

Real Analysis Qualifying Exam Notes

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Contents

| | | |
|----------|---|-----------|
| 1 | Practice Exam 2 (November 2014) | 2 |
| 1.1 | 1: Fubini-Tonelli | 2 |
| 1.1.1 | a | 2 |
| 1.1.2 | b | 2 |
| 1.2 | 2: Convolutions and the Fourier Transform | 2 |
| 1.2.1 | a | 2 |
| 1.2.2 | b | 2 |
| 1.2.3 | c | 3 |
| 1.3 | 3: Hilbert Spaces | 3 |
| 1.3.1 | a | 3 |
| 1.3.2 | b | 3 |
| 1.3.3 | c | 4 |
| 1.4 | 4: L_p Spaces | 4 |
| 1.4.1 | a | 4 |
| 1.4.2 | c | 6 |
| 1.5 | 5: Dual Spaces | 7 |
| 1.5.1 | a | 7 |
| 1.5.2 | b | 7 |
| 1.5.3 | c | 7 |
| 2 | Qual: Fall 2019 | 7 |
| 2.1 | 1 | 7 |
| 2.2 | 2 | 7 |
| 2.3 | 3 | 7 |
| 2.4 | 4 | 7 |
| 2.5 | 5 | 8 |
| 3 | Basics | 8 |
| 4 | Uniform Convergence | 10 |
| 5 | Measure Theory | 12 |
| 6 | Integration | 16 |
| 6.1 | Convergence Theorems | 16 |

| | | |
|-----------|---|-----------|
| 6.2 | L^1 Facts | 19 |
| 6.3 | L^p Spaces | 22 |
| 7 | Fourier Transform and Convolution | 22 |
| 7.1 | The Fourier Transform | 22 |
| 7.2 | Approximate Identities | 24 |
| 8 | Functional Analysis | 26 |
| 8.1 | Definitions | 26 |
| 8.2 | Theorems | 27 |
| 9 | Extra Problems | 30 |
| 10 | Inequalities and Equalities | 33 |
| 10.1 | Less Explicitly Used Inequalities | 35 |

1 Practice Exam 2 (November 2014)

1.1 1: Fubini-Tonelli

1.1.1 a

Carefully state Tonelli's theorem for a nonnegative function $F(x, t)$ on $\mathbb{R}^n \times \mathbb{R}$.

1.1.2 b

Let $f : \mathbb{R}^n \rightarrow [0, \infty]$ and define

$$\mathcal{A} := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid 0 \leq t \leq f(x)\}.$$

Prove the validity of the following two statements:

1. f is Lebesgue measurable on $\mathbb{R}^n \iff \mathcal{A}$ is a Lebesgue measurable subset of \mathbb{R}^{n+1} .
2. If f is Lebesgue measurable on \mathbb{R}^n then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x) dx = \int_0^\infty m(\{x \in \mathbb{R}^n \mid f(x) \geq t\}) dt.$$

1.2 2: Convolutions and the Fourier Transform

1.2.1 a

Let $f, g \in L^1(\mathbb{R}^n)$ and give a definition of $f * g$.

1.2.2 b

Prove that if f, g are integrable and bounded, then

$$(f * g)(x) \xrightarrow{|x| \rightarrow \infty} 0.$$

1.2.3 c

1. Define the *Fourier transform* of an integrable function f on \mathbb{R}^n .
2. Give an outline of the proof of the Fourier inversion formula.
3. Give an example of a function $f \in L^1(\mathbb{R}^n)$ such that \hat{f} is not in $L^1(\mathbb{R}^n)$.

1.3 3: Hilbert Spaces

Let $\{u_n\}_{n=1}^\infty$ be an orthonormal sequence in a Hilbert space H .

1.3.1 a

Let $x \in H$ and verify that

$$\left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|_H^2 = \|x\|_H^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2.$$

for any $N \in \mathbb{N}$ and deduce that

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|_H^2.$$

1.3.2 b

Let $\{a_n\}_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$ and prove that there exists an $x \in H$ such that $a_n = \langle x, u_n \rangle$ for all $n \in \mathbb{N}$, and moreover x may be chosen such that

$$\|x\|_H = \left(\sum_{n \in \mathbb{N}} |a_n|^2 \right)^{\frac{1}{2}}.$$

Proof .

- Take $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
- Define $x := \lim_{N \rightarrow \infty} S_N$ where $S_N = \sum_{k=1}^N a_k u_k$
- $\{S_N\}$ is Cauchy and H is complete, so $x \in H$.
- By construction,

$$\langle x, u_n \rangle = \left\langle \sum_k a_k u_k, u_n \right\rangle = \sum_k a_k \langle u_k, u_n \rangle = a_n$$

since the u_k are all orthogonal.

- By Pythagoras since the u_k are normal,

$$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2.$$

■

1.3.3 c

Prove that if $\{u_n\}$ is *complete*, Bessel's inequality becomes an equality.

Proof .

Let x and u_n be arbitrary.

$$\begin{aligned}
 \left\langle x - \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k, u_n \right\rangle &= \langle x, u_n \rangle - \left\langle \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k, u_n \right\rangle \\
 &= \langle x, u_n \rangle - \sum_{k=1}^{\infty} \langle \langle x, u_k \rangle u_k, u_n \rangle \\
 &= \langle x, u_n \rangle - \sum_{k=1}^{\infty} \langle x, u_k \rangle \langle u_k, u_n \rangle \\
 &= \langle x, u_n \rangle - \langle x, u_n \rangle = 0 \\
 \implies x - \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k &= 0 \quad \text{by completeness.}
 \end{aligned}$$

So

$$x = \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k \implies \|x\|^2 = \sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2. \blacksquare$$

■

1.4 4: Lp Spaces

1.4.1 a

Theorem (Holder's Inequality):

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Proof:

It suffices to show this when $\|f\|_p = \|g\|_q = 1$, since

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \iff \int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \leq 1.$$

Using $AB \leq \frac{1}{p}A^p + \frac{1}{q}B^q$, we have

$$\int |f| |g| \leq \int \frac{|f|^p}{p} \frac{|g|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1. \blacksquare$$

Theorem (Minkowski's Inequality):

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof:

We first note

$$|f + g|^p = |f + g| |f + g|^{p-1} \leq (|f| + |g|) |f + g|^{p-1}.$$

Note that if p, q are conjugate exponents then

$$\begin{aligned} \frac{1}{q} &= 1 - \frac{1}{p} = \frac{p-1}{p} \\ q &= \frac{p}{p-1}. \end{aligned}$$

Then taking integrals yields

$$\begin{aligned} \|f + g\|_p^p &= \int |f + g|^p \\ &\leq \int (|f| + |g|) |f + g|^{p-1} \\ &= \int |f| |f + g|^{p-1} + \int |g| |f + g|^{p-1} \\ &= \|f(f + g)^{p-1}\|_1 + \|g(f + g)^{p-1}\|_1 \\ &\leq \|f\|_p \|(f + g)^{p-1}\|_q + \|g\|_p \|(f + g)^{p-1}\|_q \\ &= (\|f\|_p + \|g\|_p) \|(f + g)^{p-1}\|_q \\ &= (\|f\|_p + \|g\|_p) \left(\int |f + g|^{(p-1)q} \right)^{\frac{1}{q}} \\ &= (\|f\|_p + \|g\|_p) \left(\int |f + g|^p \right)^{1 - \frac{1}{p}} \\ &= (\|f\|_p + \|g\|_p) \frac{\int |f + g|^p}{(\int |f + g|^p)^{\frac{1}{p}}} \\ &= (\|f\|_p + \|g\|_p) \frac{\|f + g\|_p^p}{\|f + g\|_p} \end{aligned}$$

and canceling common terms yields

$$\begin{aligned} 1 &\leq (\|f\|_p + \|g\|_p) \frac{1}{\|f + g\|_p} \\ \implies \|f + g\|_p &\leq \|f\|_p + \|g\|_p \blacksquare. \end{aligned}$$

