Real Analysis Qualifying Exam Questions

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Contents

1 Undergraduate Analysis: Uniform Convergence

1.1 Fall 2018 # 1

Let $f(x) = \dfrac{1}{x}$. Show that f is uniformly continuous on $(1, \infty)$ but not on $(0, \infty)$.

1.2 Fall 2017 # 1

Let

$$f(x) = s \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which f does and does not converge uniformly.

1.3 Fall 2014 # 1

Let $\{f_n\}$ be a sequence of continuous functions such that $\sum f_n$ converges uniformly.

Prove that $\sum f_n$ is also continuous.

1.4 Spring 2017 # 4

Let $f(x, y)$ on $[-1, 1]^2$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Determine if f is integrable.

1.5 Spring 2015 # 1

Let (X, d) and (Y, ρ) be metric spaces, $f : X \longrightarrow Y$, and $x_0 \in X$.

Prove that the following statements are equivalent:

1. For every $\varepsilon > 0 \quad \exists \delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$.
2. The sequence $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow f(x_0)$ for every sequence $\{x_n\} \longrightarrow x_0$ in X.

1.6 Fall 2014 # 2

Let I be an index set and $\alpha : I \longrightarrow (0, \infty)$.

1. Show that

$$\sum_{i \in I} a(i) := \sup_{\substack{J \subset I \\ J \text{ finite}}} \sum_{i \in J} a(i) < \infty \implies I \text{ is countable.}$$

2. Suppose $I = \mathbb{Q}$ and $\sum_{q \in \mathbb{Q}} a(q) < \infty$. Define

$$f(x) := \sum_{\substack{q \in \mathbb{Q} \\ q \leq x}} a(q).$$

Show that f is continuous at $x \iff x \notin \mathbb{Q}$.

1.7 Spring 2014 # 2

Let $\{a_n\}$ be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that $\sum a_n^2 < \infty$.

Note: Assume a_n, b_n are all non-negative.

2 General Analysis

2.1 Spring 2020 # 1

Prove that if $f : [0, 1] \longrightarrow \mathbb{R}$ is continuous then

$$\lim_{k \longrightarrow \infty} \int_0^1 k x^{k-1} f(x)\, dx = f(1).$$

Solution.
Concepts used:

- DCT
- Weierstrass Approximation Theorem

Solution:

- Suppose p is a polynomial, then

$$\begin{aligned}
\lim_{k \longrightarrow \infty} \int_0^1 k x^{k-1} p(x)\, dx &= \lim_{k \longrightarrow \infty} \int_0^1 \left(\frac{\partial}{\partial x} x^k \right) p(x)\, dx \\
&= \lim_{k \longrightarrow \infty} \left[x^k p(x) \Big|_0^1 - \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx \right] \quad \text{integrating by parts} \\
&= p(1) - \lim_{k \longrightarrow \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx,
\end{aligned}$$

- Thus it suffices to show that

$$\lim_{k \longrightarrow \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx = 0.$$

- Integrating by parts a second time yields

$$\begin{aligned}
\lim_{k\longrightarrow\infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x)\right) dx &= \lim_{k\longrightarrow\infty} \frac{x^{k+1}}{k+1} p'(x)\Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x)\right) dx \\
&= -\lim_{k\longrightarrow\infty} \int_0^1 \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x)\right) dx \\
&= -\int_0^1 \lim_{k\longrightarrow\infty} \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x)\right) dx \quad \text{by DCT} \\
&= -\int_0^1 0 \left(\frac{\partial^2}{\partial x^2} p(x)\right) dx \\
&= 0.
\end{aligned}$$

  - The DCT can be applied here because f'' is continuous and $[0,1]$ is compact, so f'' is bounded on $[0,1]$ by a constant M and

$$\int_0^1 \left| x^k f''(x) \right| \leq \int_0^1 1 \cdot M = M < \infty.$$

- Now use the Weierstrass approximation theorem:
  - If $f : [a,b] \longrightarrow \mathbb{R}$ is continuous, then for every $\varepsilon > 0$ there exists a polynomial $p_\varepsilon(x)$ such that $\|f - p_\varepsilon\|_\infty < \varepsilon$.
- Thus

$$\begin{aligned}
\left| \int_0^1 kx^{k-1} p_\varepsilon(x)\, dx - \int_0^1 kx^{k-1} f(x)\, dx \right| &= \left| \int_0^1 kx^{k-1} (p_\varepsilon(x) - f(x))\, dx \right| \\
&\leq \left| \int_0^1 kx^{k-1} \|p_\varepsilon - f\|_\infty \, dx \right| \\
&= \|p_\varepsilon - f\|_\infty \cdot \left| \int_0^1 kx^{k-1}\, dx \right| \\
&= \|p_\varepsilon - f\|_\infty \cdot x^k \Big|_0^1 \\
&= \|p_\varepsilon - f\|_\infty \overset{\varepsilon \longrightarrow 0}{\Longrightarrow} 0
\end{aligned}$$

  and the integrals are equal.
- By the first argument,

$$\int_0^1 kx^{k-1} p_\varepsilon(x)\, dx = p_\varepsilon(1) \text{ for each } \varepsilon$$

- Since uniform convergence implies pointwise convergence, $p_\varepsilon(1) \overset{\varepsilon \longrightarrow 0}{\Longrightarrow} f(1)$.

2.2 Fall 2019 # 1.

Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers.

a. Prove that if $\lim_{n \longrightarrow \infty} a_n = 0$, then

$$\lim_{n \to \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

b. Prove that if $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges, then

$$\lim_{n \to \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

Solution.

Cesaro mean/summation. Break series apart into pieces that can be handled separately.

2.2.1 a

Prove a stronger result:

$$a_k \longrightarrow S \implies S_N := \frac{1}{N} \sum_{k=1}^{N} a_k \longrightarrow S.$$

Idea: once N is large enough, $a_k \approx S$, and all smaller terms will die off as $N \longrightarrow \infty$. See this MSE answer.

- Use convergence $a_k \longrightarrow S$: choose M large enough such that

$$k \geq M + 1 \implies |a_k - S| < \varepsilon.$$

Then

$$\begin{aligned}
\left|\left(\frac{1}{N}\sum_{k=1}^{N} a_k\right) - S\right| &= \frac{1}{N}\left|\left(\sum_{k=1}^{N} a_k\right) - NS\right| \\
&= \frac{1}{N}\left|\left(\sum_{k=1}^{N} a_k\right) - \sum_{k=1}^{N} S\right| \\
&= \frac{1}{N}\left|\sum_{k=1}^{N} (a_k - S)\right| \\
&\leq \frac{1}{N}\sum_{k=1}^{N} |a_k - S| \\
&= \frac{1}{N}\sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} |a_k - S| \\
&\leq \frac{1}{N}\sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} \frac{\varepsilon}{2} \\
&= \frac{1}{N}\sum_{k=1}^{M} |a_k - S| + (N-M)\frac{\varepsilon}{2} \\
&\overset{\varepsilon \longrightarrow 0}{\longrightarrow} \frac{1}{N}\sum_{k=1}^{M} |a_k - S| + 0 \\
&\overset{N \longrightarrow \infty}{\longrightarrow} 0 + 0.
\end{aligned}$$

Note: M is fixed, so the last sum is some constant c, and $c/N \longrightarrow 0$ as $N \longrightarrow \infty$ for any constant. To be more careful, choose M first to get $\varepsilon/2$ for the tail, then choose $N(M) > M$ for the remaining truncated part of the sum.

2.2.2 b

- Define

$$\Gamma_n := \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

- $\Gamma_1 = \sum_{k=1}^{n} \frac{a_k}{k}$ is the original series and each Γ_n is a tail of Γ_1, so by assumption $\Gamma_n \overset{n \longrightarrow \infty}{\longrightarrow} 0$.
- Compute

$$\frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n}(\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \mathbf{\Gamma_{n+1}})$$

.

- This comes from consider the following summation:

$$
\begin{array}{l|cccccccc}
\Gamma_1 : & a_1 & +\frac{a_2}{2} & +\frac{a_3}{3} & +\cdots & & & \\
\Gamma_2 : & & \frac{a_2}{2} & +\frac{a_3}{3} & +\cdots & & & \\
\Gamma_3 : & & & \frac{a_3}{3} & +\cdots & & & \\
\hline
\sum_{i=1}^{n} \Gamma_i : & a_1 & +a_2 & +a_3 & +\cdots & a_n & +\frac{a_{n+1}}{n+1} & +\cdots
\end{array}
$$

- Use part (a): since $\Gamma_n \overset{n\longrightarrow\infty}{\Longrightarrow} 0$, we have $\frac{1}{n}\sum_{k=1}^{n} \Gamma_k \overset{n\longrightarrow\infty}{\Longrightarrow} 0$.
- Also a minor check: $\Gamma_n \longrightarrow 0 \implies \frac{1}{n}\Gamma_n \longrightarrow 0$.
- Then

$$
\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} a_k &= \frac{1}{n}(\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \mathbf{\Gamma_{n+1}}) \\
&= \left(\frac{1}{n}\sum_{k=0}^{n} \Gamma_k\right) - \left(\frac{1}{n}\Gamma_{n+1}\right) \\
&\overset{n\longrightarrow\infty}{\Longrightarrow} 0.
\end{aligned}
$$

2.3 Fall 2018 # 4

Let $f \in L^1([0,1])$. Prove that

$$
\lim_{n\longrightarrow\infty} \int_0^1 f(x)|\sin nx| \, dx = \frac{2}{\pi}\int_0^1 f(x) \, dx
$$

Hint: Begin with the case that f is the characteristic function of an interval.

2.4 Fall 2017 # 4

Let

$$
f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.
$$

1. Show that $f_n \longrightarrow 0$ pointwise but not uniformly on $[0,1]$.

Hint: Consider the maximum of f_n.

