Complex Analysis Qualifying Exam Notes

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Contents

Preface

References

- Simon

1 Theorems

1.1 Basics

Theorem 1.1 ***(Green's Theorem).***
If $\Omega \subseteq \mathbb{C}$ is bounded with $\partial\Omega$ piecewise smooth and $f, g \in C^1(\overline{\Omega})$, then

$$\int_{\partial\Omega} f\,dx + g\,dy = \iint_{\Omega} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA.$$

Theorem 1.2 ***(Summation by Parts).***
Define the forward difference operator $\Delta f_k = f_{k+1} - f_k$, then

$$\sum_{k=m}^{n} f_k \Delta g_k + \sum_{k=m}^{n-1} g_{k+1} \Delta f_k = f_n g_{n+1} - f_m g_m$$

Note: compare to $\int_a^b f\,dg + \int_a^b g\,df = f(b)g(b) - f(a)g(a)$.

1.2 Integrals and Residues

Theorem 1.3 ***(Cauchy Integral Formula).***
Suppose f is holomorphic on Ω, then

$$f(z) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(z)}{z-a}\,dz$$

and

$$\frac{\partial^n f}{\partial z^n}(z) - \frac{n!}{2\pi i} \oint_{\partial\Omega} \frac{f\xi}{(\xi - z)^{n+1}}\,d\xi.$$

The nth Taylor coefficient of an analytic function is at most $\sup\limits_{|z|=R} |f|/R^n$: :::{.theorem title="Cauchy's Inequality"} For $z_o \in D_R(z_0) \subset \Omega$, we have

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{\|f\|_{C_R}}{R^{n+1}} R d\theta = \frac{n! \|f\|_{C_R}}{R^n}.$$

:::

These don't quite match up.

1.3 Holomorphic and Entire Functions

Integrals of holomorphic functions vanish: :::{.theorem title="Cauchy's Theorem"} If f is holomorphic on Ω, then

$$\int_{\partial\Omega} f(z)\,dz = 0.$$

:::

All integrals vanishing along every triangle implies holomorphic: :::{.theorem title="Morera's Theorem"} If f is continuous on a domain Ω and $\int_T f = 0$ for every triangle $T \subset \Omega$, then f is holomorphic. :::

Theorem 1.4 ***(Liouville).***
If f is entire and bounded, f is constant.

1.4 Rouché

The logarithmic derivative measures the difference of zeros and poles: :::{.theorem title="Argument Principle"} Todo :::

Argument principle.

Theorem 1.5 ***(Rouché).***
If f, g are analytic on a domain Ω with finitely many zeros in Ω and $\gamma \subset \Omega$ is a closed curve surrounding each point exactly once, where $|g| < |f|$ on γ, then f and $f+g$ have the same number of zeros.

Example 1.1.
- Take $P(z) = z^4 + 6z + 3$.
- On $|z| < 2$:
  - Set $f(z) = z^4$ and $g(z) = 6z + 3$, then $|g(z)| \leq 6|z| + 3 = 15 < 16 = |f(z)|$.
  - So P has 4 zeros here.
- On $|z| < 1$:
  - Set $f(z) = 6z$ and $g(z) = z^4 + 3$.
  - Check $|g(z)| \leq |z|^4 + 3 = 4 < 6 = |f(z)|$.
  - So P has 1 zero here.

Example 1.2.
- Claim: the equation $\alpha z e^z = 1$ where $|\alpha| > e$ has exactly one solution in $\mathbb{D}$.
- Set $f(z) = \alpha z$ and $g(z) = e^{-z}$.
- Estimate at $|z| = 1$ we have $|g| = |e^{-z}| = e^{-\Re(z)} \leq e^1 < |\alpha| = |f(z)|$
- f has one zero at $z_0 = 0$, thus so does $f + g$.

Holomorphic functions preserve open sets: :::{.theorem title="Open Mapping"} Any holomorphic non-constant map is an open map. :::

Theorem 1.6 ***(Maximum Modulus).***
If f is holomorphic and nonconstant on an open region Ω, then $|f|$ can not attain a maximum on Ω.
If Ω is bounded and f is continuous on $\overline{\Omega}$, then $\max_{\overline{\Omega}} |f|$ occurs on $\partial\Omega$.
Conversely, if f attains a local maximum at $z_0 \in \Omega$, then f is constant on Ω.

