

Title

D. Zack Garza

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1 Modules

1.1 General Questions

1.1.1 Fall 2019 Final #2

Consider the \mathbb{Z} -submodule N of \mathbb{Z}^3 spanned by $f_1 = [-1, 0, 1]$, $f_2 = [2, -3, 1]$, $f_3 = [0, 3, 1]$, $f_4 = [3, 1, 5]$. Find a basis for N and describe \mathbb{Z}^3/N .

1.1.2 Spring 2018 #6.

Let

$$M = \{(w, x, y, z) \in \mathbb{Z}^4 \mid w + x + y + z \in 2\mathbb{Z}\},$$

and

$$N = \{(w, x, y, z) \in \mathbb{Z}^4 \mid 4 \mid (w - x), 4 \mid (x - y), 4 \mid (y - z)\}.$$

- Show that N is a \mathbb{Z} -submodule of M .
- Find vectors $u_1, u_2, u_3, u_4 \in \mathbb{Z}^4$ and integers d_1, d_2, d_3, d_4 such that

$$\{u_1, u_2, u_3, u_4\}$$

is a free basis for M , and

$$\{d_1 u_1, d_2 u_2, d_3 u_3, d_4 u_4\}$$

is a free basis for N .

- Use the previous part to describe M/N as a direct sum of cyclic \mathbb{Z} -modules.

1.1.3 Fall 2018 #6 ∞

Let R be a commutative ring, and let M be an R -module. An R -submodule N of M is maximal if there is no R -module P with $N \subsetneq P \subsetneq M$.

- Show that an R -submodule N of M is maximal $\iff M/N$ is a simple R -module: i.e., M/N is nonzero and has no proper, nonzero R -submodules.
- Let M be a \mathbb{Z} -module. Show that a \mathbb{Z} -submodule N of M is maximal $\iff \#M/N$ is a prime number.
- Let M be the \mathbb{Z} -module of all roots of unity in \mathbb{C} under multiplication. Show that there is no maximal \mathbb{Z} -submodule of M .

Solution.

a

By the correspondence theorem, submodules of M/N biject with submodules A of M containing N .

So

- M is maximal:
- \iff no such (proper, nontrivial) submodule A exists
- \iff there are no (proper, nontrivial) submodules of M/N
- $\iff M/N$ is simple.

b

Identify \mathbb{Z} -modules with abelian groups, then by (a), N is maximal $\iff M/N$ is simple $\iff M/N$ has no nontrivial proper subgroups.

By Cauchy's theorem, if $|M/N| = ab$ is a composite number, then $a \mid ab \implies$ there is an element (and thus a subgroup) of order a . In this case, M/N contains a nontrivial proper cyclic subgroup, so M/N is not simple. So $|M/N|$ can not be composite, and therefore must be prime.

c

Let $G = \{x \in \mathbb{C} \mid x^n = 1 \text{ for some } n \in \mathbb{N}\}$, and suppose $H < G$ is a proper subgroup.

Then there must be a prime p such that the $\zeta_{p^k} \notin H$ for all k greater than some constant m – otherwise, we can use the fact that if $\zeta_{p^k} \in H$ then $\zeta_{p^\ell} \in H$ for all $\ell \leq k$, and if $\zeta_{p^k} \in H$ for all p and all k then $H = G$.

But this means there are infinitely many elements in $G \setminus H$, and so $\infty = [G : H] = |G/H|$ is not a prime. Thus by (b), H can not be maximal, a contradiction.