QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 16 2008 (Day 1)

1. (a) Prove that the Galois group G of the polynomial $X^6 + 3$ over $\mathbb{Q}$ is of order 6.

(b) Show that in fact G is isomorphic to the symmetric group S_3.

(c) Is there a prime number p such that $X^6 + 3$ is irreducible over the finite field of order p?

Solution. We initially work over any field k in which the polynomial $X^6 + 3$ is irreducible. Clearly k cannot have characteristic 2 or 3. Let α be a root of X^6+3 in an algebraic closure $\bar{k}$ of k, and set $\omega = (-1+\alpha^3)/2$. Then a simple calculation gives $\omega^2 + \omega + 1 = 0$, so $\omega^3 = 1$ but $\omega \neq 1$. In fact, 1, ω, ω^2, -1, $-\omega$, $-\omega^2$ are all distinct elements of $\bar{k}$; they are the six roots of $X^6 + 1 = 0$, so α, $\omega\alpha$, $\omega^2\alpha$, $-\alpha$, $-\omega\alpha$, $-\omega^2\alpha$ are the six roots of $X^6 + 3 = 0$. These roots all lie in the extension $k(\alpha)$, which has degree 6 because α is a root of an irreducible degree 6 polynomial. So the Galois group of $X^6 + 3$ over k is of order 6.

The Galois group acts transitively on the roots of the polynomial $X^6 + 3$, so there are elements σ and τ of the Galois group sending α to $\omega\alpha$ and $-\alpha$ respectively. Then

$$\sigma(\omega) = \frac{-1 + \sigma(\alpha)^3}{2} = \frac{-1 + (\omega\alpha)^3}{2} = \frac{-1 + \alpha^3}{2} = \omega$$

and

$$\tau(\omega) = \frac{-1 + \tau(\alpha)^3}{2} = \frac{-1 + (-\alpha)^3}{2} = \frac{-1 - \alpha^3}{2} = -1 - \omega = \omega^2.$$

Therefore $\tau(\sigma(\alpha)) = \tau(\omega\alpha) = -\omega^2\alpha$ while $\sigma(\tau(\alpha)) = \sigma(-\alpha) = -\omega\alpha$, so σ and τ do not commute. So G is a nonabelian group of order 6, and thus must be isomorphic to the symmetric group S_3.

We now finish the problem.

(a) The polynomial $X^6 + 3$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion at the prime 3. So the preceding arguments show that the Galois group of $X^6 + 3$ over $\mathbb{Q}$ is of order 6.

(b) Similarly, we also showed under the same assumption that the Galois group was isomorphic to S_3.

(c) No, there is no prime p such that X^6+3 is irreducible over the finite field of order p. If there was, then by the preceding arguments, the extension formed by adjoining a root of X^6+3 would be a Galois extension with Galois group S_3. But the Galois groups of finite extensions of the field of order p are all cyclic groups, a contradiction.

2. Evaluate the integral

$$\int_0^\infty \frac{\sqrt{t}}{(1+t)^2}dt.$$

Solution. Write $\sqrt{z}$ for the branch of the square root function defined on $\mathbb{C}-[0,\infty)$ such that $\sqrt{z}$ has positive real part when $z=r+\epsilon i$, ϵ small and positive. Using the identity $(\sqrt{z})^2=z$ one can check that $\frac{d\sqrt{z}}{dz}=\frac{1}{2\sqrt{z}}$.

Define the meromorphic function f on $\mathbb{C}-[0,\infty)$ by $f(z)=\sqrt{z}/(1+z)^2$. Let $\epsilon>0$ be small and R large, and let γ be the contour which starts at ϵi, travels along the ray $z=[0,\infty)+\epsilon i$ until it reaches the circle $|z|=R$, traverses most of that circle counterclockwise stopping at the ray $z=[0,\infty)-\epsilon i$, then travels along that ray backwards, and finally traverses the semicircle $|z|=\epsilon$ in the left half-plane to get back to ϵi. Consider the contour integral $\int_\gamma f(z)\,dz$. The contribution from the first ray is approximately the desired integral $I=\int_0^\infty \sqrt{t}/(1+t^2)\,dt$; the contribution from the large circle is small, because when $|z|=R$, $|\sqrt{z}/(1+z)^2|$ is about $R^{-3/2}$, and the perimeter of the circle is only about $2\pi R$; the contribution from the second ray is about I again, because the sign from traveling in the opposite direction cancels the sign coming from the branch cut in $\sqrt{z}$; and the contribution from the small circle is small because $f(z)$ is bounded in a neighborhood of 0. So

$$2I=\lim_{\epsilon\to 0,R\to\infty}\int_\gamma \frac{\sqrt{z}}{(1+z)^2}\,dz=2\pi i\left.\frac{d\sqrt{z}}{dz}\right|_{z=-1}=2\pi i\frac{1}{2\sqrt{-1}}=\pi$$

and thus $I=\pi$.

3. For $X\subset\mathbb{R}^3$ a smooth oriented surface, we define the *Gauss map* $g:X\to S^2$ to be the map sending each point $p\in X$ to the unit normal vector to X at p. We say that a point $p\in X$ is *parabolic* if the differential $dg_p:T_p(X)\to T_{g(p)}(S^2)$ of the map g at p is singular.

(a) Find an example of a surface X such that every point of X is parabolic.

(b) Suppose now that the locus of parabolic points is a smooth curve $C\subset X$, and that at every point $p\in C$ the tangent line $T_p(C)\subset T_p(X)$ coincides with the kernel of the map dg_p. Show that C is a planar curve, that is, each connected component lies entirely in some plane in $\mathbb{R}^3$.

Solution.

(a) Let X be the xy-plane; then the Gauss map $g : X \to S^2$ is constant, so its differential is everywhere zero and hence singular.

(b) Consider the Gauss map of X restricted to C, $g|_C : C \to S^2$. Then for any point $p \in C$, $d(g|_C)_p = (dg_p)|_{T_p(C)}$, which is 0 by assumption. Hence $g|_C$ is locally constant on C. That is, on each connected component C_0 of C there is a fixed vector (the value of $g|_C$ at any point of the component) normal to all of C_0. Hence C_0 lies in a plane in $\mathbb{R}^3$ normal to this vector.

4. Let $X = (S^1 \times S^1) \setminus \{p\}$ be a once-punctured torus.

(a) How many connected, 3-sheeted covering spaces $f : Y \to X$ are there?

(b) Show that for any of these covering spaces, Y is either a 3-times punctured torus or a once-punctured surface of genus 2.

Solution.

(a) By covering space theory, the number of connected, 3-sheeted covering spaces of a space Z is the number of conjugacy classes of subgroups of index 3 in the fundamental group $\pi_1(Z)$. (We consider two covering spaces of Z isomorphic only when they are related by an homeomorphism over the identity on Z, not one over any homeomorphism of Z.) So we may replace X by the homotopy equivalent space $X' = S^1 \vee S^1$. If we view this new space X' as a graph with one vertex and two directed loops labeled a and b, then a connected 3-sheeted cover of X' is a connected graph with three vertices and some directed edges labeled a or b such that each vertex has exactly one incoming and one outgoing edge with each of the labels a and b. Temporarily treating the three vertices as having distinct labels x, y, z, we find six ways the a edges can be placed: loops at x, y and z; a loop at x and edges from y to z and from z to y; similarly but with the loop at y; similarly but with the loop at z; edges from x to y, y to z, and z to x; and edges from x to z, z to y, and y to x. Analogously there are six possible placements for the b edges. Considering all possible combinations, throwing out the disconnected ones, and then treating two graphs as the same if they differ only in the labels x, y, z, we arrive at seven distinct possibilities.

(b) Let C be a small loop in $S^1 \times S^1$ around the removed point p, and let $X_0 \subset X$ be the torus with the interior of C removed, so that X_0 is a compact manifold with boundary $C = S^1$. Now let Y be any connected, 3-sheeted covering space of X. Pull back the covering map $Y \to X$ along the inclusion $X_0 \to X$ to obtain a 3-sheeted covering space Y_0 of X_0. Since $X_0 \to X$ is a homotopy equivalence, so is $Y_0 \to Y$ and in particular Y_0 is still connected. We can recover Y from Y_0 by gluing a strip $D \times [0, 1)$ along the preimage D of C in Y_0. So, it will suffice to show that Y is either a torus with three small disks removed, or a surface of genus two with one small disk removed.

Since Y_0 is a 3-sheeted cover of X_0, it is a compact oriented surface with boundary. By the classification of compact oriented surfaces with boundary, Y_0 can be formed by taking a surface of some genus g and removing some number d of small disks. The boundary of Y_0 is D, the preimage of C, which is a (not necessarily connected) 3-sheeted cover of C. So Y_0 has either one or three boundary circles, i.e., $d = 1$ or $d = 3$. Moreover, we can compute using the Euler characteristic that

$$2 - 2g - d = \chi(Y_0) = 3\chi(X_0) = -3.$$

If $d = 3$, then $g = 1$; if $d = 1$, then $g = 2$. So Y is correspondingly either a 3-times punctured torus or a once-punctured surface of genus two.

5. Let X be a complete metric space with metric ρ. A map $f : X \to X$ is said to be *contracting* if for any two distinct points $x, y \in X$,

$$\rho(f(x), f(y)) < \rho(x, y).$$

The map f is said to be *uniformly contracting* if there exists a constant $c < 1$ such that for any two distinct points $x, y \in X$,

$$\rho(f(x), f(y)) < c \cdot \rho(x, y).$$

(a) Suppose that f is uniformly contracting. Prove that there exists a unique point $x \in X$ such that $f(x) = x$.

(b) Give an example of a contracting map $f : [0, \infty) \to [0, \infty)$ such that $f(x) \neq x$ for all $x \in [0, \infty)$.

Solution.

(a) We first show there exists at least one fixed point of f. Let $x_0 \in X$ be arbitrary and define a sequence $x_1, x_2, \ldots,$ by $x_n = f(x_{n-1})$. Let $d = \rho(x_0, x_1)$. By the uniformly contracting property of f, $\rho(x_n, x_{n+1}) \leq dc^n$ for every n. Now observe

$$\begin{aligned} \rho(x_n, x_{n+k}) &\leq \rho(x_n, x_{n+1}) + \cdots + \rho(x_{n+k-1}, x_{n+k}) \\ &\leq dc^n + \cdots + dc^{n+k-1} \\ &\leq dc^n/(1-c). \end{aligned}$$

This expression tends to 0 as n increases, so (x_n) is a Cauchy sequence and thus has a limit x by the completeness of X. Now f is continuous, because it is uniformly contracting, so

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x,$$

and x is a fixed point of f, as desired.

To show that f has at most one fixed point, suppose x and y were distinct points of X with $f(x) = x$ and $f(y) = y$. Then

$$\rho(x, y) = \rho(f(x), f(y)) < c\rho(x, y),$$

which is impossible since $\rho(x, y) > 0$ and $c < 1$.

(b) Let $f(x) = x + e^{-x}$. Then $f'(x) = 1 - e^{-x} \in [0, 1)$ for all $x \geq 0$, so by the Mean Value Theorem $0 \leq f(x) - f(y) < x - y$ for any $x > y \geq 0$. Thus f is contracting. But f has no fixed points, because $x + e^{-x/2} \neq x$ for all x.

6. Let K be an algebraically closed field of characteristic other than 2, and let $Q \subset \mathbb{P}^3$ be the surface defined by the equation

$$X^2 + Y^2 + Z^2 + W^2 = 0.$$

(a) Find equations of all lines $L \subset \mathbb{P}^3$ contained in Q.

(b) Let $\mathbb{G} = \mathbb{G}(1, 3) \subset \mathbb{P}^5$ be the Grassmannian of lines in $\mathbb{P}^3$, and $F \subset \mathbb{G}$ the set of lines contained in Q. Show that $F \subset \mathbb{G}$ is a closed subvariety.

Solution.

(a) Since K is algebraically closed and of characteristic other than 2, we may replace the quadratic form $X^2 + Y^2 + Z^2 + W^2$ with any other nondegenerate one, such as $AB + CD$. More explicitly, set $A = X + \sqrt{-1}Y$, $B = X - \sqrt{-1}Y$, $C = Z + \sqrt{-1}W$, $D = -Z + \sqrt{-1}W$; this change of coordinates is invertible because we can divide by 2, and $AB - CD = X^2 + Y^2 + Z^2 + W^2$.

A line contained in the surface in $\mathbb{P}^3$ defined by $AB - CD = 0$ is the same as a plane in the subset of the vector space K^4 defined by $v_1v_2 - v_3v_4 = 0$. Define a bilinear form $(\cdot, \cdot)$ on K^4 by $(v, w) = v_1w_2 + v_2w_1 - v_3w_4 - v_4w_3$. Then we want to find all the planes $V \subset K^4$ such that $(v, v) = 0$ for every $v \in V$. Observe that

$$(v + w, v + w) - (v, v) - (w, w) = (v, w) + (w, v) = 2(v, w),$$

so it is equivalent to require that $(v, w) = 0$ for all v and $w \in V$.

Suppose now that V is such a plane inside K^4. Then V has nontrivial intersection with the subspace $\{v_1 = 0\}$; let $v \in V$ be a nonzero vector with $v_1 = 0$. Since $v_1v_2 - v_3v_4 = 0$, we must have either $v_3 = 0$ or $v_4 = 0$. Assume without loss of generality that $v_3 = 0$. Write $u = v_2$, $t = v_4$; then $(u, t) \neq (0, 0)$. Now consider any vector $w \in V$; then

$$0 = (w, v) = w_1v_2 + w_2v_1 - w_3v_4 - w_4v_3 = uw_1 - tw_3.$$

So there exists $r \in K$ such that $w_1 = rt$ and $w_3 = ru$. We also have

$$0 = \frac{1}{2}(w, w) = w_1 w_2 - w_3 w_4 = rtw_2 - ruw_4.$$

Hence either $r = 0$ or there exists $s \in K$ such that $w_2 = su$ and $w_4 = st$. So

$$V \subset \{ (w_1, 0, w_3, 0) \mid w_1, w_3 \in K \} \cup \{ (rt, su, ru, st) \mid r, s \in K \}.$$

Since V has dimension 2, we conclude that V must be equal to one of these two planes.

This discussion was under the assumption that $v_3 = 0$ rather than $v_4 = 0$; in the second case, we find that V is of one of the forms $\{ (w_1, 0, 0, w_4) \mid w_1, w_4 \in K \}$ or $\{ (rt, su, st, ru) \mid r, s \in K \}$ for $(u, t) \neq (0, 0)$. But we obtain $\{ (w_1, 0, w_3, 0) \mid w_1, w_3 \in K \}$ by setting $(u, t) = (0, 1)$ in $\{ (rt, su, st, ru) \mid r, s \in K \}$ and $\{ (w_1, 0, 0, w_4) \mid w_1, w_4 \in K \}$ by setting $(u, t) = (0, 1)$ in $\{ (rt, su, ru, st) \mid r, s \in K \}$. Hence all such planes V are of one of the forms

$$V^{(1)}_{u,t} = \{ (rt, su, ru, st) \mid r, s \in K \}$$

or

$$V^{(2)}_{u,t} = \{ (rt, su, st, ru) \mid r, s \in K \}$$

for some $(u, t) \neq (0, 0)$. And it is easy to see conversely that each of these subspaces is two-dimensional and lies in the subset of K^4 determined by $(v, v) = 0$.

Translating this back into equations for the lines on the surface Q, we obtain two families of lines:

$$L^{(1)}_{u,t} = \left\{ \left[\frac{rt + su}{2} : \frac{rt - su}{2\sqrt{-1}} : \frac{ru - st}{2} : \frac{ru + st}{2\sqrt{-1}} \right] \mid r, s \in K \right\},$$

$$L^{(2)}_{u,t} = \left\{ \left[\frac{rt + su}{2} : \frac{rt - su}{2\sqrt{-1}} : \frac{st - ru}{2} : \frac{st + ru}{2\sqrt{-1}} \right] \mid r, s \in K \right\},$$

where (u, t) ranges over $K^2 \setminus \{(0, 0)\}$. The families $L^{(1)}_{*,*}$ and $L^{(2)}_{*,*}$ are disjoint, and two pairs (u, t) and (u', t') yield the same line in a given family if and only if one pair is a nonzero scalar multiple of the other.

(b) By the result of the previous part, F is the image of a regular map $\mathbb{P}^1 \amalg \mathbb{P}^1 \to \mathbb{G}$, so F is a closed subvariety of $\mathbb{G}$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday September 17 2008 (Day 2)

1. (a) Show that the ring $\mathbb{Z}[i]$ is Euclidean.

(b) What are the units in $\mathbb{Z}[i]$?

(c) What are the primes in $\mathbb{Z}[i]$?

(d) Factorize $11 + 7i$ into primes in $\mathbb{Z}[i]$.

Solution.

(a) We define a norm on $\mathbb{Z}[i]$ in the usual way, $|a+bi| = \sqrt{a^2+b^2}$. Then we must show that for any a and b in $\mathbb{Z}[i]$ with $b \neq 0$, there exist q and r in $\mathbb{Z}[i]$ with $a = qb + r$ and $|r| < |b|$. Let $q_0 = a/b \in \mathbb{C}$ and let $q \in \mathbb{Z}[i]$ be one of the Gaussian integers closest to q_0; the real and imaginary parts of q differ by at most $\frac{1}{2}$ from those of q_0, so $|q - q_0| \leq \sqrt{2}/2 < 1$. Now let $r = a - qb$. Then

$$|r| = |a - qb| = |(q_0 - q)b| = |q_0 - q||b| < |b|$$

as desired.

(b) If $u \in \mathbb{Z}[i]$ is a unit, then there exists $u' \in \mathbb{Z}[i]$ such that $uu' = 1$, so $|u||u'| = 1$ and hence $|u| = 1$ (since $|z| > 0$ for every $z \in \mathbb{Z}[i]$). Writing $u = a + bi$, we obtain $1 = |u| = \sqrt{a^2+b^2}$ so either $a = \pm 1$ and $b = 0$ or $a = 0$ and $b = \pm 1$. The four possibilities $u = 1, -1, i, -i$ are all clearly units.

(c) Since $\mathbb{Z}[i]$ is Euclidean, it contains a greatest common divisor of any two elements, and it follows that irreducibles and primes are the same: if z is irreducible and $z \nmid x$ and $n \nmid y$, then $\gcd(x,z) = \gcd(y,z) = 1$, so $1 \in (x,z)$ and $1 \in (y,z)$; hence $1 \in (xy, z)$, so $z \nmid xy$.

Let $z \in \mathbb{Z}[i]$. If $|z| \leq 1$, then z is either zero or a unit so is not prime. If $|z| = \sqrt{p}$, $p \in \mathbb{Z}$ a prime, then u must be a prime in $\mathbb{Z}[i]$, because $|\cdot|$ is multiplicative and $|z|^2 \in \mathbb{Z}$ for all $z \in \mathbb{Z}[i]$. It remains to consider z for which $|z|^2$ is composite.

Write $\sqrt{N} = |z|$, and factor $N = p_1 p_2 \cdots p_r$ in $\mathbb{Z}$. Note that

$$z \mid z\bar{z} = N = p_1 p_2 \cdots p_r$$

so if z is prime, then z divides one of the primes $p = p_i$ in $\mathbb{Z}[i]$. Moreover $\bar{z}$ also divides p so $N = z\bar{z}$ divides p^2; since N is composite we must have $N = p^2$. That is, $z\bar{z} = p^2$; by assumption the left side is a factorization into irreducibles, so up to units each p on the right hand side must be a product of some terms on the left; the only possibility is $z = pu$, $\bar{z} = p\bar{u}$ for some unit u. Now when $p \equiv 3 \pmod 4$, p is indeed a prime in $\mathbb{Z}[i]$, because then $p \mid a^2 + b^2 \implies p \mid a, b \implies p^2 \mid a^2 + b^2$, so there are no elements of $\mathbb{Z}[i]$ with norm $\sqrt{p}$. If $p \equiv 1 \pmod 4$, then p can be written in the form $p = a^2 + b^2$, so $p = (a + bi)(a - bi)$ and p is not in fact a prime.

In conclusion, the primes of $\mathbb{Z}[i]$ are

- elements $z \in \mathbb{Z}[i]$ with $z = \sqrt{p}$, $p \in \mathbb{Z}$ prime (necessarily congruent to 1 mod 4);
- elements of the form pu with $p \in \mathbb{Z}$ a prime congruent to 3 mod 4 and $u \in \mathbb{Z}[i]$ a unit.

(d) We first compute $|11 + 7i| = \sqrt{121 + 49} = \sqrt{170}$; so $11 + 7i$ will be a product of primes with norms $\sqrt{2}$, $\sqrt{5}$ and $\sqrt{17}$. There is only one prime with norm $\sqrt{2}$ up to units and only two with a norm $\sqrt{5}$; a quick calculation yields

$$11 + 7i = (1 + i)(1 + 2i)(1 - 4i).$$

2. Let $U \subset \mathbb{C}$ be the open region

$$U = \{z : |z - 1| < 1 \text{ and } |z - i| < 1\}.$$

Find a conformal map $f : U \to \Delta$ of U onto the unit disc $\Delta = \{z : |z| < 1\}$.

Solution. The map $z \mapsto 1/z$ takes the open discs $\{ z : |z - 1| < 1 \}$ and $\{ z : |z - i| < 1 \}$ holomorphically to the open half-planes $\{ z : \Re z \geq \frac{1}{2} \}$ and $\{ z : \Im z \leq -\frac{1}{2} \}$ respectively, so it takes U to their intersection. So we can define a conformal isomorphism f_0 from U to the interior U' of the fourth quadrant by

$$f_0(z) = \frac{1}{z} - \frac{1 - i}{2}.$$

Now we can send U' to the lower half plane by the squaring map, and that to Δ by the Möbius transformation $z \mapsto \frac{1}{z - i/2} - i$. Thus the composite

$$\frac{1}{(\frac{1}{z} - \frac{1-i}{2})^2 + \frac{i}{2}} - i$$

is actually a conformal isomorphism from U to Δ.

3. Let n be a positive integer, A a symmetric $n \times n$ matrix and Q the quadratic form

$$Q(x) = \sum_{1 \leq i,j \leq n} A_{i,j} x_i x_j.$$

Define a metric on $\mathbb{R}^n$ using the line element whose square is

$$ds^2 = e^{Q(x)} \sum_{1 \leq i \leq n} dx^i \otimes dx^i.$$

(a) Write down the differential equation satisfied by the geodesics of this metric

(b) Write down the Riemannian curvature tensor of this metric at the origin in $\mathbb{R}^n$.

Solution. We first compute the Christoffel symbols ${\Gamma^m}_{ij}$ with respect to the standard basis for the tangent space $(\partial/\partial x_k$. The metric tensor in these coordinates is

$$g_{ij} = \delta_{ij} e^{Q(x)} \quad \text{with inverse} \quad g^{ij} = \delta_{ij} e^{-Q(x)}.$$

Its partial derivatives are

$$\frac{\partial}{\partial x_k} g_{ij} = \delta_{ij} e^{Q(x)} \frac{\partial}{\partial x_k} Q(x) = 2\delta_{ij} e^{Q(x)} \sum_l A_{lk} x_l.$$

Then (using implicit summation notation)

$$\begin{aligned}
{\Gamma^m}_{ij} &= \frac{1}{2} g^{km} \left(\frac{\partial}{\partial x_i} g_{kj} + \frac{\partial}{\partial x_j} g_{ik} - \frac{\partial}{\partial x_k} g_{ij} \right) \\
&= \frac{1}{2} \delta_{km} e^{-Q(x)} (2\delta_{kj} e^{Q(x)} A_{li} x_l + 2\delta_{ik} e^{Q(x)} A_{lj} x_l - 2\delta_{ij} e^{Q(x)} A_{lk} x_l) \\
&= (\delta_{mj} A_{li} + \delta_{im} A_{lj} - \delta_{ij} A_{lm}) x_l.
\end{aligned}$$

(a) The geodesic equation is

$$\begin{aligned}
0 &= \frac{d^2 x_m}{dt^2} + {\Gamma^m}_{ij} \frac{dx_i}{dt} \frac{dx_j}{dt} \\
&= \frac{d^2 x_m}{dt^2} + (\delta_{mj} A_{li} + \delta_{im} A_{lj} - \delta_{ij} A_{lm}) x_l \frac{dx_i}{dt} \frac{dx_j}{dt} \\
&= \frac{d^2 x_m}{dt^2} + 2 \sum_{i,l} A_{li} x_l \frac{dx_i}{dt} \frac{dx_m}{dt} - \sum_l A_{lm} x_l \sum_i \left(\frac{dx_i}{dt} \right)^2
\end{aligned}$$

(where we have written summations explicitly on the last line).

(b) The Riemannian curvature tensor is given by

$$\begin{aligned}
{R^l}_{ijk} &= \frac{\partial}{\partial x_j} {\Gamma^l}_{ik} - \frac{\partial}{\partial x_k} {\Gamma^l}_{ij} + {\Gamma^l}_{js} {\Gamma^s}_{ik} - {\Gamma^l}_{ks} \Gamma^s_{ij} \\
&= (\delta_{lk} A_{ri} + \delta_{il} A_{rk} - \delta_{ik} A_{rl}) - (\delta_{lj} A_{ri} + \delta_{il} A_{rj} - \delta_{ij} A_{rl}) \\
&\quad + (\delta_{ls} A_{tj} + \delta_{jl} A_{ts} - \delta_{js} A_{tl}) x_t (\delta_{sk} A_{ui} + \delta_{is} A_{uk} - \delta_{ik} A_{us}) x_u \\
&\quad - (\delta_{ls} A_{tk} + \delta_{kl} A_{ts} - \delta_{ks} A_{tl}) x_t (\delta_{sj} A_{ui} + \delta_{is} A_{uj} - \delta_{ij} A_{us}) x_u.
\end{aligned}$$

4. Let H be a separable Hilbert space and $b : H \to H$ a bounded linear operator.

(a) Prove that there exists $r > 0$ such that $b + r$ is invertible.

(b) Suppose that H is infinite dimensional and that b is compact. Prove that b is not invertible.

Solution.

(a) It is equivalent to show that there exists $\epsilon > 0$ such that $1-\epsilon b$ is invertible. Since b is bounded there is a constant C such that $||bv|| \leq C||v||$ for all $v \in H$. Choose $\epsilon < 1/C$ and consider the series

$$a = 1 + \epsilon b + \epsilon^2 b^2 + \cdots .$$

For any v the sequence $v + \epsilon bv + \epsilon^2 b^2 v + \cdots$ converges by comparison to a geometric series. So this series converges to a linear operator a and $a(1-\epsilon b) = (1-\epsilon b)a = 1$, that is, $a = (1-\epsilon b)^{-1}$.

(b) Suppose for the sake of contradiction that b is invertible. Then the open mapping theorem applies to b, so if $U \subset H$ is the unit ball, then $b(U)$ contains the ball around 0 of radius ε for some $\varepsilon > 0$. By the definition of a compact operator, the closure V of $b(U)$ is a compact subset of H. But H is infinite dimensional, so there is an infinite orthonormal set v_1, v_2, $\ldots$, and the sequence εv_1, εv_2, $\ldots$ is contained in V but has no limit point, a contradiction. Hence b cannot be invertible.

5. Let $X \subset \mathbb{P}^n$ be a projective variety.

(a) Define the *Hilbert function* $h_X(m)$ and the *Hilbert polynomial* $p_X(m)$ of X.

(b) What is the significance of the degree of p_X? Of the coefficient of its leading term?

(c) For each m, give an example of a variety $X \subset \mathbb{P}^n$ such that $h_X(m) \neq p_X(m)$.

Solution.

(a) The homogeneous coordinate ring $S(X)$ is the graded ring $S(\mathbb{P}^n)/I$, where $S(\mathbb{P}^n)$ is the ring of polynomials in $n+1$ variables and I is the ideal generated by those homogeneous polynomials which vanish on X. Then $h_X(m)$ is the dimension of the mth graded piece of this ring. The Hilbert polynomial $p_X(m)$ is the unique polynomial such that $p_X(m) = h_X(m)$ for all sufficiently large integers m.

(b) The degree of p_X is the dimension d of the variety $X \subset \mathbb{P}^n$, and its leading term is $\deg X/d!$.

(c) Let X consist of any k distinct points of $\mathbb{P}^n$. Then X is a variety of dimension 0 and degree k, so by the previous part $p_X(m) = k$. But $h_X(m)$ is at most the dimension of the space of homogeneous degree m polynomials in $n+1$ variables, so for sufficiently large k, $h_X(m) < k = p_X(m)$.

6. Let $X = S^2 \vee \mathbb{RP}^2$ be the wedge of the 2-sphere and the real projective plane. (This is the space obtained from the disjoint union of the 2-sphere and the real projective plane by the equivalence relation that identifies a given point in S^2 with a given point in $\mathbb{RP}^2$, with the quotient topology.)

(a) Find the homology groups $H_n(X, \mathbb{Z})$ for all n.

(b) Describe the universal covering space of X.

(c) Find the fundamental group $\pi_1(X)$.

Solution.

(a) The wedge $A \vee B$ of two spaces satisfies $\tilde{H}_n(A \vee B, \mathbb{Z}) = \tilde{H}_n(A, \mathbb{Z}) \oplus \tilde{H}_n(B, \mathbb{Z})$ for all n, so

$$H_0(X, \mathbb{Z}) = \mathbb{Z}, \quad H_1(X, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}, \quad H_2(X, \mathbb{Z}) = \mathbb{Z}.$$

(b) The universal covering space $\tilde{X}$ of X can be constructed as the union of the unit spheres centered at $(-2, 0, 0)$, $(0, 0, 0)$ and $(2, 0, 0)$ in $\mathbb{R}^3$; the group $\mathbb{Z}/2\mathbb{Z}$ acts freely on $\tilde{X}$ by sending x to $-x$, and the quotient is X. Topologically, $\tilde{X}$ is the wedge sum $S^2 \vee S^2 \vee S^2$.

(c) Since X is the quotient of the simply connected space $\tilde{X}$ by a free action of the group $\mathbb{Z}/2\mathbb{Z}$, we have $\pi_1(X) = \mathbb{Z}/2\mathbb{Z}$.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Thursday January 31 2008 (Day 3)

1. For $z \in \mathbb{C} \setminus \mathbb{Z}$, set

$$f(z) = \lim_{N\to\infty} \left(\sum_{n=-N}^{N} \frac{1}{z+n} \right)$$

(a) Show that this limit exists, and that the function f defined in this way is meromorphic.

(b) Show that $f(z) = \pi \cot \pi z$.

Solution.

(a) We can rewrite f as

$$f(z) = \frac{1}{z} + \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{z+n} + \frac{1}{z-n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}.$$

For any $z \in \mathbb{C} \setminus \mathbb{R}$, the terms of this sum are uniformly bounded near z by a convergent series. So this sum of analytic functions converges uniformly near z and thus f is analytic near z. We can apply a similar argument to $f(z) - \frac{1}{z-n}$ to conclude that f has a simple pole at each integer n (with residue 1).

(b) The meromorphic function $\pi \cot \pi z$ also has a simple pole at each integer n with residue $\lim_{z\to n}(z-n)(\pi \cot \pi z) = 1$, so $f(z) - \pi \cot \pi z$ is a global analytic function. Moreover

$$\begin{aligned} f(z+1) - f(z) &= \lim_{N\to\infty} \left(\sum_{n=-N}^{N} \frac{1}{z+1+n} - \frac{1}{z+n} \right) \\ &= \lim_{N\to\infty} \left(\frac{1}{z+1+N} - \frac{1}{z-N} \right) \\ &= 0 \end{aligned}$$

for all $z \in \mathbb{C} \setminus \mathbb{Z}$, and $\cot \pi(z+1) = \cot \pi z$, so $f(z) - \pi \cot \pi z$ is periodic with period 1. Its derivative is

$$f'(z) - \frac{d}{dz}\pi \cot \pi z = -\frac{1}{z^2} + \sum_{n=1}^{\infty} \left(-\frac{1}{(z+n)^2} - \frac{1}{(z-n)^2} \right) + \pi^2 \sin^2 \pi z.$$

This is again an analytic function with period 1, and it approaches 0 as the imaginary part of z goes to ∞, so it must be identically 0. So

$f(z) - \pi \cot \pi z$ is constant; since it is an odd function, that constant must be 0.

2. Let p be an odd prime.

(a) What is the order of $GL_2(\mathbb{F}_p)$?

(b) Classify the finite groups of order p^2.

(c) Classify the finite groups G of order p^3 such that every element has order p.

Solution.

(a) To choose an invertible 2×2 matrix over $\mathbb{F}_p$, we first choose its first column to be any nonzero vector in $p^2 - 1$, then its second column to be any vector not a multiple of the first in $p^2 - p$ ways. So $GL_2(\mathbb{F}_p)$ has $(p^2 - 1)(p^2 - p)$ elements.

(b) Let G be a group with p^2 elements. As a p-group, G must have nontrivial center Z. If $Z = G$, then G is abelian and so $G = (\mathbb{Z}/p\mathbb{Z})^2$ or $G = \mathbb{Z}/p^2\mathbb{Z}$. Otherwise Z has order p. So there is a short exact sequence

$$1 \to Z \to G \to \mathbb{Z}/p\mathbb{Z} \to 1.$$

The sequence splits, because we can pick a generator for $\mathbb{Z}/p\mathbb{Z}$ and choose a preimage for it in G; this preimage has order p (G cannot contain an element of order p^2 or it would be cyclic) so it determines a splitting $\mathbb{Z}/p\mathbb{Z} \to G$. Hence G is the direct product of Z and $\mathbb{Z}/p\mathbb{Z}$ (because Z is central in G). So there are no new groups in this case.

(c) Let G be a group with p^3 elements in which every element has order p, and let Z be the center of G; again Z is nontrivial. If Z has order p^3, then G is abelian, and since every element has order p, G must be $(\mathbb{Z}/p\mathbb{Z})^3$. If Z has order p^2, then Z must be isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$, and there is a short exact sequence

$$1 \to Z \to G \to \mathbb{Z}/p\mathbb{Z} \to 1.$$

Again, we can split this sequence by choosing a preimage of a generator of $\mathbb{Z}/p\mathbb{Z}$, so G is the direct product $Z \times \mathbb{Z}/p\mathbb{Z}$. Hence Z is not really the center of G, and there are no groups in this case. Finally, suppose Z has order p; then there is a short exact sequence

$$1 \to Z \to G \to (\mathbb{Z}/p\mathbb{Z})^2 \to 1.$$

Let a and b be elements of G whose images together generate $(\mathbb{Z}/p\mathbb{Z})^2$. Then the image of $c = bab^{-1}a^{-1}$ is $0 \in (\mathbb{Z}/p\mathbb{Z})^2$, so c lies in Z. If a and b commuted, we could split this sequence which would lead to a contradiction as before. Hence c is a generator of Z. We can write every element of G uniquely in the form $a^i b^j c^k$ with $0 \leq i,\ j,\ k < p$, and we know the commutation relations between a, b and c; it's easy to see that G is isomorphic to the group of upper-triangular 3×3 matrices over $\mathbb{F}_p$ with ones on the diagonal via the isomorphism

$$a^i b^j c^k \leftrightarrow \begin{pmatrix} 1 & j & k \\ 0 & 1 & i \\ 0 & 0 & 1 \end{pmatrix}.$$

It remains to check that in this group every element really has order p. But one can check by induction that

$$\begin{pmatrix} 1 & j & k \\ 0 & 1 & i \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & nj & nk + \frac{n(n-1)}{2} ij \\ 0 & 1 & ni \\ 0 & 0 & 1 \end{pmatrix}$$

and setting $n = p$, the right hand side is the identity because p is odd.

3. Let X and Y be compact, connected, oriented 3-manifolds, with

$$\pi_1(X) = (\mathbb{Z}/3\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z} \qquad \text{and} \qquad \pi_1(Y) = (\mathbb{Z}/6\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

(a) Find $H_n(X, \mathbb{Z})$ and $H_n(Y, \mathbb{Z})$ for all n.

(b) Find $H_n(X \times Y, \mathbb{Q})$ for all n.

Solution.

(a) (We omit the coefficient group $\mathbb{Z}$ from the notation in this part.) By the Hurewicz theorem, $H_1(X)$ is the abelianization of $\pi_1(X)$, so $H_1(X) = (\mathbb{Z}/3\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z}$. By Poincaré duality, $H^2(X) = (\mathbb{Z}/3\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z}$ as well. Now by the universal coefficient theorem for cohomology, $H^1(X)$ is (noncanonically isomorphic to) the free part of $H_1(X)$. So $H^1(X) = \mathbb{Z} \oplus \mathbb{Z}$, and by Poincaré duality again $H_2(X) = \mathbb{Z} \oplus \mathbb{Z}$ too. Of course, $H_3(X) = \mathbb{Z}$ because X is a connected oriented 3-manifold. So the homology groups of X are

$$H_0(X) = \mathbb{Z}, \quad H_1(X) = (\mathbb{Z}/3\mathbb{Z}) \oplus \mathbb{Z}^2, \quad H_2(X) = \mathbb{Z}^2, \quad H_3(X) = \mathbb{Z}.$$

Entirely analogous arguments for Y yield

$$H_0(Y) = \mathbb{Z}, \quad H_1(Y) = (\mathbb{Z}/6\mathbb{Z}) \oplus \mathbb{Z}^3, \quad H_2(Y) = \mathbb{Z}^3, \quad H_3(Y) = \mathbb{Z}.$$

(b) The module $\mathbb{Q}$ is flat over $\mathbb{Z}$ ($\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Q}, -) = 0$ for $n > 0$) so for any space A, $H_n(A, \mathbb{Q}) = \mathbb{Q} \otimes H_n(A, \mathbb{Z})$. In particular,

$$H_0(X, \mathbb{Q}) = \mathbb{Q}, \quad H_1(X, \mathbb{Q}) = \mathbb{Q}^2, \quad H_2(X, \mathbb{Q}) = \mathbb{Q}^2, \quad H_3(X, \mathbb{Q}) = \mathbb{Q},$$

$$H_0(Y,\mathbb{Q}) = \mathbb{Q}, \quad H_1(Y,\mathbb{Q}) = \mathbb{Q}^3, \quad H_2(Y,\mathbb{Q}) = \mathbb{Q}^3, \quad H_3(Y,\mathbb{Q}) = \mathbb{Q}.$$

The Künneth theorem over a field k states that $H_*(A\times B, k) = H_*(A,k)\otimes H_*(B,k)$ for any spaces A and B. So the homology groups $H_n(X\times Y,\mathbb{Q})$ for $n = 0, \ldots, 6$ are

$$\mathbb{Q}, \quad \mathbb{Q}^5, \quad \mathbb{Q}^{11}, \quad \mathbb{Q}^{14}, \quad \mathbb{Q}^{11}, \quad \mathbb{Q}^5, \quad \mathbb{Q}.$$

Note. Actually, there are no compact connected 3-manifolds M with $\pi_1(M) = (\mathbb{Z}/3\mathbb{Z})\oplus\mathbb{Z}\oplus\mathbb{Z}$ or $\pi_1(M) = (\mathbb{Z}/6\mathbb{Z})\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus Z$. The only abelian groups which are the fundamental groups of compact connected 3-manifolds are $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$, and $(\mathbb{Z}/2\mathbb{Z})\oplus\mathbb{Z}$.

4. Let $\mathcal{C}_c^\infty(\mathbb{R})$ be the space of differentiable functions on $\mathbb{R}$ with compact support, and let $L^1(\mathbb{R})$ be the completion of $\mathcal{C}_c^\infty(\mathbb{R})$ with respect to the L^1 norm. Let $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{h\to 0} \frac{1}{h}\int_{|y-x|<h} |f(y) - f(x)| dy = 0$$

for almost every x.

Solution. Let X_k be the set of $x \in \mathbb{R}$ such that

$$\limsup_{h\to 0} \frac{1}{h}\int_{|y-x|<h} |f(y) - f(x)|\, dy > \frac{1}{k}.$$

We will show that X_k has measure 0 for each $k = 1, 2, \ldots$. The union of these sets is the set of x for which the displayed equation in the problem statement does not hold; if it is the union of countably many sets of measure 0, it also has measure 0, proving the desired statement.

Fix a positive integer k, and let $\varepsilon > 0$. By the given definition of $L^1(\mathbb{R})$, there is a differentiable function g on $\mathbb{R}$ with compact support such that $||f - g||_1 \leq \varepsilon/4k$. Write $f_1 = f - g$. I claim that

$$\limsup_{h\to 0} \frac{1}{h}\int_{|y-x|<h} |f(y) - f(x)|\, dy = \limsup_{h\to 0} \frac{1}{h}\int_{|y-x|<h} |f_1(y) - f_1(x)|\, dy,$$

so we may replace f by f_1. Indeed, by the triangle inequality, the difference between the two sides is at most

$$\limsup_{h\to 0} \frac{1}{h}\int_{|y-x|<h} |g(y) - g(x)|\, dy.$$

Since g is continuous, we may choose h small enough so that the integrand is bounded by δ for any $\delta > 0$, hence this $\limsup$ is 0.

So now suppose $f \in L^1(\mathbb{R})$ is such that $||f||_1 < \epsilon/4k$. Observe that

$$\begin{aligned}
\limsup_{h\to 0} \frac{1}{h}\int_{|y-x|<h} |f(y)-f(x)|\,dy &\leq \limsup_{h\to 0} \frac{1}{h}\int_{|y-x|<h} |f(x)|+|f(y)|\,dy \\
&= 2\,|f(x)| + \limsup_{h\to 0} \frac{1}{h}\int_{|y-x|<h} |f(y)|\,dy.
\end{aligned}$$

Now define $F(x) = \int_{-\infty}^{x} |f(y)|\,dy$. Then by the Lebesgue differentiation theorem F is differentiable with $F'(x) = |f(x)|$ for almost every x. The last term on the second line above equals $2F'(x)$ wherever the latter is defined, so for almost every x,

$$\limsup_{h\to 0} \frac{1}{h}\int_{|y-x|<h} |f(y)-f(x)|\,dy \leq 4\,|f(x)|.$$

The measure of the set of points x such that $4\,|f(x)| \geq 1/k$ is at most $4k\,||f||_1 < \varepsilon$, so the measure of X_k is at most ε. Since ε was arbitrary, X_k has measure 0 as claimed.

5. Let $\mathbb{P}^5$ be the projective space of homogeneous quadratic polynomials $F(X,Y,Z)$ over $\mathbb{C}$, and let $\Phi \subset \mathbb{P}^5$ be the set of those polynomials that are products of linear factors. Similarly, let $\mathbb{P}^9$ be the projective space of homogeneous cubic polynomials $F(X,Y,Z)$, and let $\Psi \subset \mathbb{P}^9$ be the set of those polynomials that are products of linear factors.

(a) Show that $\Phi \subset \mathbb{P}^5$ and $\Psi \subset \mathbb{P}^9$ are closed subvarieties.

(b) Find the dimensions of Φ and Ψ.

(c) Find the degrees of Φ and Ψ.

Solution.

(a) Identify $\mathbb{P}^2$ with the projective space of linear polynomials $F(X,Y,Z)$ over $\mathbb{C}$. Then there is a map $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^5$ given by multiplying the two linear polynomials to get a homogeneous quadratic polynomial. Its image is exactly Φ. Since $\mathbb{P}^2 \times \mathbb{P}^2$ is a projective variety, Φ is a closed subvariety of $\mathbb{P}^5$. Similarly, there is a map $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^9$ with image Ψ, showing that Ψ is a closed subvariety of $\mathbb{P}^9$.

(b) The fibers of the maps $\mathbb{P}^2 \times \mathbb{P}^2 \to \Phi$ and $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \Psi$ are all 0-dimensional by unique factorization, so $\dim \Phi = 4$ and $\dim \Psi = 6$.

(c) We will show that the degree of Ψ is 15. The degree of Φ can be shown to be 3 by a similar argument, or by noting that $\Phi \subset \mathbb{P}^5$ is defined by the vanishing of the determinant.

The dimension of Ψ is 6, so we could compute the degree of Ψ by intersecting Ψ with 6 generic hyperplanes in $\mathbb{P}^9$. Instead, we will choose 6 hyperplanes which are not generic. Each $f \in \Psi$ has a zero locus which is the union of three lines in $\mathbb{P}^2$. If x is a point of $\mathbb{P}^2$, the set of $g \in \mathbb{P}^9$ for which $g(x) = 0$ is a hyperplane. Pick 6 generic points $x_1, \ldots, x_6$ of $\mathbb{P}^2$, and consider those $f \in \Psi$ whose zero loci pass through all of these points. Such an f has a zero locus consisting of three lines whose union contains $x_1, \ldots, x_6$; there is exactly one way to choose those lines for each partition of $\{x_1, \ldots, x_6\}$ into three parts of size two. We can easily count that there are 15 such partitions. So Ψ meets this intersection of 6 hyperplanes set-theoretically in 15 points. Without verifying that the intersection is transverse, we can only conclude that the degree of Ψ is at least 15.

We next use the Hilbert polynomial to show that the degree of Ψ is at most 15. Let V_l be the vector space of degree-l homogeneous polynomials on Ψ, and W_l the vector space of degree-(l, l, l) tri-homogeneous polyonimals on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ which are invariant under the action of S_3 given by permuting the three $\mathbb{P}^2$ factors. Name the multiplication map $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^9$ from part (a) m. Pullback along m gives a map m^* from V_l to W_l, because $m \circ \sigma = m$ for any $\sigma \in S_3$ acting on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. Moreover m^* is injective, since m is surjective. Therefore $\dim V_l \leq \dim W_l$. The dimension of W_l is the number of monomials of tridegree (l, l, l) up to symmetry, or equivalently the number of 3×3 matrices of nonnegative integers with columns summing to l up to permutation of columns. There are $\binom{l+2}{2}$ possible columns and thus $\binom{\binom{l+2}{2}+2}{3} = \frac{l^6}{2^3 \cdot 6} + O(l^3)$ such matrices. So $\dim V_l \leq \frac{l^6}{2^3 \cdot 6} + O(l^3)$ and it follows that the degree of Ψ is at most $\frac{6!}{2^3 \cdot 6} = 15$. Together with the previous bound, this shows that $\deg \Psi = 15$.

(Note: m^* is not always surjective. The dimension of V_l is at most the dimension of the space of degree-l homogeneous polynomials on $\mathbb{P}^9$, namely $\binom{9+l}{l}$. When $l = 2$ this is only $\binom{11}{2} = 55$, while $\dim W_l = \binom{\binom{4}{2}+2}{3} = \binom{8}{3} = 56$. Thus, additional care would be needed to show that $\deg \Psi = 15$ using only the Hilbert polynomial.)

6. Realize S^1 as the quotient $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and consider the following two line bundles over S^1:

L is the subbundle of $S^1 \times \mathbb{R}^2$ given by

$$L = \{(\theta, (x, y)) : \cos(\theta) \cdot x + \sin(\theta) \cdot y = 0\}; \text{and}$$

M is the subbundle of $S^1 \times \mathbb{R}^2$ given by

$$M = \{(\theta, (x, y)) : \cos(\theta/2) \cdot x + \sin(\theta/2) \cdot y = 0\}.$$

(You should verify for yourself that M is well-defined.) Which of the following are trivial as vector bundles on S^1?

(a) L
(b) M
(c) $L \oplus M$
(d) $M \oplus M$
(e) $M \otimes M$

Solution.

(a) Since L is a line bundle, to show that L is trivial, it suffices to give a section of L which is everywhere nonzero. Take $s(\theta) = (-\sin(\theta), \cos(\theta))$.

(b) Let $B \subset M$ be the subbundle of vectors of unit length (so B is an S^0 bundle over S^1). Consider the map $\gamma : S^1 = \mathbb{R}/2\pi\mathbb{Z} \to B$ defined by $\gamma(\theta) = (2\theta, (-\sin(\theta), \cos(\theta)))$. Then γ is a homeomorphism, so in particular, B is not homeomorphic to $S^0 \times S^1$, and M cannot be a trivial line bundle.

(c) Let $C \subset L \oplus M$ be the subbundle of vectors of unit length (so C is an S^1 bundle over S^1). We will write $v \oplus w$ for a vector in $L \oplus M$ over $x \in S^1$, where v and w are vectors in L and M over x respectively. Consider the map $h : S^1 \times [0, 2\pi] \to C$ given by

$$h(\phi, \theta) = (\theta, (\cos\phi(-\sin\theta, \cos\theta) \oplus \sin\phi(-\sin(\theta/2), \cos(\theta/2)))).$$

This is a homotopy between the maps $S^1 \to C$ given by

$$h(\phi, 0) = (0, ((0, \cos\phi) \oplus (0, \sin\phi)))$$

and

$$h(\phi, 2\pi) = (0, ((0, \cos\phi) \oplus (0, -\sin\phi))).$$

If $L \oplus M \to S^1$ were a trivial plane bundle, then C would be the torus and these two paths would not be homotopic. Hence $L \oplus M$ is not a trivial plane bundle over S^1.

(d) Define $s : [0, 2\pi] \to M \oplus M$ by

$$s(\theta) = (\theta, (\cos(\theta/2)(-\sin(\theta/2), \cos(\theta/2)) \oplus \sin(\theta/2)(-\sin(\theta/2), \cos(\theta/2)))).$$

Observe that s is nowhere 0 and $s(0) = (0, ((0, 1) \oplus (0, 0)))$ is equal to $s(2\pi) = (0, (-(0, -1) \oplus (0, 0)))$. So s factors through S^1, and thus is a global nonvanishing section of $M \oplus M$. We can get a second, linearly independent section of $M \oplus M$ by applying the map $A : M \oplus M \to M \oplus M$,

$$A(\theta, (v \oplus w)) = (\theta, ((-w) \oplus v))$$

to s. So s and $A \circ s$ form a basis for $M \oplus M$ at every point, and $M \oplus M$ is a trivial plane bundle over S^1.

(e) Consider the map $s : [0, 2\pi] \to M$ given by

$$s(\theta) = (\theta, (-\sin(\theta/2), \cos(\theta/2))).$$

Since $s(0) = (0, (0, 1))$ while $s(2\pi) = (0, (0, -1))$, s does not factor through S^1. However, if we define $s' : [0, 2\pi] \to M \otimes M$ by

$$s'(\theta) = (\theta, v \otimes v) \quad \text{where} \quad (\theta, v) = s(\theta),$$

then $s'(0) = (0, (0, 1) \otimes (0, 1)) = (0, (0, -1) \otimes (0, -1)) = s'(2\pi)$. So s' is a global nonvanishing section of the line bundle $M \otimes M$, and thus $M \otimes M$ is trivial.

Note: Parts (c)–(e) can be solved more systematically using the theory of vector bundles. For X a pointed compact space, an n-dimensional vector bundle on the suspension of X is determined up to isomorphism by a homotopy class of pointed maps from X to the orthogonal group $O(n)$. For a map $f : X \to O(n)$, the corresponding vector bundle is obtained by taking trivial bundles on two copies of the cone on X and identifying them at a point $x \in X$ via the map $f(x)$. In our case $X = S^0$ and so a homotopy class of pointed maps from X to $O(n)$ is just a connected component of $O(n)$. The bundles L and M correspond to the connected components of the matrices (1) and (-1) respectively. It follows that the bundles $L \oplus M$, $M \oplus M$, and $M \otimes M$ correspond to

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 \end{pmatrix},$$

respectively, so $L \oplus M$ is nontrivial but $M \oplus M$ and $M \otimes M$ are trivial.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 1, 2009 (Day 1)

1. (RA) Let H be a Hilbert space and $\{u_i\}$ an orthonormal basis for H. Assume that $\{x_i\}$ is a sequence of vectors such that

$$\sum ||x_n - u_n||^2 < 1.$$

Prove that the linear span of $\{x_i\}$ is dense in H.

Solution. To show that the linear span $L = \text{span}\{x_i\}$ is dense, it suffices to show $\bar{L}^\perp = 0$. Suppose not, then there exists $v \neq 0$ with $v \perp x_i$ for all i. We may assume $\|v\| = 1$. Then

$$v = \sum_{i=0}^{\infty} (v, u_i) u_i$$

so $\|v\|^2 = \sum |(v, u_i)|^2$. On the other hand, by the Cauchy-Schwarz inequality

$$|(v, u + i)|^2 = |(v, x_i - u_i)|^2 \leq \|v\| \|x_i - u_i\| = \|x_i - u_i\|.$$

Thus

$$1 = \|v\|^2 = \sum |(v, u_i)|^2 \leq \sum \|x_i - u_i\|^2 < 1$$

a contradiction.

2. (T) Let $\mathbb{CP}^n$ be complex projective n-space.

(a) Describe the cohomology ring $H^*(\mathbb{CP}^n, \mathbb{Z})$ and, using the Kunneth formula, the cohomology ring $H^*(\mathbb{CP}^n \times \mathbb{CP}^n, \mathbb{Z})$.

(b) Let $\Delta \subset \mathbb{CP}^n \times \mathbb{CP}^n$ be the diagonal, and $\delta = i_*[\Delta] \in H_{2n}(\mathbb{CP}^n \times \mathbb{CP}^n, \mathbb{Z})$ the image of the fundamental class of Δ under the inclusion $i : \Delta \to \mathbb{CP}^n \times \mathbb{CP}^n$. In terms of your description of $H^*(\mathbb{CP}^n \times \mathbb{CP}^n, \mathbb{Z})$ above, find the Poincaré dual $\delta^* \in H^{2n}(\mathbb{CP}^n \times \mathbb{CP}^n, \mathbb{Z})$ of δ.

Solution.

(a) The cohomology ring $H^*(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}[\alpha]/\alpha^{n+1}$, where deg α=2. Since $H^*(\mathbb{CP}^n, \mathbb{Z})$ are $\mathbb{Z}$-free, the Künneth formula implies

$$H^*(\mathbb{CP}^n \times \mathbb{CP}^n, \mathbb{Z}) \cong H^*(\mathbb{CP}^n, \mathbb{Z}) \otimes H^*(\mathbb{CP}^n, \mathbb{Z})$$

(here $\otimes$ is the graded tensor product)

$$\cong \mathbb{Z}[\alpha]/\alpha^{n+1} \otimes \mathbb{Z}[\beta]/\beta^{n+1} = \mathbb{Z}[\alpha, \beta]/(\alpha^{n+1}, \beta^{n+1}),$$

with deg α = deg $\beta = 2$.

(b) Put $X = \mathbb{CP}^n \times \mathbb{CP}^n$.
Poincaré duality allows us to define a pushforward map $I_* : H^k(\Delta) \to H^{k+2n}(X)$ by the diagram

$$\begin{tikzcd} H^k(\Delta) \arrow[r, "i_*"] \arrow[d, "PD"] & H^{k+2n}(X) \\ H_{2n-k}(\Delta) \arrow[r, "i_*"] & H_{2n-k}(X) \arrow[u, "PD"'] \end{tikzcd}$$

were the lower horizontal map is the usual i_* by functoriality. This construction satisfies the projection formula

$$i_*(a \cup i^*b) = i_*a \cup b$$

for cohomology classes a, b. Observe that by construction $i_*1 = \delta^*$. Now put $i_*1 = \delta^* = \sum c_k \alpha^k \cup \beta^{n-k}$. Then

$$c_{n-k}\alpha^n\beta^n = i_*1 \cup \alpha^k \cup \beta^{n-k} = i_*\gamma^n$$

where we denote by γ the generator of of the cohomology algebra $H^*(\Delta) = H^*(\mathbb{CP}^n)$ as in the previous part of the question (note that $i^*\alpha = i^*\beta = \gamma$, since the composition of the diagonal with each projection is the identity). But γ^n is the Poincaré dual of the class of a point in $H_0(\Delta)$, so that $i_*\gamma^n$ is the dual of the class of a point in $H^*(X)$, which is $\alpha^n\beta^n$. Thus $c_{n-k} = 1$, and $\delta^* = \sum \alpha^k \beta^{n-k}$.

3. (AG) Let $X \subset \mathbb{P}^n$ be an irreducible projective variety, $\mathbb{G}(1, n)$ the Grassmannian of lines in $\mathbb{P}^n$, and $F \subset \mathbb{G}(1, n)$ the variety of lines contained in X.

(a) If X has dimension k, show that

$$\dim F \leq 2k - 2,$$

with equality holding if and only if $X \subset \mathbb{P}^n$ is a k-plane.

(b) Find an example of a projective variety $X \subset \mathbb{P}^n$ with $\dim X = \dim F = 3$.

Solution.

(a) Consider the incidence correspondence $\Sigma = \{(l, x) : x \in l\} \subset \mathbb{G}(1, n) \times \mathbb{P}^n$.
Since $pr_1^{-1}(F) \to F$ has fiber isomorphic to $\mathbb{P}^1$, we have

$$\dim\ \Sigma \cap pr_1^{-1}(F) = \dim\ F + 1.$$

We now estimate the dimension of the fibers of $pr_2 : pr_1^{-1}(F) \to X$, noting that pr_2 maps $pr_1^{-1}(F)$ into X by definition.
Let L_x be the fiber over x. Then $\Sigma \cap pr_1^{-1}(L_x)$ projects via pr_2 into X

with finite fibers over points in $X \setminus \{x\}$ (in fact, generically one-to-one), because a line through $x \neq y \in X$ is uniquely determined.
Thus

$$\dim\, pr_1^{-1}(L_x) \cap \Sigma = \dim\, L_x + 1 \leq \dim\, X = k\dim\, F + 1.$$

It follows that $\dim\, \Sigma \cap pr_1^{-1}(F) \leq \dim\, X +\dim\, X - 1$, so $\dim\, F \leq 2k-2$. Equality can occur only if for generic $x \in X$, the projection $pr_2 : \Sigma \cap pr_1^{-1}(L_x) \to X$ is dominant (hence surjective). In particular this implies that for all $x \neq y \in X$, the line xy is in X. The only X with such property is are linear subspaces of $\mathbb{P}^n$. (To see the last point, take a maximal set of independent points in X, then X must be contained in the plane spanned by them, but also contains the plane spanned by them).

(b) Let X be the cone over the smooth quadric $Q \subset \mathbb{P}^3$ with apex P. Clearly $\dim\, X = 2 + 1 = 3$.
We know Q is ruled by two $\mathbb{P}^1$ family of lines and any line in Q lies in one of those families.
It follows that a line in $X = C(Q)$ must lie in the plane spanned by P and a line in Q since the only rational curves on Q are the ruling lines. But the dimension of the variety of lines in $\mathbb{P}^2$ is 2, and we have two $\mathbb{P}^1$ family of planes which intersect each other only at P, hence $\dim\, F(X) = 2 + 1 = 3 = \dim\, X$.

4. (CA) Let $\Omega \subset \mathbb{C}$ be the open set

$$\Omega = \{z : |z| < 2 \ \text{ and } \ |z-1| > 1\}.$$

Give a conformal isomorphism between Ω and the unit disc $\Delta = \{z : |z| < 1\}$.

Solution. The Möbius map $z \mapsto \frac{2z}{2-z}$ sends the disk $\{|z| < 2\}$ to $\{\Re z < 1\}$, the disk $\{|z-1| < 1\}$ to $\{\Re z < 0\}$, hence it sends Ω biholomorphically to the strip $\{0 < \Re z < 1\}$.
Now the map $z \mapsto e^{2\pi i z}$ sends this strip biholomorphically to the upper half-plane, since we can write down an inverse by taking a branch of $\frac{1}{2\pi i}\log$ in the complement of the negative imaginary axis.
Finally $z \mapsto \frac{-z+i}{z+i}$ maps the upper half-plane biholomorphically to the unit disk Δ.
Thus the map

$$z \mapsto \frac{-e^{\frac{2\pi i z}{z-2}} + i}{e^{\frac{2\pi i z}{z-2}} + i}$$

defines a conformal isomorphism between Ω and Δ.

5. (A) Suppose ϕ is an endomorphism of a 10-dimensional vector space over $\mathbb{Q}$ with the following properties.

1. The characteristic polynomial is $(x-2)^4(x^2-3)^3$.
2. The minimal polynomial is $(x-2)^2(x^2-3)^2$.
3. The endomorphism $\phi - 2I$, where I is the identity map, is of rank 8.

Find the Jordan canonical form for ϕ.

Solution.

(a) Denote the underlying vector space by V. We will determine V as a $\mathbb{Q}[x]-$module, where x acts via ϕ. As such, V splits into a direct sum of cyclic modules, of form $\mathbb{Q}[x]/P(x)^k$ for irreducible P.
Since ϕ has characteristic polynomial $(x-2)^4(x^2-3)^2$, dim $V = 10$.
Since ϕ has minimal polynomial $(x-2)^2(x^2-3)^2$, V must be a direct sum of factros $\mathbb{Q}[x]/(x-2)^k$ with $k \leq 2$ and $\mathbb{Q}[x]/(x^2-3)^l$ with $l \leq 2$, and at least one factor for which $k = 2$, $l = 2$.
This gives only two possibilities:

$$V \cong \mathbb{Q}[x]/(x^2-3)^2 \oplus \mathbb{Q}[x]/(x^2-3) \oplus \mathbb{Q}[x]/(x-2)^2 \oplus \mathbb{Q}[x]/(x-2)^2$$

or

$$V \cong \mathbb{Q}[x]/(x^2-3)^2 \oplus \mathbb{Q}[x]/(x^2-3) \oplus \mathbb{Q}[x]/(x-2)^2 \oplus \mathbb{Q}[x]/(x-2) \oplus \mathbb{Q}[x]/(x-2).$$

Noting the $\phi - 2I$ has rank $6+1+1 = 8$ in the first case and $6+1+0+0 = 7$ in the second case, we conclude that the first case occurs. Thus

$$V \otimes \mathbb{C} \cong \mathbb{C}[x]/(x-\sqrt{3})^2 \oplus \mathbb{C}[x]/(x+\sqrt{3})^2 \oplus \mathbb{C}[x]/(x-\sqrt{3}) \oplus \mathbb{C}[x]/(x+\sqrt{3}) \oplus (\mathbb{C}[x]/(x-2)^2)^2$$

so that the Jordan normal form of ϕ is

$$\begin{pmatrix}
\sqrt{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{3} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}.$$

6. (DG) Let $\gamma : (0,1) \to \mathbb{R}^3$ be a smooth arc, with $\gamma' \neq 0$ everywhere.

(a) Define the *curvature* and *torsion* of the arc.

(b) Characterize all such arcs for which the curvature and torsion are constant.

Solution. We assume γ is parameterized by arc-length, so that $|\gamma'(t)| = 1$.

(a) The curvature of γ is

$$\kappa(t) = |\gamma''(t)|.$$

Put $v = \gamma'$. Because $(\gamma', \gamma') = 1$, differentiating with respect to t gives $(\gamma', \gamma'') = 0$, hence $v \perp \gamma''$. Define the normal n to be the unit vector in the direction of γ'', so

$$n(t) = \frac{\gamma''(t)}{|\gamma''(t)|}.$$

(If $\gamma'' = 0$, γ lies on a line and has curvature 0 and we do not define n in this case).