2.

$$\lim_{n\longrightarrow\infty}\int_0^1 n(1-x)^n \sin x\, dx = 0$$

2.5 Spring 2017 # 3

Let

$$f_n(x) = ae^{-nax} - be^{-nbx} \qquad \text{where } 0 < a < b.$$

Show that

a. $\sum_{n=1}^{\infty} |f_n|$ is not in $L^1([0,\infty), m)$

Hint: $f_n(x)$ has a root x_n.

b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0,\infty), m) \quad \text{and} \quad \int_0^\infty \sum_{n=1}^{\infty} f_n(x)\, dm = \ln\frac{b}{a}$$

2.6 Fall 2016 # 1

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that f converges to a differentiable function on $(1,\infty)$ and that

$$f'(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n^x}\right)'.$$

Hint:

$$\left(\frac{1}{n^x}\right)' = -\frac{1}{n^x}\ln n$$

2.7 Fall 2016 # 5

Let $\varphi \in L^\infty(\mathbb{R})$. Show that the following limit exists and satisfies the equality

$$\lim_{n\longrightarrow\infty}\left(\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2}\, dx\right)^{\frac{1}{n}} = \|\varphi\|_\infty.$$

2.8 Fall 2016 # 6

Let $f, g \in L^2(\mathbb{R})$. Show that

$$\lim_{n \longrightarrow \infty} \int_{\mathbb{R}} f(x)g(x+n)\,dx = 0$$

2.9 Spring 2016 # 1

For $n \in \mathbb{N}$, define

$$]e_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad E_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

Show that $e_n < E_n$, and prove Bernoulli's inequality:

$$(1+x)^n \geq 1 + nx \text{ for } -1 < x < \infty \text{ and } n \in \mathbb{N}$$

Use this to show the following:

1. The sequence e_n is increasing.
2. The sequence E_n is decreasing.
3. $2 < e_n < E_n < 4$.
4. $\lim_{n \longrightarrow \infty} e_n = \lim_{n \longrightarrow \infty} E_n$.

2.10 Fall 2015 # 1

Define

$$f(x) = c_0 + c_1 x^1 + c_2 x^2 + \ldots + c_n x^n \text{ with } n \text{ even and } c_n > 0.$$

Show that there is a number x_m such that $f(x_m) \leq f(x)$ for all $x \in \mathbb{R}$.

3 Measure Theory: Sets

3.1 Spring 2020 # 2

Let m_* denote the Lebesgue outer measure on $\mathbb{R}$.

3.1.1 a.

Prove that for every $E \subseteq \mathbb{R}$ there exists a Borel set B containing E such that

$$m_*(B) = m_*(E).$$

3.1.2 b.

Prove that if $E \subseteq \mathbb{R}$ has the property that

$$m_*(A) = m_*(A \bigcap E) + m_*(A \bigcap E^c)$$

for every set $A \subseteq \mathbb{R}$, then there exists a Borel set $B \subseteq \mathbb{R}$ such that $E = B \setminus N$ with $m_*(N) = 0$.

Be sure to address the case when $m_*(E) = \infty$.

Solution.
Concepts used:

- Definition of outer measure: $m_*(E) = \inf_{\{Q_j\} \rightrightarrows E} \sum |Q_j|$ where $\{Q_j\}$ is a countable collection of closed cubes.
- Break $\mathbb{R}$ into $\coprod_{n \in \mathbb{Z}} [n, n+1)$, each with finite measure.
- Theorem: $m_*(Q) = |Q|$ for Q a closed cube (i.e. the outer measure equals the volume).

Proof (of Theorem) Statement: if Q is a closed cube, then $m_*(Q) = |Q|$, the usual volume.

- $m_*(Q) \leq |Q|$:
  - Since $Q \subseteq Q$, $Q \rightrightarrows Q$ and $m_*(Q) \leq |Q|$ since m_* is an infimum over such coverings.
- $|Q| \leq m_*(Q)$:
  - Fix $\varepsilon > 0$.
  - Let $\{Q_i\}_{i=1}^{\infty} \rightrightarrows Q$ be arbitrary, it suffices to show that

$$|Q| \leq \left(\sum_{i=1}^{\infty} |Q_i| \right) + \varepsilon.$$

  - Pick open cubes S_i such that $Q_i \subseteq S_i$ and $|Q_i| \leq |S_i| \leq (1+\varepsilon)|Q_i|$.
  - Then $\{S_i\} \rightrightarrows Q$, so by compactness of Q pick a finite subcover with N elements.
  - Note

$$Q \subseteq \bigcup_{i=1}^{N} S_i \implies |Q| \leq \sum_{i=1}^{N} |S_i| \leq \sum_{i=1}^{N} (1+\varepsilon)|Q_j| \leq (1+\varepsilon) \sum_{i=1}^{\infty} |Q_i|.$$

  - Taking an infimum over coverings on the RHS preserves the inequality, so

$$|Q| \leq (1+\varepsilon) m_*(Q)$$

  - Take $\varepsilon \longrightarrow 0$ to obtain final inequality.

3.1.3 a

- If $m_*(E) = \infty$, then take $B = \mathbb{R}^n$ since $m(\mathbb{R}^n) = \infty$.
- Suppose $N := m_*(E) < \infty$.
- Since $m_*(E)$ is an infimum, by definition, for every $\varepsilon > 0$ there exists a covering by closed cubes $\{Q_i(\varepsilon)\}_{i=1}^{\infty} \rightrightarrows E$ depending on ε such that

$$\sum_{i=1}^{\infty} |Q_i(\varepsilon)| < N + \varepsilon.$$

- For each fixed n, set $\varepsilon_n = \frac{1}{n}$ to produce such a covering $\{Q_i(\varepsilon_n)\}_{i=1}^{\infty}$ and set $B_n \coloneqq \bigcup_{i=1}^{\infty} Q_i(\varepsilon_n)$.
- The outer measure of cubes is *equal* to the sum of their volumes, so

$$m_*(B_n) = \sum_{i=1}^{\infty} |Q_i(\varepsilon_n)| < N + \varepsilon_n = N + \frac{1}{n}.$$

- Now set $B \coloneqq \bigcap_{n=1}^{\infty} B_n$.
  - Since $E \subseteq B_n$ for every n, $E \subseteq B$
  - Since B is a countable intersection of countable unions of closed sets, B is Borel.
  - Since $B_n \subseteq B$ for every n, we can apply subadditivity to obtain the inequality

$$E \subseteq B \subseteq B_n \implies N \leq m_*(B) \leq m_*(B_n) < N + \frac{1}{n} \quad \text{for all} \quad n \in \mathbb{Z}^{\geq 1}.$$

- This forces $m_*(E) = m_*(B)$.

3.1.4 b

Suppose $m_*(E) < \infty$.

- By (a), find a Borel set $B \supseteq E$ such that $m_*(B) = m_*(E)$
- Note that $E \subseteq B \implies B \bigcap E = E$ and $B \bigcap E^c = B \setminus E$.
- By assumption,

$$\begin{aligned}
m_*(B) &= m_*(B \textstyle\bigcap E) + m_*(B \textstyle\bigcap E^c) \\
m_*(E) &= m_*(E) + m_*(B \setminus E) \\
m_*(E) - m_*(E) &= m_*(B \setminus E) \qquad\qquad \text{since } m_*(E) < \infty \\
\implies m_*(B \setminus E) &= 0.
\end{aligned}$$

- So take $N = B \setminus E$; this shows $m_*(N) = 0$ and $E = B \setminus (B \setminus E) = B \setminus N$.

If $m_*(E) = \infty$:

- Apply result to $E_R \coloneqq E \bigcap [R, R+1)^n \subset \mathbb{R}^n$ for $R \in \mathbb{Z}$, so $E = \coprod_R E_R$
- Obtain B_R, N_R such that $E_R = B_R \setminus N_R$, $m_*(E_R) = m_*(B_R)$, and $m_*(N_R) = 0$.
- Note that
  - $B \coloneqq \bigcup_R B_R$ is a union of Borel sets and thus still Borel
  - $E = \bigcup_R E_R$
  - $N \coloneqq B \setminus E$
  - $N' \coloneqq \bigcup_R N_R$ is a union of null sets and thus still null
- Since $E_R \subset B_R$ for every R, we have $E \subset B$
- We can compute

$$N = B \setminus E = \left(\bigcup_R B_R \right) \setminus \left(\bigcup_R E_R \right) \subseteq \bigcup_R (B_R \setminus E_R) = \bigcup_R N_R \coloneqq N'$$

where $m_*(N') = 0$ since N' is null, and thus subadditivity forces $m_*(N) = 0$.

3.2 Fall 2019 # 3.

Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) = 1$ and $\{B_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{B}$-measurable subsets of X, and

$$B \coloneqq \left\{ x \in X \;\middle|\; x \in B_n \text{ for infinitely many } n \right\}.$$

a. Argue that B is also a $\mathcal{B}$-measurable subset of X.

b. Prove that if $\displaystyle\sum_{n=1}^{\infty} \mu(B_n) < \infty$ then $\mu(B) = 0$.

c. Prove that if $\displaystyle\sum_{n=1}^{\infty} \mu(B_n) = \infty$ **and** the sequence of set complements $\{B_n^c\}_{n=1}^{\infty}$ satisfies

$$\mu\left(\bigcap_{n=k}^{K} B_n^c \right) = \prod_{n=k}^{K} \left(1 - \mu\left(B_n\right)\right)$$

for all positive integers k and K with $k < K$, then $\mu(B) = 1$.

Hint: Use the fact that $1 - x \leq e^{-x}$ for all x.

Solution.
Concepts used:

- Borel-Cantelli: for a sequence of sets X_n,

$$\limsup_n X_n = \left\{ x \;\middle|\; x \in X_n \text{ for infinitely many } n \right\} = \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} X_n$$
$$\liminf_n X_n = \left\{ x \;\middle|\; x \in X_n \text{ for all but finitely many } n \right\} = \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} X_n.$$

- Properties of logs and exponentials:

$$\prod_n e^{x_n} = e^{\Sigma_n x_n} \quad \text{and} \quad \sum_n \log(x_n) = \log\left(\prod_n x_n \right).$$

- Tails of convergent sums vanish.
- Continuity of measure: $B_n \searrow B$ and $\mu(B_0) < \infty$ implies $\lim_n \mu(B_n) = \mu(B)$, and $B_n \nearrow B \implies \lim_n \mu(B_n) = \mu(B)$.

