Real Analysis Qualifying Exam Solutions

D. Zack Garza

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1 Spring 2020

1.1 1

Concepts used:

- DCT
- Weierstrass Approximation Theorem

Suppose p is a polynomial, then

$$\begin{aligned}
\lim_{k\longrightarrow\infty}\int_0^1 kx^{k-1}p(x)\,dx &= \lim_{k\longrightarrow\infty}\int_0^1 \left(\frac{\partial}{\partial x}\,x^k\right)p(x)\,dx \\
&= \lim_{k\longrightarrow\infty}\left[x^k p(x)\Big|_0^1 - \int_0^1 x^k\left(\frac{\partial}{\partial x}\,p(x)\right)dx\right] \quad \text{integrating by parts} \\
&= p(1) - \lim_{k\longrightarrow\infty}\int_0^1 x^k\left(\frac{\partial}{\partial x}\,p(x)\right)dx,
\end{aligned}$$

and thus it suffices to show that

$$\lim_{k\longrightarrow\infty}\int_0^1 x^k\left(\frac{\partial}{\partial x}\,p(x)\right)dx = 0.$$

Integrating by parts a second time yields

$$\begin{aligned}
\lim_{k\longrightarrow\infty}\int_0^1 x^k\left(\frac{\partial}{\partial x}\,p(x)\right)dx &= \lim_{k\longrightarrow\infty}\frac{x^{k+1}}{k+1}p'(x)\Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1}\left(\frac{\partial^2}{\partial x^2}\,p(x)\right)dx \\
&= -\lim_{k\longrightarrow\infty}\int_0^1 \frac{x^{k+1}}{k+1}\left(\frac{\partial^2}{\partial x^2}\,p(x)\right)dx \\
&= -\int_0^1 \lim_{k\longrightarrow\infty}\frac{x^{k+1}}{k+1}\left(\frac{\partial^2}{\partial x^2}\,p(x)\right)dx \quad \text{by DCT} \\
&= -\int_0^1 0\left(\frac{\partial^2}{\partial x^2}\,p(x)\right)dx \\
&= 0.
\end{aligned}$$

The DCT can be applied here because f'' is continuous and $[0,1]$ is compact, so f'' is bounded on $[0,1]$ by a constant M and $\int_0^1 \left|x^k f''(x)\right| \leq \int_0^1 1 \cdot M = M < \infty$.

We now use the Weierstrass approximation theorem: if $f : [a,b] \longrightarrow \mathbb{R}$ is continuous, then for every $\varepsilon > 0$ there exists a polynomial $p_\varepsilon(x)$ such that $\|f - p_\varepsilon\|_\infty < \varepsilon$. Thus

$$\begin{aligned}
\left|\int_0^1 kx^{k-1} p_\varepsilon(x)\,dx - \int_0^1 kx^{k-1} f(x)\,dx\right| &= \left|\int_0^1 kx^{k-1}(p_\varepsilon(x) - f(x))\,dx\right| \\
&\leq \left|\int_0^1 kx^{k-1} \|p_\varepsilon - f\|_\infty\,dx\right| \\
&= \|p_\varepsilon - f\|_\infty \cdot \left|\int_0^1 kx^{k-1}\,dx\right| \\
&= \|p_\varepsilon - f\|_\infty \cdot x^k\Big|_0^1 \\
&= \|p_\varepsilon - f\|_\infty \overset{\varepsilon \longrightarrow 0}{\longrightarrow} 0
\end{aligned}$$

and the integrals are equal. Finally, the first integral is equal to $p_\varepsilon(1)$ for each ε, which converges to $f(1)$ since uniform convergence implies pointwise convergence.

1.2 2

Concepts used:

- Definition of outer measure: $m_*(E) = \inf_{\{Q_j\} \rightrightarrows E} \sum |Q_j|$ where $\{Q_j\}$ is a countable collection of closed cubes.
- Break $\mathbb{R}$ into $\coprod_{n \in \mathbb{Z}} [n, n+1)$, each with finite measure.

1.2.1 a

Suppose $m_*(E) = N < \infty$.

Since $m_*(E)$ is an infimum, by definition, for every $\varepsilon > 0$ there exists a covering by closed cubes $\{Q_i(\varepsilon)\}_{i=1}^\infty \rightrightarrows E$ such that

$$\sum_{i=1}^\infty |Q_i(\varepsilon)| < N + \varepsilon.$$

Set $\varepsilon_n = \frac{1}{n}$ to produce such a collection $\{Q_i(\varepsilon_n)\}$ and set $B_n := \bigcup_{i=1}^\infty Q_i(\varepsilon_n)$. Then (theorem) the outer measure of cubes is *equal* to the sum of their volumes, so

$$m_*(B_n) = \sum_{i=1}^\infty |Q_i(\varepsilon_n)| < N + \varepsilon_n = N + \frac{1}{n}.$$

Now set $B := \bigcap_{n=1}^\infty B_n$.

- Since $E \subseteq B_n$ for every n, $E \subseteq B$
- Since B is a countable intersection of countable unions of closed sets, B is Borel.
- Since $B_n \subseteq B$ for every n, we can apply subadditivity to obtain the inequality

$$E \subseteq B \subseteq B_n \implies N \leq m_*(B) \leq m_*(B_n) < N + \frac{1}{n} \quad \text{for all} \quad n \in \mathbb{Z}^{\geq 1}.$$

This forces $m_*(E) = m_*(B)$.

If $m_*(E) = \infty$, then take $B = \mathbb{R}^n$ since $m(\mathbb{R}^n) = \infty$.

1.2.2 b

Suppose $m_*(E) < \infty$.

- By (a), find a Borel set $B \supseteq E$ such that $m_*(B) = m_*(E)$
- Note that $E \subseteq B \implies B \bigcap E = E$ and $B \bigcap E^c = B \setminus E$.
- By assumption,

$$\begin{aligned}
m_*(B) &= m_*(B \textstyle\bigcap E) + m_*(B \textstyle\bigcap E^c) \\
m_*(E) &= m_*(E) + m_*(B \setminus E) \\
m_*(E) - m_*(E) &= m_*(B \setminus E) \qquad\qquad \text{since } m_*(E) < \infty \\
\implies m_*(B \setminus E) &= 0.
\end{aligned}$$

- So take $N = B \setminus E$; this shows $m_*(N) = 0$ and $E = B \setminus (B \setminus E) = B \setminus N$.
- If $m_*(E) = \infty$
  - Apply result to $E_R \coloneqq E \bigcap [R, R+1)^n \subset \mathbb{R}^n$ for $R \in \mathbb{Z}$, so $E = \coprod_R E_R$
  - Obtain B_R, N_R such that $E_R = B_R \setminus N_R$, $m_*(E_R) = m_*(B_R)$, and $m_*(N_R) = 0$.
  - Note that
    * $B \coloneqq \bigcup_R B_R$, which is a union of Borel sets and thus still Borel
    * $E = \bigcup_R E_R$
    * $N \coloneqq B \setminus E$
    * $N' \coloneqq \bigcup_R N_R$ which is a union of null sets and thus still null
  - Since $E_R \subset B_R$ for every R, we have $E \subset B$
  - We can compute

$$N = B \setminus E = \left(\bigcup_R B_R \right) \setminus \left(\bigcup_R E_R \right) \subseteq \bigcup_R (B_R \setminus E_R) = N'$$

  where $m_*(N) = 0$ and thus subadditivity forces $m_*(N) = 0$.

1.3 3

1.3.1 a

Stated integral equality:

- Let $\varepsilon > 0$
- $C_c(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$ is dense so choose $\{f_n\} \longrightarrow f$ with $\|f_n - f\|_1 \longrightarrow 0$.
- Since $\{f_n\}$ are compactly supported, choose $N_0 \gg 1$ such that f_n is zero outside of $B_{N_0}(\mathbf{0})$.
- Then

$$\begin{aligned}
N \geq N_0 \implies \int_{|x|>N} |f| &= \int_{|x|>N} |f - f_n + f_n| \\
&\leq \int_{|x|>N} |f - f_n| + \int_{|x|>N} |f_n| \\
&= \int_{|x|>N} |f - f_n| \\
&\leq \int_{|x|>N} \|f - f_n\|_1 \\
&= \|f_n - f\|_1 \left(\int_{|x|>N} 1 \right) \\
&\overset{n\longrightarrow\infty}{\longrightarrow} 0 \left(\int_{|x|>N} 1 \right) \\
&= 0 \\
&\overset{N\longrightarrow\infty}{\longrightarrow} 0.
\end{aligned}$$

To see that this doesn't force $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$:

- Take $f(x)$ to be a train of boxes of height 1 and area $1/2^j$ centered on even integers.
- Then the $\displaystyle\int_{|x|>N} |f| = \sum_{j=N}^{\infty} 1/2^j \overset{N\longrightarrow\infty}{\longrightarrow} 0$ as the tail of a convergent sum.
- However $f(x) = 1$ for infinitely many even integers $x > N$, so $f(x) \not\longrightarrow 0$ as $|x| \longrightarrow \infty$.

