

Topology Qualifying Exam Notes

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1 Definitions

- Topology: Closed under arbitrary unions and finite intersections.
- Basis: A subset $\{B_i\}$ is a basis iff
 - $x \in X \implies x \in B_i$ for some i .
 - $x \in B_i \cap B_j \implies x \in B_k \subset B_i \cap B_j$.
 - Topology generated by this basis: $x \in N_x \implies x \in B_i \subset N_x$ for some i .
- Dense: A subset $Q \subset X$ is dense iff $y \in N_y \subset X \implies N_y \cap Q \neq \emptyset$ iff $\bar{Q} = X$.
- Neighborhood: A neighborhood of a point x is any open set containing x .
- Hausdorff
- Second Countable: admits a countable basis.
- Closed (several characterizations)
- Closure in a subspace: $Y \subset X \implies \text{cl}_Y(A) := \text{cl}_X(A) \cap Y$.
- Bounded
- Compact: A topological space (X, τ) is **compact** if every open cover has a *finite* subcover.

That is, if $\{U_j \mid j \in J\} \subset \tau$ is a collection of open sets such that $X \subseteq \bigcup_{j \in J} U_j$, then there exists a *finite* subset $J' \subset J$ such that $X \subseteq \bigcup_{j \in J'} U_j$.

- Locally compact For every $x \in X$, there exists a $K_x \ni x$ such that K_x is compact.
- Connected: There does not exist a disconnecting set $X = A \amalg B$ such that $\emptyset \neq A, B \subsetneq X$, i.e. X is the union of two proper disjoint nonempty sets.

Equivalently, X contains no proper nonempty clopen sets.

– Additional condition for a subspace $Y \subset X$: $\text{cl}_Y(A) \cap V = A \cap \text{cl}_Y(B) = \emptyset$.

- Locally connected: A space is locally connected at a point x iff $\forall N_x \ni x$, there exists a $U \subset N_x$ containing x that is connected.
- Retract: A subspace $A \subset X$ is a *retract* of X iff there exists a continuous map $f : X \rightarrow A$ such that $f|_A = \text{id}_A$. Equivalently it is a *left* inverse to the inclusion.
- Uniform Continuity: For $f : (X, d_x) \rightarrow (Y, d_Y)$ metric spaces,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

- Lebesgue number: For (X, d) a compact metric space and $\{U_\alpha\} \Rightarrow X$, there exist $\delta_L > 0$ such that

$$A \subset X, \text{diam}(A) < \delta_L \implies A \subseteq U_\alpha \text{ for some } \alpha.$$

- Paracompact
- Components: Set $x \sim y$ iff there exists a connected set $U \ni x, y$ and take equivalence classes.
- Path Components: Set $x \sim y$ iff there exists a path-connected set $U \ni x, y$ and take equivalence classes.
- Separable: Contains a countable dense subset.
- Limit Point: For $A \subset X$, x is a limit point of A if every punctured neighborhood P_x of x satisfies $P_x \cap A \neq \emptyset$, i.e. every neighborhood of x intersects A in some point other than x itself.

Equivalently, x is a limit point of A iff $x \in \text{cl}_X(A \setminus \{x\})$.

2 Theorems

2.1 Point-Set

- Closed subsets of Hausdorff spaces are compact? (check)
- Cantor's intersection theorem?
- Tube lemma
- Properties pushed forward through continuous maps:

- Compactness?
- Connectedness (when surjective)
- Separability
- Density **only when** f is surjective
- **Not** openness
- **Not** closedness

- A retract of a Hausdorff/connected/compact space is closed/connected/compact respectively.

Proposition 2.1.

A continuous function on a compact set is uniformly continuous.

Proof .

