Topology Qualifying Exam Notes

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Contents

0.1 Conventions

- $\pi_0(X)$ is the set of path components of X, and I write $\pi_0(X) = \mathbb{Z}$ if X is path-connected (although it is not a group). Similarly, $H_0(X)$ is a free abelian group on the set of path components of X.
- Lists start at entry 1, since all spaces are connected here and thus $\pi_0 = H_0 = \mathbb{Z}$. That is,
  - $\pi_*(X) = [\pi_1(X), \pi_2(X), \pi_3(X), \cdots]$
  - $H_*(X) = [H_1(X), H_2(X), H_3(X), \cdots]$
- For a finite index set I, it is the case that $\prod_I G = \bigoplus_I G$ in **Grp**, i.e. the finite direct product and finite direct sum coincide. Otherwise, if I is infinite, the direct sum requires cofinitely many zero entries (i.e. finitely many nonzero entries), so here we always use $\prod$.

In other words, there is an injective map

$$\bigoplus_I G \hookrightarrow \prod_I G$$

which is an isomorphism when $|I| < \infty$

- $\mathbb{Z}^n := \prod_{i=1}^{n} \mathbb{Z} = \mathbb{Z} \times \mathbb{Z} \times \ldots \mathbb{Z}$ is the free abelian group of rank n.
  - $x \in \mathbb{Z}^n = \langle a_1, \cdots, a_n \rangle \implies x = \sum_n c_i a_i$ for some $c_i \in \mathbb{Z}$, i.e. a_i form a basis.
  - Example: $x = 2a_1 + 4a_2 + a_1 - a_2 = 3a_1 + 3a_2$.
- $\mathbb{Z}^{*n} := \mathbb{Z} * \mathbb{Z} * \ldots \mathbb{Z}$ is the free product of n free abelian groups, i.e. a free (nonabelian) group on n generators.
  - $x \in \mathbb{Z}^{*n} = \langle a_1, \ldots, a_n \rangle$ implies that x is a finite word in the noncommuting symbols a_i^k for $k \in \mathbb{Z}$.
  - Example: $x = a_1^2 a_2^4 a_1 a_2^{-2}$
- $K(G, n)$ is an Eilenberg-MacLane space, the homotopy-unique space satisfying

$$\pi_k(K(G,n)) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

  - $K(\mathbb{Z}, 1) = S^1$
  - $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$
  - $K(\mathbb{Z}_2, 1) = \mathbb{RP}^\infty$
- $M(G, n)$ is a Moore space, the homotopy-unique space satisfying

$$H_k(M(G,n); G) = \begin{cases} G & k = n, \\ 0 & k \neq n. \end{cases}$$

  - $M(\mathbb{Z}, n) = S^n$
  - $M(\mathbb{Z}_2, 1) = \mathbb{RP}^2$
  - $M(\mathbb{Z}_p, n)$ is made by attaching e^{n+1} to S^n via a degree p map.
- $T^n = \prod_n S^1$ is the n-torus
- $D(k, X)$ is the space X with $k \in \mathbb{N}$ distinct points deleted, i.e. the punctured space $X - \{x_1, x_2, \ldots x_k\}$ where each $x_i \in X$.
- $\mathbb{RP}^n = S^n / S^0 = S^n / \mathbb{Z}_2$
- $\mathbb{CP}^n = S^{2n+1} / S^1$
- $B^n = \left\{ \mathbf{v} \in \mathbb{R}^n \;\middle|\; \|\mathbf{v}\| \leq 1 \right\} \subset \mathbb{R}^n$
- $S^{n-1} = \partial B^n = \left\{ \mathbf{v} \in \mathbb{R}^n \;\middle|\; \|\mathbf{v}\| = 1 \right\} \subset \mathbb{R}^n$

