Real Analysis Qualifying Exam Solutions

D. Zack Garza

Monday 20^{th} July, 2020

Contents

1 Spring 2020

1.1 1

Concepts used:

- DCT
- Weierstrass Approximation Theorem

Solution:

- Suppose p is a polynomial, then

$$\begin{aligned}
\lim_{k\longrightarrow\infty} \int_0^1 kx^{k-1} p(x)\,dx &= \lim_{k\longrightarrow\infty} \int_0^1 \left(\frac{\partial}{\partial x} x^k\right) p(x)\,dx \\
&= \lim_{k\longrightarrow\infty} \left[x^k p(x) \Big|_0^1 - \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x)\right) dx \right] \quad \text{integrating by parts} \\
&= p(1) - \lim_{k\longrightarrow\infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x)\right) dx,
\end{aligned}$$

- Thus it suffices to show that

$$\lim_{k\longrightarrow\infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x)\right) dx = 0.$$

- Integrating by parts a second time yields

$$\begin{aligned}
\lim_{k\longrightarrow\infty}\int_0^1 x^k\left(\frac{\partial}{\partial x}\,p(x)\right)dx &= \lim_{k\longrightarrow\infty}\frac{x^{k+1}}{k+1}p'(x)\Big|_0^1-\int_0^1\frac{x^{k+1}}{k+1}\left(\frac{\partial^2}{\partial x^2}\,p(x)\right)dx\\
&= -\lim_{k\longrightarrow\infty}\int_0^1\frac{x^{k+1}}{k+1}\left(\frac{\partial^2}{\partial x^2}\,p(x)\right)dx\\
&= -\int_0^1\lim_{k\longrightarrow\infty}\frac{x^{k+1}}{k+1}\left(\frac{\partial^2}{\partial x^2}\,p(x)\right)dx \quad \text{by DCT}\\
&= -\int_0^1 0\left(\frac{\partial^2}{\partial x^2}\,p(x)\right)dx\\
&= 0.
\end{aligned}$$

  - The DCT can be applied here because f'' is continuous and $[0,1]$ is compact, so f'' is bounded on $[0,1]$ by a constant M and

$$\int_0^1\left|x^k f''(x)\right| \leq \int_0^1 1\cdot M = M < \infty.$$

- Now use the Weierstrass approximation theorem:
  - If $f:[a,b]\longrightarrow\mathbb{R}$ is continuous, then for every $\varepsilon>0$ there exists a polynomial $p_\varepsilon(x)$ such that $\|f-p_\varepsilon\|_\infty<\varepsilon$.
- Thus

$$\begin{aligned}
\left|\int_0^1 kx^{k-1}p_\varepsilon(x)\,dx-\int_0^1 kx^{k-1}f(x)\,dx\right| &= \left|\int_0^1 kx^{k-1}(p_\varepsilon(x)-f(x))\,dx\right|\\
&\leq \left|\int_0^1 kx^{k-1}\|p_\varepsilon-f\|_\infty\,dx\right|\\
&= \|p_\varepsilon-f\|_\infty\cdot\left|\int_0^1 kx^{k-1}\,dx\right|\\
&= \|p_\varepsilon-f\|_\infty\cdot x^k\Big|_0^1\\
&= \|p_\varepsilon-f\|_\infty \overset{\varepsilon\longrightarrow 0}{\longrightarrow} 0
\end{aligned}$$

  and the integrals are equal.
- By the first argument,

$$\int_0^1 kx^{k-1}p_\varepsilon(x)\,dx = p_\varepsilon(1) \text{ for each } \varepsilon$$

- Since uniform convergence implies pointwise convergence, $p_\varepsilon(1)\overset{\varepsilon\longrightarrow 0}{\longrightarrow} f(1)$.

1.2 2

Concepts used:

- Definition of outer measure: $m_*(E) = \inf_{\{Q_j\} \rightrightarrows E} \sum |Q_j|$ where $\{Q_j\}$ is a countable collection of closed cubes.
- Break $\mathbb{R}$ into $\coprod_{n \in \mathbb{Z}} [n, n+1)$, each with finite measure.
- Theorem: $m_*(Q) = |Q|$ for Q a closed cube (i.e. the outer measure equals the volume).

Proof (of Theorem).
Statement: if Q is a closed cube, then $m_*(Q) = |Q|$, the usual volume.

- $m_*(Q) \leq |Q|$:
  - Since $Q \subseteq Q$, $Q \rightrightarrows Q$ and $m_*(Q) \leq |Q|$ since m_* is an infimum over such coverings.
- $|Q| \leq m_*(Q)$:
  - Fix $\varepsilon > 0$.
  - Let $\{Q_i\}_{i=1}^\infty \rightrightarrows Q$ be arbitrary, it suffices to show that

$$|Q| \leq \left(\sum_{i=1}^\infty |Q_i| \right) + \varepsilon.$$

  - Pick open cubes S_i such that $Q_i \subseteq S_i$ and $|Q_i| \leq |S_i| \leq (1+\varepsilon)|Q_i|$.
  - Then $\{S_i\} \rightrightarrows Q$, so by compactness of Q pick a finite subcover with N elements.
  - Note

$$Q \subseteq \bigcup_{i=1}^N S_i \implies |Q| \leq \sum_{i=1}^N |S_i| \leq \sum_{i=1}^N (1+\varepsilon)|Q_j| \leq (1+\varepsilon) \sum_{i=1}^\infty |Q_i|.$$

  - Taking an infimum over coverings on the RHS preserves the inequality, so

$$|Q| \leq (1+\varepsilon) m_*(Q)$$

  - Take $\varepsilon \longrightarrow 0$ to obtain final inequality.

■

1.2.1 a

- If $m_*(E) = \infty$, then take $B = \mathbb{R}^n$ since $m(\mathbb{R}^n) = \infty$.
- Suppose $N \coloneqq m_*(E) < \infty$.
- Since $m_*(E)$ is an infimum, by definition, for every $\varepsilon > 0$ there exists a covering by closed cubes $\{Q_i(\varepsilon)\}_{i=1}^\infty \rightrightarrows E$ depending on ε such that

$$\sum_{i=1}^\infty |Q_i(\varepsilon)| < N + \varepsilon.$$

- For each fixed n, set $\varepsilon_n = \frac{1}{n}$ to produce such a covering $\{Q_i(\varepsilon_n)\}_{i=1}^\infty$ and set $B_n \coloneqq \bigcup_{i=1}^\infty Q_i(\varepsilon_n)$.
- The outer measure of cubes is *equal* to the sum of their volumes, so

$$m_*(B_n) = \sum_{i=1}^\infty |Q_i(\varepsilon_n)| < N + \varepsilon_n = N + \frac{1}{n}.$$

- Now set $B \coloneqq \bigcap_{n=1}^{\infty} B_n$.
  - Since $E \subseteq B_n$ for every n, $E \subseteq B$
  - Since B is a countable intersection of countable unions of closed sets, B is Borel.
  - Since $B_n \subseteq B$ for every n, we can apply subadditivity to obtain the inequality

$$E \subseteq B \subseteq B_n \implies N \leq m_*(B) \leq m_*(B_n) < N + \frac{1}{n} \quad \text{for all} \quad n \in \mathbb{Z}^{\geq 1}.$$

- This forces $m_*(E) = m_*(B)$.

1.2.2 b

Suppose $m_*(E) < \infty$.

- By (a), find a Borel set $B \supseteq E$ such that $m_*(B) = m_*(E)$
- Note that $E \subseteq B \implies B \bigcap E = E$ and $B \bigcap E^c = B \setminus E$.
- By assumption,

$$\begin{aligned} m_*(B) &= m_*(B \bigcap E) + m_*(B \bigcap E^c) \\ m_*(E) &= m_*(E) + m_*(B \setminus E) \\ m_*(E) - m_*(E) &= m_*(B \setminus E) \qquad \text{since } m_*(E) < \infty \\ \implies m_*(B \setminus E) &= 0. \end{aligned}$$

- So take $N = B \setminus E$; this shows $m_*(N) = 0$ and $E = B \setminus (B \setminus E) = B \setminus N$.

If $m_*(E) = \infty$:

- Apply result to $E_R \coloneqq E \bigcap [R, R+1)^n \subset \mathbb{R}^n$ for $R \in \mathbb{Z}$, so $E = \coprod_R E_R$
- Obtain B_R, N_R such that $E_R = B_R \setminus N_R$, $m_*(E_R) = m_*(B_R)$, and $m_*(N_R) = 0$.
- Note that
  - $B \coloneqq \bigcup_R B_R$ is a union of Borel sets and thus still Borel
  - $E = \bigcup_R E_R$
  - $N \coloneqq B \setminus E$
  - $N' \coloneqq \bigcup_R N_R$ is a union of null sets and thus still null
- Since $E_R \subset B_R$ for every R, we have $E \subset B$
- We can compute

$$N = B \setminus E = \left(\bigcup_R B_R \right) \setminus \left(\bigcup_R E_R \right) \subseteq \bigcup_R (B_R \setminus E_R) = \bigcup_R N_R \coloneqq N'$$

  where $m_*(N') = 0$ since N' is null, and thus subadditivity forces $m_*(N) = 0$.

1.3 3

Concepts used:

- Limits

- Cauchy Criterion for Integrals: $\int_a^\infty f(x)\,dx$ converges iff for every $\varepsilon > 0$ there exists an M_0 such that $A, B \geq M_0$ implies $\left|\int_A^B f\right| < \varepsilon$, i.e. $\left|\int_A^B f\right| \overset{A\longrightarrow\infty}{\longrightarrow} 0$.
- Integrals of L^1 functions have vanishing tails: $\int_N^\infty |f| \overset{N\longrightarrow\infty}{\longrightarrow} 0$.
- Mean Value Theorem for Integrals: $\int_a^b f(t)\,dt = (b-a)f(c)$ for some $c \in [a,b]$.

1.3.1 a

Stated integral equality:

- Let $\varepsilon > 0$
- $C_c(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$ is dense so choose $\{f_n\} \longrightarrow f$ with $\|f_n - f\|_1 \longrightarrow 0$.
- Since $\{f_n\}$ are compactly supported, choose $N_0 \gg 1$ such that f_n is zero outside of $B_{N_0}(\mathbf{0})$.
- Then

$$
\begin{aligned}
N \geq N_0 \implies \int_{|x|>N} |f| &= \int_{|x|>N} |f - f_n + f_n| \\
&\leq \int_{|x|>N} |f - f_n| + \int_{|x|>N} |f_n| \\
&= \int_{|x|>N} |f - f_n| \\
&\leq \int_{|x|>N} \|f - f_n\|_1 \\
&= \|f_n - f\|_1 \left(\int_{|x|>N} 1 \right) \\
&\overset{n\longrightarrow\infty}{\longrightarrow} 0 \left(\int_{|x|>N} 1 \right) \\
&= 0 \\
&\overset{N\longrightarrow\infty}{\longrightarrow} 0.
\end{aligned}
$$

To see that this doesn't force $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$:

- Take $f(x)$ to be a train of rectangles of height 1 and area $1/2^j$ centered on even integers.
- Then

$$
\int_{|x|>N} |f| = \sum_{j=N}^{\infty} 1/2^j \overset{N\longrightarrow\infty}{\longrightarrow} 0
$$

  as the tail of a convergent sum.
- However $f(x) = 1$ for infinitely many even integers $x > N$, so $f(x) \not\longrightarrow 0$ as $|x| \longrightarrow \infty$.

1.3.2 b

Solution 1 ("Trick")

- Since f is decreasing on $[1,\infty)$, for any $t \in [x-n, x]$ we have

$$x - n \leq t \leq x \implies f(x) \leq f(t) \leq f(x-n).$$

- Integrate over $[x, 2x]$, using monotonicity of the integral:

$$\begin{aligned}
\int_x^{2x} f(x)\,dt \leq \int_x^{2x} f(t)\,dt \leq \int_x^{2x} f(x-n)\,dt \\
\implies f(x)\int_x^{2x} dt \leq \int_x^{2x} f(t)\,dt \leq f(x-n)\int_x^{2x} dt \\
\implies xf(x) \leq \int_x^{2x} f(t)\,dt \leq xf(x-n).
\end{aligned}$$

- By the Cauchy Criterion for integrals, $\lim_{x\longrightarrow\infty} \int_x^{2x} f(t)\ dt = 0$.
- So the LHS term $xf(x) \overset{x\longrightarrow\infty}{\longrightarrow} 0$.
- Since $x > 1$, $|f(x)| \leq |xf(x)|$
- Thus $f(x) \overset{x\longrightarrow\infty}{\longrightarrow} 0$ as well.

Solution 2 (Variation on the Trick)

- Use mean value theorem for integrals:

$$\int_x^{2x} f(t)\,dt = xf(c_x) \quad \text{for some } c_x \in [x, 2x] \text{ depending on } x.$$

- Since f is decreasing,

$$\begin{aligned}
x \leq c_x \leq 2x &\implies f(2x) \leq f(c_x) \leq f(x) \\
&\implies 2xf(2x) \leq 2xf(c_x) \leq 2xf(x) \\
&\implies 2xf(2x) \leq 2x\int_x^{2x} f(t)\,dt \leq 2xf(x)
\end{aligned}$$

.

- By Cauchy Criterion, $\int_x^{2x} f \longrightarrow 0$.
- So $2xf(2x) \longrightarrow 0$, which by a change of variables gives $uf(u) \longrightarrow 0$.
- Since $u \geq 1$, $f(u) \leq uf(u)$ so $f(u) \longrightarrow 0$ as well.

Solution 3 (Contradiction)

Just showing $f(x) \overset{x\longrightarrow\infty}{\longrightarrow} 0$:

- Toward a contradiction, suppose not.
- Since f is decreasing, it can not diverge to $+\infty$

- If $f(x) \longrightarrow -\infty$, then $f \notin L^1(\mathbb{R})$: choose $x_0 \gg 1$ so that $t \geq x_0 \implies f(t) < -1$, then
- Then $t \geq x_0 \implies |f(t)| \geq 1$, so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f(t)|\,dt \geq \int_{x_0}^\infty 1 = \infty.$$

- Otherwise $f(x) \longrightarrow L \neq 0$, some finite limit.
- If $L > 0$:
  - Fix $\varepsilon > 0$, choose $x_0 \gg 1$ such that $t \geq x_0 \implies L - \varepsilon \leq f(t) \leq L$
  - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L-\varepsilon)\,dt = \infty$$

- If $L < 0$:
  - Fix $\varepsilon > 0$, choose $x_0 \gg 1$ such that $t \geq x_0 \implies L \leq f(t) \leq L + \varepsilon$.
  - Then

$$\int_1^\infty f \geq \int_{x_0}^\infty f \geq \int_{x_0}^\infty (L)\,dt = \infty$$

Showing $xf(x) \overset{x\longrightarrow\infty}{\longrightarrow} 0$.