Take $\{B_{\frac{\varepsilon}{2}}(y) \mid y \in Y\} \rightrightarrows Y$, pull back to an open cover of X , has Lebesgue number $\delta_L > 0$, then $x' \in B_{\delta_L}(x) \implies f(x), f(x') \in B_{\frac{\varepsilon}{2}}(y)$ for some y . ■

- Lipschitz continuity implies uniform continuity (take $\delta = \varepsilon/C$)
 - Counterexample to converse: $f(x) = \sqrt{x}$ on $[0, 1]$ has unbounded derivative.
- Extreme Value Theorem: for $f : X \rightarrow Y$ continuous with X compact and Y ordered in the order topology, there exist points $c, d \in X$ such that $f(x) \in [f(c), f(d)]$ for every x .

Theorem 2.2.

Points are closed in T_1 spaces.

Theorem 2.3.

A metric space X is sequentially compact iff it is complete and totally bounded.

Theorem 2.4.

A metric space is totally bounded iff every sequence has a Cauchy subsequence.

Theorem 2.5.

A metric space is compact iff it is complete and totally bounded.

Theorem 2.6 (Baire).

If X is a complete metric space, then the intersection of countably many dense open sets is dense in X .

Theorem 2.7.

A continuous bijective open map is a homeomorphism.

Theorem 2.8.

A closed subset A of a compact set B is compact.

Proof .

- Let $\{A_i\} \rightrightarrows A$ be a covering of A by sets open in A .
- Each $A_i = B_i \cap A$ for some B_i open in B (definition of subspace topology)
- Define $V = \{B_i\}$, then $V \rightrightarrows A$ is an open cover.
- Since A is closed, $W := B \setminus A$ is open
- Then $V \cup W$ is an open cover of B , and has a finite subcover $\{V_i\}$
- Then $\{V_i \cap A\}$ is a finite open cover of A .

■

Theorem 2.9.

The continuous image of a compact set is compact.

Theorem 2.10.

A closed subset of a Hausdorff space is compact.

2.2 Algebraic

Todo: Merge the two van Kampen theorems.

Theorem 2.11 (Van Kampen).

The pushout is the northwest colimit of the following diagram

$$\begin{array}{ccc} A \amalg_Z B & \longleftarrow & A \\ & & \uparrow \iota_A \\ B & \xrightarrow{\iota_B} & Z \end{array}$$

For groups, the pushout is given by the amalgamated free product: if $A = \langle G_A \mid R_A \rangle$, $B = \langle G_B \mid R_B \rangle$, then $A *_Z B = \langle G_A, G_B \mid R_A, R_B, T \rangle$ where T is a set of relations given by $T = \{ \iota_A(z) \iota_B(z)^{-1} \mid z \in Z \}$.

Example: $A = \mathbb{Z}/4\mathbb{Z} = \langle x \mid x^4 \rangle$, $B = \mathbb{Z}/6\mathbb{Z} = \langle y \mid y^6 \rangle$, $Z = \mathbb{Z}/2\mathbb{Z} = \langle z \mid z^2 \rangle$. Then we can identify Z as a subgroup of A, B using $\iota_A(z) = x^2$ and $\iota_B(z) = y^3$. So

$$A *_Z B = \langle x, y \mid x^4, y^6, x^2 y^{-3} \rangle$$

Suppose $X = U_1 \cup U_2$ such that $U_1 \cap U_2 \neq \emptyset$ is path connected. Then taking $x_0 \in U := U_1 \cap U_2$ yields a pushout of fundamental groups

$$\pi_1(X; x_0) = \pi_1(U_1; x_0) *_{\pi_1(U; x_0)} \pi_1(U_2; x_0).$$

Theorem 2.12 (Van Kampen).

If $X = U \cup V$ where $U, V, U \cap V$ are all path-connected then

$$\pi_1(X) = \pi_1 U *_{\pi_1(U \cap V)} \pi_1 V,$$

where the amalgamated product can be computed as follows: If we have presentations

$$\begin{aligned}\pi_1(U, w) &= \langle u_1, \dots, u_k \mid \alpha_1, \dots, \alpha_l \rangle \\ \pi_1(V, w) &= \langle v_1, \dots, v_m \mid \beta_1, \dots, \beta_n \rangle \\ \pi_1(U \cap V, w) &= \langle w_1, \dots, w_p \mid \gamma_1, \dots, \gamma_q \rangle\end{aligned}$$

