Complex Analysis Qualifying Exam Review

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A great deal of content borrowed from the following: https://web.stanford.edu/~chriseur/notes_pdf/Eur_ComplexAnalysis_Notes.pdf

1 Useful Techniques

1.1 Notation

Notation	Definition
$\mathbb{D} := \left\{ z \;\middle\vert\; \lvert z \rvert \leq 1 \right\}$	The unit disc
$\mathbb{H} := \left\{ x + iy \;\middle\vert\; y > 0 \right\}$	The upper half-plane
$X_{\frac{1}{2}}$	A "half version of X", see examples
$\mathbb{H}_{\frac{1}{2}}$	The first quadrant
$\mathbb{D}_{\frac{1}{2}}$	The portion of the first quadrant inside the unit disc
$S := \left\{ x + iy \;\middle\vert\; x \in \mathbb{R},\, 0 < y < \pi \right\}$	The horizontal strip

1.2 Greatest Hits

Things to know well:

- Estimates for derivatives, mean value theorem
- **??**CauchyTheorem]Cauchy's Theorem
- **??**CauchyIntegral]Cauchy's Integral Formula
- **??**CauchyInequality]Cauchy's Inequality
- **??**Morera]Morera's Theorem
- **??**Liouville]Liouville's Theorem
- **??**MaximumModulus]Maximum Modulus Principle
- **??**Rouche]Rouché's Theorem

- ??SchwarzReflection]The Schwarz Reflection Principle
- ??SchwarzLemma]The Schwarz Lemma
- ??Casorati]Casorati-Weierstrass Theorem
- Properties of linear fractional transformations
- Automorphisms of $\mathbb{D}, \mathbb{C}, \mathbb{CP}^1$.

1.3 Common Tricks

- Virtually any time: consider $1/f(z)$ and $f(1/z)$.

Remark 1.3.1 ***(Showing a function is constant):*** If you want to show that a function f is constant, try one of the following:

- Write $f = u + iv$ and use Cauchy-Riemann to show $u_x, u_y = 0$, etc.
- Show that f is entire and bounded.
  - If you additionally want to show f is zero, show $\lim_{z \to \infty} f(z) = 0$.

1.4 Basic but Useful Facts

Fact 1.4.1 (Some useful facts about basic complex algebra)

- $z\bar{z} = |z|^2$

$$\Re(z) = \frac{z + \bar{z}}{2} \qquad \Im(z) = \frac{z - \bar{z}}{2i}.$$

- $\operatorname{Arg}(z/w) = \operatorname{Arg}(z) - \operatorname{Arg}(w)$.

- Exponential forms of cosine and sine, where it's sometimes useful to set $w \coloneqq e^{iz}$:

$$\begin{aligned}\cos(z) &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \frac{1}{2}(w + w^{-1}) \\ \sin(z) &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = \frac{1}{2i}(w - w^{-1})\end{aligned}$$

$$\begin{aligned}\cosh(z) = \cos(iz) &= \frac{1}{2}\left(e^{z} + e^{-z}\right) \\ \sinh(z) = \sin(iz) &= \frac{1}{2}\left(e^{z} - e^{-z}\right).\end{aligned}$$

- Setting $w = e^z$ is useful:
- Various differentials:

$$\begin{aligned} dz &= dx + i\ dy \\ d\bar{z} &= dx - i\ dy \end{aligned}$$

$$f_z = f_x = f_y/i.$$

- Integral of a complex exponential:

$$\int_0^{2\pi} e^{i\ell x} dx = \begin{cases} 2\pi & \ell = 0 \\ 0 & \text{else} \end{cases}.$$

Fact 1.4.2 (Generalized Binomial Theorem)
Define $(n)_k$ to be the falling factorial

$$\prod_{j=0}^{k-1} (n-k) = n(n-1)\cdots(n-k+1)$$

and set $\binom{n}{k} := (n)_k/k!$, then

$$(x+y)^n = \sum_{k\geq 0} \binom{n}{k} x^k y^{n-k}.$$

Fact 1.4.3 (Some useful series)

$$\begin{aligned} \sum_{k=1}^{n} k &= \frac{n(n+1)}{2} \\ \sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{k=1}^{n} k^3 &= \frac{n^2(n+1)^2}{4} \\ \log(1-x) &= \sum_{n\geq 0} \frac{x^n}{n} && x \in (\\ \frac{\partial}{\partial z} \sum_{j=0}^{\infty} a_j z^j &= \sum_{j=0}^{\infty} a_{j+1} z^j \\ \sqrt{1+x} &= (1+x)^{1/2} && = 1 + (1/2)x + \frac{(1/2)(-1/2)}{2!} x^2 + \frac{(1/2)(-1/2)(-3/2)}{3!} x^3 + \cdots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots \end{aligned}$$

Fact 1.4.4
Useful trick for expanding square roots:

$$\sqrt{z} = \sqrt{z_0 + z - z_0} = \sqrt{z_0\left(1 + \frac{z - z_0}{z}\right)} = \sqrt{z_0}\sqrt{1+u}, \quad u := \frac{z - z_0}{z}$$
$$\implies \sqrt{z} = \sqrt{z_0}\sum_{k\geq 0}\binom{1/2}{k}\left(\frac{z-z_0}{z}\right)^k.$$

2 | Calculus Preliminaries

2.1 Definitions

Definition 2.1.1 (Locally uniform convergence)
A sequence of functions f_n is said to converge **locally uniformly** on $\Omega \subseteq \mathbb{C}$ iff $f_n \to f$ uniformly on every compact subset $K \subseteq \Omega$.

Definition 2.1.2 (Equicontinuous Family)
A family of functions f_n is **equicontinuous** iff for every ε there exists a $\delta = \delta(\varepsilon)$ (not depending on n or f_n) such that $|x - y| < \varepsilon \implies |f_n(x) - f_n(y)| < \varepsilon$ for all n.

Remark 2.1.3: Recall Arzelà-Ascoli, an analog of Heine-Borel: for X compact Hausdorff, consider the the Banach space $C(X;\mathbb{R})$ equipped with the *uniform norm* $\|f\|_{\infty,X} := \sup_{x\in X}|f(x)|$. Then a subset $A \subseteq X$ is compact iff A is closed, uniformly bounded, and equicontinuous. As a consequence, if A is a sequence, it contains a subsequence converging uniformly to a continuous function. The proof is an $\varepsilon/3$ argument.

Definition 2.1.4 (Normal Family)

Remark 2.1.5: A continuous function on a compact set is uniformly continuous.

Definition 2.1.6 (Univalent functions)
A function $f \in \mathrm{Hol}(U;\mathbb{C})$ is called **univalent** if f is injective.

Remark 2.1.7: If $f : \Omega \to \Omega'$ is a univalent surjection, f is invertible on Ω and f^{-1} is holomorphic. Compare to real functions: $f(x) = x^3$ is injective on $(-c, c)$ for any c but $f'(0) = 0$ and $f^{-1}(x) := x^{1/3}$ is not differentiable at zero.

2.2 Theorems

Theorem 2.2.1 *(Implicit Function Theorem).*

Theorem 2.2.2 *(Inverse Function Theorem).*
For $f \in C^1(\mathbb{R};\mathbb{R})$ with $f'(a) \neq 0$, then f is invertible in a neighborhood $U \ni a$, $g := f^{-1} \in C^1(U;\mathbb{R})$, and at $b := f(a)$ the derivative of g is given by

$$g'(b) = \frac{1}{f'(a)}.$$

For $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with D_f invertible in a neighborhood of a, so $\det(J_f) \neq 0$, then setting $b := F(a)$,

$$J_{F^{-1}}(q) = \left(J_F(p)\right)^{-1}.$$

The version for holomorphic functions: if $f \in \mathrm{Hol}(\mathbb{C};\mathbb{C})$ with $f'(p) \neq 0$ then there is a neighborhood $V \ni p$ with that $f \in \mathrm{BiHol}(V, f(V))$.

Theorem 2.2.3 *(Green's Theorem).*
If $\Omega \subseteq \mathbb{C}$ is bounded with $\partial\Omega$ piecewise smooth and $f, g \in C^1(\overline{\Omega})$, then

$$\int_{\partial\Omega} f\,dx + g\,dy = \iint_{\Omega} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.$$

2.3 Convergence

Remark 2.3.1: Recall that absolutely convergent implies convergent, but not conversely: $\sum k^{-1} = \infty$ but $\sum (-1)^k k^{-1} < \infty$. This converges because the even (odd) partial sums are monotone increasing/decreasing respectively and in $(0,1)$, so they converge to a finite number. Their difference converges to 0, and their common limit is the limit of the sum.

Proposition 2.3.2 *(Uniform Convergence of Series).*
A series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly iff

$$\lim_{n\to\infty} \left\| \sum_{k \geq n} f_k \right\|_{\infty} = 0.$$

Theorem 2.3.3 *(Weierstrass M-Test).*
If $\{f_n\}$ with $f_n : \Omega \to \mathbb{C}$ and there exists a sequence $\{M_n\}$ with $\|f_n\|_\infty \leq M_n$ and $\sum_{n\in\mathbb{N}} M_n < \infty$,

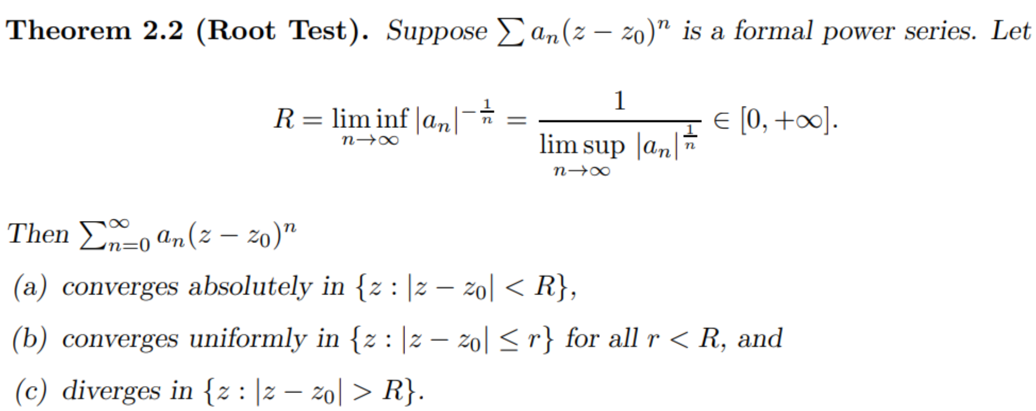
then $f(x) := \sum_{n\in\mathbb{N}} f_n(x)$ converges absolutely and uniformly on Ω. Moreover, if the f_n are continuous, by the uniform limit theorem, f is again continuous.

2.4 Series and Sequences

Remark 2.4.1: Note that if a power series converges uniformly, then summing commutes with integrating or differentiating.

Proposition 2.4.2 ***(Ratio Test).***
Consider $\sum c_k z^k$, set $R = \lim \left|\frac{c_{k+1}}{c_k}\right|$, and recall the **ratio test**:

- $R \in (0,1) \implies$ convergence.
- $R \in (1,\infty] \implies$ divergence.
- $R = 1$ yields no information.


Theorem 2.2 (Root Test). *Suppose $\sum a_n(z-z_0)^n$ is a formal power series. Let*

$$R = \liminf_{n\to\infty} |a_n|^{-\frac{1}{n}} = \frac{1}{\limsup\limits_{n\to\infty} |a_n|^{\frac{1}{n}}} \in [0, +\infty].$$

Then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$

(a) converges absolutely in $\{z : |z-z_0| < R\}$,

(b) converges uniformly in $\{z : |z-z_0| \le r\}$ for all $r < R$, and

(c) diverges in $\{z : |z-z_0| > R\}$.

Figure 1: image_2021-05-27-15-40-58

Proposition 2.4.3 ***(Root Test).***

Proposition 2.4.4 ***(Radius of Convergence by the Root Test).***
For $f(z) = \sum_{k\in\mathbb{N}} c_k z^k$, defining

$$\frac{1}{R} := \limsup_k |a_k|^{\frac{1}{k}},$$

then f converges absolutely and uniformly for $D_R := |z| < R$ and diverges for $|z| > R$.

Moreover f is holomorphic in D_R, can be differentiated term-by-term, and $f' = \sum_{k\in\mathbb{N}} nc_k z^k$.

