Topology Qualifying Exam Notes

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Contents

1 Theorems

1.1 Van Kampen

If $X = U \bigcup V$ where $U, V, U \bigcap V$ are all path-connected then

$$\pi_1(X) = \pi_1 U *_{\pi_1(U \bigcap V)} \pi_1 V,$$

where the amalgamated product can be computed as follows: If we have presentations

$$\begin{aligned}\pi_1(U,w) &= \left\langle u_1,\cdots,u_k \;\middle|\; \alpha_1,\cdots,\alpha_l \right\rangle\\ \pi_1(V,w) &= \left\langle v_1,\cdots,v_m \;\middle|\; \beta_1,\cdots,\beta_n \right\rangle\\ \pi_1(U\cap V,w) &= \left\langle w_1,\cdots,w_p \;\middle|\; \gamma_1,\cdots,\gamma_q \right\rangle\end{aligned}$$

then

$$\begin{aligned}\pi_1(X,w) &= \langle u_1,\cdots,u_k,v_1,\cdots,v_m\rangle\\ &\qquad \bmod \left\langle \alpha_1,\cdots,\alpha_l,\beta_1,\cdots,\beta_n, I(w_1)\,J(w_1)^{-1},\cdots,I(w_p)\,J(w_p)^{-1}\right\rangle\\ &= \frac{\pi_1(U)*\pi_1(B)}{\left\langle\left\{I(w_i)J(w_i)^{-1} \;\middle|\; 1\le i\le p\right\}\right\rangle}\end{aligned}$$

where

$$\begin{aligned}I &: \pi_1(U\cap V,w)\to\pi_1(U,w)\\ J &: \pi_1(U\cap V,w)\to\pi_1(V,w).\end{aligned}$$

Revised May 2006.

The weight of topics on the exam should be about 1/3 general topology and 2/3 algebraic topology.

2 General Topology

- Topological spaces, continuous functions, product and quotient topology [1, ch. 2]
- Connectedness and compactness [1, ch. 3]
- Countability and separation axioms, Urysohn lemma, Tietze theorem [1, ch. 4, except §36]
- Complete metric spaces and function spaces [1, §43, 45]

3 Algebraic Topology

- Classification of surfaces [2, ch. I]
- Fundamental group [2, ch. II], [3, §1.1]
- van Kampen's theorem [2, ch. III, IV], [3, §1.2]
- Classification of covering spaces [2, ch. V], [3 §1.3]

3.1 Homology:

- Simplicial, singular, cellular; computations and applications [3, ch. 2], [4, ch. 4]
- Degree of a map $S^n \longrightarrow S^n$ [3, p. 134], [4, §21]
- Euler characteristic [3, p. 146]
- Lefschetz fixed point theorem [3, p. 179], [4, §22]

4 References

- [1] J. Munkres, Topology , second edition, Prentice-Hall, 2000.
- [2] W. Massey, A Basic Course in Algebraic Topology , Springer-Verlag, 1991.
- [3] A. Hatcher, Algebraic Topology , Cambridge U. Press, 2002.
  - Revisions and corrections `http://www.math.cornell.edu/~hatcher/AT/ATpage.html`
- [4] J. Munkres, Elements of Algebraic Topology , Addison-Wesley, 1984.

5 Definitions

Basis: A subset $\{B_i\}$ is a basis iff

- $x \in X \implies x \in B_i$ for some i.
- $x \in B_i \bigcap B_j \implies x \in B_k \subset B_i \bigcap B_k$.
- Topology generated by this basis: $x \in N_x \implies x \in B_i \subset N_x$ for some i.

Dense: A subset $Q \subset X$ is dense iff $y \in N_y \subset X \implies N_y \bigcap Q \neq \emptyset$ iff $\overline{Q} = X$.

6 Examples

The cofinite topology:

- Non-Hausdorff
- Compact

The discrete topology:

- Discrete iff points are open
- Always Hausdorff
- Compact iff finite
- Totally disconnected
- If the domain, every map is continuous

The indiscrete topology:

- Only open sets are $\emptyset, X$
- Non-Hausdorff
- If the codomain, every map is continuous
- Compact

Non-Hausdorff spaces:

- The cofinite topology on any infinite set.
- $\mathbb{R}/\mathbb{Q}$
- The line with two origins.