then

$$\begin{aligned}\pi_1(X, w) &= \langle u_1, \dots, u_k, v_1, \dots, v_m \rangle \\ &\quad \text{mod } \langle \alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_n, I(w_1)J(w_1)^{-1}, \dots, I(w_p)J(w_p)^{-1} \rangle \\ &= \frac{\pi_1(U) * \pi_1(V)}{\langle \{I(w_i)J(w_i)^{-1} \mid 1 \leq i \leq p\} \rangle}\end{aligned}$$

where

$$\begin{aligned}I : \pi_1(U \cap V, w) &\rightarrow \pi_1(U, w) \\ J : \pi_1(U \cap V, w) &\rightarrow \pi_1(V, w).\end{aligned}$$

3 Examples

3.1 Common Spaces and Operations

Point-Set:

- Finite discrete sets with the discrete topology
- Subspaces of \mathbb{R} : (a, b) , $(a, b]$, (a, ∞) , etc.
 - $\{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{Z}^{\geq 1} \right\}$
- \mathbb{Q}
- The topologist's sine curve
- One-point compactifications
- \mathbb{R}^ω
- Hawaiian earring
- Cantor set

Non-Hausdorff spaces:

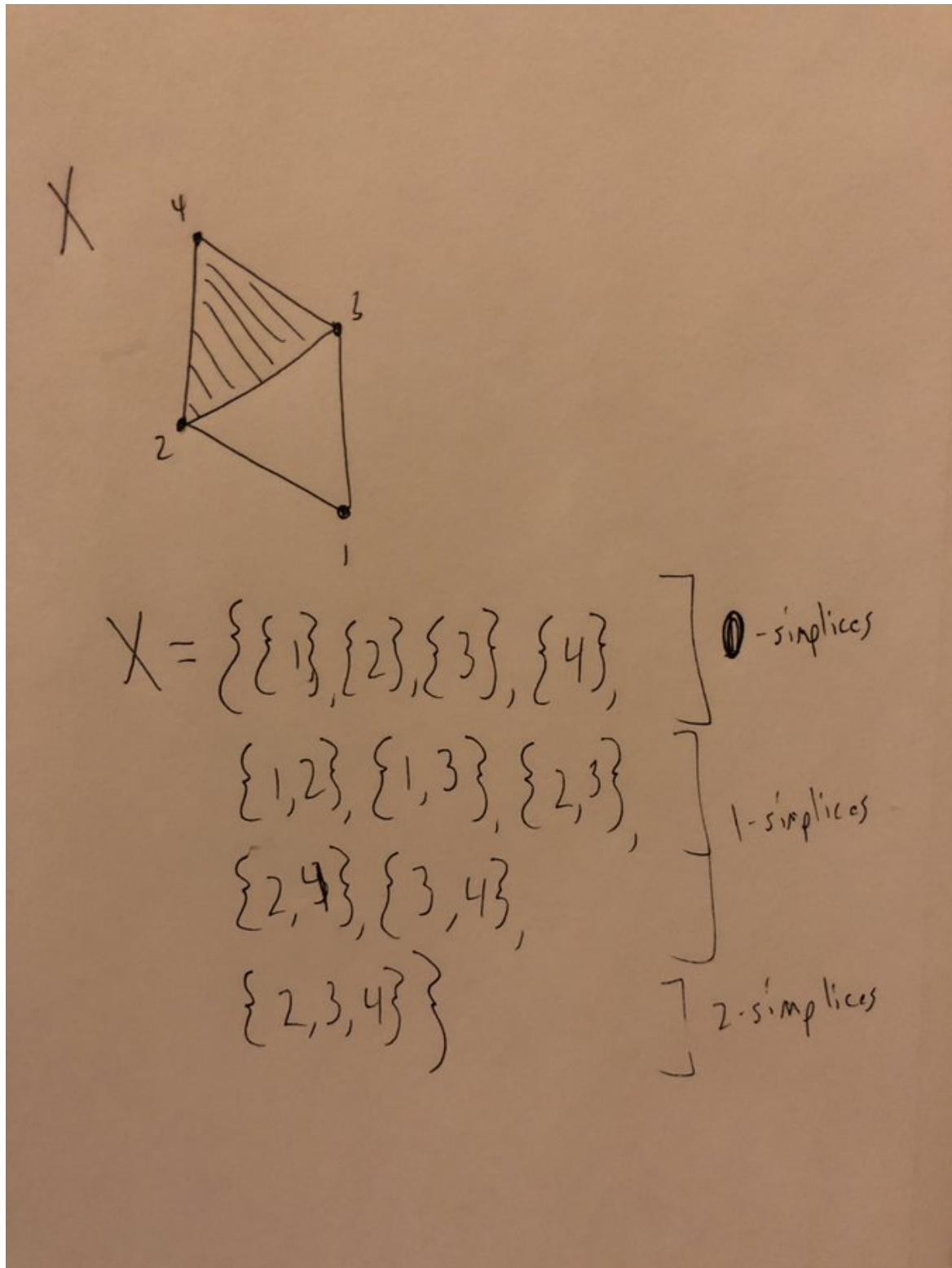
- The cofinite topology on any infinite set.
- \mathbb{R}/\mathbb{Q}
- The line with two origins.

