Complex Analysis Qualifying Exam Solutions

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4.1 5

Show that there is no sequence of polynomials converging uniformly to $f(z) = 1/z$ on S^1.

Solution

- By Cauchy's integral formula, $\displaystyle\int_{S^1} f = 2\pi i$
- If p_j is any polynomial, then p_j is holomorphic in $\mathbb{D}$, so $\displaystyle\int_{S^1} p_j = 0$.
- Contradiction: compact sets in $\mathbb{C}$ are bounded, so

$$\left|\int f - \int p_j\right| \leq \int |p_j - f| \leq \int \|p_j - f\|_\infty = \|p_j - f\|_\infty \int_{S^1} 1\,dz = \|p_j - f\|_\infty \cdot 2\pi \longrightarrow 0$$

  which forces $\displaystyle\int f = \int p_j = 0$.

4.2 9

- Note f is continuous on $\mathbb{C}$ since analytic implies continuous (f equals its power series, where the partials sums uniformly converge to it, and uniform limit of continuous is continuous).
- Strategy: take D a disc centered at a point $x \in \mathbb{R}$, show f is holomorphic in D by Morera's theorem.
- Let $\Delta \subset D$ be a triangle in D.
- Case 1: If $\Delta \bigcap \mathbb{R} = 0$, then f is holomorphic on Δ and $\displaystyle\int_\Delta f = 0$.
- Case 2: one side or vertex of Δ intersects $\mathbb{R}$, and wlog the rest of Δ is in $\mathbb{H}^+$.
  - Then let Δ_ε be the perturbation $\Delta + i\varepsilon = \left\{z + i\varepsilon \;\middle|\; z \in \Delta\right\}$; then $\Delta_\varepsilon \bigcap \mathbb{R} = 0$ and $\displaystyle\int_{\Delta_\varepsilon} f = 0$.
  - Now let $\varepsilon \longrightarrow 0$ and conclude by continuity of f (???)
    * We want

$$\int_{\Delta_\varepsilon} f = \int_a^b f(\gamma_\varepsilon(t))\gamma_\varepsilon'(t)\,dt \overset{\varepsilon \longrightarrow 0}{\longrightarrow} \int_a^b f(\gamma(t))\gamma_\varepsilon'(t)\,dt = \int_\Delta f$$

      where $\gamma_\varepsilon, \gamma$ are curves parametrizing $\Delta_\varepsilon, \Delta$ respectively.
    * Since $\gamma, \gamma_\varepsilon$ are closed and bounded in $\mathbb{C}$, they are compact subsets. Thus it suffices to show that $f(\gamma_\varepsilon(t))\gamma_\varepsilon'(t)$ converges uniformly to $f(\gamma(t))\gamma'(t)$.
    * ??
- Case 3: Δ intersects both $\mathbb{H}^+$ and $\mathbb{H}^-$.
  - Break into smaller triangles, each of which falls into one of the previous two cases.

4.3 10

Suppose $f : \mathbb{C} \longrightarrow \mathbb{C}$ is entire and bounded, and use Cauchy's theorem to prove that $f' \equiv 0$ and thus f is constant.

Solution

- Suffices to prove $f' = 0$ because $\mathbb{C}$ is connected (see Stein Ch 1, 3.4)
  - Idea: Fix w_0, show $f(w) = f(w_0)$ for any $w \neq w_0$
  - Connected = Path connected in $\mathbb{C}$, so take γ joining w to w_0.
  - f is a primitive for f', and $\displaystyle\int_\gamma f' = f(w) - f(w_0)$, but $f' = 0$.
- Fix $z_0 \in \mathbb{C}$, let B be the bound for f, so $|f(z)| \leq B$ for all z.
- Apply Cauchy inequalities: if f is holomorphic on $U \supset \overline{D}_R(z_0)$ then setting $\|f\|_C := \sup\limits_{z \in C} |f(z)|$,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n! \|f\|_C}{R^n}.$$

  - Yields $|f'(z_0)| \leq B/R$
- Take $R \longrightarrow \infty$, QED.