1 Table of Homotopy and Homology Structures

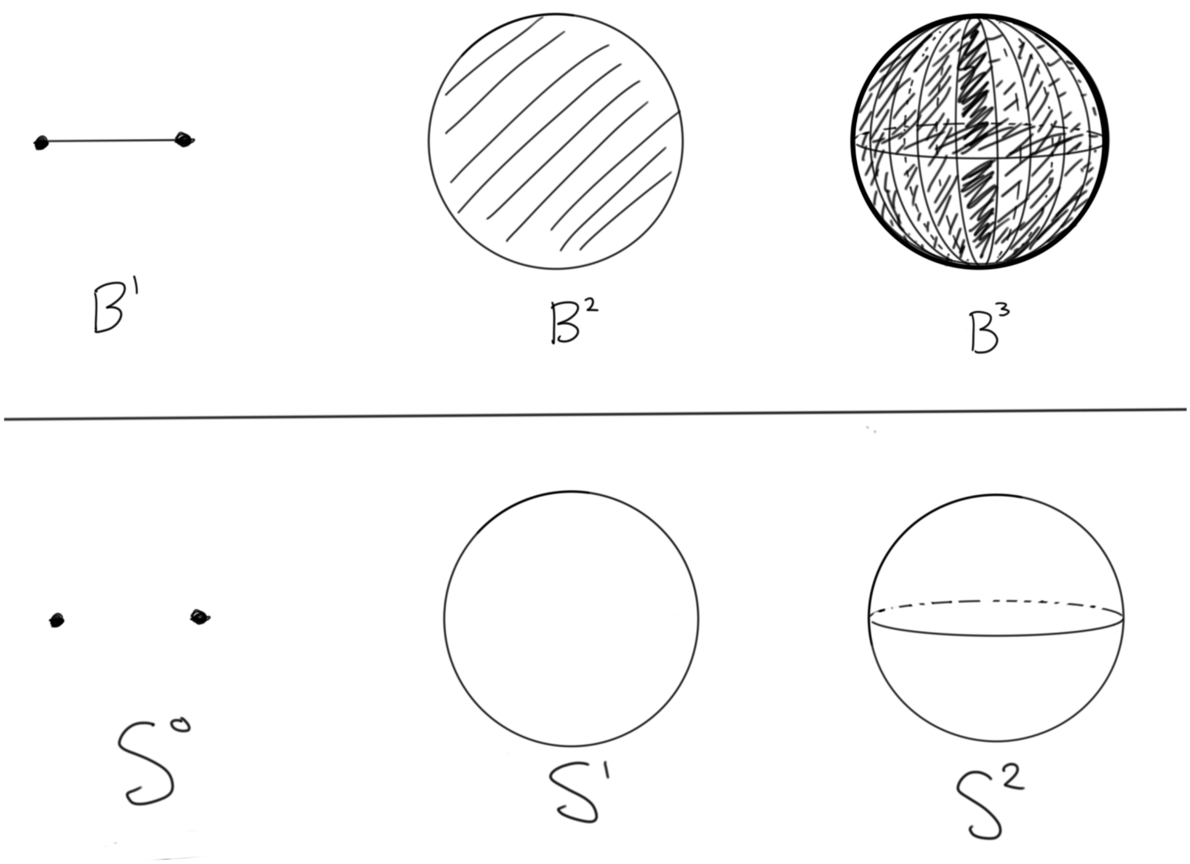


Figure 1: sphere ball correct

X	$\pi_*(X)$	$H_*(X)$	CW Structure	$H^*(X)$
$\mathbb{R}^1$	0	0	$\mathbb{Z}\cdot 1+\mathbb{Z}\cdot x$	0
$\mathbb{R}^n$	0	0	$(\mathbb{Z}\cdot 1+\mathbb{Z}\cdot x)^n$	0
$D(k,\mathbb{R}^n)$	$\pi_*\bigvee^k S^1$	$\bigoplus_k H_*M(\mathbb{Z},1)$	$1+kx$	?
B^n	$\pi_*(\mathbb{R}^n)$	$H_*(\mathbb{R}^n)$	$1+x^n+x^{n+1}$	0
S^n	$[0\ldots,\mathbb{Z},?\ldots]$	$H_*M(\mathbb{Z},n)$	$1+x^n$	$\mathbb{Z}[_nx]/(x^2)$
$D(k,S^n)$	$\pi_*\bigvee^{k-1} S^1$	$\bigoplus_{k-1} H_*M(\mathbb{Z},1)$	$1+(k-1)x^1$	?
T^2	$\pi_*S^1\times\pi_*S^1$	$(H_*M(\mathbb{Z},1))^2\times H_*M(\mathbb{Z},2)$	$1+2x+x^2$	$\Lambda(_1x_1,{}_1x_2)$
T^n	$\prod^n\pi_*S^1$	$\prod_{i=1}^n(H_*M(\mathbb{Z},i))^{\binom{n}{i}}$	$(1+x)^n$	$\Lambda(_1x_1,{}_1x_2,\ldots{}_1x_n)$
$D(k,T^n)$	$[0,0,0,0,\ldots]$?	$[0,0,0,0,\ldots]$?	$1+x$	?
$S^1\vee S^1$	$\pi_*S^1 * \pi_*S^1$	$(H_*M(\mathbb{Z},1))^2$	$1+2x$	?
$\bigvee^n S^1$	$*^n\pi_*S^1$	$\prod H_*M(\mathbb{Z},1)$	$1+x$	?
$\mathbb{RP}^1$	π_*S^1	$H_*M(\mathbb{Z},1)$	$1+x$	$_0\mathbb{Z}\times{}_1\mathbb{Z}$
$\mathbb{RP}^2$	$\pi_*K(\mathbb{Z}/2\mathbb{Z},1)+\pi_*S^2$	$H_*M(\mathbb{Z}_2,1)$	$1+x+x^2$	$_0\mathbb{Z}\times{}_2\mathbb{Z}/2\mathbb{Z}$
$\mathbb{RP}^3$	$\pi_*K(\mathbb{Z}/2\mathbb{Z},1)+\pi_*S^3$	$H_*M(\mathbb{Z}_2,1)+H_*M(\mathbb{Z},3)$	$1+x+x^2+x^3$	$_0\mathbb{Z}\times{}_2\mathbb{Z}/2\mathbb{Z}\times{}_3\mathbb{Z}$
$\mathbb{RP}^4$	$\pi_*K(\mathbb{Z}/2\mathbb{Z},1)+\pi_*S^4$	$H_*M(\mathbb{Z}_2,1)+H_*M(\mathbb{Z}_2,3)$	$1+x+x^2+x^3+x^4$	$_0\mathbb{Z}\times(_2\mathbb{Z}/2\mathbb{Z})^2$
