Complex Analysis Qualifying Exam Review

D. Zack Garza

Table of Contents

Contents

A great deal of content borrowed from the following: https://web.stanford.edu/~chriseur/notes_pdf/Eur_ComplexAnalysis_Notes.pdf

1 | General Info / Tips / Techniques

1.1 Greatest Hits

Things to know well:

- Estimates for derivatives, mean value theorem
- **??**CauchyTheorem]Cauchy's Theorem
- **??**CauchyIntegral]Cauchy's Integral Formula
- **??**CauchyInequality]Cauchy's Inequality
- **??**Morera]Morera's Theorem
- **??**Liouville]Liouville's Theorem
- **??**MaximumModulus]Maximum Modulus Principle
- **??**Rouche]Rouché's Theorem
- **??**SchwarzReflection]The Schwarz Reflection Principle
- **??**SchwarzLemma]The Schwarz Lemma
- **??**Casorati]Casorati-Weierstrass Theorem
- Properties of linear fractional transformations
- Automorphisms of $\mathbb{D}, \mathbb{C}, \mathbb{CP}^1$.

1.2 Common Tricks

- Virtually any time: consider $1/f(z)$ and $f(1/z)$.

Remark 1.2.1 ***(Showing a function is constant):*** If you want to show that a function f is constant, try one of the following:

- Write $f = u + iv$ and use Cauchy-Riemann to show $u_x, u_y = 0$, etc.
- Show that f is entire and bounded.
  - If you additionally want to show f is zero, show $\lim_{z\to\infty} f(z) = 0$.

Fact 1.2.2
To show a function is holomorphic,

- Use Morera's theorem
- Find a primitive (sufficient but not necessary)

Fact 1.2.3
To count zeros:

- Rouche's theorem
- The argument principle

- Setting $w = e^z$ is useful.

1.3 Basic but Useful Facts

1.3.1 Arithmetic

Fact 1.3.1 (Some useful facts about basic complex algebra)

$$z\bar{z} = |z|^2 \qquad \operatorname{Arg}(z/w) = \operatorname{Arg}(z) - \operatorname{Arg}(w)$$
$$\Re(z) = \frac{z+\bar{z}}{2} \qquad \Im(z) = \frac{z-\bar{z}}{2i}.$$

Exponential forms of cosine and sine, where it's sometimes useful to set $w := e^{iz}$:

$$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \frac{1}{2}(w + w^{-1})$$
$$\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = \frac{1}{2i}(w - w^{-1})$$

$$\cosh(z) = \cos(iz) = \frac{1}{2}\left(e^{z} + e^{-z}\right)$$
$$\sinh(z) = \sin(iz) = \frac{1}{2}\left(e^{z} - e^{-z}\right).$$

Fact 1.3.2
Some computations that come up frequently:

$$|z \pm w|^2 = |z|^2 + |w|^z + 2\Re(\overline{w}z)$$
$$(a+bi)(c+di) = (ac-bd) + (ad+bc)$$
$$\frac{1}{|a+b|} \leq \frac{1}{|a|-|b|}$$

$$|e^z| = e^{\Re(z)}, \quad \arg(e^z) = \Im(z).$$

1.3.2 Calculus

Fact 1.3.3
Various differentials:

$$dz = dx + i\ dy$$
$$d\bar{z} = dx - i\ dy$$

$$f_z = f_x = f_y/i.$$

Integral of a complex exponential:

$$\int_0^{2\pi} e^{i\ell x} dx = \begin{cases} 2\pi & \ell = 0 \\ 0 & \text{else} \end{cases}.$$

1.4 Series

Fact 1.4.1 (Generalized Binomial Theorem)
Define $(n)_k$ to be the falling factorial

$$\prod_{j=0}^{k-1}(n-k) = n(n-1)\cdots(n-k+1)$$

and set $\binom{n}{k} := (n)_k/k!$, then

$$(x+y)^n = \sum_{k\geq 0} \binom{n}{k} x^k y^{n-k}.$$

Fact 1.4.2 (Some useful series)

$$
\begin{aligned}
\sum_{k=1}^{n} k &= \frac{n(n+1)}{2} \\
\sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6} \\
\sum_{k=1}^{n} k^3 &= \frac{n^2(n+1)^2}{4} \\
\sum_{0\le k\le N} z^k &= \frac{1-z^{N+1}}{1-z} \\
\frac{1}{1-z} &= \sum_{k\ge 0} z^k \\
e^z &= \sum_{k\ge 0} \frac{z^k}{k!} \\
\sin(z) &= \sum_{\substack{k\ge 1 \\ \text{odd}}} (-1)^{\frac{k+1}{2}} \frac{z^k}{k!} \\
&= z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots \\
\cos(z) &= \sum_{\substack{k\ge 0 \\ \text{even}}} (-1)^{\frac{k}{2}} \frac{z^k}{k!} \\
&= 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \cdots \\
\\
\cosh(z) &= \sum_{k\ge 0} \frac{z^{2k}}{(2k)!} \\
\sinh(z) &= \sum_{k\ge 0} \frac{z^{2k+1}}{(2k+1)!} \\
\log(1-x) &= \sum_{k\ge 0} \frac{z^k}{k} \quad |z| < 1 \\
\frac{\partial}{\partial z} \sum_{k=0}^{\infty} a_k z^k &= \sum_{k=0}^{\infty} a_{k+1} z^k \\
\sqrt{1+x} = (1+x)^{1/2} &= 1 + (1/2)x + \frac{(1/2)(-1/2)}{2!}x^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}x^3 + \cdots \\
&= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots
\end{aligned}
$$

Fact 1.4.3

Useful trick for expanding square roots:

$$\sqrt{z} = \sqrt{z_0 + z - z_0} = \sqrt{z_0\left(1 + \frac{z - z_0}{z}\right)} = \sqrt{z_0}\sqrt{1+u}, \quad u := \frac{z - z_0}{z}$$
$$\implies \sqrt{z} = \sqrt{z_0}\sum_{k\geq 0}\binom{1/2}{k}\left(\frac{z - z_0}{z}\right)^k.$$

2 | Calculus Preliminaries

2.1 Definitions

Definition 2.1.1 (Locally uniform convergence)
A sequence of functions f_n is said to converge **locally uniformly** on $\Omega \subseteq \mathbb{C}$ iff $f_n \to f$ uniformly on every compact subset $K \subseteq \Omega$.

Definition 2.1.2 (Equicontinuous Family)
A family of functions f_n is **equicontinuous** iff for every ε there exists a $\delta = \delta(\varepsilon)$ (not depending on n or f_n) such that $|x - y| < \varepsilon \implies |f_n(x) - f_n(y)| < \varepsilon$ for all n.

Remark 2.1.3: Recall Arzelà-Ascoli, an analog of Heine-Borel: for X compact Hausdorff, consider the the Banach space $C(X; \mathbb{R})$ equipped with the *uniform norm* $\|f\|_{\infty,X} := \sup_{x\in X}|f(x)|$. Then a subset $A \subseteq X$ is compact iff A is closed, uniformly bounded, and equicontinuous. As a consequence, if A is a sequence, it contains a subsequence converging uniformly to a continuous function. The proof is an $\varepsilon/3$ argument.

Definition 2.1.4 (Normal Family)

Remark 2.1.5: A continuous function on a compact set is uniformly continuous.

Definition 2.1.6 (Univalent functions)
A function $f \in \operatorname{Hol}(U; \mathbb{C})$ is called **univalent** if f is injective.

Remark 2.1.7: If $f : \Omega \to \Omega'$ is a univalent surjection, f is invertible on Ω and f^{-1} is holomorphic. Compare to real functions: $f(x) = x^3$ is injective on $(-c, c)$ for any c but $f'(0) = 0$ and $f^{-1}(x) := x^{1/3}$ is not differentiable at zero.

2.2 Theorems

Theorem 2.2.1 ***(Implicit Function Theorem).***

Theorem 2.2.2 ***(Inverse Function Theorem).***
For $f \in C^1(\mathbb{R};\mathbb{R})$ with $f'(a) \neq 0$, then f is invertible in a neighborhood $U \ni a$, $g := f^{-1} \in C^1(U;\mathbb{R})$, and at $b := f(a)$ the derivative of g is given by

$$g'(b) = \frac{1}{f'(a)}.$$

For $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with D_f invertible in a neighborhood of a, so $\det(J_f) \neq 0$, then setting $b := F(a)$,

$$J_{F^{-1}}(q) = (J_F(p))^{-1}.$$

The version for holomorphic functions: if $f \in \mathrm{Hol}(\mathbb{C};\mathbb{C})$ with $f'(p) \neq 0$ then there is a neighborhood $V \ni p$ with that $f \in \mathrm{BiHol}(V, f(V))$.

