neq z_2$ , then the following equation represents a circle:

$$\left| \frac{z - z_1}{z - z_2} \right| = c.$$

Find its center and radius.

Hint: use part (a)

**10.5 5**

a. Let  $z, w \in \mathbb{C}$  with  $\bar{z}w \neq 1$ . Prove that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| < 1 \quad \text{if } |z| < 1, |w| < 1$$

with equality when  $|z| = 1$  or  $|w| = 1$ .

b. Prove that for a fixed  $w \in \mathbb{D}$ , the mapping  $F : z \mapsto \frac{w - z}{1 - \bar{w}z}$  satisfies

- $F$  maps  $\mathbb{D}$  to itself and is holomorphic.
- $F(0) = w$  and  $F(w) = 0$ .
- $|z| = 1$  implies  $|F(z)| = 1$ .

**10.6 6**

Use  $n$ th roots of unity to show that

$$2^{n-1} \sin\left(\frac{\pi}{n}\right) \sin\left(\frac{2\pi}{n}\right) \cdots \sin\left(\frac{(n-1)\pi}{n}\right) = n.$$

Hint:

$$\begin{aligned}1 - \cos(2\theta) &= 2\sin^2(\theta) \\ 2\sin(2\theta) &= 2\sin(\theta)\cos(\theta).\end{aligned}$$

**10.7 7**

Prove that  $f(z) = |z|^2$  has a derivative at  $z = 0$  and nowhere else.

**10.8 8**

Let  $f(z)$  be analytic in a domain, and prove that  $f$  is constant if it satisfies any of the following conditions:

- a.  $|f(z)|$  is constant.
- b.  $\Re(f(z))$  is constant.
- c.  $\arg(f(z))$  is constant.
- d.  $\overline{f(z)}$  is analytic.

How do you generalize (a) and (b)?

**10.9 9**

Prove that if  $z \mapsto f(z)$  is analytic, then  $z \mapsto \overline{f(\bar{z})}$  is analytic.

**10.10 10**

- a. Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

- b. Use (a) to show that the logarithm function, defined as

$$\log z = \log r + i\theta \text{ where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi.$$

is holomorphic on the region  $r > 0, -\pi < \theta < \pi$ .

Also show that this function is not continuous in  $r > 0$ .

**10.11 11**

Prove that the distinct complex numbers  $z_1, z_2, z_3$  are the vertices of an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

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## 11 Spring 2020 Homework 2

Note on notation: I sometimes use  $f_x := \frac{\partial f}{\partial x}$  to denote partial derivatives, and  $\partial_z^n f$  as  $f^{(n)}(z)$ .

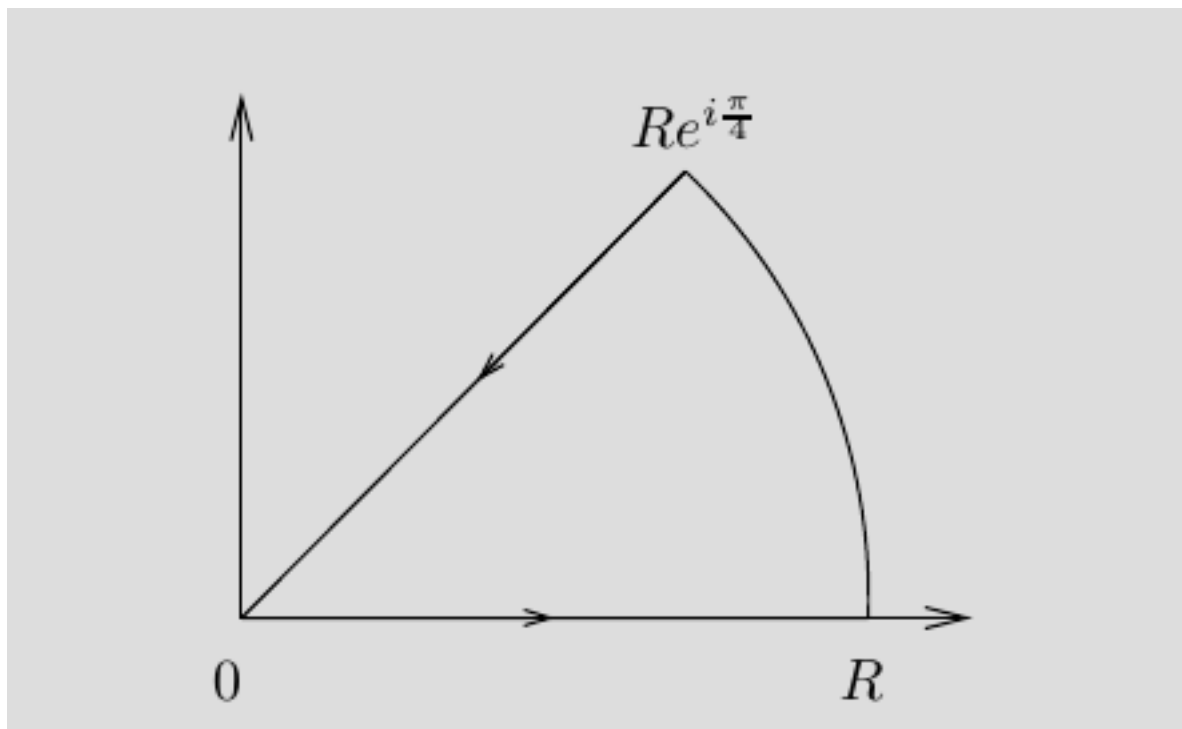
### 11.1 Stein And Shakarchi

#### 11.1.1 2.6.1

Show that

$$\int_0^\infty \sin(x^2) dx = \int_0^\infty \cos(x^2) dx = \frac{\sqrt{2\pi}}{4}.$$

Hint: integrate  $e^{-x^2}$  over the following contour, using the fact that  $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$ :



#### 11.1.2 2.6.2

Show that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Hint: use the fact that this integral equals  $\frac{1}{2i} \int_{-\infty}^\infty \frac{e^{ix} - 1}{x} dx$ , and integrate around an indented semicircle.

**11.1.3 2.6.5**

Suppose  $f \in C^1_{\mathbb{C}}(\Omega)$  and  $T \subset \Omega$  is a triangle with  $T^\circ \subset \Omega$ . Apply Green's theorem to show that  $\int_T f(z) dz = 0$ .

Assume that  $f'$  is continuous and prove Goursat's theorem.

Hint: Green's theorem states

$$\int_T Fdx + Gdy = \int_{T^\circ} \left( \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dxdy.$$

**11.1.4 2.6.6**

Suppose that  $f$  is holomorphic on a punctured open set  $\Omega \setminus \{w_0\}$  and let  $T \subset \Omega$  be a triangle containing  $w_0$ . Prove that if  $f$  is bounded near  $w_0$ , then  $\int_T f(z) dz = 0$ .

**11.1.5 2.6.7**

Suppose  $f : \mathbb{D} \rightarrow \mathbb{C}$  is holomorphic and let  $d := \sup_{z, w \in \mathbb{D}} |f(z) - f(w)|$  be the diameter of the image of  $f$ . Show that  $2|f'(0)| \leq d$ , and that equality holds iff  $f$  is linear, so  $f(z) = a_1z + a_2$ .

Hint:  $2f'(0) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi) - f(-\xi)}{\xi^2} d\xi$  whenever  $0 < r < 1$ .

**11.1.6 2.6.8**

Suppose that  $f$  is holomorphic on the strip  $S = \{x + iy \mid x \in \mathbb{R}, -1 < y < 1\}$  with  $|f(z)| \leq A(1 + |z|)^\nu$  for  $\nu$  some fixed real number. Show that for all  $z \in S$ , for each integer  $n \geq 0$  there exists an  $A_n \geq 0$  such that  $|f^{(n)}(x)| \leq A_n(1 + |x|)^\nu$  for all  $x \in \mathbb{R}$ .

Hint: Use the Cauchy inequalities.

**11.1.7 2.6.9**

Let  $\Omega \subset \mathbb{C}$  be open and bounded and  $\varphi : \Omega \rightarrow \Omega$  holomorphic. Prove that if there exists a point  $z_0 \in \Omega$  such that  $\varphi(z_0) = z_0$  and  $\varphi'(z_0) = 1$ , then  $\varphi$  is linear.

Hint: assume  $z_0 = 0$  (explain why this can be done) and write  $\varphi(z) = z + a_n z^n + O(z^{n+1})$  near 0. Let  $\varphi_k = \varphi \circ \varphi \circ \dots \circ \varphi$  and prove that  $\varphi_k(z) = z + k a_n z^n + O(z^{n+1})$ . Apply Cauchy's inequalities and let  $k \rightarrow \infty$  to conclude.

