

# Algebra Problems

UGA

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# 1 Problem Set One

## 1.1 Exercises

**Problem 1.1** (Hungerford 1.6.3).

If  $\sigma = (i_1 i_2 \cdots i_r) \in S_n$  and  $\tau \in S_n$ , then show that  $\tau \sigma \tau^{-1} = (\tau(i_1) \tau(i_2) \cdots \tau(i_r))$ .

**Problem 1.2** (Hungerford 1.6.4).

Show that  $S_n \cong \langle (12), (123 \cdots n) \rangle$  and also that  $S_n \cong \langle (12), (23 \cdots n) \rangle$

**Problem 1.3** (Hungerford 2.2.1).

Let  $G$  be a finite abelian group that is not cyclic. Show that  $G$  contains a subgroup isomorphic to  $\mathbb{Z}_p \oplus \mathbb{Z}_p$  for some prime  $p$ .

**Problem 1.4** (Hungerford 2.2.12.b).

Determine (up to isomorphism) all abelian groups of order 64; do the same for order 96.

**Problem 1.5** (Hungerford 2.4.1).

Let  $G$  be a group and  $A \trianglelefteq G$  be a normal abelian subgroup. Show that  $G/A$  acts on  $A$  by conjugation and construct a homomorphism  $\varphi : G/A \rightarrow \text{Aut}(A)$ .

**Problem 1.6** (Hungerford 2.4.9).

Let  $Z(G)$  be the center of  $G$ . Show that if  $G/Z(G)$  is cyclic, then  $G$  is abelian.

*Note that Hungerford uses the notation  $C(G)$  for the center.*

**Problem 1.7** (Hungerford 2.5.6).

Let  $G$  be a finite group and  $H \trianglelefteq G$  a normal subgroup of order  $p^k$ . Show that  $H$  is contained in every Sylow  $p$ -subgroup of  $G$ .

**Problem 1.8** (Hungerford 2.5.9).

Let  $|G| = p^n q$  for some primes  $p > q$ . Show that  $G$  contains a unique normal subgroup of index  $q$ .

## 1.2 Qual Problems

### Problem 1.9.

Let  $G$  be a finite group and  $p$  a prime number. Let  $X_p$  be the set of Sylow- $p$  subgroups of  $G$  and  $n_p$  be the cardinality of  $X_p$ . Let  $\text{Sym}(X)$  be the permutation group on the set  $X_p$ .

1. Construct a homomorphism  $\rho : G \rightarrow \text{Sym}(X_p)$  with image a transitive subgroup (i.e. with a single orbit).
2. Deduce that if  $G$  is simple then the order of  $G$  divides  $n_p!$ .
3. Show that for any  $1 \leq a \leq 4$  and any prime power  $p^k$ , no group of order  $ap^k$  is simple.

### Problem 1.10.

Let  $G$  be a finite group and let  $N \trianglelefteq G$ , and let  $p$  be a prime number and  $Q$  a subgroup of  $G$  such that  $N \subset Q$  and  $Q/N$  is a Sylow  $p$ -subgroup of  $G/N$ .

1. Prove that  $Q$  contains a Sylow  $p$ -subgroup of  $G$ .
2. Prove that every Sylow  $p$ -subgroup of  $G/N$  is the image of a Sylow  $p$ -subgroup of  $G$ .

### Problem 1.11.

Let  $G$  be a finite group and  $H < G$  a subgroup. Let  $n_H$  be the number of subgroups of  $G$  that are conjugate to  $H$ . Show that  $n_H$  divides the order of  $G$ .

### Problem 1.12.

Let  $G = S_5$ , the symmetric group on 5 elements. Identify all conjugacy classes of elements in  $G$ , provide a representative from each class, and prove that this list is complete.

## 2 Problem Set Two

### 2.1 Exercises

**Problem 2.1** (Hungerford 2.1.9).

Let  $G$  be a finitely generated abelian group in which no element (except 0) has finite order. Show that  $G$  is a free abelian group.

**Problem 2.2** (Hungerford 2.1.10).

1. Show that the additive group of rationals  $\mathbb{Q}$  is not finitely generated.
2. Show that  $\mathbb{Q}$  is not free.
3. Conclude that Exercise 9 is false if the hypothesis “finitely generated” is omitted.

**Problem 2.3** (Hungerford 2.5.8).

Show that if every Sylow  $p$ -subgroup of a finite group  $G$  is normal for every prime  $p$ , then  $G$  is the direct product of its Sylow subgroups.

**Problem 2.4** (Hungerford 2.6.4).

What is the center of the quaternion group  $Q_8$ ? Show that  $Q_8/Z(Q_8)$  is abelian.

**Problem 2.5** (Hungerford 2.6.9).

Classify up to isomorphism all groups of order 18. Do the same for orders 20 and 30.

