

# Title

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## 1 Spring 2017

### 1.1 1

Concepts used:

- Definition:  $A$  is *nowhere dense*  $\iff$  every interval  $I$  contains a subinterval  $S \subseteq A^c$ .
  - Equivalently, the interior of the closure is empty,  $(\overline{K})^\circ = \emptyset$ .

### Solution

Claim:  $K$  is **compact**.

- It suffices to show that  $K^c := [0, 1] \setminus K$  is open; Then  $K$  will be a closed and bounded subset of  $\mathbb{R}$  and thus compact by Heine-Borel.
- Strategy: write  $K^c$  as the union of open balls (since these form a basis for the Euclidean topology on  $\mathbb{R}$ ).
  - Do this by showing every point  $x \in K^c$  is an interior point, i.e.  $x$  admits a neighborhood  $N_x$  such that  $N_x \subseteq K^c$ .
- Identify  $K^c$  as the set of real numbers in  $[0, 1]$  whose decimal expansion **does** contain a 4.
  - We will show that there exists a neighborhood small enough such that all points in it contain a 4 in their decimal expansions.

- Let  $x \in K^c$ , suppose a 4 occurs as the  $k$ th digit, and write

$$x = 0.d_1d_2 \cdots d_{k-1} 4 d_{k+1} \cdots = \left( \sum_{j=1}^k d_j 10^{-j} \right) + (4 \cdot 10^{-k}) + \left( \sum_{j=k+1}^{\infty} d_j 10^{-j} \right).$$

- Set  $r_x < 10^{-k}$  and let  $y \in [0, 1] \cap B_{r_x}(x)$  be arbitrary and write

$$y = \sum_{j=1}^{\infty} c_j 10^{-j}.$$

- Thus  $|x - y| < r_x < 10^{-k}$ , and the first  $k$  digits of  $x$  and  $y$  must agree:

– We first compute the difference:

$$x - y = \sum_{i=1}^{\infty} d_i 10^{-i} - \sum_{i=1}^{\infty} c_i 10^{-i} = \sum_{i=1}^{\infty} (d_i - c_i) 10^{-i}$$

– Thus (claim)

$$|x - y| \leq \sum_{j=1}^{\infty} |d_j - c_j| 10^{-j} < 10^{-k} \iff |d_j - c_j| = 0 \quad \forall j \leq k.$$

– Otherwise we can note that any term  $|d_j - c_j| \geq 1$  and there is a contribution to  $|x - y|$  of at least  $1 \cdot 10^{-j}$  for some  $j < k$ , whereas

$$j < k \iff 10^{-j} > 10^{-k},$$

a contradiction.

- This means that for all  $j \leq k$  we have  $d_j = c_j$ , and in particular  $d_k = 4 = c_k$ , so  $y$  has a 4 in its decimal expansion.
- But then  $K^c = \bigcup_x B_{r_x}(x)$  is a union of open sets and thus open.

Claim:  $K$  is nowhere dense and  $m(K) = 0$ :

- Strategy: Show  $(\overline{K})^\circ = \emptyset$ .
- Since  $K$  is closed,  $\overline{K} = K$ , so it suffices to show that  $K$  does not properly contain any interval.
- It suffices to show  $m(K^c) = 1$ , since this implies  $m(K) = 0$  and since any interval has strictly positive measure, this will mean  $K$  can not contain an interval.
- As in the construction of the Cantor set, let

–  $K_0$  denote  $[0, 1]$  with 1 interval  $\left(\frac{4}{10}, \frac{5}{10}\right)$  of length  $\frac{1}{10}$  deleted, so

$$m(K_0^c) = \frac{1}{10}.$$

- $K_1$  denote  $K_0$  with 9 intervals  $\left(\frac{1}{100}, \frac{5}{100}\right), \left(\frac{14}{100}, \frac{15}{100}\right), \dots, \left(\frac{94}{100}, \frac{95}{100}\right)$  of length  $\frac{1}{100}$  deleted, so

$$m(K_1^c) = \frac{1}{10} + \frac{9}{100}.$$

- $K_n$  denote  $K_{n-1}$  with  $9^n$  such intervals of length  $\frac{1}{10^{n+1}}$  deleted, so

$$m(K_n^c) = \frac{1}{10} + \frac{9}{100} + \dots + \frac{9^n}{10^{n+1}}.$$

- Then compute

$$m(K^c) = \sum_{j=0}^{\infty} \frac{9^j}{10^{j+1}} = \frac{1}{10} \sum_{j=0}^{\infty} \left(\frac{9}{10}\right)^j = \frac{1}{10} \left(\frac{1}{1 - \frac{9}{10}}\right) = 1.$$

Claim:  $K$  has no isolated points:

- A point  $x \in K$  is isolated iff there is an open ball  $B_r(x)$  containing  $x$  such that  $B_r(x) \subsetneq K^c$ .
  - So every point in this ball **should** have a 4 in its decimal expansion.
- Strategy: show that if  $x \in K$ , every neighborhood of  $x$  intersects  $K$ .
- Note that  $m(K_n) = \left(\frac{9}{10}\right)^n \xrightarrow{n \rightarrow \infty} 0$
- Also note that we deleted open intervals, and the endpoints of these intervals are never deleted.
  - Thus endpoints of deleted intervals are elements of  $K$ .
- Fix  $x$ . Then for every  $\varepsilon$ , by the Archimedean property of  $\mathbb{R}$ , choose  $n$  such that  $\left(\frac{9}{10}\right)^n < \varepsilon$ .
- Then there is an endpoint  $x_n$  of some deleted interval  $I_n$  satisfying

$$|x - x_n| \leq \left(\frac{9}{10}\right)^n < \varepsilon.$$

- So every ball containing  $x$  contains some endpoint of a removed interval, and thus an element of  $K$ .