1.3.2 b

- Since f is decreasing on $[1, \infty)$, for any $t \in [x - n, x]$ we have

$$x - n \leq t \leq x \implies f(x) \leq f(t) \leq f(x - n).$$

- Integrate over $[x, 2x]$, using monotonicity of the integral:

$$\int_x^{2x} f(x)\,dt \leq \int_x^{2x} f(t)\,dt \leq \int_x^{2x} f(x-n)\,dt \implies xf(x) \leq \int_x^{2x} f(t)\,dt \leq xf(x-n).$$

- By (a), $\displaystyle\lim_{x\longrightarrow\infty} \int_x^{2x} f(t)\ dt = 0$ (?)
- So the LHS term $\displaystyle\lim_{x\longrightarrow\infty} xf(x) = 0$.
- Since $x > 1$, $|f(x)| \leq |xf(x)|$
- Thus $f(x) \longrightarrow 0$ as well.

Alternatively showing $f(x) \overset{x\longrightarrow\infty}{0}$:

- Toward a contradiction, suppose not.
- If $f(x) \longrightarrow -\infty$, then $f \notin L^1(\mathbb{R})$: choose $x \gg 1$ so that $|f(x)| > 1$, then
$$\int_{\mathbb{R}} |f| \geq \int_x^\infty f(t)\,dt \geq \int_x^\infty 1 = \infty.$$
- WLOG replace f with $-f$ to make f increasing (since $\|f\|_1 = \|f\|_2$).
- Otherwise $f(x) \longrightarrow L$ with $L < \infty$. Fix $\varepsilon > 0$.
- Choose $x \gg 1$ so that $t \geq x \implies L - \varepsilon \leq f(t) \leq L$
- Then $\displaystyle\int_x^\infty f \geq \int_x^\infty (L-\varepsilon) = \infty$.

1.3.3 c

- No: take $f(x) = \dfrac{1}{x \ln x}$
- Then $\displaystyle\int f = \ln(\ln(x)) \longrightarrow \infty$ is unbounded, so $f \notin L^1([1,\infty))$.
- But $xf(x) = 1/\ln(x) \longrightarrow 0$

1.4 4

Relevant concepts:

- Tonelli: non-negative and measurable yields measurability of slices and equality of iterated integrals
- Fubini:i $f(x,y) \in L^1$ yields *integrable* slices and equality of iterated integrals
- F/T: apply Tonelli to $|f|$; if finite, $f \in L^1$ and apply Fubini to f

$$\begin{aligned}
\|(f*g)(x)\|_1 &= \left\| \int_{\mathbb{R}} H(x,y)\,dy \right\|_1 \\
&\coloneqq \left\| \int_{\mathbb{R}} f(y)g(x-y)\,dy \right\|_1 \\
&\leq \int_{\mathbb{R}} \|f(y)g(x-y)\|_1\,dy \\
&\leq \int_{\mathbb{R}} |f(y)| \cdot \|g(x-y)\|_1\,dy \\
&\leq \int_{\mathbb{R}} |f(y)| \cdot \|g\|_1\,dy \\
&\leq \|g\|_1 \int_{\mathbb{R}} |f(y)|\,dy \\
&\leq \|g\|_1 \|f\|_1 \\
&< \infty \quad \text{by assumption} \quad .
\end{aligned}$$

Todo:

- Change of variables for $x - y$?
- How to justify F/T?

1.5 5

Note that

$$\begin{aligned}
\lim_n \left(1+\frac{x^2}{n}\right)^{-(n+1)} &= \frac{1}{\lim_n \left(1+\frac{x^2}{n}\right)^1 \left(1+\frac{x^2}{n}\right)^n} \\
&= \frac{1}{1\cdot e^{x^2}} \\
&= e^{-x^2}.
\end{aligned}$$

If passing the limit through the integral is justified, we will have $\displaystyle\int_0^\infty e^{-x^2} = \frac{\sqrt{\pi}}{2}$.

Computing the integral:

$$\begin{aligned}
\left(\int_{\mathbb{R}} e^{-x^2}\,dx\right)^2 &= \left(\int_{\mathbb{R}} e^{-x^2}\,dx\right)\left(\int_{\mathbb{R}} e^{-y^2}\,dx\right) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(x+y)^2}\,dx \\
&= \int_0^{2\pi}\int_0^\infty e^{-r^2} r\,dr\,d\theta \qquad u = r^2 \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^\infty e^{-u}\,du\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 1 \\
&= \pi,
\end{aligned}$$

and now use the fact that the function is even.

Todo:

- Justify, MCT? Possible have

$$f_n(x) \coloneqq \frac{1}{\left(1+\frac{x^2}{n}\right)^{n+1}} \nearrow e^{-x^2}.$$

1.6 6

Concepts used:

- For $e_n(x) \coloneqq 2^{2\pi i n x}$, the set $\{e_n\}$ is an orthonormal basis for $L^2([0,1])$.
- For any orthonormal set in a Hilbert space, Bessel's inequality:

$$\sum_{k=1}^\infty |\langle x, e_k\rangle|^2 \leq \|x\|^2.$$

- When a basis, the above is an inequality (Parseval)
- Arguing uniform convergence: since $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$, we should be able to apply the M test.

Want to show

$$\int_0^1 |f(x)|^2 \, dx \leq \int_0^1 |f(x)| \, dx$$
$$\sum_{n=1}^{\infty} |c_n|^2 \leq \sum_{n=1}^{\infty} |c_n|.$$

- First inequality: ?
- Second inequality: break into 2 pieces?

2 Spring 2019

2.1 1

2.1.1 a

Let $\{f_k\}$ be a Cauchy sequence in $C(I)$. For each fixed $x \in [0,1]$, the sequence of real numbers $\{f_k(x)\}$ is Cauchy in $\mathbb{R}$, which is complete, since

$$x_0 \in I \implies |f_k(x_0) - f_j(x_0)| \leq \sup_{x \in I} |f_k(x) - f_j(x)| = \|f_k - f_j\|_\infty \longrightarrow 0,$$

so we can define $f(x) \coloneqq \lim_k f_k(x)$.

We also have

$$\|f_k - f\|_\infty = \left\| f_k - \lim_{j \longrightarrow \infty} f_j \right\|_\infty = \lim_{j \longrightarrow \infty} \|f_k - f_j\|_\infty \longrightarrow 0.$$

Finally, f is the uniform limit of continuous functions and thus continuous.

$\blacksquare$

2.1.2 b

It suffices to produce a Cauchy sequence that does not converge to a continuous function. Take

$$f_k(x) = \begin{cases} (x + \frac{1}{2})^k & x \in [0, \frac{1}{2}) \\ 1 & x \in [\frac{1}{2}, 1] \end{cases} \xrightarrow{k \longrightarrow \infty} f(x) = \begin{cases} 0 & x \in [0, \frac{1}{2}) \\ 1 & x \in [\frac{1}{2}, 1] \end{cases},$$

which is Cauchy, but there is no $g \in L^1$ that is continuous such that $\|f - g\|_1 = 0$.

2.2 2

2.2.1 a

Lemma 1: $\mu(\coprod_{k=1}^\infty E_k) = \lim_{N \longrightarrow \infty} \sum_{k=1}^N \mu(E_k)$.

Lemma 2: $A = A \setminus B \coprod A \bigcap B$.

Let $A_k = F_k \setminus F_{k+1}$, so the A_k are disjoint, and let $A = \coprod_k A_k$.