Theorem 2.2.3 ***(Green's Theorem).***
If $\Omega \subseteq \mathbb{C}$ is bounded with $\partial\Omega$ piecewise smooth and $f, g \in C^1(\overline{\Omega})$, then

$$\int_{\partial\Omega} f\,dx + g\,dy = \iint_{\Omega} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.$$

2.3 Convergence

Remark 2.3.1: Recall that absolutely convergent implies convergent, but not conversely: $\sum k^{-1} = \infty$ but $\sum(-1)^k k^{-1} < \infty$. This converges because the even (odd) partial sums are monotone increasing/decreasing respectively and in $(0,1)$, so they converge to a finite number. Their difference converges to 0, and their common limit is the limit of the sum.

Proposition 2.3.2 ***(Uniform Convergence of Series).***
A series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly iff

$$\lim_{n\to\infty} \left\| \sum_{k\geq n} f_k \right\|_\infty = 0.$$

Theorem 2.3.3 ***(Weierstrass M-Test).***
If $\{f_n\}$ with $f_n : \Omega \to \mathbb{C}$ and there exists a sequence $\{M_n\}$ with $\|f_n\|_\infty \leq M_n$ and $\sum_{n\in\mathbb{N}} M_n < \infty$,

then $f(x) := \sum_{n \in \mathbb{N}} f_n(x)$ converges absolutely and uniformly on Ω. Moreover, if the f_n are continuous, by the uniform limit theorem, f is again continuous.

2.4 Series and Sequences

Remark 2.4.1: Note that if a power series converges uniformly, then summing commutes with integrating or differentiating.

Proposition 2.4.2 ***(Ratio Test).***
Consider $\sum c_k z^k$, set $R = \lim \left|\frac{c_{k+1}}{c_k}\right|$, and recall the **ratio test**:

- $R \in (0,1) \implies$ convergence.
- $R \in (1, \infty] \implies$ divergence.
- $R = 1$ yields no information.

Theorem 2.2 (Root Test). *Suppose $\sum a_n(z - z_0)^n$ is a formal power series. Let*

$$R = \liminf_{n \to \infty} |a_n|^{-\frac{1}{n}} = \frac{1}{\limsup\limits_{n \to \infty} |a_n|^{\frac{1}{n}}} \in [0, +\infty].$$

Then $\sum_{n=0}^{\infty} a_n(z - z_0)^n$

(a) *converges absolutely in $\{z : |z - z_0| < R\}$,*

(b) *converges uniformly in $\{z : |z - z_0| \leq r\}$ for all $r < R$, and*

(c) *diverges in $\{z : |z - z_0| > R\}$.*

Figure 1: image_2021-05-27-15-40-58

Proposition 2.4.3 ***(Root Test).***

Proposition 2.4.4 ***(Radius of Convergence by the Root Test).***
For $f(z) = \sum_{k \in \mathbb{N}} c_k z^k$, defining

$$\frac{1}{R} := \limsup_k |a_k|^{\frac{1}{k}},$$

then f converges absolutely and uniformly for $D_R := |z| < R$ and diverges for $|z| > R$.

Moreover f is holomorphic in D_R, can be differentiated term-by-term, and $f' = \sum_{k\in\mathbb{N}} nc_k z^k$.

Fact 2.4.5
Recall the p-**test**:

$$\sum n^{-p} < \infty \iff p \in (1, \infty).$$

Fact 2.4.6
The product of two sequences is given by the Cauchy product

$$\sum a_k z^k \cdot \sum b_k z^k = \sum c_k z^k, \quad c_k := \sum_{j\leq k} a_k b_{k-j}.$$

Fact 2.4.7
Recall how to carry out polynomial long division:

Polynomial long division

Fact 2.4.8 (Partial Fraction Decomposition)

- For every root r_i of multiplicity 1, include a term $A/(x - r_i)$.
- For any factors $g(x)$ of multiplicity k, include terms $A_1/g(x), A_2/g(x)^2, \cdots, A_k/g(x)^k$.
- For irreducible quadratic factors $h_i(x)$, include terms of the form $\dfrac{Ax+B}{h_i(x)}$.

2.5 Exercises

Exercise 2.5.1 (?)
Find the radius of convergences for the power series expansion of $\sqrt{z}$ about $z_0 = 4 + 3i$.

3 | Preliminaries

Definition 3.0.1 (Toy contour)
A closed Jordan curve that separates $\mathbb{C}$ into an exterior and interior region is referred to as a **toy contour**.

Fact 3.0.2 (Complex roots of a number)
The complex nth roots of $z := re^{i\theta}$ are given by

$$\left\{\omega_k := r^{1/n}e^{i\left(\frac{\theta+2k\pi}{n}\right)} \;\middle|\; 0 \leq k \leq n-1\right\}.$$

Note that one root is $r^{1/n} \in \mathbb{R}$, and the rest are separated by angles of $2\pi/n$. Mnemonic:

$$z = re^{i\theta} = re^{i(\theta+2k\pi)} \implies z^{1/n} = \cdots.$$

3.1 Complex Log

Fact 3.1.1 (Complex Log)
For $z = re^{i\theta} \neq 0$, θ is of the form $\Theta + 2k\pi$ where $\Theta = \operatorname{Arg} z$ We define

$$\log(z) = \ln(|z|) + i\operatorname{Arg}(z)$$

and $z^c := e^{c\log(z)}$. Thus

$$\log(re^{i\theta}) = \ln|r| + i\theta.$$

Fact 3.1.2
Common trick:

$$f^{1/n} = e^{\frac{1}{n}\log(f)},$$

taking (say) a principal branch of log given by $\mathbb{C} \setminus (-\infty, 0] \times 0$.

Proposition 3.1.3 ***(Existence of complex log).***
Suppose Ω is a simply-connected region such that $1 \in \Omega, 0 \notin \Omega$. Then there exists a branch of $F(z) := \operatorname{Log}(z)$ such that

- F is holomorphic on Ω,
- $e^{F(z)} = z$ for all $z \in \Omega$
- $F(x) = \log(x)$ for $x \in \mathbb{R}$ in a neighborhood of 1.

Definition 3.1.4 (Principal branch and exponential)
Take $\mathbb{C}$ and delete $\mathbb{R}^{\leq 0}$ to obtain the **principal branch** of the logarithm. Equivalently, this is define for all $z = re^{i\theta}$ where $\theta \in (-\pi, \pi)$.
Here the log is defined as

$$\operatorname{Log}(z) := \log(r) + i\theta \qquad\qquad |\theta| < \pi.$$

Similarly define

$$z^\alpha := e^{\alpha \operatorname{Log}(z)}.$$

Warning 3.1.5
It's tempting to define

$$z^{\frac{1}{n}} := (re^{i\theta})^{\frac{1}{n}} = r^{\frac{1}{n}} e^{\frac{i\theta}{n}},$$

but this requires a branch cut to ensure continuity.

Remark 3.1.6: Note the problem: for $z := x + i0 \in \mathbb{R}^{\leq 0}$, just above the axis consider $z_+ := x + i\varepsilon$ and $z_- := x - i\varepsilon$. Then

- $\log(z_+) = \log|x| + i\pi$, and
- $\log(z_-) = \log|x| - i\pi$.

So log can't even be made continuous if one crosses the branch. The issue is the **branch point** or **branch singularity** at $z = 0$.

Theorem 3.1.7 ***(Existence of log of a function).***
If f is holomorphic and nonvanishing on a simply-connected region Ω, then there exists a holomorphic G on Ω such that

$$f(z) = e^{G(z)}.$$

3.2 Complex Calculus

Remark 3.2.1: When parameterizing integrals $\int_\gamma f(z)\,dz$, parameterize γ by θ and write $z = re^{i\theta}$ so $dz = ire^{i\theta}\,d\theta$.

Warning 3.2.2
$f(z) = \sin(z), \cos(z)$ are unbounded on $\mathbb{C}$! An easy way to see this: they are nonconstant and entire, thus unbounded by Liouville.

Example 3.2.3 ***(?):*** You can show $f(z) = \sqrt{z}$ is not holomorphic by showing its integral over S^1

is nonzero. This is a direct computation:

$$
\begin{aligned}
\int_{S^1} z^{1/2}\,dz &= \int_0^{2\pi} (e^{i\theta})^{1/2} i e^{i\theta}\,d\theta \\
&= i\int_0^{2\pi} e^{\frac{i3\theta}{2}}\,d\theta \\
&= i\left(\frac{2}{3i}\right) e^{\frac{i3\theta}{2}}\Big|_0^{2\pi} \\
&= \frac{2}{3}\left(e^{3\pi i - 1}\right) \\
&= -\frac{4}{3}.
\end{aligned}
$$

Note an issue: a different parameterization yields a different (still nonzero) number

$$
\begin{aligned}
\cdots &= \int_{-\pi}^{\pi} (e^{i\theta})^{1/2} i e^{i\theta}\,d\theta \\
&= \frac{2}{3}\left(e^{\frac{3\pi i}{2}} - e^{\frac{-3\pi i}{2}}\right) \\
&= -\frac{4i}{3}.
\end{aligned}
$$

This is these are paths that don't lift to closed loops on the Riemann surface defined by $z \mapsto z^2$.

