# Real Analysis Qualifying Exam Notes

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Monday 18th May, 2020

## Contents

# 1 Inequalities and Equalities

**AM-GM Inequality**:

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

**Reverse Triangle Inequality**

$$\|\|x\| - \|y\|\| \leq \|x - y\|.$$

**Chebyshev's Inequality**

$$\mu(\{x : |f(x)| > \alpha\}) \leq \left(\frac{\|f\|_p}{\alpha}\right)^p.$$

**Holder's Inequality:**

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \|fg\|_1 \leq \|f\|_p \|g\|_q.$$

*Application:* For finite measure spaces,

$$1 \leq p < q \leq \infty \implies L^q \subset L^p \quad (\text{ and } \ell^p \subset \ell^q)$$

*Proof .*
Fix $p, q$, let $r = \frac{q}{p}$ and $s = \frac{r}{r-1}$ so $r^{-1} + s^{-1} = 1$. Then let $h = |f|^p$:

$$\|f\|_p^p = \|h \cdot 1\|_1 \leq \|1\|_s \|h\|_r = \mu(X)^{\frac{1}{s}} \|f\|_q^{\frac{q}{r}} \implies \|f\|_p \leq \mu(X)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q.$$

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Note: doesn't work for $\ell$ spaces, but just note that $\sum |x_n| < \infty \implies x_n < 1$ for large enough $n$, and thus $p < q \implies |x_n|^q \leq |x_n|^q$.

**Cauchy-Schwarz**:

$$|\langle f, \ g \rangle| = \|fg\|_1 \leq \|f\|_2 \|g\|_2 \quad \text{with equality} \quad \iff f = \lambda g.$$

Relates inner product to norm, and only happens to relate norms in $L^1$.

*Proof* .
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**Minkowski's Inequality:**

$$1 \le p < \infty \implies \|f+g\|_p \le \|f\|_p + \|g\|_p$$

Note: does not handle $p = \infty$ case. Use to prove $L^p$ is a normed space.

**Young's Inequality**:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \implies \|f * g\|_r \le \|f\|_p \|g\|_q.$$

**Application**: Some useful specific cases:

$$\begin{aligned} \|f * g\|_1 &\le \|f\|_1 \|g\|_1 \\ \|f * g\|_p &\le \|f\|_1 \|g\|_p, \\ \|f * g\|_\infty &\le \|f\|_2 \|g\|_2 \\ \|f * g\|_\infty &\le \|f\|_p \|g\|_q. \end{aligned}$$

**Bessel's Inequality:**

For $x \in H$ a Hilbert space and $\{e_k\}$ an orthonormal sequence,

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \le \|x\|^2.$$

Note: this does not need to be a basis.

**Parseval's Identity:**

Equality in Bessel's inequality, attained when $\{e_k\}$ is a *basis*, i.e. it is complete, i.e. the span of its closure is all of $H$.

## 2 Basics

**Useful Technique:** $\lim f_n = \limsup f_n = \liminf f_n$ iff the limit exists, so $\limsup f_n \le g \le \liminf f_n$ implies that $g = \lim f$. Similarly, a limit does not exist iff $\liminf f_n > \limsup f_n$.

**Lemma 2.1** ***(Convergent Sums Have Small Tails).***

$$\sum a_n < \infty \implies a_n \longrightarrow 0 \quad \text{and} \quad \sum_{k=N}^{\infty} \overset{N \longrightarrow \infty}{\longrightarrow} 0$$

**Theorem 2.2** ***(Heine-Borel).***
$X \subseteq \mathbb{R}^n$ is compact $\iff$ $X$ is closed and bounded.

**Lemma 2.3** ***(Geometric Series).***

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \iff |x| < 1.$$

*Corollary:* $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k} = 1.$

**Definition 2.3.1.**
A set $S$ is **nowhere dense** iff the closure of $S$ has empty interior iff every interval contains a subinterval that does not intersect $S$.

**Definition 2.3.2.**
A set is **meager** if it is a *countable* union of nowhere dense sets.

Note that a *finite* union of nowhere dense is still nowhere dense.

**Lemma 2.4.**
The Cantor set is closed with empty interior.

*Proof .*
Its complement is a union of open intervals, and can't contain an interval since intervals have positive measure and $m(C_n)$ tends to zero.
**Corollary:** The Cantor set is nowhere dense.
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**Definition 2.4.1.**
An $F_\sigma$ set is a union of closed sets, and a $G_\delta$ set is an intersection of opens.
*Mnemonic: "F" stands for* ferme*, which is "closed" in French, and $\sigma$ corresponds to a "sum", i.e. a union.*

**Lemma 2.5.**
Singleton sets in $\mathbb{R}$ are closed, and thus $\mathbb{Q}$ is an $F_\sigma$ set.

**Theorem 2.6** ***(Baire).***
$\mathbb{R}$ is a **Baire space** (countable intersections of open, dense sets are still dense). Thus $\mathbb{R}$ can not be written as a countable union of nowhere dense sets.

**Lemma 2.7.**
There is a function discontinuous precisely on $\mathbb{Q}$.

*Proof .*
$f(x) = \dfrac{1}{n}$ if $x = r_n \in \mathbb{Q}$ is an enumeration of the rationals, and zero otherwise. The limit at every point is 0.

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**Lemma 2.8.**
There *do not* exist functions that are discontinuous precisely on $\mathbb{R} \setminus \mathbb{Q}$.

*Proof .*
$D_f$ is always an $F_\sigma$ set, which follows by considering the oscillation $\omega_f$. $\omega_f(x) = 0 \iff f$ is continuous at $x$, and $D_f = \bigcup_n A_{\frac{1}{n}}$ where $A_\varepsilon = \{\omega_f \geq \varepsilon\}$ is closed.

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**Lemma 2.9.**
Any nonempty set which is bounded from above (resp. below) has a well-defined supremum (resp. infimum).

# 3 Uniform Convergence

**Theorem 3.1** ***(Egorov).***
Let $E \subseteq \mathbb{R}^n$ be measurable with $m(E) > 0$ and $\{f_k : E \longrightarrow \mathbb{R}\}$ be measurable functions such that $f(x) := \lim\limits_{k \longrightarrow \infty} f_k(x) < \infty$ exists almost everywhere.
Then $f_k \longrightarrow f$ *almost uniformly*, i.e.

$$\forall \varepsilon > 0, \ \exists F \subseteq E \text{ closed such that } m(E \setminus F) < \varepsilon \quad \text{and} \quad f_k \xrightarrow{u} f \text{ on } F.$$

**Proposition 3.2.**
The space $X = C([0,1])$, continuous functions $f : [0,1] \longrightarrow \mathbb{R}$, equipped with the norm $\|f\| = \sup\limits_{x \in [0,1]} |f(x)|$, is a **complete** metric space.

