Complex Analysis Qualifying Exam Questions

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1 | Preface

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- Mentzelos Melistas, for explaining and documenting many solutions to these questions.
- Jingzhi Tie, for supplying many additional problems and solutions.

2 Topology and Functions of One Variable (8155a)

Let $x_0 = a, x_1 = b$, and set

$$x_n \coloneqq \frac{x_{n-1} + x_{n-2}}{2} \quad n \ge 2.$$

Show that $\{x_n\}$ is a Cauchy sequence and find its limit in terms of a and b.

Suppose $f: \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{x \to \pm \infty} f(x) = 0$. Prove that f is uniformly continuous.

Give an example of a function $f: \mathbb{R} \to \mathbb{R}$ that is everywhere differentiable but f' is not continuous at 0.

Preface 10

2.4 4

Suppose $\{g_n\}$ is a uniformly convergent sequence of functions from \mathbb{R} to \mathbb{R} and $f: \mathbb{R} \to \mathbb{R}$ is uniformly continuous. Prove that the sequence $\{f \circ g_n\}$ is uniformly convergent.

Let f be differentiable on [a, b]. Say that f is uniformly differentiable iff

$$\forall \varepsilon > 0, \ \exists \delta > 0 \text{ such that } |x - y| < \delta \implies \left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \varepsilon.$$

Prove that f is uniformly differentiable on $[a, b] \iff f'$ is continuous on [a, b].

Suppose $A, B \subseteq \mathbb{R}^n$ are disjoint and compact. Prove that there exist $a \in A, b \in B$ such that

$$||a - b|| = \inf \{ ||x - y|| \mid x \in A, y \in B \}.$$

Suppose $A, B \subseteq \mathbb{R}^n$ are connected and not disjoint. Prove that $A \cup B$ is also connected.

$$\sim$$
 2.8 8 $\stackrel{\triangleright}{}$

Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of continuous functions $f_n:[0,1]\to\mathbb{R}$ such that

$$f_n(x) \ge f_{n+1}(x) \ge 0 \quad \forall n \in \mathbb{N}, \, \forall x \in [0, 1].$$

Prove that if $\{f_n\}$ converges pointwise to 0 on [0,1] then it converges to 0 uniformly on [0,1].

2.9 9

Show that if $E \subset [0,1]$ is uncountable, then there is some $t \in \mathbb{R}$ such that $E \cap (-\infty,t)$ and $E \cap (t,\infty)$ are also uncountable.

3 | Several Variables (8155h)

 \sim 3.1 1

Is the following function continuous, differentiable, continuously differentiable?

$$f: \mathbb{R}^2 \to \mathbb{R}$$

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & (x,y) \neq (0,0) \\ 0 & \text{else.} \end{cases}$$

 \sim 3.2 2

3.2.1 a

Complete this definition: " $f: \mathbb{R}^n \to \mathbb{R}^m$ is real-differentiable a point $p \in \mathbb{R}^n$ iff there exists a linear transformation..."

3.2.2 b

Give an example of a function $f: \mathbb{R}^2 \to \mathbb{R}$ whose first-order partial derivatives exist everywhere but f is not differentiable at (0,0).

3.2.3 с

Give an example of a function $f: \mathbb{R}^2 \to \mathbb{R}$ which is real-differentiable everywhere but nowhere complex-differentiable.

Let $f: \mathbb{R}^2 \to \mathbb{R}$.

3.3.1 a

Define in terms of linear transformations what it means for f to be differentiable at a point $(a,b) \in \mathbb{R}^2$.

3.3.2 b

State a version of the inverse function theorem in this setting.

3.3.3 c

Identify \mathbb{R}^2 with \mathbb{C} and give a necessary and sufficient condition for a real-differentiable function at (a,b) to be complex differentiable at the point a+ib.

Let f = u + iv be complex-differentiable with continuous partial derivatives at a point $z = re^{i\theta}$ with $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta} \,.$$

$$\sim$$
 3.5 5 $\stackrel{\triangleright}{}$

Let $P = (1,3) \in \mathbb{R}^2$ and define

$$f(s,t) := ps^3 - 6st + t^2.$$

3.5.1 a

State the conclusion of the implicit function theorem concerning f(s,t) = 0 when f is considered a function $\mathbb{R}^2 \to \mathbb{R}$.

3.5.2 b

State the above conclusion when f is considered a function $\mathbb{C}^2 \to \mathbb{C}$.

3.5.3 c

Use the implicit function theorem for a function $\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ to prove (b).

There are various approaches: using the definition of the complex derivative, the Cauchy-Riemann equations, considering total derivatives, etc.

Let $F: \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable with F(0,0) = 0 and $\|\nabla F(0,0)\| < 1$.

Prove that there is some real number r > 0 such that |F(x,y)| < r whenever ||(x,y)|| < r.



State the most general version of the implicit function theorem for real functions and outline how it can be proved using the inverse function theorem.

4 | Several Variables: Extra Questions



Let f = u + iv be differentiable (i.e. f'(z) exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$.

Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

4.2 ?

Give an example of a

Show that $f(z) = z^2$ is uniformly continuous in any open disk |z| < R, where R > 0 is fixed, but it is not uniformly continuous on \mathbb{C} .

4.2.1 1

Show that the function u = u(x, y) given by

$$u(x,y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx$$
 for $n \in \mathbb{N}$

is the solution on $D = \{(x,y) \mid x^2 + y^2 < 1\}$ of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,0) = \frac{\sin nx}{n}.$$

4.2.2 2

Show that there exist points $(x,y) \in D$ such that $\limsup_{n \to \infty} |u(x,y)| = \infty$.

5 | Conformal Maps (8155c)

Notation: \mathbb{D} is the open unit disc, \mathbb{H} is the open upper half-plane.

 \sim 5.1 1

Find a conformal map from \mathbb{D} to \mathbb{H} .

 \sim 5.2 2 $\stackrel{\triangleright}{}$

Find a conformal map from the strip $\{z \in \mathbb{C} \mid 0 < \Im(z) < 1\}$ to \mathbb{H} .

Find a fractional linear transformation T which maps \mathbb{H} to \mathbb{D} , and explicitly describe the image of the first quadrant under T.

Find a conformal map from $\{z \in \mathbb{C} \mid |z-i| > 1, \Re(z) > 0\}$ to \mathbb{H} .

Find a conformal map from $\left\{z\in\mathbb{C}\ \Big|\ |z|<1,\ \left|z-\frac{1}{2}\right|>\frac{1}{2}\right\}$ to $\mathbb{D}.$

Find a conformal map from $\{|z-1|<2\}\cap\{|z+1|<2\}$ to $\mathbb{H}.$

Let Ω be the region inside the unit circle |z|=1 and outside the circle $\left|z-\frac{1}{4}\right|=\frac{1}{4}$.

Find an injective conformal map from Ω onto some annulus $\{r < |z| < 1\}$ for gonstant r.

$$\sim$$
 5.8 8

Let D be the region obtained by deleting the real interval [0,1) from \mathbb{D} ; find a conformal map from D to \mathbb{D} .

Find a conformal map from $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$ to \mathbb{D} .

$$\sim$$
 5.10 10 $hrightarrow$

Find a conformal map from $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \geq 1\}$ to \mathbb{D} .

$$\sim$$
 5.11 11 $ightharpoonup$

Find a bijective conformal map from G to \mathbb{H} , where

$$G \coloneqq \left\{z \in \mathbb{C} \;\middle|\; |z-1| < \sqrt{2},\, |z+1| < \sqrt{2}\right\} \setminus [0,i).$$

5.12 12

Prove that TFAE for a Möbius transformation T given by $T(z) = \frac{az+b}{cz+d}$:

- a. T maps $\mathbb{R} \cup \{\infty\}$ to itself.
- b. It is possible to choose a, b, c, d to be real numbers.
- c. $\overline{T(z)} = T(\overline{z})$ for every $z \in \mathbb{CP}^1$.
- d. There exist $\alpha \in \mathbb{R}, \beta \in \mathbb{C} \setminus \mathbb{R}$ such that $T(\alpha) = \alpha$ and $T(\overline{\beta}) = \overline{T(\beta)}$.

6 | Extra Questions

6.1 12.

Find a conformal map from $D=\{z:\ |z|<1,\ |z-1/2|>1/2\}$ to the unit disk $\Delta=\{z:\ |z|<1\}.$

7 Integrals and Cauchy's Theorem (8155d)

Some interesting problems: 3, 4, 9, 10.

7.1 1

Suppose $f, g : [0, 1] \to \mathbb{R}$ where f is Riemann integrable and for $x, y \in [0, 1]$,

$$|g(x) - g(y)| \le |f(x) - f(y)|.$$

Prove that g is Riemann integrable.

 \sim 7.2 2 $^{\triangleright}$

State and prove Green's Theorem for rectangles.

Then use it to prove Cauchy's Theory for functions that are analytic in a rectangle.



Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of analytic functions on $\mathbb{D}:=\Big\{z\in\mathbb{C}\ \Big|\ |z|<1\Big\}.$

Show that if $f_n \to g$ for some $g: \mathbb{D} \to \mathbb{C}$ uniformly on every compact $K \subset \mathbb{D}$, then g is analytic on \mathbb{D} .



Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of entire functions where

- $f_n \to g$ pointwise for some $g: \mathbb{C} \to \mathbb{C}$.
- On every line segment in \mathbb{C} , $f_n \to g$ uniformly.

Show that

- g is entire, and
- $f_n \to g$ uniformly on every compact subset of \mathbb{C} .

Prove that there is no sequence of polynomials that uniformly converge to $f(z) = \frac{1}{z}$ on S^1 .

Solution omitted.

$$\sim$$
 7.6 6

Suppose that $f: \mathbb{R} \to \mathbb{R}$ is a continuous function that vanishes outside of some finite interval. For each $z \in \mathbb{C}$, define

$$g(z) = \int_{-\infty}^{\infty} f(t)e^{-izt} dt.$$

Show that g is entire.

Suppose $f:\mathbb{C}\to\mathbb{C}$ is entire and

$$|f(z)| \le |z|^{\frac{1}{2}}$$
 when $|z| > 10$.

Prove that f is constant.

Let γ be a smooth curve joining two distinct points $a, b \in \mathbb{C}$.

Prove that the function

$$f(z) \coloneqq \int_{\gamma} \frac{g(w)}{w - z} \, dw$$

is analytic in $\mathbb{C} \setminus \gamma$.

