Topology Qualifying Exam Notes

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Contents

0.1 Conventions

- $\pi_0(X)$ is the set of path components of X, and I write $\pi_0(X) = \mathbb{Z}$ if X is path-connected (although it is not a group). Similarly, $H_0(X)$ is a free abelian group on the set of path components of X.
- Lists start at entry 1, since all spaces are connected here and thus $\pi_0 = H_0 = \mathbb{Z}$. That is,

  - $\pi_*(X) = [\pi_1(X), \pi_2(X), \pi_3(X), \cdots]$
  - $H_*(X) = [H_1(X), H_2(X), H_3(X), \cdots]$

- For a finite index set I, it is the case that $\prod_I G = \bigoplus_I G$ in **Grp**, i.e. the finite direct product and finite direct sum coincide. Otherwise, if I is infinite, the direct sum requires cofinitely many zero entries (i.e. finitely many nonzero entries), so here we always use $\prod$.

  In other words, there is an injective map

$$\bigoplus_I G \hookrightarrow \prod_I G$$

  which is an isomorphism when $|I| < \infty$

- $\mathbb{Z}^n := \prod_{i=1}^n \mathbb{Z} = \mathbb{Z} \times \mathbb{Z} \times \ldots \mathbb{Z}$ is the free abelian group of rank n.
  - $x \in \mathbb{Z}^n = \langle a_1, \cdots, a_n \rangle \implies x = \sum_n c_i a_i$ for some $c_i \in \mathbb{Z}$, i.e. a_i form a basis.
  - Example: $x = 2a_1 + 4a_2 + a_1 - a_2 = 3a_1 + 3a_2$.

- $\mathbb{Z}^{*n} := \mathbb{Z} * \mathbb{Z} * \ldots \mathbb{Z}$ is the free product of n free abelian groups, i.e. a free (nonabelian) group on n generators.
  - $x \in \mathbb{Z}^{*n} = \langle a_1, \ldots, a_n \rangle$ implies that x is a finite word in the noncommuting symbols a_i^k for $k \in \mathbb{Z}$.
  - Example: $x = a_1^2 a_2^4 a_1 a_2^{-2}$

- $K(G, n)$ is an Eilenberg-MacLane space, the homotopy-unique space satisfying

$$\pi_k(K(G,n)) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

  - $K(\mathbb{Z}, 1) = S^1$
  - $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$
  - $K(\mathbb{Z}_2, 1) = \mathbb{RP}^\infty$

- $M(G, n)$ is a Moore space, the homotopy-unique space satisfying

$$H_k(M(G,n); G) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

  - $M(\mathbb{Z}, n) = S^n$
  - $M(\mathbb{Z}_2, 1) = \mathbb{RP}^2$
  - $M(\mathbb{Z}_p, n)$ is made by attaching e^{n+1} to S^n via a degree p map.

- $T^n = \prod_n S^1$ is the n-torus

- $D(k, X)$ is the space X with $k \in \mathbb{N}$ distinct points deleted, i.e. the punctured space $X - \{x_1, x_2, \ldots x_k\}$ where each $x_i \in X$.

- $\mathbb{RP}^n = S^n / S^0 = S^n / \mathbb{Z}_2$

- $\mathbb{CP}^n = S^{2n+1} / S^1$

- $B^n = \left\{ \succapprox \in \mathbb{R}^n \,\middle|\, \|\succapprox\| \leq 1 \right\} \subset \mathbb{R}^n$
- $S^{n-1} = \partial B^n = \left\{ \succapprox \in \mathbb{R}^n \,\middle|\, \|\succapprox\| = 1 \right\} \subset \mathbb{R}^n$

sphere ball correct

1 Table of Homotopy and Homology Structures

X	$\pi_*(X)$	$H_*(X)$	CW Structure	$H^*(X)$
$\mathbb{R}^1$	0	0	$\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot x$	0
$\mathbb{R}^n$	0	0	$(\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot x)^n$	0
$D(k, \mathbb{R}^n)$	$\pi_* \bigvee^k S^1$	$\bigoplus_k H_*M(\mathbb{Z}, 1)$	$1 + kx$	
B^n	$\pi_*(\mathbb{R}^n)$	$H_*(\mathbb{R}^n)$	$1 + x^n + x^{n+1}$	0
S^n	$[0 \ldots, \mathbb{Z}, ? \ldots]$	$H_*M(\mathbb{Z}, n)$	$1 + x^n$	$\mathbb{Z}[_n x]/(x^2)$
$D(k, S^n)$	$\pi_* \bigvee^{k-1} S^1$	$\bigoplus_{k-1} H_*M(\mathbb{Z}, 1)$	$1 + (k-1)x^1$	
T^2	$\pi_* S^1 \times \pi_* S^1$	$(H_*M(\mathbb{Z}, 1))^2 \times H_*M(\mathbb{Z}, 2)$	$1 + 2x + x^2$	$\Lambda(_1x_1, {_1x_2})$
T^n	$\prod^n \pi_* S^1$	$\prod_{i=1}^n (H_*M(\mathbb{Z}, i))^{\binom{n}{i}}$	$(1+x)^n$	$\Lambda(_1x_1, {_1x_2}, \ldots {_1x_n})$
$D(k, T^n)$	$[0, 0, 0, 0, \ldots]?$	$[0, 0, 0, 0, \ldots]?$	$1 + x$	
$S^1 \vee S^1$	$\pi_* S^1 * \pi_* S^1$	$(H_*M(\mathbb{Z}, 1))^2$	$1 + 2x$	
$\bigvee^n S^1$	$*^n \pi_* S^1$	$\prod H_*M(\mathbb{Z}, 1)$	$1 + x$	
$\mathbb{RP}^1$	$\pi_* S^1$	$H_*M(\mathbb{Z}, 1)$	$1 + x$	$\mathbb{Z}[_1x]/(x^2)$
$\mathbb{RP}^2$	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^2$	$H_*M(\mathbb{Z}_2, 1)$	$1 + x + x^2$	$_0\mathbb{Z} \times {_2\mathbb{Z}_2}$
$\mathbb{RP}^3$	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^3$	$H_*M(\mathbb{Z}_2, 1) \times H_*M(\mathbb{Z}, 3)$	$1 + x + x^2 + x^3$	$_0\mathbb{Z} \times {_2\mathbb{Z}/2\mathbb{Z}} \times {_3\mathbb{Z}}$
$\mathbb{RP}^4$	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^4$	$H_*M(\mathbb{Z}_2, 1) \times H_*M(\mathbb{Z}_2, 3)$	$1 + x + x^2 + x^3 + x^4$	$_0\mathbb{Z} \times (_2\mathbb{Z}/2\mathbb{Z})^2$
$\mathbb{RP}^n, n \geq 4$ even	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^n$	$\prod_{\text{odd } i<n} H_*M(\mathbb{Z}_2, i)$	$\sum_{i=1}^n x^i$	$_0\mathbb{Z} \times \prod_{i=1}^{n/2} {_2\mathbb{Z}_2}$
$\mathbb{RP}^n, n \geq 4$ odd	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^n$	$\prod_{\text{odd } i \leq n-2} H_*M(\mathbb{Z}_2, i) \times H_* S^n$	$\sum_{i=1}^n x^i$	$H^*(\mathbb{RP}^{n-1}) \times {_n\mathbb{Z}}$