*Proof .*

1. Let $\{f_k\}$ be Cauchy in $X$.
2. Define a candidate limit using pointwise convergence:
   Fix an $x$; since
   $$|f_k(x) - f_j(x)| \leq \|f_k - f_k\| \longrightarrow 0$$
   the sequence $\{f_k(x)\}$ is Cauchy in $\mathbb{R}$. So define $f(x) := \lim\limits_k f_k(x)$.
3. Show that $\|f_k - f\| \longrightarrow 0$:
   $$|f_k(x) - f_j(x)| < \varepsilon \ \forall x \implies \lim_j |f_k(x) - f_j(x)| < \varepsilon \ \forall x$$

Alternatively, $\|f_k - f\| \leq \|f_k - f_N\| + \|f_N - f_j\|$, where $N, j$ can be chosen large enough to bound each term by $\varepsilon/2$.

4. Show that $f \in X$:
   The uniform limit of continuous functions is continuous. (Note: in other cases, you may need to show the limit is bounded, or has bounded derivative, or whatever other conditions define $X$.)

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**Lemma 3.3.**
Metric spaces are compact iff they are sequentially compact, (i.e. every sequence has a convergent subsequence).

**Corollary:** The unit ball in $C([0,1])$ with the sup norm is not compact.

*Proof:* Take $f_k(x) = x^n$, which converges to a dirac delta at 1. The limit is not continuous, so no subsequence can converge.

**Lemma 3.4.**
A uniform limit of continuous functions is continuous.

**Lemma (Testing Uniform Convergence):** $f_n \longrightarrow f$ uniformly iff there exists an $M_n$ such that $\|f_n - f\|_\infty \leq M_n \longrightarrow 0$.

Negating: find an $x$ which depends on $n$ for which the norm is bounded below.

**Useful Technique**: If $f_n$ has a global maximum (computed using $f_n'$ and the first derivative test) $M_n \longrightarrow 0$, then $f_n \longrightarrow 0$ uniformly.

**Lemma 3.5** ***(Baby Commuting Limits with Integrals).***
If $f_n \longrightarrow f$ uniformly, then $\displaystyle\int f_n = \int f$.

**Lemma 3.6** ***(Uniform Convergence and Derivatives).***
If $f_n' \longrightarrow g$ uniformly for some $g$ and $f_n \longrightarrow f$ pointwise (or at least at one point), then $g = f'$.

**Lemma 3.7** ***(Uniform Convergence of Series).***
If $f_n(x) \leq M_n$ **for a fixed** $x$ where $\sum M_n < \infty$, then the series $f(x) = \sum f_n(x)$ converges pointwise.

**Lemma 3.8.**
If $\sum f_n$ converges then $f_n \longrightarrow 0$ uniformly.

**Useful Technique:** For a fixed $x$, if $f = \sum f_n$ converges *uniformly* on some $B_r(x)$ and each $f_n$ is continuous at $x$, then $f$ is also continuous at $x$ .

****Lemma (M-test for Series)** If $|f_n(x)| \leq M_n$ which does not depend on $x$, then $\sum f_n$ converges uniformly.

**Lemma (p-tests):** Let $n$ be a fixed dimension and set $B = \left\{x \in \mathbb{R}^n \;\middle|\; \|x\| \leq 1\right\}$.

$$\begin{aligned}
\sum \frac{1}{n^p} < \infty &\iff p > 1 \\
\int_\varepsilon^\infty \frac{1}{x^p} < \infty &\iff p > 1 \\
\int_0^1 \frac{1}{x^p} < \infty &\iff p < 1 \\
\int_B \frac{1}{|x|^p} < \infty &\iff p < n \\
\int_{B^c} \frac{1}{|x|^p} < \infty &\iff p > n
\end{aligned}$$

.

# 4 Measure Theory

**Useful Technique:** $s = \inf\{x \in X\} \implies$ for every $\varepsilon$ there is an $x \in X$ such that $x \leq s + \varepsilon$.

**Useful Techniques**: Always consider bounded sets, and if $E$ is unbounded write $E = \bigcup_n B_n(0) \bigcap E$ and use countable subadditivity or continuity of measure.

**Lemma 4.1.**
Every open subset of $\mathbb{R}$ (resp $\mathbb{R}^n$) can be written as a unique countable union of disjoint (resp. almost disjoint) intervals (resp. cubes).

**Definition**: The outer measure of a set is given by

$$m_*(E) = \inf_{\substack{\{Q_i\} \rightrightarrows E \\ \text{closed cubes}}} \sum |Q_i|.$$

**Lemma 4.2** ***(Properties of [Outer).***
Measure]

- Montonicity: $E \subseteq F \implies m_*(E) \leq m_*(F)$.
- Countable Subadditivity: $m_*(\bigcup E_i) \leq \sum m_*(E_i)$.
- Approximation: For all $E$ there exists a $G \supseteq E$ such that $m_*(G) \leq m_*(E) + \varepsilon$.
- Disjoint* Additivity: $m_*(A \coprod B) = m_*(A) + m_*(B)$.

Note: this holds for outer measure **iff** $\operatorname{dist}(A, B) > 0$.

**Lemma 4.3** ***(Subtraction of Measure):).***
$m(A) = m(B) + m(C)$ and $m(C) < \infty$ implies that $m(A) - m(C) = m(B)$.

**Lemma 4.4** ***(Continuity of Measure).***

$$
\begin{aligned}
E_i \nearrow E &\implies m(E_i) \longrightarrow m(E) \\
m(E_1) < \infty \text{ and } E_i \searrow E &\implies m(E_i) \longrightarrow m(E).
\end{aligned}
$$

*Proof .*

1. Break into disjoint annuli $A_2 = E_2 \setminus E_1$, etc then apply countable disjoint additivity to $E = \coprod A_i$.

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2. Use $E_1 = (\coprod E_j \setminus E_{j+1}) \coprod (\bigcap E_j)$, taking measures yields a telescoping sum,and use countable disjoint additivity.

**Lemma 4.5.**
Lebesgue measure is translation and dilation invariant.

*Proof .*
Obvious for cubes; if $Q_i \rightrightarrows E$ then $Q_i + k \rightrightarrows E + k$, etc.

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**Theorem 4.6** ***(Non-Measurable Sets).***
There is a non-measurable set.

*Proof .*

- Use AOC to choose one representative from every coset of $\mathbb{R}/\mathbb{Q}$ on $[0, 1)$, which is countable, and assemble them into a set $N$
- Enumerate the rationals in $[0, 1]$ as $q_j$, and define $N_j = N + q_j$. These intersect trivially.
- Define $M := \coprod N_j$, then $[0, 1) \subseteq M \subseteq [-1, 2)$, so the measure must be between 1 and 3. By translation invariance, $m(N_j) = m(N)$, and disjoint additivity forces $m(M) = 0$, a contradiction.

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**Lemma (Borel Characterization of Measurable Sets)**

If $E$ is Lebesgue measurable, then $E = H \coprod N$ where $H \in F_\sigma$ and $N$ is null.

**Useful technique:** $F_\sigma$ sets are Borel, so establish something for Borel sets and use this to extend it to Lebesgue.

**Proof:** For every $\frac{1}{n}$ there exists a closed set $K_n \subset E$ such that $m(E \setminus K_n) \leq \frac{1}{n}$. Take $K = \bigcup K_n$, wlog $K_n \nearrow K$ so $m(K) = \lim m(K_n) = m(E)$. Take $N := E \setminus K$, then $m(N) = 0$.

