Algebra Qualifying Exam Review

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1 | Basics

Notation:

- $\|f\|_\infty \coloneqq \sup\limits_{x \in \mathrm{dom}(f)} |f(x)|$

- $\|f\|_{L^\infty} \coloneqq \inf\left\{ M \geq 0 \,\middle|\, |f(x)| \leq M \text{for a.e. } x \right\}$.

- "f vanishes at infinity" means $f(x) \overset{|x|\to\infty}{\to} 0$.

- "f has small tails" means $\displaystyle\int_{|x|\geq N} f \overset{N\to\infty}{\to} 0$.

- H denotes a Hilbert space.

1.1 Useful Techniques

- General advice: try swapping the orders of limits, sums, integrals, etc.

- Limits:

  – Take the $\limsup$ or $\liminf$, which always exist, and aim for an inequality like

  $$c \leq \liminf a_n \leq \limsup a_n \leq c.$$

  – $\lim f_n = \limsup f_n = \liminf f_n$ iff the limit exists, so to show some g is a limit, show

  $$\limsup f_n \leq g \leq \liminf f_n \qquad (\implies g = \lim f).$$

  – A limit does *not* exist if $\liminf a_n > \limsup a_n$.

- Sequences and Series

- If f_n has a global maximum (computed using f_n' and the first derivative test) $M_n \to 0$, then $f_n \to 0$ uniformly.
- For a fixed x, if $f = \sum f_n$ converges *uniformly* on some $B_r(x)$ and each f_n is continuous at x, then f is also continuous at x .

- Equalities

  - Split into upper and lower bounds:

  $$a = b \iff a \le b \,\mathrm{and}\, a \ge b.$$

  - Use an epsilon of room:

  $$a < b + \varepsilon \,\forall \varepsilon \implies a \le b.$$

  - Showing something is zero:

  $$|a| \le \varepsilon \,\forall \varepsilon \implies a = 0.$$

- Simplifications:

  - To show something for a measurable set, show it for bounded/compact/elementary sets/
  - To show something for a function, show it for continuous, bounded, compactly supported, simple, indicator functions, L^1, etc
  - Replace a continuous sequence ($\varepsilon \to 0$) with an arbitrary countable sequence ($x_n \to 0$)
  - Intersect with a ball $B_r(\mathbf{0}) \subset \mathbb{R}^n$.

- Integrals

  - Break up $\mathbb{R}^n = \{|x| \le 1\} \coprod \{|x| > 1\}$.
  - Break up into $\{f > g\} \coprod \{f = g\} \coprod \{f < g\}$.
  - Tail estimates!

- Continuity / differentiability: show it holds on $[-M, M]$ for all M to get it to hold on $\mathbb{R}$.

- F_σ sets are Borel, so establish something for Borel sets and use this to extend it to Lebesgue.

- $s = \inf \{x \in X\} \implies$ for every ε there is an $x \in X$ such that $x \le s + \varepsilon$.

- Always consider bounded sets, and if E is unbounded write $E = \cup_n B_n(0) \cap E$ and use countable subadditivity or continuity of measure.

- F_σ sets are Borel, so establish something for Borel sets and use this to extend it to Lebesgue.

- Break integration domain up into disjoint annuli.

- Break integrals or sums into $x < 1$ and $x \ge 1$.

- Calculus techniques: Taylor series, IVT, . . .

- Approximate by dense subsets of functions

- Useful facts about compactly supported continuous functions:

    - Uniformly continuous
    - Bounded

Useful facts about C_c functions:

- Bounded almost everywhere
- Uniformly continuous

1.2 Definitions

Definition 1.2.1 (Uniform Continuity)
f is uniformly continuous iff

$$\forall \varepsilon \quad \exists \delta(\varepsilon) \,\Big|\quad \forall x, y, \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

$$\iff \forall \varepsilon \quad \exists \delta(\varepsilon) \,\Big|\quad \forall x, y, \quad |y| < \delta \implies |f(x - y) - f(y)| < \varepsilon.$$

Definition 1.2.2 (Nowhere Dense Sets)
A set S is **nowhere dense** iff the closure of S has empty interior iff every interval contains a subinterval that does not intersect S.

Definition 1.2.3 (Meager Sets)
A set is **meager** if it is a *countable* union of nowhere dense sets.

Definition 1.2.4 (F_σ and G_δ Sets)
An F_σ set is a union of closed sets, and a G_δ set is an intersection of opens. [a]

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[a] Mnemonic: "F" stands for *ferme*, which is "closed" in French, and σ corresponds to a "sum", i.e. a union.

Definition 1.2.5 (Limsup/Liminf)

$$\limsup_n a_n = \lim_{n \to \infty} \sup_{j \geq n} a_j = \inf_{n \geq 0} \sup_{j \geq n} a_j$$

$$\liminf_n a_n = \lim_{n \to \infty} \inf_{j \geq n} a_j = \sup_{n \geq 0} \inf_{j \geq n} a_j.$$

Definition 1.2.6 (Topological Notions)
Let X be a metric space and A a subset. Let A' denote the limit points of A, and $\overline{A} \coloneqq A \cup A'$ to be its closure.

- A **neighborhood** of p is an open set U_p containing p.

- An ε-**neighborhood** of p is an open ball $B_r(p) \coloneqq \left\{ q \mid d(p,q) < r \right\}$ for some $r > 0$.

- A point $p \in X$ is an **accumulation point** of A iff every neighborhood U_p of p contains a point $q \in Q$

- A point $p \in X$ is a **limit point** of A iff every *punctured* neighborhood $U_p \smallsetminus \{p\}$ contains a point $q \in A$.

- If $p \in A$ and p is not a limit point of A, then p is an **isolated point** of A.

- A is **closed** iff $A' \subset A$, so A contains all of its limit points.

- A point $p \in A$ is **interior** iff there is a neighborhood $U_p \subset A$ that is strictly contained in A.

- A is **open** iff every point of A is interior.

- A is **perfect** iff A is closed and $A \subset A'$, so every point of A is a limit point of A.

- A is **bounded** iff there is a real number M and a point $q \in X$ such that $d(p,q) < M$ for all $p \in A$.

- A is **dense** in X iff every point $x \in X$ is either a point of A, so $x \in A$, or a limit point of A, so $x \in A'$. I.e., $X \subset A \cup A'$.

    - Alternatively, $\overline{A} = X$, so the closure of A is X.

Definition 1.2.7 (Uniform Convergence)

$$(\forall \varepsilon > 0)\,(\exists n_0 = n_0(\varepsilon))\,(\forall x \in S)\,(\forall n > n_0)\,(|f_n(x) - f(x)| < \varepsilon).$$

Negated:[a]

$$(\exists \varepsilon > 0)\,(\forall n_0 = n_0(\varepsilon))\,(\exists x = x(n_0) \in S)\,(\exists n > n_0)\,(|f_n(x) - f(x)| \geq \varepsilon).$$

---
[a]Slogan: to negate, find a bad x depending on n_0 that are larger than some ε.

Definition 1.2.8 (Pointwise Convergence)
A sequence of functions $\{f_j\}$ is said to **converge pointwise** to f if and only if

$$(\forall \varepsilon > 0)(\forall x \in S)\,(\exists n_0 = n_0(x,\varepsilon))\,(\forall n > n_0)\,(|f_n(x) - f(x)| < \varepsilon).$$

Definition 1.2.9 (Outer Measure)

The **outer measure** of a set is given by

$$m_*(E) \coloneqq \inf_{\substack{\{Q_i\} \rightrightarrows E \\ \text{closed cubes}}} \sum |Q_i|.$$

Definition 1.2.10 (Limsup and Liminf of Sets)

$$\limsup_n A_n \coloneqq \cap_n \cup_{j \geq n} A_j = \left\{ x \mid x \in A_n \text{ for inf. many } n \right\}$$

$$\liminf_n A_n \coloneqq \cup_n \cap_{j \geq n} A_j = \left\{ x \mid x \in A_n \text{ for all except fin. many } n \right\}$$

.

Definition 1.2.11 (Lebesgue Measurable Sets)
A subset $E \subseteq \mathbb{R}^n$ is **Lebesgue measurable** iff for every $\varepsilon > 0$ there exists an open set $O \supseteq E$ such that $m_*(O \smallsetminus E) < \varepsilon$. In this case, we define $m(E) \coloneqq m_*(E)$.

Definition 1.2.12 (L^+: Measurable non-negative functions.)
$f \in L^+$ iff f is measurable and non-negative.

Definition 1.2.13 (Integrability)
A measurable function is **integrable** iff $\|f\|_1 < \infty$.

Definition 1.2.14 (The Infinity Norm)

$$\|f\|_\infty \coloneqq \inf_{\alpha \geq 0} \left\{ \alpha \mid m\{|f| \geq \alpha\} = 0 \right\}.$$

Definition 1.2.15 (Essentially Bounded Functions)
A function $f : X \to \mathbb{C}$ is **essentially bounded** iff there exists a real number c such that $\mu(\{|f| > x\}) = 0$, i.e. $\|f\|_\infty < \infty$.

Definition 1.2.16 (L^∞)

$$L^\infty(X) \coloneqq \left\{ f : X \to \mathbb{C} \mid f \text{ is essentially bounded } \right\} \coloneqq \left\{ f : X \to \mathbb{C} \mid \|f\|_\infty < \infty \right\}.$$

Definition 1.2.17 (Dual Norm)
For X a normed vector space and $\Lambda \in X^\vee$,

$$\|\Lambda\|_{X^\vee} \coloneqq \sup_{\left\{ x \in X \mid \|x\|_X \leq 1 \right\}} |f(x)|.$$

Definition 1.2.18 (Convolution)

$$f * g(x) = \int f(x-y)g(y)dy.$$

Definition 1.2.19 (Fourier Transform)

$$\widehat{f}(\xi) = \int f(x)\ e^{2\pi ix\cdot\xi}\ dx.$$

Definition 1.2.20 (Dilation)

$$\varphi_t(x) = t^{-n}\varphi\left(t^{-1}x\right).$$

Definition 1.2.21 (Approximations to the identity)
For $\varphi \in L^1$, the dilations satisfy $\int \varphi_t = \int \varphi$, and if $\int \varphi = 1$ then φ is an **approximate identity**.

Definition 1.2.22 (Baire Space)
A space X is a **Baire space** if and only if every countable intersections of open, dense sets is still dense.

1.2.1 Functional Analysis

Definition 1.2.23 (Orthonormal sequence)
A countable collection of elements $\{u_i\}$ is **orthonormal** if and only if

1. $\langle u_i,\ u_j\rangle = 0$ for all $j \neq k$ and
2. $\|u_j\|^2 := \langle u_j,\ u_j\rangle = 1$ for all j.