Suppose that $f: \mathbb{C} \to \mathbb{C}$ is continuous everywhere and analytic on $\mathbb{C} \setminus \mathbb{R}$ and prove that f is entire.

Something missing?

Solution omitted.

7.10 10
 ↑

Prove Liouville's theorem: suppose $f:\mathbb{C}\to\mathbb{C}$ is entire and bounded. Use Cauchy's formula to prove that $f'\equiv 0$ and hence f is constant.

Solution omitted.

8 | Extra



Assume f is continuous in the region: $0 < |z - a| \le R$, $0 \le \arg(z - a) \le \beta_0$ $(0 < \beta_0 \le 2\pi)$ and the limit $\lim_{z \to a} (z - a) f(z) = A$ exists. Show that

$$\lim_{r \to 0} \int_{\gamma_r} f(z) dz = iA\beta_0 ,$$

where $\gamma_r := \{ z \mid z = a + re^{it}, \ 0 \le t \le \beta_0 \}.$

9 | Liouville's Theorem, Power Series (8155e)



Suppose f is analytic on a region Ω such that $\mathbb{D} \subseteq \Omega \subseteq \mathbb{C}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series with radius of convergence exactly 1.

9.1.1 a 🐪

Give an example of such an f that converges at every point of S^1 .

9.1.2 b

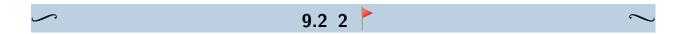
Give an example of such an f which is analytic at 1 but $\sum_{n=0}^{\infty} a_n$ diverges.

9.1.3 c

Prove that f can not be analytic at *every* point of S^1 .

Missing part (c)

Solution omitted.



Suppose f is entire and has Taylor series $\sum a_n z^n$ about 0.

9.2.1 a

Express a_n as a contour integral along the circle |z| = R.

9.2.2 b

Apply (a) to show that the above Taylor series converges uniformly on every bounded subset of \mathbb{C} .

9.2.3 c

Determine those functions f for which the above Taylor series converges uniformly on all of \mathbb{C} .

Suppose D is a domain and f, g are analytic on D.

Prove that if fg = 0 on D, then either $f \equiv 0$ or $g \equiv 0$ on D.

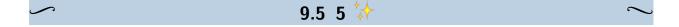
Suppose f is analytic on \mathbb{D}° . Determine with proof which of the following are possible:

a.
$$f\left(\frac{1}{n}\right) = (-1)^n$$
 for each $n > 1$.

b.
$$f\left(\frac{1}{n}\right) = e^{-n}$$
 for each even integer $n > 1$ while $f\left(\frac{1}{n}\right) = 0$ for each odd integer $n > 1$.

c.
$$f\left(\frac{1}{n^2}\right) = \frac{1}{n}$$
 for each integer $n > 1$.

d.
$$f\left(\frac{1}{n}\right) = \frac{n-2}{n-1}$$
 for each integer $n > 1$.



Prove the Fundamental Theorem of Algebra (using complex analysis).

Solution omitted.

$$\sim$$
 9.6 6 \updownarrow

Find all entire functions that satisfy

$$|f(z)| \ge |z| \quad \forall z \in \mathbb{C}.$$

Prove this list is complete.

Solution omitted.

Suppose $\sum_{n=0}^{\infty} a_n z^n$ converges for some $z_0 \neq 0$.

9.7.1 a

Prove that the series converges absolutely for each z with $|z| < |z|_0$.

9.7.2 b

Suppose $0 < r < |z_0|$ and show that the series converges uniformly on $|z| \le r$.

10

9.8 8

Suppose f is entire and suppose that for some integer $n \geq 1$,

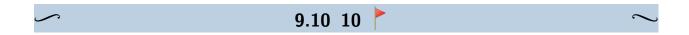
$$\lim_{z \to \infty} \frac{f(z)}{z^n} = 0.$$

Prove that f is a polynomial of degree at most n-1.



Find all entire functions satisfying

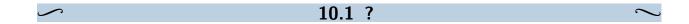
$$|f(z)| \le |z|^{\frac{1}{2}}$$
 for $|z| > 10$.



Prove that the following series converges uniformly on the set $\{z \mid \Im(z) < \ln 2\}$:

$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{2^n}.$$

10 Extra



Let f(z) be entire and assume values of f(z) lie outside a bounded open set Ω . Show without using Picard's theorems that f(z) is a constant.

Let f(z) be entire and assume values of f(z) lie outside a bounded open set Ω .

Show without using Picard's theorems that f(z) is a constant.

9.8 8

10.2 ?

10.2.1 1

Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in |z| < R.

Show that for r < R,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

10.2.2 2

Deduce Liouville's theorem from (1).

$oxed{11}$ Laurent Expansions and Singularities (8155f)



Find the Laurent expansion of

$$f(z) = \frac{z+1}{z(z-1)}$$

Solution omitted.

about z = 0 and z = 1 respectively.

11.2 2 🦙

Find the Laurent expansions about z=0 of the following functions:

$$\exp \frac{1}{z}$$
 $\cos \left(\frac{1}{z}\right)$.

Solution omitted.

Find the Laurent expansion of

$$f(z) = \frac{z+1}{z(z-1)^2}$$

about z = 0 and z = 1 respectively.

Hint: recall that power series can be differentiated.

For the following functions, find the Laurent series about 0 and classify their singularities there:

$$\frac{\sin^2(z)}{z}$$

$$z \exp \frac{1}{z^2}$$

$$\frac{1}{z(4-z)}.$$

Find all entire functions with have poles at ∞ .

 \sim 11.6 6

Find all functions on the Riemann sphere that have a simple pole at z=2 and a double pole at $z=\infty$, but are analytic elsewhere.



Let f be entire, and discuss (with proofs and examples) the types of singularities f might have (removable, pole, or essential) at $z = \infty$ in the following cases:

- 1. f has at most finitely many zeros in \mathbb{C} .
- 2. f has infinitely many zeros in \mathbb{C} .

Define

$$f(z) = \frac{\pi^2}{\sin^2(\pi z)}$$
$$g(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}.$$

- a. Show that f and g have the same singularities in $\mathbb{C}.$
- b. Show that f and g have the same singular parts at each of their singularities.
- c. Show that f, g each have period one and approach zero uniformly on $0 \le x \le 1$ as $|y| \to \infty$.
- d. Conclude that f = g.

Not finished.

Solution omitted.

12 | Residues (8155g)

 \sim 12.1 1

Calculate

$$\int_0^\infty \frac{1}{(1+z)^2(z+9x^2)} \, dx.$$

 \sim 12.2 2

Let a > 0 and calculate

$$\int_0^\infty \frac{x \sin(x)}{x^2 + a^2} \, dx.$$

12.3 3

~

Calculate

$$\int_0^\infty \frac{\sqrt{x}}{(x+1)^2} \, dx.$$

12.4 4

~

Calculate

$$\int_0^\infty \frac{\cos(x) - \cos(4x)}{x^2} \, dx.$$

12.5 5

~

Let a > 0 and calculate

$$\int_0^\infty \frac{x^2}{(x^2 + a^2)^2} \, dx.$$

12.6 6

\

Calculate

$$\int_0^\infty \frac{\sin(x)}{x} \, dx.$$

12.7 7

 \sim

Calculate

$$\int_0^\infty \frac{\sin(x)}{x(x^2+1)} \, dx.$$

∽ 12.8 8 ∤

Calculate

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2} \, dx.$$

Calculate

$$\int_{-\infty}^{\infty} \frac{1+x^2}{1+x^4} \, dx.$$

Let a > 0 and calculate

$$\int_0^\infty \frac{\cos(x)}{(x^2 + a^2)^2} \, dx.$$

Calculate

$$\int_0^\infty \frac{\sin^3(x)}{x^3} \, dx.$$

Let $n \in \mathbb{Z}^{\geq 1}$ and $0 < \theta < \pi$ and show that

$$\frac{1}{2\pi i} \int_{|z|=2} \frac{z^n}{1 - 3z \cos(\theta) + z^2} dz = \frac{\sin(n\theta)}{\sin(\theta)}.$$

12.13 13

Suppose a > b > 0 and calculate

$$\int_0^{2\pi} \frac{1}{(a+b\cos(\theta))^2} \, d\theta.$$

13 Residue Theorem: Extra Questions

3.1 ?

Suppose that f is an analytic function in the region D which contains the point a. Let

F(z) = z - a - qf(z), where q is a complex parameter.

13.1.1 1

Let $K \subset D$ be a circle with the center at point a and also we assume that $f(z) \neq 0$ for $z \in K$. Prove that the function F has one and only one zero z = w on the closed disc \overline{K} whose boundary is the circle K if $|q| < \min_{z \in K} \frac{|z - a|}{|f(z)|}$.

13.1.2 2

Let G(z) be an analytic function on the disk \overline{K} . Apply the residue theorem to prove that

$$\frac{G(w)}{F'(w)} = \frac{1}{2\pi i} \int_K \frac{G(z)}{F(z)} dz,$$

where w is the zero from (1).



Evaluate

$$\int_0^\infty \frac{x \sin x}{x^2 + a^2} \, dx.$$

12.13 13

 \sim 13.3 ? \sim

Show that

$$\int_0^\infty \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$$

using complex analysis, 0 < a < n. Here n is a positive integer.

14 Rouche's Theorem (8155h)



Prove that for every $n \in \mathbb{Z}^{\geq 0}$ the following polynomial has no roots in the open unit disc:

$$f_n(x) \coloneqq \sum_{k=0}^n \frac{z^k}{k!}.$$

Hint: check n = 1, 2 directly.

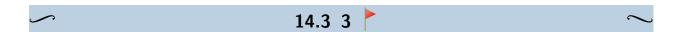
Solution omitted.

$$\sim$$
 14.2 2 $^{\updownarrow}$

Assume that |b| < 1 and show that the following polynomial has exactly two roots (counting multiplicity) in |z| < 1:

$$f(z) \coloneqq z^3 + 3z^2 + bz + b^2.$$

Solution omitted.



Let $c \in \mathbb{C}$ with $|c| < \frac{1}{3}$. Show that on the open set $\{z \in \mathbb{C} \mid \Re(z) < 1\}$, the function $f(z) \coloneqq ce^z$ has exactly one fixed point.

14.4 4 🦙

How many roots does the following polynomial have in the open disc |z| < 1?

$$f(z) = z^7 - 4z^3 - 1.$$

Solution omitted.