Fact 2.4.5
Recall the p-**test**:

$$\sum n^{-p} < \infty \iff p \in (1, \infty).$$

Fact 2.4.6
The product of two sequences is given by the Cauchy product

$$\sum a_k z^k \cdot \sum b_k z^k = \sum c_k z^k, \quad c_k := \sum_{j\leq k} a_k b_{k-j}.$$

Fact 2.4.7
Recall how to carry out polynomial long division:

Polynomial long division

Fact 2.4.8 (Partial Fraction Decomposition)

- For every root r_i of multiplicity 1, include a term $A/(x - r_i)$.
- For any factors $g(x)$ of multiplicity k, include terms $A_1/g(x), A_2/g(x)^2, \cdots, A_k/g(x)^k$.
- For irreducible quadratic factors $h_i(x)$, include terms of the form $\dfrac{Ax + B}{h_i(x)}$.

2.5 Exercises

Exercise 2.5.1 (?)
Find the radius of convergences for the power series expansion of $\sqrt{z}$ about $z_0 = 4 + 3i$.

3 Preliminaries

Definition 3.0.1 (Toy contour)
A closed Jordan curve that separates $\mathbb{C}$ into an exterior and interior region is referred to as a **toy contour**.

3.1 Complex Arithmetic

Fact 3.1.1 (Complex roots of a number)
The complex nth roots of $z := re^{i\theta}$ are given by

$$\left\{\omega_k := r^{1/n}e^{i\left(\frac{\theta+2k\pi}{n}\right)} \;\middle|\; 0 \leq k \leq n-1\right\}.$$

Note that one root is $r^{1/n} \in \mathbb{R}$, and the rest are separated by angles of $2\pi/n$. Mnemonic:

$$z = re^{i\theta} = re^{i(\theta+2k\pi)} \implies z^{1/n} = \cdots.$$

Fact 3.1.2
Common trick:

$$f^{1/n} = e^{\frac{1}{n}\log(f)},$$

taking (say) a principal branch of log given by $\mathbb{C} \setminus (-\infty, 0] \times 0$.

Fact 3.1.3
Some computations that come up frequently:

$$\begin{aligned} |z \pm w|^2 &= |z|^2 + |w|^z + 2\Re(\overline{w}z) \\ (a+bi)(c+di) &= (ac-bd) + (ad+bc). \end{aligned}$$

Fact 3.1.4
Some useful facts:

$$|e^z| = e^{\Re(z)}.$$

On S^1,

$$
\begin{aligned}
1/z &= e^{-i\theta} \\
\cos(\theta) = \frac{\mathrm{e}^{i\theta}+\mathrm{e}^{-i\theta}}{2} &= \frac{z+1/z}{2} \\
\sin(\theta) = \frac{\mathrm{e}^{i\theta}-\mathrm{e}^{-i\theta}}{2i} &= \frac{z-1/z}{2i}.
\end{aligned}
$$

3.2 Complex Log

Fact 3.2.1 (Complex Log)
For $z = re^{i\theta} \neq 0$, θ is of the form $\Theta + 2k\pi$ where $\Theta = \operatorname{Arg} z$ We define

$$\log(z) = \ln(|z|) + i \operatorname{Arg}(z)$$

and $z^c := e^{c\log(z)}$.

Proposition 3.2.2 ***(Existence of complex log).***
Suppose Ω is a simply-connected region such that $1 \in \Omega, 0 \notin \Omega$. Then there exists a branch of $F(z) := \operatorname{Log}(z)$ such that

- F is holomorphic on Ω,
- $e^{F(z)} = z$ for all $z \in \Omega$
- $F(x) = \log(x)$ for $x \in \mathbb{R}$ in a neighborhood of 1.

Definition 3.2.3 (Principal branch and exponential)
Take $\mathbb{C}$ and delete $\mathbb{R}^{\leq 0}$ to obtain the **principal branch** of the logarithm, defined as

$$\operatorname{Log}(z) := \log(r) + i\theta \qquad |\theta| < \pi.$$

Similarly define

$$z^\alpha := e^{\alpha \operatorname{Log}(z)}.$$

Theorem 3.2.4 ***(Existence of log).***
If f is holomorphic and nonvanishing on a simply-connected region Ω, then there exists a holomorphic G on Ω such that

$$f(z) = e^{G(z)}.$$

3.3 Complex Calculus

Remark 3.3.1: When parameterizing integrals $\int_\gamma f(z)\,dz$, parameterize γ by θ and write $z = re^{i\theta}$ so $dz = ire^{i\theta}\,d\theta$.

⚠ **Warning 3.3.2**
$f(z) = \sin(z), \cos(z)$ are unbounded on $\mathbb{C}$! An easy way to see this: they are nonconstant and entire, thus unbounded by Liouville.

Example 3.3.3 ***(?):*** You can show $f(z) = \sqrt{z}$ is not holomorphic by showing its integral over S^1 is nonzero. This is a direct computation:

$$\begin{aligned}
\int_{S^1} z^{1/2}\,dz &= \int_0^{2\pi} (e^{i\theta})^{1/2} ie^{i\theta}\,d\theta \\
&= i\int_0^{2\pi} e^{\frac{i3\theta}{2}}\,d\theta \\
&= i\left(\frac{2}{3i}\right) e^{\frac{i3\theta}{2}}\Big|_0^{2\pi} \\
&= \frac{2}{3}\left(e^{3\pi i - 1}\right) \\
&= -\frac{4}{3}.
\end{aligned}$$

Note an issue: a different parameterization yields a different (still nonzero) number

$$\begin{aligned}
\cdots &= \int_{-\pi}^{\pi} (e^{i\theta})^{1/2} ie^{i\theta}\,d\theta \\
&= \frac{2}{3}\left(e^{\frac{3\pi i}{2}} - e^{\frac{-3\pi i}{2}}\right) \\
&= -\frac{4i}{3}.
\end{aligned}$$

This is these are paths that don't lift to closed loops on the Riemann surface defined by $z \mapsto z^2$.

3.3.1 Holomorphy and Cauchy-Riemann

Definition 3.3.4 (Analytic)
A function $f : \Omega \to \mathbb{C}$ is *analytic* at $z_0 \in \Omega$ iff there exists a power series $g(z) = \sum a_n(z - z_0)^n$ with radius of convergence $R > 0$ and a neighborhood $U \ni z_0$ such that $f(z) = g(z)$ on U.

Definition 3.3.5 (Complex differentiable / holomorphic /entire)
A function $f : \mathbb{C} \to \mathbb{C}$ is **complex differentiable** or **holomorphic** at z_0 iff the following limit exists:

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(h)}{h}.$$

A function that is holomorphic on $\mathbb{C}$ is said to be **entire**.
Equivalently, there exists an $\alpha \in \mathbb{C}$ such that

$$f(z_0 + h) - f(z_0) = \alpha h + R(h) \qquad\qquad R(h) \xrightarrow{h \to 0} 0.$$

In this case, $\alpha = f'(z_0)$.

Example 3.3.6 ***(Holomorphic vs non-holomorphic):***

- $f(z) = \dfrac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$.
- $f(z) = \bar{z}$ is *not* holomorphic, since $\dfrac{\bar{h}}{h}$ does not converge (but is real differentiable).

Definition 3.3.7 (Real (multivariate) differentiable)
A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is **real-differentiable** at $\mathbf{p}$ iff there exists a linear transformation A such that

$$\frac{\|F(\mathbf{p}+\mathbf{h}) - F(\mathbf{p}) - A(\mathbf{h})\|}{\|\mathbf{h}\|} \xrightarrow{\|\mathbf{h}\| \to 0} 0.$$

Rewriting,

$$\|F(\mathbf{p}+\mathbf{h}) - F(\mathbf{p}) - A(\mathbf{h})\| = \|\mathbf{h}\|\|R(\mathbf{h})\| \qquad\qquad \|R(\mathbf{h})\| \xrightarrow{\|\mathbf{h}\| \to 0} 0.$$

Equivalently,

$$F(\mathbf{p}+\mathbf{h}) - F(\mathbf{p}) = A(\mathbf{h}) + \|\mathbf{h}\|R(\mathbf{h}) \qquad\qquad \|R(\mathbf{h})\| \xrightarrow{\|\mathbf{h}\| \to 0} 0.$$

Or in a slightly more useful form,

$$F(\mathbf{p}+\mathbf{h}) = F(\mathbf{p}) + A(\mathbf{h}) + R(\mathbf{h}) \qquad\qquad R \in o(\|\mathbf{h}\|), \text{ i.e. } \frac{\|R(\mathbf{h})\|}{\|\mathbf{h}\|} \xrightarrow{\mathbf{h} \to 0} 0.$$

Proposition 3.3.8 ***(Complex differentiable implies Cauchy-Riemann).***
If f is differentiable at z_0, then the limit defining $f'(z_0)$ must exist when approaching from any direction. Identify $f(z) = f(x, y)$ and write $z_0 = x + iy$, then first consider $h \in RR$, so $h = h_1 + ih_2$ with $h_2 = 0$. Then

$$f'(z_0) = \lim_{h_1 \to 0} \frac{f(x+h_1, y) - f(x, y)}{h_1} := \frac{\partial f}{\partial x}(x, y).$$

Taking $h \in i\mathbb{R}$ purely imaginary, so $h = ih_2$,

$$f'(z_0) = \lim_{ih_2 \to 0} \frac{f(x, y+h_2) - f(x, y)}{ih_2} := \frac{1}{i}\frac{\partial f}{\partial y}(x, y).$$

Equating,

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y},$$

and writing $f = u + iv$ and $1/i = -i$ yields

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ \frac{1}{i}\frac{\partial f}{\partial y} &= \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \end{aligned}$$

Thus

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Proposition 3.3.9 ***(Polar Cauchy-Riemann equations).***

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Proof .
Setting

$$z = re^{i\theta} = r(\cos(\theta) + i\sin(\theta)) = x + iy$$

yields $x = r\cos(\theta), y = r\sin(\theta)$, one can identify

$$\begin{aligned} x_r &= \cos(\theta), x_\theta = -r\sin(\theta) \\ y_r &= \sin(\theta), y_\theta = r\cos(\theta). \end{aligned}$$

Now apply the chain rule:

$$\begin{aligned} u_r &= u_x x_r + u_y y_r \\ &= v_y x_r - v_x y_r && \text{CR} \\ &= v_y \cos(\theta) - v_x \sin(\theta) \\ &= \frac{1}{r}(v_y r\cos(\theta) - v_x r\sin(\theta)) \\ &= \frac{1}{r}(v_y y_\theta + v_x x_\theta) \\ &= \frac{1}{r}v_\theta. \end{aligned}$$

Similarly,

$$\begin{aligned} v_r &= v_x x_r + v_y y_r \\ &= v_x \cos(\theta) + v_y \sin(\theta) \\ &= -u_y \cos(\theta) + u_x \sin(\theta) && \text{CR} \\ &= \frac{1}{r}(-u_y r\cos(\theta) + u_x r\sin(\theta)) \\ &= \frac{1}{r}(-u_y y_\theta - u_x x_0) \\ &= -\frac{1}{r}u_\theta. \end{aligned}$$

Thus

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

.

■

Proposition 3.3.10 ***(Holomorphic functions are continuous.).***
f is holomorphic at z_0 iff there exists an $a \in \mathbb{C}$ such that

$$f(z_0 + h) - f(z_0) - ah = h\psi(h), \quad \psi(h) \overset{h\to 0}{\to} 0.$$

In this case, $a = f'(z_0)$.

Prove

3.3.2 Delbar and the Laplacian

Definition 3.3.11 (del and delbar operators)

$$\partial := \partial_z := \frac{1}{2}\left(\partial_x - i\partial_y\right) \quad \text{and} \quad \bar{\partial} := \partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right).$$

Moreover, $f' = \partial f + \bar{\partial} f$.

