Algebra Qualifying Exam Notes

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Contents

1 Study Guide for Algebra Qualifying Exam

References:

[1]. David Dummit and Richard Foote, Abstract Algebra, Wiley, 2003.

[2]. Kenneth Hoffman and Ray Kunze, Linear Algebra, Prentice-Hall, 1971.

[3]. Thomas W. Hungerford, Algebra, Springer, 1974.

[4]. Roy Smith, Algebra Course Notes (843-1 through 845-3), `http://www.math.uga.edu/~roy/`,

As a general rule, students are responsible for knowing both the theory (proofs) and practical applications (e.g. **how to find the Jordan or rational canonical form** of a given matrix, **or the Galois group of a given polynomial**) of the topics mentioned.
A supplement to this study guide is available at:
`http://www.math.uga.edu/sites/default/files/PDFs/Graduate/QualsStudyGuides/AlgebraPhDqualremarks.pdf`

1.1 Group Theory

- Subgroups and quotient groups
- Lagrange's Theorem
- Fundamental homomorphism theorems
- Group actions with applications to the structure of groups such as
  - The Sylow Theorems
- Group constructions such as:
  - Direct and semi-direct products
- Structures of special types of groups such as:
  - p-groups
  - Dihedral,
  - Symmetric and Alternating groups
    - Cycle decompositions
- The simplicity of A_n, for $n \geq 5$
- Free groups, generators and relations
- Solvable groups

References: [1,3,4]

1.2 Linear Algebra

- Determinants
- Eigenvalues and eigenvectors
- Cayley-Hamilton Theorem
- Canonical forms for matrices
- Linear groups $(\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{O}_n, \mathrm{U}_n)$
- Duality
  - Dual spaces,
  - Dual bases,
  - Induced dual map,
  - Double duals

- Finite-dimensional spectral theorem

References: [1,2,4]

1.3 Rings and Modules

- Zorn's Lemma
  - Every vector space has a basis
  - Maximal ideals exist
- Properties of ideals and quotient rings
- Fundamental homomorphism theorems for rings and modules
- Characterizations and properties of special domains such as:
  - Euclidean $\implies$ PID $\implies$ UFD
- Classification of finitely generated modules over PIDs *(with emphasis on Euclidean Domains)*
- Applications to the structure of:
  - Finitely generated abelian groups
  - Canonical forms of matrices

References: [1,3,4]

1.4 Field Theory

- Algebraic extensions of fields
- Fundamental theorem of Galois theory
- Properties of finite fields
- Separable extensions
- Computations of Galois groups of polynomials of small degree and cyclotomic
- Polynomials
- Solvability of polynomials by radicals

References: [1,3,4]

2 Remarks

Adapted from remark written by Roy Smith, August 2006

2.1 Group theory:

The first 6 chapters (220 pages) of DF are excellent.

All the definitions and proofs of these theorems on groups are given in Smith's web based lecture notes for math 843 part 1.

Key topics:

- Sylow theorems
- Simplicity of A_n for $n > 4$.
- The first isomorphism theorem,
- The Jordan Holder theorem,

The last two (one easy, one hard) are left as exercises.

The proof JH is seldom tested on the qual, but proofs are always of interest.

- Fundamental theorem of finite abelian groups

  DF Exercises 12.1.16-19

- The simple groups of order between 60 and 168 have prime order

2.2 Rings:

- DF Chapters 7,8,9.
- Gauss's important theorem on unique factorization of polynomials:
  - $\mathbb{Z}[x]$ is a UFD
  - $R[x]$ is a UFD when R is a UFD
- The fundamental isomorphism theorems for rings (easy and useful exercise)
- How to use Zorn's lemma
  - To find maximal ideals
  - Construct algebraic field closures
  - Why it is unnecessary in countable or noetherian rings.

  Smith discusses extensively in 844-1.

- Results about PIDs

  (DF Section 8.2)

  - Example of a PID that is not a Euclidean domain

    (DF p.277)

  - Proof that a Euclidean domain is a PID and hence a UFD
  - Proof that $\mathbb{Z}$ and $k[x]$ are UFDs

    (p.289 Smith, p.300 DF)

- A polynomial ring in infinitely many variables over a UFD is still a ufd

  (Easy, DF, p.305)

- Eisenstein's criterion

  (DF p.309)

  - Stated only for monic polynomials – proof of general case identical.
  - See Smith's notes for the full version.

- Cyclic product structure of $(\mathbb{Z}/n\mathbb{Z})^{\times}$

  (exercise in DF, Smith 844-2, section 18)

- Grobner bases and division algorithms for polynomials in several variables

  (DF 9.6.)

- Modules over pid's and Canonical forms of matrices.

  DF sections 10.1, 10.2, 10.3, and 12.1, 12.2, 12.3.

  - Constructive proof of decomposition: DF Exercises 12.1.16-19
  - Smith 845-1 and 845-2: Detailed discussion of the constructive proof.

2.3 Field Theory / Galois Theory.

- DF chapters 13,14 (about 145 pages).
- Smith:
  - 843-2, sections 11,12, and 16-21 (39 pages)
  - 844-1, sections 7-9 (20 pages)
  - 844-2, sections 10-16, (37 pages)

3 Outline of Topics: UCSD Qual Algebra, Fall 2018

Chapters 1-9 of Dummit and Foote

- Groups
  - Left and right cosets
  - Lagrange's theorem
  - Isomorphism theorems
  - Group generated by a subset
  - Structure of cyclic groups
  - Composite groups
  - Normalizer
  - Symmetric groups
  - Cayley's theorem
  - Orbit stabilizer theorem
  - Orbits act on left cosets of subgroups
  - Subgroups of index p, the smallest prime dividing $|G|$, are normal

  - Action of G on itself by conjugation
  - Class equation
  - p-groups
  - p^2 groups are abelian
  - Automorphisms
    * Inner automorphisms
  - Proof of Sylow theorems
  - A_n is simple for $n \geq 5$
  - Recognition of internal direct product
  - Recognition of semi-direct product
  - Classification of groups of order pq
  - Free group & presentations
  - Commutator subgroup
  - Solvable groups
  - Derived series
  - Nilpotent groups
  - Upper central series
  - Lower central series
  - Fratini's argument
- Rings
  - I maximal iff R/I is a field
  - Zorn's lemma
  - Chinese Remainder Theorem
  - Localization of a domain
  - Field of fractions
  - Factorization in domains
  - Euclidean algorithm
  - Gaussian integers
  - Primes and irreducibles
  - Domains
    * Primes are irreducible
  - UFDs
    * Have GCDs
    * Sometimes PIDs
  - PIDs
    * Noetherian
    * Irreducibles are prime
    * Are UFDs
    * Have GCDs
  - Euclidean domains
    * Are PIDs
  - Factorization in $Z[i]$
  - Polynomial rings
  - Gauss' lemma
  - Remainder and factor theorem
  - Polynomials
  - Reducibility
  - Rational root test

– Eisenstein's criterion

4 Group Theory

4.1 Random References

4.2 Big List of Notation

$C_G(x) =$	$\left\{g \in G \,\middle\vert\, [g,x] = 1\right\}$	$\subseteq G$	Centralizer (Element)
$C_G(H) =$	$\left\{g \in G \,\middle\vert\, [g,h] = 1 \ \forall h \in H\right\} = \bigcap_{h \in H} C_G(h)$	$\leq G$	Centralizer (Subgroup)
$? =$	$\left\{ghg^{-1} \,\middle\vert\, g \in G\right\}$	$\subseteq G$	Conjugacy Class
$\mathcal{O}_x, G \cdot x =$	$\left\{g.x \,\middle\vert\, x \in X\right\}$	$\subseteq X$	Orbit
$\mathrm{Stab}_G(x), G_x =$	$\left\{g \in G \,\middle\vert\, g.x = x\right\}$	$\subseteq G$	Stabilizer
$X^g =$	$\left\{x \in X \,\middle\vert\, \forall g \in G,\ g.x = x\right\}$	$\subseteq X$	Fixed Points
$Z(G) =$	$\left\{x \in G \,\middle\vert\, \forall g \in G,\ gxg^{-1} = x\right\}$	$\subseteq G$	Center
$N_G(H) =$	$\left\{g \in G \,\middle\vert\, gHg^{-1} = H\right\}$	$\subseteq G$	Normalizer
$\mathrm{Inn}(G) =$	$\left\{\varphi_g(x) = gxg^{-1}\right\}$	$\subseteq \mathrm{Aut}(G)$	Inner Aut.
$\mathrm{Out}(G) =$	$\mathrm{Aut}(G)/\mathrm{Inn}(G)$	$\hookrightarrow \mathrm{Aut}(G)$	Outer Aut.
$[g,h] =$	$ghgh^{-1}$	$\in G$	Commutator (Element)
$[G,H] =$	$\langle [g,h] : g \in G, h \in H \rangle$	$\leq G$	Commutator (Subgroup)

Definition 4.0.1 (Normal Closure of a subgroup)**.**
The smallest normal subgroup of G containing H:

$$H^G \coloneqq \{gHg^{-1} : g \in G\} = \bigcap \{N : H \leq N \trianglelefteq G\}.$$

Definition 4.0.2 (Normal Core of a subgroup)**.**
The largest normal subgroup of G containing H:

$$H_G = \bigcap_{g \in G} gHg^{-1} = \langle N : N \trianglelefteq G \ \&\ N \leq H \rangle = \ker \psi.$$

where

$$\begin{aligned} \psi : G &\longrightarrow \mathrm{Aut}(G/H) \\ g &\mapsto (xH \mapsto gxH). \end{aligned}$$

Definition 4.0.3 (Characteristic subgroup)**.**
$H \leq G$ is *characteristic* iff H is fixed by every element of $\mathrm{Aut}(G)$.

Definition 4.0.4 (Subgroup Generated by a Subset)**.**
If $H \subset G$, then $\langle H \rangle$ is the smallest subgroup containing H:

$$\langle H \rangle = \bigcap_{H \subseteq M \leq G} M = \left\{ h_1^{\pm 1} \cdots h_n^{\pm 1} \;\middle|\; n \geq 0, h_i \in H \right\}.$$

Definition 4.0.5 (Centralizer):)**.**

$$C_G(H) = \left\{ g \in G \;\middle|\; ghg^{-1} = h \;\forall h \in H \right\}$$

Definition 4.0.6 (Normalizer)**.**

$$N_G(H) = \left\{ g \in G \;\middle|\; gHg^{-1} = H \right\} = \bigcup_{H \;\trianglelefteq\; M \leq G} M$$

Lemma 4.1.
The size of the conjugacy class of H is the index of its centralizer, i.e.

$$\left| \left\{ gHg^{-1} \;\middle|\; g \in G \right\} \right| = [G : C_G(H)].$$

Proof: Orbit-stabilizer.