1.4.2 c**Definition (Infinity Norm):**

$$L^\infty(X) = \{f : X \longrightarrow \mathbb{C} \mid \|f\|_\infty < \infty\}$$

where

$$\|f\|_\infty = \inf_{\alpha \geq 0} \left\{ \alpha \mid m\{|f| \geq \alpha\} = 0 \right\}.$$

Theorem:

$$m(X) < \infty \implies \lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty.$$

Proof: Let $M = \|f\|_\infty$. For any $L < M$, let $S = \{|f| \geq L\}$. Then $m(S) > 0$ and

$$\begin{aligned} \|f\|_p &= \left(\int_X |f|^p \right)^{\frac{1}{p}} \\ &\geq \left(\int_S |f|^p \right)^{\frac{1}{p}} \\ &\geq L m(S)^{\frac{1}{p}} \xrightarrow{p \rightarrow \infty} L \\ &\implies \liminf_p \|f\|_p \geq M. \end{aligned}$$

We also have

$$\begin{aligned} \|f\|_p &= \left(\int_X |f|^p \right)^{\frac{1}{p}} \\ &\leq \left(\int_X M^p \right)^{\frac{1}{p}} \\ &= M m(X)^{\frac{1}{p}} \xrightarrow{p \rightarrow \infty} M \\ &\implies \limsup_p \|f\|_p \leq M \blacksquare. \end{aligned}$$

Note: this doesn't help with this problem at all.

Solution:

$$\begin{aligned} \int f^p &= \int_{x \leq 1} f^p + \int_{x=1} f^p + \int_{x \geq 1} f^p \\ &= \int_{x \leq 1} f^p + \int_{x=1} 1 + \int_{x \geq 1} f^p \\ &= \int_{x \leq 1} f^p + m(\{f = 1\}) + \int_{x \geq 1} f^p \\ &\xrightarrow{p \rightarrow \infty} 0 + m(\{f = 1\}) + \begin{cases} 0 & m(\{x \geq 1\}) = 0 \\ \infty & m(\{x \geq 1\}) > 0. \end{cases} \end{aligned}$$

1.5 5: Dual Spaces

Let X be a normed vector space.

1.5.1 a

Give the definition of what it means for a map $L : X \rightarrow \mathbb{C}$ to be a *linear functional*.

1.5.2 b

Prove Minkowski's Inequality:

$$1 \leq p < \infty \implies \|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

1.5.3 c**2 Qual: Fall 2019****2.1 1**

See phone photo?

2.2 2

DCT?

2.3 3

"Follow your nose."

2.4 4

See Problem Set 8.

Bessel's Inequality: For any orthonormal set in a Hilbert space (not necessarily a basis), we have

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

Proof:

$$0 \leq \left\| x - \sum_{k=1}^n \langle x, e_k \rangle e_k \right\|^2$$

Corollary (Parseval's Identity): If $\text{span}\{u_n\}$ is dense in \mathcal{H} , so it is a basis, then this is an equality.

Riesz-Fischer: Let $U = \{u_n\}_{n=1}^{\infty}$ be an orthonormal set (not necessarily a basis), then

1. There is an isometric surjection

$$\begin{aligned}\mathcal{H} &\longrightarrow \ell^2(\mathbb{N}) \\ \mathbf{x} &\mapsto \{\langle \mathbf{x}, \mathbf{u}_n \rangle\}_{n=1}^{\infty}\end{aligned}$$

i.e. if $\{a_n\} \in \ell^2(\mathbb{N})$, so $\sum |a_n|^2 < \infty$, then there exists a $\mathbf{x} \in \mathcal{H}$ such that

$$a_n = \langle \mathbf{x}, \mathbf{u}_n \rangle \quad \forall n.$$

2. \mathbf{x} can be chosen such that

$$\|\mathbf{x}\|^2 = \sum |a_n|^2$$

Note: the choice of \mathbf{x} is unique $\iff \{u_n\}$ is **complete**, i.e. $\langle \mathbf{x}, \mathbf{u}_n \rangle = 0$ for all n implies $\mathbf{x} = \mathbf{0}$.

Proof:

- Given $\{a_n\}$, define $S_N = \sum_{n=1}^N a_n \mathbf{u}_n$.
- S_N is Cauchy in \mathcal{H} and so $S_N \longrightarrow \mathbf{x}$ for some $\mathbf{x} \in \mathcal{H}$.
- $\langle x, u_n \rangle = \langle x - S_N, u_n \rangle + \langle S_N, u_n \rangle \longrightarrow a_n$
- By construction, $\|x - S_N\|^2 = \|x\|^2 - \sum_{n=1}^N |a_n|^2 \longrightarrow 0$, so $\|x\|^2 = \sum_{n=1}^{\infty} |a_n|^2$.

2.5 5

See Problem Set 5.

Heine-Cantor theorem: Every continuous function on a compact set is uniformly continuous.

Uniform continuity:

$$\begin{aligned} &\forall \varepsilon \quad \exists \delta(\varepsilon) \mid \quad \forall x, y, \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \\ \iff &\forall \varepsilon \quad \exists \delta(\varepsilon) \mid \quad \forall x, y, \quad |y| < \delta \implies |f(x - y) - f(y)| < \varepsilon \end{aligned}$$

Fubini-Tonelli interchange of integrals, where the change of bounds becomes very important.

Continuity in L^1 :

$$\lim_{y \rightarrow 0} \|\tau_y f - f\|_1 = 0.$$

3 Basics

Useful Technique: $\lim f_n = \limsup f_n = \liminf f_n$ iff the limit exists, so $\limsup f_n \leq g \leq \liminf f_n$ implies that $g = \lim f$. Similarly, a limit does not exist iff $\liminf f_n > \limsup f_n$.

Lemma 3.1 (*Convergent Sums Have Small Tails*).

$$\sum a_n < \infty \implies a_n \longrightarrow 0 \quad \text{and} \quad \sum_{k=N}^{\infty} a_k \xrightarrow{N \rightarrow \infty} 0$$

Theorem 3.2 (*Heine-Borel*).

$X \subseteq \mathbb{R}^n$ is compact $\iff X$ is closed and bounded.

Lemma 3.3 (*Geometric Series*).

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \iff |x| < 1.$$

Corollary: $\sum_{k=0}^{\infty} \frac{1}{2^k} = 1.$

Definition 3.3.1.

A set S is **nowhere dense** iff the closure of S has empty interior iff every interval contains a subinterval that does not intersect S .

Definition 3.3.2.

A set is **meager** if it is a *countable* union of nowhere dense sets.

Note that a *finite* union of nowhere dense is still nowhere dense.

Lemma 3.4.

The Cantor set is closed with empty interior.

Proof .

Its complement is a union of open intervals, and can't contain an interval since intervals have positive measure and $m(C_n)$ tends to zero.

Corollary: The Cantor set is nowhere dense. ■

Definition 3.4.1.

An F_σ set is a union of closed sets, and a G_δ set is an intersection of opens.

Mnemonic: “F” stands for ferme, which is “closed” in French, and σ corresponds to a “sum”, i.e. a union.

Lemma 3.5.

Singleton sets in \mathbb{R} are closed, and thus \mathbb{Q} is an F_σ set.

Theorem 3.6 (Baire).

\mathbb{R} is a **Baire space** (countable intersections of open, dense sets are still dense). Thus \mathbb{R} can not be written as a countable union of nowhere dense sets.

Lemma 3.7.

There is a function discontinuous precisely on \mathbb{Q} .

Proof.

$f(x) = \frac{1}{n}$ if $x = r_n \in \mathbb{Q}$ is an enumeration of the rationals, and zero otherwise. The limit at every point is 0. ■

Lemma 3.8.

There *do not* exist functions that are discontinuous precisely on $\mathbb{R} \setminus \mathbb{Q}$.

Proof.