$\mathbb{RP}^n, n\geq 4$ even	$\pi_*K(\mathbb{Z}/2\mathbb{Z},1)+\pi_*S^n$	$\prod_{\text{odd } i<n} H_*M(\mathbb{Z}_2,i)$	$\sum_{i=1}^n x^i$	$_0\mathbb{Z}\times\prod_{i=1}^{n/2}{}_2\mathbb{Z}/2\mathbb{Z}$
$\mathbb{RP}^n, n\geq 4$ odd	$\pi_*K(\mathbb{Z}/2\mathbb{Z},1)+\pi_*S^n$	$\prod_{\text{odd } i\leq n-2} H_*M(\mathbb{Z}_2,i)\times H_*S^n$	$\sum_{i=1}^n x^i$	$H^*(\mathbb{RP}^{n-1})\times{}_n\mathbb{Z}$
$\mathbb{CP}^1$	$\pi_*K(\mathbb{Z},2)+\pi_*S^3$	H_*S^2	x^0+x^2	$\mathbb{Z}[_2x]/(_2x^2)$
$\mathbb{CP}^2$	$\pi_*K(\mathbb{Z},2)+\pi_*S^5$	$H_*S^2\times H_*S^4$	$x^0+x^2+x^4$	$\mathbb{Z}[_2x]/(_2x^3)$
$\mathbb{CP}^n, n\geq 2$	$\pi_*K(\mathbb{Z},2)+\pi_*S^{2n+1}$	$\prod_{i=1}^n H_*S^{2i}$	$\sum_{i=1}^n x^{2i}$	$\mathbb{Z}[_2x]/(_2x^{n+1})$
Mobius Band	π_*S^1	H_*S^1	$1+x$	?
Klein Bottle	$K(\mathbb{Z}\rtimes_{-1}\mathbb{Z},1)$	$H_*S^1\times H_*\mathbb{RP}^\infty$	$1+2x+x^2$	?

Facts used to compute the above table:

- $\mathbb{R}^n$ is a contractible space, and so $[S^m,\mathbb{R}^n]=0$ for all n,m which makes its homotopy groups all zero.
- $D(k,\mathbb{R}^n)=\mathbb{R}^n-\{x_1\ldots x_k\}\simeq\bigvee_{i=1}^k S^1$ by a deformation retract.
- $S^n\cong B^n/\partial B^n$ and employs an attaching map

$$\phi:(D^n,\partial D^n)\longrightarrow S^n$$
$$(D^n,\partial D^n)\mapsto(e^n,e^0)$$

- $B^n\simeq\mathbb{R}^n$ by normalizing vectors.
- Use the inclusion $S^n=B^{n+1}$ as the attaching map.
- $\mathbb{CP}^1\simeq S^2$.
- $\mathbb{RP}^1\simeq S^1$.
- Use $\pi_1\prod=\prod\pi_1$ and the universal cover $\mathbb{R}^1\twoheadrightarrow S^1$ to yield the cover $\mathbb{R}^n\twoheadrightarrow T^n$.