We have

$$\gamma''(t) = \kappa(t)\gamma'(t),$$

that is

$$n = \kappa v.$$

The binormal is defined to be $b = v \wedge n$. Note that v, b, n is a positively oriented orthonormal frame at each point of γ.

Finally the torsion τ is defined by

$$n' = -\kappa v + \tau b.$$

(b) We have

$$B' = v' \wedge n + v \wedge n' = v \wedge (-\kappa v + \tau b) = -\tau n.$$

So

$$\begin{pmatrix} v' \\ n' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} v \\ n \\ b \end{pmatrix}.$$

Suppose that κ, τ are constant along γ. We will show that γ is a helix (or a circle or a line for degenerate values of κ, τ) by setting up a differential equation that γ satisfies. Assume κ, $\tau \neq 0$.

We have $v = \gamma'$, $v' = \gamma'' = \kappa n$, so $n = \frac{\gamma''}{\kappa}$, $b = \frac{\gamma' \wedge \gamma''}{\kappa}$. Also $n' = \frac{\gamma'''}{\kappa} = -\kappa\gamma''' + \tau\frac{\gamma' \wedge \gamma''}{\kappa}$. This gives

$$\begin{pmatrix} \gamma_1''' \\ \gamma_2''' \\ \gamma_3''' \end{pmatrix} = -\kappa^2 \begin{pmatrix} \gamma_1'' \\ \gamma_2'' \\ \gamma_3'' \end{pmatrix} + \tau \begin{pmatrix} \gamma_2'\gamma_3'' - \gamma_3'\gamma_2'' \\ \gamma_3'\gamma_1'' - \gamma_1'\gamma_3'' \\ \gamma_1'\gamma_2'' - \gamma_2'\gamma_1'' \end{pmatrix}$$

This is a third order system of ODE, so it suffices to find helices $\gamma(z) = \frac{1}{r}(a\cos z, a\sin z, bz)$ (here $a^2 + b^2 = r^2$) with arbitrary given constant curvature κ, torsion τ and initial vectors $\gamma(0)$, $\gamma'(0)$, $\gamma''(0)$ such that $\gamma'(0) \perp \gamma''(0)$.

Looking at the helix $\gamma(z) = (a\cos z, a\sin z, bz)$, $a^2 + b^2 = 1$, $a > 0$ we have

$$v = \gamma'(z) = (-a\sin z, a\cos z, b)$$

$$\gamma'(0) = (0, a, b)$$

$$\gamma''(z) = (-a\cos z, a\sin z, b)$$

$$\gamma''(0) = (-a, 0, 0).$$

This gives $\kappa = a$.

$$n = (\cos z, \sin z, 0)$$

$$n' = (-\sin z, \cos z, 0)$$

$$\underline{b} = v \wedge n = (-b\sin z, b\cos z, -a)$$

so $\tau = b$.

This shows that we can arrange the helix to have arbitrary constant curvature and torsion κ, τ.

It is clear that by shifting z and scaling we can arrange the helix so that $\gamma'(0)$, $\gamma''(0)$ are an arbitrary pair of orthogonal vectors. By a translation, we can arrange for $\gamma(0)$ to be any given point. This shows that we can arrange our initial conditions to be arbitrary, and hence all space curves with non-zero constant curvature and torsion are helices. One can also easily show that curves with torsion 0 are circles, and curves with curvature 0 are lines.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Wednesday September 2, 2009 (Day 2)

1. (CA) Let $\Delta = \{z : |z| < 1\} \subset \mathbb{C}$ be the unit disc, and $\Delta^* = \Delta \setminus \{0\}$ the punctured disc. A holomorphic function f on Δ^* is said to have an *essential singularity* at 0 if $z^n f(z)$ does not extend to a holomorphic function on Δ for any n.

Show that if f has an essential singularity at 0, then f assumes values arbitrarily close to every complex number in any neighborhood of 0—that is, for any $w \in \mathbb{C}$ and $\forall \epsilon$ and $\delta > 0$, there exists $z \in \Delta^*$ with

$$|z| < \delta \quad \text{and} \quad |f(z) - w| < \epsilon.$$

Solution. Suppose the contrary, so there exists $\delta > 0$, $\epsilon > 0$, $w \in \mathbb{C}$ such that $|f(z) - w| > \epsilon$ for all $|z| < \delta$.
Then the function $g(z) = \frac{1}{f(z)-w}$ is defined and holomorphic in $|z| < \delta$.
Furthermore $|g(z)| < \frac{1}{\epsilon}$ for such z. Hence the singularity of g at 0 is removable, so $g(z)$ is holomorphic at 0. But that means $g(z) = z^n h(z)$ for holomorphic h in $|z| < \delta$ and $h(0) \neq 0$, hence $\frac{1}{h}$ is holomorphic at 0.
Now

$$f(z) = w + \frac{1}{g(z)} = w + \frac{1}{z^n h(z)}$$

so $z^n f(z)$ extends to a holomorphic function on Δ, a contradiction.

2. (AG) Let $S \subset \mathbb{P}^3$ be a smooth algebraic surface of degree d, and $S^* \subset \mathbb{P}^{3^*}$ the *dual surface*, that is, the locus of tangent planes to S.

(a) Show that no plane $H \subset \mathbb{P}^3$ is tangent to S everywhere along a curve, and deduce that S^* is indeed a surface.

(b) Assuming that a general tangent plane to S is tangent at only one point (this is true in characteristic 0), find the degree of S^*.

Solution. We assume $d > 1$ throughout.
The projective tangent plane to $F(X, Y, Z, T) = 0$ at $P \in \{F = 0\}$ has equation $\frac{\partial F}{\partial X} X + ... + \frac{\partial F}{\partial T} T = 0$ (the partial derivatives are evaluated at P).
In terms of coordinates, the Gauss map $S \mapsto S^* \subset \mathbb{P}^{3*}$ is given by

$$[X : Y : Z : T] \mapsto [\frac{\partial F}{\partial X} : \frac{\partial F}{\partial Y} : \frac{\partial F}{\partial Z} : \frac{\partial F}{\partial T}]$$

which is a morphism because S is smooth.

(a) Suppose $H \subset \mathbb{P}^3$ is a plane tangent to S everywhere along a curve γ. We can arrange so that H is $T = 0$. Then H is tangent to $F = 0$ along γ means $\frac{\partial F}{\partial X} = \frac{\partial F}{\partial Y} = \frac{\partial F}{\partial Z} = 0$ along γ.
But dim $\gamma = 1$, dim $(\frac{\partial F}{\partial X} = 0) \geq 2$, so $\frac{\partial F}{\partial X} = 0$ must intersect γ at some point P. But for this P we have $\frac{\partial F}{\partial X} = ... = \frac{\partial F}{\partial T} = 0$, so P is a singular point of S, a contradiction.

(b) Put $\phi : S \to S^* \subset \mathbb{P}^{3*}$ for the Gauss map. Since a general tangent plane to S is tangent at only one point of S, ϕ is generically one-to-one.
We can find deg S^* by intersecting S^* with a generic line $l \in \mathbb{P}^{3*}$.
Arrange coordinates so that our line has equations $Z^* = T^* = 0$ for dual coordinates $[X^* : Y^* : Z^* : T^*]$ of $\mathbb{P}^{3*}$.
The image of $[X : Y : Z : T] \in S$ is in the intersection iff $\frac{\partial F}{\partial Z} = \frac{\partial F}{\partial T} = F = 0$ at that point. Since $\frac{\partial F}{\partial Z} = 0$, $\frac{\partial F}{\partial T} = 0$ are generically hypersurfaces of degree $d-1$, and F has degree d, a generic choice of the line (reflected in the generic choice of coordinates) makes the hypersurfaces intersect at $(d-1)^2 d$ points. We can arrange so that $l \cap S^*$ lies in the open dense subset where ϕ^{-1} is a singleton, because a generic point Q will be the only point in S with image $\phi(Q)$.
Hence this shows deg $S^* = (d-1)^2 d$.

(Alternatively, one has $\phi^* \mathcal{O}(1) \cong \mathcal{O}(d-1)$. Computing the Hilbert function we have $\chi(\mathcal{O}_{S^*}(n)) = \chi(\phi_* \mathcal{O}_S(n))$ up to terms of degree < 2 in n, because ϕ is generically one-to-one.
But $\phi_*(\mathcal{O}_S \otimes \phi^* \mathcal{O}(1)) \cong \phi_* \mathcal{O}_S \otimes \mathcal{O}(1)$, so

$$\chi(\phi_* \mathcal{O}_S(n)) = \chi(\mathcal{O}_S((d+1)n)).$$

Since $\chi(\mathcal{O}_S(n)) = \frac{1}{2!} dn^2 + ...$, we have $\chi(\mathcal{O}_{S^*}(n)) = \frac{1}{2!} d(d-1)^2 n^2 + ...$, giving the degree of S^* to be $d(d-1)^2$.)

3. (T) Let $X = S^1 \vee S^1$ be a figure 8, $p \in X$ the point of attachment, and let α and $\beta : [0,1] \to X$ be loops with base point p (that is, such that $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = p$) tracing out the two halves of X. Let Y be the CW complex formed by attaching two 2-discs to X, with attaching maps homotopic to

$$\alpha^2 \beta \quad \text{and} \quad \alpha \beta^2.$$

(a) Find the homology groups $H_i(Y, \mathbb{Z})$.

(b) Find the homology groups $H_i(Y, \mathbb{Z}/3)$.

Solution. Y has cell structure with one 0-cell p, two 1-cells α, β and two 2-cells A, B. The cellular chain complex of Y is

$$0 \longrightarrow \mathbb{Z}A \oplus \mathbb{Z}B \xrightarrow{d_2} \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \xrightarrow{0} \mathbb{Z}p \longrightarrow 0$$

where $d_1 = 0$ because we must get a $\mathbb{Z}$ in H_0, as Y is connected.

To compute d_2, we have $A = n_{A\alpha} + n_{A\beta}\beta$, where $n_{A\alpha}$, $n_{A\beta}\beta$ are degrees of the maps $\partial A \to X/\beta$, $\partial A \to X/\alpha$, hence $d_2(A) = 2\alpha+\beta$. Similarly $d_2(B) = \alpha+2\beta$. Thus the cellular chain complex if Y is

$$0 \longrightarrow \mathbb{Z}A \oplus \mathbb{Z}B \xrightarrow{\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)} \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \xrightarrow{0} \mathbb{Z}p \longrightarrow 0$$

The cellular chain complex for $\mathbb{Z}/3$ coefficient is obtained by reducing the above mod 3. This gives:

(a) We already know $H_0(Y, \mathbb{Z}) = \mathbb{Z}$.
$H_1(Y, \mathbb{Z}) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta/(2\alpha + \beta = 0, \alpha + 2\beta = 0) = \mathbb{Z}/3$.
$H_2(Y, \mathbb{Z}) = 0$ since $2x + y = x + 2y = 0$ imples $x = y = 0$ in $\mathbb{Z}$.
All other homology groups vanish.

(b) $H_0(Y/\mathbb{Z}/3) = \mathbb{Z}/3$.
$H_1(Y, \mathbb{Z}/3)$ is the cokernel of $(\mathbb{Z}/3)^2 \to (\mathbb{Z}/3)^2$ given by $(x, y) \mapsto (-x + y, x - y)$, so $H_1(Y, \mathbb{Z}/3) = \mathbb{Z}/3$.
$H_2(Y, \mathbb{Z}/3) = \mathbb{Z}/3$, since $2x + y = x + 2y = 0$ imples $x = y$ in $\mathbb{Z}/3$.
All other homology groups vanish.

4. (DG) Let $f = f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be smooth, and let $S \subset \mathbb{R}^3$ be the graph of f, with the Riemannian metric ds^2 induced by the standard metric on $\mathbb{R}^3$. Denote the volume form on S by ω.

(a) Show that

$$\omega = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}.$$

(b) Find the curvature of the metric ds^2 on S

Solution.

(a) We have a parameterization of S given by $\phi : (x, y) \mapsto (x, y, f(x, y))$.
We have (the lower index denotes the variable with respect to which we differentiate)

$$\phi_x = (1, 0, f_x)$$

$$\phi_x = (0, 1, f_y)$$

The first fundamental form is $Edx^2 + 2Fdxdy + Gdy^2$ with

$$E = (\phi_x, \phi_x) = 1 + f_x^2$$

$$F = (\phi_x, \phi_y) = f_x f_y$$

$$G = (\phi_y, \phi_y) = 1 + f_y^2$$

hence the volume form

$$\omega = \sqrt{EF - G^2} = \sqrt{(1+f_x^2)(1+f_y)^2 - (f_x f_y)^2} = \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}.$$

(b) The normal vector is

$$\underline{N} = \frac{\phi_x \wedge \phi_y}{|\phi_x \wedge \phi_y|} = \frac{1}{1 + f_x^2 + f_y^2}(-f_x, -f_y, 1).$$

$$\phi_{xx} = (0, 0, f_{xx})$$
$$\phi_{xy} = (0, 0, f_{xy})$$
$$\phi_{yy} = (0, 0, f_{yy})$$

The second fundamental form $Ldx^2 + 2Mdxdy + Ndy^2$ is given by

$$L = -(\underline{N}, \phi_{xx}) = -f_{xx}$$
$$M = -(\underline{N}, \phi_{xy}) = -f_{xy}$$
$$N = -(\underline{N}, \phi_{yy}) = -f_{yy}$$

The Gaussian curvature is given by

$$K(x,y) = \frac{LN - M^2}{EG - F^2} = \frac{\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - (\frac{\partial^2 f}{\partial x \partial y})^2}{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}$$

5. (RA) Suppose that $\mathcal{O} \subset \mathbb{R}^2$ is an open set with finite Lebesgue measure. Prove that the boundary of the closure of $\mathcal{O}$ has Lebesgue measure 0.

Solution. We will construct a counter-example. First we will construct a "fat" Cantor set X in the interval $I = [0,1]$. Let X_1 be I with the interval of length $\frac{1}{4}$ in the middle removed. X_1 consists of two intervals of length $\frac{1}{2}(1 - \frac{1}{4})$. Inductively suppose we get X_n consisting of 2^n intervals of length $\frac{1}{2^n}(1 - \frac{1}{4} - ...\frac{2^{n-1}}{4^n})$. Then X_{n+1} is obtained by removing the middle intervals of length $\frac{1}{4^{n+1}}$ in each interval of X_n. Thus X_{n+1} consists of 2^{n+1} intervals of length $\frac{1}{2^{n+1}}(1 - \frac{1}{4} - ... - \frac{1}{4^{n+1}})$. The X_n form a decreasing sequence of non-empty compact subset of I, hence $X = \cap X_n$ is a non-empty compact subset of I. Note that $\mu(X) = 1 - \frac{1}{4} - \frac{2}{4^2} - ... = \frac{1}{2}$. X is also nowhere dense, since X_n can not contain an interval of size $> \frac{1}{2^n}$.

We now construct $\mathcal{O}$ as the complement of $X \times [0,1]$ inside the square $[-2,2] \times [2,2]$, which is open as $X \times [0,1]$ is compact. It has finite measure. Because X is nowhere dense, $X \times [0,1]$ can not contain any rectangle, so $\mathcal{O}$ is dense in $[-2,2] \times [-2,2]$. It follows that the boundary of $\mathcal{O}$ will contain $X \times [0,1]$ which has measure $\frac{1}{2}$. This gives the desired counter-example.

6. (A) Let R be the ring of integers in the field $\mathbb{Q}(\sqrt{-5})$, and S the ring of integers in the field $\mathbb{Q}(\sqrt{-19})$.

(a) Show that R is not a principal ideal domain

(b) Show that S is a principal ideal domain

Solution.

(a) We have $\alpha = a + b\sqrt{-5} \in R$ iff $Tr_{\mathbb{Q}(\sqrt{-5})/\mathbb{Q}}(\alpha) \in \mathbb{Z}$, $N_{\mathbb{Q}(\sqrt{-5})/\mathbb{Q}}(\alpha) \in \mathbb{Z}$ iff $2a \in \mathbb{Z}$, $a^2 + 5b^2 \in \mathbb{Z}$. This implies $2a, 2b \in \mathbb{Z}$ and $4a^2 + 20b^2 \in 4\mathbb{Z}$, so $2a, 2b \in 2\mathbb{Z}$, so $a, b \in \mathbb{Z}$.
Hence $R = \mathbb{Z}[\sqrt{-5}]$.

If R is a PID, it must be UFD. But in R we have $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$.
We claim these are two essentially different factorizations. Indeed any unit $\epsilon \in R$ has norm 1, and $N(2) = 4$, $N(3) = 9$, $N(\pm\sqrt{-5}) = 6$ (we write N as a shorthand for the norm). There are no elements of norm 2 or 3 in R (because $a^2 + 5b^2 = 2$ or 3 has no solution mod 5), hence all factors in the above factorization are non-associated irreducible elements of R.
Thus R is not a UFD, hence not a PID.

(b) By the Minkowski bound, every class in the ideal class group of $\mathcal{O}_K$ of a number field K contains an integral ideal of norm $\leq M_K = \sqrt{|D|}(\frac{4}{\pi})^{r_2}\frac{n!}{n^n}$, where $n = [K : \mathbb{Q}]$, $2r_2$ is the number of complex embeddings of K and D the discriminant of K.
For $K = \mathbb{Q}(\sqrt{-19})$ we have $n = 2$, $r_2 = 1$. To compute D, an argument as in (a) shows that $S = \mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$. Thus a $\mathbb{Z}-$basis for S is given by 1, $\frac{1+\sqrt{-19}}{2}$. Hence $D = -19$. The Minkowski bound is thus $M_K = \frac{2}{\pi}\sqrt{19} < 4$.
To show that S is a PID it suffices to show that all ideal classes are trivial. By the Minkowski bound, it suffices to check that all prime ideals in S of norm < 4 are principal. A prime ideal $\mathfrak{p}$ of norm 2, 3 must lie above 2. 3 respectively. Note that $S \cong \mathbb{Z}[x]/(x^2 - x + 5)$, and has discriminant -19 which is coprime to 2, 3, so the splitting behavior of 2, 3 in S are determined by the factorization of $x^2 - x + 5$ in $\mathbb{F}_2$, $\mathbb{F}_3$. But one easily checks that $x^2 - x + 5$ has no solution in $\mathbb{F}_2$, $\mathbb{F}_3$, hence stay irreducible there. It follows that 2, 3 are inert in S, hence there are no prime ideals in S of norm 2, 3, hence we are done.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday September 3, 2009 (Day 3)

1. (A) Let $c \in \mathbb{Z}$ be an integer not divisible by 3.

(a) Show that the polynomial $f(x) = x^3 - x + c \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$.

(b) Show that the Galois group of f is the symmetric group $\mathfrak{S}_3$.

Solution.

(a) If f is reducible in $\mathbb{Q}[x]$, it is reducible in $\mathbb{Z}[x]$ by Gauss' lemma, hence reducible ober $\mathbb{F}_3[x]$. But f mod 3 has no zeroes in $\mathbb{F}_3$, hence can not factorize there. Thus f is irreducible in $\mathbb{Q}[x]$.

(b) Because f has degree 3, its splitting field K has degree at most 6 over $\mathbb{Q}$. Since f is irreducible, $3|[K : \mathbb{Q}]$, hence $[K : \mathbb{Q}] = 3$ or 6, depending on whether $\mathrm{Gal}(K/\mathbb{Q})$ is A_3 or S_3 (it is a subgroup of S_3 via the transitive action on the roots of f in $\bar{\mathbb{Q}}$.

If $[K : \mathbb{Q}] = 3$, the discriminant $\Delta(f)$ must be a square in $\mathbb{Q}$. But $\Delta(f) = 27c^2 - 4 = -1 \bmod 3$, hence $\Delta(f) \in \mathbb{Z}$ is not s square in $\mathbb{Z}$, hence not a square in $\mathbb{Q}$. Hence the Galois group is S_3.

2. (CA) Let τ_1 and $\tau_2 \in \mathbb{C}$ be a pair of complex numbers, independent over $\mathbb{R}$, and $\Lambda = \mathbb{Z}\langle\tau_1, \tau_2\rangle \subset \mathbb{C}$ the lattice of integral linear combinations of τ_1 and τ_2. An entire meromorphic function f is said to be *doubly periodic* with respect to Λ if

$$f(z + \tau_1) = f(z + \tau_2) = f(z) \quad \forall z \in \mathbb{C}.$$

(a) Show that an entire holomorphic function doubly periodic with respect to Λ is constant.

(b) Suppose now that f is an entire meromorphic function doubly periodic with respect to Λ, and that f is either holomorphic or has one simple pole in the closed parallelogram

$$\{a\tau_1 + b\tau_2 : a, b \in [0, 1] \subset \mathbb{R}\}.$$

Show that f is constant.

Solution.

(a) The lattice Λ has fundamental domain $D = \{x\tau_1 + y\tau_2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ which is compact. If f is doubly periodic with respect to Λ, put $M = \max_D |f|$.

For any $z \in \mathbb{C}$, there is a $z_0 \in D$ with $f(z) = f(z_0)$, hence $f(z) \leq M$ for all z. Hence f is a bounded entire function, hence constant by Liouville's theorem.

(b) Suppose f is not constant. Translating a fundamental domain if necessary and using (a), we can assume that f has a simple pole inside the fundamental domain D. By the residue theorem

$$\int_{\partial D} f dz = \sum_{z \in Int(D)} 2\pi i \mathrm{Res}_z(f)$$

with the right-hand side non-zero, because f can have non-zero residue only at the unique pole, and the residue there is non-zero since the pole is simple. But $\Lambda-$periodicity implies that in the integral on the left the integral along opposite edges of D cancel each other, so the left-hand side is 0, a contradiction.

3. (DG) Let M and N be smooth manifolds, and let $\pi : M \times N \to N$ be the projection; let α be a differential k-form on $M \times N$. Show that α has the form $\pi^*\omega$ for some k-form ω on N if and only if the contraction $\iota_X(\alpha) = 0$ and the derivative $\mathcal{L}_X(\alpha) = 0$ for any vector field X on $M \times N$ whose value at every point is in the kernel of the differential $d\pi$.

Solution. If $\alpha = \pi^*\omega$

$$\iota_X(\alpha)(Y_1, ..., Y_{k-1}) = \pi^*\omega(X, , Y_1, ..., Y_{k-1}) = \omega(d\pi(X), ..., d\pi(Y_{k-1}) = 0$$

for arbitrary vector fields $Y_1, ..., Y_{k-1}$ on $M \times N$ and X whose value at every point is in ker $d\pi$.

Also

$$\mathcal{L}_X(\alpha)(Y_1, ..., Y_k) = X(\pi^*\omega(Y_1, ...Y_k)) = \pi_*X(\omega(Y_1, ..., Y_k)) = 0$$

because $\pi_*X = 0$ for X as above.

We now show the converse.

The problem is local on both M and N (by taking partitions of unity on M, N and take their product as a partition of unity on $M \times N$), so we work in a neighborhood of $M \times N$ which is of the form $U \times V$ for coordinate neighborhoods U, V of M, N. Call the corresponding local coordinates x_1,...,x_m amd y_1,...,y_n. We will also use standard multi-index notation.
Let

$$\alpha = \sum_{|I|+|J|=k, I \subset \{1,...,m\}, J \subset \{1,...,n\}} f_I dx_I \wedge dy_J$$

be a $k-$form satisfying the necessary conditions. A general vector field X killed by π_* is of form $\sum a_i \frac{\partial}{\partial x_i}$.

We have

$$\iota_{\frac{\partial}{\partial x_i}}(f dx_I \wedge dy_J) = z = \begin{cases} 0 & if \;\; i \notin I \\ \pm f dx_{I\setminus\{i\}} \wedge dy_J & if \;\; i \in I \end{cases}$$

Hence the condition $\iota_{\frac{\partial}{\partial x_i}}\alpha = 0$ implies $f_{IJ} = 0$ for all $I \ni i$. Since i is arbitrary, $f_{IJ} = 0$ for all $I \neq \emptyset$. This means

$$\alpha = \sum_{|J|=k, J\subset\{frm[o]--,...,n\}} f_J(x_1, ..., x_m, y_1, ...y_n)dy_J.$$

But

$$0 = \mathcal{L}_{\frac{\partial}{\partial x_i}}\alpha(\frac{\partial}{\partial y_{i_1}}, ..., \frac{\partial}{\partial y_{i_k}}) = \frac{\partial}{\partial x_i}(f_{i_1...i_k})$$

for $i_1 < ... < i_k$, hence the functions f_J are functions on the y_j only, so they are of form $\pi^* g_J$ for smooth functions g_J on N. Since the dy_J are pulled back from N, we have $\alpha = \pi^*\omega$ for some k-form ω on N.

4. (RA) Show that the Banach space ℓ^p can be embedded as a summand in $L^p(0,1)$—in other words, that $L^p(0,1)$ is isomorphic as a Banach space to the direct sum of ℓ^p and another Banach space.

Solution. Choose disjoint intervals $I_n \subset (0,1)$ and let f_n be a positive multiple of the characteristic function of I_n, normalized by $\|f_n\|_p = 1$. Define an embedding $\iota : l^p \to L^p(0,1)$ by $(a_n) \mapsto \sum a_n f_n$. Observe that

$$\int |\sum_r^s a_n f_n|^p = \sum_r^s \int_{I_n} |a_n|^p |f_n|^p = \sum_r^s |a_n|^p$$

This shows that if $(a_n) \in l^p$ then $\sum a_n f_n$ converges in $L^p(0,1)$ and has the same norm. This shows that our map is defined and is an isometric embedding. Therefore it remains to write down a continuous projection P splitting it. Define $P : L^p(0,1) \to l^p$ by $f \mapsto (\int f|f_n|^{p-1})_n$. The map is defined since $f \in L^1(0,1)$. Now we have by Hölder's inequality (note that $(p-1)q = p$)

$$\sum_n |\int_{I_n} f|f_n|^{p-1}|^p \leq \sum_n (\int_{I_n} |f|^p)(\int |f_n|^{(p-1)q})^{\frac{p}{q}} = \sum_n \int_{I_n} |f|^p \leq \|f\|_p^p$$

This shows that P is continuous (indeed of norm ≤ 1), and clearly $P\iota = id$. This gives the desired decomposition of $L^p(0,1)$ with one summand l^p.

5. (T) Find the fundamental groups of the following spaces:

(a) $SL_2(\mathbb{R})$

(b) $SL_2(\mathbb{C})$

(c) $SO_3(\mathbb{C})$

Solution.

(a) By the polar decomposition, any $A \in GL_2(\mathbb{R})$ can be written uniquely as $A = PU$ with $U \in O_2(\mathbb{R})$ and P a positive definite symmetric matrix. If $A \in SL_2(\mathbb{R})$ thn $U \in SO_2(\mathbb{R})$.
This gives a homeomorphism

$$SL_2(\mathbb{R}) \cong P_+ \times SO_2(\mathbb{R}).$$

The space P_+ of positive definite symmetric matrices is contractible since it is an open cone in a real vector space while $SO_2(\mathbb{R}) \cong S^1$. Thus $\pi_1(SL_2(\mathbb{R})) \cong \pi_1(S^1) \cong \mathbb{Z}$.

(b) Similar to (a), the complex polar decomposition gives a unique decomposition $A = PU$ for $A \in SL_2(\mathbb{C})$, where $U \in SU_2(\mathbb{C})$ and P a positive definite Hermitian matrix. Again, the space of positive definite Hermitian matrix is contractible, hence

$$\pi_1(SL_2(\mathbb{C})) \cong \pi_1(SU_2(\mathbb{C})) \cong \pi_1(S^3) = 0$$

noting that $SU_2(\mathbb{C}) \cong S^3$ via $(a, b) \mapsto \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}$ where a, b are complex numbers such that $|a|^2 + |b|^2 = 1$.

(c) $SL_2(\mathbb{C})$ acts on $\mathfrak{sl}_2 \cong \mathbb{C}^3$ (the subspace of $M_2(\mathbb{C})$ consisting of trace 0 matrices). This action preserves the non-degenerate symmetric bilinear form given by $K(A, B) = Tr(ad(A).ad(B))$ where $ad(A)$ is the operator $X \mapsto [A, X] = AX - XA$ on sl_2. This gives a morphism $SL_2(\mathbb{C}) \to SO_3(\mathbb{C})$ whose kernel is $\pm I$. Hence we get $PSL_2(\mathbb{C}) \hookrightarrow SO_3(\mathbb{C})$. Since both sides are connected Lie groups of the same complex dimension, the map is an isomorphism. From (b) we know that $SL_2(\mathbb{C})$ is simply connected, hence is the universal cover of $SO_3(\mathbb{C})$, so $\pi_1(SO_3(\mathbb{C})) \cong \mathbb{Z}/2$.

6. (AG) Let $X \subset \mathbb{A}^n$ be an affine algebraic variety of pure dimension r over a field K of characteristic 0.

(a) Show that the locus $X_{\text{sing}} \subset X$ of singular points of X is a closed subvariety.

(b) Show that X_{sing} is a proper subvariety of X.

Solution.

(a) Let $I(X) = (f_1, ..., f_m)$. Then $x \in X$ is singular iff the Jacobian matrix $J = (\frac{\partial f_i}{x_j})$ has rank $<$ codim $(X) = n - r$ at x. This happens iff every $(n-r) \times (n-r)$ minors of $J(x)$ vanish. Since these are regular functions, X_{sing} is a closed subvariety of X.

(b) It suffices to treat the case X irreducible. In characteristic 0, X is birational to a hypersurface $F = 0$ in some affine space $\mathbb{A}^n$. To see this, observe that the function field $K(X)$ is a simple extension of a purely transcendental field $k(t_1, ...t_r)$, by the primitive element theorem.

Hence $K(X) = k(t_1, ...t_r, u)$ with u algebraic over $k(t_1, ...t_r)$. Note $t_1, ...t_r$ is a transcendental basis of $K(X)$. If G is the minimal polynomial of u over $k(t_1, ..t_r)$, after clearing denominators we see that $K(X)$ is the function field of a hypersurface $F = 0$ in $\mathbb{A}^{r+1}$. In particular they have some isomorphic dense open subsets.

Thus we are reduced to the case X is a hypersurface $F = 0$ in $\mathbb{A}^{r+1}$. In this case X_{sing} is the locus $\frac{\partial F}{\partial X_i} = 0$ and $F = 0$. If $X_{sing} = X$, using the UFD property of $k[X_1, ..., X_{r+1}]$ and the fact F is irreducible, we deduce that $F | \frac{\partial F}{\partial X_i}$. This forces $\frac{\partial F}{\partial X_i} = 0$ for degree reasons. But this can not happen in characteristic 0, as can be seen by looking at a maximal monomial appearing in F with respect to the lexicographic order. This shows that X must contain non-singular points.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday August 31, 2010 (Day 1)

1. (CA) Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx$$

Solution. Let C be the curve on the complex plane from $-\infty$ to $+\infty$, which is along the real line for most part but gets around the origin by going upwards (clockwise). We are integrating

$$\int_C \frac{\sin^2 z}{z^2} dz = \int_C \frac{2 - e^{2iz} - e^{-2iz}}{4z^2} dz = \int_C \frac{1 - e^{2iz}}{4z^2} dz + \int_C \frac{1 - e^{-2iz}}{4z^2} dz.$$

Let C' be the curve from $-\infty$ to $+\infty$, along the real line for most part but now goes downwards around the origin. Then

$$\int_C \frac{1 - e^{-2iz}}{4z^2} dz - \int_{C'} \frac{1 - e^{-2iz}}{4z^2} dz = -2\pi i \cdot \mathrm{Res}_{z=0} \left(\frac{1 - e^{-2iz}}{4z^2} \right) = \pi.$$

As $1 - e^{2iz}$ is bounded when $\mathrm{Im}(z) \geq 0$, $\int_C \frac{1-e^{2iz}}{4z^2} dz = 0$ as we can push the integral up to infinity. Similarly $\int_{C'} \frac{1-e^{-2iz}}{4z^2} dz = 0$. This shows the original integral has value π.

2. (A) Let b be any integer with $(7, b) = 1$ and consider the polynomial

$$f_b(x) = x^3 - 21x + 35b.$$

(a) Show that f_b is irreducible over $\mathbb{Q}$.

(b) Let P denote the set of $b \in \mathbb{Z}$ such that $(7, b) = 1$ and the Galois group of f_b is the alternating group A_3. Find P.

Solution.

(a) This follows from the Eisenstein criterion on the prime 7.

(b) From (a), the Galois group is A_3 if the discriminant is a square (in $\mathbb{Q}$), and S_3 if otherwise. The discriminant of $x^3 + ax + b$ is $-4a^3 - 27b^2$, and the discriminant of f_b is $4 \cdot 21^3 - 27 \cdot 35^2 \cdot b^2 = 3^3 7^2 (28 - 25b)$. Thus we're looking for all b such that $3(28 - 25b)$ is a square. Such square must be divisible by 9 and are congruent to 9 modulo 25, hence of the form $(75n \pm 3)^2$, i.e. $3(28 - 25b) = 5625n^2 \pm 450n + 9 \Leftrightarrow b = -75n^2 \pm 6n + 1$. Thus $P = \{-75n^2 + 6n + 1 \,|\, n \in \mathbb{Z}\}$.

3. (T) Let X be the Klein bottle, obtained from the square $I^2 = \{(x, y) : 0 \le x, y \le 1\} \subset \mathbb{R}^2$ by the equivalence relation $(0, y) \sim (1, y)$ and $(x, 0) \sim (1-x, 1)$.

(a) Compute the homology groups $H_n(X, \mathbb{Z})$.

(b) Compute the homology groups $H_n(X, \mathbb{Z}/2)$.

(c) Compute the homology groups $H_n(X \times X, \mathbb{Z}/2)$.

Solution. X has the following cellular decomposition: the square F, the edges $E_1 = \{0\} \times [0, 1]$ and $E_2 = [0, 1] \times \{0\}$, and the vertex $V = (0, 0)$. We have $\delta F = 2E_1$ and $\delta E_1 = \delta E_2 = \delta V = 0$.

(a) $H_2(X, \mathbb{Z}) = \{c \cdot F | \delta(c \cdot F) = 0\} = 0$. As all other boundary maps are zero, $H_1(X, \mathbb{Z}) = (\mathbb{Z}E_1 + \mathbb{Z}E_2)/2\mathbb{Z}E_1 \cong (\mathbb{Z}/2) \oplus \mathbb{Z}$ and $H_0(X, \mathbb{Z}) = \mathbb{Z}$.

(b) All boundary maps are zero in $\mathbb{Z}/2$-coefficient. Thus $H_2(X, \mathbb{Z}/2) = \mathbb{Z}/2$, $H_1(X, \mathbb{Z}/2) = (\mathbb{Z}/2)^2$ and $H_0(X, \mathbb{Z}/2) = \mathbb{Z}/2$.

(c) $\mathbb{Z}/2$ may be seen as a field. Thus $H^i(X, \mathbb{Z}/2) = H_i(X, \mathbb{Z}/2)$ for any i and by the Kunneth formula $H_*(X \times X, \mathbb{Z}/2) = H^*(X \times X, \mathbb{Z}/2) = H^*(X, \mathbb{Z}/2)^{\otimes 2}$. Explicitly

$$H^i(X \times X, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & i = 0, 4 \\ (\mathbb{Z}/2)^4 & i = 1, 3 \\ (\mathbb{Z}/2)^6 & i = 2 \\ 0 & \text{else} \end{cases}$$

4. (RA) Let f be a Lebesgue integrable function on the closed interval $[0, 1] \subset \mathbb{R}$.

(a) Suppose g is a continuous function on $[0, 1]$ such that the integral of $|f - g|$ is less than ϵ^2. Prove that the set where $|f - g| > \epsilon$ has measure less than ϵ.

(b) Show that for every $\epsilon > 0$, there is a continuous function g on $[0, 1]$ such that the integral of $|f - g|$ is less than ϵ^2.

Solution.

(a) This is obvious.

(b) We have to prove that continuous functions are dense As f is Lebesgue integrable, f can be L^1-approximated by step functions, i.e. for any $\delta > 0$, there exist real numbers $c_1, ..., c_n$ and measurable sets $E_1, ..., E_n \subset [0, 1]$ such that the integral of $|f - c_1\chi_{E_1} - ... - c_n\chi_{E_n}|$ is smaller than δ, where we denote by χ_E the characteristic function of E. By picking small enough δ and replace f by $c_1\chi_{E_1} + ... + c_n\chi_{E_n}$, it suffices to prove that for any $\epsilon > 0$ and any characteristic function χ_E of a measurable set $E \subset [0, 1]$, there is a continuous function g_E such that the integral of $|g_E - \chi_E|$ is smaller than ϵ

As the Lebesgue measure is inner and outer regular, we may find compact K and open U such that $K \subset E \subset U$ and the measure of $U - K$ is arbitrarily small. Urysohn lemma now gives us a continuous function that is 1 on K, 0 on $[0,1] - U$ and between 0 and 1 in $U - K$. This gives the required function g_E.

5. (DG) Let v denote a vector field on a smooth manifold M and let $p \in M$ be a point. An *integral curve* of v through p is a smooth map $\gamma : U \to M$ from a neighborhood U of $0 \in \mathbb{R}$ to M such that $\gamma(0) = p$ and the differential $d\gamma$ carries the tangent vector $\partial/\partial t$ to $v(\gamma(t))$ for all $t \in U$.

(a) Prove that for any $p \in M$ there is an integral curve of v through p.

(b) Prove that any two integral curves of v through any given point p agree on some neighborhood of $0 \in \mathbb{R}$.

(c) A *complete* integral curve of v through p is one whose associated map has domain the whole of $\mathbb{R}$. Give an example of a nowhere zero vector field on $\mathbb{R}^2$ that has a complete integral curve through any given point. Then, give an example of a nowhere zero vector field on $\mathbb{R}^2$ and a point which has no complete integral curve through it.

Solution. Pick a local chart of the manifold M at the considered point p. The chart may be seen as a neighborhood of a point $p \in \mathbb{R}^n$, and the vector field v is also given on the neighborhood. To give an integral curve through a point p is then to solve the ordinary differential equation (system) $x'(t) = v(x(t))$ and $x(0) = p$. As v is smooth and thus C^1, (a) and (b) follows from the (local) existence and uniqueness of solutions for ordinary differential equations.

For (c), constant vector field $v(x, y) = (1, 0)$ on $\mathbb{R}^2$ gives the first required example. Horizontal curves parametrized by arc length are all possible integral curves. For the second required example, we may consider $v(x, y) = (x^2, 0)$. A integral curve with respect to such a vector field is a solution to the ODE $x'(t) = x(t)^2$. Such a solution is of the form $\frac{1}{t-a}$, and always blows up in finite time (either forward or backward), i.e. there is no complete integral curve for this vector field.

6. (AG) Show that a general hypersurface $X \subset \mathbb{P}^n$ of degree $d > 2n - 3$ contains no lines $L \subset \mathbb{P}^n$.

Solution. A hypersurface in $\mathbb{P}^n$ of degree d is given by a homogeneous polynomial in $n + 1$ variable $x_0, x_1, ..., x_n$ of degree d up to a constant. There are $k(d, n) = \begin{pmatrix} d+n \\ n \end{pmatrix}$ such monomials, and thus the space of such polynomials is $\mathbb{P}^{k(d,n)-1}$. After a change of coordinate a line may be expressed as $x_2 = x_3 = ... = x_n = 0$. A hypersurface that contains this line then corresponds to a polynomial with no $x_0^d, x_0^{d-1}x_1, ..., x_1^d$ terms, which constitutes a codimension $d + 1$ subplane. On the other hand, the grassmannian of lines is a variety of dimension $2 \cdot ((n+1)-2) = 2n-2 < d+1$. This proves the assertion.

To be more rigerous, let G be the grassmannian of lines in $\mathbb{P}^n$, $H \cong \mathbb{P}^{k(d,n)-1}$ the space of hypersurfaces of degree d in $\mathbb{P}^n$. We may consider

$$X = \{(l, S) \mid l \in G,\, S \in H \text{ such that } l \subset S\}.$$

Then what we have learned is that G has dimension $2n-2$ and the fiber of the projection map $X \to G$ has dimension $d+1$ less than the dimension of H. Thus the dimension of X is the sum of the dimension of G and the dimension of the fiber H, which is smaller than the dimension of H exactly when $d > 2n-3$. It follows that the projection $X \to H$ cannot be surjective, which is the assertion to be proved.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday September 1, 2010 (Day 2)

1. (T) If M_g denotes the closed orientable surface of genus g, show that continuous maps $M_g \to M_h$ of degree 1 exist if and only if $g \geq h$.

Solution. A closed orientable surface of genus $g \geq 1$ may be described as a polygon of $4g$ edges, some pairs identified in a certain way. In particular, all vertices are identified together under this identification. For $g > h$, by further identify $4g - 4h$ edges to the point, we can construct a map from M_g to M_h which is a homeomorphism on the interior of the polygon (2-cell). Since the 2-cell is the generator of $H_2(\cdot, \mathbb{Z})$, the map constructed has degree 1.

If $g < h$, then any map $f : M_g \to M_h$ induces $f^* : H^1(M_h, \mathbb{Z}) \to H^1(M_g, \mathbb{Z})$, which cannot be injective since the former is a free abelian group of rank $2h$ and the latter has rank $2g$. Pick $0 \neq \alpha \in H^1(M_h, \mathbb{Z})$ such that $f^*(\alpha) = 0$, there always exists β with $\alpha.\beta \neq 0 \in H^2(M_h, \mathbb{Z})$. However $f^*(\alpha.\beta) = f^*(\alpha).f^*(\beta) = 0$, and thus f must has degree 0.

2. (RA) Let $f \in C(S^1)$ be continuous function with a continuous first derivative $f'(x)$. Let $\{a_n\}$ be the Fourier coefficient of f. Prove that $\sum_n |a_n| < \infty$.

Solution. f' has n-th Fourier coefficient equal to na_n. We thus have

$$||f'||_{L^2}^2 = \sum_n n^2 a_n^2 < \infty.$$

Then $(\sum_n |a_n|)^2 \leq (\sum_n n^2 a_n^2)(\sum_n 1/n^2) < \infty$ by Cauchy's inequality.

3. (DG) Let $S \subset \mathbb{R}^3$ be the surface given as aa graph

$$z = ax^2 + 2bxy + cy^2$$

where a, b and c are constants.

(a) Give a formula for the curvature at $(x, y, z) = (0, 0, 0)$ of the induced Riemannian metric on S.

(b) Give a formula for the second fundamental form at $(x, y, z) = (0, 0, 0)$.

(c) Give necessary and sufficient conditions on the constants a, b and c that any curve in S whose image under projection to the (x, y)-plane is a straight line through $(0, 0)$ is a geodesic on S.

Solution. Let normal vectors to the surface may be expressed as $n(x, y, z) = l(x, y, z) \cdot (ax + by, bx + cy, -1)$, where $l(x, y, z)$ is the inverse of the length of the vector. Note that $l(x,y,z) = 0$ and the first derivative of $l(x,y,z)$ is zero at $(0,0,0) \in S$. When we compute the second fundamental form we only have to compute the first derivative of $n(x,y,z)$. Therefore to compute the second fundamental form at $(0,0,0)$ we can treat $l(x,y,z) \equiv 1$ and the second fundamental form is thus

$$\begin{pmatrix} \frac{\partial}{\partial x}(ax+by) & \frac{\partial}{\partial y}(ax+by) \\ \frac{\partial}{\partial x}(bx+cy) & \frac{\partial}{\partial y}(bx+cy) \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

The curvature of the surface at $(0,0,0)$ is the determinant at the point, i.e. $ac - b^2$. For any curve whose projection to the (x,y)-plane is a straight line through the point to be a geodesic, the corresponding vector has to be an eigenvector of the matrix. Thus it is necessary that $a = c$, $b = 0$, i.e. the matrix being a multiple of the identity matrix for that to happen. On the other hand, when $a = c$, $b = 0$, the surface is radially symmetric and thus all such curves must be geodesics.

4. (AG) Let V and W be complex vector spaces of dimensions m and n respectively and $A \subset V$ a subspace of dimension l. Let $\mathbb{P}\mathrm{Hom}(V,W) \cong \mathbb{P}^{mn-1}$ be the projective space of nonzero linear maps $\phi : V \to W$ mod scalars, and for any integer $k \le l$ let

$$\Psi_k = \{\phi : V \to W \ : \ \mathrm{rank}(\phi|_A) \le k\} \subset \mathbb{P}^{mn-1}.$$

Show that Ψ_k is an irreducible subvariety of $\mathbb{P}^{mn-1}$, and find its dimension.

Solution. An $n \times m$ matrix of rank $\le k$ can be decomposed into the product of an $n \times k$ matrix and a $k \times m$ matrix. Let $X \cong \mathbb{P}^{nk-1}$ and $Y \cong \mathbb{P}^{km-1}$ be the space of nonzero such matrices mod scalars. Then we have a surjection $X \times Y \to \Psi_k$ by the multiplication map. This shows that Ψ_k, as the image of the complete irreducible variety $X \times Y$, is irreducible and closed in $\mathbb{P}^{mn-1}$.

When a matrix has rank exactly k, the decomposition has a $GL(k)$ freedom of choice, i.e. each fiber of this map over a point in $\Psi_k - \Psi_{k-1}$ has dimension $k^2 - 1$. As Ψ_{k-1} is closed in irreducible Ψ_k, $\dim \Psi_k = \dim(\Psi_k - \Psi_{k-1}) = \dim(X \times Y) - (k^2-1) = (nk-1) + (mk-1) - (k^2-1) = k(n+m-k) - 1$. (We used the fact that $k \le l$, in which case $\Psi_k - \Psi_{k-1}$ is obviously non-empty.)

5. (CA) Find a conformal map from the region

$$\Omega = \{z : |z-1| > 1 \text{ and } |z-2| < 2\} \subset \mathbb{C}$$

between the two circles $|z-1| = 1$ and $|z-2| = 2$ onto the upper-half plane.

Solution. Let $S = \{\frac{1}{4} \le \mathrm{Re}(z) \le \frac{1}{2}\} \subset \mathbb{C}$. Then we have $\Omega \cong S$ by $z \mapsto \frac{1}{z}$ and $S \cong$ upper-half plane by $z \mapsto e^{2\pi i(z - \frac{1}{4})}$.

6. (A) Let G be a finite group with an automorphism $\sigma : G \to G$. If $\sigma^2 = id$ and the only element fixed by σ is the identity of G, show that G is abelian.

Solution. Define $\tau(x) := \sigma(x)x^{-1}$, then by assumption $\tau(x) \neq e$, $\forall x \neq e$. For any $x \neq x'$, $\tau(x)\tau(x')^{-1} = \sigma(x)x^{-1}x'\sigma(x')^{-1}$ is conjugate to $\sigma(x')^{-1}\sigma(x)x^{-1}x' = \tau(x'^{-1}x) \neq e$, i.e. $\tau(x) \neq \tau(x')$. Thus $\tau : G \to G$ is a surjective function. But we have $\sigma(\tau(x)) = x\sigma(x)^{-1} = \tau(x)^{-1}$, hence $\sigma(x) = x^{-1}$ and G is abelian.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday September 2, 2010 (Day 3)

1. (DG) Let $D \subset \mathbb{R}^2$ be the closed unit disk, with boundary $\partial D \cong S^1$. For any smooth map $\gamma : D \to \mathbb{R}^2$, let $A(\gamma)$ denote the integral over D of the pull-back $\gamma^*(dx \wedge dy)$ of the area 2-form $dx \wedge dy$ on $\mathbb{R}^2$.

(a) Prove that $A(\gamma) = A(\gamma')$ if $\gamma = \gamma'$ on the boundary of D.

(b) Let $\alpha : \partial D \to \mathbb{R}^2$ denote a smooth map, and let $\gamma : D \to \mathbb{R}^2$ denote a smooth map such that $\gamma|_{\partial D} = \alpha$. Give an expression for $A(\gamma)$ as an integral over ∂D of a function that is expressed only in terms of α and its derivatives to various orders.

(c) Give an example of a map γ such that $\gamma^*(dx \wedge dy)$ is a positive multiple of $dx \wedge dy$ at some points and a negative multiple at others.

Solution. Consider the differential $\omega = ydx$ on $\mathbb{R}^2$, so $d\omega = dx \wedge dy$. We have $\gamma^*(d\omega) = d\gamma^*(\omega)$. Thus if $\gamma|_{\partial D} = \alpha$, the by Stoke's theorem

$$\int_D \gamma^*(dx \wedge dy) = \int_D d\gamma^*(\omega) = \int_{\partial D} \alpha^*(\omega).$$

and depends only on α instead of γ. This finishes both (a) and (b).

For (c), one take for example $\gamma(x, y) = (x^2, y)$, then $\gamma^*(dx \wedge dy) = 2x(dx \wedge dy)$.

2. (T) Compute the fundamental group of the space X obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

Solution. The space X is $S^1 \times Y$, where Y is obtained from two circle S^1 by identifying a point $x_0 \in S^1$ with the corresponding point on the other circle. Thus $\pi_1(X) = \pi_1(S^1) \times \pi_1(Y)$, the product of $\mathbb{Z}$ and a free group on two generators.

3. (CA) Let u be a positive harmonic function on $\mathbb{C}$. Show that u is a constant.

Solution. There exists a holomorphic function f on $\mathbb{C}$ such that u is the real part of it. Then e^{-f} has image in the unit disk. By Liouville's theorem e^{-f} must be a constant, hence so is u.

4. (A) Let $R = \mathbb{Z}[\sqrt{-5}]$. Express the ideal $(6) = 6R \subset R$ as a product of prime ideals in R.

Solution. $(6) = (2)(3)$ and $(2) = (2, 1 + \sqrt{-5})^2$, $(3) = (3, 1 + \sqrt{-5})(3, 1 - \sqrt{-5})$. The final resulting ideals are prime because their indices (to R) are prime numbers.

5. (AG) Let $Q \subset \mathbb{P}^5$ be a smooth quadric hypersurface, and $L \subset Q$ a line. Show that there are exactly two 2-planes $\Lambda \cong \mathbb{P}^2 \subset \mathbb{P}^5$ contained in Q and containing L.

Solution. By a linear change of coordinate we may assume the line is $x_2 = x_3 = x_4 = x_5 = 0$, where $x_0, ..., x_5$ are coordinates of the projective space $\mathbb{P}^5$. Then the degree two homogeneous polynomial defining Q may be written as $F = f_0(x_2, x_3, x_4, x_5)x_0 + f_1(x_2, x_3, x_4, x_5)x_1 + q(x_2, x_3, x_4, x_5)$, where neither f_0 nor f_1 are a constant multiple of the other since $\frac{\partial F}{\partial x_0}$ and $\frac{\partial F}{\partial x_1}$ have to be independent for Q to be smooth. We may thus arrange another change of coordinate among x_2, x_3, x_4, x_5 so that $f_0 = x_2$, $f_1 = x_3$. The $F = x_0x_2 + x_1x_3 + q(x_2, x_3, x_4, x_5)$.

Any plane that lies within Q and contains L is then of the form $(x_0 = x_1 = 0, ax_4 + bx_5 = 0)$, where $ax_4 + bx_5$ is nontrivial and divides $q(0, 0, x_4, x_5)$. Note $\frac{\partial F}{\partial x_0} = x_2$ and $\frac{\partial F}{\partial x_1} = x_3$. For Q to be smooth we need $\frac{\partial F}{\partial x_i}$ to be independent, thus $\frac{\partial F}{\partial x_4}q(0, 0, x_4, x_5)$ and $\frac{\partial F}{\partial x_5}q(0, 0, x_4, x_5)$ have to be independent, which is equivalent to $q(0, 0, x_4, x_5)$ is non-degenerate, in which case it has two linear divisors.

6. (RA) Let $\mathcal{C}^\infty$ denote the space of smooth, real-valued functions on the closed interval $I = [0, 1]$. Let $\mathbb{H}$ denote the completion of $\mathcal{C}^\infty$ using the norm whose square is the functional

$$f \mapsto \int_I \left((\frac{df}{dt})^2 + f^2 \right) dt.$$

(a) Prove that the map of $\mathcal{C}^\infty$ to itself given by $f \mapsto T(f)$ with

$$T(f)(t) = \int_0^t f(s)ds$$

extends to give a bounded map from $\mathbb{H}$ to $\mathbb{H}$, and prove that the norm of T is 1. *(Remark: Its norm is actually not* 1*)*

(b) Prove that T is a compact mapping from $\mathbb{H}$ to $\mathbb{H}$.

(c) Let $\mathcal{C}^{1/2}$ be the Banach space obtained by completing $\mathcal{C}^\infty$ using the norm given by

$$f \mapsto \sup_{t \neq t'} \frac{|f(t) - f(t')|}{|t - t'|^{1/2}} + \sup_t |f(t)|.$$

Prove that the inclusion of $\mathcal{C}^\infty$ into $\mathbb{H}$ and into $\mathcal{C}^{1/2}$ extends to give a bounded, linear map from $\mathbb{H}$ to $\mathcal{C}^{1/2}$.

(d) Give an example of a sequence in $\mathbb{H}$ such that all elements have norm 1 and such that there are no convergent subsequences in $\mathcal{C}^{1/2}$.

Solution.

(a) To prove the linear map T extends to a bounded map, it suffices to prove that it is bounded on the dense $\mathcal{C}^\infty$. We have, for any $t \in [0,1]$,

$$T(f)(t)^2 = \left(\int_0^t f(s)ds\right)^2 \leq t\left(\int_0^t f(s)^2 ds\right) \leq \int_0^t f(s)^2 ds$$

and therefore also

$$\int_0^1 T(f)(t)^2 dt \leq \int_0^1 f(s)^2 ds.$$

Thus we have $||T(f)||_{\mathbb{H}}^2 \leq 2||f||_{L^2}^2 \leq 2||f||_{\mathbb{H}}^2$.
If one consider the constant function $f \equiv 1$, then $||f||_{\mathbb{H}} = 1$ but $||T(f)||_{\mathbb{H}} > 1$. This shows the norm must be greater than 1.

(b) The plan is to apply the Arzela-Ascoli theorem. For a bounded sequence $f_1, ..., f_n, ...$ in $\mathbb{H}$, as the operator is bounded by further approximation we are free to assume each $f_i \in \mathcal{C}^\infty$ and we have to prove $\{T(f_i)\}$ has a convergent subsequence. We have, for any $t_1, t_2 \in [0,1]$,

$$|f_i(t_1) - f_i(t_2)| = |\int_{t_1}^{t_2} f_i'(s)ds| \leq \left(|t_1 - t_2| \int_0^1 f_i'(s)^2 ds\right)^{1/2}$$

is bounded. Also

$$\inf_{t\in[0,1]} f_i(t) \leq \left(\int_0^1 f_i(s)^2 ds\right)^{1/2}.$$

These toghther show that f_i are uniformly bounded and equicontinuous. Thus by the Arzela-Ascoli theorem these f_i have a uniformly convergent subsequence, thus a L^2 convergent subsequence. As we've seen in (a) that the $\mathbb{H}$-norm of $T(f)$ is bounded by the L^2-norm of f, this gives us a convergent subsequence $T(f_i)$ in $\mathbb{H}$.

(c) The second last inequality just used is just

$$\frac{|f(t_1) - f(t_2)|}{|t_1 - t_2|^{1/2}} \leq \left(\int_0^1 f'(s)^2 ds\right)^{1/2}.$$

Also by the two inequalities in (b) $\sup f$ is bounded when f has bounded $\mathbb{H}$-norm. Thus the map from $\mathcal{C}^\infty$ to $\mathcal{C}^{1/2}$ is bounded with respect to the $\mathbb{H}$-norm on $\mathcal{C}^\infty$, and therefore extends.

(d) Let $g_0 : [0, +\infty) \to \mathbb{R}$ be any nonzero smooth function supported only on $[\frac{1}{2}, 1]$. Let $g_{n+1}(t) = \frac{1}{2}g_n(4t)$ for any $n \geq 0$. Then these g_i have disjoint support. Note that $||g_{n+1}||_{L^2} = \frac{1}{4}||g_n||_{L^2}$ and $||g'_{n+1}||_{L^2} = ||g'_n||_{L^2}$. Thus $||g_n||_{\mathbb{H}}$ converges to $||g'_0||_{L^2} \neq 0$. Similarly,

$$\sup_{t_1 \neq t_2} \frac{|g_{n+1}(t_1) - g_{n+1}(t_2)|}{|t_1 - t_2|^{1/2}} = \sup_{t_1 \neq t_2} \frac{|g_n(t_1) - g_n(t_2)|}{|t_1 - t_2|^{1/2}} \neq 0$$

and $\sup g_{n+1} = \frac{1}{2} \sup g_n$. Thus $||g_n||_{\mathcal{C}^{1/2}}$ also converges to some positive number (which is finite by (c)).

We can now normalize each g_n so that $||g_n||_{\mathbb{H}} = 1$, and still have $||g_n||_{\mathcal{C}^{1/2}}$ converges to a positive number. As these g_n have disjoint support, $||g_n - g_m||_{\mathcal{C}^{1/2}} \geq \max(||g_n||_{\mathcal{C}^{1/2}}, ||g_m||_{\mathcal{C}^{1/2}})$ and thus they have no convergent subsequence.

FALL 2012 - Solutions for the Algebraic Geometry Questions

PROPOSED ANSWERS IN AG

Day 1

(1) (a) If X is a linear subspace of $\mathbb{P}^n$, it intersects transversely with a generic linear subspace at a single point. Thus $\deg X = 1$.
Now consider the converse. Let $k = \dim X$. Consider the projection $\pi : X \to \mathbb{P}^{k+1}$ from a general $(n-k-2)$-plane in $\mathbb{P}^n$ such that X is birational onto its image $\bar{X} \subset \mathbb{P}^{k+1}$. Since X is of degree one, $\bar{X}$ is simply a hyperplane in $\mathbb{P}^{k+1}$. Then the inverse image of $\bar{X}$ under the projection is a hyperplane in $\mathbb{P}^n$. That means X is contained in a hyperplane P^{n-1} in $\mathbb{P}^n$. Doing the same thing for $X \subset P^{n-1}$ and continue the inductive process, we obtain $X \subset P^{k+1} \subset P^{k+2} \ldots \subset P^{n-1} \subset \mathbb{P}^n$ where one sits in the next one as a hyperplane. Thus X is a linear subspace of $\mathbb{P}^n$.

(b) Bézout's theorem states that if X and Y are two subvarieties of $\mathbb{P}^n$ which intersect generically transversely, then

$$\deg(X \cap Y) = (\deg X)(\deg Y).$$

(c) Recall that the Veronese map $v_d : \mathbb{P}^n \to \mathbb{P}^N$ of degree d is defined by sending $[Z_0, \ldots, Z_n]$ to $[P_0(Z_0, \ldots, Z_n), \ldots, P_N(Z_0, \ldots, Z_n)]$, where P_i, $i = 0, \ldots, N$ are all the monomials of degree d in $n+1$ variables, where $N = \left(\begin{array}{c} n+d \\ d \end{array} \right) - 1$.
The degree of Z equals to the number of intersection points of Z with a generic linear subspace L of $\mathbb{P}^N$ of dimension $N-k$, which is the zero set of k linear functions $l_1, \ldots, l_k$ on $\mathbb{P}^N$. Since v_d is an embedding, it suffices to compute the number of intersection points in the inverse image. The inverse image of the intersection between Z and L equals to the intersection of Y and the zero set of $v_d^* l_1, \ldots, v_d^* l_k$, which are degree d polynomials. By Bézout's theorem, the number of intersection points equals to $(\deg Y)(\deg v_d^* l_1) \ldots (\deg v_d^* l_k) = d^k a$.

Day 2

(2) (a) Recall that the Hilbert function $h_X : \mathbb{N} \to \mathbb{N}$ is defined by

$$h_X(m) = \dim \big((K[z_0, \ldots, z_N]/I_X)_m \big)$$

where $z_0, \ldots, z_N$ are the homogeneous coordinates of $\mathbb{P}^N$, I_X is the defining ideal of X, and $(K[z_0, \ldots, z_N]/I_X)_m$ denotes its m-th graded piece.
Consider the linear map

$$\mathrm{ev} : (K[z_0, \ldots, z_N]/I_X)_m \to K^d$$

defined by sending $f \in (K[z_0, \ldots, z_N]/I_X)_m$ to $(f(v_1), \ldots, f(v_d))$, where $v_i \in K^{N+1}$ are chosen representatives of $x_i \in \mathbb{P}^N$ for all $i = 1, \ldots, d$. It is well-defined and one-one since $f \in I_X$ if and only if $f(v_i) = 0$ for all $i = 1, \ldots, d$. When $m \geq d-1$, It is also surjective: Let $\{e_j\}_{j=1}^d$ be the standard basis of K^{N+1}. For every $j = 1, \ldots, d$, there exists $P \in K[z_0, \ldots, z_N]_m$ such that $\mathrm{ev}([P]) = e_j \in K^d$ defined as follows. For each $i \in \{1, \ldots, d\} - \{j\}$, choose a linear function l_i on K^{N+1} such that $l_i(v_i) = 0$ and $l_i(v_j) = 1$. (This is possible since v_i and v_j cannot be linearly dependent, or otherwise $p_i = p_j$.) For $i = d+1, \ldots, m+1$, choose a linear function l_i with $l_i(v_j) = 1$. Then $P = \prod_{i \neq j} l_i$ has degree m and satisfies $P(v_k) = 0$ for $k \in \{1, \ldots, d\} - \{j\}$ and $P(v_j) = 1$. Thus $(K[z_0, \ldots, z_N]/I_X)_m \cong K^d$ as vector spaces, and hence $h_X(m) = \dim(K[z_0, \ldots, z_N]/I_X)_m = d$.

(b) Consider the pull-back $K[z_0, \ldots, z_N]_m \to K[Z_0, Z_1]_{Nm}$ by the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^N$. It is surjective: Every monomial in Z_0 and Z_1 of degree Nm can be written as $Z_0^{jN} Z_1^{kN} Z_0^p Z_1^q$ for some $j, k, p, q \in \mathbb{Z}_{\geq 0}$ with $p, q < N$. Then it is the image of $z_0^j z_N^k z_q$ when $q \neq 0$, or $z_0^j z_N^k$ when $q = 0$. Moreover, the kernel is exactly those polynomials in $K[z_0, \ldots, z_N]_m$ which vanish on the rational normal curve. Thus $(K[z_0, \ldots, z_N]/I_X)_m \cong K[Z_0, Z_1]_{Nm}$ as vector spaces, which has dimension $Nm + 1$.