D_f is always an F_σ set, which follows by considering the oscillation ω_f . $\omega_f(x) = 0 \iff f$ is continuous at x , and $D_f = \bigcup_n A_{\frac{1}{n}}$ where $A_\varepsilon = \{\omega_f \geq \varepsilon\}$ is closed. ■

Lemma 3.9.

Any nonempty set which is bounded from above (resp. below) has a well-defined supremum (resp. infimum).

4 Uniform Convergence

Theorem 4.1 (Egorov).

Let $E \subseteq \mathbb{R}^n$ be measurable with $m(E) > 0$ and $\{f_k : E \rightarrow \mathbb{R}\}$ be measurable functions such that

$$f(x) := \lim_{k \rightarrow \infty} f_k(x) < \infty$$

exists almost everywhere.

Then $f_k \rightarrow f$ *almost uniformly*, i.e.

$$\forall \varepsilon > 0, \exists F \subseteq E \text{ closed such that } m(E \setminus F) < \varepsilon \text{ and } f_k \xrightarrow{u} f \text{ on } F.$$

Proposition 4.2.

The space $X = C([0, 1])$, continuous functions $f : [0, 1] \rightarrow \mathbb{R}$, equipped with the norm $\|f\| = \sup_{x \in [0, 1]} |f(x)|$, is a **complete** metric space.

Proof .

1. Let $\{f_k\}$ be Cauchy in X .
2. Define a candidate limit using pointwise convergence:

Fix an x ; since

$$|f_k(x) - f_j(x)| \leq \|f_k - f_j\| \longrightarrow 0$$

the sequence $\{f_k(x)\}$ is Cauchy in \mathbb{R} . So define $f(x) := \lim_k f_k(x)$.

3. Show that $\|f_k - f\| \longrightarrow 0$:

$$|f_k(x) - f_j(x)| < \varepsilon \quad \forall x \implies \lim_j |f_k(x) - f_j(x)| < \varepsilon \quad \forall x$$

Alternatively, $\|f_k - f\| \leq \|f_k - f_N\| + \|f_N - f_j\|$, where N, j can be chosen large enough to bound each term by $\varepsilon/2$.

4. Show that $f \in X$:

The uniform limit of continuous functions is continuous. (Note: in other cases, you may need to show the limit is bounded, or has bounded derivative, or whatever other conditions define X .)

■

Lemma 4.3.

Metric spaces are compact iff they are sequentially compact, (i.e. every sequence has a convergent subsequence).

Proposition 4.4.

The unit ball in $C([0, 1])$ with the sup norm is not compact.

Proof .

Take $f_k(x) = x^n$, which converges to a dirac delta at 1. The limit is not continuous, so no subsequence can converge.

■

Lemma 4.5.

A uniform limit of continuous functions is continuous.

Lemma 4.6 (Testing Uniform Convergence).

$f_n \longrightarrow f$ uniformly iff there exists an M_n such that $\|f_n - f\|_\infty \leq M_n \longrightarrow 0$.

Negating: find an x which depends on n for which the norm is bounded below.

Useful Technique: If f_n has a global maximum (computed using f'_n and the first derivative test) $M_n \longrightarrow 0$, then $f_n \longrightarrow 0$ uniformly.

Lemma 4.7 (Baby Commuting Limits with Integrals).

If $f_n \longrightarrow f$ uniformly, then $\int f_n = \int f$.

Lemma 4.8 (Uniform Convergence and Derivatives).

If $f'_n \rightarrow g$ uniformly for some g and $f_n \rightarrow f$ pointwise (or at least at one point), then $g = f'$.

Lemma 4.9 (Uniform Convergence of Series).

If $f_n(x) \leq M_n$ for a fixed x where $\sum M_n < \infty$, then the series $f(x) = \sum f_n(x)$ converges pointwise.

Lemma 4.10.

If $\sum f_n$ converges then $f_n \rightarrow 0$ uniformly.

Useful Technique: For a fixed x , if $f = \sum f_n$ converges *uniformly* on some $B_r(x)$ and each f_n is continuous at x , then f is also continuous at x .

Lemma 4.11 (M-test for Series).

If $|f_n(x)| \leq M_n$ which does not depend on x , then $\sum f_n$ converges uniformly.

Lemma 4.12 (p-tests).

Let n be a fixed dimension and set $B = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.

$$\begin{aligned}\sum \frac{1}{n^p} < \infty &\iff p > 1 \\ \int_{\varepsilon}^{\infty} \frac{1}{x^p} < \infty &\iff p > 1 \\ \int_0^1 \frac{1}{x^p} < \infty &\iff p < 1 \\ \int_B \frac{1}{|x|^p} < \infty &\iff p < n \\ \int_{B^c} \frac{1}{|x|^p} < \infty &\iff p > n\end{aligned}$$

Proposition 4.13.

A function $f : (a, b) \rightarrow \mathbb{R}$ is Lipschitz $\iff f$ is differentiable and f' is bounded. In this case, $|f'(x)| \leq C$, the Lipschitz constant.

Proposition 4.14.

There exist smooth compactly supported functions, e.g. take

$$f(x) = e^{\frac{-1}{x^2}} \chi_{(0, \infty)}(x).$$

5 Measure Theory

Useful Technique: $s = \inf \{x \in X\} \implies$ for every ε there is an $x \in X$ such that $x \leq s + \varepsilon$.

Useful Techniques: Always consider bounded sets, and if E is unbounded write $E = \bigcup_n B_n(0) \cap E$ and use countable subadditivity or continuity of measure.

Lemma 5.1.

Every open subset of \mathbb{R} (resp \mathbb{R}^n) can be written as a unique countable union of disjoint (resp. almost disjoint) intervals (resp. cubes).

Definition 5.1.1.

The outer measure of a set is given by

$$m_*(E) = \inf_{\substack{\{Q_i\} \rightrightarrows E \\ \text{closed cubes}}} \sum |Q_i|.$$

Lemma 5.2 (Properties of Outer Measure).

- Monotonicity: $E \subseteq F \implies m_*(E) \leq m_*(F)$.
- Countable Subadditivity: $m_*(\bigcup E_i) \leq \sum m_*(E_i)$.
- Approximation: For all E there exists a $G \supseteq E$ such that $m_*(G) \leq m_*(E) + \varepsilon$.
- Disjoint^a Additivity: $m_*(A \coprod B) = m_*(A) + m_*(B)$.

^aThis holds for outer measure iff $\text{dist}(A, B) > 0$.

Lemma 5.3 (Subtraction of Measure).

$$m(A) = m(B) + m(C) \quad \text{and} \quad m(C) < \infty \implies m(A) - m(C) = m(B).$$

Lemma 5.4 (Continuity of Measure).

$$\begin{aligned} E_i \nearrow E &\implies m(E_i) \longrightarrow m(E) \\ m(E_1) < \infty \text{ and } E_i \searrow E &\implies m(E_i) \longrightarrow m(E). \end{aligned}$$

Proof.

1. Break into disjoint annuli $A_2 = E_2 \setminus E_1$, etc then apply countable disjoint additivity to $E = \coprod A_i$.
2. Use $E_1 = (\coprod E_j \setminus E_{j+1}) \coprod (\bigcap E_j)$, taking measures yields a telescoping sum, and use countable disjoint additivity. ■

Theorem 5.5.

Suppose E is measurable; then for every $\varepsilon > 0$,

1. There exists an open $O \supset E$ with $m(O \setminus E) < \varepsilon$
2. There exists a closed $F \subset E$ with $m(E \setminus F) < \varepsilon$

3. There exists a compact $K \subset E$ with $m(E \setminus K) < \varepsilon$.

Proof .

- (1): Take $\{Q_i\} \rightrightarrows E$ and set $O = \bigcup Q_i$.
- (2): Since E^c is measurable, produce $O \supset E^c$ with $m(O \setminus E^c) < \varepsilon$.
 - Set $F = O^c$, so F is closed.
 - Then $F \subset E$ by taking complements of $O \supset E^c$
 - $E \setminus F = O \setminus E^c$ and taking measures yields $m(E \setminus F) < \varepsilon$
- (3): Pick $F \subset E$ with $m(E \setminus F) < \varepsilon/2$.
 - Set $K_n = F \cap \mathbb{D}_n$, a ball of radius n about 0.
 - Then $E \setminus K_n \searrow E \setminus F$
 - Since $m(E) < \infty$, there is an N such that $n \geq N \implies m(E \setminus K_n) < \varepsilon$.