Let $F = \bigcap_k F_k$. Then $F_1 = F \coprod A$ by lemma 2, so

$$\begin{aligned}
\mu(F_1) &= \mu(F) + \mu(A) \\
&= \mu(F) + \lim_{N \longrightarrow \infty} \sum_k^N \mu(A_k) \quad \text{by Lemma 1} \\
&= \mu(F) + \lim_{N \longrightarrow \infty} \sum_k^N \mu(F_k) - \mu(F_{k+1}) \\
&= \mu(F) + \lim_{N \longrightarrow \infty} (\mu(F_1) - \mu(F_N)) \quad \text{(Telescoping)} \\
&= \mu(F) + \mu(F_1) - \lim_{N \longrightarrow \infty} \mu(F_N),
\end{aligned}$$

and since the measure is finite, $\mu(F_1) < \infty$ and can be subtracted, yielding

$$\begin{aligned}
\mu(F_1) &= \mu(F) + \mu(F_1) - \lim_{N \longrightarrow \infty} \mu(F_N) \\
\implies \mu(F) &= \lim_{N \longrightarrow \infty} \mu(F_N).
\end{aligned}$$

2.2.2 b

Suppose toward a contradiction that there is some $\varepsilon > 0$ for which no such δ exists.

This means that we can take any sequence $\delta_n \longrightarrow 0$ and produce sets A_n such $m(A) < \delta_n$ but $\mu(A) > \varepsilon$.

So choose the sequence $\delta_n = \dfrac{1}{2^n}$ and define A_n accordingly, and let

$$A = \limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

Since

$$\mu\left(\bigcup_{k=n}^{\infty} A_k\right) \leq \sum_{k=n}^{\infty} \mu(A_k) \approx \frac{1}{2^n} \longrightarrow 0,$$

by part (a) we have $m(A) = 0$. Now by assumption, we should thus have $\mu(A) = 0$ as well.

However, again by part (a), we have

$$\mu(A) = \lim_n \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \geq \lim_n \mu(A_n) = \lim_n \varepsilon = \varepsilon > 0.$$

2.3 3

Since $f_k \longrightarrow f$ almost everywhere, we have $\liminf_k f_k(x) = f(x)$ and since $|f|^2 \in L^+$ we can apply Fatou:

$$\begin{aligned}\|f\|_2^2 &= \int |f(x)|^2 \\ &= \int \liminf_k |f_k(x)|^2 \\ &\underset{\text{Fatou}}{\leq} \liminf_k \int |f_k(x)|^2 \\ &= M^2,\end{aligned}$$

so $\|f\| \leq M < \infty$ and $f \in L^2$.

Let $I = [0,1]$. Applying Egorov's theorem to produce sets F_ε such that $f_k \xrightarrow{u} f$ on F_ε and taking $F = \bigcap F_\varepsilon$, we have

$$\int_I f_k = \int_{F_\varepsilon} f_k + \int_{F_\varepsilon^c} f_k \quad \overset{\varepsilon \longrightarrow 0}{\longrightarrow} \quad \int_F f_k + 0 \quad \overset{k \longrightarrow \infty}{\longrightarrow} \quad \int_F f,$$

using that fact that uniform converges allows commuting limits and integrals.

2.4 4

2.4.1 a

$\implies$:

> Idea: $\mathcal{A} = \{f(x) - t \geq 0\} \bigcap \{t \geq 0\}$.

Define $F(x,t) = f(x)$, $G(x,t) = t$, and $H(x,y) = F(x,t) - G(x,t)$, which are all measurable functions.

Then $\mathcal{A} = \{H \geq 0\} \bigcap \{G \geq 0\}$ which is an intersection of measurable sets.

$\impliedby$:

By F.T., for almost every $x \in \mathbb{R}^n$, the x-slices are measurable, so

$$\mathcal{A}_x := \left\{ t \in \mathbb{R} \;\middle|\; (x,t) \in \mathcal{A} \right\} = [0, f(x)] \implies m(\mathcal{A}_x) = f(x)$$

But $x \mapsto m(\mathcal{A}_x)$ is a measurable function, and is exactly to $x \mapsto f(x)$, so f is measurable.

2.4.2 b

We first note

$$\begin{aligned}\mathcal{A} &= \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; 0 \leq t \leq f(x) \right\} \\ \mathcal{A}_t &= \left\{ x \in \mathbb{R}^n \;\middle|\; t \leq f(x) \right\}.\end{aligned}$$

Then,

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)\ dx &= \int_{\mathbb{R}^n} \int_0^{f(x)} 1\ dt\ dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}} \chi_{\mathcal{A}}\ dt\ dx \\
&\overset{F.T.}{=} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_{\mathcal{A}}\ dx\ dt \\
&= \int_0^{\infty} \int_{\mathbb{R}^n} \chi_{\mathcal{A}}\ dx\ dt \\
&= \int_0^{\infty} m(\mathcal{A}_t)\ dt,
\end{aligned}$$

where we just note that $\int \int \chi_{\mathcal{A}} = m(\mathcal{A})$, and by F.T., all of these integrals are equal.

2.5 5

2.5.1 a

By Holder's inequality with $p = q = 2$, we have

$$\|f\|_1 = \|f \cdot 1\|_1 \leq \|f\|_2 \|1\|_2 = \|f\|_2 m(X)^{\frac{1}{2}} = \|f\|_2,$$

since $X = [0, 1] \implies m(X) = 1$.

So $L^2(X) \subseteq L^1(X)$, and since simple functions are dense in both spaces, L^2 is dense in L^1.

2.5.2 b

Step 1 Let $\Lambda \in L^1(X)^\vee$; we'll show that in fact $\Lambda \in L^2(X)^\vee$, and by Riesz Representation for L^2 there will be a $g \in L^2$ such that $\Lambda(f) = \langle f,\ g \rangle$.

Lemma: $m(X) < \infty \implies L^p(X) \subset L^2(X)$.

Proof: Write Holder's inequality as $\|fg\|_1 \leq \|f\|_a \|g\|_b$ where $\frac{1}{a} + \frac{1}{b} = 1$, then

$$\|f\|_p^p = \| |f|^p \|_1 \leq \| |f|^p \|_a\ \|1\|_b.$$

Now take $a = \frac{2}{p}$ and this reduces to

$$\begin{aligned}
&\|f\|_p^p \leq \|f\|_2^p\ m(X)^{\frac{1}{b}} \\
\implies &\|f\|_p \leq \|f\|_2 \cdot O(m(X)) < \infty.
\end{aligned}$$

Let $f \in L^2$ be arbitrary – by the lemma, $\|f\|_1 \leq C\|f\|_2$ for some constant $C = O(m(X))$.

Since $\|\Lambda\|_{1^\vee} := \sup_{\|f\|_1 = 1} |\Lambda(f)|$, given an arbitrary $f \in L^1$, we can define $\widehat{f} = f / \|f\|_1$, so $\left\|\widehat{f}\right\|_1 = 1$, and obtain

$$\left|\Lambda(\widehat{f})\right| \leq \|\Lambda\|_{1^\vee},$$

since $\|\Lambda\|_{1^\vee}$ is the *least* such bound over all $f \in L^1$, and thus

$$\begin{aligned} \frac{|\Lambda(f)|}{\|f\|_1} &= \left|\Lambda(\widehat{f})\right| \leq \|\Lambda\|_{1^\vee} \\ \implies |\Lambda(f)| &\leq \|\Lambda\|_{1^\vee} \cdot \|f\|_1 \\ &\leq \|\Lambda\|_{1^\vee} \cdot C\|f\|_2, \end{aligned}$$

which is finite by assumption. So $\Lambda \in (L^2)^\vee$ since it is bounded and thus continuous.

By Riesz Representation for L^2, there is a $g \in L^2$ such that for all $f \in L^2$, $\Lambda(f) = \langle f,\ g\rangle$

Step 2 By Holder, we already have

$$\begin{aligned} \|\Lambda\|_{1^\vee} &= \sup_{\|f\|_1=1} |\Lambda(f)| \\ &= \sup_{\|f\|_1=1} \left| \int_X fg \right| \\ &\leq \sup_{\|f\|_1=1} \|fg\|_1 \\ &\leq \sup_{\|f\|_1=1} \|f\|_1 \|g\|_\infty \\ &= \|g\|_\infty, \end{aligned}$$

so it just remains to show that $\|g\|_\infty \leq \|\Lambda\|_{1^\vee}$.