**Problem 2.6** (Hungerford 1.9.1).

Show that every non-identity element in a free group  $F$  has infinite order.

**Problem 2.7** (Hungerford 1.9.3).

Let  $F$  be a free group and for a fixed integer  $n$ , let  $H_n$  be the subgroup generated by the set  $\{x^n \mid x \in F\}$ . Show that  $H_n \trianglelefteq F$ .

## 2.2 Qual Problems

**Problem 2.8.**

List all groups of order 14 up to isomorphism.

**Problem 2.9.**

Let  $G$  be a group of order  $p^3$  for some prime  $p$ . Show that either  $G$  is abelian, or  $|Z(G)| = p$ .

**Problem 2.10.**

Let  $p, q$  be distinct primes, and let  $k$  denote the smallest positive integer such that  $p$  divides  $q^k - 1$ . Show that no group of order  $pq^k$  is simple.

**Problem 2.11.**

Show that  $S_4$  is a solvable, nonabelian group.

## 3 Problem Set Three

### 3.1 Exercises

**Problem 3.1** (Hungerford 2.7.10).

Show that  $S_n$  is solvable for  $n \leq 4$  but  $S_3$  and  $S_4$  are not nilpotent.

**Problem 3.2** (Hungerford 2.8.3).

Show that if  $N$  is a simple normal subgroup of a group  $G$  and  $G/N$  has a composition series, then  $G$  has a composition series.

**Problem 3.3** (Hungerford 2.8.9).

Show that any group of order  $p^2q$  (for primes  $p, q$ ) is solvable.

**Problem 3.4** (Hungerford 5.1.1).

Let  $F/K$  be a field extension. Show that

1.  $[F : K] = 1$  iff  $F = K$ .
2. If  $[F : K]$  is prime, then there are no intermediate fields between  $F$  and  $K$ .
3. If  $u \in F$  has degree  $n$  over  $K$ , then  $n$  divides  $[F : K]$ .

**Problem 3.5** (Hungerford 5.1.8).

Show that if  $u \in F$  is algebraic of odd degree over  $K$ , then so is  $u^2$ , and moreover  $K(u) = K(u^2)$ .

**Problem 3.6** (Hungerford 5.1.14). 1. If  $F = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ , compute  $[F : \mathbb{Q}]$  and find a basis of  $F/\mathbb{Q}$ .

2. Do the same for  $\mathbb{Q}(i, \sqrt{3}, \zeta_3)$  where  $\zeta_3$  is a complex third root of 1.

**Problem 3.7** (Hungerford 5.1.16).

Show that in  $\mathbb{C}$ , the fields  $\mathbb{Q}(i) \cong \mathbb{Q}(\sqrt{2})$  as vector spaces, but not as fields.

## 3.2 Qual Problems

### Problem 3.8.

Let  $R$  and  $S$  be commutative rings with multiplicative identity.

1. Prove that when  $R$  is a field, every non-zero ring homomorphism  $\phi : R \rightarrow S$  is injective.
2. Does (a) still hold if we only assume that  $R$  is a domain? If so, prove it, and if not provide a counterexample.

### Problem 3.9.

Determine for which integers the ring  $\mathbb{Z}/n\mathbb{Z}$  is a direct sum of fields. Carefully prove your answer.

### Problem 3.10.

Suppose that  $R$  is a commutative ring. Show that an element  $r \in R$  is not invertible iff it is contained in a maximal ideal.

### Problem 3.11.

1. Give the definition that a group  $G$  must satisfy to be solvable.
2. Show that every group  $G$  of order 36 is solvable.

*Hint: You may assume that  $S^4$  is solvable.*



## 4 Problem Set Four

### 4.1 Exercises

**Problem 4.1** (Hungerford 5.3.7).

If  $F$  is algebraically closed and  $E$  is the set of all elements in  $F$  that are algebraic over a field  $K$ , then  $E$  is an algebraic closure of  $K$ .

**Problem 4.2** (Hungerford 5.3.8).

Show that no finite field is algebraically closed.

Hint: if  $K = \{a_i\}_{i=0}^n$ , consider

$$f(x) = a_1 + \prod_{i=0}^n (x - a_i) \in K[x]$$

where  $a_1 \neq 0$ .

**Problem 4.3** (Hungerford 5.5.2).

Show that if  $p \in \mathbb{Z}$  is prime, then  $a^p = a$  for all  $a \in \mathbb{Z}_p$ , or equivalently  $c^p \equiv c \pmod{p}$  for all  $c \in \mathbb{Z}$ .

**Problem 4.4** (Hungerford 5.5.3).

Show that if  $|K| = p^n$ , then every element of  $K$  has a unique  $p$ th root in  $K$ .