Let $n \in \mathbb{Z}^{\geq 0}$ and show that the equation

$$e^z = az^n$$

has n solutions in the open unit disc if |a| > e, and no solutions if $|a| < \frac{1}{e}$.

 \sim 14.6 6

Let f be analytic in a domain D and fix $z_0 \in D$ with $w_0 := f(z_0)$. Suppose z_0 is a zero of $f(z) - w_0$ with finite multiplicity m. Show that there exists $\delta > 0$ and $\varepsilon > 0$ such that for each w such that $0 < |w - w_0| < \varepsilon$, the equation f(z) - w = 0 has exactly m distinct solutions inside the disc $|z - z_0| < \delta$.

 \sim 14.7 7

For $k = 1, 2, \dots, n$, suppose $|a_k| < 1$ and

$$f(z) := \left(\frac{z - a_1}{1 - \bar{a}_q z}\right) \left(\frac{z - a_2}{1 - \bar{a}_2 z}\right) \cdots \left(\frac{z - a_n}{1 - \bar{a}_n z}\right).$$

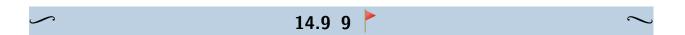
Show that f(z) = b has n solutions in |z| < 1.

15 Extras

For each $n \in \mathbb{Z}^{\geq 1}$, let

$$P_n(z) = 1 + z + \frac{1}{2!}z^2 + \dots + \frac{1}{n!}z^n.$$

Show that for sufficiently large n, the polynomial P_n has no zeros in |z| < 10, while the polynomial $P_n(z) - 1$ has precisely 3 zeros there.



Prove that

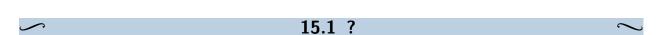
$$\max_{|z|=1} \left| a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + z^n \right| \ge 1.$$

Hint: the first part of the problem asks for a statement of Rouche's theorem.



Use Rouche's theorem to prove the Fundamental Theorem of Algebra.

15 Extras



Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra:

If

$$P_n(z) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree n, then it has n zeros in \mathbb{C} .

14.8 8

15.2 ?

Suppose f is entire and there exist A, R > 0 and natural number N such that

$$|f(z)| \ge A|z|^N$$
 for $|z| \ge R$.

Show that (i) f is a polynomial and (ii) the degree of f is at least N.

16 Schwarz Lemma and Reflection Principle (8155i)

 \sim 16.1 1

Suppose $f: \mathbb{D} \to \mathbb{D}$ is analytic and admits a continuous extension $\tilde{f}: \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that $|z| = 1 \Longrightarrow |f(z)| = 1$.

16.1.1 a

Prove that f is a rational function.

16.1.2 b

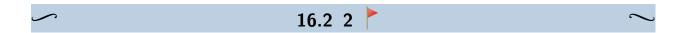
Suppose that z = 0 is the unique zero of f. Show that

$$\exists n \in \mathbb{N}, \lambda \in S^1$$
 such that $f(z) = \lambda z^n$.

16.1.3 c

Suppose that $a_1, \dots, a_n \in \mathbb{D}$ are the zeros of f and prove that

$$\exists \lambda \in S^1$$
 such that $f(z) = \lambda \prod_{j=1}^n \frac{z - a_j}{1 - \overline{a_j} z}$.



Let $\overline{B}(a,r)$ denote the closed disc of radius r about $a \in \mathbb{C}$. Let f be holomorphic on an open set containing $\overline{B}(a,r)$ and let

$$M\coloneqq \sup_{z\in \overline{B}(a,r)}|f(z)|.$$

Prove that

$$z \in \overline{B}\left(a, \frac{r}{2}\right), z \neq a, \qquad \frac{|f(z) - f(a)|}{|z - a|} \leq \frac{2M}{r}.$$

Define

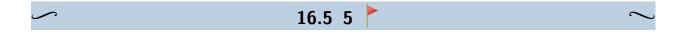
$$G := \left\{ z \in \mathbb{C} \mid \Re(z) > 0, \, |z - 1| > 1 \right\}.$$

Find all of the injective conformal maps $G \to \mathbb{D}$. These may be expressed as compositions of maps, but explain why this list is complete.

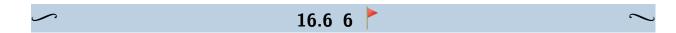
Suppose $f: \mathbb{H} \cup \mathbb{R} \to \mathbb{C}$ satisfies the following:

- f(i) = i
- f is continuous
- f is analytic on \mathbb{H}
- $f(z) \in \mathbb{R} \iff z \in \mathbb{R}$.

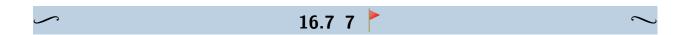
Show that $f(\mathbb{H})$ is a dense subset of \mathbb{H} .



Suppose $f: \mathbb{D} \to \mathbb{H}$ is analytic and satisfies f(0) = 2. Find a sharp upper bound for |f'(0)|, and prove it is sharp by example.



Suppose $f: \mathbb{D} \to \mathbb{D}$ is analytic, has a single zero of order k at z=0, and satsifies $\lim_{|z|\to 1} |f(z)|=1$. Give with proof a formula for f(z).



16.7.1 a

State the standard Schwarz reflection principle involving reflection across the real axis.

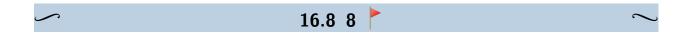
16.7.2 b

Give a linear fractional transformation T mapping \mathbb{D} to \mathbb{H} . Let $g(z) = \bar{z}$, and show

$$(T^{-1} \circ g \circ T)(z) = 1/\bar{z}.$$

16.7.3 c

Suppose that f is holomorphic on \mathbb{D} , continuous on $\overline{\mathbb{D}}$, and real on S^1 . Show that f must be constant.



Suppose $f, g : \mathbb{D} \to \Omega$ are holomorphic with f injective and f(0) = g(0).

Show that

$$\forall 0 < r < 1, \qquad g(\{|z| < r\}) \subseteq f(\{|z| < r\}).$$

The first part of this problem asks for a statement of the Schwarz lemma. 16.9 9

Let $S := \{z \in \mathbb{D} \mid \Im(z) \geq 0\}$. Suppose $f: S \to \mathbb{C}$ is continuous on S, real on $S \cap \mathbb{R}$, and holomorphic on S° .

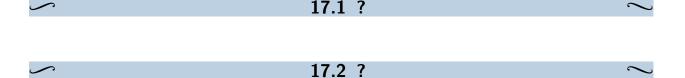
Prove that f is the restriction of a holomorphic function on \mathbb{D} .



Suppose $f: \mathbb{D} \to \mathbb{D}$ is analytic. Prove that

$$\forall a \in \mathbb{D}, \qquad \frac{|f'(a)|}{1 - |f(a)|^2} \le \frac{1}{1 - |a|^2}.$$

17 Unsorted/Unknown



Let f be a continuous function in the region

$$D = \{z \mid |z| > R, 0 \le \arg Z \le \theta\} \quad \text{where} \quad 0 \le \theta \le 2\pi.$$

If there exists k such that $\lim_{z\to\infty}zf(z)=k$ for z in the region D.

Show that

$$\lim_{R' \to \infty} \int_L f(z) dz = i\theta k,$$

where L is the part of the circle |z| = R' which lies in the region D.

17.2.1 3

If $z \in K$, prove that the function $\frac{1}{F(z)}$ can be represented as a convergent series with respect to q:

$$\frac{1}{F(z)} = \sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z-a)^{n+1}}.$$

17.3 ?

Show that

$$\int_0^\infty \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$$

using complex analysis, 0 < a < n.

Here n is a positive integer.

17.4 11.

Let g be analytic for $|z| \le 1$ and |g(z)| < 1 for |z| = 1.

17.4.1 a

Show that g has a unique fixed point in |z| < 1.

17.4.2 b

What happens if we replace |g(z)| < 1 with $|g(z)| \le 1$ for |z| = 1?

Give an example if (a) is not true or give an proof if (a) is still true.

17.4.3 c

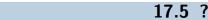
What happens if we simply assume that f is analytic for |z| < 1 and |f(z)| < 1 for |z| < 1? Suppose that $f(z) \not\equiv z$.

Can f have more than one fixed point in |z| < 1?

Hint: The map

$$\psi_{\alpha}(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$$

may be useful.



Let f(z) be entire and assume that $f(z) \leq M|z|^2$ outside some disk for some constant M.

Show that f(z) is a polynomial in z of degree ≤ 2 .

17.6 ?

Let $a_n(z)$ be an analytic sequence in a domain D such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D.

Show that $\sum_{n=0}^{\infty} |a'_n(z)|$ converges uniformly on bounded and closed sub-regions of D.

17.7 16

Let f(z) be analytic in an open set Ω except possibly at a point z_0 inside Ω .

Show that if f(z) is bounded in near z_0 , then $\int_{\Delta} f(z)dz = 0$ for all triangles Δ in Ω .

18 Riemann Mapping and Casorati-Weierstrass

18.1 10.

Let $f: \mathbb{C} \to \mathbb{C}$ be an injective analytic (also called univalent) function. Show that there exist complex numbers $a \neq 0$ and b such that f(z) = az + b.

$oldsymbol{19}|$ Spring 2020 Homework 1

19.1 1

Geometrically describe the following subsets of \mathbb{C} :

- a. |z 1| = 1
- b. |z-1| = 2|z-2|
- c. $1/z = \bar{z}$
- d. $\Re(z) = 3$
- e. $\Im(z) = a$ with $a \in \mathbb{R}$.
- f. $\Re(z) > a$ with $a \in \mathbb{R}$.
- g. |z-1| < 2|z-2|

19.2 2

Prove the following inequality, and explain when equality holds:

$$|z+w| \ge ||z| - |w||.$$

19.3 3

Prove that the following polynomial has its roots outside of the unit circle:

$$p(z) = z^3 + 2z + 4.$$

Hint: What is the maximum value of the modulus of the first two terms if $|z| \le 1$?

19.4 4

a. Prove that if c > 0,

$$|w_1| = c|w_2| \implies |w_1 - c^2w_2| = c|w_1 - w_2|.$$

b. Prove that if c > 0 and $c \neq 1$, with $z_1 \neq z_2$, then the following equation represents a circle:

$$\left|\frac{z-z_1}{z-z_2}\right| = c.$$

Find its center and radius.