Suppose otherwise, so $\|g\|_\infty > \|\Lambda\|_{1^\vee}$.

Then there exists some $E \subseteq X$ with $m(E) > 0$ such that $x \in E \implies |g(x)| > \|\Lambda\|_{1^\vee}$.

Define

$$h = \frac{1}{m(E)} \frac{\overline{g}}{|g|} \chi_E.$$

$$
\begin{aligned}
\Lambda(h) &= \int_X hg \\
&= \int_X \frac{1}{m(E)} \frac{g\overline{g}}{|g|} \chi_E \\
&= \frac{1}{m(E)} \int_E |g| \\
&\geq \frac{1}{m(E)} \|g\|_\infty m(E) \\
&= \|g\|_\infty \\
&> \|\Lambda\|_{1^\vee},
\end{aligned}
$$

a contradiction.

■

3 Fall 2019

3.1 1

Cesaro mean/summation. Break series apart into pieces that can be handled separately.

3.2 a

Prove a stronger result:

$$
a_n \longrightarrow A \implies \frac{1}{N} \sum^{N} a_k \longrightarrow A.
$$

Idea: once N is large enough, $a_k \approx A$, and all smaller terms will die off as $N \longrightarrow \infty$.
See this MSE answer.

Suppose $S_k \longrightarrow S$. Choose ℓ large enough such that

$$
k \geq \ell \implies |S_k - S| < \varepsilon.
$$

With ℓ fixed, choose N large enough such that

$$
k \leq \ell \implies \frac{|S_k - S|}{N} < \varepsilon.
$$

Then

$$\begin{aligned}\left|\left(\frac{1}{N}\sum_{k=1}^{N} S_k\right) - S\right| &= \frac{1}{N}\left|\sum_{k=1}^{N}(S_k - S)\right| \\ &\le \frac{1}{N}\sum_{k=1}^{N}|S_k - S| \\ &= \sum_{k=1}^{\ell}\frac{|S_k - S|}{N} + \sum_{k=\ell+1}^{N}\frac{|S_k - S|}{N} \\ &\longrightarrow 0.\end{aligned}$$

3.3 b

Define

$$\Gamma_n \coloneqq \sum_{k=n}^{\infty}\frac{a_k}{k}.$$

Then $\Gamma_1 = \sum_k \frac{a_k}{k}$ and each Γ_n is a tail of this series, so by assumption $\Gamma_n \longrightarrow 0$.

Then

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} a_k &= \frac{1}{n}(\Gamma_0 + \Gamma_1 + \cdots + \Gamma_n - \mathbf{\Gamma_{n+1}}) \\ &\longrightarrow 0.\end{aligned}$$

This comes from consider the following summation:

$$\begin{array}{l|lllllll}
\Gamma_1 : & a_1 & +\frac{a_2}{2} & +\frac{a_3}{3} & +\cdots & & & \\
\Gamma_2 : & & \frac{a_2}{2} & +\frac{a_3}{3} & +\cdots & & & \\
\Gamma_3 : & & & \frac{a_3}{3} & +\cdots & & & \\
\hline
\sum_{i=1}^{n}\Gamma_i : & a_1 & +a_2 & +a_3 & +\cdots & a_n & +\frac{a_{n+1}}{n+1} & +\cdots
\end{array}$$

■

3.4 2

DCT, and bounding in the right place. Don't evaluate the actual integral!

Use the fact that $\int_0^1 \cos(tx)\ dt = \sin(x)/x$, then

$$\begin{aligned}
\left|\frac{\partial^n}{\partial x} \sin(x)/x\right| &= \left|\frac{\partial^n}{\partial x} \int_0^1 \cos(tx)\ dt\right| \\
&=_? \left|\int_0^1 \frac{\partial^n}{\partial x} \cos(tx)\ dt\right| \\
&= \left|\int_0^1 -t^n \sin(tx)\ dt\right| \quad \text{for } n \text{ odd} \\
&\leq \int_0^1 |t^n \sin(tx)|\ dt \\
&\leq \int_0^1 t^n\ dt \\
&= \frac{1}{n+1} \\
&< \frac{1}{n}.
\end{aligned}$$

Where the DCT is justified by noting that $f(t) = \cos(tx)$ is dominated by $g(t) = 1$ on $[0, 1]$, which integrates to 1.

■

3.5 3

Borel-Cantelli.
Use the following observation: for a sequence of sets X_n,

$$\begin{aligned}
\limsup_n X_n &= \left\{x \;\middle|\; x \in X_n \text{ for infinitely many } n\right\} &&= \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} X_n \\
\liminf_n X_n &= \left\{x \;\middle|\; x \in X_n \text{ for all but finitely many } n\right\} &&= \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} X_n.
\end{aligned}$$

And recall

$$\prod_n e^{x_n} = e^{\sum_n x_n} \quad \text{and} \quad \sum_n \log(x_n) = \log\left(\prod_n x_n\right).$$

3.5.1 a

The Borel σ-algebra is closed under countable unions/intersections/complements, and $B = \limsup_n B_n$ is an intersection of unions of measurable sets.

3.5.2 b

We'll use the fact that tails of convergent sums go to zero, so $\sum_{n\geq M} \mu(B_n) \xrightarrow{M\longrightarrow\infty} 0$, and $B_M \coloneqq \bigcap_{m=1}^{M} \bigcup_{n\geq m} B_n \searrow B$.

$$\begin{aligned}
\mu(B_M) &= \mu\left(\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} B_n\right) \\
&\leq \mu\left(\bigcup_{n\geq m} B_n\right) \quad \text{for all } m \in \mathbb{N} \\
&\longrightarrow 0,
\end{aligned}$$

and the result follows by continuity of measure.

3.5.3 c

To show $\mu(B) = 1$, we'll show $\mu(B^c) = 0$.

Let $B_k = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{K} B_n$. Then

$$\begin{aligned}
\mu(B_K^c) &= \mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{K} B_n^c\right) \\
&\leq \sum_{m=1}^{\infty} \mu\left(\bigcap_{n=m}^{K} B_n^c\right) \quad \text{by subadditivity} \\
&= \sum_{m=1}^{\infty} \prod_{n=m}^{K} 1 - \mu(B_n) \\
&\leq \sum_{m=1}^{\infty} \prod_{n=m}^{K} e^{-\mu(B_n^c)} \quad \text{by hint} \\
&= \sum_{m=1}^{\infty} e^{-\sum_{n=m}^{K} \mu(B_n^c)} \\
&\longrightarrow 0
\end{aligned}$$

since $\sum_{n=m}^{K} \mu(B_n^c) \longrightarrow \infty$, and we can apply continuity of measure since $B_K^c \xrightarrow{K\longrightarrow\infty} B^c$.

■

3.6 4

Bessel's Inequality, surjectivity of Riesz map, and Parseval's Identity.
Trick – remember to write out finite sum S_N, and consider $\|x - S_N\|$.

3.6.1 a

Claim:

$$0 \le \left\| x - \sum_{n=1}^{N} \langle x, u_n \rangle u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n \rangle|^2$$
$$\implies \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \le \|x\|^2.$$

Proof: Let $S_N = \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n$. Then

$$\begin{aligned} 0 &\le \|x - S_N\|^2 \\ &= \langle x - S_n,\ x - S_N \rangle \\ &= \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\ &\xrightarrow{N \longrightarrow \infty} \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2. \end{aligned}$$

3.6.2 b

1. Fix $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
2. Define
$$x \coloneqq \lim_{N \longrightarrow \infty} S_N = \lim_{N \longrightarrow \infty} \sum_{k=1}^{N} a_k u_k$$
3. $\{S_N\}$ Cauchy (by 1) and H complete $\implies x \in H$.
4. $$\langle x,\ u_n \rangle = \left\langle \sum_k a_k u_k,\ u_n \right\rangle = \sum_k a_k \langle u_k,\ u_n \rangle = a_n \quad \forall n \in \mathbb{N}$$
since the u_k are all orthogonal.
5. $$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$$
by Pythagoras since the u_k are normal.

Bonus: We didn't use completeness here, so the Fourier series may not actually converge to x. If $\{u_n\}$ is **complete** (so $x = 0 \iff \langle x,\ u_n \rangle = 0\ \forall n$) then the Fourier series *does* converge to x and $\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 = \|x\|^2$ for all $x \in H$.