**Problem 4.5** (Hungerford 5.5.10).

Show that every element in a finite field can be written as the sum of two squares.

**Problem 4.6** (Hungerford 5.6.1).

Let  $F/K$  be a field extension. Let  $\text{char}K = p \neq 0$  and let  $n \geq 1$  be an integer such that  $(p, n) = 1$ . If  $v \in F$  and  $nv \in K$ , then  $v \in K$ .

**Problem 4.7** (Hungerford 5.6.8).

If  $\text{char}K = p \neq 0$  and  $[F : K]$  is finite and not divisible by  $p$ , then  $F$  is separable over  $K$ .

## 4.2 Qual Problems

### Problem 4.8.

Suppose that  $\alpha$  is a root in  $\mathbb{C}$  of  $P(x) = x^{17} - 2$ . How many field homomorphisms are there from  $\mathbb{Q}(\alpha)$  to:

1.  $\mathbb{C}$ ,
2.  $\mathbb{R}$ ,
3.  $\overline{\mathbb{Q}}$ , the algebraic closure of  $\mathbb{Q}$ ?

### Problem 4.9.

Let  $C/F$  be an algebraic field extension. Prove that the following are equivalent:

1. Every non-constant polynomial  $f \in F[x]$  factors into linear factors over  $C[x]$ .
2. For every (not necessarily finite) algebraic extension  $E/F$ , there is a ring homomorphism  $\alpha : E \rightarrow C$  such that  $\alpha|_F$  is the identity on  $F$ .

Hint: use Zorn's Lemma.

### Problem 4.10.

Let  $R$  be a commutative ring containing a field  $k$ , and suppose that  $\dim_k R < \infty$ . Let  $\alpha \in R$ .

1. Show that there exist  $n \in \mathbb{N}$  and  $\{c_0, c_1, \dots, c_{n-1}\} \subseteq k$  such that

$$a^n + c_{n-1}a^{n-1} + \dots + c_1a + c_0 = 0.$$

2. Suppose that (a) holds and show that if  $c_0 \neq 0$  then  $a$  is a unit in  $R$ .
3. Suppose that (a) holds and show that if  $a$  is not a zero divisor in  $R$ , then  $a$  is invertible.

## 5 Problem Set Five

### 5.1 Exercises

**Problem 5.1** (Hungerford 5.3.5).

Show that if  $f \in K[x]$  has degree  $n$  and  $F$  is a splitting field of  $f$  over  $K$ , the  $[F : K]$  divides  $n!$ .

**Problem 5.2** (Hungerford 5.3.12).

Let  $E$  be an intermediate field extension in  $K \leq E \leq F$ .

1. Show that if  $u \in F$  is separable over  $K$ , then  $u$  is separable over  $E$ .
2. Show that if  $F$  is separable over  $K$ , then  $F$  is separable over  $E$  and  $E$  is separable over  $K$ .

**Problem 5.3** (Hungerford 5.3.13).

Show that if  $[F : K] < \infty$ , then the following conditions are equivalent:

1.  $F$  is Galois over  $K$
2.  $F$  is separable over  $K$  and  $F$  is a splitting field of some polynomial  $f \in K[x]$ .
3.  $F$  is a splitting field over  $K$  of some polynomial  $f \in K[x]$  whose irreducible factors are separable.

**Problem 5.4** (Hungerford 5.4.1).

Suppose that  $f \in K[x]$  splits in  $F$  as

$$f = \prod_{i=1}^k (x - u_i)^{n_i}$$

with the  $u_i$  distinct and each  $n_i \geq 1$ . Let

$$g(x) = \prod_{i=1}^k (x - u_i) = \sum_{i=1}^k v_i x^i$$

and let  $E = K(\{v_i\}_{i=1}^k)$ . Then show that the following hold:

1.  $F$  is a splitting field of  $g$  over  $E$ .
2.  $F$  is Galois over  $E$ .
3.  $\text{Aut}_E(F) = \text{Aut}_K(F)$ .

**Problem 5.5** (Hungerford 5.4.10 a/g/h).

Determine the Galois groups of the following polynomials over the corresponding fields:

1.  $x^4 - 5$  over  $\mathbb{Q}, \mathbb{Q}(\sqrt{5}), \mathbb{Q}(i\sqrt{5})$ .
2.  $x^3 - 2$  over  $\mathbb{Q}$ .
3.  $(x^3 - 2)(x^2 - 5)$  over  $\mathbb{Q}$ .

**Problem 5.6** (Hungerford 5.6.11).

If  $f \in K[x]$  is irreducible of degree  $m > 0$  and  $\text{char}(K)$  does not divide  $m$ , then  $f$  is separable.