■

Lemma 5.6.

Lebesgue measure is translation and dilation invariant.

Proof .

Obvious for cubes; if $Q_i \rightrightarrows E$ then $Q_i + k \rightrightarrows E + k$, etc.

■

Theorem 5.7 (Non-Measurable Sets).

There is a non-measurable set.

Proof .

- Use AOC to choose one representative from every coset of \mathbb{R}/\mathbb{Q} on $[0, 1]$, which is countable, and assemble them into a set N
- Enumerate the rationals in $[0, 1]$ as q_j , and define $N_j = N + q_j$. These intersect trivially.
- Define $M := \bigsqcup N_j$, then $[0, 1] \subseteq M \subseteq [-1, 2]$, so the measure must be between 1 and 3. By translation invariance, $m(N_j) = m(N)$, and disjoint additivity forces $m(M) = 0$, a contradiction.

■

Proposition 5.8 (Borel Characterization of Measurable Sets).

If E is Lebesgue measurable, then $E = H \bigsqcup N$ where $H \in F_\sigma$ and N is null.

Useful technique: F_σ sets are Borel, so establish something for Borel sets and use this to extend it to Lebesgue.

Proof .

For every $\frac{1}{n}$ there exists a closed set $K_n \subset E$ such that $m(E \setminus K_n) \leq \frac{1}{n}$. Take $K = \bigcup K_n$, wlog $K_n \nearrow K$ so $m(K) = \lim m(K_n) = m(E)$. Take $N := E \setminus K$, then $m(N) = 0$.

■

Definition 5.8.1.

$$\limsup_n A_n := \bigcap_n \bigcup_{j \geq n} A_j = \left\{ x \mid x \in A_n \text{ for inf. many } n \right\}$$
$$\liminf_n A_n := \bigcup_n \bigcap_{j \geq n} A_j = \left\{ x \mid x \in A_n \text{ for all except fin. many } n \right\}$$

Lemma 5.9.

If A_n are all measurable, $\limsup A_n$ and $\liminf A_n$ are measurable.

Proof .

Measurable sets form a sigma algebra, and these are expressed as countable unions/intersections of measurable sets. ■

Theorem 5.10 (Borel-Cantelli).

Let $\{E_k\}$ be a countable collection of measurable sets. Then

$$\sum_k m(E_k) < \infty \implies \text{almost every } x \in \mathbb{R} \text{ is in at most finitely many } E_k.$$

Proof .

- If $E = \limsup_j E_j$ with $\sum m(E_j) < \infty$ then $m(E) = 0$.
- If E_j are measurable, then $\limsup_j E_j$ is measurable.
- If $\sum_j m(E_j) < \infty$, then $\sum_{j=N}^{\infty} m(E_j) \xrightarrow{N \rightarrow \infty} 0$ as the tail of a convergent sequence.
- $E = \limsup_j E_j = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j \implies E \subseteq \bigcup_{j=k}^{\infty} E_j$ for all k
- $E \subseteq \bigcup_{j=k}^{\infty} E_j \implies m(E) \leq \sum_{j=k}^{\infty} m(E_j) \xrightarrow{k \rightarrow \infty} 0$. ■

Lemma 5.11.

- Characteristic functions are measurable
- If f_n are measurable, so are $|f_n|$, $\limsup f_n$, $\liminf f_n$, $\lim f_n$,
- Sums and differences of measurable functions are measurable,
- Cones $F(x, y) = f(x)$ are measurable,
- Compositions $f \circ T$ for T a linear transformation are measurable,
- “Convolution-ish” transformations $(x, y) \mapsto f(x - y)$ are measurable

Proof (Convolution).

Take the cone on f to get $F(x, y) = f(x)$, then compose F with the linear transformation $T = [1, -1; 1, 0]$. ■

6 Integration

Definition 6.0.1.

$f \in L^+$ iff f is measurable and non-negative.

Useful techniques:

- Break integration domain up into disjoint annuli.
- Break real integrals up into $x < 1$ and $x > 1$.

Definition 6.0.2.

A measurable function is integrable iff $\|f\|_1 < \infty$.

Useful facts about C_c functions:

- Bounded almost everywhere
- Uniformly continuous

Theorem (p-Test for Integrals) :

$$\begin{aligned}\int_0^1 x^{-p} < \infty &\iff p < 1 \\ \int_1^\infty x^{-p} < \infty &\iff p > 1.\end{aligned}$$

Yields a general technique: break integrals apart at $x = 1$.

6.1 Convergence Theorems

Theorem 6.1 (Monotone Convergence).

If $f_n \in L^+$ and $f_n \nearrow f$ a.e., then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \quad \int f_n \longrightarrow \int f.$$

Needs to be positive and increasing.

Theorem 6.2 (Dominated Convergence).

If $f_n \in L^1$ and $f_n \longrightarrow f$ a.e. with $|f_n| \leq g$ for some $g \in L^1$, then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \quad \int f_n \longrightarrow \int f,$$

and more generally,

$$\int |f_n - f| \longrightarrow 0.$$

Positivity *not* needed.

Generalized DCT: can relax $|f_n| < g$ to $|f_n| < g_n \longrightarrow g \in L^1$.

Lemma 6.3.

If $f \in L^1$, then

$$\int |f_n - f| \longrightarrow 0 \iff \int |f_n| \longrightarrow \int |f|.$$

Proof.

Let $g_n = |f_n| - |f_n - f|$, then $g_n \longrightarrow |f|$ and

$$|g_n| = ||f_n| - |f_n - f|| \leq |f_n - (f_n - f)| = |f| \in L^1,$$

so the DCT applies to g_n and

$$\begin{aligned} \|f_n - f\|_1 &= \int |f_n - f| + |f_n| - |f_n| = \int |f_n| - g_n \\ &\longrightarrow_{DCT} \lim \int |f_n| - \int |f|. \end{aligned}$$

■

Fatou's Lemma If $f_n \in L^+$, then

$$\begin{aligned} \int \liminf_n f_n &\leq \liminf_n \int f_n \\ \limsup_n \int f_n &\leq \int \limsup_n f_n. \end{aligned}$$

Only need positivity.

Theorem 6.4(Tonelli).

For $f(x, y)$ **non-negative and measurable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is a **measurable** function
- $F(x) = \int f(x, y) dy$ is a **measurable** function,
- For E measurable, the slices $E_x := \{y \mid (x, y) \in E\}$ are measurable.
- $\int f = \int \int F$, i.e. any iterated integral is equal to the original.

Theorem 6.5(Fubini).

For $f(x, y)$ **integrable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is an **integrable** function

- $F(x) := \int f(x, y) dy$ is an **integrable** function,
- For E measurable, the slices $E_x := \{y \mid (x, y) \in E\}$ are measurable.
- $\int f = \int \int f(x, y)$, i.e. any iterated integral is equal to the original

Theorem 6.6 (Fubini/Tonelli).

If any iterated integral is **absolutely integrable**, i.e. $\int \int |f(x, y)| < \infty$, then f is integrable and $\int f$ equals any iterated integral.

Corollary 6.7 (Measurable Slices).

Let E be a measurable subset of \mathbb{R}^n . Then

- For almost every $x \in \mathbb{R}^{n_1}$, the slice $E_x := \{y \in \mathbb{R}^{n_2} \mid (x, y) \in E\}$ is measurable in \mathbb{R}^{n_2} .
- The function

$$F : \mathbb{R}^{n_1} \longrightarrow \mathbb{R}$$

$$x \mapsto m(E_x) = \int_{\mathbb{R}^{n_2}} \chi_{E_x} dy$$

is measurable and

$$m(E) = \int_{\mathbb{R}^{n_1}} m(E_x) dx = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{E_x} dy dx$$

Proof (Measurable Slices).