**11.1.8 2.6.10**

Can every continuous function on  $\overline{\mathbb{D}}$  be uniformly approximated by polynomials in the variable  $z$ ?

Hint: compare to Weierstrass for the real interval.

**11.1.9 2.6.13**

Suppose  $f$  is analytic, defined on all of  $\mathbb{C}$ , and for each  $z_0 \in \mathbb{C}$  there is at least one coefficient in the expansion  $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$  is zero. Prove that  $f$  is a polynomial.

Hint: use the fact that  $c_n n! = f^{(n)}(z_0)$  and use a countability argument.

**11.1.10 2.6.14**

Suppose that  $f$  is holomorphic in an open set containing  $\mathbb{D}$  except for a pole  $z_0 \in \partial\mathbb{D}$ . Let  $\sum_{n=0}^{\infty} a_n z^n$  be the power series expansion of  $f$  in  $\mathbb{D}$ , and show that  $\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} = z_0$ .

**11.1.11 2.6.15**

Suppose  $f$  is continuous and nonvanishing on  $\bar{\mathbb{D}}$ , and holomorphic in  $\mathbb{D}$ . Prove that if  $|z| = 1 \implies |f(z)| = 1$ , then  $f$  is constant.

Hint: Extend  $f$  to all of  $\mathbb{C}$  by  $f(z) = 1/\overline{f(1/\bar{z})}$  for any  $|z| > 1$ , and argue as in the Schwarz reflection principle.

**11.2 Additional Problems****11.2.1 1**

Let  $a_n \neq 0$  and show that

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L \implies \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = L.$$

In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

**11.2.2 2**

Let  $f$  be a power series centered at the origin. Prove that  $f$  has a power series expansion about any point in its disc of convergence.

**11.2.3 3**

Prove the following:

- $\sum_n n z^n$  does not converge at any point of  $S^1$
- $\sum_n \frac{z^n}{n^2}$  converges at every point of  $S^1$ .

c.  $\sum_n \frac{z^n}{n}$  converges at every point of  $S^1$  except  $z = 1$ .

**11.2.4 4**

Without using Cauchy's integral formula, show that if  $|a| < r < |b|$ , then

$$\int_{\gamma} \frac{dz}{(z - \alpha)(z - \beta)} = \frac{2\pi i}{\alpha - \beta}$$

where  $\gamma$  denotes the circle centered at the origin of radius  $r$  with positive orientation.

**11.2.5 5**

Assume  $f$  is continuous in the region  $\{x + iy \mid x \geq x_0, 0 \leq y \leq b\}$ , and the following limit exists independent of  $y$ :

$$\lim_{x \rightarrow +\infty} f(x + iy) = A.$$

Show that if  $\gamma_x := \{z = x + it \mid 0 \leq t \leq b\}$ , then

$$\lim_{x \rightarrow +\infty} \int_{\gamma_x} f(z) dz = iAb.$$

**11.2.6 6**

Show by example that there exists a function  $f(z)$  that is holomorphic on  $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$  and for all  $r < 1$ ,

$$\int_{|z|=r} f(z) dz = 0,$$

but  $f$  is not holomorphic at  $z = 0$ .

**11.2.7 7**

Let  $f$  be analytic on a region  $R$  and suppose  $f'(z_0) \neq 0$  for some  $z_0 \in R$ . Show that if  $C$  is a circle of sufficiently small radius centered at  $z_0$ , then

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{dz}{f(z) - f(z_0)}.$$

Hint: use the inverse function theorem.

**11.2.8 8**

Assume two functions  $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$  have continuous partial derivatives at  $(x_0, y_0)$ . Show that  $f := u + iv$  has derivative  $f'(z_0)$  at  $z_0 = x_0 + iy_0$  if and only if

$$\lim_{r \rightarrow 0} \frac{1}{\pi r^2} \int_{|z - z_0| = r} f(z) dz = 0.$$

**11.2.9 9 (Cauchy's Formula for Exterior Regions)**

Let  $\gamma$  be a piecewise smooth simple closed curve with interior  $\Omega_1$  and exterior  $\Omega_2$ . Assume  $f'$  exists in an open set containing  $\gamma$  and  $\Omega_2$  with  $\lim_{z \rightarrow \infty} f(z) = A$ . Show that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1 \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}.$$

**11.2.10 10**

Let  $f(z)$  be bounded and analytic in  $\mathbb{C}$ . Let  $a \neq b$  be any fixed complex numbers. Show that the following limit exists:

$$\lim_{R \rightarrow \infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

Use this to show that  $f(z)$  must be constant.

**11.2.11 11**

Suppose  $f(z)$  is entire and

$$\lim_{z \rightarrow \infty} \frac{f(z)}{z} = 0.$$

Show that  $f(z)$  is a constant.

**11.2.12 12**

Let  $f$  be analytic in a domain  $D$  and  $\gamma$  be a closed curve in  $D$ . For any  $z_0 \in D$  not on  $\gamma$ , show that

$$\int_{\gamma} \frac{f'(z)}{(z-z_0)} dz = \int_{\gamma} \frac{f(z)}{(z-z_0)^2} dz.$$

Give a generalization of this result.

**11.2.13 13**

Compute

$$\int_{|z|=1} \left( z + \frac{1}{z} \right)^{2n} \frac{dz}{z}$$

and use it to show that

$$\int_0^{2\pi} \cos^{2n}(\theta) d\theta = 2\pi \left( \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \right).$$

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## 12 Spring 2020 Homework 3

### 12.1 Stein and Shakarchi

#### 12.1.1 3.8.1

Use the following formula to show that the complex zeros of  $\sin(\pi z)$  are exactly the integers, and they are each of order 1:

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}.$$

Calculate the residue of  $\frac{1}{\sin(\pi z)}$  at  $z = n \in \mathbb{Z}$ .

#### 12.1.2 3.8.2

Evaluate the integral

$$\int_{\mathbb{R}} \frac{dx}{1+x^4}.$$

What are the poles of  $\frac{1}{1+z^4}$  ?

#### 12.1.3 3.8.4

Show that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}, \quad \text{for all } a > 0.$$

#### 12.1.4 3.8.5

Show that if  $\xi \in \mathbb{R}$ , then

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{(1+x^2)^2} dx = \frac{\pi}{2} (1 + 2\pi |\xi|) e^{-2\pi |\xi|}.$$

#### 12.1.5 3.8.6

Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \pi.$$

#### 12.1.6 3.8.7

Show that

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos \theta)^2} = \frac{2\pi a}{(a^2 - 1)^{3/2}}, \quad \text{whenever } a > 1.$$



**12.1.7 3.8.8**

Show that if  $a, b \in \mathbb{R}$  with  $a > |b|$ , then

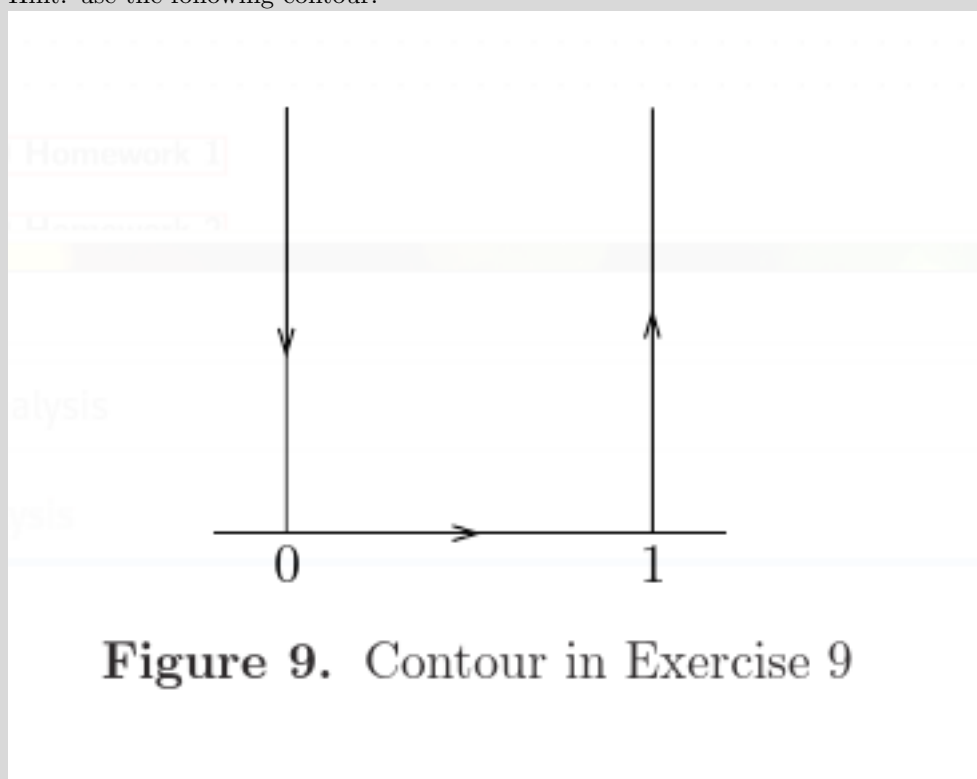
$$\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