General Spaces:

$$S^n, \mathbb{D}^n, T^n, \mathbb{RP}^n, \mathbb{CP}^n, \mathbb{M}, \mathbb{K}, \Sigma_g, \mathbb{RP}^\infty, \mathbb{CP}^\infty.$$

“Constructed” Spaces

- Knot complements in S^3
- Covering spaces (hyperbolic geometry)
- Lens spaces
- Matrix groups
- Prism spaces
- Pair of pants
- Seifert surfaces
- Surgery
- Simplicial Complexes
 - Nice minimal example:



Exotic/Pathological Spaces

- $\mathbb{H}\mathbb{P}^n$
- Dunce Cap

- Horned sphere

Operations

- Cartesian product $A \times B$
- Wedge product $A \vee B$
- Connect Sum $A \# B$
- Quotienting A/B
- Puncturing $A \setminus \{a_i\}$
- Smash product
- Join
- Cones
- Suspension
- Loop space
- Identifying a finite number of points

3.2 Alternative Topologies

- Discrete
- Cofinite
- Discrete and Indiscrete
- Uniform

The cofinite topology:

- Non-Hausdorff
- Compact

The discrete topology:

- Discrete iff points are open
- Always Hausdorff
- Compact iff finite
- Totally disconnected
- If the domain, every map is continuous

The indiscrete topology:

- Only open sets are \emptyset, X
- Non-Hausdorff
- If the codomain, every map is continuous
- Compact
- Acyclic
- Alexander duality
- Basis
 - For an R -module M , a basis B is a linearly independent generating set.
- Boundary
- Boundary of a manifold

- Points $x \in M^n$ defined by

$$\partial M = \{x \in M : H_n(M, M - \{x\}; \mathbb{Z}) = 0\}$$

- Cap Product

- Denoting $\Delta^p \xrightarrow{\sigma} X \in C_p(X; G)$, a map that sends pairs $(p\text{-chains}, q\text{-cochains})$ to $(p - q)\text{-chains}$ $\Delta^{p-q} \rightarrow X$ by

$$H_p(X; R) \times H^q(X; R) \xrightarrow{\cap} H_{p-q}(X; R)$$

$$\sigma \cap \psi = \psi(F_0^q(\sigma))F_q^p(\sigma)$$

where F_i^j is the face operator, which acts on a simplicial map σ by restriction to the face spanned by $[v_i \dots v_j]$, i.e. $F_i^j(\sigma) = \sigma|_{[v_i \dots v_j]}$.

- Cellular Homology

- CW Cell

- An n -cell of X , say e^n , is the image of a map $\Phi : B^n \rightarrow X$. That is, $e^n = \Phi(B^n)$. Attaching an n -cell to X is equivalent to forming the space $B^n \coprod_f X$ where $f : \partial B^n \rightarrow X$.