Hint: use part (a)

19.5 5

a. Let $z, w \in \mathbb{C}$ with $\bar{z}w \neq 1$. Prove that

$$\left| \frac{w - z}{1 - \overline{w}z} \right| < 1 \quad \text{if } |z| < 1, \ |w| < 1$$

with equality when |z| = 1 or |w| = 1.

b. Prove that for a fixed $w \in \mathbb{D}$, the mapping $F: z \mapsto \frac{w-z}{1-\overline{w}z}$ satisfies

- F maps \mathbb{D} to itself and is holomorphic.
- F(0) = w and F(w) = 0.
- |z| = 1 implies |F(z)| = 1.

 \sim 19.6 6

Use nth roots of unity to show that

$$2^{n-1}\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\cdots\sin\left(\frac{(n-1)\pi}{n}\right)=n.$$

Hint:

$$1 - \cos(2\theta) = 2\sin^2(\theta)$$
$$2\sin(2\theta) = 2\sin(\theta)\cos(\theta).$$

19.7 7

Prove that $f(z) = |z|^2$ has a derivative at z = 0 and nowhere else.

✓ 19.8 8 ~

Let f(z) be analytic in a domain, and prove that f is constant if it satisfies any of the following conditions:

- a. |f(z)| is constant.
- b. $\Re(f(z))$ is constant.
- c. arg(f(z)) is constant.
- d. $\overline{f(z)}$ is analytic.

How do you generalize (a) and (b)?

 \sim 19.9 9 \sim

Prove that if $z\mapsto f(z)$ is analytic, then $z\mapsto \overline{f(\bar{z})}$ is analytic.

 \sim 19.10 10 \sim

a. Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$
 and $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$.

b. Use (a) to show that the logarithm function, defined as

$$\log z = \log r + i\theta$$
 where $z = re^{i\theta}$ with $-\pi < \theta < \pi$.

is holomorphic on the region $r > 0, -\pi < \theta < \pi$.

Also show that this function is not continuous in r > 0.

19.11 11

Prove that the distinct complex numbers z_1, z_2, z_3 are the vertices of an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

20 | Spring 2020 Homework 2

Note on notation: I sometimes use $f_x := \frac{\partial f}{\partial x}$ to denote partial derivatives, and $\partial_z^n f$ as $f^{(n)}(z)$.

20.1 Stein And Shakarchi

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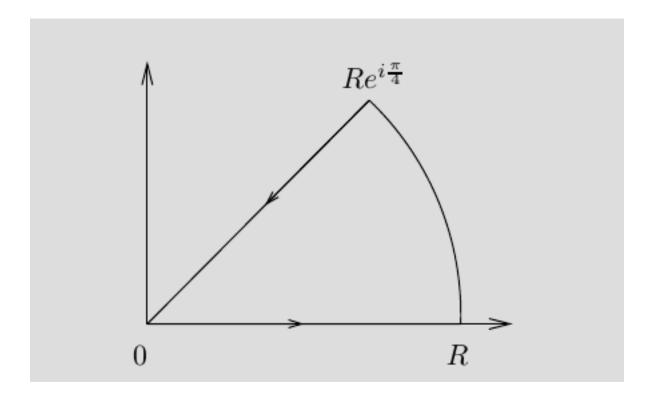
20.1.1 2.6.1

Show that

$$\int_0^\infty \sin\left(x^2\right) dx = \int_0^\infty \cos\left(x^2\right) dx = \frac{\sqrt{2\pi}}{4}.$$

Hint: integrate e^{-x^2} over the following contour, using the fact that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$:

19.11 11



20.1.2 2.6.2

Show that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Hint: use the fact that this integral equals $\frac{1}{2i}\int_{-\infty}^{\infty}\frac{e^{ix}-1}{x}dx$, and integrate around an indented semicircle.

20.1.3 2.6.5

Suppose $f \in C^1_{\mathbb{C}}(\Omega)$ and $T \subset \Omega$ is a triangle with $T^{\circ} \subset \Omega$. Apply Green's theorem to show that $\int_T f(z) \ dz = 0$.

Assume that f' is continuous and prove Goursat's theorem.

Hint: Green's theorem states

20.1 Stein And Shakarchi 4-

$$\int_T F dx + G dy = \int_{T^{\circ}} \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx dy.$$

20.1.4 2.6.6

Suppose that f is holomorphic on a punctured open set $\Omega \setminus \{w_0\}$ and let $T \subset \Omega$ be a triangle containing w_0 . Prove that if f is bounded near w_0 , then $\int_T f(z) dz = 0$.

20.1.5 2.6.7

Suppose $f: \mathbb{D} \to \mathbb{C}$ is holomorphic and let $d := \sup_{z,w \in \mathbb{D}} |f(z) - f(w)|$ be the diameter of the image of f. Show that $2|f'(0)| \leq d$, and that equality holds iff f is linear, so $f(z) = a_1z + a_2$.

Hint:
$$2f'(0) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi) - f(-\xi)}{\xi^2} d\xi$$
 whenever $0 < r < 1$.

20.1.6 2.6.8

Suppose that f is holomorphic on the strip $S = \{x + iy \mid x \in \mathbb{R}, -1 < y < 1\}$ with $|f(z)| \le A(1 + |z|)^{\nu}$ for ν some fixed real number. Show that for all $z \in S$, for each integer $n \ge 0$ there exists an $A_n \ge 0$ such that $|f^{(n)}(x)| \le A_n(1 + |x|)^{\nu}$ for all $x \in \mathbb{R}$.

Hint: Use the Cauchy inequalities.

20.1.7 2.6.9

Let $\Omega \subset \mathbb{C}$ be open and bounded and $\varphi : \Omega \to \Omega$ holomorphic. Prove that if there exists a point $z_0 \in \Omega$ such that $\varphi(z_0) = z_0$ and $\varphi'(z_0) = 1$, then φ is linear.

Hint: assume $z_0 = 0$ (explain why this can be done) and write $\varphi(z) = z + a_n z^n + O(z^{n+1})$ near 0. Let $\varphi_k = \varphi \circ \varphi \circ \cdots \circ \varphi$ and prove that $\varphi_k(z) = z + ka_n z^n + O(z^{n+1})$. Apply Cauchy's inequalities and let $k \to \infty$ to conclude.

20.1 Stein And Shakarchi 45

20.1.8 2.6.10

Can every continuous function on $\overline{\mathbb{D}}$ be uniformly approximated by polynomials in the variable z?

Hint: compare to Weierstrass for the real interval.

20.1.9 2.6.13

Suppose f is analytic, defined on all of \mathbb{C} , and for each $z_0 \in \mathbb{C}$ there is at least one coefficient in the expansion $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$ is zero. Prove that f is a polynomial.

Hint: use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.

20.1.10 2.6.14

Suppose that f is holomorphic in an open set containing \mathbb{D} except for a pole $z_0 \in \partial \mathbb{D}$. Let $\sum_{n=0}^{\infty} a_n z^n$ be the power series expansion of f in \mathbb{D} , and show that $\lim \frac{a_n}{a_{n+1}} = z_0$.

20.1.11 2.6.15

Suppose f is continuous and nonvanishing on $\overline{\mathbb{D}}$, and holomorphic in \mathbb{D} . Prove that if $|z| = 1 \Longrightarrow |f(z)| = 1$, then f is constant.

Hint: Extend f to all of \mathbb{C} by $f(z) = 1/\overline{f(1/\overline{z})}$ for any |z| > 1, and argue as in the Schwarz reflection principle.

20.2 Additional Problems

20.2.1 1

Let $a_n \neq 0$ and show that

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=L\implies \lim_{n\to\infty}|a_n|^{\frac{1}{n}}=L.$$

In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

20.2.2 2

Let f be a power series centered at the origin. Prove that f has a power series expansion about any point in its disc of convergence.

20.2.3 3

Prove the following:

- a. $\sum_{n} nz^{n}$ does not converge at any point of S^{1}
- b. $\sum_{n} \frac{z^n}{n^2}$ converges at every point of S^1 .
- c. $\sum_{n} \frac{z^n}{n}$ converges at every point of S^1 except z = 1.

20.2.4 4

Without using Cauchy's integral formula, show that if |a| < r < |b|, then

$$\int_{\gamma} \frac{dz}{(z-\alpha)(z-\beta)} = \frac{2\pi i}{\alpha - \beta}$$

where γ denotes the circle centered at the origin of radius r with positive orientation.

20.2.5 5

Assume f is continuous in the region $\{x+iy \mid x \geq x_0, \ 0 \leq y \leq b\}$, and the following limit exists independent of y:

$$\lim_{x \to +\infty} f(x + iy) = A.$$

Show that if $\gamma_x := \{z = x + it \mid 0 \le t \le b\}$, then

$$\lim_{x \to +\infty} \int_{\gamma_x} f(z) \, dz = iAb.$$

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20.2.6 6

Show by example that there exists a function f(z) that is holomorphic on $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$ and for all r < 1,

$$\int_{|z|=r} f(z) \, dz = 0,$$

but f is not holomorphic at z = 0.

20.2.7 7

Let f be analytic on a region R and suppose $f'(z_0) \neq 0$ for some $z_0 \in R$. Show that if C is a circle of sufficiently small radius centered at z_0 , then

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{dz}{f(z) - f(z_0)}.$$

Hint: use the inverse function theorem.

20.2.8 8

Assume two functions $u, b : \mathbb{R}^2 \to \mathbb{R}$ have continuous partial derivatives at (x_0, y_0) . Show that f := u + iv has derivative $f'(z_0)$ at $z_0 := x_0 + iy_0$ if and only if

$$\lim_{r \to 0} \frac{1}{\pi r^2} \int_{|z-z_0|=r} f(z) dz = 0.$$

20.2.9 9 (Cauchy's Formula for Exterior Regions)

Let γ be a piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2 . Assume f' exists in an open set containing γ and Ω_2 with $\lim_{z\to\infty} f(z) = A$. Show that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1 \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}.$$

20.2.10 10

Let f(z) be bounded and analytic in \mathbb{C} . Let $a \neq b$ be any fixed complex numbers. Show that the following limit exists:

$$\lim_{R \to \infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

Use this to show that f(z) must be constant.

20.2.11 11

Suppose f(z) is entire and

$$\lim_{z \to \infty} \frac{f(z)}{z} = 0.$$

Show that f(z) is a constant.