**12.1.8 3.8.9**

Show that

$$\int_0^1 \log(\sin \pi x) dx = -\log 2.$$

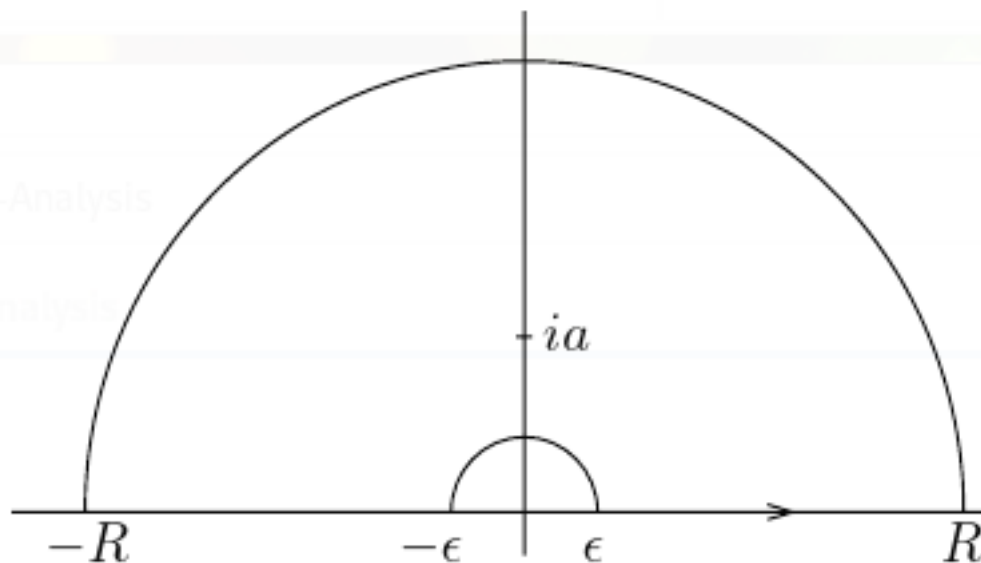
Hint: use the following contour.

**12.1.9 3.8.10**

Show that if  $a > 0$ , then

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx = \frac{\pi}{2a} \log a.$$

Hint: use the following contour.



### 12.1.10 3.8.14

Prove that all entire functions that are injective are of the form  $f(z) = az + b$  with  $a, b \in \mathbb{C}$  and  $a \neq 0$ .

Hint: Apply the Casorati-Weierstrass theorem to  $f(1/z)$ .

### 12.1.11 3.8.15

Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

- a. Prove that if  $f$  is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \leq AR^k + B$$

for all  $R > 0$ , some integer  $k \geq 0$ , and some constants  $A, B > 0$ , then  $f$  is a polynomial of degree  $\leq k$ .

- b. Show that if  $f$  is holomorphic in the unit disc, is bounded, and converges uniformly to zero in the sector  $\theta < \arg(z) < \varphi$  as  $|z| \rightarrow 0$ , then  $f \equiv 0$ .
- c. Let  $w_1, \dots, w_n$  be points on  $S^1 \subset \mathbb{C}$ . Prove that there exists a point  $z \in S^1$  such that the product of the distances from  $z$  to the points  $w_j$  is at least 1.

Conclude that there exists a point  $w \in S^1$  such that the product of the above distances is *exactly* 1.

- d. Show that if the real part of an entire function is bounded, then  $f$  is constant.

**12.1.12 3.8.17**

Let  $f$  be non-constant and holomorphic in an open set containing the closed unit disc.

- a. Show that if  $|f(z)| = 1$  whenever  $|z| = 1$ , then the image of  $f$  contains the unit disc.

Hint: Show that  $f(z) = w_0$  has a root for every  $w_0 \in \mathbb{D}$ , for which it suffices to show that  $f(z) = 0$  has a root. Conclude using the maximum modulus principle.

- b. If  $|f(z)| \geq 1$  whenever  $|z| = 1$  and there exists a  $z_0 \in \mathbb{D}$  such that  $|f(z_0)| < 1$ , then the image of  $f$  contains the unit disc.

**12.1.13 3.8.19**

Prove that maximum principle for harmonic functions, i.e.

- a. If  $u$  is a non-constant real-valued harmonic function in a region  $\Omega$ , then  $u$  can not attain a maximum or a minimum in  $\Omega$ .
- b. Suppose  $\Omega$  is a region with compact closure  $\bar{\Omega}$ . If  $u$  is harmonic in  $\Omega$  and continuous in  $\bar{\Omega}$ , then

$$\sup_{z \in \Omega} |u(z)| \leq \sup_{z \in \bar{\Omega} - \Omega} |u(z)|.$$

Hint: to prove (a), assume  $u$  attains a local maximum at  $z_0$ . Let  $f$  be holomorphic near  $z_0$  with  $\Re(f) = u$ , and show that  $f$  is not an open map. Then (a) implies (b).

**12.2 Problems From Tie****12.2.1 1**

**Problem** Prove that if  $f$  has two Laurent series expansions,

$$f(z) = \sum c_n(z-a)^n \quad \text{and} \quad f(z) = \sum c'_n(z-a)^n$$

then  $c_n = c'_n$ .

**12.2.2 2**

**Problem** Find Laurent series expansions of

$$\frac{1}{1-z^2} + \frac{1}{3-z}$$

How many such expansions are there? In what domains are each valid?

**12.2.3 3**

**Problem** Let  $P, Q$  be polynomials with no common zeros. Assume  $a$  is a root of  $Q$ . Find the principal part of  $P/Q$  at  $z = a$  in terms of  $P$  and  $Q$  if  $a$  is (1) a simple root, and (2) a double root.

**12.2.4 4**

**Problem** Let  $f$  be non-constant, analytic in  $|z| > 0$ , where  $f(z_n) = 0$  for infinitely many points  $z_n$  with  $\lim_{n \rightarrow \infty} z_n = 0$ .

Show that  $z = 0$  is an essential singularity for  $f$ .

Example:  $f(z) = \sin(1/z)$ .

**12.2.5 5**

**Problem** Show that if  $f$  is entire and  $\lim_{z \rightarrow \infty} f(z) = \infty$ , then  $f$  is a polynomial.

**12.2.6 6**

**Problem**

- a. Show (without using 3.8.9 in the S&S) that

$$\int_0^{2\pi} \log |1 - e^{i\theta}| d\theta = 0$$

- b. Show that this identity is equivalent to S&S 3.8.9:

$$\int_0^1 \log(\sin(\pi x)) dx = -\log 2.$$

**12.2.7 7**

**Problem** Let  $0 < a < 4$  and evaluate

$$\int_0^\infty \frac{x^{a-1}}{1+x^3} dx$$

**12.2.8 8**

**Problem** Prove the fundamental theorem of Algebra using

- Rouche's Theorem.
- The maximum modulus principle.

**12.2.9 9**

**Problem** Let  $f$  be analytic in a region  $D$  and  $\gamma$  a rectifiable curve in  $D$  with interior in  $D$ .

Prove that if  $f(z)$  is real for all  $z \in \gamma$ , then  $f$  is constant.

**12.2.10 10**

**Problem** For  $a > 0$ , evaluate

$$\int_0^{\pi/2} \frac{d\theta}{a + \sin^2 \theta}$$

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**12.2.11 11**

**Problem** Find the number of roots of  $p(z) = 4z^4 - 6z + 3$  in  $|z| < 1$  and  $1 < |z| < 2$  respectively.

**12.2.12 12**

**Problem** Prove that  $z^4 + 2z^3 - 2z + 10$  has exactly one root in each open quadrant.

**12.2.13 13**

**Problem** Prove that for  $a > 0$ ,  $z \tan z - a$  has only real roots.

**12.2.14 14**

**Problem** Let  $f$  be nonzero, analytic on a bounded region  $\Omega$  and continuous on its closure  $\overline{\Omega}$ .