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## 5.2 Qual Problems

### Problem 5.7.

Let  $E/F$  be a Galois field extension, and let  $K/F$  be an intermediate field of  $E/F$ . Show that  $K$  is normal over  $F$  iff  $\text{Gal}(E/K) \trianglelefteq \text{Gal}(E/F)$ .

### Problem 5.8.

Let  $F \subset L$  be fields such that  $L/F$  is a Galois field extension with Galois group equal to  $D_8 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1, \sigma\tau = \tau\sigma^3 \rangle$ . Show that there are fields  $F \subset E \subset K \subset L$  such that  $E/F$  and  $K/E$  are Galois field extensions, but  $K/F$  is not Galois.

### Problem 5.9.

Let  $f(x) = x^3 - 7$ .

1. Let  $K$  be the splitting field for  $f$  over  $\mathbb{Q}$ . Describe the Galois group of  $K/\mathbb{Q}$  and the intermediate fields between  $\mathbb{Q}$  and  $K$ . Which intermediate fields are not Galois over  $\mathbb{Q}$ ?
2. Let  $L$  be the splitting field for  $f$  over  $\mathbb{R}$ . What is the Galois group  $L/\mathbb{R}$ ?
3. Let  $M$  be the splitting field for  $f$  over  $\mathbb{F}_{13}$ , the field with 13 elements. What is the Galois group of  $M/\mathbb{F}_{13}$ ?

## 6 Problem Set Six

### 6.1 Exercises

**Problem 6.1** (Hungerford 5.4.11).

Determine all subgroups of the Galois group and all intermediate fields of the splitting (over  $\mathbb{Q}$ ) of the polynomial  $(x^3 - 2)(x^2 - 3) \in \mathbb{Q}[x]$ .

**Problem 6.2** (Hungerford 5.4.12).

Let  $K$  be a subfield of  $\mathbb{R}$  and let  $f \in K[x]$  be an irreducible quartic. If  $f$  has exactly 2 real roots, the Galois group of  $f$  is either  $S_4$  or  $D_4$ .

**Problem 6.3** (Hungerford 5.8.3).

Let  $\phi$  be the Euler function.

1.  $\phi(n)$  is even for  $n > 2$ .
2. find all  $n > 0$  such that  $\phi(n) = 2$ .

**Problem 6.4** (Hungerford 5.8.9).

If  $n > 2$  and  $\zeta$  is a primitive  $n$ -th root of unity over  $\mathbb{Q}$ , then  $[\mathbb{Q}(\zeta + \zeta^{-1}) : \mathbb{Q}] = \phi(n)/2$ .

**Problem 6.5** (Hungerford 5.9.1).

If  $F$  is a radical extension field of  $K$  and  $E$  is an intermediate field, then  $F$  is a radical extension of  $E$ .

**Problem 6.6** (Hungerford 5.9.3).

Let  $K$  be a field,  $f \in K[x]$  an irreducible polynomial of degree  $n \geq 5$  and  $F$  a splitting field of  $f$  over  $K$ . Assume that  $\text{Aut}_K(F) \simeq S_n$ . Let  $u$  be a root of  $f$  in  $F$ . Then,

1.  $K(u)$  is not Galois over  $K$ ;  $[K(u) : K] = n$  and  $\text{Aut}_K(K(u)) = 1$  (and hence solvable).
2. Every normal closure over  $K$  that contains  $u$  also contains an isomorphic copy of  $F$ .
3. There is no radical extension field  $E$  of  $K$  such that  $K \subset K(u) \subset E$ .

## 6.2 Qual Problems

**Problem 6.7.** 1. Let  $K$  be a field. State the main theorem of Galois theory for a finite field extension  $L/K$

2. Let  $\zeta_{43} := e^{2\pi i/43}$ . Describe the group of all field automorphisms  $\sigma : \mathbb{Q}(\zeta_{43}) \rightarrow \mathbb{Q}(\zeta_{43})$ .

3. How many proper subfields are there in the field  $\mathbb{Q}(\zeta_{43})$ ?

**Problem 6.8.**

Let  $F$  be a field and let  $f(x) \in F[x]$ .

1. Define what is a splitting field of  $f(x)$  over  $F$ .

2. Let  $F$  be a finite field with  $q$  elements. Let  $E/F$  be a finite extension of degree  $n > 0$ . Exhibit an explicit polynomial  $g(x) \in F[x]$  such that  $E/F$  is a splitting of  $g(x)$  over  $F$ . Fully justify your answer.

3. Show that the extension  $E/F$  in (2) is a Galois extension.

**Problem 6.9.**

Let  $K \subset L \subset M$  be a tower of finite degree field extensions. In each of the following parts, either prove the assertion or give a counterexample (with justification).