**Lemma 4.7.**

$$\limsup_n A_n = \bigcap_n \bigcup_{j \geq n} A_j = \left\{ x \,\middle|\, x \in A_n \text{ for inf. many } n \right\}$$

$$\liminf_n A_n = \bigcup_n \bigcap_{j \geq n} A_j = \left\{ x \,\middle|\, x \in A_n \text{ for all except fin. many } n \right\}$$

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**Lemma 4.8.**
If $A_n$ are all measurable, $\limsup A_n$ and $\liminf A_n$ are measurable.

*Proof:* Measurable sets form a sigma algebra, and these are expressed as countable unions/intersections of measurable sets.

**Theorem 4.9** ***(Borel-Cantelli).***
Let $\{E_k\}$ be a countable collection of measurable sets. Then

$$\sum_k m(E_k) < \infty \implies \text{almost every } x \in \mathbb{R} \text{ is in at most finitely many } E_k.$$

**Application:**

$$m\left( \left\{ x \text{ such that } \exists \text{ inf. many } \frac{p}{q} \text{ with } \left| x - \frac{p}{q} \right| \leq \frac{1}{q^3} \right\} \right) = 0.$$

*Proof* .
Idea: write $E_j$ to be the above set with $p, q$ replaced by $p_j, q_j$ where $r_j = \dfrac{p_j}{q_j}$ is an enumeration of $\mathbb{Q}$, then $m(E_j) \leq \dfrac{2}{q^3}$ and $\sum \dfrac{1}{q^3} < \infty$.

- If $E = \limsup_j E_j$ with $\sum m(E_j) < \infty$ then $m(E) = 0$.
- If $E_j$ are measurable, then $\limsup_j E_j$ is measurable.
- If $\sum_j m(E_j) < \infty$, then $\sum_{j=N}^{\infty} m(E_j) \overset{N \longrightarrow \infty}{\longrightarrow} 0$ as the tail of a convergent sequence.
- $E = \limsup_j E_j = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j \implies E \subseteq \bigcup_{j=k}^{\infty}$ for all $k$
- $E \subset \bigcup_{j=k}^{\infty} \implies m(E) \leq \sum_{j=k}^{\infty} m(E_j) \overset{k \longrightarrow \infty}{\longrightarrow} 0.$

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**Lemma 4.10.**

- Characteristic functions are measurable
- If $f_n$ are measurable, so are $|f_n|, \limsup f_n, \liminf f_n, \lim f_n$,
- Sums and differences of measurable functions are measurable,

- Cones $F(x, y) = f(x)$ are measurable,
- Compositions $f \circ T$ for $T$ a linear transformation are measurable,
- "Convolution-ish" transformations $(x, y) \mapsto f(x - y)$ are measurable

**Proof (Convolution):** Take the cone on $f$ to get $F(x, y) = f(x)$, then compose $F$ with the linear transformation $T = [1, -1; 1, 0]$.

# 5 Integration

**Definition:** $f \in L^+$ iff $f$ is measurable and non-negative.

Useful techniques:

- Break integration domain up into disjoint annuli.
- Break real integrals up into $x < 1$ and $x > 1$.

**Definition:** A measurable function is integrable iff $\|f\|_1 < \infty$.

Useful facts about $C_c$ functions:

- Bounded almost everywhere
- Uniformly continuous

## 5.1 Convergence Theorems

**Monotone Convergence Theorem (MCT):**

If $f_n \in L^+$ and $f_n \nearrow f$ a.e., then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \int f_n \longrightarrow \int f.$$

Needs to be positive and increasing.

**Dominated Convergence Theorem (DCT):**

If $f_n \in L^1$ and $f_n \longrightarrow f$ a.e. with $|f_n| \leq g$ for some $g \in L^1$, then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \int f_n \longrightarrow \int f,$$

and more generally,

$$\int |f_n - f| \longrightarrow 0.$$

Positivity *not* needed.

Generalized DCT: can relax $|f_n| < g$ to $|f_n| < g_n \longrightarrow g \in L^1$.

**Lemma:** If $f \in L^1$, then

$$\int |f_n - f| \longrightarrow 0 \iff \int |f_n| \longrightarrow |f|.$$

*Proof:* Let $g_n = |f_n| - |f_n - f|$, then $g_n \longrightarrow |f|$ and

$$|g_n| = ||f_n| - |f_n - f|| \leq |f_n - (f_n - f)| = |f| \in L^1,$$

so the DCT applies to $g_n$ and

$$\begin{aligned}\|f_n - f\|_1 = \int |f_n - f| + |f_n| - |f_n| &= \int |f_n| - g_n \\ &\longrightarrow_{DCT} \lim \int |f_n| - \int |f|.\end{aligned}$$

**Fatou's Lemma**:

If $f_n \in L^+$, then

$$\begin{aligned}\int \liminf_n f_n &\leq \liminf_n \int f_n \\ \limsup_n \int f_n &\leq \int \limsup_n f_n.\end{aligned}$$

Only need positivity.

**Theorem (Tonelli):** For $f(x,y)$ **non-negative and measurable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is a **measurable** function
- $F(x) = \displaystyle\int f(x,y)\ dy$ is a **measurable** function,
- For $E$ measurable, the slices $E_x \coloneqq \left\{y \;\middle|\; (x,y) \in E\right\}$ are measurable.
- $\displaystyle\int f = \int\int F$, i.e. any iterated integral is equal to the original.

**Theorem (Fubini):** For $f(x,y)$ **integrable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is an **integrable** function
- $F(x) = \displaystyle\int f(x,y)\ dy$ is an **integrable** function,
- For $E$ measurable, the slices $E_x \coloneqq \left\{y \;\middle|\; (x,y) \in E\right\}$ are measurable.
- $\displaystyle\int f = \int\int f(x,y)$, i.e. any iterated integral is equal to the original

**Theorem (Fubini/Tonelli):** If any iterated integral is **absolutely integrable**, i.e. $\displaystyle\int\int |f(x,y)| < \infty$, then $f$ is integrable and $\displaystyle\int f$ equals any iterated integral.

**Differentiating under the integral**:

If $\left|\dfrac{\partial}{\partial t} f(x,t)\right| \leq g(x) \in L^1$, then letting $F(t) = \displaystyle\int f(x,t)\ dt$,

$$\begin{aligned}\frac{\partial}{\partial t} F(t) &\coloneqq \lim_{h\to 0} \int \frac{f(x,t+h) - f(x,t)}{h} dx \\ &\overset{\text{DCT}}{=} \int \frac{\partial}{\partial t} f(x,t)\ dx.\end{aligned}$$

To justify passing the limit, let $h_k \longrightarrow 0$ be any sequence and define

$$f_k(x,t) = \frac{f(x, t+h_k) - f(x,t)}{h_k},$$

so $f_k \overset{\text{pointwise}}{\longrightarrow} \dfrac{\partial}{\partial t} f$.

Apply the MVT to $f_k$ to get $f_k(x,t) = f_k(\xi, t)$ for some $\xi \in [0, h_k]$, and show that $f_k(\xi, t) \in L_1$.

**Lemma (Swapping Sum and Integral)** If $f_n$ are non-negative and $\sum \int |f|_n < \infty$, then $\sum \int f_n = \int \sum f_n$.