Theorem 4.2 ***(The Fundamental Theorem of Cosets).***
$aH = bH \iff a^{-1}b \in H$ or $aH \bigcap bH = \emptyset$

Definition 4.2.1 (Commutator)**.**
$[x, y] = x^{-1}y^{-1}xy$ is the **commutator**, and $[G, G] := \left\{ [x, y] \;\middle|\; x, y \in G \right\}$ is the **commutator subgroup**.

Lemma 4.3.
$[G, G] \leq H$ and $H \trianglelefteq G \implies G/H$ is abelian.

Lemmas:

- Every subgroup of a cyclic group is itself cyclic.
- Intersections of subgroups are still subgroups
  - Intersections of distinct coprime-order subgroups are trivial
  - Intersections of subgroups of the same prime order are either trivial or equality

- The Quaternion group has only one element of order 2, namely -1.
  - They also have the presentation

$$\begin{aligned} Q &= \left\langle x, y, z \;\middle|\; x^2 = y^2 = z^2 = xyz = -1 \right\rangle \\ &= \left\langle x, y \;\middle|\; x^4 = y^4 = e, x^2 = y^2, yxy^{-1} = x^{-1} \right\rangle. \end{aligned}$$

- A dihedral group always has a presentation of the form

$$D_n = \left\langle x, y \;\middle|\; x^n = y^2 = (xy)^2 = e \right\rangle,$$

  yielding at least 2 distinct elements of order 2.

4.3 Finitely Generated Abelian Groups

Invariant factor decomposition:

$$G \cong \mathbb{Z}^r \times \prod_{j=1}^m \mathbb{Z}/(n_j) \quad \text{where } n_1 \;\Big|\; \cdots \;\Big|\; n_m.$$

Going from invariant divisors to elementary divisors:

- Take prime factorization of each factor
- Split into coprime pieces

Example:

$$\begin{aligned} &\mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(2^3 \cdot 5^2 \cdot 7) \\ \cong &\mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(2^3) \oplus \mathbb{Z}/(5^2) \oplus \mathbb{Z}/(7) \\ . \end{aligned}$$

Going from elementary divisors to invariant factors:

- Bin up by primes occurring (keeping exponents)
- Take highest power from each prime as *last* invariant factor
- Take highest power from all remaining primes as next, etc

Example: Given the invariant factor decomposition

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}, .$$

$p = 2$	$p = 3$	$p = 5$
$2, 2, 2$	$3, 3$	5^2

$\implies n_m = 5^2 \cdot 3 \cdot 2$

$p=2$	$p=3$	$p=5$
$2,2$	3	$\emptyset$

$\implies n_{m-1} = 3 \cdot 2$

$p=2$	$p=3$	$p=5$
2	$\emptyset$	$\emptyset$

$\implies n_{m-2} = 2$

and thus

$$G \cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(3 \cdot 2) \oplus \mathbb{Z}/(5^2 \cdot 3 \cdot 2).$$

Classifying Abelian Groups of a Given Order:

Let $p(x)$ be the integer partition function.
Example: $p(6) = 11$, given by $6, 5+1, 4+2, \cdots$.

Write $G = p_1^{k_1} p_2^{k_2} \cdots$; then there are $p(k_1)p(k_2)\cdots$ choices, each yielding a distinct group.

4.4 The Symmetric Group

Definitions:

- A cycle is **even** $\iff$ product of an *even* number of transpositions.
  - A cycle of even *length* is **odd**
  - A cycle of odd *length* is **even**

Mnemonic: the parity of a k-cycle is the parity of $k-1$.

Definition The **alternating group** is the subgroup of **even** permutations, i.e. $A_n := \left\{ \sigma \in S_n \;\middle|\; \operatorname{sign}(\sigma) = 1 \right\}$ where $\operatorname{sign}(\sigma) = (-1)^m$ where m is the number of cycles of even length.

Corollary: Every $\sigma \in A_n$ has an even number of *odd* cycles (i.e. an even number of *even-length* cycles).

Example:

$$\begin{aligned} A_4 = \{ &\mathrm{id}, \\ &(1,3)(2,4), (1,2)(3,4), (1,4)(2,3), \\ &(1,2,3), (1,3,2), \\ &(1,2,4), (1,4,2), \\ &(1,3,4), (1,4,3), \\ &(2,3,4), (2,4,3)\}. \end{aligned}$$

Definition 4.3.1 (Dihedral Groups)**.**
$\left\langle a, b \,\middle|\, a^n = b^2 = 1, bab^{-1} = a^{-1} \right\rangle \cong \langle r, s \rangle$

Useful Facts:

- Conjugacy classes are determined by cycle type
- The order of a cycle is its length.
- The order of an element is the least common multiple of the sizes of its cycles.
- The transitive subgroups of S_3 are S_3, A_3
- The transitive subgroups of S_4 are $S_4, A_4, D_4, \mathbb{Z}_2^2, \mathbb{Z}_4$.
- S_4 has two normal subgroups: $A_4, \mathbb{Z}_2^2$.
- $S_{n\geq 5}$ has one normal subgroup: A_n.
- $Z(S_n) = 1$ for $n \geq 3$
- $Z(A_n) = 1$ for $n \geq 4$
- $[S_n, S_n] = A_n$
- $[A_4, A_4] \cong \mathbb{Z}_2^2$
- $[A_n, A_n] = A_n$ for $n \geq 5$, so $A_{n\geq 5}$ is nonabelian.
- $A_{n\geq 5}$ is *simple*.
- $\sigma \circ (a_1 \cdots a_k) \circ \sigma^{-1} = (\sigma(a_1), \cdots \sigma(a_k))$

4.5 Counting Theorems

Theorem 4.4 ***(Lagrange's Theorem).***
$H \leq G \implies |H| \,\Big|\, |G|$.

Corollary 4.5.
The order of every element divides the size of G, i.e.

$$g \in G \implies o(g) \,\Big|\, o(G) \implies g^{|G|} = e.$$

Warning: There does **not** necessarily exist $H \leq G$ with $|H| = n$ for every $n \,\Big|\, |G|$.

Counterexample: $|A_4| = 12$ but has no subgroup of order 6.

Theorem 4.6 ***(Cauchy's Theorem).***
For every prime p dividing $|G|$. there is an element (and thus a subgroup) of order p.

This is a partial converse to Lagrange's theorem, and strengthened by Sylow's theorem.

4.5.1 Group Actions

Definition 4.6.1 (Group Action)**.**

An action of G on X is a group morphism

$$\begin{aligned}\varphi : G \times X &\to X \\ (g, x) &\mapsto g \cdot x\end{aligned}$$

or equivalently

$$\begin{aligned}\varphi : G &\longrightarrow \mathrm{Aut}(X) \\ g &\mapsto (x \mapsto \varphi_g(x) := g \cdot x)\end{aligned}$$

satisfying

1. $e \cdot x = x$
2. $g \cdot (h \cdot x) = (gh) \cdot x$

Note that $\ker \psi = \bigcap_{x \in X} G_x$ is the intersection of all stabilizers.

Definition 4.6.2 (Transitive)**.**
A group action $G \curvearrowright X$ is *transitive* iff for all $x, y \in X$ there exists a $g \in G$ such that $g \cdot x = x$. Equivalently, the action has a single orbit.

Notation: For a group G acting on a set X,

- $G \cdot x = \left\{ g \curvearrowright x \;\middle|\; g \in G \right\} \subseteq X$ is the orbit
- $G_x = \left\{ g \in G \;\middle|\; g \curvearrowright x = x \right\} \subseteq G$ is the stabilizer
- $X/G \subset \mathcal{P}(X)$ is the set of orbits
- $X^g = \left\{ x \in X \;\middle|\; g \curvearrowright x = x \right\} \subseteq X$ are the fixed points

Note that being in the same orbit is an equivalence relation which partitions X, and G acts transitively if restricted to any single orbit.

Orbit-Stabilizer:

$$|G \cdot x| = [G : G_x] = |G|/|G_x| \quad \text{if } G \text{ is finite}$$

Mnemonic: $G/G_x \cong G \cdot x$.

4.5.2 Examples of Orbit-Stabilizer

1. Let G act on itself by left translation, where $g \mapsto (h \mapsto gh)$.

- The orbit $G \cdot x = G$ is the entire group
- The stabilizer G_x is only the identity.
- The fixed points X^g are only the identity.

1. Let G act on *itself* by conjugation.

- $G \cdot x$ is the **conjugacy class** of x (so not generally transitive)

- $G_x = Z(x) \coloneqq C_G(x) = \left\{ g \;\middle|\; [g,x] = e \right\}$, the **centralizer** of x.
- G^g (the fixed points) is the **center** $Z(G)$.

Corollary 4.7.
The number of conjugates of an element (i.e. the size of its conjugacy class) is the index of its centralizer, $[G : C_G(x)]$.

Corollary 4.8 ***(Class Equation).***

$$|G| = |Z(G)| + \sum_{\substack{\text{One } x_i \text{ from} \\ \text{each conjugacy} \\ \text{class}}} [G : C_G(x_i)]$$

Note that $[G : C_G(x_i)]$ is the number of elements in the conjugacy class of x_i, and each $x_i \in Z(G)$ has a singleton conjugacy class.

1. Let G act on X, its set of *subgroups*, by conjugation.

- $G \cdot H = \left\{ gHg^{-1} \right\}$ is the **set of conjugate subgroups** of H
- $G_H = N_G(H)$ is the **normalizer** of in G of H
- X^g is the set of **normal subgroups** of G

Corollary: Given $H \leq G$, the number of conjugate subgroups is $[G : N_G(H)]$.

1. For a fixed proper subgroup $H < G$, let G act on its cosets $G/H = \left\{ gH \;\middle|\; g \in G \right\}$ by left translation.

- $G \cdot gH = G/H$, i.e. this is a *transitive* action.
- $G_{gH} = gHg^{-1}$ is a *conjugate subgroup* of H
- $(G/H)^G = \emptyset$

Application: If G is simple, $H < G$ proper, and $[G : H] = n$, then there exists an injective map $\varphi : G \hookrightarrow S_n$.
Proof: This action induces φ; it is nontrivial since $gH = H$ for all g implies $H = G$; $\ker \varphi \trianglelefteq G$ and G simple implies $\ker \varphi = 1$.

Theorem 4.9 ***(Burnside's Formula).***

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

4.5.3 Sylow Theorems

Notation: For any p, let $\mathrm{Syl}_p(G)$ be the set of Sylow-p subgroups of G.

Write

- $|G| = p^k m$ where $(p, m) = 1$,
- S_p a Sylow-p subgroup, and
- n_p the number of Sylow-p subgroups.

Definition 4.9.1.
A p-group is a group G such that every element is order p^k for some k. If G is a finite p-group, then $|G| = p^j$ for some j.