Proposition 3.3.12 ***(Holomorphic iff delbar vanishes).***
f is holomorphic at z_0 iff $\bar{\partial} f(z_0) = 0$:

$$\begin{aligned}
2\bar{\partial} f &:= (\partial_x + i\partial_y)(u + iv) \\
&= u_x + iv_x + iu_y - v_y \\
&= (u_x - v_y) + i(u_y + v_x) \\
&= 0 && \text{by Cauchy-Riemann.}
\end{aligned}$$

3.3.3 Harmonic Functions and the Laplacian

Definition 3.3.13 (Laplacian and Harmonic Functions)
A real function of two variables $u(x, y)$ is **harmonic** iff it is in the kernel of the Laplacian operator:

$$\Delta u := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0.$$

Proposition 3.3.14 ***(Cauchy-Riemann implies holomorphic).***
If $f = u + iv$ with $u, v \in C^1(\mathbb{R})$ satisfying the Cauchy-Riemann equations on Ω, then f is holomorphic on Ω and

$$f'(z) = \partial f = \frac{1}{2}\left(u_x + iv_x\right).$$

Proposition 3.3.15 ***(Holomorphic functions have harmonic components).***
If $f(z) = u(x, y) + iv(x, y)$ is holomorphic, then u, v are harmonic.

Proof (?).

- By CR,

$$u_x = v_y \qquad u_y = -v_x.$$

- Differentiate with respect to x:

$$u_{xx} = v_{yx} \qquad u_{yx} = -v_{xx}.$$

- Differentiate with respect to y:

$$u_{xy} = v_{yy} \qquad u_{yy} = -v_{xy}.$$

- Clairaut's theorem: partials are equal, so

$$u_{xx} - v_{yx} = 0 \implies u_{xx} + u_{yy} = 0$$

$$v_{xx} + u_{yx} = 0 \implies v_{xx} + v_{yy} = 0$$

.

■

3.3.4 Exercises

Proposition 3.3.16 ***(Injectivity Relates to Derivatives).***
If z_0 is a zero of f' of order n, then f is $(n+1)$-to-one in a neighborhood of z_0.

Proof .
?

■

proof

Exercise 3.3.17 (Zero derivative implies constant)
Show that if $f' = 0$ on a domain Ω, then f is constant on Ω

Solution:
Write $f = u + iv$, then $0 = 2f' = u_x + iv_x = u_y - iu_y$, so $\operatorname{grad} u = \operatorname{grad} v = 0$. Show f is constant along every straight line segment L by computing the directional derivative $\operatorname{grad} u \cdot \mathbf{v} = 0$ along L connecting p, q. Then $u(p) = u(q) = a$ some constant, and $v(p) = v(q) = b$, so $f(z) = a + bi$ everywhere.

Exercise 3.3.18 (f and fbar holomorphic implies constant)
Show that if f and $\overline{f}$ are both holomorphic on a domain Ω, then f is constant on Ω.

Solution:

- Strategy: show $f' = 0$.
- Write $f = u + iv$. Since f is analytic, it satisfies CR, so

$$u_x = v_y \qquad u_y = -v_x.$$

- Similarly write $\overline{f} = U + iV$ where $U = u$ and $V = -v$. Since $\overline{f}$ is analytic, it also satisfies CR , so

$$U_x = V_y \qquad U_y = -V_x$$

$$\implies u_x = -v_y \qquad u_y = v_x.$$

- Add the LHS of these two equations to get $2u_x = 0 \implies u_x = 0$. Subtract the right-hand side to get $-2v_x = 0 \implies v_x = 0$
- Since f is analytic, it is holomorphic, so f' exists and satisfies $f' = u_x + iv_x$. But by above, this is zero.
- By the previous exercise, $f' = 0 \implies f$ is constant.

Exercise 3.3.19 (SS 1.13: Constant real/imaginary/magnitude implies constant)
If f is holomorphic on Ω and any of the following hold, then f is constant:

1. $\Re(f)$ is constant.
2. $\Im(f)$ is constant.
3. $|f|$ is constant.

Solution:
Part 3:

- Write $|f| = c \in \mathbb{R}$.
- If $c = 0$, done, so suppose $c > 0$.

- Use $f\overline{f} = |f|^2 = c^2$ to write $\overline{f} = c^2/f$.
- Since $|f(z)| = 0 \iff f(z) = 0$, we have $f \neq 0$ on Ω, so $\overline{f}$ is analytic.
- Similarly f is analytic, and $f, \overline{f}$ analytic implies $f' = 0$ implies f is constant.

Finish

3.4 Power Series

Theorem 3.4.1 ***(Improved Taylor's Theorem).***
If f is holomorphic on a region Ω with $\overline{D_R(z_0)} \subseteq \Omega$, and for every $z \in D_r(z_0)$, f has a power series expansion of the following form:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{where } a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta}\, d\theta.$$

Proposition 3.4.2 ***(Power Series are Smooth).***
Any power series is smooth (and thus holomorphic) on its disc of convergence, and its derivatives can be obtained using term-by-term differentiation:

$$\frac{\partial}{\partial z} f(z) = \frac{\partial}{\partial z} \sum_{k \geq 0} c_k (z - z_0)^k = \sum_{k \geq 1} k c_k (z - z_0)^k.$$

Moreover, the coefficients are given by

$$c_k = \frac{f^{(n)}(z_0)}{n!}.$$

Remark 3.4.3: By an application of the Cauchy integral formula (see S&S 7.1) if f is holomorphic on $D_R(z_0)$ there is a formula for all $k \geq 0$ and all $0 < r < R$:

$$c_k = \frac{1}{2\pi r^k} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta}\, d\theta.$$

Proposition 3.4.4 ***(Exponential is uniformly convergent in discs).***
$f(z) = e^z$ is uniformly convergent in any disc in $\mathbb{C}$.

Proof .
Apply the estimate

$$|e^z| \leq \sum \frac{|z|^n}{n!} = e^{|z|}.$$

Now by the M-test,

$$|z| \leq R < \infty \implies \left| \sum \frac{z^n}{n!} \right| \leq e^R < \infty.$$

■

Lemma 3.4.5 ***(Dirichlet's Test).***
Given two sequences of real numbers $\{a_k\}, \{b_k\}$ which satisfy

1. The sequence of partial sums $\{A_n\}$ is bounded,
2. $b_k \searrow 0$.

then

$$\sum_{k\geq 1} a_k b_k < \infty.$$

Proof (?).

See http://www.math.uwaterloo.ca/~krdavids/Comp/Abel.pdf

Use summation by parts. For a fixed $\sum a_k b_k$, write

$$\sum_{n=1}^m x_n Y_n + \sum_{n=1}^m X_n y_{n+1} = X_m Y_{m+1}.$$

Set $x_n \coloneqq a_n, y_N \coloneqq b_n - b_{n-1}$, so $X_n = A_n$ and $Y_n = b_n$ as a telescoping sum. Importantly, all y_n are negative, so $|y_n| = |b_n - b_{n-1}| = b_{n-1} - b_n$, and moreover $a_n b_n = x_n Y_n$ for all n. We have

$$\begin{aligned}
\sum_{n\geq 1} a_n b_n &= \lim_{N\to\infty} \sum_{n\leq N} x_n Y_n \\
&= \lim_{N\to\infty} \sum_{n\leq N} X_N Y_N - \sum_{n\leq N} X_n y_{n+1} \\
&= -\sum_{n\geq 1} X_n y_{n+1},
\end{aligned}$$

where in the last step we've used that

$$|X_N| = |A_N| \leq M \implies |X_N Y_N| = |X_N||b_{n+1}| \leq M b_{n+1} \to 0.$$

So it suffices to bound the latter sum:

$$\begin{aligned}
\sum_{k\geq n} |X_k y_{k+1}| &\leq M \sum_{k\geq 1} |y_{k+1}| \\
&\leq M \sum_{k\geq 1} b_k - b_{k+1} \\
&\leq 2M(b_1 - b_{n+1}) \\
&\leq 2M b_1.
\end{aligned}$$

■

Theorem 3.4.6 ***(Abel's Theorem).***
If $\sum_{k=1}^{\infty} c_k z^j$ converges on $|z| < 1$ then

$$\lim_{z \to 1^-} \sum_{k \in \mathbb{N}} c_k z^k = \sum_{k \in \mathbb{N}} c_k.$$

Lemma 3.4.7 ***(Abel's Test).***
If $f(z) := \sum c_k z^k$ is a power series with $c_k \in \mathbb{R}^{\geq 0}$ and $c_k \searrow 0$, then f converges on S^1 except possibly at $z = 1$.

Example 3.4.8 ***(application of Abel's theorem):*** What is the value of the alternating harmonic series? Integrate a geometric series to obtain

$$\sum \frac{(-1)^k z^k}{n} = \log(z+1) \qquad\qquad |z| < 1.$$

Since $c_k := (-1)^k / k \searrow 0$, this converges at $z = 1$, and by Abel's theorem $f(1) = \log(2)$.

Remark 3.4.9: The converse to Abel's theorem is false: take $f(z) = \sum (-z)^n = 1/(1+z)$. Then $f(1) = 1 - 1 + 1 - \cdots$ diverges at 1, but $1/1 + 1 = 1/2$. So the limit $s := \lim_{x \to 1^-} f(x) 1/2$, but $\sum a_n$ doesn't converge to s.

Proposition 3.4.10 ***(Summation by Parts).***
Setting $A_n := \sum_{k=1}^{n} b_k$ and $B_0 := 0$,

$$\sum_{k=m}^{n} a_k b_k = A_n b_n - A_{m-1} b_m - \sum_{k=m}^{n-1} A_k (b_{k+1} - b_k).$$

Compare this to integrating by parts:

$$\int_a^b fg = F(b)g(b) - F(a)g(a) - \int_a^b Fg'.$$

Note there is a useful form for taking the product of sums:

$$A_n B_n = \sum_{k=1}^{n} A_k b_k + \sum_{k=1}^{n} a_k B_{k-1}.$$

Proof (?).
An inelegant proof: define $A_n := \sum_{k \leq n} a_k$, use that $a_k = A_k - A_{k-1}$, reindex, and peel a top/bottom term off of each sum to pattern-match.

Behold:

$$\begin{aligned}
\sum_{m\leq k\leq n} a_k b_k &= \sum_{m\leq k\leq n} (A_k - A_{k-1}) b_k \\
&= \sum_{m\leq k\leq n} A_k b_k - \sum_{m\leq k\leq n} A_{k-1} b_k \\
&= \sum_{m\leq k\leq n} A_k b_k - \sum_{m-1\leq k\leq n-1} A_k b_{k+1} \\
&= A_n b_n + \sum_{m\leq k\leq n-1} A_k b_k - \sum_{m-1\leq k\leq n-1} A_k b_{k+1} \\
&= A_n b_n - A_{m-1} b_m + \sum_{m\leq k\leq n-1} A_k b_k - \sum_{m\leq k\leq n-1} A_k b_{k+1} \\
&= A_n b_n - A_{m-1} b_m + \sum_{m\leq k\leq n-1} A_k (b_k - b_{k+1}) \\
&= A_n b_n - A_{m-1} b_m - \sum_{m\leq k\leq n-1} A_k (b_{k+1} - b_k).
\end{aligned}$$

■

Proposition 3.4.11 *(?).*
If f is non-constant, then f' is analytic and the zeros of f' are isolated. If f, g are analytic with $f' = g'$, then $f - g$ is constant.

3.4.1 Exercises: Series

Exercise 3.4.12 (Application of summation by parts)
Use summation by parts to show that $\sin(n)/n$ converges.

Exercise 3.4.13 (1.20: Series convergence on the circle)
Show that

1. $\sum kz^k$ diverges on S^1.
2. $\sum k^{-2}z^k$ converges on S^1.
3. $\sum k^{-1}z^k$ converges on $S^1 \setminus \{1\}$ and diverges at 1.

Solution: 1. Use that $\left|z^k\right| = 1$ and $\sum c_k z^k < \infty \implies |c_k| \to 0$, but $\left|kz^k\right| = |k| \to \infty$ here.

2. Use that absolutely convergent implies convergent, and $\sum \left|k^{-2}z^k\right| = \sum \left|k^{-2}\right|$ converges by the p-test.
3. If $z = 1$, this is the harmonic series. Otherwise take $a_k = 1/k, b_k = e^{ik\theta}$ where $\theta \in (0, 2\pi)$ is some constant, and apply Dirichlet's test. It suffices to bound the partial sums of the

b_k. Recalling that $\sum_{k\leq N} r^k = (1-r^{N+1})/(1-r)$,

$$\left\|\sum_{k\leq m} e^{ik\theta}\right\| = \left\|\frac{1-e^{i(m+1)\theta}}{1-e^{i\theta}}\right\| \leq \frac{2}{\|1-e^{i\theta}\|} := M,$$

which is a constant. Here we've used that two points on S^1 are at most distance 2 from each other.