$\implies :$

- Let f be measurable on \mathbb{R}^n .
- Then the cylinders $F(x, y) = f(x)$ and $G(x, y) = f(y)$ are both measurable on \mathbb{R}^{n+1} .
- Write $\mathcal{A} = \{G \leq F\} \cap \{G \geq 0\}$; both are measurable.

$\impliedby :$

- Let A be measurable in \mathbb{R}^{n+1} .
- Define $A_x = \{y \in \mathbb{R} \mid (x, y) \in A\}$, then $m(A_x) = f(x)$.
- By the corollary, A_x is measurable set, $x \mapsto m(A_x)$ is a measurable function, and $m(A) = \int f(x) dx$.
- Then explicitly, $f(x) = \chi_A$, which makes f a measurable function. ■

Proposition 6.8 (Differentiating Under an Integral).

If $\left| \frac{\partial}{\partial t} f(x, t) \right| \leq g(x) \in L^1$, then letting $F(t) = \int f(x, t) dt$,

$$\begin{aligned} \frac{\partial}{\partial t} F(t) &:= \lim_{h \rightarrow 0} \int \frac{f(x, t+h) - f(x, t)}{h} dx \\ &\stackrel{\text{DCT}}{=} \int \frac{\partial}{\partial t} f(x, t) dx. \end{aligned}$$

To justify passing the limit, let $h_k \rightarrow 0$ be any sequence and define

$$f_k(x, t) = \frac{f(x, t + h_k) - f(x, t)}{h_k},$$

so $f_k \xrightarrow{\text{pointwise}} \frac{\partial}{\partial t} f$.

Apply the MVT to f_k to get $f_k(x, t) = f_k(\xi, t)$ for some $\xi \in [0, h_k]$, and show that $f_k(\xi, t) \in L^1$.

Proposition 6.9 (Swapping Sum and Integral).

If f_n are non-negative and $\sum \int |f|_n < \infty$, then $\sum \int f_n = \int \sum f_n$.

Proof.

MCT. Let $F_N = \sum_{n=1}^N f_n$ be a finite partial sum; then there are simple functions $\varphi_n \nearrow f_n$ and so $\sum_{n=1}^N \varphi_n \nearrow F_N$, so apply MCT. ■

Lemma 6.10.

If $f_k \in L^1$ and $\sum \|f_k\|_1 < \infty$ then $\sum f_k$ converges almost everywhere and in L^1 .

Proof.

Define $F_N = \sum_{k=1}^N f_k$ and $F = \lim_N F_N$, then $\|F_N\|_1 \leq \sum_{k=1}^N \|f_k\|_1 < \infty$ so $F \in L^1$ and $\|F_N - F\|_1 \rightarrow 0$ so the sum converges in L^1 . Almost everywhere convergence: ? ■

6.2 L^1 Facts

Lemma 6.11 (Translation Invariance).

The Lebesgue integral is translation invariant, i.e. $\int f(x) dx = \int f(x+h) dx$ for any h .

Proof.

- For characteristic functions, $\int_E f(x+h) = \int_{E+h} f(x) = m(E+h) = m(E) = \int_E f$ by translation invariance of measure.

- So this also holds for simple functions by linearity
- For $f \in L^+$, choose $\varphi_n \nearrow f$ so $\int \varphi_n \rightarrow \int f$.
- Similarly, $\tau_h \varphi_n \nearrow \tau_h f$ so $\int \tau_h f \rightarrow \int f$
- Finally $\left\{ \int \tau_h \varphi \right\} = \left\{ \int \varphi \right\}$ by step 1, and the suprema are equal by uniqueness of limits. ■

Lemma 6.12 (Integrals Distribute Over Disjoint Sets).

If $X \subseteq A \cup B$, then $\int_X f \leq \int_A f + \int_{A^c} f$ with equality iff $X = A \sqcup B$.

Lemma 6.13 (Unif Cts L^1 Functions Decay Rapidly).

If $f \in L^1$ and f is uniformly continuous, then $f(x) \xrightarrow{|x| \rightarrow \infty} 0$.

Doesn't hold for general L^1 functions, take any train of triangles with height 1 and summable areas.

Lemma 6.14 (L^1 Functions Have Small Tails).

If $f \in L^1$, then for every ε there exists a radius R such that if $A = B_R(0)^c$, then $\int_A |f| < \varepsilon$.

Proof.

Approximate with compactly supported functions. Take $g \xrightarrow{L^1} f$ with $g \in C_c$, then choose N large enough so that $g = 0$ on $E := B_N(0)^c$, then $\int_E |f| \leq \int_E |f - g| + \int_E |g|$. ■

Lemma 6.15 (L^1 Functions Have Absolutely Continuity).

$m(E) \rightarrow 0 \implies \int_E f \rightarrow 0$.

Proof.

Approximate with compactly supported functions. Take $g \xrightarrow{L^1} f$, then $g \leq M$ so $\int_E f \leq \int_E f - g + \int_E g \rightarrow 0 + M \cdot m(E) \rightarrow 0$. ■

Lemma 6.16 (L^1 Functions Are Finite a.e.).

If $f \in L^1$, then $m(\{f(x) = \infty\}) = 0$.

Proof.

Idea: Split up domain Let $A = \{f(x) = \infty\}$, then $\infty > \int f = \int_A f + \int_{A^c} f = \infty \cdot m(A) + \int_{A^c} f \implies m(A) = 0$.

■

Proposition 6.17 (Continuity in L^1).

$$\|\tau_h f - f\|_1 \xrightarrow{h \rightarrow 0} 0$$

Proof.

Approximate with compactly supported functions. Take $g \xrightarrow{L^1} f$ with $g \in C_c$.

$$\begin{aligned} & \int f(x+h) - f(x) \leq \\ & \int f(x+h) - g(x+h) + \int g(x+h) - g(x) + \int g(x) - f(x) \\ & \longrightarrow 2\varepsilon + \int g(x+h) - g(x) \\ & = \int_K g(x+h) - g(x) + \int_{K^c} g(x+h) - g(x) \longrightarrow 0, \end{aligned}$$

which follows because we can enlarge the support of g to K where the integrand is zero on K^c , then apply uniform continuity on K .

■

Proposition 6.18 (Integration by Parts, Special Case).

$$\begin{aligned} F(x) &:= \int_0^x f(y)dy \quad \text{and} \quad G(x) := \int_0^x g(y)dy \\ \implies \int_0^1 F(x)g(x)dx &= F(1)G(1) - \int_0^1 f(x)G(x)dx. \end{aligned}$$

Proof.

Fubini-Tonelli, and sketch region to change integration bounds.

■

Theorem 6.19 (Lebesgue Density).

$$A_h(f)(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(y)dy \implies \|A_h(f) - f\| \xrightarrow{h \rightarrow 0} 0.$$

Proof.

Fubini-Tonelli, and sketch region to change integration bounds, and continuity in L^1 .

■

6.3 L^p Spaces

Lemma 6.20.

The following are dense subspaces of $L^2([0, 1])$:

- Simple functions
- Step functions
- $C_0([0, 1])$
- Smoothly differentiable functions $C_0^\infty([0, 1])$
- Smooth compactly supported functions C_c^∞

Theorem 6.21 (Dual L^p Spaces).

For $p \neq \infty$, $(L^p)^\vee \cong L^q$.

Proof ($p=1$).

?

■

Proof ($p=2$) > Use Riesz Representation for Hilbert spaces.

For the $p = \infty$ case: $L^1 \subset (L^\infty)^\vee$, since the isometric mapping is always injective, but *never* surjective. So this containment is always proper (requires Hahn-Banach Theorem).

7 Fourier Transform and Convolution

7.1 The Fourier Transform

Definition 7.0.1 (Convolution).

$$f * g(x) = \int f(x - y)g(y)dy.$$

Definition 7.0.2 (The Fourier Transform).

$$\hat{f}(\xi) = \int f(x) e^{2\pi i x \cdot \xi} dx.$$

Lemma 7.1.