Some useful facts:

- Coprime order subgroups are disjoint, or more generally $\mathbb{Z}_p, \mathbb{Z}_q \subset G \implies \mathbb{Z}_p \bigcap \mathbb{Z}_q = \mathbb{Z}_{(p,q)}$.
- The Chinese Remainder theorem: $(p, q) = 1 \implies \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$

4.5.4 Sylow 1 (Cauchy for Prime Powers)

Idea: Sylow p-subgroups exist for any p dividing $|G|$, and are maximal in the sense that every p-subgroup of G is contained in a Sylow p-subgroup.

$$\forall p^n \text{ dividing } |G| \quad \text{there exists a subgroup of size } p^n.$$

If $|G| = \prod p_i^{\alpha_i}$, then there exist subgroups of order $p_i^{\beta_i}$ for every i and every $0 \leq \beta_i \leq \alpha_i$.

In particular, Sylow p-subgroups always exist.

4.5.5 Sylow 2 (Sylows are Conjugate)

All sylow-p subgroups S_p are conjugate, i.e.

$$S_p^1, S_p^2 \in \mathrm{Syl}_p(G) \implies \exists g \text{ such that } g S_p^1 g^{-1} = S_p^2.$$

Corollary: $n_p = 1 \iff S_p \trianglelefteq G$

4.5.6 Sylow 3 (Numerical Constraints)

1. $n_p \mid m$ (in particular, $n_p \leq m$),
2. $n_p \equiv 1 \mod p$,
3. $n_p = [G : N_G(S_p)]$ where N_G is the normalizer.

Corollary: p does not divide n_p.

Lemma: Every p-subgroup of G is contained in a Sylow p-subgroup.

Proof: Let $H \leq G$ be a p-subgroup. If H is not *properly* contained in any other p-subgroup, it is a Sylow p-subgroup by definition.
Otherwise, it is contained in some p-subgroup H^1. Inductively this yields a chain $H \subsetneq H^1 \subsetneq \cdots$, and by Zorn's lemma $H := \bigcup_i H^i$ is maximal and thus a Sylow p-subgroup.

Theorem 4.10 ***(Fratini's Argument).***
If $H \trianglelefteq G$ and $P \in \mathrm{Syl}_p(G)$, then $HN_G(P) = G$ and $[G : H]$ divides $|N_G(P)|$.

4.6 Products

Theorem 4.11 ***(Recognizing Direct Products).***
We have $G \cong H \times K$ when

- $H, K \trianglelefteq G$
- $G = HK$.
- $H \bigcap K = \{e\} \subset G$

Note: can relax to $[h, k] = 1$ for all h, k.

Theorem 4.12 ***(Recognizing Generalized Direct Products).***
We have $G = \prod_{i=1}^{n} H_i$ when

- $H_i \trianglelefteq G$ for all i.
- $G = H_1 \cdots H_n$
- $H_k \bigcap H_1 \cdots \widehat{H_k} \cdots H_n = \emptyset$

Note on notation: intersect H_k with the amalgam *leaving out* H_k.

Theorem 4.13 ***(Recognizing Semidirect Products).***
We have $G = N \rtimes_\psi H$ when

- $G = NH$
- $N \trianglelefteq G$
- $H \curvearrowright N$ by conjugation via a map

$$\begin{aligned} \psi : H &\longrightarrow \mathrm{Aut}(N) \\ h &\mapsto h(\,\cdot\,)h^{-1}. \end{aligned}$$

Note relaxed conditions compared to direct product: $H \trianglelefteq G$ and $K \leq G$ to get a semidirect product instead

Useful Facts

- If $\sigma \in \mathrm{Aut}(H)$, then $N \rtimes_\psi H \cong N \rtimes_{\psi \circ \sigma} H$.

- $\mathrm{Aut}((\mathbb{Z}/(p)^n) \cong \mathrm{GL}(n, \mathbb{F}_p)$, which has size $|\mathrm{Aut}(\mathbb{Z}/(p)^n)| = (p^n - 1)(p^n - p) \cdots (p^n - p^{n-1})$.
  - If this occurs in a semidirect product, it suffices to consider similarity classes of matrices (i.e. just use canonical forms)
- $\mathrm{Aut}(\mathbb{Z}/(n)) \cong \mathbb{Z}/(n)^\times \cong \mathbb{Z}/(\varphi(n))$ where φ is the totient function.
  - $\varphi(p^k) = p^{k-1}(p-1)$
- If G, H have coprime order then $\mathrm{Aut}(G \oplus H) \cong \mathrm{Aut}(G) \oplus \mathrm{Aut}(H)$.

4.7 Isomorphism Theorems

Theorem 4.14 ***(1st Isomorphism Theorem).***
If $\varphi : G \longrightarrow H$ is a group morphism then $G/\ker\varphi \cong \mathrm{im}\ \varphi$.
Note: for this to make sense, we also have

- $\ker\varphi \trianglelefteq G$
- $\mathrm{im}\ \varphi \leq G$

Corollary 4.15.
If $\varphi : G \longrightarrow H$ is surjective then $H \cong G/\ker\varphi$.

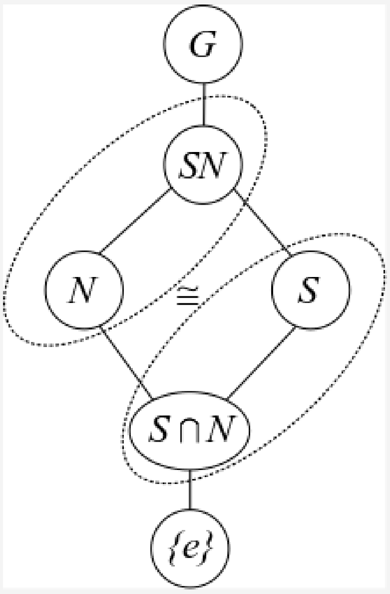
Lemma 4.16.
If $H, K \leq G$ and $H \leq N_G(K)$ (or $K \trianglelefteq G$) then $HK \leq G$ is a subgroup.

Theorem 4.17 ***(Diamond Theorem / 2nd Isomorphism Theorem).***
If $S \leq G$ and $N \trianglelefteq G$, then

$$\frac{SN}{N} \cong \frac{S}{S \bigcap N} \quad \text{and} \quad |SN| = \frac{|S||N|}{|S \bigcap N|}$$

Note: for this to make sense, we also have

- $SN \leq G$,
- $S \bigcap N \trianglelefteq S$,



Corollary 4.18.
If we relax the conditions to $S, N \leq G$ with $S \in N_G(N)$, then $S \bigcap N \trianglelefteq S$ (but is not normal in G) and the theorem still applies.

Theorem 4.19 ***(Cancellation / 3rd Isomorphism Theorem).***
Suppose $N, K \leq G$ with $N \trianglelefteq G$ and $N \subseteq K \subseteq G$.

1. If $K \leq G$ then $K/N \leq G/N$ is a subgroup
2. If $K \trianglelefteq G$ then $K/N \trianglelefteq G/N$.
3. Every subgroup of G/N is of the form K/N for some such $K \leq G$.
4. Every *normal* subgroup of G/N is of the form K/N for some such $K \trianglelefteq G$.
5. If $K \trianglelefteq G$, then we can cancel normal subgroups:

$$\frac{G/N}{K/N} \cong \frac{G}{K}.$$

Theorem 4.20 ***(The Correspondence Theorem / 4th Isomorphism Theorem).***
Suppose $N \trianglelefteq G$, then there exists a correspondence:

$$\left\{H < G \;\middle|\; N \subseteq H\right\} \iff \left\{H \;\middle|\; H < \frac{G}{N}\right\}$$
$$\left\{\begin{matrix}\text{Subgroups of } G \\ \text{containing } N\end{matrix}\right\} \iff \left\{\begin{matrix}\text{Subgroups of the} \\ \text{quotient } G/N\end{matrix}\right\}.$$

In words, subgroups of G containing N correspond to subgroups of the quotient group G/N. This is given by the map $H \mapsto H/N$.

Note: $N \trianglelefteq G$ and $N \subseteq H < G \implies N \trianglelefteq H$.

4.8 Special Classes of Groups

Definition 4.20.1 (2 out of 3 Property)**.**
The **"2 out of 3 property"** is satisfied by a class of groups $\mathcal{C}$ iff whenever $G \in \mathcal{C}$, then $N, G/N \in \mathcal{C}$ for any $N \trianglelefteq G$.

Definition 4.20.2 (p-groups)**.**
If $|G| = p^k$, then G is a **p-group.**

Definition 4.20.3 (Normalizers Grow)**.**
If for every proper $H < G$, $H \trianglelefteq N_G(H)$ is again proper, then "normalizers grow" in G.

4.9 Classification of Groups

General strategy: find a normal subgroup (usually a Sylow) and use recognition of semidirect products.

- Keith Conrad: Classifying Groups of Order 12
- Order p: cyclic.
- Order p^2q: ?

4.10 Groups of Small Order

4.11 Series of Groups

Definition 4.20.4.
A **normal series** of a group G is a sequence $G \longrightarrow G^1 \longrightarrow G^2 \longrightarrow \cdots$ such that $G^{i+1} \trianglelefteq G_i$ for every i.

Definition 4.20.5.
A **central series** for a group G is a terminating normal series $G \longrightarrow G^1 \longrightarrow \cdots \longrightarrow \{e\}$ such that each quotient is **central**, i.e. $[G, G^i] \leq G^{i-1}$ for all i.

Definition 4.20.6 (Composition Series)**.**
A **composition series** of a group G is a finite normal series such that G^{i+1} is a *maximal proper* normal subgroup of G^i.

Theorem 4.21 ***(Jordan-Holder).***
Any two composition series of a group have the same length and isomorphic composition factors (up to permutation).

Definition 4.21.1 (Simple Groups)**.**
A group G is **simple** iff $H \trianglelefteq G \implies H = \{e\}, G$, i.e. it has no non-trivial proper subgroups.

Lemma 4.22.
If G is *not* simple, then for any $N \trianglelefteq G$, it is the case that $G \cong E$ for an extension of the form $N \longrightarrow E \longrightarrow G/N$.

Definition 4.22.1 (Lower Central Series)**.**
Set $G^0 = G$ and $G^{i+1} = [G, G^i]$, then $G^0 \geq G^1 \geq \cdots$ is the *lower central series* of G.

Mnemonic: "lower" because the chain is descending. Iterate the adjoint map $[\,\cdot\,, G]$, if this terminates then the map is nilpotent, so call G nilpotent!

Definition 4.22.2 (Upper Central Series)**.**
Set $Z_0 = 1$, $Z_1 = Z(G)$, and $Z_{i+1} \leq G$ to be the subgroup satisfying $Z_{i+1}/Z_i = Z(G/Z_i)$. Then $Z_0 \leq Z_1 \leq \cdots$ is the *upper central series* of G.

Equivalently, since $Z_i \trianglelefteq G$, there is a quotient map $\pi : G \longrightarrow G/Z_i$, so define $Z_{i+1} \coloneqq \pi^{-1}(Z(G/Z_i))$ (?).

Mnemonic: "upper" because the chain is ascending. "Take higher centers".

Definition 4.22.3 (Derived Series)**.**
Set $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$, then $G^{(0)} \geq G^{(1)} \geq \cdots$ is the *derived series* of G.