The image of a disc punctured at an essential singularity is dense in $\mathbb{C}$: :::{.theorem title="Casorati-Weierstrass"} If f is holomorphic on $\Omega \setminus \{z_0\}$ where z_0 is an essential singularity, then for every $V \subset \Omega \setminus \{z_0\}$, $f(V)$ is dense in $\mathbb{C}$. :::

Theorem 1.7 ***(Cayley Transform).***
The fractional linear transformation given by $F(z) = \dfrac{i-z}{i+z}$ maps $\mathbb{D} \longrightarrow \mathbb{H}$ with inverse $G(w) = i\dfrac{1-w}{1+w}$.

Two functions agreeing on a set with a limit point are equal on a domain: :::{.theorem title="Continuation Principle / Identity Theorem"} If f is holomorphic on a bounded connected domain Ω and there exists a sequence $\{z_i\}$ with a limit point in Ω such that $f(z_i) = 0$, then $f \equiv 0$ on Ω. :::

Theorem 1.8 ***(Schwarz Reflection).***
If f is continuous and holomorphic on $\mathbb{H}^+$ and real-valued on $\mathbb{R}$, then the extension defined by $F(z) = \overline{f(\overline{z})}$ for $z \in \mathbb{H}^-$ is a well-defined holomorphic function on $\mathbb{C}$.

Remark 1.
$\mathbb{H}^+, \mathbb{H}^-$ can be replaced with any region symmetric about a line segment $L \subseteq \mathbb{R}$.

Theorem 1.9 ***(Schwarz Lemma).***
If $f : \mathbb{D} \longrightarrow \mathbb{D}$ is holomorphic with $f(0) = 0$, then

1. $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$
2. $|f'(0)| \leq 1$.

Moreover, if $|f(z_0)| = |z_0|$ for any $z_0 \in \mathbb{D}$ or $|f'(0)| = 1$, then f is a rotation

Theorem 1.10 ***(Riemann Mapping).***
If Ω is simply connected, nonempty, and not $\mathbb{C}$, then for every $z_0 \in \Omega$ there exists a unique conformal map $F : \Omega \longrightarrow \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$.
Thus any two such sets Ω_1, Ω_2 are conformally equivalent.

2 Stuff

2.0.1 Fundamental Theorem of Algebra: Argument Principle

- Let $P(z) = a_n z^n + \cdots + a_0$ and $g(z) = P'(z)/P(z)$, note P is holomorphic
- Since $\lim_{|z| \longrightarrow \infty} P(z) = \infty$, there exist an $R > 0$ such that P has no roots in $\{|z| \geq R\}$.
- Apply the argument principle:

$$N(0) = \frac{1}{2\pi i} \oint_{|\xi| = R} g(\xi)\, d\xi.$$

- Check that $\lim_{|z \longrightarrow \infty|} zg(z) = n$, so g has a simple pole at ∞
- Then g has a Laurent series $\dfrac{n}{z} + \dfrac{c_2}{z^2} + \cdots$
- Integrate term-by-term to get $N(0) = n$.

2.0.2 Fundamental Theorem of Algebra: Rouche's Theorem

- Let $P(z) = a_n z^n + \cdots + a_0$
- Set $f(z) = a_n z^n$ and $g(z) = P(z) - f(z) = a_{n-1}z^{n-1} + \cdots + a_0$, so $f + g = P$.
- Choose $R > \max\left(\dfrac{|a_{n-1}| + \cdots + |a_0|}{|a_n|}, 1\right)$, then