3.2.1 Holomorphy and Cauchy-Riemann

Definition 3.2.4 (Analytic)
A function $f : \Omega \to \mathbb{C}$ is *analytic* at $z_0 \in \Omega$ iff there exists a power series $g(z) = \sum a_n (z - z_0)^n$ with radius of convergence $R > 0$ and a neighborhood $U \ni z_0$ such that $f(z) = g(z)$ on U.

Definition 3.2.5 (Complex differentiable / holomorphic /entire)
A function $f : \mathbb{C} \to \mathbb{C}$ is **complex differentiable** or **holomorphic** at z_0 iff the following limit exists:

$$
\lim_{h \to 0} \frac{f(z_0 + h) - f(h)}{h}.
$$

A function that is holomorphic on $\mathbb{C}$ is said to be **entire**.
Equivalently, there exists an $\alpha \in \mathbb{C}$ such that

$$
f(z_0 + h) - f(z_0) = \alpha h + R(h) \qquad\qquad R(h) \xrightarrow{h \to 0} 0.
$$

In this case, $\alpha = f'(z_0)$.

Example 3.2.6 ***(Holomorphic vs non-holomorphic):***

- $f(z) := |z|$ is not holomorphic.
- $f(z) := \arg z$ is not holomorphic.

- $f(z) := \Re z$ is not holomorphic.
- $f(z) := \Im z$ is not holomorphic.
- $f(z) = \dfrac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ but not holomorphic on $\mathbb{C}$
- $f(z) = \bar{z}$ is *not* holomorphic, but is real differentiable:
$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{\overline{z_0}+\bar{h}-\overline{z_0}}{h} = \frac{\bar{h}}{h} = \frac{re^{-i\theta}}{re^{i\theta}} = e^{-2i\theta} \overset{h\to 0}{\longrightarrow} e^{-2i\theta},$$
which is a complex number that depends on θ and is thus not a single value.

Definition 3.2.7 (Real (multivariate) differentiable)
A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is **real-differentiable** at $\mathbf{p}$ iff there exists a linear transformation A such that

$$\frac{\|F(\mathbf{p}+\mathbf{h})-F(\mathbf{p})-A(\mathbf{h})\|}{\|\mathbf{h}\|} \overset{\|\mathbf{h}\|\to 0}{\longrightarrow} 0.$$

Rewriting,

$$\|F(\mathbf{p}+\mathbf{h})-F(\mathbf{p})-A(\mathbf{h})\| = \|\mathbf{h}\|\|R(\mathbf{h})\| \qquad \|R(\mathbf{h})\| \overset{\|\mathbf{h}\|\to 0}{\longrightarrow} 0.$$

Equivalently,

$$F(\mathbf{p}+\mathbf{h})-F(\mathbf{p}) = A(\mathbf{h}) + \|\mathbf{h}\|R(\mathbf{h}) \qquad \|R(\mathbf{h})\| \overset{\|\mathbf{h}\|\to 0}{\longrightarrow} 0.$$

Or in a slightly more useful form,

$$F(\mathbf{p}+\mathbf{h}) = F(\mathbf{p}) + A(\mathbf{h}) + R(\mathbf{h}) \qquad R \in o(\|\mathbf{h}\|), \text{ i.e. } \frac{\|R(\mathbf{h})\|}{\|\mathbf{h}\|} \overset{\mathbf{h}\to 0}{\longrightarrow} 0.$$

Proposition 3.2.8 ***(Complex differentiable implies Cauchy-Riemann).***
If f is differentiable at z_0, then the limit defining $f'(z_0)$ must exist when approaching from any direction. Identify $f(z) = f(x,y)$ and write $z_0 = x+iy$, then first consider $h \in RR$, so $h = h_1 + ih_2$ with $h_2 = 0$. Then

$$f'(z_0) = \lim_{h_1\to 0} \frac{f(x+h_1,y)-f(x,y)}{h_1} := \frac{\partial f}{\partial x}(x,y).$$

Taking $h \in i\mathbb{R}$ purely imaginary, so $h = ih_2$,

$$f'(z_0) = \lim_{ih_2\to 0} \frac{f(x,y+h_2)-f(x,y)}{ih_2} := \frac{1}{i}\frac{\partial f}{\partial y}(x,y).$$

Equating,

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y},$$

and writing $f = u+iv$ and $1/i = -i$ yields

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ \frac{1}{i}\frac{\partial f}{\partial y} &= \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \end{aligned}$$

Thus

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,.$$

Proposition 3.2.9 ***(Polar Cauchy-Riemann equations).***

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Proof .
Setting

$$z = re^{i\theta} = r(\cos(\theta) + i\sin(\theta)) = x + iy$$

yields $x = r\cos(\theta), y = r\sin(\theta)$, one can identify

$$\begin{aligned} x_r &= \cos(\theta), x_\theta = -r\sin(\theta) \\ y_r &= \sin(\theta), y_\theta = r\cos(\theta). \end{aligned}$$

Now apply the chain rule:

$$\begin{aligned} u_r &= u_x x_r + u_y y_r \\ &= v_y x_r - v_x y_r && \text{CR} \\ &= v_y \cos(\theta) - v_x \sin(\theta) \\ &= \frac{1}{r}\left(v_y r\cos(\theta) - v_x r\sin(\theta)\right) \\ &= \frac{1}{r}\left(v_y y_\theta + v_x x_\theta\right) \\ &= \frac{1}{r} v_\theta. \end{aligned}$$

Similarly,

$$\begin{aligned} v_r &= v_x x_r + v_y y_r \\ &= v_x \cos(\theta) + v_y \sin(\theta) \\ &= -u_y \cos(\theta) + u_x \sin(\theta) && \text{CR} \\ &= \frac{1}{r}\left(-u_y r\cos(\theta) + u_x r\sin(\theta)\right) \\ &= \frac{1}{r}\left(-u_y y_\theta - u_x x_0\right) \\ &= -\frac{1}{r} u_\theta. \end{aligned}$$

Thus

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

.

■

Proposition 3.2.10 ***(Holomorphic functions are continuous.).***
f is holomorphic at z_0 iff there exists an $a \in \mathbb{C}$ such that

$$f(z_0 + h) - f(z_0) - ah = h\psi(h), \quad \psi(h) \overset{h \to 0}{\to} 0.$$

In this case, $a = f'(z_0)$.

Prove

3.2.2 Delbar and the Laplacian

Definition 3.2.11 (del and delbar operators)

$$\partial := \partial_z := \frac{1}{2}\left(\partial_x - i\partial_y\right) \quad \text{and} \quad \bar{\partial} := \partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right).$$

Moreover, $f' = \partial f + \bar{\partial} f$.

Proposition 3.2.12 ***(Holomorphic iff delbar vanishes).***
f is holomorphic at z_0 iff $\bar{\partial} f(z_0) = 0$:

$$\begin{aligned} 2\bar{\partial} f &:= (\partial_x + i\partial_y)(u + iv) \\ &= u_x + iv_x + iu_y - v_y \\ &= (u_x - v_y) + i(u_y + v_x) \\ &= 0 && \text{by Cauchy-Riemann.} \end{aligned}$$

3.2.3 Harmonic Functions and the Laplacian

Definition 3.2.13 (Laplacian and Harmonic Functions)
A real function of two variables $u(x, y)$ is **harmonic** iff it is in the kernel of the Laplacian operator:

$$\Delta u := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0.$$

Proposition 3.2.14 ***(Cauchy-Riemann implies holomorphic).***
If $f = u + iv$ with $u, v \in C^1(\mathbb{R})$ satisfying the Cauchy-Riemann equations on Ω, then f is holomorphic on Ω and

$$f'(z) = \partial f = \frac{1}{2}\left(u_x + iv_x\right).$$

Proposition 3.2.15 ***(Holomorphic functions have harmonic components).***
If $f(z) = u(x,y) + iv(x,y)$ is holomorphic, then u, v are harmonic.

Proof (?).

- By CR,
$$u_x = v_y \qquad u_y = -v_x.$$
- Differentiate with respect to x:
$$u_{xx} = v_{yx} \qquad u_{yx} = -v_{xx}.$$
- Differentiate with respect to y:
$$u_{xy} = v_{yy} \qquad u_{yy} = -v_{xy}.$$
- Clairaut's theorem: partials are equal, so
$$u_{xx} - v_{yx} = 0 \implies u_{xx} + u_{yy} = 0$$
$$v_{xx} + u_{yx} = 0 \implies v_{xx} + v_{yy} = 0$$
$$.$$

■

3.2.4 Exercises

Proposition 3.2.16 ***(Injectivity Relates to Derivatives).***
If z_0 is a zero of f' of order n, then f is $(n+1)$-to-one in a neighborhood of z_0.

