

Title

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1 Modules

1.1 General Questions

1.1.1 Fall 2019 Final #2

Consider the \mathbb{Z} -submodule N of \mathbb{Z}^3 spanned by $f_1 = [-1, 0, 1], f_2 = [2, -3, 1], f_3 = [0, 3, 1], f_4 = [3, 1, 5]$. Find a basis for N and describe \mathbb{Z}^3/N .

1.1.2 Spring 2018 #6.

Let

$$M = \{(w, x, y, z) \in \mathbb{Z}^4 \mid w + x + y + z \in 2\mathbb{Z}\},$$

and

$$N = \{(w, x, y, z) \in \mathbb{Z}^4 \mid 4 \mid (w - x), 4 \mid (x - y), 4 \mid (y - z)\}.$$

- a. Show that N is a \mathbb{Z} -submodule of M .
- b. Find vectors $u_1, u_2, u_3, u_4 \in \mathbb{Z}^4$ and integers d_1, d_2, d_3, d_4 such that

$$\{u_1, u_2, u_3, u_4\}$$

is a free basis for M , and

$$\{d_1u_1, d_2u_2, d_3u_3, d_4u_4\}$$

is a free basis for N .

- c. Use the previous part to describe M/N as a direct sum of cyclic \mathbb{Z} -modules.

1.1.3 Fall 2018 #6 \bowtie

Let R be a commutative ring, and let M be an R -module. An R -submodule N of M is maximal if there is no R -module P with $N \subsetneq P \subsetneq M$.

- a. Show that an R -submodule N of M is maximal $\iff M/N$ is a simple R -module: i.e., M/N is nonzero and has no proper, nonzero R -submodules.
- b. Let M be a \mathbb{Z} -module. Show that a \mathbb{Z} -submodule N of M is maximal $\iff \#M/N$ is a prime number.
- c. Let M be the \mathbb{Z} -module of all roots of unity in \mathbb{C} under multiplication. Show that there is no maximal \mathbb{Z} -submodule of M .

Solution.

a

By the correspondence theorem, submodules of M/N biject with submodules A of M containing N .

So

- M is maximal:
- \iff no such (proper, nontrivial) submodule A exists
- \iff there are no (proper, nontrivial) submodules of M/N
- $\iff M/N$ is simple.

b

Identify \mathbb{Z} -modules with abelian groups, then by (a), N is maximal $\iff M/N$ is simple $\iff M/N$ has no nontrivial proper subgroups.

By Cauchy's theorem, if $|M/N| = ab$ is a composite number, then $a \mid ab \implies$ there is an element (and thus a subgroup) of order a . In this case, M/N contains a nontrivial proper cyclic subgroup, so M/N is not simple. So $|M/N|$ can not be composite, and therefore must be prime.

c

Let $G = \{x \in \mathbb{C} \mid x^n = 1 \text{ for some } n \in \mathbb{N}\}$, and suppose $H < G$ is a proper subgroup.

Then there must be a prime p such that the $\zeta_{p^k} \notin H$ for all k greater than some constant m – otherwise, we can use the fact that if $\zeta_{p^k} \in H$ then $\zeta_{p^\ell} \in H$ for all $\ell \leq k$, and if $\zeta_{p^k} \in H$ for all p and all k then $H = G$.

But this means there are infinitely many elements in $G \setminus H$, and so $\infty = [G : H] = |G/H|$ is not a prime. Thus by (b), H can not be maximal, a contradiction.

1.1.4 Spring 2018 #7.

Let R be a PID and M be an R -module. Let p be a prime element of R . The module M is called $\langle p \rangle$ -primary if for every $m \in M$ there exists $k > 0$ such that $p^k m = 0$.

- Suppose M is $\langle p \rangle$ -primary. Show that if $m \in M$ and $t \in R$, $t \notin \langle p \rangle$, then there exists $a \in R$ such that $atm = m$.
- A submodule S of M is said to be *pure* if $S \cap rM = rS$ for all $r \in R$. Show that if M is $\langle p \rangle$ -primary, then S is pure if and only if $S \cap p^k M = p^k S$ for all $k \geq 0$.

1.1.5 Fall 2016 #6

Let R be a ring and $f : M \rightarrow N$ and $g : N \rightarrow M$ be R -module homomorphisms such that $g \circ f = \text{id}_M$. Show that $N \cong \text{im } f \oplus \ker g$.

1.1.6 Spring 2016 #4

Let R be a ring with the following commutative diagram of R -modules, where each row represents a short exact sequence of R -modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

Prove that if α and γ are isomorphisms then β is an isomorphism.

1.1.7 Spring 2015 #8

Let R be a PID and M a finitely generated R -module.

- Prove that there are R -submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that for all $0 \leq i \leq n-1$, the module M_{i+1}/M_i is cyclic.

- Is the integer n in part (a) uniquely determined by M ? Prove your answer.

1.1.8 Fall 2012 #6

Let R be a ring and M an R -module. Recall that M is *Noetherian* iff any strictly increasing chain of submodule $M_1 \subsetneq M_2 \subsetneq \cdots$ is finite. Call a proper submodule $M' \subsetneq M$ *intersection-decomposable* if it can not be written as the intersection of two proper submodules $M' = M_1 \cap M_2$ with $M_i \subsetneq M$.

Prove that for every Noetherian module M , any proper submodule $N \subsetneq M$ can be written as a finite intersection $N = N_1 \cap \cdots \cap N_k$ of intersection-indecomposable modules.

1.1.9 Fall 2019 Final #1

Let A be an abelian group, and show A is a \mathbb{Z} -module in a unique way.

1.2 Torsion and the Structure Theorem**1.2.1 ★ Fall 2019 #5** ∞

Let R be a ring and M an R -module.

Recall that the set of torsion elements in M is defined by

$$\text{Tor}(M) = \{m \in M \mid \exists r \in R, r \neq 0, rm = 0\}.$$

- a. Prove that if R is an integral domain, then $\text{Tor}(M)$ is a submodule of M .
- b. Give an example where $\text{Tor}(M)$ is not a submodule of M .
- c. If R has zero-divisors, prove that every non-zero R -module has non-zero torsion elements.