

Title

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1 Group Theory

1.1 Spring 2020 #1

- Show that any group of order 2020 is solvable.
- Give (without proof) a classification of all abelian groups of order 2020.
- Describe one nonabelian group of order 2020.

1.2 Spring 2020 #2

Let H be a normal subgroup of a finite group G where the order of H and the index of H in G are relatively prime. Prove that no other subgroup of G has the same order as H .

1.3 Fall 2019 #1

Let G be a finite group with n distinct conjugacy classes. Let $g_1 \cdots g_n$ be representatives of the conjugacy classes of G .

Prove that if $g_i g_j = g_j g_i$ for all i, j then G is abelian.

1.4 Fall 2019 #2

Let G be a group of order 105 and let P, Q, R be Sylow 3, 5, 7 subgroups respectively.

- Prove that at least one of Q and R is normal in G .
- Prove that G has a cyclic subgroup of order 35.
- Prove that both Q and R are normal in G .
- Prove that if P is normal in G then G is cyclic.

1.5 Spring 2019 #3

How many isomorphism classes are there of groups of order 45?

Describe a representative from each class.

1.6 Spring 2019 #4

For a finite group G , let $c(G)$ denote the number of conjugacy classes of G .

- (a) Prove that if two elements of G are chosen uniformly at random, then the probability they commute is precisely

$$\frac{c(G)}{|G|}.$$

- (b) State the class equation for a finite group.

- (c) Using the class equation (or otherwise) show that the probability in part (a) is at most

$$\frac{1}{2} + \frac{1}{2[G : Z(G)]}.$$

Here, as usual, $Z(G)$ denotes the center of G .

1.7 Fall 2012 #1

Let G be a finite group and X a set on which G acts.

- Let $x \in X$ and $G_x := \{g \in G \mid g \cdot x = x\}$. Show that G_x is a subgroup of G .
- Let $x \in X$ and $G \cdot x := \{g \cdot x \mid g \in G\}$. Prove that there is a bijection between elements in $G \cdot x$ and the left cosets of G_x in G .

1.8 Fall 2012 #2

Let G be a group of order 30.

- Show that G contains normal subgroups of orders 3, 5, and 15.
- Give all possible presentations and relations for G .
- Determine how many groups of order 30 there are up to isomorphism.

1.9 Spring 2012 #2

Let G be a finite group and p a prime number such that there is a normal subgroup $H \trianglelefteq G$ with $|H| = p^i > 1$.

- Show that H is a subgroup of any Sylow p -subgroup of G .
- Show that G contains a nonzero abelian normal subgroup of order divisible by p .

1.10 Spring 2012 #3

Let G be a group of order 70.

- Show that G is not simple.
- Exhibit 3 nonisomorphic groups of order 70 and prove that they are not isomorphic.

1.11 Fall 2018 #1

Let G be a finite group whose order is divisible by a prime number p . Let P be a normal p -subgroup of G (so $|P| = p^c$ for some c).

- (a) Show that P is contained in every Sylow p -subgroup of G .
- (b) Let M be a maximal proper subgroup of G . Show that either $P \subseteq M$ or $|G/M| = p^b$ for some $b \leq c$.

1.12 Fall 2018 #2

- (a) Suppose the group G acts on the set X . Show that the stabilizers of elements in the same orbit are conjugate.
- (b) Let G be a finite group and let H be a proper subgroup. Show that the union of the conjugates of H is strictly smaller than G , i.e.

$$\bigcup_{g \in G} gHg^{-1} \subsetneq G$$

- (c) Suppose G is a finite group acting transitively on a set S with at least 2 elements. Show that there is an element of G with no fixed points in S .

1.13 Spring 2018 #1

- (a) Use the Class Equation (equivalently, the conjugation action of a group on itself) to prove that any p -group (a group whose order is a positive power of a prime integer p) has a nontrivial center.
- (b) Prove that any group of order p^2 (where p is prime) is abelian.
- (c) Prove that any group of order $5^2 \cdot 7^2$ is abelian.
- (d) Write down exactly one representative in each isomorphism class of groups of order $5^2 \cdot 7^2$.