Exercise 3.4.14 (Laurent expansions inside and outside of a disc)
Expand $f(z) = \dfrac{1}{z(z-1)}$ in both

- $|z| < 1$
- $|z| > 1$

Solution:

$$\frac{1}{z(z-1)} = -\frac{1}{z}\frac{1}{1-z} = -\frac{1}{z}\sum z^k.$$

and

$$\frac{1}{z(z-1)} = \frac{1}{z^2(1-\frac{1}{z})} = \frac{1}{z^2}\sum\left(\frac{1}{z}\right)^k.$$

Exercise 3.4.15 (Laurent expansions about different points)
Find the Laurent expansion about $z = 0$ and $z = 1$ respectively of the following function:

$$f(z) := \frac{z+1}{z(z-1)}.$$

Solution:
Note: once you see that everything is in terms of powers of $(z-z_0)$, you're essentially done.
For $z = 0$:

$$\begin{aligned}\frac{z+1}{z(z-1)} &= \frac{1}{z}\frac{z+1}{z-1} \\ &= -\frac{z+1}{z}\frac{1}{1-z} \\ &= -\left(1+\frac{1}{z}\right)\sum_{k\geq 0} z^k.\end{aligned}$$

For $z = 1$:

$$\begin{aligned}\frac{z+1}{z(z-1)} &= \frac{1}{z-1}\left(1+\frac{1}{z}\right)\\ &= \frac{1}{z-1}\left(1+\frac{1}{1-(1-z)}\right)\\ &= \frac{1}{z-1}\left(1+\sum_{k\geq 0}(1-z)^k\right)\\ &= \frac{1}{z-1}\left(1+\sum_{k\geq 0}(-1)^k(z-1)^k\right).\end{aligned}$$

4 Cauchy's Theorem

4.1 Complex Integrals

Definition 4.1.1 (Complex Integral)

$$\int_\gamma f dz := \int_I f(\gamma(t))\gamma'(t)\,dt = \int_\gamma (u+iv)\,dx \wedge (-v+iu)\,dy.$$

Theorem 4.1.2 ***(Cauchy-Goursat Theorem).***
If f is holomorphic on a region Ω with $\pi_1\Omega = 1$, then for any closed path $\gamma \subseteq \Omega$,

$$\int_\gamma f(z)\,dz = 0.$$

Slogan 4.1.3
Closed path integrals of holomorphic functions vanish.

4.2 Applications of Cauchy's Theorem

4.2.1 Integral Formulas and Estimates

See reference

Theorem 4.2.1 ***(Cauchy Integral Formula).***
Suppose f is holomorphic on Ω, then for any $z_0 \in \Omega$ and any open disc $\overline{D_R(z_0)}$ such that $\gamma := \partial \overline{D_R(z_0)} \subseteq \Omega$,

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z_0} \, d\xi$$

and

$$\frac{\partial^n f}{\partial z^n}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi.$$

Proof. It follows from a consequence of Cauchy's theorem (see above) that if $C(z_0, r)$ denotes the circle of radius r around z_0 for a sufficiently small $r > 0$ then

$$\begin{aligned} \left|\frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz - f(z_0)\right| &= \left|\frac{1}{2\pi i} \int_{C(z_0,r)} \frac{f(z) - f(z_0)}{z - z_0} dz\right| \\ &= \left|\frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta}) - f(z_0)}{re^{i\theta}} ire^{i\theta} d\theta\right| \\ &\leq \frac{1}{2\pi} 2\pi \times \sup_{\theta \in [0,2\pi)} |f(z_0 + re^{i\theta}) - f(z_0)| \\ &\quad (\text{ by } ML \text{ inequality}). \end{aligned}$$

As f is continuous it follows that the righthand side goes to zero as r tends to zero. This completes the proof. □

Figure 2: image_2021-05-27-16-54-06

Proof (?). ∎

Proof (?).

Proof. (*) Using Cauchy's integral formula we can write that

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \to 0} \frac{1}{2\pi i h} \int_C \left(\frac{f(z)}{z - z_0 - h} - \frac{f(z)}{z - z_0}\right) dz \\ &\quad (C \text{ is so chosen that the point } z_0 + h \text{ is enclosed by } C) \\ &= \lim_{h \to 0} \frac{1}{2\pi i h} \int_C \frac{f(z)h}{(z - z_0 - h)(z - z_0)} dz. \end{aligned}$$

So we need to prove that

$$\begin{aligned}
&|\int_C \frac{f(z)}{(z-z_0-h)(z-z_0)}dz - \int_C \frac{f(z)}{(z-z_0)^2}dz| \\
= &\left|\int_C \frac{f(z)h}{(z-z_0-h)(z-z_0)^2}dz\right| \to 0, \text{ as } \quad h \to 0.
\end{aligned}$$

We will basically use ML inequality to prove this. Note that, as f is continuous it is bounded on C by M (say). Let $\alpha = \min\{|z-z_0| : z \in C\}$. Then $|z-z_0|^2 \geq \alpha^2$ and $\alpha \leq |z-z_0| = |z-z_0-h+h| \leq |z-z_0-h| + |h|$ and hence for $|h| \leq \frac{\alpha}{2}$ (after all h is going to be small) we get $|z-z_0-h| \geq \alpha - |h| \geq \frac{\alpha}{2}$. Therefore

$$\left|\int_C \frac{f(z)h}{(z-z_0-h)(z-z_0)^2}dz\right| \leq \frac{M|h|l}{\frac{\alpha}{2}\alpha^2} = \frac{2M|h|l}{\alpha^3} \to 0,$$

as $h \to 0$. By repeating exactly the same technique we get $f^2(z_0) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^3}dz$ and so on.

□

■

Theorem 4.2.2 ***(Cauchy's Inequality / Cauchy's Estimate).***
For $z_0 \in D_R(z_0) \subset \Omega$, setting $M := \sup_{z\in\gamma} |f(z)|$ so $|f(z)| \leq M$ on γ

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{2\pi}\int_0^{2\pi} \frac{M}{R^{n+1}} R\, d\theta = \frac{Mn!}{R^n}.$$

Proof (of Cauchy's inequality).

- Given $z_0 \in \Omega$, pick the largest disc $D_R(z_0) \subset \Omega$ and let $C = \partial D_R$.
- Then apply the integral formula.

$$\begin{aligned}
\left|f^{(n)}(z_0)\right| &= \left|\frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\, d\zeta\right| \\
&= \left|\frac{n!}{2\pi i}\int_0^{2\pi} \frac{f\left(z_0+re^{i\theta}\right) rie^{i\theta}}{\left(re^{i\theta}\right)^{n+1}}\, d\theta\right| \\
&\leq \frac{n!}{2\pi}\int_0^{2\pi} \left|\frac{f\left(z_0+re^{i\theta}\right) rie^{i\theta}}{\left(re^{i\theta}\right)^{n+1}}\right| d\theta \\
&= \frac{n!}{2\pi}\int_0^{2\pi} \frac{\left|f\left(z_0+re^{i\theta}\right)\right|}{r^n}\, d\theta \\
&\leq \frac{n!}{2\pi}\int_0^{2\pi} \frac{M}{r^n}\, d\theta \\
&= \frac{Mn!}{r^n}.
\end{aligned}$$

■

Slogan 4.2.3
The nth Taylor coefficient of an analytic function is at most $\sup_{|z|=R} |f|/R^n$.

Theorem 4.2.4 ***(Mean Value Property for Holomorphic Functions).***
If f is holomorphic on $D_r(z_0)$

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta = \frac{1}{\pi r^2} \iint_{D_r(z_0)} f(z)\, dA.$$

Taking the real part of both sides, one can replace $f = u + iv$ with u.

4.2.2 Liouville

Theorem 4.2.5 ***(Liouville's Theorem).***
If f is entire and bounded, f is constant.

Proof (of Liouville).

- Since f is bounded, $f(z) \leq M$ uniformly on $\mathbb{C}$.
- Apply Cauchy's estimate for the 1st derivative:

$$|f'(z)| \leq \frac{1!\|f\|_{C_R}}{R} \leq \frac{M}{R} \xrightarrow{R \to \infty} 0,$$

so $f'(z) = 0$ for all z.

■

Exercise 2.E. [SSh03, 2.15] *Suppose f is continuous and non-zero on $\overline{\mathbb{D}}$ and holomorphic on $\mathbb{D}$ such that $|f(z)| = 1$ for all $|z| = 1$. Show that f is then constant.*

Figure 3: image_2021-05-17-11-54-14

Exercise 4.2.6 (?)

4.2.3 Continuation Principle

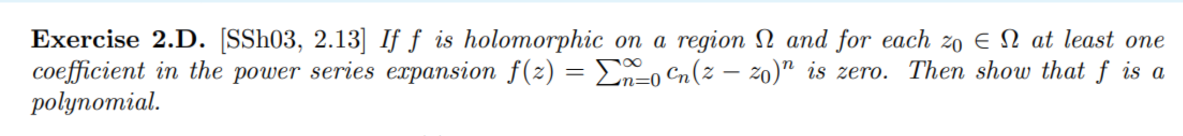
Theorem 4.2.7 ***(Continuation Principle / Identity Theorem).***
If f is holomorphic on a bounded connected domain Ω and there exists a sequence $\{z_i\}$ with a limit point in Ω such that $f(z_i) = 0$, then $f \equiv 0$ on Ω.

Slogan 4.2.8
Two functions agreeing on a set with a limit point are equal on a domain.

Proof (?).
Apply Improved Taylor Theorem?

todo

∎



Exercise 2.D. [SSh03, 2.13] *If f is holomorphic on a region Ω and for each $z_0 \in \Omega$ at least one coefficient in the power series expansion $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ is zero. Then show that f is a polynomial.*

Figure 4: image_2021-05-17-11-53-33

Exercise 4.2.9 (?)

4.3 Exercises

Exercise 4.3.1 (Primitives imply vanishing integral)
Show that if f has a primitive F on Ω then $\displaystyle\int_\gamma f = 0$ for every closed curve $\gamma \subseteq \Omega$.

Exercise 4.3.2 (?)
Prove the uniform limit theorem for holomorphic functions: if $f_n \to f$ locally uniformly and each f_n is holomorphic then f is holomorphic.

Solution:
This is S&S Theorem 5.2. Statement: if $f_n \to f$ uniformly locally uniformly on Ω then f is holomorphic on Ω.

- Let $D \subset \Omega$ with $\overline{\mathbb{D}} \subset \Omega$ and $\Delta \subset D$ be a triangle.

- Apply Goursat: $\int_\Delta f_n = 0$.
- $f_n \to f$ uniformly on Δ since it is closed and bounded and thus compact by Heine-Borel, so f is continuous and
$$\lim_n \int_\Delta f_n = \int_\Delta \lim_n f_n := \int_\Delta f.$$
- Apply Morera's theorem: $\int_\Delta f$ vanishes on every triangle in Ω, so f is holomorphic on Ω.

Exercise 4.3.3 (?)
Prove that if $f_n \to f$ locally uniformly with f_n holomorphic, then $f_n' \to f'$ locally uniformly and f' is holomorphic.

Solution:

- Simplifying step: for some reason, it suffices to assume $f_n \to f$ uniformly on all of Ω?
- Take Ω_R to be Ω with a buffer of R, so $d(z, \partial\Omega) > R$ for every $z \in \overline{\Omega_R}$.
- It suffices to show the following bound for F any holomorphic function on Ω:
$$\sup_{z\in\Omega_R} |F'(z)| \leq \frac{1}{R} \sup_{\zeta\in\Omega} |F(\zeta)| \qquad \forall R,$$
where on the right we take the sup over all Ω.
  - Then take $F := f_n - f$ and $R \to 0$ to conclude, since the right-hand side is a constant not depending on Ω_R.
- For any $z \in \Omega_R$, we have $\overline{D_R(z)} \subseteq \Omega_R$, so Cauchy's integral formula can be applied:

- $$\begin{aligned} |F'(z)| &= \left| \frac{1}{2\pi i} \int_{\partial D_R(z)} \frac{F(\xi)}{(\xi - z)^2} \, d\xi \right| \\ &\leq \frac{1}{2\pi} \int_{\partial D_R(z)} \frac{|F(\xi)|}{|\xi - z|^2} \, d\xi \\ &\leq \frac{1}{2\pi} \int_{\partial D_R(z)} \frac{\sup_{\zeta \in \Omega} |F(\zeta)|}{|\xi - z|^2} \, d\xi \\ &= \frac{1}{2\pi} \sup_{\zeta \in \Omega} |F(\zeta)| \int_{\partial D_R(z)} \frac{1}{R^2} \, d\xi \\ &= \frac{1}{2\pi} \sup_{\zeta \in \Omega} |F(\zeta)| \frac{1}{R^2} \int_{\partial D_R(z)} \, d\xi \\ &= \frac{1}{2\pi} \sup_{\zeta \in \Omega} |F(\zeta)| \frac{1}{R^2} 2\pi R \\ &\leq \frac{1}{2\pi} \sup_{\zeta \in \Omega} |F(\zeta)| \frac{1}{R^2} (2\pi R) \\ &= \frac{1}{R} \sup_{\zeta \in \Omega} |F(\zeta)|. \end{aligned}$$

- Now
$$\|f_n' - f'\|_{\infty, \Omega_R} \leq \frac{1}{R} \|f_n - f\|_{\infty, \Omega},$$
where if R is fixed then by uniform convergence of $f_n \to f$, for n large enough $\|f_n - f\| < \varepsilon / R$.