5 Liouville's Theorem, Power Series (8155e)

5.1 1

Suppose f is analytic on $\Omega \supseteq \mathbb{D}$ whose power series $\sum a_n z^n$ has radius of convergence 1.

a. Give an example of an f which converges at every point on S^1.
b. Give an example of an f which is analytic at $z = 1$ but $\sum a_n$ diverges.
c. Prove that f can not be analytic at every point of S^1.

Solution:

a. Take $\sum \frac{z^n}{n^2}$; then $|z| \leq 1 \implies \left|\frac{z^n}{n^2}\right| \leq \frac{1}{n^2}$ which is summable, so the series converges for $|z| \leq 1$.

b. Take $\sum \frac{z^n}{n}$; then $z = 1$ yields the harmonic series, which diverges.

   - For $z \in S^1 \setminus \{1\}$, we have $z = e^{2\pi i t}$ for $0 < t < 2\pi$.
   - So fix t.
   - Toward applying the Dirichlet test, set $a_n = 1/n, b_n = z^n$.
   - Then for all N,

$$\left|\sum_{n=1}^N b_n\right| = \left|\sum_{n=1}^N b_n\right| = \left|\sum_{n=1}^N z^n\right| = \left|\frac{z - z^{N+1}}{|1-z|}\right| \leq \frac{2}{1-z} < \infty.$$

   - Thus $\sum a_n b_n < \infty$ and $\sum z^n/n$ converges.

c. ?

5.2 5

Prove the Fundamental Theorem of Algebra: every non-constant polynomial $p(z) = a_n z^n + \cdots + a_0 \in \mathbb{C}[x]$ has a root in $\mathbb{C}$.

Solution:

- Strategy: By contradiction with Liouville's Theorem
- Suppose p is non-constant and has no roots.
- Claim: $1/p(z)$ is a bounded holomorphic function on $\mathbb{C}$.
  - Holomorphic: clear? Since p has no roots.
  - Bounded: for $z \neq 0$, write

$$\frac{P(z)}{z^n} = a_n + \left(\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right).$$

  - The term in parentheses goes to 0 as $|z| \longrightarrow \infty$
  - Thus there exists an $R > 0$ such that

$$|z| > R \implies \left|\frac{P(z)}{z^n}\right| \geq c := \frac{|a_n|}{2}.$$

  - So p is bounded below when $|z| > R$
  - Since p is continuous and has no roots in $|z| \leq R$, it is bounded below when $|z| \leq R$.
  - Thus p is bounded below on $\mathbb{C}$ and thus $1/p$ is bounded above on $\mathbb{C}$.
- By Liouville's theorem, $1/p$ is constant and thus p is constant, a contradiction.

5.3 6

Find all entire functions f which satisfy the following inequality, and prove the list is complete:

$$|f(z)| \geq |z|.$$

Solution:

- Suppose f is entire and define $g(z) := \dfrac{z}{f(z)}$.
- By the inequality, $|g(z)| \leq 1$, so g is bounded.
- g potentially has singularities at the zeros $Z_f := f^{-1}(0)$, but since f is entire, g is holomorphic on $\mathbb{C} \setminus Z_f$.
- Claim: $Z_f = \{0\}$.
  - If $f(z) = 0$, then $|z| \leq |f(z)| = 0$ which forces $z = 0$.
- We can now apply Riemann's removable singularity theorem:
  - Check g is bounded on some open subset $D \setminus \{0\}$, clear since it's bounded everywhere
  - Check g is holomorphic on $D \setminus \{0\}$, clear since the only singularity of g is $z = 0$.
- By Riemann's removable singularity theorem, the singularity $z = 0$ is removable and g has an extension to an entire function $\tilde{g}$.
- By continuity, we have $|\tilde{g}(z)| \leq 1$ on all of $\mathbb{C}$
  - If not, then $|\tilde{g}(0)| = 1 + \varepsilon > 1$, but then there would be a domain $\Omega \subseteq \mathbb{C} \setminus \{0\}$ such that $1 < |\tilde{g}(z)| \leq 1 + \varepsilon$ on Ω, a contradiction.
- By Liouville, $\tilde{g}$ is constant, so $\tilde{g}(z) = c_0$ with $|c_0| \leq 1$
- Thus $f(z) = c_0^{-1} z := cz$ where $|c| \geq 1$

Thus all such functions are of the form $f(z) = cz$ for some $c \in \mathbb{C}$ with $|c| \geq 1$.