## 1.2 2

Concepts used:

- Absolute continuity of measures:  $\lambda \ll \mu \iff E \in \mathcal{M}, \mu(E) = 0 \implies \lambda(E) = 0$ .
- Radon-Nikodym: if  $\lambda \ll \mu$ , then there exists a measurable function  $\frac{\partial \lambda}{\partial \mu} := f$  where  $\lambda(E) = \int_E f d\mu$ .
- Chebyshev's inequality:

$$A_c := \left\{x \in X \mid |f(x)| \geq c\right\} \implies \mu(A_c) \leq c^{-p} \int_{A_c} |f|^p d\mu \quad \forall 0 < p < \infty.$$

## 1.2.1 a

- Strategy: use approximation by simple functions to show absolute continuity and apply Radon-Nikodym
- Claim:  $\lambda \ll \mu$ , i.e.  $\mu(E) = 0 \implies \lambda(E) = 0$ .

– Note that if this holds, by Radon-Nikodym,  $f = \frac{\partial \lambda}{\partial \mu} \implies d\lambda = f d\mu$ , which would yield

$$\int g d\lambda = \int gf d\mu.$$

- So let  $E$  be measurable and suppose  $\mu(E) = 0$ .
- Then

$$\lambda(E) := \int_E f d\mu = \lim_{n \rightarrow \infty} \left\{ \int_E s_n d\mu \mid s_n := \sum_{j=1}^{\infty} c_j \mu(E_j), s_n \nearrow f \right\}$$

where we take a sequence of simple functions increasing to  $f$ .

- But since each  $E_j \subseteq E$ , we must have  $\mu(E_j) = 0$  for any such  $E_j$ , so every such  $s_n$  must be zero and thus  $\lambda(E) = 0$ .

What is the final step in this approximation?

## 1.2.2 b

- Set  $g(x) = x^2$ , note that  $g$  is positive and measurable.
- By part (a), there exists a positive  $f$  such that for any  $E \subseteq \mathbb{R}$ ,

$$\int_E g dm = \int_E gf d\mu$$

- The LHS is zero by assumption and thus so is the RHS.
- $m \ll \mu$  by construction.
- Note that  $gf$  is positive.
- Define  $A_k = \left\{ x \in X \mid gf \cdot \chi_E > \frac{1}{k} \right\}$ , for  $k \in \mathbb{Z}^{\geq 0}$
- Then by Chebyshev with  $p = 1$ , for every  $k$  we have

$$\mu(A_k) \leq k \int_E gf d\mu = 0$$

- Then noting that  $A_k \searrow A := \left\{ x \in E \mid x^2 > 0 \right\}$ , we have  $\mu(A) = 0$ .
- Since  $gf$  is positive, we have

$$x \in E \iff gf\chi_E(x) > 0 \iff x \in A$$

so  $E = A$  and  $\mu(E) = \mu(A)$ .

- But  $m \ll \mu$  and  $\mu(E) = 0$ , so we can conclude that  $m(E) = 0$ .

**1.3 3****1.3.1 a**

Letting  $x_n := \frac{1}{n}$ , we have

$$\sum_{k=1}^{\infty} |f_k(x)| \geq |f_n(x_n)| = |ae^{-ax} - be^{-bx}| := M.$$

In particular,  $\sup_x |f_n(x)| \not\rightarrow 0$ , so the terms do not go to zero and the sum can not converge.

**1.3.2 b**

?

**1.4 4**

Switching to polar coordinates and integrating over a half-circle contained in  $I^2$ , we have

$$\int_{I^2} f \geq \int_0^\pi \int_0^1 \frac{\cos(\theta) \sin(\theta)}{r^2} dr d\theta = \infty,$$

so  $f$  is not integrable.

**1.5 5**

See <https://math.stackexchange.com/questions/507263/prove-that-c1a-b-with-the-c1-norm-is-a-banach-space>

This is clearly a norm, which we'll write  $\|\cdot\|_u$

Let  $f_n$  be a Cauchy sequence and define a candidate limit  $f(x) = \lim_n f_n(x)$ .

Then noting that  $\|f_n\|_\infty, \|f'_n\|_\infty \leq \|f_n\|_u < \infty$ , both  $f_n, f'_n$  are Cauchy sequences in  $C^0([a, b], \|\cdot\|_\infty)$ , which is a Banach space.

So  $f_n \rightarrow f$  uniformly, and  $f'_n \rightarrow g$  uniformly for some  $g$ , and moreover  $f, g \in C^0([a, b])$ .

We thus have

$$\begin{aligned} f_n(x) - f_n(a) &\xrightarrow{u} f(x) - f(a) \\ \int_a^x f'_n &\xrightarrow{u} \int_a^x g, \end{aligned}$$

and by the FTC, the left-hand sides are equal, and by uniqueness of limits so are the right-hand sides, so  $f' = g$ .

Since  $f, f' \in C^0([a, b])$ , they are bounded, and so  $\|f\|_u < \infty$ . This means that  $\|f_n - f\|_u \rightarrow 0$ , so  $f_n$  converges to  $f$ , which is in the same space.

■