- * A 0-cell is a point.
- * A 1-cell is an interval $[-1, 1] = B^1 \subset \mathbb{R}^1$. Attaching requires a map from $S^0 = \{-1, +1\} \rightarrow X$
- * A 2-cell is a solid disk $B^2 \subset \mathbb{R}^2$ in the plane. Attaching requires a map $S^1 \rightarrow X$.
- * A 3-cell is a solid ball $B^3 \subset \mathbb{R}^3$. Attaching requires a map from the sphere $S^2 \rightarrow X$.

- Cellular Map

- A map $X \xrightarrow{f} Y$ is said to be cellular if $f(X^{(n)}) \subseteq Y^{(n)}$ where $X^{(n)}$ denotes the n -skeleton.

- Chain

- An element $c \in C_p(X; R)$ can be represented as the singular p simplex $\Delta^p \rightarrow X$.

- Chain Homotopy

- Given two maps between chain complexes $(C_*, \partial_C) \xrightarrow{f, g} (D_*, \partial_D)$, a chain homotopy is a family $h_i : C_i \rightarrow B_{i+1}$ satisfying

$$f_i - g_i = \partial_{B,i-1} \circ h_n + h_{i+1} \circ \partial_{A,i}$$

$$\begin{array}{ccccccc}
 \dots & \xleftarrow{d_{A,n-1}} & A_{n-1} & \xleftarrow{d_{A,n}} & A_n & \xleftarrow{d_{A,n+1}} & A_{n+1} & \xleftarrow{d_{A,n+2}} & \dots \\
 & \searrow h_{n-2} & \downarrow f_{n-1} & \searrow g_{n-1} & \downarrow h_{n-1} & \downarrow f_n & \downarrow g_n & \searrow h_n & \downarrow f_{n+1} & \downarrow g_{n+1} & \searrow h_{n+1} \\
 & & B_{n-1} & \xleftarrow{d_{B,n-1}} & B_n & \xleftarrow{d_{B,n}} & B_{n+1} & \xleftarrow{d_{B,n+1}} & B_{n+2} & \xleftarrow{d_{B,n+2}} & \dots
 \end{array}$$

- Chain Map

- A map between chain complexes $(C_*, \partial_C) \xrightarrow{f} (D_*, \partial_D)$ is a chain map iff each component $C_i \xrightarrow{f_i} D_i$ satisfies

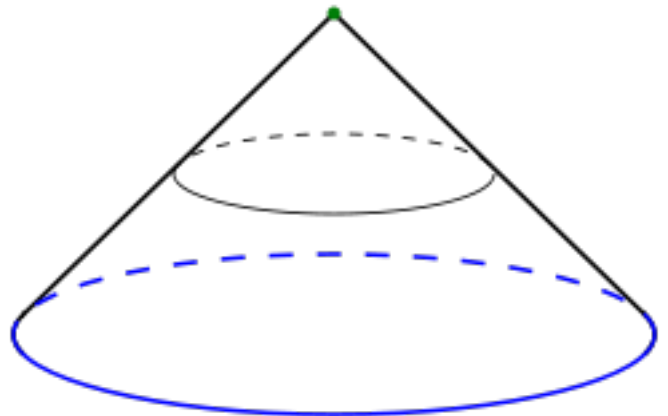
$$f_{i-1} \circ \partial_{C,i} = \partial_{D,i} \circ f_i$$

$$\begin{array}{ccccccc}
 \dots & \xleftarrow{d_{A,n-1}} & A_{n-1} & \xleftarrow{d_{A,n}} & A_n & \xleftarrow{d_{A,n+1}} & A_{n+1} \\
 & & \downarrow f_{n-1} & & \downarrow f_n & & \downarrow f_{n+1} \\
 & \xleftarrow{d_{B,n-1}} & B_{n-1} & \xleftarrow{d_{B,n}} & B_n & \xleftarrow{d_{B,n+1}} & B_{n+1}
 \end{array}$$

(i.e this forms a commuting ladder) $\dots \xleftarrow{d_{B,n-1}} B_{n-1} \xleftarrow{d_{B,n}} B_n \xleftarrow{d_{B,n+1}} B_{n+1}$