3.2.1 a

- The Borel σ-algebra is closed under countable unions/intersections/complements,
- $B = \limsup_n B_n$ is an intersection of unions of measurable sets.

3.2.2 b

- Tails of convergent sums go to zero, so $\displaystyle\sum_{n \geq M} \mu(B_n) \xrightarrow{M \longrightarrow \infty} 0$,
- $\displaystyle B_M \coloneqq \bigcap_{m=1}^{M} \bigcup_{n \geq m} B_n \searrow B.$

$$\begin{aligned}
\mu(B_M) &= \mu\left(\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} B_n\right) \\
&\leq \mu\left(\bigcup_{n\geq m} B_n\right) \quad \text{for all } m\in\mathbb{N} \text{ by countable subadditivity} \\
&\longrightarrow 0,
\end{aligned}$$

- The result follows by continuity of measure.

3.2.3 c

- To show $\mu(B) = 1$, we'll show $\mu(B^c) = 0$.
- Let $B_k = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{K} B_n$. Then

$$\begin{aligned}
\mu(B_K^c) &= \mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{K} B_n^c\right) \\
&\leq \sum_{m=1}^{\infty} \mu\left(\bigcap_{n=m}^{K} B_n^c\right) \quad \text{by subadditivity} \\
&= \sum_{m=1}^{\infty} \prod_{n=m}^{K} (1-\mu(B_n)) \quad \text{by assumption} \\
&\leq \sum_{m=1}^{\infty} \prod_{n=m}^{K} e^{-\mu(B_n^c)} \quad \text{by hint} \\
&= \sum_{m=1}^{\infty} \exp\left(-\sum_{n=m}^{K} \mu(B_n^c)\right) \\
&\overset{K\longrightarrow\infty}{\longrightarrow} 0
\end{aligned}$$

since $\sum_{n=m}^{K} \mu(B_n^c) \overset{K\longrightarrow\infty}{\longrightarrow} \infty$ by assumption

- We can apply continuity of measure since $B_K^c \xrightarrow{K\longrightarrow\infty} B^c$.

Proving the hint: ?

3.3 Spring 2019 # 2

Let $\mathcal{B}$ denote the set of all Borel subsets of $\mathbb{R}$ and $\mu : \mathcal{B} \longrightarrow [0,\infty)$ denote a finite Borel measure on $\mathbb{R}$.

3.3.1 a

Prove that if $\{F_k\}$ is a sequence of Borel sets for which $F_k \supseteq F_{k+1}$ for all k, then

$$\lim_{k\to\infty} \mu\left(F_k\right) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right)$$

3.3.2 b

Suppose μ has the property that $\mu(E) = 0$ for every $E \in \mathcal{B}$ with Lebesgue measure $m(E) = 0$.

Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $E \in \mathcal{B}$ with $m(E) < \delta$, then $\mu(E) < \varepsilon$.

Solution.

3.3.3 a

See Folland p.26

- Lemma 1: $\mu(\coprod_{k=1}^{\infty} E_k) = \lim_{N\longrightarrow\infty} \sum_{k=1}^{N} \mu(E_k)$.
- Suppose $F_0 \supseteq F_1 \supseteq \cdots$.
- Let $A_k = F_k \setminus F_{k+1}$, since the F_k are nested the A_k are disjoint
- Set $A := \coprod_{k=1}^{\infty} A_k$ and $F := \bigcap_{k=1}^{\infty} F_k$.
- Note $X = X \setminus Y \coprod X \bigcap Y$ for any two sets (just write $X \setminus Y := X \bigcap Y^c$)
- Note that A contains anything that was removed from F_0 when passing from any F_j to F_{j+1}, while F contains everything that is never removed at any stage, and these are disjoint possibilities.
- Thus $F_0 = F \coprod A$, so

$$\begin{aligned}
\mu(F_0) &= \mu(F) + \mu(A) \\
&= \mu(F) + \mu(\textstyle\coprod_{k=1}^{\infty} A_k) \\
&= \mu(F) + \lim_{n\longrightarrow\infty} \sum_{k=0}^{n} \mu(A_k) \quad \text{by countable additivity} \\
&= \mu(F) + \lim_{n\longrightarrow\infty} \sum_{k=0}^{n} \mu(F_k) - \mu(F_{k+1}) \\
&= \mu(F) + \lim_{n\longrightarrow\infty} (\mu(F_1) - \mu(F_n)) \quad \text{(Telescoping)} \\
&= \mu(F) + \mu(F_1) - \lim_{N\longrightarrow\infty} \mu(F_n),
\end{aligned}$$

- Since μ is a finite measure, $\mu(F_1) < \infty$ and can be subtracted, yielding

$$\begin{aligned}
\mu(F_1) &= \mu(F) + \mu(F_1) - \lim_{n\longrightarrow\infty} \mu(F_n) \\
\implies \mu(F) &= \lim_{n\longrightarrow\infty} \mu(F_n) \\
\implies \mu\left(\bigcap_{k=1}^{\infty} F_k\right) &= \lim_{n\longrightarrow\infty} \mu(F_n).
\end{aligned}$$

3.3.4 b

- Toward a contradiction, negate the implication: suppose there exists an $\varepsilon > 0$ such that for all δ, we have $m(E) < \delta$ but $\mu(E) > \varepsilon$.
- The sequence $\left\{\delta_n := \frac{1}{2^n}\right\}_{n\in\mathbb{N}}$ and produce sets $A_n \in \mathcal{B}$ such $m(A_n) < \frac{1}{2^n}$ but $\mu(A_n) > \varepsilon$.
- Define

$$
\begin{aligned}
F_n &:= \bigcup_{j\geq n} A_j \\
C_m &:= \bigcap_{k=1}^{m} F_k \\
A := C_\infty &:= \bigcap_{k=1}^{\infty} F_k.
\end{aligned}
$$

- Note that $F_1 \supseteq F_2 \supseteq \cdots$, since each increase in index unions fewer sets.
- By continuity for the Lebesgue measure,

$$
m(A) = m\left(\bigcap_{k=1}^{\infty} F_k\right) = \lim_{k\longrightarrow\infty} m(F_k) = \lim_{k\longrightarrow\infty} m\left(\bigcup_{j\geq k} A_j\right) \leq \lim_{k\longrightarrow\infty} \sum_{j\geq k} m(A_j) = \lim_{k\longrightarrow\infty} \sum_{j\geq k} \frac{1}{2^n} = 0,
$$

  which follows because this is the tail of a convergent sum
- Thus $m(A) = 0$ and by assumption, this implies $\mu(A) = 0$.
- However, by part (a),

$$
\mu(A) = \lim_n \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \geq \lim_n \mu(A_n) = \lim_n \varepsilon = \varepsilon > 0.
$$

All messed up.

3.4 Fall 2018 # 2

Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Show that there is a Borel set $B \subset E$ such that $m(E \setminus B) = 0$.

3.5 Spring 2018 # 1

Define

$$
E := \left\{x \in \mathbb{R} : \left|x - \frac{p}{q}\right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N}\right\}.
$$

Prove that $m(E) = 0$.

3.6 Fall 2017 # 2

Let $f(x) = x^2$ and $E \subset [0, \infty) := \mathbb{R}^+$.

1. Show that
$$m^*(E) = 0 \iff m^*(f(E)) = 0.$$
2. Deduce that the map

$$\begin{aligned} \varphi : \mathcal{L}(\mathbb{R}^+) &\longrightarrow \mathcal{L}(\mathbb{R}^+) \\ E &\mapsto f(E) \end{aligned}$$

is a bijection from the class of Lebesgue measurable sets of $[0, \infty)$ to itself.

3.7 Spring 2017 # 2

a. Let μ be a measure on a measurable space $(X, \mathcal{M})$ and f a positive measurable function.

Define a measure λ by
$$\lambda(E) := \int_E f \, d\mu, \quad E \in \mathcal{M}$$
Show that for g any positive measurable function,
$$\int_X g \, d\lambda = \int_X fg \, d\mu$$

b. Let $E \subset \mathbb{R}$ be a measurable set such that
$$\int_E x^2 \, dm = 0.$$
Show that $m(E) = 0$.

3.8 Fall 2016 # 4

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\{E_n\} \subset \mathcal{M}$ satisfies
$$\lim_{n \to \infty} \mu\left(X \setminus E_n\right) = 0.$$
Define
$$G := \left\{ x \in X \;\middle|\; x \in E_n \text{ for only finitely many } n \right\}.$$
Show that $G \in \mathcal{M}$ and $\mu(G) = 0$.

3.9 Spring 2016 # 3

Let f be Lebesgue measurable on $\mathbb{R}$ and $E \subset \mathbb{R}$ be measurable such that
$$0 < A = \int_E f(x) dx < \infty.$$
Show that for every $0 < t < 1$, there exists a measurable set $E_t \subset E$ such that
$$\int_{E_t} f(x) dx = tA.$$

3.10 Spring 2016 # 5

Let $(X, \mathcal{M}, \mu)$ be a measure space. For $f \in L^1(\mu)$ and $\lambda > 0$, define

$$\varphi(\lambda) = \mu(\{x \in X | f(x) > \lambda\}) \quad \text{and} \quad \psi(\lambda) = \mu(\{x \in X | f(x) < -\lambda\})$$

Show that φ, ψ are Borel measurable and

$$\int_X |f| \ d\mu = \int_0^\infty [\varphi(\lambda) + \psi(\lambda)] \ d\lambda$$

3.11 Fall 2015 # 2

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be Lebesgue measurable.

1. Show that there is a sequence of simple functions $s_n(x)$ such that $s_n(x) \longrightarrow f(x)$ for all $x \in \mathbb{R}$.
2. Show that there is a Borel measurable function g such that $g = f$ almost everywhere.