- Toward a contradiction, suppose not.
- (How to show that $xf(x) \not\longrightarrow +\infty$?)
- If $xf(x) \longrightarrow -\infty$
  - Choose a sequence $\Gamma = \{\widehat{x}_i\}$ such that $x_i \longrightarrow \infty$ and $x_i f(x_i) \longrightarrow -\infty$.
  - Choose a subsequence $\Gamma' = \{x_i\}$ such that $x_i f(x_i) \leq -1$ for all i and $x_i \leq x_{i+1}$.
  - Choose a further subsequence $S = \left\{x_i \in \Gamma' \;\middle|\; 2x_i < x_{i+1}\right\}$.
  - Then since f is always decreasing, for $t \geq x_0$, $|f|$ is increasing, and $|f(x_i)| \leq |f(2x_i)|$, so

$$\int_1^\infty |f| \geq \int_{x_0}^\infty |f| \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(t)|\,dt \geq \sum_{x_i \in S} \int_{x_i}^{2x_i} |f(x_i)| = \sum_{x_i \in S} x_i f(x_i) \longrightarrow \infty.$$

- If $xf(x) \longrightarrow L \neq 0$ for $0 < L < \infty$:
  - Fix $\varepsilon > 0$, choose an infinite sequence $\{x_i\}$ such that $L - \varepsilon \leq x_i f(x_i) \leq L$ for all i.

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)|\,dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i)\,dt = \sum_S x_i f(x_i) \geq \sum_S (L-\varepsilon) \longrightarrow \infty.$$

- If $xf(x) \longrightarrow L \neq 0$ for $-\infty < L < 0$:
  - Fix $\varepsilon > 0$, choose an infinite sequence $\{x_i\}$ such that $L \leq x_i f(x_i) \leq L + \varepsilon$ for all i.

$$\int_1^\infty |f| \geq \sum_S \int_{x_i}^{2x_i} |f(t)|\,dt \geq \sum_S \int_{x_i}^{2x_i} f(x_i)\,dt = \sum_S x_i f(x_i) \geq \sum_S (L) \longrightarrow \infty.$$

Solution 4 (Akos's Suggestion) For $x \geq 1$,

$$|xf(x)| = \left| \int_x^{2x} f(x)\, dt \right| \leq \int_x^{2x} |f(x)|\, dt \leq \int_x^{2x} |f(t)|\, dt \leq \int_x^{\infty} |f(t)|\, dt \overset{x \longrightarrow \infty}{\longrightarrow} 0$$

where we've used

- Since f is decreasing and $\lim_{x \longrightarrow \infty} f(x) = 0$ from part (a), f is non-negative.
- Since f is positive and decreasing, for every $t \in [a, b]$ we have $|f(a)| \leq |f(t)|$.
- By part (a), the last integral goes to zero.

Solution 5 (Peter's)

- Toward a contradiction, produce a sequence $x_i \longrightarrow \infty$ with $x_i f(x_i) \longrightarrow \infty$ and $x_i f(x_i) > \varepsilon > 0$, then

$$\begin{aligned}
\int f(x)\, dx &\geq \sum_{i=1}^{\infty} \int_{x_i}^{x_{i+1}} f(x)\, dx \\
&\geq \sum_{i=1}^{\infty} \int_{x_i}^{x_{i+1}} f(x_{i+1})\, dx \\
&= \sum_{i=1}^{\infty} f(x_{i+1}) \int_{x_i}^{x_{i+1}} dx \\
&\geq \sum_{i=1}^{\infty} (x_{i+1} - x_i) f(x_{i+1}) \\
&\geq \sum_{i=1}^{\infty} (x_{i+1} - x_i) \frac{\varepsilon}{x_{i+1}} \\
&= \varepsilon \sum_{i=1}^{\infty} \left(1 - \frac{x_{i-1}}{x_i} \right) \longrightarrow \infty
\end{aligned}$$

  which can be ensured by passing to a subsequence where $\sum \frac{x_{i-1}}{x_i} < \infty$.

1.3.3 c

- No: take $f(x) = \frac{1}{x \ln x}$
- Then by a u-substitution,

$$\int_0^x f = \ln(\ln(x)) \overset{x \longrightarrow \infty}{\longrightarrow} \infty$$

  is unbounded, so $f \notin L^1([1, \infty))$.
- But

$$xf(x) = \frac{1}{\ln(x)} \overset{x \longrightarrow \infty}{\longrightarrow} 0.$$

1.4 4

Relevant concepts:

- Tonelli: non-negative and measurable yields measurability of slices and equality of iterated integrals
- Fubini: $f(x,y) \in L^1$ yields *integrable* slices and equality of iterated integrals
- F/T: apply Tonelli to $|f|$; if finite, $f \in L^1$ and apply Fubini to f

$$\begin{aligned}
\|H(x)\|_1 &= \int_{\mathbb{R}} |H(x,y)|\,dx \\
&= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y)g(x-y)\,dy \right| dx \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(x-y)|\,dy \right) dx \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(x-y)|\,dx \right) dy \quad \text{by Tonelli} \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)g(t)|\,dt \right) dy \quad \text{setting } t = x-y,\, dt = -dx \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)| \cdot |g(t)|\,dt \right) dy \\
&= \int_{\mathbb{R}} |f(y)| \cdot \left(\int_{\mathbb{R}} |g(t)|\,dt \right) dy \\
&\coloneqq \int_{\mathbb{R}} |f(y)| \cdot \|g\|_1\,dy \\
&= \|g\|_1 \int_{\mathbb{R}} |f(y)|\,dy \\
&\coloneqq \|g\|_1 \|f\|_1 \\
&< \infty \quad \text{by assumption} \quad .
\end{aligned}$$

- H is measurable on $\mathbb{R}^2$:
  - If we can show $\tilde{f}(x,y) \coloneqq f(y)$ and $\tilde{g}(x,y) \coloneqq g(x-y)$ are both measurable on $\mathbb{R}^2$, then $H = \tilde{f} \cdot \tilde{g}$ is a product of measurable functions and thus measurable.
  - $f \in L^1$, and L^1 functions are measurable by definition.
  - The function $(x,y) \mapsto g(x-y)$ is measurable on $\mathbb{R}^2$:
    * Let g be measurable on $\mathbb{R}$, then the cylinder function $G(x,y) = g(x)$ on $\mathbb{R}^2$ is always measurable
    * Define a linear transformation $T \coloneqq [1,-1;0,1]$ which sends $(x,y) \longrightarrow (x-y,y)$, then $T \in \mathrm{GL}(2,\mathbb{R})$ is linear and thus measurable.
    * Then $(G \circ T)(x,y) = G(x-y,y) = \tilde{g}(x-y)$, so $\tilde{g}$ is a composition of measurable functions and thus measurable.
- Apply **Tonelli** to $|H|$
  - H measurable implies $|H|$ is measurable
  - $|H|$ is non-negative
  - So the iterated integrals are equal in the extended sense
  - The calculation shows the iterated integral is finite, to $\displaystyle\int |H|$ is finite and H is thus integrable on $\mathbb{R}^2$.

Note: Fubini is not needed, since we're not calculating the actual integral, just showing H is integrable.

1.5 5

Concepts used:

- DCT
- Passing limits through products and quotients

Note that

$$\begin{aligned}
\lim_n \left(1+\frac{x^2}{n}\right)^{-(n+1)} &= \frac{1}{\lim_n \left(1+\frac{x^2}{n}\right)^1 \left(1+\frac{x^2}{n}\right)^n} \\
&= \frac{1}{1\cdot e^{x^2}} \\
&= e^{-x^2}.
\end{aligned}$$

If passing the limit through the integral is justified, we will have

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \int_0^n \left(1+\frac{x^2}{n}\right)^{-(n+1)} dx &= \lim_{n\longrightarrow\infty} \int_{\mathbb{R}} \chi_{[0,n]} \left(1+\frac{x^2}{n}\right)^{-(n+1)} dx \\
&= \int_{\mathbb{R}} \lim_{n\longrightarrow\infty} \chi_{[0,n]} \left(1+\frac{x^2}{n}\right)^{-(n+1)} dx \quad \text{by the DCT} \\
&= \int_{\mathbb{R}} \lim_{n\longrightarrow\infty} \left(1+\frac{x^2}{n}\right)^{-(n+1)} dx \\
&= \int_0^\infty e^{-x^2} \\
&= \frac{\sqrt{\pi}}{2}.
\end{aligned}$$

Computing the last integral:

$$\begin{aligned}
\left(\int_{\mathbb{R}} e^{-x^2}\, dx\right)^2 &= \left(\int_{\mathbb{R}} e^{-x^2}\, dx\right)\left(\int_{\mathbb{R}} e^{-y^2}\, dx\right) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(x+y)^2}\, dx \\
&= \int_0^{2\pi}\int_0^\infty e^{-r^2} r\, dr\, d\theta \qquad u = r^2 \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^\infty e^{-u}\, du\, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 1 \\
&= \pi,
\end{aligned}$$

and now use the fact that the function is even so $\int_0^\infty f = \frac{1}{2}\int_{\mathbb{R}} f$.

Justifying the DCT:

- Apply Bernoulli's inequality:

$$1+\frac{x^2}{n}^{n+1} \geq 1+\frac{x^2}{n}\left(1+x^2\right) \geq 1+x^2,$$

  where the last inequality follows from the fact that $1+\frac{x^2}{n} \geq 1$ Flesh out

1.6 6

Concepts used:

- For $e_n(x) := e^{2\pi i n x}$, the set $\{e_n\}$ is an orthonormal basis for $L^2([0,1])$.
- For any orthonormal sequence in a Hilbert space, we have Bessel's inequality:

$$\sum_{k=1}^\infty |\langle x, e_k\rangle|^2 \leq \|x\|^2.$$

- When $\{e_n\}$ is a basis, the above is an *equality* (Parseval)
- Arguing uniform convergence: since $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$, we should be able to apply the M test.

1.6.1 a

Claim: $\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$.

- Set $\mathbf{c} = \left\{c_k \;\middle|\; k \in \mathbb{Z}\right\} \in \ell^1(\mathbb{Z})$.
- It suffices to show that if $\sum_{k\in\mathbb{Z}} |c_k| < \infty$ then $\sum_{k\in\mathbb{Z}} |c_k|^2 < \infty$.
- Let $S = \left\{c_k \;\middle|\; |c_k| \leq 1\right\}$, then $c_k \in S \implies |c_k|^2 \leq |c_k|$
- Claim: S^c can only contain finitely many elements, all of which are finite.
  - If not, either $S^c := \{c_j\}_{j=1}^\infty$ is infinite with every $|c_j| > 1$, which forces

$$\sum_{c_k \in S^c} |c_k| = \sum_{j=1}^\infty |c_j| > \sum_{j=1}^\infty 1 = \infty.$$

  - If any $c_j = \infty$, then $\sum_{k\in\mathbb{Z}} |c_k| \geq c_j = \infty$.
- So S^c is a finite set of finite integers, let $N = \max\left\{|c_j|^2 \;\middle|\; c_j \in S^c\right\} < \infty$.

- Rewrite the sum

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |c_k|^2 &= \sum_{c_k\in S} |c_k|^2 + \sum_{c_k\in S^c} |c_k|^2 \\
&\leq \sum_{c_k\in S} |c_k| + \sum_{c_k\in S^c} |c_k|^2 \\
&\leq \sum_{k\in\mathbb{Z}} |c_k| + \sum_{c_k\in S^c} |c_k|^2 \quad \text{since the } |c_k| \text{ are all positive} \\
&= \|\mathbf{c}\|_{\ell^1} + \sum_{c_k\in S^c} |c_k|^2 \\
&\leq \|\mathbf{c}\|_{\ell^1} + |S^c| \cdot N \\
&< \infty.
\end{aligned}$$

Claim: $L^2([0,1]) \subseteq L^1([0,1])$.

- It suffices to show that $\displaystyle\int |f|^2 < \infty \implies \int |f| < \infty$.
- Define $S = \left\{ x \in [0,1] \;\middle|\; |f(x)| \leq 1 \right\}$, then $x \in S^c \implies |f(x)|^2 \geq |f(x)|$.
- Break up the integral:

$$\begin{aligned}
\int_{\mathbb{R}} |f| &= \int_S |f| + \int_{S^c} |f| \\
&\leq \int_S |f| + \int_{S^c} |f|^2 \\
&\leq \int_S |f| + \|f\|_2 \\
&\leq \sup_{x\in S} \{|f(x)|\} \cdot \mu(S) + \|f\|_2 \\
&= 1 \cdot \mu(S) + \|f\|_2 \quad \text{by definition of } S \\
&\leq 1 \cdot \mu([0,1]) + \|f\|_2 \quad \text{since } S \subseteq [0,1] \\
&= 1 + \|f\|_2 \\
&< \infty.
\end{aligned}$$

Note: this proof shows $L^2(X) \subseteq L^1(X)$ whenever $\mu(X) < \infty$.

2 Fall 2019

2.1 1

Cesaro mean/summation. Break series apart into pieces that can be handled separately.

2.2 a

Prove a stronger result:

$$a_k \longrightarrow S \implies S_N := \frac{1}{N} \sum_{k=1}^{N} a_k \longrightarrow S.$$

Idea: once N is large enough, $a_k \approx S$, and all smaller terms will die off as $N \longrightarrow \infty$.
See this MSE answer.

- Use convergence $a_k \longrightarrow S$: choose M large enough such that

$$k \geq M+1 \implies |a_k - S| < \varepsilon.$$

Then

$$\begin{aligned}
\left|\left(\frac{1}{N}\sum_{k=1}^{N} a_k\right) - S\right| &= \frac{1}{N}\left|\left(\sum_{k=1}^{N} a_k\right) - NS\right| \\
&= \frac{1}{N}\left|\left(\sum_{k=1}^{N} a_k\right) - \sum_{k=1}^{N} S\right| \\
&= \frac{1}{N}\left|\sum_{k=1}^{N} (a_k - S)\right| \\
&\leq \frac{1}{N}\sum_{k=1}^{N} |a_k - S| \\
&= \frac{1}{N}\sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} |a_k - S| \\
&\leq \frac{1}{N}\sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} \frac{\varepsilon}{2} \\
&= \frac{1}{N}\sum_{k=1}^{M} |a_k - S| + (N-M)\frac{\varepsilon}{2} \\
&\overset{\varepsilon \longrightarrow 0}{\longrightarrow} \frac{1}{N}\sum_{k=1}^{M} |a_k - S| + 0 \\
&\overset{N \longrightarrow \infty}{\longrightarrow} 0 + 0.
\end{aligned}$$

Note: M is fixed, so the last sum is some constant c, and $c/N \longrightarrow 0$ as $N \longrightarrow \infty$ for any constant. To be more careful, choose M first to get $\varepsilon/2$ for the tail, then choose $N(M) > M$ for the remaining truncated part of the sum.