- Closed manifold
 - A manifold that is compact, with or without boundary.
- Coboundary
- Cochain
 - An cochain $c \in C^p(X; R)$ is a map $c \in \text{hom}(C_p(X; R), R)$ on chains.
- Cocycle
- Colimit
- Compact
 - A space X is compact iff every open cover of X has a finite subcover.
- Cone
 - For a space X , defined as

$$CX = \frac{X \times I}{X \times \{0\}}$$



Example: The cone on the circle CS^1

Note that the cone embeds X in a contractible space CX .

- Contractible

- A space is contractible if its identity map is nullhomotopic.
- Contractible
- Coproduct
- Covering Space
- Cup Product
 - A map taking pairs (p -cocycles, q -cocycles) to $(p + q)$ -cocycles by

$$H^p(X; R) \times H^q(X; R) \xrightarrow{\sim} H^{p+q}(X; R)$$

$$(a \cup b)(\sigma) = a(\sigma \circ I_0^p) \ b(\sigma \circ I_p^{p+q})$$

where $\Delta^{p+q} \xrightarrow{\sigma} X$ is a singular $p + q$ simplex and

$$I_i^j : [i, \dots, j] \hookrightarrow \Delta^{p+q}$$

is an embedding of the $(j - i)$ -simplex into a $(p + q)$ -simplex. On a manifold, the cup product is Poincare dual to the intersection of submanifolds.

- Applications
 - * $T^2 \not\cong S^2 \vee S^1 \vee S^1$.

Proof: todo

- CW Complex
- Cycle
- Deck Transformation
- Deformation
- Deformation Retract
 - A map r in $A \xleftarrow{\iota} X$ that is a retraction (so $r \circ \iota = \text{id}_A$) **that also satisfies** $\iota \circ r \simeq \text{id}_X$.
 - Note that this is equality in one direction, but only homotopy equivalence in the other.
- Degree of a Map
- Derived Functor
 - For a functor T and an R -module A , a *left derived functor* $(L_n T)$ is defined as $h_n(TP_A)$, where P_A is a projective resolution of A .
- Dimension of a manifold
 - For $x \in M$, the only nonvanishing homology group $H_i(M, M - \{x\}; \mathbb{Z})$
- Direct Limit
- Direct Product
- Direct Sum
- Eilenberg-MacLane Space
- Euler Characteristic

- Exact Functor

- A functor T is *right exact* if a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

yields an exact sequence

$$\dots TA \longrightarrow TB \longrightarrow TC \longrightarrow 0,$$

and is *left exact* if it yields

$$0 \longrightarrow TA \longrightarrow TB \longrightarrow TC \longrightarrow \dots$$

Thus a functor is exact iff it is both left and right exact, yielding

$$0 \longrightarrow TA \longrightarrow TB \longrightarrow TC \longrightarrow 0$$

- Examples:

$\ast \cdot \otimes_R \cdot$ is a right exact bifunctor.

- Exact Sequence

- Excision

- Ext Group

- Flat

- An R -module is flat if $A \otimes_R \cdot$ is an exact functor.

- Free and Properly Discontinuous

- Free module

- A R -module M with a basis $S = \{s_i\}$ of generating elements. Every such module is the image of a unique map $\mathcal{F}(S) = R^S \twoheadrightarrow M$, and if $M = \langle S \mid \mathcal{R} \rangle$ for some set of relations \mathcal{R} , then $M \cong R^S / \mathcal{R}$.

- Free Product

- Free product with amalgamation

- Fundamental Class

- For a connected, closed, orientable manifold, $[M]$ is a generator of $H_n(M; \mathbb{Z}) = \mathbb{Z}$.

- Fundamental classes

- Fundamental Group

- Generating Set

- $S = \{s_i\}$ is a generating set for an R -module M iff

$$x \in M \implies x = \sum r_i s_i$$

for some coefficients $r_i \in R$ (where this sum may be infinite).