Use the fact that $D(1, S^n) \cong \mathbb{R}^n$ and thus $D(k, S^n) \cong D(k-1, \mathbb{R}^n) \cong \bigvee^{k-1} S^1$ # Euler Characteristics - Only surfaces with positive χ: - $\chi S^2 = 2$ - $\chi \mathbb{RP}^2 = 1$ - $\chi B^2 = 1$ - Manifolds with zero χ - $T^2, K, M, S^1 \times I$ - Manifolds with negative χ - $\Sigma_{g \geq 2}$ by $\chi(X) = 2 - 2g$.

2 Useful Facts and Techniques

- Fundamental group:
  - Van Kampen
- Homotopy Groups
  - Hurewicz map
- Homology
  - Mayer-Vietoris
    - $(X = A \bigcup B) \mapsto (\bigcap, \oplus, \bigcup)$ in homology
  - LES of a pair
    - $(A \hookrightarrow X) \mapsto (A, X, X/A)$
  - Excision
- $\pi_{i \geq 2}(X)$ is always abelian.
- The ranks of π_0 and H_0 are the number of path components, and $\pi_0(X) = \mathbb{Z}$ iff X is simply connected.
  - X simply connected implies $\pi_k(X) \cong H_k(X)$ up to and including the first nonvanishing H_k
  - $H_1(X) = \pi_1 X / [\pi_1 X, \pi_1 X]$, the abelianization.
- General mantra: homotopy plays nicely with products, homology with wedge products.[1]

In general, homotopy groups behave nicely under homotopy pull-backs (e.g., fibrations and products), but not homotopy push-outs (e.g., cofibrations and wedges). Homology is the opposite.

- $\pi_k \prod X = \prod \pi_k X$ by LES.[2]
- $H_k \prod X \neq \prod H_k X$ due to torsion.

[1] More generally, in **Top**, we can look at $A \leftarrow \{\text{pt}\} \longrightarrow B$ – then $A \times B$ is the pullback and $A \vee B$ is the pushout. In this case, homology $h : \textbf{Top} \longrightarrow \textbf{Grp}$ takes pushouts to pullbacks but doesn't behave well with pullbacks. Similarly, while π takes pullbacks to pullbacks, it doesn't behave nicely with pushouts.

[2] This follows because $X \times Y \twoheadrightarrow X$ is a fiber bundle, so use LES in homotopy and the fact that $\pi_{i \geq 2} \in \textbf{Ab}$.

- Nice case: $H_k(A\times B)=\prod_{i+j=k} H_iA\otimes H_jB$ by Kunneth when all groups are torsion-free.[3]

- $H_k\bigvee X=\prod H_kX$ by Mayer-Vietoris.[4]
- $\pi_k\bigvee X\neq\prod\pi_kX$ (counterexample: $S^1\vee S^2$)
  - Nice case: $\pi_1\bigvee X=*\pi_1X$ by Van Kampen.
- $\pi_i(\widehat{X})\cong\pi_i(X)$ for $i\geq 2$ whenever $\widehat{X}\twoheadrightarrow X$ is a universal cover.
- Groups and Group Actions
  - $\pi_0(G)=G$ for G a discrete topological group.
  - $\pi_k(G/H)=\pi_k(G)$ if $\pi_k(H)=\pi_{k-1}(H)=0$.
  - $\pi_1(X/G)=\pi_0(G)$ when G acts freely/transitively on X.
- Manifolds
  - $H^n(M^n)=\mathbb{Z}$ if M^n is orientable and zero if M^n is nonorientable.
  - Poincare Duality: $H_iM^n=\cong H^{n-i}M^n$ iff M^n is closed and orientable.

3 Other Interesting Things To Consider

- The "generalized uniform bouquet"? $\mathcal{B}^n(m)=\bigvee_{i=1}^{n} S^m$
- Lie Groups
  - The real general linear group, $\mathrm{GL}_n(\mathbb{R})$
    - The real special linear group $SL_n(\mathbb{R})$
    - The real orthogonal group, $O_n(\mathbb{R})$
      - The real special orthogonal group, $SO_n(\mathbb{R})$
    - The real unitary group, $U_n(\mathbb{R})$
      - The real special unitary group, $SU_n(\mathbb{R})$
    - The real symplectic group $Sp(n)$
- "Geometric" Stuff
  - Affine n-space over a field $\mathbb{A}^n(k)=k^n\rtimes GL_n(k)$
  - The projective space $\mathbb{P}^n(k)$
    - The projective linear group over a ring R, $PGL_n(R)$
    - The projective special linear group over a ring R, $PSL_n(R)$