3.12 Spring 2015 # 3

Let μ be a finite Borel measure on $\mathbb{R}$ and $E \subset \mathbb{R}$ Borel. Prove that the following statements are equivalent:

1. $\forall \varepsilon > 0$ there exists G open and F closed such that
$$F \subseteq E \subseteq G \quad \text{and} \quad \mu(G \setminus F) < \varepsilon.$$
2. There exists a $V \in G_\delta$ and $H \in F_\sigma$ such that
$$H \subseteq E \subseteq V \quad \text{and} \quad \mu(V \setminus H) = 0$$

3.13 Spring 2014 # 3

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and suppose

$$\forall x \in \mathbb{R}, \quad f(x) \geq \limsup_{y \to x} f(y)$$

Prove that f is Borel measurable.

3.14 Spring 2014 # 4

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose f is a measurable function on X. Show that

$$\lim_{n \to \infty} \int_X f^n \ d\mu = \begin{cases} \infty & \text{or} \\ \mu(f^{-1}(1)), \end{cases}$$

and characterize the collection of functions of each type.

3.15 Spring 2017 # 1

Let K be the set of numbers in $[0,1]$ whose decimal expansions do not use the digit 4.

> We use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 with $399\cdots$. For example, $0.8754 = 0.8753999\cdots$.

Show that K is a compact, nowhere dense set without isolated points, and find the Lebesgue measure $m(K)$.

3.16 Spring 2016 # 2

Let $0 < \lambda < 1$ and construct a Cantor set C_λ by successively removing middle intervals of length λ.

Prove that $m(C_\lambda) = 0$.

4 Measure Theory: Functions

4.1 Fall 2016 # 2

Let $f, g : [a,b] \longrightarrow \mathbb{R}$ be measurable with

$$\int_a^b f(x)\ dx = \int_a^b g(x)\ dx.$$

Show that either

1. $f(x) = g(x)$ almost everywhere, or
2. There exists a measurable set $E \subset [a,b]$ such that

$$\int_E f(x)\ dx > \int_E g(x)\ dx$$

4.2 Spring 2016 # 4

Let $E \subset \mathbb{R}$ be measurable with $m(E) < \infty$. Define

$$f(x) = m(E \cap (E + x)).$$

Show that

1. $f \in L^1(\mathbb{R})$.
2. f is uniformly continuous.
3. $\lim\limits_{|x| \longrightarrow \infty} f(x) = 0$.

> Hint:
>
> $$\chi_{E\cap(E+x)}(y) = \chi_E(y)\chi_E(y - x)$$

5 Integrals: Convergence

5.1 Fall 2019 # 2.

Prove that

$$\left|\frac{d^n}{dx^n}\frac{\sin x}{x}\right| \leq \frac{1}{n}$$

for all $x \neq 0$ and positive integers n.

Hint: Consider $\int_0^1 \cos(tx)dt$

Solution.
Concepts used:

- DCT
- Bounding in the right place. Don't evaluate the actual integral!

Solution:

- By induction on the number of limits we can pass through the integral.
- For $n = 1$ we first pass one derivative into the integral: let $x_n \longrightarrow x$ be any sequence converging to x, then

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\sin(x)}{x} &= \frac{\partial}{\partial x}\int_0^1 \cos(tx)\,dt \\
&= \lim_{x_n \longrightarrow x} \frac{1}{x_n - x}\left(\int_0^1 \cos(tx_n)\,dt - \int_0^1 \cos(tx)\,dt\right) \\
&= \lim_{x_n \longrightarrow x} \left(\int_0^1 \frac{\cos(tx_n) - \cos(tx)}{x_n - x}\,dt\right) \\
&= \lim_{x_n \longrightarrow x} \left(\int_0^1 \left(t\sin(tx)\,\Big|_{x=\xi_n}\right)dt\right) \quad \text{where} \quad \xi_n \in [x_n, x] \text{ by MVT}, \xi_n \longrightarrow x \\
&= \lim_{\xi_n \longrightarrow x} \left(\int_0^1 t\sin(t\xi_n)\,dt\right) \\
&=_{\text{DCT}} \int_0^1 \lim_{\xi_n \longrightarrow x} t\sin(t\xi_n)\,dt \\
&= \int_0^1 t\sin(tx)\,dt
\end{aligned}$$

.

- Taking absolute values we obtain an upper bound

$$\begin{aligned}
\left|\frac{\partial}{\partial x}\frac{\sin(x)}{x}\right| &= \left|\int_0^1 t\sin(tx)\,dt\right| \\
&\leq \int_0^1 |t\sin(tx)|\,dt \\
&\leq \int_0^1 1\,dt = 1,
\end{aligned}$$

since $t \in [0, 1] \implies |t| < 1$, and $|\sin(xt)| \leq 1$ for any x and t.

- Note that this bound also justifies the DCT, since the functions $f_n(t) = t\sin(t\xi_n)$ are uniformly dominated by $g(t) = 1$ on $L^1([0,1])$.

Note: integrating by parts here yields the actual formula:

$$\begin{aligned}
\int_0^1 t\sin(tx)\,dt &=_{\text{IBP}} \left(\frac{-t\cos(tx)}{x}\right)\Big|_{t=0}^{t=1} - \int_0^1 \frac{\cos(tx)}{x}\,dt \\
&= \frac{-\cos(x)}{x} - \frac{\sin(x)}{x^2} \\
&= \frac{x\cos(x) - \sin(x)}{x^2}.
\end{aligned}$$

- For the inductive step, we assume that we can pass $n-1$ limits through the integral and show we can pass the nth through as well.

$$\begin{aligned}
\frac{\partial^n}{\partial x^n}\frac{\sin(x)}{x} &= \frac{\partial^n}{\partial x^n}\int_0^1 \cos(tx)\,dt \\
&= \frac{\partial}{\partial x}\int_0^1 \frac{\partial^{n-1}}{\partial x^{n-1}}\cos(tx)\,dt \\
&= \frac{\partial}{\partial x}\int_0^1 t^{n-1} f_{n-1}(x,t)\,dt
\end{aligned}$$

  - Note that $f_n(x,t) = \pm\sin(tx)$ when n is odd and $f_n(x,t) = \pm\cos(tx)$ when n is even, and a constant factor of t is multiplied when each derivative is taken.

- We continue as in the base case:

$$\begin{aligned}
\frac{\partial}{\partial x}\int_0^1 t^{n-1} f_{n-1}(x,t)\,dt &= \lim_{x_k \longrightarrow x}\int_0^1 t^{n-1}\left(\frac{f_{n-1}(x_n,t) - f_{n-1}(x,t)}{x_n - x}\right)dt \\
&=_{\text{IVT}} \lim_{x_k \longrightarrow x}\int_0^1 t^{n-1}\frac{\partial f_{n-1}}{\partial x}(\xi_k,t)\,dt \quad \text{where } \xi_k \in [x_k, x],\ \xi_k \longrightarrow x \\
&=_{\text{DCT}} \int_0^1 \lim_{x_k \longrightarrow x} t^{n-1}\frac{\partial f_{n-1}}{\partial x}(\xi_k,t)\,dt \\
&:= \int_0^1 \lim_{x_k \longrightarrow x} t^n f_n(\xi_k,t)\,dt \\
&:= \int_0^1 t^n f_n(x,t)\,dt.
\end{aligned}$$

  - We've used the fact that $f_0(x) = \cos(tx)$ is smooth as a function of x, and in particular continuous
  - The DCT is justified because the functions $h_{n,k}(x,t) = t^n f_n(\xi_k,t)$ are again uniformly (in k) bounded by 1 since $t \leq 1 \implies t^n \leq 1$ and each f_n is a sin or cosine.

- Now take absolute values

$$\begin{aligned}
\left|\frac{\partial^n}{\partial x^n}\frac{\sin(x)}{x}\right| &= \left|\int_0^1 -t^n f_n(x,t)\ dt\right| \\
&\leq \int_0^1 |t^n f_n(x,t)|\ dt \\
&\leq \int_0^1 |t^n||f_n(x,t)|\ dt \\
&\leq \int_0^1 |t^n| \cdot 1\ dt \\
&\leq \int_0^1 t^n\ dt \quad \text{since } t \text{ is positive} \\
&= \frac{1}{n+1} \\
&< \frac{1}{n}.
\end{aligned}$$

  - We've again used the fact that $f_n(x,t)$ is of the form $\pm\cos(tx)$ or $\pm\sin(tx)$, both of which are bounded by 1.

5.2 Spring 2020 # 5

Compute the following limit and justify your calculations:

$$\lim_{n\longrightarrow\infty}\int_0^n \left(1+\frac{x^2}{n}\right)^{-(n+1)} dx.$$

Not finished, flesh out.

Solution.
Concepts used:

- DCT
- Passing limits through products and quotients

Note that

$$\begin{aligned}
\lim_n \left(1+\frac{x^2}{n}\right)^{-(n+1)} &= \frac{1}{\lim_n \left(1+\frac{x^2}{n}\right)^1 \left(1+\frac{x^2}{n}\right)^n} \\
&= \frac{1}{1\cdot e^{x^2}} \\
&= e^{-x^2}.
\end{aligned}$$

If passing the limit through the integral is justified, we will have

$$\begin{aligned}
\lim_{n\longrightarrow\infty}\int_0^n\left(1+\frac{x^2}{n}\right)^{-(n+1)}\,dx &= \lim_{n\longrightarrow\infty}\int_{\mathbb{R}}\chi_{[0,n]}\left(1+\frac{x^2}{n}\right)^{-(n+1)}\,dx\\
&= \int_{\mathbb{R}}\lim_{n\longrightarrow\infty}\chi_{[0,n]}\left(1+\frac{x^2}{n}\right)^{-(n+1)}\,dx \quad \text{by the DCT}\\
&= \int_{\mathbb{R}}\lim_{n\longrightarrow\infty}\left(1+\frac{x^2}{n}\right)^{-(n+1)}\,dx\\
&= \int_0^\infty e^{-x^2}\\
&= \frac{\sqrt{\pi}}{2}.
\end{aligned}$$

Computing the last integral:

$$\begin{aligned}
\left(\int_{\mathbb{R}} e^{-x^2}\,dx\right)^2 &= \left(\int_{\mathbb{R}} e^{-x^2}\,dx\right)\left(\int_{\mathbb{R}} e^{-y^2}\,dx\right)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(x+y)^2}\,dx\\
&= \int_0^{2\pi}\int_0^\infty e^{-r^2}r\,dr\,d\theta \qquad u=r^2\\
&= \frac{1}{2}\int_0^{2\pi}\int_0^\infty e^{-u}\,du\,d\theta\\
&= \frac{1}{2}\int_0^{2\pi}1\\
&= \pi,
\end{aligned}$$

and now use the fact that the function is even so $\displaystyle\int_0^\infty f=\frac{1}{2}\int_{\mathbb{R}}f$.