Show that if  $|f(z)| \equiv M$  is constant for  $z \in \partial\Omega$ , then  $f(z) \equiv Me^{i\theta}$  for some real constant  $\theta$ .

**13 Extra Questions from Jingzhi Tie****13.1 Fall 2009****13.1.1 ?**

(1) Assume  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  converges in  $|z| < R$ . Show that for  $r < R$ ,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

(2) Deduce Liouville's theorem from (1).

**13.1.2 ?**

Let  $f$  be a continuous function in the region

$$D = \{z \mid |z| > R, 0 \leq \arg z \leq \theta\} \quad \text{where } 1 \leq \theta \leq 2\pi.$$

If there exists  $k$  such that  $\lim_{z \rightarrow \infty} zf(z) = k$  for  $z$  in the region  $D$ . Show that

$$\lim_{R' \rightarrow \infty} \int_L f(z) dz = i\theta k,$$

where  $L$  is the part of the circle  $|z| = R'$  which lies in the region  $D$ .

**13.1.3 ?**

Suppose that  $f$  is an analytic function in the region  $D$  which contains the point  $a$ . Let

$$F(z) = z - a - qf(z), \quad \text{where } q \text{ is a complex parameter.}$$

- (1) Let  $K \subset D$  be a circle with the center at point  $a$  and also we assume that  $f(z) \neq 0$  for  $z \in K$ . Prove that the function  $F$  has one and only one zero  $z = w$  on the closed disc  $\bar{K}$  whose boundary is the circle  $K$  if  $|q| < \min_{z \in K} \frac{|z - a|}{|f(z)|}$ .
- (2) Let  $G(z)$  be an analytic function on the disk  $\bar{K}$ . Apply the residue theorem to prove that  $\frac{G(w)}{F'(w)} = \frac{1}{2\pi i} \int_K \frac{G(z)}{F(z)} dz$ , where  $w$  is the zero from (1).
- (3) If  $z \in K$ , prove that the function  $\frac{1}{F(z)}$  can be represented as a convergent series with respect to  $q$ :  $\frac{1}{F(z)} = \sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z - a)^{n+1}}$ .

**13.1.4 ?**

Evaluate

$$\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx.$$

**13.1.5 ?**

Let  $f = u + iv$  be differentiable (i.e.  $f'(z)$  exists) with continuous partial derivatives at a point  $z = re^{i\theta}$ ,  $r \neq 0$ . Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

**13.1.6 ?**

Show that  $\int_0^{\infty} \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$  using complex analysis,  $0 < a < n$ . Here  $n$  is a positive integer.

**13.1.7 ?**

For  $s > 0$ , the **gamma function** is defined by  $\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt$ .

1. Show that the gamma function is analytic in the half-plane  $\Re(s) > 0$ , and is still given there by the integral formula above.
2. Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need  $\Gamma(1-s) = t \int_0^{\infty} e^{-vt} (vt)^{-s} dv$  for  $t > 0$ .

**13.1.8 ?**

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n \quad (a_n \neq 0)$$

is a polynomial of degree  $n$ , then it has  $n$  zeros in  $\mathbb{C}$ .

**13.1.9 ?**

Suppose  $f$  is entire and there exist  $A, R > 0$  and natural number  $N$  such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that (i)  $f$  is a polynomial and (ii) the degree of  $f$  is at least  $N$ .

**13.1.10 ?**

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be an injective analytic (also called *univalent*) function. Show that there exist complex numbers  $a \neq 0$  and  $b$  such that  $f(z) = az + b$ .

**13.1.11 ?**

Let  $g$  be analytic for  $|z| \leq 1$  and  $|g(z)| < 1$  for  $|z| = 1$ .

1. Show that  $g$  has a unique fixed point in  $|z| < 1$ .
2. What happens if we replace  $|g(z)| < 1$  with  $|g(z)| \leq 1$  for  $|z| = 1$ ? Give an example if (a) is not true or give an proof if (a) is still true.
3. What happens if we simply assume that  $f$  is analytic for  $|z| < 1$  and  $|f(z)| < 1$  for  $|z| < 1$ ? Suppose that  $f(z) \neq z$ . Can  $f$  have more than one fixed point in  $|z| < 1$ ?

Hint: The map  $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$  may be useful.

**13.1.12 ?**

Find a conformal map from  $D = \{z : |z| < 1, |z - 1/2| > 1/2\}$  to the unit disk  $\Delta = \{z : |z| < 1\}$ .

**13.1.13 ?**

Let  $f(z)$  be entire and assume values of  $f(z)$  lie outside a *bounded* open set  $\Omega$ . Show without using Picard's theorems that  $f(z)$  is a constant.

**13.1.14 ?**

- (1) Assume  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  converges in  $|z| < R$ . Show that for  $r < R$ ,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

(2) Deduce Liouville's theorem from (1).

**13.1.15 ?**

Let  $f(z)$  be entire and assume that  $f(z) \leq M|z|^2$  outside some disk for some constant  $M$ . Show that  $f(z)$  is a polynomial in  $z$  of degree  $\leq 2$ .

**13.1.16 ?**

Let  $a_n(z)$  be an analytic sequence in a domain  $D$  such that  $\sum_{n=0}^{\infty} |a_n(z)|$  converges uniformly on bounded and closed sub-regions of  $D$ . Show that  $\sum_{n=0}^{\infty} |a'_n(z)|$  converges uniformly on bounded and closed sub-regions of  $D$ .

**13.1.17 ?**

Let  $f(z)$  be analytic in an open set  $\Omega$  except possibly at a point  $z_0$  inside  $\Omega$ . Show that if  $f(z)$  is bounded in near  $z_0$ , then  $\int_{\Delta} f(z)dz = 0$  for all triangles  $\Delta$  in  $\Omega$ .

**13.1.18 ?**

Assume  $f$  is continuous in the region:  $0 < |z - a| \leq R$ ,  $0 \leq \arg(z - a) \leq \beta_0$  ( $0 < \beta_0 \leq 2\pi$ ) and the limit  $\lim_{z \rightarrow a} (z - a)f(z) = A$  exists. Show that

$$\lim_{r \rightarrow 0} \int_{\gamma_r} f(z)dz = iA\beta_0,$$

where  $\gamma_r := \{z \mid z = a + re^{it}, 0 \leq t \leq \beta_0\}$ .

**13.1.19 ?**

Show that  $f(z) = z^2$  is uniformly continuous in any open disk  $|z| < R$ , where  $R > 0$  is fixed, but it is not uniformly continuous on  $\mathbb{C}$ .

**13.1.20 ?**

(1) Show that the function  $u = u(x, y)$  given by

$$u(x, y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx \quad \text{for } n \in \mathbf{N}$$

is the solution on  $D = \{(x, y) \mid x^2 + y^2 < 1\}$  of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = \frac{\sin nx}{n}.$$

(2) Show that there exist points  $(x, y) \in D$  such that  $\limsup_{n \rightarrow \infty} |u(x, y)| = \infty$ .



**13.2 Fall 2011****13.2.1 ?**

- (1) Assume  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  converges in  $|z| < R$ . Show that for  $r < R$ ,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

- (2) Deduce Liouville's theorem from (1).

**13.2.2 ?**

Let  $f$  be a continuous function in the region

$$D = \{z \mid |z| > R, 0 \leq \arg z \leq \theta\} \quad \text{where} \quad 0 \leq \theta \leq 2\pi.$$

If there exists  $k$  such that  $\lim_{z \rightarrow \infty} z f(z) = k$  for  $z$  in the region  $D$ . Show that

$$\lim_{R' \rightarrow \infty} \int_L f(z) dz = i\theta k,$$

where  $L$  is the part of the circle  $|z| = R'$  which lies in the region  $D$ .

**13.2.3 ?**

Suppose that  $f$  is an analytic function in the region  $D$  which contains the point  $a$ . Let

$$F(z) = z - a - qf(z), \quad \text{where} \quad q \text{ is a complex parameter.}$$

- (1) Let  $K \subset D$  be a circle with the center at point  $a$  and also we assume that  $f(z) \neq 0$  for  $z \in K$ . Prove that the function  $F$  has one and only one zero  $z = w$  on the closed disc  $\bar{K}$  whose boundary is the circle  $K$  if  $|q| < \min_{z \in K} \frac{|z - a|}{|f(z)|}$ .
- (2) Let  $G(z)$  be an analytic function on the disk  $\bar{K}$ . Apply the residue theorem to prove that  $\frac{G(w)}{F'(w)} = \frac{1}{2\pi i} \int_K \frac{G(z)}{F(z)} dz$ , where  $w$  is the zero from (1).
- (3) If  $z \in K$ , prove that the function  $\frac{1}{F(z)}$  can be represented as a convergent series with respect to  $q$ :  $\frac{1}{F(z)} = \sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z - a)^{n+1}}$ .