Definition 1.2.24 (Basis of a Hilbert space)
A set $\{u_n\}$ is a **basis** for a Hilbert space $\mathcal{H}$ iff it is dense in $\mathcal{H}$.

Definition 1.2.25 (Completeness of a Hilbert space)
A collection of vectors $\{u_n\} \subset H$ is **complete** iff $\langle x,\ u_n\rangle = 0$ for all $n \iff x = 0$ in H.

Definition 1.2.26 (Dual of a Hilbert space)
The **dual** of a Hilbert space H is defined as

$$H^\vee := \left\{ L : H \to \mathbb{C} \ \middle|\ L \text{ is continuous} \right\}.$$

Definition 1.2.27 (Linear functionals)
A map $L : X \to \mathbb{C}$ is a **linear functional** iff

$$L(\alpha \mathbf{x} + \mathbf{y}) = \alpha L(\mathbf{x}) + L(\mathbf{y})..$$

Definition 1.2.28 (Operator norm)
The **operator norm** of an operator L is defined as

$$\|L\|_{X^\vee} := \sup_{\substack{x \in X \\ \|x\|=1}} |L(x)|.$$

Definition 1.2.29 (Banach Space)
A space is a **Banach space** if and only if it is a complete normed vector space.

Definition 1.2.30 (Hilbert Space)
A **Hilbert space** is an inner product space which is a Banach space under the induced norm.

1.3 Theorems

1.3.1 Topology / Sets

Theorem 1.3.1 *(Heine-Cantor).*
Every continuous function on a compact space is uniformly continuous.

Proposition 1.3.2 *(Compact if and only if sequentially compact for metric spaces).*
Metric spaces are compact iff they are sequentially compact, (i.e. every sequence has a convergent subsequence).

Proposition 1.3.3 *(A unit ball that is not compact).*
The unit ball in $C([0,1])$ with the sup norm is not compact.

Proof (?).
Take $f_k(x) = x^n$, which converges to $\chi(x = 1)$. The limit is not continuous, so no subsequence can converge.

∎

Proposition 1.3.4 *(?).*
A *finite* union of nowhere dense is again nowhere dense.

Proposition 1.3.5 *(Convergent Sums Have Small Tails).*

$$\sum a_n < \infty \implies a_n \to 0 \quad \text{and} \quad \sum_{k=N}^{\infty} a_n \overset{N \to \infty}{\to} 0$$

Theorem 1.3.6*(Heine-Borel).*
$X \subseteq \mathbb{R}^n$ is compact $\iff$ X is closed and bounded.

Proposition 1.3.7*(Geometric Series).*

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \iff |x| < 1.$$

Corollary 1.3.8*(?).*

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = 1.$$

Proposition 1.3.9*(?).*
The Cantor set is closed with empty interior.

Proof (?).
Its complement is a union of open intervals, and can't contain an interval since intervals have positive measure and $m(C_n)$ tends to zero.

■

Corollary 1.3.10*(?).*
The Cantor set is nowhere dense.

Proposition 1.3.11*(?).*
Singleton sets in $\mathbb{R}$ are closed, and thus $\mathbb{Q}$ is an F_σ set.

Theorem 1.3.12*(Baire).*
$\mathbb{R}$ is a **Baire space** Thus $\mathbb{R}$ can not be written as a countable union of nowhere dense sets.

Lemma 1.3.13*(?).*
Any nonempty set which is bounded from above (resp. below) has a well-defined supremum (resp. infimum).

1.3.2 Functions

> **Proposition 1.3.14***(Existence of Smooth Compactly Supported Functions).*
> There exist smooth compactly supported functions, e.g. take
>
> $$f(x) = e^{-\frac{1}{x^2}} \chi_{(0,\infty)}(x).$$

> **Lemma 1.3.15***(Function discontinuous on the rationals).*
> There is a function discontinuous precisely on $\mathbb{Q}$.

> *Proof (?).*
> $f(x) = \dfrac{1}{n}$ if $x = r_n \in \mathbb{Q}$ is an enumeration of the rationals, and zero otherwise. The limit at every point is 0.
> ∎

> **Proposition 1.3.16***(No functions discontinuous on the irrationals).*
> There *do not* exist functions that are discontinuous precisely on $\mathbb{R} \setminus \mathbb{Q}$.

> *Proof (?).*
> D_f is always an F_σ set, which follows by considering the oscillation ω_f. $\omega_f(x) = 0 \iff f$ is continuous at x, and $D_f = \cup_n A_{\frac{1}{n}}$ where $A_\varepsilon = \{\omega_f \geq \varepsilon\}$ is closed.
> ∎

> **Proposition 1.3.17***(Lipschitz $\iff$ differentiable with bounded derivative.).*
> A function $f : (a,b) \to \mathbb{R}$ is Lipschitz $\iff f$ is differentiable and f' is bounded. In this case, $|f'(x)| \leq C$, the Lipschitz constant.

1.4 Uniform Convergence

> **Proposition 1.4.1***(Testing Uniform Convergence: The Sup Norm Test).*
> $f_n \to f$ uniformly iff there exists an M_n such that $\|f_n - f\|_\infty \leq M_n \to 0$.

Remark 1.4.2*(Negating the Sup Norm test):* **Negating**: find an x which depends on n for which $\|f_n\|_\infty > \varepsilon$ (negating small tails) or $\|f_n - f_m\| > \varepsilon$ (negating the Cauchy criterion).

1.4.1 Example: Completeness of a Normed Function Space

:::{.proposition title=" $C(I)$ is complete"} The space $X = C([0,1])$, continuous functions $f : [0,1] \to \mathbb{R}$, equipped with the norm

$$\|f\|_\infty \coloneqq \sup_{x\in[0,1]} |f(x)|$$

is a **complete** metric space. :::

Proof .

1. Let $\{f_k\}$ be Cauchy in X.

2. Define a candidate limit using pointwise convergence:

   Fix an x; since

   $$|f_k(x) - f_j(x)| \le \|f_k - f_k\| \to 0$$

   the sequence $\{f_k(x)\}$ is Cauchy in $\mathbb{R}$. So define $f(x) \coloneqq \lim_k f_k(x)$.

3. Show that $\|f_k - f\| \to 0$:

   $$|f_k(x) - f_j(x)| < \varepsilon \ \forall x \implies \lim_j |f_k(x) - f_j(x)| < \varepsilon \ \forall x$$

   Alternatively, $\|f_k - f\| \le \|f_k - f_N\| + \|f_N - f_j\|$, where N, j can be chosen large enough to bound each term by $\varepsilon/2$.

4. Show that $f \in X$:

   The uniform limit of continuous functions is continuous.

   ∎

Remark 1.4.3: In other cases, you may need to show the limit is bounded, or has bounded derivative, or whatever other conditions define X.

Theorem 1.4.4 *(Uniform Limit Theorem).*
If $f_n \to f$ pointwise and uniformly with each f_n continuous, then f is continuous. [a]

---
[a] Slogan: a uniform limit of continuous functions is continuous.

Proof .

- Follows from an $\varepsilon/3$ argument:

  $$|F(x) - F(y)| \le |F(x) - F_N(x)| + |F_N(x) - F_N(y)| + |F_N(y) - F(y)| \le \varepsilon \to 0.$$

  - The first and last $\varepsilon/3$ come from uniform convergence of $F_N \to F$.
  - The middle $\varepsilon/3$ comes from continuity of each F_N.

- So just need to choose N large enough and δ small enough to make all 3 ε bounds hold.

■

Proposition 1.4.5 *(Uniform Limits Commute with Integrals).*
If $f_n \to f$ uniformly, then $\int f_n = \int f$.

1.4.2 Series

Proposition 1.4.6 *(p-tests).*
Let n be a fixed dimension and set $B = \left\{ x \in \mathbb{R}^n \mid \|x\| \le 1 \right\}$.

$$\sum \frac{1}{n^p} < \infty \iff p > 1$$
$$\int_\varepsilon^\infty \frac{1}{x^p} < \infty \iff p > 1$$
$$\int_0^1 \frac{1}{x^p} < \infty \iff p < 1$$
$$\int_B \frac{1}{|x|^p} < \infty \iff p < n$$
$$\int_{B^c} \frac{1}{|x|^p} < \infty \iff p > n$$

.

Proposition 1.4.7 *(Comparison Test).*
If $0 \le a_n \le b_n$, then

- $\sum b_n < \infty \implies \sum a_n < \infty$, and
- $\sum a_n = \infty \implies \sum b_n = \infty$.

Proposition 1.4.8 *(Small Tails for Series of Functions).*
If $\sum f_n$ converges then $f_n \to 0$ uniformly.

Corollary 1.4.9 *(Term by Term Continuity Theorem).*
If f_n are continuous and $\sum f_n \to f$ converges uniformly, then f is continuous.

Proposition 1.4.10 *(Weak M-Test).*
If $f_n(x) \le M_n$ **for a fixed** x where $\sum M_n < \infty$, then the series $f(x) = \sum f_n(x)$ converges.[a]

---
[a]Note that this is only pointwise convergence of f, whereas the full M-test gives uniform convergence.

Proposition 1.4.11 *(The Weierstrass M-Test).*
If $\sup\limits_{x \in A} |f_n(x)| \le M_n$ for each n where $\sum M_n < \infty$, then $\sum\limits_{n=1}^\infty f_n(x)$ converges uniformly and

absolutely on A. [a] Conversely, if $\sum f_n$ converges uniformly on A then $\sup_{x \in A} |f_n(x)| \to 0$.

---
[a]It suffices to show $|f_n(x)| \le M_n$ for some M_n not depending on x.

Proposition 1.4.12*(Cauchy criterion for sums).*
f_n are uniformly Cauchy (so $\|f_n - f_m\|_\infty < \varepsilon$) iff f_n is uniformly convergent.

Derivatives

Theorem 1.4.13*(Term by Term Differentiability Theorem).*
If f_n are differentiable, $\sum f_n' \to g$ uniformly, and there exists one point[a] x_0 such that $\sum f_n(x)$ converges, then there exist an f such that $\sum f_n \to f$ uniformly and $f' = g$.[b]

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[a]So this implicitly holds if f is the pointwise limit of f_n.
[b]See Abbott theorem 6.4.3, pp 168.