1. If  $M/K$  is Galois, then  $L/K$  is Galois

2. If  $M/K$  is Galois, then  $M/L$  is Galois.

## 7 Problem Set Seven

### 7.1 Exercises

**Problem 7.1** (Hungerford 4.1.3).

Let  $I$  be a left ideal of a ring  $R$ , and let  $A$  be an  $R$ -module.

1. Show that if  $S$  is a nonempty subset of  $A$ , then

$$IS := \left\{ \sum_{i=1}^n r_i a_i \mid n \in \mathbb{N}^*; r_i \in I; a_i \in S \right\}$$

is a submodule of  $A$ .

*Note that if  $S = \{a\}$ , then  $IS = Ia = \{ra \mid r \in I\}$ .*

2. If  $I$  is a two-sided ideal, then  $A/IA$  is an  $R/I$  module with the action of  $R/I$  given by

$$(r + I)(a + IA) = ra + IA.$$

**Problem 7.2** (Hungerford 4.1.5).

If  $R$  has an identity, then a nonzero unitary  $R$ -module is **simple** if its only submodules are 0 and  $A$ .

1. Show that every simple  $R$ -module is cyclic.
2. If  $A$  is simple, every  $R$ -module endomorphism is either the zero map or an isomorphism.

**Problem 7.3** (Hungerford 4.1.7). 1. Show that if  $A, B$  are  $R$ -modules, then the set  $\text{Hom}_R(A, B)$  is all  $R$ -module homomorphisms  $A \rightarrow B$  is an abelian group with  $f + g$  given on  $a \in A$  by

$$(f + g)(a) := f(a) + g(a) \in B.$$

Also show that the identity element is the zero map.

2. Show that  $\text{Hom}_R(A, A)$  is a ring with identity, where multiplication is given by composition of functions.

*Note that  $\text{Hom}_R(A, A)$  is called the **endomorphism ring** of  $A$ .*

3. Show that  $A$  is a left  $\text{Hom}_R(A, A)$ -module with an action defined by

$$a \in A, f \in \text{Hom}_R(A, A) \implies f \curvearrowright a := f(a).$$

**Problem 7.4** (Hungerford 4.1.12).

Let the following be a commutative diagram of  $R$ -modules and  $R$ -module homomorphisms with exact rows:



$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
\downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \downarrow \alpha_4 & & \downarrow \alpha_5 \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
\end{array}$$

Prove the following:

1. If  $\alpha_1$  is an epimorphism and  $\alpha_2, \alpha_4$  are monomorphisms then  $\alpha_3$  is a monomorphism.
2. If  $\alpha_5$  is a monomorphism and  $\alpha_2, \alpha_4$  are epimorphisms then  $\alpha_3$  is an epimorphism.

**Problem 7.5** (Hungerford 4.2.4).

Let  $R$  be a principal ideal domain,  $A$  a unitary left  $R$ -module, and  $p \in R$  a prime (and thus irreducible) element. Define

$$\begin{aligned}
pA &:= \{pa \mid a \in A\} \\
A[p] &:= \{a \in A \mid pa = 0\}.
\end{aligned}$$

Show the following:

1.  $R/(p)$  is a field.
2.  $pA$  and  $A[p]$  are submodules of  $A$ .
3.  $A/pA$  is a vector space over  $R/(p)$ , with

$$(r + (p))(a + pA) = ra + pA.$$

4.  $A[p]$  is a vector space over  $R/(p)$  with

$$(r + (p))a = ra.$$

**Problem 7.6** (Hungerford 4.2.8).

If  $V$  is a finite dimensional vector space and

$$V^m := V \oplus V \oplus \cdots \oplus V \quad (m \text{ summands}),$$

then for each  $m \geq 1$ ,  $V^m$  is finite dimensional and  $\dim V^m = m(\dim V)$ .

**Problem 7.7** (Hungerford 4.2.9).

If  $F_1, F_2$  are free modules of a ring with the invariant dimension property, then

$$\text{rank}(F_1 \oplus F_2) = \text{rank} F_1 + \text{rank} F_2.$$

## 7.2 Qual Problems

### Problem 7.8.

Let  $F$  be a field and let  $f(x) \in F[x]$ .

1. State the definition of a splitting field of  $f(x)$  over  $F$ .
2. Let  $F$  be a finite field with  $q$  elements. Let  $E/F$  be a finite extension of degree  $n > 0$ . Exhibit an explicit polynomial  $g(x) \in F[x]$  such that  $E/F$  is a splitting field of  $g$  over  $F$ . Fully justify your answer.
3. Show that the extension in (b) is a Galois extension.