**13.2.4 ?**

Evaluate  $\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx$ .

**13.2.5 ?**

Let  $f = u + iv$  be differentiable (i.e.  $f'(z)$  exists) with continuous partial derivatives at a point  $z = re^{i\theta}$ ,  $r \neq 0$ . Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

**13.2.6 ?**

Show that  $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$  using complex analysis,  $0 < a < n$ . Here  $n$  is a positive integer.

**13.2.7 ?**

For  $s > 0$ , the **gamma function** is defined by  $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$ .

1. Show that the gamma function is analytic in the half-plane  $\Re(s) > 0$ , and is still given there by the integral formula above.
2. Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need  $\Gamma(1-s) = t \int_0^\infty e^{-vt} (vt)^{-s} dv$  for  $t > 0$ .

**13.2.8 ?**

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree  $n$ , then it has  $n$  zeros in  $\mathbb{C}$ .

**13.2.9 ?**

Suppose  $f$  is entire and there exist  $A, R > 0$  and natural number  $N$  such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that (i)  $f$  is a polynomial and (ii) the degree of  $f$  is at least  $N$ .

**13.2.10 ?**

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be an injective analytic (also called univalent) function. Show that there exist complex numbers  $a \neq 0$  and  $b$  such that  $f(z) = az + b$ .

**13.2.11 ?**

Let  $g$  be analytic for  $|z| \leq 1$  and  $|g(z)| < 1$  for  $|z| = 1$ .

- Show that  $g$  has a unique fixed point in  $|z| < 1$ .
- What happens if we replace  $|g(z)| < 1$  with  $|g(z)| \leq 1$  for  $|z| = 1$ ? Give an example if (a) is not true or give an proof if (a) is still true.
- What happens if we simply assume that  $f$  is analytic for  $|z| < 1$  and  $|f(z)| < 1$  for  $|z| < 1$ ? Suppose that  $f(z) \neq z$ . Can  $f$  have more than one fixed point in  $|z| < 1$ ?

Hint: The map  $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$  may be useful.

**13.2.12 ?**

Find a conformal map from  $D = \{z : |z| < 1, |z - 1/2| > 1/2\}$  to the unit disk  $\Delta = \{z : |z| < 1\}$ .

**13.2.13 ?**

Let  $f(z)$  be entire and assume values of  $f(z)$  lie outside a *bounded* open set  $\Omega$ . Show without using Picard's theorems that  $f(z)$  is a constant.

**13.2.14 ?**

Let  $f(z)$  be entire and assume values of  $f(z)$  lie outside a *bounded* open set  $\Omega$ . Show without using Picard's theorems that  $f(z)$  is a constant.

**13.2.15 ?**

- (1) Assume  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  converges in  $|z| < R$ . Show that for  $r < R$ ,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

- (2) Deduce Liouville's theorem from (1).

**13.2.16 ?**

Let  $f(z)$  be entire and assume that  $f(z) \leq M|z|^2$  outside some disk for some constant  $M$ . Show that  $f(z)$  is a polynomial in  $z$  of degree  $\leq 2$ .

**13.2.17 ?**

Let  $a_n(z)$  be an analytic sequence in a domain  $D$  such that  $\sum_{n=0}^{\infty} |a_n(z)|$  converges uniformly on bounded and closed sub-regions of  $D$ . Show that  $\sum_{n=0}^{\infty} |a'_n(z)|$  converges uniformly on bounded and closed sub-regions of  $D$ .

**13.2.18 ?**

Let  $f(z)$  be analytic in an open set  $\Omega$  except possibly at a point  $z_0$  inside  $\Omega$ . Show that if  $f(z)$  is bounded in near  $z_0$ , then  $\int_{\Delta} f(z)dz = 0$  for all triangles  $\Delta$  in  $\Omega$ .

**13.2.19 ?**

Assume  $f$  is continuous in the region:  $0 < |z - a| \leq R$ ,  $0 \leq \arg(z - a) \leq \beta_0$  ( $0 < \beta_0 \leq 2\pi$ ) and the limit  $\lim_{z \rightarrow a} (z - a)f(z) = A$  exists. Show that

$$\lim_{r \rightarrow 0} \int_{\gamma_r} f(z)dz = iA\beta_0,$$

where  $\gamma_r := \{z \mid z = a + re^{it}, 0 \leq t \leq \beta_0\}$ .

**13.2.20 ?**

Show that  $f(z) = z^2$  is uniformly continuous in any open disk  $|z| < R$ , where  $R > 0$  is fixed, but it is not uniformly continuous on  $\mathbb{C}$ .

(1) Show that the function  $u = u(x, y)$  given by

$$u(x, y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx \quad \text{for } n \in \mathbf{N}$$

is the solution on  $D = \{(x, y) \mid x^2 + y^2 < 1\}$  of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = \frac{\sin nx}{n}.$$

(2) Show that there exist points  $(x, y) \in D$  such that  $\limsup_{n \rightarrow \infty} |u(x, y)| = \infty$ .

**13.3 Spring 2014****13.3.1 ?**

The question provides some insight into Cauchy's theorem. Solve the problem without using the Cauchy theorem.

1. Evaluate the integral  $\int_{\gamma} z^n dz$  for all integers  $n$ . Here  $\gamma$  is any circle centered at the origin with the positive (counterclockwise) orientation.

2. Same question as (a), but with  $\gamma$  any circle not containing the origin.
3. Show that if  $|a| < r < |b|$ , then  $\int_{\gamma} \frac{dz}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$ . Here  $\gamma$  denotes the circle centered at the origin, of radius  $r$ , with the positive orientation.

**13.3.2 ?**

- (1) Assume the infinite series  $\sum_{n=0}^{\infty} c_n z^n$  converges in  $|z| < R$  and let  $f(z)$  be the limit. Show that for  $r < R$ ,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

- (2) Deduce Liouville's theorem from (1). Liouville's theorem: If  $f(z)$  is entire and bounded, then  $f$  is constant.

**13.3.3 ?**

Let  $f$  be a continuous function in the region

$$D = \{z \mid |z| > R, 0 \leq \arg Z \leq \theta\} \quad \text{where } 0 \leq \theta \leq 2\pi.$$

If there exists  $k$  such that  $\lim_{z \rightarrow \infty} z f(z) = k$  for  $z$  in the region  $D$ . Show that

$$\lim_{R' \rightarrow \infty} \int_L f(z) dz = i\theta k,$$

where  $L$  is the part of the circle  $|z| = R'$  which lies in the region  $D$ .

**13.3.4 ?**

Evaluate  $\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx$ .

**13.3.5 ?**

Let  $f = u + iv$  be differentiable (i.e.  $f'(z)$  exists) with continuous partial derivatives at a point  $z = re^{i\theta}$ ,  $r \neq 0$ . Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

**13.3.6 ?**

Show that  $\int_0^{\infty} \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$  using complex analysis,  $0 < a < n$ . Here  $n$  is a positive integer.

**13.3.7 ?**

For  $s > 0$ , the **gamma function** is defined by  $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$ .

- Show that the gamma function is analytic in the half-plane  $\Re(s) > 0$ , and is still given there by the integral formula above.
- Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need  $\Gamma(1-s) = t \int_0^\infty e^{-vt} (vt)^{-s} dv$  for  $t > 0$ .

**13.3.8 ?**

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree  $n$ , then it has  $n$  zeros in  $\mathbb{C}$ .

**13.3.9 ?**

Suppose  $f$  is entire and there exist  $A, R > 0$  and natural number  $N$  such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that (i)  $f$  is a polynomial and (ii) the degree of  $f$  is at least  $N$ .

**13.3.10 ?**

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be an injective analytic (also called univalent) function. Show that there exist complex numbers  $a \neq 0$  and  $b$  such that  $f(z) = az + b$ .

**13.3.11 ?**

Let  $g$  be analytic for  $|z| \leq 1$  and  $|g(z)| < 1$  for  $|z| = 1$ .

- Show that  $g$  has a unique fixed point in  $|z| < 1$ .
- What happens if we replace  $|g(z)| < 1$  with  $|g(z)| \leq 1$  for  $|z| = 1$ ? Give an example if (a) is not true or give a proof if (a) is still true.
- What happens if we simply assume that  $f$  is analytic for  $|z| < 1$  and  $|f(z)| < 1$  for  $|z| < 1$ ? Suppose that  $f(z) \neq z$ . Can  $f$  have more than one fixed point in  $|z| < 1$ ?