$$
\begin{aligned}
|g(z)| &\coloneqq |a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| \\
&\leq |a_{n-1}z^{n-1}| + \cdots + |a_1 z| + |a_0| \quad \text{by the triangle inequality} \\
&= |a_{n-1}| \cdot |z^{n-1}| + \cdots + |a_1| \cdot |z| + |a_0| \\
&= |a_{n-1}| \cdot R^{n-1} + \cdots + |a_1| R + |a_0| \\
&\leq |a_{n-1}| \cdot R^{n-1} + |a_{n-2}| \cdot R^{n-1} + \cdots + |a_1| \cdot R^{n-1} + |a_0| \cdot R^{n-1} \quad \text{since } R > 1 \implies R^{a+b} \geq R^a \\
&= R^{n-1}\left(|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0|\right) \\
&\leq R^{n-1}\left(|a_n| \cdot R\right) \quad \text{by choice of } R \\
&= R^n |a_n| \\
&= |a_n z^n| \\
&\coloneqq |f(z)|
\end{aligned}
$$

- Then $a_n z^n$ has n zeros in $|z| < R$, so $f + g$ also has n zeros.

2.0.3 Fundamental Theorem of Algebra: Liouville's Theorem

- Suppose p is nonconstant and has no roots, then $\dfrac{1}{p}$ is entire
- Write $g(z) \coloneqq \dfrac{p(z)}{z^n} = a_n \left(\dfrac{a_{n-1}}{z} + \cdots + \dfrac{a_0}{z^n}\right)$
- Outside a disc:
  - Note $\lim\limits_{z \longrightarrow \infty} = 0$ for the parenthesized terms, so there exists an R large enough such that $|g(z)| \geq \dfrac{1}{2}|a_n|$
  - Then $|p(z)| \geq \dfrac{R^n}{2}|a_n|$ implies $\dfrac{1}{p}$ is bounded in $|z| > R$
- Inside a disc:
  - p is continuous with no roots so p is bounded below on $|z| < R$.
  - p is continuous on a compact set and thus achieves a min A
  - Set $B = \min(A, \dfrac{R^n}{2}|a_n|)$, then $p \geq B$ on $|z| < R$.
- Thus p is bounded below everywhere and thus $\dfrac{1}{p}$ is bounded above everywhere, thus bounded.
- Thus $\dfrac{1}{p}$ is constant, forcing p to be constant.

2.0.4 Fundamental Theorem of Algebra: Open Mapping Theorem

- p induces a continuous map $\mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$
- The continuous image of compact space is compact;
- Since the codomain is Hausdorff space, the image is closed.

- p is holomorphic and non-constant, so by the Open Mapping Theorem, the image is open.
- Thus the image is clopen in $\mathbb{CP}^1$.
- The image is nonempty, since $p(1) = \sum a_i \in \mathbb{C}$
- $\mathbb{CP}^1$ is connected
- But the only nonempty clopen subset of a connected space is the entire space.
- So p is surjective, and $p^{-1}(0)$ is nonempty.
- So p has a root.

3 Conformal Maps

Conformal maps $\mathbb{D} \longrightarrow \mathbb{D}$ have the form

$$g(z) = \lambda \frac{1-a}{1-\bar{a}z}, \quad |a| < 1, \quad |\lambda| = 1.$$

3.1 Plane to Disc

$$\begin{aligned} \varphi &: \mathbb{H} \longrightarrow \mathbb{D} \\ \varphi(z) &= \frac{z-i}{z+i} \qquad f^{-1}(z) = i\left(\frac{1+w}{1-w}\right). \end{aligned}$$