4.4 Morera's Theorem

Theorem 4.4.1 ***(Morera's Theorem).***
If f is continuous on a domain Ω and $\displaystyle\int_T f = 0$ for every triangle $T \subset \Omega$, then f is holomorphic.

Slogan 4.4.2
If every integral along a triangle vanishes, implies holomorphic.

Corollary 4.4.3 ***(Sufficient condition for a sequence to converge to a holomorphic function).***
If $\{f_n\}_{n \in \mathbb{N}}$ is a holomorphic sequence on a region Ω which uniformly converges to f on every compact subset $K \subseteq \Omega$, then f is holomorphic, and $f_n' \to f'$ uniformly on every such compact subset K.

Proof (?).
Commute limit with integral and apply Morera's theorem. ■

Remark 4.4.4: This can be applied to series of the form $\sum_k f_k(z)$.

4.4.1 Symmetric Regions

In this section, take Ω to be a region symmetric about the real axis, so $z \in \Omega \iff \bar{z} \in \Omega$. Partition this set as $\Omega^+ \subseteq \mathbb{H}, I \subseteq \mathbb{R}, \Omega^- \subseteq \overline{\mathbb{H}}$.

Theorem 4.4.5 ***(Symmetry Principle).***
Suppose that f^+ is holomorphic on Ω^+ and f^- is holomorphic on Ω^-, and f extends continuously to I with $f^+(x) = f^-(x)$ for $x \in I$. Then the following piecewise-defined function is holomorphic on Ω:

$$f(z) := \begin{cases} f^+(z) & z \in \Omega^+ \\ f^-(z) & z \in \Omega^- \\ f^+(z) = f^-(z) & z \in I. \end{cases}$$

Proof (?).
Apply Morera? ■

Theorem 4.4.6 ***(Schwarz Reflection).***
If f is continuous and holomorphic on $\mathbb{H}^+$ and real-valued on $\mathbb{R}$, then the extension defined by $F^-(z) = \overline{f(\bar{z})}$ for $z \in \mathbb{H}^-$ is a well-defined holomorphic function on $\mathbb{C}$.

Proof (?).
Apply the symmetry principle. ■

Remark 4.4.7: $\mathbb{H}^+, \mathbb{H}^-$ can be replaced with any region symmetric about a line segment $L \subseteq \mathbb{R}$.

5 | Zeros and Singularities

Definition 5.0.1 (Singularity)
A point z_0 is an **isolated singularity** if $f(z_0)$ is undefined but $f(z)$ is defined in a punctured neighborhood $D(z_0) \setminus \{z_0\}$ of z_0.
There are three types of isolated singularities:

- Removable singularities
- Poles
- Essential singularities

Example 5.0.2 ***(?):*** The singularities of a rational function are always isolated, since there are finitely many zeros of any polynomial. The function $F(z) := \operatorname{Log}(z)$ has a singularity at $z = 0$ that is **not** isolated, since every neighborhood intersects the branch cut $(-\infty, 0) \times \{0\}$, where F is not even defined. The function $G(z) := 1/\sin(\pi/z)$ has a non-isolated singularity at 0 and isolated singularities at $1/n$ for all n.

Definition 5.0.3 (Removable Singularities)
If z_0 is a singularity of f. then z_0 is a **removable singularity** iff there exists a holomorphic function g such that $f(z) = g(z)$ in a punctured neighborhood of z_0. Equivalently,

$$\lim_{z \to z_0} (z - z_0) f(z) = 0.$$

Equivalently, f is bounded on a neighborhood of z_0.

Theorem 5.0.4 ***(Extension over removable singularities).***
If f is holomorphic on $\Omega \setminus \{z_0\}$ where z_0 is a removable singularity, then there is a unique holomorphic extension of f to all of Ω.

Proof (?).
Take γ to be a circle centered at z_0 and use

$$f(z) := \int_\gamma \frac{f(\xi)}{\xi - z}\, dx.$$

This is valid for $z \neq z_0$, but the right-hand side is analytic. (?)

■

Revisit

Theorem 5.0.5 ***(Improved Taylor Remainder Theorem).***
If f is analytic on a region Ω containing z_0, then f can be written as

$$f(z) = \left(\sum_{k=0}^{n-1} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right) + R_n(z) (z - z_0)^n,$$

where R_n is analytic.

Definition 5.0.6 (Zeros)
If f is analytic and not identically zero on Ω with $f(z_0) = 0$, then there exists a nonvanishing holomorphic function g such that

$$f(z) = (z - z_0)^n g(z).$$

We refer to z_0 as a **zero of order** n.

Definition 5.0.7 (Poles (and associated terminology))
A *pole* z_0 of a function $f(z)$ is a zero of $g(z) := \frac{1}{f(z)}$. Equivalently, $\lim_{z \to z_0} f(z) = \infty$. In this case there exists a minimal n and a holomorphic h such that

$$f(z) = (z - z_0)^{-n} h(z).$$

Such an n is the *order* of the pole. A pole of order 1 is said to be a *simple pole*.

Definition 5.0.8 (Principal Part and Residue)
If f has a pole of order n at z_0, then there exist a holomorphic G in a neighborhood of z_0 such that

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0} + G(z) := P(z) + G(z).$$

The term $P(z)$ is referred to as the *principal part of f at z_0* consists of terms with negative degree, and the *residue* of f at z_0 is the coefficient a_{-1}.

Definition 5.0.9 (Essential Singularity)
A singularity z_0 is *essential* iff it is neither removable nor a pole. Equivalently, a Laurent series expansion about z_0 has a principal part with infinitely many terms.

Theorem 5.0.10 ***(Casorati-Weierstrass).***
If f is holomorphic on $\Omega \setminus \{z_0\}$ where z_0 is an essential singularity, then for every $V \subset \Omega \setminus \{z_0\}$, $f(V)$ is dense in $\mathbb{C}$.

Slogan 5.0.11
The image of a punctured disc at an essential singularity is dense in $\mathbb{C}$.

Proof (of Casorati-Weierstrass).
Pick $w \in \mathbb{C}$ and suppose toward a contradiction that $D_R(w) \cap f(V)$ is empty. Consider

$$g(z) := \frac{1}{f(z) - w},$$

and use that it's bounded to conclude that z_0 is either removable or a pole for f. ■

Definition 5.0.12 (Singularities at infinity)
For any f holomorphic on an unbounded region, we say $z = \infty$ is a singularity (of any of the above types) of f if $g(z) := f(1/z)$ has a corresponding singularity at $z = 0$.

Definition 5.0.13 (Meromorphic)
A function $f : \Omega \to \mathbb{C}$ is *meromorphic* iff there exists a sequence $\{z_n\}$ such that

- $\{z_n\}$ has no limit points in Ω.

- f is holomorphic in $\Omega \setminus \{z_n\}$.
- f has poles at the points $\{z_n\}$.

Equivalently, f is holomorphic on Ω with a discrete set of points delete which are all poles of f.

Theorem 5.0.14 ***(Meromorphic implies rational).***
Meromorphic functions on $\mathbb{C}$ are rational functions.

Proof (?).
Consider $f(z) - P(z)$, subtracting off the principal part at each pole z_0, to get a bounded entire function and apply Liouville. ■

Theorem 5.0.15 ***(Riemann Extension Theorem).***
A singularity of a holomorphic function is removable if and only if the function is bounded in some punctured neighborhood of the singular point.

6 Residues

6.1 Basics

Remark 6.1.1: Check: do you need residues at all?? You may be able to just compute an integral!

- Directly by parameterization:
$$\int_\gamma f\,dz = \int_a^b f(z(t))\,z'(t)\,dt \qquad \text{for } z(t) \text{ a parameterization of } \gamma,$$
- Finding a primitive F, then
$$\int_\gamma f = F(b) - F(a).$$
  - Note: you can parameterize a circle around z_0 using
$$z = z_0 + re^{i\theta}.$$

Fact 6.1.2 (Integrating z^k around S^1 powers residues)
The major fact that reduces integrals to residues:
$$\int_\gamma z^k\,dz = \int_0^{2\pi} e^{ik\theta} ie^{i\theta}\,d\theta = \int_0^{2\pi} e^{i(k+1)\theta\,d\theta} = \begin{cases} 2\pi i & k=-1 \\ 0 & \text{else.} \end{cases}.$$

Thus

$$\int \sum_{k\geq -M} c_k z^k = \sum_{k\geq -M} \int c_k z^k = 2\pi i c_{-1},$$

i.e. the integral picks out the c_{-1} coefficient in a Laurent series expansion.

Example 6.1.3 *(?):* Consider

$$f(z) := \frac{e^{iz}}{1+z^2}$$

where $z \neq \pm i$, and attempt to integrate

$$\int_{\mathbb{R}} f(z)\, dz.$$

Use a semicircular contour γ_R where $z = Re^{it}$ and check

$$\begin{aligned}\sup_{z\in\gamma_R} |f(z)| &= \max_{t\in[0,\pi} \frac{1}{1+(Re^{it})^2} \\ &= \max_{t\in[0,\pi} \frac{1}{1+R^2e^{2it}} \\ &\qquad\qquad\qquad\qquad = \frac{1}{R^2-1}.\end{aligned}$$

6.2 Estimates

Proposition 6.2.1 ***(Length bound / ML Estimate).***

$$\left|\int_\gamma f\right| \leq ML := \sup_{z\in\gamma} |f| \cdot \operatorname{length}(\gamma).$$

Proof (?).

$$\left|\int_\gamma f(z)dz\right| \leq \sup_{t\in[a,b]} |f(z(t))| \int_a^b |z'(t)|\, dt \leq \sup_{z\in\gamma} |f(z)| \cdot \operatorname{length}(\gamma).$$

■

Proposition 6.2.2 ***(Jordan's Lemma).***

Suppose that $f(z) = e^{iaz} g(z)$ for some g, and let $C_R \coloneqq \left\{ z = Re^{it} \;\middle|\; t \in [0, \pi] \right\}$. Then

$$\left| \int_{C_R} f(z)\, dz \right| \leq \frac{\pi M_R}{a}$$

where $M_R \coloneqq \sup_{t \in [0,\pi]} \left| g(Re^{it}) \right|$.