20.2.12 12

Let f be analytic in a domain D and γ be a closed curve in D. For any $z_0 \in D$ not on γ , show that

$$\int_{\gamma} \frac{f'(z)}{(z - z_0)} dz = \int_{\gamma} \frac{f(z)}{(z - z_0)^2} dz.$$

Give a generalization of this result.

20.2.13 13

Compute

$$\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z}$$

and use it to show that

$$\int_0^{2\pi} \cos^{2n}(\theta) d\theta = 2\pi \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \right).$$

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21 | Spring 2020 Homework 3

21.1 Stein and Shakarchi



21.1.1 3.8.1

Use the following formula to show that the complex zeros of $\sin(\pi z)$ are exactly the integers, and they are each of order 1:

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}.$$

Calculate the residue of $\frac{1}{\sin(\pi z)}$ at $z = n \in \mathbb{Z}$.

21.1.2 3.8.2

Evaluate the integral

$$\int_{\mathbb{R}} \frac{dx}{1+x^4}.$$

What are the poles of $\frac{1}{1+z^4}$?

21.1.3 3.8.4

Show that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}, \quad \text{for all } a > 0.$$

21.1.4 3.8.5

Show that if $\xi \in \mathbb{R}$, then

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{(1+x^2)^2} dx = \frac{\pi}{2} (1+2\pi |\xi|) e^{-2\pi |\xi|}.$$

21.1.5 3.8.6

Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \pi.$$

21.1.6 3.8.7

Show that

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos \theta)^2} = \frac{2\pi a}{(a^2 - 1)^{3/2}}, \quad \text{whenever } a > 1.$$

21.1.7 3.8.8

Show that if $a, b \in \mathbb{R}$ with a > |b|, then

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

21.1.8 3.8.9

Show that

$$\int_0^1 \log(\sin \pi x) dx = -\log 2.$$

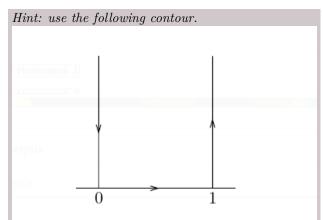


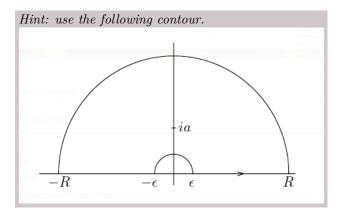
Figure 9. Contour in Exercise 9

21.1 Stein and Shakarchi

21.1.9 3.8.10

Show that if a > 0, then

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx = \frac{\pi}{2a} \log a.$$



21.1.10 3.8.14

Prove that all entire functions that are injective are of the form f(z) = az + b with $a, b \in \mathbb{C}$ and $a \neq 0$.

 $\it Hint: Apply the Casorati-Weierstrass theorem to f(1/z).$

21.1.11 3.8.15

Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

a. Prove that if f is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \le AR^k + B$$

for all R > 0, some integer $k \ge 0$, and some constants A, B > 0, then f is a polynomial of degree $\le k$.

- b. Show that if f is holomorphic in the unit disc, is bounded, and converges uniformly to zero in the sector $\theta < \arg(z) < \varphi$ as $|z| \to 0$, then $f \equiv 0$.
- c. Let $w_1, \dots w_n$ be points on $S^1 \subset \mathbb{C}$. Prove that there exists a point $z \in S^1$ such that the product of the distances from z to the points w_j is at least 1.

21.1 Stein and Shakarchi 52

Conclude that there exists a point $w \in S^1$ such that the product of the above distances is exactly 1.

d. Show that if the real part of an entire function is bounded, then f is constant.

21.1.12 3.8.17

Let f be non-constant and holomorphic in an open set containing the closed unit disc.

a. Show that if |f(z)| = 1 whenever |z| = 1, then the image of f contains the unit disc.

Hint: Show that $f(z) = w_0$ has a root for every $w_0 \in \mathbb{D}$, for which it suffices to show that f(z) = 0 has a root. Conclude using the maximum modulus principle.

b. If $|f(z)| \ge 1$ whenever |z| = 1 and there exists a $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$, then the image of f contains the unit disc.

21.1.13 3.8.19

Prove that maximum principle for harmonic functions, i.e.

- a. If u is a non-constant real-valued harmonic function in a region Ω , then u can not attain a maximum or a minimum in Ω .
- b. Suppose Ω is a region with compact closure $\overline{\Omega}$. If u is harmonic in Ω and continuous in $\overline{\Omega}$, then

$$\sup_{z\in\Omega}|u(z)|\leq \sup_{z\in\overline{\Omega}-\Omega}|u(z)|.$$

Hint: to prove (a), assume u attains a local maximum at z_0 . Let f be holomorphic near z_0 with $\Re(f) = u$, and show that f is not an open map. Then (a) implies (b).

21.2 Extra Problems



21.1 Stein and Shakarchi 53

21.2.1 1

Problem Prove that if f has two Laurent series expansions,

$$f(z) = \sum c_n(z-a)^n$$
 and $f(z) = \sum c'_n(z-a)^n$

then $c_n = c'_n$.

21.2.2 2

Problem Find Laurent series expansions of

$$\frac{1}{1 - z^2} + \frac{1}{3 - z}$$

How many such expansions are there? In what domains are each valid?

21.2.3 3

Problem Let P,Q be polynomials with no common zeros. Assume a is a root of Q. Find the principal part of P/Q at z=a in terms of P and Q if a is (1) a simple root, and (2) a double root.

21.2.4 4

Problem Let f be non-constant, analytic in |z| > 0, where $f(z_n) = 0$ for infinitely many points z_n with $\lim_{n \to \infty} z_n = 0$.

Show that z = 0 is an essential singularity for f.

Example:
$$f(z) = \sin(1/z)$$
.

21.2.5 5

Problem Show that if f is entire and $\lim_{z\to\infty} f(z) = \infty$, then f is a polynomial.

21.2 Extra Problems 54

21.2.6 6

Problem

a. Show (without using 3.8.9 in the S&S) that

$$\int_0^{2\pi} \log \left| 1 - e^{i\theta} \right| \, d\theta = 0$$

b. Show that this identity is equivalent to S&S 3.8.9:

$$\int_0^1 \log(\sin(\pi x)) \ dx = -\log 2.$$

21.2.7 7

Problem Let 0 < a < 4 and evaluate

$$\int_0^\infty \frac{x^{\alpha - 1}}{1 + x^3} \ dx$$

21.2.8 8

Problem Prove the fundamental theorem of Algebra using

- a. Rouche's Theorem.
- b. The maximum modulus principle.

21.2.9 9

Problem Let f be analytic in a region D and γ a rectifiable curve in D with interior in D.

Prove that if f(z) is real for all $z \in \gamma$, then f is constant.

21.2.10 10

Problem For a > 0, evaluate

$$\int_0^{\pi/2} \frac{d\theta}{a + \sin^2 \theta}$$

21.2.11 11

Problem Find the number of roots of $p(z) = 4z^4 - 6z + 3$ in |z| < 1 and 1 < |z| < 2 respectively.

21.2.12 12

Problem Prove that $z^4 + 2z^3 - 2z + 10$ has exactly one root in each open quadrant.

21.2.13 13

Problem Prove that for a > 0, $z \tan z - a$ has only real roots.

21.2.14 14

Problem Let f be nonzero, analytic on a bounded region Ω and continuous on its closure $\overline{\Omega}$.

Show that if $|f(z)| \equiv M$ is constant for $z \in \partial \Omega$, then $f(z) \equiv Me^{i\theta}$ for some real constant θ .

22 | Extra Questions from Jingzhi Tie



22.1.1 ?

(1) Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in |z| < R. Show that for r < R,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} .$$

(2) Deduce Liouville's theorem from (1).

22.1.2 ?

Let f be a continuous function in the region

$$D = \{z \mid |z| > R, 0 \le \arg z \le \theta\} \quad \text{where} \quad 1 \le \theta \le 2\pi.$$

If there exists k such that $\lim_{z\to\infty}zf(z)=k$ for z in the region D. Show that

$$\lim_{R' \to \infty} \int_L f(z) dz = i\theta k,$$

where L is the part of the circle |z| = R' which lies in the region D.

22.1.3 ?

Suppose that f is an analytic function in the region D which contains the point a. Let

$$F(z) = z - a - qf(z)$$
, where q is a complex parameter.

- (1) Let $K \subset D$ be a circle with the center at point a and also we assume that $f(z) \neq 0$ for $z \in K$. Prove that the function F has one and only one zero z = w on the closed disc \overline{K} whose boundary is the circle K if $|q| < \min_{z \in K} \frac{|z - a|}{|f(z)|}$.
- (2) Let G(z) be an analytic function on the disk \overline{K} . Apply the residue theorem to prove that $\frac{G(w)}{F'(w)} = \frac{1}{2\pi i} \int_K \frac{G(z)}{F(z)} dz$, where w is the zero from (1).
- (3) If $z \in K$, prove that the function $\frac{1}{F(z)}$ can be represented as a convergent series with respect to q: $\frac{1}{F(z)} = \sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z-a)^{n+1}}$.

22.1.4 ?

Evaluate

$$\int_0^\infty \frac{x \sin x}{x^2 + a^2} \, dx.$$

22.1 Fall 2009 57

22.1.5 ?

Let f = u + iv be differentiable (i.e. f'(z) exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

22.1.6 ?

Show that $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$ using complex analysis, 0 < a < n. Here n is a positive integer.

22.1.7 ?

For s > 0, the **gamma function** is defined by $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$.

- 1. Show that the gamma function is analytic in the half-plane $\Re(s) > 0$, and is still given there by the integral formula above.
- 2. Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need $\Gamma(1-s) = t \int_0^\infty e^{-vt} (vt)^{-s} dv$ for t > 0.

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22.1.8 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree n, then it has n zeros in \mathbb{C} .

22.1.9 ?

Suppose f is entire and there exist A, R > 0 and natural number N such that

$$|f(z)| \ge A|z|^N$$
 for $|z| \ge R$.

Show that (i) f is a polynomial and (ii) the degree of f is at least N.

22.1 Fall 2009

22.1.10 ?

Let $f: \mathbb{C} \to \mathbb{C}$ be an injective analytic (also called *univalent*) function. Show that there exist complex numbers $a \neq 0$ and b such that f(z) = az + b.

22.1.11 ?

Let g be analytic for $|z| \le 1$ and |g(z)| < 1 for |z| = 1.