If $\hat{f} = \hat{g}$ then $f = g$ almost everywhere.

Lemma 7.2 (Riemann-Lebesgue: Fourier transforms have small tails).

$$f \in L^1 \implies \hat{f}(\xi) \rightarrow 0 \text{ as } |\xi| \rightarrow \infty.$$

Lemma 7.3.

If $f \in L^1$, then \hat{f} is continuous and bounded.

Proof .

- Boundedness:

$$|\hat{f}(\xi)| \leq \int |f| \cdot |e^{2\pi i x \cdot \xi}| = \|f\|_1.$$

- Continuity:

– Apply DCT to show $|\hat{f}(\xi_n) - \hat{f}(\xi)| \xrightarrow{n \rightarrow \infty} 0$.

■

Theorem (Fourier Inversion) :

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

Proof .

Idea: Fubini-Tonelli doesn't work directly, so introduce a convergence factor, take limits, and use uniqueness of limits.

■

- Take the modified integral:

$$\begin{aligned} I_t(x) &= \int \hat{f}(\xi) e^{2\pi i x \cdot \xi} e^{-\pi t^2 |\xi|^2} \\ &= \int \hat{f}(\xi) \varphi(\xi) \\ &= \int f(\xi) \hat{\varphi}(\xi) \\ &= \int f(\xi) \hat{g}(\xi - x) \\ &= \int f(\xi) g_t(x - \xi) d\xi \\ &= \int f(y - x) g_t(y) dy \quad (\xi = y - x) \\ &= (f * g_t) \\ &\longrightarrow f \text{ in } L^1 \text{ as } t \longrightarrow 0. \end{aligned}$$

- We also have

$$\begin{aligned}
 \lim_{t \rightarrow 0} I_t(x) &= \lim_{t \rightarrow 0} \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi} e^{-\pi t^2 |\xi|^2} \\
 &= \lim_{t \rightarrow 0} \int \widehat{f}(\xi) \varphi(\xi) \\
 &=_{DCT} \int \widehat{f}(\xi) \lim_{t \rightarrow 0} \varphi(\xi) \\
 &= \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi}
 \end{aligned}$$

- So

$$I_t(x) \longrightarrow \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi} \text{ pointwise and } \|I_t(x) - f(x)\|_1 \longrightarrow 0.$$

- So there is a subsequence I_{t_n} such that $I_{t_n}(x) \longrightarrow f(x)$ almost everywhere
- Thus $f(x) = \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi}$ almost everywhere by uniqueness of limits.

Proposition (Eigenfunction of the Fourier Transform) :

$$\begin{aligned}
 g(x) := e^{-\pi |x|^2} &\implies \widehat{g}(\xi) = g(\xi) \quad \text{and} \\
 \widehat{g}_t(x) = g(tx) &= e^{-\pi t^2 |x|^2}, .
 \end{aligned}$$

Proposition 7.4 (*Properties of the Fourier Transform*).

?????.

7.2 Approximate Identities

Definition 7.4.1 (Dilation).

$$\varphi_t(x) = t^{-n} \varphi(t^{-1}x).$$

Definition 7.4.2 (Approximation to the Identity).

For $\varphi \in L^1$, the dilations satisfy $\int \varphi_t = \int \varphi$, and if $\int \varphi = 1$ then φ is an *approximate identity*.

Example: $\varphi(x) = e^{-\pi x^2}$

Theorem 7.5 (*Convolution Against Approximate Identities Converge in L^1*).

$$\|f * \varphi_t - f\|_1 \xrightarrow{t \rightarrow 0} 0.$$

Proof .

$$\begin{aligned}
\|f - f * \varphi_t\|_1 &= \int f(x) - \int f(x-y)\varphi_t(y) \, dy dx \\
&= \int f(x) \int \varphi_t(y) \, dy - \int f(x-y)\varphi_t(y) \, dy dx \\
&= \int \int \varphi_t(y)[f(x) - f(x-y)] \, dy dx \\
&=_{FT} \int \int \varphi_t(y)[f(x) - f(x-y)] \, dx dy \\
&= \int \varphi_t(y) \int f(x) - f(x-y) \, dx dy \\
&= \int \varphi_t(y) \|f - \tau_y f\|_1 \, dy \\
&= \int_{y < \delta} \varphi_t(y) \|f - \tau_y f\|_1 \, dy + \int_{y \geq \delta} \varphi_t(y) \|f - \tau_y f\|_1 \, dy \\
&\leq \int_{y < \delta} \varphi_t(y) \varepsilon + \int_{y \geq \delta} \varphi_t(y) (\|f\|_1 + \|\tau_y f\|_1) \, dy \quad \text{by continuity in } L^1 \\
&\leq \varepsilon + 2\|f\|_1 \int_{y \geq \delta} \varphi_t(y) \, dy \\
&\leq \varepsilon + 2\|f\|_1 \cdot \varepsilon \quad \text{since } \varphi_t \text{ has small tails} \\
&\xrightarrow{\varepsilon \rightarrow 0} 0.
\end{aligned}$$

■

Theorem 7.6 (Convolutions Vanish at Infinity).

$$f, g \in L^1 \text{ and bounded} \implies \lim_{|x| \rightarrow \infty} (f * g)(x) = 0.$$

Proof .

- Choose $M \geq f, g$.
- By small tails, choose N such that $\int_{B_N^c} |f|, \int_{B_N^c} |g| < \varepsilon$
- Note

$$|f * g| \leq \int |f(x-y)| |g(y)| \, dy := I.$$

- Use $|x| \leq |x-y| + |y|$, take $|x| \geq 2N$ so either

$$|x-y| \geq N \implies I \leq \int_{\{|x-y| \geq N\}} |f(x-y)| M \, dy \leq \varepsilon M \longrightarrow 0$$

then

$$|y| \geq N \implies I \leq \int_{\{|y| \geq N\}} M |g(y)| \, dy \leq M \varepsilon \longrightarrow 0.$$

■

Proposition (Young's Inequality?) :

$$\frac{1}{r} := \frac{1}{p} + \frac{1}{q} - 1 \implies \|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Corollary 7.7.

Take $q = 1$ to obtain

$$\|f * g\|_p \leq \|f\|_p \|g\|_1.$$

Corollary 7.8.

If $f, g \in L^1$ then $f * g \in L^1$.

8 Functional Analysis

8.1 Definitions

Notation: H denotes a Hilbert space.

Definition 8.0.1 (Orthonormal Sequence).

?

Definition 8.0.2 (Basis).

?

Definition 8.0.3 (Complete).

A collection of vectors $\{u_n\} \subset H$ is *complete* iff $\langle x, u_n \rangle = 0$ for all $n \iff x = 0$ in H .

Definition 8.0.4 (Dual Space).

$$X^\vee := \left\{ L : X \longrightarrow \mathbb{C} \mid L \text{ is continuous} \right\}.$$

Definition 8.0.5.

A map $L : X \longrightarrow \mathbb{C}$ is a linear functional iff

$$L(\alpha \mathbf{x} + \mathbf{y}) = \alpha L(\mathbf{x}) + L(\mathbf{y}).$$

Definition 8.0.6 (Operator Norm).

$$\|L\|_{X^\vee} := \sup_{\substack{x \in X \\ \|x\|=1}} |L(x)|.$$

Definition 8.0.7 (Banach Space).

A complete normed vector space.

Definition 8.0.8 (Hilbert Space).

An inner product space which is a Banach space under the induced norm.

8.2 Theorems

Theorem (Bessel's Inequality) :

$$\left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2$$

and thus

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2.$$

Proof .