Proof .
?

■

proof

Exercise 3.2.17 (Zero derivative implies constant)
Show that if $f' = 0$ on a domain Ω, then f is constant on Ω

Solution:
Write $f = u+iv$, then $0 = 2f' = u_x + iv_x = u_y - iu_y$, so $\operatorname{grad} u = \operatorname{grad} v = 0$. Show f is constant

along every straight line segment L by computing the directional derivative $\operatorname{grad} u \cdot \mathbf{v} = 0$ along L connecting p, q. Then $u(p) = u(q) = a$ some constant, and $v(p) = v(q) = b$, so $f(z) = a + bi$ everywhere.

Exercise 3.2.18 (f and fbar holomorphic implies constant)
Show that if f and $\overline{f}$ are both holomorphic on a domain Ω, then f is constant on Ω.

Solution:

- Strategy: show $f' = 0$.
- Write $f = u + iv$. Since f is analytic, it satisfies CR, so

$$u_x = v_y \qquad\qquad u_y = -v_x.$$

- Similarly write $\overline{f} = U + iV$ where $U = u$ and $V = -v$. Since $\overline{f}$ is analytic, it also satisfies CR , so

$$U_x = V_y \qquad\qquad U_y = -V_x$$

$$\implies u_x = -v_y \qquad\qquad u_y = v_x.$$

- Add the LHS of these two equations to get $2u_x = 0 \implies u_x = 0$. Subtract the right-hand side to get $-2v_x = 0 \implies v_x = 0$
- Since f is analytic, it is holomorphic, so f' exists and satisfies $f' = u_x + iv_x$. But by above, this is zero.
- By the previous exercise, $f' = 0 \implies f$ is constant.

Exercise 3.2.19 (SS 1.13: Constant real/imaginary/magnitude implies constant)
If f is holomorphic on Ω and any of the following hold, then f is constant:

1. $\Re(f)$ is constant.
2. $\Im(f)$ is constant.
3. $|f|$ is constant.

Solution:
Part 3:

- Write $|f| = c \in \mathbb{R}$.
- If $c = 0$, done, so suppose $c > 0$.
- Use $f\overline{f} = |f|^2 = c^2$ to write $\overline{f} = c^2/f$.
- Since $|f(z)| = 0 \iff f(z) = 0$, we have $f \neq 0$ on Ω, so $\overline{f}$ is analytic.
- Similarly f is analytic, and $f, \overline{f}$ analytic implies $f' = 0$ implies f is constant.

Finish

3.3 Power Series

Theorem 3.3.1 ***(Improved Taylor's Theorem).***
If f is holomorphic on a region Ω with $\overline{D_R(z_0)} \subseteq \Omega$, and for every $z \in D_r(z_0)$, f has a power series expansion of the following form:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{where } a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta}\, d\theta.$$

Proposition 3.3.2 ***(Power Series are Smooth).***
Any power series is smooth (and thus holomorphic) on its disc of convergence, and its derivatives can be obtained using term-by-term differentiation:

$$\frac{\partial}{\partial z} f(z) = \frac{\partial}{\partial z} \sum_{k \geq 0} c_k (z - z_0)^k = \sum_{k \geq 1} k c_k (z - z_0)^k.$$

Moreover, the coefficients are given by

$$c_k = \frac{f^{(n)}(z_0)}{n!}.$$

Remark 3.3.3: By an application of the Cauchy integral formula (see S&S 7.1) if f is holomorphic on $D_R(z_0)$ there is a formula for all $k \geq 0$ and all $0 < r < R$:

$$c_k = \frac{1}{2\pi r^k} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta}\, d\theta.$$

Proposition 3.3.4 ***(Exponential is uniformly convergent in discs).***
$f(z) = e^z$ is uniformly convergent in any disc in $\mathbb{C}$.

Proof .
Apply the estimate

$$|e^z| \leq \sum \frac{|z|^n}{n!} = e^{|z|}.$$

Now by the M-test,

$$|z| \leq R < \infty \implies \left| \sum \frac{z^n}{n!} \right| \leq e^R < \infty.$$

■

Lemma 3.3.5 ***(Dirichlet's Test).***
Given two sequences of real numbers $\{a_k\}, \{b_k\}$ which satisfy

1. The sequence of partial sums $\{A_n\}$ is bounded,

2. $b_k \searrow 0$.

then

$$\sum_{k\geq 1} a_k b_k < \infty.$$

Proof (?).

See http://www.math.uwaterloo.ca/~krdavids/Comp/Abel.pdf

Use summation by parts. For a fixed $\sum a_k b_k$, write

$$\sum_{n=1}^{m} x_n Y_n + \sum_{n=1}^{m} X_n y_{n+1} = X_m Y_{m+1}.$$

Set $x_n := a_n, y_N := b_n - b_{n-1}$, so $X_n = A_n$ and $Y_n = b_n$ as a telescoping sum. Importantly, all y_n are negative, so $|y_n| = |b_n - b_{n-1}| = b_{n-1} - b_n$, and moreover $a_n b_n = x_n Y_n$ for all n. We have

$$\begin{aligned}
\sum_{n\geq 1} a_n b_n &= \lim_{N\to\infty} \sum_{n\leq N} x_n Y_n \\
&= \lim_{N\to\infty} \sum_{n\leq N} X_N Y_N - \sum_{n\leq N} X_n y_{n+1} \\
&= -\sum_{n\geq 1} X_n y_{n+1},
\end{aligned}$$

where in the last step we've used that

$$|X_N| = |A_N| \leq M \implies |X_N Y_N| = |X_N||b_{n+1}| \leq M b_{n+1} \to 0.$$

So it suffices to bound the latter sum:

$$\begin{aligned}
\sum_{k\geq n} |X_k y_{k+1}| &\leq M \sum_{k\geq 1} |y_{k+1}| \\
&\leq M \sum_{k\geq 1} b_k - b_{k+1} \\
&\leq 2M(b_1 - b_{n+1}) \\
&\leq 2M b_1.
\end{aligned}$$

■

Theorem 3.3.6 ***(Abel's Theorem).***

If $\sum_{k=1}^{\infty} c_k z^j$ converges on $|z| < 1$ then

$$\lim_{z\to 1^-} \sum_{k\in\mathbb{N}} c_k z^k = \sum_{k\in\mathbb{N}} c_k.$$

Lemma 3.3.7 ***(Abel's Test).***
If $f(z) := \sum c_k z^k$ is a power series with $c_k \in \mathbb{R}^{\geq 0}$ and $c_k \searrow 0$, then f converges on S^1 except possibly at $z = 1$.

Example 3.3.8 ***(application of Abel's theorem):*** What is the value of the alternating harmonic series? Integrate a geometric series to obtain

$$\sum \frac{(-1)^k z^k}{n} = \log(z+1) \qquad\qquad |z| < 1.$$

Since $c_k := (-1)^k/k \searrow 0$, this converges at $z = 1$, and by Abel's theorem $f(1) = \log(2)$.

Remark 3.3.9: The converse to Abel's theorem is false: take $f(z) = \sum(-z)^n = 1/(1+z)$. Then $f(1) = 1 - 1 + 1 - \cdots$ diverges at 1, but $1/1+1 = 1/2$. So the limit $s := \lim_{x \to 1^-} f(x)1/2$, but $\sum a_n$ doesn't converge to s.

Proposition 3.3.10 ***(Summation by Parts).***
Setting $A_n := \sum_{k=1}^{n} b_k$ and $B_0 := 0$,

$$\sum_{k=m}^{n} a_k b_k = A_n b_n - A_{m-1} b_m - \sum_{k=m}^{n-1} A_k (b_{k+1} - b_k).$$

Compare this to integrating by parts:

$$\int_a^b fg = F(b)g(b) - F(a)g(a) - \int_a^b Fg'.$$

Note there is a useful form for taking the product of sums:

$$A_n B_n = \sum_{k=1}^{n} A_k b_k + \sum_{k=1}^{n} a_k B_{k-1}.$$

Proof (?).
An inelegant proof: define $A_n := \sum_{k \leq n} a_k$, use that $a_k = A_k - A_{k-1}$, reindex, and peel a top/bottom term off of each sum to pattern-match.