> *Proof: MCT.* Let $F_N = \sum^N f_n$ be a finite partial sum; then there are simple functions $\phi_n \nearrow f_n$ and so $\sum^N \phi_n \nearrow F_N$, so apply MCT.

**Lemma:** If $f_k \in L^1$ and $\sum \|f_k\|_1 < \infty$ then $\sum f_k$ converges almost everywhere and in $L^1$.

> *Proof:* Define $F_N = \sum^N f_k$ and $F = \lim_N F_N$, then $\|F_N\|_1 \leq \sum^N \|f_k\| < \infty$ so $F \in L^1$ and $\|F_N - F\|_1 \longrightarrow 0$ so the sum converges in $L^1$. Almost everywhere convergence: ?

## 5.2 $L^1$ Facts

**Lemma (Translation Invariance):** The Lebesgue integral is translation invariant, i.e. $\int f(x)\, dx = \int f(x+h)\, dx$ for any $h$.

> *Proof:*
>
> - For characteristic functions, $\int_E f(x+h) = \int_{E+h} f(x) = m(E+h) = m(E) = \int_E f$ by translation invariance of measure.
> - So this also holds for simple functions by linearity
> - For $f \in L^+$, choose $\phi_n \nearrow f$ so $\int \phi_n \longrightarrow \int f$.
> - Similarly, $\tau_h \phi_n \nearrow \tau_h f$ so $\int \tau_h f \longrightarrow \int f$
> - Finally $\left\{ \int \tau_h \phi \right\} = \left\{ \int \phi \right\}$ by step 1, and the suprema are equal by uniqueness of limits.

**Lemma (Integrals Distribute Over Disjoint Sets):**

If $X \subseteq A \bigcup B$, then $\int_X f \leq \int_A f + \int_{A^c} f$ with equality iff $X = A \coprod B$.

**Lemma ($L^1$ functions may Decay Rapidly):**

If $f \in L^1$ and $f$ is uniformly continuous, then $f(x) \overset{|x| \longrightarrow \infty}{\longrightarrow} 0$.

> Doesn't hold for general $L^1$ functions, take any train of triangles with height 1 and summable areas.

**Lemma ($L^1$ functions have Small Tails):**

If $f \in L^1$, then for every $\varepsilon$ there exists a radius $R$ such that if $A = B_R(0)^c$, then $\int_A |f| < \varepsilon$.

*Proof: Approximate with compactly supported functions.* Take $g \xrightarrow{L_1} f$ with $g \in C_c$, then choose $N$ large enough so that $g = 0$ on $E := B_N(0)^c$, then $\int_E |f| \leq \int_E |f-g| + \int_E |g|$.

**Lemma ($L^1$ functions have absolutely continuity):**

$m(E) \longrightarrow 0 \implies \int_E f \longrightarrow 0$.

*Proof: Approximate with compactly supported functions.* Take $g \xrightarrow{L_1} f$, then $g \leq M$ so $\int_E f \leq \int_E f - g + \int_E g \longrightarrow 0 + M \cdot m(E) \longrightarrow 0$.

**Lemma ($L^1$ functions are finite almost everywhere):**

If $f \in L^1$, then $m(\{f(x) = \infty\}) = 0$.

*Proof (Split up domain2):* Let $A = \{f(x) = \infty\}$, then $\infty > \int f = \int_A f + \int_{A^c} f = \infty \cdot m(A) + \int_{A^c} f \implies m(X) = 0$.

**Lemma (Continuity in $L^1$):** $\|\tau_h f - f\|_1 \longrightarrow 0$ as $h \longrightarrow 0$.

*Proof: Approximate with compactly supported functions.* Take $g \xrightarrow{L_1} f$ with $g \in C_c$.

$$\begin{aligned}
&\int f(x+h) - f(x) \leq \\
&\int f(x+h) - g(x+h) + \int g(x+h) - g(x) + \int g(x) - f(x) \\
&\longrightarrow 2\varepsilon + \int g(x+h) - g(x) \\
&= \int_K g(x+h) - g(x) + \int_{K^c} g(x+h) - g(x) \longrightarrow 0,
\end{aligned}$$

which follows because we can enlarge the support of $g$ to $K$ where the integrand is zero on $K^c$, then apply uniform continuity on $K$.

**Theorem (Integration by Parts, Special Case):**

$$F(x) := \int_0^x f(y)dy \quad \text{and} \quad G(x) := \int_0^x g(y)dy$$
$$\implies \int_0^1 F(x)g(x)dx = F(1)G(1) - \int_0^1 f(x)G(x)dx.$$

*Proof:* Fubini-Tonelli, and sketch region to change integration bounds.

**Theorem (Lebesgue Density):**

$$A_h(f)(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(y)dy \implies \|A_h(f) - f\| \overset{h \longrightarrow 0}{\longrightarrow} 0.$$

*Proof*: Fubini-Tonelli, and sketch region to change integration bounds, and continuity in $L^1$.

### 5.3 $L^p$ Spaces

**Lemma:** The following are dense subspaces of $L^2([0,1])$:

- Simple functions
- Step functions
- $C_0([0,1])$
- Smoothly differentiable functions $C_0^\infty([0,1])$
- Smooth compactly supported functions $C_c^\infty$

**Dual Spaces:** In general, $(L^p)^\vee \cong L^q$

- For qual, supposed to know the $p = 1$ case, i.e. $(L^1)^\vee \cong L^\infty$
  - For the analogous $p = \infty$ case: $L^1 \subset (L^\infty)^\vee$, since the isometric mapping is always injective, but *never* surjective. So this containment is always proper (requires Hahn-Banach Theorem).
- The $p = 2$ case: Easy by the Riesz Representation for Hilbert spaces.

## 6 Fourier Series and Convolution

**Definition (Convolution)**

$$f * g(x) = \int f(x-y)g(y)dy.$$

**Definition (Dilation)**

$$\phi_t(x) = t^{-n}\phi\left(t^{-1}x\right).$$

**Definition (The Fourier Transform):**

$$\widehat{f}(\xi) = \int f(x)\ e^{2\pi i x\cdot\xi}\ dx.$$

**Lemma:** $\widehat{f} = \widehat{g} \implies f = g$ almost everywhere.

**Lemma (Riemann-Lebesgue)**

$$f \in L^1 \implies \widehat{f}(\xi) \to 0 \text{ as } |\xi| \to \infty.$$

Motto: Fourier transforms decay.

**Lemma:** If $f \in L^1$, then $\widehat{f}$ is continuous and bounded.

*Proof:* $\left|\widehat{f}\right| \leq \int |f|\cdot|e^{\cdots}| \leq \|f\|_1$, and the DCT shows that $\left|\widehat{f}(\xi_n) - \widehat{f}(\xi)\right| \longrightarrow 0$.

Todo: search qual alerts.

# 7 Exam 2 (Practice)

Link to PDF File

Proving uniform continuity: show

$$\|f - \tau_h f\|_1 \xrightarrow{h \longrightarrow 0} 0$$

Notation: $C_0$ is the set of functions that vanish at infinity.