### Problem 7.9.

Let  $R$  be a commutative ring and let  $M$  be an  $R$ -module. Recall that for  $\mu \in M$ , the *annihilator* of  $\mu$  is the set

$$\text{Ann}(\mu) = \{r \in R \mid r\mu = 0\}.$$

Suppose that  $I$  is an ideal in  $R$  which is maximal with respect to the property there exists a nonzero element  $\mu \in M$  such that  $I = \text{Ann}(\mu)$ .

Prove that  $I$  is a *prime* ideal in  $R$ .

### Problem 7.10.

Suppose that  $R$  is a principal ideal domain and  $I \leq R$  is an ideal. If  $a \in I$  is an irreducible element, show that  $I = Ra$ .

## 8 Problem Set Eight

### 8.1 Exercises

**Problem 8.1** (Hungerford 4.4.1).

Show the following:

1. For any abelian group  $A$  and any positive integer  $m$ ,

$$\text{Hom}(\mathbb{Z}_m, A) \cong A[m] := \{a \in A \mid ma = 0\}.$$

2.  $\text{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{\gcd(m,n)}$ .
3. As a  $\mathbb{Z}$ -module,  $\mathbb{Z}_m^* = 0$ .
4. For each  $k \geq 1$ ,  $\mathbb{Z}_m$  is a  $\mathbb{Z}_{mk}$ -module, and as a  $\mathbb{Z}_{mk}$  module,  $\mathbb{Z}_m^* \cong \mathbb{Z}_m$ .

**Problem 8.2** (Hungerford 4.4.3).

Let  $\pi : \mathbb{Z} \rightarrow \mathbb{Z}_2$  be the canonical epimorphism. Show that the induced map  $\bar{\pi} : \text{Hom}(\mathbb{Z}_2, \mathbb{Z}) \rightarrow \text{Hom}(\mathbb{Z}_2, \mathbb{Z}_2)$  is the zero map. Conclude that  $\bar{\pi}$  is not an epimorphism.

**Problem 8.3** (Hungerford 4.4.5).

Let  $R$  be a unital ring, show that there is a ring homomorphism  $\text{Hom}_R(R, R) \rightarrow R^{op}$  where  $\text{Hom}_R$  denotes left  $R$ -module homomorphisms. Conclude that if  $R$  is commutative, then there is a ring isomorphism  $\text{Hom}_R(R, R) \cong R$ .

**Problem 8.4** (Hungerford 4.4.9).

Show that for any homomorphism  $f : A \rightarrow B$  of left  $R$ -modules the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\theta_A} & A^{**} \\ \downarrow f & & \downarrow f^* \\ B & \xrightarrow{\theta_B} & B^{**} \end{array}$$

where  $\theta_A, \theta_B$  are as in Theorem 4.12 and  $f^*$  is the map induced on  $A^{**} := \text{Hom}_R(\text{Hom}(A, R), R)$  by the map

$$\bar{f} : \text{Hom}(B, R) \rightarrow \text{Hom}_R(A, R).$$

**Problem 8.5** (Hungerford 4.6.2).

Show that every free module over a unital integral domain is torsion-free. Show that the converse is false.

**Problem 8.6** (Hungerford 4.6.3).

Let  $A$  be a cyclic  $R$ -module of order  $r \in R$ .

1. Show that if  $s$  is relatively prime to  $r$ , then  $sA = A$  and  $A[s] = 0$ .
2. If  $s$  divides  $r$ , so  $sk = r$ , then  $sA \cong R/(k)$  and  $A[s] \cong R/(s)$ .

**Problem 8.7** (Hungerford 4.6.6).

Let  $A, B$  be cyclic modules over  $R$  of nonzero orders  $r, s$  respectively, where  $r$  is *not* relatively prime to  $s$ . Show that the invariant factors of  $A \oplus B$  are  $\gcd(r, s)$  and  $\text{lcm}(r, s)$ .

## 8.2 Qual Problems

### Problem 8.8.

Let  $R$  be a PID. Let  $n > 0$  and  $A \in M_n(R)$  be a square  $n \times n$  matrix with coefficients in  $R$ .

Consider the  $R$ -module  $M := R^n/\text{im}(A)$ .

1. Give a necessary and sufficient condition for  $M$  to be a torsion module (i.e. every nonzero element is torsion). Justify your answer.
2. Let  $F$  be a field and now let  $R := F[x]$ . Give an example of an integer  $n > 0$  and an  $n \times n$  square matrix  $A \in M_n(R)$  such that  $M := R^n/\text{im}(A)$  is isomorphic as an  $R$ -module to  $R \times F$ .

**Problem 8.9.** 1. State the structure theorem for finitely generated modules over a PID.

2. Find the decomposition of the  $\mathbb{Z}$ -module  $M$  generated by  $w, x, y, z$  satisfying the relations

$$3w + 12y + 3x + 6z = 0$$

$$6y = 0$$

$$-3w - 3x + 6y = 0.$$

### Problem 8.10.

Let  $R$  be a commutative ring and  $M$  an  $R$ -module.