- 1. Show that g has a unique fixed point in |z| < 1.
- 2. What happens if we replace |g(z)| < 1 with $|g(z)| \le 1$ for |z| = 1? Give an example if (a) is not true or give an proof if (a) is still true.
- 3. What happens if we simply assume that f is analytic for |z| < 1 and |f(z)| < 1 for |z| < 1? Suppose that $f(z) \not\equiv z$. Can f have more than one fixed point in |z| < 1?

Hint: The map
$$\psi_{\alpha}(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$$
 may be useful.

22.1.12 ?

Find a conformal map from $D=\{z:\ |z|<1,\ |z-1/2|>1/2\}$ to the unit disk $\Delta=\{z:\ |z|<1\}.$

22.1.13 ?

Let f(z) be entire and assume values of f(z) lie outside a bounded open set Ω . Show without using Picard's theorems that f(z) is a constant.

22.1.14 ?

(1) Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in |z| < R. Show that for r < R,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} .$$

(2) Deduce Liouville's theorem from (1).

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22.1.15 ?

Let f(z) be entire and assume that $f(z) \leq M|z|^2$ outside some disk for some constant M. Show that f(z) is a polynomial in z of degree ≤ 2 .

22.1.16 ?

Let $a_n(z)$ be an analytic sequence in a domain D such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D. Show that $\sum_{n=0}^{\infty} |a'_n(z)|$ converges uniformly on bounded and closed sub-regions of D.

22.1.17 ?

Let f(z) be analytic in an open set Ω except possibly at a point z_0 inside Ω . Show that if f(z) is bounded in near z_0 , then $\int_{\Delta} f(z)dz = 0$ for all triangles Δ in Ω .

22.1.18 ?

Assume f is continuous in the region: $0 < |z - a| \le R$, $0 \le \arg(z - a) \le \beta_0$ $(0 < \beta_0 \le 2\pi)$ and the limit $\lim_{z \to a} (z - a) f(z) = A$ exists. Show that

$$\lim_{r \to 0} \int_{\gamma_r} f(z) dz = iA\beta_0 ,$$

where $\gamma_r := \{ z \mid z = a + re^{it}, \ 0 \le t \le \beta_0 \}.$

22.1.19 ?

Show that $f(z) = z^2$ is uniformly continuous in any open disk |z| < R, where R > 0 is fixed, but it is not uniformly continuous on \mathbb{C} .

22.1.20 ?

(1) Show that the function u = u(x, y) given by

$$u(x,y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx$$
 for $n \in \mathbb{N}$

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is the solution on $D = \{(x,y) | x^2 + y^2 < 1\}$ of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,0) = \frac{\sin nx}{n}.$$

(2) Show that there exist points $(x,y) \in D$ such that $\limsup_{n \to \infty} |u(x,y)| = \infty$.

22.2 Fall 2011

22.2.1 ?

(1) Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in |z| < R. Show that for r < R,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} .$$

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22.2.2 ?

Let f be a continuous function in the region

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 where $0 \le \theta \le 2\pi$.

If there exists k such that $\lim_{z\to\infty}zf(z)=k$ for z in the region D. Show that

$$\lim_{R' \to \infty} \int_L f(z) dz = i\theta k,$$

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Suppose that f is an analytic function in the region D which contains the point a. Let

$$F(z) = z - a - qf(z)$$
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- (2) Let G(z) be an analytic function on the disk \overline{K} . Apply the residue theorem to prove that $\frac{G(w)}{F'(w)} = \frac{1}{2\pi i} \int_K \frac{G(z)}{F(z)} dz$, where w is the zero from (1).
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Evaluate $\int_0^\infty \frac{x \sin x}{x^2 + a^2} dx$.

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Let f = u + iv be differentiable (i.e. f'(z) exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

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Show that $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$ using complex analysis, 0 < a < n. Here n is a positive integer.

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22.2.9 ?

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Show that (i) f is a polynomial and (ii) the degree of f is at least N.

22.2.10 ?

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(2) Deduce Liouville's theorem from (1).

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Let f(z) be entire and assume that $f(z) \leq M|z|^2$ outside some disk for some constant M. Show that f(z) is a polynomial in z of degree ≤ 2 .

22.2.17 ?

Let $a_n(z)$ be an analytic sequence in a domain D such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D. Show that $\sum_{n=0}^{\infty} |a_n'(z)|$ converges uniformly on bounded and closed sub-regions of D.

22.2.18 ?

Let f(z) be analytic in an open set Ω except possibly at a point z_0 inside Ω . Show that if f(z) is bounded in near z_0 , then $\int_{\Lambda} f(z)dz = 0$ for all triangles Δ in Ω .

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22.2.19 ?

Assume f is continuous in the region: $0 < |z - a| \le R$, $0 \le \arg(z - a) \le \beta_0$ $(0 < \beta_0 \le 2\pi)$ and the limit $\lim_{z \to a} (z - a) f(z) = A$ exists. Show that

$$\lim_{r \to 0} \int_{\gamma_r} f(z) dz = iA\beta_0 ,$$

where $\gamma_r := \{ z \mid z = a + re^{it}, \ 0 \le t \le \beta_0 \}.$

22.2.20 ?

Show that $f(z) = z^2$ is uniformly continuous in any open disk |z| < R, where R > 0 is fixed, but it is not uniformly continuous on \mathbb{C} .

(1) Show that the function u = u(x, y) given by

$$u(x,y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx$$
 for $n \in \mathbb{N}$

is the solution on $D = \{(x,y) \mid x^2 + y^2 < 1\}$ of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,0) = \frac{\sin nx}{n}.$$

(2) Show that there exist points $(x,y) \in D$ such that $\limsup_{n \to \infty} |u(x,y)| = \infty$.

22.3 Spring 2014

22.3.1 ?

The question provides some insight into Cauchy's theorem. Solve the problem without using the Cauchy theorem.

- 1. Evaluate the integral $\int_{\gamma} z^n dz$ for all integers n. Here γ is any circle centered at the origin with the positive (counterclockwise) orientation.
- 2. Same question as (a), but with γ any circle not containing the origin.
- 3. Show that if |a| < r < |b|, then $\int_{\gamma} \frac{dz}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$. Here γ denotes the circle centered at the origin, of radius r, with the positive orientation.

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22.3.2 ?

(1) Assume the infinite series $\sum_{n=0}^{\infty} c_n z^n$ converges in |z| < R and let f(z) be the limit. Show that for r < R,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} .$$

(2) Deduce Liouville's theorem from (1). Liouville's theorem: If f(z) is entire and bounded, then f is constant.

22.3.3 ?

Let f be a continuous function in the region

$$D = \{z \mid |z| > R, 0 \le \arg Z \le \theta\}$$
 where $0 \le \theta \le 2\pi$.

If there exists k such that $\lim_{z\to\infty}zf(z)=k$ for z in the region D. Show that

$$\lim_{R' \to \infty} \int_L f(z) dz = i\theta k,$$

where L is the part of the circle |z| = R' which lies in the region D.

22.3.4 ?

Evaluate $\int_0^\infty \frac{x \sin x}{x^2 + a^2} dx$.

22.3.5 ?

Let f = u + iv be differentiable (i.e. f'(z) exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

22.3.6 ?

Show that $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$ using complex analysis, 0 < a < n. Here n is a positive integer.

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22.3.7 ?

For s > 0, the **gamma function** is defined by $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$.

- Show that the gamma function is analytic in the half-plane $\Re(s) > 0$, and is still given there by the integral formula above.
- Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need $\Gamma(1-s) = t \int_0^\infty e^{-vt} (vt)^{-s} dv$ for t > 0.

22.3.8 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree n, then it has n zeros in C.

22.3.9 ?

Suppose f is entire and there exist A, R > 0 and natural number N such that

$$|f(z)| \ge A|z|^N$$
 for $|z| \ge R$.

Show that (i) f is a polynomial and (ii) the degree of f is at least N.

22.3.10 ?

Let $f: \mathbb{C} \to \mathbb{C}$ be an injective analytic (also called univalent) function. Show that there exist complex numbers $a \neq 0$ and b such that f(z) = az + b.

22.3.11 ?

Let g be analytic for $|z| \le 1$ and |g(z)| < 1 for |z| = 1.

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- Show that g has a unique fixed point in |z| < 1.
- What happens if we replace |g(z)| < 1 with $|g(z)| \le 1$ for |z| = 1? Give an example if (a) is not true or give an proof if (a) is still true.
- What happens if we simply assume that f is analytic for |z| < 1 and |f(z)| < 1 for |z| < 1? Suppose that $f(z) \not\equiv z$. Can f have more than one fixed point in |z| < 1?

Hint: The map
$$\psi_{\alpha}(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$$
 may be useful.

22.3.12 ?

Find a conformal map from $D=\{z:\ |z|<1,\ |z-1/2|>1/2\}$ to the unit disk $\Delta=\{z:\ |z|<1\}.$



\sim

22.4.1 ?

Let $a_n \neq 0$ and assume that $\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = L$. Show that $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L$. In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

22.4.2 ?

(a) Let z, w be complex numbers, such that $\bar{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right|<1 \quad \text{if } |z|<1 \text{ and } |w|<1,$$

and also that

$$\left|\frac{w-z}{1-\overline{w}z}\right|=1 \quad \text{if } |z|=1 \text{ or } |w|=1.$$

(b) Prove that for fixed w in the unit disk \mathbb{D} , the mapping

$$F: z \mapsto \frac{w-z}{1-\overline{w}z}$$

satisfies the following conditions:

(c) F maps \mathbb{D} to itself and is holomorphic.

- (ii) F interchanges 0 and w, namely, F(0) = w and F(w) = 0.
- (iii) |F(z)| = 1 if |z| = 1.
- (iv) $F: \mathbb{D} \to \mathbb{D}$ is bijective.

Hint: Calculate $F \circ F$.

22.4.3 ?

Use *n*-th roots of unity (i.e. solutions of $z^n - 1 = 0$) to show that

$$2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n}=n.$$

Hint: $1 - \cos 2\theta = 2\sin^2 \theta$, $\sin 2\theta = 2\sin \theta \cos \theta$.

(a) Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$
 and $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$

(b) Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta$$
 where $z = re^{i\theta}$ with $-\pi < \theta < \pi$

is a holomorphic function in the region r > 0, $-\pi < \theta < \pi$. Also show that $\log z$ defined above is not continuous in r > 0.

22.4.4 ?