Behold:

$$\begin{aligned}
\sum_{m\le k\le n} a_k b_k &= \sum_{m\le k\le n} (A_k - A_{k-1}) b_k \\
&= \sum_{m\le k\le n} A_k b_k - \sum_{m\le k\le n} A_{k-1} b_k \\
&= \sum_{m\le k\le n} A_k b_k - \sum_{m-1\le k\le n-1} A_k b_{k+1} \\
&= A_n b_n + \sum_{m\le k\le n-1} A_k b_k - \sum_{m-1\le k\le n-1} A_k b_{k+1} \\
&= A_n b_n - A_{m-1} b_m + \sum_{m\le k\le n-1} A_k b_k - \sum_{m\le k\le n-1} A_k b_{k+1} \\
&= A_n b_n - A_{m-1} b_m + \sum_{m\le k\le n-1} A_k (b_k - b_{k+1}) \\
&= A_n b_n - A_{m-1} b_m - \sum_{m\le k\le n-1} A_k (b_{k+1} - b_k).
\end{aligned}$$

■

Proposition 3.3.11 *(?).*
If f is non-constant, then f' is analytic and the zeros of f' are isolated. If f, g are analytic with $f' = g'$, then $f - g$ is constant.

3.3.1 Exercises: Series

Exercise 3.3.12 (Application of summation by parts)
Use summation by parts to show that $\sin(n)/n$ converges.

Exercise 3.3.13 (1.20: Series convergence on the circle)
Show that

1. $\sum kz^k$ diverges on S^1.
2. $\sum k^{-2} z^k$ converges on S^1.
3. $\sum k^{-1} z^k$ converges on $S^1 \setminus \{1\}$ and diverges at 1.

Solution: 1. Use that $\left|z^k\right| = 1$ and $\sum c_k z^k < \infty \implies |c_k| \to 0$, but $\left|kz^k\right| = |k| \to \infty$ here.

2. Use that absolutely convergent implies convergent, and $\sum \left|k^{-2} z^k\right| = \sum \left|k^{-2}\right|$ converges by the p-test.
3. If $z = 1$, this is the harmonic series. Otherwise take $a_k = 1/k, b_k = e^{ik\theta}$ where $\theta \in (0, 2\pi)$ is some constant, and apply Dirichlet's test. It suffices to bound the partial sums of the

b_k. Recalling that $\sum_{k \le N} r^k = (1 - r^{N+1})/(1 - r)$,

$$\left\| \sum_{k \le m} e^{ik\theta} \right\| = \left\| \frac{1 - e^{i(m+1)\theta}}{1 - e^{i\theta}} \right\| \le \frac{2}{\|1 - e^{i\theta}\|} := M,$$

which is a constant. Here we've used that two points on S^1 are at most distance 2 from each other.

Exercise 3.3.14 (Laurent expansions inside and outside of a disc)
Expand $f(z) = \dfrac{1}{z(z-1)}$ in both

- $|z| < 1$
- $|z| > 1$

Solution:

$$\frac{1}{z(z-1)} = -\frac{1}{z}\frac{1}{1-z} = -\frac{1}{z}\sum z^k.$$

and

$$\frac{1}{z(z-1)} = \frac{1}{z^2(1-\frac{1}{z})} = \frac{1}{z^2}\sum \left(\frac{1}{z}\right)^k.$$

Exercise 3.3.15 (Laurent expansions about different points)
Find the Laurent expansion about $z = 0$ and $z = 1$ respectively of the following function:

$$f(z) := \frac{z+1}{z(z-1)}.$$

Solution:
Note: once you see that everything is in terms of powers of $(z - z_0)$, you're essentially done. For $z = 0$:

$$\begin{aligned} \frac{z+1}{z(z-1)} &= \frac{1}{z}\frac{z+1}{z-1} \\ &= -\frac{z+1}{z}\frac{1}{1-z} \\ &= -\left(1 + \frac{1}{z}\right)\sum_{k \ge 0} z^k. \end{aligned}$$

For $z = 1$:

$$\begin{aligned}
\frac{z+1}{z(z-1)} &= \frac{1}{z-1}\left(1 + \frac{1}{z}\right) \\
&= \frac{1}{z-1}\left(1 + \frac{1}{1-(1-z)}\right) \\
&= \frac{1}{z-1}\left(1 + \sum_{k\geq 0}(1-z)^k\right) \\
&= \frac{1}{z-1}\left(1 + \sum_{k\geq 0}(-1)^k(z-1)^k\right).
\end{aligned}$$

Exercise 3.3.16 (?)
Show that a real-valued holomorphic function must be constant.

4 Cauchy's Theorem

4.1 Complex Integrals

Definition 4.1.1 (Complex Integral)

$$\int_\gamma f dz := \int_I f(\gamma(t))\gamma'(t)\,dt = \int_\gamma (u+iv)\,dx \wedge (-v+iu)\,dy.$$

Theorem 4.1.2 ***(Cauchy-Goursat Theorem).***
If f is holomorphic on a region Ω with $\pi_1\Omega = 1$, then for any closed path $\gamma \subseteq \Omega$,

$$\int_\gamma f(z)\,dz = 0.$$

Slogan 4.1.3
Closed path integrals of holomorphic functions vanish.

4.2 Applications of Cauchy's Theorem

4.2.1 Integral Formulas and Estimates

See reference

Theorem 4.2.1 ***(Cauchy Integral Formula).***
Suppose f is holomorphic on Ω, then for any $z_0 \in \Omega$ and any open disc $\overline{D_R(z_0)}$ such that $\gamma := \partial \overline{D_R(z_0)} \subseteq \Omega$,

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z_0} \, d\xi$$

and

$$\frac{\partial^n f}{\partial z^n}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi.$$

Proof. It follows from a consequence of Cauchy's theorem (see above) that if $C(z_0, r)$ denotes the circle of radius r around z_0 for a sufficiently small $r > 0$ then

$$\begin{aligned} \left|\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz - f(z_0)\right| &= \left|\frac{1}{2\pi i}\int_{C(z_0,r)} \frac{f(z)-f(z_0)}{z-z_0}dz\right| \\ &= \left|\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{i\theta})-f(z_0)}{re^{i\theta}} ire^{i\theta}d\theta\right| \\ &\le \frac{1}{2\pi}2\pi \times \sup_{\theta\in[0,2\pi)}|f(z_0+re^{i\theta})-f(z_0)| \\ &\quad (\text{ by } ML \text{ inequality}). \end{aligned}$$

As f is continuous it follows that the righthand side goes to zero as r tends to zero. This completes the proof. □

Figure 2: image_2021-05-27-16-54-06

Proof (?). ■

Proof (?).

Proof. (*) Using Cauchy's integral formula we can write that

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} = \lim_{h\to 0}\frac{1}{2\pi i h}\int_C\left(\frac{f(z)}{z-z_0-h}-\frac{f(z)}{z-z_0}\right)dz \\ &\quad (C \text{ is so chosen that the point } z_0+h \text{ is enclosed by } C) \\ &= \lim_{h\to 0}\frac{1}{2\pi i h}\int_C\frac{f(z)h}{(z-z_0-h)(z-z_0)}dz. \end{aligned}$$

So we need to prove that

$$\begin{aligned} &\left|\int_C\frac{f(z)}{(z-z_0-h)(z-z_0)}dz-\int_C\frac{f(z)}{(z-z_0)^2}dz\right| \\ &= \left|\int_C\frac{f(z)h}{(z-z_0-h)(z-z_0)^2}dz\right| \to 0, \text{ as } \quad h\to 0. \end{aligned}$$

We will basically use ML inequality to prove this. Note that, as f is continuous it is bounded on C by M (say). Let $\alpha = \min\{|z-z_0| : z\in C\}$. Then $|z-z_0|^2 \geq \alpha^2$ and $\alpha \leq |z-z_0| = |z-z_0-h+h| \leq |z-z_0-h|+|h|$ and hence for $|h|\leq \frac{\alpha}{2}$ (after all h is going to be small) we get $|z-z_0-h| \geq \alpha-|h| \geq \frac{\alpha}{2}$. Therefore

$$\left|\int_C\frac{f(z)h}{(z-z_0-h)(z-z_0)^2}dz\right| \leq \frac{M|h|l}{\frac{\alpha}{2}\alpha^2} = \frac{2M|h|l}{\alpha^3} \to 0,$$

as $h\to 0$. By repeating exactly the same technique we get $f^2(z_0) = \frac{2!}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^3}dz$ and so on.

□

■

Theorem 4.2.2 ***(Cauchy's Inequality / Cauchy's Estimate).***
For $z_0 \in D_R(z_0) \subset \Omega$, setting $M := \sup_{z\in\gamma}|f(z)|$ so $|f(z)| \leq M$ on γ

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{2\pi}\int_0^{2\pi}\frac{M}{R^{n+1}}R\,d\theta = \frac{Mn!}{R^n}.$$

Proof (of Cauchy's inequality).