[3] The generalization of Kunneth is as follows: write $\mathcal{P}(n,k)$ be the set of partitions of n into k parts, i.e. $\backsim\in\mathcal{P}(n,k)\implies\backsim=(x_1,x_2,\ldots,x_k)$ where $\sum x_i=n$. Then

$$H_n(\prod_{j=1}^{k}X_j)=\bigoplus_{\backsim\in\mathcal{P}(n,k)}\bigotimes_{i=1}^{k}H_{x_i}(X_i).$$

[4] $\bigvee$ is the coproduct in the category $\mathbf{Top}_0$ of pointed topological spaces, and alternatively, $X\vee Y$ is the pushout in $\mathbf{Top}$ of $X\leftarrow\{\mathrm{pt}\}\longrightarrow Y$

    * The modular groups $PSL_n(\mathbb{Z})$
        · Specifically $PSL_2(\mathbb{Z})$

- The real Grassmannian, $Gr(n, k, \mathbb{R})$, i.e. the set of k dimensional subspaces of $\mathbb{R}^n$
- The Stiefel manifold $V_n(k)$
- Possible modifications to a space X:
  - Remove k points by taking $D(k, X)$
  - Remove a line segment
  - Remove an entire line/axis
  - Remove a hole
  - Quotient by a group action (e.g. antipodal map, or rotation)
  - Remove a knot
  - Take complement in ambient space
- Assorted info about other Lie Groups:
- $O_n, U_n, SO_n, SU_n, Sp_n$
- $\pi_k(U_n) = \mathbb{Z} \cdot \mathbb{1}[k \text{ odd}]$
  - $\pi_1(U_n) = 1$
- $\pi_k(SU_n) = \mathbb{Z} \cdot \mathbb{1}[k \text{ odd}]$
  - $\pi_1(SU_n) = 0$
- $\pi_k(U_n) = \mathbb{Z}_2 \cdot \mathbb{1}[k = 0, 1 \mod 8] + \mathbb{Z} \cdot \mathbb{1}[k = 3, 7 \mod 8]$
- $\pi_k(SP_n) = \mathbb{Z}_2 \cdot \mathbb{1}[k = 4, 5 \mod 8] + \mathbb{Z} \cdot \mathbb{1}[k = 3, 7 \mod 8]$

4 Spheres

- $\pi_i(S^n) = 0$ for $i < n$, $\pi_n(S^n) = \mathbb{Z}$
  - Not necessarily true that $\pi_i(S^n) = 0$ when $i > n$!!!
    * E.g. $\pi_3(S^2) = \mathbb{Z}$ by Hopf fibration
- $H_i(S^n) = \mathbb{1}[i \in \{0, n\}]$
- $H_n(\bigvee_i X_i) \cong \prod_i H_n(X_i)$ for "good pairs"
  - Corollary: $H_n(\bigvee_k S^n) = \mathbb{Z}^k$
- $S^n/S^k \simeq S^n \vee \Sigma S^k$
  - $\Sigma S^n = S^{n+1}$
- S^n has the CW complex structure of 2 k-cells for each $0 \leq k \leq n$.

5 Definitions

- Topology: Closed under arbitrary unions and finite intersections.
- Basis: A subset $\{B_i\}$ is a basis iff
  - $x \in X \implies x \in B_i$ for some i.
  - $x \in B_i \bigcap B_j \implies x \in B_k \subset B_i \bigcap B_k$.
  - Topology generated by this basis: $x \in N_x \implies x \in B_i \subset N_x$ for some i.
- Dense: A subset $Q \subset X$ is dense iff $y \in N_y \subset X \implies N_y \bigcap Q \neq \emptyset$ iff $\overline{Q} = X$.
- Neighborhood: A neighborhood of a point x is any open set containing x.
- Hausdorff
- Second Countable: admits a countable basis.
- Closed (several characterizations)
- Closure in a subspace: $Y \subset X \implies \mathrm{cl}_Y(A) := \mathrm{cl}_X(A) \bigcap Y$.
- Bounded
- Compact: A topological space (X, τ) is **compact** if every open cover has a *finite* subcover.

  That is, if $\left\{U_j \;\middle|\; j \in J\right\} \subset \tau$ is a collection of open sets such that $X \subseteq \bigcup_{j \in J} U_j$, then there exists a *finite* subset $J' \subset J$ such that $X \subseteq \bigcup_{j \in J'} U_j$.
- Locally compact For every $x \in X$, there exists a $K_x \ni x$ such that K_x is compact.
- Connected: There does not exist a disconnecting set $X = A \coprod B$ such that $\emptyset \neq A, B \subsetneq$, i.e. X is the union of two proper disjoint nonempty sets.