Hint: The map  $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$  may be useful.

**13.3.12 ?**

Find a conformal map from  $D = \{z : |z| < 1, |z - 1/2| > 1/2\}$  to the unit disk  $\Delta = \{z : |z| < 1\}$ .

**13.4 Fall 2015****13.4.1 ?**

Let  $a_n \neq 0$  and assume that  $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L$ . Show that  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$ . In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

**13.4.2 ?**

(a) Let  $z, w$  be complex numbers, such that  $\bar{z}w \neq 1$ . Prove that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

(b) Prove that for fixed  $w$  in the unit disk  $\mathbb{D}$ , the mapping

$$F : z \mapsto \frac{w - z}{1 - \bar{w}z}$$

satisfies the following conditions:

- (c)  $F$  maps  $\mathbb{D}$  to itself and is holomorphic.
- (ii)  $F$  interchanges 0 and  $w$ , namely,  $F(0) = w$  and  $F(w) = 0$ .
- (iii)  $|F(z)| = 1$  if  $|z| = 1$ .
- (iv)  $F : \mathbb{D} \mapsto \mathbb{D}$  is bijective.

Hint: Calculate  $F \circ F$ .

**13.4.3 ?**

Use  $n$ -th roots of unity (i.e. solutions of  $z^n - 1 = 0$ ) to show that

$$2^{n-1} \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \cdots \sin \frac{(n-1)\pi}{n} = n.$$

Hint:  $1 - \cos 2\theta = 2 \sin^2 \theta$ ,  $\sin 2\theta = 2 \sin \theta \cos \theta$ .

(a) Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

(b) Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \quad \text{where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

is a holomorphic function in the region  $r > 0$ ,  $-\pi < \theta < \pi$ . Also show that  $\log z$  defined above is not continuous in  $r > 0$ .

#### 13.4.4 ?

Assume  $f$  is continuous in the region:  $x \geq x_0$ ,  $0 \leq y \leq b$  and the limit

$$\lim_{x \rightarrow +\infty} f(x + iy) = A$$

exists uniformly with respect to  $y$  (independent of  $y$ ). Show that

$$\lim_{x \rightarrow +\infty} \int_{\gamma_x} f(z) dz = iAb,$$

where  $\gamma_x := \{z \mid z = x + it, 0 \leq t \leq b\}$ .

#### 13.4.5 ?

(Cauchy's formula for "exterior" region) Let  $\gamma$  be piecewise smooth simple closed curve with interior  $\Omega_1$  and exterior  $\Omega_2$ . Assume  $f'(z)$  exists in an open set containing  $\gamma$  and  $\Omega_2$  and  $\lim_{z \rightarrow \infty} f(z) = A$ . Show that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

#### 13.4.6 ?

Let  $f(z)$  be bounded and analytic in  $\mathbb{C}$ . Let  $a \neq b$  be any fixed complex numbers. Show that the following limit exists

$$\lim_{R \rightarrow \infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

Use this to show that  $f(z)$  must be a constant (Liouville's theorem).

#### 13.4.7 ?

Prove by *justifying all steps* that for all  $\xi \in \mathbb{C}$  we have  $e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x \xi} dx$ .

Hint: You may use that fact in Example 1 on p. 42 of the textbook without proof, i.e., you may assume the above is true for real values of  $\xi$ .



**13.4.8 ?**

Suppose that  $f$  is holomorphic in an open set containing the closed unit disc, except for a pole at  $z_0$  on the unit circle. Let  $\sum c_n z^n$  denote the power series in the open disc. Show that (1)  $c_n \neq 0$  for all large enough  $n$ 's, and (2)  $\lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}} = z_0$ .

**13.4.9 ?**

Let  $f(z)$  be a non-constant analytic function in  $|z| > 0$  such that  $f(z_n) = 0$  for infinite many points  $z_n$  with  $\lim_{n \rightarrow \infty} z_n = 0$ . Show that  $z = 0$  is an essential singularity for  $f(z)$ . (An example of such a function is  $f(z) = \sin(1/z)$ .)

**13.4.10 ?**

Let  $f$  be entire and suppose that  $\lim_{z \rightarrow \infty} f(z) = \infty$ . Show that  $f$  is a polynomial.

**13.4.11 ?**

Expand the following functions into Laurent series in the indicated regions:

(a)  $f(z) = \frac{z^2 - 1}{(z + 2)(z + 3)}, \quad 2 < |z| < 3, \quad 3 < |z| < +\infty.$

(b)  $f(z) = \sin \frac{z}{1 - z}, \quad 0 < |z - 1| < +\infty$

**13.4.12 ?**

Assume  $f(z)$  is analytic in region  $D$  and  $\Gamma$  is a rectifiable curve in  $D$  with interior in  $D$ . Prove that if  $f(z)$  is real for all  $z \in \Gamma$ , then  $f(z)$  is a constant.

**13.4.13 ?**

Find the number of roots of  $z^4 - 6z + 3 = 0$  in  $|z| < 1$  and  $1 < |z| < 2$  respectively.

**13.4.14 ?**

Prove that  $z^4 + 2z^3 - 2z + 10 = 0$  has exactly one root in each open quadrant.

**13.4.15 ?**

(1) Let  $f(z) \in H(\mathbb{D})$ ,  $\operatorname{Re}(f(z)) > 0$ ,  $f(0) = a > 0$ . Show that

$$\left| \frac{f(z) - a}{f(z) + a} \right| \leq |z|, \quad |f'(0)| \leq 2a.$$

(2) Show that the above is still true if  $\operatorname{Re}(f(z)) > 0$  is replaced with  $\operatorname{Re}(f(z)) \geq 0$ .

**13.4.16 ?**

Assume  $f(z)$  is analytic in  $\mathbb{D}$  and  $f(0) = 0$  and is not a rotation (i.e.  $f(z) \neq e^{i\theta}z$ ). Show that  $\sum_{n=1}^{\infty} f^n(z)$  converges uniformly to an analytic function on compact subsets of  $\mathbb{D}$ , where  $f^{n+1}(z) = f(f^n(z))$ .

**13.4.17 ?**

Let  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  be analytic and one-to-one in  $|z| < 1$ . For  $0 < r < 1$ , let  $D_r$  be the disk  $|z| < r$ . Show that the area of  $f(D_r)$  is finite and is given by

$$S = \pi \sum_{n=1}^{\infty} n |c_n|^2 r^{2n}.$$

(Note that in general the area of  $f(D_1)$  is infinite.)

**13.4.18 ?**

Let  $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$  be analytic and one-to-one in  $r_0 < |z| < R_0$ . For  $r_0 < r < R < R_0$ , let  $D(r, R)$  be the annulus  $r < |z| < R$ . Show that the area of  $f(D(r, R))$  is finite and is given by

$$S = \pi \sum_{n=-\infty}^{\infty} n |c_n|^2 (R^{2n} - r^{2n}).$$

**13.5 Spring 2015****13.5.1 ?**

Let  $a_n(z)$  be an analytic sequence in a domain  $D$  such that  $\sum_{n=0}^{\infty} |a_n(z)|$  converges uniformly on bounded and closed sub-regions of  $D$ . Show that  $\sum_{n=0}^{\infty} |a'_n(z)|$  converges uniformly on bounded and closed sub-regions of  $D$ .

**13.5.2 ?**

Let  $f_n, f$  be analytic functions on the unit disk  $\mathbb{D}$ . Show that the following are equivalent.

- (i)  $f_n(z)$  converges to  $f(z)$  uniformly on compact subsets in  $\mathbb{D}$ .
- (ii)  $\int_{|z|=r} |f_n(z) - f(z)| |dz|$  converges to 0 if  $0 < r < 1$ .

**13.5.3 ?**

Let  $f$  and  $g$  be non-zero analytic functions on a region  $\Omega$ . Assume  $|f(z)| = |g(z)|$  for all  $z$  in  $\Omega$ . Show that  $f(z) = e^{i\theta}g(z)$  in  $\Omega$  for some  $0 \leq \theta < 2\pi$ .

**13.5.4 ?**

Suppose  $f$  is analytic in an open set containing the unit disc  $\mathbb{D}$  and  $|f(z)| = 1$  when  $|z|=1$ . Show that either  $f(z) = e^{i\theta}$  for some  $\theta \in \mathbb{R}$  or there are finite number of  $z_k \in \mathbb{D}$ ,  $k \leq n$  and  $\theta \in \mathbb{R}$  such that  $f(z) = e^{i\theta} \prod_{k=1}^n \frac{z - z_k}{1 - \bar{z}_k z}$ .