Justifying the DCT:

- Apply Bernoulli's inequality:

$$1+\frac{x^2}{n}^{n+1}\geq 1+\frac{x^2}{n}\left(1+x^2\right)\geq 1+x^2,$$

 where the last inequality follows from the fact that $1+\dfrac{x^2}{n}\geq 1$

5.3 Spring 2019 # 3

Let $\{f_k\}$ be any sequence of functions in $L^2([0,1])$ satisfying $\|f_k\|_2\leq M$ for all $k\in\mathbb{N}$.

Prove that if $f_k\longrightarrow f$ almost everywhere, then $f\in L^2([0,1])$ with $\|f\|_2\leq M$ and

$$\lim_{k\to\infty}\int_0^1 f_k(x)dx=\int_0^1 f(x)dx$$

Hint: Try using Fatou's Lemma to show that $\|f\|_2 \leq M$ and then try applying Egorov's Theorem.

5.4 Fall 2018 # 6

Compute the following limit and justify your calculations:

$$\lim_{n\to\infty} \int_1^n \frac{dx}{(1+\frac{x}{n})^n \sqrt[n]{x}}$$

5.5 Fall 2018 # 3

Suppose $f(x)$ and $xf(x)$ are integrable on $\mathbb{R}$. Define F by

$$F(t) := \int_{-\infty}^{\infty} f(x)\cos(xt)dx$$

Show that

$$F'(t) = -\int_{-\infty}^{\infty} xf(x)\sin(xt)dx.$$

5.6 Spring 2018 # 5

Suppose that

- $f_n, f \in L^1$,
- $f_n \longrightarrow f$ almost everywhere, and
- $\displaystyle\int |f_n| \to \int |f|$.

Show that $\displaystyle\int f_n \to \int f$

5.7 Spring 2018 # 2

Let

$$f_n(x) := \frac{x}{1+x^n}, \quad x \geq 0.$$

a. Show that this sequence converges pointwise and find its limit. Is the convergence uniform on $[0,\infty)$?

b. Compute

$$\lim_{n\to\infty} \int_0^{\infty} f_n(x)dx$$

5.8 Fall 2016 # 3

Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{x\longrightarrow 0} \int_{\mathbb{R}} |f(y-x) - f(y)|\, dy = 0$$

5.9 Fall 2015 # 3

Compute the following limit:

$$\lim_{n\to\infty} \int_1^n \frac{ne^{-x}}{1+nx^2} \sin\left(\frac{x}{n}\right) dx$$

5.10 Fall 2015 # 4

Let $f : [1, \infty) \longrightarrow \mathbb{R}$ such that $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Show that the following limit exists and satisfies the equality

$$\lim_{x\to\infty} f(x) \leq 1 + \frac{\pi}{4}$$

6 Integrals: Approximation

6.1 Spring 2018 # 3

Let f be a non-negative measurable function on $[0, 1]$.

Show that

$$\lim_{p\to\infty} \left(\int_{[0,1]} f(x)^p dx \right)^{\frac{1}{p}} = \|f\|_\infty.$$

6.2 Spring 2018 # 4

Let $f \in L^2([0, 1])$ and suppose

$$\int_{[0,1]} f(x)x^n dx = 0 \text{ for all integers } n \geq 0.$$

Show that $f = 0$ almost everywhere.

6.3 Spring 2015 # 2

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be continuous with period 1. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(t)dt \quad \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

Hint: show this first for the functions $f(t) = e^{2\pi ikt}$ for $k \in \mathbb{Z}$.

6.4 Fall 2014 # 4

Let $g \in L^\infty([0,1])$ Prove that

$$\int_{[0,1]} f(x)g(x)\,dx = 0 \quad \text{for all continuous } f:[0,1] \longrightarrow \mathbb{R} \implies g(x) = 0 \text{ almost everywhere.}$$

7 L^1

7.1 Spring 2020 # 3

a. Prove that if $g \in L^1(\mathbb{R})$ then

$$\lim_{N \longrightarrow \infty} \int_{|x| \geq N} |f(x)|\,dx = 0,$$

and demonstrate that it is not necessarily the case that $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.

b. Prove that if $f \in L^1([1,\infty])$ and is decreasing, then $\lim_{x \longrightarrow \infty} f(x) = 0$ and in fact $\lim_{x \longrightarrow \infty} x f(x) = 0$.

c. If $f : [1,\infty) \longrightarrow [0,\infty)$ is decreasing with $\lim_{x \longrightarrow \infty} x f(x) = 0$, does this ensure that $f \in L^1([1,\infty))$?

Solution.
Concepts used:

- Limits
- Cauchy Criterion for Integrals: $\int_a^\infty f(x)\,dx$ converges iff for every $\varepsilon > 0$ there exists an M_0 such that $A, B \geq M_0$ implies $\left| \int_A^B f \right| < \varepsilon$, i.e. $\left| \int_A^B f \right| \overset{A \longrightarrow \infty}{\longrightarrow} 0$.
- Integrals of L^1 functions have vanishing tails: $\int_N^\infty |f| \overset{N \longrightarrow \infty}{\longrightarrow} 0$.
- Mean Value Theorem for Integrals: $\int_a^b f(t)\,dt = (b-a)f(c)$ for some $c \in [a,b]$.

7.1.1 a

Stated integral equality:

- Let $\varepsilon > 0$
- $C_c(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$ is dense so choose $\{f_n\} \longrightarrow f$ with $\|f_n - f\|_1 \longrightarrow 0$.
- Since $\{f_n\}$ are compactly supported, choose $N_0 \gg 1$ such that f_n is zero outside of $B_{N_0}(\mathbf{0})$.

- Then

$$
\begin{aligned}
N \geq N_0 \implies \int_{|x|>N} |f| &= \int_{|x|>N} |f - f_n + f_n| \\
&\leq \int_{|x|>N} |f - f_n| + \int_{|x|>N} |f_n| \\
&= \int_{|x|>N} |f - f_n| \\
&\leq \int_{|x|>N} \|f - f_n\|_1 \\
&= \|f_n - f\|_1 \left(\int_{|x|>N} 1 \right) \\
&\overset{n\longrightarrow\infty}{\longrightarrow} 0 \left(\int_{|x|>N} 1 \right) \\
&= 0 \\
&\overset{N\longrightarrow\infty}{\longrightarrow} 0.
\end{aligned}
$$

To see that this doesn't force $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$:

- Take $f(x)$ to be a train of rectangles of height 1 and area $1/2^j$ centered on even integers.
- Then

$$
\int_{|x|>N} |f| = \sum_{j=N}^{\infty} 1/2^j \overset{N\longrightarrow\infty}{\longrightarrow} 0
$$

  as the tail of a convergent sum.
- However $f(x) = 1$ for infinitely many even integers $x > N$, so $f(x) \not\longrightarrow 0$ as $|x| \longrightarrow \infty$.

7.1.2 b

Solution 1 ("Trick")

- Since f is decreasing on $[1, \infty)$, for any $t \in [x - n, x]$ we have

$$
x - n \leq t \leq x \implies f(x) \leq f(t) \leq f(x - n).
$$

- Integrate over $[x, 2x]$, using monotonicity of the integral:

$$
\begin{aligned}
\int_x^{2x} f(x)\, dt &\leq \int_x^{2x} f(t)\, dt \leq \int_x^{2x} f(x-n)\, dt \\
\implies f(x) \int_x^{2x} dt &\leq \int_x^{2x} f(t)\, dt \leq f(x-n) \int_x^{2x} dt \\
\implies xf(x) &\leq \int_x^{2x} f(t)\, dt \leq xf(x-n).
\end{aligned}
$$

- By the Cauchy Criterion for integrals, $\lim_{x\longrightarrow\infty} \int_x^{2x} f(t)\ dt = 0$.
- So the LHS term $xf(x) \overset{x\longrightarrow\infty}{\longrightarrow} 0$.
- Since $x > 1$, $|f(x)| \leq |xf(x)|$
- Thus $f(x) \overset{x\longrightarrow\infty}{\longrightarrow} 0$ as well.

Solution 2 (Variation on the Trick)

- Use mean value theorem for integrals:

$$\int_x^{2x} f(t)\,dt = xf(c_x) \quad \text{for some } c_x \in [x, 2x] \text{ depending on } x.$$

- Since f is decreasing,

$$\begin{aligned} x \leq c_x \leq 2x &\implies f(2x) \leq f(c_x) \leq f(x) \\ &\implies 2xf(2x) \leq 2xf(c_x) \leq 2xf(x) \\ &\implies 2xf(2x) \leq 2x \int_x^{2x} f(t)\,dt \leq 2xf(x) \end{aligned}$$

.

- By Cauchy Criterion, $\int_x^{2x} f \longrightarrow 0$.
- So $2xf(2x) \longrightarrow 0$, which by a change of variables gives $uf(u) \longrightarrow 0$.
- Since $u \geq 1$, $f(u) \leq uf(u)$ so $f(u) \longrightarrow 0$ as well.