■

3.7 5

Continuity in L^1 (recall that DCT won't work! Notes 19.4, prove it for a dense subset first). Lebesgue differentiation in 1-dimensional case. See HW 5.6.

3.8 a

Choose $g \in C_c^0$ such that $\|f-g\|_1 \longrightarrow 0$.

By translation invariance, $\|\tau_h f - \tau_h g\|_1 \longrightarrow 0$.

Write

$$\begin{aligned}
\|\tau f - f\|_1 &= \|\tau_h f - g + g - \tau_h g + \tau_h g - f\|_1 \\
&\leq \|\tau_h f - \tau_h g\| + \|g - f\| + \|\tau_h g - g\| \\
&\longrightarrow \|\tau_h g - g\|,
\end{aligned}$$

so it suffices to show that $\|\tau_h g - g\| \longrightarrow 0$ for $g \in C_c^0$.

Fix $\varepsilon > 0$. Enlarge the support of g to K such that

$$|h| \leq 1 \text{ and } x \in K^c \implies |g(x-h) - g(x)| = 0.$$

By uniform continuity of g, pick $\delta \leq 1$ small enough such that

$$x \in K, \ |h| \leq \delta \implies |g(x-h) - g(x)| < \varepsilon,$$

then

$$\int_K |g(x-h) - g(x)| \leq \int_K \varepsilon = \varepsilon \cdot m(K) \longrightarrow 0.$$

3.9 b

We have

$$\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x)| \ dx &= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y) \ dy \right| \ dx \\
&\leq \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)| \ dy \ dx \\
&=_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)| \ \mathbf{dx\ dy} \\
&= \int_{\mathbb{R}} |f(y)| \ dy \\
&= \|f\|_1.
\end{aligned}$$

and (rough sketch)

$$
\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x) - f(x)|\ dx &= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y)\ dy \right) - f(x) \right| dx \\
&= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y)\ dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x)\ dy \right| dx \\
&\leq_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{B(h,x)} |f(y-x) - f(x)|\ \mathbf{dx\ dy} \\
&\leq \frac{1}{2h} \int_{\mathbb{R}} \|\tau_x f - f\|_1\ dy \\
&\longrightarrow 0 \quad \text{by (a).}
\end{aligned}
$$

■

4 Spring 2018

4.1 1

We'll show that $m(E)\bigcap[n, n+1] = 0$ for all $n \in \mathbb{Z}$; then the result will follow from that fact that

$$
m(E) = m\left(\bigcup_{n\in\mathbb{Z}} E \bigcap [n, n+1] \right) \leq \sum m(E \bigcap [n, n+1]) = 0
$$

By translation invariance of measure, it suffices to show $m(E\bigcap[0,1]) = 0$.

Define

$$
E_j := \left\{ x \in [0,1] \;\middle|\; \exists p \in \mathbb{Z}^{\geq 0} \text{ s.t. } \left| x - \frac{p}{j} \right| < \frac{1}{j^3} \right\}.
$$

Note that we can write E_j is a union of intervals

$$
\begin{aligned}
E_j =& (1, \frac{1}{j^3}) \\
&\coprod B_{\frac{1}{j^3}}\left(\frac{1}{j}\right) \coprod B_{\frac{1}{j^3}}\left(\frac{2}{j}\right) \coprod \cdots \coprod B_{\frac{1}{j^3}}\left(\frac{j-1}{j}\right) \\
&\coprod (1 - \frac{1}{j^3}, 1),
\end{aligned}
$$

from which we can conclude that E_j is Borel and thus Lebesgue measurable, and that for each j, there are exactly $j+1$ possible choices for a numerator (corresponding to the $j+1$ sets appearing above.)

The first and last intervals are length $\frac{1}{j^3}$ and the remaining $(j+1) - 2 = j - 1$ intervals are length $\frac{2}{j^3}$, so we find that

$$
m(E_j) = 2\left(\frac{1}{j^3}\right) + (j-1)\left(\frac{2}{j^3}\right) = \frac{2}{j^2}
$$

We can then note that

$$\sum_{j\in\mathbb{N}} m(E_j) \leq 2\sum_{j\in\mathbb{N}} \frac{1}{j^2} < \infty,$$

which converges by the p-test for sums.

Since $\{E_j\}$ is a countable collection of measurable sets such that $\sum m(E_j) < \infty$, Borel-Cantelli applies and $m(\limsup_j E_j) = 0$, where we can just note that $\limsup_j E_j = E\bigcap[0,1]$.

■

4.2 2

4.2.1 a

Since $x < 1 \implies x^n \longrightarrow 0$ and $x > 1 \implies x^n \longrightarrow \infty$, we have

$$f_n(x) = \frac{x}{1+x^n} \overset{n\longrightarrow\infty}{\Longrightarrow} f(x) = \begin{cases} 0, & x = 0 \\ x, & x < 1 \\ \frac{1}{2}, & x = 1 \\ 0, & x > 0 \end{cases}.$$

If $f_n \longrightarrow f$ uniformly on $[0,\infty)$, it would converge uniformly on every subset.

Butach $f_n(x)$ is clearly continuous on $(0,\infty)$, and if the convergence was uniform then f would be continuous. However f has a clear discontinuity at $x = 1$.

4.2.2 b

If the DCT applies, we can interchange the limit and integral, and the value would be the area under the graph of f which is $\int_0^1 x\ dx = \frac{1}{2}$.

To justify the DCT, write

$$\int_0^\infty f_n(x) = \int_0^1 f_n(x) + \int_1^\infty f_n(x).$$

Then

$$x \in [0,1] \implies \frac{x}{1+x^n} < \frac{1}{1+x^n} < 1$$

and $\int_0^1 1\ dx = 1 < \infty$.

On the other hand,

$$x \in (1,\infty) \implies \frac{x}{1+x^n} \approx O\left(\frac{1}{x^{n-1}}\right),$$

and so for $n > 2$ the integral will converge by the p-test.

4.3 3

Since $|f(x)| \leq \|f\|_\infty$ almost everywhere, we have

$$\|f\|_p^p = \int_X |f(x)|^p \, dx \leq \int_X \|f\|_\infty^p \, dx = \|f\|_\infty^p \cdot m(X) = \|f\|_\infty^p,$$

so $\|f\|_p \leq \|f\|_\infty$ for all p and taking $\lim\limits_{p \longrightarrow \infty}$ preserves this inequality.

Conversely, let $\varepsilon > 0$. Define

$$S_\varepsilon := \left\{ x \in \mathbb{R} \;\middle|\; |f(x)| \geq \|f\|_\infty - \varepsilon \right\}.$$

Then

$$\begin{aligned}
\|f\|_p^p &= \int_X |f(x)|^p \, dx \\
&\geq \int_{S_\varepsilon} |f(x)|^p \, dx \\
&\geq \int_{S_\varepsilon} |\|f\|_\infty - \varepsilon|^p \, dx \\
&= |\|f\|_\infty - \varepsilon|^p \cdot m(S_\varepsilon) \\
\implies \|f\|_p &\geq |\|f\|_\infty - \varepsilon| \cdot m(S_\varepsilon)^{\frac{1}{p}} \\
&\overset{p \longrightarrow \infty}{\longrightarrow} |\|f\|_\infty - \varepsilon| \\
&\overset{\varepsilon \longrightarrow 0}{\longrightarrow} \|f\|_\infty.
\end{aligned}$$

So $\|f\|_p \geq \|f\|_\infty$.

$\blacksquare$

4.4 4

Fix $k \in \mathbb{Z}$. Since $e^{2\pi i k x}$ is continuous on the compact interval $[0, 1]$, it is uniformly continuous, and is thus there is a sequence of polynomials P_ℓ such that

$$P_{\ell,k} \overset{\ell \longrightarrow \infty}{\longrightarrow} e^{2\pi i k x} \text{ uniformly on } [0, 1].$$

Note that by linearity,

$$\int f(x) x^n = 0 \; \forall n \implies \int f(x) P_{\ell,k}(x) = 0 \quad \forall \ell \in \mathbb{N}$$

But then the kth Fourier coefficient of f is given by

$$\begin{aligned}
\langle f,\ e_k \rangle &= \int_0^1 f(x) e^{-2\pi i k x}\ dx \\
&= \int_0^1 f(x) \lim_{\ell \longrightarrow \infty} P_\ell(x) \\
&= \lim_{\ell \longrightarrow \infty} \int_0^1 f(x) P_\ell(x) \qquad \text{by uniform convergence} \\
&= \lim_{\ell \longrightarrow \infty} 0 \\
&= 0 \quad \forall k \in \mathbb{Z},
\end{aligned}$$

so $\widehat{f}$ is the zero function, and $\widehat{f} = 0 \iff f = 0$ almost everywhere.