7 Theorems

Theorem: Points are closed in T_1 spaces.

Theorem: A metric space X is sequentially compact iff it is complete and totally bounded.

Theorem: A metric space is totally bounded iff every sequence has a Cauchy subsequence.

Theorem: A metric space is compact iff it is complete and totally bounded.

Theorem (Baire): If X is a complete metric space, then the intersection of countably many dense open sets is dense in X.

Theorem: A continuous bijective open map is a homeomorphism.

Theorem: A closed subset A of a compact set B is compact.

Proof:

- Let $\{A_i\} \rightrightarrows A$ be a covering of A by sets open in A.
- Each $A_i = B_i \bigcap A$ for some B_i open in B (definition of subspace topology)
- Define $V = \{B_i\}$, then $V \rightrightarrows A$ is an open cover.
- Since A is closed, $W \coloneqq B \setminus A$ is open
- Then $V \bigcup W$ is an open cover of B, and has a finite subcover $\{V_i\}$
- Then $\left\{V_i \bigcap A\right\}$ is a finite open cover of A.

Theorem: The continuous image of a compact set is compact.

Theorem: A closed subset of a Hausdorff space is compact.

8 Unsorted

8.1 Definitions

- Neighborhood: A neighborhood of a point x is any open set containing x.
- Hausdorff
- Second Countable: admits a countable basis.
- Closed (several characterizations)
- Closure in a subspace: $Y \subset X \implies \mathrm{cl}_Y(A) \coloneqq \mathrm{cl}_X(A) \bigcap Y$.
- Bounded
- Compact: A topological space (X, τ) is **compact** if every open cover has a *finite* subcover. That is, if $\left\{U_j \;\middle|\; j \in J\right\} \subset \tau$ is a collection of open sets such that $X \subseteq \bigcup_{j \in J} U_j$, then there exists a *finite* subset $J' \subset J$ such that $X \subseteq \bigcup_{j \in J'} U_j$.
- Locally compact For every $x \in X$, there exists a $K_x \ni x$ such that K_x is compact.
- Connected: There does not exist a disconnecting set $X = A \coprod B$ such that $\emptyset \neq A, B \subsetneq$, i.e. X is the union of two proper disjoint nonempty sets. Equivalently, X contains no proper nonempty clopen sets.
  - Additional condition for a subspace $Y \subset X$: $\mathrm{cl}_Y(A) \bigcap V = A \bigcap \mathrm{cl}_Y(B) = \emptyset$.

- Locally connected: A space is locally connected at a point x iff $\forall N_x \ni x$, there exists a $U \subset N_x$ containing x that is connected.
- Retract: A subspace $A \subset X$ is a *retract* of X iff there exists a continuous map $f : X \longrightarrow A$ such that $f \Big|_A = \mathrm{id}_A$. Equivalently it is a *left* inverse to the inclusion.
- Uniform Continuity: For $f : (X, d_x) \longrightarrow (Y, d_Y)$ metric spaces,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \quad d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

- Lebesgue number: For (X, d) a compact metric space and $\{U_\alpha\} \rightrightarrows X$, there exist $\delta_L > 0$ such that

$$A \subset X,\ \mathrm{diam}(A) < \delta_L \implies A \subseteq U_\alpha \text{ for some } \alpha.$$

- Paracompact
- Components: Set $x \sim y$ iff there exists a connected set $U \ni x, y$ and take equivalence classes.
- Path Components: Set $x \sim y$ iff there exists a path-connected set $U \ni x, y$ and take equivalence classes.
- Separable: countable dense subset.
- Limit Point: For $A \subset X$, x is a limit point of A if every punctured neighborhood P_x of x satisfies $P_x \bigcap A \neq \emptyset$, i.e. every neighborhood of x intersects A in some point other than x itself. Equivalently, x is a limit point of A iff $x \in \mathrm{cl}_X(A \setminus \{x\})$.