(c) Let $n = \dim X$. Let P be a linear subspace of dimension $N - n$ which intersects X transversely. P is the zero set of n linear functions $l_1, \ldots, l_n$. The number of intersection points between X and P is $d = \deg X$. Let $X^{(i)} = X \cap \{l_1 = \ldots = l_i = 0\}$ for $i = 0, \ldots, n$. Then for all $i = 0, \ldots, n-1$, we have a homomorphism

$$A(X^{(i)})_m \to A(X^{(i)})_{m+1}$$

given by multiplication by l_{i+1}. Here $A(X^{(i)}) := K[z_0, \ldots, z_N]/I_{X^{(i)}}$ denotes the coordinate ring of $X^{(i)}$. Since the intersection is transverse, this homomorphism is injective. Moreover

$$A(X^{(i)})_{m+1}/\mathrm{Im}(A(X^{(i)})_m) \cong A(X^{(i+1)})_{m+1}$$

given by restriction. Thus

$$h_{X^{(i)}}(m+1) - h_{X^{(i)}}(m) = h_{X^{(i+1)}}(m+1)$$

for all $m \in \mathbb{N}$.

By (b), $h_{X^{(n)}}(m) = d$ for all sufficiently large m. We conclude that $h_X = h_{X^{(0)}}$ is a polynomial.

Day 3

(3) (a) By Riemann-Roch, $h^0(K_X) - h^0(O_X) = \deg(K_X) - g + 1$. Since $h^0(O_X) = 1$ and $h^0(K_X) = g$, $\deg(K_X) = 2g - 2$.

(b) $\deg K_X = 0$ while $h^0(K_X) = 1$. Thus there exists a meromorphic function f such that $K_X + (f)$ is an effective divisor. $K_X + (f)$ is of degree 0 because (f) and K_X are. This forces $K_X + (f) = 0$. It follows that $K_X \sim 0$.

(c) Since D is effective,

$$\begin{aligned}\dim |D| &= h^0(D) - 1 \\ &= h^0(K - D) + \deg D - g \\ &\leq h^0(K) + \deg D - g \\ &= \deg D.\end{aligned}$$

Equality holds if and only if $h^0(K - D) = h^0(K)$. Obviously $D = 0$ implies this equality. When $g = h^0(K) = 0$, $0 \leq h^0(K-D) \leq h^0(K) = 0$. Thus $h^0(K - D) = h^0(K) = 0$ and equality holds.

Conversely, suppose equality holds, yet $g \neq 0$. Then $D \sim 0$, and since D is effective, $D = 0$.

FALL 2012 - Qualifying Exams Solutions for Algebra

Day 1

A1. The characteristic polynomial is $T^n - 1$. The Galois group of $K = \mathbb{F}_{p^n}$ over $\mathbb{F}_p$ is the cyclic group of order n generated by F. By the normal basis theorem there is an element $v \in K$ such that

$$v, Fv, \cdots, F^{n-1}v$$

forms an $\mathbb{F}_p$-basis of K. So $\det(T \cdot I_n - F) = T^n - 1$.

Day 2

A2.

(a) It fixes the generator of I, hence preserves the ideal I.

(b) If P is a prime ideal, and if $s, t \in S$ satisfy $s \cdot t \in \tau(P)$, then $\tau^{-1}(s) \cdot \tau^{-1}(t) \in P$ and either $\tau^{-1}(s) \in P$ or $\tau^{-1}(t) \in P$, hence $s \in \tau(P)$ or $t \in \tau(P)$. In addition, by definition $1 \notin P$, which implies $1 \notin \tau(P)$.

The image under τ of the principal ideal generated by α is that generated by $\tau(\alpha)$.

(c) We have $R/\mathfrak{p} = \mathbb{Q}[x, y]/(x, y) = \mathbb{Q}$, which is an integral domain.

(d) First, we have $\mathfrak{p}^2 = (\bar{x})$. For $\subseteq$, note $\bar{y}^2 = \bar{x}(\bar{x}^2 - 1)$. For $\supseteq$, note that $\mathfrak{p}^2$ contains both $\bar{x}^2$ and $-\bar{y}^2 = \bar{x} - \bar{x}^3$.

Suppose $\mathfrak{p} = (\alpha)$, and write $\alpha = P_1(x) + P_2(x)\bar{y}$. Since $\mathfrak{p}$ is fixed by σ, we have

$$\mathfrak{p}^2 = \mathfrak{p}\sigma(\mathfrak{p}) = (P_1(x)+P_2(x)\bar{y})(P_1(x)-P_2(x)\bar{y}) = (P_1(x)^2-P_2(x)^2(x^3-x)).$$

Because this ideal contains x, there are $Q_1(x)$ and $Q_2(x)$ such that

$$x = (P_1^2 - P_2^2 \cdot (x^3 - x))(Q_1 + \bar{y}Q_2)$$

in R. First, by taking half the trace $((1 + \sigma)/2)$, we may assume $Q_2 = 0$. Then the degree consideration in $\mathbb{Q}[x]$ leads to $P_2 = 0$ and $P_1(x) \in \mathbb{Q}^\times$, which is absurd.

Day 3

A3.

(a) $|G| = (p^2 - 1)(p^2 - p) = p(p - 1)^2(p + 1)$, $|G'| = p(p - 1)(p + 1)$.

(b) The upper unitriangular matrices.

(c) The units act trivially on X. Conversely, any $g \in G$ fixing every line fixes both $[1 : 0]$ and $[0 : 1]$, hence g is diagonal. If the entries were distinct, it would fail to fix $[1 : 1]$.

(d) From (c) we get an injection $\mathrm{PGL}_2(\mathbb{F}_3) \hookrightarrow S_4$, since when $p = 3$, X has 4 elements. The two groups have the same order, by (a).

FALL 2012

Solutions of Qualifying Exam Problems in Algebraic Topology in September 2012

Day 1

Problem 1 *(Third Homotopy Group of 2-Sphere).* Let $\Phi : \mathbb{C}^2 - \{0\} \to \mathbb{CP}^1$ be defined by mapping the inhomogneous coordinates (z_1, z_2) of $\mathbb{C}^2$ to the homogeneous coordinates $[z_1, z_2]$ of the complex projective line $\mathbb{CP}^1$. Let $f : S^3 \to S^2$ be defined by restricting Φ to the unit 3-sphere in $\mathbb{C}^2$. Define the group homomorphism $\gamma : \mathbb{Z} \to \pi_3(S^2)$ by setting $\gamma(1)$ to be the element of $\pi_3(S^2)$ defined by f. Compute the kernel and the cokernel of the group homomorphism $\gamma : \mathbb{Z} \to \pi_3(S^2)$. Justify each step of your computation.

Solution. The action of $\mathbb{C} - \{0\}$ on $\mathbb{C}^2 - \{0\}$ defined by scalar multiplication of vectors in $\mathbb{C}^2$ makes $\Phi : \mathbb{C}^2 - \{0\} \to \mathbb{CP}^1$ a principal bundle with structure group $\mathbb{C} - \{0\}$. The map $f : S^3 \to S^2$ defined by restricting Φ to the unit 3-sphere in $\mathbb{C}^2$ is the principal bundle with circle group S^1 as the structure group and is, in particular, a fiber bundle over S^2 with fiber S^1. The exact sequence of homotopy groups for the fiber bundle $f : S^3 \to S^2$ is

$$\cdots \to \pi_i\left(S^1\right) \to \pi_i\left(S^3\right) \to \pi_i\left(S^2\right) \to \pi_{i-1}\left(S^1\right) \to \cdots$$

and, in particular, the sequence

$$\pi_3\left(S^1\right) \to \pi_3\left(S^3\right) \to \pi_3\left(S^2\right) \to \pi_2\left(S^1\right)$$

is exact. Since the universal cover of S^1 is $\mathbb{R}$ which is contractible and since any continuous map from the simply-connected 3-sphere S^3 or 2-sphere S^2 to S^1 can be lifted to the contractible universal cover $\mathbb{R}$ of S^1, it follows that both $\pi_3(S^1)$ and $\pi_2(S^1)$ vanish and from the above exact sequence of four terms the map $\pi_3(S^3) \to \pi_3(S^2)$ induced by $f : S^3 \to S^2$ is an isomorphism.

Since $\pi_j(S^3)$ is trivial for $j = 1, 2$ (as every element of $\pi_j(S^3)$ for $j = 1, 2$ can be represented by a continuous map $S^j \to S^3$ whose image misses some point of S^3), it follows from Hurewicz's theorem (relating homotopy groups to homology groups) that the map $\pi_3(S^3) \to H_3(S^3)$ is an isomorphism and, in particular, the homotopy group $\pi_3(S^3)$ is the cyclic group $\mathbb{Z}$ whose generator is represented by the identity map of S^3. Hence the group homomorphism $\gamma : \mathbb{Z} \to \pi_3(S^2)$ is an isomorphism and both its kernel and cokernel are trivial.

Day 2 **Problem 2** *(Fundamental Groups of Spaces Obtained by Glueing).* Denote by $\mathbb{RP}^2$ the real projective plane (which is the quotient of the 2-sphere with antipodal points identified). Denote by T^2 the real 2-dimensional torus (which is the quotient of a closed rectangle with opposite sides identified). Let D be the interior of a closed disk in T^2 whose boundary is C. Let G be the interior of a closed disk in $\mathbb{RP}^2$ whose boundary is E. Let X be the space obtained by glueing $T^2 - D$ to $\mathbb{RP}^2 - G$ along a homeomorphism between the two circles C and E. Compute the fundamental group of X by describing a presentation of it. Then compute $H_1(X, \mathbb{Z})$.

Solution. Let $f : C \to E$ be the homeomorphism used to glue together $T^2 - D$ and $\mathbb{RP}^2 - G$ to construct X. The fundamental group $\pi_1(X)$ of X will be computed by applying the theorem of van Kampen to $X = (T^2 - D) \cup_f (\mathbb{RP}^2 - G)$. The fundamental group $\pi_1 (T^2 - D)$ of $T^2 - D$ is the free group generated by two elements a and b which are represented by two standard basis loops of T^2 avoiding the topological closure of D. The removal of D from T^2 makes the relation $aba^{-1}b^{-1} = 1$ in $\pi_1 (T^2 - D)$ disappear, because when T^2 is represented by identifying the opposite sides of a rectangle, the removal of a disk in the center of the rectangle makes the relation obtained by going around the boundary of the rectangle impossible. From this picture of removing a disk in the center of rectangle, we know that going around the boundary of the rectangle shows that $aba^{-1}b^{-1}$ is homotopic in $T^2 - D$ to a loop going once around the circle C (or E under identification by f). The space $\mathbb{RP}^2 - G$ is the same as the Möbius band, as one can easily see by considering the map from S^2 minus two antipodal disks to $\mathbb{RP}^2 - G$ defined by identifying antipodal points. The generator c of the fundamental group $\pi_1 (\mathbb{RP}^2 - G)$ of the Möbius band $\mathbb{RP}^2 - G$ is the loop represented by going around the center line of the Möbius band once. The loop c^2 is homotopic in $\mathbb{RP}^2 - G$ to going once around the circle E (or C identified by f), because going around the edge of the Möbius band once is the same as going around the the centerline of the Möbius band twice. By van Kampen's theorem, c^2 needs to be identified with $aba^{-1}b^{-1}$, because both represent going around C or E once (which are identified by the glueing homeomorphism f). Hence the fundamental group $\pi_1(X)$ of X is equal to the free group generated by three elements a, b, c subject to one single relation $c^{-2}aba^{-1}b^{-1} = 1$. We can compute $H_1(X, \mathbb{Z})$ by abelianizing $\pi_1(X)$. In the abelianization of $\pi_1(X)$ the element $aba^{-1}b^{-1}$ of $\pi_1(X)$ becomes 1 and the single relation $c^{-2}aba^{-1}b^{-1} = 1$

in $\pi_1(X)$ becomes the single relation $c^{-2}=1$. Hence

$$H_1(X,\mathbb{Z}) \approx \mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}).$$

Day 3 **Problem 3** *(Universal Cover of One-Point Union of Two Real Projective Planes).* Let $\mathbb{RP}^2$ denote the real projective plane (which is the quotient of the 2-sphere with antipodal points identified). Let X be the one point union $\mathbb{RP}^2 \vee \mathbb{RP}^2$ (or wedge sum) of two real projective planes (*i.e.,* the result obtained by identifying, in the disjoint union of two real projective planes, one point on one identified with one point on the other). Find the universal cover of X.

Solution. Let $\tilde{X} = \bigcup_{n\in\mathbb{Z}} (S^2 + (2n+1,0,0))$, where S^2 is the unit 2-sphere (centered at the origin of radius 1) in $\mathbb{R}^3$ and $S^2 + (2n+1,0,0)$ means the translate of S^2 by the vector $(2n+1,0,0)$ so that $S^2 + (2n+1,0,0)$ is the 2-sphere in $\mathbb{R}^3$ centered at the origin of radius 1. The space $\tilde{X}$ is an infinite string of touching 2-spheres of radius 1 centered at $(2n+1,0,0)$ touching the two adjacent 2-spheres. Let $\varphi:\mathbb{R}^3\to\mathbb{R}^3$ be the map defined by $\varphi(x,y,z)=(-x+2,-y,-z)$ and $\psi:\mathbb{R}^3\to\mathbb{R}^3$ be the map with $\psi(x,y,z)=(-x-2,-y,-z)$. The map φ, when restricted to the 2-sphere of radius 1 centered at $(1,0,0)$, is simply the antipodal map on that 2-sphere. The map ψ, when restricted to the 2-sphere of radius 1 centered at $(-1,0,0)$, is simply the antipodal map on that 2-sphere. The group G of transformations in $\mathbb{R}^3$ generated by φ and ψ acts free on $\tilde{X}$ to give the quotient X. Since $\tilde{X}$ is simply connected, $\tilde{X}$ is the universal cover of X. The argument given up to this point is already the complete rigorous solution of the problem of finding the universal cover $\tilde{X}$ of X.

If one wants to know how one arrives at the candidate $\tilde{X}$ as the universal cover of X, one can do it either geometrically or algebraically.

The geometric way to arrive at the candidate $\tilde{X}$ is to find all liftings to $\tilde{X}$ of the universal cover $S^2\to\mathbb{RP}^2$ of the first summand $\mathbb{P}^2$ of $\mathbb{RP}^2\vee\mathbb{RP}^2$ (with the image for each individual lifting a 2-sphere in $\tilde{X}$) and also all the liftings of the universal cover $S^2\to\mathbb{RP}^2$ of the second summand $\mathbb{P}^2$ of $\mathbb{RP}^2\vee\mathbb{RP}^2$ (again with the image for each individual lifting a 2-sphere in $\tilde{X}$). The first sequence of 2-spheres alternates between the second sequence of 2-spheres with points of touching being the inverse images of the common point of the two copies of $\mathbb{P}^2$ in $\mathbb{RP}^2\vee\mathbb{RP}^2$. The universal cover is the union of the two sequences of 2-spheres alternatingly touching each other to form an infinite sting of touching 2-spheres. For the rigorous proof that such a construction of $\tilde{X}$ is indeed the universal cover of X, one goes back to the argument in the preceding paragraph.

The algebraic way to arrive at the candidate $\tilde{X}$ is to use the theorem of van Kampen to conclude that the fundamental group of $\mathbb{RP}^2 \vee \mathbb{RP}^2$ is equal to the group $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ amalgamated from the fundamental group $\mathbb{Z}/2\mathbb{Z}$ of the first summand of $\mathbb{RP}^2 \vee \mathbb{RP}^2$ and the fundamental group $\mathbb{Z}/2\mathbb{Z}$ of the second summand of $\mathbb{RP}^2 \vee \mathbb{RP}^2$. The map $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\varphi(x, y, z) = (-x + 2, -y, -z)$ can be used to represent the generator of the fundamental group $\mathbb{Z}/2\mathbb{Z}$ of the right summand of $\mathbb{RP}^2 \vee \mathbb{RP}^2$, because it represents the antipodal map of the 2-sphere in $\mathbb{R}^3$ of radius 1 centered at $(1, 0, 0)$. The map $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\psi(x, y, z) = (-x - 2, -y, -z)$ can be used to represent the generator of the fundamental group $\mathbb{Z}/2\mathbb{Z}$ of the left summand of $\mathbb{RP}^2 \vee \mathbb{RP}^2$, because it represents the antipodal map of the 2-sphere in $\mathbb{R}^3$ of radius 1 centered at $(-1, 0, 0)$. The group G generated by φ and ψ is the fundamental group $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ of $\mathbb{RP}^2 \vee \mathbb{RP}^2$. The orbit of the union of the two unit 2-spheres centered respectively at $(1, 0, 0)$ and $(0, 0, 1)$ under the group G is the universal cover $\tilde{X}$ of X. Again, for the rigorous proof that such an orbit $\tilde{X}$ is indeed the universal cover of X, one goes back to the argument in the first paragraph of this solution of the problem.

FALL 2012 - Qualifying Exam Solutions for Complex Analysis

1. Differential Geometry

Day 1

Exam I, Complex analysis. The integrand $z^5 \sin(\frac{1}{z^2})$ is an analytic function on the punctured complex plane ($0 < |z| < \infty$). The Taylor series for $\sin(u)$ is

$$\sin(u) = \sum_{n=0}^{\infty} \frac{(-)^n}{(2n+1)!} u^{2n+1}.$$

As we are in the domain $0 < |z| < \infty$, we can substitute with $u = z^{-2}$:

$$z^5 \sin(\frac{1}{z^2}) = \sum_{k=0}^{\infty} \frac{(-)^n}{(2n+1)!} z^{-4n+3},$$

The pole for $z = 0$ is at $n = 1$. It follows that the residue is $-\frac{1}{3!}$. For the integral, we see that the function is analytic everywhere within the first circle $|z| < 1$, with the exception of the $z = 0$. There are no singularities on the boundary. The second contour is a deformation of the first one without meeting a singularity. So it has the same value. It follows that the residue at $z = 0$ will contribute twice and the final answer is $-2\frac{2\pi i}{3!} = -\frac{2\pi i}{3}$.

Day 2

Exam II, Q2, Complex analysis. The residue of the Γ function at $z = -n$ is $\frac{(-1)^n}{n!}$.

Defining the Γ function by the integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt,$$

we have

$$\Gamma(z) = \int_1^\infty t^{z-1} e^{-t} dt + \int_0^1 dt e^{-t} t^{z-1} = \int_1^\infty t^{z-1} e^{-t} dt + \int_0^1 dt t^{z-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n$$

$$= \int_1^\infty t^{z-1} e^{-t} dt + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z+n}.$$

The first integral is an analytic function of z and the second terms shows that the residue at $z = -n$ is $\frac{(-1)^n}{n!}$.

Alternatively, you can the functional relation $\Gamma(z+1) = z\Gamma(z)$ ti show that the residue at $z = 0$ is $\Gamma(1) = 1$. You then reduce the residue of $\Gamma(z)$ at $z = -n$ to the residue at $z = 0$ using recursively the functional relation $\Gamma(z) = \frac{\Gamma(z+1)}{z}$.

Day 3

Exam III, Q2, Complex analysis. We consider analytic functions such that

$$||f||_2^2 := \int_U f(z)\overline{f(z)} dz \leq \infty.$$

Using Cauchy integral formula, an analytic function f of $L^2(U)$ admits the following estimate for every compact K strictly included in U:

$$sup_{z\in K}|f(z)| \leq C_K||f||_2,$$

where C_K is a constant depending on the compact space K. You can use this estimate together with Cauchy-Schwarz inequality to prove the uniform convergence of the sequence (f_n). To show that the limit is analytic, you can use Moreva theorem.

FALL 2012 - Solutions for Differential Geometry

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Differential geometry, Paul

Day **1.** (a) Prove that SU_N (the set of $N \times N$ unitary matrices with determinant 1) is a submanifold of $M_N(\mathbb{C})$ (the set of $N \times N$ matrices with entries in $\mathbb{C}$).

(b) Precise the dimension of SU_N and its tangent space at identity.

(c) Prove that the submanifolds SL_N (the set of $N \times N$ matrices with determinant 1) and U_N (the set of $N \times N$ unitary matrices) of $M_N(\mathbb{C})$ do not intersect transversally.

Solution. (a) Assume first that SU_N is a submanifold in a neighborhood of Id: it is locally the zero set of a submersion F. Let $M \in SU_N$. Let $L_M : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ be the left multiplication by M. It is a diffeomorphism with inverse $L_{M^{-1}}$. Then SU_N is locally, in a neighborhood of M, the zero set for the submersion $F \circ L_{M^{-1}}$.

Hence we just need to consider the case of a neighborhood of Id. Let $E = \{M \in M_N(\mathbb{C}) : M^t = \overline{M}\}$, and consider the map

$$\Phi : \left\{ \begin{array}{ccc} M_N(\mathbb{C}) & \to & E \times \mathbb{R} \\ M & \mapsto & (M^t\overline{M}, \Im(\det(M))) \end{array} \right. .$$

We have $\Phi^{(-1)}(Id, 0) = \{M \in M_N(\mathbb{C}) : \det M = \pm 1\}$, so in a neighborhood of identity $\Phi(M) = (Id, 0)$ is an equation for SU_N: we need to check that Φ is a submersion at Id.

A calculation yields $d\Phi_{Id}(H) = (H^t + \overline{H}, \Im(Tr(H)))$. For any $(M, \lambda) \in E \times \mathbb{R}$, one therefore can find H such that $d\Phi_{Id}(H) = (M, \lambda)$ (choose for example H_0 with a real trace such that $M = H_0 + \overline{H_0}$, and H to be H_0 with $\mathrm{i}\lambda$ added in the upper left entry). This proves that $d\Phi_{Id}$ is surjective, concluding the proof.

(b) The tangent space at identity for SU_N is the kernel of $d\Phi_{Id}(H) = (H^t + \overline{H}, \Im(Tr(H)))$, that is to say matrices with trace 0 and equal to the opposite of transpose of their conjugate. The dimension of $SU(N)$ is therefore $2N(N-1)/2 + N - 1 = N^2 - 1$.

(c) SL_N and U_N do not intersect transversally at Id. Indeed the tangent spaces of these two varieties at Id are both included in the sub-vector space of $M_N(\mathbb{C})$ consisting in matrices with purely imaginary trace.

Day **2.** Let M be a dimension 2 Riemannian manifold. We write its metric in polar coordinates as $dr^2 + f(r, \theta)^2 d\theta^2$. Prove that its Gaussian curvature is

$K = -f^{-1}\frac{\partial^2 f}{\partial r^2}$.

Solution. Let $f' = \partial f/\partial r$ and $f'' = \partial^2 f/\partial r^2$. Then

$$\begin{aligned}(R_{e_r,e_\theta}e_r)\cdot e_\theta &= \nabla_{e_r}\nabla_{e_\theta}e_r\cdot e_\theta - \nabla_{e_\theta}\nabla_{e_r}e_r\cdot e_\theta - \nabla_{[e_r,e_\theta]}e_r\cdot e_\theta\\ &= \nabla_{e_r}\left(\frac{f'}{f}e_\theta\right)\cdot e_\theta - 0 - \nabla_{-\frac{f'}{f}e_\theta}e_r\cdot e_\theta\\ &= \frac{ff''-f'^2}{f^2} + \frac{f'}{f}\nabla_{e_r}e_\theta\cdot e_\theta + \frac{f'}{f}\nabla_{e_\theta}e_\theta\\ &= \frac{ff''-f'^2}{f^2} + \frac{f'^2}{f^2}e_\theta\cdot e_\theta\\ &= \frac{f''}{f},\end{aligned}$$

so $K = -\frac{f''}{f}$.

Day **3.** Exercise about calculating the Levi-Civita connection for the n-dimensional hyperbolic space.

Solution. Remember that $\nabla_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k\partial_k$, where

$$\Gamma_{ij}^k = \frac{1}{2}\sum_n g^{kn}(\partial_i g_{nj} + \partial_j g_{ni} - \partial_n g_{ij}) = \frac{1}{2}g^{kk}(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}),$$

the last equality because g is diagonal. A calculation then yields, on $\{i = k, j = n\} \cup \{j = k, i = n\}$, $\Gamma_{ij}^k = -x_n^{-1}$, and on $\{i = j, k = n, i \neq n\}$, $\Gamma_{ij}^k = x_n^{-1}$. For all other indices, $\Gamma_{ij}^k = 0$.

FALL 2012 - Qualifying Exam Solutions for Real Analysis

Solution:

Day 1. (a) Let $f = 0$ and

$$f_n(x) = \sqrt{n}, \quad 0 \le x \le 1/n$$

and $f_n(x) = 0$ otherwise. Then $f_n(x) \to 0$ a.e. but $\|f_n\|_2 = 1$ for all n. This is a counterexample.
(b) Let $g_n = f_n - f$ and for $M > 0$ rewrite $g_n = h_n + k_n$ where

$$h_n(x) = g_n(x)\mathbf{1}(|g_n(x)| > M).$$

Then by the dominated convergence theorem, we have

$$\lim_{n\to\infty} \int |k_n(x)| dx = 0.$$

Also,

$$\int |h_n(x)| dx \le M^{-1} \int |h_n(x)|^2 dx \le M^{-1} \int |g_n(x)|^2 dx \le M^{-1}(\|f_n\|_2 + \|f\|_2)^2 \le 4/M$$

Since M can be arbitrary large, this proves that

$$\lim_{n\to\infty} \int |g_n(x)| dx = 0.$$

Day 2. Define the Fourier transform by

$$f(p) = \int e^{-ixp} f(x) \mathrm{d}x$$

Take the Fourier transform in x to get

$$\partial_t \hat{u}(t,p) = -\frac{p^2}{2}\hat{u}(t,p)$$

Here the assumptions on $u(t,x)$ make sure that the Fourier transforms can be taken on both sides of the equation. Hence

$$\hat{u}(t,p) = e^{-tp^2/2}\hat{u}(0,p) = e^{-tp^2/2}\hat{f}(p)$$

Take the inverse Fourier transform to get

$$u(t,x) = [g(t,\cdot) * f](x)$$

where $g(t,\cdot)$ is the inverse transform of $e^{-tp^2/2}$. Using the well-known formula of the Fourier transform of Gaussian, i.e.,

$$g(t,x) = \frac{C_1}{\sqrt{t}} e^{-C_2 x^2/t}$$

we prove (a).

To prove (b), we have

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C \int \mathrm{d}x\, t^{-1} \left[\int_{\mathbb{R}} \exp\left(-C\frac{(x-y)^2}{t}\right) f(y) \mathrm{d}y \right]^2$$

Since $f \in L_1$, we can use Holder (or Jensen) inequality to have

$$\left[\int_{\mathbb{R}} \exp\left(-C\frac{(x-y)^2}{t}\right) f(y)\mathrm{d}y\right]^2 \le \int_{\mathbb{R}} \exp\left(-2C\frac{(x-y)^2}{t}\right) f(y)\mathrm{d}y \int_{\mathbb{R}} f(y)\mathrm{d}y$$

Combining these two inequalities, we have

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C\int \mathrm{d}x\, t^{-1}\int_{\mathbb{R}} \exp\left(-2C\frac{(x-y)^2}{t}\right) f(y)\mathrm{d}y\|f\|_{L^1(\mathbb{R})} \le Ct^{-1/2}\|f\|^2_{L^1(\mathbb{R})}.$$

This proves (b).

Day

3. Let $Y_j = X_j - 1$. We have

$$\mathbb{E}\Big(n^{-1}\sum_{j=1}^n Y_j\Big)^2 = n^{-2}\mathbb{E}\sum_{i,j=1}^n Y_iY_j \le n^{-2}\mathbb{E}\sum_{i,j=1}^n f(|i-j|) \le n^{-1}A$$

By the Chebyshev's inequality, we have

$$\mathbb{P}(n^{-1}\sum_{j=1}^n Y_j \ge 1) \le A/n$$

Hence $A = B$.

Solutions of Qualifying Exams I, 2013 Fall

1. (ALGEBRA) Consider the algebra $M_2(k)$ of 2×2 matrices over a field k. Recall that an *idempotent* in an algebra is an element e such that $e^2 = e$.

(a) Show that an idempotent $e \in M_2(k)$ different from 0 and 1 is conjugate to

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

by an element of $GL_2(k)$.

(b) Find the stabilizer in $GL_2(k)$ of $e_1 \in M_2(k)$ under the conjugation action.

(c) In case $k = \mathbb{F}_p$ is the prime field with p elements, compute the number of idempotents in $M_2(k)$. (Count 0 and 1 in.)

Solution. (a) Since $e \neq 0, 1$, the image and the kernel of e are both one-dimensional. Let v_1 be a nonzero element in the image, so $v_1 = e(v_0)$ for some $v_0 \in k^{\oplus 2}$. Then

$$e(v_1) = e(e(v_0)) = e^2(v_0) = e(v_0) = v_1.$$

Pick a nonzero element v_2 in the kernel of e, and we get a basis of $k^{\oplus 2}$ in which e takes the form e_1.

(b) For a general element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to be in the stabilizer, it must satisfy $ge_1 = e_1 g$. Writing the equation in four entries out, one sees that it means $b = c = 0$ (and a, d arbitrary). So the centralizer is the subgroup of diagonal matrices.

(c) By (a) and (b), the set of rank 1 idempotents is in bijection with $GL_2(\mathbb{F}_p)/T(\mathbb{F}_p)$, whose cardinality is

$$\frac{(p^2-1)(p^2-p)}{(p-1)(p-1)} = (p+1)p.$$

So the total number of idempotents is equal to $p^2 + p + 2$.

2. (Algebraic Geometry) (a) Find an everywhere regular differential n-form on the affine n-space $\mathbb{A}^n$.

(b) Prove that the canonical bundle of the projective n-dimensional space $\mathbb{P}^n$ is $\mathcal{O}(-n-1)$.

Solution *(Sketch).* Part (a) is really a hint for Part (b). Letting $x_1, x_2, \ldots, x_n$ be affine ($\mathbb{A}^n$) coordinates, put $\omega := dx_1 \wedge dx_2 \cdots \wedge dx_n$ giving (a) . Denoting the corresponding homogenous $\mathbb{P}^n$ coordinates $t_0, t_1, \ldots, t_n$, with $x_i := t_i/t_0$ for $i = 1, 2, \ldots, n$ extend ω to $\mathbb{P}^n$ writing $dx_i = dt_i/t_0 - t_i/t_0^2 dt_0$ and wedging to discover that the divisor of poles of ω is $(n+1)H$ where H is the hyperplane at infinity ($t_0 = 0$) and then conclude (appropriately).

3. (Complex Analysis) *(Bol's Theorem of 1949).* Let $\tilde{W}$ be a domain in $\mathbb{C}$ and W be a relatively compact nonempty subdomain of $\tilde{W}$. Let $\varepsilon > 0$ and G_ε be the set of all $(a, b, c, d) \in \mathbb{C}$ such that $\max(|a-1|, |b|, |c|, |d-1|) < \varepsilon$. Assume that $cz + d \neq 0$ and $\frac{az+b}{cz+d} \in \tilde{W}$ for $z \in W$ and $(a, b, c, d) \in G_\varepsilon$. Let $m \geq 2$ be an integer. Prove that there exists a positive integer ℓ (depending on m) with the property that for any holomorphic function φ on $\tilde{W}$ such that

$$\varphi(z) = \varphi\left(\frac{az+b}{cz+d}\right) \frac{(cz+d)^{2m}}{(ad-bc)^m}$$

for $z \in W$ and $(a, b, c, d) \in G_\varepsilon$, the ℓ-th derivative $\psi(z) = \varphi^{(\ell)}(z)$ of $\varphi(z)$ on $\tilde{W}$ satisfies the equation

$$\psi(z) = \psi\left(\frac{az+b}{cz+d}\right) \frac{(ad-bc)^{\ell-m}}{(cz+d)^{2(\ell-m)}}$$

for $z \in W$ and $(a, b, c, d) \in G_\varepsilon$. Express ℓ in terms of m.

Hint: Use Cauchy's integral formula for derivatives.

Solution. Let

$$Az = \frac{az+b}{cz+d}$$

for $A \in G_\varepsilon$. We take a positive integer ℓ which we will determine later as a function of n. We use Cauchy's integral formula for derivatives to take the ℓ-th derivative $\psi(z)$ of $\varphi(z)$. For $z \in \tilde{W}$ we use $U(z)$ to denote an open neighborhood of z in $\tilde{W}$ and use $\partial U(z)$ to denote its boundary. The ℓ-th derivative ψ of φ at $z \in \tilde{W}$ is given by the formula

$$\psi(z) = \frac{\ell!}{2\pi\sqrt{-1}} \int_{\zeta \in \partial U(z)} \frac{\varphi(\zeta) d\zeta}{(\zeta - z)^{\ell+1}}$$

and

$$\psi(Az) = \frac{\ell!}{2\pi\sqrt{-1}} \int_{\zeta \in \partial U(Az)} \frac{\varphi(\zeta) d\zeta}{(\zeta - Az)^{\ell+1}} \quad \text{when } Az \in \tilde{W}.$$

It follows from

$$\zeta \in U(z) \Longleftrightarrow A\zeta \in U(Az),$$
$$\zeta \in \partial U(z) \Longleftrightarrow A\zeta \in \partial U(Az),$$

with the change of variable $\zeta \mapsto A\zeta$, that

$$\int_{\zeta \in \partial U(Az)} \frac{\varphi(\zeta) d\zeta}{(\zeta - Az)^{\ell+1}} = \int_{A\zeta \in \partial U(Az)} \frac{\varphi(A\zeta) d(A\zeta)}{(A\zeta - Az)^{\ell+1}}.$$

From the following straightforward direct computation of the discrete version of the formula for the derivative of fractional linear transformation

$$\begin{aligned}
A\zeta - Az &= \frac{a\zeta + b}{c\zeta + d} - \frac{az + b}{cz + d} \\
&= \frac{(a\zeta + b)(cz + d) - (az + b)(c\zeta + d)}{(c\zeta + d)(cz + d)} \\
&= \frac{(ac\zeta z + bcz + ad\zeta + bd) - (ac\zeta z + adz + bc\zeta + bd)}{(c\zeta + d)(cz + d)} \\
&= \frac{(ad - bc)(\zeta - z)}{(c\zeta + d)(cz + d)}
\end{aligned}$$

we obtain

$$\begin{aligned}
\int_{A\zeta \in \partial U(Az)} \frac{\varphi(A\zeta) d(A\zeta)}{(A\zeta - Az)^{\ell+1}} &= \int_{\zeta \in \partial U(z)} \frac{\varphi\left(\frac{a\zeta+b}{c\zeta+d}\right) \frac{ad-bc}{(c\zeta+d)^2} d\zeta}{\frac{(ad-bc)^{\ell+1}(\zeta-z)^{\ell+1}}{(c\zeta+d)^{\ell+1}(cz+d)^{\ell+1}}} \\
&= \int_{\zeta \in \partial U(z)} \frac{\varphi(\zeta) \frac{(ad-bc)^m}{(c\zeta+d)^{2m}} \frac{ad-bc}{(c\zeta+d)^2} d\zeta}{\frac{(ad-bc)^{\ell+1}(\zeta-z)^{\ell+1}}{(c\zeta+d)^{\ell+1}(cz+d)^{\ell+1}}} \\
&= \frac{(cz + d)^{\ell+1}}{(ad - bc)^{\ell-m}} \int_{\zeta \in \partial U(z)} \frac{\varphi(\zeta) d\zeta}{(\zeta - z)^{\ell+1}} (c\zeta + d)^{\ell-1-2m}.
\end{aligned}$$

The extra factor $(c\zeta + d)^{\ell-1-2m}$ inside the integrand on the extreme right-hand side becomes 1 and can be dropped if $\ell - 1 - 2m = 0$, that is, if $\ell = 2m + 1$. Thus, if $\ell = 2m + 1$, then

$$\psi(Az) = \frac{(cz+d)^{\ell+1}}{(ad-bc)^{\ell-m}}\psi(z).$$

That is,

$$\psi(z) = \psi\left(\frac{az+b}{cz+d}\right)\frac{(ad-bc)^{\ell-m}}{(cz+d)^{2(\ell-m)}},$$

because $\ell = 2m+1$ implies $\ell + 1 = 2(\ell - m)$.

4. (Algebraic Topology) (a) Show that the Euler characteristic of any contractible space is 1.

(b) Let B be a connected CW complex made of finitely many cells so that its Euler characteristic is defined. Let $E \to B$ be a covering map whose fibers are discrete, finite sets of cardinality N. Show the Euler characteristic of E is N times the Euler characteristic of B.

(c) Let G be a finite group with cardinality > 2. Show that BG (the classifying space of G) cannot have homology groups whose direct sum has finite rank.

Solution. (a) The homology of a point with coefficients in a field k is $H_0 = k$, $H_i = 0$ for $i > 0$. Hence its Euler characteristic is $\sum(-1)^i \dim H_i = 1$. All contractible spaces are homotopy equivalent so their Euler characteristic is that of the point.

(b) For any open cover $\{U_i\}$, we know that the chain complex of singular chains living in U_i for some i has equivalent homology to the chain complex of all chains. Taking the cover of B by trivializing neighborhoods U_i, the chain complex of chains living in U_i receives a map from chains in E living in $\pi^{-1}(U_i)$. The latter is simply $|G|$ direct sums of the former, and the chain map between them is the "add every component" map. This shows the ranks of homology of E is N times the rank of homology of B.

(c) Strictly speaking, this problem cannot be solved based on easy machinery (as far as I know). A much more reasonable problem would be: Prove BG is not homotopy equivalent to anything made up of only finitely many cells. I did not take off points for people not distinguishing between this condition, and the condition stated in the problem itself. We know $BG = EG/G$, but EG is contractible. So $\chi(EG) = 1$. If BG has finite homology, $\chi(BG) = 1/|G|$, which cannot be an integer unless $|G| = 1$.

5. (Differential Geometry) Let $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ be the upper half plane. Let g be the Riemannian metric on H given by

$$g = \frac{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}{y^2}.$$

(H, g) is known as the half-plane model of the hyperbolic plane.

(a) Let $\gamma(\theta) = (\cos\theta, \sin\theta)$ and $\eta(\theta) = (\cos\theta + 1, \sin\theta)$ for $\theta \in (0, \pi)$ be two paths in H. Compute the angle A at their intersection point shown in Figure 1, measured by the metric g.

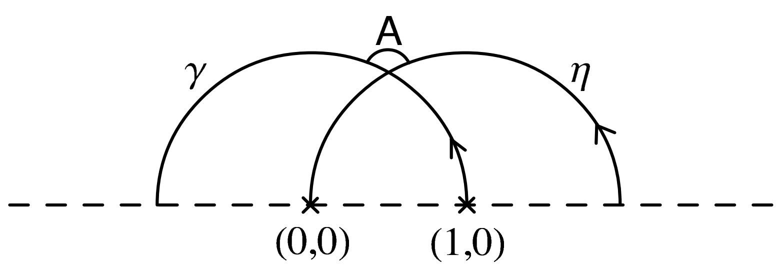

Figure 1: Angle A between the two curves γ and η in the upper half plane H.

(b) By computing the Levi-Civita connection

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{2} \Gamma_{ij}^{k} \frac{\partial}{\partial x_k}$$

of g or otherwise (where $(x_1, x_2) = (x, y)$), show that the path γ, *after arc-length reparametrization*, is a geodesic with respect to the metric g.

Solution. (a) The intersection point is $(1/2, \sqrt{3}/2)$: solving for

$$\gamma(\theta) = (\cos\theta, \sin\theta) = (\cos\phi + 1, \sin\phi) = \eta(\phi)$$

we obtain $\theta = \pi/3$, $\phi = 2\pi/3$.

The angle A satisfies

$$\begin{aligned}\cos A &= \frac{\langle \gamma'(\pi/3), -\eta'(2\pi/3)\rangle_g}{||\gamma'(\pi/3)||_g \; ||-\eta'(2\pi/3)||_g} \\ &= \frac{\langle(-\sqrt{3}/2, 1/2), (\sqrt{3}/2, 1/2)\rangle_g}{||(-\sqrt{3}/2, 1/2)||_g \; ||(\sqrt{3}/2, 1/2)||_g} \\ &= \frac{-\frac{1}{2}\frac{1}{y^2}}{\frac{1}{y^2}} \\ &= -\frac{1}{2}\end{aligned}$$

and so $A = 2\pi/3$.

(b) Using the formula

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(g_{jl,k} + g_{jl,j} - g_{jk,l})$$

one obtains

$$\Gamma^i_{jk} = \frac{-1}{y}(\delta_{ij}\delta_{k,2} + \delta_{ki}\delta_{j,2} - \delta_{jk}\delta_{i,2}).$$

After arc-length reparametrization, the tangent vectors of the path are

$$v(\theta) = \frac{\gamma'(\theta)}{||\gamma'(\theta)||_g} = (-\sin^2\theta, \sin\theta\cos\theta).$$

Then

$$\nabla_{v(\theta)} v(\theta) = v'(\theta) + \begin{pmatrix} \Gamma^1_1 & \Gamma^1_2 \\ \Gamma^2_1 & \Gamma^2_2 \end{pmatrix} \cdot v(\theta)$$

where

$$\begin{aligned}\Gamma^1_1 &= (-\sin\theta)\Gamma^1_{11} + (\cos\theta)\Gamma^1_{21} = -\cot\theta; \\ \Gamma^1_2 &= (-\sin\theta)\Gamma^1_{12} + (\cos\theta)\Gamma^1_{22} = 1; \\ \Gamma^2_1 &= (-\sin\theta)\Gamma^2_{11} + (\cos\theta)\Gamma^2_{21} = -1; \\ \Gamma^2_2 &= (-\sin\theta)\Gamma^2_{12} + (\cos\theta)\Gamma^2_{22} = -\cot\theta.\end{aligned}$$

Thus one has $\nabla_{v(\theta)} v(\theta) = 0$.

6. (Real Analysis) For any positive integer n let M_n be a positive number such that the series $\sum_{n=1}^{\infty} M_n$ of positive numbers is convergent and its limit is M. Let $a < b$ be real numbers and $f_n(x)$ be a real-valued continuous function on $[a,b]$ for any positive integer n such that its derivative $f_n'(x)$ exists for every $a < x < b$ with $|f_n'(x)| \leq M_n$ for $a < x < b$. Assume that the series $\sum_{n=1}^{\infty} f_n(a)$ of real numbers converges. Prove that

(a) the series $\sum_{n=1}^{\infty} f_n(x)$ converges to some real-valued function $f(x)$ for every $a \leq x \leq b$,

(b) $f'(x)$ exists for every $a < x < b$, and

(c) $|f'(x)| \leq M$ for $a < x < b$.

Hint for (b): For fixed $x \in (a,b)$ consider the series of functions

$$\sum_{n=1}^{\infty} \frac{f_n(y) - f_n(x)}{y - x}$$

of the variable y and its uniform convergence.

Solution. **(a)** Fix $x \in (a,b]$. For $q > p \geq 1$, by the Mean Value Theorem applied to the function $\sum_{n=p}^{q} f_n$ on $[a,x]$ we can find $a < \xi_{p,q} < x$ such that

$$\sum_{n=p}^{q} f_n(x) - \sum_{n=p}^{q} f_n(a) = (x-a) \sum_{n=p}^{q} f_n'\left(\xi_{p,q}\right),$$

which implies that

$$\begin{aligned} \left|\sum_{n=p}^{q} f_n(x)\right| &\leq \left|\sum_{n=p}^{q} f_n(a)\right| + (x-a)\left|\sum_{n=p}^{q} f_n'\left(\xi_{p,q}\right)\right| \\ &\leq \left|\sum_{n=p}^{q} f_n(a)\right| + (x-a)\sum_{n=p}^{q} M_n. \end{aligned}$$

Since both series $\sum_{n=1}^{\infty} f_n(a)$ and $\sum_{n=1}^{\infty} M_n$ are convergent and therefore Cauchy, for any $\varepsilon > 0$ we can find a positive integer N_1 such that

$$\left|\sum_{n=p}^{q} f_n(a)\right| < \frac{\varepsilon}{2}$$

for $q > p \geq N_1$ and we can find a positive integer N_2 such that

$$\left|\sum_{n=p}^{q} M_n\right| < \frac{\varepsilon}{2(x-a)}$$

for $q > p \geq N_2$. Thus for $n \geq \max(N_1, N_2)$ we have

$$\left|\sum_{n=p}^{q} f_n(x)\right| < \varepsilon$$

and the series $\sum_{n=1}^{\infty} f_n(x)$ is Cauchy. Hence the series $\sum_{n=1}^{\infty} f_n(x)$ converges to some real-valued function $f(x)$ for every $a \leq x \leq b$.

(b) Before the proof of the statement in (b), we would like to state that the uniform limit of continuous functions is continuous. That is, if $h_n(x)$ is a sequence of functions on a metric space E which converges to a function $h(x)$ on E uniformly on E and if for some $x_0 \in E$ and for every n the function $h_n(x)$ is continuous at $x = x_0$, then $h(x)$ is continuous at x_0. This results from the so-called 3ε argument as follows. Given any $\varepsilon > 0$. The uniform convergence of $h_n \to h$ on E implies that there exists some positive integer N such that $|h_N(x) - h(x)| < \varepsilon$ for all $x \in E$. Since h_N is continuous at $x = x_0$, there exists some $\delta > 0$ such that $|h_N(x) - h_N(x_0)| < \varepsilon$ for $d_E(x, x_0) < \delta$ (where $d_E(\cdot, \cdot)$ is the metric of the metric space E). Thus for $d_E(x, x_0) < \delta$ we have

$$|h(x) - h(x_0)| \leq |h(x) - h_N(x)| + |h_N(x) - h_N(x_0)| + |h_N(x_0) - h(x_0)| < 3\varepsilon,$$

which implies the continuity of h at $x = x_0$.

We now prove the statement in (b). Take $x_0 \in (a, b)$. We introduce the function $g_{n,x_0}(x)$ on $[a, b]$ which is defined by

$$\begin{cases} g_{n,x_0}(x) = \frac{f_n(x) - f_n(x_0)}{x - x_0} & \text{for } x \neq x_0 \\ g_{n,x_0}(x_0) = f_n'(x_0). \end{cases}$$

It follows from the continuity of f_n on $[a, b]$ and the existence of $f_n'(x_0)$ that g_{n,x_0} is a continuous function on $[a, b]$.

When $x \in [a,b]$ with $x \neq x_0$, by the Mean Value Theorem

$$\frac{f_n(x) - f_n(x_0)}{x - x_0} = f_n'(\xi_x)$$

for some ξ_x strictly between x_0 and x and as a consequence

$$|g_{n,x_0}(x)| = |f_n'(x_0)| \leq M_n.$$

When $x = x_0$,

$$|g_{n,x_0}(x)| = |f_n'(x_0)| \leq M_n.$$

Thus $|g_{n,x_0}(x)| \leq M_n$ for $x \in [a,b]$. From $\sum_{n=1}^{\infty} M_n \leq M < \infty$ it follows that the series $\sum_{n=1}^{\infty} g_{n,x_0}$ is uniformly convergent on $[a,b]$. It follows that the uniform limit $\sum_{n=1}^{\infty} g_{n,x_0}$ is a continuous function on $[a,b]$ by the 3ε argument given above. For $x \neq x_0$

$$\sum_{n=1}^{\infty} g_{n,x_0}(x) = \sum_{n=1}^{\infty} \frac{f_n(x) - f_n(x_0)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0}.$$

The continuity of $\sum_{n=1}^{\infty} g_{n,x_0}(x)$ at $x = x_0$ means that the limit of

$$\frac{f(x) - f(x_0)}{x - x_0}$$

exists as $x \to x_0$, which implies that $f'(x_0)$ exists and is equal to

$$\sum_{n=1}^{\infty} g_{n,x_0}(x_0) = \sum_{n=1}^{\infty} f_n'(x_0).$$

(c) From

$$f'(x_0) = \sum_{n=1}^{\infty} g_{n,x_0}(x_0) = \sum_{n=1}^{\infty} f_n'(x_0)$$

and $|f_n'(x_0)| \leq M_n$, it follows that

$$|f'(x_0)| \leq \sum_{n=1}^{\infty} M_n = M.$$

Solutions of Qualifying Exams II, 2013 Fall

1. (Algebra) Find all the field automorphisms of the real numbers $\mathbb{R}$.
Hint: Show that any automorphism maps a positive number to a positive number, and deduce from this that it is continuous.

Solution. If $t > 0$, there exists an element $s \neq 0$ such that $t = s^2$. If φ is any field automorphism of $\mathbb{R}$, then

$$\varphi(t) = \varphi(s^2) = (\varphi(s))^2 > 0.$$

It follows that φ preserves the order on $\mathbb{R}$: If $t < t'$, then

$$\varphi(t') = \varphi(t + (t' - t)) = \varphi(t) + \varphi(t' - t) > \varphi(t).$$

Any real number α is determined by the set (Dedekind's cut) of rational numbers that are less than α, and any field automorphism fixes each rational number. Therefore φ is the identity automorphism.

2. (Algebraic Geometry) What is the maximum number of ramification points that a mapping of finite degree from one smooth projective curve over $\mathbb{C}$ of genus 1 to another (smooth projective curve of genus 1) can have? Give an explanation for your answer.

Solution *(Sketch).* By the Riemann-Hurwitz formula, if we have a mapping f of finite degree d from one smooth projective (irreducible, say) curve onto another the Euler characteristic of the source curve is d times the Euler characteristic of the target *minus* a certain nonnegative number e, and moreover e is zero if and only if the mapping is unramified. Now compute: the Euler characterstic of our source and target curves is, by hypothesis, 0 and so this e is zero, and therefore the mapping is unramified.

3. (Complex Analysis) Let ω and η be two complex numbers such that $\mathrm{Im}\left(\frac{\omega}{\eta}\right) > 0$. Let G be the closed parallelogram consisting of all $z \in \mathbb{C}$ such that $z = \lambda\omega + \rho\eta$ for some $0 \leq \lambda, \rho \leq 1$. Let ∂G be the boundary of G and Let $G^0 = G - \partial G$ be the interior of G. Let $P_1, \cdots, P_k, Q_1, \cdots, Q_\ell$ be points in G^0 and let $m_1, \cdots, m_k, n_1, \cdots, n_\ell$ be positive integers. Let f be a function on G such that

$$\frac{f(z)\prod_{j=1}^{\ell}(z - Q_j)^{n_j}}{\prod_{p=1}^{k}(z - P_p)^{m_p}}$$

is continuous and nowhere zero on G and is holomorphic on G^0. Let $\varphi(z)$ and $\psi(z)$ be two polynomials on $\mathbb{C}$. Assume that $f(z+\omega) = e^{\varphi(z)} f(z)$ if both z and $z+\omega$ are in G. Assume also that $f(z+\eta) = e^{\psi(z)} f(z)$ if both z and $z+\eta$ are in G. Express $\sum_{p=1}^{k} m_p - \sum_{j=1}^{\ell} n_j$ in terms of ω and η and the coefficients of $\varphi(z)$ and $\psi(z)$.

Solution. Let $A = 0$, $B = \eta$, $C = \eta + \omega$, and $D = \omega$. Since $\operatorname{Im}\left(\frac{\omega}{\eta}\right) > 0$, it follows that going from A to B, to C, to D and then back to A is in the counterclockwise direction. By the argument principle

$$\begin{aligned}
\sum_{p=1}^{k} m_p - \sum_{j=1}^{\ell} n_j &= \frac{1}{2\pi\sqrt{-1}} \oint_{\partial G} d\log f \\
&= \frac{1}{2\pi\sqrt{-1}} \left(\int_{\overrightarrow{AB}} d\log f + \int_{\overrightarrow{BC}} d\log f + \int_{\overrightarrow{CD}} d\log f + \int_{\overrightarrow{DA}} d\log f \right) \\
&= \frac{1}{2\pi\sqrt{-1}} \left(\int_{\overrightarrow{AB}} d\log f - \int_{\overrightarrow{CD}} d\log f + \int_{\overrightarrow{BC}} d\log f - \int_{\overrightarrow{AD}} d\log f \right) \\
&= \frac{1}{2\pi\sqrt{-1}} \left(-\int_{\overrightarrow{AB}} d\varphi(z) + \int_{\overrightarrow{AD}} d\psi(z) \right) \\
&= \frac{1}{2\pi\sqrt{-1}} \left(-\varphi(\eta) + \varphi(0) + \psi(\omega) - \psi(0) \right).
\end{aligned}$$

Thus, the answer is

$$\sum_{p=1}^{k} m_p - \sum_{j=1}^{\ell} n_j = \frac{1}{2\pi\sqrt{-1}} \left(-\varphi(\eta) + \varphi(0) + \psi(\omega) - \psi(0) \right).$$

4. (Algebraic Topology) (a) Fix a basis for H_1 of the two-torus (with integer coefficients). Show that for every element $x \in SL(2, \mathbb{Z})$, there is an automorphism of the two-torus such that the induced map on H_1 acts by x. *Hint:* $SL(2, \mathbb{Z})$ also acts on the universal cover of the torus.

(b) Fix an embedding $j : D^2 \times S^1 \to S^3$. Remove its interior from S^3 to obtain a manifold X with boundary T^2. Let f be an automorphism of the two-torus and consider the glued space

$$X_f := (D^2 \times S^1) \cup_f X.$$

If X is homotopy equivalent to $D^2 \times S^1$, compute the homology groups of X_f.

Solution. (a) Given $g \in SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$ let $x : \mathbb{R}^2 \to \mathbb{R}^2$ be the induced action. Since g is in $SL(2,\mathbb{Z})$ it respects the relationship of whether two vectors in $\mathbb{R}^2$ differ by integer coordinates. So the map on the torus $[(x_1, x_2)] \mapsto [g(x_1, x_2)]$ is well-defined. This clearly sends a homology generating pair given by the curves $(x_1, 0)$ and $(0, x_2)$ to the expected images via g.

(b) There is an ambiguity in the problem about how f glues X and $D^2 \times S^1$ together; so I gave full credit regardless of whether you identified this ambiguity or not. Note $X_f = (D^2 \times S^1) \cup_{S^1 \times S^1} X$. Write $U = D^2 \times S^1$ and $V = X$. The Mayer-Vietoris sequence gives

$$\begin{tikzcd}
{} \arrow[r] & H_0(U \cap V) \arrow[r] & H_0(U) \oplus H_0(V) \arrow[r] & H_0(U \cup V) \\
{} \arrow[r] & H_1(U \cap V) \arrow[r] & H_1(U) \oplus H_1(V) \arrow[r] & H_1(U \cup V) \\
{} \arrow[r] & H_2(U \cap V) \arrow[r] & H_2(U) \oplus H_2(V) \arrow[r] & H_2(U \cup V) \\
{} \arrow[r] & H_3(U \cap V) \arrow[r] & H_3(U) \oplus H_3(V) \arrow[r] & H_3(U \cup V)
\end{tikzcd}$$

but because we know the homology of $D^2 \times S^1 \simeq S^1$ and $S^1 \times S^1$, we can fill in various groups in the long exact sequence:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & H_0(U \cup V) \\
\mathbb{Z}^2 \arrow[r, "g"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & H_1(U \cup V) \arrow[ull] \\
\mathbb{Z} \arrow[r] & 0 \oplus 0 \arrow[r] & H_2(U \cup V) \arrow[ull] \\
0 \arrow[r] & 0 \oplus 0 \arrow[r] & H_3(U \cup V) \arrow[ull]
\end{tikzcd}$$

Since g is an isomorphism, we know H_1 must inject into $\mathbb{Z}$, but the inclusion map $H_0(U \cap V) \to H_0(U) \oplus H_0(V)$ is an injection, so $H_1(U \cup V) = 0$.

We know H_0 is either equal to $\mathbb{Z}$ from the long exact sequence above, or by observing that X_f is path-connected.

If f induces an isomorphism, we see H_2 must be zero; this was the intent of the problem, but you can get a different answer based on how you interpreted the "gluing" by f.

Finally, H_3 is also isomorphic to $\mathbb{Z}$ by the exactness of the above sequence.

5. (Differential Geometry) Let $M = U(n)/O(n)$ for $n \geq 1$, where $U(n)$ is the group of $n \times n$ unitary matrices and $O(n)$ is the group of $n \times n$ orthogonal matrices. M is a real manifold called the *Lagrangian Grassmannian.*

(a) Compute and state the dimension of M.

(b) Construct a Riemannian metric which is invariant under the left action of $U(n)$ on M.

(c) Let ∇ be the corresponding Levi-Civita connection on the tangent bundle TM, and X, Y, Z be any $U(n)$-invariant vector fields on M. Using the given identity (which you are not required to prove)

$$\nabla_X Y = \frac{1}{2}[X, Y],$$

show that the Riemannian curvature tensor R of ∇ satisfies the formula

$$R(X, Y)Z = \frac{1}{4}[Z, [X, Y]].$$

Solution. (a)

$$T_{[I]}M \cong \mathfrak{u}(n)/\mathfrak{o}(n) \cong \mathrm{Sym}^2(\mathbb{R}^n)$$

where $\mathrm{Sym}^2(\mathbb{R}^n)$ denotes the space of real $n \times n$ symmetric matrices. Thus

$$\dim M = \frac{n(n+1)}{2}.$$

(b) Define a metric on $\mathrm{Sym}^2(\mathbb{R}^n)$ by

$$\langle A, B \rangle = \mathrm{tr}(AB^t) = \mathrm{tr}(AB).$$

$g \in O(n)$ acts on $T_{[I]}M \cong \mathrm{Sym}^2(\mathbb{R}^n)$ by $g \cdot A = gAg^{-1}$. Then

$$\langle g \cdot A, g \cdot B \rangle = \mathrm{tr}(g \cdot ABg^{-1}) = \langle A, B \rangle.$$

Hence this metric is invariant under the action of $O(n)$. By translating the metric to tangent spaces at other points by the action of $U(n)$, this gives a well-defined invariant metric on $U(n)/O(n)$.

(c)

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

Then

$$\begin{aligned} R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= \frac{1}{4}\left([X,[Y,Z]] - [Y,[X,Z]]\right) - \frac{1}{2}[[X,Y],Z] \\ &= \frac{1}{4}[Z,[X,Y]] \end{aligned}$$

where the last equality follows from Jacobi identity.

6. (Real Analysis) Show that there is no function $f\colon \mathbb{R} \to \mathbb{R}$ whose set of continuous points is precisely the set $\mathbb{Q}$ of all rational numbers.

Solution. For fixed $\delta > 0$ let $C(\delta)$ be the set of points $x \in \mathbb{R}$ such that for some $\varepsilon > 0$ we have $|f(x') - f(x'')| < \delta$ for all $x', x'' \in (x - \varepsilon, x + \varepsilon)$. Clearly $C(\delta)$ is open since for every $x \in C(\delta)$, we have $(x - \varepsilon, x + \varepsilon) \subset C(\delta)$. Now let C denote the set of continuous points of f. From the definitions, we have that

$$C = \bigcap_{n=1}^{\infty} C(1/n).$$

Now suppose that $C = \mathbb{Q}$. Then

$$\mathbb{R} - \mathbb{Q} = \bigcup_{n=1}^{\infty} X_n,$$

where $X_n = \mathbb{R} - C(1/n)$. Since $C(1/n)$ is open, X_n is closed. Also $\mathbb{Q}$ is countable, say $\mathbb{Q} = \{q_1, q_2, \dots\}$. Let $Y_n = \{q_n\}$. Then

$$\mathbb{R} = \left(\bigcup_{n=1}^{\infty} X_n\right) \cup \left(\bigcup_{n=1}^{\infty} Y_n\right),$$

i.e. we have written $\mathbb{R}$ as a countable union of closed sets. Then by Baire's theorem, some X_n or Y_n has nonempty interior. Clearly it cannot be one of the Y_n. So there exists X_n containing an interval (a, b). But this is impossible because $X_n \subset \mathbb{R} - \mathbb{Q}$ and every interval contains a rational number. Thus, we obtain a contradiction, which shows that $C \neq \mathbb{Q}$.

Solutions of Qualifying Exams III, 2013 Fall

1. (Algebra) Consider the function fields $K = \mathbb{C}(x)$ and $L = \mathbb{C}(y)$ of one variable, and regard L as a finite extension of K via the $\mathbb{C}$-algebra inclusion

$$x \mapsto \frac{-(y^5 - 1)^2}{4y^5}$$

Show that the extension L/K is Galois and determine its Galois group.

Solution. Consider the intermediate extension $K' = \mathbb{C}(y^5)$. Then clearly $[L : K'] = 5$ and $[K' : K] = 2$, therefore $[L : K] = 10$.

Thus, to prove that L/K is Galois it is enough to find 10 field automorphisms of L over K. Choose a primitive 5th root of 1, say $\zeta = e^{2\pi i/5}$. For $i \in \mathbb{Z}/5$ and $s \in \{\pm 1\}$, the $\mathbb{C}$-algebra automorphism $\sigma_{i,s}$ of L defined by

$$y \mapsto \zeta^i y^s$$

leaves x, hence K, fixed.

There can be many ways to determine the group, here's one.

Looking at the law of composition of these automorphisms, one sees that the subgroup $\mathrm{Gal}(L/K') \simeq \mathbb{Z}/5$, (which is necessarily normal, being of index 2) is not central, for conjugation by $\sigma_{0,-1}$ acts as -1 on it.

So the group is the dihedral group of 10 elements.

2. (Algebraic Geometry) Is every smooth projective curve of genus 0 defined over the field of complex numbers isomorphic to a conic in the projective plane? Give an explanation for your answer.

Solution *(Sketch)*. Yes. Apply the Riemann-Roch theorem which guarantees the existence of a nonconstant meromorphic function with a simple pole at exactly one point. Argue that this meromorphic function identifies the curve with $\mathbb{P}^1$, and using that fact, embed the curve as a conic in the plane in any convenient way, e.g., If t_0, t_1 are projective ($\mathbb{P}^1$) coordinates, let $z_0 = t_0^2$, $z_1 = t_0 t_1$ $z_2 = t_1^2$ be the map to $\mathbb{P}^2$. The conic, then, would be $z_0 z_2 = z_1^2$. (Alternatively: one can consider the complete linear system attached to the anticanonical divisor.)

3. (COMPLEX ANALYSIS) Let $f(z) = z + e^{-z}$ for $z \in \mathbb{C}$ and let $\lambda \in \mathbb{R}$, $\lambda > 1$. Prove or disprove the statement that $f(z)$ takes the value λ exactly once in the open right half-plane $H_r = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Solution. First, let us consider the real function $f(x) = x + e^{-x}$. Since f is continuous, $f(0) = 1$ and $\lim_{x\to\infty} f(x) = \infty$, by the intermediate value theorem, there exists $u \in \mathbb{R}$ such that $f(u) = \lambda$. Now let us show that such u is unique. Let $R > 2\lambda$ and let Γ be the closed right half disk of radius R centered at the origin

$$\{z = x + iy \in \mathbb{C} : x = 0, |y| \leq R\} \cup \left\{z \in \mathbb{C} : |z| = R, -\frac{\pi}{2} \leq \arg(z) \leq \frac{\pi}{2}\right\}.$$

Let $F(z) = \lambda - z$ and $G(z) = -e^{-z}$. Then for $z \in \Gamma$, we have $|G(z)| = |e^{-\mathrm{Re} z}| \leq 1$ since Re $z \geq 0$, while $|F(z)| > 1$ by construction. Hence by Rouché's theorem, $\lambda - f(z) = F(z) + G(z)$ has the same number of zeros inside Γ as $F(z)$, namely 1. Since this is true for all R large enough, we conclude that the point u is unique.