X	$\pi_*(X)$	$H_*(X)$	CW Structure	$H^*(X)$
$\mathbb{CP}^1$ [a] [a] $\mathbb{CP}^1 \simeq S^2$.	$\pi_* K(\mathbb{Z},2) \times \pi_* S^3$	$H_* S^2$	$x^0 + x^2$	$\mathbb{Z}[{}_2x]/({}_2x^2)$
$\mathbb{CP}^2$	$\pi_* K(\mathbb{Z},2) \times \pi_* S^5$	$H_* S^2 \times H_* S^4$	$x^0 + x^2 + x^4$	$\mathbb{Z}[{}_2x]/({}_2x^3)$
$\mathbb{CP}^n, n \geq 2$	$\pi_* K(\mathbb{Z},2) \times \pi_* S^{2n+1}$	$\prod_{i=1}^{n} H_* S^{2i}$	$\sum_{i=1}^{n} x^{2i}$	$\mathbb{Z}[{}_2x]/({}_2x^{n+1})$
Mobius Band [a] [a] Uses the fact that $M \simeq S^1$ by deformation-retracting onto the center circle.	$\pi_* S^1$	$H_* S^1$	$1 + x$	
Klein Bottle	$K(\mathbb{Z} \rtimes_{x \mapsto -x} \mathbb{Z}, 1)$	$H_* S^1 \times H_* \mathbb{RP}^\infty$	$1 + 2x + x^2$	

- $\mathbb{R}^n$ is a contractible space, and so $[S^m, \mathbb{R}^n] = 0$ for all n, m which makes its homotopy groups all zero.
- All calculations follow from the fact that $D(k, \mathbb{R}^n) = \mathbb{R}^n - \{x_1 \dots x_k\} \simeq \bigvee_{i=1}^{k} S^1$ by a deformation retract.
- This uses the fact that $S^n \cong B^n / \partial B^n$ and employs an attaching map

$$\begin{gathered}\phi : (D^n, \partial D^n) \longrightarrow S^n \\ (D^n, \partial D^n) \mapsto (e^n, e^0)\end{gathered}$$

- $B^n \simeq \mathbb{R}^n$ by normalizing vectors.
- Use the inclusion $S^n = B^{n+1}$ as the attaching map.
- $\mathbb{CP}^1 \simeq S^2$.
- $\mathbb{RP}^1 \simeq S^1$.
- Use $\pi_1 \prod = \prod \pi_1$ and the universal cover $\mathbb{R}^1 \twoheadrightarrow S^1$ to yield the cover $\mathbb{R}^n \twoheadrightarrow T^n$.

2 Useful Facts and Techniques

- Fundamental group:
  - Van Kampen
- Homotopy Groups

- Hurewicz map

- Homology
  - Mayer-Vietoris
    * $(X = A\bigcup B) \mapsto (\bigcap, \oplus, \bigcup)$ in homology
  - LES of a pair
    * $(A \hookrightarrow X) \mapsto (A, X, X/A)$
  - Excision

- $\pi_{i\geq 2}(X)$ is always abelian.

- The ranks of π_0 and H_0 are the number of path components, and $\pi_0(X) = \mathbb{Z}$ iff X is simply connected.
  - X simply connected implies $\pi_k(X) \cong H_k(X)$ up to and including the first nonvanishing H_k
  - $H_1(X) = \pi_1 X/[\pi_1 X, \pi_1 X]$, the abelianization.

- General mantra: homotopy plays nicely with products, homology with wedge products.[1]

> In general, homotopy groups behave nicely under homotopy pull-backs (e.g., fibrations and products), but not homotopy push-outs (e.g., cofibrations and wedges). Homology is the opposite.

- $\pi_k \prod X = \prod \pi_k X$ by LES.[2]

- $H_k \prod X \neq \prod H_k X$ due to torsion.
  - Nice case: $H_k(A \times B) = \displaystyle\prod_{i+j=k} H_i A \otimes H_j B$ by Kunneth when all groups are torsion-free.[3]

- $H_k \bigvee X = \prod H_k X$ by Mayer-Vietoris.[4]

- $\pi_k \bigvee X \neq \prod \pi_k X$ (counterexample: $S^1 \vee S^2$)
  - Nice case: $\pi_1 \bigvee X = *\pi_1 X$ by Van Kampen.

- $\pi_i(\widehat{X}) \cong \pi_i(X)$ for $i \geq 2$ whenever $\widehat{X} \twoheadrightarrow X$ is a universal cover.

- Groups and Group Actions

[1] More generally, in **Top**, we can look at $A \leftarrow \{\text{pt}\} \longrightarrow B$ – then $A \times B$ is the pullback and $A \vee B$ is the pushout. In this case, homology $h : \textbf{Top} \longrightarrow \textbf{Grp}$ takes pushouts to pullbacks but doesn't behave well with pullbacks. Similarly, while π takes pullbacks to pullbacks, it doesn't behave nicely with pushouts.

[2] This follows because $X \times Y \twoheadrightarrow X$ is a fiber bundle, so use LES in homotopy and the fact that $\pi_{i\geq 2} \in \textbf{Ab}$.