Proof (?).

$$\begin{aligned}
\left| \int_{C_R} f(z)\, dz \right| &= \left| \int_{C_R} e^{iaz} g(z)\, dz \right| \\
&= \left| \int_{[0,\pi]} e^{ia\left(Re^{it}\right)} g(Re^{it}) iRe^{it}\, dt \right| \\
&\leq \int_{[0,\pi]} \left| e^{ia\left(Re^{it}\right)} g(Re^{it}) iRe^{it} \right| dt \\
&= R \int_{[0,\pi]} \left| e^{ia\left(Re^{it}\right)} g(Re^{it}) \right| dt \\
&\leq RM_R \int_{[0,\pi]} \left| e^{ia\left(Re^{it}\right)} \right| dt \\
&= RM_R \int_{[0,\pi]} e^{\Re\left(iaRe^{it}\right)}\, dt \\
&= RM_R \int_{[0,\pi]} e^{\Re(iaR(\cos(t) + i\sin(t)))}\, dt \\
&= RM_R \int_{[0,\pi]} e^{-aR\sin(t)}\, dt \\
&= 2RM_R \int_{[0,\pi/2]} e^{-aR\sin(t)}\, dt \\
&\leq 2RM_R \int_{[0,\pi/2]} e^{-aR\left(\frac{2t}{\pi}\right)}\, dt \\
&= 2RM_R \left(\frac{\pi}{2aR} \right) \left(1 - e^{-aR} \right) \\
&= \frac{\pi M_R}{a}.
\end{aligned}$$

where we've used that on $[0, \pi/2]$, there is an inequality $2t/\pi \leq \sin(t)$. This is obvious from a picture, since $\sin(t)$ is a height on S^1 and $2t/\pi$ is a height on a diagonal line:

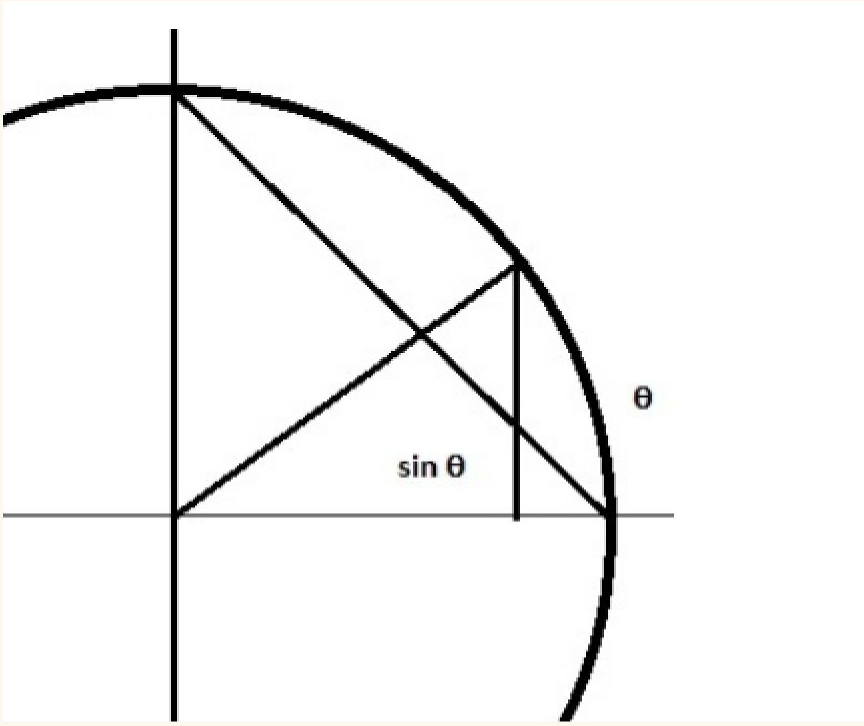


Figure 5: image_2021-06-09-01-29-22

■

6.3 Residue Formulas

Theorem 6.3.1 ***(The Residue Theorem).***
Let f be meromorphic on a region Ω with poles $\{z_1, z_2, \cdots, z_N\}$. Then for any $\gamma \in \Omega \setminus \{z_1, z_2, \cdots, z_N\}$,

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz = \sum_{j=1}^{N} n_\gamma(z_j) \operatorname*{Res}_{z=z_j} f.$$

If γ is a toy contour, then

$$\frac{1}{2\pi i} \int_\gamma f\, dz = \sum_{j=1}^{N} \operatorname*{Res}_{z=z_j} f.$$

Proposition 6.3.2 ***(Residue formula for higher order poles).***

If f has a pole z_0 of order n, then

$$\operatorname*{Res}_{z=z_0} f = \lim_{z\to z_0} \frac{1}{(n-1)!} \left(\frac{\partial}{\partial z}\right)^{n-1} (z-z_0)^n f(z).$$

Proposition 6.3.3 ***(Residue formula for simple poles).***
As a special case, if z_0 is a simple pole of f, then

$$\operatorname*{Res}_{z=z_0} f = \lim_{z\to z_0} (z-z_0) f(z).$$

Corollary 6.3.4 ***(Better derivative formula that sometimes works for simple poles).***
If additionally $f = g/h$ where $h(z_0) = 0$ and $h'(z_0) \neq 0$,

$$\operatorname*{Res}_{z=z_0} \frac{g(z)}{h(z)} = \frac{g(z_0)}{h'(z_0)}.$$

Proof (?).
Apply L'Hopital:

$$(z-z_0)\frac{g(z)}{h(z)} = \frac{(z-z_0)g(z)}{h(z)} \overset{LH}{=} \frac{g(z)+(z-z_0)g'(z)}{h'(z)} \xrightarrow{z\to z_0} \frac{g(z_0)}{h'(z_0)}.$$

■

Example 6.3.5 ***(Residue of a simple pole (order 1)):*** Let $f(z) = \dfrac{1}{1+z^2}$, then $g(z) = 1, h(z) = 1+z^2$, and $h'(z) = 2z$ so that $h'(i) = 2i \neq 0$. Thus

$$\operatorname*{Res}_{z=i} \frac{1}{1+z^2} = \frac{1}{2i}.$$

6.3.1 Exercises

Some good computations here.

Exercise 6.3.6
Show that the complex zeros of $f(z) := \sin(\pi z)$ are exactly $\mathbb{Z}$, and each is order 1. Calculate the residue of $1/\sin(\pi x)$ at $z = n \in \mathbb{Z}$.

Exerci

Exercise 3.A. [SSh03, 3.1] *Show that the complex zeros of $\sin \pi z$ are exactly at the integers, and are each of order 1. Calculate the residue of $1/\sin \pi x$ are $z = n \in \mathbb{Z}$.*

Exercise 3.C. [SSh03, 3.8] *Prove that*

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Exe

Exercise 6.3.7 (?)

$$\int_{\mathbb{R}} \frac{1}{(1+x^2)^2}\,dx.$$

Solution:

- Factor $(1+z^2)^2 = ((z-i)(z+i))^2$, so f has poles at $\pm i$ of order 2.
- Take a semicircular contour $\gamma \coloneqq I_R \cup D_R$, then $f(z) \approx 1/z^4 \to 0$ for large R and $\displaystyle\int_{D_R} f \to 0$.
- Note $\displaystyle\int_{I_R} f \to \int_{\mathbb{R}} f$, so $\displaystyle\int_{\gamma} f \to \int_{\mathbb{R}} f$.
- $\displaystyle\int_{\gamma} f = 2\pi i \sum_{z_0} \operatorname*{Res}_{z=z_0} f$, and $z_0 = i$ is the only pole in this region.
- Compute

$$\begin{aligned}
\operatorname*{Res}_{z=i} f &= \lim_{z\to i} \frac{1}{(2-1)!}\frac{\partial}{\partial z}(z-i)^2 f(z) \\
&= \lim_{z\to i} \frac{\partial}{\partial z}\frac{1}{(z+i)^2} \\
&= \lim_{z\to i} \frac{-2}{(z+i)^3} \\
&= -\frac{2}{(2i)^3} \\
&= \frac{1}{4i}
\end{aligned}$$

$$\implies \int_{\gamma} f = \frac{2\pi i}{4i} = \pi/2,$$

Exercise 6.3.8 (?)
Use a direct Laurent expansion to show

$$\operatorname*{Res}_{z=0} \frac{1}{z - \sin(z)} = \frac{3!}{5\cdot 4}.$$

Note the necessity: one doesn't know the order of the pole at zero, so it's unclear how many derivatives to take.

Solution:
Expand:

$$\begin{aligned}
\frac{1}{z-\sin(z)} &= z^{-1}\left(1-z^{-1}\sin(z)\right)^{-1} \\
&= z^{-1}\left(1-z^{-1}\left(z-\frac{1}{3!}z^3+\frac{1}{5!}z^5-\cdots\right)\right)^{-1} \\
&= z^{-1}\left(1-\left(1-\frac{1}{3!}z^2+\frac{1}{5!}z^4-\cdots\right)\right)^{-1} \\
&= z^{-1}\left(\frac{1}{3!}z^2-\frac{1}{5!}z^4+\cdots\right)^{-1} \\
&= z^{-1}\cdot 3!z^{-2}\left(1-\frac{1}{5!/3!}z^2+\cdots\right)^{-1} \\
&= \frac{3!}{z^3}\left(\frac{1}{1-\left(\frac{1}{5\cdot 4}z^2+\cdots\right)}\right) \\
&= \frac{3!}{z^3}\left(1+\left(\frac{1}{5\cdot 4}z^2\right)+\left(\frac{1}{5\cdot 4}z^2\right)^2+\cdots\right) \\
&= 3!z^{-3}+\frac{3!}{5\cdot 4}z^{-1}+O(z)
\end{aligned}$$

.

Exercise 6.3.9 (?)
Compute

$$\operatorname*{Res}_{z=0}\frac{1}{z^2\sin(z)}.$$

Solution:
First expand $(\sin(z))^{-1}$:

$$\begin{aligned}
\frac{1}{\sin(z)} &= \left(z-\frac{1}{3!}z^3+\frac{1}{5!}z^5-\cdots\right)^{-1} \\
&= z^{-1}\left(1-\frac{1}{3!}z^2+\frac{1}{5!}z^4-\cdots\right)^{-1} \\
&= z^{-1}\left(1+\left(\frac{1}{3!}z^2-\frac{1}{5!}z^4+\cdots\right)+\left(\frac{1}{3!}z^2-\cdots\right)^2+\cdots\right) \\
&= z^{-1}\left(1+\frac{1}{3!}z^2\pm O(z^4)\right),
\end{aligned}$$

using that $(1-x)^{-1}=1+x+x^2+\cdots$.
Thus

$$\begin{aligned}
z^{-2}\left(\sin(z)\right)^{-1} &= z^{-2}\cdot z^{-1}\left(1+\frac{1}{3!}z^2\pm O(z^4)\right) \\
&= z^{-3}+\frac{1}{3!}z^{-1}+O(z).
\end{aligned}$$

Exercise 6.3.10 (Keyhole contour and ML estimate)
Compute

$$\int_{[0,\infty]} \frac{\log(x)}{(1+x^2)^2}\, dx.$$

Solution:
Factor $(1+z^2)^2 = (z+i^2(z-i)^2$. Take a keyhole contour similar to the following:

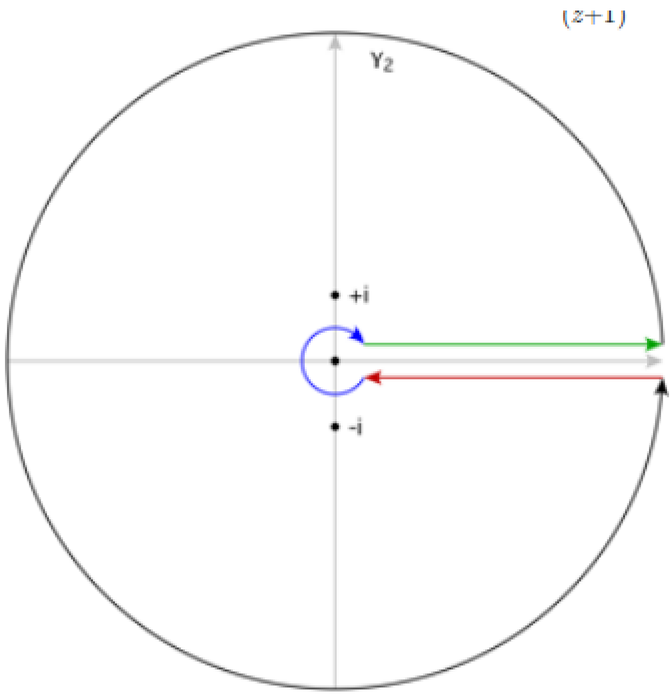


Figure 6: image_2021-06-09-02-11-59

Show that outer radius R and inner radius ρ circles contribute zero in the limit by the ML estimate? Compute the residues by just applying the formula and manually computing derivatives:

$$\begin{aligned}
\operatorname*{Res}_{z=\pm i} f(z) &= \lim_{z\to\pm i} \frac{\partial}{\partial z} \frac{\log^2(z)}{(z\pm i)^2} \\
&= \lim_{z\to\pm i} \frac{2\log(z)(z\pm i)^2 - 2(z\pm i)^2 \log^2(z)}{((z\pm i)^2)^2} \\
&= \frac{2\log(\pm i)(\pm 2i)^2 - 2(\pm 2i)^2 \log^2(\pm i)}{(\pm 2i)^4} \\
&=_? \frac{\pi}{4} \pm \frac{i\pi^2}{16}.
\end{aligned}$$

See p.4: http://www.math.toronto.edu/mmica/complex1.pdf

Exercise 6.3.11 (Sinc Function)
Show

$$\int_{(0,\infty)} \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}.$$

Solution:
Take an indented semicircle. Let I be the original integral, then

$$I = \frac{1}{2i}\int_{\mathbb{R}} \frac{e^{iz}-1}{z}\,dz.$$

6.4 Argument Principle

Definition 6.4.1 (Winding Number)
For $\gamma \subseteq \Omega$ a closed curve not passing through a point z_0, the **winding number of** γ **about** z_0 is defined as

$$n_\gamma(z_0) := \frac{1}{2\pi i}\int_\gamma \frac{1}{\xi - z_0}\,d\xi.$$

Theorem 6.4.2 ***(Argument Principle).***
For f meromorphic in γ° with zeros $\{z_j\}$ and poles $\{p_k\}$ repeated with multiplicity where γ does not intersect any zeros or poles, then

$$\Delta_\gamma \arg f(z) := \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_j n_\gamma(z_j) - \sum_k n_\gamma(p_k) = Z_f - P_f,$$

where Z_f and P_f are the number of zeros and poles respectively enclosed by γ, counted with multiplicity.