■

4.5 5

> Moral: $\int |f_n - f| \longrightarrow \iff \int f_n = \int f$.

Since if $\int |f_n| \longrightarrow \int |f|$ then we can define

$$\begin{aligned}
h_n &= |f_n - f| && \longrightarrow 0 \ a.e. \\
g_n &= |f_n| + |f| && \longrightarrow 2|f| \ a.e.
\end{aligned}$$

$$\begin{aligned}
\int 2|f| &= \int \liminf\ (g_n - h_n) \\
&= \int \liminf\ g_n - \int \liminf\ h_n \\
&= \int 2|f| - \int \liminf\ h_n \\
&\overset{Fatou}{\leq} \int 2|f| + \limsup \int h_n,
\end{aligned}$$

which forces $\int h_n = \int |f_n - f| \longrightarrow 0$.

But then

$$\left| \int f_n - \int f \right| = \left| \int f_n - f \right| \leq \int |f_n - f| \longrightarrow 0,$$

so $\int f_n \longrightarrow \int f$.

■

5 Fall 2018

Note: this is considered. . . not the most useful or representative exam of all time.

5.1 1

We'll show a stronger statement: $f(x) = \frac{1}{x}$ is uniformly continuous on any interval of the form (c, ∞) where $c > 0$.

We can use that fact that $x, y > c \implies xy > c^2 \implies \frac{1}{xy} < \frac{1}{c^2}$.

Letting ε be arbitrary, choose $\delta < \varepsilon c^2$. Then

$$\begin{aligned}
|f(x) - f(y)| &= \left| \frac{1}{x} - \frac{1}{y} \right| \\
&= \frac{|x-y|}{xy} \\
&\leq \frac{\delta}{xy} \\
&< \frac{\delta}{c^2} \\
&< \varepsilon,
\end{aligned}$$

which shows uniform continuity since δ does not depend on x or y.

To see that f is not uniformly continuous when $c = 0$, let $\varepsilon < 1$ be arbitrary.

Let $x_n = \frac{1}{n}$. Then choose n large enough such that $|x_n - x_{n+1}| = \frac{1}{n} - \frac{1}{n+1} < \delta$. Then just note that $f(x_n) = n$ and thus $|f(x_n) - f(x_{n+1})| = n - (n+1) = 1 > \varepsilon$.

5.2 2

First consider the bounded case where $m(E) < \infty$.

E is measurable $\iff$ for every ε there exist $F_\varepsilon \subset E \subset G_\varepsilon$ with F_ε closed and G_ε open and $m(G_\varepsilon \setminus E) < \varepsilon$ and $m(E \setminus F_\varepsilon) < \varepsilon$.

So take the sequence $\varepsilon_n = \frac{1}{n} \longrightarrow 0$ to produce a sequence of closed sets F_n such that $m(E \setminus F_n) < \frac{1}{n}$ for all n, and let $F \coloneqq \bigcup_n$, which is clear an F_σ and thus Borel set.

Since $F_n \subseteq F_{n+1}$, we have $F_n \nearrow F$ and so by continuity of measure,

$$m(F) = \lim_n m(F_n) < \lim_n \left(\frac{1}{n} \right) \longrightarrow 0.$$

If E is not bounded, let $E_N = B_N(0) \bigcap E$ which is bounded. Then $E_N \nearrow E$, and for each N we can find an F_N by the previous case such that $m(E_N \setminus F_N) = 0$.

So take $F \coloneqq \bigcup_N F_N$ so $F_N \nearrow F$. Then

$$E_N \setminus F_N \nearrow E \setminus F \implies m(E \setminus F) = \lim_N m(E_N \setminus F_N) = 0.$$

5.3 3

$$\begin{aligned}
\frac{\partial}{\partial t} F(t) &= \frac{\partial}{\partial t} \int_{\mathbb{R}} f(x)\cos(xt)\ dx \\
&\overset{DCT}{=} \int_{\mathbb{R}} f(x) \frac{\partial}{\partial t} \cos(xt)\ dx \\
&= \int_{\mathbb{R}} x f(x) \cos(xt)\ dx,
\end{aligned}$$

so it only remains to justify the DCT.

Fix t, then let $t_n \longrightarrow t$ be any sequence. Then

$$\begin{aligned}
\frac{\partial}{\partial t} \cos(tx) &\coloneqq \lim_{t_n \longrightarrow t} \frac{\cos(tx) - \cos(t_n x)}{t_n - t} \\
&\overset{MVT}{=} \frac{\partial}{\partial t} \cos(tx) \Big|_{t=\xi_n} \qquad \text{for some } \xi_n \in [t, t_n] \text{ or } [t_n, t] \\
&= x \sin(\xi_n x).
\end{aligned}$$

So we can define

$$h_n(x,t) = f(x) \left(\frac{\cos(tx) - \cos(t_n x)}{t_n - t} \right)$$

and note that $h_n \longrightarrow \frac{\partial}{\partial t} [f(x)\cos(xt)]$ pointwise.

We then have $|h_n| = |f(x) x \sin(\xi_n x)| \leq |x f(x)|$ for every n by the above argument, and since $g(x) \coloneqq x f(x) \in L^1(\mathbb{R})$ by assumption, the DCT can be applied.

5.4 4

???

Apparently "easy" part: let $f(x) = \chi_{[0,\pi]}$, then $\int_{\mathbb{R}} f(x)|\sin(nx)| = \int_0^\pi |\sin(nx)| = 2$, and so $\int_0^1 |\sin(nx)| = \frac{2}{\pi}$, none of which depend on n.

Now approximate f by step functions.

5.5 5

???

6 Spring 2017

6.1 1

A is nowhere dense $\iff$ every interval I contains a subinterval $S \subseteq A^c$.

K **is compact:**

It suffices to show that $K^c := [0,1] \setminus K$ is open; then K will be a closed and bounded subset of $\mathbb{R}$ and thus compact by Heine-Borel.

We can identify K^c as the set of real numbers in $[0,1]$ whose decimal expansion **does** use a 4. Let $x \in K^c$, and suppose a 4 occurs as the kth digit and write

$$x = 0.d_1 d_2 \cdots d_{k-1}\ 4\ d_{k+1} \cdots = \sum_{j=1}^{k} d_j 10^{-j} + 4 \cdot 10^{-k} + \sum_{j=k+1}^{\infty} d_j 10^{-j}.$$

Then if we set $r < 10^{-k}$ and pick any $y \in [0,1]$ such that $y \in B_r(x)$, then $|x - y| < 10^{-k}$. If we write $y = \displaystyle\sum_{j=1}^{\infty} c_j 10^{-j}$, this means that for all $j \leq k$ we have $d_j = c_j$, and in particular $d_k = 4 = c_k$, so y has a 4 in its decimal expansion.

But then $K^c = \displaystyle\bigcup_x B_r(x)$ is a union of open sets and thus open.

K **is nowhere dense and** $m(K) = 0$**:**

Since K is closed, we'll show that K can not properly contain any interval, so $(\overline{K})^\circ = \emptyset$.

As in the construction of the Cantor set, let

- K_1 denote $[0,1]$ with 1 interval $[0.4, 0.5]$ of length $\dfrac{1}{10}$ deleted
- K_2 denote K_1 with 9 intervals $[0.04, 0.05],\ [0.14, 0.15], \cdots [0.94, 0.95]$ length $\dfrac{1}{100}$ deleted
- K_n denote K_{n-1} with 9^{n-1} such intervals of length 10^{-n} deleted.

Then $K = \bigcap K_n$, and

$$m(K) = 1 - m(K^c) = 1 - \sum_{j=0}^{\infty} \frac{9^n}{10^{n+1}} = 1 - \frac{1}{10}\left(\frac{1}{1 - \frac{9}{10}}\right) = 0,$$

and since any interval has strictly positive measure, K can not contain any interval.