1.5 Commuting Limiting Operations

Proposition 1.5.1*(Limits of bounded functions need not be bounded).*

$$\lim_{n \to \infty} \sup_{x \in X} |f_n(x)| \ne \sup_{x \in X} \left| \lim_{n \to \infty} f_n(x) \right|.$$

Proposition 1.5.2*(Limits of continuous functions need not be continuous).*

$$\lim_{k \to \infty} \lim_{n \to \infty} f_n(x_k) \ne \lim_{n \to \infty} \lim_{k \to \infty} f_n(x_k).$$

Proposition 1.5.3*(Limits of differentiable functions need not be differentiable).*

$$\lim_{n \to \infty} \frac{\partial}{\partial x} f_n \ne \frac{\partial}{\partial n} \left(\lim_{n \to \infty} f_n \right).$$

Proposition 1.5.4*(?).*

$$\lim_{n \to \infty} \int_a^b f_n(x)\, dx \ne \int_a^b \lim_{n \to \infty} (f_n(x))\, dx.$$

1.6 Slightly Advanced Stuff

Theorem 1.6.1 *(Weierstrass Approximation).*
If $[a, b] \subset \mathbb{R}$ is a closed interval and f is continuous, then for every $\varepsilon > 0$ there exists a polynomial p_ε such that $\|f - p_\varepsilon\|_{L^\infty([a,b])} \overset{\varepsilon \to 0}{\to} 0$.
Equivalently, polynomials are dense in the Banach space $C([0, 1], \|\cdot\|_\infty)$.

Theorem 1.6.2 *(Egorov).*
Let $E \subseteq \mathbb{R}^n$ be measurable with $m(E) > 0$ and $\{f_k : E \to \mathbb{R}\}$ be measurable functions such that

$$f(x) \coloneqq \lim_{k \to \infty} f_k(x) < \infty$$

exists almost everywhere.
Then $f_k \to f$ *almost uniformly*, i.e.

$$\forall \varepsilon > 0, \ \exists F \subseteq E \text{ closed such that } m(E \smallsetminus F) < \varepsilon \ \text{ and } \ f_k \to f \text{ uniformly on } F.$$

1.7 Examples

Example 1.7.1 *(?):* A series of continuous functions that does *not* converge uniformly but is still continuous:

$$g(x) \coloneqq \sum \frac{1}{1 + n^2 x}.$$

Take $x = 1/n^2$.

2 | Measure Theory

2.1 Theorems

Proposition 2.1.1 *(Opens are unions of almost disjoint intervals.).*
Every open subset of $\mathbb{R}$ (resp $\mathbb{R}^n$) can be written as a unique countable union of disjoint (resp. almost disjoint) intervals (resp. cubes).

Proposition 2.1.2 *(Properties of Outer Measure).*

1. Monotonicity: $E \subseteq F \implies m_*(E) \leq m_*(F)$.

2. Countable Subadditivity: $m_*(\cup E_i) \le \sum m_*(E_i)$.
3. Approximation: For all E there exists a $G \supseteq E$ such that $m_*(G) \le m_*(E) + \varepsilon$.
4. Disjoint[a] Additivity: $m_*(A \bigsqcup B) = m_*(A) + m_*(B)$.

---

[a]This holds for outer measure **iff** $\operatorname{dist}(A, B) > 0$.

Proposition 2.1.3*(Subtraction of Measures).*

$$m(A) = m(B) + m(C) \quad \text{and} \quad m(C) < \infty \implies m(A) - m(C) = m(B).$$

Proposition 2.1.4*(Continuity of Measure).*

$$E_i \nearrow E \implies m(E_i) \to m(E)$$
$$m(E_1) < \infty \text{ and } E_i \searrow E \implies m(E_i) \to m(E).$$

Proof (of continuity of measure).

1. Break into disjoint annuli $A_2 = E_2 \smallsetminus E_1$, etc then apply countable disjoint additivity to $E = \bigsqcup A_i$.

2. Use $E_1 = (\bigsqcup E_j \smallsetminus E_{j+1}) \bigsqcup (\cap E_j)$, taking measures yields a telescoping sum, and use countable disjoint additivity.

■

Theorem 2.1.5*(Measurable sets can be approximated by open/closed/compact sets.).*
Suppose E is measurable; then for every $\varepsilon > 0$,

1. There exists an open $O \supset E$ with $m(O \smallsetminus E) < \varepsilon$
2. There exists a closed $F \subset E$ with $m(E \smallsetminus F) < \varepsilon$
3. There exists a compact $K \subset E$ with $m(E \smallsetminus K) < \varepsilon$.

Proof (that measurable sets can be approximated).

- (1): Take $\{Q_i\} \rightrightarrows E$ and set $O = \cup Q_i$.
- (2): Since E^c is measurable, produce $O \supset E^c$ with $m(O \smallsetminus E^c) < \varepsilon$.

  - Set $F = O^c$, so F is closed.
  - Then $F \subset E$ by taking complements of $O \supset E^c$
  - $E \smallsetminus F = O \smallsetminus E^c$ and taking measures yields $m(E \smallsetminus F) < \varepsilon$

- (3): Pick $F \subset E$ with $m(E \smallsetminus F) < \varepsilon/2$.

  - Set $K_n = F \cap \mathbb{D}_n$, a ball of radius n about 0.
  - Then $E \smallsetminus K_n \searrow E \smallsetminus F$

– Since $m(E) < \infty$, there is an N such that $n \geq N \implies m(E \smallsetminus K_n) < \varepsilon$.

∎

Proposition 2.1.6 *(Translation and Dilation Invariance).*
Lebesgue measure is translation and dilation invariant.

Proof ((Todo) of translation/dilation invariance).
Obvious for cubes; if $Q_i \rightrightarrows E$ then $Q_i + k \rightrightarrows E + k$, etc.

∎

Theorem 2.1.7 *(Non-measurable sets exist).*
There is a non-measurable set.

Proof (Constructing a non-measurable set).

- Use AOC to choose one representative from every coset of $\mathbb{R}/\mathbb{Q}$ on $[0,1)$, which is countable, and assemble them into a set N
- Enumerate the rationals in $[0,1]$ as q_j, and define $N_j = N + q_j$. These intersect trivially.
- Define $M \coloneqq \coprod N_j$, then $[0,1) \subseteq M \subseteq [-1,2)$, so the measure must be between 1 and 3.
- By translation invariance, $m(N_j) = m(N)$, and disjoint additivity forces $m(M) = 0$, a contradiction.

∎

Proposition 2.1.8 *(Borel Characterization of Measurable Sets).*
If E is Lebesgue measurable, then $E = H \coprod N$ where $H \in F_\sigma$ and N is null.

Proof (of Borel characterization).
For every $\dfrac{1}{n}$ there exists a closed set $K_n \subset E$ such that $m(E \smallsetminus K_n) \leq \dfrac{1}{n}$. Take $K = \cup K_n$, wlog $K_n \nearrow K$ so $m(K) = \lim m(K_n) = m(E)$. Take $N \coloneqq E \smallsetminus K$, then $m(N) = 0$.

∎

Proposition 2.1.9 *(Limsups/infs of measurable sets are measurable.).*
If A_n are all measurable, $\limsup A_n$ and $\liminf A_n$ are measurable.

Proof (That limsups/infs are measurable).
Measurable sets form a sigma algebra, and these are expressed as countable unions/intersections of measurable sets.

∎

Theorem 2.1.10 *(Borel-Cantelli).*
Let $\{E_k\}$ be a countable collection of measurable sets. Then

$$\sum_k m(E_k) < \infty \implies \text{ almost every } x \in \mathbb{R} \text{ is in at most finitely many } E_k.$$

Proof (of Borel-Cantelli).

- If $E = \limsup_j E_j$ with $\sum m(E_j) < \infty$ then $m(E) = 0$.
- If E_j are measurable, then $\limsup_j E_j$ is measurable.
- If $\sum_j m(E_j) < \infty$, then $\sum_{j=N}^{\infty} m(E_j) \overset{N\to\infty}{\to} 0$ as the tail of a convergent sequence.
- $E = \limsup_j E_j = \cap_{k=1}^{\infty} \cup_{j=k}^{\infty} E_j \implies E \subseteq \cup_{j=k}^{\infty}$ for all k
- $E \subset \cup_{j=k}^{\infty} \implies m(E) \le \sum_{j=k}^{\infty} m(E_j) \overset{k\to\infty}{\to} 0$.

∎

Proposition 2.1.11 *(Extending the class of measurable functions.).* •
     Characteristic functions are measurable
- If f_n are measurable, so are $|f_n|, \limsup f_n, \liminf f_n, \lim f_n$,
- Sums and differences of measurable functions are measurable,
- Cones $F(x, y) = f(x)$ are measurable,
- Compositions $f \circ T$ for T a linear transformation are measurable,
- "Convolution-ish" transformations $(x, y) \mapsto f(x - y)$ are measurable

Proof (Convolution).
Take the cone on f to get $F(x, y) = f(x)$, then compose F with the linear transformation $T = [1, -1; 1, 0]$.

∎

3 | Integration

3.1 Theorems

Remark 3.1.1: If $f \in L^\infty(X)$, then f is equal to some bounded function g almost everywhere.

Example 3.1.2 *(?):* $f(x) = x\chi_\mathbb{Q}(x)$ is essentially bounded but not bounded.

Theorem 3.1.3 *(p-Test for Integrals).*

$$\int_0^1 \frac{1}{x^p} < \infty \iff p < 1$$
$$\int_1^\infty \frac{1}{x^p} < \infty \iff p > 1.$$

Slogan 3.1.4

Large powers of x help us in neighborhoods of infinity and hurt around zero

3.1.1 Convergence Theorems

Theorem 3.1.5 *(Monotone Convergence).*
If $f_n \in L^+$ and $f_n \nearrow f$ almost everywhere, then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \quad \int f_n \to \int f.$$

Needs to be positive and increasing.

Theorem 3.1.6 *(Dominated Convergence).*
If $f_n \in L^1$ and $f_n \to f$ almost everywhere with $|f_n| \le g$ for some $g \in L^1$, then $f \in L^1$ and

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \quad \int f_n \to \int f < \infty,$$

and more generally,

$$\int |f_n - f| \to 0.$$

Positivity not *needed.*

Theorem 3.1.7 *(Generalized DCT).*
If

- $f_n \in L^1$ with $f_n \to f$ almost everywhere,
- There exist $g_n \in L^1$ with $|f_n| \le g_n$, $g_n \ge 0$.
- $g_n \to g$ almost everywhere with $g \in L^1$, and
- $\lim \int g_n = \int g$,

then $f \in L^1$ and $\lim \int f_n = \int f < \infty$.

Note that this is the DCT with $|f_n| < |g|$ relaxed to $|f_n| < g_n \to g \in L^1$.

Proof .
Proceed by showing $\limsup \int f_n \le \int f \le \liminf \int f_n$:

- $\int f \geq \limsup \int f_n$:

$$\int g - \int f = \int (g - f)$$
$$\leq \liminf \int (g_n - f_n) \quad \text{Fatou}$$
$$= \lim \int g_n + \liminf \int (-f_n)$$
$$= \lim \int g_n - \limsup \int f_n$$
$$= \int g - \limsup \int f_n$$

$$\implies \int f \geq \limsup \int f_n.$$

   - Here we use $g_n - f_n \xrightarrow{n \to \infty} g - f$ with $0 \leq |f_n| - f_n \leq g_n - f_n$, so $g_n - f_n$ are nonnegative (and measurable) and Fatou's lemma applies.