[3] The generalization of Kunneth is as follows: write $\mathcal{P}(n,k)$ be the set of partitions of n into k parts, i.e. $\curvearrowleft \in \mathcal{P}(n,k) \implies \curvearrowleft = (x_1, x_2, \ldots, x_k)$ where $\sum x_i = n$. Then

$$H_n(\prod_{j=1}^{k} X_j) = \bigoplus_{\curvearrowleft \in \mathcal{P}(n,k)} \bigotimes_{i=1}^{k} H_{x_i}(X_i).$$

[4] $\bigvee$ is the coproduct in the category $\textbf{Top}_0$ of pointed topological spaces, and alternatively, $X \vee Y$ is the pushout in **Top** of $X \leftarrow \{\text{pt}\} \longrightarrow Y$

- $\pi_0(G) = G$ for G a discrete topological group.
- $\pi_k(G/H) = \pi_k(G)$ if $\pi_k(H) = \pi_{k-1}(H) = 0$.
- $\pi_1(X/G) = \pi_0(G)$ when G acts freely/transitively on X.

- Manifolds
  - $H^n(M^n) = \mathbb{Z}$ if M^n is orientable and zero if M^n is nonorientable.
  - Poincare Duality: $H_i M^n = \cong H^{n-i} M^n$ iff M^n is closed and orientable.

3 Other Interesting Things To Consider

- The "generalized uniform bouquet"? $\mathcal{B}^n(m) = \bigvee_{i=1}^{n} S^m$
- Lie Groups
  - The real general linear group, $\mathrm{GL}_n(\mathbb{R})$
    - The real special linear group $SL_n(\mathbb{R})$
    - The real orthogonal group, $O_n(\mathbb{R})$
      - The real special orthogonal group, $SO_n(\mathbb{R})$
    - The real unitary group, $U_n(\mathbb{R})$
      - The real special unitary group, $SU_n(\mathbb{R})$
    - The real symplectic group $Sp(n)$
- "Geometric" Stuff
  - Affine n-space over a field $\mathbb{A}^n(k) = k^n \rtimes GL_n(k)$
  - The projective space $\mathbb{P}^n(k)$
    - The projective linear group over a ring R, $PGL_n(R)$
    - The projective special linear group over a ring R, $PSL_n(R)$
    - The modular groups $PSL_n(\mathbb{Z})$
      - Specifically $PSL_2(\mathbb{Z})$
- The real Grassmannian, $Gr(n, k, \mathbb{R})$, i.e. the set of k dimensional subspaces of $\mathbb{R}^n$
- The Stiefel manifold $V_n(k)$
- Possible modifications to a space X:
  - Remove k points by taking $D(k, X)$
  - Remove a line segment
  - Remove an entire line/axis
  - Remove a hole
  - Quotient by a group action (e.g. antipodal map, or rotation)
  - Remove a knot
  - Take complement in ambient space
- Assorted info about other Lie Groups:
- $O_n, U_n, SO_n, SU_n, Sp_n$
- $\pi_k(U_n) = \mathbb{Z} \cdot \mathbb{1}\,[k \text{ odd}]$
  - $\pi_1(U_n) = 1$

- $\pi_k(SU_n) = \mathbb{Z} \cdot \mathbb{1}\,[k \text{ odd}]$
  - $\pi_1(SU_n) = 0$
- $\pi_k(U_n) = \mathbb{Z}_2 \cdot \mathbb{1}\,[k = 0, 1 \mod 8] + \mathbb{Z} \cdot \mathbb{1}\,[k = 3, 7 \mod 8]$
- $\pi_k(SP_n) = \mathbb{Z}_2 \cdot \mathbb{1}\,[k = 4, 5 \mod 8] + \mathbb{Z} \cdot \mathbb{1}\,[k = 3, 7 \mod 8]$

4 Spheres

- $\pi_i(S^n) = 0$ for $i < n$, $\pi_n(S^n) = \mathbb{Z}$
  - Not necessarily true that $\pi_i(S^n) = 0$ when $i > n$!!!
    - E.g. $\pi_3(S^2) = \mathbb{Z}$ by Hopf fibration
- $H_i(S^n) = \mathbb{1}\,[i \in \{0, n\}]$
- $H_n(\bigvee_i X_i) \cong \prod_i H_n(X_i)$ for "good pairs"
  - Corollary: $H_n(\bigvee_k S^n) = \mathbb{Z}^k$
- $S^n/S^k \simeq S^n \vee \Sigma S^k$
  - $\Sigma S^n = S^{n+1}$
- S^n has the CW complex structure of 2 k-cells for each $0 \le k \le n$.
- $\pi_0(X)$ is the set of path components of X, and I write $\pi_0(X) = \mathbb{Z}$ if X is path-connected (although it is not a group). Similarly, $H_0(X)$ is a free abelian group on the set of path components of X.
- Lists start at entry 1, since all spaces are connected here and thus $\pi_0 = H_0 = \mathbb{Z}$. That is,
  - $\pi_*(X) = [\pi_1(X), \pi_2(X), \pi_3(X), \cdots]$
  - $H_*(X) = [H_1(X), H_2(X), H_3(X), \cdots]$
- For a finite index set I, it is the case that $\prod_I G = \bigoplus_I G$ in **Grp**, i.e. the finite direct product and finite direct sum coincide. Otherwise, if I is infinite, the direct sum requires cofinitely many zero entries (i.e. finitely many nonzero entries), so here we always use $\prod$.

  In other words, there is an injective map

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  which is an isomorphism when $|I| < \infty$
- $\mathbb{Z}^n := \prod_{i=1}^n \mathbb{Z} = \mathbb{Z} \times \mathbb{Z} \times \ldots \mathbb{Z}$ is the free abelian group of rank n.
  - $x \in \mathbb{Z}^n = \langle a_1, \cdots, a_n \rangle \implies x = \sum_n c_i a_i$ for some $c_i \in \mathbb{Z}$, i.e. a_i form a basis.
  - Example: $x = 2a_1 + 4a_2 + a_1 - a_2 = 3a_1 + 3a_2$.