- Let $S_N = \sum_{n=1}^N \langle x, u_n \rangle u_n$

$$\begin{aligned} \|x - S_N\|^2 &= \langle x - S_N, x - S_N \rangle \\ &= \|x\|^2 + \|S_N\|^2 - 2\Re \langle x, S_N \rangle \\ &= \|x\|^2 + \|S_N\|^2 - 2\Re \left\langle x, \sum_{n=1}^N \langle x, u_n \rangle u_n \right\rangle \\ &= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^N \langle x, \langle x, u_n \rangle u_n \rangle \\ &= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^N \overline{\langle x, u_n \rangle} \langle x, u_n \rangle \\ &= \|x\|^2 + \left\| \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|^2 - 2 \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\ &= \|x\|^2 + \sum_{n=1}^N |\langle x, u_n \rangle|^2 - 2 \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\ &= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2. \end{aligned}$$

- By continuity of the norm and inner product, we have

$$\begin{aligned}
\lim_{N \rightarrow \infty} \|x - S_N\|^2 &= \lim_{N \rightarrow \infty} \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\
\Rightarrow \left\| x - \lim_{N \rightarrow \infty} S_N \right\|^2 &= \|x\|^2 - \lim_{N \rightarrow \infty} \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\
\Rightarrow \left\| x - \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2.
\end{aligned}$$

- Then noting that $0 \leq \|x - S_N\|^2$,

$$\begin{aligned}
0 &\leq \|x\|^2 - \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \\
\Rightarrow \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 &\leq \|x\|^2 \blacksquare.
\end{aligned}$$

■

Theorem 8.1 (Riesz Representation for Hilbert Spaces).

If Λ is a continuous linear functional on a Hilbert space H , then there exists a unique $y \in H$ such that

$$\forall x \in H, \quad \Lambda(x) = \langle x, y \rangle.$$

Proof .

- Define $M := \ker \Lambda$.
- Then M is a closed subspace and so $H = M \oplus M^\perp$.
- There is some $z \in M^\perp$ such that $\|z\| = 1$.
- Set $u := \Lambda(x)z - \Lambda(z)x$
- Check

$$\Lambda(u) = \Lambda(\Lambda(x)z - \Lambda(z)x) = \Lambda(x)\Lambda(z) - \Lambda(z)\Lambda(x) = 0 \implies u \in M$$

- Compute

$$\begin{aligned}
0 &= \langle u, z \rangle \\
&= \langle \Lambda(x)z - \Lambda(z)x, z \rangle \\
&= \langle \Lambda(x)z, z \rangle - \langle \Lambda(z)x, z \rangle \\
&= \Lambda(x)\langle z, z \rangle - \Lambda(z)\langle x, z \rangle \\
&= \Lambda(x)\|z\|^2 - \Lambda(z)\langle x, z \rangle \\
&= \Lambda(x) - \Lambda(z)\langle x, z \rangle \\
&= \Lambda(x) - \langle x, \overline{\Lambda(z)z} \rangle,
\end{aligned}$$

- Choose $y := \overline{\Lambda(z)}z$.
- Check uniqueness:

$$\begin{aligned}
& \langle x, y \rangle = \langle x, y' \rangle \quad \forall x \\
& \implies \langle x, y - y' \rangle = 0 \quad \forall x \\
& \implies \langle y - y', y - y' \rangle = 0 \\
& \implies \|y - y'\| = 0 \\
& \implies y - y' = \mathbf{0} \implies y = y'.
\end{aligned}$$

■

Theorem 8.2 (Continuous iff Bounded).

Let $L : X \longrightarrow \mathbb{C}$ be a linear functional, then the following are equivalent:

1. L is continuous
2. L is continuous at zero
3. L is bounded, i.e. $\exists c \geq 0 \mid |L(x)| \leq c\|x\|$ for all $x \in H$

Proof .

2 \implies 3: Choose $\delta < 1$ such that

$$\|x\| \leq \delta \implies |L(x)| < 1.$$

Then

$$\begin{aligned}
|L(x)| &= \left| L \left(\frac{\|x\|}{\delta} \frac{\delta}{\|x\|} x \right) \right| \\
&= \frac{\|x\|}{\delta} \left| L \left(\delta \frac{x}{\|x\|} \right) \right| \\
&\leq \frac{\|x\|}{\delta} 1,
\end{aligned}$$

so we can take $c = \frac{1}{\delta}$.

3 \implies 1:

We have $|L(x - y)| \leq c\|x - y\|$, so given $\varepsilon \geq 0$ simply choose $\delta = \frac{\varepsilon}{c}$.

■

■

Theorem: (Operator Norm is a Norm) If H is a Hilbert space, then $(H^\vee, \|\cdot\|_{\text{op}})$ is a normed space.

Proof .

The only nontrivial property is the triangle inequality, but

$$\|L_1 + L_2\| = \sup |L_1(x) + L_2(x)| \leq \sup |L_1(x)| + \sup |L_2(x)| = \|L_1\| + \|L_2\|.$$

■

Theorem 8.3 (Completeness in Operator Norm).

If X is a normed vector space, then $(X^\vee, \|\cdot\|_{\text{op}})$ is a Banach space.

Proof.

- Let $\{L_n\}$ be Cauchy in X^\vee .
- Then for all $x \in C$, $\{L_n(x)\} \subset \mathbb{C}$ is Cauchy and converges to something denoted $L(x)$.
- Need to show L is continuous and $\|L_n - L\| \rightarrow 0$.
- Since $\{L_n\}$ is Cauchy in X^\vee , choose N large enough so that

$$n, m \geq N \implies \|L_n - L_m\| < \varepsilon \implies |L_m(x) - L_n(x)| < \varepsilon \quad \forall x \mid \|x\| = 1.$$

- Take $n \rightarrow \infty$ to obtain

$$\begin{aligned} m \geq N &\implies |L_m(x) - L(x)| < \varepsilon \quad \forall x \mid \|x\| = 1 \\ &\implies \|L_m - L\| < \varepsilon \rightarrow 0. \end{aligned}$$

- Continuity:

$$\begin{aligned} |L(x)| &= |L(x) - L_n(x) + L_n(x)| \\ &\leq |L(x) - L_n(x)| + |L_n(x)| \\ &\leq \varepsilon \|x\| + c \|x\| \\ &= (\varepsilon + c) \|x\| \blacksquare. \end{aligned}$$

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9 Extra Problems

Integration

- Show that if $f \in C^1(\mathbb{R})$ and $\lim_{x \rightarrow \infty} f(x), f'(x)$ exist, then $\lim_{x \rightarrow \infty} f'(x) = 0$.

Basics

- If f is continuous, is it necessarily the case that f' is continuous?
- If $f_n \rightarrow f$, is it necessarily the case that f'_n converges to f' (or at all)?
- Is it true that the sum of differentiable functions is differentiable?
- Is it true that the limit of integrals equals the integral of the limit?
- Is it true that a limit of continuous functions is continuous?
- Prove that uniform convergence implies pointwise convergence implies a.e. convergence, but none of the implications may be reversed.
- Show that if K is compact and F is closed with K, F disjoint then $\text{dist}(K, F) > 0$.
- Show that if $f_n \rightarrow f$ uniformly with each f_n continuous then f is continuous.
- Show that a subset of a metric space is closed iff it is complete.
- Show that if a subset of a metric space is complete and totally bounded, then it is compact.
- Show that every compact set is closed and bounded.
- Show that a uniform limit of bounded functions is bounded.

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- Show that a uniform limit of continuous function is continuous.
 - Show that if $f_n \rightarrow f$ pointwise, $f'_n \rightarrow g$ uniformly for some f, g , then f is differentiable and $g = f'$.

Measure Theory

- \star : Show that for $E \subseteq \mathbb{R}^n$, TFAE:
 1. E is measurable
 2. $E = H \cup Z$ here H is F_σ and Z is null
 3. $E = V \setminus Z'$ where $V \in G_\delta$ and Z' is null.
- Show that continuity of measure from above/below holds for outer measures.
- \star : Show that if $E \subseteq \mathbb{R}^n$ is measurable then $m(E) = \sup_{K \subset E \text{ compact}} m(K)$ iff for all $\varepsilon > 0$ there exists a compact $K \subseteq E$ such that $m(K) \geq m(E) - \varepsilon$.
- Show that a countable union of null sets is null.