  Equivalently, X contains no proper nonempty clopen sets.
  - Additional condition for a subspace $Y \subset X$: $\mathrm{cl}_Y(A) \bigcap V = A \bigcap \mathrm{cl}_Y(B) = \emptyset$.
- Locally connected: A space is locally connected at a point x iff $\forall N_x \ni x$, there exists a $U \subset N_x$ containing x that is connected.
- Retract: A subspace $A \subset X$ is a *retract* of X iff there exists a continuous map $f : X \longrightarrow A$ such that $f \Big|_A = \mathrm{id}_A$. Equivalently it is a *left* inverse to the inclusion.
- Uniform Continuity: For $f : (X, d_x) \longrightarrow (Y, d_Y)$ metric spaces,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \quad d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

- Lebesgue number: For (X, d) a compact metric space and $\{U_\alpha\} \rightrightarrows X$, there exist $\delta_L > 0$ such that

$$A \subset X,\ \mathrm{diam}(A) < \delta_L \implies A \subseteq U_\alpha \text{ for some } \alpha.$$

- Paracompact

- Components: Set $x \sim y$ iff there exists a connected set $U \ni x, y$ and take equivalence classes.
- Path Components: Set $x \sim y$ iff there exists a path-connected set $U \ni x, y$ and take equivalence classes.
- Separable: Contains a countable dense subset.
- Limit Point: For $A \subset X$, x is a limit point of A if every punctured neighborhood P_x of x satisfies $P_x \bigcap A \neq \emptyset$, i.e. every neighborhood of x intersects A in some point other than x itself.

  Equivalently, x is a limit point of A iff $x \in \mathrm{cl}_X(A \setminus \{x\})$.

6 Examples

6.1 Common Spaces and Operations

Point-Set:

- Finite discrete sets with the discrete topology
- Subspaces of $\mathbb{R}$: $(a, b), (a, b], (a, \infty)$, etc.
  - $\{0\} \bigcup \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{Z}^{\geq 1} \right\}$
- $\mathbb{Q}$
- The topologist's sine curve
- One-point compactifications
- $\mathbb{R}^\omega$
- Hawaiian earring
- Cantor set

Non-Hausdorff spaces:

- The cofinite topology on any infinite set.
- $\mathbb{R}/\mathbb{Q}$
- The line with two origins.

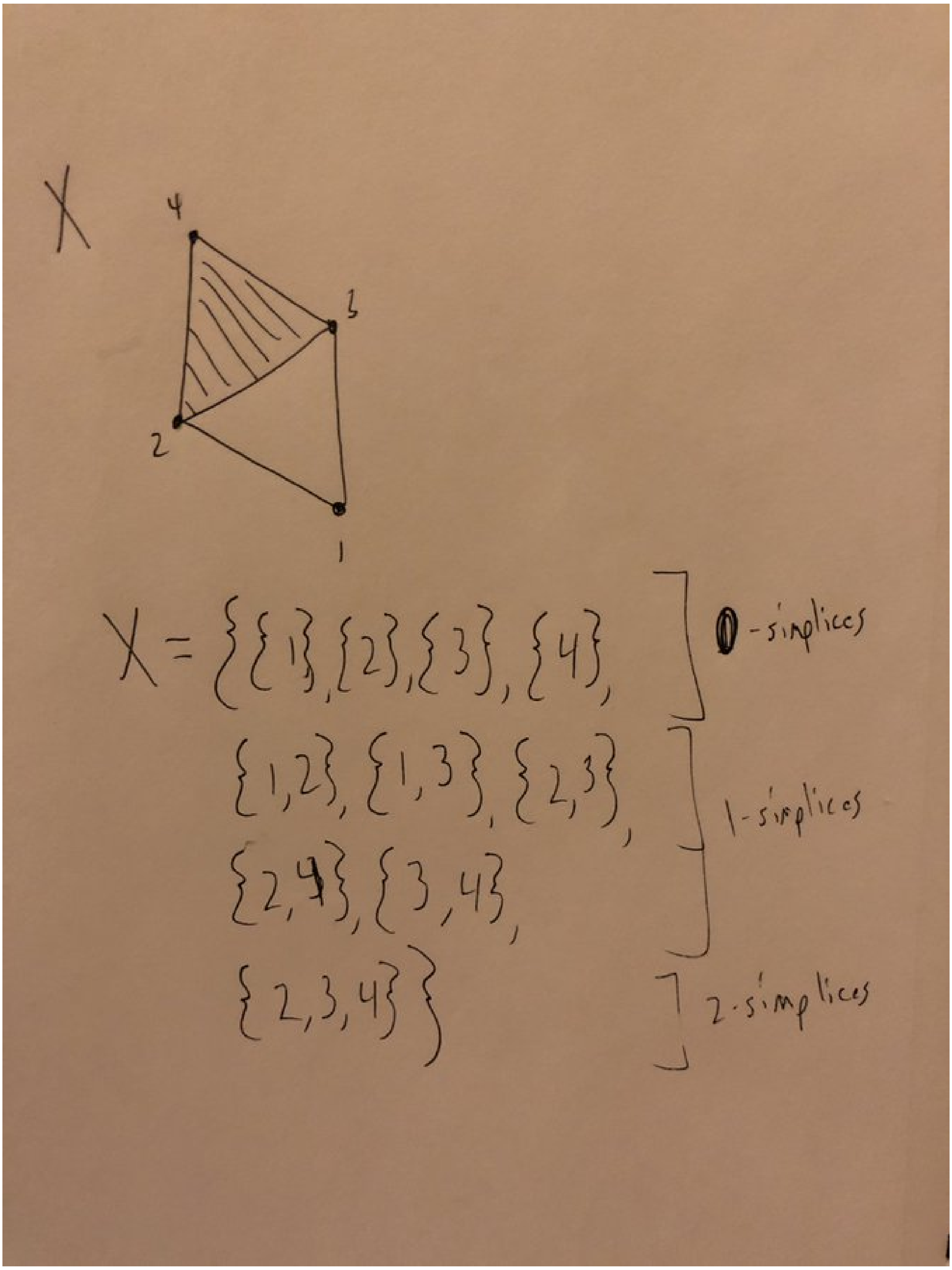
General Spaces:

$$S^n, \mathbb{D}^n, T^n, \mathbb{RP}^n, \mathbb{CP}^n, \mathbb{M}, \mathbb{K}, \Sigma_g, \mathbb{RP}^\infty, \mathbb{CP}^\infty.$$

"Constructed" Spaces

- Knot complements in S^3
- Covering spaces (hyperbolic geometry)
- Lens spaces
- Matrix groups
- Prism spaces
- Pair of pants
- Seifert surfaces

- Surgery
- Simplicial Complexes
    - Nice minimal example:



Exotic/Pathological Spaces

- $\mathbb{HP}^n$
- Dunce Cap
- Horned sphere

Operations

- Cartesian product $A \times B$
- Wedge product $A \vee B$
- Connect Sum $A \# B$
- Quotienting A/B
- Puncturing $A \setminus \{a_i\}$
- Smash product
- Join
- Cones
- Suspension
- Loop space
- Identifying a finite number of points

6.2 Alternative Topologies

- Discrete
- Cofinite
- Discrete and Indiscrete
- Uniform

The cofinite topology:

- Non-Hausdorff
- Compact

The discrete topology:

- Discrete iff points are open
- Always Hausdorff
- Compact iff finite
- Totally disconnected
- If the domain, every map is continuous

The indiscrete topology:

- Only open sets are $\emptyset, X$
- Non-Hausdorff
- If the codomain, every map is continuous
- Compact

7 Theorems

7.1 Point-Set

- Closed subsets of Hausdorff spaces are compact? (check)

- Cantor's intersection theorem?
- Tube lemma
- Properties pushed forward through continuous maps:
  - Compactness?
  - Connectedness (when surjective)
  - Separability
  - Density **only when** f is surjective
  - **Not** openness
  - **Not** closedness
- A retract of a Hausdorff/connected/compact space is closed/connected/compact respectively.

Proposition 7.1.
A continuous function on a compact set is uniformly continuous.

Proof .
Take $\left\{ B_{\frac{\varepsilon}{2}}(y) \;\middle|\; y \in Y \right\} \rightrightarrows Y$, pull back to an open cover of X, has Lebesgue number $\delta_L > 0$, then $x' \in B_{\delta_L}(x) \implies f(x), f(x') \in B_{\frac{\varepsilon}{2}}(y)$ for some y. ∎

- Lipschitz continuity implies uniform continuity (take $\delta = \varepsilon/C$)
  - Counterexample to converse: $f(x) = \sqrt{x}$ on $[0,1]$ has unbounded derivative.
- Extreme Value Theorem: for $f : X \longrightarrow Y$ continuous with X compact and Y ordered in the order topology, there exist points $c, d \in X$ such that $f(x) \in [f(c), f(d)]$ for every x.

Theorem 7.2.
Points are closed in T_1 spaces.

Theorem 7.3.
A metric space X is sequentially compact iff it is complete and totally bounded.

Theorem 7.4.
A metric space is totally bounded iff every sequence has a Cauchy subsequence.

Theorem 7.5.
A metric space is compact iff it is complete and totally bounded.