Also cf. Stein et al, 1.4.7, 3.8.17

**13.5.5 ?**

- (1) Let  $p(z)$  be a polynomial,  $R > 0$  any positive number, and  $m \geq 1$  an integer. Let  $M_R = \sup\{|z^m p(z) - 1| : |z| = R\}$ . Show that  $M_R > 1$ .
- (2) Let  $m \geq 1$  be an integer and  $K = \{z \in \mathbb{C} : r \leq |z| \leq R\}$  where  $r < R$ . Show (i) using (1) as well as, (ii) without using (1) that there exists a positive number  $\varepsilon_0 > 0$  such that for each polynomial  $p(z)$ ,

$$\sup\{|p(z) - z^{-m}| : z \in K\} \geq \varepsilon_0.$$

**13.5.6 ?**

Let  $f(z) = \frac{1}{z} + \frac{1}{z^2 - 1}$ . Find all the Laurent series of  $f$  and describe the largest annuli in which these series are valid.

**13.5.7 ?**

Suppose  $f$  is entire and there exist  $A, R > 0$  and natural number  $N$  such that  $|f(z)| \leq A|z|^N$  for  $|z| \geq R$ . Show that (i)  $f$  is a polynomial and (ii) the degree of  $f$  is at most  $N$ .

**13.5.8 ?**

Suppose  $f$  is entire and there exist  $A, R > 0$  and natural number  $N$  such that  $|f(z)| \geq A|z|^N$  for  $|z| \geq R$ . Show that (i)  $f$  is a polynomial and (ii) the degree of  $f$  is at least  $N$ .

**13.5.9 ?**

- (1) Explicitly write down an example of a non-zero analytic function in  $|z| < 1$  which has infinitely zeros in  $|z| < 1$ .
- (2) Why does not the phenomenon in (1) contradict the uniqueness theorem?

**13.5.10 ?**

- (1) Assume  $u$  is harmonic on open set  $O$  and  $z_n$  is a sequence in  $O$  such that  $u(z_n) = 0$  and  $\lim z_n \in O$ . Prove or disprove that  $u$  is identically zero. What if  $O$  is a region?
- (2) Assume  $u$  is harmonic on open set  $O$  and  $u(z) = 0$  on a disc in  $O$ . Prove or disprove that  $u$  is identically zero. What if  $O$  is a region?

- (3) Formulate and prove a Schwarz reflection principle for harmonic functions

cf. Theorem 5.6 on p.60 of Stein et al.

Hint: Verify the mean value property for your new function obtained by Schwarz reflection principle.

### 13.5.11 ?

Let  $f$  be holomorphic in a neighborhood of  $D_r(z_0)$ . Show that for any  $s < r$ , there exists a constant  $c > 0$  such that

$$\|f\|_{(\infty,s)} \leq c \|f\|_{(1,r)},$$

where  $\|f\|_{(\infty,s)} = \sup_{z \in D_s(z_0)} |f(z)|$  and  $\|f\|_{(1,r)} = \int_{D_r(z_0)} |f(z)| dx dy$ .

Note: Exercise 3.8.20 on p.107 in Stein et al is a straightforward consequence of this stronger result using the integral form of the Cauchy-Schwarz inequality in real analysis.

### 13.5.12 ?

- (1) Let  $f$  be analytic in  $\Omega : 0 < |z - a| < r$  except at a sequence of poles  $a_n \in \Omega$  with  $\lim_{n \rightarrow \infty} a_n = a$ . Show that for any  $w \in \mathbb{C}$ , there exists a sequence  $z_n \in \Omega$  such that  $\lim_{n \rightarrow \infty} f(z_n) = w$ .
- (2) Explain the similarity and difference between the above assertion and the Weierstrass-Casorati theorem.

### 13.5.13 ?

Compute the following integrals.

$$\begin{aligned} & i \int_0^\infty \frac{1}{(1+x^n)^2} dx, n \geq 1 \quad (\text{ii}) \int_0^\infty \frac{\cos x}{(x^2+a^2)^2} dx, a \in \mathbb{R} \quad (\text{iii}) \int_0^\pi \frac{1}{a+\sin \theta} d\theta, a > 1 \\ & iv \int_0^{\frac{\pi}{2}} \frac{d\theta}{a+\sin^2 \theta}, a > 0. \quad (\text{v}) \int_{|z|=2} \frac{1}{(z^5-1)(z-3)} dz \quad (\text{vi}) \int_{-\infty}^\infty \frac{\sin \pi a}{\cosh \pi x + \cos \pi a} e^{-ix\xi} dx, 0 < a < 1, \\ & \xi \in \mathbb{R} \quad (\text{vii}) \int_{|z|=1} \cot^2 z dz. \end{aligned}$$

### 13.5.14 ?

Compute the following integrals.

$$\begin{aligned} & i \int_0^\infty \frac{\sin x}{x} dx \quad (\text{ii}) \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx \quad (\text{iii}) \int_0^\infty \frac{x^{a-1}}{(1+x)^2} dx, 0 < a < 2 \\ & i \int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx, a, b > 0 \quad (\text{ii}) \int_0^\infty \frac{x^{a-1}}{1+x^n} dx, 0 < a < n \\ & iii \int_0^\infty \frac{\log x}{1+x^n} dx, n \geq 2 \quad (\text{iv}) \int_0^\infty \frac{\log x}{(1+x^2)^2} dx \quad (\text{v}) \int_0^\pi \log |1 - a \sin \theta| d\theta, a \in \mathbb{C} \end{aligned}$$

**13.5.15 ?**

Let  $0 < r < 1$ . Show that polynomials  $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$  have no zeros in  $|z| < r$  for all sufficiently large  $n$ 's.

**13.5.16 ?**

Let  $f$  be an analytic function on a region  $\Omega$ . Show that  $f$  is a constant if there is a simple closed curve  $\gamma$  in  $\Omega$  such that its image  $f(\gamma)$  is contained in the real axis.

**13.5.17 ?**

(1) Show that  $\frac{\pi^2}{\sin^2 \pi z}$  and  $g(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$  have the same principal part at each integer point.

(2) Show that  $h(z) = \frac{\pi^2}{\sin^2 \pi z} - g(z)$  is bounded on  $\mathbb{C}$  and conclude that  $\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ .

**13.5.18 ?**

Let  $f(z)$  be an analytic function on  $\mathbb{C} \setminus \{z_0\}$ , where  $z_0$  is a fixed point. Assume that  $f(z)$  is bijective from  $\mathbb{C} \setminus \{z_0\}$  onto its image, and that  $f(z)$  is bounded outside  $D_r(z_0)$ , where  $r$  is some fixed positive number. Show that there exist  $a, b, c, d \in \mathbb{C}$  with  $ad - bc \neq 0$ ,  $c \neq 0$  such that  $f(z) = \frac{az + b}{cz + d}$ .

**13.5.19 ?**

Assume  $f(z)$  is analytic in  $\mathbb{D} : |z| < 1$  and  $f(0) = 0$  and is not a rotation (i.e.  $f(z) \neq e^{i\theta}z$ ). Show that  $\sum_{n=1}^{\infty} f^n(z)$  converges uniformly to an analytic function on compact subsets of  $\mathbb{D}$ , where  $f^{n+1}(z) = f(f^n(z))$ .

**13.5.20 ?**

Let  $f$  be a non-constant analytic function on  $\mathbb{D}$  with  $f(\mathbb{D}) \subseteq \mathbb{D}$ . Use  $\psi_a(f(z))$  (where  $a = f(0)$ ,  $\psi_a(z) = \frac{a-z}{1-\bar{a}z}$ ) to prove that  $\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}$ .

**13.5.21 ?**

Find a conformal map

1. from  $\{z : |z - 1/2| > 1/2, \operatorname{Re}(z) > 0\}$  to  $\mathbb{H}$
2. from  $\{z : |z - 1/2| > 1/2, |z| < 1\}$  to  $\mathbb{D}$
3. from the intersection of the disk  $|z + i| < \sqrt{2}$  with  $\mathbb{H}$  to  $\mathbb{D}$ .

4. from  $\mathbb{D} \setminus [a, 1)$  to  $\mathbb{D} \setminus [0, 1)$  ( $0 < a < 1$ ).

*Short solution possible using Blaschke factor*

5. from  $\{z : |z| < 1, \operatorname{Re}(z) > 0\} \setminus (0, 1/2]$  to  $\mathbb{H}$ .

