Title

D. Zack Garza

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1 Basics

1.1 Useful Techniques

- $\lim f_n = \limsup f_n = \liminf f_n$ iff the limit exists, so $\limsup f_n \leq g \leq \liminf f_n$ implies that $g = \lim f$.
- A limit does not exist iff $\liminf f_n > \limsup f_n$.
- If f_n has a global maximum (computed using f_n' and the first derivative test) $M_n \longrightarrow 0$, then $f_n \longrightarrow 0$ uniformly.
- For a fixed x, if $f = \sum f_n$ converges *uniformly* on some $B_r(x)$ and each f_n is continuous at x, then f is also continuous at x .

1.2 Definitions

Definition (Uniform Continuity) f is uniformly continuous iff

$$\begin{aligned} &\forall \varepsilon \quad \exists \delta(\varepsilon) \,\middle|\, \quad \forall x, y, \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \\ \iff &\forall \varepsilon \quad \exists \delta(\varepsilon) \,\middle|\, \quad \forall x, y, \quad |y| < \delta \implies |f(x - y) - f(y)| < \varepsilon \\ &. \end{aligned}$$

Definition (Nowhere Dense Sets) A set S is **nowhere dense** iff the closure of S has empty interior iff every interval contains a subinterval that does not intersect S.

Definition (Meager Sets) A set is **meager** if it is a *countable* union of nowhere dense sets.

Definition (F_σ and G_δ) An F_σ set is a union of closed sets, and a G_δ set is an intersection of opens.

> Mnemonic: "F" stands for *ferme*, which is "closed" in French, and σ corresponds to a "sum", i.e. a union.

Theorem (Heine-Cantor) Every continuous function on a compact space is uniformly continuous.

1.3 Theorems

1.3.1 Topology / Sets

Lemma Metric spaces are compact iff they are sequentially compact, (i.e. every sequence has a convergent subsequence).

Proposition The unit ball in $C([0,1])$ with the sup norm is not compact.

Proof Take $f_k(x) = x^n$, which converges to a dirac delta at 1. The limit is not continuous, so no subsequence can converge.

Proposition A *finite* union of nowhere dense is again nowhere dense.

Lemma (Convergent Sums Have Small Tails)

$$\sum a_n < \infty \implies a_n \longrightarrow 0 \quad \text{and} \quad \sum_{k=N}^{\infty} \overset{N\longrightarrow\infty}{\longrightarrow} 0$$

Theorem (Heine-Borel) $X \subseteq \mathbb{R}^n$ is compact $\iff$ X is closed and bounded.

Lemma (Geometric Series)

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \iff |x| < 1.$$

Corollary: $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k} = 1.$

Lemma The Cantor set is closed with empty interior.

Proof Its complement is a union of open intervals, and can't contain an interval since intervals have positive measure and $m(C_n)$ tends to zero.

Corollary The Cantor set is nowhere dense.

Lemma Singleton sets in $\mathbb{R}$ are closed, and thus $\mathbb{Q}$ is an F_σ set.

Theorem (Baire) $\mathbb{R}$ is a **Baire space** (countable intersections of open, dense sets are still dense). Thus $\mathbb{R}$ can not be written as a countable union of nowhere dense sets.

Lemma Any nonempty set which is bounded from above (resp. below) has a well-defined supremum (resp. infimum).

1.3.2 Functions

Proposition (Existence of Smooth Compactly Supported Functions) There exist smooth compactly supported functions, e.g. take

$$f(x) = e^{-\frac{1}{x^2}}\chi_{(0,\infty)}(x).$$

Lemma There is a function discontinuous precisely on $\mathbb{Q}$.

Proof $f(x) = \dfrac{1}{n}$ if $x = r_n \in \mathbb{Q}$ is an enumeration of the rationals, and zero otherwise. The limit at every point is 0.

Lemma There *do not* exist functions that are discontinuous precisely on $\mathbb{R} \setminus \mathbb{Q}$.

Proof D_f is always an F_σ set, which follows by considering the oscillation ω_f. $\omega_f(x) = 0 \iff f$ is continuous at x, and $D_f = \bigcup_n A_{\frac{1}{n}}$ where $A_\varepsilon = \{\omega_f \geq \varepsilon\}$ is closed.