## 7.1 1: Fubini-Tonelli

**Theorem (Fubini):**

Suppose

- $f : \mathbb{R}^{n_1+n_2} \longrightarrow \mathbb{R}$ is measurable on its domain
- $f$ is non-negative

Then for almost every $x \in \mathbb{R}^{n_1}$,

1. Every slice

$$\begin{aligned} f_x : \mathbb{R}^{n_2} &\longrightarrow \mathbb{R} \\ y &\mapsto f(x,y) \end{aligned}$$

is measurable on $\mathbb{R}^{n_2}$.

2. The function

$$\begin{aligned} F : \mathbb{R}^{n_1} &\longrightarrow \mathbb{R} \\ x &\mapsto \int_{\mathbb{R}^{n_2}} f_x(y)\ dy \end{aligned}$$

   is measurable on $\mathbb{R}^{n_1}$

3. The function

$$G(y) = \int_{\mathbb{R}^n} F(x)\ dx$$

   is measurable and

$$G(y) = \int_{\mathbb{R}^n} f = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} f(x,y)\ dy\ dx$$

   for any iterated version of this integral.

**Corollary (Measurable Slices):**

Let $E$ be a measurable subset of $\mathbb{R}^n$. Then

- For almost every $x \in \mathbb{R}^{n_1}$, the slice $E_x \coloneqq \left\{ y \in \mathbb{R}^{n_2} \;\middle|\; (x,y) \in E \right\}$ is measurable in $\mathbb{R}^{n_2}$.
- The function

$$\begin{aligned} F : \mathbb{R}^{n_1} &\longrightarrow \mathbb{R} \\ x &\mapsto m(E_x) = \int_{\mathbb{R}^{n_2}} \chi_{E_x} \, dy \end{aligned}$$

is measurable and

$$m(E) = \int_{\mathbb{R}^{n_1}} m(E_x) \, dx = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{E_x} \, dy \, dx$$

$\implies$:

- Let $f$ be measurable on $\mathbb{R}^n$.
- Then the cylinders $F(x,y) = f(x)$ and $G(x,y) = f(y)$ are both measurable on $\mathbb{R}^{n+1}$.
- Write $\mathcal{A} = \{G \leq F\} \bigcap \{G \geq 0\}$; both are measurable.

$\impliedby$:

- Let $A$ be measurable in $\mathbb{R}^{n+1}$.
- Define $A_x = \left\{y \in \mathbb{R} \;\middle|\; (x,y) \in \mathcal{A}\right\}$, then $m(A_x) = f(x)$.
- By the corollary, $A_x$ is measurable set, $x \mapsto A_x$ is a measurable function, and $m(A) = \int f(x) \, dx$.
- Then explicitly, $f(x) = \chi_A$, which makes $f$ a measurable function.

### 7.1.1 b

- Define $A_y = \left\{x \in \mathbb{R}^n \;\middle|\; (x,y) \in A\right\}$, and notice that $A_y = \left\{x \in \mathbb{R}^n \;\middle|\; 0 \leq y \leq f(x)\right\}$.
- By the corollary, $A_y$ is measurable and

$$m(\mathcal{A}) = \int m\left(\mathcal{A}_y\right) dy = \int_0^y m\left(\{x \in \mathbb{R}^n \ni f(x) \geq y\}\right) dy$$

## 7.2 2: Convolutions and the Fourier Transform

### 7.2.1 a

**Definition (Convolution):**

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) \, dy.$$

**Facts:**

- $f, g \in L^1 \implies f * g \in L^1$
- $f \in L^1, g \leq M \implies f * g \leq M'$ and is uniformly continuous.
- $f, g \in L^1, f \leq M, g \leq M' \implies f * g \xrightarrow{x \longrightarrow \infty} 0.2$
- $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$
- $f \in L^1, g'$ exists, $\dfrac{\partial g}{\partial x_i}$ all bounded $\implies \dfrac{\partial}{\partial x_i}(f * g) = f * \dfrac{\partial g}{\partial x_i}$
- $f, g \in C_c^\infty \implies f * g \in C^\infty$ and $f * g \xrightarrow{x \longrightarrow \infty} 0$.

### 7.2.2 b

**Definition (Approximation to the Identity):**

$$\phi(x) = e^{-\pi x^2}$$
$$\phi_t(x) = t^{-n}\phi(\frac{x}{t}).$$

**Facts:**

- $\int \phi = \int \phi_t = 1$
- $f$ bounded and uniformly continuous $\implies f * \phi_t \rightrightarrows f$

**Theorem (Norm Convergence of Approximate Identities):**

$$\|f * \phi_t - f\|_1 \xrightarrow{t \longrightarrow 0} 0.$$

*Proof:*

$$\begin{aligned}
\|f - f * \phi_t\|_1 &= \int f(x) - \int f(x-y)\phi_t(y)\ dydx \\
&= \int f(x) \int \phi_t(y)\ dy - \int f(x-y)\phi_t(y)\ dydx \\
&= \int\int \phi_t(y)[f(x) - f(x-y)]\ dydx \\
&=_{FT} \int\int \phi_t(y)[f(x) - f(x-y)]\ dxdy \\
&= \int \phi_t(y) \int f(x) - f(x-y)\ dxdy \\
&= \int \phi_t(y) \|f - \tau_y f\|_1 dy \\
&= \int_{y<\delta} \phi_t(y)\|f - \tau_y f\|_1 dy + \int_{y \geq \delta} \phi_t(y)\|f - \tau_y f\|_1 dy \\
&\leq \int_{y<\delta} \phi_t(y)\varepsilon + \int_{y \geq \delta} \phi_t(y)\left(\|f\|_1 + \|\tau_y f\|_1\right) dy \quad \text{by continuity in } L^1 \\
&\leq \varepsilon + 2\|f\|_1 \int_{y \geq \delta} \phi_t(y) dy \\
&\leq \varepsilon + 2\|f\|_1 \varepsilon \quad \text{since } \phi_t \text{ has small tails} \\
&\longrightarrow 0 \blacksquare.
\end{aligned}$$

**Theorem (Convolutions Vanish at Infinity)**

$$f, g \in L^1 \text{ and bounded } \implies \lim_{|x| \to \infty} (f * g)(x) = 0.$$

*Proof:*

- Choose $M \geq f, g$.
- By small tails, choose $N$ such that $\int_{B_N^c} |f|, \int_{B_n^c} |g| < \varepsilon$

- Note
$$|f * g| \leq \int |f(x-y)|\ |g(y)|\ dy := I$$

- Use $|x| \leq |x-y| + |y|$, take $|x| \geq 2N$ so either
  -
  $$|x-y| \geq N \implies I \leq \int_{\{x-y \geq N\}} |f(x-y)| M\ dy \leq \varepsilon M \longrightarrow 0$$
  -
  $$|y| \geq N \implies I \leq \int_{\{y \geq N\}} M |g(y)|\ dy \leq M\varepsilon \longrightarrow 0$$

■

### 7.2.3 c

**Definition (The Fourier Transform):**

$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi}\ dx.$$

**Facts:**

- *Riemann-Lebesgue lemma:* $\widehat{f}$ vanishes at infinity
- $f \in L^1 \implies \widehat{f}$ is bounded and uniformly continuous
- $f, \widehat{f} \in L^1 \implies f$ is bounded, uniformly continuous, and vanishes at infinity
- $f \in L^1$ and $\widehat{f} = 0$ almost everywhere $\implies f = 0$ almost everywhere.