- $\mathbb{Z}^{*n} := \mathbb{Z} * \mathbb{Z} * \ldots \mathbb{Z}$ is the free product of n free abelian groups, i.e. a free (nonabelian) group on n generators.
  - $x \in \mathbb{Z}^{*n} = \langle a_1, \ldots, a_n \rangle$ implies that x is a finite word in the noncommuting symbols a_i^k for $k \in \mathbb{Z}$.
  - Example: $x = a_1^2 a_2^4 a_1 a_2^{-2}$
- $K(G, n)$ is an Eilenberg-MacLane space, the homotopy-unique space satisfying

$$\pi_k(K(G,n)) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

  - $K(\mathbb{Z}, 1) = S^1$
  - $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$
  - $K(\mathbb{Z}_2, 1) = \mathbb{RP}^\infty$
- $M(G, n)$ is a Moore space, the homotopy-unique space satisfying

$$H_k(M(G,n); G) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

  - $M(\mathbb{Z}, n) = S^n$
  - $M(\mathbb{Z}_2, 1) = \mathbb{RP}^2$
  - $M(\mathbb{Z}_p, n)$ is made by attaching e^{n+1} to S^n via a degree p map.
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- $\mathbb{RP}^n = S^n / S^0 = S^n / \mathbb{Z}_2$
- $\mathbb{CP}^n = S^{2n+1} / S^1$
- $B^n = \left\{ \underset{\approx}{\succsim} \in \mathbb{R}^n \;\middle|\; \|\underset{\approx}{\succsim}\| \leq 1 \right\} \subset \mathbb{R}^n$
- $S^{n-1} = \partial B^n = \left\{ \underset{\approx}{\succsim} \in \mathbb{R}^n \;\middle|\; \|\underset{\approx}{\succsim}\| = 1 \right\} \subset \mathbb{R}^n$

sphere ball correct

5 Table of Homotopy and Homology Structures

X	$\pi_*(X)$	$H_*(X)$	CW	$H^*(X)$
$\mathbb{R}^n$ [a] [a] $\mathbb{R}^n$ is a contractible space, and so $[S^m, \mathbb{R}^n] = 0$ for all n, m which makes its homotopy groups all zero.	0	0	$(\bigcup_{\mathbb{Z}} e^0 + \bigcup_{\mathbb{Z}} e^1)^n$	0
$D(k, \mathbb{R}^n)$ [a] [a] All calculations follow from the fact that $D(k, \mathbb{R}^n) = \mathbb{R}^n - \{x_1 \dots x_k\} \simeq \bigvee_{i=1}^{k} S^1$ by a deformation retract.	$\pi_* \bigvee^{k} S^1$	$\bigoplus_k H_* M(\mathbb{Z}, 1)$	$e^0 + ke^1$	
B^n [a] [a] $B^n \simeq \mathbb{R}^n$ by normalizing vectors.	$\pi_*(\mathbb{R}^n)$	$H_*(\mathbb{R}^n)$	$e^0 + e^n + e^{n+1}$ [a] [a] Use the inclusion $S^n = B^{n+1}$ as the attaching map.	0
S^n	$[0 \dots, \mathbb{Z}, ? \dots]$	$H_* M(\mathbb{Z}, n)$	$e^0 + e^n$ [a] [a] This uses the fact that $S^n \cong B^n / \partial B^n$ and employs an attaching map $\phi : (D^n, \partial D^n) \longrightarrow S^n$ $(D^n, \partial D^n) \mapsto (e^n, e^0)$	$\mathbb{Z}[_n x]/(x^2)$

X	$\pi_*(X)$	$H_*(X)$	CW	$H^*(X)$
$D(k, S^n)$[a] [a] Use the fact that $D(1, S^n) \cong \mathbb{R}^n$ and thus $D(k, S^n) \cong D(k-1, \mathbb{R}^n) \cong \bigvee^{k-1} S^1$ Conventions	$\pi_* \bigvee^{k-1} S^1$	$\bigoplus_{k-1} H_*M(\mathbb{Z}, 1)$	$e^0 + (k-1)e^1$	
T^2	$\pi_* S^1 \times \pi_* S^1$	$(H_*M(\mathbb{Z}, 1))^2 \times H_*M(\mathbb{Z}, 2)$	$e^0 + 2e^1 + e^2$	$\Lambda({}_1x_1, {}_1x_2)$
T^n	$\prod^n \pi_* S^1$[a] [a] Use $\pi_1 \prod = \prod \pi_1$ and the universal cover $\mathbb{R}^1 \twoheadrightarrow S^1$ to yield the cover $\mathbb{R}^n \twoheadrightarrow T^n$.	$\prod_{i=1}^{n} (H_*M(\mathbb{Z}, i))^{\binom{n}{i}}$	$(e^0 + e^1)^n = \sum_{i=1}^{n} \binom{n}{i} e^i$	$\Lambda({}_1x_1, {}_1x_2, \dots {}_1x_n)$
$D(k, T^n)$	$[0, 0, 0, 0, \dots]$?	$[0, 0, 0, 0, \dots]$?	$e^0 + e^1$	
$S^1 \vee S^1$	$\pi_* S^1 * \pi_* S^1$	$(H_*M(\mathbb{Z}, 1))^2$	$e^0 + 2e^1$	
$\bigvee^n S^1$[a] [a] ?	$*^n \pi_* S^1$	$\prod H_*M(\mathbb{Z}, 1)$	$e^0 + e^1$	
$\mathbb{RP}^1$[a] [a] $\mathbb{RP}^1 \simeq S^1$.	$\pi_* S^1$	$H_*M(\mathbb{Z}, 1)$	$e^0 + e^1$	${}_0\mathbb{Z} \times {}_1\mathbb{Z}$
$\mathbb{RP}^2$	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^2$	$H_*M(\mathbb{Z}_2, 1)$	$e^0 + e^1 + e^2$	${}_0\mathbb{Z} \times {}_2\mathbb{Z}_2$
$\mathbb{RP}^3$	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^3$	$H_*M(\mathbb{Z}_2, 1) \times H_*M(\mathbb{Z}, 3)$	$e^0 + e^1 + e^2 + e^3$	${}_0\mathbb{Z} \times {}_2\mathbb{Z}_2 \times {}_3\mathbb{Z}$
$\mathbb{RP}^4$	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^4$	$H_*M(\mathbb{Z}_2, 1) \times H_*M(\mathbb{Z}_2, 3)$	$e^0 + e^1 + e^2 + e^3 + e^4$	${}_0\mathbb{Z} \times ({}_2\mathbb{Z}_2)^2$
$\mathbb{RP}^n, n \geq 4$ even	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^n$[a] [a] Take the universal double cover $S^n \twoheadrightarrow^{\times 2} \mathbb{RP}^n$ to get equality in $\pi_{i \geq 2}$.	$\prod_{\text{odd } i<n} H_*M(\mathbb{Z}_2, i)$	$\sum_{i=1}^{n} e^i$	${}_0\mathbb{Z} \times \prod_{i=1}^{n/2} {}_2\mathbb{Z}_2$