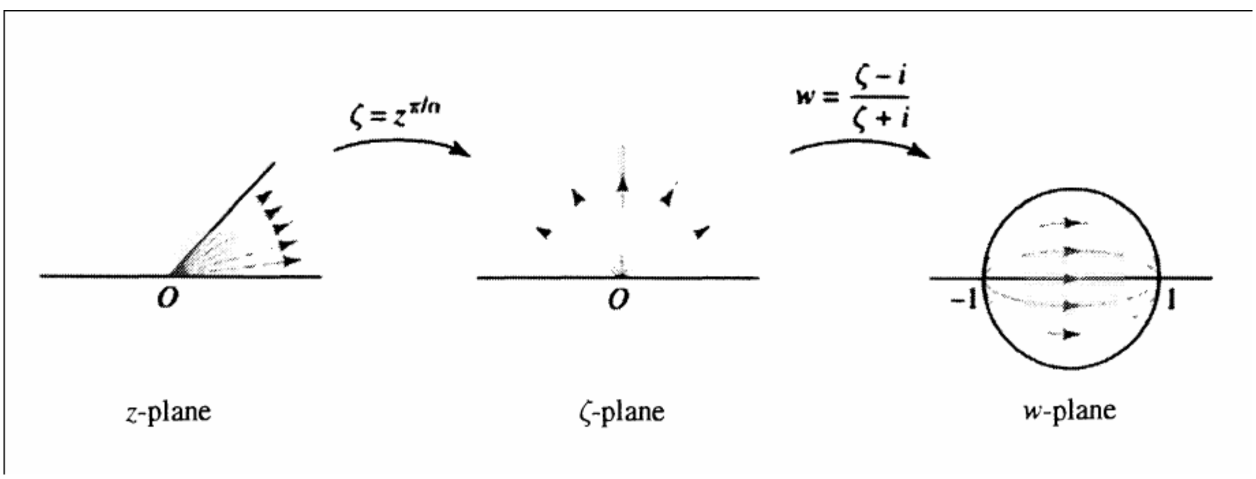
3.2 Sector to Disc

For $S_\alpha := \left\{ z \in \mathbb{C} \;\middle|\; 0 < \arg(z) < \alpha \right\}$ an open sector for α some angle, first map the sector to the half-plane:

$$\begin{aligned} g &: S_\alpha \longrightarrow \mathbb{H} \\ g(z) &= z^{\frac{\pi}{\alpha}}. \end{aligned}$$

Then compose with a map $\mathbb{H} \longrightarrow \mathbb{D}$:

$$\begin{aligned} f &: S_\alpha \longrightarrow \mathbb{D} \\ f(z) &= (\varphi \circ g)(z) = \frac{z^{\frac{\pi}{\alpha}} - i}{z^{\frac{\pi}{\alpha}} + i}. \end{aligned}$$



3.3 Strip to Disc

- Map to horizontal strip by rotation $z \mapsto \lambda z$.
- Map horizontal strip to sector by $z \mapsto e^z$
- Map sector to $\mathbb{H}$ by $z \mapsto z^{\frac{\pi}{\alpha}}$.
- Map $\mathbb{H} \longrightarrow \mathbb{D}$.

4 Appendix

$$
\begin{aligned}
dz &= dx + i\ dy \\
d\bar{z} &= dx - i\ dy \\
f_z &= f_x = i^{-1} f_y \\
\int_0^{2\pi} e^{i\ell x} dx &= \begin{cases} 2\pi & (\ell = 0) \\ 0 & (\ell \neq 0) \end{cases}.
\end{aligned}
$$

- Holomorphic: once complex differentiable in neighborhoods of every point.
- Analytic: equal to its Taylor series expansion

Collection of facts used on problem sets

4.1 Things to know well:

- Cauchy Integral Formula
- Estimates for derivatives, mean value theorem
- Rouché's theorem
- Casorati-Weierstrass
- The 8 types of conformal maps

4.2 Theorems

4.2.1 The Argument Principle

Theorem (Statement 1) For f meromorphic in γ°,

$$\Delta_\gamma \arg f(z) = 2\pi(Z_f - P_f).$$

4.2.2 Rouche

Theorem (Statement 1) Suppose $f = g + h$ with $g \neq 0, \infty$ on γ with $|g| > |h|$ on γ. Then

$$\Delta_\gamma \arg(f) = \Delta_\gamma \arg(h) \quad \text{and } Z_f - P_f = Z_g - P_g.$$

4.3 Misc Prereq

Standard forms of conic sections:

- Circle: $x^2 + y^2 = r^2$

- Ellipse: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$
- Hyperbola: $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$
  - Rectangular Hyperbola: $xy = \frac{c^2}{2}$.
- Parabola: $-4ax + y^2 = 0$.

