

# Complex Analysis Qualifying Exam Solutions

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## Contents

|          |   |          |
|----------|---|----------|
| <b>1</b> | <b>Week 1</b>   | <b>1</b> |
| 1.1      | Integrals and Cauchy's Theorem . . . . .                              | 1        |
| 1.1.1    | 5 . . . . .   | 1        |
| 1.1.2    | 9 . . . . .   | 1        |
| 1.1.3    | 10 . . . . .  | 1        |
| 1.2      | Liouville. The Fundamental Theorem of Algebra, Power Series . . . . . | 2        |
| 1.2.1    | 1 . . . . .   | 2        |
| 1.2.2    | 5 . . . . .   | 2        |
| 1.2.3    | 6 . . . . .   | 3        |

## 1 Week 1

### 1.1 Integrals and Cauchy's Theorem

#### 1.1.1 5

Show that there is no sequence of polynomials converging uniformly to  $f(z) = 1/z$  on  $S^1$ .

Solution

- By Cauchy's integral formula,  $\int_{S^1} f = 2\pi i$
- If  $p_j$  is any polynomial, then  $p_j$  is holomorphic in  $\mathbb{D}$ , so  $\int_{S^1} p_j = 0$ .
- Contradiction: compact sets in  $\mathbb{C}$  are bounded, so

$$\left| \int f - \int p_j \right| \leq \int |p_j - f| \leq \int \|p_j - f\|_\infty = \|p_j - f\|_\infty \int_{S^1} 1 dz = \|p_j - f\|_\infty \cdot 2\pi \longrightarrow 0$$

which forces  $\int f = \int p_j = 0$ .

#### 1.1.2 9

#### 1.1.3 10

Suppose  $f : \mathbb{C} \longrightarrow \mathbb{C}$  is entire and bounded, and use Cauchy's theorem to prove that  $f' \equiv 0$  and thus  $f$  is constant.

Solution

- Suffices to prove  $f' = 0$  because  $\mathbb{C}$  is connected (see Stein Ch 1, 3.4)
  - Idea: Fix  $w_0$ , show  $f(w) = f(w_0)$  for any  $w \neq w_0$
  - Connected = Path connected in  $\mathbb{C}$ , so take  $\gamma$  joining  $w$  to  $w_0$ .
  - $f$  is a primitive for  $f'$ , and  $\int_{\gamma} f' = f(w) - f(w_0)$ , but  $f' = 0$ .
- Fix  $z_0 \in \mathbb{C}$ , let  $B$  be the bound for  $f$ , so  $|f(z)| \leq B$  for all  $z$ .
- Apply Cauchy inequalities: if  $f$  is holomorphic on  $U \supset \bar{D}_R(z_0)$  then setting  $\|f\|_C := \sup_{z \in C} |f(z)|$ ,

$$|f^{(n)}(z_0)| \leq \frac{n! \|f\|_C}{R^n}.$$

- Yields  $|f'(z_0)| \leq B/R$
- Take  $R \rightarrow \infty$ , QED.

## 1.2 Liouville. The Fundamental Theorem of Algebra, Power Series

### 1.2.1 1

Suppose  $f$  is analytic on  $\Omega \supseteq \mathbb{D}$  whose power series  $\sum a_n z^n$  has radius of convergence 1.

- Give an example of an  $f$  which converges at every point on  $S^1$ .
- Give an example of an  $f$  which is analytic at  $z = 1$  but  $\sum a_n$  diverges.
- Prove that  $f$  can not be analytic at every point of  $S^1$ .

Solution:

- Take  $\sum \frac{z^n}{n^2}$ ; then  $|z| \leq 1 \implies \left| \frac{z^n}{n^2} \right| \leq \frac{1}{n^2}$  which is summable, so the series converges for  $|z| \leq 1$ .

- Take  $\sum \frac{z^n}{n}$ ; then  $z = 1$  yields the harmonic series, which diverges.

- For  $z \in S^1 \setminus \{1\}$ , we have  $z = e^{2\pi i t}$  for  $0 < t < 2\pi$ .
- So fix  $t$ .
- Toward applying the Dirichlet test, set  $a_n = 1/n, b_n = z^n$ .
- Then for all  $N$ ,

$$\left| \sum_{n=1}^N b_n \right| = \left| \sum_{n=1}^N z^n \right| = \left| \sum_{n=1}^N z^n \right| = \left| \frac{z - z^{N+1}}{1 - z} \right| \leq \frac{2}{1 - |z|} < \infty.$$

- Thus  $\sum a_n b_n < \infty$  and  $\sum z^n/n$  converges.

c. ?

### 1.2.2 5

Prove the Fundamental Theorem of Algebra: every non-constant polynomial  $p(z) = a_n z^n + \dots + a_0 \in \mathbb{C}[x]$  has a root in  $\mathbb{C}$ .

Solution:

- Strategy: By contradiction with Liouville's Theorem
- Suppose  $p$  is non-constant and has no roots.
- Claim:  $1/p(z)$  is a bounded holomorphic function on  $\mathbb{C}$ .
  - Holomorphic: clear? Since  $p$  has no roots.
  - Bounded: for  $z \neq 0$ , write

$$\frac{P(z)}{z^n} = a_n + \left( \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right).$$

- The term in parentheses goes to 0 as  $|z| \rightarrow \infty$
- Thus there exists an  $R > 0$  such that

$$|z| > R \implies \left| \frac{P(z)}{z^n} \right| \geq c := \frac{|a_n|}{2}.$$

- So  $p$  is bounded below when  $|z| > R$
- Since  $p$  is continuous and has no roots in  $|z| \leq R$ , it is bounded below when  $|z| \leq R$ .
- Thus  $p$  is bounded below on  $\mathbb{C}$  and thus  $1/p$  is bounded above on  $\mathbb{C}$ .
- By Liouville's theorem,  $1/p$  is constant and thus  $p$  is constant, a contradiction.

### 1.2.3 6

Find all entire functions  $f$  which satisfy the following inequality, and prove the list is complete:

$$|f(z)| \geq z.$$

Solution:

- Suppose  $f$  is entire and define  $g(z) := \frac{z}{f(z)}$ .
- By the inequality,  $|g(z)| \leq 1$ , so  $g$  is bounded.
- $g$  potentially has singularities at the zeros  $Z_f := f^{-1}(0)$ , but since  $f$  is entire,  $g$  is holomorphic on  $\mathbb{C} \setminus Z_f$ .
- Claim:  $Z_f \subset \mathbb{C}$  is closed and discrete
  - ???
- Thus the singularities  $Z_f$  are isolated
- By Riemann's removable singularity theorem, the singularities  $Z_f$  are removable and  $g$  has an extension to an entire function  $\tilde{g}$ .
- By continuity, we have  $|\tilde{g}(z)| \leq 1$  on all of  $\mathbb{C}$
- By Liouville,  $\tilde{g}$  is constant, so  $\tilde{g}(z) = c_0$  with  $|c_0| \leq 1$
- Thus  $f(z) = c_0^{-1}z$

Thus all such functions are of the form  $f(z) = cz$  for some  $c \neq 0 \in \mathbb{C}$ .