- $\int f \leq \liminf \int f_n$:

$$\int g + \int f = \int (g + f)$$
$$\leq \liminf \int (g_n + f_n)$$
$$= \lim \int g_n + \liminf \int f_n$$
$$= \int g + \liminf f_n$$

$$\int f \leq \liminf \int f_n.$$

   - Here we use that $g_n + f_n \to g + f$ with $0 \leq |f_n| + f_n \leq g_n + f_n$ so Fatou's lemma again applies.

$\blacksquare$

Proposition 3.1.8*(Convergence in L^1 implies convergence of L^1 norm).*
If $f \in L^1$, then

$$\int |f_n - f| \to 0 \iff \int |f_n| \to \int |f|.$$

Proof .
Let $g_n = |f_n| - |f_n - f|$, then $g_n \to |f|$ and

$$|g_n| = ||f_n| - |f_n - f|| \leq |f_n - (f_n - f)| = |f| \in L^1,$$

so the DCT applies to g_n and

$$\|f_n - f\|_1 = \int |f_n - f| + |f_n| - |f_n| = \int |f_n| - g_n$$

$$\rightarrow_{DCT} \lim \int |f_n| - \int |f|.$$

$\blacksquare$

Theorem 3.1.9 *(Fatou).*
If f_n is a sequence of nonnegative measurable functions, then

$$\int \liminf_n f_n \le \liminf_n \int f_n$$
$$\limsup_n \int f_n \le \int \limsup_n f_n.$$

Theorem 3.1.10 *(Tonelli (Non-Negative, Measurable)).*
For $f(x,y)$ **non-negative and measurable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is a **measurable** function
- $F(x) = \int f(x,y)\,dy$ is a **measurable** function,
- For E measurable, the slices $E_x := \left\{ y \mid (x,y) \in E \right\}$ are measurable.
- $\int f = \int \int F$, i.e. any iterated integral is equal to the original.

Theorem 3.1.11 *(Fubini (Integrable)).*
For $f(x,y)$ **integrable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is an **integrable** function
- $F(x) := \int f(x,y)\,dy$ is an **integrable** function,
- For E measurable, the slices $E_x := \left\{ y \mid (x,y) \in E \right\}$ are measurable.
- $\int f = \int \int f(x,y)$, i.e. any iterated integral is equal to the original

Theorem 3.1.12 *(Fubini-Tonelli).*
If any iterated integral is **absolutely integrable**, i.e. $\int \int |f(x,y)| < \infty$, then f is integrable and $\int f$ equals any iterated integral.

Proposition 3.1.13 *(Measurable Slices).*
Let E be a measurable subset of $\mathbb{R}^n$. Then

- For almost every $x \in \mathbb{R}^{n_1}$, the slice $E_x := \left\{ y \in \mathbb{R}^{n_2} \mid (x,y) \in E \right\}$ is measurable in $\mathbb{R}^{n_2}$.
- The function

$$F : \mathbb{R}^{n_1} \to \mathbb{R}$$

$$x \mapsto m(E_x) = \int_{\mathbb{R}^{n_2}} \chi_{E_x} \, dy$$

is measurable and

$$m(E) = \int_{\mathbb{R}^{n_1}} m(E_x) \, dx = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{E_x} \, dy \, dx.$$

Proof .

$\implies$:

- Let f be measurable on $\mathbb{R}^n$.
- Then the cylinders $F(x,y) = f(x)$ and $G(x,y) = f(y)$ are both measurable on $\mathbb{R}^{n+1}$.
- Write $\mathcal{A} = \{G \le F\} \cap \{G \ge 0\}$; both are measurable.

$\impliedby$:

- Let A be measurable in $\mathbb{R}^{n+1}$.
- Define $A_x = \left\{ y \in \mathbb{R} \mid (x,y) \in \mathcal{A} \right\}$, then $m(A_x) = f(x)$.
- By the corollary, A_x is measurable set, $x \mapsto A_x$ is a measurable function, and $m(A) = \int f(x) \, dx$.
- Then explicitly, $f(x) = \chi_A$, which makes f a measurable function.

∎

Proposition 3.1.14 *(Differentiating Under an Integral).*
If $\left| \dfrac{\partial}{\partial t} f(x,t) \right| \le g(x) \in L^1$, then letting $F(t) = \int f(x,t) \, dt$,

$$\frac{\partial}{\partial t} F(t) := \lim_{h \to 0} \int \frac{f(x, t+h) - f(x,t)}{h} dx$$
$$\overset{\text{DCT}}{=} \int \frac{\partial}{\partial t} f(x,t) \, dx.$$

To justify passing the limit, let $h_k \to 0$ be any sequence and define

$$f_k(x,t) = \frac{f(x, t + h_k) - f(x,t)}{h_k},$$

so $f_k \overset{\text{pointwise}}{\to} \dfrac{\partial}{\partial t} f$.
Apply the MVT to f_k to get $f_k(x,t) = f_k(\xi, t)$ for some $\xi \in [0, h_k]$, and show that $f_k(\xi, t) \in L_1$.

Proposition 3.1.15 *(Commuting Sums with Integrals (non-negative)).*
If f_n are non-negative and $\sum \int |f|_n < \infty$, then $\sum \int f_n = \int \sum f_n$.

Proof .   • Idea: MCT.

- Let $F_N = \sum\limits^{N} f_n$ be a finite partial sum;
- Then there are simple functions $\varphi_n \nearrow f_n$
- So $\sum\limits^{N} \varphi_n \nearrow F_N$ and MCT applies

∎

Theorem 3.1.16 *(Commuting Sums with Integrals (integrable)).*
If $\{f_n\}$ integrable with either $\sum \int |f_n| < \infty$ or $\int \sum |f_n| < \infty$, then

$$\int \sum f_n = \sum \int f_n.$$

Proof .   • By Tonelli, if $f_n(x) \geq 0$ for all n, taking the counting measure allows interchanging the order of "integration".
- By Fubini on $|f_n|$, if either "iterated integral" is finite then the result follows.

∎

Proposition 3.1.17 *(?).*
If $f_k \in L^1$ and $\sum \|f_k\|_1 < \infty$ then $\sum f_k$ converges almost everywhere and in L^1.

Proof (?).
Define $F_N = \sum\limits^{N} f_k$ and $F = \lim\limits_{N} F_N$, then $\|F_N\|_1 \leq \sum\limits^{N} \|f_k\| < \infty$ so $F \in L^1$ and $\|F_N - F\|_1 \to 0$ so the sum converges in L^1. Almost everywhere convergence: ?

∎

3.2 Examples of (Non)Integrable Functions

Example 3.2.1 *(Examples of integrable functions):*

- $\int \dfrac{1}{1 + x^2} = \arctan(x) \overset{x \to \infty}{\to} \pi/2 < \infty$

- Any bounded function (or continuous on a compact set, by EVT)

- $\int_0^1 \dfrac{1}{\sqrt{x}} < \infty$

- $\int_0^1 \dfrac{1}{x^{1-\varepsilon}} < \infty$

- $\int_1^{\infty} \dfrac{1}{x^{1+\varepsilon}} < \infty$

Example 3.2.2*(Examples of non-integrable functions):*

- $\displaystyle\int_0^1 \frac{1}{x} = \infty.$
- $\displaystyle\int_1^\infty \frac{1}{x} = \infty.$
- $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}} = \infty$
- $\displaystyle\int_1^\infty \frac{1}{x^{1-\varepsilon}} = \infty$
- $\displaystyle\int_0^1 \frac{1}{x^{1+\varepsilon}} = \infty$

3.3 L^1 Facts

Proposition 3.3.1*(Zero in L^1 iff zero almost everywhere).*
For $f \in L^+$,

$$\int f = 0 \quad \Longleftrightarrow \quad f \equiv 0 \text{ almost everywhere.}$$

Proof .

- Obvious for simple functions:

    - If $f(x) = \sum_{j=1}^n c_j \chi_{E_j}$, then $\int f = 0$ iff for each j, either $c_j = 0$ or $m(E_j) = 0$.
    - Since nonzero c_j correspond to sets where $f \neq 0$, this says $m(\{f \neq 0\}) = 0$.

- $\Longleftarrow$:

    - If $f = 0$ almost everywhere and $\varphi \nearrow f$, then $\varphi = 0$ almost everywhere since $\varphi(x) \leq f(x)$ -Then

    $$\int f = \sup_{\varphi \leq f} \int \varphi = \sup_{\varphi \leq f} 0 = 0.$$

- $\Longrightarrow$:

    - Instead show negating "$f = 0$ almost everywhere" implies $\int f \neq 0$.
    - Write $\{f \neq 0\} = \cup_{n \in \mathbb{N}} S_n$ where $S_n := \left\{ x \mid f(x) \geq \frac{1}{n} \right\}$.
    - Since "not $f = 0$ almost everywhere", there exists an n such that $m(S_n) > 0$.
    - Then

    $$0 < \frac{1}{n} \chi_{E_n} \leq f \implies 0 < \int \frac{1}{n} \chi_{E_n} \leq \int f.$$

    ∎

> **Proposition 3.3.2***(Translation Invariance).*
> The Lebesgue integral is translation invariant, i.e.
> $$\int f(x) \, dx = \int f(x+h) \, dx \qquad\qquad \text{for any} h.$$

Proof .

- Let $E \subseteq X$; for characteristic functions,
$$\int_X \chi_E(x+h) = \int_X \chi_{E+h}(x) = m(E+h) = m(E) = \int_X \chi_E(x)$$
  by translation invariance of measure.
- So this also holds for simple functions by linearity.
- For $f \in L^+$, choose $\varphi_n \nearrow f$ so $\int \varphi_n \to \int f$.
- Similarly, $\tau_h \varphi_n \nearrow \tau_h f$ so $\int \tau_h f \to \int f$
- Finally $\left\{\int \tau_h \varphi\right\} = \left\{\int \varphi\right\}$ by step 1, and the suprema are equal by uniqueness of limits.

∎

> **Proposition 3.3.3***(Integrals distribute over disjoint sets).*
> If $X \subseteq A \cup B$, then $\int_X f \le \int_A f + \int_{A^c} f$ with equality iff $X = A \bigsqcup B$.

> **Proposition 3.3.4***(Uniformly continuous L^1 functions vanish at infinity.).*
> If $f \in L^1$ and f is uniformly continuous, then $f(x) \overset{|x| \to \infty}{\to} 0$.

