Title

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1.1 Unsorted

- $H_n(X/A) \cong \tilde{H}_n(X, A)$ when $A \subset X$ has a neighborhood that deformation retracts onto it.
- $H_n(\bigvee_\alpha X_\alpha) = \bigoplus_\alpha H_n X_\alpha$
- Useful fact: since $\mathbb{Z}$ is free, any exact sequence of the form $0 \to \mathbb{Z}^n \to A \to \mathbb{Z}^m \to 0$ splits and $A \cong \mathbb{Z}^n \times \mathbb{Z}^m$.
- Useful fact: $\tilde{H}_*(A \vee B) \cong H_*(A) \times H_*(B)$.
- $H_n(\bigvee_\alpha X_\alpha) = \bigoplus_\alpha H_n X_\alpha$
- $H_n(X, A) \cong H_n(X/A)$
- $H_n(X) = 0 \iff X$ has no n-cells.
- $C^0 X = \{\mathrm{pt}\} \implies d_1 : C^1 \to C^0$ is the zero map.
- $H^*(X; \mathbb{F}) = \hom(H_*(X; \mathbb{F}), \mathbb{F})$ for a field.
- Useful tools:
    - Mayer-Vietoris

    * $(X = A \cup B) \mapsto (\cap, \oplus, \cup)$ in homology
  - LES of a pair
    * $(A \hookrightarrow X) \mapsto (A, X, X/A)$
  - Excision

- $H_k \prod X \neq \prod H_k X$ due to torsion.
  - Nice case: $H_k(A \times B) = \prod_{i+j=k} H_i A \otimes H_j B$ by Kunneth when all groups are torsion-free.[1]
- $H_k \bigvee X = \prod H_k X$ by Mayer-Vietoris.[2]
- $H_i(S^n) = \mathbb{1}\left[i \in \{0, n\}\right]$
- $H_n(\bigvee_i X_i) \cong \prod_i H_n(X_i)$ for "good pairs"
  - Corollary: $H_n(\bigvee_k S^n) = \mathbb{Z}^k$

$$
\begin{aligned}
X = A \cup B &\implies A \cap B \to A \oplus B \to A \cup B \xrightarrow{\delta} \cdots (X, A) \implies \qquad A \to X \to X, A \xrightarrow{\delta} \cdots \\
A \to B \to C &\implies \operatorname{Tor}(A, G) \to \operatorname{Tor}(B, G) \to \operatorname{Tor}(C, G) \xrightarrow{\delta_\downarrow} \cdots \\
A \to B \to C &\implies \operatorname{Ext}(A, G) \to \operatorname{Ext}(B, G) \to \operatorname{Ext}(C, G) \xrightarrow{\delta_\uparrow} \cdots
\end{aligned}
$$

1.2 Constructing a CW Complex with Prescribed Homology

- Given $G = \bigoplus G_i$, and want a space such that $H_i X = G$? Construct $X = \bigvee X_i$ and then $H_i(\bigvee X_i) = \bigoplus H_i X_i$. Reduces problem to: given a group H, find a space Y such that $H_n(Y) = G$.
  - Attach an e^n to a point to get $H_n = \mathbb{Z}$
  - Then attach an e^{n+1} with attaching map of degree d to get $H_n = \mathbb{Z}_d$

1.3 Mayer-Vietoris

Theorem (Mayer Vietoris) Let $X = A^\circ \cup B^\circ$; then there is a SES of chain complexes

$$
0 \to C_n(A \cap B) \xrightarrow{x \mapsto (x, -x)} C_n(A) \oplus C_n(B) \xrightarrow{(x,y) \mapsto x+y} C_n(A + B) \to 0
$$

where $C_n(A+B)$ denotes the chains that are sums of chains in A and chains in B. This yields a LES in homology:

[1] The generalization of Kunneth is as follows: write $\mathcal{P}(n, k)$ be the set of partitions of n into k parts, i.e. $\mathbf{x} \in \mathcal{P}(n, k) \implies \mathbf{x} = (x_1, x_2, \ldots, x_k)$ where $\sum x_i = n$. Then

$$
H_n\left(\prod_{j=1}^k X_j\right) = \bigoplus_{\mathbf{x} \in \mathcal{P}(n,k)} \bigotimes_{i=1}^k H_{x_i}(X_i).
$$

[2] $\bigvee$ is the coproduct in the category $\mathbf{Top}_0$ of pointed topological spaces, and alternatively, $X \vee Y$ is the pushout in $\mathbf{Top}$ of $X \leftarrow \{\mathrm{pt}\} \to Y$

$$\cdots \to H_n(A \cap B) \xrightarrow{x \mapsto (x,-x)} H_n(A) \oplus H_n(B) \xrightarrow{(x,y) \mapsto x+y} H_n(X) \to \cdots.$$