2.3 b

- Define

$$\Gamma_n := \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

- $\Gamma_1 = \sum_{k=1}^{n} \frac{a_k}{k}$ is the original series and each Γ_n is a tail of Γ_1, so by assumption $\Gamma_n \overset{n \longrightarrow \infty}{\longrightarrow} 0$.

- Compute

$$\frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n}(\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \mathbf{\Gamma_{n+1}})$$

.

- This comes from consider the following summation:

$$\begin{array}{l|lllllll}
\Gamma_1 : & a_1 & +\frac{a_2}{2} & +\frac{a_3}{3} & +\cdots & & & \\
\Gamma_2 : & & \frac{a_2}{2} & +\frac{a_3}{3} & +\cdots & & & \\
\Gamma_3 : & & & \frac{a_3}{3} & +\cdots & & & \\
\hline
\sum_{i=1}^{n}\Gamma_i : & a_1 & +a_2 & +a_3 & +\cdots & a_n & +\frac{a_{n+1}}{n+1} & +\cdots
\end{array}$$

- Use part (a): since $\Gamma_n \overset{n\longrightarrow\infty}{\Longrightarrow} 0$, we have $\frac{1}{n}\sum_{k=1}^{n}\Gamma_k \overset{n\longrightarrow\infty}{\Longrightarrow} 0$.
- Also a minor check: $\Gamma_n \longrightarrow 0 \implies \frac{1}{n}\Gamma_n \longrightarrow 0$.
- Then

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} a_k &= \frac{1}{n}(\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \mathbf{\Gamma_{n+1}}) \\
&= \left(\frac{1}{n}\sum_{k=0}^{n}\Gamma_k\right) - \left(\frac{1}{n}\Gamma_{n+1}\right) \\
&\overset{n\longrightarrow\infty}{\Longrightarrow} 0.
\end{aligned}$$

$\blacksquare$

2.4 2

DCT, and bounding in the right place. Don't evaluate the actual integral!

- By induction on the number of limits we can pass through the integral.

- For $n = 1$ we first pass one derivative into the integral: let $x_n \longrightarrow x$ be any sequence converging to x, then

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\sin(x)}{x} &= \frac{\partial}{\partial x}\int_0^1 \cos(tx)\,dt \\
&= \lim_{x_n \longrightarrow x} \frac{1}{x_n - x}\left(\int_0^1 \cos(tx_n)\,dt - \int_0^1 \cos(tx)\,dt\right) \\
&= \lim_{x_n \longrightarrow x} \left(\int_0^1 \frac{\cos(tx_n) - \cos(tx)}{x_n - x}\,dt\right) \\
&= \lim_{x_n \longrightarrow x} \left(\int_0^1 \left(t\sin(tx)\,\Big|_{x=\xi_n}\right)dt\right) \quad \text{where} \quad \xi_n \in [x_n, x] \text{ by MVT}, \xi_n \longrightarrow x \\
&= \lim_{\xi_n \longrightarrow x} \left(\int_0^1 t\sin(t\xi_n)\,dt\right) \\
&=_{\text{DCT}} \int_0^1 \lim_{\xi_n \longrightarrow x} t\sin(t\xi_n)\,dt \\
&= \int_0^1 t\sin(tx)\,dt
\end{aligned}$$

.

- Taking absolute values we obtain an upper bound

$$\begin{aligned}
\left|\frac{\partial}{\partial x}\frac{\sin(x)}{x}\right| &= \left|\int_0^1 t\sin(tx)\,dt\right| \\
&\leq \int_0^1 |t\sin(tx)|\,dt \\
&\leq \int_0^1 1\,dt = 1,
\end{aligned}$$

since $t \in [0,1] \implies |t| < 1$, and $|\sin(xt)| \leq 1$ for any x and t.

- Note that this bound also justifies the DCT, since the functions $f_n(t) = t\sin(t\xi_n)$ are uniformly dominated by $g(t) = 1$ on $L^1([0,1])$.

Note: integrating by parts here yields the actual formula:

$$\begin{aligned}
\int_0^1 t\sin(tx)\,dt &=_{\text{IBP}} \left(\frac{-t\cos(tx)}{x}\right)\Big|_{t=0}^{t=1} - \int_0^1 \frac{\cos(tx)}{x}\,dt \\
&= \frac{-\cos(x)}{x} - \frac{\sin(x)}{x^2} \\
&= \frac{x\cos(x) - \sin(x)}{x^2}.
\end{aligned}$$

- For the inductive step, we assume that we can pass $n-1$ limits through the integral and show we can pass the nth through as well.

$$\begin{aligned}
\frac{\partial^n}{\partial x^n}\frac{\sin(x)}{x} &= \frac{\partial^n}{\partial x^n}\int_0^1 \cos(tx)\,dt \\
&= \frac{\partial}{\partial x}\int_0^1 \frac{\partial^{n-1}}{\partial x^{n-1}}\cos(tx)\,dt \\
&= \frac{\partial}{\partial x}\int_0^1 t^{n-1} f_{n-1}(x,t)\,dt
\end{aligned}$$

- Note that $f_n(x,t) = \pm\sin(tx)$ when n is odd and $f_n(x,t) = \pm\cos(tx)$ when n is even, and a constant factor of t is multiplied when each derivative is taken.

- We continue as in the base case:

$$\begin{aligned}
\frac{\partial}{\partial x}\int_0^1 t^{n-1} f_{n-1}(x,t)\,dt &= \lim_{x_k \longrightarrow x}\int_0^1 t^{n-1}\left(\frac{f_{n-1}(x_n,t) - f_{n-1}(x,t)}{x_n - x}\right) dt \\
&=_{\text{IVT}} \lim_{x_k \longrightarrow x}\int_0^1 t^{n-1}\frac{\partial f_{n-1}}{\partial x}(\xi_k,t)\,dt \quad \text{where } \xi_k \in [x_k, x],\ \xi_k \longrightarrow x \\
&=_{\text{DCT}} \int_0^1 \lim_{x_k \longrightarrow x} t^{n-1}\frac{\partial f_{n-1}}{\partial x}(\xi_k,t)\,dt \\
&\coloneqq \int_0^1 \lim_{x_k \longrightarrow x} t^n f_n(\xi_k,t)\,dt \\
&\coloneqq \int_0^1 t^n f_n(x,t)\,dt.
\end{aligned}$$

  - We've used the fact that $f_0(x) = \cos(tx)$ is smooth as a function of x, and in particular continuous
  - The DCT is justified because the functions $h_{n,k}(x,t) = t^n f_n(\xi_k,t)$ are again uniformly (in k) bounded by 1 since $t \le 1 \implies t^n \le 1$ and each f_n is a sin or cosine.

- Now take absolute values

$$\begin{aligned}
\left|\frac{\partial^n}{\partial x^n}\frac{\sin(x)}{x}\right| &= \left|\int_0^1 -t^n f_n(x,t)\ dt\right| \\
&\le \int_0^1 |t^n f_n(x,t)|\ dt \\
&\le \int_0^1 |t^n||f_n(x,t)|\ dt \\
&\le \int_0^1 |t^n| \cdot 1\ dt \\
&\le \int_0^1 t^n\ dt \quad \text{since } t \text{ is positive} \\
&= \frac{1}{n+1} \\
&< \frac{1}{n}.
\end{aligned}$$

  - We've again used the fact that $f_n(x,t)$ is of the form $\pm\cos(tx)$ or $\pm\sin(tx)$, both of which are bounded by 1.

■

2.5 3

Concepts used:

- Borel-Cantelli: for a sequence of sets X_n,

$$\limsup_n X_n = \left\{ x \;\middle|\; x \in X_n \text{ for infinitely many } n \right\} = \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} X_n$$
$$\liminf_n X_n = \left\{ x \;\middle|\; x \in X_n \text{ for all but finitely many } n \right\} = \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} X_n.$$

- Properties of logs and exponentials:

$$\prod_n e^{x_n} = e^{\Sigma_n x_n} \quad \text{and} \quad \sum_n \log(x_n) = \log\left(\prod_n x_n\right).$$

- Tails of convergent sums vanish.
- Continuity of measure: $B_n \searrow B$ and $\mu(B_0) < \infty$ implies $\lim_n \mu(B_n) = \mu(B)$, and $B_n \nearrow B \implies \lim_n \mu(B_n) = \mu(B)$.

2.5.1 a

- The Borel σ-algebra is closed under countable unions/intersections/complements,
- $B = \limsup_n B_n$ is an intersection of unions of measurable sets.

2.5.2 b

- Tails of convergent sums go to zero, so $\sum_{n\geq M} \mu(B_n) \xrightarrow{M\longrightarrow\infty} 0$,
- $B_M := \bigcap_{m=1}^{M} \bigcup_{n\geq m} B_n \searrow B$.

$$\begin{aligned}
\mu(B_M) &= \mu\left(\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} B_n\right) \\
&\leq \mu\left(\bigcup_{n\geq m} B_n\right) \quad \text{for all } m \in \mathbb{N} \text{ by countable subadditivity} \\
&\longrightarrow 0,
\end{aligned}$$

- The result follows by continuity of measure.

2.5.3 c

- To show $\mu(B) = 1$, we'll show $\mu(B^c) = 0$.

- Let $B_k = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{K} B_n$. Then

$$\begin{aligned}
\mu(B_K^c) &= \mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{K} B_n^c\right) \\
&\leq \sum_{m=1}^{\infty} \mu\left(\bigcap_{n=m}^{K} B_n^c\right) \quad \text{by subadditivity} \\
&= \sum_{m=1}^{\infty} \prod_{n=m}^{K} (1-\mu(B_n)) \quad \text{by assumption} \\
&\leq \sum_{m=1}^{\infty} \prod_{n=m}^{K} e^{-\mu(B_n^c)} \quad \text{by hint} \\
&= \sum_{m=1}^{\infty} \exp\left(-\sum_{n=m}^{K} \mu(B_n^c)\right) \\
&\overset{K\longrightarrow\infty}{\longrightarrow} 0
\end{aligned}$$

since $\sum_{n=m}^{K} \mu(B_n^c) \overset{K\longrightarrow\infty}{\longrightarrow} \infty$ by assumption

- We can apply continuity of measure since $B_K^c \xrightarrow{K\longrightarrow\infty} B^c$.

Proving the hint: ?

■

2.6 4

Concepts used:

- Bessel's Inequality
- Pythagoras
- Surjectivity of the Riesz map
- Parseval's Identity
- Trick – remember to write out finite sum S_N, and consider $\|x - S_N\|$.

2.6.1 a

Claim:

$$\begin{aligned}
0 \leq \left\| x - \sum_{n=1}^{N} \langle x, u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n \rangle|^2 \\
&\implies \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2.
\end{aligned}$$

Proof: Let $S_N = \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n$. Then

$$\begin{aligned} 0 &\leq \|x - S_N\|^2 \\ &= \langle x - S_n,\ x - S_N \rangle \\ &= \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\ &\xrightarrow{N \longrightarrow \infty} \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2. \end{aligned}$$

2.6.2 b

1. Fix $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
2. Define

$$x \coloneqq \lim_{N \longrightarrow \infty} S_N = \lim_{N \longrightarrow \infty} \sum_{k=1}^{N} a_k u_k$$

3. $\{S_N\}$ Cauchy (by 1) and H complete $\implies x \in H$.
4.

$$\langle x,\ u_n \rangle = \left\langle \sum_k a_k u_k,\ u_n \right\rangle = \sum_k a_k \langle u_k,\ u_n \rangle = a_n \quad \forall n \in \mathbb{N}$$

since the u_k are all orthogonal.

5.

$$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$$

by Pythagoras since the u_k are normal.

Bonus: We didn't use completeness here, so the Fourier series may not actually converge to x. If $\{u_n\}$ is **complete** (so $x = 0 \iff \langle x,\ u_n \rangle = 0\ \forall n$) then the Fourier series *does* converge to x and $\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 = \|x\|^2$ for all $x \in H$.

■

2.7 5

Continuity in L^1 (recall that DCT won't work! Notes 19.4, prove it for a dense subset first). Lebesgue differentiation in 1-dimensional case. See HW 5.6.

2.8 a

Choose $g \in C_c^0$ such that $\|f-g\|_1 \longrightarrow 0$.

By translation invariance, $\|\tau_h f - \tau_h g\|_1 \longrightarrow 0$.

Write

$$\begin{aligned}
\|\tau f - f\|_1 &= \|\tau_h f - g + g - \tau_h g + \tau_h g - f\|_1 \\
&\leq \|\tau_h f - \tau_h g\| + \|g - f\| + \|\tau_h g - g\| \\
&\longrightarrow \|\tau_h g - g\|,
\end{aligned}$$

so it suffices to show that $\|\tau_h g - g\| \longrightarrow 0$ for $g \in C_c^0$.