- Given $z_0\in\Omega$, pick the largest disc $D_R(z_0)\subset\Omega$ and let $C=\partial D_R$.
- Then apply the integral formula.

$$\begin{aligned}
\left|f^{(n)}(z_0)\right| &= \left|\frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,d\zeta\right| \\
&= \left|\frac{n!}{2\pi i}\int_0^{2\pi} \frac{f\left(z_0 + re^{i\theta}\right) rie^{i\theta}}{\left(re^{i\theta}\right)^{n+1}}\,d\theta\right| \\
&\le \frac{n!}{2\pi}\int_0^{2\pi} \left|\frac{f\left(z_0 + re^{i\theta}\right) rie^{i\theta}}{\left(re^{i\theta}\right)^{n+1}}\right| d\theta \\
&= \frac{n!}{2\pi}\int_0^{2\pi} \frac{\left|f\left(z_0 + re^{i\theta}\right)\right|}{r^n}\,d\theta \\
&\le \frac{n!}{2\pi}\int_0^{2\pi} \frac{M}{r^n}\,d\theta \\
&= \frac{Mn!}{r^n}.
\end{aligned}$$

■

Slogan 4.2.3
The nth Taylor coefficient of an analytic function is at most $\sup\limits_{|z|=R} |f|/R^n$.

Theorem 4.2.4 ***(Mean Value Property for Holomorphic Functions).***
If f is holomorphic on $D_r(z_0)$

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta = \frac{1}{\pi r^2}\iint_{D_r(z_0)} f(z)\,dA.$$

Taking the real part of both sides, one can replace $f = u + iv$ with u.

4.2.2 Liouville

Theorem 4.2.5 ***(Liouville's Theorem).***
If f is entire and bounded, f is constant.

Proof (of Liouville).

- Since f is bounded, $f(z) \le M$ uniformly on $\mathbb{C}$.
- Apply Cauchy's estimate for the 1st derivative:

$$|f'(z)| \le \frac{1!\|f\|_{C_R}}{R} \le \frac{M}{R} \overset{R\to\infty}{\longrightarrow} 0,$$

so $f'(z) = 0$ for all z.

■

Exercise 2.E. [SSh03, 2.15] *Suppose f is continuous and non-zero on $\overline{\mathbb{D}}$ and holomorphic on $\mathbb{D}$ such that $|f(z)| = 1$ for all $|z| = 1$. Show that f is then constant.*

Figure 3: image_2021-05-17-11-54-14

Exercise 4.2.6 (?)

4.2.3 Continuation Principle

Theorem 4.2.7 ***(Continuation Principle / Identity Theorem).***
If f is holomorphic on a bounded connected domain Ω and there exists a sequence $\{z_i\}$ with a limit point in Ω such that $f(z_i) = 0$, then $f \equiv 0$ on Ω.

Slogan 4.2.8
Two functions agreeing on a set with a limit point are equal on a domain.

Proof (?).
Apply Improved Taylor Theorem?

todo

■

Exercise 2.D. [SSh03, 2.13] *If f is holomorphic on a region Ω and for each $z_0 \in \Omega$ at least one coefficient in the power series expansion $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ is zero. Then show that f is a polynomial.*

Figure 4: image_2021-05-17-11-53-33

Exercise 4.2.9 (?)

4.3 Exercises

Exercise 4.3.1 (Primitives imply vanishing integral)
Show that if f has a primitive F on Ω then $\displaystyle\int_\gamma f = 0$ for every closed curve $\gamma \subseteq \Omega$.

Exercise 4.3.2 (?)
Prove the uniform limit theorem for holomorphic functions: if $f_n \to f$ locally uniformly and each f_n is holomorphic then f is holomorphic.

Solution:
This is S&S Theorem 5.2. Statement: if $f_n \to f$ uniformly locally uniformly on Ω then f is holomorphic on Ω.

- Let $D \subset \Omega$ with $\overline{\mathbb{D}} \subset \Omega$ and $\Delta \subset D$ be a triangle.
- Apply Goursat: $\displaystyle\int_\Delta f_n = 0$.
- $f_n \to f$ uniformly on Δ since it is closed and bounded and thus compact by Heine-Borel, so f is continuous and

$$\lim_n \int_\Delta f_n = \int_\Delta \lim_n f_n := \int_\Delta f.$$

- Apply Morera's theorem: $\displaystyle\int_\Delta f$ vanishes on every triangle in Ω, so f is holomorphic on Ω.

Exercise 4.3.3 (?)
Prove that if $f_n \to f$ locally uniformly with f_n holomorphic, then $f_n' \to f'$ locally uniformly and f' is holomorphic.

Solution:

- Simplifying step: for some reason, it suffices to assume $f_n \to f$ uniformly on all of Ω?
- Take Ω_R to be Ω with a buffer of R, so $d(z, \partial\Omega) > R$ for every $z \in \overline{\Omega_R}$.
- It suffices to show the following bound for F any holomorphic function on Ω:

$$\sup_{z\in\Omega_R} |F'(z)| \leq \frac{1}{R} \sup_{\zeta\in\Omega} |F(\zeta)| \qquad \forall R,$$

  where on the right we take the sup over all Ω.
  - Then take $F := f_n - f$ and $R \to 0$ to conclude, since the right-hand side is a constant not depending on Ω_R.
- For any $z \in \Omega_R$, we have $\overline{D_R(z)} \subseteq \Omega_R$, so Cauchy's integral formula can be applied:

- $$\begin{aligned}
|F'(z)| &= \left| \frac{1}{2\pi i} \int_{\partial D_R(z)} \frac{F(\xi)}{(\xi - z)^2}\, d\xi \right| \\
&\le \frac{1}{2\pi} \int_{\partial D_R(z)} \frac{|F(\xi)|}{|\xi - z|^2}\, d\xi \\
&\le \frac{1}{2\pi} \int_{\partial D_R(z)} \frac{\sup_{\zeta\in\Omega} |F(\zeta)|}{|\xi - z|^2}\, d\xi \\
&= \frac{1}{2\pi} \sup_{\zeta\in\Omega} |F(\zeta)| \int_{\partial D_R(z)} \frac{1}{R^2}\, d\xi \\
&= \frac{1}{2\pi} \sup_{\zeta\in\Omega} |F(\zeta)| \frac{1}{R^2} \int_{\partial D_R(z)} d\xi \\
&= \frac{1}{2\pi} \sup_{\zeta\in\Omega} |F(\zeta)| \frac{1}{R^2} 2\pi R \\
&\le \frac{1}{2\pi} \sup_{\zeta\in\Omega} |F(\zeta)| \frac{1}{R^2} (2\pi R) \\
&= \frac{1}{R} \sup_{\zeta\in\Omega} |F(\zeta)|.
\end{aligned}$$

- Now
$$\|f_n' - f'\|_{\infty,\Omega_R} \le \frac{1}{R} \|f_n - f\|_{\infty,\Omega},$$
where if R is fixed then by uniform convergence of $f_n \to f$, for n large enough $\|f_n - f\| < \varepsilon/R$.

4.4 Morera's Theorem

Theorem 4.4.1 ***(Morera's Theorem).***
If f is continuous on a domain Ω and $\displaystyle\int_T f = 0$ for every triangle $T \subset \Omega$, then f is holomorphic.

Slogan 4.4.2
If every integral along a triangle vanishes, implies holomorphic.

Corollary 4.4.3 ***(Sufficient condition for a sequence to converge to a holomorphic function).***
If $\{f_n\}_{n\in\mathbb{N}}$ is a holomorphic sequence on a region Ω which uniformly converges to f on every compact subset $K \subseteq \Omega$, then f is holomorphic, and $f_n' \to f'$ uniformly on every such compact subset K.

Proof (?).
Commute limit with integral and apply Morera's theorem. ■

Remark 4.4.4: This can be applied to series of the form $\sum_k f_k(z)$.

4.4.1 Symmetric Regions

In this section, take Ω to be a region symmetric about the real axis, so $z \in \Omega \iff \bar{z} \in \Omega$. Partition this set as $\Omega^+ \subseteq \mathbb{H}, I \subseteq \mathbb{R}, \Omega^- \subseteq \overline{\mathbb{H}}$.

Theorem 4.4.5 ***(Symmetry Principle).***
Suppose that f^+ is holomorphic on Ω^+ and f^- is holomorphic on Ω^-, and f extends continuously to I with $f^+(x) = f^-(x)$ for $x \in I$. Then the following piecewise-defined function is holomorphic on Ω:

$$f(z) := \begin{cases} f^+(z) & z \in \Omega^+ \\ f^-(z) & z \in \Omega^- \\ f^+(z) = f^-(z) & z \in I. \end{cases}$$

Proof (?).
Apply Morera? ■

Theorem 4.4.6 ***(Schwarz Reflection).***
If f is continuous and holomorphic on $\mathbb{H}^+$ and real-valued on $\mathbb{R}$, then the extension defined by $F^-(z) = \overline{f(\bar{z})}$ for $z \in \mathbb{H}^-$ is a well-defined holomorphic function on $\mathbb{C}$.