Proof (?).
Residue formula applied to $\frac{f'}{f}$? ∎

Corollary 6.4.3 ***(Rouché's Theorem).***
If f, g are analytic on a domain Ω with finitely many zeros in Ω and $\gamma \subset \Omega$ is a closed curve surrounding each point exactly once, where $|g| < |f|$ on γ, then f and $f+g$ have the same number of zeros.
Alternatively:
Suppose $f = g + h$ with $g \neq 0, \infty$ on γ with $|g| > |h|$ on γ. Then

$$\Delta_\gamma \arg(f) = \Delta_\gamma \arg(h) \quad \text{and } Z_f - P_f = Z_g - P_g.$$

Prove

Corollary 6.4.4 ***(Open Mapping).***
Any holomorphic non-constant map is an open map.

Prove

Corollary 6.4.5 ***(Maximum Modulus).***
If f is holomorphic and nonconstant on an open connected region Ω, then $|f|$ can not attain a maximum on Ω. If Ω is bounded and f is continuous on $\bar{\Omega}$, then $\max_{\bar{\Omega}} |f|$ occurs on $\partial\Omega$.
Conversely, if f attains a local supremum at $z_0 \in \Omega$, then f is constant on Ω.

Prove

Corollary 6.4.6 ***(?).***
If f is nonzero on Ω, then f attains a minimum on $\partial\Omega$. This follows from applying the MMP to $1/f$.

6.4.1 Exercises

Exercise 3.E. [SSh03, 3.14] *Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$.*

Figure 7: image_2021-05-17-13-33-55

7 Rouche

7.1 Counting Zeros

Example 7.1.1:

- Take $P(z) = z^4 + 6z + 3$.
- On $|z| < 2$:
  - Set $f(z) = z^4$ and $g(z) = 6z + 3$, then $|g(z)| \leq 6|z| + 3 = 15 < 16 = |f(z)|$.
  - So P has 4 zeros here.
- On $|z| < 1$:
  - Set $f(z) = 6z$ and $g(z) = z^4 + 3$.
  - Check $|g(z)| \leq |z|^4 + 3 = 4 < 6 = |f(z)|$.

– So P has 1 zero here.

Example 7.1.2:

- Claim: the equation $\alpha z e^z = 1$ where $|\alpha| > e$ has exactly one solution in $\mathbb{D}$.
- Set $f(z) = \alpha z$ and $g(z) = e^{-z}$.
- Estimate at $|z| = 1$ we have $|g| = |e^{-z}| = e^{-\Re(z)} \leq e^1 < |\alpha| = |f(z)|$
- f has one zero at $z_0 = 0$, thus so does $f + g$.

8 Conformal Maps

8.1 Linear Fractional Transformations

Definition 8.1.1 (Conformal Map / Biholomorphism)
A map f is **conformal** on Ω iff f is complex-differentiable, $f'(z) \neq 0$ for $z \in \Omega$, and f preserves signed angles (so f is orientation-preserving). Conformal implies holomorphic, and a bijective conformal map has a holomorphic inverse. A bijective conformal map $f : U \to V$ is called a **biholomorphism**, and we say U and V are **biholomorphic**. Self-biholomorphisms of a domain Ω form a group $\mathrm{Aut}_{\mathbb{C}}(\Omega)$.

Remark 8.1.2: There is an oft-used weaker condition that $f'(z) \neq 0$ for any point. Note that that this condition alone doesn't necessarily imply f is holomorphic, since anti-holomorphic maps may be nonzero derivative. For example, take $f(z) = \bar{z}$, so $f(x + iy) = x - iy$ – this does not satisfy the Cauchy-Riemann equations.

Remark 8.1.3: A bijective holomorphic map automatically has a holomorphic inverse. This can be weakened: an injective holomorphic map satisfies $f'(z) \neq 0$ and f^{-1} is well-defined on its range and holomorphic.

Definition 8.1.4 (Linear fractional transformation / Mobius transformation)
A map of the following form is a *linear fractional transformation*:

$$T(z) = \frac{az + b}{cz + d},$$

where the denominator is assumed to not be a multiple of the numerator. These have inverses given by

$$T^{-1}(w) = \frac{dw - b}{-cw + a}.$$

Proposition 8.1.5 *(?).*
Given any three points z_1, z_2, z_3, the following Mobius transformation sends them to $1, 0, \infty$

respectively:

$$f(z) := \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}.$$

Such a map is sometimes denoted (z, z_1, z_2, z_3).

Example 8.1.6 ***(?):***

- $(z, i, 1, -1) : \mathbb{D} \to \mathbb{H}$
- $(z, 0, -1, 1) : \mathbb{D} \cap \mathbb{H} \to Q_1$.

Theorem 8.1.7 ***(Cayley Transform).***

The fractional linear transformation given by $F(z) = \frac{i - z}{i + z}$ maps $\mathbb{D} \to \mathbb{H}$ with inverse $G(w) = i\frac{1 - w}{1 + w}$.

Theorem 8.1.8 ***(Classification of Conformal Maps).***

There are 8 major types of conformal maps:

Type/Domains	Formula
Translation/Dilation/Rotation	$z \mapsto e^{i\theta}(cz + h)$
Sectors to sectors	$z \mapsto z^n$
$\mathbb{D}_{\frac{1}{2}} \to \mathbb{H}_{\frac{1}{2}}$, the first quadrant	$z \mapsto \frac{1 + z}{1 - z}$
$\mathbb{H} \to S$	$z \mapsto \log(z)$
$\mathbb{D}_{\frac{1}{2}} \to S_{\frac{1}{2}}$	$z \mapsto \log(z)$
$S_{\frac{1}{2}} \to \mathbb{D}_{\frac{1}{2}}$	$z \mapsto e^{iz}$
$\mathbb{D}_{\frac{1}{2}} \to \mathbb{H}$	$z \mapsto \frac{1}{2}\left(z + \frac{1}{z}\right)$
$S_{\frac{1}{2}} \to \mathbb{H}$	$z \mapsto \sin(z)$

Theorem 8.1.9 ***(Characterization of conformal maps).***

Conformal maps $\mathbb{D} \to \mathbb{D}$ have the form

$$g(z) = \lambda \frac{1 - a}{1 - \bar{a}z}, \quad |a| < 1, \quad |\lambda| = 1.$$

8.2 Schwarz

Theorem 8.2.1 ***(Schwarz Lemma).***

If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic with $f(0) = 0$, then

1. $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$

2. $|f'(0)| \leq 1$.

Moreover, if

- $|f(z_0)| = |z_0|$ for any $z_0 \in \mathbb{D}$, or
- $|f'(0)| = 1$,

then f is a rotation.

Proof (?).
Apply the maximum modulus principle to $f(z)/z$. ■

Exercise 8.2.2 (?)
Show that $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}) = \left\{ z \mapsto az + b \;\middle|\; a \in \mathbb{C}^\times, b \in \mathbb{C} \right\}$.

Theorem 8.2.3 ***(Biholomorphisms of the disc).***

$$\mathrm{Aut}_{\mathbb{C}}(\mathbb{D}) = \left\{ z \mapsto e^{i\theta} \left(\frac{\alpha - z}{1 - \bar{\alpha} z} \right) \right\}.$$

Proof (?).
Schwarz lemma. ■

Theorem 8.2.4 ***(?).***

$$\mathrm{Aut}_{\mathbb{C}}(\mathbb{H}) = \left\{ z \mapsto \frac{az + b}{cz + d} \;\middle|\; a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\} \cong \mathrm{PSL}_2(\mathbb{R}).$$

8.3 By Type

8.3.1 Plane to Disc

$$\begin{aligned} &\varphi : \mathbb{H} \to \mathbb{D} \\ &\varphi(z) = \frac{z - i}{z + i} \qquad f^{-1}(z) = i \left(\frac{1 + w}{1 - w} \right). \end{aligned}$$

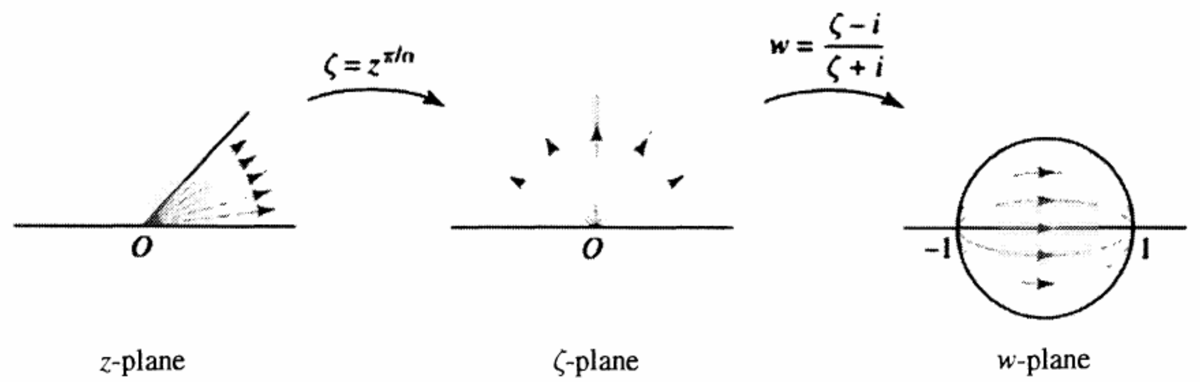
8.3.2 Sector to Disc

For $S_\alpha := \left\{ z \in \mathbb{C} \,\middle|\, 0 < \arg(z) < \alpha \right\}$ an open sector for α some angle, first map the sector to the half-plane:

$$\begin{aligned} g &: S_\alpha \to \mathbb{H} \\ g(z) &= z^{\frac{\pi}{\alpha}}. \end{aligned}$$

Then compose with a map $\mathbb{H} \to \mathbb{D}$:

$$\begin{aligned} f &: S_\alpha \to \mathbb{D} \\ f(z) &= (\varphi \circ g)(z) = \frac{z^{\frac{\pi}{\alpha}} - i}{z^{\frac{\pi}{\alpha}} + i}. \end{aligned}$$



8.3.3 Strip to Disc

- Map to horizontal strip by rotation $z \mapsto \lambda z$.
- Map horizontal strip to sector by $z \mapsto e^z$
- Map sector to $\mathbb{H}$ by $z \mapsto z^{\frac{\pi}{\alpha}}$.
- Map $\mathbb{H} \to \mathbb{D}$.

$$e^z : \mathbb{R} \times (0, \pi) \to \mathbb{R} \times (0, \infty).$$

9 | Schwarz Reflection

?

10 Schwarz Lemma

?

Montel's theorem

Normal families

Schwarz lemma

Equicontinuity

11 Linear Fractional Transformations

12 Montel's Theorem

13 Unsorted Theorems

Theorem 13.0.1 ***(Riemann Mapping).***
If Ω is simply connected, nonempty, and not $\mathbb{C}$, then for every $z_0 \in \Omega$ there exists a unique conformal map $F : \Omega \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$.
Thus any two such sets Ω_1, Ω_2 are conformally equivalent.