Fix $\varepsilon > 0$. Enlarge the support of g to K such that

$$|h| \leq 1 \text{ and } x \in K^c \implies |g(x-h) - g(x)| = 0.$$

By uniform continuity of g, pick $\delta \leq 1$ small enough such that

$$x \in K,\ |h| \leq \delta \implies |g(x-h) - g(x)| < \varepsilon,$$

then

$$\int_K |g(x-h) - g(x)| \leq \int_K \varepsilon = \varepsilon \cdot m(K) \longrightarrow 0.$$

2.9 b

We have

$$\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x)|\ dx &= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y)\ dy \right| dx \\
&\leq \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)|\ dy\ dx \\
&=_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)|\ \mathbf{dx\ dy} \\
&= \int_{\mathbb{R}} |f(y)|\ dy \\
&= \|f\|_1.
\end{aligned}$$

and (rough sketch)

$$
\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x) - f(x)| \; dx &= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y) \; dy \right) - f(x) \right| \; dx \\
&= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y) \; dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x) \; dy \right| \; dx \\
&\leq_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{B(h,x)} |f(y-x) - f(x)| \; \mathbf{dx} \; \mathbf{dy} \\
&\leq \frac{1}{2h} \int_{\mathbb{R}} \|\tau_x f - f\|_1 \; dy \\
&\longrightarrow 0 \quad \text{by (a).}
\end{aligned}
$$

■

3 Spring 2019

3.1 1

3.1.1 a

- Let $\{f_n\}$ be a Cauchy sequence in $C(I, \|\cdot\|_\infty)$, so $\lim_n \lim_m \|f_m - f_n\|_\infty = 0$, we will show it converges to some f in this space.
- For each fixed $x_0 \in [0,1]$, the sequence of real numbers $\{f_n(x_0)\}$ is Cauchy in $\mathbb{R}$ since

$$
x_0 \in I \implies |f_m(x_0) - f_n(x_0)| \leq \sup_{x \in I} |f_m(x) - f_n(x)| \coloneqq \|f_m - f_n\|_\infty \overset{m>n \longrightarrow \infty}{\longrightarrow} 0,
$$

- Since $\mathbb{R}$ is complete, this sequence converges and we can define $f(x) \coloneqq \lim_{k \longrightarrow \infty} f_n(x)$.
- Thus $f_n \longrightarrow f$ pointwise by construction
- Claim: $\|f - f_n\| \overset{n \longrightarrow \infty}{\longrightarrow} 0$, so f_n converges to f in $C([0,1], \|\cdot\|_\infty)$.
  - Proof:
    - Fix $\varepsilon > 0$; we will show there exists an N such that $n \geq N \implies \|f_n - f\| < \varepsilon$
    - Fix an $x_0 \in I$. Since $f_n \longrightarrow f$ pointwise, choose N_1 large enough so that

$$
n \geq N_1 \implies |f_n(x_0) - f(x_0)| < \varepsilon/2.
$$

    - Since $\|f_n - f_m\|_\infty \longrightarrow 0$, choose and N_2 large enough so that

$$
n, m \geq N_2 \implies \|f_n - f_m\|_\infty < \varepsilon/2.
$$

- $*$ Then for $n, m \geq \max(N_1, N_2)$, we have

$$
\begin{aligned}
|f_n(x_0) - f(x_0)| &= |f_n(x_0) - f(x_0) + f_m(x_0) - f_m(x_0)| \\
&= |f_n(x_0) - f_m(x_0) + f_m(x_0) - f(x_0)| \\
&\leq |f_n(x_0) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \\
&< |f_n(x_0) - f_m(x_0)| + \frac{\varepsilon}{2} \\
&\leq \sup_{x \in I} |f_n(x) - f_m(x)| + \frac{\varepsilon}{2} \\
&< \|f_n - f_m\|_\infty + \frac{\varepsilon}{2} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
\implies |f_n(x_0) - f(x_0)| &< \varepsilon \\
\implies \sup_{x \in I} |f_n(x_0) - f(x_0)| &\leq \sup_{x \in I} \varepsilon \quad \text{by order limit laws} \\
\implies \|f_n - f\| &\leq \varepsilon
\end{aligned}
$$

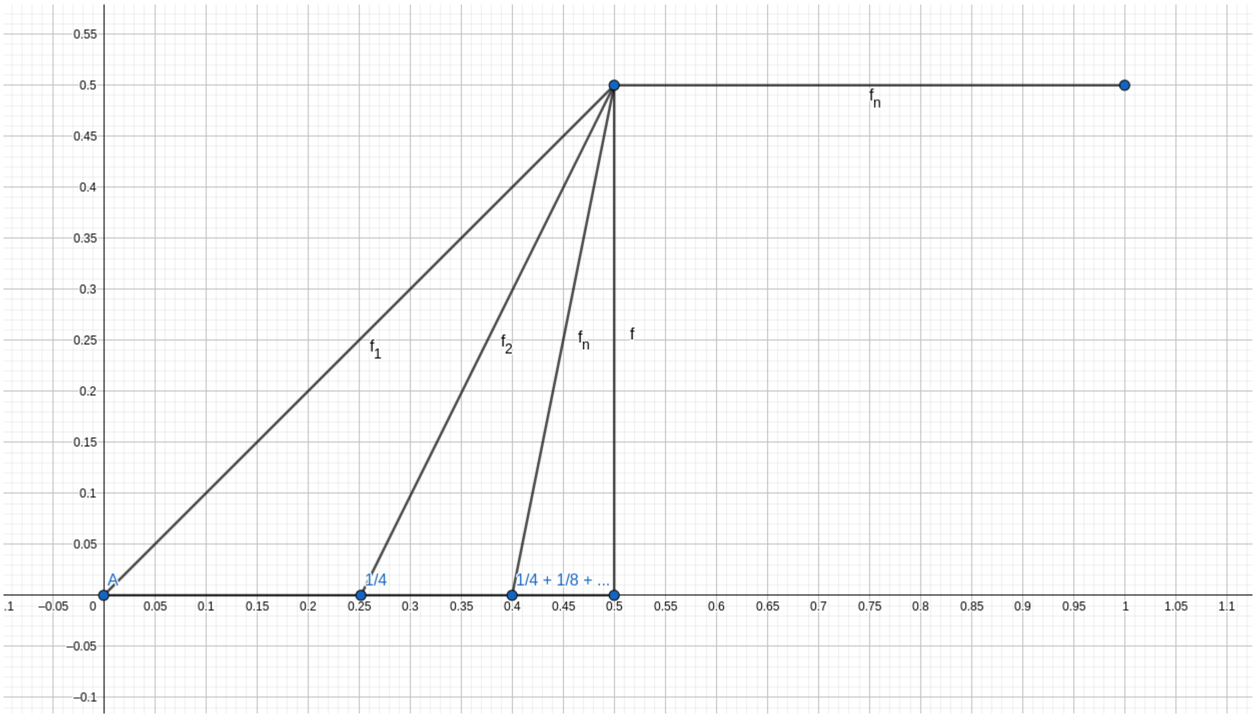
.

- f is the uniform limit of continuous functions and thus continuous, so $f \in C([0,1])$.

3.1.2 b

- It suffices to produce a Cauchy sequence that does not converge to a continuous function.
- Take the following sequence of functions:
  - f_1 increases linearly from 0 to 1 on $[0, 1/2]$ and is 1 on $[1/2, 1]$
  - f_2 is 0 on $[0, 1/4]$ increases linearly from 0 to 1 on $[1/4, 1/2]$ and is 1 on $[1/2, 1]$
  - f_3 is 0 on $[0, 3/8]$ increases linearly from 0 to 1 on $[3/8, 1/2]$ and is 1 on $[1/2, 1]$
  - f_3 is 0 on $[0, (1/2 - 3/8)/2]$ increases linearly from 0 to 1 on $[(1/2 - 3/8)/2, 1/2]$ and is 1 on $[1/2, 1]$

> Idea: take sequence starting points for the triangles: $0, 0 + \frac{1}{4}, 0 + \frac{1}{4} + \frac{1}{8}, \cdots$ which converges to 1/2 since $\sum_{k=1}^{\infty} \frac{1}{2^k} = -\frac{1}{2} + \sum_{k=0}^{\infty} \frac{1}{2^k}$.



- Then each f_n is clearly integrable, since its graph is contained in the unit square.
- $\{f_n\}$ is Cauchy: geometrically subtracting areas yields a single triangle whose area tends to 0.
- But f_n converges to $\chi_{[\frac{1}{2},1]}$ which is discontinuous.

Todo: show that $\int_0^1 |f_n(x) - f_m(x)|\, dx \longrightarrow 0$ rigorously, show that no $g \in L^1([0,1])$ can converge to this indicator function.

3.2 2

3.2.1 a

See Folland p.26

- Lemma 1: $\mu(\coprod_{k=1}^{\infty} E_k) = \lim_{N \longrightarrow \infty} \sum_{k=1}^{N} \mu(E_k)$.
- Suppose $F_0 \supseteq F_1 \supseteq \cdots$.
- Let $A_k = F_k \setminus F_{k+1}$, since the F_k are nested the A_k are disjoint
- Set $A := \coprod_{k=1}^{\infty} A_k$ and $F := \bigcap_{k=1}^{\infty} F_k$.
- Note $X = X \setminus Y \coprod X \bigcap Y$ for any two sets (just write $X \setminus Y := X \bigcap Y^c$)
- Note that A contains anything that was removed from F_0 when passing from any F_j to F_{j+1}, while F contains everything that is never removed at any stage, and these are disjoint possibilities.

- Thus $F_0 = F \coprod A$, so

$$\begin{aligned}
\mu(F_0) &= \mu(F) + \mu(A) \\
&= \mu(F) + \mu(\textstyle\coprod_{k=1}^{\infty} A_k) \\
&= \mu(F) + \lim_{n \longrightarrow \infty} \sum_{k=0}^{n} \mu(A_k) \quad \text{by countable additivity} \\
&= \mu(F) + \lim_{n \longrightarrow \infty} \sum_{k=0}^{n} \mu(F_k) - \mu(F_{k+1}) \\
&= \mu(F) + \lim_{n \longrightarrow \infty} (\mu(F_1) - \mu(F_n)) \quad \text{(Telescoping)} \\
&= \mu(F) + \mu(F_1) - \lim_{N \longrightarrow \infty} \mu(F_n),
\end{aligned}$$

- Since μ is a finite measure, $\mu(F_1) < \infty$ and can be subtracted, yielding

$$\begin{aligned}
\mu(F_1) &= \mu(F) + \mu(F_1) - \lim_{n \longrightarrow \infty} \mu(F_n) \\
\implies \mu(F) &= \lim_{n \longrightarrow \infty} \mu(F_n) \\
\implies \mu\left(\bigcap_{k=1}^{\infty} F_k\right) &= \lim_{n \longrightarrow \infty} \mu(F_n).
\end{aligned}$$

3.2.2 b

- Toward a contradiction, negate the implication: suppose there exists an $\varepsilon > 0$ such that for all δ, we have $m(E) < \delta$ but $\mu(E) > \varepsilon$.
- The sequence $\left\{\delta_n := \frac{1}{2^n}\right\}_{n \in \mathbb{N}}$ and produce sets $A_n \in \mathcal{B}$ such $m(A_n) < \frac{1}{2^n}$ but $\mu(A_n) > \varepsilon$.
- Define

$$\begin{aligned}
F_n &:= \bigcup_{j \geq n} A_j \\
C_m &:= \bigcap_{k=1}^{m} F_k \\
A := C_\infty &:= \bigcap_{k=1}^{\infty} F_k.
\end{aligned}$$

- Note that $F_1 \supseteq F_2 \supseteq \cdots$, since each increase in index unions fewer sets.
- By continuity for the Lebesgue measure,

$$m(A) = m\left(\bigcap_{k=1}^{\infty} F_k\right) = \lim_{k \longrightarrow \infty} m(F_k) = \lim_{k \longrightarrow \infty} m\left(\bigcup_{j \geq k} A_j\right) \leq \lim_{k \longrightarrow \infty} \sum_{j \geq k} m(A_j) = \lim_{k \longrightarrow \infty} \sum_{j \geq k} \frac{1}{2^n} = 0,$$

  which follows because this is the tail of a convergent sum
- Thus $m(A) = 0$ and by assumption, this implies $\mu(A) = 0$.

- However, by part (a),

$$\mu(A) = \lim_n \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \geq \lim_n \mu(A_n) = \lim_n \varepsilon = \varepsilon > 0.$$

All messed up.

3.3 3

Concepts used:

- Definition of L^+: space of measurable function $X \longrightarrow [0, \infty]$.
- Fatou: For any sequence of L^+ functions, $\displaystyle\int \liminf f_n \leq \liminf \int f_n$.
- Egorov's Theorem: If $E \subseteq \mathbb{R}^n$ is measurable, $m(E) > 0$, $f_k : E \longrightarrow \mathbb{R}$ a sequence of measurable functions where $\displaystyle\lim_{n\longrightarrow\infty} f_n(x)$ exists and is finite a.e., then $f_n \longrightarrow f$ *almost uniformly*: for every $\varepsilon > 0$ there exists a closed subset $F_\varepsilon \subseteq E$ with $m(E \setminus F) < \varepsilon$ and $f_n \longrightarrow f$ uniformly on F.

L^2 bound:

- Since $f_k \longrightarrow f$ almost everywhere, $\liminf_n f_n(x) = f(x)$ a.e.
- $\|f_n\|_2 < \infty$ implies each f_n is measurable and thus $|f_n|^2 \in L^+$, so we can apply Fatou:

$$\begin{aligned}
\|f\|_2^2 &= \int |f(x)|^2 \\
&= \int \liminf_n |f_n(x)|^2 \\
&\underset{\text{Fatou}}{\leq} \liminf_n \int |f_n(x)|^2 \\
&\leq \liminf_n M \\
&= M.
\end{aligned}$$

- Thus $\|f\|_2 \leq \sqrt{M} < \infty$ implying $f \in L^2$.

Equality of Integrals:

What is the "right" proof here that uses the first part?

- Take the sequence $\varepsilon_n = \dfrac{1}{n}$
- Apply Egorov's theorem: obtain a set F_ε such that $f_n \longrightarrow f$ uniformly on F_ε and $m(I \setminus F_\varepsilon) < \varepsilon$.

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \left|\int_0^1 f_n - f\right| &\leq \lim_{n\longrightarrow\infty} \int_0^1 |f_n - f| \\
&= \lim_{n\longrightarrow\infty} \left(\int_{F_\varepsilon} |f_n - f| + \int_{I\setminus F_\varepsilon} |f_n - f|\right) \\
&= \int_{F_\varepsilon} \lim_{n\longrightarrow\infty} |f_n - f| + \lim_{n\longrightarrow\infty} \int_{I\setminus F_\varepsilon} |f_n - f| \quad \text{by uniform convergence} \\
&= 0 + \lim_{n\longrightarrow\infty} \int_{I\setminus F_\varepsilon} |f_n - f|,
\end{aligned}$$

so it suffices to show $\displaystyle\int_{I\setminus F_\varepsilon} |f_n - f| \overset{n\longrightarrow\infty}{\longrightarrow} 0$.

- We can obtain a bound using Holder's inequality with $p = q = 2$:

$$\begin{aligned}
\int_{I \setminus F_\varepsilon} |f_n - f| &\leq \left(\int_{I \setminus F_\varepsilon} |f_n - f|^2\right)\left(\int_{I \setminus F_\varepsilon} |1|^2\right) \\
&= \left(\int_{I \setminus F_\varepsilon} |f_n - f|^2\right)\mu(F_\varepsilon) \\
&\leq \|f_n - f\|_2 \mu(F_\varepsilon) \\
&\leq (\|f_n\|_2 + \|f\|_2)\mu(F_\varepsilon) \\
&\leq 2M \cdot \mu(F_\varepsilon)
\end{aligned}$$

  where M is now a constant not depending on ε or n.

- Now take a nested sequence of sets F_ε with $\mu(F_\varepsilon) \longrightarrow 0$ and applying continuity of measure yields the desired statement.

3.4 4

See S&S p.82.

3.4.1 a

$\implies$:

- Suppose f is a measurable function.
- Note that $\mathcal{A} = \{f(x) - t \geq 0\} \bigcap \{t \geq 0\}$.
- Define $F(x,t) = f(x)$, $G(x,t) = t$, which are cylinders on measurable functions and thus measurable.
- Define $H(x,y) = F(x,t) - G(x,t)$, which are linear combinations of measurable functions and thus measurable.
- Then $\mathcal{A} = \{H \geq 0\} \bigcap \{G \geq 0\}$ as a countable intersection of measurable sets, which is again measurable.