4. (ALGEBRAIC TOPOLOGY) (a) Let X and Y be locally contractible, connected spaces with fixed basepoints. Let $X \vee Y$ be the wedge sum at the basepoints. Show that $\pi_1(X \vee Y)$ is the free product of $\pi_1 X$ with $\pi_1 Y$.

(b) Show that $\pi_1(X \times Y)$ is the direct product of $\pi_1 X$ with $\pi_1 Y$.

(c) Note the canonical inclusion $f : X \vee Y \to X \times Y$. Assume that X and Y have abelian fundamental groups. Show that the map f_* on fundamental groups exhibits $\pi_1(X \times Y)$ as the abelianization of $\pi_1(X \vee Y)$.
Hint: The Hurewicz map is natural.

Solution. (a) This follows form the Van Kampen theorem: Writing $X \vee Y$ as the union

$$X \cup_* Y$$

we have that $\pi_1(X \vee Y) \cong \pi_1(X) *_{\pi_1(*)} \pi_1(Y) = \pi_1(X) * \pi_1 Y$.

(b) There is the obvious continuous map

$$Maps_*(S^1, X) \times Maps_*(S^1, Y) \to Maps_*(S^1, X \times Y)$$

given by sending $(t \mapsto \gamma_X(t), t \mapsto \gamma_Y(t)) \mapsto (t \mapsto (\gamma_X(t), \gamma_Y(t)))$. This map is a continuous so it induces a map

$$\pi_0(Maps_*(S^1, X) \times Maps_*(S^1, Y)) \to \pi_0 Maps_*(S^1, X \times Y)$$

where the lefthand side is isomorphic to $\pi_0 Maps_*(S^1, X) \times \pi_0 Maps_*(S^1, Y))$. Further, the above map is clearly a bijection, so it induces an injection and a surjection on π_0.

(c) The Hurewicz map is natural so we have a commutative diagram

$$\begin{tikzcd}
\pi_1(X \vee Y) \arrow[r, "f_*"] \arrow[d, "q"] & \pi_1(X \times Y) \arrow[d] \\
H_1(X \vee Y) \arrow[r, "f_*"] & H_1(X \times Y)
\end{tikzcd}$$

where the vertical maps are abelianizations by the Hurewicz theorem. But the lower-right corner is equal to $H_1(X) \times H_1(Y)$ by the Kunneth theorem (since X and Y are connected), and the bottom copy of f_* is the obvious isomorphism on H_1. Since q is an abelianization by definition, but the bottom arrow and rightmost arrow are both isomorphisms, the top arrow must also be an abelianization.

5. (Differential Geometry) (a) Let $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ be a circle and consider the connection

$$\nabla := \mathrm{d} + \pi\sqrt{-1}\mathrm{d}\theta$$

defined on the trivial complex line bundle over $\mathbb{S}^1$, where θ is the standard coordinate on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ descended from $\mathbb{R}$. By solving the differential equation for flat sections $f(\theta)$

$$\nabla f = \mathrm{d}f + \pi\sqrt{-1}f\mathrm{d}\theta = 0$$

or otherwise, show that there does not exist global flat sections with respect to ∇ over $\mathbb{S}^1$.

(b) Let $T = V/\Lambda$ be a torus, where Λ is a lattice and $V = \Lambda \otimes \mathbb{R}$ is the real vector space containing Λ. Let L be the trivial complex line bundle equipped with the standard Hermitian metric. By identifying flat $U(1)$ connections with $U(1)$ representations of the fundamental group $\pi_1(T)$ or otherwise, show that the space of flat unitary connections on L is the dual torus $T^* = V^*/\Lambda^*$, where $\Lambda^* := \mathrm{Hom}(\Lambda, \mathbb{Z})$ is the dual lattice and $V^* := \mathrm{Hom}(V, \mathbb{R})$ is the dual vector space.

Solution. (a) The differential equation

$$f'(\theta) + \pi\sqrt{-1}f(\theta) = 0$$

has a unique solution

$$f(\theta) = A\mathrm{e}^{-\pi\sqrt{-1}\theta}$$

up to a constant $A \in \mathbb{C}$. This is not a well-defined function over $\mathbb{S}^1$ because $f(0) \neq f(1)$.

(b) The space of flat G-connections over T can be identified as

$$\mathrm{Hom}(\pi_1(T), G)/\mathrm{Ad}G.$$

Since $\pi_1(T) = \Lambda$ and for the abelian group $G = U(1)$ the adjoint action is trivial, we have

$$\mathrm{Hom}(\pi_1(T), G)/\mathrm{Ad}G = \mathrm{Hom}(\Lambda, U(1)) = T^*.$$

6. (Real Analysis) *(Fundamental Solutions of Linear Partial Differential Equations with Constant Coefficients).* Let Ω be an open interval $(-M, M)$ in $\mathbb{R}$ with $M > 0$. Let n be a positive integer and $L = \sum_{\nu=0}^{n} a_\nu \frac{d^\nu}{dx^\nu}$ be a linear differential operator of order n on $\mathbb{R}$ with constant coefficients, where the coefficients $a_0, \cdots, a_{n-1}, a_n \neq 0$ are complex numbers and x is the coordinate of $\mathbb{R}$. Let $L^* = \sum_{\nu=0}^{n}(-1)^\nu \overline{a_\nu} \frac{d^\nu}{dx^\nu}$. Prove, by using Plancherel's identity, that there exists a constant $c > 0$ which depends only on M and a_n and is independent of $a_0, a_1, \cdots, a_{n-1}$ such that for any $f \in L^2(\Omega)$ a weak solution u of $Lu = f$ exists with $\|u\|_{L^2(\Omega)} \leq c\,\|f\|_{L^2(\Omega)}$. Give one explicit expression for c as a function of M and a_n.

Hint: A *weak* solution u of $Lu = f$ means that $(f, \psi)_{L^2(\Omega)} = (u, L^*\psi)_{L^2(\Omega)}$ for every infinitely differentiable function ψ on Ω with compact support. For the solution of this problem you can consider as known and given the following three statements.

(I) If there exists a positive number $c > 0$ such that $\|\psi\|_{L^2(\Omega)} \leq c\,\|L^*\psi\|_{L^2(\Omega)}$ for all infinitely differentiable complex-valued functions ψ on Ω with compact support, then for any $f \in L^2(\Omega)$ a weak solution u of $Lu = f$ exists with $\|u\|_{L^2(\Omega)} \leq c\,\|f\|_{L^2(\Omega)}$.

(II) Let $P(z) = z^m + \sum_{k=0}^{m-1} b_k z^k$ be a polynomial with leading coefficient 1. If F is a holomorphic function on $\mathbb{C}$, then

$$|F(0)|^2 \leq \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \left|P\left(e^{i\theta}\right) F\left(e^{i\theta}\right)\right|^2 d\theta.$$

(III) For an L^2 function f on $\mathbb{R}$ which is zero outside $\Omega = (-M, M)$ its Fourier transform

$$\hat{f}(\xi) = \int_{-M}^{M} f(x) e^{-2\pi i x \xi} dx$$

as a function of $\xi \in \mathbb{R}$ can be extended to a holomorphic function

$$\hat{f}\left(\xi + i\eta\right) = \int_{-M}^{M} f(x) e^{-2\pi i x(\xi + i\eta)} dx$$

on $\mathbb{C}$ as a function of $\xi + i\eta$.

Solution. This problem is to compute the constant c in Lemma 3.3 on p.225 of the book of Stein and Shakarchi on *Real Analysis* by going over its arguments and keeping track of the constants involved in each step.

Introduce the polynomial

$$Q(\zeta) = \sum_{k=0}^{n} (-1)^k \, \overline{a_k} \, (2\pi\zeta)^k$$

so that

$$(\#) \qquad \left(\widehat{L^*\psi}\right)(\zeta) = Q(\zeta) \hat{\psi}(\zeta)$$

any $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$, where $\widehat{\ }$ denotes taking the Fourier transform. Consider first the special case where $a_n = \frac{1}{(2\pi i)^n}$ so that the coefficient of ξ^n in the polynomial $Q(\zeta)$ of degree n in ζ is 1. Writing $\zeta = \xi + \sqrt{-1}\eta$ (with both ξ and η real) and taking the L^2 of both sides of $(\#)$ over $\mathbb{R}$ as functions of η. Then

$$(\flat) \qquad \int_{-\infty}^{\infty} \left|Q\left(\xi + i\eta\right) \hat{\psi}\left(\xi + i\eta\right)\right|^2 d\xi = \int_{-\infty}^{\infty} \left|\left(\widehat{L^*\psi}\right)\left(\xi + i\eta\right)\right|^2 d\xi.$$

Since from the definition of Fourier transform

$$\left(\widehat{L^*\psi}\right)(\xi + i\eta) = \int_{x=-\infty}^{\infty} \left(L^*\psi\right)(x) e^{-2\pi i(\xi + i\eta)x} dx = \int_{x=-\infty}^{\infty} \left(\left(L^*\psi\right)(x) e^{2\pi\eta x}\right) e^{-2\pi i \xi x} dx,$$

it follows that $\left(\widehat{L^*\psi}\right)(\xi+i\eta)$ is equal to the value at ξ of the Fourier transform of the function $(L^*\psi)(x)e^{2\pi\eta x}$. Thus, by applying Plancherel's identity to the function $(L^*\psi)(x)e^{2\pi\eta x}$, we get

$$\int_{\xi=-\infty}^{\infty}\left|\left(\widehat{L^*\psi}\right)(\xi+i\eta)\right|^2 d\xi$$

$$=\int_{x=-\infty}^{\infty}\left|(L^*\psi)(x)e^{2\pi\eta x}\right|^2 dx \le e^{4\pi|\eta|M}\int_{-\infty}^{\infty}\left|(L^*\psi)(x)\right|^2 dx,$$

because the support of $\psi(x)$ (as well as the support of $(L^*\psi)(x)$) is in the interval $\Omega=(-M,M)$. Thus from ($\flat$) it follows that

$$(\sharp)\qquad \int_{-\infty}^{\infty}\left|Q(\xi+i\eta)\,\hat{\psi}(\xi+i\eta)\right|^2 d\xi \le e^{4\pi|\eta|M}\int_{-\infty}^{\infty}\left|(L^*\psi)(x)\right|^2 dx.$$

Setting $\eta=\sin\theta$ in ($\sharp$), we get from $|\eta|\le 1$ that

$$(\dagger)\qquad \int_{-\infty}^{\infty}\left|Q(\xi+i\sin\theta)\,\hat{\psi}(\xi+i\sin\theta)\right|^2 d\xi \le e^{4\pi M}\int_{-\infty}^{\infty}\left|(L^*\psi)(x)\right|^2 dx.$$

Replacing ξ by $\xi+\cos\theta$ in the integrand on the left-hand side of ($\dagger$), we get

$$(\ddagger)\qquad \int_{-\infty}^{\infty}\left|Q(\xi+\cos\theta+i\sin\theta)\,\hat{\psi}(\xi+\cos\theta+i\sin\theta)\right|^2 d\xi$$

$$\le e^{4\pi M}\int_{-\infty}^{\infty}\left|(L^*\psi)(x)\right|^2 dx.$$

By Statement (III) given above the function $\hat{\psi}(\xi+i\eta)$ as a function of $\xi+i\eta\in\mathbb{C}$ is holomorphic on $\mathbb{C}$. Since $Q(\xi+i\eta)$ as a function of $\xi+i\eta\in\mathbb{C}$ is a polynomial of degree n with leading coefficient 1, it follows from Statement (II) applied to $F(z)=\hat{\psi}(\xi+z)$ and $P(z)=Q(\xi+z)$ that

$$\left|\hat{\psi}(\xi)\right|^2 \le \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\left|Q(\xi+\cos\theta+i\sin\theta)\,\hat{\psi}(\xi+\cos\theta+i\sin\theta)\right|^2 d\theta.$$

Integrating both sides over $\xi\in(-\infty,\infty)$ and using ($\ddagger$), we get

$$\int_{\xi=-\infty}^{\infty}\left|\hat{\psi}(\xi)\right|^2 \le \int_{\xi=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\left|Q(\xi+\cos\theta+i\sin\theta)\,\hat{\psi}(\xi+\cos\theta+i\sin\theta)\right|^2 d\theta\right)d\xi$$

$$=\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\left(\int_{\xi=-\infty}^{\infty}\left|Q(\xi+\cos\theta+i\sin\theta)\,\hat{\psi}(\xi+\cos\theta+i\sin\theta)\right|^2 d\xi\right)d\theta$$

$$\le\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\left(e^{4\pi M}\int_{-\infty}^{\infty}\left|(L^*\psi)(x)\right|^2 dx\right)d\theta = e^{4\pi M}\int_{-\infty}^{\infty}\left|(L^*\psi)(x)\right|^2 dx.$$

By applying Plancherel's formula to ψ, we conclude that

$$\|\psi(\xi)\|_{L^2(\Omega)}^2 \leq e^{4\pi M} \|(L^*\psi)(x)\|_{L^2(\Omega)}^2$$

under the additional assumption that $a_n = \frac{1}{(2\pi i)^n}$. When this additional assumption is not satisfied, we can apply the argument for the special case to

$$\frac{1}{a_n (2\pi i)^n} L$$

instead of to L to conclude that

$$\|\psi(\xi)\|_{L^2(\Omega)}^2 \leq \frac{e^{4\pi M}}{|a_n (2\pi)^n|^2} \|(L^*\psi)(x)\|_{L^2(\Omega)}^2,$$

or

$$\|\psi(\xi)\|_{L^2(\Omega)} \leq c \|(L^*\psi)(x)\|_{L^2(\Omega)},$$

with

$$c = \frac{e^{2\pi M}}{|a_n| (2\pi)^n}.$$

By Statement (I) given above, when we set

$$c = \frac{e^{2\pi M}}{|a_n| (2\pi)^n},$$

we can conclude that for any $f \in L^2(\Omega)$ a weak solution u of $Lu = f$ exists with $\|u\|_{L^2(\Omega)} \leq c \|f\|_{L^2(\Omega)}$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 1, 2015 (Day 1)

1. (A) The integer 8871870642308873326043363 is the 13^{th} power of an integer n. Find n.

Solution. Counting digits, we see that $n < 100$, so is determined by its residue class (mod 99).Both (mod 11) and (mod 9) raising to the 13^{th} power is a bijection. After a little computation we find that $n \equiv 2 (\text{mod } 9)$ and $n \equiv 6 (\text{mod } 11)$. This implies, by the Chinese remainder theorem that $n \equiv 83 (\text{mod } 99)$. Hence $n = 83$.

2. (AG) Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree 4.

(a) Describe the canonical bundle of C in terms of line bundles on $\mathbb{P}^2$. What are the effective canonical divisors on C?

(b) What is the genus of C? Explain how you obtain this formula.

(c) Prove that C is not hyperelliptic.

Solution: By the adjunction formula, the canonical divisor class of a curve of degree d is $K_C = \mathcal{O}_C(d-3)$, that is, plane curves of degree $d-3$ cut out canonical divisors on C. It follows that effective canonical divisors on C are the intersection with lines in the plane, so have degree 4. Since the degree of the canonical class is $2g-2$, the genus $g = 3$. Furthermore, any two points $p, q \in C$ impose independent conditions on the canonical series $|K_C|$; that is, $h^0(K_C(-p-q)) = g-2$, so by the Riemann-Roch formula $h^0(\mathcal{O}_C(p+q)) = 1$, i.e., C is not hyperelliptic.

3. (DG) Let M be a C^∞ manifold, TM its tangent bundle, and $T^{\mathbb{C}}M = \mathbb{C} \otimes_{\mathbb{R}} TM$ the complexified tangent bundle. An almost complex structure on M is a C^∞ bundle map $J : TM \to TM$ such that $J^2 = -1$.

(a) Show that an almost complex structure naturally determines, and is determined by, each of the following two structures:
i) the structure of a complex C^∞ vector bundle – i.e., with fibres that are complex vector spaces – on TM compatible with its real structure.
ii) a C^∞ direct sum decomposition $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ with $T^{0,1}M$ = complex conjugate of $T^{1,0}M$.

(b) Show that every almost complex manifold is orientable.

(c) If S is a C^∞, orientable, 2-dimensional, Riemannian manifold, construct a natural almost complex structure on S in terms of its Riemannian structure, but one that depends only on the underlying conformal structure of S.

(d) Does the almost complex structure constructed in (c) determine the conformal structure of S? You need NOT give a detailed answer to this question; a heuristic one- or two-sentence answer suffices.

Solution: The correspondence between J and the structure of complex vector bundle on TM is given by $J \leftrightarrow$ multiplication by i; this is well defined and bijective because both are C^∞ bundle maps, defined over $\mathbb{R}$, of square -1. For the same reason, $J \leftrightarrow$ bijectively corresponds to decompositions $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}$ with $T^{0,1}M =$ complex conjugate of $T^{1,0}M$, via

$$T^{1,0} \;=\; i - \text{eigenspace of } J\,, \;\text{ and }\; T^{0,1} \;=\; (-i) - \text{eigenspace of } J\,,$$

on each fiber. That is the assertion (a). To establish (b), let $\{s_1, s_2, \ldots, s_n\}$ denote a local C^∞ frame of $T^{1,0}M$. The complex conjugate frame is then a frame of $T^{0,1}M$. it follows that

is a local C^∞ generator of $\wedge^{\rm top} T^{\mathbb{C}}M$. It is defined over $\mathbb{R}$, as can be checked by an easy calculation, and hence can be regarded as a local C^∞ generator of $\wedge^{\rm top} TM$. Now let $\{t_1, t_2, \ldots, t_n\}$ be another local C^∞ frame of $T^{1,0}M$. On the overlap of the domains, the two frames are related by a C^∞ matrix valued function $(a_{i,j})$. But then

$$i^{-n}\, t_1 \wedge \cdots \wedge t_n \wedge \overline{t_1} \wedge \ldots \overline{t_n} \;=\; |\det(a_{i,j})|^2\, i^{-n}\, s_1 \wedge \cdots \wedge s_n \wedge \overline{s_1} \wedge \ldots \overline{s_n}\,,$$

so any two local frames induce the same orientation on M. This proves (b). On a 2-dimensional Riemannian manifold S one has the notion of an angle between any two tangent vectors at a point, which depends only on the underlying conformal structure, and if S is oriented, one even has the notion of a directed angle. In this situation it makes sense to define $J =$ rotation through an angle $\pi/2$. This is a C^∞ bundle map because the metric is smooth, and $J^2 = -1$ by definition. That implies (c). Finally, for (d), note that on the tangent spaces of a Riemannian surface one can make sense of a rotation though any angle if one knows the effect of a rotation about the angle $\pi/2$.

4. (RA) In this problem V denotes a Banach space over $\mathbb{R}$ or $\mathbb{C}$.

(a) Show that any finite dimensional subspace $U_0 \subset V$ is closed in V.

(b) Now let $U_1 \subset V$ a closed subspace, and $U_2 \subset V$ a finite dimensional subspace. Show that $U_1 + U_2$ is closed in V.

Solution: For definiteness suppose V is a Banach space over $\mathbb{R}$. Let $\{u_k \mid 1 \leq k \leq n\}$ be a basis of U_0, and use this basis to identify $U_0 \cong \mathbb{R}^n$. Then, for $u = \sum_k a_k\, u_k \in U_0 \subset V$,

$$\|u\| \;\leq\; \textstyle\sum_{0\leq k\leq n} |a_k|\, \|u_k\| \;\leq\; C\; \max_{0\leq k\leq n} |a_k|\,, \;\text{ with }\; C \;=\; \max_{0\leq k\leq n}\; \|u_k\|\,.$$

It follows that $\mathbb{R}^n \cong U_0 \hookrightarrow V$ is bounded with respect to the sup norm on $\mathbb{R}^n$ (and hence with respect to any other Banach norm on $\mathbb{R}^n$). Now let $\{v_m\}$ be a convergent sequence in V, all of whose terms lie in the subspace U_0. But then the inverse image of this sequence in $\mathbb{R}^n$ must be bounded, and has a convergent subsequence. Its limit, viewed as a vector in V, must coincide with the original limit, of course. This implies (a). In establishing (b) we can replace U_2 by a linear complement, in U_2, of $U_1 \cap U_2$. In other words, we may assume $U_1 \cap U_2 = 0$. Any convergent sequence $\{v_m\}$ whose terms lie in $U_1 + U_2$ can now be written uniquely as $\{v_m = v'_m + v''_m\}$, with $v'_m \in U_1$ and $v''_m \in U_2$. Let's distinguish two cases:

i) The sequence $\{v''_m\}$ has a bounded subsequence, which by (a) in turn has a subsequence that converges in U_2. But then the corresponding subsequence of $\{v'_m\}$ must converge, necessarily to a point in the closed subspace U_1. It follows that the limit of the original series must lie in $U_1 + U_2$.

ii) $\|v''_m\| \to \infty$ as $m \to \infty$. Going to an appropriate subsequence of the original series, we may then assume that $v''_m \neq 0$ for all m and $\|v''_m\|^{-1}\, v''_m \to \tilde{v}'' \in U_2$, $\|\tilde{v}''\| = 1$. Because of the hypotheses, $\|v''_m\|^{-1}\, v_m \to 0 \in V$, which now implies the convergence of $\|v''_m\|^{-1}\, v'_m$ to some point $\tilde{v}'$ in the closed subspace U_1. At this point, we know that

$$\begin{aligned} 0 \;&=\; \lim_{m\to\infty}\; \|v''_m\|^{-1}\, v_m \\ &\qquad\qquad =\; \lim_{m\to\infty}\; \|v''_m\|^{-1}\, v'_m \;+\; \lim_{m\to\infty}\; \|v''_m\|^{-1}\, v''_m \;\;=\;\; \tilde{v}' + \tilde{v}''\,. \end{aligned}$$

That is a contradiction because $0 \neq \tilde{v}'' = -\tilde{v}' \in U_1 \cap U_2 = 0$.

5. (AT) Consider the following three topological spaces:

$$A = \mathbb{H}\mathrm{P}^3, \qquad\qquad B = S^4 \times S^8, \qquad\qquad C = S^4 \vee S^8 \vee S^{12}.$$

($\mathbb{H}\mathrm{P}^3$ denotes quaternionic projective 3-space.)

(a) Calculate the cohomology groups (with integer coefficients) of all three.

(b) Show that A and B are not homotopy equivalent.

(c) Show that C is not homotopy equivalent to any compact manifold.

Solution:

1. The cohomology rings of the three spaces are as follows:

$$\begin{aligned}
&H^*A = \mathbb{Z}[x]/x^4, && |x| = 4,\\
&H^*B = \mathbb{Z}[a,b]/(a^2,b^2), && |a| = 4, |b| = 8,\\
&H^*C = \mathbb{Z}\{r,s,t\}, && |r| = 4, |s| = 8, |t| = 12,\\
& && \text{with all products zero.}
\end{aligned}$$

2. The ring structures differ: $x \cdot x = x^2 \neq 0$, but $a \cdot a = 0$.
3. If C were homotopy equivalent to a compact manifold, then it would enjoy Poincaré duality. In particular, r could be taken to be Poincaré dual to s and t to be the volume form, so that $r \cdot s = t$. However, $r \cdot s = 0$ in H^*C.

6. (CA) Let $f(z)$ be a function which is analytic in the unit disc $D = \{|z| < 1\}$, and assume that $|f(z)| \leq 1$ in D. Also assume that $f(z)$ has at least two fixed points z_1 and z_2. Prove that $f(z) = z$ for all $z \in D$.

Solution: First observe that we can find a fractional linear transformation S mapping D to itself and 0 to z_1. Now consider $g = S^{-1} \circ f \circ S$. The function g is also analytic on D, and satisfies $|g(z)| \leq 1$ on D. One of the fixed points of g is 0, hence the function $h(z) = g(z)/z$ is analytic; call p the other fixed point of g. We claim that $|h(z)| \leq 1$. Before proving the claim, note that this implies the desired result, since $|h(p)| = 1$, hence h is identically 1 on D by the maximum principle.

To prove the claim, we also use the maximum principle. Fix some small $\epsilon > 0$. On $\{|z| = 1 - \epsilon\}$, we have $|h(z)| = |g(z)|/|z| \leq 1/(1 - \epsilon)$, hence $|h(z)| \leq 1/(1 - \epsilon)$ on $\{|z| \leq 1 - \epsilon\}$ by the maximum principle. Letting ϵ tend to 0 gives $|h(z)| \leq 1$ on D, as desired.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday September 2, 2015 (Day 2)

1. (AT) Let $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ be n dimensional complex projective space.

(a) Show that every map $f : \mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$ has a fixed point. (Hint: Use the ring structure on cohomology.)

(b) For every $n \geq 0$, give an example of a map $f : \mathbb{CP}^{2n+1} \to \mathbb{CP}^{2n+1}$ without any fixed points and describe its induced map on cohomology.

Solution: We have $H^*(\mathbb{CP}^{2n};\mathbb{Z}) = \mathbb{Z}[x]/x^{2n+1}$ with x in degree 2. The map f induces a ring endomorphism f^* given by $f^*(x) = kx$ for some $k \in \mathbb{Z}$. Thus, the trace of f^* is

$$\mathrm{Tr}(f^*) = 1 + \sum_{i=1}^{2n} k^i \equiv 1 \pmod 2.$$

In particular, the trace is non-zero and hence f has a fixed point by the Lefschetz fixed-point theorem.

For the second part we can take the map $f : \mathbb{CP}^{2n+1} \to \mathbb{CP}^{2n+1}$ with

$$f([z_0, \ldots, z_{2n+1}]) = [-\bar{z}_1, \bar{z}_0, \ldots, -\bar{z}_{2n+1}, \bar{z}_{2n}]$$

since $f(z) = z$ implies $z_j = -|\lambda|^2 z_j$ for some $\lambda \in \mathbb{C}^*$ and all $0 \leq j \leq 2n+1$, a contradiction. Note that f sends the line $\mathbb{CP}^1 = [z_0, z_1, 0, \ldots, 0] \subset \mathbb{CP}^{2n+1}$ to itself, but with reverse orientation, so $f^*(x) = -x$.

2. (A) Let A be a commutative ring with unit. Define what it means for A to be *Noetherian*. Prove that the ring of continuous functions $f : [0,1] \to \mathbb{R}$ (with pointwise addition and multiplication) is *not* Noetherian.

Solution: A is *Noetherian* if it has no sequence of ideals I_1, I_2, I_3, … such that $I_n \subset I_{n+1}$ and $I_n \neq I_{n+1}$ for each n. If A is the ring of continuous functions $[0,1] \to \mathbb{R}$ then we get a counterexample by taking for I_n the ideal of functions $f \in A$ such that there exists $B \in \mathbb{R}$ with $|f(x)| \leq Bx^{1/n}$ for all $x \in [0,1]$. The inclusions are strict because $x^{1/n}$ is in I_n but not in I_{n+1}. [Alternatively, let I_n consist of the functions supported on $[1/n, 1]$, or of the functions vanishing at $1/m$ for all integers $m \geq n$.]

3. (CA) Let $S \subset \mathbb{C}$ be the open half-disc $\{x + iy : y > 0,\ x^2 + y^2 < 1\}$.

(a) Construct a surjective conformal mapping $f : S \to D$, where D is the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$.

(b) Construct a harmonic function $h : S \to \mathbb{R}$ such that:
- $h(x + iy) \to 0$ as $y \to 0$ from above, for all real x with $|x| < 1$, and
- $h(re^{i\theta}) \to 1$ as $r \to 1$ from below, for all real θ with $0 < |x| < \pi$.

Solution: We construct f as a composition $f_3 \circ f_2 \circ f_1$ of conformal maps where f_1 and f_3 are Möbius transformations and $f_2(z) = z^2$. Set $f_1(z) = (1+z)/(1-z)$, which transforms D conformally into the first quadrant $\{(x, y) : x > 0, y > 0\}$, taking $(-1, 1)$ to the positive real axis and the semicircular boundary of S to the imaginary real axis. Thus $f_2 \circ f_1$ conformally transforms D to the upper half-plane $\{(x, y) : y > 0\}$, and finally $f_3(z) := (z - i)/(z + i)$ takes the upper half-plane to D, whence $f_3 \circ f_2 \circ f_1$ is a conformal map $S \to D$ as demanded.

[Of course there are other variations such as $f_1(z) = (z - 1)/(z + 1)$ etc., any of which earns full credit as long as f_1 fits into f_2 fits into f_3 correctly.]

The function $h(z) = (2/\pi)\Im \log f_1(z)$ [a.k.a. $h(z) = (1/\pi)\Im \log f_2(f_1(z))$] is harmonic because it is the imaginary part of an analytic function, and has the requisite limiting behavior by our description of f_1 in part (i) (the principal value of $\log(z)$ has imaginary part 0 for $z = x > 0$, and imaginary part $\pi/2$ when $z = iy$ with $y > 0$).

4. (AG) Let Q be the complex quadric surface in $\mathbb{P}^3$ defined by the homogeneous equation $x_0x_3 - x_1x_2 = 0$.

(a) Show that Q is non-singular.

(b) Show that through each point of Q there are exactly two lines which lie on Q.

(c) Show that Q is rational, but not isomorphic to $\mathbb{P}^2$.

Solution: The partial derivatives of $F(X, Y, Z, W) = XY - ZW$ have no common zeroes in $\mathbb{P}^3$. A line which lies on Q corresponds to an isotropic plane V in the quadratic space $\mathbb{C}^4$, whereas a point on Q corresponds to an isotropic line L. The quadratic space $L^\perp/L$ is split of dimension 2, so contains exactly two isotropic lines. These give the two isotropic planes V which contain L.

Q which is the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$. Since $\mathbb{A}^2 \subset \mathbb{P}^1 \times \mathbb{P}^1$ as Zariski-dense subset, X is rational. To see that $X \not\cong \mathbb{P}^2$ one can use $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}^2 \not\cong \mathbb{Z} = \mathrm{Pic}(\mathbb{P}^2)$.

5. (DG) Let Ω be the 2-form on $\mathbb{R}^3 - \{0\}$ defined by

$$\Omega = \frac{1}{x^2 + y^2 + z^2}(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy).$$

(a) Prove that Ω is closed.

(b) Let $f : \mathbb{R}^3 - \{0\} \to S^2$ be the map which sends (x, y, z) to $(\frac{1}{x^2+y^2+z^2})^{1/2}(x, y, z)$. Show that Ω is the pull-back via f of a 2-form on S^2.

(c) Prove that Ω is not exact.

Solution: Introduce spherical coordinates (r, θ, ϕ) by writing $x = r\sin(\theta)\cos(\phi)$, $y = r\sin(\theta)\sin(\phi)$ and $z = r\cos(\theta)$. Written with these coordinates,

$$\Omega = \sin^2(\theta)d\theta d\phi$$

$$d\Omega = 2\sin(\theta)\cos(\theta)d\theta d\theta d\phi + \sin^2(\theta)(dd\theta d\phi - d\theta dd\phi)$$

This is zero because $d^2 = 0$ and because the wedge of a one-form with itself is zero. The map in these coordinates sends (r, θ, ϕ) to the point on $(1, \theta, \phi)$. A differential form Θ is the pull-back of a form on S^2 via f if and only if both Θ and $d\Theta$ annihilate the vector fields in the kernel of the differential of f. Since these vector fields are proportional $\partial/\partial r$ in this case, both of the conditions are obeyed by Ω. If Ω were exact, then its integral over S^2 would be 0, but this integral is equal to 4π.

6. (RA) Consider the linear ODE $f'' + P\,f' + Q\,f = 0$ on the interval $(a, b) \subset \mathbb{R}$, with P, Q denoting C^∞ real valued functions on (a, b). Recall the definition of the Wronskian $W(f_1, f_2) = f_1\,f_2' - f_1'\,f_2$ associated to any two solutions f_1, f_2 of this differential equation.

(a) Show that $W(f_1, f_2)$ either vanishes identically or is everywhere nonzero, depending on whether the two solutions f_1, f_2 are linearly dependent or not.

(b) Now suppose that f_1, f_2 are linearly independent, real valued solutions. Show that they have at most first order zeroes, and that the zeroes occur in an alternating fashion: between any two zeroes of one of the solutions there must be a zero of the other solution.

Solution:

$$\begin{aligned} W'(f_1, f_2) &= f_1'\,f_2' + f_1\,f_2'' - f_1''\,f_2 - f_1'\,f_2' = \\ &= f_2(Pf_1' + Qf_1) - f_1(Pf_2' + Qf_2) = -P\,W(f_1, f_2)\,, \end{aligned}$$

which implies $W(f_1, f_2) = c\,e^{-P}$. In particular, $W(f_1, f_2)$ either vanishes identically or not at all. The Wronskian vanishes at some $x_0 \in (a, b)$ if and only if the initial conditions for (f_1', f_1) and (f_2', f_2) are proportional at x_0, which is the case if and only if the global solutions are proportional. This implies (a). Next suppose that f_1, f_2 are real valued, linearly independent solutions. Since $W(f_1, f_2)$ never vanishes, neither solution can have a double zero; moreover, if $f_1(x_0) = 0$ at some x_0 then $f_2(x_0) \neq 0$, and vice versa. Finally suppose that $f_1(x_0) = 0$, $f_1(x_1) = 0$, with $x_0 < x_1$ and $f_1(x) \neq 0$ for $x \in (x_0, x_1)$. Since the zeroes are first order, the derivatives of f_1 at the two points must have opposite signs. Since the Wronskian has the same sign globally, f_2 cannot have the same sign at the two points. It follows that f_2 vanishes somewhere between x_0 and x_1. Similarly, between any two zeros of f_2 there must be a zero of f_1. That is the assertion (b).

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday September 3, 2015 (Day 3)

1. (DG) Consider the graph S of the function $F(x, y) = \cosh(x)\cos(y)$ in $\mathbb{R}^3$ and let

$$\Phi : \mathbb{R}^2 \to S \subset R^3$$

be its parametrization: $\Phi(x, y) = (x, y, \cosh(x)\cos(y))$.

(a) Write down the metric on $\mathbb{R}^2$ that is defined by the rule that the inner product of two vectors v and w at the point (x, y) is equal to the inner product of $\Phi_*(v)$ and $\Phi_*(w)$ at the point $\Phi(x, y)$ in $\mathbb{R}^3$.

(b) Define the Gaussian curvature of a general surface embedded in $\mathbb{R}^3$.

(c) Compute the Gaussian curvature of the surface S at the point $(0, 0, 1)$.

Solution: The push-forward via Φ_* of the vectors $\partial/\partial x$ and $\partial/\partial y$ at a given point (x, y) are the vectors in $\mathbb{R}^3$ at $(x, y, F(x, y))$ given by $\Phi_*(\partial/\partial x) = (1, 0, F_x)$ and $\Phi_*(\partial/d\partial y) = (0, 1, F_y)$. The metric is

$$g = (1 + F_x^2)dx \otimes dx + F_xF_y(dx \otimes dy + dy \otimes dx) + (1 + F_y^2)dy \otimes dy.$$

In this problem, $F_x = \sinh(x)\cos(y)$ and $F_y = -\cosh(x)\sin(y)$.

The Gauss curvature is the determinant of the second fundamental form as computed using an orthonormal frame for the metric whereby the inner product of any two tangent vectors is their $\mathbb{R}^3$ inner product. The second fundamental form is defined as follows: Let n denote a unit length normal to the surface and let (e^1, e^2) denote an orthonormal frame at a given point. The second fundamental form has components (m_{ab}) defined by $m_{ab} = \langle e^a, \nabla_{e^b}(n)\rangle$ It is also defined by writing the Riemann curvature tensor R for this metric using an orthornormal frame (e^1, e^2) for T^*S as $R = \kappa(e^1 \wedge e^2) \otimes (e^1 \wedge e^2)$. In the case of the surface S, the normal vector is $n = (-F_x, -F_y, 1)/(1 + F_x^2 + F_y^2)^{1/2}$. At the point $(0, 0, 1)$ in S, the vectors d/dx and d/dy are orthonormal. A computation then finds that the Gauss curvature is -1.

2. (RA) Let $f(x) \in C(\mathbb{R}/\mathbb{Z})$ be a continuous $\mathbb{C}$-valued function on $\mathbb{R}/\mathbb{Z}$ and let $\sum_{n=-\infty}^{\infty} a_n\, e^{2\pi i n x}$ be its Fourier series.

(a) Show that f is C^∞ if and only if $|a_n| = O(|n|^{-k})$ for all $k \in \mathbb{N}$.

(b) Prove that a sequence of functions $\{\, f_n\}_{n\geq 1}$ in $C^\infty(\mathbb{R}/\mathbb{Z})$ converges in the C^∞ topology (uniform convergence of functions and their derivatives of all orders) if and only if the sequences of k-th derivatives $\{\, f_n^{(k)}\}_{n\geq 1}$, for all $k \geq 0$, converge in the L^2-norm on $\mathbb{R}/\mathbb{Z}$.

Solution A simple integration by parts argument shows that $f \in C^1(\mathbb{R}/\mathbb{Z})$ implies

$$f'(x) \;=\; 2\pi i \sum\nolimits_{n=-\infty}^{\infty} n\, a_n\, e^{2\pi i n x}\,.$$

Hence for all $k \in \mathbb{N}$ and $f \in C^\infty(\mathbb{R}/\mathbb{Z})$,

$$f^{(k)}(x) \;=\; (2\pi i)^k \sum\nolimits_{n=-\infty}^{\infty} n^k\, a_n\, e^{2\pi i n x}$$

is the Fourier series of a continuous, hence L^2 function, with squared L^2 norm

$$\|f^{(k)}\|_{L^2}^2 \;=\; (2\pi)^{2k} \sum\nolimits_{n=-\infty}^{\infty} n^{2k}\, |a_n|^2 \;<\; \infty\,.$$

It follows that for fixed k, $|n|^k\,|a_n|$ is bounded.

The topology of $C^\infty(\mathbb{R}/\mathbb{Z})$ is defined by the family of norms $f \mapsto \|f^{(k)}\|_{\sup}$, and according to (a), also by the family of seminorms $f \mapsto \|f^{(k)}\|_{L^2}$, because

$$\begin{aligned}\|f^{(k)}\|_{L^2} \;&\le\; \|f^{(k)}\|_{\sup} \;\le\; (2\pi)^k \sum\nolimits_{n=-\infty}^{\infty} |n|^k\,|a_n| \\ &=\; (2\pi)^k \sum\nolimits_{n\neq 0} |n|^{k+1}\,|a_n|\,|n|^{-1} \;\le\; \tfrac{1}{2\pi}(\textstyle\sum_{n\neq 0}|n|^{-2})^{1/2}\|f^{(k+1)}\|_{L^2}\,.\end{aligned}$$

3. (AG) Let C be a smooth projective curve over $\mathbb{C}$ and $\omega_C^{\otimes 2}$ the square of its canonical sheaf.

(a) What is the dimension of the space of sections $\Gamma(C, \omega_C^{\otimes 2})$?

(b) Suppose $g(C) \ge 2$ and $s \in \Gamma(C, \omega_C^{\otimes 2})$ is a section with simple zeros. Compute the genus of $\Sigma = \{x^2 = s\}$ in the total space of the line bundle ω_C, i.e. the curve defined by the 2-valued 1-form $\sqrt{s}$.

Solution: Write $L = \omega_C^{\otimes 2}$ and $g = g(C)$, then $\deg(L) = 4g-4$. For $g = 0$: $h^0(L) = 0$ since L is negative, for $g = 1$: $h^0(L) = 1$ since L is trivial, and for $g \ge 2$: $h^0(L) = 3g-3$ by Riemann–Roch.

For the second part note that the projection $T^*C \to C$ gives a natural 2:1 covering $\Sigma \to C$ which is ramified at the $4g-4$ zeros of s. The Riemann–Hurwitz formula gives $\chi(\Sigma) = 2\chi(C) - (4g-4)$, thus $g(\Sigma) = 4g-3$.

4. (AT) Show (using the theory of covering spaces) that every subgroup of a free group is free.

Solution: For a set I of generators we let $X = \bigvee_I S^1$, then $F = \pi_1(X)$ is the free group on I. Let $G \subset F$ be a subgroup, then there is a covering $p : Y \to X$ with $p_*(\pi_1(Y)) = G$ and p_* is injective. Note that Y has the stucture of a connected 1-dimensional CW complex and is thus homotopy equivalent to a wedge of S^1's by contracting a maximal subtree. It follows that $G \cong \pi_1(Y)$ is free.

5. (CA)

(a) Define Euler's Gamma function $\Gamma(z)$ in the half plane $Re(z) > 0$ and show that it is holomorphic in this half plane.

(b) Show that $\Gamma(z)$ has a meromorphic continuation to the entire complex plane.

(c) Where are the poles of $\Gamma(z)$?

(d) Show that these poles are all simple and determine the residue at each pole.

Solution:

$$\Gamma(z) = \int_0^\infty t^z e^{-t} dt/t$$

The identity $\Gamma(z+1) = z\Gamma(z)$ then follows from integration by parts. Rewriting this identity as

$$\Gamma(z) = \Gamma(z+1)/z$$

at $z = 0$ with residue 1. Using this identity again, we can extend to the half plane $Re(z) > -2$ with a simple pole at $z = -1$. Continuing in this manner, we get a meromorphic continuation to the entire plane with simple poles at the negative integers. The residue at $z = -n$ is $(-1)^n/n!$.

6. (A) Let G be a finite group, and $\rho : G \to GL_n(\mathbb{C})$ a linear representation. Then for each integer $i \geq 0$ there is a representation $\wedge^i \rho$ of G on the exterior power $\wedge^i(\mathbb{C}^n)$. Let W_i be the subspace $(\wedge^i(\mathbb{C}^n))^G$ of $\wedge^i(\mathbb{C}^n)$ fixed under this action of G.

Prove that $\dim W_i$ is the T^i coefficient of the polynomial

$$\frac{1}{|G|} \sum_{g \in G} \det(\mathbf{1}_n + T\rho(g))$$

where $\mathbf{1}_n$ is the $n \times n$ identity matrix.

Solution: It is a standard consequence of Schur's lemma that if (V, ϱ) is any finite-dimensional representation of G then

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday August 30, 2016 (Day 1)

1. (DG)

(a) Show that if V is a $\mathcal{C}^\infty$-vector bundle over a compact manifold X, then there exists a vector bundle W over X such that $V \oplus W$ is trivializable, i.e. isomorphic to a trivial bundle.

(b) Find a vector bundle W on S^2, the 2-sphere, such that $T^*S^2 \oplus W$ is trivializable.

Solution: Since V is locally trivializable and M is compact, one can find a finite open cover U_i, $i = 1, \dots, n$, of M and trivializations $T_i : V|_{U_i} \to \mathbb{R}^k$. Thus, each T_i is a smooth map which restricts to a linear isomorphism on each fiber of $V|_{U_i}$. Next, choose a smooth partition of unity $\{f_i\}_{i=1,\dots,n}$ subordinate to the cover $\{U_i\}_{i=1,\dots,n}$. If $p : V \to M$ is the projection to the base, then there are maps

$$V|_{U_i} \to \mathbb{R}^k, \qquad v \mapsto f_i(p(v))T_i(v)$$

which extend (by zero) to all of V and which we denote by f_iT_i. Together, the f_iT_i give a map $T : V \to \mathbb{R}^{nk}$ which has maximal rank k everywhere, because at each point of X at least one of the f_i is non-zero. Thus V is isomorphic to a subbundle, $T(V)$, of the trivial bundle, $\mathbb{R}^{nk}$. Using the standard inner product on $\mathbb{R}^{nk}$ we get an orthogonal bundle $W = T(V)^\perp$ which has the desired property.

For the second part, embed S^2 into $\mathbb{R}^3$ in the usual way, then

$$TS^2 \oplus N_{S^2} = T\mathbb{R}^3|_{S^2}$$

where N_{S^2} is the normal bundle to S^2 in $\mathbb{R}^3$. Dualizing we get

$$T^*S^2 \oplus (N_{S^2})^* = T^*\mathbb{R}^3|_{S^2}$$

which solves the problem with $W = (N_{S^2})^*$.

2. (RA) Let (X, d) be a metric space. For any subset $A \subset X$, and any $\epsilon > 0$ we set

$$B_\epsilon(A) = \bigcup_{p \in A} B_\epsilon(p).$$

(This is the "ϵ-fattening" of A.) For Y, Z bounded subsets of X define the *Hausdorff distance* between Y and Z by

$$d_H(Y,Z) := \inf\{\epsilon > 0 \mid Y \subset B_\epsilon(Z), \quad Z \subset B_\epsilon(Y)\}.$$

Show that d_H defines a metric on the set $\tilde{X} := \{A \subset X \big| \, A \text{ is closed and bounded}\}$.

Solution: We need to show that $(\tilde{X}, d_H)$ is a metric space. First, since the sets are bounded, $d_H(Y,Z)$ is well defined for any closed sets Y, Z. Secondly, $d_H(Y,Z) = d_H(Z,Y) \geq 0$ is obvious from the definition. We need to prove that the distance is positive when $Y \neq Z$, and that d_H satisfies the triangle inequality. First, let us show that $d_H(Y,Z) > 0$ if $Y \neq Z$. Without loss of generality, we can assume there is a point $p \in Y \cap Z^c$. Since Z is closed, so there exists $r > 0$ such that $B_r(p) \subset Z^c$. In particular, p is not in $B_r(Z)$. Thus Y is not contained in $B_r(Z)$ and so $d_H(Y,Z) \geq r > 0$.

It remains to prove the triangle inequality. To this end, suppose that Y, Z, W are relevant subsets of X. Fix $\epsilon_1 > d_H(Y,Z), \epsilon_2 > d_H(Z,W)$, then

$$Y \subset B_{\epsilon_1}(Z), \quad Z \subset B_{\epsilon_1}(Y), \quad Z \subset B_{\epsilon_2}(W), \quad W \subset B_{\epsilon_2}(Z)$$

Then $d_H(Y,Z) < \epsilon_1, d_H(Z,W) < \epsilon_2$. Let us prove that $Y \subset B_{\epsilon_1+\epsilon_2}(W)$, the other containment being identical. Fix a point $y \in Y$. By our choice of ϵ_1 there exists a point $z \in Z$ such that $y \in N_{\epsilon_1}(z)$. By our choice of ϵ_2 there exists a point $w \in W$ such that $z \in B_{\epsilon_2}(w)$. Then

$$d(y,w) \leq d(y,z) + d(z,w) \leq \epsilon_1 + \epsilon_2$$

so $y \in B_{\epsilon_1+\epsilon_2}(w)$. This proves the containment. The other containment is identical, by just swapping Y, W. Thus

$$d_H(Y,W) \leq \epsilon_1 + \epsilon_2$$

But this holds for all ϵ_1, ϵ_2 as above. Taking the infimum we obtain the result.

3. (AT) Let $T^n = \mathbb{R}^n/\mathbb{Z}^n$, the n-torus. Prove that any path-connected covering space $Y \to T^n$ is homeomorphic to $T^m \times \mathbb{R}^{n-m}$, for some m.

Solution: The universal covering space of T^n is $\mathbb{R}^n$, so that any path connected covering space of X is of the form $\mathbb{R}^n/G$, for some subgroup $G \subseteq \pi_1(T^n)$. We have $\pi_1(T^n) = \pi_1(S^1) \times \cdots \times \pi_1(S^1) = \mathbb{Z}^n$, and $\mathbb{Z}^n$ is acting on $\mathbb{R}^n$ by translation. Thus, $G \subseteq \mathbb{Z}^n$ is free. Choose a $\mathbb{Z}$-basis $(v_1, \ldots, v_m)$ of G, and consider the (real!) change of basis taking $(v_1, \ldots, v_m)$ to the first m standard basis vectors $(e_1, \ldots, e_m)$. Hence, G is acting on $\mathbb{R}^n$ by translation in the first m coordinates. Thus,

$$\mathbb{R}^n/G \simeq \mathbb{R}^m/\mathbb{Z}^m \times \mathbb{R}^{n-m} \simeq T^m \times \mathbb{R}^{n-m}.$$

4. (CA)

Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant holomorphic function. Show that the image of f is dense in $\mathbb{C}$.

Solution: Suppose that for some $w_0 \in \mathbb{C}$ and some $\epsilon > 0$, the image of f lies outside the ball $B_\epsilon(w_0) = \{w \in \mathbb{C} \mid |w - w_0| < \epsilon\}$. Then the function

$$g(z) \;=\; \frac{1}{f(z) - w_0}$$

is bounded and homomorphic in the entire plane, hence constant.

5. (A) Let $F \supset \mathbb{Q}$ be a splitting field for the polynomial $f = x^n - 1$.

(a) Let $A \subset F^\times = \{z \in F \mid z \neq 0\}$ be a finite (multiplicative) subgroup. Prove that A is cyclic.

(b) Prove that $G = \mathrm{Gal}(F/\mathbb{Q})$ is abelian.

Solution: For the first part, let $m = |A|$. Suppose that A is not cyclic, so that the order of any element in A is less than m. A is a finite abelian group so it is isomorphic to a product of cyclic groups $A \simeq \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$, where $n_i | n_{i+1}$. In particular, the order of any element in A divides n_k. Hence, for any $z \in A$, $z^{n_k} = 1$. However, the polynomial $x^{n_k} - 1 \in F[x]$ admits at most $n_k < m$ roots in F, which is a contradiction. So, there must be some element in A with order m.

For the second part, since $f' = nx^{n-1}$ and f are relatively prime, f admits n distinct roots $1 = z_0, \ldots, z_{n-1}$. As F is a splitting field of f we can assume that $F = \mathbb{Q}(z_0, \ldots, z_{n-1}) \subseteq \mathbb{C}$. $U = \{z_0, \ldots, z_{n-1}\} \subset F^\times$ is a subgroup of the multiplicative group of units in F and is cyclic; moreover, $Aut(U)$ is isomorphic to the (multiplicative) group of units $(\mathbb{Z}/n\mathbb{Z})^*$. Restriction defines a homomorphism $G \to Aut(U)$, $\alpha \mapsto \alpha_{|U}$; this homomorphism is injective because $F = \mathbb{Q}(z_0, \ldots, z_{n-1})$. In particular, G is isomorphic to a subgroup of the abelian group $(\mathbb{Z}/n\mathbb{Z})^*$.

6. (AG) Let C and $D \subset \mathbb{P}^2$ be two plane cubics (that is, curves of degree 3), intersecting transversely in 9 points $\{p_1, p_2, \ldots, p_9\}$. Show that $p_1, \ldots, p_6$ lie on a conic (that is, a curve of degree 2) if and only if p_7, p_8 and p_9 are colinear.

Solution: First, observe that we can replace $C = V(F)$ and $D = V(G)$ by any two independent linear combinations $C' = V(a_0F + a_1G)$ and $D' = V(b_0F + b_1G)$. Now suppose that $p_1, \ldots, p_6$ lie on a conic $Q \subset \mathbb{P}^2$. Picking a seventh point $q \in Q$, we see that some linear combination C_0 of C and D contains q and hence contains Q; thus $C_0 = Q \cup L$ for some line $L \subset \mathbb{P}^2$. Replacing C or D with C_0, we see that p_7, p_8 and $p_9 \in L$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday August 31, 2016 (Day 2)

1. (A) Let R be a commutative ring with unit. If $I \subsetneq R$ is a proper ideal, we define the *radical* of I to be

$$\sqrt{I} \;=\; \{a \in R \;\mid\; a^m \in I \quad \text{for some } m > 0\}.$$

Prove that

$$\sqrt{I} = \bigcap_{\substack{\mathfrak{p} \supseteq I \\ \mathfrak{p}\text{ prime}}} \mathfrak{p}.$$

Solution: First, we prove for the case $I = 0$. Let $f \in \sqrt{0}$ so that $f^n = 0$, and $f^n \in \mathfrak{p}$, for any prime ideal $\mathfrak{p} \subseteq R$. Let $\mathfrak{p}$ be a prime ideal in R. The quotient ring $R/\mathfrak{p}$ is an integral domain and, in particular, contains no nonzero nilpotent elements. Hence, $f^n + \mathfrak{p} = 0 \in R/\mathfrak{p}$ so that $f \in \mathfrak{p}$.

Now, suppose that $f \notin \sqrt{0}$. The set $S = \{1, f, f^2, \ldots\}$ does not contain 0 so that the localisation R_f is not the zero ring. Let $\mathfrak{m} \subset R_f$ be a maximal ideal. Denote the canonical homomorphism $j : R \to R_f$. As $j(f) \in R_f$ is a unit, $j(f) \notin \mathfrak{m}$. Then $j^{-1}(\mathfrak{m}) \subset R$ is a prime ideal that does not contain f. Hence, $f \notin \bigcap_{\mathfrak{p} \subseteq R\text{ prime}} \mathfrak{p}$.

If $I \subsetneq R$ is a proper ideal, we consider the quotient ring $\pi : R \to S = R/I$. Recall the bijective correspondence

$$\{\text{prime ideals in } S\} \leftrightarrow \{\text{prime ideals in } R \text{ containing } I\}\ ,\ \mathfrak{p} \leftrightarrow \pi^{-1}(\mathfrak{p})$$

Then,

$$\sqrt{I} = \pi^{-1}(\sqrt{0_S}) = \pi^{-1}\left(\bigcap_{\mathfrak{p} \subseteq S\text{ prime}} \mathfrak{p}\right) = \bigcap_{\mathfrak{p} \subseteq S\text{ prime}} \pi^{-1}(\mathfrak{p}) = \bigcap_{\substack{\mathfrak{q} \supseteq I \\ \mathfrak{q}\text{ prime}}} \mathfrak{q}.$$

2. (DG) Let $c(s) = (r(s), z(s))$ be a curve in the (x, z)-plane which is parameterized by arc length s. We construct the corresponding rotational surface, S, with parametrization

$$\varphi : (s, \theta) \mapsto (r(s)\cos\theta, r(s)\sin\theta, z(s)).$$

Find an example of a curve c such that S has constant negative curvature -1.

Solution:

$$\frac{\partial\varphi}{\partial s}(s,\theta) = (r'(s)\cos\theta, r'(s)\sin\theta, z'(s))$$
$$\frac{\partial\varphi}{\partial\theta}(s,\theta) = (-r(s)\sin\theta, r(s)\cos\theta, 0)$$

The coefficients of the first fundamental form are:

$$E = r'(s)^2 + z'(s)^2 = 1, \qquad F = 0, \qquad G = r(s)^2$$

Curvature:

$$K = -\frac{1}{\sqrt{G}}\frac{\partial^2}{\partial s^2}\sqrt{G} = -\frac{r''(s)}{r(s)}$$

To get $K = -1$ we need to find $r(s)$, $z(s)$ such that

$$r''(s) = r(s),$$
$$r'(s)^2 + z'(s)^2 = 1.$$

A possible solution is $r(s) = e^{-s}$ with

$$z(s) = \int \sqrt{1 - e^{-2t}}dt = \operatorname{Arcosh}(r^{-1}) - \sqrt{1 - r^2}.$$

3. (RA) Let $f \in L^2(0,\infty)$ and consider

$$F(z) = \int_0^\infty f(t)e^{2\pi izt}dt$$

for z in the upper half-plane.

(a) Check that the above integral converges absolutely and uniformly in any region $\operatorname{Im}(z) \geq C > 0$.

(b) Show that

$$\sup_{y>0}\int_0^\infty |F(x+iy)|^2dx = \|f\|^2_{L^2(0,\infty)}.$$

Solution: For $\operatorname{Im}(z) \geq C > 0$ we have

$$|f(t)e^{2\pi izt}| \leq |f(t)|e^{-2C\pi t}$$

thus with the Cauchy–Schwarz inequality

$$\int_0^\infty |f(t)e^{2\pi izt}|dt \leq \left(\int_0^\infty |f(t)|^2dt\right)^{1/2}\left(\int_0^\infty e^{-4C\pi t}dt\right)^{1/2}$$

which proves the claim.

For the second part, Plancherel's theorem gives

$$\int_0^\infty |F(x+iy)|^2 dx = \int_0^\infty |f(t)|^2 e^{-4\pi yt} dt \le \|f\|^2_{L^2(0,\infty)}$$

and

$$\sup_{y>0} \int_0^\infty |f(t)|^2 e^{-4\pi yt} dt = \int_0^\infty |f(t)|^2 dt$$

by the monotone convergence theorem.

4. (CA) Given that $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$, use contour integration to prove that each of the improper integrals $\int_0^\infty \sin(x^2)\,dx$ and $\int_0^\infty \cos(x^2)\,dx$ converges to $\sqrt{\pi/8}$.

Solution: We integrate $e^{-z^2}\,dz$ along a triangular contour with vertices at 0, M, and $(1+i)M$, and let $M \to \infty$. Since e^{-z^2} is holomorphic on $\mathbb{C}$, the integral vanishes. The integral from 0 to M is $\int_0^M e^{-x^2}\,dx$, which approaches $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$. The vertical integral approaches zero, because it is bounded in absolute value by

$$\int_0^M |e^{-(M+yi)^2}|\,dy = \int_0^M e^{y^2-M^2}\,dy < \int_0^M e^{M(y-M)}\,dy$$

$$= \int_0^M e^{-Mt}\,dt < \int_0^\infty e^{-Mt}\,dt = \frac{1}{M} \to 0$$

(substituting $t = M - y$ in the middle step). Thus the diagonal integral (with direction reversed, from 0 to $(1+i)\infty$) equals $\frac{1}{2}\sqrt{\pi}$. The change of variable $z = e^{\pi i/4}x$ converts this integral to $e^{\pi i/4}\int_0^\infty e^{-ix^2}\,dx$. Hence

$$\int_0^\infty (\cos x^2 - i\sin x^2)\,dx = \int_0^\infty e^{-ix^2}\,dx = \frac{1}{2}e^{-\pi i/4}\sqrt{\pi} = \frac{1-i}{2\sqrt{2}}\sqrt{\pi}.$$

equating real and imaginary parts yields the required result.

5. (AT)

(a) Let $X = \mathbb{R}P^3 \times S^2$ and $Y = \mathbb{R}P^2 \times S^3$. Show that X and Y have the same homotopy groups but are not homotopy equivalent.

(b) Let $A = S^2 \times S^4$ and $B = \mathbb{C}P^3$. Show that A and B have the same singular homology groups with $\mathbb{Z}$-coefficients but are not homotopy equivalent.

Solution: The universal covers of $\mathbb{R}P^2$ and $\mathbb{R}P^3$ are S^2 and S^3, respectively. Moreover, these covers are both 2-sheeted. Hence, we have

$$\pi_1(X) = \pi_1(\mathbb{R}P^3) \times \pi_1(S^2) = \pi_1(\mathbb{R}P^3) = \mathbb{Z}/2\mathbb{Z}$$

$$\pi_1(Y) = \pi_1(\mathbb{R}P^2) \times \pi_1(S^3) = \pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}.$$

Also, $\pi_k(\mathbb{R}P^j) = \pi_k(S^j)$, for $k > 1$, $j = 2, 3$ so that

$$\pi_k(X) = \pi_k(S^2) \times \pi_k(S^3) = \pi_k(Y), \quad k > 1.$$

To show that X and Y are not homotopy equivalent, we show that they have nonisomorphic homology groups. We make use of the following well-known singular homology groups (with integral coefficients)

$$H_0(S^n) = H_n(S^n) = \mathbb{Z}, \quad H_i(S^k) = 0, \ i \neq 0, n,$$

$$H_0(\mathbb{R}P^2) = H_2(\mathbb{R}P^2) = \mathbb{Z}, \ H_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}, \ H_i(\mathbb{R}P^2) = 0, i \neq 0, 1, 2$$
$$H_0(\mathbb{R}P^3) = \mathbb{Z}, \ H_1(\mathbb{R}P^3) = \mathbb{Z}/2\mathbb{Z}, \ H_i(\mathbb{R}P^3) = 0, i \neq 0, 1$$

Now, the Kunneth theorem in singular homology (with $\mathbb{Z}$-coefficients) gives an exact sequence

$$0 \to \bigoplus_{i+j=2} H_i(\mathbb{R}P^3) \otimes_{\mathbb{Z}} H_j(S^2) \to H_2(X) \to \bigoplus_{i+j=1} Tor_1(H_i(\mathbb{R}P^3), H_j(S^2)) \to 0$$

Since $H_k(S^2)$ is free, for every k, we have

$$H_2(X) \simeq \bigoplus_{i+j=2} H_i(\mathbb{R}P^3) \otimes_{\mathbb{Z}} H_j(S^2) = \mathbb{Z}$$

Similarly, we compute

$$H_2(Y) \simeq \bigoplus_{i+j=2} H_i(\mathbb{R}P^2) \otimes_{\mathbb{Z}} H_j(S^3) = \mathbb{Z}/2\mathbb{Z}.$$

In particular, X and Y are not homotopy equivalent.

For the second part, B can be constructed as a cell complex with a single cell in dimensions $0, 2, 4, 6$. Therefore, the homology of B is $H_{2i}(B) = \mathbb{Z}$, for $i = 0, \ldots, 3$, and $H_k(B) = 0$ otherwise.

The Kunneth theorem for singular cohomology (with $\mathbb{Z}$-coefficients), combined with the fact that $H_k(S^n)$ is free, for any k, gives

$$H_k(A) \simeq \bigoplus_{i+j=k} H_i(S^2) \otimes H_j(S^4).$$

Hence, $H_{2i}(A) = \mathbb{Z}$, for $i = 0, \ldots, 3$, and $H_k(A) = 0$ otherwise.

In order to show that A and B are not homotopy equivalent we will show that they have nonisomorphic homotopy groups.

Consider the canonical quotient map $\mathbb{C}^4 - \{0\} \to \mathbb{C}P^3$. This restricts to give a fiber bundle $S^1 \to S^7 \to \mathbb{C}P^3$. The associated long exact sequence in homotopy

$$\cdots \to \pi_{k+1}(\mathbb{C}P^3) \to \pi_k(S^1) \to \pi_k(S^7) \to \pi_k(\mathbb{C}P^3) \to \cdots$$

together with the fact that $\pi_3(S_1) = \pi_4(S^7)$, shows that $\pi_4(\mathbb{C}P^3) = 0$. However, $\pi_4(A) = \pi_4(S^4) = \mathbb{Z}$.

6. (AG)

Let C be the smooth projective curve over $\mathbb{C}$ with affine equation $y^2 = f(x)$, where $f \in \mathbb{C}[x]$ is a square-free monic polynomial of degree $d = 2n$.

(a) Prove that the genus of C is $n - 1$.