**Theorem (Fourier Inversion):**

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(x) e^{2\pi i x \cdot \xi} d\xi.$$

*Proof:* Idea: Fubini-Tonelli doesn't work directly, so introduce a convergence factor, take limits, and use uniqueness of limits.

Use the following facts:

- $f, g \in L^1 \implies \displaystyle\int \widehat{f} g = \int f \widehat{g}.$
- $g(x) := e^{-\pi |t|^2} \implies \widehat{g}(\xi) = g(\xi)$
- $g_t(x) = t^{-n} g(x/t) = t^{-n} e^{-\pi |x|^2 / t^2}$
- $\widehat{g}_t(x) = g(tx) = e^{-\pi t^2 |x|^2}$
- $\phi(\xi) := e^{2\pi i x \cdot \xi}\ \widehat{g}_t(\xi)$
- $\widehat{\phi}(\xi) = \mathcal{F}(\widehat{g}_t(\xi - x)) = g_t(x - \xi)$
- $\displaystyle\lim_{t \longrightarrow 0} \phi(\xi) = e^{2\pi i x \cdot \xi}$

Take the modified integral:

$$\begin{aligned}
I_t(x) &= \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}\ e^{-\pi t^2 |\xi|^2} \\
&= \int \widehat{f}(\xi)\phi(\xi) \\
&= \int f(\xi)\widehat{\phi}(\xi) \\
&= \int f(\xi)\widehat{\widehat{g}}(\xi - x) \\
&= \int f(\xi) g_t(x-\xi)\ d\xi \\
&= \int f(y-x) g_t(y)\ dy \quad (\xi = y - x) \\
&= (f * g_t) \\
&\longrightarrow f \text{ in } L^1 \text{ as } t \longrightarrow 0,
\end{aligned}$$

but we also have

$$\begin{aligned}
\lim_{t\longrightarrow 0} I_t(x) &= \lim_{t\longrightarrow 0} \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}\ e^{-\pi t^2|\xi|^2} \\
&= \lim_{t\longrightarrow 0} \int \widehat{f}(\xi)\phi(\xi) \\
&=_{DCT} \int \widehat{f}(\xi) \lim_{t\longrightarrow 0} \phi(\xi) \\
&= \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}
\end{aligned}$$

.

So

$$I_t(x) \longrightarrow \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi} \text{ pointwise and } \left\| I_t(x) - f(x) \right\|_1 \longrightarrow 0.$$

So there is a subsequence $I_{t_n}$ such that $I_{t_n}(x) \longrightarrow f(x)$ almost everywhere, so $f(x) = \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}$ almost everywhere by uniqueness of limits. ■

## 7.3 3: Hilbert Spaces

### 7.3.1 a

**Theorem (Bessel's Inequality):**

$$\left\| x - \sum_{n=1}^{N} \langle x, u_n\rangle\, u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n\rangle|^2$$

and thus

$$\sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 \leq \|x\|^2$$

*Proof:* Let $S_N = \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n$

$$\begin{aligned}
\|x - S_N\|^2 &= \langle x - S_n,\ x - S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\langle x,\ S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\left\langle x,\ \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n \right\rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^{N} \langle x,\ \langle x,\ u_n \rangle u_n \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^{N} \overline{\langle x,\ u_n \rangle} \langle x,\ u_n \rangle \\
&= \|x\|^2 + \left\| \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n \right\|^2 - 2 \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 - 2 \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
&= \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2.
\end{aligned}$$

And by continuity of the norm and inner product, we have

$$\begin{aligned}
\lim_{N \longrightarrow \infty} \|x - S_N\|^2 &= \lim_{N \longrightarrow \infty} \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
\implies \left\| x - \lim_{N \longrightarrow \infty} S_N \right\|^2 &= \|x\|^2 - \lim_{N \longrightarrow \infty} \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
\implies \left\| x - \sum_{n=1}^{\infty} \langle x,\ u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2.
\end{aligned}$$

Then noting that $0 \leq \|x - S_N\|^2$, we have

$$\begin{aligned}
0 &\leq \|x\|^2 - \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2 \\
\implies \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2 &\leq \|x\|^2 \blacksquare.
\end{aligned}$$

### 7.3.2 b

- Take $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
- Define $x = \lim_{N \longrightarrow \infty} S_N$ where $S_N = \sum_{k=1}^{N} a_k u_k$

- $\{S_N\}$ is Cauchy and $H$ is complete, so $x \in H$.
- By construction, $\langle x,\ u_n \rangle = \left\langle \sum_k a_k u_k,\ u_n \right\rangle = \sum_k a_k \langle u_k,\ u_n \rangle = a_n$ since the $u_k$ are all orthogonal.
- $\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$ by Pythagoras since the $u_k$ are normal.

### 7.3.3 c

Let $x$ and $u_n$ be arbitrary. Then

$$\begin{aligned}
\left\langle x - \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k,\ u_n \right\rangle &= \langle x,\ u_n \rangle - \left\langle \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k,\ u_n \right\rangle \\
&= \langle x,\ u_n \rangle - \sum_{k=1}^{\infty} \langle \langle x,\ u_k \rangle u_k,\ u_n \rangle \\
&= \langle x,\ u_n \rangle - \sum_{k=1}^{\infty} \langle x,\ u_k \rangle \langle u_k,\ u_n \rangle \\
&= \langle x,\ u_n \rangle - \langle x,\ u_n \rangle = 0 \\
\implies x - \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k = 0 \quad & \text{by completeness.}
\end{aligned}$$

So

$$x = \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k \implies \|x\|^2 = \sum_{k=1}^{\infty} |\langle x,\ u_k \rangle|^2 . \blacksquare$$

## 7.4 4: Lp Spaces

p-test for integrals:

$$\begin{aligned}
\int_0^1 x^{-p} < \infty &\iff p < 1 \\
\int_1^{\infty} x^{-p} < \infty &\iff p > 1.
\end{aligned}$$

Yields a general technique: break integrals apart at $x = 1$.

Inclusions and relationships:

$$\begin{aligned}
m(X) < \infty \implies & L^{\infty} \subset L^2 \subset L^1 \\
& \ell^2(\mathbb{Z}) \subset \ell^1(\mathbb{Z}) \subset \ell^{\infty}(\mathbb{Z}).
\end{aligned}$$

### 7.4.1 a

**Theorem (Holder's Inequality):**

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

*Proof:*

It suffices to show this when $\|f\|_p = \|g\|_q = 1$, since

$$\|fg\|_1 \leq \|f\|_p \|f\|_q \Longleftrightarrow \int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \leq 1.$$

Using $AB \leq \frac{1}{p}A^p + \frac{1}{q}B^q$, we have

$$\int |f||g| \leq \int \frac{|f|^p}{p} \frac{|g|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1 \blacksquare.$$

**Theorem (Minkowski's Inequality):**

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

*Proof:*

We first note

$$|f+g|^p = |f+g||f+g|^{p-1} \leq (|f|+|g|)\,|f+g|^{p-1}.$$

Note that if $p, q$ are conjugate exponents then

$$\begin{aligned} \frac{1}{q} &= 1 - \frac{1}{p} = \frac{p-1}{p} \\ q &= \frac{p}{p-1}. \end{aligned}$$