⚠ **Warning 3.3.5**
This doesn't hold for general L^1 functions, take any train of triangles with height 1 and summable areas.

> **Theorem 3.3.6***(Small Tails in L^1).*
> If $f \in L^1$, then for every ε there exists a radius R such that if $A = B_R(0)^c$, then $\int_A |f| < \varepsilon$.

Proof .

- Approximate with compactly supported functions.
- Take $g \overset{L_1}{\to} f$ with $g \in C_c$
- Then choose N large enough so that $g = 0$ on $E := B_N(0)$
- Then
$$\int_E |f| \le \int_E |f - g| + \int_E |g|.$$

∎

Proposition 3.3.7 *(L^1 functions are absolutely continuous.)*.
$m(E) \to 0 \implies \int_E f \to 0.$

Proof (?).
Approximate with compactly supported functions. Take $g \overset{L_1}{\to} f$, then $g \leq M$ so $\int_E f \leq$
$\int_E f - g + \int_E g \to 0 + M \cdot m(E) \to 0.$ ∎

Proposition 3.3.8 *(L^1 functions are finite almost everywhere.)*.
If $f \in L^1$, then $m(\{f(x) = \infty\}) = 0.$

Proof (?).
Idea: Split up domain Let $A = \{f(x) = \infty\}$, then $\infty > \int f = \int_A f + \int_{A^c} f = \infty \cdot m(A) + \int_{A^c} f \implies$
$m(X) = 0.$ ∎

Theorem 3.3.9 *(Continuity in L^1)*.

$$\|\tau_h f - f\|_1 \overset{h \to 0}{\to} 0$$

Proof .
Approximate with compactly supported functions. Take $g \overset{L_1}{\to} f$ with $g \in C_c$.

$$\int f(x+h) - f(x) \leq \int f(x+h) - g(x+h) + \int g(x+h) - g(x) + \int g(x) - f(x)$$
$$\overset{?\to?}{\to} 2\varepsilon + \int g(x+h) - g(x)$$
$$= \int_K g(x+h) - g(x) + \int_{K^c} g(x+h) - g(x)$$
$$\overset{??}{\to} 0,$$

which follows because we can enlarge the support of g to K where the integrand is zero on K^c, then apply uniform continuity on K. ∎

Proposition 3.3.10 *(Integration by parts, special case)*.

$$F(x) := \int_0^x f(y)dy \quad \text{and} \quad G(x) := \int_0^x g(y)dy$$
$$\implies \int_0^1 F(x)g(x)dx = F(1)G(1) - \int_0^1 f(x)G(x)dx.$$

Proof (?).
Fubini-Tonelli, and sketch region to change integration bounds.

∎

Theorem 3.3.11 *(Lebesgue Density).*

$$A_h(f)(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(y) dy \implies \|A_h(f) - f\| \overset{h \to 0}{\to} 0.$$

Proof (?).
Fubini-Tonelli, and sketch region to change integration bounds, and continuity in L^1.

∎

3.4 Lp Facts

Proposition 3.4.1 *(Dense subspaces of $L^2(I)$).*
The following are dense subspaces of $L^2([0,1])$:

- Simple functions
- Step functions
- $C_0([0,1])$
- Smoothly differentiable functions $C_0^\infty([0,1])$
- Smooth compactly supported functions C_c^∞

Theorem 3.4.2 *(?).*

$$m(X) < \infty \implies \lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$

Proof (?).
Let $M = \|f\|_\infty$.

- For any $L < M$, let $S = \{|f| \geq L\}$.
- Then $m(S) > 0$ and

$$\|f\|_p = \left(\int_X |f|^p \right)^{\frac{1}{p}}$$

$$\geq \left(\int_S |f|^p \right)^{\frac{1}{p}}$$

$$\geq L \ m(S)^{\frac{1}{p}} \overset{p \to \infty}{\to} L$$

$$\implies \liminf_p \|f\|_p \geq M.$$

We also have

$$\|f\|_p = \left(\int_X |f|^p \right)^{\frac{1}{p}}$$

$$\leq \left(\int_X M^p \right)^{\frac{1}{p}}$$

$$= M \ m(X)^{\frac{1}{p}} \xrightarrow{p \to \infty} M$$

$$\implies \limsup_p \|f\|_p \leq M \ \blacksquare.$$

∎

Theorem 3.4.3 *(Duals for L^p spaces).*
For $1 \leq p < \infty$, $(L^p)^\vee \cong L^q$.

Proof (p = 1 case).
?

∎

todo

Proof (p = 2 case).
Use Riesz Representation for Hilbert spaces.

∎

Proposition 3.4.4 *(L^1 is not quite the dual of L^∞.).*
$L^1 \subset (L^\infty)^\vee$, since the isometric mapping is always injective, but *never* surjective.

4 | Fourier Transform and Convolution

4.1 The Fourier Transform

Proposition 4.1.1*(?).*
If $\widehat{f} = \widehat{g}$ then $f = g$ almost everywhere.

Proposition 4.1.2*(Riemann-Lebesgue: Fourier transforms have small tails.).*

$$f \in L^1 \implies \widehat{f}(\xi) \to 0 \text{ as } |\xi| \to \infty,$$

if $f \in L^1$, then $\widehat{f}$ is continuous and bounded.

Proof (?).

- Boundedness:

$$\left|\widehat{f}(\xi)\right| \leq \int |f| \cdot \left|e^{2\pi i x \cdot \xi}\right| = \|f\|_1.$$

- Continuity:
- $\left|\widehat{f}(\xi_n) - \widehat{f}(\xi)\right|$
- Apply DCT to show $a \overset{n \to \infty}{\to} 0$.

∎

Theorem 4.1.3*(Fourier Inversion).*

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(x) e^{2\pi i x \cdot \xi} d\xi.$$

⚠ **Warning 4.1.4**
Fubini-Tonelli does not work here!

Proof (?).
Idea: Fubini-Tonelli doesn't work directly, so introduce a convergence factor, take limits, and use uniqueness of limits.

- Take the modified integral:

$$I_t(x) = \int \widehat{f}(\xi)\, e^{2\pi i x \cdot \xi}\, e^{-\pi t^2 |\xi|^2}$$

$$= \int \widehat{f}(\xi)\varphi(\xi)$$

$$= \int f(\xi)\widehat{\varphi}(\xi)$$

$$= \int f(\xi)\widehat{\widehat{g}}(\xi - x)$$

$$= \int f(\xi)g_t(x - \xi)\, d\xi$$

$$= \int f(y - x)g_t(y)\, dy \quad (\xi = y - x)$$

$$= (f * g_t)$$

$$\to f \text{ in } L^1 \text{ as } t \to 0.$$

- We also have

$$\lim_{t \to 0} I_t(x) = \lim_{t \to 0} \int \widehat{f}(\xi)\, e^{2\pi i x \cdot \xi}\, e^{-\pi t^2 |\xi|^2}$$

$$= \lim_{t \to 0} \int \widehat{f}(\xi)\varphi(\xi)$$

$$=_{DCT} \int \widehat{f}(\xi) \lim_{t \to 0} \varphi(\xi)$$

$$= \int \widehat{f}(\xi)\, e^{2\pi i x \cdot \xi}$$

.

- So

$$I_t(x) \to \int \widehat{f}(\xi)\, e^{2\pi i x \cdot \xi} \quad \text{pointwise and} \quad \|I_t(x) - f(x)\|_1 \to 0.$$

- So there is a subsequence I_{t_n} such that $I_{t_n}(x) \to f(x)$ almost everywhere

- Thus $f(x) = \int \widehat{f}(\xi)\, e^{2\pi i x \cdot \xi}$ almost everywhere by uniqueness of limits.

∎

Proposition 4.1.5 *(Eigenfunction of the Fourier transform).*

$$g(x) := e^{-\pi |t|^2} \implies \widehat{g}(\xi) = g(\xi) \quad \text{and} \quad \widehat{g_t}(x) = g(tx) = e^{-\pi t^2 |x|^2}.$$

4.2 Approximate Identities

Example 4.2.1*(of an approximation to the identity.):*

$$\varphi(x) := e^{-\pi x^2}.$$

Theorem 4.2.2*(Convolving against an approximate identity converges in L^1.).*

$$\|f * \varphi_t - f\|_1 \overset{t \to 0}{\to} 0.$$

Proof (?).

$$
\begin{aligned}
\|f - f * \varphi_t\|_1 &= \int f(x) - \int f(x-y)\varphi_t(y) \, dydx \\
&= \int f(x) \int \varphi_t(y) \, dy - \int f(x-y)\varphi_t(y) \, dydx \\
&= \int \int \varphi_t(y)[f(x) - f(x-y)] \, dydx \\
&=_{FT} \int \int \varphi_t(y)[f(x) - f(x-y)] \, dxdy \\
&= \int \varphi_t(y) \int f(x) - f(x-y) \, dxdy \\
&= \int \varphi_t(y)\|f - \tau_y f\|_1 dy \\
&= \int_{y < \delta} \varphi_t(y)\|f - \tau_y f\|_1 dy + \int_{y \geq \delta} \varphi_t(y)\|f - \tau_y f\|_1 dy \\
&\leq \int_{y < \delta} \varphi_t(y)\varepsilon + \int_{y \geq \delta} \varphi_t(y)\left(\|f\|_1 + \|\tau_y f\|_1\right) dy \quad \text{by continuity in } L^1 \\
&\leq \varepsilon + 2\|f\|_1 \int_{y \geq \delta} \varphi_t(y) dy \\
&\leq \varepsilon + 2\|f\|_1 \cdot \varepsilon \quad \text{since } \varphi_t \text{ has small tails} \\
&\overset{\varepsilon \to 0}{\to} 0.
\end{aligned}
$$

∎

Theorem 4.2.3*(Convolutions vanish at infinity).*

$$f, g \in L^1 \text{ and bounded} \implies \lim_{|x| \to \infty} (f * g)(x) = 0.$$

Proof (?). • Choose $M \geq f, g$.

• By small tails, choose N such that $\int_{B_N^c} |f|, \int_{B_n^c} |g| < \varepsilon$

- Note

$$|f * g| \le \int |f(x-y)| \, |g(y)| \, dy \coloneqq I.$$

- Use $|x| \le |x-y| + |y|$, take $|x| \ge 2N$ so either

$$|x-y| \ge N \implies I \le \int_{\{x-y \ge N\}} |f(x-y)| M \, dy \le \varepsilon M \to 0$$

then

$$|y| \ge N \implies I \le \int_{\{y \ge N\}} M|g(y)| \, dy \le M\varepsilon \to 0.$$

∎

Proposition 4.2.4 *(Corollary of Young's inequality).*
Take $q = 1$ in Young's inequality to obtain

$$\|f * g\|_p \le \|f\|p\|g\|1.$$

Proposition 4.2.5 *(L^1 is closed under convolution.).*
If $f, g \in L^1$ then $f * g \in L^1$.