- Gluing Along a Map
- Group Ring
- Homologous
- Homotopic
- Homotopy
- Homotopy Class
- Homotopy Equivalence
- Homotopy Extension Property
- Homotopy Groups
- Homotopy Lifting Property
- Injection
 - A map ι with a **left** inverse f satisfying $f \circ \iota = \text{id}$
- Intersection Pairing For a manifold M , a map on homology defined by

$$\begin{aligned} H_i M \otimes H_j M &\longrightarrow H_{i+j} X \\ \alpha \otimes \beta &\mapsto \langle \alpha, \beta \rangle \end{aligned}$$

obtained by conjugating the cup product with Poincare Duality, i.e.

$$\langle \alpha, \beta \rangle = [M] \frown ([\alpha]^\vee \smile [\beta]^\vee)$$

Then, if $[A], [B]$ are transversely intersecting submanifolds representing α, β , then

$$\langle \alpha, \beta \rangle = [A \bigcap B]$$

.

If $\hat{i} = j$ then $\langle \alpha, \beta \rangle \in H_0 M = \mathbb{Z}$ is the signed number of intersection points.

- Inverse Limit
- Intersection Pairing
 - The pairing obtained from dualizing Poincare Duality to obtain

$$F(H_i M) \otimes F(H_{n-i} M) \longrightarrow \mathbb{Z}$$

Computed as an oriented intersection number between two homology classes (perturbed to be transverse).

- Intersection Form
 - The nondegenerate bilinear form cohomology induced by the Kronecker Pairing:

$$I : H^k(M_n) \times H^{n-k}(M^n) \longrightarrow \mathbb{Z}$$

where $n = 2k$.

- * When k is odd, I is skew-symmetric and thus a *symplectic form*.
- * When k is even (and thus $n \equiv 0 \pmod{4}$) this is a symmetric form.
- * Satisfies $I(x, y) = (-1)^{k(n-k)} I(y, x)$

- Kronecker Pairing

- A map pairing a chain with a cochain, given by

$$\begin{aligned} H^n(X; R) \times H_n(X; R) &\longrightarrow R \\ ([\psi, \alpha]) &\mapsto \psi(\alpha) \end{aligned}$$

which is a nondegenerate bilinear form.

- Kronecker Product

- Lefschetz duality

- Lefschetz Number

- Lens Space

- Local Degree

- At a point $x \in V \subset M$, a generator of $H_n(V, V - \{x\})$. The degree of a map $S^n \rightarrow S^n$ is the sum of its local degrees.

- Local Orientation

- Limit

- Linear Independence

- A generating S for a module M is linearly independent if $\sum r_i s_i = 0_M \implies \forall i, r_i = 0$ where $s_i \in S, r_i \in R$.

- Local homology

- $H_n(X, X - A; \mathbb{Z})$ is the local homology at A , also denoted $H_n(X \mid A)$

- Local Homology

- Local orientation of a manifold

- At a point $x \in M^n$, a choice of a generator μ_x of $H_n(M, M - \{x\}) = \mathbb{Z}$.

- Long exact sequence

- Loop Space

- Manifold

- An n -manifold is a Hausdorff space in which each neighborhood has an open neighborhood homeomorphic to \mathbb{R}^n .

- Manifold with boundary

- A manifold in which open neighborhoods may be isomorphic to either \mathbb{R}^n or a half-space $\{\mathbf{x} \in \mathbb{R}^n \mid x_i > 0\}$.

- Mapping Cone
- Mapping Cylinder
- Mapping Path Space
- Mayer-Vietoris Sequence
- Monodromy
- Moore Space
- N-cell
- N-connected
- Nullhomotopic
 - A map $X \xrightarrow{f} Y$ is nullhomotopic if it is homotopic to a constant map $X \xrightarrow{c} \{y_0\}$; that is, there exists a homotopy
- Orientable manifold
 - A manifold for which an orientation exists, see “Orientation of a Manifold”.
- Orientation Cover
 - For any manifold M , a two sheeted orientable covering space \tilde{M}_o . M is orientable iff \tilde{M} is disconnected. Constructed as

$$\tilde{M} = \coprod_{x \in M} \left\{ \mu_x \mid \mu_x \text{ is a local orientation} \right\}$$