Assume f is continuous in the region: $x \ge x_0, \ 0 \le y \le b$ and the limit

$$\lim_{x \to +\infty} f(x + iy) = A$$

exists uniformly with respect to y (independent of y). Show that

$$\lim_{x \to +\infty} \int_{\gamma_x} f(z) dz = iAb \;,$$

where $\gamma_x := \{ z \mid z = x + it, \ 0 \le t \le b \}.$

22.4.5 ?

(Cauchy's formula for "exterior" region) Let γ be piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2 . Assume f'(z) exists in an open set containing γ and Ω_2 and $\lim_{z\to\infty} f(z) = A$. Show that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

22.4.6 ?

Let f(z) be bounded and analytic in \mathbb{C} . Let $a \neq b$ be any fixed complex numbers. Show that the following limit exists

$$\lim_{R \to \infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

Use this to show that f(z) must be a constant (Liouville's theorem).

22.4.7 ?

Prove by justifying all steps that for all $\xi \in \mathbb{C}$ we have $e^{-\pi \xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x \xi} dx$.

Hint: You may use that fact in Example 1 on p. 42 of the textbook without proof, i.e., you may assume the above is true for real values of ξ .

22.4.8 ?

Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Let denote the power series in the open disc. Show that (1) $c_n \neq 0$ for all large enough n's, and (2) $\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = z_0$.

22.4.9 ?

Let f(z) be a non-constant analytic function in |z| > 0 such that $f(z_n) = 0$ for infinite many points z_n with $\lim_{n \to \infty} z_n = 0$. Show that z = 0 is an essential singularity for f(z). (An example of such a function is $f(z) = \sin(1/z)$.)

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22.4.10 ?

Let f be entire and suppose that $\lim_{z\to\infty} f(z) = \infty$. Show that f is a polynomial.

22.4.11 ?

Expand the following functions into Laurent series in the indicated regions:

(a)
$$f(z) = \frac{z^2 - 1}{(z+2)(z+3)}$$
, $2 < |z| < 3$, $3 < |z| < +\infty$.

(b)
$$f(z) = \sin \frac{z}{1-z}$$
, $0 < |z-1| < +\infty$

22.4.12 ?

Assume f(z) is analytic in region D and Γ is a rectifiable curve in D with interior in D. Prove that if f(z) is real for all $z \in \Gamma$, then f(z) is a constant.

22.4.13 ?

Find the number of roots of $z^4 - 6z + 3 = 0$ in |z| < 1 and 1 < |z| < 2 respectively.

22.4.14 ?

Prove that $z^4 + 2z^3 - 2z + 10 = 0$ has exactly one root in each open quadrant.

22.4.15 ?

(1) Let $f(z) \in H(\mathbb{D})$, Re(f(z)) > 0, f(0) = a > 0. Show that

$$\left| \frac{f(z) - a}{f(z) + a} \right| \le |z|, \quad |f'(0)| \le 2a.$$

(2) Show that the above is still true if Re(f(z)) > 0 is replaced with $Re(f(z)) \ge 0$.

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22.4.16 ?

Assume f(z) is analytic in \mathbb{D} and f(0) = 0 and is not a rotation (i.e. $f(z) \neq e^{i\theta}z$). Show that $\sum_{n=1}^{\infty} f^n(z)$ converges uniformly to an analytic function on compact subsets of \mathbb{D} , where $f^{n+1}(z) = f(f^n(z))$.

22.4.17 ?

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic and one-to-one in |z| < 1. For 0 < r < 1, let D_r be the disk |z| < r. Show that the area of $f(D_r)$ is finite and is given by

$$S = \pi \sum_{n=1}^{\infty} n|c_n|^2 r^{2n}.$$

(Note that in general the area of $f(D_1)$ is infinite.)

22.4.18 ?

Let $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ be analytic and one-to-one in $r_0 < |z| < R_0$. For $r_0 < r < R < R_0$, let D(r,R) be the annulus r < |z| < R. Show that the area of f(D(r,R)) is finite and is given by

$$S = \pi \sum_{n = -\infty}^{\infty} n |c_n|^2 (R^{2n} - r^{2n}).$$

22.5 Spring 2015

22.5.1 ?

Let $a_n(z)$ be an analytic sequence in a domain D such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D. Show that $\sum_{n=0}^{\infty} |a'_n(z)|$ converges uniformly on bounded and closed sub-regions of D.

22.5.2 ?

Let f_n, f be analytic functions on the unit disk \mathbb{D} . Show that the following are equivalent.

(i) $f_n(z)$ converges to f(z) uniformly on compact subsets in \mathbb{D} .

(ii)
$$\int_{|z|=r} |f_n(z) - f(z)| |dz|$$
 converges to 0 if $0 < r < 1$.

22.5.3 ?

Let f and g be non-zero analytic functions on a region Ω . Assume |f(z)| = |g(z)| for all z in Ω . Show that $f(z) = e^{i\theta}g(z)$ in Ω for some $0 \le \theta < 2\pi$.

22.5.4 ?

Suppose f is analytic in an open set containing the unit disc \mathbb{D} and |f(z)|=1 when |z|=1. Show that either $f(z)=e^{i\theta}$ for some $\theta\in\mathbb{R}$ or there are finite number of $z_k\in\mathbb{D},\,k\leq n$ and $\theta\in\mathbb{R}$ such that $f(z)=e^{i\theta}\prod_{k=1}^n\frac{z-z_k}{1-\bar{z}_kz}$.

Also cf. Stein et al, 1.4.7, 3.8.17

22.5.5 ?

- (1) Let p(z) be a polynomial, R > 0 any positive number, and $m \ge 1$ an integer. Let $M_R = \sup\{|z^m p(z) 1| : |z| = R\}$. Show that $M_R > 1$.
- (2) Let $m \ge 1$ be an integer and $K = \{z \in \mathbb{C} : r \le |z| \le R\}$ where r < R. Show (i) using (1) as well as, (ii) without using (1) that there exists a positive number $\varepsilon_0 > 0$ such that for each polynomial p(z),

$$\sup\{|p(z)-z^{-m}|:z\in K\}\geq \varepsilon_0.$$

22.5.6 ?

Let $f(z) = \frac{1}{z} + \frac{1}{z^2 - 1}$. Find all the Laurent series of f and describe the largest annuli in which these series are valid

22.5.7 ?

Suppose f is entire and there exist A, R > 0 and natural number N such that $|f(z)| \le A|z|^N$ for $|z| \ge R$. Show that (i) f is a polynomial and (ii) the degree of f is at most N.

22.5.8 ?

Suppose f is entire and there exist A, R > 0 and natural number N such that $|f(z)| \ge A|z|^N$ for $|z| \ge R$. Show that (i) f is a polynomial and (ii) the degree of f is at least N.

22.5.9 ?

- (1) Explicitly write down an example of a non-zero analytic function in |z| < 1 which has infinitely zeros in |z| < 1.
- (2) Why does not the phenomenon in (1) contradict the uniqueness theorem?

22.5.10 ?

- (1) Assume u is harmonic on open set O and z_n is a sequence in O such that $u(z_n) = 0$ and $\lim z_n \in O$. Prove or disprove that u is identically zero. What if O is a region?
- (2) Assume u is harmonic on open set O and u(z) = 0 on a disc in O. Prove or disprove that u is identically zero. What if O is a region?
- (3) Formulate and prove a Schwarz reflection principle for harmonic functions

cf. Theorem 5.6 on p.60 of Stein et al.

Hint: Verify the mean value property for your new function obtained by Schwarz reflection principle.

22.5.11 ?

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any s < r, there exists a constant c > 0 such that

$$||f||_{(\infty,s)} \le c||f||_{(1,r)},$$

where $|f||_{(\infty,s)} = \sup_{z \in D_s(z_0)} |f(z)|$ and $||f||_{(1,r)} = \int_{D_r(z_0)} |f(z)| dx dy$.

Note: Exercise 3.8.20 on p.107 in Stein et al is a straightforward consequence of this stronger result using the integral form of the Cauchy-Schwarz inequality in real analysis.

22.5.12 ?

- (1) Let f be analytic in $\Omega: 0 < |z-a| < r$ except at a sequence of poles $a_n \in \Omega$ with $\lim_{n \to \infty} a_n = a$. Show that for any $w \in \mathbb{C}$, there exists a sequence $z_n \in \Omega$ such that $\lim_{n \to \infty} f(z_n) = w$.
- (2) Explain the similarity and difference between the above assertion and the Weierstrass-Casorati theorem.

22.5.13 ?

Compute the following integrals.

$$i \int_{0}^{\infty} \frac{1}{(1+x^{n})^{2}} dx, \ n \ge 1 \ (ii) \int_{0}^{\infty} \frac{\cos x}{(x^{2}+a^{2})^{2}} dx, \ a \in \mathbb{R} \ (iii) \int_{0}^{\pi} \frac{1}{a+\sin\theta} d\theta, \ a > 1$$

$$iv \int_{0}^{\frac{\pi}{2}} \frac{d\theta}{a+\sin^{2}\theta}, \ a > 0. \ (v) \int_{|z|=2}^{1} \frac{1}{(z^{5}-1)(z-3)} dz \ (v) \int_{-\infty}^{\infty} \frac{\sin \pi a}{\cosh \pi x + \cos \pi a} e^{-ix\xi} dx, \ 0 < a < 1,$$

$$\xi \in \mathbb{R} \ (vi) \int_{|z|=1} \cot^{2} z \, dz.$$

22.5.14 ?

Compute the following integrals.

$$\begin{split} & i \int_0^\infty \frac{\sin x}{x} \, dx \text{ (ii) } \int_0^\infty (\frac{\sin x}{x})^2 \, dx \text{ (iii) } \int_0^\infty \frac{x^{a-1}}{(1+x)^2} \, dx, \, 0 < a < 2 \\ & i \int_0^\infty \frac{\cos ax - \cos bx}{x^2} \, dx, \, a, b > 0 \text{ (ii) } \int_0^\infty \frac{x^{a-1}}{1+x^n} \, dx, \, 0 < a < n \\ & iii \int_0^\infty \frac{\log x}{1+x^n} \, dx, \, n \ge 2 \text{ (iv) } \int_0^\infty \frac{\log x}{(1+x^2)^2} \, dx \text{ (v) } \int_0^\pi \log |1-a\sin\theta| \, d\theta, \, a \in \mathbb{C} \end{split}$$

22.5.15 ?