---

5 | Functional Analysis

5.1 Theorems

Theorem 5.1.1 *(Bessel's Inequality).*
For any orthonormal set $\{u_n\} \subseteq \mathcal{H}$ a Hilbert space (not necessarily a basis),

$$\left\| x - \sum_{n=1}^{N} \langle x, u_n \rangle \, u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n \rangle|^2$$

and thus

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \le \|x\|^2.$$

Proof (of Bessel's inequality).

- Let $S_N = \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n$

$$
\begin{aligned}
\|x - S_N\|^2 &= \langle x - S_n,\ x - S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\langle x,\ S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\left\langle x,\ \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n \right\rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^{N} \langle x,\ \langle x,\ u_n \rangle u_n \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^{N} \overline{\langle x,\ u_n \rangle}\langle x,\ u_n \rangle \\
&= \|x\|^2 + \left\|\sum_{n=1}^{N} \langle x,\ u_n \rangle u_n \right\|^2 - 2 \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 - 2 \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
&= \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2.
\end{aligned}
$$

- By continuity of the norm and inner product, we have

$$
\lim_{N \to \infty} \|x - S_N\|^2 = \lim_{N \to \infty} \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2
$$

$$
\implies \left\|x - \lim_{N \to \infty} S_N \right\|^2 = \|x\|^2 - \lim_{N \to \infty} \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2
$$

$$
\implies \left\|x - \sum_{n=1}^{\infty} \langle x,\ u_n \rangle u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2.
$$

- Then noting that $0 \le \|x - S_N\|^2$,

$$
0 \le \|x\|^2 - \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2
$$

$$
\implies \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2 \le \|x\|^2 \ \blacksquare\ .
$$

$\blacksquare$

Theorem 5.1.2*(Riesz Representation for Hilbert Spaces).*
If Λ is a continuous linear functional on a Hilbert space H, then there exists a unique $y \in H$ such that

$$
\forall x \in H, \quad \Lambda(x) = \langle x,\ y \rangle..
$$

Proof (?).

- Define $M \coloneqq \ker \Lambda$.
- Then M is a closed subspace and so $H = M \oplus M^\perp$
- There is some $z \in M^\perp$ such that $\|z\| = 1$.
- Set $u \coloneqq \Lambda(x)z - \Lambda(z)x$
- Check

$$\Lambda(u) = \Lambda(\Lambda(x)z - \Lambda(z)x) = \Lambda(x)\Lambda(z) - \Lambda(z)\Lambda(x) = 0 \implies u \in M$$

- Compute

$$\begin{aligned}
0 &= \langle u, \; z \rangle \\
&= \langle \Lambda(x)z - \Lambda(z)x, \; z \rangle \\
&= \langle \Lambda(x)z, \; z \rangle - \langle \Lambda(z)x, \; z \rangle \\
&= \Lambda(x)\langle z, \; z \rangle - \Lambda(z)\langle x, \; z \rangle \\
&= \Lambda(x)\|z\|^2 - \Lambda(z)\langle x, \; z \rangle \\
&= \Lambda(x) - \Lambda(z)\langle x, \; z \rangle \\
&= \Lambda(x) - \left\langle x, \; \overline{\Lambda(z)}z \right\rangle,
\end{aligned}$$

- Choose $y \coloneqq \overline{\Lambda(z)}z$.
- Check uniqueness:

$$\begin{aligned}
\langle x, \; y \rangle &= \left\langle x, \; y' \right\rangle \quad \forall x \\
\implies \left\langle x, \; y - y' \right\rangle &= 0 \quad \forall x \\
\implies \left\langle y - y', \; y - y' \right\rangle &= 0 \\
\implies \left\| y - y' \right\| &= 0 \\
\implies y - y' &= \mathbf{0} \implies y = y'.
\end{aligned}$$

∎

Theorem 5.1.3 *(Functionals are continuous if and only if bounded).*
Let $L : X \to \mathbb{C}$ be a linear functional, then the following are equivalent:

1. L is continuous
2. L is continuous at zero
3. L is bounded, i.e. $\exists c \geq 0 \mid |L(x)| \leq c\|x\|$ for all $x \in H$

Proof (?).

$2 \implies 3$: Choose $\delta < 1$ such that

$$\|x\| \leq \delta \implies |L(x)| < 1.$$

Then

$$
\begin{aligned}
|L(x)| &= \left| L\left(\frac{\|x\|}{\delta} \frac{\delta}{\|x\|} x \right) \right| \\
&= \frac{\|x\|}{\delta} \left| L\left(\delta \frac{x}{\|x\|} \right) \right| \\
&\leq \frac{\|x\|}{\delta} 1,
\end{aligned}
$$

so we can take $c = \dfrac{1}{\delta}$. ∎

$3 \implies 1$:

We have $|L(x - y)| \leq c\|x - y\|$, so given $\varepsilon \geq 0$ simply choose $\delta = \dfrac{\varepsilon}{c}$. ∎

Theorem 5.1.4*(The operator norm is a norm).*

If H is a Hilbert space, then $(H^{\vee}, \|\cdot\|_{\mathrm{op}})$ is a normed space.

Proof (?).

The only nontrivial property is the triangle inequality, but

$$\|L_1 + L_2\|_{\mathrm{op}} = \sup |L_1(x) + L_2(x)| \leq \sup |L_1(x)| + |\sup L_2(x)| = \|L_1\|_{\mathrm{op}} + \|L_2\|_{\mathrm{op}}.$$

∎

Theorem 5.1.5*(The operator norm on $X^{\vee}$ yields a Banach space).*

If X is a normed vector space, then $(X^{\vee}, \|\cdot\|_{\mathrm{op}})$ is a Banach space.

Proof (?).

- Let $\{L_n\}$ be Cauchy in $X^{\vee}$.

- Then for all $x \in C$, $\{L_n(x)\} \subset \mathbb{C}$ is Cauchy and converges to something denoted $L(x)$.

- Need to show L is continuous and $\|L_n - L\| \to 0$.

- Since $\{L_n\}$ is Cauchy in $X^{\vee}$, choose N large enough so that

$$n, m \geq N \implies \|L_n - L_m\| < \varepsilon \implies |L_m(x) - L_n(x)| < \varepsilon \quad \forall x \,\Big|\, \|x\| = 1.$$

- Take $n \to \infty$ to obtain

$$
\begin{aligned}
m \geq N &\implies |L_m(x) - L(x)| < \varepsilon \quad \forall x \,\Big|\, \|x\| = 1 \\
&\implies \|L_m - L\| < \varepsilon \to 0.
\end{aligned}
$$

- Continuity:

$$\begin{aligned}
|L(x)| &= |L(x) - L_n(x) + L_n(x)| \\
&\leq |L(x) - L_n(x)| + |L_n(x)| \\
&\leq \varepsilon \|x\| + c\|x\| \\
&= (\varepsilon + c)\|x\| \ \blacksquare \ .
\end{aligned}$$

$\blacksquare$

Theorem 5.1.6 *(Riesz-Fischer).*
Let $U = \{u_n\}_{n=1}^{\infty}$ be an orthonormal set (not necessarily a basis), then

1. There is an isometric surjection

$$\mathcal{H} \to \ell^2(\mathbb{N})$$
$$\mathbf{x} \mapsto \{\langle \mathbf{x}, \ \mathbf{u}_n \rangle\}_{n=1}^{\infty}$$

   i.e. if $\{a_n\} \in \ell^2(\mathbb{N})$, so $\sum |a_n|^2 < \infty$, then there exists a $\mathbf{x} \in \mathcal{H}$ such that

$$a_n = \langle \mathbf{x}, \ \mathbf{u}_n \rangle \quad \forall n.$$

2. $\mathbf{x}$ can be chosen such that

$$\|\mathbf{x}\|^2 = \sum |a_n|^2$$

> *Note: the choice of $\mathbf{x}$ is unique $\iff$ $\{u_n\}$ is* ***complete****, i.e. $\langle \mathbf{x}, \ \mathbf{u}_n \rangle = 0$ for all n implies $\mathbf{x} = \mathbf{0}$.*
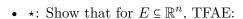

Proof (?).

- Given $\{a_n\}$, define $S_N = \sum\limits^{N} a_n \mathbf{u}_n$.
- S_N is Cauchy in $\mathcal{H}$ and so $S_N \to \mathbf{x}$ for some $\mathbf{x} \in \mathcal{H}$.
- $\langle x, \ u_n \rangle = \langle x - S_N, \ u_n \rangle + \langle S_N, \ u_n \rangle \to a_n$
- By construction, $\|x - S_N\|^2 = \|x\|^2 - \sum\limits^{N} |a_n|^2 \to 0$, so $\|x\|^2 = \sum\limits^{\infty} |a_n|^2$.

$\blacksquare$

6 | **Extra Problems**

6.1 Greatest Hits

- ⋆: Show that for $E \subseteq \mathbb{R}^n$, TFAE:

  1. E is measurable
  2. $E = H \cup Z$ here H is F_σ and Z is null
  3. $E = V \smallsetminus Z'$ where $V \in G_\delta$ and Z' is null.

- ⋆: Show that if $E \subseteq \mathbb{R}^n$ is measurable then $m(E) = \sup \left\{ m(K) \mid K \subset E \text{ compact} \right\}$ iff for all $\varepsilon > 0$ there exists a compact $K \subseteq E$ such that $m(K) \geq m(E) - \varepsilon$.

- ⋆: Show that cylinder functions are measurable, i.e. if f is measurable on $\mathbb{R}^s$, then $F(x, y) \coloneqq f(x)$ is measurable on $\mathbb{R}^s \times \mathbb{R}^t$ for any t.

- ⋆: Prove that the Lebesgue integral is translation invariant, i.e. if $\tau_h(x) = x + h$ then $\displaystyle\int \tau_h f = \int f$.

- ⋆: Prove that the Lebesgue integral is dilation invariant, i.e. if $f_\delta(x) = \dfrac{f(\frac{x}{\delta})}{\delta^n}$ then $\displaystyle\int f_\delta = \int f$.

- ⋆: Prove continuity in L^1, i.e.

$$ f \in L^1 \implies \lim_{h \to 0} \int |f(x+h) - f(x)| = 0. $$

- ⋆: Show that

$$ f, g \in L^1 \implies f * g \in L^1 \quad \text{and} \quad \|f * g\|_1 \leq \|f\|_1 \|g\|_1. $$

- ⋆: Show that if $X \subseteq \mathbb{R}$ with $\mu(X) < \infty$ then

$$ \|f\|_p \overset{p \to \infty}{\to} \|f\|_\infty. $$

6.2 By Topic

6.2.1 Topology

- Show that every compact set is closed and bounded.