X	$\pi_*(X)$	$H_*(X)$	CW	$H^*(X)$
$\mathbb{RP}^n, n \geq 4$ odd	$\pi_* K(\mathbb{Z}_2, 1) \times \pi_* S^n$ [a] [a] Take the universal double cover $S^n \twoheadrightarrow^{\times 2} \mathbb{RP}^n$ to get equality in $\pi_{i \geq 2}$.	$\prod_{\text{odd } i \leq n-2} H_* M(\mathbb{Z}_2, i) \times H_* S^n$	$\sum_{i=1}^{n} e^i$	$H^*(\mathbb{RP}^{n-1}) \times {}_n\mathbb{Z}$
$\mathbb{CP}^1$ [a] [a] $\mathbb{CP}^1 \simeq S^2$.	$\pi_* K(\mathbb{Z}, 2) \times \pi_* S^3$	$H_* S^2$	$e^0 + e^2$	$\mathbb{Z}[{}_2x]/({}_2x^2)$
$\mathbb{CP}^2$	$\pi_* K(\mathbb{Z}, 2) \times \pi_* S^5$	$H_* S^2 \times H_* S^4$	$e^0 + e^2 + e^4$	$\mathbb{Z}[{}_2x]/({}_2x^3)$
$\mathbb{CP}^n, n \geq 2$	$\pi_* K(\mathbb{Z}, 2) \times \pi_* S^{2n+1}$ [a] [a] Use $\mathbb{CP}^n = S^{2n+1}/S^1$	$\prod_{i=1}^{n} H_* S^{2i}$	$\sum_{i=1}^{n} e^{2i}$	$\mathbb{Z}[{}_2x]/({}_2x^{n+1})$
Mobius Band [a] [a] Uses the fact that $M \simeq S^1$ by deformation-retracting onto the center circle.	$\pi_* S^1$	$H_* S^1$	$e^0 + e^1$	
Klein Bottle	$K(\mathbb{Z} \rtimes_{x \mapsto -x} \mathbb{Z}, 1)$ [a] [a] Alternatively, the fundamental group is $\mathbb{Z} * \mathbb{Z}/bab^{-1}a$. Use the fact the $\tilde{K} = \mathbb{R}^2$.	$H_* S^1 \times H_* \mathbb{RP}^\infty$	$e^0 + 2e^1 + e^2$	

6 Euler Characteristics

- Only surfaces with positive χ:
  - $\chi S^2 = 2$
  - $\chi \mathbb{RP}^2 = 1$
  - $\chi B^2 = 1$
- Manifolds with zero χ
  - $T^2, K, M, S^1 \times I$
- Manifolds with negative χ
  - $\Sigma_{g \geq 2}$ by $\chi(X) = 2 - 2g$.

7 Useful Facts and Techniques

- Fundamental group:
  - Van Kampen
- Homotopy Groups
  - Hurewicz map
- Homology
  - Mayer-Vietoris
    - $(X = A \bigcup B) \mapsto (\bigcap, \oplus, \bigcup)$ in homology
  - LES of a pair
    - $(A \hookrightarrow X) \mapsto (A, X, X/A)$
  - Excision
- $\pi_{i\geq 2}(X)$ is always abelian.
- The ranks of π_0 and H_0 are the number of path components, and $\pi_0(X) = \mathbb{Z}$ iff X is simply connected.
  - X simply connected implies $\pi_k(X) \cong H_k(X)$ up to and including the first nonvanishing H_k
  - $H_1(X) = \pi_1 X/[\pi_1 X, \pi_1 X]$, the abelianization.
- General mantra: homotopy plays nicely with products, homology with wedge products.[5]

> In general, homotopy groups behave nicely under homotopy pull-backs (e.g., fibrations and products), but not homotopy push-outs (e.g., cofibrations and wedges). Homology is the opposite.

- $\pi_k \prod X = \prod \pi_k X$ by LES.[6]
- $H_k \prod X \neq \prod H_k X$ due to torsion.
  - Nice case: $H_k(A \times B) = \displaystyle\prod_{i+j=k} H_i A \otimes H_j B$ by Kunneth when all groups are torsion-free.[7]
- $H_k \bigvee X = \prod H_k X$ by Mayer-Vietoris.[8]
- $\pi_k \bigvee X \neq \prod \pi_k X$ (counterexample: $S^1 \vee S^2$)

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[5] More generally, in **Top**, we can look at $A \leftarrow \{\mathrm{pt}\} \longrightarrow B$ – then $A \times B$ is the pullback and $A \vee B$ is the pushout. In this case, homology $h : \mathbf{Top} \longrightarrow \mathbf{Grp}$ takes pushouts to pullbacks but doesn't behave well with pullbacks. Similarly, while π takes pullbacks to pullbacks, it doesn't behave nicely with pushouts.

[6] This follows because $X \times Y \twoheadrightarrow X$ is a fiber bundle, so use LES in homotopy and the fact that $\pi_{i\geq 2} \in \mathbf{Ab}$.

[7] The generalization of Kunneth is as follows: write $\mathcal{P}(n,k)$ be the set of partitions of n into k parts, i.e. $\curvearrowleft \in \mathcal{P}(n,k) \implies \curvearrowleft = (x_1, x_2, \ldots, x_k)$ where $\sum x_i = n$. Then

$$H_n(\prod_{j=1}^{k} X_j) = \bigoplus_{\curvearrowleft \in \mathcal{P}(n,k)} \bigotimes_{i=1}^{k} H_{x_i}(X_i).$$

[8] $\bigvee$ is the coproduct in the category $\mathbf{Top}_0$ of pointed topological spaces, and alternatively, $X \vee Y$ is the pushout in **Top** of $X \leftarrow \{\mathrm{pt}\} \longrightarrow Y$

  - Nice case: $\pi_1 \bigvee X = *\pi_1 X$ by Van Kampen.

- $\pi_i(\widehat{X}) \cong \pi_i(X)$ for $i \geq 2$ whenever $\widehat{X} \twoheadrightarrow X$ is a universal cover.

- Groups and Group Actions

  - $\pi_0(G) = G$ for G a discrete topological group.
  - $\pi_k(G/H) = \pi_k(G)$ if $\pi_k(H) = \pi_{k-1}(H) = 0$.
  - $\pi_1(X/G) = \pi_0(G)$ when G acts freely/transitively on X.