Theorem 13.0.2 ***(Riemann's Removable Singularity Theorem).***
If f is holomorphic on Ω except possibly at z_0 and f is bounded on $\Omega \setminus \{z_0\}$, then z_0 is a removable singularity.

Theorem 13.0.3 ***(Little Picard).***
If $f : \mathbb{C} \to \mathbb{C}$ is entire and nonconstant, then $\operatorname{im}(f)$ is either $\mathbb{C}$ or $\mathbb{C} \setminus \{z_0\}$ for some point z_0.

???

Corollary 13.0.4.
The ring of holomorphic functions on a domain in $\mathbb{C}$ has no zero divisors.

Proof .
???
∎

Find the proof!

Morera

Proposition 13.0.5 ***(Bounded Complex Analytic Functions form a Banach Space).***
For $\Omega \subseteq \mathbb{C}$, show that $A(\mathbb{C}) := \left\{ f : \Omega \to \mathbb{C} \;\middle|\; f \text{ is bounded} \right\}$ is a Banach space.

Proof .
?

Apply Morera's Theorem and Cauchy's Theorem

■

14 Appendix: Proofs of the Fundamental Theorem of Algebra

14.0.1 Fundamental Theorem of Algebra: Argument Principle

- Let $P(z) = a_n z^n + \cdots + a_0$ and $g(z) = P'(z)/P(z)$, note P is holomorphic
- Since $\lim_{|z| \to \infty} P(z) = \infty$, there exist an $R > 0$ such that P has no roots in $\{|z| \geq R\}$.
- Apply the argument principle:

$$N(0) = \frac{1}{2\pi i} \oint_{|\xi|=R} g(\xi) \, d\xi.$$

- Check that $\lim_{|z \to \infty|} z g(z) = n$, so g has a simple pole at ∞
- Then g has a Laurent series $\frac{n}{z} + \frac{c_2}{z^2} + \cdots$
- Integrate term-by-term to get $N(0) = n$.

14.0.2 Fundamental Theorem of Algebra: Rouche's Theorem

- Let $P(z) = a_n z^n + \cdots + a_0$
- Set $f(z) = a_n z^n$ and $g(z) = P(z) - f(z) = a_{n-1} z^{n-1} + \cdots + a_0$, so $f + g = P$.
- Choose $R > \max\left(\frac{|a_{n-1}| + \cdots + |a_0|}{|a_n|}, 1 \right)$, then

$$\begin{aligned}
|g(z)| &\coloneqq |a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| \\
&\leq |a_{n-1}z^{n-1}| + \cdots + |a_1 z| + |a_0| \quad \text{by the triangle inequality} \\
&= |a_{n-1}| \cdot |z^{n-1}| + \cdots + |a_1| \cdot |z| + |a_0| \\
&= |a_{n-1}| \cdot R^{n-1} + \cdots + |a_1| R + |a_0| \\
&\leq |a_{n-1}| \cdot R^{n-1} + |a_{n-2}| \cdot R^{n-1} + \cdots + |a_1| \cdot R^{n-1} + |a_0| \cdot R^{n-1} \quad \text{since } R > 1 \implies R^{a+b} \geq R^a \\
&= R^{n-1}\left(|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0|\right) \\
&\leq R^{n-1}\left(|a_n| \cdot R\right) \quad \text{by choice of } R \\
&= R^n |a_n| \\
&= |a_n z^n| \\
&\coloneqq |f(z)|
\end{aligned}$$

- Then $a_n z^n$ has n zeros in $|z| < R$, so $f + g$ also has n zeros.

14.0.3 Fundamental Theorem of Algebra: Liouville's Theorem

- Suppose p is nonconstant and has no roots, then $\frac{1}{p}$ is entire. We will show it is also bounded and thus constant, a contradiction.
- Write $p(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right)$
- Outside a disc:
  - Note that $p(z) \overset{z\to\infty}{\to} \infty$. so there exists an R large enough such that $|p(z)| \geq \frac{1}{A}$ for any fixed chosen constant A.
  - Then $|1/p(z)| \leq A$ outside of $|z| > R$, i.e. $1/p(z)$ is bounded there.
- Inside a disc:
  - p is continuous with no roots and thus must be bounded below on $|z| < R$.
  - p is entire and thus continuous, and since $\overline{D}_r(0)$ is a compact set, p achieves a min A there
  - Set $C \coloneqq \min(A, B)$, then $|p(z)| \geq C$ on all of $\mathbb{C}$ and thus $|1/p(z)| \leq C$ everywhere.
  - So $1/p(z)$ is bounded an entire and thus constant by Liouville's theorem – but this forces p to be constant. $\lightning$

14.0.4 Fundamental Theorem of Algebra: Open Mapping Theorem

- p induces a continuous map $\mathbb{CP}^1 \to \mathbb{CP}^1$
- The continuous image of compact space is compact;
- Since the codomain is Hausdorff space, the image is closed.
- p is holomorphic and non-constant, so by the Open Mapping Theorem, the image is open.
- Thus the image is clopen in $\mathbb{CP}^1$.

- The image is nonempty, since $p(1) = \sum a_i \in \mathbb{C}$
- $\mathbb{CP}^1$ is connected
- But the only nonempty clopen subset of a connected space is the entire space.
- So p is surjective, and $p^{-1}(0)$ is nonempty.
- So p has a root.

15 | Appendix

15.1 Misc Basic Algebra

Fact 15.1.1 (Standard forms of conic sections)

- Circle: $x^2 + y^2 = r^2$
- Ellipse: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$
- Hyperbola: $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$
  - Rectangular Hyperbola: $xy = \frac{c^2}{2}$.
- Parabola: $-4ax + y^2 = 0$.

Mnemonic: Write $f(x, y) = Ax^2 + Bxy + Cy^2 + \cdots$, then consider the discriminant $\Delta = B^2 - 4AC$:

- $\Delta < 0 \iff$ ellipse
  - $\Delta < 0$ and $A = C, B = 0 \iff$ circle
- $\Delta = 0 \iff$ parabola
- $\Delta > 0 \iff$ hyperbola

Fact 15.1.2 (Completing the square)

$$
\begin{aligned}
x^2 - bx = (x - s)^2 - s^2 \quad \text{where} s = \frac{b}{2} \\
x^2 + bx = (x + s)^2 - s^2 \quad \text{where} s = \frac{b}{2}.
\end{aligned}
$$

Fact 15.1.3
The sum of the interior angles of an n-gon is $(n-2)\pi$, where each angle is $\frac{n-2}{n}\pi$.

Definition 15.1.4 (The Dirichlet Problem)
Given a bounded piecewise continuous function $u : S^1 \to \mathbb{R}$, is there a unique extension to a continuous harmonic function $\tilde{u} : \mathbb{D} \to \mathbb{R}$?

Remark 15.1.5: More generally, this is a boundary value problem for a region where the *values* of the function on the boundary are given. Compare to prescribing conditions on the normal vector on the boundary, which would be a Neumann BVP. Why these show up: a harmonic function on a simply connected region has a harmonic conjugate, and solutions of BVPs are always analytic functions with harmonic real/imaginary parts.

Example 15.1.6 ***(Dirichlet problem on the strip):*** See section 27, example 1 in Brown and Churchill. On the strip $(x, y) \in (0, \pi) \times (0, \infty)$, set up the BVP for temperature on a thin plate with no sinks/sources:

$$\Delta T = 0 \qquad T(0, y) = 0,\ T(\pi, y) = 0\ \forall y$$

$$T(x, 0) = \sin(x) \qquad T(x, y) \overset{y\to\infty}{\longrightarrow} 0.$$

Then the following function is harmonic on $\mathbb{R}^2$ and satisfies that Dirichlet problem:

$$T(x, y) = e^{-y}\sin(x) = \Re(-ie^{iz}) = \Im(e^{iz}).$$

Definition 15.1.7 (Logarithmic Derivative)
The **logarithmic derivative** of f is $(\ln f)' = f'/f$.

Remark 15.1.8: Why this is useful: deriving the argument principle. If f has a pole of order n at z_0, then write $f(z) = (z - z_0)^{-n} g(z)$ with g analytic in a neighborhood of z_0. Then a direct computation of the derivatives will show

$$(\ln f)' := \frac{f'(z)}{f(z)} = -\frac{n}{z - z_0} + \frac{g'(z)}{g(z)}.$$

Thus

$$\frac{1}{2\pi i}\int_\gamma (\ln f)' = -n,$$

for γ a small circle about z_0. A similar argument for z_0 a **zero** of f yields

$$\frac{1}{2\pi i}\int_\gamma h = +n.$$

Exercise 15.1.9 (?)
Show that there is no continuous square root function defined on all of $\mathbb{C}$.

Solution:
Suppose $f(z)^2 = z$. Then f is a section to the covering map

$$\begin{aligned} p : \mathbb{C}^\times &\to \mathbb{C}^\times \\ z &\mapsto z^2, \end{aligned}$$

so $p \circ f = \mathrm{id}$. Using $\pi_1(\mathbb{C}^\times) = \mathbb{Z}$, the induced maps are $p_*(1) = 2$ and $f_*(1) = n$ for some $n \in \mathbb{Z}$. But then $p_* \circ f_*$ is multiplication by $2n$, contradicting $p_* \circ f_* = \mathrm{id}$ by functoriality.

Theorem 15.1.10 ***(Uniformization).***
Every Riemann surface S is the quotient of a free proper holomorphic action of a group G on the universal cover $\tilde{S}$ of S, so $S \cong \tilde{S}/G$ is a biholomorphism. Moreover, $\tilde{S}$ is biholomorphic to either

- $\mathbb{CP}^1$
- $\mathbb{C}$
- $\mathbb{D}$

Basics

- Show that $\dfrac{1}{z}\displaystyle\sum_{k=1}^{\infty} \dfrac{z^k}{k}$ converges on $S^1 \setminus \{1\}$ using summation by parts.
- Show that any power series is continuous on its domain of convergence.
- Show that a uniform limit of continuous functions is continuous.

??

- Show that if f is holomorphic on $\mathbb{D}$ then f has a power series expansion that converges uniformly on every compact $K \subset \mathbb{D}$.

- Show that any holomorphic function f can be uniformly approximated by polynomials.

- Show that if f is holomorphic on a connected region Ω and $f' \equiv 0$ on Ω, then f is constant on Ω.

- Show that if $|f| = 0$ on $\partial\Omega$ then either f is constant or f has a zero in Ω.

- Show that if $\{f_n\}$ is a sequence of holomorphic functions converging uniformly to a function f on every compact subset of Ω, then f is holomorphic on Ω and $\{f_n'\}$ converges uniformly to f' on every such compact subset.

- Show that if each f_n is holomorphic on Ω and $F := \sum f_n$ converges uniformly on every compact subset of Ω, then F is holomorphic.

- Show that if f is once complex differentiable at each point of Ω, then f is holomorphic.

16 | Draft of Problem Book

- Prove the triangle inequality
- Prove the reverse triangle inequality
- Show that $\sum z^{k-1}/k$ converges for all $z \in S^1$ except $z = 1$.
- What is an example of a noncontinuous limit of continuous functions?
- Show that the uniform limit of continuous functions is continuous.
- Show that f is holomorphic if and only if $\bar{\partial} f = 0$.
- Show $n^{\frac{1}{n}} \stackrel{n \to \infty}{\to} 1$.
- Show that if f is holomorphic with $f' = 0$ on Ω then f is constant.
- Show that holomorphic implies analytic.
- Use Cauchy's inequality to prove Liouville's theorem

Problem 16.0.1 (?)
What is a pair of conformal equivalences between $\mathbb{H}$ and $\mathbb{D}$?

Solution:

$$
\begin{aligned}
F : HH &\to \mathbb{D} \\
z &\mapsto \frac{i - z}{i + z}
\end{aligned}
$$

$$
\begin{aligned}
G : \mathbb{D} &\to \mathbb{H} \\
w &\mapsto i\frac{1 - w}{1 + w}.
\end{aligned}
$$

Mnemonic: any point in $\mathbb{H}$ is closer to i than $-i$, so $|F(z)| < 1$.

- Maps $\mathbb{R} \to S^1 \setminus \{-1\}$.

Problem 16.0.2 (?)
What is conformal equivalence $\mathbb{H} \rightleftharpoons S := \left\{ w \in \mathbb{C} \;\middle|\; 0 < \arg(w) < \alpha\pi \right\}$?

Solution:

$$
f(z) = z^{\alpha}.
$$