- Show that if a subset of a metric space is complete and totally bounded, then it is compact.
- Show that if K is compact and F is closed with K, F disjoint then $\operatorname{dist}(K, F) > 0$.

6.2.2 Continuity

- Show that a continuous function on a compact set is uniformly continuous.

6.2.3 Differentiation

- Show that if $f \in C^1(\mathbb{R})$ and both $\lim\limits_{x \to \infty} f(x)$ and $\lim\limits_{x \to \infty} f'(x)$ exist, then $\lim\limits_{x \to \infty} f'(x)$ must be zero.

6.2.4 Advanced Limitology

- If f is continuous, is it necessarily the case that f' is continuous?
- If $f_n \to f$, is it necessarily the case that f'_n converges to f' (or at all)?
- Is it true that the sum of differentiable functions is differentiable?
- Is it true that the limit of integrals equals the integral of the limit?
- Is it true that a limit of continuous functions is continuous?
- Show that a subset of a metric space is closed iff it is complete.
- Show that if $m(E) < \infty$ and $f_n \to f$ uniformly, then $\lim \int_E f_n = \int_E f$.

Uniform Convergence

- Show that a uniform limit of bounded functions is bounded.
- Show that a uniform limit of continuous function is continuous.

  - I.e. if $f_n \to f$ uniformly with each f_n continuous then f is continuous.

- Show that if $f_n \to f$ pointwise, $f'_n \to g$ uniformly for some f, g, then f is differentiable and $g = f'$.
- Prove that uniform convergence implies pointwise convergence implies a.e. convergence, but none of the implications may be reversed.
- Show that $\sum \dfrac{x^n}{n!}$ converges uniformly on any compact subset of $\mathbb{R}$.

Measure Theory

- Show that continuity of measure from above/below holds for outer measures.

- Show that a countable union of null sets is null.

Measurability

- Show that $f = 0$ a.e. iff $\int_E f = 0$ for every measurable set E.

Integrability

- Show that if f is a measurable function, then $f = 0$ a.e. iff $\int f = 0$.
- Show that a bounded function is Lebesgue integrable iff it is measurable.
- Show that simple functions are dense in L^1.
- Show that step functions are dense in L^1.
- Show that smooth compactly supported functions are dense in L^1.

Convergence

- Prove Fatou's lemma using the Monotone Convergence Theorem.
- Show that if $\{f_n\}$ is in L^1 and $\sum \int |f_n| < \infty$ then $\sum f_n$ converges to an L^1 function and

$$\int \sum f_n = \sum \int f_n.$$

Convolution

- Show that if $f \in L^1$ and g is bounded, then $f * g$ is bounded and uniformly continuous.
- If f, g are compactly supported, is it necessarily the case that $f * g$ is compactly supported?
- Show that under any of the following assumptions, $f * g$ vanishes at infinity:

  - $f, g \in L^1$ are both bounded.
  - $f, g \in L^1$ with just g bounded.
  - f, g smooth and compactly supported (and in fact $f * g$ is smooth)
  - $f \in L^1$ and g smooth and compactly supported (and in fact $f * g$ is smooth)

- Show that if $f \in L^1$ and g' exists with $\dfrac{\partial g}{\partial x_i}$ all bounded, then

$$\frac{\partial}{\partial x_i} (f * g) = f * \frac{\partial g}{\partial x_i}$$

Fourier Analysis

- Show that if $f \in L^1$ then $\widehat{f}$ is bounded and uniformly continuous.
- Is it the case that $f \in L^1$ implies $\widehat{f} \in L^1$?
- Show that if $f, \widehat{f} \in L^1$ then f is bounded, uniformly continuous, and vanishes at infinity.

  - Show that this is not true for arbitrary L^1 functions.

- Show that if $f \in L^1$ and $\widehat{f} = 0$ almost everywhere then $f = 0$ almost everywhere.

  - Prove that $\widehat{f} = \widehat{g}$ implies that $f = g$ a.e.

- Show that if $f, g \in L^1$ then

$$\int \widehat{f} g = \int f \widehat{g}.$$

    – Give an example showing that this fails if g is not bounded.

- Show that if $f \in C^1$ then f is equal to its Fourier *series*.

Approximate Identities

- Show that if φ is an approximate identity, then

$$\|f * \varphi_t - f\|_1 \overset{t\to 0}{\to} 0.$$

    – Show that if additionally $|\varphi(x)| \leq c(1 + |x|)^{-n-\varepsilon}$ for some $c, \varepsilon > 0$, then this converges is almost everywhere.

- Show that is f is bounded and uniformly continuous and φ_t is an approximation to the identity, then $f * \varphi_t$ uniformly converges to f.

L^p Spaces

- Show that if $E \subseteq \mathbb{R}^n$ is measurable with $\mu(E) < \infty$ and $f \in L^p(X)$ then

$$\|f\|_{L^p(X)} \overset{p\to\infty}{\to} \|f\|_\infty.$$

- Is it true that the converse to the DCT holds? I.e. if $\int f_n \to \int f$, is there a $g \in L^p$ such that $f_n < g$ a.e. for every n?
- Prove continuity in L^p: If f is uniformly continuous then for all p,

$$\|\tau_h f - f\|_p \overset{h\to 0}{\to} 0.$$

- Prove the following inclusions of L^p spaces for $m(X) < \infty$:

$$L^\infty(X) \subset L^2(X) \subset L^1(X)$$
$$\ell^2(\mathbb{Z}) \subset \ell^1(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z}).$$

7 | Practice Exam (November 2014)

7.1 1: Fubini-Tonelli

7.1.1 a

Carefully state Tonelli's theorem for a nonnegative function $F(x, t)$ on $\mathbb{R}^n \times \mathbb{R}$.

7.1.2 b

Let $f : \mathbb{R}^n \to [0, \infty]$ and define

$$\mathcal{A} := \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \ \middle|\ 0 \le t \le f(x) \right\}.$$

Prove the validity of the following two statements:

1. f is Lebesgue measurable on $\mathbb{R}^n$ $\iff$ $\mathcal{A}$ is a Lebesgue measurable subset of $\mathbb{R}^{n+1}$.
2. If f is Lebesgue measurable on $\mathbb{R}^n$ then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)dx = \int_0^\infty m\left(\left\{ x \in \mathbb{R}^n \ \middle|\ f(x) \ge t \right\} \right) dt.$$

7.2 2: Convolutions and the Fourier Transform

7.2.1 a

Let $f, g \in L^1(\mathbb{R}^n)$ and give a definition of $f * g$.

7.2.2 b

Prove that if f, g are integrable and bounded, then

$$(f * g)(x) \overset{|x| \to \infty}{\to} 0.$$

7.2.3 c

1. Define the *Fourier transform* of an integrable function f on $\mathbb{R}^n$.
2. Give an outline of the proof of the Fourier inversion formula.
3. Give an example of a function $f \in L^1(\mathbb{R}^n)$ such that $\widehat{f}$ is not in $L^1(\mathbb{R}^n)$.

7.3 3: Hilbert Spaces

Let $\{u_n\}_{n=1}^\infty$ be an orthonormal sequence in a Hilbert space H.

7.3.1 a

Let $x \in H$ and verify that

$$\left\| x - \sum_{n=1}^{N} \langle x, u_n \rangle \, u_n \right\|_H^2 = \|x\|_H^2 - \sum_{n=1}^{N} |\langle x, u_n \rangle|^2 .$$

for any $N \in \mathbb{N}$ and deduce that

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|_H^2 .$$

7.3.2 b

Let $\{a_n\}_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$ and prove that there exists an $x \in H$ such that $a_n = \langle x, \ u_n \rangle$ for all $n \in \mathbb{N}$, and moreover x may be chosen such that

$$\|x\|_H = \left(\sum_{n \in \mathbb{N}} |a_n|^2 \right)^{\frac{1}{2}} .$$

Proof

- Take $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.

- Define $x := \lim_{N \to \infty} S_N$ where $S_N = \sum_{k=1}^{N} a_k u_k$

- $\{S_N\}$ is Cauchy and H is complete, so $x \in H$.

- By construction,

$$\langle x, \ u_n \rangle = \left\langle \sum_k a_k u_k, \ u_n \right\rangle = \sum_k a_k \langle u_k, \ u_n \rangle = a_n$$

  since the u_k are all orthogonal.

- By Pythagoras since the u_k are normal,

$$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2 .$$

7.3.3 c

Prove that if $\{u_n\}$ is *complete*, Bessel's inequality becomes an equality.

Proof Let x and u_n be arbitrary.

$$\left\langle x - \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k,\ u_n \right\rangle = \langle x,\ u_n \rangle - \left\langle \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k,\ u_n \right\rangle$$

$$= \langle x,\ u_n \rangle - \sum_{k=1}^{\infty} \langle \langle x,\ u_k \rangle u_k,\ u_n \rangle$$

$$= \langle x,\ u_n \rangle - \sum_{k=1}^{\infty} \langle x,\ u_k \rangle \langle u_k,\ u_n \rangle$$

$$= \langle x,\ u_n \rangle - \langle x,\ u_n \rangle = 0$$

$$\implies x - \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k = 0 \quad \text{by completeness.}$$

So

$$x = \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k \implies \|x\|^2 = \sum_{k=1}^{\infty} |\langle x,\ u_k \rangle|^2. \ \blacksquare \ .$$

7.4 4: L^p Spaces

7.4.1 a

Prove Holder's inequality: let $f \in L^p, g \in L^q$ with p, q conjugate, and show that

$$\|fg\|_p \le \|f\|_p \cdot \|g\|_q.$$

7.4.2 b

Prove Minkowski's Inequality:

$$1 \le p < \infty \implies \|f + g\|_p \le \|f\|_p + \|g\|_p.$$

Conclude that if $f, g \in L^p(\mathbb{R}^n)$ then so is $f + g$.

7.4.3 c

Let $X = [0, 1] \subset \mathbb{R}$.

1. Give a definition of the Banach space $L^{\infty}(X)$ of essentially bounded functions of X.

2. Let f be non-negative and measurable on X, prove that

$$\int_X f(x)^p \, dx \overset{p \to \infty}{\to} \begin{cases} \infty & \text{or} \\ m\left(\left\{f^{-1}(1)\right\}\right) \end{cases},$$

and characterize the functions of each type

Solution:

$$\begin{aligned} \int f^p &= \int_{x<1} f^p + \int_{x=1} f^p + \int_{x>1} f^p \\ &= \int_{x<1} f^p + \int_{x=1} 1 + \int_{x>1} f^p \\ &= \int_{x<1} f^p + m(\{f=1\}) + \int_{x>1} f^p \\ &\overset{p \to \infty}{\to} 0 + m(\{f=1\}) + \begin{cases} 0 & m(\{x \geq 1\}) = 0 \\ \infty & m(\{x \geq 1\}) > 0. \end{cases} \end{aligned}$$

7.5 5: Dual Spaces

Let X be a normed vector space.