Definition 4.22.4 (Solvable)**.**
A group G is **solvable** iff G has a terminating normal series with abelian composition factors, i.e.

$$G \longrightarrow G^1 \longrightarrow \cdots \longrightarrow \{e\} \text{ with } G^i/G^{i+1} \text{ abelian for all } i.$$

Theorem 4.23.
A group G is solvable iff its derived series terminates.

Theorem 4.24.
If $n \geq 4$ then S_n is solvable.

Lemmas

- G is solvable iff G has a terminating *derived series.*
- Solvable groups satisfy the 2 out of 3 property
- Abelian $\implies$ solvable
- Every group of order less than 60 is solvable.

Definition 4.24.1 (Nilpotent)**.**
A group G is **nilpotent** iff G has a terminating upper central series.

Moral: the adjoint map is nilpotent.

Theorem 4.25.
A group G is nilpotent iff all of its Sylow p-subgroups are normal for every p dividing $|G|$.

Theorem 4.26.
A group G is nilpotent iff every maximal subgroup is normal.

Theorem 4.27.
G is nilpotent iff G has an upper central series terminating at G.

Theorem 4.28.
G is nilpotent iff G has a lower central series terminating at 1.

Lemma: For G a finite group, TFAE:

- G is nilpotent
- Normalizers grow (i.e.$H < N_G(H)$ whenever H is proper)
- Every Sylow-p subgroup is normal
- G is the direct product of its Sylow p-subgroups
- Every maximal subgroup is normal
- G has a terminating *Lower* Central Series
- G has a terminating *Upper* Central Series

Lemmas:

- G nilpotent $\implies$ G solvable
- Nilpotent groups satisfy the 2 out of 3 property.
- G has normal subgroups of order d for *every* d dividing $|G|$
- G nilpotent $\implies$ $Z(G) \neq 0$
- Abelian $\implies$ nilpotent
- p-groups $\implies$ nilpotent

5 Rings

5.1 Definitions

Definition 5.0.1 (Irreducible Element)**.**
An element $r \in R$ is **irreducible** iff $r = ab \implies a$ is a unit or b is a unit.

Definition 5.0.2 (Prime Element)**.**
An element $r \in R$ is **prime** iff $ab \mid r \implies a \mid r$ or $b \mid r$ whenever a, b are nonzero and not units.

Definition 5.0.3 (Integral Domain)**.**
?

Definition 5.0.4 (Principal Ideal Domain)**.**
?

Definition 5.0.5 (Unique Factorization Domain).
?

Definition 5.0.6 (Noetherian).
A ring R is Noetherian if the ACC holds: every ascending chain of ideals $I_1 \leq I_2 \cdots$ stabilizes.

Theorem 5.1 ***(Zorn's Lemma).***
If P is a poset in which every chain has an upper bound, then P has a maximal element.

Definition 5.1.1 (Principal Ideals).
$I \trianglelefteq R$ *principal* when $\exists a \in R : I = \langle a \rangle$

Definition 5.1.2 (Irreducible Ideal).
$I \trianglelefteq R$ *irreducible* when $\nexists \{J \trianglelefteq R : I \subset J\} : I = \bigcap J$

Definition 5.1.3 (Primary Ideal).
An ideal $I \trianglelefteq R$ is *primary* iff whenever $pq \in I$, $p \in I$ and $q^n \in I$ for some n.

Definition 5.1.4 (Simple Ring).
A ring R is **simple** iff every ideal $I \trianglelefteq R$ is either 0 or R.

Definition 5.1.5 (Local Ring).
A ring R is *local* iff it contains a unique maximal ideal.

Definition 5.1.6 (Prime Ideal).
$\mathfrak{p}$ is a **prime** ideal $\iff ab \in \mathfrak{p} \implies a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

Definition 5.1.7 (Prime Spectrum).
$\text{Spec}\ (R) = \left\{ \mathfrak{p} \trianglelefteq R \;\middle|\; \mathfrak{p} \text{ is prime} \right\}$ is the **spectrum** of R.

Definition 5.1.8 (Maximal Ideal).
$\mathfrak{m}$ is **maximal** $\iff I \ntrianglelefteq R \implies I \subseteq \mathfrak{m}$.

Examples:

- Maximal ideals of $R[x]$ are of the form $I = (x - a_i)$ for some $a_i \in R$.

Definition 5.1.9 (Max Spectrum).
$\text{maxSpec}\ (R) = \left\{ \mathfrak{m} \trianglelefteq R \;\middle|\; \mathfrak{m} \text{ is maximal} \right\}$ is the **max-spectrum** of R.

Definition 5.1.10 (Nilradical).
$\mathfrak{N}(R) := \left\{ x \in R \;\middle|\; x^n = 0 \text{ for some } n \right\}$ is the **nilradical** of R.

Definition 5.1.11 (Jacobson Radical)**.**
The **Jacobson radical** $\mathfrak{J}(R)$ is the intersection of all maximal ideals, i.e.

$$\mathfrak{J}(R) = \bigcap_{\mathfrak{m} \in \operatorname{Spec}_{\max}} \mathfrak{m}$$

Definition (Semisimple)

A nonzero unital ring R is **semisimple** iff $R \cong \bigoplus_{i=1}^{n} M_i$ with each M_i a simple module.

Definition 5.1.12 (Radical of an Ideal)**.**
For an ideal $I \trianglelefteq R$, the radical $\operatorname{rad}(I) := \left\{ r \in R \;\middle|\; r^n \in I \text{ for some } n \geq 0 \right\}$, so $x^n \in I \iff x \in I$.

Definition 5.1.13 (Radical Ideal)**.**
An ideal is *radical* iff $\operatorname{rad}(I) = I$.

Lemma (Characterizations of Rings):

- R a commutative division ring $\implies R$ is a field
- R a finite integral domain $\implies R$ is a field.
- $\mathbb{F}$ a field $\implies \mathbb{F}[x]$ is a Euclidean domain.
- $\mathbb{F}$ a field $\implies \mathbb{F}[x]$ is a PID.
- $\mathbb{F}$ is a field $\iff \mathbb{F}$ is a commutative simple ring.
- R is a UFD $\iff R[x]$ is a UFD.
- R a PID $\implies R[x]$ is a UFD
- R a PID $\implies R$ Noetherian
- $R[x]$ a PID $\implies R$ is a field.

Lemma: Fields $\subset$ Euclidean domains $\subset$ PIDs $\subset$ UFDs $\subset$ Integral Domains $\subset$ Rings

- A Euclidean Domain that is not a field: $\mathbb{F}[x]$ for $\mathbb{F}$ a field
  - *Proof*: Use previous lemma, and x is not invertible
- A PID that is not a Euclidean Domain: $\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$.
  - *Proof*: complicated.
- A UFD that is not a PID: $\mathbb{F}[x, y]$.
  - *Proof*: $\langle x, y \rangle$ is not principal
- An integral domain that is not a UFD: $\mathbb{Z}[\sqrt{-5}]$
  - *Proof*: $(2+\sqrt{-5})(2-\sqrt{-5}) = 9 = 3 \cdot 3$, where all factors are irreducible (check norm).
- A ring that is not an integral domain: $\mathbb{Z}/(4)$
  - *Proof*: $2 \mod 4$ is a zero divisor.

Lemma 5.2.
In R a UFD, an element $r \in R$ is prime $\iff r$ is irreducible.

Note: For R an integral domain, prime $\implies$ irreducible, but generally not the converse.

Example of a prime that is not irreducible: $x^2 \mod (x^2+x) \in \mathbb{Q}[x]/(x^2+x)$. Check that x is prime directly, but $x = x \cdot x$ and x is not a unit.

Example of an irreducible that is not prime: $3 \in \mathbb{Z}[\sqrt{-5}]$. Check norm to see irreducibility, but $3 \mid 9 = (2+\sqrt{-5})(2-\sqrt{-5})$ and doesn't divide either factor.

Lemma 5.3.
If R is a PID, then every element in R has a unique prime factorization.

Theorem 5.4 *(Krull)*.
Every ring has proper maximal ideals, and any proper ideal is contained in a maximal ideal.

Theorem 5.5 *(Artin-Wedderubrn)*.
If R is a nonzero, unital, *semisimple* ring then $R \cong \bigoplus_{i=1}^{m} \mathrm{Mat}(n_i, D_i)$, a finite sum of matrix rings over division rings.

Corollary 5.6.
If M is a simple ring over R a division ring, the M is isomorphic to a matrix ring.

5.1.1 Zorn's Lemma

Lemma 5.7.
Fields are simple rings.

Lemma 5.8.
If $I \trianglelefteq R$ is a proper ideal $\iff$ I contains no units.

Proof .
$r \in R^\times \bigcap I \implies r^{-1} r \in I \implies 1 \in I \implies x \cdot 1 \in I \quad \forall x \in R.$

■

Lemma 5.9.
If $I_1 \subseteq I_2 \subseteq \cdots$ are ideals then $\bigcup_j I_j$ is an ideal.

Example Application: Every proper ideal is contained in a maximal ideal.

Proof .
Let $0 < I < R$ be a proper ideal, and consider the set

$$S = \left\{ J \;\middle|\; I \subseteq J < R \right\}.$$

Note $I \in S$, so S is nonempty. The claim is that S contains a maximal element M.
S is a poset, ordered by set inclusion, so if we can show that every chain has an upper bound, we can apply Zorn's lemma to produce M.

Let $C \subseteq S$ be a chain in S, so $C = \{C_1 \subseteq C_2 \subseteq \cdots\}$ and define $\widehat{C} = \bigcup_i C_i$.

$\widehat{C}$ is an upper bound for C: This follows because every $C_i \subseteq \widehat{C}$.
$\widehat{C}$ is in S: Use the fact that $I \subseteq C_i < R$ for every C_i and since no C_i contains a unit, $\widehat{C}$ doesn't contain a unit, and is thus proper.

■

6 Fields

Let k denote a field.

Lemmas:

- The characteristic of any field k is either 0 or p a prime.
- All fields are simple rings (no proper nontrivial ideals).
- If L/k is algebraic, then $\min(\alpha, L)$ divides $\min(\alpha, k)$.
- Every field morphism is either zero or injective.

Theorem 6.1.
Every finite extension is algebraic.

Proof .
Todo?

■

Theorem 6.2 *(Gauss' Lemma)*.
Let R be a UFD and F its field of fractions. Then a primitive $p \in R[x]$ is irreducible in $R[x] \iff p$ is irreducible in $F[x]$.

Corollary 6.3.
A primitive polynomial $p \in \mathbb{Q}[x]$ is irreducible $\iff$ p is irreducible in $\mathbb{Z}[x]$.

Theorem 6.4 *(Eisenstein's Criterion)*.
If $f(x) = \sum_{i=0}^{n} \alpha_i x^i \in \mathbb{Q}[x]$ and $\exists p$ such that

- p divides every coefficient *except* a_n and
- p^2 does not divide a_0,

then f is irreducible over $\mathbb{Q}[x]$, and by Gauss' lemma, over $\mathbb{Z}[x]$.