- Manifolds

  - $H^n(M^n) = \mathbb{Z}$ if M^n is orientable and zero if M^n is nonorientable.
  - Poincare Duality: $H_i M^n =\cong H^{n-i} M^n$ iff M^n is closed and orientable.

8 Other Interesting Things To Consider

- The "generalized uniform bouquet"? $\mathcal{B}^n(m) = \bigvee_{i=1}^{n} S^m$

- Lie Groups

  - The real general linear group, $\mathrm{GL}_n(\mathbb{R})$
    - The real special linear group $SL_n(\mathbb{R})$
    - The real orthogonal group, $O_n(\mathbb{R})$
      - The real special orthogonal group, $SO_n(\mathbb{R})$
    - The real unitary group, $U_n(\mathbb{R})$
      - The real special unitary group, $SU_n(\mathbb{R})$
    - The real symplectic group $Sp(n)$

- "Geometric" Stuff

  - Affine n-space over a field $\mathbb{A}^n(k) = k^n \rtimes GL_n(k)$
  - The projective space $\mathbb{P}^n(k)$
    - The projective linear group over a ring R, $PGL_n(R)$
    - The projective special linear group over a ring R, $PSL_n(R)$
    - The modular groups $PSL_n(\mathbb{Z})$
      - Specifically $PSL_2(\mathbb{Z})$

- The real Grassmannian, $Gr(n, k, \mathbb{R})$, i.e. the set of k dimensional subspaces of $\mathbb{R}^n$

- The Stiefel manifold $V_n(k)$

- Possible modifications to a space X:

  - Remove k points by taking $D(k, X)$
  - Remove a line segment
  - Remove an entire line/axis
  - Remove a hole
  - Quotient by a group action (e.g. antipodal map, or rotation)
  - Remove a knot
  - Take complement in ambient space

- Assorted info about other Lie Groups:
- $O_n, U_n, SO_n, SU_n, Sp_n$
- $\pi_k(U_n) = \mathbb{Z} \cdot \mathbb{1}\left[k \text{ odd}\right]$
  - $\pi_1(U_n) = 1$
- $\pi_k(SU_n) = \mathbb{Z} \cdot \mathbb{1}\left[k \text{ odd}\right]$
  - $\pi_1(SU_n) = 0$
- $\pi_k(U_n) = \mathbb{Z}_2 \cdot \mathbb{1}\left[k = 0, 1 \mod 8\right] + \mathbb{Z} \cdot \mathbb{1}\left[k = 3, 7 \mod 8\right]$
- $\pi_k(SP_n) = \mathbb{Z}_2 \cdot \mathbb{1}\left[k = 4, 5 \mod 8\right] + \mathbb{Z} \cdot \mathbb{1}\left[k = 3, 7 \mod 8\right]$

9 Spheres

- $\pi_i(S^n) = 0$ for $i < n$, $\pi_n(S^n) = \mathbb{Z}$
  - Not necessarily true that $\pi_i(S^n) = 0$ when $i > n$!!!
    - E.g. $\pi_3(S^2) = \mathbb{Z}$ by Hopf fibration
- $H_i(S^n) = \mathbb{1}\left[i \in \{0, n\}\right]$
- $H_n(\bigvee_i X_i) \cong \prod_i H_n(X_i)$ for "good pairs"
  - Corollary: $H_n(\bigvee_k S^n) = \mathbb{Z}^k$
- $S^n/S^k \simeq S^n \vee \Sigma S^k$
  - $\Sigma S^n = S^{n+1}$
- S^n has the CW complex structure of 2 k-cells for each $0 \leq k \leq n$.

10 Definitions

- Topology: Closed under arbitrary unions and finite intersections.
- Basis: A subset $\{B_i\}$ is a basis iff
  - $x \in X \implies x \in B_i$ for some i.
  - $x \in B_i \bigcap B_j \implies x \in B_k \subset B_i \bigcap B_k$.
  - Topology generated by this basis: $x \in N_x \implies x \in B_i \subset N_x$ for some i.
- Dense: A subset $Q \subset X$ is dense iff $y \in N_y \subset X \implies N_y \bigcap Q \neq \emptyset$ iff $\overline{Q} = X$.
- Neighborhood: A neighborhood of a point x is any open set containing x.
- Hausdorff
- Second Countable: admits a countable basis.
- Closed (several characterizations)

- Closure in a subspace: $Y \subset X \implies \mathrm{cl}_Y(A) := \mathrm{cl}_X(A) \bigcap Y$.
- Bounded
- Compact: A topological space (X, τ) is **compact** if every open cover has a *finite* subcover.

  That is, if $\left\{U_j \;\middle|\; j \in J\right\} \subset \tau$ is a collection of open sets such that $X \subseteq \bigcup\limits_{j \in J} U_j$, then there exists a *finite* subset $J' \subset J$ such that $X \subseteq \bigcup\limits_{j \in J'} U_j$.
- Locally compact For every $x \in X$, there exists a $K_x \ni x$ such that K_x is compact.
- Connected: There does not exist a disconnecting set $X = A \coprod B$ such that $\emptyset \neq A, B \subsetneq$, i.e. X is the union of two proper disjoint nonempty sets.

  Equivalently, X contains no proper nonempty clopen sets.

  - Additional condition for a subspace $Y \subset X$: $\mathrm{cl}_Y(A) \bigcap V = A \bigcap \mathrm{cl}_Y(B) = \emptyset$.
- Locally connected: A space is locally connected at a point x iff $\forall N_x \ni x$, there exists a $U \subset N_x$ containing x that is connected.
- Retract: A subspace $A \subset X$ is a *retract* of X iff there exists a continuous map $f : X \longrightarrow A$ such that $f \Big|_A = \mathrm{id}_A$. Equivalently it is a *left* inverse to the inclusion.
- Uniform Continuity: For $f : (X, d_x) \longrightarrow (Y, d_Y)$ metric spaces,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \quad d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

- Lebesgue number: For (X, d) a compact metric space and $\{U_\alpha\} \rightrightarrows X$, there exist $\delta_L > 0$ such that

$$A \subset X,\ \mathrm{diam}(A) < \delta_L \implies A \subseteq U_\alpha \text{ for some } \alpha.$$

- Paracompact
- Components: Set $x \sim y$ iff there exists a connected set $U \ni x, y$ and take equivalence classes.
- Path Components: Set $x \sim y$ iff there exists a path-connected set $U \ni x, y$ and take equivalence classes.
- Separable: Contains a countable dense subset.
- Limit Point: For $A \subset X$, x is a limit point of A if every punctured neighborhood P_x of x satisfies $P_x \bigcap A \neq \emptyset$, i.e. every neighborhood of x intersects A in some point other than x itself.