7.5.1 a

Give the definition of what it means for a map $L : X \to \mathbb{C}$ to be a *linear functional.*

7.5.2 b

Define what it means for L to be *bounded* and show L is bounded $\iff$ L is continuous.

7.5.3 c

Prove that $(X^\vee, \| \cdot \|_{\mathrm{op}})$ is a Banach space.

8 | Midterm Exam 2 (November 2018)

8.1 1 (Integration by Parts)

Let $f, g \in L^1([0,1])$, define $F(x) = \int_0^x f$ and $G(x) = \int_0^x g$, and show

$$\int_0^1 F(x)g(x)\,dx = F(1)G(1) - \int_0^1 f(x)G(x)\,dx.$$

8.2 2

Let $\varphi \in L^1(\mathbb{R}^n)$ such that $\int \varphi = 1$ and define $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$.

Show that if f is bounded and uniformly continuous then $f * \varphi_t \overset{t \to 0}{\to} f$ uniformly.

8.3 3

Let $g \in L^\infty([0,1])$.

a. Prove

$$\|g\|_{L^p([0,1])} \overset{p \to \infty}{\to} \|g\|_{L^\infty([0,1])}.$$

b. Prove that the map

$$\Lambda_g : L^1([0,1]) \to \mathbb{C}$$

$$f \mapsto \int_0^1 fg$$

defines an element of $L^1([0,1])^\vee$ with $\|\Lambda_g\|_{L^1([0,1])^\vee} = \|g\|_{L^\infty([0,1])}$.

> *Note: 4 is a repeat.*

9 │ **Midterm Exam 2 (December 2014)**

9.1 1

- Let $\Lambda \in L^2(X)^{\vee}$.

  - Show that $M \coloneqq \left\{ f \in L^2(X) \mid \Lambda(f) = 0 \right\} \subseteq L^2(X)$ is a closed subspace, and $L^2(X) = M \oplus M \perp$.
  - Prove that there exists a unique $g \in L^2(X)$ such that $\Lambda(f) = \int_X g\overline{f}$.

9.2 2

a. In parts:

- Given a definition of $L^{\infty}(\mathbb{R}^n)$.
- Verify that $\|\cdot\|_{\infty}$ defines a norm on $L^{\infty}(\mathbb{R}^n)$.
- Carefully proved that $(L^{\infty}(\mathbb{R}^n), \|\cdot\|_{\infty})$ is a Banach space.

b. Prove that for any measurable $f : \mathbb{R}^n \to \mathbb{C}$,

$$L^1(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \quad \text{and} \quad \|f\|_2 \le \|f\|_1^{\frac{1}{2}} \cdot \|f\|_{\infty}^{\frac{1}{2}}.$$

9.3 3

a. Prove that if $f, g : \mathbb{R}^n \to \mathbb{C}$ is both measurable then $F(x, y) \coloneqq f(x)$ and $h(x, y) \coloneqq f(x - y)g(y)$ is measurable on $\mathbb{R}^n \times \mathbb{R}^n$.

b. Show that if $f \in L^1(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, then $f * g \in L^1(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$ is well defined, and carefully show that it satisfies the following properties:

$$\|f * g\|_{\infty} \le \|g\|_1 \|f\|_{\infty} \|f * g\|_1 \qquad \le \|g\|_1 \|f\|_1 \|f * g\|_2 \le \|g\|_1 \|f\|_2.$$

Note: (a) is a repeat.

Let $f : [0,1] \to \mathbb{R}$ be continuous, and prove the Weierstrass approximation theorem: for any $\varepsilon > 0$ there exists a polynomial P such that $\|f - P\|_\infty < \varepsilon$.

10 | Common Inequalities

Proposition 10.0.1 *(Reverse Triangle Inequality).*

$$\big|\|x\| - \|y\|\big| \le \|x - y\|.$$

Proposition 10.0.2 *(Chebyshev's Inequality).*

$$\mu(\{x : |f(x)| > \alpha\}) \le \left(\frac{\|f\|_p}{\alpha}\right)^p.$$

Proposition 10.0.3 *(Holder's Inequality).*

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \|fg\|_1 \le \|f\|_p \|g\|_q.$$

Proof (of Holder's inequality).
It suffices to show this when $\|f\|_p = \|g\|_q = 1$, since

$$\|fg\|_1 \le \|f\|_p \|f\|_q \iff \int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \le 1.$$

Using $AB \le \frac{1}{p}A^p + \frac{1}{q}B^q$, we have

$$\int |f||g| \le \int \frac{|f|^p}{p} \frac{|g|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1.$$

■

Example 10.0.4 *(Application of Holder's inequality: containment of L^p spaces):* For

finite measure spaces,

$$1 \le p < q \le \infty \implies L^q \subset L^p \quad (\text{ and } \ell^p \subset \ell^q).$$

> *Proof (of containment of L^p spaces).*
> Fix p, q, let $r = \dfrac{q}{p}$ and $s = \dfrac{r}{r-1}$ so $r^{-1} + s^{-1} = 1$. Then let $h = |f|^p$:
>
> $$\|f\|_p^p = \|h \cdot 1\|_1 \le \|1\|_s \|h\|_r = \mu(X)^{\frac{1}{s}} \|f\|_q^{\frac{q}{r}} \implies \|f\|_p \le \mu(X)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q.$$
>
> > *Note: doesn't work for ℓ_p spaces, but just note that $\sum |x_n| < \infty \implies x_n < 1$ for large enough n, and thus $p < q \implies |x_n|^q \le |x_n|^q$.*
>
> ∎

> **Proposition 10.0.5** *(Cauchy-Schwarz Inequality).*
>
> $$|\langle f, \ g \rangle| = \|fg\|_1 \le \|f\|_2 \|g\|_2 \qquad\qquad \text{with equality} \iff f = \lambda g.$$

Remark 10.0.6: In general, Cauchy-Schwarz relates inner product to norm, and only happens to relate norms in L^1.

> **Proposition 10.0.7** *(Minkowski's Inequality).*
>
> $$1 \le p < \infty \implies \|f + g\|_p \le \|f\|_p + \|g\|_p.$$

Remark 10.0.8: This does not handle $p = \infty$ case. Use to prove L^p is a normed space.

> *Proof (?).*
>
> - We first note
>
> $$|f + g|^p = |f + g||f + g|^{p-1} \le (|f| + |g|) |f + g|^{p-1}.$$
>
> - Note that if p, q are conjugate exponents then
>
> $$\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$$
> $$q = \frac{p}{p-1}.$$

- Then taking integrals yields

$$
\begin{aligned}
\|f + g\|_p^p &= \int |f + g|^p \\
&\leq \int (|f| + |g|) \, |f + g|^{p-1} \\
&= \int |f| |f + g|^{p-1} + \int |g| |f + g|^{p-1} \\
&= \left\| f(f + g)^{p-1} \right\|_1 + \left\| g(f + g)^{p-1} \right\|_1 \\
&\leq \|f\|_p \left\| (f + g)^{p-1} \right\|_q + \|g\|_p \left\| (f + g)^{p-1} \right\|_q \\
&= \left(\|f\|_p + \|g\|_p \right) \left\| (f + g)^{p-1} \right\|_q \\
&= \left(\|f\|_p + \|g\|_p \right) \left(\int |f + g|^{(p-1)q} \right)^{\frac{1}{q}} \\
&= \left(\|f\|_p + \|g\|_p \right) \left(\int |f + g|^p \right)^{1 - \frac{1}{p}} \\
&= \left(\|f\|_p + \|g\|_p \right) \frac{\int |f + g|^p}{\left(\int |f + g|^p \right)^{\frac{1}{p}}} \\
&= \left(\|f\|_p + \|g\|_p \right) \frac{\|f + g\|_p^p}{\|f + g\|_p}.
\end{aligned}
$$

- Cancelling common terms yields

$$
1 \leq \left(\|f\|_p + \|g\|_p \right) \frac{1}{\|f + g\|_p}
$$
$$
\implies \|f + g\|_p \leq \|f\|_p + \|g\|_p.
$$

∎

Proposition 10.0.9 *(Young's Inequality).*

$$
\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \implies \|f * g\|_r \leq \|f\|_p \|g\|_q
$$

Remark 10.0.10 *(some useful special cases):*

$$
\begin{aligned}
\|f * g\|_1 &\leq \|f\|_1 \|g\|_1 \\
\|f * g\|_p &\leq \|f\|_1 \|g\|_p, \\
\|f * g\|_\infty &\leq \|f\|_2 \|g\|_2 \\
\|f * g\|_\infty &\leq \|f\|_p \|g\|_q.
\end{aligned}
$$

Proposition 10.0.11 *(Bessel's Inequality).*

For $x \in H$ a Hilbert space and $\{e_k\}$ an orthonormal sequence,

$$\sum_{k=1}^{\infty} \|\langle x,\ e_k \rangle\|^2 \leq \|x\|^2.$$

Note that this does not need to be a basis.

Proposition 10.0.12*(Parseval's Identity).*
Equality in Bessel's inequality, attained when $\{e_k\}$ is a *basis*, i.e. it is complete, i.e. the span of its closure is all of H.

11 | Less Explicitly Used Inequalities

Proposition 11.0.1*(AM-GM Inequality).*

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Proposition 11.0.2*(Jensen's Inequality).*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Proposition 11.0.3*(Young's Product Inequality).*

$$AB \leq \frac{A^p}{p} + \frac{B^q}{q}.$$

Proposition 11.0.4*(?).*

$$(a+b)^p \leq 2^{p-1}(a^p + b^p).$$

Proposition 11.0.5*(Bernoulli's Inequality).*

$$(1+x)^n \geq 1 + nx \quad x \geq -1, \text{ or } n \in 2\mathbb{Z} \text{ and } \forall x.$$

Proposition 11.0.6*(Exponential Inequality).*

$$\forall t \in \mathbb{R}, \quad 1 + t \le e^t.$$

Proof .

- It's an equality when $t = 0$.
- $\dfrac{\partial}{\partial t} 1 + t < \dfrac{\partial t}{\partial e}^{\,t} \iff t < 0$

∎

Proposition 11.0.7 *(Young's Convolution Inequality).*

$$\frac{1}{r} := \frac{1}{p} + \frac{1}{q} - 1 \implies \|f * g\|_r \le \|f\|_p \|g\| q.$$