Definition 6.4.1 (Primitive)**.**
For R a UFD, a polynomial $p \in R[x]$ is **primitive** iff the greatest common divisors of its coefficients is a unit.

6.1 Finite Fields

Definition 6.4.2.
The **prime subfield** of a field F is the subfield generated by 1.

Lemma 6.5 ***(Characterization of Prime Subfields).***
The prime subfield of any field is isomorphic to either $\mathbb{Q}$ or $\mathbb{F}_p$ for some p.

Proposition 6.6 ***(Freshman's Dream).***
If $\operatorname{char} k = p$ then $(a+b)^p = a^p + b^p$ and $(ab)^p = a^p b^p$.

Proof .
Todo

■

Theorem 6.7 ***(Construction of Finite Fields).***
$\mathbb{GF}(p^n) \cong \dfrac{\mathbb{F}_p}{(f)}$ where $f \in \mathbb{F}_p[x]$ is any irreducible of degree n, and $\mathbb{GF}(p^n) \cong \mathbb{F}[\alpha] \cong \operatorname{span}_{\mathbb{F}} \left\{1, \alpha, \cdots, \alpha^{n-1}\right\}$ for any root α of f.

Lemma 6.8 ***(Prime Subfields of Finite Fields).***
Every finite field F is isomorphic to a unique field of the form $\mathbb{GF}(p^n)$ and if $\operatorname{char} F = p$, it has prime subfield $\mathbb{F}_p$.

Lemma 6.9 ***(Containment of Finite Fields).***
$\mathbb{GF}(p^\ell) \leq \mathbb{GF}(p^k) \iff \ell$ divides k.

Lemma 6.10 ***(Identification of Finite Fields as Splitting Fields).***
$\mathbb{GF}(p^n)$ is the splitting field of $\rho(x) = x^{p^n} - x$, and the elements are exactly the roots of ρ.

Proof .
Todo. Every element is a root by Cauchy's theorem, and the p^n roots are distinct since its derivative is identically -1.

■

Lemma 6.11 ***(Splits Product of Irreducibles).***
Let $\rho_n := x^{p^n} - x$. Then $f(x) \mathrel{\Big|} \rho_n(x) \iff \deg f \mathrel{\Big|} n$ and f is irreducible.

Corollary 6.12.
$x^{p^n} - x = \prod f_i(x)$ over all irreducible monic $f_i \in \mathbb{F}_p[x]$ of degree d dividing n.

Proof .

$\Longleftarrow$: Suppose f is irreducible of degree d. Then $f \mid x^{p^d} - x$ (consider $F[x]/\langle f\rangle$) and $x^{p^d} - x \mid x^{p^n} - x \iff d \mid n$.

$\Longrightarrow$:

- $\alpha \in \mathbb{GF}(p^n) \iff \alpha^{p^n} - \alpha = 0$, so every element is a root of φ_n and $\deg \min(\alpha, \mathbb{F}_p) \mid n$ since $\mathbb{F}_p(\alpha)$ is an intermediate extension.
- So if f is an irreducible factor of φ_n, f is the minimal polynomial of some root α of φ_n, so $\deg f \mid n$. $\varphi_n'(x) = p^n x^{p^{n-1}} \neq 0$, so φ_n has distinct roots and thus no repeated factors. So φ_n is the product of all such irreducible f.

■

Lemma 6.13.
No finite field is algebraically closed.

Proof .
Todo?

■

6.2 Galois Theory

Definition 6.13.1.
A field extension L/k is **algebraic** iff every $\alpha \in L$ is the root of some polynomial $f \in k[x]$.

Definition 6.13.2.
Let L/k be a finite extension. Then TFAE:

- L/k is **normal**.
- Every irreducible $f \in k[x]$ that has one root in L has *all* of its roots in L
  - i.e. every polynomial splits into linear factors
- Every embedding $\sigma : L \hookrightarrow \overline{k}$ that is a lift of the identity on k satisfies $\sigma(L) = L$.
- If L is separable: L is the splitting field of some irreducible $f \in k[x]$.

Definition 6.13.3.
Let L/k be a field extension, $\alpha \in L$ be arbitrary, and $f(x) := \min(\alpha, k)$. TFAE:

- L/k is **separable**
- f has no repeated factors/roots
- $\gcd(f, f') = 1$, i.e. f is coprime to its derivative
- $f' \not\equiv 0$

Lemma 6.14.
If $\operatorname{char} k = 0$ or k is finite, then every *algebraic* extension L/k is separable.

Definition 6.14.1.
$\mathrm{Aut}(L/k) = \left\{ \sigma : L \longrightarrow L \;\middle|\; \sigma|_k = \mathrm{id}_k \right\}$.

Lemma 6.15.
If L/k is algebraic, then $\mathrm{Aut}(L/k)$ permutes the roots of irreducible polynomials.

Lemma 6.16.
$|\mathrm{Aut}(L/k)| \leq [L : k]$ with equality precisely when L/k is normal.

Definition 6.16.1.
If L/k is Galois, we define $\mathrm{Gal}(L/k) := \mathrm{Aut}(L/k)$.

6.2.1 Lemmas About Towers

Let $L/F/k$ be a finite tower of field extensions

- Multiplicativity: $[L : k] = [L : F][F : k]$
- L/k normal/algebraic/Galois $\implies$ L/F normal/algebraic/Galois.
  - *Proof (normal):* $\min(\alpha, F) \,\Big|\, \min(\alpha, k)$, so if the latter splits in L then so does the former.
  - *Corollary:* $\alpha \in L$ algebraic over $k \implies \alpha$ algebraic over F.
  - *Corollary:* E_1/k normal and E_2/k normal $\implies E_1E_2/k$ normal and $E_1 \bigcap E_2/k$ normal.

$$\begin{tikzcd} L \arrow[d, dotted, bend right] \arrow[dd, bend left] \\ F \\ k \end{tikzcd}$$

- F/k algebraic and L/F algebraic $\implies$ L/k algebraic.
- If L/k is algebraic, then F/k separable and L/F separable $\iff$ L/k separable

$$\begin{tikzcd} L \arrow[d, bend right] \arrow[dd, dashed, bend left] \\ F \arrow[d, bend right] \\ k \end{tikzcd}$$

- F/k Galois and L/K Galois $\implies$ F/k Galois **only if** $\mathrm{Gal}(L/F) \trianglelefteq \mathrm{Gal}(L/k)$
  - $\implies \mathrm{Gal}(F/k) \cong \dfrac{\mathrm{Gal}(L/k)}{\mathrm{Gal}(L/F)}$

$$\begin{tikzcd} L \arrow[d, dotted, bend right] \arrow[dd, bend left] \\ F \arrow[d, bend right] \\ k \end{tikzcd}$$

Common Counterexamples:

- $\mathbb{Q}(\zeta_3, 2^{1/3})$ is normal but $\mathbb{Q}(2^{1/3})$ is not since the irreducible polynomial $x^3 - 2$ has only one root in it.

Definition 6.16.2 (Characterizations of Galois Extensions)**.**
Let L/k be a finite field extension. TFAE:
- L/k is **Galois**
- L/k is finite, normal, and separable.
- L/k is the splitting field of a separable polynomial
- $|\mathrm{Aut}(L/k)| = [L : k]$
- The fixed field of $\mathrm{Aut}(L/k)$ is exactly k.

Theorem 6.17 ***(Fundamental Theorem of Galois Theory).***
Let L/k be a Galois extension, then there is a correspondence:

$$\left\{\text{Subgroups } H \leq \mathrm{Gal}(L/k)\right\} \iff \left\{\begin{matrix}\text{Fields } F \text{ such} \\ \text{that } L/F/k\end{matrix}\right\}$$
$$H \to \left\{E^H := \text{The fixed field of } H\right\}$$
$$\left\{\mathrm{Gal}(L/F) := \left\{\sigma \in \mathrm{Gal}(L/k) \;\middle|\; \sigma(F) = F\right\}\right\} \leftarrow F.$$

- This is contravariant with respect to subgroups/subfields.
- $[F : k] = [G : H]$, so degrees of extensions over the base field correspond to indices of subgroups.
- $[K : F] = |H|$
- L/F is Galois and $Gal(K/F) = H$
- F/k is Galois $\iff$ H is normal, and $\mathrm{Gal}(F/k) = \mathrm{Gal}(L/k)/H$.
- The compositum F_1F_2 corresponds to $H_1 \bigcap H_2$.
- The subfield $F_1 \bigcap F_2$ corresponds to H_1H_2.

6.2.2 Examples

1. $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong \mathbb{Z}/(n)^\times$ and is generated by maps of the form $\zeta_n \mapsto \zeta_n^j$ where $(j, n) = 1$.

   I.e., the following map is an isomorphism:

$$\begin{aligned} \mathbb{Z}/(n)^\times &\longrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_n), \mathbb{Q}) \\ r \mod n &\mapsto (\varphi_r : \zeta_n \mapsto \zeta_n^r). \end{aligned}$$

2. $\mathrm{Gal}(\mathbb{GF}(p^n)/\mathbb{F}_p) \cong \mathbb{Z}/(n)$, a cyclic group generated by powers of the Frobenius automorphism:

$$\begin{aligned} \varphi_p : \mathbb{GF}(p^n) &\longrightarrow \mathbb{GF}(p^n) \\ x &\mapsto x^p. \end{aligned}$$

Lemma 6.18.
Every quadratic extension is Galois.

Lemma 6.19.
If K is the splitting field of an irreducible polynomial of degree n, then $\mathrm{Gal}(K/\mathbb{Q}) \leq S_n$ is a transitive subgroup.

Corollary 6.20.
n divides the order $|\mathrm{Gal}(K/\mathbb{Q})|$.

Definition 6.20.1.
TFAE:

- k is a **perfect** field.
- Every irreducible polynomial $p \in k[x]$ is separable
- Every finite extension F/k is separable.
- If char $k > 0$, the Frobenius is an automorphism of k.

Theorem 6.21.

- If char $k = 0$ or k is finite, then k is perfect.
- $k = \mathbb{Q}, \mathbb{F}_p$ are perfect, and any finite normal extension is Galois.
- Every splitting field of a polynomial over a perfect field is Galois.

Proposition 6.22 ***(Composite Extensions).***
If F/k is finite and Galois and L/k is arbitrary, then FL/L is Galois and

$$\mathrm{Gal}(FL/L) = \mathrm{Gal}(F/F \bigcap L) \subset \mathrm{Gal}(F/k).$$

6.3 Cyclotomic Polynomials

Definition 6.22.1 (Cyclotomic Polynomials)**.**
Let $\zeta_n = e^{2\pi i/n}$, then the n**th cyclotomic polynomial** is given by

$$\Phi_n(x) = \prod_{\substack{k=1 \\ (j,n)=1}}^{n} \left(x - \zeta_n^k\right),$$

which is a product over primitive roots of unity. It is the unique irreducible polynomial which is a divisor of $x^n - 1$ but *not* a divisor of $x^k - 1$ for any $k < n$.