  Equivalently, x is a limit point of A iff $x \in \mathrm{cl}_X(A \setminus \{x\})$.

11 Examples

11.1 Common Spaces and Operations

Point-Set:

- Finite discrete sets with the discrete topology
- Subspaces of $\mathbb{R}$: $(a,b),(a,b],(a,\infty)$, etc.
  - $\{0\}\bigcup\left\{\frac{1}{n} \mid n\in\mathbb{Z}^{\geq 1}\right\}$
- $\mathbb{Q}$
- The topologist's sine curve
- One-point compactifications
- $\mathbb{R}^{\omega}$
- Hawaiian earring
- Cantor set

Non-Hausdorff spaces:

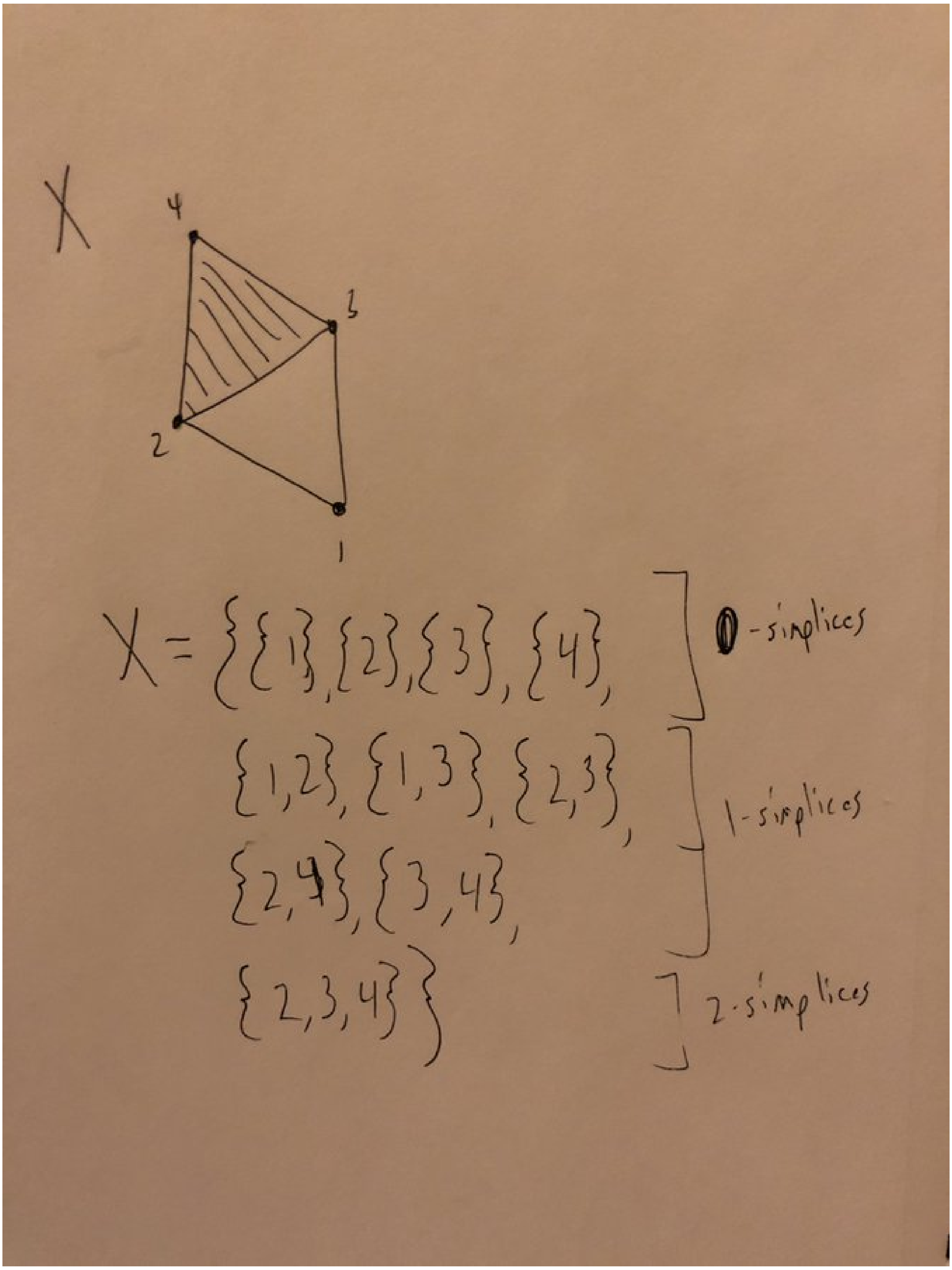
- The cofinite topology on any infinite set.
- $\mathbb{R}/\mathbb{Q}$
- The line with two origins.

General Spaces:

$$S^n,\mathbb{D}^n,T^n,\mathbb{RP}^n,\mathbb{CP}^n,\mathbb{M},\mathbb{K},\Sigma_g,\mathbb{RP}^\infty,\mathbb{CP}^\infty.$$

"Constructed" Spaces

- Knot complements in S^3
- Covering spaces (hyperbolic geometry)
- Lens spaces
- Matrix groups
- Prism spaces
- Pair of pants
- Seifert surfaces
- Surgery
- Simplicial Complexes
  - Nice minimal example:



Exotic/Pathological Spaces

- $\mathbb{HP}^n$
- Dunce Cap

- Horned sphere

Operations

- Cartesian product $A \times B$
- Wedge product $A \vee B$
- Connect Sum $A\#B$
- Quotienting A/B
- Puncturing $A \setminus \{a_i\}$
- Smash product
- Join
- Cones
- Suspension
- Loop space
- Identifying a finite number of points

11.2 Alternative Topologies

- Discrete
- Cofinite
- Discrete and Indiscrete
- Uniform

The cofinite topology:

- Non-Hausdorff
- Compact

The discrete topology:

- Discrete iff points are open
- Always Hausdorff
- Compact iff finite
- Totally disconnected
- If the domain, every map is continuous

The indiscrete topology:

- Only open sets are $\emptyset, X$
- Non-Hausdorff
- If the codomain, every map is continuous
- Compact

12 Theorems

12.1 Point-Set

- Closed subsets of Hausdorff spaces are compact? (check)
- Cantor's intersection theorem?
- Tube lemma

- Properties pushed forward through continuous maps:
  - Compactness?
  - Connectedness (when surjective)
  - Separability
  - Density **only when** f is surjective
  - **Not** openness
  - **Not** closedness
- A retract of a Hausdorff/connected/compact space is closed/connected/compact respectively.