Mnemonic: Write $f(x, y) = Ax^2 + Bxy + Cy^2 + \cdots$, then consider the discriminant $\Delta = B^2 - 4AC$:

- $\Delta < 0 \iff$ ellipse
  - $\Delta < 0$ and $A = C, B = 0 \iff$ circle
- $\Delta = 0 \iff$ parabola
- $\Delta > 0 \iff$ hyperbola

Completing the square:

$$x^2 - bx = (x - s)^2 - s^2 \quad \text{where} s = \frac{b}{2}$$
$$x^2 + bx = (x + s)^2 - s^2 \quad \text{where} s = \frac{b}{2}.$$

Useful Properties

- $\Re(z) = \frac{1}{2}(z + \bar{z})$ and $\Im(z) = \frac{1}{2i}(z - \bar{z})$.
- $z\bar{z} = |z|^2$
- $\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$
- $\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$.

Useful Series

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$
$$\log(z) = \sum_{j=0}^{\infty} (-1)^j \frac{(z-a)^j}{j}$$

Cauchy-Riemann Equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$
.

4.4 Useful Techniques

Showing a function is constant:

- Write $f = u + iv$ and use Cauchy-Riemann to show $u_x, u_y = 0$, etc.
- Show that f is entire and bounded.

Showing a function is zero: Show f is entire, bounded, and $\lim_{z \longrightarrow \infty} f(z) = 0$.

Deriving Polar Cauchy-Riemann: See walkthrough here. Take derivative along two paths, along a ray with constant angle θ_0 and along a circular arc of constant radius r_0. Then equate real and imaginary parts. See problem set 1.

Computing Arguments: $\text{Arg}(z/w) = \text{Arg}(z) - \text{Arg}(w)$.

The sum of the interior angles of an n-gon is $(n-2)\pi$, where each angle is $\frac{n-2}{n}\pi$.

4.5 Residues

If p is a simple pole, $\text{Res}(p, f) = \lim_{z \longrightarrow p} (z-p)f(z)$. Example: Let $f(z) = \frac{1}{1+z^2}$, then $\text{Res}(i, f) = \frac{1}{2i}$.

Green's Theorem: Todo

$\frac{\partial}{\partial z} \sum_{j=0}^{\infty} a_j z^j = \sum_{j=0}^{\infty} a_{j+1} z^j$.

4.6 Pithy Statements

- Little Picard: f misses at most one point and is a homeomorphism onto its image.
- Baire's Theorem: The intersection of open dense sets is open.
- Morera: Integrals vanishing along every rectangle implies holomorphic.
- Schwarz Reflection: ???
- The ring of holomorphic functions on a domain in $\mathbb{C}$ has no zero divisors (by the identity principle).

4.7 Precise Refinements

Cauchy Inequality: Given $z_0 \in \Omega$, pick the largest disc $D_R(z_0) \subset \Omega$ and let $C_R = \partial D_R$. Using the integral formula, defining $\|f\|_{C_R} = \max_{|z-z_0|=R} |f(z)|$

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{\|f\|_{C_R}}{R^{n+1}} R \, d\theta = \frac{n! \|f\|_{C_R}}{R^n}.$$

Basics

- Show that $\frac{1}{z}\sum_{k=1}^{\infty}\frac{z^k}{k}$ converges on $S^1 \setminus \{1\}$ using summation by parts.
- Show that any power series is continuous on its domain of convergence.
- Show that a uniform limit of continuous functions is continuous.

??

- Show that if f is holomorphic on $\mathbb{D}$ then f has a power series expansion that converges uniformly on every compact $K \subset \mathbb{D}$.
- Show that any holomorphic function f can be uniformly approximated by polynomials.
- Show that if f is holomorphic on a connected region Ω and $f' \equiv 0$ on Ω, then f is constant on Ω.
- Show that if $|f| = 0$ on $\partial\Omega$ then either f is constant or f has a zero in Ω.
- Show that if $\{f_n\}$ is a sequence of holomorphic functions converging uniformly to a function f on every compact subset of Ω, then f is holomorphic on Ω and $\{f_n'\}$ converges uniformly to f' on every such compact subset.
- Show that if each f_n is holomorphic on Ω and $F := \sum f_n$ converges uniformly on every compact subset of Ω, then F is holomorphic.
- Show that if f is once complex differentiable at each point of Ω, then f is holomorphic.