(b) Write down an explicit basis for the space of global differentials on C.

Solution: For the first part, use Riemann-Hurwitz: the $2:1$ map from C to the x-line is ramified above the roots of f and nowhere else (not even at infinity because $\deg f$ is even), so

$$2 - 2g(C) = \chi(C) = 2\chi(\mathbb{P}^1) - \deg P = 4 - 2n,$$

whence $g(C) = n - 1$.

For the second, let $\omega_0 = dx/y$. This differential is holomorphic, with zeros of order $g - 1$ at the two points at infinity. (Proof by local computation around those points and the roots of P, which are the only places where holomorphy is not immediate; dx has a pole of order -2 at infinity but $1/y$ has zeros of order n at the points above $x = \infty$, while $2y\,dy = P'(x)\,dx$ takes care of the Weierstrass points.) Hence the space of holomorphic differentials contains

$$\Omega := \{P(x)\,\omega_0 \mid \deg P < g\},$$

which has dimension g. Thus Ω is the full space of differentials, with basis $\{\omega_k = x^k\omega_0,\ k = 0, \ldots, g - 1\}$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday September 1, 2016 (Day 3)

1. (AT) Model S^{2n-1} as the unit sphere in $\mathbb{C}^n$, and consider the inclusions

$$\begin{array}{ccccccc} \cdots & \to & S^{2n-1} & \to & S^{2n+1} & \to & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \to & \mathbb{C}^n & \to & \mathbb{C}^{n+1} & \to & \cdots . \end{array}$$

Let S^∞ and $\mathbb{C}^\infty$ denote the union of these spaces, using these inclusions.

(a) Show that S^∞ is a contractible space.

(b) The group S^1 appears as the unit norm elements of $\mathbb{C}^\times$, which acts compatibly on the spaces $\mathbb{C}^n$ and S^{2n-1} in the systems above. Calculate *all* the homotopy groups of the homogeneous space S^∞/S^1.

Solution: The shift operator gives a norm-preserving injective map $T : \mathbb{C}^\infty \to \mathbb{C}^\infty$ that sends S^∞ into the hemisphere where the first coordinate is zero. The line joining $x \in S^\infty$ to $T(x)$ cannot pass through zero, since x and $T(x)$ cannot be scalar multiples, and hence the linear homotopy joining x to $T(x)$ shows that T is homotopic to the identity. However, since $T(S^\infty)$ forms an equatorial hemisphere, there is a also a linear homotopy from T to the constant map at either of the poles.

For the second part, because S^1 acts properly discontinuously on S^∞, the quotient sequence

$$S^1 \to S^\infty \to S^\infty/S^1$$

forms a fiber bundle. The homotopy groups of S^1 are known: $\pi_1 S^1 \cong \mathbb{Z}$ and $\pi_{\neq 1} S^1 = 0$ otherwise. Since S^∞ is contractible, the long exact sequence of higher homotopy groups shows that $\pi_2(S^\infty/S^1) = \mathbb{Z}$ and $\pi_{\neq 2}(S^\infty/S^1) = 0$ otherwise.

2. (AG) Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d. Show that if

$$\binom{k+d}{k} \; > \; (k+1)(n-k)$$

then X does not contain any k-plane $\Lambda \subset \mathbb{P}^n$.

Solution: For the first, let $\mathbb{P}^N$ be the space of all hypersurfaces of degree d in $\mathbb{P}^n$, and let

$$\Gamma \;=\; \{(X, \Lambda) \in \mathbb{P}^N \times \mathbb{G}(k,n) \;\mid\; \Lambda \subset X\}.$$

The fiber of Γ over the point $[\Lambda] \in \mathbb{G}(k,n)$ is just the subspace of $\mathbb{P}^N$ corresponding to the vector space of polynomials vanishing on Λ; since the space of polynomials on $\mathbb{P}^n$ surjects onto the space of polynomials on $\Lambda \cong \mathbb{P}^k$, this is a subspace of codimension $\binom{k+d}{k}$ in $\mathbb{P}^N$. We deduce that

$$\dim \Gamma \;=\; (k+1)(n-k) + N - \binom{k+d}{k};$$

in particular, if the inequality of the problem holds, then $\dim \Gamma < N$, so that Γ cannot dominate $\mathbb{P}^N$.

3. (DG) Let $\mathcal{H}^2 := \{(x,y) \in \mathbb{R}^2 : y > 0\}$. Equip $\mathcal{H}^2$ with a metric

$$g_\alpha := \frac{dx^2 + dy^2}{y^\alpha}$$

where $\alpha \in \mathbb{R}$.

(a) Show that $(\mathcal{H}^2, g_\alpha)$ is incomplete if $\alpha \neq 2$.

(b) Identify $z = x + iy$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$, consider the map $z \mapsto \frac{az+b}{cz+d}$. Show that this defines an isometry of $(\mathcal{H}^2, g_2)$.

(c) Show that $(\mathcal{H}^2, g_2)$ is complete. (Hint: Show that the isometry group acts transitively on the tangent space at each point.)

Solution: For the first part, consider the geodesic $\gamma(t)$ with $\gamma(0) = (0,1)$, and $\gamma'(0) = \frac{\partial}{\partial y}$. In order for $(\mathcal{H}^2, g_\alpha)$ to be complete, this geodesics must exist for all $t \in (-\infty, \infty)$. By symmetry, this geodesic must be given by

$$\mathbf{x}(t) = (0, y(t)).$$

Furthermore, $\mathbf{x}(t)$ must have constant speed, which we may as well take to be 1. Thus $\frac{(\dot{y})^2}{y^\alpha} = 1$, or in other words,

$$\dot{y} = y^{\alpha/2}.$$

If $\alpha \neq 2$, then the solution to this ODE is

$$y(t) = \Big((1 - \frac{\alpha}{2})t + 1\Big)^{1/(1-\frac{\alpha}{2})}$$

thus, this geodesics persists only as long as $(1-\frac{\alpha}{2})t + 1 \geq 0$. This set is always bounded from one side. Note that when $\alpha = 2$, we get $\mathbf{x}(t) = (0, e^t)$, which exists for all time.

(b) To begin, note that $dz \otimes d\bar{z} = dx \otimes dx + dy \otimes dy$, so we can write the metric as

$$g_2 = \frac{4dz \otimes d\bar{z}}{|z-\bar{z}|^2}$$

Let $A \in SL(2,\mathbb{R})$, we compute

$$A^*dz = \frac{adz}{cz+d} - c\frac{(az+b)dz}{(cz+d)^2} = (ad-bc)\frac{dz}{(cz+d)^2} = \frac{dz}{(cz+d)^2}$$

and so $A^*d\bar{z} = \frac{d\bar{z}}{(c\bar{z}+d)^2}$. It remains to compute

$$A^*z - A^*\bar{z} = \frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d} = \frac{z-\bar{z}}{|cz+d|^2},$$

where we have used that $A \in SL(2,\mathbb{R})$. Putting everything together we get

$$A^*g_2 = \frac{4dz \otimes d\bar{z}}{|cz+d|^4} \cdot \frac{|cz+d|^4}{|z-\bar{z}|^2} = g_2,$$

and so $SL(2,\mathbb{R})$ acts by isometry.

(c) By the computation from part (a), we know that the geodesic– let's call it $\gamma_0(t)$– through the point $(0,1)$ in the direction $(0,1)$ exists for all time. Let $z = x+iy$ be any point in $\mathcal{H}^2$. By an isometry, we can map this point to $z = iy$. Without loss of generality, let us assume $y = 1$. It suffices to show that we can find $A \in SL(2,\mathbb{R})$ so that $A(i) = i$, and $A_*V = (0,1)$, where V is any unit vector in the tangent space $T_i\mathcal{H}^2$, for then the geodesic through i with tangent vector V will be nothing but $A^{-1}(\gamma_0(t))$, and hence will exist for all time. First, observe that $A(i) = i$, if and only if $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Consider the rotation matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

A straightforward computation shows that, in complex coordinates,

$$A_*V = \frac{1}{(\cos\theta + i\sin\theta)^2}V = e^{-2\sqrt{-1}\theta}V,$$

that is, $A_* : T_i\mathcal{H}^2 \to T_i\mathcal{H}^2$ acts as a rotation. Since θ is arbitrary, and the rotations act transitively on S^2, we're done.

4. (RA)

(a) Let H be a Hilbert space, $K \subset H$ a closed subspace, and x a point in H. Show that there exists a unique y in K that minimizes the distance $\|x - y\|$ to x.

(b) Give an example to show that the conclusion can fail if H is an inner product space which is not complete.

Solution: (a): If $y, y' \in K$ both minimize distance to x, then by the parallelogram law:

$$\|x - \frac{y+y'}{2}\|^2 + \|\frac{y-y'}{2}\|^2 = \frac{1}{2}(\|x-y\|^2 + \|x-y'\|^2) = \|x-y\|^2$$

But $\frac{y+y'}{2}$ cannot be closer to x than y, by assumption, so $y = y'$.

Let $C = \inf_{y \in K} \|x - y\|$, then $0 \leq C < \infty$ because K is non-empty. We can find a sequence $y_n \in K$ such that $\|x - y_n\| \to C$, which we want to show is Cauchy. The midpoints $\frac{y_n + y_m}{2}$ are in K by convexity, so $\|x - \frac{y_n+y_m}{2}\| \geq C$ and using the parallelogram law as above one sees that $\|y_n - y_m\| \to 0$ as $n, m \to \infty$. By completeness of H the sequence y_n converges to a limit y, which is in K, since K is closed. Finally, continuity of the norm implies that $\|x - y\| = C$.

(b): For example choose $H = C([0,1]) \subset L^2([0,1])$, K the subspace of functions with support contained in $[0, \frac{1}{2}]$, and and $x = 1$ the constant function.

If f_n is a sequence in K converging to $f \in H$ in L^2-norm, then

$$\int_{1/2}^{1} |f|^2 = 0$$

thus f vanishes on $[1/2, 1]$, showing that K is closed. The distance $\|x - y\|$ can be made arbitrarily close to $1/\sqrt{2}$ for $y \in K$ by approximating $\chi_{[0,1/2]}$ by continuous functions, but the infimum is not attained.

5. (A)

(a) Prove that there exists a unique (up to isomorphism) nonabelian group of order 21.

(b) Let G be this group. How many conjugacy classes does G have?

(c) What are the dimensions of the irreducible representations of G?

Solution: Let G be a group of order 21, and select elements g_3 and g_7 of orders 3 and 7 respectively. The subgroup generated by g_7 is normal — if it weren't, then g_7 and xg_7x^{-1} witnessing nonnormality would generate a group of order

49. In particular, we have $g_3 g_7 g_3^{-1} = g_7^j$ for some nonzero $j \in \mathbb{Z}/7$. Now we use the order of g_3:

$$\begin{aligned} g_7 &= g_3 g_3 g_3 \cdot g_7 \cdot g_3^{-1} g_3^{-1} g_3^{-1} \\ &= g_3 g_3 (g_7^j) g_3^{-1} g_3^{-1} \\ &= g_3 (g_7^{j^2}) g_3^{-1} \\ &= g_7^{j^3}, \end{aligned}$$

and hence $j^3 \equiv 1 \pmod 7$. This is nontrivially solved by $j = 2$ and $j = 4$, and these two cases coincide: if for instance $g_3 g_7 g_3^{-1} = g_7^2$, then by replacing the generator g_3 with g_3^2 we instead see

$$g_3^2 g_7 (g_3^2)^{-1} = g_3 g_7^2 g_3^{-1} = g_7^4.$$

We have the following conjugacy classes of elements:

- $\{e\}$ forms a class of its own.
- $\{g_7, g_7^4, g_7^2\}$ and $\{g_7^3, g_7^5, g_7^6\}$ form classes by our choice of j.
- Any element of order 3 generates a Sylow 3–subgroup, all of which are conjugate as subgroups. However, there cannot be an x with $x g_3 x^{-1} = g_3^2$, since G has only elements of odd order. Hence, there are two final conjugacy classes, each of size 7: those elements conjugate to g_3 and those conjugate to g_3^2.

These five conjugacy sets give rise to five irreducible representations, which must be of dimensions 1, 1, 1, 3, and 3 (since these square-sum to $|G| = 21$).

6. (CA) Find (with proof) all entire holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ satisfying the conditions:

1. $f(z+1) = f(z)$ for all $z \in \mathbb{C}$; and
2. There exists M such that $|f(z)| \leq M \exp(10|z|)$ for all $z \in \mathbb{C}$.

Solution: The functions satisfying these conditions are precisely the $\mathbb{C}$-linear combinations of $e^{-2\pi i z}$, 1, and $e^{2\pi i z}$. Indeed such f is readily seen to satisfy the two conditions. Conversely (1) means that f descends to a function of $q := e^{2\pi i z} \in \mathbb{C}^*$, say $f(z) = F(q)$, and then by (2) there is some M' such that $|F(q)| \leq M' \max(|q|^{-5/\pi}, |q|^{5/\pi})$ for all q, whence qF and $q^{-1}F$ have removable singularities at $q = 0$ and $q = \infty$ respectively, etc.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 4, 2018 (Day 1)

1. (AT)

(a) Let X and Y be compact, oriented manifolds of the same dimension n. Define the *degree* of a continuous map $f : X \to Y$.

(b) Let $f : \mathbb{CP}^3 \to \mathbb{CP}^3$ be any continuous map. Show that the degree of f is of the form m^3 for some integer m.

(c) Show that conversely for any $m \in \mathbb{Z}$ there is a continuous map $f : \mathbb{CP}^3 \to \mathbb{CP}^3$ of degree m^3.

Solution: For the first part, the induced map $f^* : H^n(Y, \mathbb{Z}) \cong \mathbb{Z} \to H^n(X, \mathbb{Z}) \cong \mathbb{Z}$ (where the isomorphisms with $\mathbb{Z}$ are given by the orientation) is multiplication by some integer d; this is the degree of f.

For the second part, note that $H^*(\mathbb{CP}^3, \mathbb{Z}) \cong \mathbb{Z}[\zeta]/(\zeta^4)$ and that f^* is a ring homomorphism. If $f^*(\zeta) = m\zeta$, then $f^*(\zeta^3) = m^3\zeta^3$ and so the degree must be a cube. To see that all cubes occur, just consider the map $[X, Y, Z, W] \mapsto [X^m, Y^m, Z^m, W^m]$ for positive $d = m^3$; take complex conjugates to exhibit maps with negative degrees.

2. (A) Let G be a group.

(a) Prove that, if V and W are irreducible G-representations defined over a field $\mathbb{F}$, then a G-homomorphism $f : V \to W$ is either zero or an isomorphism.

(b) Let $G = D_8$ be the dihedral group with 8 elements. What are the dimensions of its irreducible representations over $\mathbb{C}$?

Solution: (a) If f is nonzero, then its image in W is a nontrivial subrepresentation of W and hence W itself by irreducibility; therefore, f is surjective. Similarly, the kernel of f is a subrepresentation of V, which, if nontrivial, must be V itself, contradicting the assumption that f is nonzero; therefore f is injective. (b) There are five irreducible representations of D_8, four one-dimensional ones coming from characters of the quotient $\mathbb{Z}/2 \times \mathbb{Z}/2$ of D_8 by its center, and one two-dimensional representation corresponding to the realization of D_8 as the group of automorphisms of the plane preserving a square of D_8. The fact that the irreducible representations have dimensions 1, 1, 1, 1 and 2 can also be seen by arguing that the number of irreducible representations is the same as the number of conjugacy classes in D_8, which is 5, and that the sum of the squares of their dimensions must be 8.

3. (CA) Let f_n be a sequence of analytic functions on the unit disk $\Delta \subset \mathbb{C}$ such that $f_n \to f$ uniformly on compact sets and such that f is not identically zero. Prove that $f(0) = 0$ if and only if there is a sequence $z_n \to 0$ such that $f_n(z_n) = 0$ for n large enough.

Solution: $\Longleftarrow$ We begin by observing that there must be some ϵ such that for n large enough, f_n is not zero in a neighborhood of the circle. By uniform convergence, since $f_n \to f$, it must also be that $f_n' \to f'$. thus

$$\lim \int_{C_\epsilon} \frac{f_n'}{f_n'} dz = \int \frac{f'}{f} dz$$

The right handside must, eventually, be larger than 1, so the left hand side must be as well. As this holds for every ϵ, we see that f has a zero in $B_\epsilon(0)$ for every ϵ. Since f is not identically zero it must be the only zero.

$\Longrightarrow$ By the argument principle,

$$\frac{1}{2\pi} \int_{C_\epsilon} \frac{f'}{f} dz = 1$$

where C_ϵ is the circle of radius ϵ around zero for some ϵ suffiently small. On the otherhand,

$$\lim \int_{C_\epsilon} \frac{f_n'}{f_n'} dz = \int \frac{f'}{f} dz \geq 1.$$

so by the argument principle again, then for every ϵ, there is an N large enoug that $n \geq N$ yields f_n has a at least one zero. applying this for each $\epsilon \to 0$ yields the result.

4. (AG) Let K be an algebraically closed field of characteristic 0, and let $\mathbb{P}^n$ be the projective space of homogeneous polynomials of degree n in two variables over K. Let $X \subset \mathbb{P}^n$ be the locus of n^{th} powers of linear forms, and let $Y \subset \mathbb{P}^n$ be the locus of polynomials with a multiple root (that is, a repeated factor).

(a) Show that X and $Y \subset \mathbb{P}^n$ are closed subvarieties.

(b) What is the degree of X?

(c) What is the degree of Y?

Solution: First, X is the image of the map $\mathbb{P}^1 \to \mathbb{P}^n$ sending $[a, b] \in \mathbb{P}^1$ to $(ax + by)^n \in \mathbb{P}^n$. This is projectively equivalent (in characteristic 0!) to the degree n Veronese map, whose image is a closed curve of degree n. Second, Y is the zero locus of the discriminant, which is a polynomial of degree $2n - 2$ in the coefficients of a polynomial of degree n (this number can be deduced from the Riemann-Hurwitz formula, which says that a degree n map from $\mathbb{P}^1$ to $\mathbb{P}^1$ has $2n-2$ branch points; that is, a general line in $\mathbb{P}^n$ meets Y in $2n-2$ points). Thus $Y \subset \mathbb{P}^n$ is a hypersurface of degree $2n - 2$.

5. (DG) Given a smooth function $f : \mathbb{R}^{n-1} \to \mathbb{R}$, define $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(x_1, \ldots, x_n) := f(x_1, \ldots, x_{n-1}) - x_n$$

and consider the preimage $X_f = F^{-1}(0) \subset \mathbb{R}^n$.

(a) Prove that X_f is a smooth manifold which is diffeomorphic to $\mathbb{R}^{n-1}$.

(b) Consider the two examples X_f and $X_g \subset \mathbb{R}^3$ with $f(x_1, x_2) = x_1^2 + x_2^2$ and $g(x_1, x_2) = x_1^2 - x_2^2$. Compute their normal vectors at every point $(x_1, x_2, x_3) \in X_f$ and $(x_1, x_2, x_3) \in X_g$.

Solution.
Part (a). The last row of the Jacobian of $F : \mathbb{R}^n \to \mathbb{R}$ is $(0, \ldots, 0, 1)$ and so the Jacobian has rank 1 everywhere. This implies that $F^{-1}(0)$ is a smooth manifold. It has the global chart $\psi : F^{-1}(0) \to \mathbb{R}^{n-1}$ defined by

$$\psi(x_1, \ldots, x_n) := (x^1, \ldots, x_{n-1})$$

and is therefore diffeomorphic to $\mathbb{R}^{n-1}$.

Part (b). The first example is a paraboloid. Its normal vector is

$$\frac{(-2x, -2y, 1)}{\sqrt{1 + 4x^2 + 4y^2}}.$$

The second example is a hyperbolic paraboloid or “saddle surface”. Its normal vector is

$$\frac{(-2x, 2y, 1)}{\sqrt{1 + 4x^2 + 4y^2}}.$$

6. (RA) Let $K \subset \mathbb{R}^n$ be a compact set. Show that for any measurable function $f : K \to \mathbb{C}$, it holds that

$$\lim_{p\to\infty} \|f\|_{L^p(K)} = \|f\|_{L^\infty(K)}.$$

(Recall that $\|f\|_{L^p(K)} = \left(\int_K |f|^p \mathrm{d}x\right)^{1/p}$ and that $\|f\|_{L^\infty(K)}$ is the essential supremum of f, i.e., the smallest upper bound if the behavior of f on null sets is ignored.)

Solution.
Let $p > 1$. Since $|f| \leq \|f\|_{L^\infty(K)}$ holds almost everywhere, we have

$$\|f\|_{L^p(K)} = \left(\int_K |f|^p \mathrm{d}x\right)^{1/p} \leq |K|^{1/p}\|f\|_{L^\infty(K)} \overset{p\to\infty}{\to} \|f\|_{L^\infty(K)}.$$

It remains to prove the lower bound. Let $\epsilon > 0$. By definition of the essential supremum, there exists a set $A \subset K$ of Lebesgue measure $|A| > 0$, such that $|f| \geq (1-\epsilon)\|f\|_{L^\infty(K)}$ holds on A. Hence,

$$\|f\|_{L^p(K)} \geq \left(\int_A |f|^p \mathrm{d}x\right)^{1/p} \geq (1-\epsilon)\|f\|_{L^\infty(K)}|A|^{1/p} \overset{p\to\infty}{\to} (1-\epsilon)\|f\|_{L^\infty(K)}.$$

Sending $\epsilon \to 0$ proves the claim.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday September 5, 2018 (Day 2)

1. (AG) Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree d.

(a) Let K_C be the canonical bundle of C. For what integer n is it the case that $K_C \cong \mathcal{O}_C(n)$?

(b) Prove that if $d \geq 4$ then C is not hyperelliptic.

(c) Prove that if $d \geq 5$ then C is not trigonal (that is, expressible as a 3-sheeted cover of $\mathbb{P}^1$).

Solution: By the adjunction formula, the canonical divisor class is $K_C = \mathcal{O}_C(d-3)$, that is, plane curves of degree $d-3$ cut out canonical divisors on C.

Now, if C were hyperelliptic—meaning that there exists a degree 2 map $\pi : C \to \mathbb{P}^1$—a general fiber of π would consist of two points $p, q \in C$ moving in a pencil, that is, such that $h^0(\mathcal{O}_C(p+q)) = 2$. But if $d \geq 4$ then any two points $p, q \in C$ impose independent conditions on the canonical series $|K_C|$; that is, $h^0(K_C(-p-q)) = g-2$, so by Riemann-Roch $h^0(\mathcal{O}_C(p+q)) = 1$, and hence C is not hyperelliptic. Similarly, if $d \geq 5$ then any three points $p, q, r \in C$ impose independent conditions on the canonical series $|K_C|$; by Riemann-Roch it follows that $h^0(\mathcal{O}_C(p+q+r)) = 1$ so C is not trigonal.

2. (CA) (The 1/4 theorem). Let $\mathcal{S}$ denote the class of functions that are analytic on the disk and one-to-one with $f(0) = 0$ and $f'(0) = 1$.

(a) Prove that if $f \in \mathcal{S}$ and w is not in the range of f then

$$g(z) = \frac{wf(z)}{(w - f(z))}$$

is also in $\mathcal{S}$.

(b) Show that for any $f \in \mathcal{S}$, the image of f contains the ball of radius 1/4 around the origin. You may use the elementary result (Bieberbach) that if $f(z) = z + \sum_{k\geq 2} a_k z^k$ in $\mathcal{S}$ then $|a_2| \leq 2$.

Solution: The proof of the first part is by checking. Since $w \notin R(f)$ it is analytic on the disk. now observe that map $h(z)$ which is

$$h(z) = \frac{wz}{(w-z)}$$

is one-to-one. thus $g(z) = h \circ f$ so it is one-to-one. Finally since

$$g'(z) = \frac{w^2 f'(z)}{(w - f(z))^2}$$

it follows that $g(0) = 0$ and $g'(0) = 1$ as desired.

For the second part, Suppose that w is not in the image of f. Then we may look at

$$g(z) = \frac{wf(z)}{(w - f(z))^2}.$$

Observe that

$$|g''(0)| = |a_2 + \frac{1}{w}| \leq 2$$

and $a_2 \leq 2$. From this it follows that

$$|1/w| \leq |a_2| + |a_2 + 1/w| \leq 4,$$

from which it follows that $|w| \geq 1/4$. Thus the set $|w| \leq 1/4$ is in the image of f as desired.

3. (A) Find a polynomial $f \in \mathbb{Q}[x]$ whose Galois group (over $\mathbb{Q}$) is D_8, the dihedral group of order 8.

Solution: There are lots of ways to find examples. Here is one: consider a quartic polynomial whose cubic resolvent has exactly one rational root and discriminant is nonsquare. Indeed, ordering the roots as α_1 through α_4, suppose $\alpha_1\alpha_2 + \alpha_3\alpha_4$ is rational so that the Galois group is contained in the dihedral group generated by $(1324), (13)(24)$. We want to ensure that the Galois group is no smaller: that the other roots of the resolvent are not rational ensures that the Galois group is not contained in the Klein subgroup generated by $(12)(34), (13)(24)$; equivalently, this is the restriction that the discriminant be nonsquare and so the Galois group not be contained in the alternating group. But now, if K represents the splitting field, we have the exact sequence $1 \to \mathrm{Gal}(K/\mathbb{Q}(\sqrt{D})) \to \mathrm{Gal}(K/\mathbb{Q}) \to \mathbb{Z}/2 \to 1$ and if $\mathrm{Gal}(K/\mathbb{Q})$ were any smaller than the D_8 in which it is already contained, $\mathrm{Gal}(K/\mathbb{Q}(\sqrt{D}))$ would have order at most 2 and hence the polynomial would have (multiple) roots over $\mathbb{Q}(\sqrt{D})$. Hence it suffices to find a quartic polynomial with a cubic resolvent with exactly one rational root that stays irreducible over

$\mathbb{Q}(\sqrt{D})$. After some experimentation, $f(x) = x^4 + 3x + 3$ with cubic resolvent $(x+3)(x^2 - 3x + 3)$, discriminant $3^3 \cdot 5^2 \cdot 7$, and which over $\mathbb{Q}(\sqrt{21})$ has no roots, as one can check manually: suppose $\alpha + \beta\sqrt{21}$ were a root. We know the ring of integers of $\mathbb{Q}(\sqrt{21})$ and will use that $\tilde{\alpha} = 2\alpha, \tilde{\beta} = 2\beta$ are (usual) integers. Expanding the equation, we find that $4\alpha(\alpha^2 + 21\beta^2) = -3$, or $\tilde{\alpha}(\tilde{\alpha}^2 + 21\tilde{\beta}^2) = -6$. This cannot happen without $\tilde{\beta} = 0$, which is impossible as f is irreducible over $\mathbb{Q}$.

4. (RA)

(a) Let $a_k \geq 0$ be a monotone increasing sequence with $a_k \to \infty$, and consider the ellipse,

$$E(a_k) = \{v \in \ell^2(\mathbb{Z}) : \sum a_k v_k^2 \leq 1\}.$$

Show that $E(a_n)$ is a compact subset of $\ell^2(\mathbb{Z})$.

(b) Let $\mathbb{T}$ denote the one-dimensional torus; that is, $\mathbb{R}/2\pi\mathbb{Z}$, or $[0, 2\pi]$ with the ends identified. Recall that the space $H^1(\mathbb{T})$ is the closure of $C^\infty(\mathbb{T})$ in the norm

$$\|f\|_{H^1(\mathbb{T})} = \sqrt{\|f\|_{L^2(\mathbb{T})} + \|\frac{d}{dx}f\|_{L^2(\mathbb{T})}}.$$

Use part (a) to conclude that the inclusion $i : H^1(\mathbb{T}) \hookrightarrow L^2(\mathbb{T})$ is a compact operator.

Solution: Part a. Firstly, since a_k is monotone,

$$\sum a_k v_k^2 \leq 1$$

implies that for any L,

$$\sum_{k \geq L} v_k^2 \leq \frac{1}{a_L}.$$

Thus E is is norm bounded.

Suppose that we have sequence $v^n \in E(a_k)$. Observe that

$$v_k^n \leq \frac{1}{\sqrt{a_k}}.$$

Thus we may diagonalize this sequence pointwise to obtain a sequence v_k with $v_k \in \ell^\infty$. Passing to this subsequence, we see that by fatou's lemma,

$$\sum v_k^2 \leq \frac{1}{a_1},$$

i.e., v is in ℓ^2. It remains to show that $v^n \to v$ in ℓ^2. To see this, observe that

$$\begin{aligned}\sum |v_k^n - v_k|^2 &\le \sum_{k\ge L} |v_k^n - v_k|^2 + \sum_{k\le L} \cdots \\ &\le \frac{2}{a_L} + \sum_{k\le L} |v_k^n - v_k|^2.\end{aligned}$$

Sending $n \to \infty$ and then $L \to \infty$ yields the result.

Part b. It suffices to show that the unit ball of $H^1(\mathbb{T})$ is a compact subset of $L^2(\mathbb{T})$. By Parseval's identity/the fourier transform, it follows that

$$\sum k^2 |\hat{f}_n(k)|^2 \le C.$$

for some positive constant. This is a compact subset of ℓ_2 by part a. Thus by fourier inversion, the ball of H^1 is as well.

5. (AT) Consider the following topological spaces:

$$A = S^1 \times S^1 \qquad\qquad B = S^1 \vee S^1 \vee S^2.$$

(a) Compute the fundamental group of each space.

(b) Compute the integral cohomology ring of each space.

(c) Show that B is not homotopy equivalent to any compact orientable manifold.

Solution: (a) The fundamental group construction preserves products, so

$$\pi_1(A) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}.$$

By the Van Kampen theorem,

$$\pi_1(B) \cong \pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z},$$

where the first step uses that S^2 is simply connected. (b) By the Künneth theorem,

$$H^*(S^1 \times S^1) \cong H^*(S^1) \otimes H^*(S^1) \cong \Lambda[x, y], \quad |x| = |y| = 1.$$

Since the reduced cohomology ring construction takes wedges of spaces to products of (nonunital) rings,

$$H^*(S^1 \vee S^1 \vee S^2) \cong \frac{\Lambda[x, y, z]}{xy = yz = zx = 0}, \quad |x| = |y| = 1,\ |z| = 2.$$

(c) Suppose that B is homotopy equivalent to the compact orientable manifold M. Choosing a fundamental class $[M]$, Poincaré duality guarantees that the assignment

$$(a, b) \mapsto \langle a \smile b, [M] \rangle$$

defines a symplectic form on $H^1(M)$. Since $xy = 0$, this pairing is degenerate, a contradiction.

6. (DG) Consider the set

$$G := \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathbb{R}_+,\ y \in \mathbb{R} \right\},$$

and equip it with a smooth structure via the global chart that sends $(x, y) \in \mathbb{R}_+ \times \mathbb{R}$ to the corresponding element of G.

(a) Show that G is a Lie subgroup of the Lie group $GL(\mathbb{R}, 3)$.

(b) Prove that the set

$$\left\{ x\frac{\partial}{\partial x}, x\frac{\partial}{\partial y} \right\}$$

forms a basis of left-invariant vector fields on G.

(c) Find the structure constants of the Lie algebra $\mathfrak{g}$ of G with respect to the basis in (b).

Solution.
Part (a). We consider the multiplication and inverse operations on G:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} ax & 0 & 0 \\ 0 & ax & ay + b \\ 0 & 0 & 1 \end{pmatrix} \in G$$

and

$$\begin{pmatrix} x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/x & 0 & 0 \\ 0 & 1/x & -y/x \\ 0 & 0 & 1 \end{pmatrix} \in G.$$

We see that G is closed under these operations, and that they are smooth. This proves that G is a Lie group itself.

Moreover: (a) Since the inverse of any element in G exists, G is a subset of $GL(\mathbb{R}, 3)$. (b) The inclusion map $G \to GL(\mathbb{R}, 3)$ is trivially a group homomorphism. (c) The inclusion map $G \to GL(\mathbb{R}, 3)$ is an immersion. To see this, recall that the smooth structure on $GL(\mathbb{R}, 3)$ is that of $\mathbb{R}^9$ and note that the map $(x, y) \mapsto (x, 0, 0, 0, x, y, 0, 0, 1)$ has rank 2.

Together, (a)-(c) imply that G is a Lie subgroup of $GL(\mathbb{R}, 3)$.

Part (b). Linear independence follows from the linear independence of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$. To prove left-invariance, let us identify a vector (x, y) with the corresponding matrix in G. The formula for the product $G \times G \to G$ shows that left translation in G is given by

$$L_{(a,b)}(x, y) = (ax, ay + b),$$

and so $L_{(a,b)*} = a \operatorname{Id}_{\mathbb{R}^2}$. This shows that

$$L_{(a,b)*} X_{(x,y)} = X_{(a,b)(x,y)}$$

holds for both vector fields X in (b), hence they are left-invariant.

Part (c). Let us call the two vector fields in (b) X_1, X_2, respectively. Explicit computation shows that

$$[X_1, X_2] = X_2,$$

and so the non-zero structure constants are $f_{12}^2 = -f_{21}^2 = 1$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday September 6, 2018 (Day 3)

1. (AT) Let $p : E \to B$ be a k-fold covering space, and suppose that the Euler characteristic $\chi(E)$ is defined, nonzero, and relatively prime to k. Show that any CW decomposition of B has infinitely many cells.

Solution: Suppose that B has a finite CW decomposition; in particular, $\chi(B)$ is defined. Restricted to each of the cells of B, the covering p is trivial, and the connected components of the total spaces of these restricted covers form a CW decomposition of E. Counting cells, we find that $\chi(E) = k \cdot \chi(B)$. Since $\chi(E) \neq 0$, it follows that k divides $\chi(E)$, a contradiction.

2. (RA) Let W be Gumbel distributed, that is $P(W \leq x) = e^{-e^{-x}}$. Let X_i be independent and identically distributed Exponential random variables with mean 1; that is, X_i are independent, with $P(X_i \leq x) = \exp(-\max x, 0)$.

Let

$$M_n = \max_{i \leq n} X_i.$$

Show that there are deterministic sequences a_n, b_n such that

$$\frac{M_n - b_n}{a_n} \to W$$

in law; that is, such that for any continuous bounded function F,

$$\mathbb{E}F\left(\frac{M_n - b_n}{a_n}\right) \to \mathbb{E}F(W).$$

.

Solution: let $b_n = \log n$ and $a_n = 1$. Then

$$P(M_n - b_n \leq x) = P(X_i \leq x + \log n)^n$$

since X_i are i.i.d. Now

$$\begin{aligned} P(X \leq x + \log n)^n &= (1 - P(X > x + \log n))^n \\ &= \left(1 - \int_{x+\log n}^{\infty} e^{-w} dw\right)^n \\ &= \left(1 - \frac{1}{n}\int_{x}^{\infty} e^{-w} dw\right)^n \to e^{-e^{-x}} \end{aligned}$$

As $e^{-e^{-x}}$ is continuous every where and

$$P(M_n - b_n \leq x) \to P(W \leq x),$$

we see that $M_n - b_n \to W$ in law by the Portmanteau lemma.

3. (DG) Consider $\mathbb{R}^2$ as a Riemannian manifold equipped with the metric

$$g = e^x \mathrm{d}x^2 + \mathrm{d}y^2.$$

(i) Compute the Christoffel symbols of the Levi-Civita connection for g.

(ii) Show that the geodesics are described by the curves $x(t) = 2\log(At+B)$ and $y(t) = Ct + D$, for appropriate constants A, B, C, D.

(iii) Let $\gamma : \mathbb{R}_+ \to \mathbb{R}^2$, $\gamma(t) = (t, t)$. Compute the parallel transport of the vector $(1, 2)$ along the curve γ.

(iv) Are there two vector fields X, Y parallel to the curve γ, such that $g(X(t), Y(t))$ is non-constant?

Solution:
Part (i). We can identify

$$g^{-1} = \begin{pmatrix} e^{-x} & 0 \\ 0 & 1 \end{pmatrix}.$$

Denoting $x^1 = x$, $x^2 = y$, the only non-vanishing Christoffel symbol is

$$\Gamma^1_{11} = \frac{1}{2} g_{11}^{-1} \partial_1 g_{11} = \frac{1}{2}.$$

Part (ii). Using part (i), the two ODE describing the geodesic $(x(t), y(t))$ are given by

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = 0, \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = 0.$$

The second ODE is solved by $y(t) = Ct + D$. For the first ODE, we introduce $u(t) := \frac{\mathrm{d}x}{\mathrm{d}t}$ and obtain

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{1}{2}u^2 = 0.$$

By separation of variables, this is solved by $u(t) = \frac{2}{t+C_1}$. We integrate this to get x and find

$$x(t) = 2\log(t + C_1) + C_2 = 2\log(At + B),$$

where we redefined the constants in the second step.

Part (iii). The equation for parallel transport $\nabla_{\gamma'}(a^1, a^2) = 0$, with $\gamma(t) = (t, t)$, becomes

$$\frac{\mathrm{d}a^1}{\mathrm{d}t} + \frac{1}{2}a^1 = 0, \qquad \frac{\mathrm{d}a^2}{\mathrm{d}t} = 0.$$

These are solved by $a^1(t) = Ae^{-t/2}$ and $a^2(t) = B$, respectively. To satisfy the initial condition $(a^1(0), a^2(0)) = (1, 2)$, we take $A = 1$ and $B = 2$. The solution is thus

$$(a^1(t), a^2(t)) = (e^{-t/2}, 2).$$

Part (iv). No. Since ∇ is the Levi-Civita connection, the scalar product of two vectors is preserved by parallel transport.

4. (A) Let G be a group of order 78.

(a) Show that G contains a normal subgroup of index 6.

(b) Show by example that G may contain a subgroup of index 13 that is not normal.

Solution: (a) Sylow theory guarantees the existence of a 13-Sylow subgroup $H \leq G$, which has index 6. This Sylow subgroup is unique and hence normal; indeed, the number of such divides 6 and is congruent to 1 mod 13 by Sylow's theorems. (b) Take G to be the semidirect product $C_{13} \rtimes S_3$ of the cyclic group of order 13 and the symmetric group on 3 letters, where S_3 acts via the composite

$$S_3 \xrightarrow{\mathrm{sgn}} C_2 \xrightarrow{\mathrm{inv}} \mathrm{Aut}(C_{13})$$

of the sign homomorphism and the inversion homomorphism (we use that C_{13} is Abelian). We claim that the subgroup $S_3 \leq G$ is not normal. To see why this is so, let $\sigma \in S_3$ be an odd permutation and $\rho \in C_{13}$ a generator, and compute that

$$(\rho, e)(e, \sigma)(\rho^{-1}, e) = (\rho^2, \sigma) \notin S_3.$$

5. (AG) Let K be an algebraically closed field of characteristic 0, and consider the curve $C \subset \mathbb{A}^3$ over K given as the image of the map

$$\begin{aligned} \phi : &\mathbb{A}^1 \to \mathbb{A}^3 \\ &t \mapsto (t^3, t^4, t^5). \end{aligned}$$

Show that no neighborhood of the point $\phi(0) = (0,0,0) \in C$ can be embedded in $\mathbb{A}^2$.

Solution: Suppose $f(x,y,z)$ is any polynomial on $\mathbb{A}^3$ vanishing on C. The constant term of f must be zero, since f vanishes at $(0,0,0) \in C$, and the linear terms of f must also be zero, since the pullback to $\mathbb{A}^1$ of any monomial in x, y and z of degree 2 or more must vanish to order at least 6. In other words, the ideal $I(C)$ is contained in the square $(x,y,z)^2$ of the maximal ideal of the origin. In particular, the Zariski tangent space to C at $(0,0,0)$ is three dimensional, and hence no neighborhood of this point is embeddable in $\mathbb{A}^2$.

6. (CA) Let $f(z)$ be an entire function such that

a) $f(z)$ vanishes at all points $z = n$, $n \in \mathbb{Z}$;

b) $|f(z)| \leq e^{\pi |\operatorname{Im} z|}$ for all $z \in \mathbb{C}$.

Prove that $f(z) \;=\; c \sin \pi z$, with $c \in \mathbb{C}$, $|c| \leq 1$.

Solution: Define $h(z) = (\sin \pi z)^{-1} f(z)$. The hypotheses imply that $h(z)$ is entire. Then, for $\operatorname{Im} z > 0$,

$$|h(z)| \;=\; |\sin \pi z|^{-1}\,|f(z)| \;\leq\; |\sin \pi z|^{-1}\, e^{\pi|\operatorname{Im} z|} \;\leq\; 2(1 - e^{-2\pi \operatorname{Im} z})^{-1}\,.$$

Since the hypotheses are invariant under the substitution $z \mapsto -z$, we get the analogous bound for $\operatorname{Im} z < 0$. Thus $h(z)$ is uniformly bounded on $|\operatorname{Im} z| \geq \delta$, $\delta > 0$. On the vertical lines $\operatorname{Re} z = (n + 1/2)\pi$, $n \in \mathbb{Z}$, $|\sin \pi z|^{-1} e^{\pi|\operatorname{Im} z|} = 2(1 + e^{-2\pi|\operatorname{Im} z|})^{-1}$, which is bounded by 2. Applying the maximum principle to $h(z)$ on the rectangles with sides $\operatorname{Im} z = \pm 1$, $\operatorname{Re} z = (n \pm 1/2)\pi$, we find that $h(z)$ is a bounded entire function, hence $f(z) = c \sin \pi z$ with $c \in \mathbb{C}$. Evaluating the inequality $|f(z)| = c\,|\sin \pi z| \leq e^{\pi|\operatorname{Im} z|}$ at $z = 1/2$ leads to $|c| \leq 1$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday January 19, 2010 (Day 1)

1. Let (X, μ) be a measure space with $\mu(X) < \infty$. For $q > 0$, let $L^q = L^q(X, \mu)$ denote the Banach space completion of the space of bounded functions on X with the norm

$$||f||_q = \left(\int_X |f|^q \mu \right)^{\frac{1}{q}}.$$

Now suppose that $0 < p \leq q$. Prove that all functions in L^q are in L^p, and that the inclusion map $L^q \hookrightarrow L^p$ is continuous.

Solution. By Hölder's inequality

$$\|f\|_p^p = \left(\int_X |f|^p \cdot 1 d\mu \right) \leq \left(\int_X (|f|^p)^{\frac{q}{p}} d\mu \right)^{\frac{p}{q}} \left(\int_X (1)^{\frac{q}{q-p}} d\mu \right)^{1-\frac{p}{q}}$$

so

$$\|f\|_p^p \leq \|f\|_q^p \cdot \mu(X)^{1-\frac{p}{q}}$$

so

$$\|f\|_p \leq \|f\|_q \cdot \mu(X)^{\frac{1}{p}-\frac{1}{q}}.$$

Hence if f is in L^q, the left-hand side is finite hence so is the right-hand side, so f is in L^p. Also, the inequality shows that if $\|f\|_p$ is small then $\|f\|_q$ is also small, hence the inclusion $L^q \hookrightarrow L^p$ is continuous

2. Let $X \subset \mathbb{P}^n$ be an irreducible projective variety of dimension k, $\mathbb{G}(\ell, n)$ the Grassmannian of ℓ-planes in $\mathbb{P}^n$ for some $\ell < n - k$, and $C(X) \subset \mathbb{G}(\ell, n)$ the variety of ℓ-planes meeting X. Prove that $C(X)$ is irreducible, and find its dimension.

Solution. We have the diagram

$$\begin{tikzcd}
H \arrow[r] & \mathbb{G}(l,n) \times \mathbb{P}^n \\
V \arrow[u] \arrow[r] \arrow[d, "pr_1"] \arrow[dr, "pr_2"] & \mathbb{G}(l,n) \times X \arrow[u] \\
C(X) & X
\end{tikzcd}$$

Here H is the universal l-plane $\{(h, x) : x \in \mathbb{P}^n, h \in \mathbb{G}(l, n), x \in h\}$, and $V = H \cap (\mathbb{G}(l, n) \times X)$ For $x \in X$, the fiber over x is $V_x = \{h \in \mathbb{G}(l, n), x \in h\}$.

This can be identified with the subspace of the Grassmannian of $(l+1)$-dimensional subspaces in an $(n+1)$- dimensional vector space containing a fixed line, hence is isomorphic to the Grassmannian of l-dimensional subspaces of an n-dimensional vector space. It is therefore irreducible of dimension $l(n-l)>0$. But $pr_2 : V \twoheadrightarrow X$ is a surjective morphism with irreducible base and irreducible fibers of constant dimension $l(n-l)$, so V is also irreducible and has dimension dim V= dim $X + l(n-l)$= $k + l(n-l)$.

By definition, $C(X) = pr_1(X) \hookrightarrow \mathbb{G}(l,n)$, and hence is irreducible. If $h \in C(X)$, the fiber of V over h is $V_h = \{(h,x) \mid x \in h \cap X\} \simeq h \cap X$. Because $k+l<n$, we can find an $(n-k)$-plane that meets X at finitely many points. Then any l-plane h in this $(n-k)$-plane going through one of the intersection point will meet X at finitely many points. Hence for such h, V_h will be a finite set of points, so has dimension 0. By upper-semicontinuity of fiber dimension for the proper morphism pr_1, there is a dense open set where the fiber has dimension 0, and hence dim $C(X)$= dim V= $k + l(n-l)$.

3. Let λ be real number greater than 1. Show that the equation $ze^{\lambda - z} = 1$ has exactly one solution z with $|z|<1$, and that this solution z is real. (Hint: use Rouché's theorem.)

Solution. On $|z| = 1$ w have

$$|z \cdot e^{\lambda - z}| = e^{\lambda - Re(z)} \geq e^{\lambda - 1} > 1$$

because $\lambda > 1$. Hence by Rouché's theorem $1 - ze^{\lambda - z}$ has the same number of zeroes counted with multiplicity as $ze^{\lambda - z}$ inside $|z| = 1$, hence has exactly one zero. Observe if z is a zero of $1 - ze^{\lambda - z}$ then so is $\bar{z}$ because λ is real, hence by uniqueness the unique zero must be real.

4. Let k be a finite field, with algebraic closure $\overline{k}$.

(a) For each integer $n \geq 1$, show that there is a unique subfield $k_n \subset \overline{k}$ containing k and having degree n over k.

(b) Show that k_n is a Galois extension of k, with cyclic Galois group.

(c) Show that the norm map $k_n^\times \to k^\times$ (sending a nonzero element of k_n to the product of its Galois conjugates) is a surjective homomorphism.

Solution. Let the cardinality of k be q, a prime power.

(a) If a subfield $k_n \subset \bar{k}$ is of degree n over k, it has cardinality q^n. The multiplicative group $k_n^\times$ is a finite subgroup of the multiplicative group of a field, hence is cyclic. It has order $q^n - 1$. Hence $x^{q^n} - x = 0$ for all $x \in k_n$. On the other hand $x^{q^n} - x = 0$ has precisely q^n distinct solution in $\bar{k}$ (note $\frac{d}{dx}(x^{q^n} - x) = -1$ has no common zero with $x^{q^n} - x$), hence this forces k_n to be the set of zeroes of $x^{q^n} - x$ in $\bar{k}$. Note that in particular k is the set of zeroes of $x^q - x$. This shows there is at most one k_n. To show it exists, we must check that the zeroes of $x^{q^n} - x$ form a subfield of $\bar{k}$. But observe that it is the set of fixed point of the map $x \mapsto x^{q^n}$ defined on $\bar{k}$, which is a field endomorphism, so the set of fixed point is a subfield.

(b) We have $[k_n : k] = n$ by definition. Denote by F the map $x \mapsto x^q$ in $\bar{k}$. The characteristic of $\bar{k}$ divides q so this is an injective field endomorphism. The description of k_n above shows that it is stable under F, and F fixes k pointwise. Thus F is an automorphism of k_n over k. For any $k < n$, $x^{q^k} - x =$ has q^k solution in $\bar{k}$, so the solution set can not contain all of k_n. Hence $F^k \neq id$ on k_n, but $F^n = id$ on k_n. This shows that F generates a cyclic subgroup of $Aut(k_n/k)$. But the latter has size at most n, hence equality occurs, so k_n/k is Galois and has Galois group a cyclic group of order n, generated by F.

(c) Explicitly the norm map $k_n^\times \to k^\times$ is $x \mapsto x^{1+q+..+q^{n-1}} = x^{\frac{q^n-1}{q-1}}$. Because $\bar{k}$ is algebraically closed, for any $a \in k^\times$ there is $x \in \bar{k}^\times$ such that $x^{\frac{q^n-1}{q-1}} = a$. But then for such x we have $x^{q^n-1} = a^{q-1} = a$ because $a \in k$. Thus $x^{q^n} = x$ and $x \neq 0$, so $x \in k_n^\times$.

5. Suppose ω is a closed 2-form on a manifold M. For every point $p \in M$, let

$$R_p(\omega) = \{v \in T_pM \ : \ \omega_p(v,u) = 0 \text{ for all } u \in T_pM\}.$$

Suppose that the dimension of R_p is the same for all p. Show that the assignment $p \mapsto R_p$ as p varies in M defines an integrable subbundle of the tangent bundle TM.

Solution. We have the following identity for vector fields X, Y, Z:

$$d\omega(X,Y,Z) = -X\omega(Y,Z) + Y\omega(X,Z) - Z\omega(X,Y) + \omega([X,Y],Z) +$$

$$+\omega([Y,Z],X) - \omega([X,Z],Y) \quad (*)$$

To see this, observe that X acts derivations on $C^\infty(M)$ and that $[f \cdot X, Y]g = f \cdot [X,Y]g - Yf \cdot Xg$ for $f, g \in C^\infty(M)$, hence $[f \cdot X, Y] = [X,Y] - Yf \cdot X$. Hence

$$Y\omega(f \cdot X, Z) + \omega([f \cdot X, Y], Z)) = f \cdot Y\omega(X,Z) + f \cdot \omega([X,Y],Z) +$$

$$+Yf \cdot \omega(X,Z) - Yf \cdot \omega([X,Y],Z) = f \cdot (Y\omega(X,Z) + \omega([X,Y],Z)).$$

This and similar identities for Y, Z shows that the right-hand side of (*) is $C^\infty(M)$ linear, as is the left-hand side. Also note that both sides have the same variance under the action of S_3 via permuting X, Y, Z, and are both $\mathbb{C}$-linear in ω. To check the identity is also a local question. This reduces us to the case X, Y, Z are $\partial/\partial x_i$, $\partial/\partial x_j$, $\partial/\partial x_k$ and $\omega = f \cdot dx_1 \wedge dx_2 \wedge dx_3$ for some function f and local coordinates $x_1, ...x_n$, which quickly follows by

direct inspection (note in this case all the Lie brackets vanish).
We now show that the distribution defined by $R_p \subset T_pM$ is integrable (it is a subbundle of TM since the dimension of the fibers are constant). Indeed suppose X, Y are two vector fields belonging to it. Pick a point $p \in M$ and let Z_p be an arbitrary vector in T_pM. We can then find a global vector field Z which agrees with Z_p at p. Looking at the identity (*) at the point p, we then have (noting $d\omega = 0$)

$$0 = d\omega_p(X_p, Y_p, Z_p) = -(X\omega(Y,Z))_p + (Y\omega(X,Z))_p - (Z\omega(X,Y))_p +$$

$$+\omega_p([X,Y]_p, Z_p) + \omega_p([Y,Z]_p, X_p) - \omega_p([X,Z]_p, Y_p).$$

But $\omega(X,Z) = \omega(Y,Z) = \omega(X,Y) = \omega([X,Z],Y) = \omega([Y,Z],X) = 0$ by assumption on X, Y, hence $\omega_p([X,Y]_p, Z_p) = 0$. Since p and Z_p can be chosen arbitrarily, it follows that $[X,Y]$ also belongs to the distribution

6. Let X be a topological space. We say that two covering spaces $f : Y \to X$ and $g : Z \to X$ are isomorphic if there exists a homeomorphism $h : Y \to Z$ such that $g \circ h = f$. If X is a compact oriented surface of genus g (that is, a g-holed torus), how many connected 2-sheeted covering spaces does X have, up to isomorphism?

Solution. By covering space theory, there is a bijection between connected 2-sheeted coverings of X up to isomorphism and conjugacy classes of index 2 subgroups of $\pi_1(X)$. As any index 2 subgroup is normal, this set is in bijection with the set of index 2 subgroups of $\pi_1(X)$, which is the same as the set of surjective group homomorphisms from $\pi_1(X)$ to $\mathbb{Z}/2\mathbb{Z}$. Because $\mathbb{Z}/2\mathbb{Z}$ is commutative, all such homomorphisms factor through the abelization $\pi_1(X)^{ab}$

Now for X the compact oriented surface of genus g, $\pi_1(X)$ has a presentation $\langle a_1, ...a_g, b_1, ...b_g \mid [a_1, b_1][a_2, b_2]...[a_g, b_g] = 1\rangle$, here the bracket $[a,b] = aba^{-1}b^{-1}$ is the commutator. Hence its abelization is the free abelian group on $2g$ generators $\mathbb{Z}^{2g}$. Thus specifying a surjective homomorphism from $\pi_1(X)$ to $\mathbb{Z}/2\mathbb{Z}$ is the same thing as specifying where each generator goes to, such that not all go to the identity. The number of such homomorphisms is thus $2^{2g} - 1$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday January 20, 2010 (Day 2)

1. Let a be an arbitrary real number and b a positive real number. Evaluate the integral

$$\int_0^\infty \frac{\cos(ax)}{\cosh(bx)}dx$$

(Recall that $\cosh(x) = \cos(ix) = \frac{1}{2}(e^x + e^{-x})$ is the hyperbolic cosine.)

Solution. We will use the residue theorem for the rectangular contour bounded by the real axis, the line $\Im z = \frac{\pi i}{b}$, and the lines $\Re z = \pm A$, where A will be large, and for the function $f(z) = \frac{e^{iax}}{\cosh(bx)}$. The integrals over the horizontal edges of the contour are

$$\int_{-A}^{A} \frac{e^{iax}}{\cosh(bx)}dx$$

and

$$\int_{-A}^{A} \frac{e^{iax-\frac{\pi a}{b}}}{\cosh(bx)}dx$$

Hence the contribution of the horizontal edges are

$$(1+e^{-\frac{\pi a}{b}})\int_{-A}^{A} \frac{e^{iax}}{\cosh(bx)}dx$$

The contribution of the vertical sides are

$$\pm i\int_0^{\frac{\pi}{b}} \frac{e^{ia(\pm A+iy)}}{\cosh(b(\pm A+iy))}dy$$

Now note $|\cosh(b(\pm A+iy))| = |e^{bA\pm iy} + e^{-bA\mp iy}| \geq e^{bA}-1$, hence each side integral has norm bounded by $C\frac{1}{e^{bA}-1}$ for some constant C that does not depend on A, and hence tends to 0 when $A\to\infty$. Now the residue theorem says that the sum of the integral over the sides are the sum of $2\pi i$ times the residues of $f(z)$ inside the contour. For any A, the function $f(z)$ only has a simple pole at $z = \frac{\pi}{2b}$. We have $\cosh(b(x+\frac{\pi}{2b})) = i(e^{bx}-e^{-bx}) = 2ibx + O(x^3)$, hence $\mathrm{Res}_{z=\frac{\pi}{2b}} f = \frac{e^{-\frac{\pi a}{2b}}}{2ibx}$.

Thus letting $A\to\infty$ gives

$$(1+e^{-\frac{\pi a}{b}})\int_{-\infty}^{+\infty} \frac{e^{iax}}{\cosh(bx)}dx = 2\pi i \cdot \mathrm{Res}_{z=\frac{\pi}{2b}} f = \frac{\pi e^{-\frac{\pi a}{2b}}}{bx}$$

Taking real parts gives

$$\int_0^{+\infty} \frac{\cos(ax)}{\cosh(bx)} dx = \frac{\pi}{2bxcosh(\frac{\pi a}{2b})}.$$

2. For any irreducible plane curve $C \subset \mathbb{P}^2$ of degree $d > 1$, we define the *Gauss map* $g : C \to \mathbb{P}^{2^*}$ to be the rational map sending a smooth point $p \in C$ to its tangent line; we define the *dual curve* $C^* \subset \mathbb{P}^{2^*}$ of C to be the image of g.

(a) Show that the dual of the dual of C is C itself.

(b) Show that two irreducible conic curves $C, C' \subset \mathbb{P}^2$ are tangent if and only if their duals are.

Solution. The ground field is assumed to have characteristic 0, as the result fails otherwise. We can assume the ground field is $\mathbb{C}$.

(a) Let the plane curve C be given by $F = 0$, where F is an irreducible polynomial of degree d in X, Y, Z. We will choose coordinates on $\mathbb{P}^2$ and its dual in such a way that the dual pairing is given by $X \cdot U + Y \cdot V + Z \cdot W$. Because $d > 1$, the dual curve C^* is actually a plane curve, and hence is given by $G = 0$ for some homogenous polynomial. The Gauss map is given by $[X : Y : Z] \mapsto [\frac{\partial}{\partial X}F(X,Y,Z) : \frac{\partial}{\partial Y}F(X,Y,Z) : \frac{\partial}{\partial Z}F(X,Y,Z)]$. It follows that $G(\frac{\partial}{\partial X}F, \frac{\partial}{\partial Y}F, \frac{\partial}{\partial Z}F) = 0$ on $F = 0$. Because F is irreducible, there exists a polynomial H such that $G(\frac{\partial}{\partial X}F, \frac{\partial}{\partial Y}F, \frac{\partial}{\partial Z}F) = H \cdot F$.
Differentiating both sides with respect to X, Y, Z, we see that

$$\begin{pmatrix} \frac{\partial^2}{\partial X^2}F & \frac{\partial^2}{\partial X \partial Y}F & \frac{\partial^2}{\partial X \partial Z}F \\ \frac{\partial^2}{\partial X \partial Y}F & \frac{\partial^2}{\partial Y^2}F & \frac{\partial^2}{\partial Y \partial Z}F \\ \frac{\partial^2}{\partial X \partial Z}F & \frac{\partial^2}{\partial Z \partial Y}F & \frac{\partial^2}{\partial Z^2}F \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial X}G \\ \frac{\partial}{\partial Y}G \\ \frac{\partial}{\partial Z}G \end{pmatrix} = H \begin{pmatrix} \frac{\partial}{\partial X}F \\ \frac{\partial}{\partial Y}F \\ \frac{\partial}{\partial Z}F \end{pmatrix} + F \begin{pmatrix} \frac{\partial}{\partial X}G \\ \frac{\partial}{\partial Y}G \\ \frac{\partial}{\partial Z}G \end{pmatrix} \qquad (*)$$

We claim that the determinant of the Hessian of F cannot vanish identically along $F = 0$. Suppose this were the case. This means that all non-singular points of $F = 0$ are inflection points, that is points whose tangent to C intersects C with multiplicity ≥ 3. Choose an affine chart of $\mathbb{P}^2$ such that $(0, 0) \in C$ is a non-singular point. We can then choose an analytic parameterization of C near $(0, 0)$ given by $t \mapsto \gamma(t) = (v(t), w(t))$, for t in a small disk. Then $(v(t), w(t))$ being an inflection point of C implies $\dot\gamma(t) \neq 0$ and $\ddot\gamma(t)$ are proportional. Thus $\dot\gamma(t) \wedge \ddot\gamma(t) = 0$. Differentiating with respect to t gives $\dot\gamma(t) \wedge \gamma^{(3)}(t) = 0$, so that $\gamma^{(3)}(t)$ is also proportional to $\dot\gamma(t)$. Continuing inductively gives $\dot\gamma(t) \wedge \gamma^{(n)}(t) = 0$ for all $n > 0$. But now $\gamma(t) = \dot\gamma(0)t + \frac{1}{2!}\ddot\gamma(0)t^2 + ...$, hence $\dot\gamma(0) \wedge \gamma(t) = 0$. But this means C has infinitely many intersections with a line, hence is a line, contradiction. Thus the Hessian of F is invertible on a dense open subset of C.

Now if we evaluate (*) at a non-singular non-inflection point in $[x : y : z] \in C$, we see that $[\frac{\partial}{\partial X}G : \frac{\partial}{\partial Y}G : \frac{\partial}{\partial Z}G]$ at that point is uniquely determined by

$$\begin{pmatrix} \frac{\partial^2}{\partial X^2}F & \frac{\partial^2}{\partial X\partial Y}F & \frac{\partial^2}{\partial X\partial Z}F \\ \frac{\partial^2}{\partial X\partial Y}F & \frac{\partial^2}{\partial Y^2}F & \frac{\partial^2}{\partial Y\partial Z}F \\ \frac{\partial^2}{\partial X\partial Z}F & \frac{\partial^2}{\partial Z\partial Y}F & \frac{\partial^2}{\partial Z^2}F \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial X}G \\ \frac{\partial}{\partial Y}G \\ \frac{\partial}{\partial Z}G \end{pmatrix} = H(x,y,z) \begin{pmatrix} \frac{\partial}{\partial X}F \\ \frac{\partial}{\partial Y}F \\ \frac{\partial}{\partial Z}F \end{pmatrix}$$

But Euler's formula says that $X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z}$ is multiplication by d on the space of homogenous polynomials of degree d, hence

$$\begin{pmatrix} \frac{\partial^2}{\partial X^2}F & \frac{\partial^2}{\partial X\partial Y}F & \frac{\partial^2}{\partial X\partial Z}F \\ \frac{\partial^2}{\partial X\partial Y}F & \frac{\partial^2}{\partial Y^2}F & \frac{\partial^2}{\partial Y\partial Z}F \\ \frac{\partial^2}{\partial X\partial Z}F & \frac{\partial^2}{\partial Z\partial Y}F & \frac{\partial^2}{\partial Z^2}F \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = d(d-1) \begin{pmatrix} \frac{\partial}{\partial X}F \\ \frac{\partial}{\partial Y}F \\ \frac{\partial}{\partial Z}F \end{pmatrix}$$

It thus follows that $[x : y : z] = [\frac{\partial}{\partial X}G : \frac{\partial}{\partial Y}G : \frac{\partial}{\partial Z}G]$, hence the composition of Gauss maps $C \to C^* \to C^{**}$ is generically the identity, hence C^{**} is canonically identified with C.

(b) For an irreducible conic C the Gauss map is a linear isomorphism. Suppose two such conics C, C' are tangent at a point p. Then the image of p under the Gauss map of C, C' is a point q. By the previous part, we know that q will get sent to p under both the Gauss map of C and C'. But this means that C, C' are tangent at q.

3. Let Λ_1 and $\Lambda_2 \subset \mathbb{R}^4$ be complementary 2-planes, and let $X = \mathbb{R}^4 \setminus (\Lambda_1 \cup \Lambda_2)$ be the complement of their union. Find the homology and cohomology groups of X with integer coefficients.