K **has no isolated points**:

A point $x \in K$ is isolated iff there there is an open ball $B_r(x)$ containing x such that $B_r(x) \bigcap K = \emptyset$, so every point in this ball has a 4 in its decimal expansion.

Note that $m(K_n) = \left(\dfrac{9}{10}\right)^n \longrightarrow 0$ and that the endpoints of intervals are never removed and are thus elements of K. Then for every ε, we can choose n such that $\left(\dfrac{9}{10}\right)^n < \varepsilon$; then there is an endpoint of a removed interval e_n satisfying $|x - e_n| \leq \left(\dfrac{9}{10}\right)^n < \varepsilon$.

So every ball containing x contains some endpoint of a removed interval, and thus an element of K.

■

6.2 2

$\lambda \ll \mu \iff E \in \mathcal{M}, \mu(E) = 0 \implies \lambda(E) = 0.$

6.2.1 a

By Radon-Nikodym, if $\lambda \ll \mu$ then $d\lambda = f d\mu$, which would yield

$$\int g \, d\lambda = \int gf \, d\mu.$$

So let E be measurable and suppose $\mu(E) = 0$. Then

$$\lambda(E) := \int_E f \, d\mu = \lim_n \left\{ \varphi_n := \sum_j c_j \mu(E_j) \right\},$$

where we take a sequence of simple functions increasing to f.

But since each $E_j \subseteq E$, we must have $\mu(E_j) = 0$ for any such E_j, so every such φ_n must be zero and thus $\lambda(E) = 0$.

6.2.2 b

By Radon-Nikodym, there exists a positive f such that

$$\int g \, dm = \int gf \, d\mu,$$

where we can take $g(x) = x^2$, then the LHS is zero by assumption and thus so is the RHS.

Note that gf is positive.

Define $A_k = \left\{ x \in X \;\middle|\; gf\chi_E > \frac{1}{k} \right\}$, then by Chebyshev

$$\mu(A_k) \leq k \int_E gf \, d\mu = 0,$$

which holds for every k.

Then noting that $A_k \searrow A := \left\{ x \in E \;\middle|\; x^2 > 0 \right\}$, and gf is positive, we have

$$x \in E \iff gf\chi_E(x) > 0 \iff x \in A,$$

so $E = A$ and $\mu(E) = \mu(A)$.

But since $m \ll \mu$ by construction, we can conclude that $m(E) = 0$.

■

6.3 3

6.3.1 a

Letting $x_n := \frac{1}{n}$, we have

$$\sum_{k=1}^{\infty} |f_k(x)| \geq |f_n(x_n)| = \left| ae^{-ax} - be^{-bx} \right| := M.$$

In particular, $\sup_x |f_n(x)| \not\longrightarrow 0$, so the terms do not go to zero and the sum can not converge.

6.3.2 b

?

6.4 4

Switching to polar coordinates and integrating over a half-circle contained in I^2, we have

$$\int_{I^2} f \geq \int_0^{\pi} \int_0^1 \frac{\cos(\theta)\sin(\theta)}{r^2} \, dr \, d\theta = \infty,$$

so f is not integrable.

6.5 5

> See https://math.stackexchange.com/questions/507263/prove-that-c1a-b-with-the-c1-norm-is-a-banach-space

This is clearly a norm, which we'll write $\|\cdot\|_u$

Let f_n be a Cauchy sequence and define a candidate limit $f(x) = \lim_n f_n(x)$.

Then noting that $\|f_n\|_\infty, \|f_n'\|\infty \leq \|f_n\|_u < \infty$, both f_n, f_n are Cauchy sequences in $C^0([a,b], \|\cdot\|_\infty)$, which is a Banach space.

So $f_n \longrightarrow f$ uniformly, and $f_n' \longrightarrow g$ uniformly for some g, and moreover $f, g \in C^0([a,b])$.

We thus have

$$\begin{aligned} f_n(x) - f_n(a) &\xrightarrow{u} f(x) - f(a) \\ \int_a^x f_n' &\xrightarrow{u} \int_a^x g, \end{aligned}$$

and by the FTC, the left-hand sides are equal, and by uniqueness of limits so are the right-hand sides, so $f' = g$.

Since $f, f' \in C^0([a,b])$, they are bounded, and so $\|f\|_u < \infty$. This means that $\|f_n - f\|_u \longrightarrow 0$, so f_n converges to f, which is in the same space.

■

7 Fall 2017

7.1 1

Note that $f(x) = e^x$ is entire and thus equal to its power series. So $f(x) = \sum_{j=0}^{\infty} \frac{1}{j!} x^j$.

Letting $f_N(x) = \sum_{j=1}^{N} \frac{1}{j!} x^j$, we have $f_N(x) \longrightarrow f(x)$ pointwise on $(-\infty, \infty)$.

For any compact interval $[-M, M]$, we have

$$\begin{aligned}
\|f_N(x) - f(x)\|_\infty &= \sup_{-M \le x \le M} \left| \sum_{j=N+1}^{\infty} \frac{1}{j!} x^j \right| \\
&\le \sup_{-M \le x \le M} \sum_{j=N+1}^{\infty} \frac{1}{j!} |x|^j \\
&\le \sum_{j=N+1}^{\infty} \frac{1}{j!} M^j \\
&\le \sum_{j=0}^{\infty} \frac{1}{j!} M^j \\
&= e^M \\
&< \infty,
\end{aligned}$$

so $f_N \longrightarrow f$ uniformly on $[-M, M]$ by the M-test. Thus it converges on any bounded interval.

It does not converge on $\mathbb{R}$, since x^N is unbounded.

7.2 2

7.2.1 a

It suffices to consider the bounded case, i.e. $E \subseteq B_M(0)$ for some M. Then write $E_n = B_n(0) \bigcap E$ and apply the theorem to E_n, and by subadditivity, $m^*(E) = m^*(\bigcup_n E_n) \le \sum_n m^*(E_n) = 0$.

Lemma: $f(x) = x^2, f^{-1}(x) = \sqrt{x}$ are Lipschitz on any compact subset of $[0, \infty)$.

Proof: Let $g = f$ or f^{-1}. Then $g \in C^1([0, M])$ for any M, so g is differentiable and g' is continuous. Since g' is continuous on a compact interval, it is bounded, so $|g'(x)| \le L$ for all x. Applying the MVT,

$$|f(x) - f(y)| = f'(c)|x - y| \le L|x - y|.$$

Lemma: If g is Lipschitz on $\mathbb{R}^n$, then $m(E) = 0 \implies m(g(E)) = 0$.

Proof: If g is Lipschitz, then

$$g(B_r(x)) \subseteq B_{Lr}(x),$$

which is a dilated ball/cube, and so

$$m^*(B_{Lr}(x)) \leq L^n \cdot m^*(B_r(x)).$$

Now choose $\{Q_j\} \rightrightarrows E$; then $\{g(Q_j)\} \rightrightarrows g(E)$.

By the above observation,

$$|g(Q_j)| \leq L^n |Q_j|,$$

and so

$$m^*(g(E)) \leq \sum_j |g(Q_j)| \leq \sum_j L^n |Q_j| = L^n \sum_j |Q_j| \longrightarrow 0.$$

Now just take $g(x) = x^2$ for one direction, and $g(x) = f^{-1}(x) = \sqrt{x}$ for the other. ■

7.2.2 b

> Lemma: E is measurable iff $E = K \coprod N$ for some K compact, N null.

Write $E = K \coprod N$ where K is compact and N is null.

Then $\varphi^{-1}(E) = \varphi^{-1}(K \coprod N) = \varphi^{-1}(K) \coprod \varphi^{-1}(N)$.

Since $\varphi^{-1}(N)$ is null by part (a) and $\varphi^{-1}(K)$ is the preimage of a compact set under a continuous map and thus compact, $\varphi^{-1}(E) = K' \coprod N'$ where K' is compact and N' is null, so $\varphi^{-1}(E)$ is measurable.

So φ is a measurable function, and thus yields a well-defined map $\mathcal{L}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathbb{R})$ since it preserves measurable sets. Restricting to $[0, \infty)$, f is bijection, and thus so is φ.

■

7.3 3

> From homework: E is Lebesgue measurable iff there exists a finite union of closed cubes A such that $m(E \Delta A) < \varepsilon$.

It suffices to show that S is dense in simple functions, and since simple functions are *finite* linear combinations of characteristic functions, it suffices to show this for χ_A for A a measurable set.