$\impliedby$:

- Suppose $\mathcal{A}$ is a measurable set.
- Then FT on $\chi_{\mathcal{A}}$ implies that for almost every $x \in \mathbb{R}^n$, the x-slices $\mathcal{A}_x$ are measurable and \$

$$\mathcal{A}_x \coloneqq \left\{t \in \mathbb{R} \;\middle|\; (x,t) \in \mathcal{A}\right\} = [0, f(x)] \implies m(\mathcal{A}_x) = f(x) - 0 = f(x)$$

- But $x \mapsto m(\mathcal{A}_x)$ is a measurable function, and is exactly the function $x \mapsto f(x)$, so f is measurable.

3.4.2 b

- Note

$$\begin{aligned}
\mathcal{A} &= \left\{(x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; 0 \leq t \leq f(x)\right\} \\
\mathcal{A}_t &= \left\{x \in \mathbb{R}^n \;\middle|\; t \leq f(x)\right\}.
\end{aligned}$$

- Then

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)\,dx &= \int_{\mathbb{R}^n}\int_0^{f(x)} 1\,dt\,dx \\ &= \int_{\mathbb{R}^n}\int_0^{\infty} \chi_{\mathcal{A}}\,dt\,dx \\ &\overset{F.T.}{=} \int_0^{\infty}\int_{\mathbb{R}^n} \chi_{\mathcal{A}}\,dx\,dt \\ &= \int_0^{\infty} m(\mathcal{A}_t)\,dt,\end{aligned}$$

where we just use that $\int\int \chi_{\mathcal{A}} = m(\mathcal{A})$

- By F.T., all of these integrals are equal.

Why is FT justified.

3.5 5

Concepts used:

- Holders' inequality: $\|fg\|_1 \leq \|f\|_p \|f\|_q$
- Riesz Representation for L^2: If $\Lambda \in (L^2)^\vee$ then there exists a unique $g \in L^2$ such that $\Lambda(f) = \int fg$.
- $\|f\|_{L^\infty(X)} \coloneqq \inf \left\{ t \geq 0 \;\middle|\; |f(x)| \leq t \text{ almost everywhere} \right\}$.
- **Lemma**: $m(X) < \infty \implies L^p(X) \subset L^2(X)$.

Proof: Write Holder's inequality as $\|fg\|_1 \leq \|f\|_a \|g\|_b$ where $\frac{1}{a} + \frac{1}{b} = 1$, then

$$\|f\|_p^p = \||f|^p\|_1 \leq \||f|^p\|_a\ \|1\|_b.$$

Now take $a = \frac{2}{p}$ and this reduces to

$$\begin{aligned} &\|f\|_p^p \leq \|f\|_2^p\ m(X)^{\frac{1}{b}} \\ \implies &\|f\|_p \leq \|f\|_2 \cdot O(m(X)) < \infty.\end{aligned}$$

3.5.1 a

- Note $X = [0,1] \implies m(X) = 1$.
- By Holder's inequality with $p = q = 2$,

$$\|f\|_1 = \|f \cdot 1\|_1 \leq \|f\|_2 \cdot \|1\|_2 = \|f\|_2 \cdot m(X)^{\frac{1}{2}} = \|f\|_2,$$

- Thus $L^2(X) \subseteq L^1(X)$
- Since they share a common dense subset (simple functions) L^2 is dense in L^1

What theorem is this using?

3.5.2 b

Let $\Lambda \in L^1(X)^\vee$ be arbitrary.

(i): Existence of g Representing Λ.

- Let $f \in L^2 \subseteq L^1$ be arbitrary
- Claim: $\Lambda \in L^1(X)^\vee \implies \Lambda \in L^2(X)^\vee$.
  - Suffices to show that $\|\Gamma\|_{L^2(X)^\vee} := \sup_{\|f\|_2 = 1} |\Gamma(f)| < \infty$, since bounded implies continuous.
  - By the lemma, $\|f\|_1 \leq C\|f\|_2$ for some constant $C \approx m(X)$.
  - Note

$$\|\Lambda\|_{L^1(X)^\vee} := \sup_{\|f\|_1 = 1} |\Lambda(f)|$$

  - Define $\widehat{f} = \dfrac{f}{\|f\|_1}$ so $\left\|\widehat{f}\right\|_1 = 1$
  - Since $\|\Lambda\|_{1^\vee}$ is a supremum over *all* $f \in L^1(X)$ with $\|f\|_1 = 1$,

$$\left|\Lambda(\widehat{f})\right| \leq \|\Lambda\|_{(L^1(X))^\vee},$$

  - Then

$$\begin{aligned}
\frac{|\Lambda(f)|}{\|f\|_1} &= \left|\Lambda(\widehat{f})\right| \leq \|\Lambda\|_{L^1(X)^\vee} \\
\implies |\Lambda(f)| &\leq \|\Lambda\|_{1^\vee} \cdot \|f\|_1 \\
&\leq \|\Lambda\|_{1^\vee} \cdot C\|f\|_2 < \infty \quad \text{by assumption,}
\end{aligned}$$

  - So $\Lambda \in (L^2)^\vee$.
- Now apply Riesz Representation for L^2: there is a $g \in L^2$ such that

$$f \in L^2 \implies \Lambda(f) = \langle f,\ g\rangle := \int_0^1 f(x)\overline{g(x)}\,dx.$$

(ii): g is in L^∞

- It suffices to show $\|g\|_{L^\infty(X)} < \infty$.
- Since we're assuming $\|\Gamma\|_{L^1(X)^\vee} < \infty$, it suffices to show the stated equality. Is this assumed..? Or did we show it..?
- Claim: $\|\Lambda\|_{L^1(X)^\vee} = \|g\|_{L^\infty(X)}$
  - The result follows because Λ was assumed to be in $L^1(X)^\vee$, so $\|\Lambda\|_{L^1(X)^\vee} < \infty$.

- $\leq$:

$$\begin{aligned}\|\Lambda\|_{L^1(X)^\vee} &= \sup_{\|f\|_1=1} |\Lambda(f)| \\ &= \sup_{\|f\|_1=1} \left|\int_X f\bar{g}\right| \quad \text{by (i)} \\ &= \sup_{\|f\|_1=1} \int_X |f\bar{g}| \\ &\coloneqq \sup_{\|f\|_1=1} \|fg\|_1 \\ &\leq \sup_{\|f\|_1=1} \|f\|_1\|g\|_\infty \quad \text{by Holder with } p=1, q=\infty \\ &= \|g\|_\infty,\end{aligned}$$

- $\geq$:
  * Suppose toward a contradiction that $\|g\|_\infty > \|\Lambda\|_{1^\vee}$.
  * Then there exists some $E \subseteq X$ with $m(E) > 0$ such that

$$x \in E \implies |g(x)| > \|\Lambda\|_{L^1(X)^\vee}.$$

  * Define

$$h = \frac{1}{m(E)}\frac{\overline{g}}{|g|}\chi_E.$$

  * Note $\|h\|_{L^1(X)} = 1$.
  * Then

$$\begin{aligned}\Lambda(h) &= \int_X hg \\ &\coloneqq \int_X \frac{1}{m(E)}\frac{g\overline{g}}{|g|}\chi_E \\ &= \frac{1}{m(E)}\int_E |g| \\ &\geq \frac{1}{m(E)}\|g\|_\infty m(E) \\ &= \|g\|_\infty \\ &> \|\Lambda\|_{L^1(X)^\vee},\end{aligned}$$

  a contradiction since $\|\Lambda\|_{L^1(X)^\vee}$ is the supremum over all h_α with $\|h_\alpha\|_{L^1(X)} = 1$.

4 Fall 2018

4.1 1

Concepts used:

- Uniform continuity.

Show a stronger statement: $f(x) = \dfrac{1}{x}$ is uniformly continuous on any interval of the form (c, ∞) where $c > 0$.

- Note that

$$|x|, |y| > c > 0 \implies |xy| = |x||y| > c^2 \implies \frac{1}{|xy|} < \frac{1}{c^2}.$$

- Letting ε be arbitrary, choose $\delta < \varepsilon c^2$.
- Note that δ does not depend on x, y.
- Then

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{x} - \frac{1}{y} \right| \\ &= \frac{|x-y|}{xy} \\ &\leq \frac{\delta}{xy} \\ &< \frac{\delta}{c^2} \\ &< \varepsilon, \end{aligned}$$

  which shows uniform continuity.

To see that f is not uniformly continuous when $c = 0$:

> Note: negating uniform continuity says $\exists \varepsilon > 0$ such that $\forall \delta(\varepsilon)$ there exist x, y such that $|x - y| < \delta$ *and* $|f(x) - f(y)| > \varepsilon$.

- Let $\varepsilon < 1$.
- Let $x_n = \dfrac{1}{n}$ for $n \geq 1$.
- Choose n large enough such that $|x_n - x_{n+1}| = \dfrac{1}{n} - \dfrac{1}{n+1} < \delta$.
  - Why this can be done: by the archimedean property of $\mathbb{R}$, choose n such that $\dfrac{1}{n} < \varepsilon$.
  - Then

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \leq \frac{1}{n} < \varepsilon \quad \text{since } n+1 > 1.$$

- Note $f(x_n) = n$ and thus

$$|f(x_n) - f(x_{n+1})| = n - (n+1) = 1 > \varepsilon.$$

■

4.2 2

Concepts used:

- Definition of measurability: there exists an open $O \supset E$ such that $m_*(O \setminus E) < \varepsilon$ for all $\varepsilon > 0$.
- Theorem: E is Lebesgue measurable iff there exists a closed set $F \subseteq E$ such that $m_*(E \setminus F) < \varepsilon$ for all $\varepsilon > 0$.
- Every F_σ, G_δ is Borel.
- Claim: E is measurable $\iff$ for every ε there exist $F_\varepsilon \subset E \subset G_\varepsilon$ with F_ε closed and G_ε open and $m(G_\varepsilon \setminus E) < \varepsilon$ and $m(E \setminus F_\varepsilon) < \varepsilon$.
  - Proof: existence of G_ε is the definition of measurability.
  - Existence of F_ε: ?
- Claim: E is measurable $\implies$ there exists an open $O \supseteq E$ such that $m(O \setminus E) = 0$.
  - Since E is measurable, for each $n \in \mathbb{N}$ choose $G_n \supseteq E$ such that $m_*(G_n \setminus E) < \frac{1}{n}$.
  - Set $O_N := \bigcap_{n=1}^{N} G_n$ and $O := \bigcap_{n=1}^{\infty} G_n$.
  - Suppose E is bounded.
    * Note $O_N \searrow O$ and $m_*(O_1) < \infty$ if E is bounded, since in this case

$$m_*(G_n \setminus E) = m_*(G_1) - m_*(E) < 1 \iff m_*(G_1) < m_*(E) + \frac{1}{n} < \infty.$$

    * Note $O_N \setminus E \searrow O \setminus E$ since $O_N \setminus E := O_N \bigcap E^c \supseteq O_{N+1} \bigcap E^c$ for all N, and again $m_*(O_1 \setminus E) < \infty$.
    * So it's valid to apply continuity of measure from above:

$$\begin{aligned} m_*(O \setminus E) &= \lim_{N \longrightarrow \infty} m_*(O_N \setminus E) \\ &\leq \lim_{N \longrightarrow \infty} m_*(G_N \setminus E) \\ &= \lim_{N \longrightarrow \infty} \frac{1}{N} = 0, \end{aligned}$$

    where the inequality uses subadditivity on $\bigcap_{n=1}^{N} G_n \subseteq G_N$
  - Suppose E is unbounded.
    * Write $E^k = E \bigcap [k, k+1]^d \subset \mathbb{R}^d$ as the intersection of E with an annulus, and note that $E = \coprod_{k \in \mathbb{N}} E_k$.
    * Each E_k is bounded, so apply the previous case to obtain $O_k \supseteq E_k$ with $m(O_k \setminus E_k) = 0$.
    * So write $O_k = E_k \coprod N_k$ where $N_k := O_k \setminus E_k$ is a null set.
    * Define $O = \bigcup_{k \in \mathbb{N}} O_k$, note that $E \subseteq O$.
    * Now note

$$\begin{aligned} O \setminus E &= \left(\coprod\nolimits_k O_k\right) \setminus \left(\coprod\nolimits_K E_k\right) \\ &\subseteq \coprod\nolimits_k (O_k \setminus E_k) \\ \implies m_*(O \setminus E) &\leq m_*\left(\coprod (O_k \setminus E_k)\right) = 0, \end{aligned}$$

    since any countable union of null sets is again null.
  - So $O \supseteq E$ with $m(O \setminus E) = 0$.
- Theorem: since E is measurable, E^c is measurable

- Proof: It suffices to write E^c as the union of two measurable sets, $E^c = S\bigcup(E^c - S)$, where S is to be determined.
- We'll produce an S such that $m_*(E^c - S) = 0$ and use the fact that any subset of a null set is measurable.
- Since E is measurable, for every $\varepsilon > 0$ there exists an open $\mathcal{O}_\varepsilon \supseteq E$ such that $m_*(\mathcal{O}_\varepsilon \backslash E) < \varepsilon$.
- Take the sequence $\left\{\varepsilon_n := \dfrac{1}{n}\right\}$ to produce a sequence of sets $\mathcal{O}_n$.
- Note that each $\mathcal{O}_n^c$ is closed and

$$\mathcal{O}_n \supseteq E \iff \mathcal{O}_n^c \subseteq E^c.$$

- Set $S := \bigcup_n \mathcal{O}_n^c$, which is a union of closed sets, thus an F_σ set, thus Borel, thus measurable.
- Note that $S \subseteq E^c$ since each $\mathcal{O}_n \subseteq E^c$.
- Note that

$$\begin{aligned}
E^c \setminus S &:= E^c \setminus \left(\bigcup_{n=1}^\infty \mathcal{O}_n^c\right) \\
&:= E^c \bigcap \left(\bigcup_{n=1}^\infty \mathcal{O}_n^c\right)^c \quad \text{definition of set minus} \\
&= E^c \bigcap \left(\bigcap_{n=1}^\infty \mathcal{O}_n\right)^c \quad \text{De Morgan's law} \\
&= E^c \bigcup \left(\bigcap_{n=1}^\infty \mathcal{O}_n\right) \\
&:= \left(\bigcap_{n=1}^\infty \mathcal{O}_n\right) \setminus E \\
&\subseteq \mathcal{O}_N \setminus E \quad \text{for every } N \in \mathbb{N}.
\end{aligned}$$

- Then by subadditivity,

$$m_*(E^c \setminus S) \le m_*(\mathcal{O}_N \setminus E) \le \frac{1}{N} \quad \forall N \implies m_*(E^c \setminus S) = 0.$$

- Thus $E^c \setminus S$ is measurable.