Solution. Let

$$U = \mathbb{R}^4 \setminus \Lambda_1 \simeq S^1 \times \mathbb{R}^2$$

$$V = \mathbb{R}^4 \setminus \Lambda_2 \simeq S^1 \times \mathbb{R}^2$$

$$U \cap V = \mathbb{R}^4 \setminus (\Lambda_1 \cup \Lambda_2) = X$$

$$U \cup V = \mathbb{R}^4 \setminus pt \simeq S^3.$$

Then from the Mayer-Vietoris sequene we get

$$0 \to H_4(X) \to H_4(U) \oplus H_4(V) \to H_4(S^3) \to H_3(X) \to H_3(U) \oplus H_3(V) \to H_3(S^3) \to H_2(X) \to H_2(U)$$

$$\to H_2(S^3) \to H_1(X) \to H_1(U) \oplus H_1(V) \to H_1(S^3) \to H_0(X) \to ...$$

Since X is connected, $H_0(X) = \mathbb{Z}$. Plugging in the values of $H_*(S^1)$ and $H_*(S^3)$ we get

$$0 \to H_4(X) \to 0$$

$$0 \to H_3(X) \to 0 \to \mathbb{Z} \to H_2(X) \to 0$$
$$0 \to H_1(X) \to \mathbb{Z}^2 \to 0.$$

Hence $H_2(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}^2$, $H_0(X) = \mathbb{Z}$ and all other homology groups vanish. Note that all homology groups are $\mathbb{Z}$-free, hence the cohomology groups are just their $\mathbb{Z}$-duals. Thus $H^2(X) = \mathbb{Z}$, $H^1(X) = \mathbb{Z}^2$, $H^0(X) = \mathbb{Z}$

4. Let $X = \{(x, y, z) : x^2 + y^2 = 1\} \subset \mathbb{R}^3$ be a cylinder. Show that the geodesics on X are *helices*, that is, curves such that the angle between their tangent lines and the vertical is constant.

Solution. We have a parameterization of the cylinder given by $(\theta, z) \mapsto (\cos\theta, \sin\theta, z)$, with $z \in \mathbb{R}$, $\theta \in [0.2\pi]$. Thus $T_{(\theta,z)}$ is spanned by $(-\sin\theta, \cos\theta, 0)$ and $(0, 0, 1)$.

Suppose $t \mapsto (\theta(t), z(t))$ is a geodesic. Put $\gamma(t) = (\cos\theta(t), \sin\theta(t), z(t))$. Being a geodesic means $\ddot{\gamma}$ is orthogonal to $T_{(\theta(t),z(t))}$ (the dot denotes differentiation with respect to t). We have

$$\dot{\gamma} = (-\sin\theta \cdot \dot{\theta}, \cos\theta \cdot \dot{\theta}, \dot{z})$$
$$\ddot{\gamma} = (-\cos\theta \cdot \dot{\theta} - \sin\theta \cdot \ddot{\theta}, -\sin\theta \cdot \dot{\theta} + \cos\theta \cdot \ddot{\theta}, \ddot{z})$$

Thus the geodesic equations say

$$\sin\theta\cos\theta \cdot \dot{\theta} + \sin^2\theta \cdot \ddot{\theta} + \cos^2\theta \cdot \ddot{\theta} - \cos\theta\sin\theta \cdot \dot{\theta} = 0$$
$$\ddot{z} = 0$$

Hence $\ddot{\theta} = 0$, $\ddot{z} = 0$, so $\theta(t) = at + c$, $z = bt + d$ for some constants a, b. But this is precisely the equation of a helix (the tangent line is $(-a\sin\theta, a\cos\theta, b)$, which makes a constant angle with the vertical).

5. (a) Show that if every closed and bounded subspace of a Hilbert space E is compact, then E is finite dimensional.

(b) Show that any decreasing sequence of nonempty, closed, convex, and bounded subsets of a Hilbert space has a nonempty intersection.

(c) Show that any closed, convex, and bounded subset of a Hilbert space is the intersection of the closed balls that contain it.

(d) Deduce that any closed, convex, and bounded subset of a Hilbert space is compact in the weak topology.

Solution.

(a) Suppose every closed and bounded subset of E is compact. If E is infinite-dimensional, we can choose an infinite sequence of orthonormal vectors e_1, e_2,... Consider the set $\{e_1, e_2, ...\}$ Because $\|e_i - e_j\| = \sqrt{2}$ for $i \neq j$, this sequence does not contain any Cauchy subsequence. In particular it is closed and clearly bounded, and cannot contain a convergent subsequence, hence is not compact.

(b) It follows from (d) that a closed, convex, bounded subset of E is weakly compact. Since a decreasing sequence of closed, convex, bounded subset has the finite intersection property, it follows from compactness that the intersection of the family is non-empty. (The argument below makes no use of (b)).

(c) Let C be a closed, convex, bounded subset of E. Suppose $a \notin C$. Let $c_0 \in C$ be the point in C closest to a (such a point exists because C is closed), and let $d = \|a - c_0\| > 0$.
We claim this point is unique: If c_1 is another such point, then

$$2d^2 = \|a - c_0\|^2 + \|a - c_1\|^2 = 2\|a - \frac{c_0 + c_1}{2}\|^2 + 2\|\frac{c_0 - c_1}{2}\| \geq 2d^2.$$

Thus equalities occur, and $c_0 = c_1$. Now let H be the hyperplane through c_0 orthogonal to the segment $[a, c_0]$. For any $c \in C$, $0 \leq t \leq 1$, we have $|a-(tc+(1-t)c_0)\|^2 \geq \|a-c_0\|^2$ hence $-2t(a-c_0, c-c_0)+t^2(c-c_0, c-c_0) \geq 0$. Letting $t \to 0$ gives $(c_0 - a, c_0 - c) \leq 0$, hence H separates a and C.
Now let L be the hyperplane going through $\frac{a+c_0}{2}$ and perpendicular to $a - c_0$. We choose a point u in the line through a and c_0 such that $\|u - \frac{a+c_0}{2}\| = R$, where R is to be determined. For any $c \in C$, let c' be the orthogonal projection of c onto L, put $h = \|c - c'|$, $t = \|c' - \frac{a+c_0}{2}\|$ We have $\|u - c\|^2 - R^2 = (R - h)^2 + t^2 - R^2 = h^2 - 2Rh + t^2$. The last expression is ≤ 0 iff $R - \sqrt{R^2 - t^2} \leq h \leq R + \sqrt{R^2 - t^2}$. Now observe that as C is bounded, t and h are bounded as c varies in C, and $h \geq \frac{d}{2}$. Hence if we choose R sufficiently large we can ensure that $\|u - c\|^2 - R^2 \leq 0$. Then the closed ball centered at u of radius R will contain C but not a.

(d) In view of (c), it suffices to prove that the closed unit ball B in E is weakly compact. Consider the map $B \to \prod_{v \in E}[-\|v\|, \|v\|]$ given by $b \mapsto (b, v)$. The target is compact by Tychonoff's theorem. The map is clearly injective and the product topology on the target induces the weak topology on B by definition. Thus it suffices to check that B is closed. But this is clear because B is precisely the set of $(x_v)_v$ such that $x_{\lambda \cdot v + \mu \cdot w} = \lambda \cdot v + \mu \cdot w$ (the condition implies that $v \mapsto x_v$ is a continuous functional on E with norm ≤ 1, hence is of form $v \mapsto (b, v)$ for a unique $b \in B$).

6. Let p be a prime, and let G be the group $\mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$.

(a) How many subgroups of order p does G have?

(b) How many subgroups of order p^2 does G have? How many of these are cyclic?

Solution.

(a) A subgroup of order p of G is a one-dimensional $\mathbb{F}_p$-subspace of the p-torsion $G[p] \simeq \mathbb{F}_p^2$. There are $p^2 - 1$ non-zero vectors in $G[p]$, and each of them generate a one-dimensional $\mathbb{F}_p$-subspace. Each such line contains exactly $p - 1$ non-zero vectors, hence the number of order p subgroup is $\frac{p^2-1}{p-1} = p + 1$.

(b) Since G is abelian, any subgroup of order p^2 of G must be isomorphic to $\mathbb{Z}/p \oplus \mathbb{Z}/p$ or $\mathbb{Z}^2$. The first possibility happens precisely for the p-torsion subgroup $G[p]$, hence there is only one such subgroup.
To count the number of cyclic subgroups of G of order p^2, we count the number of elements not of order p (note G is killed by p^2). G has precisely p^2 elements of order dividing p, hence $p^4 - p^2$ elements of order p^2. Each such element generate a cyclic subgroup of order p^2. Each such subgroup contains $p^2 - p$ elements of exact order p^2. Hence G has $\frac{p^4-p^2}{p^2-p} = p(p+1)$ cyclic subgroup of order p^2.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday January 21, 2010 (Day 3)

1. Consider the ring

$$A = \mathbb{Z}[x]/(f) \quad \text{where} \quad f = x^4 - x^3 + x^2 - 2x + 4.$$

Find all prime ideals of A that contain the ideal (3).

Solution. Prime ideals of A that contains (3) are in bijection with prime ideals of

$$A/3A \cong \mathbb{F}_3[x]/(f) = \mathbb{F}_3[x]/(x-1)(x+1)(x^2-x-1).$$

Note $x^2 - x - 1$ is irreducible in $\mathbb{F}_3[x]$ because it has no zeroes in $\mathbb{F}_3$. Hence $A/3A$ has precisely 3 prime ideals, namely those generated by $x-1$, $x+1$ and x^2-x-1. Hence the primes of A containing (3) are $(3, x-1)$, $(3, x+1)$ and $(3, x^2-x-1)$.

2. Let f be a holomorphic function on a domain containing the closed disc $\{z : |z| \leq 3\}$, and suppose that

$$f(1) = f(i) = f(-1) = f(-i) = 0.$$

Show that

$$|f(0)| \leq \frac{1}{80} \max_{|z|=3} |f(z)|$$

and find all such functions for which equality holds in this inequality.

Solution. The assumption on f implies $f(z) = (z^4 - 1)g(z)$ for an analytic function $g(z)$ with the same domain as f. We have $|f(0)| = |g(0)|$. By the maximum modulus principle, $|g(0)| \leq \max_{|z|=3} |g(z)|$. On $|z| = 3$, $|f(z)| = |(z^4-1)g(z)| \geq (3^4 - 1)|g(z)|$. Hence $|f(0)| \leq \frac{1}{80} \max_{|z|=3} |f(z)|$. For equality to appear, we must have $|g(0)| = \max_{|z|=3} |g(z)|$, hence g is constant. But then $f(z) = c(z^4 - 1)$ does not make the equality hold, because $\max_{|z|=3} |z^4 - 1| = 82$.

3. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a differentiable, positive real function. Find the Gaussian curvature and mean curvature of the surface of revolution

$$S = \{(x, y, z) : y^2 + z^2 = f(x)\}.$$

Solution. The surface S has a parameterization $(x,\theta) \mapsto \Phi(x,\theta) = (x, \sqrt{f(x)}\cos\theta, \sqrt{f(x)}\sin\theta)$ for $x \in \mathbb{R}$, $\theta \in [0, 2\pi)$. Hence (the lower index signifies the variable with respect to which we differentiate)

$$\Phi_x = (1, \frac{f'}{2\sqrt{f}}\cos\theta, \frac{f'}{2\sqrt{f}}\sin\theta)$$

$$\Phi_\theta = (0, -\sqrt{f}\sin\theta, \sqrt{f}\cos\theta)$$

$$\Phi_x \wedge \Phi_\theta = (\frac{f'}{2}, -\sqrt{f}\cos\theta, -\sqrt{f}\sin\theta)$$

$$|\Phi_x \wedge \Phi_\theta| = \sqrt{f + \frac{f'^2}{4}}$$

$$\Phi_{xx} = (0, \frac{2ff'' - f'^2}{4f\sqrt{f}}\cos\theta, \frac{2ff'' - f'^2}{4f\sqrt{f}}\sin\theta)$$

$$\Phi_{x\theta} = (0, -\frac{f'}{2\sqrt{f}}\sin\theta, \frac{f'}{2\sqrt{f}}\cos\theta)$$

$$\Phi_{\theta\theta} = (0, -\sqrt{f}\cos\theta, -\sqrt{f}\sin\theta)$$

The second fundamental form is $Ldx^2 + 2Mdxd\theta + Nd\theta^2$, where

$$L = \frac{\Phi_x \wedge \Phi_\theta}{|\Phi_x \wedge \Phi_\theta|} \cdot \Phi_{xx} = -\frac{2ff'' - f'^2}{4f\sqrt{f + \frac{f'^2}{4}}}$$

$$M = \frac{\Phi_x \wedge \Phi_\theta}{|\Phi_x \wedge \Phi_\theta|} \cdot \Phi_{x\theta} = 0$$

$$N = \frac{\Phi_x \wedge \Phi_\theta}{|\Phi_x \wedge \Phi_\theta|} \cdot \Phi_{\theta\theta} = \frac{f}{\sqrt{f + \frac{f'^2}{4}}}$$

The Gaussian curvature is

$$K = L \cdot N - M^2 = -\frac{2ff'' - f'^2}{4f + f'^2}$$

The mean curvature is

$$H = L + N = \frac{4f^2 - 2ff'' + f'^2}{4f\sqrt{f + \frac{f'^2}{4}}}$$

4. Show that for any given natural number n, there exists a finite Borel measure on the interval $[0, 1] \subset \mathbb{R}$ such that for any real polynomial of degree at most n, we have

$$\int_0^1 p \, d\mu = p'(0).$$

Show, on the other hand, that there does *not* exist a finite Borel measure on the interval $[0,1] \subset \mathbb{R}$ such that for any real polynomial we have

$$\int_0^1 p\, d\mu = p'(0).$$

Solution. Note that $P_k(x) = (k+1)x^k$ with $k \leq n$ form basis for the space of polynomials of degree at most n. We will construct the desired measure μ as follows: on $[\frac{i}{n+1}, \frac{i+1}{n+1}]$ μ will be x_i times the Lebesgue measure. We show that there is a choice of x_i so that μ has the desired property. We want to have

$$\int_0^1 P_k d\mu = \sum_{i=0}^{n}((\frac{i+1}{n+1})^{k+1} - (\frac{i}{n+1})^{k+1})x_i = P_k'(0).$$

This is a system of $n+1$ linearly independent linear equation in $n+1$ variables, hence has a solution. (To see the linear independence, note the matrix $(\frac{i+1}{n+1})^{k+1} - (\frac{i}{n+1})^{k+1})_{i,k}$ has the same determinant as a Van der Monde determinant, as can be seen by adding the first column to the second column, then add the second column to the third column and so on).

Now suppose μ is a finite Borel measure on $[0,1]$ such that

$$\int_0^1 p d\mu = p'(0)$$

for all real polynomials p. Let $f(x)$ be the characteristic function of the set $\{0\}$, and put $q_n(x) = (1-x)^n$. Then

$$|\int_0^1 (f - q_n) d\mu| \leq \mu([0.1])$$

so

$$n = q_n'(0) = \int_0^1 q_n d\mu \leq \int_0^1 f d\mu + \mu([0,1])$$

for all n, a contradiction.

5. Let $X = \mathbb{RP}^2 \times \mathbb{RP}^4$.

(a) Find the homology groups $H_*(X, \mathbb{Z}/2)$

(b) Find the homology groups $H_*(X, \mathbb{Z})$

(c) Find the cohomology groups $H^*(X, \mathbb{Z})$

Solution. The Künneth formula says (for coefficient ring R)

$$H_n(X \times Y, R) = \oplus_{i+j=n} H_i(X,R) \otimes H_j(Y,R) \oplus_{i+j=n-1} \mathrm{Tor}_R(H_i(X,R), H_j(Y,R)).$$

In particular for a field

$$H_n(X \times Y, R) = \oplus_{i+j=n} H_i(X,R) \otimes H_j(Y,R)$$

We have (the groups not shown are 0)

	0	1	2	3	4
$H_*(\mathbb{RP}^2, \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0
$H_*(\mathbb{RP}^4, \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$H_*(\mathbb{RP}^2, \mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}/2$	0	0	0
$H_*(\mathbb{RP}^4, \mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0

So by the Künneth formula:

(a)

	0	1	2	3	4	5	6
$H_*(X, \mathbb{Z}/2)$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$

(b)

	0	1	2	3	4	5
$H_*(X, \mathbb{Z})$	$\mathbb{Z}$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

(c) By the universal coefficient theorem (for an abelian group G)

$$H^n(X, G) = \operatorname{Hom}(H_n(X, \mathbb{Z}), G) \oplus \operatorname{Ext}^1(H_{n-1}(X, \mathbb{Z}), G)$$

hence by the previous part

	0	1	2	3	4	5	6
$H^*(X, \mathbb{Z})$	$\mathbb{Z}$	0	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

6. By a *twisted cubic curve* we mean the image of the map $\mathbb{P}^1 \to \mathbb{P}^3$ given by

$$[X, Y] \mapsto \big[F_0(X, Y), F_1(X, Y), F_2(X, Y), F_3(X, Y)\big]$$

where $F_0, \ldots, F_3$ form a basis for the space of homogeneous cubic polynomials in X and Y.

(a) Show that if $C \subset \mathbb{P}^3$ is a twisted cubic curve, then there is a 3-dimensional vector space of homogeneous quadratic polynomials on $\mathbb{P}^3$ vanishing on C.

(b) Show that C is the common zero locus of the homogeneous quadratic polynomials vanishing on it.

(c) Suppose now that $Q, Q' \subset \mathbb{P}^3$ are two distinct quadric surfaces containing C. Describe the intersection $Q \cap Q'$.

Solution.

(a) Up to a projective automorphism, the twisted cubic is isomorphic to the parametric curve $[X : Y] \mapsto [X^3 : X^2Y : XY^2 : Y^3]$. Given a homogenous quadratic polynomial Q, Q will vanish on this curve iff $Q(X^3, X^2Y, XY^2, Y^3)$ is the *zero* polynomial. This happens iff each coefficient of this degree 6 homogenous polynomial vanish. This gives 7 linear condition on the 10 coefficients of Q, which are linearly independent because each equation involve a distinct set of coefficients. It follows that the space of homogenous quadratic polynomial vanishing on a twisted cubic has dimension $10 - 7 = 3$.

(b) As above, we assume the twisted cubic is given by $[X : Y] \mapsto [X^3 : X^2Y : XY^2 : Y^3]$. In this case we see that it lies in the 3 quadrics $AD - BC = 0$, $B^2 - AC = 0$ and $C^2 - BD = 0$ (here $[A : B : C : D]$ are homogenous coordinates for $\mathbb{P}^3$). We claim that the intersection of these three quadrics is the twisted cubic. Indeed assume $[A : B : C : D]$ lies in the intersection. Without loss of generality we assume $A \neq 0$, and put $A = X^3$ for $X \neq 0$. Put $Y = B/X^2$. Then $C = B^2/A = XY^2$ and $D = BC/A = Y^3$, so $[A : B : C : D]$ lies in the twisted cubic.

(c) From the definition, the twisted cubic C does not lie in any hyperplane. Hence it cannot lie in any reducible quadric, so all quadrics that contain it are irreducible. For quadrics Q, Q' containing C, their intersection will have all component of dimension 1. The intersection has total multiplicity 4 by Bézout's theorem. The twisted cubic part contributes a multiple of 3. So the intersection contains another component with multiplicity 1 and is a curve of degree 1, i.e. a line. Hence $Q \cap Q'$ is the union of the twisted cubic and a line (all with multiplicity 1) .

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday January 18, 2011 (Day 1)

1. (CA) Evaluate

$$\int_0^\infty \frac{x^2+1}{x^4+1}dx$$

Solution. We can consider the integration from $-\infty$ to ∞ instead. For $R > 1$, consider the contour that consists of the segment from $-R$ to R and the arc $\{Re^{i\theta} \mid \theta \in [0, \pi]\}$. Since $\frac{z^2+1}{z^4+1}$ decays as $|z|^{-2}$ on the complex plane as $|z| \to \infty$, this contour integral converges to twice of the original integral when $R \to \infty$.

The contour encloses two simple poles $e^{\pi i/4}$ and $e^{3\pi i/4}$ of the function. At $e^{\pi i/4}$ the residue of the function is $\frac{(e^{\pi i/4})^2+1}{\frac{d}{dz}(z^4+1)|_{z=e^{\pi i/4}}} = \frac{1+i}{4e^{3\pi i/4}} = \frac{-i}{2\sqrt{2}}$. Similarly the residue at $e^{3\pi i/4}$ is also $\frac{1-i}{4e^{9\pi i/4}} = \frac{-i}{2\sqrt{2}}$. The contour goes counter-clockwise, and thus the integration along the contour is $2\pi i \cdot \frac{-i}{\sqrt{2}} = \sqrt{2}\pi$, and thus the original integral is $\frac{\sqrt{2}}{2}\pi$.

2. (A) Let k be a field and V be a k-vector space of dimension n. Let $A \in End_k(V)$. Show that the following are equivalent:

(a) The minimal polynomial of A is the same as the characteristic polynomial of A.

(b) There exists a vector $v \in V$ such that $v, Av, A^2v, ..., A^{n-1}v$ is a basis of V.

Solution. The theorem on the existence of rational canonical form says that we have $V = \bigoplus_{i=1}^r V_i$ such that each V_i is invariant under T, that $T|_{V_i}$ satisfies both (a) and (b) above, and that if we denote by $p_i(x)$ the characteristic polynomial, we have $p_j(x)|p_i(x)$ for any $j > i$. One sees that the characteristic polynomial of T is then the product of that of its blocks, namely $\prod_{i=1}^r p_i(x)$, while the minimal polynomial of T is $p_1(x)$. This shows that (a) implies $r = 1$, and thus the result of (b). On the other hand, if (b) holds, then the minimal polynomial has degree equal to $\dim V$ and thus must be equal to the characteristic polynomial.

3. (T) Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

Solution. The Kunneth formula shows the homology groups of $S^1 \times S^1$ are $\mathbb{Z}$, $\mathbb{Z}^2$ and $\mathbb{Z}$ in dimension $0, 1$ and 2, respectively. Note that since the homology groups of S^1 are free, there is no contribution from the torsion part. The reduced homology groups of $S^1 \vee S^1 \vee S^2$ are the direct sums of that of two S^1 and S^2 in corresponding dimension, hence the same result.

The universal covering of $S^1 \times S^1$ is $\mathbb{R}^2$, which is contractible and thus have trivial reduced homology groups. On the other hand the universal covering space of $S^1 \vee S^1 \vee S^2$ is the universal covering space of $S^1 \vee S^1$ with each vertex attached an S^2. The second homology group H_2 of it is therefore an infinite direct sum of $\mathbb{Z}$.

4. (RA)

(a) Prove that any countable subset of the interval $[0,1] \subset \mathbb{R}$ is Lebesgue measurable, and has Lebesgue measure 0.

(b) Let $\Phi \subset [0,1]$ be the set of real numbers x that, when written as a decimal $x = 0.a_1a_2a_3...$, satisfy the rule $a_{n+2} \notin \{a_n, a_{n+1}\}$ for all $n \geq 1$. What is the Lebesgue measure of Φ?

Solution.

(a) Suppose the countable subset is $\{y_1, y_2, ..., y_n, ...\}$. For each k take the open interval $U_k = (y_k - \frac{\delta}{2^k}, y_k + \frac{\delta}{2^k})$, where $\delta > 0$ is fixed. Then U_k has length $\frac{\delta}{2^{k-1}}$ and $\bigcup U_k$ has the sum of lengths δ. As $\delta \to 0$, this shows $\{y_1, y_2, ..., y_n, ...\}$ has measure zero and in particular is Lebesgue measurable.

(b) Let E_k be the set of real numbers $x = 0.a_1a_2a_3...$ such that $a_{k+1} \neq a_{k+2}$. Let $F_k = E_k - E_k \cap (\bigcup_{i<k} E_i)$. One easily sees that F_k has measure $\frac{1}{10} \cdot \left(\frac{9}{10}\right)^{k-1}$ and they are disjoint. Thus $\bigsqcup F_k \subset [0,1]$ has measure 1. As $\Phi \subset [0,1] - \bigsqcup F_k$, Φ has measure zero.

5. (DG) Let $B \subset \mathbb{R}^4$ be the closed ball of radius 2 centered at the origin, with the metric induced from the euclidean metric on $\mathbb{R}^4$. Give an example of a smooth vector field v on B with the property that for any L there exists an integral curve of v with both endpoints on the boundary ∂B and length greater than L.

Solution. Take the vector field $(1 - x^2 - y^2)\frac{\partial}{\partial z} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$. The function $x^2 + y^2$ is invariant on any integral curve. When $x^2 + y^2$ is close to 1, the "vertical" part of the vector field is no more than $2(1 - x^2 - y^2)$ times the horizontal part of the vector field. The vertical length such an integral curve has to travel is $2\sqrt{1 - x^2 - y^2}$, and thus the total length of the integral curve is no less than $\frac{2\sqrt{1-x^2-y^2}}{2(1-x^2-y^2)} = \sqrt{1 - x^2 - y^2}^{-1/2}$. For $x^2 + y^2$ arbitrarily close to 1 we get arbitrarily long integral curves.

6. (AG) Let $C \subset \mathbb{P}^3$ be a smooth, irreducible, non-degenerate curve of degree d.

(a) Show that $d \geq 3$.

(b) Show that every point $p \in \mathbb{P}^3$ lies on a secant and tangent line to C.

(c) If $d = 3$ show that every point of $\mathbb{P}^3 \backslash C$ lies on a unique secant or tangent line to C.

Solution.

(a) We may assume C is not a line. Take any three point on C which are not collinear and intersect C with the plane passing through these three points. As C is non-degenerate, C is not contained in this plane and thus the intersection number of C and this plane is 3 or more (when they intersect at more points or intersect at these three points with higher multiplicity). This says $d \geq 3$.

(b) Consider $X = \{(x, y, z) \,|\, x, y \in C,\ z \text{ is on the secant passing through } x, y\}$. Here z should be on the tangent line if $x = y$. Then X is closed in $C \times C \times \mathbb{P}^3$ and X projects to $C \times C$ with fiber $\mathbb{P}^1$. Hence X is irreducible with dimension 3. Consider the projection map from $C \times C \times \mathbb{P}^3$ to $\mathbb{P}^3$ and denote by S the image of X. S is the so-called secant variety, and we see that it is an irreducible closed subvariety of $\mathbb{P}^3$.

The statement to be proved is then that S is the whole $\mathbb{P}^3$. If this is not the case, then the map from X to S has every fiber at least dimension 1 (by semi-continuity of the dimension of fibers). Take any four non-planar points $x_1, ..., x_4$ on C. All lines through x_i to all points on C is a 2-dimensional closed subvariety of $\mathbb{P}^3$, and thus must be S itself. This says S is a cone over x_i; if we write A_S the subvariety in $\mathbb{A}^4$ whose projectivization is S, and $V_i \subset \mathbb{A}^4$ the line through the origin corresponding to x_i, then $A_S = A_S + V_i$. But this implies $A_S = \mathbb{A}^4$ and thus $S = \mathbb{P}^3$.

(c) Suppose otherwise x lies on two tangent or secant lines. Consider the plane that contains x and this two lines. Then this plane intersect C at least four times as it intersects C twice on each line. This contradicts with that $d = 3$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday January 19, 2011 (Day 2)

1. (T) Find all the connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$ up to isomorphism of covering spaces without basepoints. Indicate which covering spaces are normal.

Solution. A connected 2-sheeted covering is given by a homomorphism $\pi_1(S^1 \vee S^1) \to S_2$ such that the image acts transitively on the set of two elements; this is given by how the path that lifts the loop interchanges different fibers. Since $\pi_1(S^1 \vee S^1)$ is freely generated by two generators, this is the same as giving two elements in S_3 so that they generates a subgroup of S_2 that acts transitively. We thus have 3 choices: $(e, (12)), ((12), (12)), ((12), e)$. They are all normal.

For 3-sheeted covering we are giving two elements in S_3 so that they generates a subgroup of S_2 that acts transitively, but *up to conjugacy of* S_3. This is because a renumbering of the underlying set of S_3 corresponds to a renumbering of fibers, which always give an isomorphism of covering spaces. There are therefore 7 possibilities: $(e, (123))$, $((12), (13))$, $((12), (123))$, $((123), e)$, $((123), (12))$, $((123), (123))$ and $((123), (132))$.

Among these possibilities 4 are normal; since a group of order 3 is always abelian, such coverings are normal iff they are abelian. A covering space described this way is abelian iff the two elements are abelian. This leaves $(e, (123))$, $((123), e)$, $((123), (123))$ and $((123), (132))$.

2. (RA) Let g be a differentiable function on $\mathbb{R}$ that is non-negative and has compact support.

(a) Prove that the Fourier transform $\hat{g}$ of g does not have compact support unless $g = 0$.

(b) Prove that there exist constants A and c such that for all $k \in \mathbb{N}$ the k-th derivative of $\hat{g}$ is bounded by cA^k.

Solution.

(a) By scaling if necessary we may assume the support of g lies inside $[-3, 3]$. If the Fourier transform of $\hat{g}$ also has support in $(-N, N)$, then this implies in particular that, if we think of g as defined on $[-\pi, \pi]$ and consider its Fourier series, then all terms after the N-th term are zero. In particular g has a finite Fourier series and therefore must be analytic on $(-\pi, \pi)$. This contradicts with that supp $g \subset [-3, 3]$ unless $g \equiv 0$.

(b) Since g is smooth with compact support, say supp $g \subset [-A, A]$, we have

$$|\frac{d^k}{dy^k}\int_{\mathbb{R}} e^{ixy} f(x)dx| = |\int_{\mathbb{R}} (ix)^k e^{ixy} f(x)dx| \leq A^k \cdot \sup_{x\in[-A,A]} f(x) \cdot 2A.$$

3. (DG) Let $S^2 \subset \mathbb{R}^3$ be the sphere of radius 1 centered at the origin, with the metric induced from the euclidean metric on $\mathbb{R}^3$. Introduce spherical coordinates $(\theta, \phi) \in [0, \pi] \times \mathbb{R}/(2\pi\mathbb{Z})$ on the complement of the north and south poles, where

$$(x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

The metric in these coordinates is given by the section

$$d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi$$

of the second symmetric power of the cotangent bundle T^*S^2; it has constant scalar curvature 1.

Now let u be a smooth function on S^2 depending only on the coordinate θ, and consider the metric given by the section

$$e^u(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi).$$

(a) Compute the scalar curvature of this new metric in terms of u and its derivative.

(b) Prove that the integral over S^2 of the function you computed in Part (a) is equal to 4π.

Solution. *For our convenience, we replace u by $2u$ throughout the solution.*

(a) We compute using the method of orthonormal frame. We have $e^\theta = e^u d\theta$ and $e^\phi = e^u \sin\theta d\phi$ is an orthonormal coframe (orthonormal basis for cotangent bundle). Write ω_j^k to be the matrix 1-form so that $\nabla(e_j) = \omega_j^k e_k$, where ∇ is the Levi-Civita connection and $e_\theta = e^{-u}\frac{\partial}{\partial\theta}$ and $e_\phi = e^{-u}\csc\theta\frac{\partial}{\partial\phi}$ is the dual frame. The torsion-free condition of the connection gives the Cartan's structure equation (using Einstein summation convention)

$$de^k + \omega_j^k \wedge e^j = 0.$$

Also that ∇ is compatible with metric gives $\omega_j^k = -\omega_k^j$. So $\omega_\theta^\theta = \omega_\phi^\phi = 0$. Next we have $\omega_\phi^\theta = -\omega_\theta^\phi$, $de^\theta = 0$ says ω_ϕ^θ at every point is a multiple of e^ϕ, and $de^\phi = (e^u\sin\theta)'d\theta \wedge d\phi = \frac{(e^u\sin\theta)'}{e^{2u}\sin\theta}e^\theta \wedge e^\phi$ gives $\omega_\theta^\phi = -\frac{(e^u\sin\theta)'}{e^{2u}\sin\theta}e^\phi = -e^{-u}(e^u\sin\theta)'d\phi$.

The curvature 2-form is $d\omega + \omega\wedge\omega$. Note that $\omega\wedge\omega$ is zero because each entry of ω has only $d\phi$ term. And $(d\omega)_\theta^\phi = -(e^{-u}(e^u\sin\theta)')'d\theta \wedge d\phi =$

$-\frac{(e^{-u}(e^u\sin\theta)')'}{e^{2u}\sin\theta}e^\theta\wedge e^\phi$. So the scalar curvature is (or twice of it, in the convention in Wikipedia for general dimension)

$$S=\iota_{e_\phi}\iota_{e_\theta}((d\omega)^\phi_\theta)-\frac{(e^{-u}(e^u\sin\theta)')'}{e^{2u}\sin\theta}.$$

(b) The volume form is $e^{2u}\sin\theta d\theta\wedge d\phi$. We thus have to integrate

$$\int_0^{2\pi}\int_0^\pi -(e^{-u}(e^u\sin\theta)')'d\theta d\phi.$$

The deduction that this gives 4π is straightforward.

4. (AG) Show that no two of the following rings are isomorphic:

1. $\mathbb{C}[x,y]/(y^2-x)$.
2. $\mathbb{C}[x,y]/(y^2-x^2)$.
3. $\mathbb{C}[x,y]/(y^2-x^3)$.
4. $\mathbb{C}[x,y]/(y^2-x^4)$.
5. $\mathbb{C}[x,y]/(y^2-x^5)$.
6. $\mathbb{C}[x,y]/(y^3-x^4)$.

Solution. These rings are reduced (no non-trivial nilpotent elements). Two rings are isomorphic if and only if their corresponding complex analytic space (variety over $\mathbb{C}$) are isomorphic. The first one is the only one that is non-singular, and therefore non-isomorphic with any others.

The second and the fourth varieties are reducible, i.e. the rings are not integral domains. To distinct them from each other, note that they both have two component, namely two minimal prime ideals, which is $(x+y)$, $(x-y)$ for the second ring and $(x+y^2)$, $(x-y^2)$ for the fourth ring. The union of this two ideals gives the maximal ideal (x,y) for the second ring, but gives the non-primary ideal (x,y^2) for the fourth ring. This reflects that fact that in one case the two component intersect transversally and in the other case they intersect twice. This shows the second ring and the fourth ring are not isomorphic.

It remains to check that the third, fifth and sixth rings are different. They all have a unique singularity at the origin, which can be resolved (the same holds for all curves) by taking normalization, i.e. integral closure of the ring. They have the same integral closure $\mathbb{C}[t]$, in which the three rings may be written as $\mathbb{C}[t^2,t^3]$, $\mathbb{C}[t^2,t^5]$ and $\mathbb{C}[t^3,t^4]$. Now note that $\dim_{\mathbb{C}}\mathbb{C}/\mathbb{C}[t^2,t^3]=1$, $\dim_{\mathbb{C}}\mathbb{C}/\mathbb{C}[t^2,t^5]=2$, $\dim_{\mathbb{C}}\mathbb{C}/\mathbb{C}[t^3,t^4]=3$, and thus these rings are non-isomorphic. This number is called the delta invariant and measure the loss of geometric genus at this singularity.

5. (CA) Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant holomorphic function. Prove that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Solution. Assume z_0 is not in the closure of the image of $f(\mathbb{C})$. $1/(f(\mathbb{C}) - z_0)$ would then be bounded, and thus constant, a contradiction.

6. (A) Let a be a positive integer, and consider the polynomial

$$f_a(x) = x^6 + 3ax^4 + 3x^3 + 3ax^2 + 1 \in \mathbb{Q}[x]$$

(a) Show that it is irreducible.

(b) Show that the Galois group of f_a is solvable.

Solution. For (b) it's the same as to prove that all roots can be written using successive radicals. Let $y = x + \frac{1}{x}$. We observe $f_a(x) = y^3 + (3a - 3)y + 3$. That $y^3+(3a-3)y+3$ is of course solvable, and thus all roots of x are solvable. This proves (b). For part (a), by Gauss lemma, we know factorization in $\mathbb{Q}[x]$ is the same as that in $\mathbb{Z}[x]$. Note that $y^3 + (3a - 3)y + 3$ is irreducible by Eisenstein criterion modulo 3. This shows that the size of Galois orbits of the roots of $f_a(x)$ is at least three; if $f_a(x)$ is reducible, it can only be factorized into two cubic polynomials.

However, by reduction modulo 3 for $f_a(x)$ again, we see that $f_a(x) \equiv (x^2+1)^3 \pmod 3$. This says $f_a(x)$ can only be factorized into even degree polynomials. Two results combined imply $f_a(x)$ is irreducible.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday January 20, 2011 (Day 3)

1. (DG) Let $(\ \cdot\ )$ be the standard inner product on $\mathbb{R}^3$, and let

$$S^2 = \{\mathbf{x} = (x_1, x_2, x_3) : (\mathbf{x} \cdot \mathbf{x}) = 1\}$$

be the sphere of radius 1 centered at the origin; identify the tangent space $T_{\mathbf{x}}S^2$ at a point $\mathbf{x} \in S^2$ with the subspace

$$T_{\mathbf{x}}S^2 = \{v \in \mathbb{R}^3 : (\mathbf{x} \cdot v) = 0\} \subset \mathbb{R}^3,$$

where $(\ \cdot\ )$ is the standard inner product on $\mathbb{R}^3$. Let $e \in \mathbb{R}^3$ be any fixed vector, and let V be the vector field on S^2 given by

$$V(\mathbf{x}) = e - (\mathbf{x} \cdot e)\mathbf{x}.$$

(a) Compute the Lie derivative by V of the 1-form $x_1 dx_2$.

(b) Define a Riemannian metric on S^2 by setting the inner product of tangent vectors $v, v' \in T_x S^2$ equal to $(v \cdot v')$, (that is, take the metric induced on S^2 by the euclidean metric on $\mathbb{R}^3$). Use the associated Levi-Civita connection to define a covariant derivative on the space of 1-forms on S^2.

(c) Compute the covariant derivative of the 1-form $x_1 dx_2$ in the direction of the vector field V.

Solution.

(a) We'll write $\mathbf{x} = (x, y, z)$, $e = (a, b, c)$ and $x_1 dx_2 = xdy$. We have $V(\mathbf{x}) = (a - Sx, b - Sy, c - Sz)$, where $S = ax + by + cz$. Using Cartan's formula, the Lie derivative

$$\mathcal{L}_V(xdy) = \iota_V(dx \wedge dy) + d(\iota_V xdy) = (a - Sx)dy - (b - Sy)dx + d((b - Sy)x)$$

$$= -axydx + (a - 2Sx - bxy)dy - cxydz.$$

(b) The covariant derivative on 1-forms can be given as follows: For 1-form $\alpha \in \Omega^1(S^2)$, we compute the dual vector field $V_\alpha\ V \in C^\infty(TS^2)$. Then $\nabla\alpha$ is defined to be the dual of ∇V_α, which is the projection of the derivative of V_α in $C^\infty(T\mathbb{R}^3)$. This can be seen as follows: For any vector field $u, v \in C^\infty(TS^2)$, we have

$$(\nabla_u \alpha)(v) = \alpha(\nabla_u v) - u.(\alpha(v)) = (V_\alpha \cdot \nabla_u v) - u.(V_\alpha \cdot v) = (\nabla_u T_\alpha \cdot v).$$

(c) Using part (b), we shall first compute the dual of xdy on S^2. On R^3 its dual is $x\frac{\partial}{\partial y}$. We then project this vector to the tangent space of S^2. Note that $x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}$ is the unit normal of S^2, and $(x\frac{\partial}{\partial y}\cdot x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}) = xy$. Hence the dual of xdy is $x\frac{\partial}{\partial y} - xy(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})$. One then computes the derivative of this vector field, project it to the tangent plane of S^2, then take the dual 1-form.

2. (T) Let D^2 be the closed unit disk in $\mathbb{R}^2$. Prove the Brouwer fixed point theorem for maps $f : D^2 \to D^2$ by applying degree theory to the $S^2 \to S^2$ that sends both the northern and southern hemisphere of S^2 to the southern hemisphere via f.

Solution. We have to prove that such an f has a fixed point. Denote by $g : S^2 \to S^2$ the map constructed in the statement of the problem. Since the image of f is in the southern hemisphere, g is homotopic to the map that sends all points on S^2 to the southern pole, and thus g has degree 0. On the other hand, if f has no fixed point, then g has no fixed point as well, and g is homotopic to the antipodal map that sends every point to the opposite point on S^2. This is an orientation-reversing homeomorphism and has degree -1 instead, a contradiction.

3. (CA) Prove that for every $\lambda > 1$, the equation $ze^{\lambda - z} = 1$ has exactly one root in the unit disk $\mathbb{D}$ and that this root is real.

Solution. On $|z| = 1$ w have

$$|z \cdot e^{\lambda - z}| = e^{\lambda - Re(z)} \geq e^{\lambda - 1} > 1$$

because $\lambda > 1$. Hence by Rouché's theorem $1 - ze^{\lambda - z}$ has the same number of zeroes counted with multiplicity as $ze^{\lambda - z}$ inside $|z| = 1$, hence has exactly one zero.

Observe if z is a zero of $1 - ze^{\lambda - z}$ then so is $\bar{z}$ because λ is real, hence by uniqueness the unique zero must be real.

4. (A) Let K be an algebraically closed field of characteristic 0, and let $f \in K[x]$ be any cubic polynomial. Show that exactly one of the following two statement is true:

1. $f = \alpha(x - \lambda)^3 + \beta(x - \lambda)^2$ for some $\alpha, \beta, \lambda \in K$; or
2. $f = \alpha(x - \lambda)^3 + \beta(x - \mu)^3$ for some $\alpha, \beta \neq 0 \in K$ and $\lambda \neq \mu \in K$.

In the second case, show that λ and μ are unique up to order.

Solution. The statement in this problem is incorrect. Take $f = x^3 - 1$. f has no repeated root therefore the first case doesn't happen. Suppose $x^3 - 1 = \alpha(x - \lambda)^3 + \beta(x - \mu)^3$. Looking at the x and x^2 terms gives $\alpha\lambda + \beta\mu = \alpha\lambda^2 + \beta\mu^2 = 0$. This is impossible when $\lambda \neq \mu$ and $\alpha, \beta \neq 0$.

5. (AG) Let $Q \subset \mathbb{P}^{2n+1}$ be a smooth quadric hypersurface in an odd-dimensional projective space over $\mathbb{C}$.

(a) What is the largest dimension of a linear subspace of $\mathbb{P}^{2n+1}$ contained in Q.

(b) What is the dimension of the family of such planes?

Solution. The quadric corresponds to a quadratic form $q(v)$ with $2(v,w) = q(v+w) - q(v) - q(w)$ on $\mathbb{C}^{2n+2}$. If q has non-trivial kernel, i.e. there is $v \neq 0 \in \mathbb{C}^{2n+2}$ s.t. $(v,w) = 0\ \forall w \in \mathbb{C}^{2n+2}$. Then $(\partial_w q)|_v = (w,v) = 0\ \forall w$ and v is a singular point on Q, contradicts to that Q is non-singular. So we'll begin with a non-degenerate quadratic form q.

(a) q cannot have a isotropic subspace of dimension $n+2$ because such a space will have a dimension n orthogonal complement. Therefore Q cannot contains a $n+1$-dimensional linear subspace. We'll construct a n-dimensional linear subspace in Q in (b). So n is the largest dimension for linear subspaces in Q.

(b) Consider $P_k = \{l \in \mathbb{G}(k-1, 2n+1) \mid l \text{ lies in } Q\}$ of $(k-1)$-dimensional (projective) subspaces that lie in Q. Also we take P_0 to be a point. Consider the incidence correspondence $Q_k = \{l_1 \in P_k, l_2 \in P_{k+1} \mid l_1 \text{ lies in } l_2\}$. The fiber of the projection map from Q_k to P_k at $l_1 \in P_k$ can be given as follows: let V_1 be the affine subspace of $\mathbb{C}^{2n+2}$ that corresponds to l_1. Then to get l_2 one has to find a line in $V_1^{\perp}/V_1$ that is isotropic with respect to the induced non-degenerate quadratic form on $V_1^{\perp}/V_1$, namely a choice of a point on a quadric in $\mathbb{P}^{2n+1-2k}$. Therefore the fiber is non-empty and has constant dimension $2n-2k$.

Each fiber of the projection map from Q_k to P_{k+1} is isomorphic to $\mathbb{P}^k$. Thus $\dim P_{k+1} = \dim P_k + (2n-2k) - k$. One then computes $\dim P_{n+1} = 2n + (2n-3) + \ldots + (-n) = n(n+1)/2$.

6. (RA) Let $\mathbb{H}$ and $\mathbb{L}$ denote a pair of Banach spaces.

(a) Prove that a linear map from $\mathbb{H}$ to $\mathbb{L}$ is continuous if and only if it's bounded

(b) Define what is meant by a compact linear map from $\mathbb{H}$ to $\mathbb{L}$.

(c) Now let $\mathbb{H}$ and $\mathbb{L}$ be the Banach spaces obtained by completing the space $C_c^\infty([0,1])$ of compactly supported C^∞ functions on $[0,1]$ using the norms with squares

$$||f||_{\mathbb{H}}^2 = \int_{[0,1]} |\frac{df}{ds}|^2 s^2 ds \text{ and } ||f||_{\mathbb{L}}^2 = \int_{[0,1]} |f|^2 ds.$$

The identity map $C_c^\infty([0,1])$ extends to a bounded linear map $\phi : \mathbb{H} \to \mathbb{L}$ (you don't need to prove this). Prove that ϕ is not compact.

Solution.

(a) If $\phi : \mathbb{H} \to \mathbb{L}$ is bounded with $||\phi|| = A$, then the preimage of the open ball of radius r centered at $\phi(x)$ contains the ball of radius r/A centered at x. This shows ϕ is continuous at x. On the other hand, if it is continuous, then the preimage of the open ball of radius 1 centered at $0 \in \mathbb{L}$ contains an open ball of radius B centered at $0 \in \mathbb{H}$. This says $||x|| < B$ implies $||\phi(x)|| < 1$ and by linearity $||x|| < r \Rightarrow ||\phi(x)|| < r/B$, i.e. $||\phi|| \leq 1/B$.

(b) A continuous linear map $\phi : \mathbb{H} \to \mathbb{L}$ is compact if for any sequence of vectors $\{x_n\}$ in a bounded subset of $\mathbb{H}$, $\{\phi(x_n)\}$ has a convergent subsequence.

(c) Take f to be any non-zero smooth function with support in $[1/2, 1]$. For any $\alpha > 1$, consider scaling $f_\alpha(x) := \alpha f(\alpha^2 x)$ (with the convention $f(x) = 0$ for $x > 1$). We observe that $||f||^2_{\mathbb{H}} = ||f_\alpha||^2_{\mathbb{H}}$ and $||f||^2_{\mathbb{L}} = ||f_\alpha||^2_{\mathbb{L}}$. Take $\alpha_i = 2, 4, 8, ...$ so that each f_α have disjoint support. So $\{f_{\alpha_i}\}$ is bounded but $\{\phi(f_{\alpha_i})\}$ has no convergent subsequence.

Questions in AG for the qualifying exam, Spring 2013 (draft version by Suh).

P1. Prove that the following complex algebraic varieties are pairwise nonisomorphic.

(a) $X_1 = \operatorname{Spec}\mathbb{C}[x,y]/(y^2-x^3)$, $X_2 = \operatorname{Spec}\mathbb{C}[x,y]/(y^2-x^3-x)$ and $X_3 = \operatorname{Spec}\mathbb{C}[x,y]/(y^2-x^3-x^2)$.

(b) $X_1 = \operatorname{Spec}\mathbb{C}[x,y]/(xy^2+x^2y)$ and $X_2 = \operatorname{Spec}\mathbb{C}[x,y,z]/(xy,yz,zx)$.

(c) $X_1 = \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, $X_2 = \mathbb{P}^2_{\mathbb{C}}$ and $X_3 =$ the blowing up of X_2 at the point $[0:0:1]$.

P2. Let f and g be irreducible homogeneous polynomials in $S = \mathbb{C}[X_0,X_1,X_2,X_3]$ of degrees 2 and 3, respectively. For parts (a) and (b), combinatorial polynomials (such as $\binom{T}{2} = T(T-1)/2$) are acceptable in the final answer.

(a) Compute the Hilbert polynomial of $X = \operatorname{Proj}(S/(g))$ embedded in $P = \mathbb{P}^3_{\mathbb{C}} = \operatorname{Proj}(S)$.

(b) Compute the Hilbert polynomial of $Y = \operatorname{Proj}(S/(f,g))$ embedded in P.

(c) Assuming in addition that Y is nonsingular, use your answer for part (b) to compute its geometric genus

$$\dim_{\mathbb{C}} \Gamma(Y, \Omega^1_{Y/\mathbb{C}}).$$

P3. Let X_0 be the affine plane curve defined by the equation

$$y^3 - 3y = x^5$$

over the complex numbers, and let X be the projective smooth model of X_0.

(a) Show that X_0 is nonsingular.

(b) Find all $a \in \mathbb{C}$ for which the polynomial $P_a(y) = y^3 - 3y - a$ has repeated roots. For each such a, factor the polynomial $P_a(y)$.

(c) Let $\pi : X \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ be the unique extension of the coordinate map $x : X_0 \longrightarrow \mathbb{A}^1_{\mathbb{C}}$. Describe the ramification divisor of π and compute its degree.

(d) Compute the genus of X by applying Hurwitz's theorem to $\pi : X \longrightarrow \mathbb{P}^1$.

A solution.

A1.

(a) Only X_2 among the three is nonsingular. The normalization map is a set-theoretic bijection in the case of X_1, but not in the case of X_3.

(b) Since X_1 embeds into the affine plane, the Zariski cotangent space at every $\mathbb{C}$-valued point of X_1 has dimension at most 2. At $(0,0,0) \in X_2$, the Zariski cotangent space has dimension 3.

(c) By Bézout's theorem, any two distinct irreducible curves on X_2 intersect; this is not the case of X_1, nor of X_3. The exceptional divisor on X_3 has self-intersection -1, while no prime divisor on X_1 has strictly negative self-intersection (one can translate any prime divisor into a *different* prime divisor, using the action of $PGL_2 \times PGL_2$ by fractional linear transformations).

A2.

(a) One has a short exact sequence of sheaves on P:

$$0 \longrightarrow \mathcal{O}_P(-3) \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

hence the Hilbert polynomial is

$$h_X(T) = \binom{T+3}{3} - \binom{T+3-3}{3} = \frac{3}{2}T^2 + \frac{3}{2}T + 1.$$

(b) By the assumptions on f and g, they are relatively prime, and we have an exact sequence

$$0 \longrightarrow \mathcal{O}_P(-5) \longrightarrow \mathcal{O}_P(-2) \oplus \mathcal{O}_P(-3) \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_Y \longrightarrow 0,$$

hence the Hilbert polynomial is

$$h_Y(T) = \binom{T+3}{3} - \binom{T+1}{3} - \binom{T}{3} + \binom{T-2}{3} = 6T - 3$$

(c) By Serre duality it is equal to the arithmetic genus, or

$$1 - \chi(\mathcal{O}_Y) = 1 - h_Y(0) = 4.$$

A3.

(a) By the Jacobian criterion of smoothness, at a singular point we have

$$y^3 - 3y = x^5, \;\; 0 = 5x^4 \;\; \text{and } 3y^2 - 3 = 0,$$

which has no solution.

(b) We have $P_a'(y) = 3(y^2 - 1)$, and

$$P_a(y) = \frac{y}{3}P_a'(y) - 2y - a,$$

so $P_a(y)$ is separable exactly when $a \neq \pm 2$. When $a = 2$, $P_a(y) = (y+1)^2(y-2)$ and when $a = -2$, $P_a(y) = (y-1)^2(y+2)$.

(c) For $x \in \mathbb{A}^1(\mathbb{C})$, the fiber $\pi^{-1}(x)$ consists of three distinct points when $x^5 \neq \pm 2$. When $x^5 = \pm 2$, the fiber consists of two points, one point P_x with multiplicity two and the other with multiplicity one.

Since 3 is prime to 5, $\pi^{-1}(\infty)$ consists of one point P_∞ with multiplicity three.

In this notation, the ramification divisor is

$$R = \sum_{x^5 = \pm 2} 1 \cdot [P_x] + 2[P_\infty],$$

and has degree 12.

(d) Let g be the genus of X, and Hurwitz formula reads

$$2g - 2 = 3 \cdot (2 \cdot 0 - 2) + 12,$$

so $g = 4$.

PROPOSED QUESTIONS FOR QUALIFYING EXAMINATION IN ALGEBRAIC TOPOLOGY (2013 SPRING)

(1) Let $\mathbb{H}$ be the space of quaternions, and denote by $\mathbb{S}^3$ the unit sphere inside $\mathbb{H}$. The quaternion group $G = \{\pm 1, \pm i, \pm j, \pm k\}$ acts on $\mathbb{H}$ by left multiplication, and the action preserves the unit sphere $\mathbb{S}^3$. Let X be the quotient space $\mathbb{S}^3/G$. Compute its fundamental group $\pi_1(X)$ and its first homology group $H_1(X, \mathbb{Z})$.

($\mathbb{H}$ is spanned by four independent unit vectors $1, i, j, k$ as a real normed vector space. The multiplication between two elements of $\mathbb{H}$ is bilinear and is determined by the rules $i^2 = j^2 = k^2 = ijk = -1$, and 1 is the multiplicative identity.)

(2) Use Z to denote the subset of $\mathbb{R}^2$ that is given using standard polar coordinates (r, θ) by the equation $r = \cos^2(2\theta)$. The set Z is depicted in Figure 1.

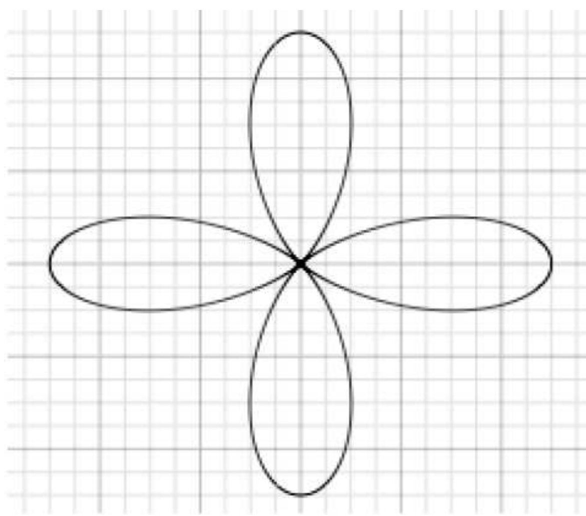

FIGURE 1. The set Z.

(a) Compute the fundamental group of Z.

(b) Let D denote the closed unit disk in $\mathbb{R}^2$ centered at the origin. The boundary of D is the unit circle, this denoted here by ∂D. Parametrize ∂D by the angle $\phi \in [0, 2\pi)$ and let f denote the map from the boundary of ∂D to Z that sends the angle ϕ to the point in Z with polar coordinates $(r = cos^2(2\phi), \theta = \phi)$. Let X denote the space that is obtained from the disjoint union of D and Z by identifying $\phi \in \partial D$ with $f(\phi) \in Z$. Give a set of generators and relations for the fundamental group of X.

(3) Let $K \subset \mathbb{R}^3$ denote a knot, this being a compact, connected, dimension 1 submanifold.
   (a) Compute the homology of the complement in $\mathbb{R}^3$ of any given knot K.
   (b) Figure 2 shows a picture of the trefoil knot.

FIGURE 2. The trefoil knot.

   Sketch on this picture a curve or curves in the complement of K that generate(s) the first homology of $\mathbb{R}^3 - K$.
   (c) A Seifert surface for a knot in $\mathbb{R}^3$ is a connected, embedded surface with boundary, with the knot being the boundary (we do not impose orientability here). By way of an example, view the unit circle in the xy plane as a knot in $\mathbb{R}^3$. This is called the unknot. The unit disk in the xy plane is a Seifert surface for the unknot.
      (i) Compute the second homology of the complement in $\mathbb{R}^3$ of any given Seifert surface for the unknot.
      (ii) Sketch a Seifert surface for the unknot whose complement is not simply connected.

Proposed Answer

(1) Since G is a finite group and it acts on $\mathbb{S}^3$ freely, the quotient map $\mathbb{S}^3 \to X$ is a covering map. $\mathbb{S}^3$ is simply connected and hence it is the universal cover of X, where G acts as Deck transformations. Thus $\pi_1(X) = G$.

The first homology group is the abelianization of the fundamental group. Thus

$$\begin{aligned} H_1(X) &= \pi_1(X)/[\pi_1(X), \pi_1(X)] \\ &= G/[G, G] = G/\{\pm 1\} \\ &= \mathbb{Z}_2 \oplus \mathbb{Z}_2. \end{aligned}$$

(2) (a) The fundamental group is $\mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.
(b) The fundamental group is generated by a, b, c, d with $abcd = 1$.

(3) (a) $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}$, $H_2 = \mathbb{Z}$, $H_3 = 0$ by Mayer Vietoris sequence.
(b) A circle surrounding a segment of the knot.
(c) (i) $H_2 = \mathbb{Z}$.
(ii) A Mobius strip.

Exercise 1. The following questions are independent.
a) For any $a \in (-1, 1)$, compute

$$\int_0^{2\pi} \frac{\mathrm{d}t}{1 + a\cos t}.$$

b) For any $p > 1$, compute

$$\int_0^{\infty} \frac{\mathrm{d}x}{x^p + 1}.$$

Exercise 2. Is there a conformal map between the following domains? If the answer is yes, give such a conformal map. If it is no, prove it.
a) From $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ to $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.
b) From the intersection of the open disks $D((0,0),3)$ and $D((0,3),2)$ to $\mathbb{C}$.
c) From $\mathbb{H}\backslash(0, i]$ to $\mathbb{H}$.
d) From $\mathbb{D}$ to $\mathbb{C}\backslash(-\infty, -\frac{1}{4}]$.

Exercise 3. The following questions are independent.
a) Describe all harmonic functions on the plane $\mathbb{R}^2$ that are bounded from above.
b) Let $h : \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} \to \mathbb{C}$ be holomorphic. Assume that $|h(z)| \leq 1$ for any $z \in \mathbb{H}$ and i is a zero of h of order $m \geq 1$. Prove that, for any $z \in \mathbb{H}$,

$$|h(z)| \leq \left|\frac{z - i}{z + i}\right|^m.$$

Solution of exercise 1.

a) The case $a = 0$ is obvious, we assume $a \neq 0$ from now. We have

$$\int_0^{2\pi} \frac{\mathrm{d}t}{1 + a\cos t} = \int_T \frac{\mathrm{d}z}{iz\left(1 + \frac{a}{2}\left(z + \frac{1}{z}\right)\right)} = \frac{1}{i}\int_T \frac{2\mathrm{d}z}{az^2 + 2z + a} = \frac{1}{i}\int_T \frac{2\mathrm{d}z}{a(z - z_1)(z - z_2)}.$$

where $z_1 = \frac{-1-\sqrt{1-a^2}}{a}$, $z_2 = \frac{-1+\sqrt{1-a^2}}{a}$. The point z_1 is outside the closed unit disk while the point z_2 is inside the open unit disk; thus the residue theorem leads to

$$\int_0^{2\pi} \frac{\mathrm{d}t}{1 + a\cos t} = 2\pi \mathrm{Res}\left(\frac{2}{a(z - z_1)(z - z_2)}, z_2\right) = \frac{2\pi}{\sqrt{1 - a^2}}.$$

b) For $z = re^{i\theta}$, $r > 0$, $-\pi < \theta < \pi$, let $\log z = \log r + i\theta$ The function $\frac{1}{1+x^p}$ is the restriction to $(0, +\infty)$ of $f(z) = \frac{1}{1+\exp(p\log z)}$, which is analytic in $\mathbb{C} - (-\infty, 0]$. For $0 < \epsilon < R < \infty$ we consider the positively oriented contour $\Gamma = \Gamma(\epsilon, R)$ made of the following four arcs:

- the closed interval $[\epsilon, R]$,
- the arc of circle of radius R and angle from 0 to $\frac{2\pi i}{p}$.
- the interval joining $Re^{i\frac{2\pi}{p}}$ and $\epsilon e^{i\frac{2\pi}{p}}$
- the arc of circle of radius ϵ and angle from $\frac{2\pi i}{p}$ to 0.

For $R > 1$ there is only one 0 of f inside the contour, $z_p = e^{i\frac{\pi}{p}}$. The residue at z_p can be computed to be

$$\mathrm{Res}(f, z_p) = -\frac{e^{i\frac{\pi}{p}}}{p}.$$

The residue theorem applied to Γ for $R \to \infty$ and $\epsilon \to 0$ allows to conclude that

$$\int_0^{\infty} \frac{\mathrm{d}x}{x^p + 1} = \frac{\pi}{p}\frac{1}{\sin(\pi/p)}.$$

Solution of exercise 2.

a) Yes, $i\frac{1+z}{1-z}$.

b) No. The inverse map would be a conformal map from the plane to a bounded set. It would therefore constant, by Liouville's theorem. This is absurd.

c) Yes, $\sqrt{z^2+1}$. To see this, $z \mapsto z^2$ sends the domain to $\mathbb{C}/[-1, \infty)$. Therefore $z^2 + 1$ sends the domain to $\mathbb{C}/[0, \infty)$. Taking the square root maps it to the upper half plane.

d) Yes, it can be deduced from questions a) and c), going from the disk to the upper half plane and then the complement of a ray. It is the Koebe function $\frac{z}{(1-z)^2}$.

Solution of exercise 3.

a) Let v be the harmonic conjugate of u. Then the function

$$H(z) = \exp(u(x, y) + iv(x, y)),\ z = x + iy,$$

is entire and bounded in the complex plane. By Liouville's theorem it is constant, which implies that u is constant.
b) The map $\Phi : z \mapsto i\frac{1+z}{1-z}$ is conformal from $\mathbb{D}$ to $\mathbb{H}$. Let $f(z) = h(\Phi(z))$.
The function $g(z) = f(z)/z^m$ is analytic on $\mathbb{D}$ and $|g(z)| \leq 1$ on $\partial\mathbb{D}$. The maximum principle implies that $|g(z)| \leq 1$ on $\mathbb{D}$ and therefore $|f(z)| \leq |z|^m$ for any z in $\mathbb{D}$. This implies the result.

Differential Geometry (Qualifying exams, Spring 2013)

(1) The Heisenberg group is the subgroup of $\mathrm{Sl}(3,\mathbb{R})$ composed of the 3×3, upper triangular matrices with 1 on the diagonal, this being the set of matrices of the form:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{with} \quad (x,y,z)\in\mathbb{R}^3.$$

This group is observably diffeomorphic to $\mathbb{R}^3$.

(a) Compute the Lie algebra of the Heisenburg group.

(b) Exhibit a left-invariant Riemannian metric on the Heisenberg group.