6 Laurent Expansions and Singularities (8155f)

6.1 1

Let $f(z) = \dfrac{z+1}{z(z-1)}$.

About $z = 0$:

$$\begin{aligned}
f(z) &= (z+1)\left(-\frac{1}{z} + \frac{1}{z-1}\right) \\
&= -(z+1)\left(\frac{1}{z} + \sum_{n=0}^{\infty} z^n\right) \\
&= -(z+1) \sum_{n=-1}^{\infty} z^n \\
&= \frac{1}{z} + 2\sum_{n=0}^{\infty} z^n \\
&= -\frac{1}{z} - 2 - 2z - 2z^2 - \cdots .
\end{aligned}$$

About $z = 1$:

$$\begin{aligned}
f(z) &= \left(\frac{(1-z)-2}{1-z}\right)\left(\frac{1}{1-(1-z)}\right) \\
&= \left(1 - \frac{2}{1-z}\right)\sum_{n=0}^{\infty}(1-z)^n \\
&= \sum_{n=0}^{\infty}(1-z)^n - 2\sum_{n=-1}^{\infty}(1-z)^n \\
&= -\frac{2}{1-z} - \sum_{n=0}^{\infty}(1-z)^n \\
&= \frac{2}{z-1} + \sum_{n=0}^{\infty}(-1)^{n+1}(z-1)^n \\
&= \frac{2}{z-1} - 1 + (z-1) - (z-1)^2 + \cdots .
\end{aligned}$$

6.2 2

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots .$$

$$\begin{aligned}\cos\left(\frac{1}{z}\right) &= \frac{1}{2}\left(e^{\frac{i}{z}} + e^{-\frac{i}{z}}\right)\\ &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\left(\frac{i}{z}\right)^n + \left(\frac{-i}{z}\right)^n\right)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{z}\right)^{2n}.\end{aligned}$$

6.3 8

Idea: show their $f - g$ is analytic by taking away all of the negative powers, and bounded by (c).

7 Residues (8155g)

8 Rouche's Theorem (8155h)

8.1 1

Note

- $f_1(z) = 1 + z$, which has the single root $z = -1$ which is not inside $|z| < 1$.
- $f_2(z) = 1 + z + \frac{1}{2}z^2 = (z - (1 + i))(z - (1 - i))$, and $|1 \pm i| = \sqrt{2} > 1$.
- Note that $p_n(z)^{n \longrightarrow \infty z} e$ uniformly on any compact set.
- Let r be arbitrary and fix $N := \mathbb{D}_r(0)$, then $p_n(z) \longrightarrow e^z$ uniformly on $\overline{N}$.
- Set $g_n(z) := p_n(z)/e^z$, then $g_n \longrightarrow 1$ uniformly on $\overline{N}$.
- Choose $n \gg 0$ so that $|f(z) - 1| < \varepsilon < 1$ for all $z \in \overline{N}$.
- So take $h(z) = 1$, then on ∂N,?

8.2 2

Multiple versions of Rouches theorem!

- Set $h(z) = 3z^2$ and $g(z) = z^3 + bz + b^2$.
- Then on $|z| = 1$,

  $$|g(z)| \leq 1 + b + b^2 < 3 = 3|z|^2 = \left|3z^2\right| = |h|,$$

  so g, h have the same number of roots in $|z| \leq_? 1$.
- But h evidently has two roots in this region.

8.3 4

- Set $h(z) = -4z^3$ and $g(z) = z^7 - 1$, then on $|z| = 1$,

$$|g(z)| = \left|z^7 - 1\right| \leq 1 + 1 = 2 < 4 = \left|-4z^3\right| = |h(z)|.$$

- So h and $h + g$ have the same number of roots, but h has three roots here.

9 Schwarz Lemma and Reflection Principle (8155i)