4.2.1 Indirect Proof

- Since E is measurable, E^c is measurable.
- Since E^c is measurable exists an open $O \supseteq E^c$ such that $m(O \setminus E^c) = 0$.
- Set $B := O^c$, then $O \supseteq E^c \iff \mathcal{O}^c \subseteq E \iff B \subseteq E$.
- Computing measures yields

$$E \setminus B := E \setminus \mathcal{O}^c := E \bigcap (\mathcal{O}^c)^c = E \bigcap \mathcal{O} = \mathcal{O} \bigcap (E^c)^c := \mathcal{O} \setminus E^c,$$

  thus $m(E \setminus B) = m(\mathcal{O} \setminus E^c) = 0$.
- Since $\mathcal{O}$ is open, B is closed and thus Borel.

4.2.2 Direct Proof

? Try to construct the set.

4.3 3

Concepts used:

- Mean Value Theorem
- DCT

$$\begin{aligned}\frac{\partial}{\partial t} F(t) &= \frac{\partial}{\partial t} \int_{\mathbb{R}} f(x)\cos(xt)\ dx \\ &\overset{DCT}{=} \int_{\mathbb{R}} f(x) \frac{\partial}{\partial t} \cos(xt)\ dx \\ &= \int_{\mathbb{R}} x f(x)\cos(xt)\ dx,\end{aligned}$$

so it only remains to justify the DCT.

- Fix t, then let $t_n \longrightarrow t$ be arbitrary.
- Define

  $$h_n(x,t) = f(x)\left(\frac{\cos(tx) - \cos(t_n x)}{t_n - t}\right) \overset{n\longrightarrow\infty}{\longrightarrow} \frac{\partial}{\partial t}\left(f(x)\cos(xt)\right)$$

  since $\cos(tx)$ is differentiable in t and this is the limit definition of differentiability.
- Note that

  $$\begin{aligned}\frac{\partial}{\partial t}\cos(tx) &:= \lim_{t_n \longrightarrow t} \frac{\cos(tx) - \cos(t_n x)}{t_n - t} \\ &\overset{MVT}{=} \frac{\partial}{\partial t}\cos(tx)\ \Big|_{t=\xi_n} && \text{for some } \xi_n \in [t, t_n] \text{ or } [t_n, t] \\ &= x\sin(\xi_n x)\end{aligned}$$

  where $\xi_n \overset{n\longrightarrow\infty}{\longrightarrow} t$ since wlog $t_n \leq \xi_n \leq t$ and $t_n \nearrow t$.
- We then have

  $$|h_n(x)| = |f(x)x\sin(\xi_n x)| \leq |xf(x)| \quad \text{since } |\sin(\xi_n x)| \leq 1$$

  for every x and every n.
- Since $xf(x) \in L^1(\mathbb{R})$ by assumption, the DCT applies.

4.4 4

Case of characteristic function

- First suppose $f(x) = \chi_{[0,1]}(x)$.

- Note that $\sin(nx)$ has a period of $2\pi/n$, and thus $\left\lfloor \frac{n}{2\pi} \right\rfloor$ full periods in $[0, 1]$.
- Taking the absolute value yields a new function with half the period, so a period of π/n and $\lfloor \pi/n \rfloor$ full periods in $[0, 1]$.
- We can compute the integral over one full period (which is independent of *which* period is chosen), and since $\sin(x)$ is positive and agrees with $|\sin(nx)|$ on the first period, we have

$$\begin{aligned}
\int_{\text{One Period}} |\sin(nx)|\, dx &= \int_0^{\pi/n} \sin(nx)\, dx \\
&= \frac{1}{n} \int_0^{\pi} \sin(u)\, du \quad u = nx \\
&= \frac{1}{n} - \cos(u) \Big|_0^{\pi} \\
&= \frac{2}{n}.
\end{aligned}$$

- Then break the integral up into integrals over periods $P_1, P_2, \cdots, P_N$ where $N \coloneqq \lfloor n/\pi \rfloor$:

$$\begin{aligned}
\int_0^1 |\sin(nx)|\, dx &= \left(\sum_{j=1}^{N} \int_{P_j} |\sin(nx)|\, dx \right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)|\, dx \\
&= \left(\sum_{j=1}^{N} \frac{2}{n} \right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)|\, dx \\
&= N \left(\frac{2}{n} \right) + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)|\, dx \\
&\coloneqq \left\lfloor \frac{n}{\pi} \right\rfloor \frac{2}{n} + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)|\, dx \\
&= \frac{2}{\pi} + \int_{N\lfloor \pi/n \rfloor}^{1} |\sin(nx)|\, dx \\
&\coloneqq \frac{2}{\pi} + R(n)
\end{aligned}$$

  so it suffices to show that $R(n) \overset{n\longrightarrow\infty}{\longrightarrow} 0$.

  Need to justify removing floor function and cancellation.

- Showing this: ????????????

  No clue how to show this.

General case

Not sure. Approximate f by simple functions...?

4.5 5

Concepts used:

- Claim: If $E \subseteq \mathbb{R}^a \times \mathbb{R}^b$ is a measurable set, then for almost every $y \in \mathbb{R}^b$, the slice E^y is measurable and

$$m(E) = \int_{\mathbb{R}^b} m(E^y)\, dy.$$

  - Set $g = \chi_E$, which is non-negative and measurable, so apply Tonelli.

- Conclude that $g^y = \chi_{E^y}$ is measurable, the function $y \mapsto \int g^y(x)\,dx$ is measurable, and $\int \int g^y(x)\,dx\,dy = \int g$.
- But $\int g = m(E)$ and $\int \int g^y(x)\,dx\,dy = \int m(E^y)\,dy$.

Solution

Note: f is a function $\mathbb{R} \longrightarrow \mathbb{R}$ in the original problem, but here I've assumed $f : \mathbb{R}^n \longrightarrow \mathbb{R}$.

- Since $f \geq 0$, set

$$E \coloneqq \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; f(x) > t \right\} = \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; 0 \leq t < f(x) \right\}.$$

- Claim: since f is measurable, E is measurable and thus $m(E)$ makes sense.
  - Since f is measurable, $F(x,t) \coloneqq t - f(x)$ is measurable on $\mathbb{R}^n \times \mathbb{R}$.
  - Then write $E = \{F < 0\} \bigcap \{t \geq 0\}$ as an intersection of measurable sets.
- We have slices

$$\begin{aligned} E^t &\coloneqq \left\{ x \in \mathbb{R}^n \;\middle|\; (x,t) \in E \right\} = \left\{ x \in \mathbb{R}^n \;\middle|\; 0 \leq t < f(x) \right\} \\ E^x &\coloneqq \left\{ t \in \mathbb{R} \;\middle|\; (x,t) \in E \right\} = \left\{ t \in \mathbb{R} \;\middle|\; 0 \leq t \leq f(x) \right\} = [0, f(x)]. \end{aligned}$$

  - E_t is precisely the set that appears in the original RHS integrand.
  - $m(E^x) = f(x)$.
- Claim: χ_E satisfies the conditions of Tonelli, and thus $m(E) = \int \chi_E$ is equal to any iterated integral.
  - Non-negative: clear since $0 \leq \chi_E \leq 1$
  - Measurable: characteristic functions of measurable sets are measurable.
- Conclude:
  1. For almost every x, E^x is a measurable set, $x \mapsto m(E^x)$ is a measurable function, and $m(E) = \int_{\mathbb{R}^n} m(E^x)\,dx$
  2. For almost every t, E^t is a measurable set, $t \mapsto m(E^t)$ is a measurable function, and $m(E) = \int_{\mathbb{R}} m(E^t)\,dt$
- On one hand,

$$\begin{aligned} m(E) &= \int_{\mathbb{R}^{n+1}} \chi_E(x,t) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_E(x,t)\,dt\,dx \quad \text{by Tonelli} \\ &= \int_{\mathbb{R}^n} m(E^x)\,dx \quad \text{first conclusion} \\ &= \int_{\mathbb{R}^n} f(x)\,dx. \end{aligned}$$

- On the other hand,

$$\begin{aligned} m(E) &= \int_{\mathbb{R}^{n+1}} \chi_E(x,t) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_E(x,t)\,dx\,dt \quad \text{by Tonelli} \\ &= \int_{\mathbb{R}} m(E^t)\,dt \quad \text{second conclusion.} \end{aligned}$$

- Thus

$$\int_{\mathbb{R}^n} f\,dx = m(E) = \int_{\mathbb{R}} m(E^t)\,dt = \int_{\mathbb{R}} m\left(\left\{x \;\middle|\; f(x) > t\right\}\right).$$

■

4.6 6

- Note that $x^{\frac{1}{n}} \overset{n\longrightarrow\infty}{\longrightarrow} 1$ for any $0 < x < \infty$.
- Thus the integrand converges to $\dfrac{1}{e^x}$, which is integrable on $(0,\infty)$ and integrates to 1.
- Break the integrand up:

$$\int_0^\infty \frac{1}{(1+\frac{x}{n})^n x^{\frac{1}{n}}}\,dx = \int_0^1 \frac{1}{(1+\frac{x}{n})^n x^{\frac{1}{n}}}\,dx = \int_1^\infty \frac{1}{(1+\frac{x}{n})^n x^{\frac{1}{n}}}\,dx.$$

5 Spring 2018

5.1 1

Concepts used:

- Borel-Cantelli: If $\{E_k\}_{k\in\mathbb{Z}} \subset 2^{\mathbb{R}}$ is a countable collection of Lebesgue measurable sets with $\sum_{k\in\mathbb{Z}} m(E_k) < \infty$, then almost every $x \in \mathbb{R}$ is in *at most finitely* many E_k.
  - Equivalently (?), $m(\limsup_{k\longrightarrow\infty} E_k) = 0$, where $\limsup_{k\longrightarrow\infty} E_k = \bigcap_{k=1}^\infty \bigcup_{j\geq k} E_j$, the elements which are in E_k for infinitely many k.

Solution:

- Strategy: Borel-Cantelli.
- We'll show that $m(E)\bigcap[n,n+1] = 0$ for all $n \in \mathbb{Z}$; then the result follows from

$$m(E) = m\left(\bigcup_{n\in\mathbb{Z}} E\bigcap[n,n+1]\right) \leq \sum_{n=1}^\infty m(E\bigcap[n,n+1]) = 0.$$

- By translation invariance of measure, it suffices to show $m(E\bigcap[0,1]) = 0$.

- So WLOG, replace E with $E\bigcap[0,1]$.

- Define
$$E_j := \left\{ x \in [0,1] \;\middle|\; \exists p \in \mathbb{Z}^{\geq 0} \text{ s.t. } \left| x - \frac{p}{j} \right| < \frac{1}{j^3} \right\}.$$
  - Note that $E_j \subseteq \coprod_{p\in\mathbb{Z}^{\geq 0}} B_{j^{-3}}\left(\frac{p}{j}\right)$, i.e. a union over integers p of intervals of radius $1/j^3$ around the points p/j. Since $1/j^3 < 1/j$, this union is in fact disjoint.

- Importantly, note that
$$\limsup_{j\longrightarrow\infty} E_j := \bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty} E_j = E$$
since

$$\begin{aligned}
x \in \limsup_j E_j &\iff x \in E_j \text{ for infinitely many } j \\
&\iff \text{ there are infinitely many } j \text{ for which there exist a } p \text{ such that } \left| x - \frac{p}{j} \right| < j^{-3} \\
&\iff \text{ there are infinitely many such pairs } p, j \\
&\iff x \in E.
\end{aligned}$$

- Intersecting with $[0,1]$, we can write E_j as a union of intervals:
$$E_j = \left(0, j^{-3}\right) \quad \coprod \quad B_{j^{-3}}\left(\frac{1}{j}\right)\coprod B_{j^{-3}}\left(\frac{2}{j}\right)\coprod\cdots\coprod B_{j^{-3}}\left(\frac{j-1}{j}\right) \quad \coprod \quad (1-j^{-3},1),$$
where we've separated out the "boundary" terms to emphasize that they are balls about 0 and 1 intersected with $[0,1]$.

- Since E_j is a union of open sets, it is Borel and thus Lebesgue measurable.

- Computing the measure of E_j:
  - For a fixed j, there are exactly $j+1$ possible choices for a numerator $(0,1,\cdots,j)$, thus there are exactly $j+1$ sets appearing in the above decomposition.
  - The first and last intervals are length $\frac{1}{j^3}$
  - The remaining $(j+1)-2 = j-1$ intervals are twice this length, $\frac{2}{j^3}$
  - Thus
$$m(E_j) = 2\left(\frac{1}{j^3}\right) + (j-1)\left(\frac{2}{j^3}\right) = \frac{2}{j^2}$$

- Note that
$$\sum_{j\in\mathbb{N}} m(E_j) = 2\sum_{j\in\mathbb{N}} \frac{1}{j^2} < \infty,$$
which converges by the p-test for sums.

- But then

$$\begin{aligned} m(E) &= m(\limsup_j E_j) \\ &= m(\bigcap_{n\in\mathbb{N}} \bigcup_{j\geq n} E_j) \\ &\leq m(\bigcup_{j\geq N} E_j) \quad \text{for every } N \\ &\leq \sum_{j\geq N} m(E_j) \\ &\overset{N\longrightarrow\infty}{\longrightarrow} 0 \quad . \end{aligned}$$

- Thus E is measurable as a subset of a null set and $m(E) = 0$.

■

5.2 2

5.2.1 a

Claim: f_n does not converge uniformly to its limit.

- Note each $f_n(x)$ is clearly continuous on $(0, \infty)$, since it is a quotient of continuous functions where the denominator is never zero.
- Note

$$x < 1 \implies x^n \overset{n\longrightarrow\infty}{\longrightarrow} 0 \quad \text{and} \quad x > 1 \implies x^n \overset{n\longrightarrow\infty}{\longrightarrow} \infty.$$

- Thus

$$f_n(x) = \frac{x}{1+x^n} \overset{n\longrightarrow\infty}{\longrightarrow} f(x) := \begin{cases} x, & 0 \leq x < 1 \\ \frac{1}{2}, & x = 1 \\ 0, & x > 1 \end{cases} \quad .$$

- If $f_n \longrightarrow f$ uniformly on $[0, \infty)$, it would converge uniformly on every subset and thus uniformly on $(0, \infty)$.
  - Then f would be a uniform limit of continuous functions on $(0, \infty)$ and thus continuous on $(0, \infty)$.
  - By uniqueness of limits, f_n would converge to the pointwise limit f above, which is not continuous at $x = 1$, a contradiction.