Let $s = \chi_A$. By regularity of the Lebesgue measure, choose an open set $O \supseteq A$ such that $m(O \setminus A) < \varepsilon$.

O is an open subset of $\mathbb{R}$, and thus $O = \coprod_{j \in \mathbb{N}} I_j$ is a disjoint union of countably many open intervals.

Now choose N large enough such that $m(O \Delta I_{N,n}) < \varepsilon = \dfrac{1}{n}$ where we define $I_{N,n} \coloneqq \coprod_{j=1}^N I_j$.

Now define $f_n = \chi_{I_{N,n}}$, then

$$\|s - f_n\|_1 = \int \left| \chi_A - \chi_{I_{N,n}} \right| = m(A \Delta I_{N,n}) \overset{n \to \infty}{\longrightarrow} 0.$$

Since any simple function is a finite linear combination of χ_{A_i}, we can do this for each i to extend this result to all simple functions. But simple functions are dense in L^1, so S is dense in L^1.

7.4 4

7.4.1 a

Let $G(x) = \sum_{n=1}^{\infty} nx(1-x)^n$. Applying the ratio test, we have

$$\left| \frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n} \right| = \frac{n+1}{n}|1-x| \overset{n\longrightarrow\infty}{\longrightarrow} |1-x| < 1 \iff 0 \le x \le 2,$$

and in particular, this series converges on $[0,2]$. Thus its terms go to zero, and $nx(1-x)^n \longrightarrow 0$ on $[0,1] \subset [0,2]$.

To see that the convergence is not uniform, let $x_n = \frac{1}{n}$ and $\varepsilon > \frac{1}{e}$, then

$$\sup_{x\in[0,1]} |nx(1-x)^n - 0| \ge |nx_n(1-x_n)^n| = \left| \left(1 - \frac{1}{n}\right)^n \right| \overset{n\longrightarrow\infty}{\longrightarrow} e^{-1} > \varepsilon.$$

7.4.2 b

Note: could use the first part with $\sin(x) \le x$, but then integral ends up more complicated.

Noting that $\sin(x) \le 1$, we have We have

$$\begin{aligned} \left| \int_0^1 n(1-x)^n \sin(x) \right| &\le \int_0^1 |n(1-x)^n \sin(x)| \\ &\le \int_0^1 |n(1-x)^n| \\ &= n\int_0^1 (1-x)^n \\ &= -\frac{n(1-x)^{n+1}}{n+1} \\ &\overset{n\longrightarrow\infty}{\longrightarrow} 0. \end{aligned}$$

7.5 5

7.5.1 a

Lemma: If $\varphi \in C_c^1$, then $(f * \varphi)' = f * \varphi'$ almost everywhere.

Silly Proof:

$$\begin{aligned}
\mathcal{F}((f * \varphi)') &= 2\pi i \xi \ \mathcal{F}(f * \varphi) \\
&= 2\pi i \xi \ \mathcal{F}(f) \ \mathcal{F}(\varphi) \\
&= \mathcal{F}(f) \cdot (2\pi i \xi \ \mathcal{F}(\varphi)) \\
&= \mathcal{F}(f) \cdot \mathcal{F}(\varphi') \\
&= \mathcal{F}(f * \varphi').
\end{aligned}$$

Actual proof:

$$\begin{aligned}
(f * \varphi)'(x) &= (\varphi * f)'(x) \\
&= \lim_{h \longrightarrow 0} \frac{(\varphi * f)'(x+h) - (\varphi * f)'(x)}{h} \\
&= \lim_{h \longrightarrow 0} \int \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \\
&\overset{DCT}{=} \int \lim_{h \longrightarrow 0} \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \\
&= \int \varphi'(x-y) f(y) \\
&= (\varphi' * f)(x) \\
&= (f * \varphi')(x).
\end{aligned}$$

To see that the DCT is justified, we can apply the MVT on the interval $[0, h]$ to f to obtain

$$\frac{\varphi(x+h-y) - \varphi(x-y)}{h} = \varphi'(c) \quad c \in [0, h],$$

and since φ' is continuous and compactly supported, φ' is bounded by some $M < \infty$ by the extreme value theorem and thus

$$\begin{aligned}
\int \left| \frac{\varphi(x+h-y) - \varphi(x-y)}{h} f(y) \right| &= \int |\varphi'(c) f(y)| \\
&\leq \int |M||f| \\
&= |M| \int |f| < \infty,
\end{aligned}$$

since $f \in L^1$ by assumption, so we can take $g := |M||f|$ as the dominating function.

Applying this theorem infinitely many times shows that $f * \varphi$ is smooth.

To see that $f * \varphi$ is compactly supported, approximate f by a *continuous* compactly supported function h, so $\|h - f\|_1 \xrightarrow{L^1} 0$.

Now let $g_x(y) = \varphi(x - y)$, and note that $\mathrm{supp}(g) = x - \mathrm{supp}(\varphi)$ which is still compact.

But since supp(h) is bounded, there is some N such that

$$|x| > N \implies A_x \coloneqq \operatorname{supp}(h) \bigcap \operatorname{supp}(g_x) = \emptyset$$

and thus

$$\begin{aligned}(h * \varphi)(x) &= \int_{\mathbb{R}} \varphi(x-y)h(y)\ dy \\ &= \int_{A_x} g_x(y)h(y) \\ &= 0,\end{aligned}$$

so $\left\{x \;\middle|\; f * g(x) = 0\right\}$ is open, and its complement is closed and bounded and thus compact.

7.5.2 b

$$\begin{aligned}\|f * K_j - f\|_1 &= \int \left| \int f(x-y)K_j(y)\ dy - f(x) \right| dx \\ &= \int \left| \int f(x-y)K_j(y)\ dy - \int f(x)K_j(y)\ dy \right| dx \\ &= \int \left| \int (f(x-y) - f(x))K_j(y)\ dy \right| dx \\ &\leq \int \int |(f(x-y) - f(x))| \cdot |K_j(y)|\ dy\ dx \\ &\overset{FT}{=} \int \int |(f(x-y) - f(x))| \cdot |K_j(y)|\ \mathbf{dx\ dy} \\ &= \int |K_j(y)| \left(\int |(f(x-y) - f(x))|\ dx \right)\ dy \\ &= \int |K_j(y)| \cdot \|f - \tau_y f\|_1\ dy.\end{aligned}$$

We now split the integral up into pieces.

1. Chose δ small enough such that $|y| < \delta \implies \|f - \tau_y f\|_1 < \varepsilon$ by continuity of translation in L^1, and
2. Since φ is compactly supported, choose J large enough such that

$$j > J \implies \int_{|y| \geq \delta} |K_j(y)|\ dy = \int_{|y| \geq \delta} |j\varphi(jy)| = 0$$

Then

$$\begin{aligned}\|f * K_j - f\|_1 &\leq \int |K_j(y)| \cdot \|f - \tau_y f\|_1\ dy \\ &= \int_{|y|<\delta} |K_j(y)| \cdot \|f - \tau_y f\|_1\ dy + \int_{|y|\geq\delta} |K_j(y)| \cdot \|f - \tau_y f\|_1\ dy \\ &= \varepsilon \int_{|y|\geq\delta} |K_j(y)| + 0 \\ &\leq \varepsilon(1) \longrightarrow 0.\end{aligned}$$

■

7.6 6

Should be supremum maybe..?

Let $\{f_k\}$ be a Cauchy sequence, so $\|f_k\| < \infty$ for all k. Then for a fixed x, the sequence $f_k(x)$ is Cauchy in $\mathbb{R}$ and thus converges to some $f(x)$, so define f by $f(x) \coloneqq \lim_{k \longrightarrow \infty} f_k(x)$.

Then $\|f_k - f\| = \max_{x \in X} |f_k(x) - f(x)| \overset{k \longrightarrow \infty}{\longrightarrow} 0$, and thus $f_k \longrightarrow f$ uniformly and thus f is continuous. It just remains to show that f has bounded norm.

Choose N large enough so that $\|f - f_N\| < \varepsilon$, and write $\|f_N\| \coloneqq M < \infty$

$$\|f\| \leq \|f - f_N\| + \|f_N\| < \varepsilon + M < \infty.$$

8 Spring 2016

8.1 1

9 Fall 2016

9.1 1

10 Spring 2014

10.1 1