1. Define what a torsion element of  $M$  is .
2. Given an example of a ring  $R$  and a cyclic  $R$ -module  $M$  such that  $M$  is infinite and  $M$  contains a nontrivial torsion element  $m$ . Justify why  $m$  is torsion.
3. Show that if  $R$  is a domain, then the subset of elements of  $M$  that are torsion is an  $R$ -submodule of  $M$ . Clearly show where the hypothesis that  $R$  is a domain is used.

## 9 Problem Set Nine

### 9.1 Exercises

**Problem 9.1** (Hungerford 7.1.3).

1. Show that the center of the ring  $M_n(R)$  consists of matrices of the form  $rI_n$  where  $r$  is in the center of  $R$ .

*Hint: Every such matrix must commute with  $\epsilon_{ij}$ , the matrix with  $1_R$  in the  $i, j$  position and zeros elsewhere.*

2. Show that  $Z(M_n(R)) \cong Z(R)$ .

**Problem 9.2** (Hungerford 7.1.5).

1. Show that if  $A, B$  are (skew)-symmetric then  $A + B$  is (skew)-symmetric.
2. Let  $R$  be commutative. Show that if  $A, B$  are symmetric, then  $AB$  is symmetric  $\iff AB = BA$ . Also show that for any matrix  $B \in M_n(R)$ , both  $BB^t$  and  $B + B^t$  are always symmetric, and  $B - B^t$  is always skew-symmetric.

**Problem 9.3** (Hungerford 7.1.7).

Show that similarity is an equivalence relation on  $M_n(R)$ , and \*equivalence\* is an equivalence relation on  $M_{m \times n}(R)$ .

**Problem 9.4** (Hungerford 7.2.2).

Show that an  $n \times m$  matrix  $A$  over a division ring  $D$  has an  $m \times n$  left inverse  $B$  (so  $BA = I_m$ )  $\iff \text{rank} A = m$ . Similarly, show  $A$  has a right  $m \times n$  inverse  $\iff \text{rank} A = n$ .

**Problem 9.5** (Hungerford 7.2.4).

1. Show that a system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= b_1 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= b_n \end{aligned}$$

has a simultaneous solution  $\iff$  the corresponding matrix equation

$AX = B$  has a solution, where  $A = (a_{ij})$ ,  $X = [x_1, \dots, x_m]^t$ , and  $B = [b_1, \dots, b_n]^t$ .

2. If  $A_1, B_1$  are matrices obtained from  $A, B$  respectively by performing the same sequence of elementary **row** operations, then  $X$  is a solution of  $AX = B \iff X$  is a solution of  $A_1X = B_1$ .
3. Let  $C$  be the  $n \times (m + 1)$  matrix given by

$$C = \begin{pmatrix} a_{11} & \cdots & a_{1m} & b_1 \\ \vdots & & & \vdots \\ a_{n1} & \cdots & a_{nm} & b_n \end{pmatrix}.$$

Then  $AX = B$  has a solution  $\iff \text{rank} A = \text{rank} C$  and the solution is unique  $\iff \text{rank}(A) = m$ .

*Hint: use part 2.*

4. If  $B = 0$ , so the system  $AX = B$  is homogeneous, then it has a nontrivial solution  $\iff A < m$  and in particular  $n < m$ .

**Problem 9.6** (Hungerford 7.2.5).

Let  $R$  be a PID. For each positive integer  $r$  and sequence of nonzero ideals  $I_1 \supset I_2 \supset \cdots \supset I_r$ , choose a sequence  $d_i \in R$  such that  $(d_i) = I_i$  and  $d_i \mid d_{i+1}$ .

For a given pair of positive integers  $n, m$ , let  $S$  be the set of all  $n \times m$  matrices of the form  $\begin{pmatrix} L_r & 0 \\ 0 & 0 \end{pmatrix}$  where  $r = 1, 2, \dots, \min(m, n)$  and  $L_r$  is a diagonal  $r \times r$  matrix with main diagonal  $d_i$ .

Show that  $S$  is a set of canonical forms under equivalence for the set of all  $n \times m$  matrices over  $R$ .

## 9.2 Qual Problems

### Problem 9.7.

Let  $R$  be a commutative ring.

1. Say what it means for  $R$  to be a unique factorization domain (UFD).
2. Say what it means for  $R$  to be a principal ideal domain (PID)
3. Give an example of a UFD that is not a PID. Prove that it is not a PID.

### Problem 9.8.

Let  $A$  be an  $n \times n$  matrix over a field  $F$  such that  $A$  is diagonalizable. Prove that the following are equivalent:

1. There is a vector  $v \in F^n$  such that  $v, Av, \dots, A^{n-1}v$  is a basis for  $F^n$ .
2. The eigenvalues of  $A$  are distinct.