Then taking integrals yields

$$
\begin{aligned}
\|f+g\|_p^p &= \int |f+g|^p \\
&\leq \int (|f|+|g|)\,|f+g|^{p-1} \\
&= \int |f||f+g|^{p-1} + \int |g||f+g|^{p-1} \\
&= \left\| f(f+g)^{p-1} \right\|_1 + \left\| g(f+g)^{p-1} \right\|_1 \\
&\leq \|f\|_p \left\| (f+g)^{p-1}) \right\|_q + \|g\|_p \left\| (f+g)^{p-1}) \right\|_q \\
&= \left( \|f\|_p + \|g\|_p \right) \left\| (f+g)^{p-1}) \right\|_q \\
&= \left( \|f\|_p + \|g\|_p \right) \left( \int |f+g|^{(p-1)q} \right)^{\frac{1}{q}} \\
&= \left( \|f\|_p + \|g\|_p \right) \left( \int |f+g|^{p} \right)^{1-\frac{1}{p}} \\
&= \left( \|f\|_p + \|g\|_p \right) \frac{\int |f+g|^p}{\left(\int |f+g|^p\right)^{\frac{1}{p}}} \\
&= \left( \|f\|_p + \|g\|_p \right) \frac{\|f+g\|_p^p}{\|f+g\|_p}
\end{aligned}
$$

and canceling common terms yields

$$
\begin{aligned}
1 &\leq \left( \|f\|_p + \|g\|_p \right) \frac{1}{\|f+g\|_p} \\
&\implies \|f+g\|_p \leq \|f\|_p + \|g\|_p \blacksquare.
\end{aligned}
$$

### 7.4.2 c

**Definition (Infinity Norm):**

$$
\begin{aligned}
L^\infty(X) = \left\{ f : X \longrightarrow \mathbb{C} \;\middle|\; \|f\|_\infty < \infty \right\} \\
\text{where} \\
\|f\|_\infty = \inf_{\alpha \geq 0} \left\{ \alpha \;\middle|\; m\left\{ |f| \geq \alpha \right\} = 0 \right\}.
\end{aligned}
$$

**Theorem:**

$$
m(X) < \infty \implies \lim_{p \longrightarrow \infty} \|f\|_p = \|f\|_\infty.
$$

*Proof:* Let $M = \|f\|_\infty$. For any $L < M$, let $S = \{|f| \geq L\}$. Then $m(S) > 0$ and

$$\begin{aligned}
\|f\|_p &= \left( \int_X |f|^p \right)^{\frac{1}{p}} \\
&\geq \left( \int_S |f|^p \right)^{\frac{1}{p}} \\
&\geq L\ m(S)^{\frac{1}{p}} \xrightarrow{p \longrightarrow \infty} L \\
&\implies \liminf_p \|f\|_p \geq M.
\end{aligned}$$

We also have

$$\begin{aligned}
\|f\|_p &= \left( \int_X |f|^p \right)^{\frac{1}{p}} \\
&\leq \left( \int_X M^p \right)^{\frac{1}{p}} \\
&= M\ m(X)^{\frac{1}{p}} \xrightarrow{p \longrightarrow \infty} M \\
&\implies \limsup_p \|f\|_p \leq M \blacksquare.
\end{aligned}$$

Note: this doesn't help with this problem at all.

Solution:

$$\begin{aligned}
\int f^p &= \int_{x \leq 1} f^p + \int_{x=1} f^p + \int_{x \geq 1} f^p \\
&= \int_{x \leq 1} f^p + \int_{x=1} 1 + \int_{x \geq 1} f^p \\
&= \int_{x \leq 1} f^p + m(\{f = 1\}) + \int_{x \geq 1} f^p \\
&\xrightarrow{p \longrightarrow \infty} 0 + m(\{f = 1\}) + \begin{cases} 0 & m(\{x \geq 1\}) = 0 \\ \infty & m(\{x \geq 1\}) > 0. \end{cases}
\end{aligned}$$

### 7.5 5: Dual Spaces

**Definition:** A map $L : X \longrightarrow \mathbb{C}$ is a linear functional iff

$$L(\alpha \mathbf{x} + \mathbf{y}) = \alpha L(\mathbf{x}) + L(\mathbf{y}).$$

**Theorem (Riesz Representation for Hilbert Spaces):** If $\Lambda$ is a continuous linear functional on a Hilbert space $H$, then there exists a unique $y \in H$ such that

$$\forall x \in H, \quad \Lambda(x) = \langle x,\ y \rangle.$$

*Proof:*

- Define $M \coloneqq \ker \Lambda$.
- Then $M$ is a closed subspace and so $H = M \oplus M^\perp$
- There is some $z \in M^\perp$ such that $\|z\| = 1$.
- Set $u \coloneqq \Lambda(x)z - \Lambda(z)x$
- Check

$$\Lambda(u) = \Lambda(\Lambda(x)z - \Lambda(z)x) = \Lambda(x)\Lambda(z) - \Lambda(z)\Lambda(x) = 0 \implies u \in M$$

- Compute

$$\begin{aligned}
0 &= \langle u,\ z\rangle \\
&= \langle \Lambda(x)z - \Lambda(z)x,\ z\rangle \\
&= \langle \Lambda(x)z,\ z\rangle - \langle \Lambda(z)x,\ z\rangle \\
&= \Lambda(x)\langle z,\ z\rangle - \Lambda(z)\langle x,\ z\rangle \\
&= \Lambda(x)\|z\|^2 - \Lambda(z)\langle x,\ z\rangle \\
&= \Lambda(x) - \Lambda(z)\langle x,\ z\rangle \\
&= \Lambda(x) - \left\langle x,\ \overline{\Lambda(z)}z\right\rangle,
\end{aligned}$$

- Choose $y \coloneqq \overline{\Lambda(z)}z$.
- Check uniqueness:

$$\begin{aligned}
\langle x,\ y\rangle = \langle x,\ y'\rangle \quad \forall x \\
\implies \langle x,\ y - y'\rangle = 0 \quad \forall x \\
\implies \langle y - y',\ y - y'\rangle = 0 \\
\implies \|y - y'\| = 0 \\
\implies y - y' = \mathbf{0} \implies y = y'.
\end{aligned}$$

### 7.5.1 b

**Theorem (Continuous iff Bounded):** Let $L : X \longrightarrow \mathbb{C}$ be a linear functional, then the following are equivalent:

1. $L$ is continuous
2. $L$ is continuous at zero
3. $L$ is bounded, i.e. $\exists c \geq 0 \mathrel{\Big|} |L(x)| \leq c\|x\|$ for all $x \in H$

$2 \implies 3$: Choose $\delta < 1$ such that

$$\|x\| \leq \delta \implies |L(x)| < 1.$$

Then

$$
\begin{aligned}
|L(x)| &= \left| L\left( \frac{\|x\|}{\delta} \frac{\delta}{\|x\|} x \right) \right| \\
&= \frac{\|x\|}{\delta} \left| L\left( \delta \frac{x}{\|x\|} \right) \right| \\
&\leq \frac{\|x\|}{\delta} 1,
\end{aligned}
$$

so we can take $c = \frac{1}{\delta}$. ■

$3 \implies 1$:

We have $|L(x-y)| \leq c\|x-y\|$, so given $\varepsilon \geq 0$ simply choose $\delta = \frac{\varepsilon}{c}$.