Given $A, B \subset X$ such that $A^\circ \cup B^\circ = X$, there is a long exact sequence in homology:

$$\begin{tikzcd}
 & & & \cdots \arrow[dlll, "\delta_3"'] \\
H_2(A \cap B) \arrow[r, "(i^*, -j^*)_2"] & H_2 A \oplus H_2 B \arrow[r, "(l^* - r^*)_2"] & H_2(A \cup B) \arrow[dll, "\delta_2"'] & \\
H_1(A \cap B) \arrow[r, "(i^*, -j^*)_1"] & H_1 A \oplus H_1 B \arrow[r, "(l^* - r^*)_1"] & H_1(A \cup B) \arrow[dll, "\delta_1"'] & \\
H_0(A \cap B) \arrow[r, "(i^*, -j^*)_0"] & H_0 A \oplus H_0 B \arrow[r, "(l^* - r^*)_0"] & H_0(A \cup B) \arrow[dll, "\delta_0"'] & \\
0 & & &
\end{tikzcd}$$

This is sometimes written in the following compact form:

$$\cdots H_n(A \cap B) \xrightarrow{(i^*,\, j^*)} H_n(A) \oplus H_n(B) \xrightarrow{l^* - r^*} H_n(X) \xrightarrow{\delta} H_{n-1}(A \cap B) \cdots$$

Where

- $i : A \cap B \hookrightarrow A$ induces $i^* : H_*(A \cap B) \to H_*(A)$
- $j : A \cap B \hookrightarrow B$ induces $j^* : H_*(A \cap B) \to H_*(B)$
- $l : A \hookrightarrow A \cup B$ induces $l^* : H_*(A) \to H_*(X)$
- $r : B \hookrightarrow A \cup B$ induces $r^* : H_*(B) \to H_*(X)$

The connecting homomorphisms $\delta_n : H_n(X) \to H_{n-1}(X)$ are defined by taking a class $[\alpha] \in H_n(X)$, writing it as an n-cycle z, then decomposing $z = \sum c_i$ where each c_i is an $x + y$ chain. Then $\partial(c_i) = \partial(x + y) = 0$, since the boundary of a cycle is zero, so $\partial(x) = -\partial(y)$. So then just define $\delta([\alpha]) = [\partial x] = [-\partial y]$.

Handy mnemonic diagram:

$$\begin{tikzcd}
 & A \cap B \arrow[dl, dash] \arrow[dr, dash] & \\
A \cup B & & A \oplus B \arrow[ll]
\end{tikzcd}.$$

1.3.1 Application: Isomorphisms in the homology of spheres.

Proposition 1.1 *(?).*

$$H^i(S^n) \cong H^{i-1}(S^{n-1}).$$

Proof .
Write $X = A \cup B$, the northern and southern hemispheres, so that $A \cap B = S^{n-1}$, the equator. In the LES, we have:

$$H^{i+1}(S^n) \to H^i(S^{n-1}) \to H^iA \oplus H^iB \to H^iS^n \to H^{i-1}(S^{n-1}) \to H^{i-1}A \oplus H^{i-1}B.$$

But A, B are contractible, so $H^iA = H^iB = 0$, so we have

$$H^{i+1}(S^n) \to H^i(S^{n-1}) \to 0 \oplus 0 \to H^i(S^n) \to H^{i-1}(S^{n-1}) \to 0.$$

In particular, we have the shape $0 \to A \to B \to 0$ in an exact sequence, which is always an isomorphism.

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Theorem 1.2 *(Eilenber-Zilber).*
Given two spaces X, Y, there are chain maps

$$\begin{aligned} F &: C_*(X \times Y; R) \to C_*(X; R) \otimes_R C_*(Y; R) \\ G &: C_*(X; R) \otimes_R C_*(Y; R) \to C_*(X \times Y; R) \end{aligned}$$

such that $FG = \mathrm{id}$ and $GF \simeq \mathrm{id}$. In particular,

$$H_*(X \times Y; R) \cong H_*(X; R) \otimes_R H_*(Y; R).$$

Theorem 1.3 *(Kunneth).*
There exists a short exact sequence

$$0 \to \bigoplus_{i+j=k} H_j(X; R) \otimes_R H_i(Y; R) \to H_k(X \times Y; R) \to \bigoplus_{i+j=k-1} \mathrm{Tor}^1_R(H_i(X; R), H_j(Y; R))$$

It has a non-canonical splitting given by

$$H_k(X \times Y) = \left(\bigoplus_{i+j=k} H_iX \oplus H_jY \right) \oplus \bigoplus_{i+j=k-1} \mathrm{Tor}(H_iX, H_jY)$$