1.14 Fall 2017 #1

Suppose the group G acts on the set A . Assume this action is faithful (recall that this means that the kernel of the homomorphism from G to $\text{Sym}(A)$ which gives the action is trivial) and transitive (for all a, b in A , there exists g in G such that $g \cdot a = b$).

- (a) For $a \in A$, let G_a denote the stabilizer of a in G . Prove that for any $a \in A$,

$$\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = \{1\}.$$

- (b) Suppose that G is abelian. Prove that $|G| = |A|$. Deduce that every abelian transitive subgroup of S_n has order n .

1.15 Fall 2017 #2

- (a) Classify the abelian groups of order 36.

For the rest of the problem, assume that G is a non-abelian group of order 36.

You may assume that the only subgroup of order 12 in S_4 is A_4 and that A_4 has no subgroup of order 6.

- (b) Prove that if the 2-Sylow subgroup of G is normal, G has a normal subgroup N such that G/N is isomorphic to A_4 .
- (c) Show that if G has a normal subgroup N such that G/N is isomorphic to A_4 and a subgroup H isomorphic to A_4 it must be the direct product of N and H .
- (d) Show that the dihedral group of order 36 is a non-abelian group of order 36 whose Sylow-2 subgroup is not normal.

1.16 Spring 2017 #1

Let G be a finite group and $\pi : G \rightarrow \text{Sym}(G)$ the Cayley representation. (Recall that this means that for an element $x \in G$, $\pi(x)$ acts by left translation on G .)

Prove that $\pi(x)$ is an odd permutation \iff the order $|\pi(x)|$ of $\pi(x)$ is even and $|G|/|\pi(x)|$ is odd.

1.17 Spring 2017 #2

- a. How many isomorphism classes of abelian groups of order 56 are there? Give a representative for one of each class.
- b. Prove that if G is a group of order 56, then either the Sylow-2 subgroup or the Sylow-7 subgroup is normal.
- c. Give two non-isomorphic groups of order 56 where the Sylow-7 subgroup is normal and the Sylow-2 subgroup is *not* normal. Justify that these two groups are not isomorphic.

1.18 Fall 2016 #1

Let G be a finite group and $s, t \in G$ be two distinct elements of order 2. Show that subgroup of G generated by s and t is a dihedral group.

Recall that the dihedral groups of order $2m$ for $m \geq 2$ are of the form

$$D_{2m} = \langle \sigma, \tau \mid \sigma^m = 1 = \tau^2, \tau\sigma = \sigma^{-1}\tau \rangle.$$

1.19 Fall 2016 #3

How many groups are there up to isomorphism of order pq where $p < q$ are prime integers?

1.20 ★ Fall 2016 #7

- a. Define what it means for a group G to be *solvable*.
- b. Show that every group G of order 36 is solvable.

Hint: you can use that S_4 is solvable.

1.21 Spring 2016 #3

- a. State the three Sylow theorems.
- b. Prove that any group of order 1225 is abelian.
- c. Write down exactly one representative in each isomorphism class of abelian groups of order 1225.

1.22 Spring 2016 #5

Let G be a finite group acting on a set X . For $x \in X$, let G_x be the stabilizer of x and $G \cdot x$ be the orbit of x .

- a. Prove that there is a bijection between the left cosets G/G_x and $G \cdot x$.
- b. Prove that the center of every finite p -group G is nontrivial by considering that action of G on $X = G$ by conjugation.

1.23 Fall 2015 #1

Let G be a group containing a subgroup H not equal to G of finite index. Prove that G has a normal subgroup which is contained in every conjugate of H which is of finite index.

1.24 Fall 2015 #2

Let G be a finite group, H a p -subgroup, and P a Sylow p -subgroup for p a prime. Let H act on the left cosets of P in G by left translation.

Prove that this is an orbit under this action of length 1.