### 7.5.2 c

**Definition (Dual Space):**

$$
X^{\vee} \coloneqq \left\{ L : X \longrightarrow \mathbb{C} \;\middle|\; L \text{ is continuous } \right\}
$$

**Definition (Operator Norm):**

$$
\|L\|_{X^{\vee}} \coloneqq \sup_{\substack{x \in X \\ \|x\|=1}} |L(x)|
$$

**Theorem: (Operator Norm is a Norm)**

*Proof:* The only nontrivial property is the triangle inequality, but

$$
\|L_1 + L_2\| = \sup |L_1(x) + L_2(x)| \leq \sup L_1(x) + \sup L_2(x) = \|L_1\| + \|L_2\|.
$$

**Theorem (Completeness in Operator Norm):**

$X^{\vee}$ equipped with the operator norm is a Banach space.

*Proof:*

- Let $\{L_n\}$ be Cauchy in $X^{\vee}$.
- Then for all $x \in C$, $\{L_n(x)\} \subset \mathbb{C}$ is Cauchy and converges to something denoted $L(x)$.
- Need to show $L$ is continuous and $\|L_n - L\| \longrightarrow 0$.
- Since $\{L_n\}$ is Cauchy in $X^{\vee}$, choose $N$ large enough so that

$$
n, m \geq N \implies \|L_n - L_m\| < \varepsilon \implies |L_m(x) - L_n(x)| < \varepsilon \quad \forall x \;\middle|\; \|x\| = 1.
$$

- Take $n \longrightarrow \infty$ to obtain

$$
\begin{aligned}
m \geq N &\implies |L_m(x) - L(x)| < \varepsilon \quad \forall x \;\middle|\; \|x\| = 1 \\
&\implies \|L_m - L\| < \varepsilon \longrightarrow 0.
\end{aligned}
$$

- Continuity:

$$\begin{aligned}
|L(x)| &= |L(x) - L_n(x) + L_n(x)| \\
&\leq |L(x) - L_n(x)| + |L_n(x)| \\
&\leq \varepsilon \|x\| + c\|x\| \\
&= (\varepsilon + c)\|x\| \blacksquare.
\end{aligned}$$

# 8 Exam 2 (2018)

Link to PDF File

# 9 Exam 2 (2014)

Link to PDF File

# 10 Qual: Fall 2019

## 10.1 1

See phone photo?

## 10.2 2

DCT?

## 10.3 3

"Follow your nose."

## 10.4 4

See Problem Set 8.

**Bessel's Inequality:** For any orthonormal set in a Hilbert space (not necessarily a basis), we have

$$\sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2 \leq \|x\|^2$$

*Proof:*

$$0 \leq \left\| x - \sum_{k=1}^{n} \langle x, e_k \rangle\, e_k \right\|^2$$

**Corollary (Parseval's Identity):** If span $\{u_n\}$ is dense in $\mathcal{H}$, so it is a basis, then this is an equality.

**Riesz-Fischer:** Let $U = \{u_n\}_{n=1}^{\infty}$ be an orthonormal set (not necessarily a basis), then

1. There is an isometric surjection

$$
\begin{aligned}
\mathcal{H} &\longrightarrow \ell^2(\mathbb{N}) \\
\mathbf{x} &\mapsto \{\langle \mathbf{x},\ \mathbf{u}_n \rangle\}_{n=1}^{\infty}
\end{aligned}
$$

i.e. if $\{a_n\} \in \ell^2(\mathbb{N})$, so $\sum |a_n|^2 < \infty$, then there exists a $\mathbf{x} \in \mathcal{H}$ such that

$$a_n = \langle \mathbf{x},\ \mathbf{u}_n \rangle \quad \forall n.$$

2. $\mathbf{x}$ can be chosen such that

$$\|\mathbf{x}\|^2 = \sum |a_n|^2$$

Note: the choice of $\mathbf{x}$ is unique $\iff$ $\{u_n\}$ is **complete**, i.e. $\langle \mathbf{x},\ \mathbf{u}_n \rangle = 0$ for all $n$ implies $\mathbf{x} = \mathbf{0}$.

*Proof:*

- Given $\{a_n\}$, define $S_N = \sum^N a_n \mathbf{u}_n$.
- $S_N$ is Cauchy in $\mathcal{H}$ and so $S_N \longrightarrow \mathbf{x}$ for some $\mathbf{x} \in \mathcal{H}$.
- $\langle x,\ u_n \rangle = \langle x - S_N,\ u_n \rangle + \langle S_N,\ u_n \rangle \longrightarrow a_n$
- By construction, $\|x - S_N\|^2 = \|x\|^2 - \sum^N |a_n|^2 \longrightarrow 0$, so $\|x\|^2 = \sum^\infty |a_n|^2$.

### 10.5 5

See Problem Set 5.

**Heine-Cantor theorem:** Every continuous function on a compact set is uniformly continuous.

Uniform continuity:

$$
\begin{aligned}
&\forall \varepsilon \quad \exists \delta(\varepsilon) \;\Big|\quad \forall x, y, \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \\
\iff &\forall \varepsilon \quad \exists \delta(\varepsilon) \;\Big|\quad \forall x, y, \quad |y| < \delta \implies |f(x - y) - f(y)| < \varepsilon \\
&.
\end{aligned}
$$

Fubini-Tonelli interchange of integrals, where the change of bounds becomes very important.

Continuity in $L^1$:

$$\lim_{y \longrightarrow 0} \|\tau_y f - f\|_1 = 0.$$

### 10.6 Definitions

- Banach Space
- $L^p$

## 10.7 Useful Results

- Cauchy-Schwarz
- Young's Inequality
- Holder's Inequality
- Minkowski's Inequality
- Jensen's Inequality:

$$r^{-1} := p^{-1} + q^{-1} - 1 \implies \|f * g\|_r \leq \|f\|_p \|g\|_q$$

  - Useful variant - take $q = 1$ to get $\|f * g\|_p \leq \|f\|_p \|g\|_1$
  - Take $p = 1$ to show $L_1$ is closed under $*$.
- The Riemann-Lebesgue Lemma
- Proving that $\delta \notin L_1(\mathbb{R})$ and that there is no such identity
  - Rather, is a distribution or measure that *acts* on $f$ and satisfies $f(x) \int_{\mathbb{R}} f(t)\delta(t-x)\, dt$
- Fubini's Theorem
- Density Results:
  - $C_c(\mathbb{R}) \subset C_0(\mathbb{R})$
- $C_c(\mathbb{R}) \bigcap C^\infty(\mathbb{R}) \neq \emptyset$, e.g. take $f(x) = e^{\frac{-1}{x^2}} \chi_{(0,\infty}(x)$.
- The Banach Algebra $L^1(\mathbb{R})$ is not a principal ideal domain.
- Every locally compact abelian group $G$ has a unique Borel measure (up to scaling) that is positive, regular, translation-invariant (the Haar measure).
  - For $\mathbb{R}, (S_1)^2$, equal to the Lebesgue measure. For $\mathbb{Z}$, the counting measure.