8.2 Sandbox of Spaces

- Finite discrete sets with the discrete topology
- Subspaces of $\mathbb{R}$: $(a, b), (a, b], (a, \infty)$, etc.
  - $\{0\} \bigcup \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{Z}^{\geq 1} \right\}$
- $\mathbb{Q}$
- The topologist's sine curve
- One-point compactifications
- $\mathbb{R}^\omega$

Alternative topologies to consider:

- Cofinite
- Discrete and Indiscrete
- Uniform

8.3 Definitions

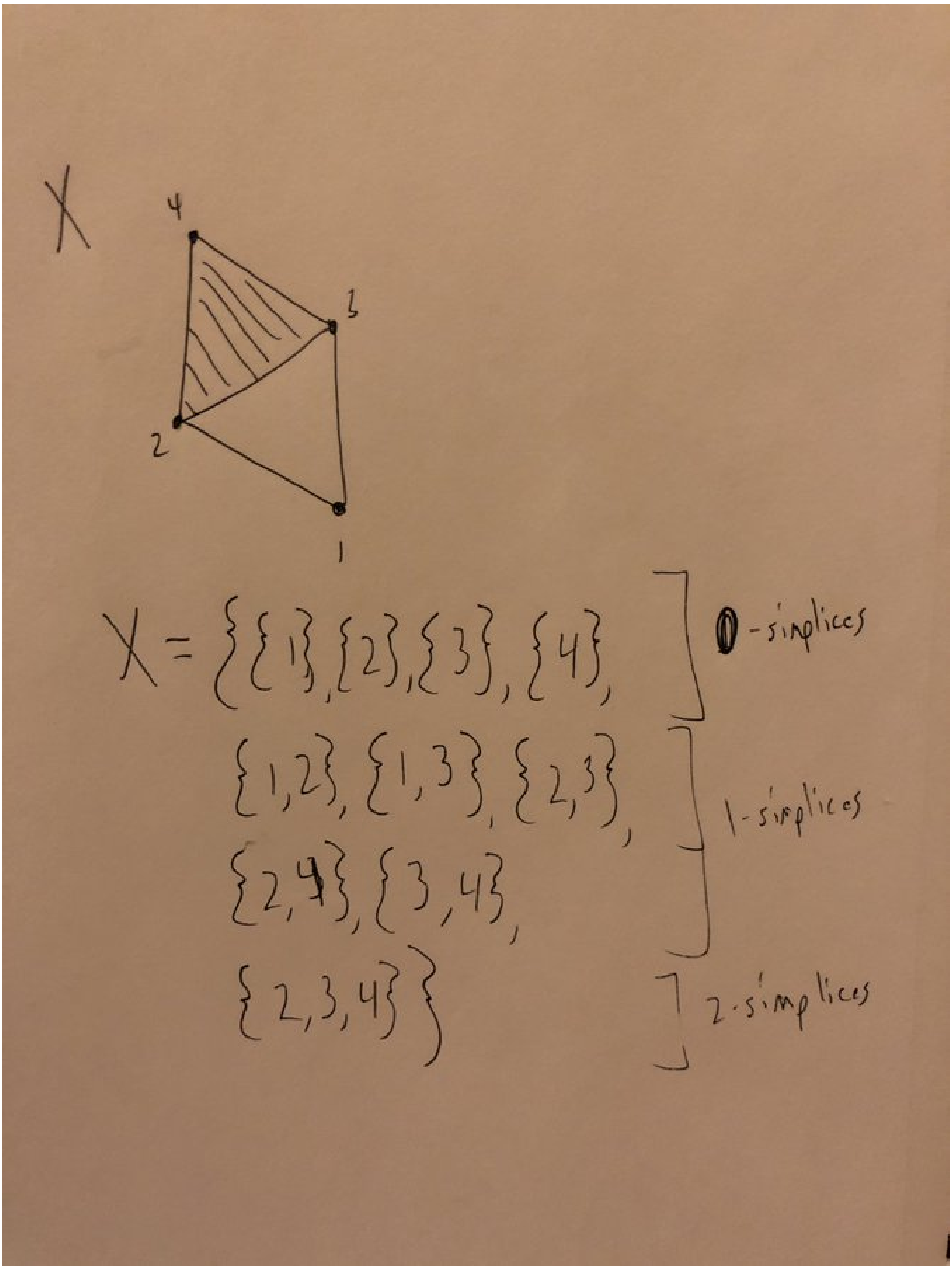
- Topology: arbitrary unions, finite intersections