Solution 3 (Contradiction)

Just showing $f(x) \overset{x\longrightarrow\infty}{\longrightarrow} 0$:

- Toward a contradiction, suppose not.
- Since f is decreasing, it can not diverge to $+\infty$
- If $f(x) \longrightarrow -\infty$, then $f \notin L^1(\mathbb{R})$: choose $x_0 \gg 1$ so that $t \geq x_0 \implies f(t) < -1$, then
- Then $t \geq x_0 \implies |f(t)| \geq 1$, so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f(t)|\,dt \geq \int_{x_0}^\infty 1 = \infty.$$

- Otherwise $f(x) \longrightarrow L \neq 0$, some finite limit.
- If $L > 0$:
  - Fix $\varepsilon > 0$, choose $x_0 \gg 1$ such that $t \geq x_0 \implies L - \varepsilon \leq f(t) \leq L$
  - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L-\varepsilon)\,dt = \infty$$

- If $L < 0$:
  - Fix $\varepsilon > 0$, choose $x_0 \gg 1$ such that $t \geq x_0 \implies L \leq f(t) \leq L + \varepsilon$.
  - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L)\,dt = \infty$$

Showing $xf(x) \overset{x\longrightarrow\infty}{\longrightarrow} 0$.

- Toward a contradiction, suppose not.
- (How to show that $xf(x) \not\longrightarrow +\infty$?)
- If $xf(x) \longrightarrow -\infty$
  - Choose a sequence $\Gamma = \{\widehat{x}_i\}$ such that $x_i \longrightarrow \infty$ and $x_i f(x_i) \longrightarrow -\infty$.
  - Choose a subsequence $\Gamma' = \{x_i\}$ such that $x_i f(x_i) \leq -1$ for all i and $x_i \leq x_{i+1}$.

- Choose a further subsequence $S = \left\{ x_i \in \Gamma' \;\middle|\; 2x_i < x_{i+1} \right\}$.
- Then since f is always decreasing, for $t \geq x_0$, $|f|$ is increasing, and $|f(x_i)| \leq |f(2x_i)|$, so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f| \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(t)|\, dt \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(x_i)| = \sum_{x_i \in S} x_i f(x_i) \longrightarrow \infty.$$

- If $xf(x) \longrightarrow L \neq 0$ for $0 < L < \infty$:
  - Fix $\varepsilon > 0$, choose an infinite sequence $\{x_i\}$ such that $L - \varepsilon \leq x_i f(x_i) \leq L$ for all i.

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)|\, dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i)\, dt = \sum_S x_i f(x_i) \geq \sum_S (L - \varepsilon) \longrightarrow \infty.$$

- If $xf(x) \longrightarrow L \neq 0$ for $-\infty < L < 0$:
  - Fix $\varepsilon > 0$, choose an infinite sequence $\{x_i\}$ such that $L \leq x_i f(x_i) \leq L + \varepsilon$ for all i.

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)|\, dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i)\, dt = \sum_S x_i f(x_i) \geq \sum_S (L) \longrightarrow \infty.$$

Solution 4 (Akos's Suggestion) For $x \geq 1$,

$$|xf(x)| = \left| \int_x^{2x} f(x)\, dt \right| \leq \int_x^{2x} |f(x)|\, dt \leq \int_x^{2x} |f(t)|\, dt \leq \int_x^\infty |f(t)|\, dt \overset{x \longrightarrow \infty}{\longrightarrow} 0$$

where we've used

- Since f is decreasing and $\lim_{x \longrightarrow \infty} f(x) = 0$ from part (a), f is non-negative.
- Since f is positive and decreasing, for every $t \in [a, b]$ we have $|f(a)| \leq |f(t)|$.
- By part (a), the last integral goes to zero.

Solution 5 (Peter's)

- Toward a contradiction, produce a sequence $x_i \longrightarrow \infty$ with $x_i f(x_i) \longrightarrow \infty$ and $x_i f(x_i) > \varepsilon > 0$, then

$$\begin{aligned}
\int f(x)\, dx &\geq \sum_{i=1}^\infty \int_{x_i}^{x_{i+1}} f(x)\, dx \\
&\geq \sum_{i=1}^\infty \int_{x_i}^{x_{i+1}} f(x_{i+1})\, dx \\
&= \sum_{i=1}^\infty f(x_{i+1}) \int_{x_i}^{x_{i+1}} dx \\
&\geq \sum_{i=1}^\infty (x_{i+1} - x_i) f(x_{i+1}) \\
&\geq \sum_{i=1}^\infty (x_{i+1} - x_i) \frac{\varepsilon}{x_{i+1}} \\
&= \varepsilon \sum_{i=1}^\infty \left(1 - \frac{x_{i-1}}{x_i} \right) \longrightarrow \infty
\end{aligned}$$

  which can be ensured by passing to a subsequence where $\sum \frac{x_{i-1}}{x_i} < \infty$.

7.1.3 c

- No: take $f(x) = \dfrac{1}{x \ln x}$
- Then by a u-substitution,

$$\int_0^x f = \ln(\ln(x)) \overset{x \longrightarrow \infty}{\longrightarrow} \infty$$

  is unbounded, so $f \notin L^1([1, \infty))$.
- But

$$x f(x) = \frac{1}{\ln(x)} \overset{x \longrightarrow \infty}{\longrightarrow} 0.$$

7.2 Fall 2019 # 5.

7.2.1 a

Show that if f is continuous with compact support on $\mathbb{R}$, then

$$\lim_{y \to 0} \int_{\mathbb{R}} |f(x - y) - f(x)| dx = 0$$

7.2.2 b

Let $f \in L^1(\mathbb{R})$ and for each $h > 0$ let

$$\mathcal{A}_h f(x) := \frac{1}{2h} \int_{|y| \leq h} f(x - y) dy$$

i. Prove that $\|\mathcal{A}_h f\|_1 \leq \|f\|_1$ for all $h > 0$.

ii. Prove that $\mathcal{A}_h f \longrightarrow f$ in $L^1(\mathbb{R})$ as $h \longrightarrow 0^+$.

Solution.

Continuity in L^1 (recall that DCT won't work! Notes 19.4, prove it for a dense subset first).

Lebesgue differentiation in 1-dimensional case. See HW 5.6.

7.3 a

Choose $g \in C_c^0$ such that $\|f - g\|_1 \longrightarrow 0$.
By translation invariance, $\|\tau_h f - \tau_h g\|_1 \longrightarrow 0$.
Write

$$\begin{aligned}
\|\tau f - f\|_1 &= \|\tau_h f - g + g - \tau_h g + \tau_h g - f\|_1 \\
&\leq \|\tau_h f - \tau_h g\| + \|g - f\| + \|\tau_h g - g\| \\
&\longrightarrow \|\tau_h g - g\|,
\end{aligned}$$

so it suffices to show that $\|\tau_h g - g\| \longrightarrow 0$ for $g \in C_c^0$.

Fix $\varepsilon > 0$. Enlarge the support of g to K such that

$$|h| \leq 1 \text{ and } x \in K^c \implies |g(x-h) - g(x)| = 0.$$

By uniform continuity of g, pick $\delta \leq 1$ small enough such that

$$x \in K, \ |h| \leq \delta \implies |g(x-h) - g(x)| < \varepsilon,$$

then

$$\int_K |g(x-h) - g(x)| \leq \int_K \varepsilon = \varepsilon \cdot m(K) \longrightarrow 0.$$

7.4 b

We have

$$\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x)| \ dx &= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y) \ dy \right| \ dx \\
&\leq \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)| \ dy \ dx \\
&=_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)| \ \mathbf{dx} \ \mathbf{dy} \\
&= \int_{\mathbb{R}} |f(y)| \ dy \\
&= \|f\|_1.
\end{aligned}$$

and (rough sketch)

$$\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x) - f(x)| \ dx &= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y) \ dy \right) - f(x) \right| \ dx \\
&= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y) \ dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x) \ dy \right| \ dx \\
&\leq_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{B(h,x)} |f(y-x) - f(x)| \ \mathbf{dx} \ \mathbf{dy} \\
&\leq \frac{1}{2h} \int_{\mathbb{R}} \|\tau_x f - f\|_1 \ dy \\
&\longrightarrow 0 \quad \text{by (a).}
\end{aligned}$$

7.5 Fall 2017 # 3

Let

$$S = \operatorname{span}_{\mathbb{C}} \left\{ \chi_{(a,b)} \ \middle| \ a, b \in \mathbb{R} \right\},$$

the complex linear span of characteristic functions of intervals of the form (a, b).

Show that for every $f \in L^1(\mathbb{R})$, there exists a sequence of functions $\{f_n\} \subset S$ such that

$$\lim_{n\to\infty} \|f_n - f\|_1 = 0$$

7.6 Spring 2015 # 4

Define

$$f(x,y) := \begin{cases} \dfrac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Carefully show that $f \in L^1(\mathbb{R}^2)$.