Proof (?).
Apply the symmetry principle. ■

Remark 4.4.7: $\mathbb{H}^+, \mathbb{H}^-$ can be replaced with any region symmetric about a line segment $L \subseteq \mathbb{R}$.

5 | Zeros and Singularities

Definition 5.0.1 (Singularity)
A point z_0 is an **isolated singularity** if $f(z_0)$ is undefined but $f(z)$ is defined in a punctured neighborhood $D(z_0) \setminus \{z_0\}$ of z_0.
There are three types of isolated singularities:

- Removable singularities
- Poles
- Essential singularities

Definition 5.0.2 (Removable Singularities)
If z_0 is a singularity of f. then z_0 is a **removable singularity** iff there exists a holomorphic function g such that $f(z) = g(z)$ in a punctured neighborhood of z_0. Equivalently,

$$\lim_{z \to z_0} (z - z_0) f(z) = 0.$$

Equivalently, f is bounded on a neighborhood of z_0.

Remark 5.0.3: Singularities can be classified by Laurent expansions $f(z) = \sum_{k \in \mathbb{Z}} c_k z^k$:

- Essential singularity: infinitely many negative terms.
- Pole of order N: truncated at $k = -N$, so $c_{N-\ell} = 0$ for all ℓ.
- Removable singularity: truncated at $k = 0$, so $c_{\leq -1} = 0$.

Example 5.0.4 ***(Removable singularities):***

- $f(z) := \sin(z)/z$ has a removable singularity at $z = 0$, and one can redefine $f(0) := 1$.
- If $f(z) = p(z)/q(z)$ with $q(z_0) = 0$ and $p(z_0) = 0$, then z_0 is removable with $f(z_0) := p'(z_0)/q'(z_0)$.

Example 5.0.5 ***(Essential singularities):*** $f(z) := e^{1/z}$ has an essential singularity at $z = 0$, since we can expand and pick up infinitely many negative terms:

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots .$$

In fact there exists a neighborhood of zero such that $f(U) = \mathbb{C} \setminus \{0\}$. Similarly $g(z) := \sin\left(\frac{1}{z}\right)$ has an essential singularity at $z = 0$, and there is a neighborhood V of zero such that $g(V) = \mathbb{C}$.

Example 5.0.6 ***(?):*** The singularities of a rational function are always isolated, since there are finitely many zeros of any polynomial. The function $F(z) := \operatorname{Log}(z)$ has a singularity at $z = 0$ that is **not** isolated, since every neighborhood intersects the branch cut $(-\infty, 0) \times \{0\}$, where F is not even defined. The function $G(z) := 1/\sin(\pi/z)$ has a non-isolated singularity at 0 and isolated singularities at $1/n$ for all n.

⚠ **Warning 5.0.7**
$f(z) := z^{\frac{1}{2}}$ has a singularity at zero that does not fall under this classification – $z = 0$ is a **branch singularity** and admits no Laurent expansion around $z = 0$.

A similar example: $(z(z-1))^{\frac{1}{2}}$ has two branch singularities at $z = 0, 1$.

Theorem 5.0.8 ***(Extension over removable singularities).***
If f is holomorphic on $\Omega \setminus \{z_0\}$ where z_0 is a removable singularity, then there is a unique holomorphic extension of f to all of Ω.

Proof (?).
Take γ to be a circle centered at z_0 and use

$$f(z) := \int_\gamma \frac{f(\xi)}{\xi - z}\, dx.$$

This is valid for $z \neq z_0$, but the right-hand side is analytic. (?) ∎

Revisit

Theorem 5.0.9 ***(Improved Taylor Remainder Theorem).***
If f is analytic on a region Ω containing z_0, then f can be written as

$$f(z) = \left(\sum_{k=0}^{n-1} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k\right) + R_n(z)(z - z_0)^n,$$

where R_n is analytic.

Definition 5.0.10 (Zeros)
If f is analytic and not identically zero on Ω with $f(z_0) = 0$, then there exists a nonvanishing holomorphic function g such that

$$f(z) = (z - z_0)^n g(z).$$

We refer to z_0 as a **zero of order** n.

Definition 5.0.11 (Poles (and associated terminology))
A *pole* z_0 of a function $f(z)$ is a zero of $g(z) := \dfrac{1}{f(z)}$. Equivalently, $\lim_{z \to z_0} f(z) = \infty$. In this case there exists a minimal n and a holomorphic h such that

$$f(z) = (z - z_0)^{-n} h(z).$$

Such an n is the *order* of the pole. A pole of order 1 is said to be a *simple pole*.

Definition 5.0.12 (Principal Part and Residue)
If f has a pole of order n at z_0, then there exist a holomorphic G in a neighborhood of z_0 such that

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0} + G(z) := P(z) + G(z).$$

The term $P(z)$ is referred to as the *principal part of f at z_0* consists of terms with negative degree, and the *residue* of f at z_0 is the coefficient a_{-1}.

Definition 5.0.13 (Essential Singularity)
A singularity z_0 is *essential* iff it is neither removable nor a pole. Equivalently, a Laurent series expansion about z_0 has a principal part with infinitely many terms.

Theorem 5.0.14 ***(Casorati-Weierstrass).***
If f is holomorphic on $\Omega \setminus \{z_0\}$ where z_0 is an essential singularity, then for every $V \subset \Omega \setminus \{z_0\}$, $f(V)$ is dense in $\mathbb{C}$.

Slogan 5.0.15
The image of a punctured disc at an essential singularity is dense in $\mathbb{C}$.

Proof (of Casorati-Weierstrass).
Pick $w \in \mathbb{C}$ and suppose toward a contradiction that $D_R(w) \cap f(V)$ is empty. Consider

$$g(z) := \frac{1}{f(z) - w},$$

and use that it's bounded to conclude that z_0 is either removable or a pole for f.

■

Definition 5.0.16 (Singularities at infinity)
For any f holomorphic on an unbounded region, we say $z = \infty$ is a singularity (of any of the above types) of f if $g(z) := f(1/z)$ has a corresponding singularity at $z = 0$.

Definition 5.0.17 (Meromorphic)
A function $f : \Omega \to \mathbb{C}$ is *meromorphic* iff there exists a sequence $\{z_n\}$ such that

- $\{z_n\}$ has no limit points in Ω.
- f is holomorphic in $\Omega \setminus \{z_n\}$.
- f has poles at the points $\{z_n\}$.

Equivalently, f is holomorphic on Ω with a discrete set of points delete which are all poles of f.

Theorem 5.0.18 ***(Meromorphic implies rational).***
Meromorphic functions on $\mathbb{C}$ are rational functions.

Proof (?).
Consider $f(z) - P(z)$, subtracting off the principal part at each pole z_0, to get a bounded entire function and apply Liouville.

■

Theorem 5.0.19 ***(Riemann Extension Theorem).***
A singularity of a holomorphic function is removable if and only if the function is bounded in some punctured neighborhood of the singular point.

6 Residues

6.1 Basics

Remark 6.1.1: Check: do you need residues at all?? You may be able to just compute an integral!

- Directly by parameterization:

$$\int_\gamma f\,dz = \int_a^b f(z(t))\,z'(t)\,dt \qquad \text{for } z(t) \text{ a parameterization of } \gamma,$$

- Finding a primitive F, then

$$\int_\gamma f = F(b) - F(a).$$

  - Note: you can parameterize a circle around z_0 using

$$z = z_0 + re^{i\theta}.$$

Fact 6.1.2 (Integrating z^k around S^1 powers residues)
The major fact that reduces integrals to residues:

$$\int_\gamma z^k\,dz = \int_0^{2\pi} e^{ik\theta} i e^{i\theta}\,d\theta = \int_0^{2\pi} e^{i(k+1)\theta\,d\theta} = \begin{cases} 2\pi i & k = -1 \\ 0 & \text{else.} \end{cases}.$$

Thus

$$\int \sum_{k\geq -M} c_k z^k = \sum_{k \geq -M} \int c_k z^k = 2\pi i c_{-1},$$

i.e. the integral picks out the c_{-1} coefficient in a Laurent series expansion.

Example 6.1.3 ***(?):*** Consider

$$f(z) := \frac{e^{iz}}{1+z^2}$$

where $z \neq \pm i$, and attempt to integrate

$$\int_{\mathbb{R}} f(z)\,dz.$$

Use a semicircular contour γ_R where $z = Re^{it}$ and check

$$\begin{aligned}\sup_{z\in\gamma_R} |f(z)| &= \max_{t\in[0,\pi} \frac{1}{1+(Re^{it})^2} \\ &= \max_{t\in[0,\pi} \frac{1}{1+R^2e^{2it}} \\ &= \frac{1}{R^2-1}.\end{aligned}$$

6.2 Estimates

Proposition 6.2.1 ***(Length bound / ML Estimate).***

$$\left|\int_\gamma f\right| \leq ML := \sup_{z\in\gamma} |f| \cdot \operatorname{length}(\gamma).$$

Proof (?).

$$\left|\int_\gamma f(z)dz\right| \leq \sup_{t\in[a,b]} |f(z(t))| \int_a^b |z'(t)|\, dt \leq \sup_{z\in\gamma} |f(z)| \cdot \operatorname{length}(\gamma).$$

■

Proposition 6.2.2 ***(Jordan's Lemma).***
Suppose that $f(z) = e^{iaz}g(z)$ for some g, and let $C_R := \left\{z = Re^{it} \;\middle|\; t \in [0,\pi]\right\}$. Then

$$\left|\int_{C_R} f(z)\, dz\right| \leq \frac{\pi M_R}{a}$$

where $M_R := \sup_{t\in[0,\pi]} \left|g(Re^{it})\right|$.