(2) View $\mathbb{R}^2\times\mathbb{C}$ as the product complex line bundle over $\mathbb{R}^2$ and let θ_0 denote the connection on this line bundle whose covariant derivative acts on a given section s as ds with d being the exterior derivative. Let A denote the connection

$$\theta_0 + \frac{i}{1+x^2+y^2}(xdy - ydx).$$

(a) Compute as a function of $r\in(0,\infty)$ the linear map from $\mathbb{C}$ to $\mathbb{C}$ that is obtained by using A to parallel transport a given non-zero vector in $\mathbb{C}$ in the clockwise direction on the circle where $x^2+y^2=r^2$ from the point $(r,0)$ to itself.

(b) Give a formula for the curvature 2-form of the connection A.

(3) Use (t,x,y,z) to denote the Euclidean coordinates for $\mathbb{R}^4$. Let $t\mapsto a(t)$ denote a strictly positive function on $\mathbb{R}$. A Riemannian metric on $\mathbb{R}^4$ is given by the quadratic form:

$$g = dt\otimes dt + a(t)^2\Big(dx\otimes dx + dy\otimes dy + dz\otimes dz\Big).$$

Compute the components of the Riemann curvature tensor of g using the orthonormal basis $\{dt, adx, ady, adz\}$ for $\mathrm{T}^\star\mathbb{R}^4$.

Solutions

Question 1. We denote an element of the Heisenberg group as

$$M = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

(1.a). To identify the generators of the Lie algebra, we compute the left Maurer-Cartan one-form:

$$\theta_L = M^{-1}dM = dxX + dyY + (dz - xdy)Z,$$

where

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

are the generators of the Lie algebra. The Lie bracket is given by the commutators of these matrices:

$$[X, Z] = [Y, Z] = 0, \quad [X, Y] = Z.$$

(1.b). Denoting the transpose of a matrix U as U^t and its trace as $\mathrm{Tr}\ U$, the following is trivially a left-invariant Riemannian metric:

$$g_L = \mathrm{Tr}(\theta_L \cdot \theta_L^t) = dx^2 + dy^2 + (dz - xdy)^2.$$

Question 2

(2.a). The parametric equation of the curve is

$$\gamma : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto \gamma_t = (r \cos\ t, -r \sin\ t).$$

Pulling back the connection to the curve gives

$$\gamma_t^\star A = \gamma_t^\star \theta_0 - \frac{ir^2 dt}{1 + r^2}.$$

To find the parallel transport of $s_0 \in \mathbb{C}$ along the curve γ_t, we solve the first order differential equation with initial condition $s(0) = s_0$:

$$\nabla_{\dot\gamma_t} s(t) = 0,$$

where $\dot\gamma_t$ is $d\gamma_t/dt$ and the covariant derivative on the curve is computed with respect to the connection one-form $\gamma_t^\star A$:

$$\nabla_{\dot\gamma_t} = \frac{d}{dt} - \frac{ir^2 dt}{1 + r^2}.$$

With the initial condition $s(0) = s_0$, the solution is

$$s(t) = s_0 \exp\left(\frac{t\ r^2}{1 + r^2} i\right).$$

The parallel transport of s_0 is then given by evaluating $s(t)$ at the end point $t = 2\pi$. This defines the following linear map:

$$\mathbb{C} \to \mathbb{C} : s_0 \mapsto s_0 \exp\left(\frac{2\pi r^2}{1 + r^2} i\right).$$

(2.b). The curvature of the connection A is :

$$\Omega(A) = dA + A \wedge A = \frac{2i}{(1+x^2+y^2)^2} dx \wedge dy.$$

Question 3

We use the following notation:

$$e^0 = dt, \quad e^1 = adx, \quad e^2 = ady, \quad e^3 = adz, \quad \dot{a} = \frac{da}{dt}, \quad \ddot{a} = \frac{d^2a}{dt^2}.$$

The metric can then be rewritten as

$$g = \sum_{m=0}^{3} e^m \otimes e^m.$$

We can compute the connection from the Cartan's structure equations with zero torsion:

$$de^m + \sum_{n=0}^{3} \omega^m{}_n \wedge e^n = 0, \quad m = 0, 1, 2, 3.$$

Since the connection is compatible with the metric, $\omega^m{}_n$ has to be antisymmetric in m and n. we can then uniquely determine its components. A direct calculation gives:

$$de^0 = 0, \quad de^i = \frac{\dot{a}}{a} e^0 \wedge e^i, \quad i = 1, 2, 3,$$

from which we get:

$$\omega = \frac{\dot{a}}{a} \begin{pmatrix} 0 & -e^1 & -e^2 & -e^3 \\ e^1 & 0 & 0 & 0 \\ e^2 & 0 & 0 & 0 \\ e^3 & 0 & 0 & 0 \end{pmatrix}.$$

The Riemann curvature is then:

$$R(\omega) = d\omega + \omega \wedge \omega = \frac{1}{a^2} \begin{pmatrix} 0 & -a\ddot{a}\ e^0 \wedge e^1 & -a\ddot{a}\ e^0 \wedge e^2 & -a\ddot{a}\ e^0 \wedge e^1 \\ a\ddot{a}\ e^0 \wedge e^1 & 0 & -\dot{a}^2\ e^1 \wedge e^2 & -\dot{a}^2\ e^1 \wedge e^2 \\ a\ddot{a}\ e^0 \wedge e^2 & \dot{a}^2\ e^1 \wedge e^2 & 0 & -\dot{a}^2\ e^1 \wedge e^3 \\ a\ddot{a}\ e^0 \wedge e^3 & \dot{a}^2\ e^1 \wedge e^3 & \dot{a}^2\ e^2 \wedge e^3 & 0 \end{pmatrix}.$$

1. Suppose $f_j, j = 1, 2, \ldots$ and f are real functions on $[0,1]$. Define $f_j \to f$ in measure if and only if for any $\varepsilon > 0$ we have

$$\lim_{j\to\infty} \mu\{x \in [0,1] : |f_j(x) - f(x)| > \varepsilon\} = 0$$

where μ is the Lebesgue measure on $[0,1]$. In this problems, all functions are assumed to be in $L^1[0,1]$.

(a) Suppose that $f_j \to f$ in measure. Does it implies that

$$\lim_{j\to\infty} \int |f_j(x) - f(x)| \mathrm{dx} = 0.$$

Prove it or give a counterexample.

(b) Suppose that $f_j \to f$ in measure. Does this imply that $f_j(x) \to f(x)$ almost everywhere in $[0,1]$? Prove it or give a counter example.

(c) Suppose that $f_j(x) \to f(x)$ almost everywhere in $[0,1]$. Does it implies that $f_j \to f$ in measure? Prove it or give a counter example.

2.

(a) For any bounded positive function f define

$$A(f) := \int_0^1 f(x)\log f(x)\mathrm{d}x, \quad B(f) := \Big(\int_0^1 f(x)\mathrm{d}x\Big)\log\Big(\int_0^1 f(y)\mathrm{d}y\Big).$$

There are three possibilities: (i) $A(f) \geq B(f)$ for all bounded positive functions, (ii) $B(f) \geq A(f)$ for all bounded positive functions, and (iii) $A(f) \geq B(f)$ for some functions while $B(f) \geq A(f)$ for some functions. Decide which possibility is correct and prove your answer. If you use any inequality, state all assumptions of the inequality precisely and clearly.

(b) Let $\hat{f}$ denote the Fourier transform of the function f on $\mathbb{R}$. Suppose that $f \in C^\infty(\mathbb{R})$ and

$$\|\hat{f}(\xi)\|_{L^2(\mathbb{R})} \leq \alpha, \qquad \||\xi|^{1+\varepsilon}\hat{f}(\xi)\|_{L^2(\mathbb{R})} \leq \beta$$

for some $\varepsilon > 0$. Find a bound on $\|f\|_{L^\infty}(\mathbb{R})$ in terms of α, β and ε.

3. Assume that $X_1, X_2, \ldots$ are independent random variables uniformly distributed on $[0,1]$. Let $Y^{(n)} = n\inf\{X_i, 1 \leq i \leq n\}$. Prove that $Y^{(n)}$ converges weakly to an exponential random variable, i.e. for any continuous bounded function $f : \mathbb{R}^+ \to \mathbb{R}$,

$$\mathbb{E}\Big(f(Y^{(n)})\Big) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^+} f(u)e^{-u}\mathrm{d}u.$$

Solutions:

1a. no. let $f = 0$ and
$$f_j(x) = j,\ 0 \le x \le 1/j; \quad = 0 \text{ otherwise}$$

1b. no. let $f = 0$ and f_j are the j-th Haar functions.

1c. yes. Fix any $\varepsilon > 0$. Let
$$E_j = \{x \in [0,1] : |f_j(x) - f(x)| > \varepsilon\}, \quad F_j = \cup_{k \ge j} E_j$$
By changing x on a set of measure zero, we have $f_j(x) \to f(x)$ for all x. Thus
$$\lim_{j\to\infty} F_j = \emptyset$$
Hence $\mu(F_j) \to 0$.

Alternatively, WLOG, we assume that $f = 0$. Let $g_j = \min(1, |f_j|)$. Then $g_j \to 0$. Thus
$$\mu(E_j) \le \varepsilon^{-1} \int_0^1 g_j(x)\mathrm{d}x \to 0.$$

2a. Since the function $x \to x \log x$ is convex, by Jensen inequality, we have $A(f) \ge B(f)$.

2b. From the Fourier inversion formula and Schwarz inequality,
$$|f(x)| \le \Big| \int \hat{f}(\xi) e^{2\pi i x \xi} \mathrm{d}\xi \Big| \le \int |\hat{f}(\xi)|^2 (1 + |\xi|)^{1+\varepsilon} \mathrm{d}\xi \int (1 + |\xi|)^{-1-\varepsilon} \mathrm{d}\xi \le \varepsilon^{-1} C[\alpha^2 + \beta^2]$$
for some constant C.

3. Up to a permutation, we can assume that $x_1 \le x_2 \ldots \le x_n$. In this case, $Y^{(n)} = nx_1$. Thus
$$\mathbb{E} f(Y^{(n)}) = n! \int_0^1 \mathrm{d}x_1 \int_{x_1}^1 \mathrm{d}x_2 \ldots \int_{x_{n-1}}^1 \mathrm{d}x_n f(nx_1)$$
$$= n! \int_0^1 \mathrm{d}x_1 \int_{x_1}^1 \mathrm{d}x_2 \ldots \int_{x_{n-2}}^1 \mathrm{d}x_{n-1} (1 - x_{n-1}) f(nx_1)$$
$$= n \int_0^1 \mathrm{d}x_1 f(nx_1)(1 - x_1)^{n-1} = \int_0^n \mathrm{d}x\, f(x)(1 - x/n)^{n-1} \to \int_{\mathbb{R}^+} f(u) e^{-u} \mathrm{d}u.$$
where we have used dominated convergence in the last step.

Solutions of Qualifying Exams I, 2014 Spring

1. (Algebra) Let $k = \mathbb{F}_q$ be a finite field with q elements. Count the number of monic irreducible polynomials of degree 12 over k.

Solution. Let $G := \mathrm{Gal}(\mathbb{F}_{q^{12}}/\mathbb{F}_q)$ act naturally on $\mathbb{F}_{q^{12}}$. The set of monic irreducible polynomials of degree 12 are in one-to-one correspondence with the set of G-orbits of order 12 in $\mathbb{F}_{q^{12}}$. An orbit $G\alpha$ has order 12 exactly when the subfield $\mathbb{F}_q(\alpha)$ coincides with $\mathbb{F}_{q^{12}}$, i.e., exactly when

$$\alpha \in \mathbb{F}_{q^{12}} \setminus \bigcup_{\mathbb{F}_q \leq K \lneq \mathbb{F}_{q^{12}}} K$$

The maximal proper subfields of $\mathbb{F}_{q^{12}}$ are $\mathbb{F}_{q^6}$ and $\mathbb{F}_{q^4}$. By inclusion-exclusion principle, the number of the polynomials sought is equal to

$$\frac{q^{12} - q^6 - q^4 + q^2}{12}.$$

2. (Algebraic Geometry) (a) Show that the set of lines $L \subset \mathbb{P}^3_{\mathbb{C}}$ may be identified with a quadric hypersurface in $\mathbb{P}^5_{\mathbb{C}}$.

(b) Let $L_0 \subset \mathbb{P}^3_{\mathbb{C}}$ be a given line. Show that the set of lines not meeting L_0 is isomorphic to the affine space $\mathbb{A}^4_{\mathbb{C}}$.

Solution. (a) If $\mathbb{P}^3 = \mathbb{P}V$ is the projective space of one-dimensional subspaces of a 4-dimensional vector space V, then we associate to the line L spanned by two vectors $v, w \in V$ the wedge product $v \wedge w \in \mathbb{P}\bigwedge^2 V \cong \mathbb{P}^5$. Since a 2-form $\eta \in \bigwedge^2 V$ is decomposable if and only if $Q(\eta) = \eta \wedge \eta = 0 \in \bigwedge^4 V \cong \mathbb{C}$, this identifies the set of lines with the zeroes of the quadratic form Q.

(b) Choose 2 planes $\Lambda, \Lambda' \subset \mathbb{P}^3$ containing L_0. Any line not meeting L_0 is determined by its points of intersection with the two planes, giving an isomorphism between the set of lines not meeting L_0 and

$$(\Lambda \setminus L_0) \times (\Lambda' \setminus L_0) \cong \mathbb{A}^2 \times \mathbb{A}^2 \cong \mathbb{A}^4.$$

3. (Complex Analysis) (a) Compute

$$\int_{|z|=1} \frac{z^{31}}{(2\bar{z}^{\,2}+3)^2\,(\bar{z}^{\,4}+2)^3}\,dz$$

Note that the integrand is not a meromorphic function.

(b) Evaluate the integral

$$\int_{x=0}^{\infty} \left(\frac{\sin x}{x}\right)^3 dx$$

by using the theory of residues. Justify carefully all the limiting processes in your computation.

Solution. (a) Since $\bar{z} = \frac{1}{z}$ for $|z| = 1$, it follows that

$$\begin{aligned}
&\int_{|z|=1} \frac{z^{31}}{(2\bar{z}^{\,2}+3)^2(\bar{z}^{\,4}+2)^3}\, dz \\
&= \int_{|z|=1} \frac{z^{31}}{\left(2\left(\frac{1}{z}\right)^2+3\right)^2\left(\left(\frac{1}{z}\right)^4+2\right)^3}\, dz.
\end{aligned}$$

Use the change of variables $z = \frac{1}{w}$ to transform the integral to

$$-\int_{|w|=1} \frac{\frac{1}{w^{31}}}{(2w^2+3)^2(w^4+2)^3} \left(-\frac{dw}{w^2}\right).$$

The negative sign in front of the integral comes from the change of orientation when the parametrization $z = e^{i\theta}$ for $0 \le \theta \le 2\pi$ is transformed to the parametrization $w = e^{-i\theta}$ for $0 \le \theta \le 2\pi$. This new integral can be rewritten as

$$\int_{|w|=1} \frac{dw}{w^{33}(3+2w^2)^2(2+w^4)^3},$$

which is equal to $2\pi i$ times the residue of the meromorphic function

$$\frac{1}{w^{33}(3+2w^2)^2(2+w^4)^3}$$

at $w = 0$. We have the power series expansion of the factor

$$\begin{aligned}
\frac{1}{(3+2w^2)^2} &= \frac{1}{9}\frac{1}{\left(1+\frac{2}{3}w^2\right)^2} \\
&= \frac{1}{9}\sum_{k=0}^{\infty} \frac{(-2)(-3)\cdots(-2-k+1)}{k!}\left(\frac{2}{3}w^2\right)^k
\end{aligned}$$

at $w = 0$ and the power series expansion of the factor

$$
\begin{aligned}
\frac{1}{(2+w^4)^3} &= \frac{1}{8}\frac{1}{\left(1+\frac{1}{2}w^4\right)^3} \\
&= \frac{1}{8}\sum_{\ell=0}^{\infty}\frac{(-3)(-4)\cdots(-3-\ell+1)}{\ell!}\left(\frac{1}{2}w^4\right)^{\ell}
\end{aligned}
$$

at $w = 0$. Contributions to the residue in question from the two power series expansions come from $2k + 4\ell = 32$, which means that k must be divisible by 2 and there are only 9 choices for ℓ from 0 to 8 inclusively (with the corresponding value $k = \frac{32-4\ell}{2} = 16 - 2\ell$). Hence the residue in question is equal to the following sum

$$
\frac{1}{72}\sum_{\ell=0}^{8}\frac{(-2)(-3)\cdots(-2-(16-2\ell)+1)}{(16-2\ell)!}\left(\frac{2}{3}\right)^{16-2\ell}\frac{(-3)(-4)\cdots(-3-\ell+1)}{\ell!}\left(\frac{1}{2}\right)^{\ell}
$$

of 9 terms. The final answer is that

$$
\int_{|z|=1}\frac{z^{31}}{\left(2\bar{z}^{\,2}+3\right)^2\left(\bar{z}^{\,4}+2\right)^3}\,dz
$$

is equal to

$$
\frac{2\pi i}{72}\sum_{\ell=0}^{8}\frac{(-2)(-3)\cdots(-2-(16-2\ell)+1)}{(16-2\ell)!}\left(\frac{2}{3}\right)^{16-2\ell}\frac{(-3)(-4)\cdots(-3-\ell+1)}{\ell!}\left(\frac{1}{2}\right)^{\ell}.
$$

(b) By Euler's formula we have $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$ and

$$
\begin{aligned}
\sin^3 x &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^3 \\
&= \frac{1}{-8i}\left(e^{3ix}-3e^{ix}+3e^{-ix}-e^{-3ix}\right) \\
&= \frac{1}{4}\left(3\,\frac{e^{ix}-e^{-ix}}{2i}-\frac{e^{3ix}-e^{-3ix}}{2i}\right).
\end{aligned}
$$

Thus $\sin^3 x$ is the imaginary part of

$$
\frac{3}{4}\,e^{ix}-\frac{1}{4}\,e^{3ix}.
$$

The power series expansion of

$$\frac{3}{4}\, e^{iz} - \frac{1}{4}\, e^{3iz}$$

is

$$\frac{3}{4}\left(1 + iz + O\left(z^2\right)\right) - \frac{1}{4}\left(1 + 3iz + O\left(z^2\right)\right) = \frac{1}{2} + O\left(z^2\right).$$

The $\mathbb{R}$-linear combination

$$\frac{3}{4}\, e^{iz} - \frac{1}{4}\, e^{3iz} - \frac{1}{2}$$

vanishes to order 2 at $z = 0$ and its imaginary part for $z = x$ real is equal to $\sin^3 x$. Let

$$f(z) = \frac{-\frac{1}{4}\, e^{3iz} + \frac{3}{4}\, e^{iz} - \frac{1}{2}}{z^3}.$$

Its behavior near $z = 0$ is given by

$$f(z) = \frac{-\frac{1}{4}\frac{(3iz)^2}{2} + \frac{3}{4}\frac{(iz)^2}{2} + O\left(z^3\right)}{z^3} = \frac{3}{4}\frac{1}{z} + O\left(z^3\right)$$

and we have a simple pole for f at $z = 0$ whose residue $\mathrm{Res}_0 f$ is $\frac{3}{4}$. Integrating

$$f(z)\, dz$$

over the boundary of the set which is equal to the upper half-disk of radius $R > 0$ minus the upper half-disk of radius r with $0 < r < R$ and letting $R \to \infty$ and $r \to 0$, we get

$$\int_{x=-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^3 dx = \mathrm{Im}\left(\pi i\, \mathrm{Res}_0 f\right) = \mathrm{Im}\left(\pi i\, \frac{3}{4}\right) = \frac{3\pi}{4}$$

and

$$\int_{x=0}^{\infty} \left(\frac{\sin x}{x}\right)^3 dx = \frac{3\pi}{8}.$$

To justify the limiting process, we have to show that the integral

$$\int_{C_R} f(z)\, dz$$

over the upper half-circle of radius R centered at the origin 0 approaches 0 as $R \to \infty$. This is a consequence of the fact that both $|e^{3iz}|$ and $|e^{iz}|$ are ≤ 1 for $\operatorname{Im} z \geq 0$ so that

$$\left| \int_{C_R} f(z)\, dz \right| \leq \frac{1}{R^3} \pi R \to 0 \quad \text{as} \quad R \to \infty.$$

We need also to compute the integral

$$\int_{C_r} f(z)\, dz$$

over the upper half-circle of radius r centered at the origin 0 in the counter-clockwise sense as $r \to 0+$. This is done by using

$$f(z) = \frac{3}{4} \frac{1}{z} + O\left(z^3\right)$$

and the parametrization $z = re^{i\theta}$ for $0 \leq \theta \leq \pi$ so that

$$\begin{aligned} \lim_{r \to 0+} \int_{C_r} f(z)\, dz &= \lim_{r \to 0+} \int_{C_r} \frac{3}{4} \frac{1}{z}\, dz \\ &= \int_{\theta=0}^{\pi} \frac{3}{4} \frac{1}{re^{i\theta}}\, ire^{i\theta} d\theta = \pi i\, \frac{3}{4}. \end{aligned}$$

4. (Algebraic Topology) Suppose that X is a finite connected CW complex such that $\pi_1(X)$ is finite and nontrivial. Prove that the universal covering $\tilde{X}$ of X cannot be contractible. (*Hint:* Lefschetz fixed point theorem.)

Solution. Since X is a finite CW complex, $\tilde{X}$ is also a finite CW complex. Suppose $\tilde{X}$ is contractible. Then $\tilde{X}$ has the same homology as a point, i.e. $H_0(\tilde{X}) = \mathbb{Z}$ and $H_i(\tilde{X}) = 0$ for $i \neq 0$. Then by the Lefschetz fixed point theorem any continuous map $f\colon \tilde{X} \to \tilde{X}$ has a fixed point. On the other hand, the group of covering transformations of $\tilde{X}$ is isomorphic to $\pi_1(X)$, hence is nontrivial. Since a non-identity covering transformation does not have fixed points, we obtain a contradiction. Thus $\tilde{X}$ cannot be contractible.

5. (Differential Geometry) Let $\mathbb{P}^2 = (\mathbb{C}^3 - \{0\})/\mathbb{C}^\times$, which is called the complex projective plane.

1. Show that $\mathbb{P}^2$ is a complex manifold by writing down its local coordinate charts and transitions.

2. Define $L \subset \mathbb{P}^2 \times \mathbb{C}^3$ to be the subset containing elements of the form $([x], \lambda x)$, where $x \in \mathbb{C}^3 - \{0\}$ and $\lambda \in \mathbb{C}$. Show that L is the total space of a holomorphic line bundle over $\mathbb{P}^2$ by writing down its local trivializations and transitions. It is called the tautological line bundle.

3. Using the standard Hermitian metric on $\mathbb{C}^3$ or otherwise, construct a Hermitian metric on the tautological line bundle. Express the metric in terms of local trivializations.

Sketched Solution.

1. The charts are $\phi_0 : U_0 = \{[x, y, z] : z \neq 0\} \to \mathbb{C}^2 = V_0$ by $[x, y, z] \mapsto (x/z, y/z)$, and ϕ_1, ϕ_2 are defined similarly. The transition from V_0 to V_1 is $(X, Y) \mapsto [X, Y, 1] \mapsto (1, Y/X, 1/X)$ for $X \neq 0$, and other transitions are computed in a similar way.

2. The local trivialization over U_0 is $([x, y, 1], \lambda(x, y, 1)) \mapsto ([x, y, 1], \lambda)$, and that over U_1 and U_2 are defined in a similar way. The transition over $U_0 \cap U_1$ is

   $$([x, y, 1], \lambda) \mapsto ([x, y, 1], \lambda(x, y, 1)) = ([1, y/x, 1/x], \lambda x(1, y/x, 1/x)) \mapsto ([x, y, 1], x\lambda).$$

   The transition over U_{12} and U_{02} are similarly defined.

3. Define a metric by $([x], \lambda x) \mapsto \|\lambda x\|$. Over U_0, it is given by $([x, y, 1], \lambda) \mapsto \|\lambda(x, y, 1)\|$. It is similar for the other trivializations U_1, U_2.

6. (Real Analysis) *(Schwartz's Theorem on Perturbation of Surjective Maps by Compact Maps Between Hilbert Spaces).* Let E, F be Hilbert spaces over $\mathbb{C}$, $S : E \to F$ be a compact $\mathbb{C}$-linear map, and $T : E \to F$ be a continuous surjective $\mathbb{C}$-linear map. Prove that the cokernel of $S+T : E \to F$ is finite-dimensional and the image of $S+T : E \to F$ is a closed subspace of F.

Here the compactness of the $\mathbb{C}$-linear map $S : E \to F$ means that for any sequence $\{x_n\}_{n=1}^{\infty}$ in E with $\|x_n\|_E \leq 1$ for all $n \in \mathbb{N}$ there exists some subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $S(x_{n_k})$ converges in F to some element of F as $k \to \infty$.

Hint: Verify first that the conclusion is equivalent to the following equivalent statement for the adjoints $T^*, S^* : F \to E$ of T, S. The kernel of $T^* + S^*$ is finite-dimensional and the image of $T^* + S^*$ is closed. Then prove the equivalent statement.

Solution. We prove first the equivalent statement for the adjoints $T^*, S^* : F \to E$ for T, S and then at the end obtain from it the original statement for $T, S : E \to F$.

The adjoint S^* of the compact operator S is again compact (see *e.g.,* p.189 of Stein and Shakarchi's *Real Analysis*). Since T is surjective, by the open mapping theorem for Banach spaces and in particular for Hilbert spaces, the map $T : E \to F$ is open. It implies that F is the quotient of E by the kernel of T. Thus T^* is the isometry between F and the orthogonal complement of the kernel of T in E, when a Hilbert space is naturally identified with its dual by using its inner product according to the Riesz representation theorem (see *e.g.,* Theorem 5.3 on p.182 of Stein and Shakarchi's *Real Analysis*).

Now we verify that the kernel of $T^* + S^*$ is finite-dimensional by showing that its closed unit ball is compact. Take a sequence of points $\{y_n\}_{n\in\mathbb{N}}$ in the kernel of $T^* + S^*$ with $\|y_n\|_F \leq 1$ for $n \in \mathbb{N}$. Then $T^* y_n + S^* y_n = 0$ for $n \in \mathbb{N}$. Since S^* is compact, there exists a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$ such that $S^* y_{n_k} \to z$ in E for some $z \in E$. From $T^* y_{n_k} \to -z$ as $k \to \infty$ and the fact that T^* is the isometry between F and the orthogonal complement of kernel of T in E, it follows that y_{n_k} converges to the unique element $\hat{z}$ in F such that $T^* \hat{z} = -z$. Since $z = \lim_{k\to\infty} S^* y_{n_k} = S^* \hat{z}$ and $-z = T^* \hat{z}$, it follows that $T^* \hat{z} + S^* \hat{z} = 0$ and z is in the kernel of $T^* + S^*$. Thus the closed unit ball of the kernel of $T^* + S^*$ is compact. Since every locally compact Hilbert space is finite dimensional, it follows that the kernel of $T^* + S^*$ is finite-dimensional.

Now we verify that the image of $T^* + S^*$ is closed. Suppose for some sequence of points $\{y_n\}_{n\in\mathbb{N}}$ in F we have the convergence of $T^* y_n + S^* y_n$ in E to some element z in E. We have to show that z belongs to the image of $T^* + S^*$. By replacing y_n by its projection onto the orthogonal complement $(\operatorname{Ker}(T^* + S^*))^{\perp}$ of the kernel of $T^* + S^*$ in F, we can assume without loss of generality that each y_n belongs to $(\operatorname{Ker}(T^* + S^*))^{\perp}$.

We claim that the sequence of points $\{y_n\}_{n\in\mathbb{N}}$ in $(\operatorname{Ker}(T^* + S^*))^{\perp}$ is bounded in the norm $\|\cdot\|_F$ of F, otherwise we can define $\hat{y}_n = \frac{y_n}{\|y_n\|_F}$ so that $T^*\hat{y}_n + S^*\hat{y}_n \to 0$ as $n \to \infty$ with $\|\hat{y}_n\|_F = 1$ for all $n \in \mathbb{N}$. Since S^* is compact, there is a subsequence $\{\hat{y}_{n_k}\}_{k\in\mathbb{N}}$ of $\{\hat{y}_n\}_{n\in\mathbb{N}}$ with $S^*\hat{y}_{n_k}$ converging to some element u in E. Thus

$$T^*\hat{y}_{n_k} = (T^*\hat{y}_{n_k} + S^*\hat{y}_{n_k}) - S^*\hat{y}_{n_k}$$

converges to the element $-u$ in E. Since T^* is the isometry between F and the orthogonal complement of kernel of T in E, it follows that $\hat{y}_{n_k}$ converges to the unique element v in F such that $T^*v = -u$. This means that $(T^* + S^*)(v) = 0$ and $v \in \mathrm{Ker}\,(T^* + S^*)$. On the other hand, v being the limit of the sequence $\hat{y}_{n_k}$ in $(\mathrm{Ker}\,(T^* + S^*))^\perp$ must be in $(\mathrm{Ker}\,(T^* + S^*))^\perp$ also. Thus, $v = 0$, which contradicts the fact that it is the limit of $\hat{y}_{n_k}$ with $\|\hat{y}_{n_k}\|_F = 1$ for all $k \in \mathbb{N}$. This finishes the proof of the claim that sequence of points $\{y_n\}_{n\in\mathbb{N}}$ in $(\mathrm{Ker}\,(T^* + S^*))^\perp$.

Since the sequence of points $\{y_n\}_{n\in\mathbb{N}}$ in $(\mathrm{Ker}\,(T^* + S^*))^\perp$ is bounded in the norm $\|\cdot\|_F$ of F, by the compactness of S^* we can select a a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$ with $S^*y_{n_k}$ converging to some element w in E. Thus

$$T^*y_{n_k} = (T^*y_{n_k} + S^*y_{n_k}) - S^*y_{n_k}$$

converges to the element $z - w$ in E. Since T^* is the isometry between F and the orthogonal complement of kernel of T in E, it follows that y_{n_k} converges to the unique element t in F such that $T^*t = z - w$. With $w = S^*t = \lim_{k\to\infty} y_{n_k}$, this implies that $(T^* + S^*)(t) = z$. This finishes the proof that the image of $T^* + S^*$ is closed.

Since we now know that the image of $T^* + S^*$ is closed, it follows from the Riesz representation theorem that the map $S + T$ maps E onto the orthogonal complement $\mathrm{Ker}(T^* + S^*)^\perp$ of the kernel $\mathrm{Ker}(T^* + S^*)$ of $T^* + S^*$ in F. Hence the image of $T + S$ is closed and the cokernel of $T + S$ is finite-dimensional.

Solutions of Qualifying Exams II, 2014 Spring

1. (ALGEBRA) Let A be a finite group of order n, and let $V_1, \cdots, V_k$ be its irreducible representations.

(a) Show that the dimensions of the vector spaces V_i satisfy the equality $\sum_{i=1}^{k} (\dim V_i)^2 = n$.

(b) What are the dimensions of the irreducible representations of the symmetric group S_6 of six elements.

Solution. (a) Use the character theory and show that V_i appears $(\dim V_i)$ times in the regular representation $\mathbb{C}[A]$.

(b) Irreducible representations of S_6 correspond to conjugacy classes in S_6, and then to partitions of 6, of which there are $p(6) = 11$. Then use the "hook-length formula",

$$\dim V_\lambda = \frac{d!}{\prod(\text{ hook lengths })}.$$

They are: 16, 10 (twice), 9 (twice), 5 (four times) and 1 (twice).

2. (ALGEBRAIC GEOMETRY) Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree ≥ 3.

(a) Show that C admits a regular map $f : C \to \mathbb{P}^1$ of degree $d - 1$.

(b) Show that C does not admit a regular map $f : C \to \mathbb{P}^1$ of degree e with $0 < e < d - 1$.

Solution. (a) Solution: Simply project from any point $p \in C$ to a complementary line.

(b) Since the canonical series of C is cut on C by plane curves of degree $d-3$, by Riemann-Roch the general fiber of any map $f : C \to \mathbb{P}^1$ of degree e must consist of e points of C that fail to impose independent conditions on curves of degree $d-3$. But any set $d-2$ or fewer points in the plane impose independent conditions on curves of degree $d-3$.

3. (COMPLEX ANALYSIS) Suppose that f is holomorphic in an open set containing the closed unit disk $\{|z| \leq 1\}$ in $\mathbb{C}$, except for a pole at z_0 on the unit circle $\{|z| = 1\}$. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denote the power series expansion of f in the open unit disk $\{|z| < 1\}$, then

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = z_0.$$

Solution. Since z_0 is the only pole of the meromorphic function f on an open set containing the closed unit disk in $\mathbb{C}$, we can express $f(z)$ in the form

$$\sum_{k=1}^{m} \frac{A_k}{(z - z_0)^k} + g(z)$$

with $A_1, \cdots, A_m \in \mathbb{C}$, where $m \geq 1$ and $A_m = h(z_0) \neq 0$ and $g(z)$ is a power series $\sum_{n=0}^{\infty} b_n z^n$ with radius of convergence $R > 1$. For any positive number r with $|z_0| < r < R$ we can find a positive number B such that

$$|b_n| \leq \frac{B}{r^n}$$

for all nonnegative integer n. By using the binomial expansion of $\frac{1}{(z-z_0)^k}$ (or differentiating the geometric series $\frac{1}{z-z_0}$ in z $(k-1)$-times) and noting that $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$, we have

$$a_n = b_n + \sum_{k=1}^{m} (-1)^k A_k \frac{(n+k-1)(n+k-2) \cdots (n+2)(n+1)}{(k-1)!\,(z_0)^{n+k}}.$$

In the computation of the limit

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}},$$

since $A_m = h(z_0) \neq 0$ and $\frac{1}{r} < \left|\frac{1}{z_0}\right|$ and $|b_n| \leq \frac{B}{r^n}$, the dominant term from a_n is

$$(-1)^m A_m \frac{(n+m-1)(n+m-2) \cdots (n+2)(n+1)}{(m-1)!\,(z_0)^{n+m}}$$

and we get

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{(-1)^m A_m \frac{(n+m-1)(n+m-2)\cdots(n+2)(n+1)}{(m-1)!(z_0)^{n+m}}}{(-1)^m A_m \frac{(n+m)(n+m-1)\cdots(n+3)(n+2)}{(m-1)!(z_0)^{n+1+m}}} = z_0.$$

The dominant term from a_n means that a_n minus the dominant term and then divided by the dominant term would have limit zero when $n \to \infty$.

4. (Algebraic Topology) Show that if $n > 1$, then every map from the real projective space $\mathbb{RP}^n$ to the n-torus T^n is null-homotopic.

Solution. Recall that $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2\mathbb{Z}$ and

$$\pi_1(T^n) = \pi_1(S^1 \times \cdots \times S^1) = \mathbb{Z}^n.$$

Now if $f\colon \mathbb{RP}^n \to T^n$ is any map, then the induced homomorphism

$$f_*\colon \pi_1(\mathbb{RP}^n) \to \pi_1(T^n)$$

must be trivial because $\mathbb{Z}^n$ has no nontrivial elements of finite order. Let $p\colon \mathbb{R}^n \to T^n$ be the standard covering map. Then, by the general lifting lemma, we obtain a continuous map $\tilde{f}\colon \mathbb{RP}^n \to \mathbb{R}^n$ such that $f = p \circ \tilde{f}$. Since $\mathbb{R}^n$ is contractible, we obtain that $\tilde{f}$ is nullhomotopic, from which it follows that f is nullhomotopic.

5. (Differential Geometry) Let $\mathbb{S}^2 := \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ be the unit sphere in the Euclidean space. Let $C = \{(r\cos t, r\sin t, h) : t \in \mathbb{R}\}$ be a circle in $\mathbb{S}^2$, where $r, h > 0$ are constants with $r^2 + h^2 = 1$.

1. Compute the holonomy of the sphere $\mathbb{S}^2$ (with the standard induced metric) around the circle C.
2. By using Gauss-Bonnet theorem or otherwise, compute the total curvature

   $$\int_D \kappa \, \mathrm{d}A$$

   where $D = \mathbb{S}^2 \cap \{z \geq h\}$ is the disc bounded by the circle C, and $\mathrm{d}A$ is the area form of $\mathbb{S}^2$.

Sketched Solution.

1. The holonomy is rotation by $2\pi h$.
2. The total curvature is $2\pi - 2\pi\sqrt{r^2h^2 + (1-r^2)^2}$.

6. (Real Analysis) *(Commutation of Differentiation and Summation of Integrals).* Let Ω be an open subset of $\mathbb{R}^d$ and $a < b$ be real numbers. For any positive integer n let $f_n(x, y)$ be a complex-valued measurable function on $\Omega \times (a, b)$. Let $a < c < b$. Assume that the following three conditions are satisfied.

(i) For each n and for almost all $x \in \Omega$ the function $f_n(x, y)$ as a function of y is absolutely continuous in y for $y \in (a, b)$.

(ii) The function $\frac{\partial}{\partial y} f_n(x, y)$ is measurable on $\Omega \times (a, b)$ for each n and the function
$$\sum_{n=1}^{\infty} \left| \frac{\partial}{\partial y} f_n(x, y) \right|$$
is integrable on $\Omega \times (a, b)$.

(iii) The function $f_n(x, c)$ is measurable on Ω for each n and the function $\sum_{n=1}^{\infty} |f_n(x, c)|$ is integrable on Ω.

Prove that the function
$$y \mapsto \int_{x \in \Omega} \sum_{n=1}^{\infty} f_n(x, y) dx$$
is a well-defined function for almost all points y of (a, b) and that
$$\frac{d}{dy} \int_{x \in \Omega} \sum_{n=1}^{\infty} f_n(x, y) dx = \sum_{n=1}^{\infty} \int_{x \in \Omega} \left(\frac{\partial}{\partial y} f_n(x, y) \right) dx$$
for almost all $y \in (a, b)$.

Hint: Use Fubini's theorem to exchange the order of integration and use convergence theorems for integrals of sequences of functions to exchange the order of summation and integration.

Solution. The theorem of Fubini which we will use states that if $F(x, y)$ on $\Omega_1 \times \Omega_2$ (with Ω_j open in $\mathbb{R}^{d_j}$ for $j = 1, 2$) and if
$$\int_{(x,y) \in \Omega_1 \times \Omega_2} |F(x, y)| < \infty,$$

then

$$\int_{x\in\Omega_1}\left(\int_{y\in\Omega_2} F(x,y)dy\right)dx = \int_{y\in\Omega_2}\left(\int_{x\in\Omega_1} F(x,y)dx\right)dy.$$

One consequence of the theorem of dominated convergence which we will use is the folloing exchange of integration and summation. If $F_n(x)$ is a sequence of measurable functions on an open subset $\tilde{\Omega}$ of $\mathbb{R}^{\tilde{d}}$ such that

$$\int_{x\in\tilde{\Omega}}\sum_{n=1}^{\infty}|F_n(x)| < \infty,$$

then

$$\int_{x\in\tilde{\Omega}}\sum_{n=1}^{\infty}F_n(x) = \sum_{n=1}^{\infty}\int_{x\in\tilde{\Omega}}F_n(x).$$

These two results make it possible for us to both exchange the order of integration and the order of summation and integration in the following equation for $a < \eta < b$,
(†)

$$\int_{y=c}^{\eta}\left(\sum_{n=1}^{\infty}\int_{x\in\Omega}\left(\frac{\partial}{\partial y}f_n(x,y)\right)dx\right)dy = \int_{x\in\Omega}\left(\sum_{n=1}^{\infty}\int_{y=c}^{\eta}\left(\frac{\partial}{\partial y}f_n(x,y)\right)dy\right)dx,$$

because the function

$$\sum_{n=1}^{\infty}\left|\frac{\partial}{\partial y}f_n(x,y)\right|$$

is integrable on $\Omega\times(a,b)$. Since for almost all $x\in\Omega$ the function $f_n(x,y)$ as a function of y is absolutely continuous in y, it follows that

$$\int_{y=c}^{\eta}\left(\frac{\partial}{\partial y}f_n(x,y)\right)dy = f_n(x,\eta)-f_n(x,c)$$

for almost all $x\in\Omega$, which implies that

$$\begin{aligned}\int_{x\in\Omega}\left(\sum_{n=1}^{\infty}\int_{y=c}^{\eta}\left(\frac{\partial}{\partial y}f_n(x,y)\right)dy\right)dx &= \int_{x\in\Omega}\left(\sum_{n=1}^{\infty}(f_n(x,\eta)-f_n(x,c))\right)dx\\ &= \int_{x\in\Omega}\left(\sum_{n=1}^{\infty}f_n(x,\eta)\right)dx - \int_{x\in\Omega}\left(\sum_{n=1}^{\infty}f_n(x,c)\right)dx,\end{aligned}$$

because $\sum_{n=1}^{\infty} |f_n(x,c)|$ is integrable on Ω. Putting this together with (†) yields
(‡)

$$\int_{y=c}^{\eta} \left(\sum_{n=1}^{\infty} \int_{x\in\Omega} \left(\frac{\partial}{\partial y} f_n(x,y) \right) dx \right) dy = \int_{x\in\Omega} \left(\sum_{n=1}^{\infty} f_n(x,\eta) \right) dx - \int_{x\in\Omega} \left(\sum_{n=1}^{\infty} f_n(x,c) \right) dx.$$

Differentiating both sides of (‡) with respect to η and applying the fundamental theorem of calculus in the theory of Lebesgue and then replacing η by y, we obtain

$$\sum_{n=1}^{\infty} \int_{x\in\Omega} \left(\frac{\partial}{\partial y} f_n(x,y) \right) dx = \frac{\partial}{\partial y} \int_{x\in\Omega} \left(\sum_{n=1}^{\infty} f_n(x,y) \right) dx$$

for almost all $y \in (a,b)$.

Solutions of Qualifying Exams III, 2014 Spring

1. (Algebra) Prove or disprove: There exists a prime number p such that the principal ideal (p) in the ring of integers $\mathcal{O}_K$ in $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ is a prime ideal.

Solution. If there were, the decomposition group and the inertia group at (p) would be isomorphic to the whole $\mathrm{Gal}\,(K/\mathbb{Q}) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ and to the trivial group, respectively, and the quotient would not be cyclic.

2. (Algebraic Geometry) Let $\Gamma = \{p_1, \cdots, p_5\} \subset \mathbb{P}^2$ be a configuration of 5 points in the plane.

(a) What is the smallest Hilbert function Γ can have?

(b) What is the largest Hilbert function Γ can have?

(c) Find all the Hilbert functions Γ can have.

Solution. (a) The smallest Hilbert function Γ can have occurs if Γ consists of 5 collinear points; the Hilbert function in this case is

$$(h_\Gamma(0), h_\Gamma(1), h_\Gamma(2), \dots) = (1, 2, 3, 4, 5, 5, \dots).$$

(b) The largest Hilbert function Γ can have occurs if Γ consists of 5 general points; the Hilbert function in this case is $(1, 3, 5, 5, \dots)$.

(c) The only other Hilbert function Γ can have occurs when Γ consists of four collinear points and one point not collinear with those; the Hilbert function in this case is $(1, 3, 4, 5, 5, \dots)$.

3. (Complex Analysis) *(Cauchy's Integral Formula for Smooth Functions and Solution of $\bar{\partial}$ Equation).* (a) Let Ω be a bounded domain in $\mathbb{C}$ with smooth boundary $\partial\Omega$. Let f be a C^∞ complex-valued function on some open neighborhood U of the topological closure $\bar{\Omega}$ of Ω in $\mathbb{C}$.

(i) Show that for $a \in \Omega$,

$$f(a) = \frac{1}{2\pi i} \int_{z \in \partial\Omega} \frac{f(z)dz}{z - a} + \frac{1}{2\pi i} \int_{\Omega} \frac{\frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z}}{z - a},$$

where

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + \sqrt{-1}\, \frac{\partial f}{\partial y} \right)$$

with $z = x + \sqrt{-1}\, y$ and x, y real.

(ii) Show that $a \in \Omega$,

$$f(a) = -\frac{1}{2\pi i}\int_{z\in\partial\Omega}\frac{f(z)d\bar{z}}{\bar{z}-\bar{a}} + \frac{1}{2\pi i}\int_{\Omega}\frac{\frac{\partial f}{\partial z}dz\wedge d\bar{z}}{\bar{z}-\bar{a}},$$

where

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \sqrt{-1}\,\frac{\partial f}{\partial y}\right).$$

(iii) For $z \in \Omega$ define

$$h(z) = \frac{1}{2\pi i}\int_{\zeta\in\Omega}\frac{f(\zeta)\,d\zeta\wedge d\bar{\zeta}}{\zeta - z}.$$

Show that $\frac{\partial h}{\partial \bar{z}}(z) = f(z)$ on Ω.

Hint: For (i), apply Stokes's theorem to $d\left(f(z)\frac{dz}{z-a}\right)$ on Ω minus a closed disk of radius $\varepsilon > 0$ centered at a and then let $\varepsilon \to 0$.

For the proof of (iii), for any fixed $z \in \Omega$, apply Stokes's theorem to $d\left(f(\zeta)\log|\zeta - z|d\bar{\zeta}\right)$ (with variable ζ) on Ω minus a closed disk of radius $\varepsilon > 0$ centered at z and then let $\varepsilon \to 0$. Then apply $\frac{\partial}{\partial \bar{z}}$ and use (ii).

(b) Let $\mathbb{D}_r$ be the open disk of radius $r > 0$ in $\mathbb{C}$ centered at 0. Prove that for any C^∞ complex-valued function g on $\mathbb{D}_1$ there exists some C^∞ complex-valued function h on $\mathbb{D}_1$ such that $\frac{\partial h}{\partial \bar{z}} = g$ on $\mathbb{D}_1$.

Hint: First use (a)(iii) to show that for $0 < r < 1$ there exists some C^∞ complex-valued function h_r on $\mathbb{D}_1$ such that $\frac{\partial h_r}{\partial \bar{z}} = g$ on $\mathbb{D}_r$. Then use some approximation and limiting process to construct h.

Solution. (a) Take an arbitrary positive number ε less than the distance from a to the boundary of Ω. Let B_ε be the closed disk of radius $\varepsilon > 0$ centered at a. Application of Stokes's theorem to

$$d\left(f(z)\frac{dz}{z-a}\right) = \frac{\frac{\partial f}{\partial \bar{z}}d\bar{z}\wedge dz}{z-a}$$

on $\Omega - B_\varepsilon$ yields

$$\int_{\Omega - B_\varepsilon}\frac{\frac{\partial f}{\partial \bar{z}}d\bar{z}\wedge dz}{z-a} = \int_{\partial\Omega} f(z)\frac{dz}{z-a} - \int_{|z-a|=\varepsilon} f(z)\frac{dz}{z-a}.$$

We use the parametrization $z = a + \varepsilon e^{i\theta}$ to evaluate the last integral and use $f\left(a + \varepsilon e^{i\theta}\right) - f(a) \to 0$ as $\varepsilon \to 0$ from the continuous differentiability of f at 0 to conclude that

$$\lim_{\varepsilon \to 0^+} \int_{|z-a|=\varepsilon} f(z) \frac{dz}{z-a} = 2\pi i f(a).$$

Hence

$$f(a) = \frac{1}{2\pi i} \int_{z \in \partial\Omega} \frac{f(z)dz}{z-a} + \frac{1}{2\pi i} \int_{\Omega} \frac{\frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z}}{z-a}.$$

This finishes the proof of the formula in (i). For the proof of the formula in (ii) we apply (i) to $\overline{f(z)}$ to get

$$\overline{f(a)} = \frac{1}{2\pi i} \int_{z \in \partial\Omega} \frac{\overline{f(z)}dz}{z-a} + \frac{1}{2\pi i} \int_{\Omega} \frac{\frac{\partial \bar{f}}{\partial \bar{z}} dz \wedge d\bar{z}}{z-a}.$$

and then we take the complex-conjugates of both sides to get

$$f(a) = -\frac{1}{2\pi i} \int_{z \in \partial\Omega} \frac{f(z)d\bar{z}}{\bar{z}-\bar{a}} + \frac{1}{2\pi i} \int_{\Omega} \frac{\frac{\partial f}{\partial z} dz \wedge d\bar{z}}{\bar{z}-\bar{a}},$$

which is the formula in (ii).

For the proof of (iii) we apply Stokes's theorem to

$$d\left(f(\zeta) \log|\zeta - z|^2 d\bar{\zeta}\right) = \frac{\partial f}{\partial \zeta} \log|\zeta - z|^2 d\zeta \wedge d\bar{\zeta} + \frac{f(\zeta) d\zeta \wedge d\bar{\zeta}}{\zeta - z}$$

on $\Omega - B_\varepsilon$ yields

$$\begin{aligned} &\int_{\Omega - B_\varepsilon} \frac{\partial f}{\partial \zeta} \log|\zeta - z|^2 d\zeta \wedge d\bar{\zeta} + \int_{\Omega - B_\varepsilon} \frac{f(\zeta) d\zeta \wedge d\bar{\zeta}}{\zeta - z} \\ &= \int_{\partial\Omega} f(\zeta) \log|\zeta - z|^2 d\bar{\zeta} - \int_{|z-a|=\varepsilon} f(\zeta) \log|\zeta - z|^2 d\bar{\zeta}. \end{aligned}$$

With its evaluation by the parametrization $z = a + \varepsilon e^{i\theta}$, the last integral

$$\int_{|z-a|=\varepsilon} f(\zeta) \log|\zeta - z|^2 d\bar{\zeta}$$

approaches 0 as $\varepsilon \to 0+$ so that

$$\int_{\Omega} \frac{\partial f}{\partial \zeta} \log |\zeta - z|^2 d\zeta \wedge d\bar{\zeta} + \int_{\Omega} \frac{f(\zeta) d\zeta \wedge d\bar{\zeta}}{\zeta - z} = \int_{\partial\Omega} f(\zeta) \log |\zeta - z|^2 d\bar{\zeta}.$$

We apply $\frac{\partial}{\partial \bar{z}}$ to both sides and separately justify the commutation of $\frac{\partial}{\partial x}$ with integration and the commutation of $\frac{\partial}{\partial y}$ with integration, because on the right-hand sides of the following two formulae both integrals over Ω after differentiation are absolutely convergent.

$$\frac{\partial}{\partial x} \int_{\Omega} \frac{\partial f}{\partial \zeta} \log |\zeta - z|^2 d\zeta \wedge d\bar{\zeta} = \int_{\Omega} \frac{\partial f}{\partial \zeta} \left(\frac{\partial}{\partial x} \log |\zeta - z|^2 \right) d\zeta \wedge d\bar{\zeta}$$

and

$$\frac{\partial}{\partial y} \int_{\Omega} \frac{\partial f}{\partial \zeta} \log |\zeta - z|^2 d\zeta \wedge d\bar{\zeta} = \int_{\Omega} \frac{\partial f}{\partial \zeta} \left(\frac{\partial}{\partial y} \log |\zeta - z|^2 \right) d\zeta \wedge d\bar{\zeta}.$$

We get

$$-\int_{\Omega} \frac{\frac{\partial f}{\partial \zeta} d\zeta \wedge d\bar{\zeta}}{\bar{\zeta} - \bar{z}} + \frac{\partial}{\partial z} \int_{\Omega} \frac{f(\zeta) d\zeta \wedge d\bar{\zeta}}{\zeta - z} = -\int_{\partial\Omega} \frac{f(\zeta)\, d\bar{\zeta}}{\bar{\zeta} - \bar{z}},$$

or

$$\frac{\partial}{\partial z} \left(\frac{1}{2\pi i} \int_{\Omega} \frac{f(\zeta) d\zeta \wedge d\bar{\zeta}}{\zeta - z} \right) = -\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)\, d\bar{\zeta}}{\bar{\zeta} - \bar{z}} + \frac{1}{2\pi i} \int_{\Omega} \frac{\frac{\partial f}{\partial \zeta} d\zeta \wedge d\bar{\zeta}}{\bar{\zeta} - \bar{z}},$$

which by the formula in (ii) is equal to $f(z)$. This finishes the proof of the formula in (iii).

For use in (b) we also observe that

$$\frac{\partial}{\partial \bar{z}} \left(\frac{1}{2\pi i} \int_{\Omega} \frac{f(\zeta) d\zeta \wedge d\bar{\zeta}}{\zeta - z} \right) = -\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)\, d\bar{\zeta}}{\zeta - z} + \frac{1}{2\pi i} \int_{\Omega} \frac{\frac{\partial f}{\partial \zeta} d\zeta \wedge d\bar{\zeta}}{\zeta - z}.$$

This implies that

$$\frac{\partial}{\partial \bar{z}} \left(\frac{1}{2\pi i} \int_{\Omega} \frac{f(\zeta) d\zeta \wedge d\bar{\zeta}}{\zeta - z} \right)$$

is uniformly bounded on compact subsets of Ω. By induction on k and by applying the argument to $\frac{\partial f}{\partial \zeta}$ on a neighborhood of $\bar{\Omega}$ in U in going from the

k-th step to the $(k+1)$-st step in the induction process, we conclude that all the k-th partial derivatives of

$$\int_\Omega \frac{f(\zeta)d\zeta \wedge d\bar{\zeta}}{\zeta - z}$$

with respect to x and y (*i.e.,* with respect to z and $\bar{z}$) are uniformly bounded on compact subsets of Ω. Hence

$$\int_\Omega \frac{f(\zeta)d\zeta \wedge d\bar{\zeta}}{\zeta - z}$$

is C^∞ on Ω as a function of z.

(b) Choose $r_n = 1 - \frac{1}{2^n}$. We can set

$$h_{r_n}(z) = \frac{1}{2\pi i}\int_{\zeta \in \mathbb{D}_{r_{n+1}}} \frac{g(\zeta)d\zeta \wedge d\bar{\zeta}}{\zeta - z}$$

on $\mathbb{D}_{r_n}$ to get $\partial_{\bar{z}} h_{r_n} = g$ on $\mathbb{D}_{r_n}$ from (a)(iii). As observed above, $h_{r_n}(z)$ is an infinitely differentiable function on $\mathbb{D}_{r_n}$.

We now look at the approximation and limiting process to construct h on all of $\mathbb{D}_1$ such that $\partial_{\bar{z}} h = g$ on $\mathbb{D}_1$.

For $n \geq 3$ the function $h_{r_n} - h_{r_{n-1}}$ is holomorphic on $\mathbb{D}_{r_{n-1}}$. By using the Taylor polynomial P_n of $h_{r_n} - h_{r_{n-1}}$ centered at 0 of degree N_n for N_n sufficiently large, we have

$$\left|\left(h_{r_n} - h_{r_{n-1}}\right) - P_n\right| \leq \frac{1}{2^n}$$

on $\mathbb{D}_{r_{n-2}}$. Let $\hat{h}_{r_n} = h_{r_n} - \sum_{k=3}^n P_n$ on $\mathbb{D}_{r_n}$. Then for any $n > k \geq 3$ from

$$\hat{h}_{r_n} - \hat{h}_{r_k} = \sum_{\ell=k+1}^{n}\left(\hat{h}_{r_\ell} - \hat{h}_{r_{\ell-1}}\right) = \sum_{\ell=k+1}^{n}\left(h_{r_\ell} - h_{r_{\ell-1}} - P_\ell\right)$$

it follows that

$$\left|\hat{h}_{r_n} - \hat{h}_{r_k}\right| \leq \sum_{\ell=k+1}^{n}\left|h_{r_\ell} - h_{r_{\ell-1}} - P_\ell\right| \leq \sum_{\ell=k+1}^{n}\frac{1}{2\ell} \leq \frac{1}{2^k}$$

on $\mathbb{D}_{r_{k-1}}$. Thus, for any fixed $k \geq 3$ the sequence $\{h_{r_n} - h_{r_k}\}_{n=k+1}^{\infty}$ is a Cauchy sequence of holomorphic functions on $\mathbb{D}_{r_{k-1}}$ and we can define $h = \lim_{n\to\infty} \hat{h}_{r_n}$ on $\mathbb{D}$ with $\frac{\partial h}{\partial \bar{z}} = g$ on $\mathbb{D}$, because $h - h_{r_k}$ is holomorphic on $\mathbb{D}_{r_{k-1}}$ and $\frac{\partial h_{r_{k-1}}}{\partial \bar{z}} = g$ on $\mathbb{D}_{r_{k-1}}$. Since $\hat{h}_{r_n}$ is infinitely differentiable on $\mathbb{D}_{r_{k-1}}$, it follows that h is infinitely differentiable on each $\mathbb{D}_{r_{k-1}}$ and hence is infinitely differentiable on all of $\mathbb{D}_1$.

4. (Algebraic Topology) Suppose that X is contractible and that some point a of X has a neighborhood homeomorphic to $\mathbb{R}^k$. Prove that $H_n(X \setminus \{a\}) \simeq H_n(S^{k-1})$ for all n.

Solution. We have the following piece of the long exact homology sequence:

$$H_k(X) \to H_k(X, X \setminus \{a\}) \to H_{k-1}(X \setminus \{a\}) \to H_{k-1}(X).$$

Now for $k > 1$, the outer two groups are 0, hence

$$H_k(X, X \setminus \{a\}) \simeq H_{k-1}(X \setminus \{a\}).$$

Let U be a neighborhood of a homeomorphic to $\mathbb{R}^m$ and let $C = X \setminus U$. Then $C \subset X \setminus \{a\}$, which is open. Hence, by excision,

$$H_k(X, X \setminus \{a\}) \simeq H_k(U, U \setminus \{a\}) \simeq H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{a\}).$$

On the other hand, we have the same piece of exact sequence of $(\mathbb{R}^m, \mathbb{R}^m \setminus \{a\}) : \psi(x, y) = (x, y)$ when $y < 0$, and

$$H_k(\mathbb{R}^m) \to H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{a\}) \to H_{k-1}(\mathbb{R}^m \setminus \{a\}) \to H_{k-1}(\mathbb{R}^m),$$

and the outer two groups are 0 for $k > 1$. Since $\mathbb{R}^m \setminus \{a\}$ deformation retracts onto S^{m-1}, putting everything together we obtain that for $k > 1$, $H_k(S^{m-1}) \simeq H_k(X \setminus \{a\})$.

5. (Differential Geometry) Let $U_+ = \mathbb{R}^2 - (\mathbb{R}_{\leq 0} \times \{0\})$, $U_- = \mathbb{R}^2 - (\mathbb{R}_{\geq 0} \times \{0\})$, and $U_0 = \mathbb{R}^2 - (\mathbb{R} \times \{0\})$. Let B be obtained by gluing U_+ and U_- over U_0 by the map $\psi : U_0 \to U_0$ defined by

$$\psi(x, y) = (x, y)$$

when $y < 0$, and

$$\psi(x, y) = (x + y, y)$$

when $y > 0$.

1. Show that B is a manifold.

2. Show that the trivial connections on the tangent bundles of U_+ and U_- glue together and give a global connection on the tangent bundle TB. Compute the curvature of this connection.

3. Compute the holonomy of the above connection around the loop $\gamma : [0, 2\pi] \to B$ determined by $\gamma|_{U_+}(\theta) = (\cos\theta, \sin\theta)$ for $\theta \in (0, 2\pi)$.

Sketched Solution.

1. U_+ and U_- already serve as charts of B, and the transition between them is affine.

2. Since the transition is affine, the differential d is preserved by the transition. The curvature is just zero.

3. The holonomy is given by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

6. (Real Analysis) *(Bernstein's Theorem on Approximation of Continuous Functions by Polynomials).* Use the probabilistic argument outlined in the two steps below to prove the following theorem of Bernstein. Let f be a real-valued continuous function on $[0, 1]$. For any positive integer n let

$$B_n(f; x) = \sum_{j=0}^{n} f\left(\frac{j}{n}\right)\binom{n}{j} x^j (1-x)^{n-j}$$

be the *Bernstein polynomial.* Then $B_n(f; x)$ converges to f uniformly on $[0, 1]$ as $n \to \infty$.

Step One. For $0 < x < 1$ consider the binomial distribution

$$b(n, x, j) = \binom{n}{j} x^j (1-x)^{n-j}$$

for $0 \le j \le n$, which is the probability of getting j heads and $n - j$ tails in tossing a coin n times if the probability of getting a head is x. Verify that the mean μ of this probability distribution is nx and its standard deviation σ is $\sqrt{nx(1-x)}$.

Step Two. Let X be the random variable which assumes the value j with probability $b(n,x,j)$ for $0 \le j \le n$. Consider the random variable $Y = \left|f(x) - f\left(\frac{X}{n}\right)\right|$ which assumes the value $\left|f(x) - f\left(\frac{j}{n}\right)\right|$ with probability $b(n,x,j)$ for $0 \le j \le n$. Prove Bernstein's theorem by bounding, for an arbitrary positive number ε, the sum which defines the expected value $E(Y)$ of the random variable Y, after breaking the sum up into two parts defined respectively by $|j-\mu| \ge \eta\sigma$ and $|j-\mu| < \eta\sigma$ for some appropriate positive number η depending on ε and the uniform bound of f.

Solution. *Step One.* From

$$j\binom{n}{j} = n\,\frac{(n-1)(n-2)\cdots(n-j+1)}{(j-1)!} = n\binom{n-1}{j-1}$$

it follows that

$$\begin{aligned}
\mu &= \sum_{j=0}^{n} j\, b(n,x,j) \\
&= \sum_{j=0}^{n} j\binom{n}{j} x^j (1-x)^{n-j} \\
&= \sum_{j=1}^{n} n\binom{n-1}{j-1} x\, x^{j-1}(1-x)^{n-j} \\
&= nx\left(x + (1-x)\right)^{n-1} \\
&= nx.
\end{aligned}$$

From

$$j(j-1)\binom{n}{j} = n(j-1)\binom{n-1}{j-1} = n(n-1)\binom{n-2}{j-2}$$

and

$$
\begin{aligned}
E(X(X-1)) = \sum_{j=0}^{n} j\,(j-1)\,b(n,x,j) &= \sum_{j=0}^{n} j\,(j-1)\binom{n}{j} x^j (1-x)^{n-j} \\
&= \sum_{j=2}^{n} n(n-1)x^2 \binom{n-2}{j-2} x^{j-2}(1-x)^{n-j} \\
&= n(n-1)x^2 \sum_{j=0}^{n-2} \binom{n-2}{j} x^j (1-x)^{n-2-j} \\
&= n(n-1)x^2\,(1+(1-x))^{n-2} \\
&= n(n-1)x^2
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\sigma^2 &= E\left((X-\mu)^2\right) \\
&= E(X^2) - 2\mu E(X) + \mu^2 \\
&= E(X^2) - \mu^2 \\
&= n(n-1)x^2 + nx - (nx)^2 \\
&= nx\,((n-1)x + 1 - nx) \\
&= nx(1-x).
\end{aligned}
$$

and $\sigma = \sqrt{nx(1-x)}$.