Proposition A function $f : (a, b) \longrightarrow \mathbb{R}$ is Lipschitz $\iff$ f is differentiable and f' is bounded. In this case, $|f'(x)| \leq C$, the Lipschitz constant.

1.4 Uniform Convergence

Theorem (Weierstrass Approximation) If $[a, b] \subset \mathbb{R}$ is a closed interval and f is continuous, then for every $\varepsilon > 0$ there exists a polynomial p_ε such that $\|f - p_\varepsilon\|_{L^\infty([a,b])} \overset{\varepsilon \longrightarrow 0}{\longrightarrow} 0$.

Theorem (Egorov) Let $E \subseteq \mathbb{R}^n$ be measurable with $m(E) > 0$ and $\{f_k : E \longrightarrow \mathbb{R}\}$ be measurable functions such that

$$f(x) := \lim_{k \longrightarrow \infty} f_k(x) < \infty$$

exists almost everywhere.

Then $f_k \longrightarrow f$ *almost uniformly*, i.e.

$$\forall \varepsilon > 0,\ \exists F \subseteq E \text{ closed such that } m(E \setminus F) < \varepsilon \quad \text{and} \quad f_k \overset{u}{\to} f \text{ on } F.$$

Proposition The space $X = C([0, 1])$, continuous functions $f : [0, 1] \longrightarrow \mathbb{R}$, equipped with the norm $\|f\| = \sup_{x \in [0,1]} |f(x)|$, is a **complete** metric space.

Proof

1. Let $\{f_k\}$ be Cauchy in X.
2. Define a candidate limit using pointwise convergence:

   Fix an x; since

   $$|f_k(x) - f_j(x)| \leq \|f_k - f_k\| \longrightarrow 0$$

   the sequence $\{f_k(x)\}$ is Cauchy in $\mathbb{R}$. So define $f(x) := \lim_k f_k(x)$.
3. Show that $\|f_k - f\| \longrightarrow 0$:

   $$|f_k(x) - f_j(x)| < \varepsilon\ \forall x \implies \lim_j |f_k(x) - f_j(x)| < \varepsilon\ \forall x$$

   Alternatively, $\|f_k - f\| \leq \|f_k - f_N\| + \|f_N - f_j\|$, where N, j can be chosen large enough to bound each term by $\varepsilon/2$.

4. Show that $f \in X$:

   The uniform limit of continuous functions is continuous.

   > Note: in other cases, you may need to show the limit is bounded, or has bounded derivative, or whatever other conditions define X.

Theorem (Uniform Limits of Continuous Functions are Continuous) A uniform limit of continuous functions is continuous.

Lemma (Testing Uniform Convergence) $f_n \longrightarrow f$ uniformly iff there exists an M_n such that $\|f_n - f\|_\infty \leq M_n \longrightarrow 0$.

Negating: find an x which depends on n for which the norm is bounded below.

Lemma (Uniform Limits Commute with Integrals) If $f_n \longrightarrow f$ uniformly, then $\displaystyle\int f_n = \int f$.

Lemma (Uniform Convergence and Derivatives) If $f_n' \longrightarrow g$ uniformly for some g and $f_n \longrightarrow f$ pointwise (or at least at one point), then $g = f'$.

1.4.1 Series

Lemma (Uniform Convergence of Series of Numbers) If $f_n(x) \leq M_n$ **for a fixed** x where $\sum M_n < \infty$, then the series $f(x) = \sum f_n(x)$ converges pointwise.

Lemma (Small Tails for Series of Functions) If $\sum f_n$ converges then $f_n \longrightarrow 0$ uniformly.

Lemma (M-test for Series) If $|f_n(x)| \leq M_n$ which does not depend on x, then $\sum f_n$ converges uniformly.

Lemma (p-tests) Let n be a fixed dimension and set $B = \left\{ x \in \mathbb{R}^n \;\middle|\; \|x\| \leq 1 \right\}$.

$$
\begin{aligned}
\sum \frac{1}{n^p} < \infty &\iff p > 1 \\
\int_\varepsilon^\infty \frac{1}{x^p} < \infty &\iff p > 1 \\
\int_0^1 \frac{1}{x^p} < \infty &\iff p < 1 \\
\int_B \frac{1}{|x|^p} < \infty &\iff p < n \\
\int_{B^c} \frac{1}{|x|^p} < \infty &\iff p > n
\end{aligned}
$$

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