Theorem 1.4 *(UCT for Change of Group).*

For changing coefficients from $\mathbb{Z}$ to G an arbitrary group, there are short exact sequences

$$0 \to H_iX \otimes G \to H_i(X;G) \to \mathrm{Tor}(H_{i-1}X, G) \to 0$$

$$0 \to \mathrm{Ext}(H_{i-1}X, G) \to H^i(X;G) \to \hom(H_iX, G) \to 0$$

which split unnaturally:

$$H_i(X;G) = (H_iX \otimes G) \oplus \mathrm{Tor}(H_{i-1}X;G)$$

$$H^i(X;G) = \hom(H_iX, G) \oplus \mathrm{Ext}(H_{i-1}X;G)$$

When H_iX are all finitely generated, write $H_i(X;\mathbb{Z}) = \mathbb{Z}^{\beta_i} \oplus T_i$. Then

$$H^i(X;\mathbb{Z}) = \mathbb{Z}^{\beta_i} \oplus T_{i-1}.$$

1.3.2 Useful long exact sequences

$$\cdots \to H^i(X) \to H^i(U) \oplus H^i(V) \to H^i(U \cap V) \xrightarrow{\delta} H^{i+1}(X) \to \cdots$$

$$\cdots \to H_i(A) \to H_i(X) \to H_i(X,A) \xrightarrow{\delta} H_{i-1}(A) \to \cdots$$

1.3.3 Useful Short Exact Sequences

Note that $\mathrm{Ext}^0_R = \hom_R$ and $\mathrm{Tor}^0_R = \otimes_R$

Homology to Cohomology

$$0 \to \mathrm{Tor}^0_{\mathbb{Z}}(H_i(X;\mathbb{Z}), A) \to H_i(X;A) \to \mathrm{Tor}^1_{\mathbb{Z}}(H_{i-1}(X;\mathbb{Z}), A) \to 0.$$

Cohomology to Dual of Homology

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}(X;\mathbb{Z}), A) \to H^i(X;A) \to \mathrm{Ext}^0_{\mathbb{Z}}(H_i(X;\mathbb{Z}), A) \to 0.$$

Product of Spaces to Tensor Product in Homology

$$0 \to \bigoplus_{i+j=k} H_i(X;R) \otimes_R H_j(Y;R) \to H_k(X \times Y;R) \to \bigoplus_{i+j=k-1} \mathrm{Tor}^R_1(H_i(X;R), H_j(Y;R)) \to 0$$

1.3.4 Useful Shortcuts

- Cohomology: If A is a field, then

$$H^i(X;A) \cong \hom(H_i(X;A), A)$$

- Kunneth: If R is a freely generated free R-module (or a PID or a field), then

$$H_k(X \times Y) \cong \bigoplus_{i+j=k} H_i(X) \otimes H_j(Y) \bigoplus_{i+j=k-1} \mathrm{Tor}(H_i(X), H_j(X))$$

- Universal Coefficients Theorem: If X is a finite CW complex then

$$H^i(X;\mathbb{Z}) = F(H_i(X;\mathbb{Z})) \times T(H_{i-1}(X;\mathbb{Z}))$$
$$H_i(X;\mathbb{Z}) = F(H^i(X;\mathbb{Z})) \times T(H^{i+1}(X;\mathbb{Z}))$$

1.4 Cellular Homology

- S^n has the CW complex structure of 2 k-cells for each $0 \leq k \leq n$.

How to compute:

1. Write cellular complex

$$0 \to C^n \to C^{n-1} \to \cdots C^2 \to C^1 \to C^0 \to 0$$

2. Compute differentials $\partial_i : C^i \to C^{i-1}$
   3. *Note: if C^0 is a point, ∂_1 is the zero map.*
   4. *Note: $H_n X = 0 \iff C^n = \emptyset$.*
   5. Compute degrees: Use $\partial_n(e_i^n) = \sum_i d_i e_i^{n-1}$ where

   $$d_i = \deg(\text{Attach } e_i^n \to \text{Collapse } X^{n-1}\text{-skeleton}),$$

   which is a map $S^{n-1} \to S^{n-1}$.
      1. Alternatively, choose orientations for both spheres. Then pick a point in the target, and look at points in the fiber. Sum them up with a weight of +1 if the orientations match and -1 otherwise.
   6. Note that $\mathbb{Z}^m \xrightarrow{f} \mathbb{Z}^n$ has an $n \times m$ matrix
   7. Row reduce, image is span of rows with pivots. Kernel can be easily found by taking RREF, padding with zeros so matrix is square and has all diagonals, then reading down diagonal - if a zero is encountered on nth element, take that column vector as a basis element with -1 substituted in for the nth entry. e.g.

$$\begin{array}{cccc} \mathbf{1} & 2 & 0 & 2 \\ 0 & 0 & \mathbf{1} & -1 \\ 0 & 0 & 0 & \mathbf{0} \end{array} \to \begin{array}{cccc} \mathbf{1} & 2 & 0 & 2 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & \mathbf{1} & -1 \\ 0 & 0 & 0 & \mathbf{0} \end{array} \begin{array}{cccc} \mathbf{1} & 2 & 0 & 2 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & \mathbf{1} & -1 \\ 0 & 0 & 0 & \mathbf{0} \end{array}$$

$$\ker = \begin{array}{cc} 2 & 3 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{array}$$

$$\mathrm{im} \ = \langle a + 2b + 2d, c - d \rangle \,.$$

6. Or look at elementary divisors, say n_i, then the image is isomorphic to $\bigoplus n_i\mathbb{Z}$