Proposition 6.23.
$\deg \Phi_n(x) = \varphi(n)$ for φ the totient function.

Proof .
$\deg \Phi_n(x)$ is the number of nth primitive roots, which is the number of numbers less than and coprime to n.

■

Computing Φ_n:

1.

$$\Phi_n(z) = \prod_{d|n, d>0} \left(z^d - 1\right)^{\mu\left(\frac{n}{d}\right)}$$

where

$$\mu(n) \equiv \begin{cases} 0 & \text{if } n \text{ has one or more repeated prime factors} \\ 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes,} \end{cases}$$

2.

$$x^n - 1 = \prod_{d|n} \Phi_d(x) \implies \Phi_n(x) = \frac{x^n - 1}{\prod_{\substack{d|n \\ d<n}} \Phi_d(x)},$$

so just use polynomial long division.

Lemma 6.24.

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$
$$\Phi_{2p}(x) = x^{p-1} - x^{p-2} + \cdots - x + 1.$$

Lemma 6.25.

$$k \,\Big|\, n \implies \Phi_{nk}(x) = \Phi_n\left(x^k\right)$$

Definition 6.25.1.
An extension F/k is **simple** if $F = k[\alpha]$ for a single element α.

Theorem 6.26 ***(Primitive Element).***
Every finite separable extension is simple.

Corollary 6.27.
$\mathbb{GF}(p^n)$ is a simple extension over $\mathbb{F}_p$.

7 Modules

7.1 General Modules

Definition: A module is **simple** iff it has no nontrivial proper submodules.

Definition: A **free** module is a module with a basis (i.e. a spanning, linearly independent set).

> *Example*: $\mathbb{Z}/(6)$ is a $\mathbb{Z}$-module that is *not* free.

Definition: A module M is **projective** iff M is a direct summand of a free module $F = M \oplus \cdots$.

> Free implies projective, but not the converse.

Definition: A sequence of homomorphisms $0 \xrightarrow{d_1} A \xrightarrow{d_2} B \xrightarrow{d_3} C \longrightarrow 0$ is *exact* iff $\operatorname{im} d_i = \ker d_{i+1}$.

Lemma: If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence, then

- C free $\implies$ the sequence splits
- C projective $\implies$ the sequence splits
- A injective $\implies$ the sequence splits

Moreover, if this sequence splits, then $B \cong A \oplus C$.

7.2 Classification of Modules over a PID

Let M be a finitely generated modules over a PID R. Then there is an invariant factor decomposition

$$M \cong F \bigoplus R/(r_i) \quad \text{where } r_1 \,\Big|\, r_2 \,\Big|\, \cdots,$$

and similarly an elementary divisor decomposition.

7.3 Minimal / Characteristic Polynomials

Fix some notation:

$$\begin{aligned} \min_A(x) &: \quad \text{The minimal polynomial of } A \\ \chi_A(x) &: \quad \text{The characteristic polynomial of } A. \end{aligned}$$

Definition: The minimal polynomial is the unique polynomial $\min_A(x)$ of minimal degree such that $\min_A(A) = 0$.

Definition: The **characteristic polynomial** of A is given by

$$\chi_A(x) = \det(A - xI)) = \det(SNF(A - xI)).$$

> *Useful lemma:* If A is upper triangular, then $\det(A) = \prod_i a_{ii}$

Theorem (Cayley-Hamilton): The minimal polynomial divides the characteristic polynomial, and in particular $\chi_A(A) = 0$.

Lemma: Writing

$$\begin{aligned} \min_A(x) &= \prod (x - \lambda_i)^{a_i} \\ \chi_A(x) &= \prod (x - \lambda_i)^{b_i} \end{aligned}$$

- $a_i \leq b_i$
- The roots both polynomials are precisely the eigenvalues of A.

Proof: By Cayley-Hamilton, $\min_A$ divides χ_A. Every λ_i is a root of μ_M:
Let $(\mathbf{v}_i, \lambda_i)$ be a nontrivial eigenpair. Then by linearity,

$$\min_A(\lambda_i)\mathbf{v}_i = \min_A(A)\mathbf{v}_i = \mathbf{0},$$

which forces $\min_A(\lambda_i) = 0$.

Definition: Two matrices A, B are **similar** (i.e. $A = PBP^{-1}$) $\iff$ A, B have the same Jordan Canonical Form (JCF).

Definition: Two matrices A, B are **equivalent** (i.e. $A = PBQ$) $\iff$

- They have the same rank,
- They have the same invariant factors, *and*
- They have the same (JCF)

Finding the minimal polynomial:

Let $m(x)$ denote the minimal polynomial A.

1. Find the characteristic polynomial $\chi(x)$; this annihilates A by Cayley-Hamilton. Then $m(x) \mathrel{\Big|} \chi(x)$, so just test the finitely many products of irreducible factors.
2. Pick any $\mathbf{v}$ and compute $T\mathbf{v}, T^2\mathbf{v}, \cdots T^k\mathbf{v}$ until a linear dependence is introduced. Write this as $p(T) = 0$; then $\min_A(x) \mathrel{\Big|} p(x)$.

Definition: Given a monic $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + x^n$, the **companion matrix** of p is given by

$$C_p \coloneqq \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}.$$

7.4 Canonical Forms

7.4.1 Rational Canonical Form

Corresponds to the **Invariant Factor Decomposition** of T.

Lemma: $RCF(A)$ is a block matrix where each block is the companion matrix of an invariant factor of A.

Derivation:

- Let $k[x] \curvearrowright V$ using T, take invariant factors a_i,
- Note that $T \curvearrowright V$ by multiplication by x
- Write $\overline{x} = \pi(x)$ where $F[x] \xrightarrow{\pi} F[x]/(a_i)$; then $\operatorname{span}\{\overline{x}\} = F[x]/(a_i)$.
- Write $a_i(x) = \sum b_i x^i$, note that $V \longrightarrow F[x]$ pushes $T \curvearrowright V$ to $T \curvearrowright k[x]$ by multiplication by $\overline{x}$
- WRT the basis $\overline{x}$, T then acts via the companion matrix on this summand.
- Each invariant factor corresponds to a block of the RCF.

7.4.2 Jordan Canonical Form

Corresponds to the **Elementary Divisor Decomposition** of T.

Lemma: The elementary divisors of A are the minimal polynomials of the Jordan blocks.

Lemma: Writing

$$\min_A(x) = \prod (x - \lambda_i)^{a_i}$$
$$\chi_A(x) = \prod (x - \lambda_i)^{b_i}$$

- $a_i \leq b_i$
- a_i tells you the size of the **largest** Jordan block associated to λ_i,
- b_i is the **sum of sizes** of all Jordan blocks associated to λ_i
- $\dim E_{\lambda_i}$ is the **number of Jordan blocks** associated to λ_i

7.5 Using Canonical Forms

Lemma: The characteristic polynomial is the *product of the invariant factors*, i.e.

$$\chi_A(x) = \prod_{j=1}^{n} f_j(x).$$

Lemma: The minimal polynomial of A is the *invariant factor of highest degree*, i.e.

$$\min_A(x) = f_n(x).$$

Lemma: For a linear operator on a vector space of nonzero finite dimension, TFAE:

- The minimal polynomial is equal to the characteristic polynomial.
- The list of invariant factors has length one.
- The Rational Canonical Form has a single block.
- The operator has a matrix similar to a companion matrix.
- There exists a *cyclic vector* $\mathbf{v}$ such that $\operatorname{span}_k \left\{ T^j \mathbf{v} \;\middle|\; j = 1, 2, \cdots \right\} = V$.
- T has $\dim V$ distinct eigenvalues

7.6 Diagonalizability

Notation: A^* denotes the conjugate tranpose of A.

Lemma: Let V be a vector space over k an algebraically closed and $A \in \operatorname{End}(V)$. Then if $W \subseteq V$ is an invariant subspace, so $A(W) \subseteq W$, the A has an eigenvector in W.

Theorem (The Spectral Theorem):

1. Hermitian matrices (i.e. $A^* = A$) are diagonalizable over $\mathbb{C}$.
2. Symmetric matrices (i.e. $A^t = A$) are diagonalizable over $\mathbb{R}$.

Proof: Suppose A is Hermitian. Since V itself is an invariant subspace, A has an eigenvector $\mathbf{v}_1 \in V$. Let $W_1 = \operatorname{span}_k \{\mathbf{v}_1\}^\perp$. Then for any $\mathbf{w}_1 \in W_1$,

$$\langle \mathbf{v}_1,\ A\mathbf{w}_1 \rangle = \langle A\mathbf{v}_1,\ \mathbf{w}_1 \rangle = \lambda \langle \mathbf{v}_1,\ \mathbf{w}_1 \rangle = 0,$$

so $A(W_1) \subseteq W_1$ is an invariant subspace, etc.
Suppose now that A is symmetric. Then there is an eigenvector of norm 1, $\mathbf{v} \in V$.

$$\lambda = \lambda \langle \mathbf{v},\ \mathbf{v} \rangle = \langle A\mathbf{v},\ \mathbf{v} \rangle = \langle \mathbf{v},\ A\mathbf{v} \rangle = \overline{\lambda} \implies \lambda \in \mathbb{R}.$$

Lemma: $\{A_i\}$ pairwise commute $\iff$ they are all simultaneously diagonalizable.

Proof: By induction on number of operators

- A_n is diagonalizable, so $V = \bigoplus E_i$ a sum of eigenspaces
- Restrict all $n-1$ operators A to E_n.
- The commute in V so they commute in E_n
- **(Lemma)** They were diagonalizable in V, so they're diagonalizable in E_n
- So they're simultaneously diagonalizable by I.H.
- But these eigenvectors for the A_i are all in E_n, so they're eigenvectors for A_n too.
- Can do this for each eigenspace. ■

Full details here

Theorem (Characterizations of Diagonalizability)

M is diagonalizable over $\mathbb{F}$ $\iff$ $\min_M(x, \mathbb{F})$ splits into distinct linear factors over $\mathbb{F}$, or equivalently iff all of the roots of $\min_M$ lie in $\mathbb{F}$.

Proof: $\implies$: If $\min_A$ factors into linear factors, so does each invariant factor, so every elementary divisor is linear and $JCF(A)$ is diagonal.
$\impliedby$: If A is diagonalizable, every elementary divisor is linear, so every invariant factor factors into linear pieces. But the minimal polynomial is just the largest invariant factor.

7.7 Matrix Counterexamples

1. A matrix that is:

- Not diagonalizable over $\mathbb{R}$ but diagonalizable over $\mathbb{C}$
- No eigenvalues in $\mathbb{R}$ but distinct eigenvalues over $\mathbb{C}$
- $\min_M(x) = \chi_M(x) = x^2 + 1$

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \sim \left[\begin{array}{r|r} -1\sqrt{-1} & 0 \\ \hline 0 & 1\sqrt{-1} \end{array}\right].$$

2.