Step Two. Given any $\varepsilon > 0$. By the uniform continuity of f on $[0,1]$ there exists some $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ for $|x_1 - x_2| < \delta$. Choose a positive number η sufficiently large so that

$$
\frac{1}{\eta^2}\, 2 \sup_{[0,1]} |f| < \frac{\varepsilon}{2}
$$

and then choose a positive N with

$$
\frac{\eta}{\sqrt{N}} < \delta.
$$

We are going to prove that $|f - B_n(f;x)| < \varepsilon$ on $[0,1]$ for $n \geq N$ by bounding the sum which defines the expected value $E(Y)$ of the random variable Y, after breaking the sum up into two parts defined respectively by $|j-\mu| \geq \eta\sigma$ and $|j-\mu| < \eta\sigma$.

First of all, for any fixed $x \in [0,1]$,

$$\begin{aligned}
|f(x) - B_n(f;x)| &= \left| \sum_{j=0}^{n} \left(f(x) - f\left(\frac{j}{n}\right) \right) \binom{n}{j} x^j (1-x)^{n-j} \right| \\
&= \left| \sum_{j=0}^{n} \left(f(x) - f\left(\frac{j}{n}\right) \right) b(n,x,j) \right| \\
&\le \sum_{j=0}^{n} \left| f(x) - f\left(\frac{j}{n}\right) \right| b(n,x,j),
\end{aligned}$$

which is the expected value $E(Y)$ of the random variable Y, because

$$\begin{aligned}
\sum_{j=0}^{n} f(x) \binom{n}{j} x^j (1-x)^{n-j} &= f(x) \sum_{j=0}^{n} \binom{n}{j} x^j (1-x)^{n-j} \\
&= f(x) \left(x + (1-x) \right)^n = f(x).
\end{aligned}$$

For the estimation of the part

$$\sum_{|j-nx|<\eta\sigma} \left| f(x) - f\left(\frac{j}{n}\right) \right| b(n,x,j)$$

of the sum

$$E(Y) = \sum_{j=0}^{n} \left| f(x) - f\left(\frac{j}{n}\right) \right| b(n,x,j)$$

we have

$$\left| x - \frac{j}{n} \right| < \frac{\eta\sigma}{n} = \frac{\eta\sqrt{nx(1-x)}}{n} \le \frac{\eta}{\sqrt{n}} \le \frac{\eta}{\sqrt{N}} < \delta,$$

which implies that $\left| f(x) - f\left(\frac{j}{n}\right) \right| < \frac{\varepsilon}{2}$ so that

$$\sum_{|j-nx|<\eta\sigma} \left| f(x) - f\left(\frac{j}{n}\right) \right| b(n,x,j) < \frac{\varepsilon}{2}\, b(n,x,j) \le \frac{\varepsilon}{2}.$$

For the estimation of the part

$$\sum_{|j-nx|\ge\eta\sigma} \left| f(x) - f\left(\frac{j}{n}\right) \right| b(n,x,j)$$

of the sum

$$E(Y) = \sum_{j=0}^{n} \left| f(x) - f\left(\frac{j}{n}\right) \right| b(n,x,j)$$

we use Chebyshev's inequality that in any probability distribution no more than $\frac{1}{\eta^2}$ of the distribution's values can be no less than η standard deviations away from the mean, which, when applied to our random variable X with mean $\mu = nx$, means that

$$\sum_{|j-nx| \geq \eta\sigma} b(n,x,j) \leq \frac{1}{\eta^2}.$$

Thus,

$$\sum_{|j-nx| \geq \eta\sigma} \left| f(x) - f\left(\frac{j}{n}\right) \right| b(n,x,j) \leq \left(2 \sup_{[0,1]} |f| \right) \frac{1}{\eta^2} < \frac{\varepsilon}{2}.$$

This finishes the verification that

$$|f(x) - B_n(f;x)| < \varepsilon$$

for $n \geq N$ and thus the proof of Bernstein's theorem.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday January 20, 2015 (Day 1)

1. (AG) Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree d.

(a) Let K_C be the canonical bundle of C. For what integer n is it the case that $K_C \cong \mathcal{O}_C(n)$?

(b) Prove that if $d \geq 4$ then C is not hyperelliptic.

(c) Prove that if $d \geq 5$ then C is not trigonal (that is, expressible as a 3-sheeted cover of $\mathbb{P}^1$).

Solution: By the adjunction formula, the canonical divisor class is $K_C = \mathcal{O}_C(d-3)$, that is, plane curves of degree $d-3$ cut out canonical divisors on C. It follows that if $d \geq 4$ then any two points $p, q \in C$ impose independent conditions on the canonical series $|K_C|$; that is, $h^0(K_C(-p-q)) = g-2$, so by Riemann-Roch $h^0(\mathcal{O}_C(p+q)) = 1$, i.e., C is not hyperelliptic. Similarly, if $d \geq 5$ then any three points $p, q, r \in C$ impose independent conditions on the canonical series $|K_C|$; by Riemann-Roch it follows that $h^0(\mathcal{O}_C(p+q+r)) = 1$ so C is not trigonal.

2. (A) Let S_4 be the group of automorphisms of a 4-element set. Give the character table for S_4 and explain how you arrived at it.

Solution: To start with, there are five conjugacy classes in S_4: $(1), (12), (123), (1234)$ and $(12)(34)$. The characters of the trivial and alternating representations U and U' are clear. The standard representation of S_4 on $\mathbb{C}^4$ splits as a direct sum of the trivial and a three-dimensional representation V, whose character is simply the character of $\mathbb{C}^4$ minus one; we see that it's irreducible because the norm of its character is 1. We get another irreducible as $V' = V \otimes U'$; its character is $\chi_{V'} = \chi_V \chi_{U'}$. The final irreducible representation W (and its character) can be found by pulling back the standard representation of S_3 via the quotient map $S_4 \to S_3$ (or by the orthogonality relations). Altogether, we have

conjugacy class	e	(12)	(123)	(1234)	(12)(34)
number of elements	1	6	8	6	3
U	1	1	1	1	1
U'	1	-1	1	-1	1
V	3	1	0	-1	-1
V'	3	-1	0	1	-1
W	2	0	1	0	2

3. (DG) Let

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 - y^2 - z^3 - z = 0\}.$$

(a) Prove that M is a smooth surface in $\mathbb{R}^3$.

(b) For what values of $c \in \mathbb{R}$ does the plane $z = c$ intersect M transversely?

Solution: See attached.

4. Define the Banach space $\mathcal{L}$ to be the completion of the space of continuous functions on the interval $[-1, 1] \subset \mathbb{R}$ using the norm

$$||f|| = \int_{-1}^{1} |f(t)|dt.$$

Suppose that $f \in \mathcal{L}$ and $t \in [-1, 1]$. For $h > 0$, let I_h be the set of points in $[-1, 1]$ with distance h or less from t. Prove that

$$\lim_{h \to 0} \int_{t \in I_h} |f(t)|dt = 0$$

Solution: See attached.

5. (AT) What are the homology groups of the 5-manifold $\mathbb{RP}^2 \times \mathbb{RP}^3$,

(a) with coefficients in $\mathbb{Z}$?

(b) with coefficients in $\mathbb{Z}/2$?

(c) with coefficients in $\mathbb{Z}/3$?

Solution: $\mathbb{RP}^2$ and $\mathbb{RP}^3$ have cell complexes with sequences

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0 \quad \text{and} \quad 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0$$

where the maps are alternately 0 and multiplication by 2; from this the homology groups of $\mathbb{RP}^2$ and $\mathbb{RP}^3$ can be calculated as $\mathbb{Z}, \mathbb{Z}/2, 0$ and $\mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z}$ respectively. The rest is just Kunneth; the answers are

(a): $\mathbb{Z}$, $(\mathbb{Z}/2)^2$, $(\mathbb{Z}/2)^2$, $\mathbb{Z}$, $\mathbb{Z}/2$, 0;

(b): $\mathbb{Z}/2$, $(\mathbb{Z}/2)^2$, $(\mathbb{Z}/2)^3$, $(\mathbb{Z}/2)^3$, $(\mathbb{Z}/2)^2$, $\mathbb{Z}/2$,

(c): $\mathbb{Z}/3$, 0, 0, $\mathbb{Z}/3$, 0, 0

6. Let Ω be an open subset of the Euclidean plane $\mathbb{R}^2$ A map $f : \Omega \to \mathbb{R}^2$ is said to be *conformal* at $p \in \Omega$ if its differential df_p preserves the angle between any two tangent vectors at p. Now view $\mathbb{R}^2$ as $\mathbb{C}$ and a map $f : \Omega \to \mathbb{R}^2$ as a $\mathbb{C}$-valued function on Ω.

(a) Supposing that f is a holomorphic function on Ω, prove that f is conformal where its differential is nonzero.

(b) Suppose that f is a nonconstant holomorphic function on Ω, and $p \in \Omega$ is a point where $df_p = 0$. Let L_1 and L_2 denote distinct lines through p. Prove that the angle at $f(p)$ between $f(L_1)$ and $f(L_2)$ is n times that between L_1 and L_2, with n being an integer greater than 1.

Solution: See attached.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday January 21, 2015 (Day 2)

1. (AT) Let $X \subset \mathbb{R}^3$ be the union of the unit sphere $S^2 = \{(x, y, z) \mid x^2+y^2+z^2 = 1\}$ and the line segment $I = \{(x, 0, 0) \mid -1 \leq x \leq 1\}$.

(a) What are the homology groups of X?

(b) What are the homotopy groups $\pi_1(X)$ and $\pi_2(X)$?

Solution: Under the attaching map $I \hookrightarrow X$, the boundary $\varphi(I)$ is homologous to 0, so attaching I simply adds one new, non-torsion generator to H^1; thus

$$H_0(X) = H^1(X) = H^2(X) = \mathbb{Z},$$

and all other homology groups are 0. Similarly, $\pi_1(X) = \mathbb{Z}$. For $\pi_2(X)$, note that the universal cover of X is a string of spheres attached in a sequence by line segments; $\pi_2(X)$ is thus the free abelian group on countably many generators.

2. (A) Let

$$f(t) = t^4 + bt^2 + c \in \mathbb{Z}[t].$$

(a) If E is the splitting field for f over $\mathbb{Q}$, show that $Gal(E/\mathbb{Q})$ is isomorphic to a subgroup of the dihedral group D_8.

(b) Given an example of b and c for which f is irreducible, and for which the Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Justify.

(c) Give an example of b and c for which f is irreducible, and for which the Galois group is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Justify.

(d) Give an example of b and c for which f is irreducible, and for which the Galois group is isomorphic to D_8.

Solution:

(a) Obviously if α is a root of f, so is $-\alpha$. So let $\pm\alpha_1, \pm\alpha_2$ be the four distinct roots of f in E. If ϕ is an element of the Galois group, it must permute the roots of f—moreover, ϕ is determined completely by its action on α_1 and α_2. Also by definition of automorphism, note that $\phi(\alpha_1)$ cannot be a rational multiple of $\phi(\alpha_2)$, while $\phi(-\alpha_1) = -\phi(\alpha_1)$. Hence any field automorphism must necessarily give rise to a symmetry of the following square:

$$\boxed{\begin{array}{cc} \alpha_1 & \alpha_2 \\ & \\ -\alpha_2 & -\alpha_1 \end{array}}$$

This gives the injection of Gal into D_8.

(b) An obvious strategy is to find a quadratic extension of a quadratic extension, then find an element whose minimal polynomial is degree 4. For instance, the element $\alpha = \sqrt{2}+\sqrt{3}$ in $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ has a degree 4 minimal polynomial which we can construct by repeatedly multiplying by conjugates: Begin with $t-\sqrt{2}-\sqrt{3}$, the multiply by $(t-\sqrt{2})+\sqrt{3}$, then multiply this by $(t^2+\sqrt{2})^2-3$. For this choice of α, we have $f(t) = t^4 - 10t^2 + 1$.

(c) Taking $b = 0$ and $c = 1$, we see that the splitting field is isomorphic to the subfield of the complex numbers generated by adjoining to $\mathbb{Q}$ the number $\alpha = e^{\pi i/4}$. This is a degree 4 field over $\mathbb{Q}$. Since we have a splitting field in characteristic zero, the Galois group has order 4. We see that the field automorphism sending

$$\alpha \mapsto \alpha^3$$

has order 4, hence the Galois group is cyclic.

(d) Take $b = 0$ and $c = 2$. Clearly we have roots $\alpha_1 = 2^{1/4}$ and $\alpha_2 = i2^{1/4}$, which together lie in an extension of at least degree 8 over $\mathbb{Q}$. By part (a), the Galois group must be D_8 itself.

3. (CA) Let $a \in (0,1)$. By using a contour integral, compute

$$\int_0^{2\pi} \frac{dx}{1-2a\cos x + a^2}.$$

Solution (HT): By the periodicity of cos, it suffices to compute the integral from $-\pi$ to π. We note that there is a pole for the function

$$f(z) = \frac{1}{1-2a\cos z + a^2}$$

at $z_0 = i\cosh^{-1}\frac{1+a^2}{2a}$. Let R_t be the rectangle bordered by the lines $x = \pm\pi$ and $y = 0, y = t$. As $t \to \infty$, the contribution from the line $y = t$ goes to zero. On the other hand, for all values of t, the contribution to the integral from $x = \pm\pi$ cancel each other out. Thus the integral along the bottom edge of the rectangle (which is what we seek) is equal to $2\pi i$ times the residue of $f(z)$ at z_0. Near z_0, we have that

$$1 - 2a\cos z + a^2 = (z-z_0)2ai\sinh iz_0 + \ldots$$

so we conclude the integral is given by

$$\frac{2\pi i}{2ai\sinh z_0}.$$

This simplifies to

$$\frac{2\pi}{1-a^2}.$$

Alternate solution (CH): Write the integral as a contour integral on the unit circle: set $dx = \frac{-idz}{z}$, so that

$$\int_0^{2\pi} \frac{1}{1 - 2a\cos x + a^2} dx = -i \int_{|z|=1} \frac{1}{z(1+a^2) - az^2 - a} dz.$$

Factor the denominator to find the poles of the latter integrand; one is inside the unit circle and one outside. Calculate the residue at the former pole and use Cauchy's theorem to evaluate the integral.

4. (AG) Let K be an algebraically closed field of characteristic 0 and let $Q \subset \mathbb{P}^n$ be a smooth quadric hypersurface over K.

(a) Show that Q is rational by exhibiting a birational map $\pi : Q \to \mathbb{P}^{n-1}$.

(b) How does the map π factor into blow-ups and blow-downs?

Solution: For the first part, we choose any point $p \in Q$ and take π to be the projection from p. Since Q has degree 2, a general line in $\mathbb{P}^n$ through p will meet Q in one other point, so that the map $\pi : Q \to \mathbb{P}^{n-1}$ has degree 1; that is, it is a birational map. This map blows up the point p, and then blows down the union of the lines on Q through p. In the other direction, starting with $\mathbb{P}^{n-1}$ we blow up the intersection $Z = S \cap H$ of a quadric hypersurface $S \subset \mathbb{P}^{n-1}$ and a hyperplane $H \subset \mathbb{P}^{n-1}$, and then blow down the proper transform of H.

5. DG Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

be the unit sphere centered at the origin in $\mathbb{R}^3$.

(a) Prove that the vector field

$$v = yz\frac{\partial}{\partial x} + zx\frac{\partial}{\partial y} - 2xy\frac{\partial}{\partial z}$$

on $\mathbb{R}^3$ is tangent to S at all points of S, and thus defines a section of the tangent bundle TS.

(b) Let g be the metric on S induced from the euclidean metric on $\mathbb{R}^3$, and let ∇ be the associated, metric compatible, torsion free covariant derivative. The tensor ∇v is a section of $TS \otimes TS^*$. Write ∇v at the point $(0, 0, 1) \in S$ using the coordinates (x_1, x_2) given by the map $(x_1, x_2) \mapsto (x_1, x_2, \sqrt{1 - x_1^2 - x_2^2})$ from the unit disc $x_1^2 + x_2^2 < 1$ to S.

Solution: See attached

6. (RA) Let L be a positive real number.

(a) Compute the Fourier expansion of the function x on the interval $[-L, L] \subset \mathbb{R}$.

(b) Prove that the Fourier transform does not converge to x pointwise on the closed interval $[-L, L]$.

Solution: See attached. One note: the second part follows immediately from the observation that whatever the Fourier expansion converges to at $-L$ must be the same as what it converges to at L.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday January 22, 2015 (Day 3)

1. (DG) The helicoid is the parametrized surface given by

$$\phi : \mathbb{R}^2 \to \mathbb{R}^3 : (u, v) \to (v \cos u, v \sin u, au)$$

where a is a real constant. Compute its induced metric.

Solution. Compute $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ and deduce that the metric is $g = (v^2 + a^2) du \otimes du + dv \otimes dv$.

2. (RA) A real valued function defined on an interval $(a, b) \subset \mathbb{R}$ is said to be *convex* if

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$$

whenever $x, y \in (a, b)$ and $t \in (0, 1)$.

(a) Give an example of a non-constant, non-linear convex function.

(b) Prove that if f is a non-constant convex function on $(a, b) \in \mathbb{R}$, then the set of local minima of f is a connected set where f is constant.

Solution: See attached

3. (AG) Let K be an algebraically closed field of characteristic 0, and let $\mathbb{P}^n$ be the projective space of homogeneous polynomials of degree n in two variables over K. Let $X \subset \mathbb{P}^n$ be the locus of n^{th} powers of linear forms, and let $Y \subset \mathbb{P}^n$ be the locus of polynomials with a multiple root (that is, a repeated factor).

(a) Show that X and $Y \subset \mathbb{P}^n$ are closed subvarieties.

(b) What is the degree of X?

(c) What is the degree of Y?

Solution: First, X is the image of the map $\mathbb{P}^1 \to \mathbb{P}^n$ sending $[a, b] \in \mathbb{P}^1$ to $(ax + by)^n \in \mathbb{P}^n$. This is projectively equivalent (in characteristic 0!) to the degree n Veronese map, whose image is a closed curve of degree n. Second, Y is the zero locus of the discriminant, which is a polynomial of degree $2n - 2$ in the coefficients of a polynomial of degree n (this number can be deduced from the Riemann-Hurwitz formula, which says that a degree n map from $\mathbb{P}^1$ to $\mathbb{P}^1$ has $2n - 2$ branch points; that is, a general line in $\mathbb{P}^n$ meets Y in $2n - 2$ points).

4. (AT) Let X be a compact, connected and locally simply connected Hausdorff space, and let $p : \tilde{X} \to X$ be its universal covering space. Prove that $\tilde{X}$ is compact if and only if the fundamental group $\pi_1(X)$ is finite.

Solution: See attached

5. (CA) Prove that if f and g are entire holomorphic functions and $|f| \leq |g|$ everywhere, then $f = \alpha \cdot g$ for some complex number α.

Solution: The conclusion trivially holds in the case $g = 0$; from now on, assume that g is not the zero function. The identity theorem implies that the zeros of g are isolated, so $h := f/g$ is meromorphic. The function h is bounded by hypothesis, so Riemann's theorem implies that h can be extended to an entire bounded function. Liouville's theorem implies that h is constant, which implies the conclusion.

6. (A) Consider the rings

$$R = \mathbb{Z}[x]/(x^2 + 1) \quad \text{and} \quad S = \mathbb{Z}[x]/(x^2 + 5).$$

(a) Show that R is a principal ideal domain.

(b) Show that S is not a principal ideal domain, by exhibiting a non-principal ideal.

Solution: For the first, the fact that R is a principal ideal domain follows from the fact that it's a Euclidean domain, with size function $|z|^2$: for any $a, b \in R$ we can write

$$b = ma + r$$

with $|r| < |a|$; carrying this out repeatedly shows that the ideal generated by two elements of R can be generated by one. For the second, the ideal $(2, 1 + x) \subset S$ is not principal.

Qualifying Examination

HARVARD UNIVERSITY

Department of Mathematics

Tuesday, January 19, 2016 (Day 1)

PROBLEM 1 (DG)

Let S denote the surface in $\mathbb{R}^3$ where the coordinates (x, y, z) obey $x^2+y^2 = 1+z^2$. This surface can be parametrized by coordinates $t \in \mathbb{R}$ and $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ by the map

$$(t, \theta) \to \psi(t,\theta) = (\sqrt{1+t^2}\cos\theta, \sqrt{1+t^2}\sin\theta, t).$$

a) Compute the induced inner product on the tangent space to S using these coordinates.
b) Compute the Gaussian curvature of the metric that you computed in Part a).
c) Compute the parallel transport around the circle in S where $z = 0$ for the Levi-Civita connection of the metric that you computed in Part a).

PROBLEM 2 (T)

Let X be path-connected and locally path-connected, and let Y be a finite Cartesian product of circles. Show that if $\pi_1(X)$ is finite, then every continuous map from X to Y is null-homotopic. (Hint: recall that there is a fiber bundle $\mathbb{Z} \to \mathbb{R} \to S^1$.)

PROBLEM 3 (AN)

Let K be the field $\mathbb{C}(z)$ of rational functions in an indeterminate z, and let $F \subset K$ be the subfield $\mathbb{C}(u)$ where $u = (z^6 + 1)/z^3$.

a) Show that the field extension K/F is normal, and determine its Galois group.
b) Find all fields E, other than F and K themselves, such that $F \subset E \subset K$. For each E, determine whether the extensions E/F and K/E are normal.

PROBLEM 4 (AG)

The nodal cubic is the curve in $\mathbb{CP}^2$ (denoted by X) given in homogeneous coordinates (x, y, z) by the locus $\{zy^2 = x^2(x+z)\}$.

a) Give a definition of a rational map between algebraic varieties.

b) Show that there is a birational map from X to $\mathbb{CP}^1$.

c) Explain how to resolve the singularity of X by blowing up a point in $\mathbb{CP}^2$.

PROBLEM 5 (RA)

Let $\mathbb{B}$ and $\mathbb{L}$ denote Banach spaces, and let $\|\cdot\|_{\mathbb{B}}$ and $\|\cdot\|_{\mathbb{L}}$ denote their norms.

a) Let L: $\mathbb{B} \to \mathbb{L}$ denote a continuous, invertible linear map and let $\mathfrak{m}$: $\mathbb{B} \otimes \mathbb{B} \to \mathbb{L}$ denote a linear map such that $\|\mathfrak{m}(\phi \otimes \psi)\|_{\mathbb{L}} \leq \|\phi\|_{\mathbb{B}} \|\psi\|_{\mathbb{B}}$ for all $\phi, \psi \in \mathbb{B}$. Prove the following assertions:

- *There exists a number* $\kappa > 1$ *depending only on* L *such that if* $a \in \mathbb{B}$ *has norm less than* κ^{-2}, *then there is a unique solution to the equation* $L\phi + \mathfrak{m}(\phi \otimes \phi) = a$ *with* $\|\phi\|_{\mathbb{B}} < \kappa^{-1}$.
- *The norm of the solution from the previous bullet is at most* $\kappa \|a\|_{\mathbb{L}}$.

b) Recall that a Banach space is defined to be a *complete*, normed vector space. Is the assertion of Part a) of the first bullet always true if $\mathbb{B}$ is normed but not complete? If not, explain where the assumption that $\mathbb{B}$ is complete enters your proof of Part a).

PROBLEM 6 (CA)

Fix $a \in \mathbb{C}$ and an integer $n \geq 2$. Show that the equation $az^n + z + 1 = 0$ for a complex number z necessarily has a solution with $|z| \leq 2$.

PROBLEM 1 SOLUTION:

Answer to a): The vector fields $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial\theta}$ along S are

$$\frac{\partial}{\partial t} = \frac{t}{1+t^2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + \frac{\partial}{\partial z} \quad \textit{and} \quad \frac{\partial}{\partial\theta} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\ .$$

Since their inner product is $\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t}\rangle = \frac{t^2}{1+t^2} + 1$ $\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial\theta}\rangle = 0$ and $\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\rangle = (1+t^2)$, it follows that the square of the line element for the induced metric is

$$ds^2 = \frac{1+2t^2}{1+t^2}\, dt\otimes dt + (1+t^2)\, d\theta\otimes d\theta.$$

Answer to b): The 1-forms $e^0 = (\frac{1+2t^2}{1+t^2})^{1/2} dt$ and $e^1 = (1+t^2)^{1/2} d\theta$ are orthonormal. Write The connection matrix of 1-forms is $\mathbb{A} = \begin{pmatrix} 0 & \Gamma \\ -\Gamma & 0 \end{pmatrix}$ with the 1-form Γ obeying

$$de^0 = -\Gamma\wedge e^1 \quad \textit{and} \quad de^1 = \Gamma\wedge e^0.$$

The unique solution is $\Gamma = -\frac{t}{\sqrt{1+2t^2}}\, d\theta$. The Gauss curvature is denoted by κ and it is defined by writing $d\Gamma$ as $\kappa e^0\wedge e^1$. Thus, $\kappa = -(\frac{1}{1+2t^2})^2$.

Answer to c): Since $\Gamma = 0$ on the $z = 0$ circle, the parallel transport is given by the identity matrix when written using the orthonormal frame $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial\theta}\}$ for TS at (1,0,0).

PROBLEM 2 SOLUTION:

Here are two solutions:

Solution 1: Let Y denote the space $\times_n S^1$. It is enough to prove that the map from X to Y factors as a map

$$X \xrightarrow{f} \mathbb{R}^n \xrightarrow{(\exp)^{\times n}} Y.$$

To prove this factorization, note that a map f: X → Y lifts through a cover p: $\tilde{Y} \to Y$ if and only if $f_*(\pi_1(X))$ is a subgroup of $p_*(\pi_1(\tilde{Y}))$ (they are both subgroups of $\pi_1(Y)$). (See, for example Proposition 1.33 in Hatcher's book on algebraic topology.) Since $\pi_1(\mathbb{R}^n) = 0$

and f_* in this case must be the zero homomorphism, this condition is satisfied and so f lifts to some $\tilde{f}$. Because $\mathbb{R}^n$ is contractible, this lift is null-homotopic and any null-homotopy pushes forward to give a null-homotopy of f.

Solution 2: Recall that S^1 (which is $K(\mathbb{Z}, 1)$) classifies integral cohomology classes of degree 1. As a consequence, a map $X \to Y$ is (up to homotopy) determined by an n-tupel of elements in $H^1(X;\mathbb{Z})$. The universal coefficient short exact sequence in this degree is

$$0 \to \mathrm{Ext}(H_0X, \mathbb{Z}) \to H^1(X;\mathbb{Z}) \to \mathrm{Hom}(H_1X, \mathbb{Z}) \to 0.$$

The two end groups are zero: The right most group is zero because $H_1(X;\mathbb{Z})$ is the Abelianization of $\pi_1(X)$ and thus it is a finite group; and finite groups have no non-trivial homomorphisms to $\mathbb{Z}$. The left most group is zero because $H_0(X:\mathbb{Z}) = \mathbb{Z}$ and $\mathrm{Ext}(\mathbb{Z};\mathbb{Z})$ is trivial since $\mathbb{Z}$ is a free group. Thus $H^1(X;\mathbb{Z}) = 0$ and so all maps from X to Y are homotopic to the constant map.

PROBLEM 3 SOLUTION:

Answer to a) One has $[K:F] = 6$ because the extension K/F is generated by the solution z of the polynomial equation $z^6 - uz^3 + 1 = 0$ which has degree 6. The Galois group contains the automorphisms $\alpha : z \to 1/z$ and $\beta : z \to \varrho z$, where $\varrho = e^{i2\pi/3} = (-1 + \sqrt{-3})/2$. Since α and β have orders 2 and 3 respectively, the group G generated by α and β has order at least 6. However, $|\mathrm{Gal}(K/F)| \le [K:F] = 6$ with equality iff K/F is normal, so K/F must be normal with Galois group G of order 6, which is readily identified with the symmetric group the symmetric group S_3 (for instance, via its permutation action on the set $\{1, \varrho, \varrho^2\}$).

Answer to b) By the fundamental theorem of Galois theory, the intermediate fields E of the Galois extension K/F correspond to subgroups $H \subset G$ by $E = K^H$ (fixed subfield); K/E is always normal with $\mathrm{Gal}(K/E) = H$, while E/F is normal iff $H \trianglelefteq G$. Since F and K are excluded, one need not consider $H = G$ and $H = \{1\}$. The remaining subgroups are $A_3 = \langle\beta\rangle$, which yields the normal extension $\mathbb{C}(z^3)$ of F, and three two-element subgroups which yield non-normal extensions $\mathbb{C}(z + 1/z)$, $\mathbb{C}(z + \varrho z)$, $\mathbb{C}(z + \varrho^2 z)$. (The fact that each of these is indeed the corresponding KE can be confirmed by computing its degree as in Part a).)

PROBLEM 4 SOLUTION:

Answer to a) A rational map from X to Y is an equivalence class of pairs (U, f) where $U \subset X$ is a Zariski dense open subset and $f : U \to Y$ is a regular map. Two pairs (U, f) and (V, g) are equivalent if $f = g$ on the intersection $U \cap V$.

Answer to b) The projection from the point $(0,0,1) \in \mathbb{CP}^2$ to the line where $z=0$ restricts to a rational map $\mathfrak{p}: X = \{zy^2 = x^2(x + z)\} \to \mathbb{CP}^1$. An inverse is given by the map given in homogeneous coordinates by the rule $(u, v) \to (x = (v^2-u^2)u, y = (v^2-u^2)v, z = u^3)$. This is an inverse since $x^3 = (y^2 - x^2)z$ on X. It follows that $\mathfrak{p}$ is a birational map.

Answer to c) Away from the line $z = 0$ the blowup of $\mathbb{CP}^2$ at $(0,0,1)$ is given by the locus $\{xt=ys\} \subset \{((x, y),(s, t))\}=\mathbb{C}^2\times\mathbb{CP}^1$. Consider the chart in $\mathbb{CP}^1$ where $s \neq 0$. The blow up of X is defined here by the equations $xt = y$ and $y^2 = x^2(x + 1)$. Substituting for y gives the equation $x^2(t^2 - x - 1) = 0$ which has one irreducible component being the locus $x = y = 0$ (which is the exceptional curve), and the other being the locus where both $t^2 = x+1$ and $xt = y$. This is the blow-up of X. In the chart where $t \neq 0$, the blow up of X is defined by the locus where $x = ys$ and $1 = s^2(sy + 1)$. By the Jacobian criterion the curve defined by these equations is nonsingular.

PROBLEM 5 SOLUTION:

Answer to a) Since L is invertible, its inverse defines a bounded linear map from $\mathbb{L}$ to $\mathbb{B}$ to be denoted by L^{-1}. Using L^{-1}, one can define a map $\mathcal{T}: \mathbb{B} \to \mathbb{B}$ by the rule

$$\mathcal{T}(\phi) = L^{-1}(a - \mathfrak{m}(\phi,\phi)).$$

This is relevant because ϕ is a fixed point of $\mathcal{T}$ (it obeys $\mathcal{T}(\phi) = \phi$) if and only if ϕ obeys the equation $L\phi + \mathfrak{m}(\phi\otimes\phi) = a$. Let c denote the norm of the operator L^{-1}. Then the following are computations:

- $\|\mathcal{T}(\phi)\|_{\mathbb{B}} \le c(\|a\|_{\mathbb{L}} + \|\phi\|_{\mathbb{B}}{}^2)$.
- $\|\mathcal{T}(\phi) - \mathcal{T}(\phi')\| \le 4c(\|\phi\|_{\mathbb{B}} + \|\phi'\|_{\mathbb{B}})\,\|\phi - \phi'\|_{\mathbb{B}}$.

Given $\delta > 0$, let $\mathbb{B}(\delta)$ denote the ball of radius δ about the origin in $\mathbb{B}$. If $E > 0$ and if $\|a\|_{\mathbb{L}} \le E$ then the top bullet implies that $\mathcal{T}$ maps $\mathbb{B}(\delta)$ to $\mathbb{B}(cE + c\delta^2)$. Thus, if $\delta < (2c)^{-1}$ and if $E < (2c)^{-1}\delta$, then $\mathcal{T}$ maps $\mathbb{B}(\delta)$ to itself. Meanwhile, if $\delta < (8c)^{-1}$ then the lower bullet implies that $\|\mathcal{T}(\phi) - \mathcal{T}(\phi')\| \le \gamma\,\|\phi - \phi'\|_{\mathbb{B}}$ for fixed $\gamma < 1$ when $\phi, \phi' \in \mathbb{B}(\delta)$. This implies in turn that $\mathcal{T}$ is a contraction mapping of $\mathbb{B}(\delta)$ to itself. The contraction mapping theorem supplies a unique fixed point of $\mathcal{T}$ in $\mathbb{B}(\delta)$ under these circumstances. Noting again that an element $\phi \in \mathbb{B}$ is a fixed point of $\mathcal{T}$ if and only if ϕ obeys $L\phi + \mathfrak{m}(\phi\otimes\phi) = a$, the top bullet follows if $\| a \|_{\mathbb{L}} \le (16c)^{-1}$. Take κ to be the maximum of $4c^{1/2}$ and $8c$ to obtaine the answer to the first bullet of Part a). The second bullet of Part a) follows directly from the fact that $\phi = \mathcal{T}(\phi)$ and $\|\phi\|_{\mathbb{B}}^2 \le \frac{1}{2}\|\phi\|_{\mathbb{B}}$ because these and the inequality $\|\mathcal{T}(\phi)\|_{\mathbb{B}} \le c(\|a\|_{\mathbb{L}} + \|\phi\|_{\mathbb{B}}^2)$ imply that $\frac{1}{2}\|\phi\|_{\mathbb{B}} \le c\|a\|_{\mathbb{L}}$.

<u>Answer</u> <u>to</u> <u>b)</u> The completeness of B is required. Here is an example: Take $\mathbb{B}$ and $\mathbb{L}$ to be the span of the polynomials functions on [-1, 1] with the norms $\|f\|_{\mathbb{B}} = \|f\|_{\mathbb{L}} = \sup_t |f(t)|$. Take the equation $\phi + \phi^2 = \delta t$ with δ being a small, non-zero number. A solution, must be either $\phi = -\frac{1}{2} + \frac{1}{2}(1 + 4\delta^2 t^2)^{1/2}$ or $\phi = -\frac{1}{2} - \frac{1}{2}(1 + 4\delta^2 t^2)^{1/2}$; but neither is in $\mathbb{B}$. Note that the contraction mapping theorem does not hold if the Banach space in question is not complete because the contraction mapping theorem constructs the desired solution as a limit of a Cauchy sequence in $\mathbb{B}$.

PROBLEM 6 SOLUTION:

There are two cases. First, assume that $|a| < 2^{-n}$. Let D denote the disk where $|z| \le 2$ and let ∂D denote the circle $|z| = 2$. Let $f(z) = az^n + z + 1$ and let $g(z) = z + 1$. On ∂D, the function g - f obeys the inequality $|g(z) - f(z)| = |a|\,|z|^n < 1$. Since this is less than $|g(z)|$ for each $z \in \partial D$, and since g has no zeros on ∂D, none of the members of the 1-parameter family of functions $\{f_\tau = f + \tau(g - f)\}_{\tau\in[0,1]}$ has a zero on ∂D. Therefore, f (which is $f_{\tau=0}$) and g (which is $f_{\tau=1}$) have the same number of zeros (counting multiplicity) in D. This number is 1 (This is Rouche's theorem). Now assume that $|a| \ge 2^{-n}$. By the fundamental theorem of algebra, the function $f(z) = az^n + z + 1$ factors as

$$f(z) = a\prod_{k=1}^{n}(z - \alpha_k)$$

where the $\{\alpha_k\}_{k=1,\ldots,n}$ are complex numbers. This implies in particular the identity

$$(-1)^n a\prod_{k=1}^{n}\alpha_k = 1.$$

hence $\prod_{k=1}^{n}|\alpha_k| \le 2^n$. This can happen only if one or more roots α_k are in D.

Qualifying Examination

HARVARD UNIVERSITY

Department of Mathematics

Wednesday, January 20, 2016 (Day 2)

PROBLEM 1 (DG)

Let k denote a positive integer. A non-optimal version of the Whitney embedding theorem states that any k-dimensional manifold can be embedded into $\mathbb{R}^{2k+1}$. Using this, show that any k-dimensional manifold can be immersed in $\mathbb{R}^{2k}$. (Hint: Compose the embedding with a projection onto an appropriate subspace.)

PROBLEM 2 (T)

Let X be a CW-complex with a single cell in each of the dimensions 0, 1, 2, 3, and 5 and no other cells.

a) What are the possible values of $H_*(X; \mathbb{Z})$? (Note: it is not sufficient to consider $H_n(X;\mathbb{Z})$ for each n independently. The value of $H_1(X;\mathbb{Z})$ may constrain the value of $H_2(X;\mathbb{Z})$, for instance.)

b) Now suppose in addition that X is its own universal cover. What extra information does this provide about $H_*(X;\mathbb{Z})$?

PROBLEM 3 (AN)

Let k be a finite field of characteristic p, and n a positive integer. Let G be the group of invertible linear transformations of the k-vector space k^n. Identify G with the group of invertible $n \times n$ matrices with entries in k (acting from the left on column vectors).

a) Prove that the order of G is $\prod_{m=0}^{n-1}(q^n - q^m)$ where q is the number of elements of k.

b) Let U be the subgroup of G consisting of upper-triangular matrices with all diagonal entries equal 1. Prove that U is a p-Sylow subgroup of G.

c) Suppose $H \subset G$ is a subgroup whose order is a power of p. Prove that there is a basis $(v_1, v_2, \ldots, v_n)$ of k^n such that for every $h \in H$ and every $m \in \{1, 2, 3, \ldots, n\}$, the vector $h(v_m) - v_m$ is in the span of $\{v_d : d < m\}$.

PROBLEM 4 (AG)

Let X be a complete intersection of surfaces of degrees a and b in $\mathbb{CP}^3$. Compute the Hilbert polynomial of X.

PROBLEM 5 (RA)

Let $\mathcal{C}^0$ denote the vector space of continuous functions on the interval $[0,1]$. Define a norm on $\mathcal{C}^0$ as follows: If $f \in \mathcal{C}^0$, then its norm (denoted by $\|f\|$) is

$$\|f\| = \sup_{t\in[0,1]}|f(t)| \ .$$

Let $\mathcal{C}^\infty$ denote the space of smooth functions on [0, 1]. View $\mathcal{C}^\infty$ as a normed, linear space with the norm defined as follows: If $f \in \mathcal{C}^\infty$, then its norm (denoted by $\|f\|_*$) is

$$\|f\|_* = \int_{[0,1]} \left(\left|\frac{d}{dt}f\right| + |f|\right) dt \ .$$

a) Prove that $\mathcal{C}^0$ is Banach space with respect to the norm $\|\cdot\|$. In particular, prove that it is complete.
b) Let ψ denote the 'forgetful' map from $\mathcal{C}^\infty$ to $\mathcal{C}^0$ that sends f to f. Prove that ψ is a bounded map from $\mathcal{C}^\infty$ to $\mathcal{C}^0$, but not a compact map from $\mathcal{C}^\infty$ to $\mathcal{C}^0$.

PROBLEM 6 (CA)

Let $\mathbb{D}$ denote the closed disk in $\mathbb{C}$ where $|z| \le 1$. Fix $R > 0$ and let $\varphi\colon \mathbb{D} \to \mathbb{C}$ denote a continuous map with the following properties:

i) φ is holomorphic on the interior of $\mathbb{D}$.
ii) $\varphi(0) = 0$ and its z-derivative, φ', obeys $\varphi'(0) = 1$.
iii) $|\varphi| \le R$ for all $z \in \mathbb{D}$.

Since $\varphi'(0) = 1$, there exists $\delta > 0$ such that φ maps the $|z| < \delta$ disk diffeomorphically onto its image. Prove the following:

a) There is a unique solution in [0, 1] to the equation $2R\delta = (1 - \delta)^3$.
b) Let δ_* denote the unique solution to this equation. If If $0 < \delta < \delta_*$, then φ maps the $|z| < \delta$ disk diffeomorphically onto its image.

PROBLEM 1 SOLUTION:

The desired immersion will come from a projection onto the orthogonal complement of a suitably chosen, nonzero vector in $\mathbb{R}^{2k}$. To find this vector, let M denote the manifold in question and let f denote the embedding of M into $\mathbb{R}^{2k}$. Let g denote the map from TM to $\mathbb{R}^{2k+1}$ that is defined as follows: Supposing that $x \in M$ and $v \in TM|_x$ set $g(x, v) = f_*|_x \cdot v$ where f_* denotes the differential of f. Sard's theorem can be invoked to see that g is not surjective. Let a denote a point that is not in the image of g. (Note that a is necessarily nonzero.) Use π to denote the projection onto the orthogonal complement of a. To see that $\pi \circ f$ is an immersion, let x denote a point in M and let v denote a nonzero vector in $TM|_x$. Suppose for the sake of argument that $(\pi \circ f)_* v$ is zero. If this is so, then the chain rule and the fact that π is linear implies that $f_*|_x \cdot v = ta$ for some nonzero $t \in \mathbb{R}$. This implies in turn that $f_*|_x(t^{-1}v) = a$ which is nonsense because a is in the complement of the image of f_*.

PROBLEM 2 SOLUTION:

<u>Answer</u> <u>to</u> <u>a)</u> The cellular chain complex for X must be of the form

$$\begin{array}{ccccccccccccccc}
0 & \to & C_5X & \to & C_4X & \to & C_3X & \to & C_2X & \to & C_1X & \to & C_0X & \to & 0 \\
\| & & \| & & \| & & \| & & \| & & \| & & \| & & \| \\
0 & \to & \mathbb{Z} & \to & 0 & \to & \mathbb{Z} & \xrightarrow{a} & \mathbb{Z} & \xrightarrow{b} & \mathbb{Z} & \xrightarrow{c} & \mathbb{Z} & \to & 0.
\end{array}$$

Since X is connected, it must have $H_0(X; \mathbb{Z}) = \mathbb{Z}$, so the map c must be zero. The only other restriction is that the sequence form a complex, so $b \circ a = 0$; but since $b \circ a$ is multiplication by some integer, either $a = 0$ or $b = 0$. In the case $a = 0$ and $b \neq 0$, the homology groups take the form

$$\begin{array}{cccccc}
H_5X & H_4X & H_3X & H_2X & H_1X & H_0X \\
\| & \| & \| & \| & \| & \| \\
\mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z}/b & \mathbb{Z}.
\end{array}$$

$$\begin{array}{cccccc}
H_5X & H_4X & H_3X & H_2X & H_1X & H_0X \\
\| & \| & \| & \| & \| & \| \\
\mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z}/b & \mathbb{Z}.
\end{array}$$

In the case $a \neq 0$ and $b = 0$, the homology groups take the form

$$\begin{array}{cccccc} H_5X & H_4X & H_3X & H_2X & H_1X & H_0X \\ \| & \| & \| & \| & \| & \| \\ \mathbb{Z} & 0 & 0 & \mathbb{Z}/a & \mathbb{Z} & \mathbb{Z}. \end{array}$$

In the remaing $a = 0 = b$ case, they take the form

$$\begin{array}{cccccc} H_5X & H_4X & H_3X & H_2X & H_1X & H_0X \\ \| & \| & \| & \| & \| & \| \\ \mathbb{Z} & 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}. \end{array}$$

Answer to b) The assertion that X is its own universal cover is the same as the assertion $\pi_1(X) = 0$. But, since $H_1(X) = \pi_1(X)^{ab}$, this means $H_1(X) = 0$. The only case where this is possible is when $a = 0$ and $b \neq 0$. Moreover, since $\mathbb{Z}/b = 0$ in this case, b must be a multiplicative unit: $b = \pm 1$.

PROBLEM 3 SOLUTION:

Answer to a) The elements of G are in bijection with ordered bases $(v_1,\dots,v_n)$ of k^n (the map takes each matrix to its columns). For each $j \in \{0, 1, 2,\dots, n-1\}$, once v_i for all $i \leq j$ has been chosen, then there are $q^n - q^j$ choices for the index $(j + 1)$ basis element because any of the q^n elements of k^n except the q^m linear combinations of $v_1, \dots, v_j$ will do. Hence the number of possible bases is $\prod_{m=0}^{n-1} (q^n - q^m)$.

Answer to b) Each factor $q^n - q^j$ is q^j times an integer not divisible by p because it is congruent to -1 modulo q, and q is a multiple of p. Hence the number of elements in G is qd times some integer not divisible by p, where $d = \sum_{j=0}^{n-1} j$. But q^d is the order of U because there are d entries above the diagonal, and a power of p. Hence U is a p-Sylow subgroup of G.

Answer to c) U consists of the matrices h that satisfy the desired property with respect to the standard basis of unit vectors. Hence the matrices h that satisfy this property for the basis $(v_1,\dots,v_n)$ constitute the subgroup of G obtained by conjugating U by the matrix with columns $v_1, \dots, v_n$. But by Sylow's second theorem H is contained in a conjugate of U.

PROBLEM 4 SOLUTION:

Let $S = \mathbb{C}[x_0, x_1, x_2, x_3]$ be the homogeneous coordinate ring of $\mathbb{CP}^3$. The coordinate ring of X is of the form S/(f, g) for some irreducible polynomials f and g of degrees a, b respectively. There is a four-term exact sequence of graded modules

$$0 \to S(-a-b) \to S(-a) \otimes S(-b) \to S \to S/(f,g) \to 0$$

with maps given by multiplication with f and g. Hence the Hilbert polynomial of X is

$$\begin{aligned} P_X(z) &= \binom{z+3}{3} - \binom{z+3-a}{3} - \binom{z+3-b}{3} + \binom{z+3-a-b}{3} \\ &= ab\left(z + \frac{4-a-b}{2}\right) \end{aligned}$$

PROBLEM 5 SOLUTION:

Answer to a) One has to show that a Cauchy sequence $\{f_n\}_{n=1,2,\ldots}$ in $\mathcal{C}^0$ converges to a continuous function. To do this, note that for each $t \in [0, 1]$, the sequence $\{f_n(t)\}_{n\in\{1,2,\ldots\}}$ is a Cauchy sequence in $\mathbb{R}$ so it converges. Let $f(t)$ denote the limit. The assignment $t \to f(t)$ defines a function on [0, 1]. The task is to prove that this function is continuous. This means the following: Given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(t) - f(t')| < \varepsilon$ when $|t - t'| < \delta$. To find δ, first fix N so that $|f_n(t) - f_m(t)| < \frac{1}{3}\varepsilon$ for all $t \in [0, 1]$ and all pairs n, m > N. This implies that $|f_n(t) - f(t)| \leq \frac{1}{3}\varepsilon$ for all t. Such N exists because $\{f_n\}_{n\in\{1,2,\ldots\}}$ is a Cauchy sequence in $\mathcal{C}^0$. To continue, take n > N and fix δ so that $|f_n(t) - f_n(t')| < \frac{1}{3}\varepsilon$ when $|t' - t| < \delta$. It then follows by the triangle inequality that

$$|f(t) - f(t')| \leq |f(t) - f_n(t)| + |f(t') - f_n(t')| + |f_n(t') - f_n(t)| < \varepsilon.$$

Answer to b) The map ψ is bounded because for all t, one has the identity

$$f(t) = \int_0^1 \Big(\int_r^t \tfrac{d}{ds} f(s)ds + f(r)\Big)\, dr \ ,$$

and thus $|f(t)| \leq \| f \|_*$ for all t. It is not a compact map. To prove this, fix a smooth function on $[0, \infty)$ that is equal to 1 near t = 0 and equal to 0 for $t > \frac{1}{2}$. Call this function

f. Define $f_n(t) = f(nt)$. This function is smooth on $[0, 1]$. The sequence $\{f_n(t)\}$ has bounded $\|\cdot\|_*$ norm but it has no convergent sequence in $\mathcal{C}^0$.

PROBLEM 6 SOLUTION:

Answer to a) The function $f(\delta) = 2R\delta/(1-\delta)^3$ has strictly positive derivative and therefore defines a diffeomorphism from $[0, 1)$ to $[0, \infty)$. It follows from this that there is a single point where f is equal to 1.

Answer to b) To obtain the asserted lower bound for δ, note that φ maps the disk where $|z| < \delta$ diffeomorphically to its image if it is 1-1 on this disk and if $|\varphi'| > 0$ on this disk. The Cauchy integral formula is used to see when this happens. Here is Cauchy's formula:

$$\varphi(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{1}{z - w}\, \varphi(w)dw\ .$$

Differentiating this, one sees that $|\varphi''|$ on the $|z| < \delta$ disk is bounded by $2R(1-\delta)^{-3}$. This implies that

$$|\varphi' - 1| < 2R\,\delta\,(1-\delta)^{-3} \quad \textit{where}\ |z| < \delta.$$

If $\varphi' > 0$, then φ is a local diffeomorphism. This is the case when $\delta < \delta_*$ with δ_* being the solution in $(0, 1)$ to the equation $2R\delta_*(1-\delta_*)^{-3} = 1$. Meanwhile, if z, z' have norm less than δ, then $|\varphi(z) - \varphi(z')| \geq (1 - 2R\,\delta\,(1-\delta)^{-3})\,|z - z'|$ which is a positive multiple of $|z - z'|$ precisely when $\delta < \delta_*$.

Qualifying Examination
HARVARD UNIVERSITY
Department of Mathematics
Thursday, January 21, 2016 (Day 3)

PROBLEM 1 (DG)
Recall that a symplectic manifold is a pair (M, ω), where M is a smooth manifold and ω is a closed nondegenerate differential 2-form on M. (The 2-form ω is called the symplectic form.)

a) Show that if $H: M \to R$ is a smooth function, then there exists a unique vector field, to be denoted by X_H, satisfying $\iota_{X_H}\omega = dH$. (Here, ι denotes the contraction operation.)
b) Supposing that $t > 0$ is given, suppose in what follows that the flow of X_H is defined for time t, and let ϕ_t denote the resulting diffeomorphism of M. Show that $\phi_t{}^*\omega = \omega$.
c) Denote the Euclidean coordinates on $\mathbb{R}^4$ by (x_1, y_1, x_2, y_2) and use these to define the symplectic form $\omega_0 = dx^1 \wedge dy^1 + dx^2 \wedge dy^2$. Find a function $H: \mathbb{R}^4 \to \mathbb{R}$ such that the diffeomorphism $\phi_{t=1}$ that is defined by the time $t = 1$ flow of X_H fixes the half space where $x_1 \leq 0$ and moves each point in the half space where $x_1 \geq 1$ by 1 in the y_2 direction.

PROBLEM 2 (T)
Let X denote a finite CW complex and let $f: X \to X$ be a self-map of X. Recall that the Lefschetz trace of f, denoted by $\tau(f)$, is defined by the rule

$$\tau(f) = \sum_{n=0}^{\infty} (-1)^n \operatorname{tr}(f_n : H_n(X;\mathbb{Q}) \to H_n(X;\mathbb{Q}))$$

with f_n denoting the induced homomorphism. Use $\tau(\cdot)$ to answer the following:

a) Does there exist a continuous map from $\mathbb{RP}^2$ to itself with no fixed points? If so, give an example; and if not, give a proof.
b) Does there exist a continuous map from $\mathbb{RP}^3$ to itself with no fixed points? If so, give an example; and if not, give a proof.

PROBLEM 3 (AN)
Let A be the ring $\mathbb{Z}[\sqrt[5]{2016}\,] = \mathbb{Z}[X]/(X^5 - 2016)$. Given that 2017 is prime in $\mathbb{Z}$, determine the factorization of $2017 \cdot A$ into prime ideals of A.

PROBLEM 4 (AG)

a) State a version of the Riemann–Roch theorem.

b) Apply this theorem to show that if X is a complete nonsingular curve and $P \in X$ is any point, there is a rational function on X which has a pole at P and is regular on $X-\{P\}$.

PROBLEM 5 (RA)

Let $\wp$ denote a probability measure for a real valued random variable with mean 0. Denote this random variable by x. Suppose that the random variable $|x|$ has mean equal to 2.

a) Given $R > 2$, state a non-trivial upper bound for event that $x \geq R$. (The trivial upper bound is 1.)

b) Give a non-zero lower bound for the standard deviation of x.

c) A function f on $\mathbb{R}$ is Lipshitz when there exists a number $c \geq 0$ such that

$$|f(p) - f(p')| \leq c|p - p'| \quad \text{for any pair } p, p' \in \mathbb{R}.$$

Let $\hat{\wp}$ denote the function on $\mathbb{R}$ whose value at a given $p \in \mathbb{R}$ is the expectation of the random variable e^{ipx}. (This is the *characteristic function of* $\wp$.) Give a rigorous proof that $\hat{\wp}$ is Lipshitz and give an upper bound for c in this case.

d) Suppose that the standard deviation of x is equal to 4. Let N denote an integer greater than 1, and let $\{x_1, \ldots, x_N\}$ denote a set of independent random variables each with probabilities given by $\wp$. Use S_N to denote the random variable $\frac{1}{N}(x_1 + \cdots + x_N)$. The central limit theorem gives an integral that approximates the probability of the event where $S_N \in [-1,1]$ when N is large. Write this integral.

PROBLEM 6 (CA)

Let $H \subset \mathbb{C}$ denote the open right half plane, thus $H = \{z = x + iy\colon x > 0\}$. Suppose that $f\colon H \to \mathbb{C}$ is a bounded, analytic function such that $f(1/n) = 0$ for each positive integer n. Prove that $f(z) = 0$ for all z.

(Hint: Consider the behavior of the sequence of functions $\{h_N(z) = \prod_{n=1}^{N} \frac{z - 1/n}{z + 1/n}\}_{N=1,2\ldots}$ on H and, in particular, on the positive real axis.}

PROBLEM 1 SOLUTION:

Answer to a) To say that ω is non-degenerate is to say that the contraction operation defines a vector bundle isomorphism between TM and T*M.

Answer to b) The definition of the Lie derivative is such that $\frac{\partial}{\partial t}(\phi_t^*\omega) = \phi_t^*(\mathfrak{L}_{X_H}\omega)$ with $\mathfrak{L}_{X_H}\omega$ denoting the Lie derivative of ω along the vector field X_H. Cartan's formula for $\mathfrak{L}_{X_H}\omega$ is $\mathfrak{L}_{X_H}\omega = d(\iota_{X_H}\omega) + \iota_{X_H}d\omega$ and both of these terms are zero. Thus, $\phi_t^*\omega$ is independent of t and thus equal to its value at t = 0 which is ω.

Answer to c) Choose a smooth function $f\colon \mathbb{R} \to [0, 1]$ so that $f(s) = 0$ for $s \le 0$ and $f(s) = 1$ for $s \ge 1$. The function sending $(x_1, y_1, x_2, y_2) \to H(x_1, y_1, x_2, y_2) = -f(x_1)\,x_2$ has the desired properties because $X_H = 0$ for $x_1 \le 0$ and $X_H = \frac{\partial}{\partial y_2}$ for $x_1 \ge 1$.

PROBLEM 2 SOLUTION:

Answer to a) The Lefschetz trace theorem states that if $\tau(f) \ne 0$, then f must have a fixed point. To see that $\tau(f)$ is never zero, note first that the rational homology of $\mathbb{RP}^2$ is zero except for $H_0(\mathbb{RP}^2;\mathbb{Q})$, which is $\mathbb{Q}$. Since f_0 is multiplication by 1, it $\tau(f)$ is never zero.

Answer to b) In this case, the non-zero rational homology is in dimensions 0 and 3, each being isomorphic to $\mathbb{Q}$. As a consequence, the argument used for $\mathbb{RP}^2$ can not be used here. In fact, there is a self-map with no fixed points and it is constructed momentarily. It is instructive to consider first the case of $\mathbb{RP}^1$ which is S^1, where a rotation by angle π has no fixed points. Now viewing $\mathbb{RP}^1$ as $(\mathbb{R}^2–0)/\mathbb{R}^*$, then this rotation through angle π is depicted using homogeneous coordinates $[x_1, x_2]$ as the map $[x_1, x_2] \to [x_2, -x_1]$ which can't have a fixed point because there is no non-zero real number λ and $(x_1, x_2) \in \mathbb{R}^2–0$ with $x_2 = \lambda x_1$ and $x_1 = -\lambda x_2$. To mimick this for $\mathbb{RP}^3$, write $\mathbb{RP}^3$ as $(\mathbb{R}^4–0)/\mathbb{R}^*$ and then define the desired self map using homogeneous coordinates $[x_1, x_2, x_3, x_4]$ by the rule whereby $[x_1, x_2, x_3, x_4] \to [x_2, -x_1, x_4, -x_3]$. This has no fixed points because there is no non-zero real number λ and $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4–0$ such that $x_2 = \lambda x_1$, $x_1 = -\lambda x_2$, $x_4 = \lambda x_3$ and $x_3 = -\lambda x_4$.

PROBLEM 3 SOLUTION:

2017A is the product of the prime ideals (2017, X + 1) and (2017, $X^4 - X^3 + X^2 - X + 1$). In general, if the polynomial P(X) factors modulo a prime p into distinct irreducibles $\{P_i\}$ then the ideal pZ[X]/(P(X)) is the product of ideals (p, P_i). In our case, p = 2017 and $P = X^5 - 2016 \equiv X^5 + 1$ mod p. The roots of $X^5 + 1$ in an algebraic closure of Z/pZ are the set $\{-1, -w, -w^2, -w^3, -w^4\}$ where w is a nontrivial 5th root of unity. The irreducible factors correspond to orbits of the permutation $x \to x^p$ of those roots. Clearly -1 is a fixed point, and since $p \equiv 2$ mod 5 the remaining roots fall in to a single orbit

$$-w \to -w^2 \to -w^4 \to -w^3 \to -w.$$

Hence the irreducible factors of $X^5 +1$ mod p are X + 1 and $(X^5 +1)/(X +1)$ which is the polynomial $X^4 - X^3 + X^2 - X + 1$.

PROBLEM 4 SOLUTION:

<u>Answer</u> <u>to</u> <u>a)</u> Let X be a complete non-singular curve of genus g. Let K denote the canonical divisor. If D is any divisor on X, let $\ell(D) = \dim(H_0(X, \mathcal{O}_X(D)))$. The Riemann-Roch theorem asserts that $\ell(D) - \ell(K-D) = \deg(D) + 1 - g$.

<u>Answer</u> <u>to</u> <u>b)</u> Fix a point $Q \neq P$ and let D denote the divisor $2P - Q$. Choose a positive integer n such that $n > \max\{2g - 2, 0\}$. Noting that $n = \deg(nD)$ and that $\deg(K) = 2g-2$, it follows that $\deg(K - nD) < 0$. This implies that $\ell(K-D) = 0$. Therefore, the Riemann–Roch theorem applied to nD implies that $\ell(nD) = n + 1 - g$ which is greater than 1. This means that there is an effective divisor (to be denoted by D´) and a rational function on X (to be denoted by f) such that $nD + (f) = D'$. Rewriting this gives $(f) = D' - 2nP + nQ$ so f has poles only at P.

PROBLEM 5 SOLUTION:

<u>Answer</u> <u>to</u> <u>a)</u> The event in question is $\int_{x\geq R} \wp$. This is no smaller than $\frac{1}{R} \int_{x\geq R} |x| \wp$ which in turn is no greater than $\frac{2}{R}$.

<u>Answer</u> <u>to</u> <u>b)</u> The square of the standard deviation is the square root of the expectation of the random variable x^2. Since

$$\int_{\mathbb{R}} |x| \wp \leq \left(\int_{\mathbb{R}} \wp\right)^{1/2} \left(\int_{\mathbb{R}} x^2 \wp\right)^{1/2} \qquad (*)$$

(which is proved momentarily), and since $\int_{\mathbb{R}} \wp = 1$, it follows that $(\int_{\mathbb{R}} x^2 \wp)^{1/2} \geq 2$. To prove (∗), note that for any $t \in (0, \infty)$, the expectation of $(t - t^{-1} x)^2$ is the sum

$$t^2 \int_{\mathbb{R}} \wp - 2 \int_{\mathbb{R}} |x| \wp + t^{-2} \int_{\mathbb{R}} x^2 \wp .$$

This is non-negative for any $t \in (0, 1)$ since it is the expectation of a positive random variable. The assertion that it is non-negative for the case $t = (\int_{\mathbb{R}} x^2 \wp)^{1/4} (\int_{\mathbb{R}} \wp)^{-1/4}$ is (∗).

<u>Answer</u> <u>to</u> <u>c)</u> Supposing that $p, p' \in \mathbb{R}$, then

$$\hat{\wp}(p) - \hat{\wp}(p') = \int_{\mathbb{R}} (e^{ixp} - e^{ixp'}) \wp . \qquad (**)$$

Noting that $e^{ixp} - e^{ixp'} = i\,x \int_{p}^{p'} e^{ixq}\, dq$ by the fundamental theorem of calculus, it follows that $|e^{ixp} - e^{ixp'}| \leq |x|\,|p - p'|$. This understood, then (∗∗) leads to the bound

$$|\hat{\wp}(p) - \hat{\wp}(p')| \leq \left(\int_{\mathbb{R}} |x| \wp\right) |p - p'| = 2\,|p - p'| .$$

<u>Answer</u> <u>to</u> <u>d)</u> The random variable S_N has mean 0 and standard deviation equal to $N^{-1/2}$ times the standard deviation of x, thus $4N^{-1/2}$. (The expecation of S_N^2 is the that of $N^{-2}\sum_{i,k=1,\ldots,N} x_i x_k$. Only the $i = k$ terms are non-zero (because x has mean zero), there are N of them and each is the expectation of x^2 which is 16.) Denote this standard deviation of S_N by σ_N for the moment. The central limit theorem approximates the probability in question by $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}\,\sigma_N} e^{-x^2/2\sigma_N^2} dx$ where σ_N again denotes $4N^{-1/2}$.

PROBLEM 6 SOLUTION

This is a form of Jensen's inequality. To elaborate, fix B so that $|f(z)| \leq B$ for all $z \in H$. For each integer N, define

$$F_N(z) = f(z)/h_N(z) = f(z)\prod_{n=1}^{N}\frac{z + 1/n}{z - 1/n} \ .$$

This function is analytic on H because the poles at z = 1, 2, 3,..., N are matched by zeros of f. Moreover, the absolute value of each of the factors $(z + 1/n)/(z - 1/n)$ approaches 1 as $\mathrm{Re}(z) \to 0$ (uniformly in Im(z)), and also approaches 1 as $|z| \to \infty$. Hence $|F_n(z)| \leq B$ for all $z \in H$ by virtue of the maximum modulus principle (the norm of an analytic function can not take on a local maximum). With the preceding understood, note that for any fixed, positive real z, the factor $\prod_{n=1}^{N}\frac{z + 1/n}{z - 1/n}$ becomes unbounded as $N \to \infty$. Hence its product with $f(z)$ cannot remain bounded unless $f(z) = 0$ on the real axis. But a holomorphic function on any domain has discrete zeros, so $f(z)$ must be everywhere 0.