7.7 Fall 2014 # 3

Let $f \in L^1(\mathbb{R})$. Show that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that} \qquad m(E) < \delta \implies \int_E |f(x)|\,dx < \varepsilon$$

7.8 Spring 2014 # 1

1. Give an example of a continuous $f \in L^1(\mathbb{R})$ such that $f(x) \not\longrightarrow 0$ as$|x| \longrightarrow \infty$.
2. Show that if f is *uniformly* continuous, then

$$\lim_{|x|\longrightarrow\infty} f(x) = 0.$$

8 Fubini-Tonelli

8.1 Spring 2020 # 4

Let $f, g \in L^1(\mathbb{R})$. Argue that $H(x,y) := f(y)g(x-y)$ defines a function in $L^1(\mathbb{R}^2)$ and deduce from this fact that

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x-y)\,dy$$

defines a function in $L^1(\mathbb{R})$ that satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

Solution.
Relevant concepts:

- Tonelli: non-negative and measurable yields measurability of slices and equality of iterated integrals
- Fubini: $f(x,y) \in L^1$ yields *integrable* slices and equality of iterated integrals

- F/T: apply Tonelli to $|f|$; if finite, $f \in L^1$ and apply Fubini to f

$$\begin{aligned}
\|H(x)\|_1 &= \int_{\mathbb{R}} |H(x,y)|\,dx \\
&= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y)g(x-y)\,dy \right| dx \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(x-y)|\,dy \right) dx \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(x-y)|\,dx \right) dy \quad \text{by Tonelli} \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(t)|\,dt \right) dy \quad \text{setting } t = x-y,\, dt = -dx \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)| \cdot |g(t)|\,dt \right) dy \\
&= \int_{\mathbb{R}} |f(y)| \cdot \left(\int_{\mathbb{R}} |g(t)|\,dt \right) dy \\
&\coloneqq \int_{\mathbb{R}} |f(y)| \cdot \|g\|_1\,dy \\
&= \|g\|_1 \int_{\mathbb{R}} |f(y)|\,dy \\
&\coloneqq \|g\|_1 \|f\|_1 \\
&< \infty \quad \text{by assumption} \quad .
\end{aligned}$$

- H is measurable on $\mathbb{R}^2$:
  - If we can show $\tilde{f}(x,y) \coloneqq f(y)$ and $\tilde{g}(x,y) \coloneqq g(x-y)$ are both measurable on $\mathbb{R}^2$, then $H = \tilde{f} \cdot \tilde{g}$ is a product of measurable functions and thus measurable.
  - $f \in L^1$, and L^1 functions are measurable by definition.
  - The function $(x,y) \mapsto g(x-y)$ is measurable on $\mathbb{R}^2$:
    * Let g be measurable on $\mathbb{R}$, then the cylinder function $G(x,y) = g(x)$ on $\mathbb{R}^2$ is always measurable
    * Define a linear transformation $T \coloneqq [1,-1;0,1]$ which sends $(x,y) \longrightarrow (x-y,y)$, then $T \in \mathrm{GL}(2,\mathbb{R})$ is linear and thus measurable.
    * Then $(G \circ T)(x,y) = G(x-y,y) = \tilde{g}(x-y)$, so $\tilde{g}$ is a composition of measurable functions and thus measurable.
- Apply **Tonelli** to $|H|$
  - H measurable implies $|H|$ is measurable
  - $|H|$ is non-negative
  - So the iterated integrals are equal in the extended sense
  - The calculation shows the iterated integral is finite, to $\int |H|$ is finite and H is thus integrable on $\mathbb{R}^2$.

Note: Fubini is not needed, since we're not calculating the actual integral, just showing H is integrable.

8.2 Spring 2019 # 4

Let f be a non-negative function on $\mathbb{R}^n$ and $\mathcal{A} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq f(x)\}$.

Prove the validity of the following two statements:

a. f is a Lebesgue measurable function on $\mathbb{R}^n \iff \mathcal{A}$ is a Lebesgue measurable subset of $\mathbb{R}^{n+1}$

b. If f is a Lebesgue measurable function on $\mathbb{R}^n$, then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)dx = \int_0^\infty m\left(\{x \in \mathbb{R}^n : f(x) \geq t\}\right) dt$$

8.3 Fall 2018 # 5

Let $f \geq 0$ be a measurable function on $\mathbb{R}$. Show that

$$\int_{\mathbb{R}} f = \int_0^\infty m(\{x : f(x) > t\})dt$$

8.4 Fall 2015 # 5

Let $f, g \in L^1(\mathbb{R})$ be Borel measurable.

1. Show that

- The function

$$F(x,y) \coloneqq f(x-y)g(y)$$

  is Borel measurable on $\mathbb{R}^2$, and
- For almost every $y \in \mathbb{R}$,

$$F_y(x) \coloneqq f(x-y)g(y)$$

  is integrable with respect to y.

2. Show that $f * g \in L^1(\mathbb{R})$ and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

8.5 Spring 2014 # 5

Let $f, g \in L^1([0,1])$ and for all $x \in [0,1]$ define

$$F(x) \coloneqq \int_0^x f(y)\,dy \quad \text{and} \quad G(x) \coloneqq \int_0^x g(y)\,dy.$$

Prove that

$$\int_0^1 F(x)g(x)\,dx = F(1)G(1) - \int_0^1 f(x)G(x)\,dx$$

9 L^2 and Fourier Analysis

9.1 Spring 2020 # 6

9.1.1 a

Show that

$$L^2([0,1]) \subseteq L^1([0,1]) \quad \text{and} \quad \ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z}).$$

9.1.2 b

For $f \in L^1([0,1])$ define

$$\widehat{f}(n) := \int_0^1 f(x) e^{-2\pi i n x}\, dx.$$

Prove that if $f \in L^1([0,1])$ and $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$ then

$$S_N f(x) := \sum_{|n| \leq N} \widehat{f}(n) e^{2\pi i n x}.$$

converges uniformly on $[0,1]$ to a continuous function g such that $g = f$ almost everywhere.

> Hint: One approach is to argue that if $f \in L^1([0,1])$ with $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$ then $f \in L^2([0,1])$.

Solution.
Concepts used:

- For $e_n(x) := e^{2\pi i n x}$, the set $\{e_n\}$ is an orthonormal basis for $L^2([0,1])$.
- For any orthonormal sequence in a Hilbert space, we have Bessel's inequality:

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

- When $\{e_n\}$ is a basis, the above is an *equality* (Parseval)
- Arguing uniform convergence: since $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$, we should be able to apply the M test.

9.1.3 a

Claim: $\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$.

- Set $\mathbf{c} = \left\{c_k \;\middle|\; k \in \mathbb{Z}\right\} \in \ell^1(\mathbb{Z})$.
- It suffices to show that if $\sum_{k \in \mathbb{Z}} |c_k| < \infty$ then $\sum_{k \in \mathbb{Z}} |c_k|^2 < \infty$.
- Let $S = \left\{c_k \;\middle|\; |c_k| \leq 1\right\}$, then $c_k \in S \implies |c_k|^2 \leq |c_k|$
- Claim: S^c can only contain finitely many elements, all of which are finite.

- If not, either $S^c := \{c_j\}_{j=1}^{\infty}$ is infinite with every $|c_j| > 1$, which forces

$$\sum_{c_k \in S^c} |c_k| = \sum_{j=1}^{\infty} |c_j| > \sum_{j=1}^{\infty} 1 = \infty.$$

- If any $c_j = \infty$, then $\displaystyle\sum_{k \in \mathbb{Z}} |c_k| \geq c_j = \infty$.

- So S^c is a finite set of finite integers, let $N = \max \left\{ |c_j|^2 \;\middle|\; c_j \in S^c \right\} < \infty$.
- Rewrite the sum

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} |c_k|^2 &= \sum_{c_k \in S} |c_k|^2 + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \sum_{c_k \in S} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \sum_{k \in \mathbb{Z}} |c_k| + \sum_{c_k \in S^c} |c_k|^2 \quad \text{since the } |c_k| \text{ are all positive} \\
&= \|\mathbf{c}\|_{\ell^1} + \sum_{c_k \in S^c} |c_k|^2 \\
&\leq \|\mathbf{c}\|_{\ell^1} + |S^c| \cdot N \\
&< \infty.
\end{aligned}$$

Claim: $L^2([0,1]) \subseteq L^1([0,1])$.

- It suffices to show that $\displaystyle\int |f|^2 < \infty \implies \int |f| < \infty$.
- Define $S = \left\{ x \in [0,1] \;\middle|\; |f(x)| \leq 1 \right\}$, then $x \in S^c \implies |f(x)|^2 \geq |f(x)|$.
- Break up the integral:

$$\begin{aligned}
\int_{\mathbb{R}} |f| &= \int_S |f| + \int_{S^c} |f| \\
&\leq \int_S |f| + \int_{S^c} |f|^2 \\
&\leq \int_S |f| + \|f\|_2 \\
&\leq \sup_{x \in S} \{|f(x)|\} \cdot \mu(S) + \|f\|_2 \\
&= 1 \cdot \mu(S) + \|f\|_2 \quad \text{by definition of } S \\
&\leq 1 \cdot \mu([0,1]) + \|f\|_2 \quad \text{since } S \subseteq [0,1] \\
&= 1 + \|f\|_2 \\
&< \infty.
\end{aligned}$$

Note: this proof shows $L^2(X) \subseteq L^1(X)$ whenever $\mu(X) < \infty$.

9.2 Fall 2017 # 5

Let φ be a compactly supported smooth function that vanishes outside of an interval $[-N, N]$ such that $\int_{\mathbb{R}} \varphi(x)\,dx = 1$.

For $f \in L^1(\mathbb{R})$, define

$$K_j(x) \coloneqq j\varphi(jx), \qquad f * K_j(x) \coloneqq \int_{\mathbb{R}} f(x-y)K_j(y)\,dy$$

and prove the following:

1. Each $f * K_j$ is smooth and compactly supported.
2. $$\lim_{j \longrightarrow \infty} \|f * K_j - f\|_1 = 0$$

> Hint:
>
> $$\lim_{y \longrightarrow 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dy = 0$$

9.3 Spring 2017 # 5

Let $f, g \in L^2(\mathbb{R})$. Prove that the formula

$$h(x) \coloneqq \int_{-\infty}^{\infty} f(t)g(x-t)\,dt$$

defines a uniformly continuous function h on $\mathbb{R}$.

9.4 Spring 2015 # 6

Let $f \in L^1(\mathbb{R})$ and g be a bounded measurable function on $\mathbb{R}$.

1. Show that the convolution $f * g$ is well-defined, bounded, and uniformly continuous on $\mathbb{R}$.
2. Prove that one further assumes that $g \in C^1(\mathbb{R})$ with bounded derivative, then $f * g \in C^1(\mathbb{R})$ and

$$\frac{d}{dx}(f * g) = f * \left(\frac{d}{dx} g\right)$$

9.5 Fall 2014 # 5

1. Let $f \in C_c^0(\mathbb{R}^n)$, and show

$$\lim_{t \longrightarrow 0} \int_{\mathbb{R}^n} |f(x+t) - f(x)|\,dx = 0.$$

2. Extend the above result to $f \in L^1(\mathbb{R}^n)$ and show that

$$f \in L^1(\mathbb{R}^n), \quad g \in L^\infty(\mathbb{R}^n) \quad \implies f * g \text{ is bounded and uniformly continuous.}$$

10 Functional Analysis: General

10.1 Fall 2019 # 4.

Let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space $\mathcal{H}$.