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

- Not diagonalizable over $\mathbb{C}$
- Eigenvalues $[1, 1]$ (repeated, multiplicity 2)
- $\min_M(x) = \chi_M(x) = x^2 - 2x + 1$

3. Non-similar matrices with the same characteristic polynomial

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

4. A full-rank matrix that is not diagonalizable:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. Matrix roots of unity:

$$\sqrt{I_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$\sqrt{-I_2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

7.8 Miscellaneous

Lemma: $I \trianglelefteq R$ is a free R-module iff I is a principal ideal.

Proof: $\implies$:
Suppose I is free as an R-module, and let $B = \{\mathbf{m}_j\}_{j \in J} \subseteq I$ be a basis so we can write $M = \langle B \rangle$.
Suppose that $|B| \geq 2$, so we can pick at least 2 basis elements $\mathbf{m}_1 \neq \mathbf{m}_2$, and consider

$$\mathbf{c} = \mathbf{m}_1\mathbf{m}_2 - \mathbf{m}_2\mathbf{m}_1,$$

which is also an element of M .
Since R is an integral domain, R is commutative, and so

$$\mathbf{c} = \mathbf{m}_1\mathbf{m}_2 - \mathbf{m}_2\mathbf{m}_1 = \mathbf{m}_1\mathbf{m}_2 - \mathbf{m}_1\mathbf{m}_2 = \mathbf{0}_M$$

However, this exhibits a linear dependence between $\mathbf{m}_1$ and $\mathbf{m}_2$, namely that there exist $\alpha_1, \alpha_2 \neq 0_R$ such that $\alpha_1\mathbf{m}_1 + \alpha_2\mathbf{m}_2 = \mathbf{0}_M$; this follows because $M \subset R$ means that we can take $\alpha_1 = -m_2, \alpha_2 = m_1$. This contradicts the assumption that B was a basis, so we must have $|B| = 1$ and so $B = \{\mathbf{m}\}$ for some $\mathbf{m} \in I$. But then $M = \langle B \rangle = \langle \mathbf{m} \rangle$ is generated by a single element, so M is principal.
$\impliedby$:
Suppose $M \trianglelefteq R$ is principal, so $M = \langle \mathbf{m} \rangle$ for some $\mathbf{m} \neq \mathbf{0}_M \in M \subset R$.
Then $x \in M \implies x = \alpha\mathbf{m}$ for some element $\alpha \in R$ and we just need to show that $\alpha\mathbf{m} = \mathbf{0}_M \implies \alpha = 0_R$ in order for $\{\mathbf{m}\}$ to be a basis for M, making M a free R-module.
But since $M \subset R$, we have $\alpha, m \in R$ and $\mathbf{0}_M = 0_R$, and since R is an integral domain, we have $\alpha m = 0_R \implies \alpha = 0_R$ or $m = 0_R$.
Since $m \neq 0_R$, this forces $\alpha = 0_R$, which allows $\{m\}$ to be a linearly independent set and thus a basis for M as an R-module.

■

Lemma 7.1.
Every $a \in R$ for a finite ring is either a unit or a zero divisor.

Proof .
Let $a \in R$ and define $\varphi(x) = ax$. If φ is injective, then it is surjective, so $1 = ax$ for some $x \implies x^{-1} = a$. Otherwise, $ax_1 = ax_2$ with $x_1 \neq x_2 \implies a(x_1 - x_2) = 0$ and $x_1 - x_2 \neq 0$, so a is a zero divisor.

■

Lemma 7.2.
Maximal $\implies$ prime, but generally not the converse.

Proof .
Suppose $\mathfrak{m}$ is maximal, $ab \in \mathfrak{m}$, and $b \notin \mathfrak{m}$. Then there is a containment of ideals $\mathfrak{m} \subsetneq \mathfrak{m} + (b) \implies \mathfrak{m} + (b) = R$.
So

$$1 = m + rb \implies a = am + r(ab),$$

but $am \in \mathfrak{m}$ and $ab \in \mathfrak{m} \implies a \in \mathfrak{m}$.

■

Counterexample: $(0) \in \mathbb{Z}$ is prime since $\mathbb{Z}$ is a domain, but not maximal since it is properly contained in any other ideal.

Lemma 7.3.
The nilradical is the intersection of all prime ideals, i.e.

$$\mathfrak{N}(R) = \bigcap_{\mathfrak{p} \in \text{Spec }(R)} \mathfrak{p}$$

Proof .
$\mathfrak{N} \subseteq \bigcap \mathfrak{p}$: $x \in \mathfrak{N} \implies x^n = 0 \in \mathfrak{p} \implies x \in \mathfrak{p}$ or $x^{n-1} \in \mathfrak{p}$.
$\mathfrak{N}^c \subseteq \bigcup \mathfrak{p}^c$: Define $S = \left\{ I \trianglelefteq R \;\middle|\; a^n \notin I \text{ for any } n \right\}$. Then apply Zorn's lemma to get a maximal ideal $\mathfrak{m}$, and maximal $\implies$ prime.

■

Lemma 7.4.
$R/\mathfrak{N}(R)$ has no nonzero nilpotent elements.

Proof .

$$\begin{aligned} a + \mathfrak{N}(R) \text{ nilpotent} &\implies (a + \mathfrak{N}(R))^n := a^n + \mathfrak{N}(R) = \mathfrak{N}(R) \\ &\implies a^n \in \mathfrak{N}(R) \\ &\implies \exists \ell \text{ such that } (a^n)^\ell = 0 \\ &\implies a \in \mathfrak{N}(R). \end{aligned}$$

■

Lemma 7.5.
$\mathfrak{N}(R) \subseteq \mathfrak{J}(R)$.

Proof .
Maximal $\implies$ prime, and so if x is in every prime ideal, it is necessarily in every maximal ideal as well.

■

8 Extra Problems

8.1 Group Theory

8.1.1 Basic Structure

Just Structure

- Show that the intersection of two subgroups is again a subgroup.
- Show that $G = H \times K$ iff the conditions for recognizing direct products hold.
- Show that if $H, K \trianglelefteq G$ and $H \bigcap K = \emptyset$, then $hk = kh$ for all $h \in H, k \in K$.

- Show that if $H, K \trianglelefteq G$ are normal subgroups that intersect trivially, then $[H, K] = 1$ (so $hk = kh$ for all k and h).
- Give a counterexample where $H, K \leq G$ but HK is not a subgroup of G.
- Show that the order of any element in a group divides the order of the group.

Cyclic Groups

- Show that any cyclic group is abelian.
- Show that every subgroup of a cyclic group is cyclic.
- Show that

$$\varphi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right).$$

- Show that $|G|/|H| = [G : H]$.
- Show that if $G/Z(G)$ is cyclic then G is abelian.
- Show that G/N is abelian iff $[G, G] \leq N$.
- Show that every normal subgroup of G is contained in $Z(G)$.
- Give an example showing that normality is not transitive: i.e. $H \trianglelefteq K \trianglelefteq G$ with H *not* normal in G.
- Show that the size of a conjugacy class divides the order of a group.

  Hint: Orbit-stabilizer

8.1.2 Centralizing and Normalizing

- Show that $C_G(H) \subseteq N_G(H) \leq G$.
- Show that $Z(G) \subseteq C_G(H) \subseteq N_G(H)$.
- Given $H \subseteq G$, let $S(H) = \bigcup_{g \in G} gHg^{-1}$, so $|S(H)|$ is the number of conjugates to H. Show that $|S(H)| = [G : N_G(H)]$.
  - That is, the number of subgroups conjugate to H equals the index of the normalizer of H.
- Show that $Z(G) = \bigcap_{a \in G} C_G(a)$.
- Show that the centralizer $G_G(H)$ of a subgroup is again a subgroup.
- Show that $C_G(H) \trianglelefteq N_G(H)$ is a normal subgroup.
- Show that $C_G(G) = Z(G)$.
- Show that for $H \leq G$, $C_H(x) = H \bigcap C_G(x)$.
- Let $H, K \leq G$ a finite group, and without using the normalizers of H or K, show that $|HK| = |H||K|/\left|H \bigcap K\right|$.

- Show that if $H \leq N_G(K)$ then $HK \leq H$, and give a counterexample showing that this condition is necessary.
- Show that HK is a subgroup of G iff $HK = KH$.
- Prove that the kernel of a homomorphism is a normal subgroup.

8.1.3 Primes in Group Theory

- Show that any group of prime order is cyclic and simple.
- Analyze groups of order pq with $q < p$.

  Hint: consider the cases when p does or does not divide $q - 1$.

  - Show that if q does not divide $p - 1$, then G is cyclic.
  - Show that G is never simple.
- Analyze groups of order $p^2 q$.

  Hint: Consider the cases when q does or does not divide $p^2 - 1$.
- Show that no group of order $p^2 q^2$ is simple for $p < q$ primes.
- Show that a group of order $p^2 q^2$ has a normal Sylow subgroup.
- Show that a group of order $p^2 q^2$ where q does not divide $p^2 - 1$ and p does not divide $q^2 - 1$ is abelian.
- Show that every group of order pqr with $p < q < r$ primes contains a normal Sylow subgroup.
  - Show that G is never simple.
- Show that any normal p- subgroup is contained in every Sylow p-subgroup of G.

8.1.4 p-Groups

- Show that every p-group has a nontrivial center.
- Show that every p-group is nilpotent.
- Show that every p-group is solvable.
- Show that every maximal subgroup of a p-group has index p.
- Show that every maximal subgroup of a p-group is normal.
- Show that every group of order p is cyclic.
- Show that every group of order p^2 is abelian and classify them.
- Show that every normal subgroup of a p-group is contained in the center.

  Hint: Consider $G/Z(G)$.
- Let $O_P(G)$ be the intersection of all Sylow p-subgroups of G. Show that $O_p(G) \trianglelefteq G$, is maximal among all normal p-subgroups of G
- Let $P \in \mathrm{Syl}_p(H)$ where $H \trianglelefteq G$ and show that $P \bigcap H \in \mathrm{Syl}_p(H)$.

- Show that Sylow p_i-subgroups S_{p_1}, S_{p_2} for distinct primes $p_1 \neq p_2$ intersect trivially.

