Real Analysis Qualifying Exam Notes

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Contents

1 Practice Exam 2 (November 2014)

1.1 1: Fubini-Tonelli

1.1.1 a

Carefully state Tonelli's theorem for a nonnegative function $F(x,t)$ on $\mathbb{R}^n \times \mathbb{R}$.

1.1.2 b

Let $f : \mathbb{R}^n \longrightarrow [0, \infty]$ and define

$$\mathcal{A} \coloneqq \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; 0 \leq t \leq f(x) \right\}.$$

Prove the validity of the following two statements:

1. f is Lebesgue measurable on $\mathbb{R}^n \iff \mathcal{A}$ is a Lebesgue measurable subset of $\mathbb{R}^{n+1}$.
2. If f is Lebesgue measurable on $\mathbb{R}^n$ then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)dx = \int_0^\infty m\left(\left\{x \in \mathbb{R}^n \;\middle|\; f(x) \geq t\right\}\right) dt.$$

1.2 2: Convolutions and the Fourier Transform

1.2.1 a

Let $f, g \in L^1(\mathbb{R}^n)$ and give a definition of $f * g$.

1.2.2 b

Prove that if f, g are integrable and bounded, then

$$(f * g)(x) \overset{|x| \longrightarrow \infty}{\longrightarrow} 0.$$

1.2.3 c

1. Define the *Fourier transform* of an integrable function f on $\mathbb{R}^n$.
2. Give an outline of the proof of the Fourier inversion formula.
3. Give an example of a function $f \in L^1(\mathbb{R}^n)$ such that $\widehat{f}$ is not in $L^1(\mathbb{R}^n)$.

1.3 3: Hilbert Spaces

Let $\{u_n\}_{n=1}^\infty$ be an orthonormal sequence in a Hilbert space H.

1.3.1 a

Let $x \in H$ and verify that

$$\left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|_H^2 = \|x\|_H^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2.$$

for any $N \in \mathbb{N}$ and deduce that

$$\sum_{n=1}^\infty |\langle x, u_n \rangle|^2 \leq \|x\|_H^2.$$

1.3.2 b

Let $\{a_n\}_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ and prove that there exists an $x \in H$ such that $a_n = \langle x,\ u_n \rangle$ for all $n \in \mathbb{N}$, and moreover x may be chosen such that

$$\|x\|_H = \left(\sum_{n\in\mathbb{N}} |a_n|^2 \right)^{\frac{1}{2}}.$$

Proof .

- Take $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
- Define $x \coloneqq \lim_{N \longrightarrow \infty} S_N$ where $S_N = \sum_{k=1}^{N} a_k u_k$
- $\{S_N\}$ is Cauchy and H is complete, so $x \in H$.
- By construction,

$$\langle x,\ u_n \rangle = \left\langle \sum_k a_k u_k,\ u_n \right\rangle = \sum_k a_k \langle u_k,\ u_n \rangle = a_n$$

  since the u_k are all orthogonal.
- By Pythagoras since the u_k are normal,

$$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2.$$

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1.3.3 c

Prove that if $\{u_n\}$ is *complete*, Bessel's inequality becomes an equality.

Proof .
Let x and u_n be arbitrary.

$$\begin{aligned}
\left\langle x - \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k,\ u_n \right\rangle &= \langle x,\ u_n \rangle - \left\langle \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k,\ u_n \right\rangle \\
&= \langle x,\ u_n \rangle - \sum_{k=1}^{\infty} \langle \langle x,\ u_k \rangle u_k,\ u_n \rangle \\
&= \langle x,\ u_n \rangle - \sum_{k=1}^{\infty} \langle x,\ u_k \rangle \langle u_k,\ u_n \rangle \\
&= \langle x,\ u_n \rangle - \langle x,\ u_n \rangle = 0 \\
\implies x - \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k = 0 \quad &\text{by completeness.}
\end{aligned}$$

So

$$x = \sum_{k=1}^{\infty} \langle x,\ u_k \rangle u_k \implies \|x\|^2 = \sum_{k=1}^{\infty} |\langle x,\ u_k \rangle|^2 . \blacksquare.$$

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1.4 4: Lp Spaces

1.4.1 a

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1.4.2 c

Definition (Infinity Norm):

$$L^\infty(X) = \left\{ f : X \longrightarrow \mathbb{C} \;\middle|\; \|f\|_\infty < \infty \right\}$$
$$\text{where}$$
$$\|f\|_\infty = \inf_{\alpha \geq 0} \left\{ \alpha \;\middle|\; m\left\{|f| \geq \alpha\right\} = 0 \right\}.$$

Theorem:

$$m(X) < \infty \implies \lim_{p \longrightarrow \infty} \|f\|_p = \|f\|_\infty.$$

Proof: Let $M = \|f\|_\infty$. For any $L < M$, let $S = \{|f| \geq L\}$. Then $m(S) > 0$ and

$$\begin{aligned}
\|f\|_p &= \left(\int_X |f|^p \right)^{\frac{1}{p}} \\
&\geq \left(\int_S |f|^p \right)^{\frac{1}{p}} \\
&\geq L \; m(S)^{\frac{1}{p}} \xrightarrow{p \longrightarrow \infty} L \\
&\implies \liminf_p \|f\|_p \geq M.
\end{aligned}$$

We also have

$$\begin{aligned}
\|f\|_p &= \left(\int_X |f|^p \right)^{\frac{1}{p}} \\
&\leq \left(\int_X M^p \right)^{\frac{1}{p}} \\
&= M \; m(X)^{\frac{1}{p}} \xrightarrow{p \longrightarrow \infty} M \\
&\implies \limsup_p \|f\|_p \leq M \blacksquare.
\end{aligned}$$

Note: this doesn't help with this problem at all.

Solution:

$$\begin{aligned}\int f^p &= \int_{x\leq 1} f^p + \int_{x=1} f^p + \int_{x\geq 1} f^p \\ &= \int_{x\leq 1} f^p + \int_{x=1} 1 + \int_{x\geq 1} f^p \\ &= \int_{x\leq 1} f^p + m(\{f=1\}) + \int_{x\geq 1} f^p \\ &\xrightarrow{p\longrightarrow\infty} 0 + m(\{f=1\}) + \begin{cases} 0 & m(\{x\geq 1\}) = 0 \\ \infty & m(\{x\geq 1\}) > 0. \end{cases}\end{aligned}$$

1.5 5: Dual Spaces

Let X be a normed vector space.

1.5.1 a

Give the definition of what it means for a map $L : X \longrightarrow \mathbb{C}$ to be a *linear functional*.

1.5.2 b

Prove Minkowski's Inequality:

$$1 \leq p < \infty \implies \|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Conclude that if $f, g \in L^p(\mathbb{R}^n)$ then so is $f+g$.