8.4 Common Spaces and Operations

Spaces

$$S^n, \mathbb{D}^n, T^n \mathbb{RP}^n, \mathbb{CP}^n, M, K, \Sigma_g, \mathbb{RP}^\infty, \mathbb{CP}^\infty.$$

- Knot complements in S^3
- Lens spaces
- Prism spaces
- $\mathbb{HP}^n$
- Dunce Cap
- Matrix groups
- Pair of pants
- Covering spaces (hyperbolic geometry)
- Seifert surfaces
- Surgery
- Hawaiian earring
- Horned sphere
- Cantor set
- Simplicial Complexes
  - Nice minimal example:



Operations

- Cartesian product $A \times B$
- Wedge product $A \vee B$

- Connect Sum $A\#B$
- Quotienting A/B
- Puncturing $A \setminus \{a_i\}$
- Smash product
- Join
- Cones
- Suspension
- Loop space
- Identifying a finite number of points

8.5 Theorems

General Topology

- Closed subsets of Hausdorff spaces are compact? (check)
- Cantor's intersection theorem?
- Tube lemma
- Properties pushed forward through continuous maps:
  - Compactness?
  - Connectedness (when surjective)
  - Separability
  - Density **only when** f is surjective
  - **Not** openness
  - **Not** closedness
- A retract of a Hausdorff/connected/compact space is closed/connected/compact respectively.

Proposition 8.1.
A continuous function on a compact set is uniformly continuous.

Proof .
Take $\left\{ B_{\frac{\varepsilon}{2}}(y) \;\middle|\; y \in Y \right\} \rightrightarrows Y$, pull back to an open cover of X, has Lebesgue number $\delta_L > 0$, then $x' \in B_{\delta_L}(x) \implies f(x), f(x') \in B_{\frac{\varepsilon}{2}}(y)$ for some y.

■

- Lipschitz continuity implies uniform continuity (take $\delta = \varepsilon/C$)
  - Counterexample to converse: $f(x) = \sqrt{x}$ on $[0, 1]$ has unbounded derivative.
- Extreme Value Theorem: for $f : X \longrightarrow Y$ continuous with X compact and Y ordered in the order topology, there exist points $c, d \in X$ such that $f(x) \in [f(c), f(d)]$ for every x.

Algebraic Topology

Theorem 8.2 ***(Van Kampen).***
The pushout is the northwest colimit of the following diagram

$$\begin{tikzcd} A\coprod_Z B & A \arrow[l] \\ B & Z \arrow[u, hook', "\iota_A"] \arrow[l, hook', "\iota_B"] \end{tikzcd}$$

For groups, the pushout is given by the amalgamated free product: if $A = \left\langle G_A \;\middle|\; R_A \right\rangle, B = \left\langle G_B \;\middle|\; R_B \right\rangle$, then $A *_Z B = \left\langle G_A, G_B \;\middle|\; R_A, R_B, T \right\rangle$ where T is a set of relations given by $T = \left\{ \iota_A(z)\iota_B(z)^{-1} \;\middle|\; z \in Z \right\}$.

Example: $A = \mathbb{Z}/4\mathbb{Z} = \left\langle x \;\middle|\; x^4 \right\rangle, B = \mathbb{Z}/6\mathbb{Z} = \left\langle y \;\middle|\; x^6 \right\rangle, Z = \mathbb{Z}/2\mathbb{Z} = \left\langle z \;\middle|\; z^2 \right\rangle$. Then we can identify Z as a subgroup of A, B using $\iota_A(z) = x^2$ and $\iota_B(z) = y^3$. So

$$A *_Z B = \left\langle x, y \;\middle|\; x^4, y^6, x^2 y^{-3} \right\rangle$$

.

Suppose $X = U_1 \bigcup U_2$ such that $U_1 \bigcap U_2 \neq \emptyset$ is path connected. Then taking $x_0 \in U \coloneqq U_1 \bigcap U_2$ yields a pushout of fundamental groups

$$\pi_1(X; x_0) = \pi_1(U_1; x_0) *_{\pi_1(U; x_0)} \pi_1(U_2; x_0).$$