Theorem 7.6 ***(Baire)*****.**
If X is a complete metric space, then the intersection of countably many dense open sets is dense in X.

Theorem 7.7.
A continuous bijective open map is a homeomorphism.

Theorem 7.8.
A closed subset A of a compact set B is compact.

Proof .

- Let $\{A_i\} \rightrightarrows A$ be a covering of A by sets open in A.
- Each $A_i = B_i \bigcap A$ for some B_i open in B (definition of subspace topology)
- Define $V = \{B_i\}$, then $V \rightrightarrows A$ is an open cover.
- Since A is closed, $W := B \setminus A$ is open
- Then $V \bigcup W$ is an open cover of B, and has a finite subcover $\{V_i\}$
- Then $\left\{V_i \bigcap A\right\}$ is a finite open cover of A.

■

Theorem 7.9.
The continuous image of a compact set is compact.

Theorem 7.10.
A closed subset of a Hausdorff space is compact.

7.2 Algebraic

Todo: Merge the two van Kampen theorems.

Theorem 7.11 ***(Van Kampen).***
The pushout is the northwest colimit of the following diagram

$$\begin{tikzcd} A\coprod_Z B & A \arrow[l] \\ B \arrow[r, leftrightarrow, "\iota_B"'] & Z \arrow[u, leftrightarrow, "\iota_A"] \end{tikzcd}$$

For groups, the pushout is given by the amalgamated free product: if $A = \left\langle G_A \;\middle|\; R_A \right\rangle, B = \left\langle G_B \;\middle|\; R_B \right\rangle$, then $A *_Z B = \left\langle G_A, G_B \;\middle|\; R_A, R_B, T \right\rangle$ where T is a set of relations given by $T = \left\{\iota_A(z)\iota_B(z)^{-1} \;\middle|\; z \in Z\right\}$.
Example: $A = \mathbb{Z}/4\mathbb{Z} = \left\langle x \;\middle|\; x^4 \right\rangle, B = \mathbb{Z}/6\mathbb{Z} = \left\langle y \;\middle|\; x^6 \right\rangle, Z = \mathbb{Z}/2\mathbb{Z} = \left\langle z \;\middle|\; z^2 \right\rangle$. Then we can identify Z as a subgroup of A, B using $\iota_A(z) = x^2$ and $\iota_B(z) = y^3$. So

$$A *_Z B = \left\langle x, y \;\middle|\; x^4, y^6, x^2y^{-3} \right\rangle$$

.
Suppose $X = U_1 \bigcup U_2$ such that $U_1 \bigcap U_2 \neq \emptyset$ is path connected. Then taking $x_0 \in U := U_1 \bigcap U_2$ yields a pushout of fundamental groups

$$\pi_1(X; x_0) = \pi_1(U_1; x_0) *_{\pi_1(U; x_0)} \pi_1(U_2; x_0).$$

Theorem 7.12 ***(Van Kampen).***
If $X = U \bigcup V$ where $U, V, U \bigcap V$ are all path-connected then

$$\pi_1(X) = \pi_1 U *_{\pi_1(U \bigcap V)} \pi_1 V,$$

where the amalgamated product can be computed as follows: If we have presentations

$$\begin{aligned} \pi_1(U, w) &= \left\langle u_1, \cdots, u_k \,\middle|\, \alpha_1, \cdots, \alpha_l \right\rangle \\ \pi_1(V, w) &= \left\langle v_1, \cdots, v_m \,\middle|\, \beta_1, \cdots, \beta_n \right\rangle \\ \pi_1(U \cap V, w) &= \left\langle w_1, \cdots, w_p \,\middle|\, \gamma_1, \cdots, \gamma_q \right\rangle \end{aligned}$$

then

$$\begin{aligned} \pi_1(X, w) = &\langle u_1, \cdots, u_k, v_1, \cdots, v_m \rangle \\ &\quad \bmod \left\langle \alpha_1, \cdots, \alpha_l, \beta_1, \cdots, \beta_n, I\left(w_1\right) J\left(w_1\right)^{-1}, \cdots, I\left(w_p\right) J\left(w_p\right)^{-1} \right\rangle \\ = &\frac{\pi_1(U) * \pi_1(B)}{\left\langle \left\{ I(w_i) J(w_i)^{-1} \,\middle|\, 1 \leq i \leq p \right\} \right\rangle} \end{aligned}$$

where

$$\begin{aligned} I &: \pi_1(U \cap V, w) \to \pi_1(U, w) \\ J &: \pi_1(U \cap V, w) \to \pi_1(V, w). \end{aligned}$$