- Orientation of a manifold
 - A family of $\{\mu_x\}_{x \in M}$ with local consistency: if $x, y \in U$ then μ_x, μ_y are related via a propagation.
 - * Formally, a function

$$M^n \longrightarrow \coprod_{x \in M} H(X \mid \{x\})$$

$$x \mapsto \mu_x$$

such that $\forall x \exists N_x$ in which $\forall y \in N_x$, the preimage of each μ_y under the map $H_n(M \mid N_x) \rightarrow H_n(M \mid y)$ is a single generator μ_{N_x} .

- TFAE:
 - * M is orientable.
 - * The map $W : (M, x) \rightarrow \mathbb{Z}_2$ is trivial.
 - * $\tilde{M}_o = M \coprod \mathbb{Z}_2$ (two sheets).
 - * \tilde{M}_o is disconnected
 - * The projection $\tilde{M}_o \rightarrow M$ admits a section.

- Oriented manifold
- Path
- Path Lifting Property

- Perfect Pairing

- A pairing alone is an R -bilinear module map, or equivalently a map out of a tensor product since $p : M \otimes_R N \rightarrow L$ can be partially applied to yield $\varphi : M \rightarrow L^N = \text{hom}_R(N, L)$.

A pairing is **perfect** when φ is an isomorphism.

* Example: $\det : \underset{M}{k^2} \times k^2 \rightarrow k$

- Poincare Duality

- For a closed, orientable n -manifold, following map $[M] \frown \cdot$ is an isomorphism:

$$D : H^k(M; R) \rightarrow H_{n-k}(M; R)$$

$$D(\alpha) = [M] \frown \alpha$$

- Projective Resolution

- Properly Discontinuous

- Pullback

- Pushout

- Quasi-isomorphism

- R-orientability

- Relative boundaries

- Relative cycles

- Relative homotopy groups

- Retraction

- A map r in $A \xleftarrow{\iota} X$ satisfying

$$r \circ \iota = \text{id}_A.$$

Equivalently $X \twoheadrightarrow_r A$ and $r|_A = \text{id}_A$. If X retracts onto A , then i_* is injective.

- Short exact sequence

- Simplicial Complex

- Simplicial Map

- For a map

$$K \xrightarrow{f} L$$

between simplicial complexes, f is a simplicial map if for any set of vertices $\{v_i\}$ spanning a simplex in K , the set $\{f(v_i)\}$ are the vertices of a simplex in L .

- Simply Connected

- Singular Chain

$$x \in C_n(x) \implies X = \sum_i n_i \sigma_i = \sum_i n_i (\Delta^n \xrightarrow{\sigma_i} X)$$

-
- Singular Cochain

$$x \in C^n(x) \implies X = \sum_i n_i \psi_i = \sum_i n_i (\sigma_i \xrightarrow{\psi_i} X)$$

- Singular Homology
- Smash Product
- Surjection

- A map π with a **right** inverse f satisfying

$$\pi \circ f = \text{id}$$

- Suspension Compact represented as $\Sigma X = CX \coprod_{\text{id}_X} CX$, two cones on X glued along X .
Explicitly given by

$$\Sigma X = \frac{X \times I}{(X \times \{0\}) \cup (X \times \{1\}) \cup (\{x_0\} \times I)}$$

- Tor Group
 - For an R -module

$$\text{Tor}_R^n(\cdot, B) = L_n(\cdot \otimes_R B)$$

where L_n denotes the n th left derived functor.

- Universal Cover
- Universal Coefficient Theorem for Cohomology
- Universal Coefficient Theorem for Change of Coefficient Ring
- Weak Homotopy Equivalence
- Weak Topology
- Wedge Product

4 Notation

- C_X
- $\Sigma(X)$
- Σ_g
- ι, π
- $\widehat{i+j}$: for an n -dimensional manifold, the “dual” dimension $\widehat{i+j} := n - (i+j)$.