Let 0 < r < 1. Show that polynomials $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ have no zeros in |z| < r for all sufficiently large n's.

22.5.16 ?

Let f be an analytic function on a region Ω . Show that f is a constant if there is a simple closed curve γ in Ω such that its image $f(\gamma)$ is contained in the real axis.

22.5.17 ?

- (1) Show that $\frac{\pi^2}{\sin^2 \pi z}$ and $g(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ have the same principal part at each integer point.
- (2) Show that $h(z) = \frac{\pi^2}{\sin^2 \pi z} g(z)$ is bounded on \mathbb{C} and conclude that $\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$.

22.5.18 ?

Let f(z) be an analytic function on $\mathbb{C}\setminus\{z_0\}$, where z_0 is a fixed point. Assume that f(z) is bijective from $\mathbb{C}\setminus\{z_0\}$ onto its image, and that f(z) is bounded outside $D_r(z_0)$, where r is some fixed positive number. Show that there exist $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$, $c \neq 0$ such that $f(z) = \frac{az + b}{cz + d}$.

22.5.19 ?

Assume f(z) is analytic in $\mathbb{D}: |z| < 1$ and f(0) = 0 and is not a rotation (i.e. $f(z) \neq e^{i\theta}z$). Show that $\sum_{n=1}^{\infty} f^n(z)$ converges uniformly to an analytic function on compact subsets of \mathbb{D} , where $f^{n+1}(z) = f(f^n(z))$.

22.5.20 ?

Let f be a non-constant analytic function on $\mathbb D$ with $f(\mathbb D)\subseteq \mathbb D$. Use $\psi_a(f(z))$ (where a=f(0), $\psi_a(z)=\frac{a-z}{1-\bar az}$) to prove that $\frac{|f(0)|-|z|}{1+|f(0)||z|}\leq |f(z)|\leq \frac{|f(0)|+|z|}{1-|f(0)||z|}$.

22.5.21 ?

Find a conformal map

1. from $\{z: |z-1/2| > 1/2, \operatorname{Re}(z) > 0\}$ to \mathbb{H}

- 2. from $\{z: |z-1/2| > 1/2, |z| < 1\}$ to \mathbb{D}
- 3. from the intersection of the disk $|z+i| < \sqrt{2}$ with \mathbb{H} to \mathbb{D} .
- 4. from $\mathbb{D}\setminus[a,1)$ to $\mathbb{D}\setminus[0,1)$ (0 < a < 1).

Short solution possible using Blaschke factor

5. from $\{z: |z| < 1, \text{Re}(z) > 0\} \setminus (0, 1/2]$ to \mathbb{H} .

22.5.22 ?

Let C and C' be two circles and let $z_1 \in C$, $z_2 \notin C$, $z_1' \in C'$, $z_2' \notin C'$. Show that there is a unique fractional linear transformation f with f(C) = C' and $f(z_1) = z_1'$, $f(z_2) = z_2'$.

22.5.23 ?

Assume $f_n \in H(\Omega)$ is a sequence of holomorphic functions on the region Ω that are uniformly bounded on compact subsets and $f \in H(\Omega)$ is such that the set $\{z \in \Omega : \lim_{n \to \infty} f_n(z) = f(z)\}$ has a limit point in Ω . Show that f_n converges to f uniformly on compact subsets of Ω .

22.5.24 ?

Let $\psi_{\alpha}(z) = \frac{\alpha - z}{1 - \overline{\alpha}z}$ with $|\alpha| < 1$ and $\mathbb{D} = \{z : |z| < 1\}$. Prove that

•
$$\frac{1}{\pi} \iint_{\mathbb{D}} |\psi_{\alpha}'|^2 dx dy = 1.$$

•
$$\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_{\alpha}| dx dy = \frac{1 - |\alpha|^2}{|\alpha|^2} \log \frac{1}{1 - |\alpha|^2}.$$

22.5.25 ?

Prove that $f(z) = -\frac{1}{2}\left(z + \frac{1}{z}\right)$ is a conformal map from half disc $\{z = x + iy : |z| < 1, y > 0\}$ to upper half plane $\mathbb{H} = \{z = x + iy : y > 0\}$.

22.5.26 ?

Let Ω be a simply connected open set and let γ be a simple closed contour in Ω and enclosing a bounded region U anticlockwise. Let $f: \Omega \to \mathbb{C}$ be a holomorphic function and $|f(z)| \leq M$ for all $z \in \gamma$. Prove that $|f(z)| \leq M$ for all $z \in U$.

22.5.27 ?

Compute the following integrals. (i) $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx$, 0 < a < n (ii) $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$

22.5.28 ?

Let 0 < r < 1. Show that polynomials $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ have no zeros in |z| < r for all sufficiently large n's.

22.5.29 ?

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any s < r, there exists a constant c > 0 such that

$$||f||_{(\infty,s)} \le c||f||_{(1,r)},$$

where $||f||_{(\infty,s)} = \sup_{z \in D_s(z_0)} |f(z)|$ and $||f||_{(1,r)} = \int_{D_r(z_0)} |f(z)| dx dy$.

22.5.30 ?

Let $\psi_{\alpha}(z) = \frac{\alpha - z}{1 - \overline{\alpha}z}$ with $|\alpha| < 1$ and $\mathbb{D} = \{z : |z| < 1\}$. Prove that

•
$$\frac{1}{\pi} \iint_{\mathbb{D}} |\psi_{\alpha}'|^2 dx dy = 1.$$

•
$$\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_{\alpha}| dx dy = \frac{1 - |\alpha|^2}{|\alpha|^2} \log \frac{1}{1 - |\alpha|^2}.$$

Prove that $f(z) = -\frac{1}{2}\left(z + \frac{1}{z}\right)$ is a conformal map from half disc $\{z = x + iy : |z| < 1, y > 0\}$ to upper half plane $\mathbb{H} = \{z = x + iy : y > 0\}$.

22.5.31 ?

Let Ω be a simply connected open set and let γ be a simple closed contour in Ω and enclosing a bounded region U anticlockwise. Let $f: \Omega \to \mathbb{C}$ be a holomorphic function and $|f(z)| \leq M$ for all $z \in \gamma$. Prove that $|f(z)| \leq M$ for all $z \in U$.

22.5.32 ?

Compute the following integrals. (i)
$$\int_0^\infty \frac{x^{a-1}}{1+x^n} dx$$
, $0 < a < n$ (ii) $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$

22.5.33 ?

Let 0 < r < 1. Show that polynomials $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ have no zeros in |z| < r for all sufficiently large n's.

22.5.34 ?

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any s < r, there exists a constant c > 0 such that

$$||f||_{(\infty,s)} \le c||f||_{(1,r)},$$

where $||f||_{(\infty,s)} = \sup_{z \in D_s(z_0)} |f(z)|$ and $||f||_{(1,r)} = \int_{D_r(z_0)} |f(z)| dx dy$.

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22.6.1 ?

Let u(x,y) be harmonic and have continuous partial derivatives of order three in an open disc of radius R > 0.

(a) Let two points (a, b), (x, y) in this disk be given. Show that the following integral is independent of the path in this disk joining these points:

$$v(x,y) = \int_{a,b}^{x,y} \left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right).$$

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- (b) {=tex} \hfill
 - (i) Prove that u(x,y) + iv(x,y) is an analytic function in this disc.
 - (ii) Prove that v(x, y) is harmonic in this disc.

22.6.2 ?

- (a) f(z) = u(x,y) + iv(x,y) be analytic in a domain $D \subset \mathbb{C}$. Let $z_0 = (x_0, y_0)$ be a point in D which is in the intersection of the curves $u(x,y) = c_1$ and $v(x,y) = c_2$, where c_1 and c_2 are constants. Suppose that $f'(z_0) \neq 0$. Prove that the lines tangent to these curves at z_0 are perpendicular.
- (b) Let $f(z) = z^2$ be defined in \mathbb{C} .
- (c) Describe the level curves of Re(f) and of Im(f).
- (ii) What are the angles of intersections between the level curves Re(f) = 0 and Im(f)? Is your answer in agreement with part a) of this question?

22.6.3 ?

(a) $f: D \to \mathbb{C}$ be a continuous function, where $D \subset \mathbb{C}$ is a domain.Let $\alpha: [a, b] \to D$ be a smooth curve. Give a precise definition of the *complex line integral*

$$\int_{\alpha} f.$$

(b) Assume that there exists a constant M such that $|f(\tau)| \leq M$ for all $\tau \in \text{Image}(\alpha)$. Prove that

$$\left| \int_{\alpha} f \right| \leq M \times \operatorname{length}(\alpha).$$

(c) Let C_R be the circle |z| = R, described in the counterclockwise direction, where R > 1. Provide an upper bound for $|\int_{C_R} \frac{\log(z)}{z^2}|$, which depends only on R and other constants.

22.6.4 ?

(a) Let Let $f: \mathbb{C} \to \mathbb{C}$ be an entire function. Assume the existence of a non-negative integer m, and of positive constants L and R, such that for all z with |z| > R the inequality

$$|f(z)| \le L|z|^m$$

holds. Prove that f is a polynomial of degree $\leq m$.

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(b) Let $f:\mathbb{C}\to\mathbb{C}$ be an entire function. Suppose that there exists a real number M such that for all $z\in\mathbb{C}$

$$\operatorname{Re}(f) \leq M$$
.

Prove that f must be a constant.

22.6.5 ?

Prove that all the roots of the complex polynomial

$$z^7 - 5z^3 + 12 = 0$$

lie between the circles |z| = 1 and |z| = 2.

22.6.6 ?

(a) Let F be an analytic function inside and on a simple closed curve C, except for a pole of order $m \ge 1$ at z = a inside C. Prove that

$$\frac{1}{2\pi i} \oint_C F(\tau) d\tau = \lim_{\tau \to a} \frac{d^{m-1}}{d\tau^{m-1}} ((\tau - a)^m F(\tau)).$$

(b) Evaluate

$$\oint_C \frac{e^{\tau}}{(\tau^2 + \pi^2)^2} d\tau$$

where C is the circle |z| = 4.

22.6.7 ?

Find the conformal map that takes the upper half-plane comformally onto the half-strip $\{w = x + iy : -\pi/2 < x < \pi/2 \ y > 0\}$.

22.6.8 ?

Compute the integral $\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{\cosh \pi x} dx$ where $\cosh z = \frac{e^z + e^{-z}}{2}$.