Continuity

- Show that a continuous function on a compact set is uniformly continuous.

Measurability

- Show that $f = 0$ a.e. iff $\int_E f = 0$ for every measurable set E .
- \star : Show that cylinder functions are measurable, i.e. if f is measurable on \mathbb{R}^s , then $F(x, y) := f(x)$ is measurable on $\mathbb{R}^s \times \mathbb{R}^t$ for any t .
- Show that if f is a measurable function, then $f = 0$ a.e. iff $\int f = 0$.

Integrability

- \star : Prove that the Lebesgue integral is translation invariant, i.e. if $\tau_h(x) = x + h$ then $\int \tau_h f = \int f$.
- \star : Prove that the Lebesgue integral is dilation invariant, i.e. if $f_\delta(x) = \frac{f(\frac{x}{\delta})}{\delta^n}$ then $\int f_\delta = \int f$.
- \star : Prove continuity in L^1 , i.e.

$$f \in L^1 \implies \lim_{h \rightarrow 0} \int |f(x+h) - f(x)| = 0.$$

- Show that a bounded function is Lebesgue integrable iff it is measurable.
- Show that simple functions are dense in L^1 .
- Show that step functions are dense in L^1 .
- Show that smooth compactly supported functions are dense in L^1 .

Convergence

- Prove Fatou's lemma using the Monotone Convergence Theorem.
- Show that if $\{f_n\}$ is in L^1 and $\sum \int |f_n| < \infty$ then $\sum f_n$ convergence to an L^1 function and $\int \sum f_n = \sum \int f_n$.

Convolution

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- Show that $f, g \in L^1 \implies f * g \in L^1$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.
 - Show that $f \in L^1, g \leq M \implies f * g \leq M'$ and is uniformly continuous.
 - Show that if $f, g \in L^1$ with $f \leq M, g \leq M'$, then $f * g \xrightarrow{x \rightarrow \infty} 0$.
 - Show that if $f \in L^1$ and g' exists with $\frac{\partial g}{\partial x_i}$ all bounded, then $\frac{\partial}{\partial x_i}(f * g) = f * \frac{\partial g}{\partial x_i}$.
 - Show that if f, g are smooth and compactly supported then $f * g$ is smooth and $f * g \xrightarrow{x \rightarrow \infty} 0$.
 - \star : show that if $f, g \in L^1$, then $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.
 - Is it the case that $f, g \in C_c$ implies that $f * g \in C_c$?
 - Show that if $f \in L^1$ and $g \in C_c^\infty$ then $f * g$ is smooth and $f * g$ vanishes at infinity.
 - Show that if $f, g \in L^1$ and g is bounded, then $\lim_{|x| \rightarrow \infty} (f * g)(x) = 0$.

Fourier Analysis

- Show that if $f \in L^1$ then \hat{f} is bounded and uniformly continuous.
- Is it the case that $f \in L^1$ implies $\hat{f} \in L^1$?
- Show that if $f, \hat{f} \in L^1$ then f is bounded, uniformly continuous, and vanishes at infinity.
 - Show that this is not true for arbitrary L^1 functions.
- Show that if $f \in L^1$ and $\hat{f} = 0$ almost everywhere then $f = 0$ almost everywhere.
 - Prove that $\hat{f} = \hat{g}$ implies that $f = g$ a.e.
- Show that if $f, g \in L^1$ then $\int \hat{f}g = \int f\hat{g}$.
 - Give an example showing that this fails if g is not bounded.
- Show that if $f \in C^1$ then f is equal to its Fourier series.

Approximate Identities

- Show that if φ is an approximate identity, then $\|f * \varphi_t - f\|_1 \xrightarrow{t \rightarrow 0} 0$.
 - Show that if additionally $|\varphi(x)| \leq c(1 + |x|)^{-n-\varepsilon}$ for some $c, \varepsilon > 0$, then this converges is almost everywhere.
- Show that if f is bounded and uniformly continuous and φ_t is an approximation to the identity, then $f * \varphi_t$ uniformly converges to f .

L^p Spaces

- Show that if $E \subseteq \mathbb{R}^n$ is measurable with $\mu(E) < \infty$ and $f \in L^p(X)$ then $\|f\|_{L^p(X)} \xrightarrow{p \rightarrow \infty} \|f\|_\infty$.
- Is it true that the converse to the DCT holds? I.e. if $\int f_n \rightarrow \int f$, is there a $g \in L^p$ such that $f_n < g$ a.e. for every n ?
- Prove continuity in L^p : If f is uniformly continuous then $\|\tau_h f - f\|_p \rightarrow 0$ as $h \rightarrow 0$ for all p .
- Prove the following inclusions of L^p spaces for $m(X) < \infty$:

$$L^\infty(X) \subset L^2(X) \subset L^1(X)$$

$$\ell^2(\mathbb{Z}) \subset \ell^1(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z}).$$

10 Inequalities and Equalities

Proposition 10.1 (*Reverse Triangle Inequality*).

$$|||x|| - ||y||| \leq \|x - y\|.$$

Proposition 10.2 (*Chebyshev's Inequality*).

$$\mu(\{x : |f(x)| > \alpha\}) \leq \left(\frac{\|f\|_p}{\alpha} \right)^p.$$

Proposition 10.3 (*Holder's Inequality (when surjective)*).

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Application: For finite measure spaces,

$$1 \leq p < q \leq \infty \implies L^q \subset L^p \quad (\text{and } \ell^p \subset \ell^q).$$

Proof (Holder's Inequality).

Fix p, q , let $r = \frac{q}{p}$ and $s = \frac{r}{r-1}$ so $r^{-1} + s^{-1} = 1$. Then let $h = |f|^p$:

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$$\|f\|_p^p = \|h \cdot 1\|_1 \leq \|1\|_s \|h\|_r = \mu(X)^{\frac{1}{s}} \|f\|_q^{\frac{q}{r}} \implies \|f\|_p \leq \mu(X)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q.$$

Note: doesn't work for ℓ_p spaces, but just note that $\sum |x_n| < \infty \implies x_n < 1$ for large enough n , and thus $p < q \implies |x_n|^q \leq |x_n|^p$.

Proposition 10.4 (*Cauchy-Schwarz Inequality*).

$$|\langle f, g \rangle| = \|fg\|_1 \leq \|f\|_2 \|g\|_2 \quad \text{with equality} \iff f = \lambda g.$$

Note: Relates inner product to norm, and only happens to relate norms in L^1 .

Proof .
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Proposition 10.5 (*Minkowski's Inequality*).

$$1 \leq p < \infty \implies \|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Note: does not handle $p = \infty$ case. Use to prove L^p is a normed space.

Proposition 10.6 (*Young's Inequality**).

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \implies \|f * g\|_r \leq \|f\|_p \|g\|_q.$$

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Application: Some useful specific cases:

$$\begin{aligned}\|f * g\|_1 &\leq \|f\|_1 \|g\|_1 \\ \|f * g\|_p &\leq \|f\|_1 \|g\|_p, \\ \|f * g\|_\infty &\leq \|f\|_2 \|g\|_2 \\ \|f * g\|_\infty &\leq \|f\|_p \|g\|_q.\end{aligned}$$

Proposition 10.7 (*? Inequality*).

$$(a + b)^p \leq 2^p(a^p + b^p).$$

Proposition 10.8 (*Bezel's Inequality*).

For $x \in H$ a Hilbert space and $\{e_k\}$ an orthonormal sequence,

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

Note: this does not need to be a basis.

Proposition 10.9 (*Parseval's Identity*).

Equality in Bessel's inequality, attained when $\{e_k\}$ is a *basis*, i.e. it is complete, i.e. the span of its closure is all of H .

10.1 Less Explicitly Used Inequalities

Proposition 10.10 (*AM-GM Inequality*).

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Proposition 10.11 (*Jensen's Inequality*).

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$