### 13.5.22 ?

Let  $C$  and  $C'$  be two circles and let  $z_1 \in C$ ,  $z_2 \notin C$ ,  $z'_1 \in C'$ ,  $z'_2 \notin C'$ . Show that there is a unique fractional linear transformation  $f$  with  $f(C) = C'$  and  $f(z_1) = z'_1$ ,  $f(z_2) = z'_2$ .

### 13.5.23 ?

Assume  $f_n \in H(\Omega)$  is a sequence of holomorphic functions on the region  $\Omega$  that are uniformly bounded on compact subsets and  $f \in H(\Omega)$  is such that the set  $\{z \in \Omega : \lim_{n \rightarrow \infty} f_n(z) = f(z)\}$  has a limit point in  $\Omega$ . Show that  $f_n$  converges to  $f$  uniformly on compact subsets of  $\Omega$ .

### 13.5.24 ?

Let  $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$  with  $|\alpha| < 1$  and  $\mathbb{D} = \{z : |z| < 1\}$ . Prove that

- $\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_\alpha|^2 dx dy = 1.$
- $\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_\alpha| dx dy = \frac{1 - |\alpha|^2}{|\alpha|^2} \log \frac{1}{1 - |\alpha|^2}.$

### 13.5.25 ?

Prove that  $f(z) = -\frac{1}{2} \left( z + \frac{1}{z} \right)$  is a conformal map from half disc  $\{z = x + iy : |z| < 1, y > 0\}$  to upper half plane  $\mathbb{H} = \{z = x + iy : y > 0\}$ .

### 13.5.26 ?

Let  $\Omega$  be a simply connected open set and let  $\gamma$  be a simple closed contour in  $\Omega$  and enclosing a bounded region  $U$  anticlockwise. Let  $f : \Omega \rightarrow \mathbb{C}$  be a holomorphic function and  $|f(z)| \leq M$  for all  $z \in \gamma$ . Prove that  $|f(z)| \leq M$  for all  $z \in U$ .

### 13.5.27 ?

Compute the following integrals. (i)  $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx$ ,  $0 < a < n$  (ii)  $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$

### 13.5.28 ?

Let  $0 < r < 1$ . Show that polynomials  $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$  have no zeros in  $|z| < r$  for all sufficiently large  $n$ 's.

**13.5.29 ?**

Let  $f$  be holomorphic in a neighborhood of  $D_r(z_0)$ . Show that for any  $s < r$ , there exists a constant  $c > 0$  such that

$$\|f\|_{(\infty,s)} \leq c\|f\|_{(1,r)},$$

where  $\|f\|_{(\infty,s)} = \sup_{z \in D_s(z_0)} |f(z)|$  and  $\|f\|_{(1,r)} = \int_{D_r(z_0)} |f(z)| dx dy$ .

**13.5.30 ?**

Let  $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$  with  $|\alpha| < 1$  and  $\mathbb{D} = \{z : |z| < 1\}$ . Prove that

- $\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_\alpha|^2 dx dy = 1$ .
- $\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_\alpha| dx dy = \frac{1 - |\alpha|^2}{|\alpha|^2} \log \frac{1}{1 - |\alpha|^2}$ .

Prove that  $f(z) = -\frac{1}{2} \left( z + \frac{1}{z} \right)$  is a conformal map from half disc  $\{z = x + iy : |z| < 1, y > 0\}$  to upper half plane  $\mathbb{H} = \{z = x + iy : y > 0\}$ .

**13.5.31 ?**

Let  $\Omega$  be a simply connected open set and let  $\gamma$  be a simple closed contour in  $\Omega$  and enclosing a bounded region  $U$  anticlockwise. Let  $f : \Omega \rightarrow \mathbb{C}$  be a holomorphic function and  $|f(z)| \leq M$  for all  $z \in \gamma$ . Prove that  $|f(z)| \leq M$  for all  $z \in U$ .

**13.5.32 ?**

Compute the following integrals. (i)  $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx$ ,  $0 < a < n$  (ii)  $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$

**13.5.33 ?**

Let  $0 < r < 1$ . Show that polynomials  $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$  have no zeros in  $|z| < r$  for all sufficiently large  $n$ 's.

**13.5.34 ?**

Let  $f$  be holomorphic in a neighborhood of  $D_r(z_0)$ . Show that for any  $s < r$ , there exists a constant  $c > 0$  such that

$$\|f\|_{(\infty,s)} \leq c\|f\|_{(1,r)},$$

where  $\|f\|_{(\infty,s)} = \sup_{z \in D_s(z_0)} |f(z)|$  and  $\|f\|_{(1,r)} = \int_{D_r(z_0)} |f(z)| dx dy$ .

**13.6 Fall 2016****13.6.1 ?**

Let  $u(x, y)$  be harmonic and have continuous partial derivatives of order three in an open disc of radius  $R > 0$ .

- (a) Let two points  $(a, b), (x, y)$  in this disk be given. Show that the following integral is independent of the path in this disk joining these points:

$$v(x, y) = \int_{a,b}^{x,y} \left( -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right).$$

(b)

- (i) Prove that  $u(x, y) + iv(x, y)$  is an analytic function in this disc.  
(ii) Prove that  $v(x, y)$  is harmonic in this disc.

**13.6.2 ?**

- (a)  $f(z) = u(x, y) + iv(x, y)$  be analytic in a domain  $D \subset \mathbb{C}$ . Let  $z_0 = (x_0, y_0)$  be a point in  $D$  which is in the intersection of the curves  $u(x, y) = c_1$  and  $v(x, y) = c_2$ , where  $c_1$  and  $c_2$  are constants. Suppose that  $f'(z_0) \neq 0$ . Prove that the lines tangent to these curves at  $z_0$  are perpendicular.
- (b) Let  $f(z) = z^2$  be defined in  $\mathbb{C}$ .
- (c) Describe the level curves of  $\operatorname{Re}(f)$  and of  $\operatorname{Im}(f)$ .
- (ii) What are the angles of intersections between the level curves  $\operatorname{Re}(f) = 0$  and  $\operatorname{Im}(f)$ ? Is your answer in agreement with part a) of this question?

**13.6.3 ?**

- (a)  $f : D \rightarrow \mathbb{C}$  be a continuous function, where  $D \subset \mathbb{C}$  is a domain. Let  $\alpha : [a, b] \rightarrow D$  be a smooth curve. Give a precise definition of the *complex line integral*

$$\int_{\alpha} f.$$

- (b) Assume that there exists a constant  $M$  such that  $|f(\tau)| \leq M$  for all  $\tau \in \operatorname{Image}(\alpha)$ . Prove that

$$\left| \int_{\alpha} f \right| \leq M \times \operatorname{length}(\alpha).$$

- (c) Let  $C_R$  be the circle  $|z| = R$ , described in the counterclockwise direction, where  $R > 1$ . Provide an upper bound for  $\left| \int_{C_R} \frac{\log(z)}{z^2} \right|$ , which depends only on  $R$  and other constants.



**13.6.4 ?**

- (a) Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be an entire function. Assume the existence of a non-negative integer  $m$ , and of positive constants  $L$  and  $R$ , such that for all  $z$  with  $|z| > R$  the inequality

$$|f(z)| \leq L|z|^m$$

holds. Prove that  $f$  is a polynomial of degree  $\leq m$ .

- (b) Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be an entire function. Suppose that there exists a real number  $M$  such that for all  $z \in \mathbb{C}$

$$\operatorname{Re}(f) \leq M.$$

Prove that  $f$  must be a constant.

**13.6.5 ?**

Prove that all the roots of the complex polynomial

$$z^7 - 5z^3 + 12 = 0$$

lie between the circles  $|z| = 1$  and  $|z| = 2$ .

**13.6.6 ?**

- (a) Let  $F$  be an analytic function inside and on a simple closed curve  $C$ , except for a pole of order  $m \geq 1$  at  $z = a$  inside  $C$ . Prove that

$$\frac{1}{2\pi i} \oint_C F(\tau) d\tau = \lim_{\tau \rightarrow a} \frac{d^{m-1}}{d\tau^{m-1}} ((\tau - a)^m F(\tau)).$$

- (b) Evaluate

$$\oint_C \frac{e^\tau}{(\tau^2 + \pi^2)^2} d\tau$$

where  $C$  is the circle  $|z| = 4$ .

**13.6.7 ?**

Find the conformal map that takes the upper half-plane conformally onto the half-strip  $\{w = x + iy : -\pi/2 < x < \pi/2, y > 0\}$ .

**13.6.8 ?**

Compute the integral  $\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{\cosh \pi x} dx$  where  $\cosh z = \frac{e^z + e^{-z}}{2}$ .