5.2.2 b

- If the DCT applies, interchange the limit and integral:

$$\begin{aligned}\lim_{n\longrightarrow\infty}\int_0^\infty f_n(x)\,dx &= \int_0^\infty \lim_{n\longrightarrow\infty} f_n(x)\,dx \quad \text{DCT}\\ &= \int_0^\infty f(x)\,dx\\ &= \int_0^1 x\,dx + \int_1^\infty 0\,dx\\ &= \frac{1}{2}x^2\Big|_0^1\\ &= \frac{1}{2}.\end{aligned}$$

- To justify the DCT, write

$$\int_0^\infty f_n(x) = \int_0^1 f_n(x) + \int_1^\infty f_n(x).$$

- f_n restricted to $(0,1)$ is uniformly bounded by $g_0(x)=1$ in the first integral, since

$$x\in[0,1] \implies \frac{x}{1+x^n} < \frac{1}{1+x^n} < 1 \coloneqq g(x)$$

  so

$$\int_0^1 f_n(x)\,dx \le \int_0^1 1\,dx = 1 < \infty.$$

  Also note that $g_0\cdot\chi_{(0,1)}\in L^1((0,\infty))$.

- The f_n restricted to $(1,\infty)$ are uniformly bounded by $g_1(x)=\dfrac{1}{x^2}$ on $[1,\infty)$, since

$$x\in(1,\infty) \implies \frac{x}{1+x^n} \le \frac{x}{x^n} = \frac{1}{x^{n-1}} \le \frac{1}{x^2} \in L^1([1,\infty) \text{ when } n\ge 3,$$

  by the p-test for integrals.

- So set

$$g \coloneqq g_0\cdot\chi_{(0,1)} + g_1\cdot\chi_{[1,\infty)},$$

  then by the above arguments $g\in L^1((0,\infty))$ and $f_n\le g$ everywhere, so the DCT applies.

5.3 3

Concepts used:

- $\|f\|_\infty \coloneqq \inf_t \left\{ t \;\middle|\; m\Big(\Big\{x\in\mathbb{R}^n \;\Big|\; f(x)>t\Big\}\Big)=0\right\}$, i.e. this is the lowest upper bound that holds almost everywhere.

Solution:

- $\|f\|_p \leq \|f\|_\infty$:
  - Note $|f(x)| \leq \|f\|_\infty$ almost everywhere and taking pth powers preserves this inequality.
  - Thus

$$\begin{aligned} & |f(x)| \leq \|f\|_\infty \quad \text{a.e. by definition} \\ \implies & |f(x)|^p \leq \|f\|_\infty^p \quad \text{for } p \geq 0 \\ \implies & \|f\|_p^p = \int_X |f(x)|^p \, dx \\ & \leq \int_X \|f\|_\infty^p \, dx \\ & = \|f\|_\infty^p \int_X 1 \, dx \\ & = \|f\|_\infty^p \cdot m(X) \quad \text{since the norm doesn't depend on } x \\ & = \|f\|_\infty^p \qquad \text{since } m(X) = 1. \end{aligned}$$

    * Thus $\|f\|_p \leq \|f\|_\infty$ for all p and taking $\lim_{p \longrightarrow \infty}$ preserves this inequality.
- $\|f\|_p \geq \|f\|_\infty$:
  - Fix $\varepsilon > 0$.
  - Define

$$S_\varepsilon := \left\{ x \in \mathbb{R}^n \;\middle|\; |f(x)| \geq \|f\|_\infty - \varepsilon \right\}.$$

    * Note that $m(S_\varepsilon) > 0$; otherwise if $m(S_\varepsilon) = 0$, then $t := \|f\|_\infty - \varepsilon < \|f\|_\varepsilon$. But this produces a *smaller* upper bound almost everywhere than $\|f\|_\varepsilon$, contradicting the definition of $\|f\|_\varepsilon$ as an infimum over such bounds.
  - Then

$$\begin{aligned} \|f\|_p^p &= \int_X |f(x)|^p \, dx \\ &\geq \int_{S_\varepsilon} |f(x)|^p \, dx \quad \text{since } S_\varepsilon \subseteq X \\ &\geq \int_{S_\varepsilon} |\|f\|_\infty - \varepsilon|^p \, dx \quad \text{since on } S_\varepsilon, |f| \geq \|f\|_\infty - \varepsilon \\ &= |\|f\|_\infty - \varepsilon|^p \cdot m(S_\varepsilon) \quad \text{since the integrand is independent of } x \\ &\geq 0 \quad \text{since } m(S_\varepsilon) > 0 \end{aligned}$$

  - Taking pth roots for $p \geq 1$ preserves the inequality, so

$$\implies \|f\|_p \geq |\|f\|_\infty - \varepsilon| \cdot m(S_\varepsilon)^{\frac{1}{p}} \overset{p \longrightarrow \infty}{\longrightarrow} |\|f\|_\infty - \varepsilon| \overset{\varepsilon \longrightarrow 0}{\longrightarrow} \|f\|_\infty$$

    where we've used the fact that above arguments work
  - Thus $\|f\|_p \geq \|f\|_\infty$.

■

5.4 4

5.4.1 Proof 1: Using Fourier Transforms

Concepts used:

- Weierstrass Approximation: A uniformly continuous function on a compact set can be uniformly approximated by polynomials.

Solution:

- Fix $k \in \mathbb{Z}$.
- Since $e^{2\pi ikx}$ is continuous on the compact interval $[0,1]$, it is uniformly continuous.
- Thus there is a sequence of polynomials P_ℓ such that

$$P_{\ell,k} \overset{\ell \longrightarrow \infty}{\longrightarrow} e^{2\pi ikx} \text{ uniformly on } [0,1].$$

- Note applying linearity to the assumption $\int f(x)\, x^n$, we have

$$\int f(x) x^n \, dx = 0 \ \forall n \implies \int f(x) p(x)\, dx = 0$$

  for any polynomial $p(x)$, and in particular for $P_{\ell,k}(x)$ for every ℓ and every k.

- But then

$$\begin{aligned}
\langle f,\ e_k \rangle &= \int_0^1 f(x) e^{-2\pi ikx}\ dx \\
&= \int_0^1 f(x) \lim_{\ell \longrightarrow \infty} P_\ell(x) \\
&= \lim_{\ell \longrightarrow \infty} \int_0^1 f(x) P_\ell(x) \qquad \text{by uniform convergence} \\
&= \lim_{\ell \longrightarrow \infty} 0 \quad \text{by assumption} \\
&= 0 \quad \forall k \in \mathbb{Z},
\end{aligned}$$

  so f is orthogonal to every e_k.

- Thus $f \in S^\perp := \operatorname{span}_{\mathbb{C}} \{e_k\}_{k\in\mathbb{Z}}^\perp \subseteq L^2([0,1])$, but since this is a basis, S is dense and thus $S^\perp = \{0\}$ in $L^2([0,1])$.
- Thus $f \equiv 0$ in $L^2([0,1])$, which implies that f is zero almost everywhere.

$\blacksquare$

5.4.2 Alternative Proof

Concepts used

- $C^1([0,1])$ is dense in $L^2([0,1])$
- Polynomials are dense in $L^p(X, \mathcal{M}, \mu)$ for any $X \subseteq \mathbb{R}^n$ compact and μ a finite measure, for all $1 \leq p < \infty$.
  - Use Weierstrass Approximation, then uniform convergence implies $L^p(\mu)$ convergence by DCT.

Solution:

- By density of polynomials, for $f \in L^2([0,1])$ choose $p_\varepsilon(x)$ such that $\|f - p_\varepsilon\| < \varepsilon$ by Weierstrass approximation.
- Then on one hand,

$$\begin{aligned}\|f(f-p_\varepsilon)\|_1 &= \left\|f^2\right\|_1 - \|f \cdot p_\varepsilon\|_1 \\ &= \left\|f^2\right\|_1 - 0 \quad \text{by assumption} \\ &= \|f\|_2^2.\end{aligned}$$

  - Where we've used that $\left\|f^2\right\|_1 = \int \left|f^2\right| = \int |f|^2 = \|f\|_2^2$.
- On the other hand

$$\begin{aligned}\|f(f-p_\varepsilon)\| \leq \|f\|_1 \|f - p_\varepsilon\|_\infty \quad \text{by Holder} \\ \leq \|f\|_1.\end{aligned}$$

5.5 5

Concepts used:

- $\int |f_n - f| \longrightarrow \iff \int f_n = \int f$.
- Fatou:

$$\begin{aligned}\int \liminf f_n \leq \liminf \int f_n \\ \int \limsup f_n \geq \limsup \int f_n.\end{aligned}$$

Solution:

- Since $\int |f_n| \overset{n\longrightarrow\infty}{\longrightarrow} \int |f|$, define

$$\begin{aligned}h_n &= |f_n - f| & \overset{n\longrightarrow\infty}{\longrightarrow} 0 \ a.e. \\ g_n &= |f_n| + |f| & \overset{n\longrightarrow\infty}{\longrightarrow} 2|f| \ a.e.\end{aligned}$$

  - Note that $g_n - h_n \overset{n\longrightarrow\infty}{\longrightarrow} 2|f| - 0 = 2|f|$.
- Then

$$\begin{aligned}\int 2|f| &= \int \liminf_n (g_n - h_n) \\ &= \int \liminf_n (g_n) + \int \liminf_n (-h_n) \\ &= \int \liminf_n (g_n) - \int \limsup_n (h_n) \\ &= \int 2|f| - \int \limsup_n (h_n) \\ &\leq \int 2|f| - \limsup_n \int h_n \quad \text{by Fatou,}\end{aligned}$$

- Since $f \in L^1$, $\int 2|f| = 2\|f\|_1 < \infty$ and it makes sense to subtract it from both sides, thus

$$\begin{aligned} 0 &\leq -\limsup_n \int h_n \\ &:= -\limsup_n \int |f_n - f|. \end{aligned}$$

  which forces $\limsup_n \int |f_n - f| = 0$, since

  - The integral of a nonnegative function is nonnegative, so $\int |f_n - f| \geq 0$.
  - So $\left(-\int |f_n - f|\right) \leq 0$.
  - But the above inequality shows $\left(-\int |f_n - f|\right) \geq 0$ as well.

- Since $\liminf_n \int h_n \leq \limsup_n \int h_n = 0$, $\lim_n \int h_n$ exists and is equal to zero.
- But then

$$\left|\int f_n - \int f\right| = \left|\int f_n - f\right| \leq \int |f_n - f|,$$

  and taking $\lim_{n \longrightarrow \infty}$ on both sides yields

$$\lim_{n \longrightarrow \infty} \left|\int f_n - \int f\right| \leq \lim_{n \longrightarrow \infty} \int |f_n - f| = 0,$$

  so $\lim_{n \longrightarrow \infty} \int f_n = \int f$.

$\blacksquare$

6 Fall 2017

6.1 1

Note that $f(x) = e^x$ is entire and thus equal to its power series. So $f(x) = \sum_{j=0}^{\infty} \frac{1}{j!} x^j$.

Letting $f_N(x) = \sum_{j=1}^{N} \frac{1}{j!} x^j$, we have $f_N(x) \longrightarrow f(x)$ pointwise on $(-\infty, \infty)$.

For any compact interval $[-M, M]$, we have

$$
\begin{aligned}
\|f_N(x) - f(x)\|_\infty &= \sup_{-M \le x \le M} \left| \sum_{j=N+1}^{\infty} \frac{1}{j!} x^j \right| \\
&\le \sup_{-M \le x \le M} \sum_{j=N+1}^{\infty} \frac{1}{j!} |x|^j \\
&\le \sum_{j=N+1}^{\infty} \frac{1}{j!} M^j \\
&\le \sum_{j=0}^{\infty} \frac{1}{j!} M^j \\
&= e^M \\
&< \infty,
\end{aligned}
$$

so $f_N \longrightarrow f$ uniformly on $[-M, M]$ by the M-test. Thus it converges on any bounded interval.

It does not converge on $\mathbb{R}$, since x^N is unbounded.

6.2 2

6.2.1 a

It suffices to consider the bounded case, i.e. $E \subseteq B_M(0)$ for some M. Then write $E_n = B_n(0) \bigcap E$ and apply the theorem to E_n, and by subadditivity, $m^*(E) = m^*(\bigcup_n E_n) \le \sum_n m^*(E_n) = 0$.

Lemma: $f(x) = x^2, f^{-1}(x) = \sqrt{x}$ are Lipschitz on any compact subset of $[0, \infty)$.

Proof: Let $g = f$ or f^{-1}. Then $g \in C^1([0, M])$ for any M, so g is differentiable and g' is continuous. Since g' is continuous on a compact interval, it is bounded, so $|g'(x)| \le L$ for all x. Applying the MVT,

$$
|f(x) - f(y)| = f'(c)|x - y| \le L|x - y|.
$$

Lemma: If g is Lipschitz on $\mathbb{R}^n$, then $m(E) = 0 \implies m(g(E)) = 0$.

Proof: If g is Lipschitz, then

$$
g(B_r(x)) \subseteq B_{Lr}(x),
$$

which is a dilated ball/cube, and so

$$
m^*(B_{Lr}(x)) \le L^n \cdot m^*(B_r(x)).
$$

Now choose $\{Q_j\} \rightrightarrows E$; then $\{g(Q_j)\} \rightrightarrows g(E)$.

By the above observation,

$$
|g(Q_j)| \le L^n |Q_j|,
$$

and so

$$m^*(g(E)) \leq \sum_j |g(Q_j)| \leq \sum_j L^n |Q_j| = L^n \sum_j |Q_j| \longrightarrow 0.$$

Now just take $g(x) = x^2$ for one direction, and $g(x) = f^{-1}(x) = \sqrt{x}$ for the other. ■

6.2.2 b

> Lemma: E is measurable iff $E = K \coprod N$ for some K compact, N null.

Write $E = K \coprod N$ where K is compact and N is null.

Then $\varphi^{-1}(E) = \varphi^{-1}(K \coprod N) = \varphi^{-1}(K) \coprod \varphi^{-1}(N)$.

Since $\varphi^{-1}(N)$ is null by part (a) and $\varphi^{-1}(K)$ is the preimage of a compact set under a continuous map and thus compact, $\varphi^{-1}(E) = K' \coprod N'$ where K' is compact and N' is null, so $\varphi^{-1}(E)$ is measurable.

So φ is a measurable function, and thus yields a well-defined map $\mathcal{L}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathbb{R})$ since it preserves measurable sets. Restricting to $[0, \infty)$, f is bijection, and thus so is φ.

■

6.3 3

> From homework: E is Lebesgue measurable iff there exists a finite union of closed cubes A such that $m(E \Delta A) < \varepsilon$.

It suffices to show that S is dense in simple functions, and since simple functions are *finite* linear combinations of characteristic functions, it suffices to show this for χ_A for A a measurable set.