Prove that xP is an orbit of length 1 $\iff H$ is contained in xPx^{-1} .

1.25 Spring 2015 #1

For a prime p , let G be a finite p -group and let N be a normal subgroup of G of order p . Prove that N is contained in the center of G .

1.26 Spring 2015 #4

Let N be a positive integer, and let G be a finite group of order N .

- a. Let $\text{Sym}G$ be the set of all bijections from $G \rightarrow G$ viewed as a group under composition. Note that $\text{Sym}G \cong S_N$. Prove that the Cayley map

$$\begin{aligned} C : G &\longrightarrow \text{Sym}G \\ g &\mapsto (x \mapsto gx) \end{aligned}$$

is an injective homomorphism.

- b. Let $\Phi : \text{Sym}G \rightarrow S_N$ be an isomorphism. For $a \in G$ define $\varepsilon(a) \in \{\pm 1\}$ to be the sign of the permutation $\Phi(C(a))$. Suppose that a has order d . Prove that $\varepsilon(a) = -1 \iff d$ is even and N/d is odd.
- c. Suppose $N > 2$ and $n \equiv 2 \pmod{4}$. Prove that G is not simple.

Hint: use part (b).

1.27 Fall 2014 #2

Let G be a group of order 96.

- a. Show that G has either one or three 2-Sylow subgroups.
- b. Show that either G has a normal subgroup of order 32, or a normal subgroup of order 16.

1.28 Fall 2014 #6

Let G be a group and $H, K < G$ be subgroups of finite index. Show that

$$[G : H \cap K] \leq [G : H] [G : K].$$

1.29 Spring 2014 #1

Let p, n be integers such that p is prime and p does not divide n . Find a real number $k = k(p, n)$ such that for every integer $m \geq k$, every group of order $p^m n$ is not simple.

1.30 Spring 2014 #2

Let $G \subset S_9$ be a Sylow-3 subgroup of the symmetric group on 9 letters.

- a. Show that G contains a subgroup H isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ by exhibiting an appropriate set of cycles.
- b. Show that H is normal in G .
- c. Give generators and relations for G as an abstract group, such that all generators have order 3. Also exhibit elements of S_9 in cycle notation corresponding to these generators.
- d. Without appealing to the previous parts of the problem, show that G contains an element of order 9.

1.31 Fall 2013 #1

Let p, q be distinct primes.

- Let $\bar{q} \in \mathbb{Z}_p$ be the class of $q \pmod p$ and let k denote the order of \bar{q} as an element of \mathbb{Z}_p^\times . Prove that no group of order pq^k is simple.
- Let G be a group of order pq , and prove that G is not simple.

1.32 Fall 2013 #2

Let G be a group of order 30.

- Show that G has a subgroup of order 15.
- Show that every group of order 15 is cyclic.
- Show that G is isomorphic to some semidirect product $\mathbb{Z}_{15} \rtimes \mathbb{Z}_2$.
- Exhibit three nonisomorphic groups of order 30 and prove that they are not isomorphic. You are not required to use your answer to (c).

1.33 Spring 2013 #3

Let P be a finite p -group. Prove that every nontrivial normal subgroup of P intersects the center of P nontrivially.

1.34 Spring 2013 #4

Define a *simple group*. Prove that a group of order 56 can not be simple.

1.35 Fall 2019 Midterm #1

Let G be a group of order p^2q for p, q prime. Show that G has a nontrivial normal subgroup.

1.36 Fall 2019 Midterm #2

Let G be a finite group and let P be a Sylow p -subgroup for p prime. Show that $N(N(P)) = N(P)$ where N is the normalizer in G .

1.37 Fall 2019 Midterm #3

Show that there exist no simple groups of order 148.

1.38 Fall 2019 Midterm #4

Let p be a prime. Show that $S_p = \langle \tau, \sigma \rangle$ where τ is a transposition and σ is a p -cycle.

1.39 Fall 2019 Midterm #5

Let G be a nonabelian group of order p^3 for p prime. Show that $Z(G) = [G, G]$