Proposition 12.1.
A continuous function on a compact set is uniformly continuous.

Proof .
Take $\left\{ B_{\frac{\varepsilon}{2}}(y) \,\middle|\, y \in Y \right\} \rightrightarrows Y$, pull back to an open cover of X, has Lebesgue number $\delta_L > 0$, then $x' \in B_{\delta_L}(x) \implies f(x), f(x') \in B_{\frac{\varepsilon}{2}}(y)$ for some y. ∎

- Lipschitz continuity implies uniform continuity (take $\delta = \varepsilon/C$)
  - Counterexample to converse: $f(x) = \sqrt{x}$ on $[0,1]$ has unbounded derivative.
- Extreme Value Theorem: for $f : X \longrightarrow Y$ continuous with X compact and Y ordered in the order topology, there exist points $c, d \in X$ such that $f(x) \in [f(c), f(d)]$ for every x.

Theorem 12.2.
Points are closed in T_1 spaces.

Theorem 12.3.
A metric space X is sequentially compact iff it is complete and totally bounded.

Theorem 12.4.
A metric space is totally bounded iff every sequence has a Cauchy subsequence.

Theorem 12.5.
A metric space is compact iff it is complete and totally bounded.

Theorem 12.6 ***(Baire).***
If X is a complete metric space, then the intersection of countably many dense open sets is dense in X.

Theorem 12.7.
A continuous bijective open map is a homeomorphism.

Theorem 12.8.
A closed subset A of a compact set B is compact.

Proof .

- Let $\{A_i\} \rightrightarrows A$ be a covering of A by sets open in A.
- Each $A_i = B_i \bigcap A$ for some B_i open in B (definition of subspace topology)
- Define $V = \{B_i\}$, then $V \rightrightarrows A$ is an open cover.
- Since A is closed, $W := B \setminus A$ is open
- Then $V \bigcup W$ is an open cover of B, and has a finite subcover $\{V_i\}$
- Then $\left\{V_i \bigcap A\right\}$ is a finite open cover of A.

∎

Theorem 12.9.
The continuous image of a compact set is compact.

Theorem 12.10.
A closed subset of a Hausdorff space is compact.

12.2 Algebraic

Todo: Merge the two van Kampen theorems.

Theorem 12.11 ***(Van Kampen).***
The pushout is the northwest colimit of the following diagram

$$\begin{tikzcd} A\coprod_Z B & A \arrow[l] \\ B \arrow[u, phantom] & Z \arrow[u, "\iota_A"'] \arrow[l, "\iota_B"] \end{tikzcd}$$

For groups, the pushout is given by the amalgamated free product: if $A = \left\langle G_A \,\middle|\, R_A \right\rangle, B = \left\langle G_B \,\middle|\, R_B \right\rangle$, then $A *_Z B = \left\langle G_A, G_B \,\middle|\, R_A, R_B, T \right\rangle$ where T is a set of relations given by $T = \left\{ \iota_A(z)\iota_B(z)^{-1} \,\middle|\, z \in Z \right\}$.
Example: $A = \mathbb{Z}/4\mathbb{Z} = \left\langle x \,\middle|\, x^4 \right\rangle, B = \mathbb{Z}/6\mathbb{Z} = \left\langle y \,\middle|\, x^6 \right\rangle, Z = \mathbb{Z}/2\mathbb{Z} = \left\langle z \,\middle|\, z^2 \right\rangle$. Then we can identify Z as a subgroup of A, B using $\iota_A(z) = x^2$ and $\iota_B(z) = y^3$. So

$$A *_Z B = \left\langle x, y \,\middle|\, x^4, y^6, x^2 y^{-3} \right\rangle$$

.
Suppose $X = U_1 \bigcup U_2$ such that $U_1 \bigcap U_2 \neq \emptyset$ is path connected. Then taking $x_0 \in U := U_1 \bigcap U_2$ yields a pushout of fundamental groups

$$\pi_1(X; x_0) = \pi_1(U_1; x_0) *_{\pi_1(U; x_0)} \pi_1(U_2; x_0).$$

Theorem 12.12 ***(Van Kampen).***
If $X = U \bigcup V$ where $U, V, U \bigcap V$ are all path-connected then

$$\pi_1(X) = \pi_1 U *_{\pi_1(U \bigcap V)} \pi_1 V,$$

where the amalgamated product can be computed as follows: If we have presentations

$$\begin{aligned} \pi_1(U, w) &= \left\langle u_1, \cdots, u_k \,\middle|\, \alpha_1, \cdots, \alpha_l \right\rangle \\ \pi_1(V, w) &= \left\langle v_1, \cdots, v_m \,\middle|\, \beta_1, \cdots, \beta_n \right\rangle \\ \pi_1(U \cap V, w) &= \left\langle w_1, \cdots, w_p \,\middle|\, \gamma_1, \cdots, \gamma_q \right\rangle \end{aligned}$$

then

$$\begin{aligned} \pi_1(X, w) = \langle u_1, \cdots, u_k, v_1, \cdots, v_m \rangle \\ \mod \left\langle \alpha_1, \cdots, \alpha_l, \beta_1, \cdots, \beta_n, I\left(w_1\right) J\left(w_1\right)^{-1}, \cdots, I\left(w_p\right) J\left(w_p\right)^{-1} \right\rangle \\ = \frac{\pi_1(U) * \pi_1(B)}{\left\langle \left\{ I(w_i) J(w_i)^{-1} \,\middle|\, 1 \leq i \leq p \right\} \right\rangle} \end{aligned}$$

where

$$\begin{aligned} I &: \pi_1(U \cap V, w) \to \pi_1(U, w) \\ J &: \pi_1(U \cap V, w) \to \pi_1(V, w). \end{aligned}$$