Let $s = \chi_A$. By regularity of the Lebesgue measure, choose an open set $O \supseteq A$ such that $m(O \setminus A) < \varepsilon$.

O is an open subset of $\mathbb{R}$, and thus $O = \coprod_{j \in \mathbb{N}} I_j$ is a disjoint union of countably many open intervals.

Now choose N large enough such that $m(O \Delta I_{N,n}) < \varepsilon = \dfrac{1}{n}$ where we define $I_{N,n} := \coprod_{j=1}^{N} I_j$.

Now define $f_n = \chi_{I_{N,n}}$, then

$$\|s - f_n\|_1 = \int \left| \chi_A - \chi_{I_{N,n}} \right| = m(A \Delta I_{N,n}) \overset{n \to \infty}{\longrightarrow} 0.$$

Since any simple function is a finite linear combination of χ_{A_i}, we can do this for each i to extend this result to all simple functions. But simple functions are dense in L^1, so S is dense in L^1.

6.4 4

6.4.1 a

Let $G(x) = \sum_{n=1}^{\infty} nx(1-x)^n$. Applying the ratio test, we have

$$\left| \frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n} \right| = \frac{n+1}{n} |1-x| \overset{n \longrightarrow \infty}{\longrightarrow} |1-x| < 1 \iff 0 \le x \le 2,$$

and in particular, this series converges on $[0,2]$. Thus its terms go to zero, and $nx(1-x)^n \longrightarrow 0$ on $[0,1] \subset [0,2]$.

To see that the convergence is not uniform, let $x_n = \frac{1}{n}$ and $\varepsilon > \frac{1}{e}$, then

$$\sup_{x \in [0,1]} |nx(1-x)^n - 0| \ge |nx_n(1-x_n)^n| = \left| \left(1 - \frac{1}{n}\right)^n \right| \overset{n \longrightarrow \infty}{\longrightarrow} e^{-1} > \varepsilon.$$

6.4.2 b

Note: could use the first part with $\sin(x) \le x$, but then integral ends up more complicated.

Noting that $\sin(x) \le 1$, we have We have

$$\begin{aligned} \left| \int_0^1 n(1-x)^n \sin(x) \right| &\le \int_0^1 |n(1-x)^n \sin(x)| \\ &\le \int_0^1 |n(1-x)^n| \\ &= n \int_0^1 (1-x)^n \\ &= -\frac{n(1-x)^{n+1}}{n+1} \\ &\overset{n \longrightarrow \infty}{\longrightarrow} 0. \end{aligned}$$

6.5 5

6.5.1 a

Lemma: If $\varphi \in C_c^1$, then $(f * \varphi)' = f * \varphi'$ almost everywhere.

Silly Proof:

$$\begin{aligned} \mathcal{F}((f * \varphi)') &= 2\pi i \xi \; \mathcal{F}(f * \varphi) \\ &= 2\pi i \xi \; \mathcal{F}(f) \; \mathcal{F}(\varphi) \\ &= \mathcal{F}(f) \cdot (2\pi i \xi \; \mathcal{F}(\varphi)) \\ &= \mathcal{F}(f) \cdot \mathcal{F}(\varphi') \\ &= \mathcal{F}(f * \varphi'). \end{aligned}$$

Actual proof:

$$\begin{aligned}
(f*\varphi)'(x) &= (\varphi * f)'(x) \\
&= \lim_{h\longrightarrow 0} \frac{(\varphi * f)'(x+h) - (\varphi * f)'(x)}{h} \\
&= \lim_{h\longrightarrow 0} \int \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \\
&\overset{DCT}{=} \int \lim_{h\longrightarrow 0} \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \\
&= \int \varphi'(x-y) f(y) \\
&= (\varphi' * f)(x) \\
&= (f * \varphi')(x).
\end{aligned}$$

To see that the DCT is justified, we can apply the MVT on the interval $[0,h]$ to f to obtain

$$\frac{\varphi(x+h-y) - \varphi(x-y)}{h} = \varphi'(c) \quad c \in [0,h],$$

and since φ' is continuous and compactly supported, φ' is bounded by some $M < \infty$ by the extreme value theorem and thus

$$\begin{aligned}
\int \left| \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \right| &= \int |\varphi'(c) f(y)| \\
&\leq \int |M||f| \\
&= |M| \int |f| < \infty,
\end{aligned}$$

since $f \in L^1$ by assumption, so we can take $g \coloneqq |M||f|$ as the dominating function.

Applying this theorem infinitely many times shows that $f * \varphi$ is smooth.

To see that $f * \varphi$ is compactly supported, approximate f by a *continuous* compactly supported function h, so $\|h - f\|_1 \xrightarrow{L^1} 0$.

Now let $g_x(y) = \varphi(x-y)$, and note that $\mathrm{supp}(g) = x - \mathrm{supp}(\varphi)$ which is still compact.

But since $\mathrm{supp}(h)$ is bounded, there is some N such that

$$|x| > N \implies A_x \coloneqq \mathrm{supp}(h) \bigcap \mathrm{supp}(g_x) = \emptyset$$

and thus

$$\begin{aligned}
(h * \varphi)(x) &= \int_{\mathbb{R}} \varphi(x-y) h(y) \, dy \\
&= \int_{A_x} g_x(y) h(y) \\
&= 0,
\end{aligned}$$

so $\left\{x \;\middle|\; f * g(x) = 0\right\}$ is open, and its complement is closed and bounded and thus compact.

6.5.2 b

$$\begin{aligned}
\|f * K_j - f\|_1 &= \int \left| \int f(x-y)K_j(y)\ dy - f(x) \right| dx \\
&= \int \left| \int f(x-y)K_j(y)\ dy - \int f(x)K_j(y)\ dy \right| dx \\
&= \int \left| \int (f(x-y) - f(x))K_j(y)\ dy \right| dx \\
&\leq \int \int |(f(x-y) - f(x))| \cdot |K_j(y)|\ dy\ dx \\
&\overset{FT}{=} \int \int |(f(x-y) - f(x))| \cdot |K_j(y)|\ \mathbf{dx\ dy} \\
&= \int |K_j(y)| \left(\int |(f(x-y) - f(x))|\ dx \right)\ dy \\
&= \int |K_j(y)| \cdot \|f - \tau_y f\|_1\ dy.
\end{aligned}$$

We now split the integral up into pieces.

1. Chose δ small enough such that $|y| < \delta \implies \|f - \tau_y f\|_1 < \varepsilon$ by continuity of translation in L^1, and
2. Since φ is compactly supported, choose J large enough such that

$$j > J \implies \int_{|y| \geq \delta} |K_j(y)|\ dy = \int_{|y| \geq \delta} |j\varphi(jy)| = 0$$

Then

$$\begin{aligned}
\|f * K_j - f\|_1 &\leq \int |K_j(y)| \cdot \|f - \tau_y f\|_1\ dy \\
&= \int_{|y|<\delta} |K_j(y)| \cdot \|f - \tau_y f\|_1\ dy + \int_{|y| \geq \delta} |K_j(y)| \cdot \|f - \tau_y f\|_1\ dy \\
&= \varepsilon \int_{|y| \geq \delta} |K_j(y)| + 0 \\
&\leq \varepsilon(1) \longrightarrow 0.
\end{aligned}$$

■

6.6 6

Should be supremum maybe..?

Let $\{f_k\}$ be a Cauchy sequence, so $\|f_k\| < \infty$ for all k. Then for a fixed x, the sequence $f_k(x)$ is Cauchy in $\mathbb{R}$ and thus converges to some $f(x)$, so define f by $f(x) \coloneqq \lim_{k \longrightarrow \infty} f_k(x)$.

Then $\|f_k - f\| = \max_{x\in X} |f_k(x) - f(x)| \overset{k\longrightarrow\infty}{\longrightarrow} 0$, and thus $f_k \longrightarrow f$ uniformly and thus f is continuous. It just remains to show that f has bounded norm.

Choose N large enough so that $\|f - f_N\| < \varepsilon$, and write $\|f_N\| \coloneqq M < \infty$

$$\|f\| \leq \|f - f_N\| + \|f_N\| < \varepsilon + M < \infty.$$

7 Spring 2017

7.1 1

A is nowhere dense $\iff$ every interval I contains a subinterval $S \subseteq A^c$.

K **is compact:**

It suffices to show that $K^c \coloneqq [0,1] \setminus K$ is open; then K will be a closed and bounded subset of $\mathbb{R}$ and thus compact by Heine-Borel.

We can identify K^c as the set of real numbers in $[0,1]$ whose decimal expansion **does** use a 4. Let $x \in K^c$, and suppose a 4 occurs as the kth digit and write

$$x = 0.d_1 d_2 \cdots d_{k-1}\ 4\ d_{k+1} \cdots = \sum_{j=1}^{k} d_j 10^{-j} + 4 \cdot 10^{-k} + \sum_{j=k+1}^{\infty} d_j 10^{-j}.$$

Then if we set $r < 10^{-k}$ and pick any $y \in [0,1]$ such that $y \in B_r(x)$, then $|x - y| < 10^{-k}$. If we write $y = \sum_{j=1}^{\infty} c_j 10^{-j}$, this means that for all $j \leq k$ we have $d_j = c_j$, and in particular $d_k = 4 = c_k$, so y has a 4 in its decimal expansion.

But then $K^c = \bigcup_x B_r(x)$ is a union of open sets and thus open.

K **is nowhere dense and** $m(K) = 0$**:**

Since K is closed, we'll show that K can not properly contain any interval, so $(\overline{K})^\circ = \emptyset$.

As in the construction of the Cantor set, let

- K_1 denote $[0,1]$ with 1 interval $[0.4, 0.5]$ of length $\frac{1}{10}$ deleted
- K_2 denote K_1 with 9 intervals $[0.04, 0.05],\ [0.14, 0.15], \cdots [0.94, 0.95]$ length $\frac{1}{100}$ deleted
- K_n denote K_{n-1} with 9^{n-1} such intervals of length 10^{-n} deleted.

Then $K = \bigcap K_n$, and

$$m(K) = 1 - m(K^c) = 1 - \sum_{j=0}^{\infty} \frac{9^n}{10^{n+1}} = 1 - \frac{1}{10}\left(\frac{1}{1 - \frac{9}{10}}\right) = 0,$$

and since any interval has strictly positive measure, K can not contain any interval.

K **has no isolated points**:

A point $x \in K$ is isolated iff there there is an open ball $B_r(x)$ containing x such that $B_r(x) \bigcap K = \emptyset$, so every point in this ball has a 4 in its decimal expansion.

Note that $m(K_n) = \left(\frac{9}{10}\right)^n \longrightarrow 0$ and that the endpoints of intervals are never removed and are thus elements of K. Then for every ε, we can choose n such that $\left(\frac{9}{10}\right)^n < \varepsilon$; then there is an endpoint of a removed interval e_n satisfying $|x - e_n| \leq \left(\frac{9}{10}\right)^n < \varepsilon$.

So every ball containing x contains some endpoint of a removed interval, and thus an element of K.

$\blacksquare$

7.2 2

$\lambda \ll \mu \iff E \in \mathcal{M}, \mu(E) = 0 \implies \lambda(E) = 0.$

7.2.1 a

By Radon-Nikodym, if $\lambda \ll \mu$ then $d\lambda = f d\mu$, which would yield

$$\int g \; d\lambda = \int gf \; d\mu.$$

So let E be measurable and suppose $\mu(E) = 0$. Then

$$\lambda(E) := \int_E f \; d\mu = \lim_n \left\{ \varphi_n := \sum_j c_j \mu(E_j) \right\},$$

where we take a sequence of simple functions increasing to f.

But since each $E_j \subseteq E$, we must have $\mu(E_j) = 0$ for any such E_j, so every such φ_n must be zero and thus $\lambda(E) = 0$.

7.2.2 b

By Radon-Nikodym, there exists a positive f such that

$$\int g \; dm = \int gf \; d\mu,$$

where we can take $g(x) = x^2$, then the LHS is zero by assumption and thus so is the RHS.

Note that gf is positive.

Define $A_k = \left\{ x \in X \;\middle|\; gf\chi_E > \frac{1}{k} \right\}$, then by Chebyshev

$$\mu(A_k) \leq k \int_E gf \; d\mu = 0,$$

which holds for every k.

Then noting that $A_k \searrow A \coloneqq \left\{x \in E \;\middle|\; x^2 > 0\right\}$, and gf is positive, we have

$$x \in E \iff gf\chi_E(x) > 0 \iff x \in A,$$

so $E = A$ and $\mu(E) = \mu(A)$.

But since $m \ll \mu$ by construction, we can conclude that $m(E) = 0$.

■

7.3 3

7.3.1 a

Letting $x_n \coloneqq \frac{1}{n}$, we have

$$\sum_{k=1}^{\infty} |f_k(x)| \geq |f_n(x_n)| = \left|ae^{-ax} - be^{-bx}\right| \coloneqq M.$$

In particular, $\sup_x |f_n(x)| \not\longrightarrow 0$, so the terms do not go to zero and the sum can not converge.

7.3.2 b

?

7.4 4

Switching to polar coordinates and integrating over a half-circle contained in I^2, we have

$$\int_{I^2} f \geq \int_0^{\pi} \int_0^1 \frac{\cos(\theta)\sin(\theta)}{r^2} \; dr \; d\theta = \infty,$$

so f is not integrable.

7.5 5

See https://math.stackexchange.com/questions/507263/prove-that-c1a-b-with-the-c1-norm-is-a-banach-space

This is clearly a norm, which we'll write $\|\cdot\|_u$

Let f_n be a Cauchy sequence and define a candidate limit $f(x) = \lim_n f_n(x)$.

Then noting that $\|f_n\|_\infty, \|f_n'\|\infty \leq \|f_n\|_u < \infty$, both f_n, f_n are Cauchy sequences in $C^0([a,b], \|\cdot\|_\infty)$, which is a Banach space.

So $f_n \longrightarrow f$ uniformly, and $f_n' \longrightarrow g$ uniformly for some g, and moreover $f, g \in C^0([a,b])$.

We thus have

$$\begin{aligned}
f_n(x) - f_n(a) &\xrightarrow{u} f(x) - f(a) \\
\int_a^x f_n' &\xrightarrow{u} \int_a^x g,
\end{aligned}$$

and by the FTC, the left-hand sides are equal, and by uniqueness of limits so are the right-hand sides, so $f' = g$.

Since $f, f' \in C^0([a,b])$, they are bounded, and so $\|f\|_u < \infty$. This means that $\|f_n - f\|_u \longrightarrow 0$, so f_n converges to f, which is in the same space.

■

8 Fall 2016

8.1 1

9 Spring 2016

9.1 1

10 Spring 2014

10.1 1