### Problem 9.9.

Let  $x, y \in \mathbb{C}$  and consider the matrix

$$M = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ y & 0 & 1 \end{bmatrix}$$

1. Show that  $[0, 1, 0]^t$  is an eigenvector of  $M$ .
2. Compute the rank of  $M$  as a function of  $x$  and  $y$ .
3. Find all values of  $x$  and  $y$  for which  $M$  is diagonalizable.



## 10 Problem Set Ten

### 10.1 Exercises

**Problem 10.1** (Hungerford 7.3.1).

Let  $B$  be an  $R$ -module. Show that if  $r + r \neq 0$  for all  $r \neq 0 \in R$ , then an  $n$ -linear form  $B^n \rightarrow R$  is alternating  $\iff$  it is skew-symmetric.

**Problem 10.2** (Hungerford 7.3.5).

If  $R$  is a field and  $A, B \in M_n(R)$  are invertible then the matrix  $A + rB$  is invertible for all but a finite number of  $r \in R$ .

**Problem 10.3** (Hungerford 7.4.4).

Show that if  $q$  is the minimal polynomial of a linear transformation  $\phi : E \rightarrow E$  with  $\dim_k E = n$  then  $\deg q \leq n$ .

**Problem 10.4** (Hungerford 7.4.8).

Show that  $A \in M_n(K)$  is similar to a diagonal matrix  $\iff$  the elementary divisors of  $A$  are all linear.

**Problem 10.5** (Hungerford 7.4.10).

Find all possible rational canonical forms for a matrix  $A \in M_n(\mathbb{Q})$  such that

1.  $A$  is  $6 \times 6$  with minimal polynomial  $q(x) = (x - 2)^2(x + 3)$ .
2.  $A$  is  $7 \times 7$  with  $q(x) = (x^2 + 1)(x - 7)$ .

Also find all such forms when  $A \in M_n(\mathbb{C})$  instead, and find all possible Jordan Canonical Forms over  $\mathbb{C}$ .

**Problem 10.6** (Hungerford 7.5.2).

Show that if  $\phi$  is an endomorphism of a free  $k$ -module  $E$  of finite rank, then  $p_\phi(\phi) = 0$ .

*Hint:*

If  $A$  is the matrix of  $\phi$  and  $B = xI_n - A$  then  $B^a B = BI_n = p_\phi I_n$  in  $M_n(k[x])$ . If  $E$  is a  $k[x]$ -module with structure induced by  $\phi$ , and  $\psi$  is the  $k[x]$ -module endomorphism  $E \rightarrow E$  with matrix given by  $B$ , then

$$\psi(u) = xu - \phi(u) = \phi(u) - \phi(u) = 0$$

for all  $u \in E$ .

**Problem 10.7** (Hungerford 7.5.7).

1. Let  $\phi, \psi$  be endomorphisms of a finite-dimensional vector space  $E$  such that  $\phi\psi = \psi\phi$ . Show that if  $E$  has a basis of eigenvectors of  $\psi$ , then it has a basis of eigenvectors for both  $\psi$  and  $\phi$  simultaneously.
2. Interpret the previous part as a statement about matrices similar to a diagonal matrix.

## 10.2 Qual Problems

### Problem 10.8.

Let  $M \in M_5(\mathbb{R})$  be a  $5 \times 5$  square matrix with real coefficients defining a linear map  $L : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ . Assume that when considered as an element of  $M_5(\mathbb{C})$ , then the scalars  $0, 1 + i, 1 + 2i$  are eigenvalues of  $M$ .

1. Show that the associated linear map  $L$  is neither injective nor surjective.
2. Compute the characteristic polynomial and minimal polynomial of  $M$ .
3. How many fixed points can  $L$  have?

*(That is, how many solutions are there to the equation  $L(v) = v$  with  $v \in \mathbb{R}^5$ ?)*

### Problem 10.9.

Let  $n$  be a positive integer and let  $B$  denote the  $n \times n$  matrix over  $\mathbb{C}$  such that every entry is 1. Find the Jordan normal form of  $B$ .

### Problem 10.10.

Suppose that  $V$  is a 6-dimensional vector space and that  $T$  is a linear transformation on  $V$  such that  $T^6 = 0$  and  $T^5 \neq 0$ .

1. Find a matrix for  $T$  in Jordan Canonical form.
2. Show that if  $S, T$  are linear transformations on a 6-dimensional vector space  $V$  which both satisfy  $T^6 = S^6 = 0$  and  $T^5, S^5 \neq 0$ , then there exists a linear transformation  $A$  from  $V$  to itself such that  $ATA^{-1} = S$ .