8.1.5 Symmetric, Alternating, Dihedral Groups

- Show that the center of S_3 is trivial.
- Show that $\mathrm{Aut}(S_3) = \mathrm{Inn}(S_3) \cong S_3$.
- Show that $\mathrm{Out}(A_4)$ is nontrivial.
- Show that an m-cycle is an odd permutation iff m is an even number.
- Show that a permutation is odd iff it has an odd number of even cycles.
- Show that the center of S_n for $n \geq 4$ is nontrivial.
- Show that disjoint cycles commute.
- Show that S_n is generated by any of the following types of cycles:

Group	Generating Set	Size
S_n, $n \geq 2$	(ij)'s	$\frac{n(n-1)}{2}$
	$(12), (13), \ldots, (1n)$	$n-1$
	$(12), (23), \ldots, (n-1\ n)$	$n-1$
	$(12), (12\ldots n)$ if $n \geq 3$	2
	$(12), (23\ldots n)$ if $n \geq 3$	2
	$(ab), (12\ldots n)$ if $(b-a, n) = 1$	2
A_n, $n \geq 3$	3-cycles	$\frac{n(n-1)(n-2)}{3}$
	$(1ij)$'s	$(n-1)(n-2)$
	$(12i)$'s	$n-2$
	$(i\ i+1\ i+2)$'s	$n-2$
	$(123), (12\ldots n)$ if $n \geq 4$ odd	2
	$(123), (23\ldots n)$ if $n \geq 4$ even	2

- Show directly that any k-cycle is a product of transpositions, and determine how many transpositions are needed.
- Show that S_n is generated by transpositions.
- Show that S_n is generated by *adjacent* transpositions.
- Show that S_n is generated by $\{(12), (12\cdots n)\}$ for $n \geq 2$
- Show that S_n is generated by $\{(12), (23\cdots n)\}$ for $n \geq 3$
- Show that S_n is generated by $\{(ab), (12\cdots n)\}$ where $1 \leq a < b \leq n$ iff $\gcd(b-a, n) = 1$.
- Show that S_p is generated by any arbitrary transposition and any arbitrary p-cycle.
- Show that A_n is generated 3-cycles.

- Show that $\mathbb{Q}$ is not finitely generated as a group.
- Show that if $N \trianglelefteq D_n$ is a normal subgroup of a dihedral group, then D_n/N is again a dihedral group.
- Prove that A_n is normal in S_n.
- Argue that A_n is simple for $n \geq 5$.
- Compute $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ for n composite.
- Compute $\mathrm{Aut}((\mathbb{Z}/p\mathbb{Z})^n)$.

8.1.6 Classification

- Show that no group of order 36 is simple.
- Show that no group of order 90 is simple.
- Show that all groups of order 45 are abelian.
- Classify all groups of order 10.
- Classify the five groups of order 12.
- Classify the four groups of order 28.

8.1.7 Group Actions

- Show that the stabilizer of an element G_x is a subgroup of G.
- Show that if x, y are in the same orbit, then their stabilizers are conjugate.
- Show that the stabilizer of an element need not be a normal subgroup?
- Show that if $G \curvearrowright X$ is a group action, then the stabilizer G_x of a point is a subgroup.

8.1.8 Series of Groups

- Show that A_n is simple for $n \geq 5$
- Give a necessary and sufficient condition for a cyclic group to be solvable.
- Prove that every simple abelian group is cyclic.
- Show that S_n is generated by disjoint cycles.
- Show that S_n is generated by transpositions.
- Show if G is finite, then G is solvable $\iff$ all of its composition factors are of prime order.
- Show that if N and G/N are solvable, then G is solvable.
- Show that if G is finite and solvable then every composition factor has prime order.
- Show that G is solvable iff its derived series terminates.
- Show that S_3 is not nilpotent.

8.1.9 Misc

- Prove Burnside's theorem.
- Show that $\mathrm{Inn}(G) \trianglelefteq Aut(G)$
- Show that $\mathrm{Inn}(G) \cong G/Z(G)$
- Show that the kernel of the map $G \longrightarrow \mathrm{Aut}(G)$ given by $g \mapsto (h \mapsto ghg^{-1})$ is $Z(G)$.

- Show that $N_G(H)/C_G(H) \cong A \leq Aut(H)$
- Show that if $|G| = 12$ and has a normal subgroup of order 4, then $G \cong A_4$.

8.1.10 Nonstandard Topics

- Show that H char $G \Rightarrow H \trianglelefteq G$

  Thus "characteristic" is a strictly stronger condition than normality

- Show that H char K char $G \Rightarrow H$ char G

  So "characteristic" is a transitive relation for subgroups.

- Show that if $H \leq G$, $K \trianglelefteq G$ is a normal subgroup, and H char K then H is normal in G.

  So normality is not transitive, but strengthening one to "characteristic" gives a weak form of transitivity.

8.2 Ring Theory

Basic Structure

- Show that if an ideal $I \trianglelefteq R$ contains a unit then $I = R$.
- Show that $R^\times$ need not be closed under addition.

Ideals

- Show that every proper ideal is contained in a maximal ideal
- Show that if $x \in R$ a PID, then x is irreducible $\iff \langle x \rangle \trianglelefteq R$ is maximal.
- Show that intersections, products, and sums of ideals are ideals.
- Show that the union of two ideals need not be an ideal.
- Show that every ring has a proper maximal ideal.
- Show that $I \trianglelefteq R$ is maximal iff R/I is a field.
- Show that $I \trianglelefteq R$ is prime iff R/I is an integral domain.
- Show that $\bigcup_{\mathfrak{m} \in \text{maxSpec }(R)} = R \setminus R^\times$.
- Show that maxSpec $(R) \subsetneq$ Spec (R) but the containment is strict.
- Show that if x is not a unit, then x is contained in some maximal ideal.
- Show that if R is a finite ring then every $a \in R$ is either a unit or a zero divisor.
- Show that $R/\mathfrak{N}(R)$ has no nonzero nilpotent elements.
- Show that the nilradical is contained in the Jacobson radical.
- Show that every prime ideal is radical.
- Show that the nilradical is given by $\mathfrak{N}(R) = \text{rad }(0)$.
- Show that $\text{rad}(IJ) = \text{rad}(I) \bigcap \text{rad}(J)$
- Show that if Spec $(R) \subseteq$ maxSpec (R) then R is a UFD.
- Show that if R is Noetherian then every ideal is finitely generated.

Characterizing Certain Ideals

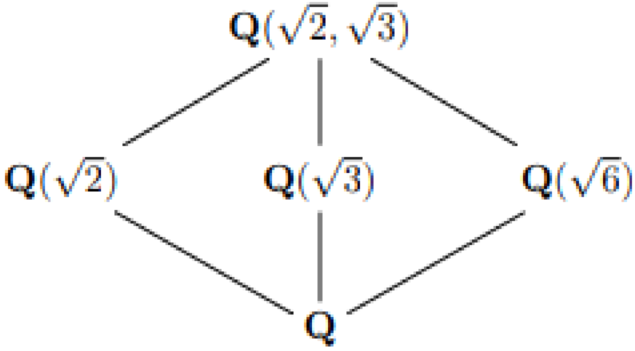
- Show that the nilradical is the intersection of all prime ideals.
- Show that for an ideal $I \trianglelefteq R$, its radical is the intersection of all prime ideals containing I.
- Show that rad (I) is the intersection of all prime ideals containing I.

Misc

- Show that localizing a ring at a prime ideal produces a local ring.
- Show that R is a local ring iff for every $x \in R$, either x or $1-x$ is a unit.
- Show that if R is a local ring then $R \setminus R^\times$ is a proper ideal that is contained in $\mathfrak{J}(R)$.
- Show that if $R \neq 0$ is a ring in which every non-unit is nilpotent then R is local.
- Show that every prime ideal is primary.

8.3 Field Theory

- What is $[\mathbb{Q}(\sqrt{2}+\sqrt{3}) : \mathbb{Q}]$?
- What is $[\mathbb{Q}(2^{\frac{3}{2}}) : \mathbb{Q}]$?
- Show that every field is simple.
- Show that any field morphism is either 0 or injective.
- Show that if $p \in \mathbb{Q}[x]$ and $r \in \mathbb{Q}$ is a rational root, then in fact $r \in \mathbb{Z}$.
- If $\{\alpha_i\}_{i=1}^n \subset F$ are algebraic over K, show that $K[\alpha_1, \cdots, \alpha_n] = K(\alpha_1, \cdots, \alpha_n)$.
- Show that the Galois group of $x^n - 2$ is D_n, the dihedral group on n vertices.
- Compute all intermediate field extensions of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, show it is equal to $\mathbb{Q}(\sqrt{2}+\sqrt{3})$, and find a corresponding minimal polynomial.



- Compute all intermediate field extensions of $\mathbb{Q}(2^{\frac{1}{4}}, \zeta_8)$.
- Show that $\mathbb{Q}(2^{\frac{1}{3}})$ and $\mathbb{Q}(\zeta_3 2^{\frac{1}{3}})$
- Show that if L/K is separable, then L is normal $\iff$ there exists a polynomial $p(x) = \prod_{i=1}^n x - \alpha_i \in K[x]$ such that $L = K(\alpha_1, \cdots, \alpha_n)$ (so L is the splitting field of p).
- Is $\mathbb{Q}(2^{\frac{1}{3}})/\mathbb{Q}$ normal?
- Show that any finite integral domain is a field.
- Prove that if R is an integral domain, then $R[t]$ is again an integral domain.
- Show that $ff(R[t]) = ff(R)(t)$.
- Prove that $x^{p^n} - x$ is the product of all monic irreducible polynomials in $\mathbb{F}_p[x]$ with degree dividing n.
- Prove that an irreducible $\pi(x) \in \mathbb{F}_p[x]$ divides $x^{p^n} - x \iff \deg \pi(x)$ divides n.
- Show that a field with p^n elements has exactly one subfield of size p^d for every d dividing n.
- Show that $\mathbb{GF}(p^n)$ is the splitting field of $x^{p^n} - x \in \mathbb{F}_p[x]$.
- Show that $x^{p^d} - x \,\Big|\, x^{p^n} - x \iff d \,\Big|\, n$

- Show that $\mathbb{GF}(p^d) \leq \mathbb{GF}(p^n) \iff d \mid n$
- Show that $x^{p^n} - x = \prod f_i(x)$ over all irreducible monic f_i of degree d dividing n.
- Compute the Galois group of $x^n - 1 \in \mathbb{Q}[x]$ as a function of n.
- Identify all of the elements of the Galois group of $x^p - 2$ for p an odd prime (note: this has a complicated presentation).
- Show that $\text{Gal}(x^{15} + 2)/\mathbb{Q} \cong S_2 \rtimes \mathbb{Z}/15\mathbb{Z}$ for S_2 a Sylow 2-subgroup.
- Show that $\text{Gal}(x^3 + 4x + 2)/\mathbb{Q} \cong S_3$, a symmetric group.

8.4 Modules and Linear Algebra

- Prove the Cayley-Hamilton theorem.
- Prove that the minimal polynomial divides the characteristic polynomial.
- Prove that the cokernel of $A \in \text{Mat}(n \times n, \mathbb{Z})$ is finite $\iff \det A \neq 0$, and show that in this case $|\text{coker}(A)| = |\det(A)|$.
- Show that a nilpotent operator is diagonalizable.
- Show that if A, B are diagonalizable and $[A, B] = 0$ then A, B are simultaneously diagonalizable.
- Does diagonalizable imply invertible? The converse?

8.5 Commutative Algebra

- Show that a finitely generated module over a Noetherian local ring is flat iff it is free using Nakayama and Tor.