10.1.1 a

Prove that for every $x \in \mathcal{H}$ one has

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

10.1.2 b

Prove that for any sequence $\{a_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$ there exists an element $x \in \mathcal{H}$ such that

$$a_n = \langle x,\ u_n \rangle \text{ for all } n \in \mathbb{N}$$

and

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$$

Solution.

Concepts used:

- Bessel's Inequality
- Pythagoras
- Surjectivity of the Riesz map
- Parseval's Identity
- Trick – remember to write out finite sum S_N, and consider $\|x - S_N\|$.

10.1.3 a

Claim:

$$\begin{aligned} 0 \leq \left\| x - \sum_{n=1}^{N} \langle x, u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n \rangle|^2 \\ &\implies \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2. \end{aligned}$$

Proof: Let $S_N = \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n$. Then

$$\begin{aligned} 0 &\leq \|x - S_N\|^2 \\ &= \langle x - S_n,\ x - S_N \rangle \\ &= \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\ &\xrightarrow{N \longrightarrow \infty} \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2. \end{aligned}$$

10.1.4 b

1. Fix $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
2. Define

$$x \coloneqq \lim_{N \longrightarrow \infty} S_N = \lim_{N \longrightarrow \infty} \sum_{k=1}^{N} a_k u_k$$

3. $\{S_N\}$ Cauchy (by 1) and H complete $\implies x \in H$.
4.

$$\langle x,\ u_n \rangle = \left\langle \sum_k a_k u_k,\ u_n \right\rangle = \sum_k a_k \langle u_k,\ u_n \rangle = a_n \quad \forall n \in \mathbb{N}$$

since the u_k are all orthogonal.

5.

$$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$$

by Pythagoras since the u_k are normal.

Bonus: We didn't use completeness here, so the Fourier series may not actually converge to x. If $\{u_n\}$ is **complete** (so $x = 0 \iff \langle x,\ u_n \rangle = 0\ \forall n$) then the Fourier series *does* converge to x and $\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 = \|x\|^2$ for all $x \in H$.

10.2 Spring 2019 # 5

10.2.1 a

Show that $L^2([0,1]) \subseteq L^1([0,1])$ and argue that $L^2([0,1])$ in fact forms a dense subset of $L^1([0,1])$.

10.2.2 b

Let Λ be a continuous linear functional on $L^1([0,1])$.

Prove the Riesz Representation Theorem for $L^1([0,1])$ by following the steps below:

i. Establish the existence of a function $g \in L^2([0,1])$ which represents Λ in the sense that

$$\Lambda(f) = f(x)g(x)dx \text{ for all } f \in L^2([0,1]).$$

Hint: You may use, without proof, the Riesz Representation Theorem for $L^2([0,1])$.

ii. Argue that the g obtained above must in fact belong to $L^\infty([0,1])$ and represent Λ in the sense that

$$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}dx \quad \text{for all } f \in L^1([0,1])$$

with

$$\|g\|_{L^\infty([0,1])} = \|\Lambda\|_{L^1([0,1])^\vee}$$

10.3 Spring 2016 # 6

Without using the Riesz Representation Theorem, compute

$$\sup\left\{\left|\int_0^1 f(x)e^x dx\right| \;\middle|\; f \in L^2([0,1],m),\ \|f\|_2 \leq 1\right\}$$

10.4 Spring 2015 # 5

Let $\mathcal{H}$ be a Hilbert space.

1. Let $x \in \mathcal{H}$ and $\{u_n\}_{n=1}^N$ be an orthonormal set. Prove that the best approximation to x in $\mathcal{H}$ by an element in $\operatorname{span}_{\mathbb{C}}\{u_n\}$ is given by

$$\widehat{x} := \sum_{n=1}^N \langle x,\ u_n\rangle u_n.$$

2. Conclude that finite dimensional subspaces of $\mathcal{H}$ are always closed.

10.5 Fall 2015 # 6

Let $f : [0,1] \longrightarrow \mathbb{R}$ be continuous. Show that

$$\sup\left\{\|fg\|_1 \;\middle|\; g \in L^1[0,1],\ \|g\|_1 \leq 1\right\} = \|f\|_\infty$$

10.6 Fall 2014 # 6

Let $1 \leq p, q \leq \infty$ be conjugate exponents, and show that

$$f \in L^p(\mathbb{R}^n) \implies \|f\|_p = \sup_{\|g\|_q = 1}\left|\int f(x)g(x)dx\right|$$

11 Functional Analysis: Banach Spaces

11.1 Spring 2019 # 1

Let $C([0,1])$ denote the space of all continuous real-valued functions on $[0,1]$.

a. Prove that $C([0,1])$ is complete under the uniform norm $\|f\|_u := \sup_{x\in[0,1]} |f(x)|$.

b. Prove that $C([0,1])$ is not complete under the L^1-norm $\|f\|_1 = \int_0^1 |f(x)|\,dx$.

Solution.

11.1.1 a

- Let $\{f_n\}$ be a Cauchy sequence in $C(I, \|\cdot\|_\infty)$, so $\lim_n \lim_m \|f_m - f_n\|_\infty = 0$, we will show it converges to some f in this space.
- For each fixed $x_0 \in [0,1]$, the sequence of real numbers $\{f_n(x_0)\}$ is Cauchy in $\mathbb{R}$ since

$$x_0 \in I \implies |f_m(x_0) - f_n(x_0)| \leq \sup_{x\in I} |f_m(x) - f_n(x)| := \|f_m - f_n\|_\infty \overset{m>n\longrightarrow\infty}{\longrightarrow} 0,$$

- Since $\mathbb{R}$ is complete, this sequence converges and we can define $f(x) := \lim_{k\longrightarrow\infty} f_n(x)$.
- Thus $f_n \longrightarrow f$ pointwise by construction
- Claim: $\|f - f_n\| \overset{n\longrightarrow\infty}{\longrightarrow} 0$, so f_n converges to f in $C([0,1], \|\cdot\|_\infty)$.
  - Proof:
    * Fix $\varepsilon > 0$; we will show there exists an N such that $n \geq N \implies \|f_n - f\| < \varepsilon$
    * Fix an $x_0 \in I$. Since $f_n \longrightarrow f$ pointwise, choose N_1 large enough so that

$$n \geq N_1 \implies |f_n(x_0) - f(x_0)| < \varepsilon/2.$$

    * Since $\|f_n - f_m\|_\infty \longrightarrow 0$, choose and N_2 large enough so that

$$n, m \geq N_2 \implies \|f_n - f_m\|_\infty < \varepsilon/2.$$

    * Then for $n, m \geq \max(N_1, N_2)$, we have

$$\begin{aligned}
|f_n(x_0) - f(x_0)| &= |f_n(x_0) - f(x_0) + f_m(x_0) - f_m(x_0)| \\
&= |f_n(x_0) - f_m(x_0) + f_m(x_0) - f(x_0)| \\
&\leq |f_n(x_0) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \\
&< |f_n(x_0) - f_m(x_0)| + \frac{\varepsilon}{2} \\
&\leq \sup_{x\in I} |f_n(x) - f_m(x)| + \frac{\varepsilon}{2} \\
&< \|f_n - f_m\|_\infty + \frac{\varepsilon}{2} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
\implies |f_n(x_0) - f(x_0)| &< \varepsilon \\
\implies \sup_{x\in I} |f_n(x_0) - f(x_0)| &\leq \sup_{x\in I} \varepsilon \quad \text{by order limit laws} \\
\implies \|f_n - f\| &\leq \varepsilon
\end{aligned}$$

.

- f is the uniform limit of continuous functions and thus continuous, so $f \in C([0,1])$.

11.1.2 b

- It suffices to produce a Cauchy sequence that does not converge to a continuous function.
- Take the following sequence of functions:
  - f_1 increases linearly from 0 to 1 on $[0, 1/2]$ and is 1 on $[1/2, 1]$
  - f_2 is 0 on $[0, 1/4]$ increases linearly from 0 to 1 on $[1/4, 1/2]$ and is 1 on $[1/2, 1]$
  - f_3 is 0 on $[0, 3/8]$ increases linearly from 0 to 1 on $[3/8, 1/2]$ and is 1 on $[1/2, 1]$
  - f_3 is 0 on $[0, (1/2 - 3/8)/2]$ increases linearly from 0 to 1 on $[(1/2 - 3/8)/2, 1/2]$ and is 1 on $[1/2, 1]$

  Idea: take sequence starting points for the triangles: $0, 0 + \frac{1}{4}, 0 + \frac{1}{4} + \frac{1}{8}, \cdots$ which converges to $1/2$ since $\sum_{k=1}^{\infty} \frac{1}{2^k} = -\frac{1}{2} + \sum_{k=0}^{\infty} \frac{1}{2^k}$.
- Then each f_n is clearly integrable, since its graph is contained in the unit square.
- $\{f_n\}$ is Cauchy: geometrically subtracting areas yields a single triangle whose area tends to 0.
- But f_n converges to $\chi_{[\frac{1}{2},1]}$ which is discontinuous.

Todo: show that $\int_0^1 |f_n(x) - f_m(x)|\,dx \longrightarrow 0$ rigorously, show that no $g \in L^1([0,1])$ can converge to this indicator function.

11.2 Spring 2017 # 5

Show that the space $C^1([a,b])$ is a Banach space when equipped with the norm

$$\|f\| := \sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |f'(x)|.$$

11.3 Fall 2017 # 6

Let X be a complete metric space and define a norm

$$\|f\| := \max\{|f(x)| : x \in X\}.$$

Show that $(C^0(\mathbb{R}), \|\cdot\|)$ (the space of continuous functions $f : X \longrightarrow \mathbb{R}$) is complete.