Proof (?).

$$
\begin{aligned}
\left|\int_{C_R} f(z)\,dz\right| &= \left|\int_{C_R} e^{iaz} g(z)\,dz\right| \\
&= \left|\int_{[0,\pi]} e^{ia\left(Re^{it}\right)} g(Re^{it}) iRe^{it}\,dt\right| \\
&\leq \int_{[0,\pi]} \left|e^{ia\left(Re^{it}\right)} g(Re^{it}) iRe^{it}\right| dt \\
&= R\int_{[0,\pi]} \left|e^{ia\left(Re^{it}\right)} g(Re^{it})\right| dt \\
&\leq RM_R \int_{[0,\pi]} \left|e^{ia\left(Re^{it}\right)}\right| dt \\
&= RM_R \int_{[0,\pi]} e^{\Re\left(iaRe^{it}\right)}\,dt \\
&= RM_R \int_{[0,\pi]} e^{\Re(iaR(\cos(t)+i\sin(t)))}\,dt \\
&= RM_R \int_{[0,\pi]} e^{-aR\sin(t)}\,dt \\
&= 2RM_R \int_{[0,\pi/2]} e^{-aR\sin(t)}\,dt \\
&\leq 2RM_R \int_{[0,\pi/2]} e^{-aR\left(\frac{2t}{\pi}\right)}\,dt \\
&= 2RM_R \left(\frac{\pi}{2aR}\right)\left(1-e^{-aR}\right) \\
&= \frac{\pi M_R}{a}.
\end{aligned}
$$

where we've used that on $[0,\pi/2]$, there is an inequality $2t/\pi \leq \sin(t)$. This is obvious from a picture, since $\sin(t)$ is a height on S^1 and $2t/\pi$ is a height on a diagonal line:

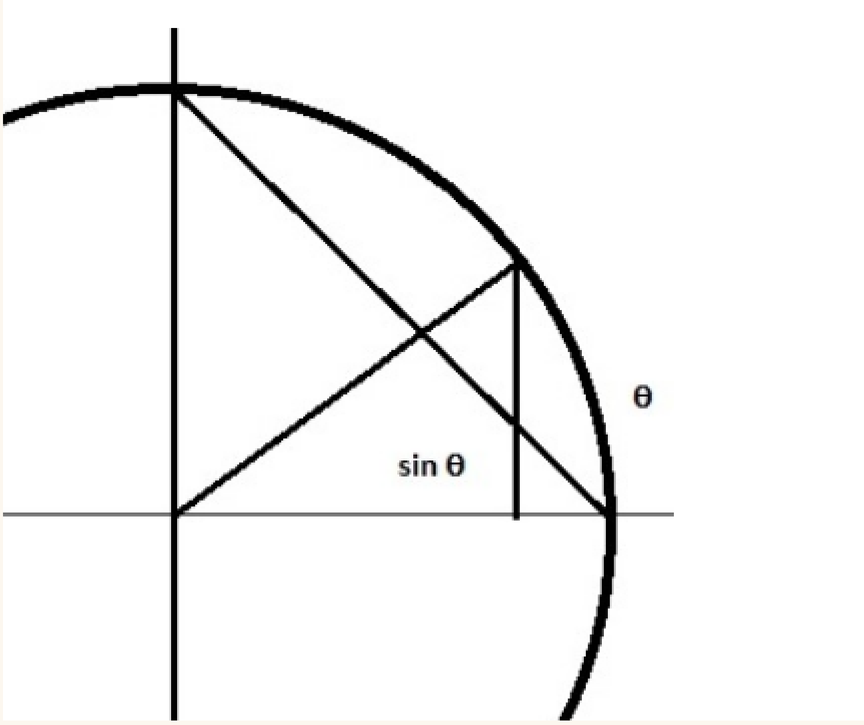


Figure 5: image_2021-06-09-01-29-22

■

6.3 Residue Formulas

Theorem 6.3.1 ***(The Residue Theorem).***
Let f be meromorphic on a region Ω with poles $\{z_1, z_2, \cdots, z_N\}$. Then for any $\gamma \in \Omega \setminus \{z_1, z_2, \cdots, z_N\}$,

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_{j=1}^{N} n_\gamma(z_j) \operatorname*{Res}_{z=z_j} f.$$

If γ is a toy contour, then

$$\frac{1}{2\pi i}\int_\gamma f\,dz = \sum_{j=1}^{N} \operatorname*{Res}_{z=z_j} f.$$

Proposition 6.3.2 ***(Residue formula for higher order poles).***

If f has a pole z_0 of order n, then

$$\operatorname*{Res}_{z=z_0} f = \lim_{z\to z_0} \frac{1}{(n-1)!}\left(\frac{\partial}{\partial z}\right)^{n-1}(z-z_0)^n f(z).$$

Proposition 6.3.3 ***(Residue formula for simple poles).***
As a special case, if z_0 is a simple pole of f, then

$$\operatorname*{Res}_{z=z_0} f = \lim_{z\to z_0}(z-z_0)f(z).$$

Corollary 6.3.4 ***(Better derivative formula that sometimes works for simple poles).***
If additionally $f = g/h$ where $h(z_0) = 0$ and $h'(z_0) \neq 0$,

$$\operatorname*{Res}_{z=z_0} \frac{g(z)}{h(z)} = \frac{g(z_0)}{h'(z_0)}.$$

Proof (?).
Apply L'Hopital:

$$(z-z_0)\frac{g(z)}{h(z)} = \frac{(z-z_0)g(z)}{h(z)} \overset{LH}{=} \frac{g(z)+(z-z_0)g'(z)}{h'(z)} \xrightarrow{z\to z_0} \frac{g(z_0)}{h'(z_0)}.$$

■

Example 6.3.5 ***(Residue of a simple pole (order 1)):*** Let $f(z) = \dfrac{1}{1+z^2}$, then $g(z) = 1, h(z) = 1+z^2$, and $h'(z) = 2z$ so that $h'(i) = 2i \neq 0$. Thus

$$\operatorname*{Res}_{z=i} \frac{1}{1+z^2} = \frac{1}{2i}.$$

Proposition 6.3.6 ***(Residue at infinity).***

$$\operatorname*{Res}_{z=\infty} f(z) = \operatorname*{Res}_{z=0} g(z) \qquad g(z) := -\frac{1}{z^2} f\left(\frac{1}{z}\right).$$

6.3.1 Exercises

Some good computations here.

Exercise 6.3.7
Show that the complex zeros of $f(z) := \sin(\pi z)$ are exactly $\mathbb{Z}$, and each is order 1. Calculate the residue of $1/\sin(\pi x)$ at $z = n \in \mathbb{Z}$.

Exerci

Exercise 3.A. [SSh03, 3.1] *Show that the complex zeros of* $\sin \pi z$ *are exactly at the integers, and are each of order 1. Calculate the residue of* $1/\sin \pi x$ *are* $z = n \in \mathbb{Z}$.

Exe

Exercise 3.C. [SSh03, 3.8] *Prove that*

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Exercise 6.3.8 (?)

$$\int_{\mathbb{R}} \frac{1}{(1+x^2)^2}\,dx.$$

Solution:

- Factor $(1+z^2)^2 = ((z-i)(z+i))^2$, so f has poles at $\pm i$ of order 2.
- Take a semicircular contour $\gamma := I_R \cup D_R$, then $f(z) \approx 1/z^4 \to 0$ for large R and $\displaystyle\int_{D_R} f \to 0$.
- Note $\displaystyle\int_{I_R} f \to \int_{\mathbb{R}} f$, so $\displaystyle\int_\gamma f \to \int_{\mathbb{R}} f$.
- $\displaystyle\int_\gamma f = 2\pi i \sum_{z_0} \operatorname*{Res}_{z=z_0} f$, and $z_0 = i$ is the only pole in this region.
- Compute

$$\begin{aligned}
\operatorname*{Res}_{z=i} f &= \lim_{z\to i} \frac{1}{(2-1)!}\frac{\partial}{\partial z}(z-i)^2 f(z) \\
&= \lim_{z\to i} \frac{\partial}{\partial z}\frac{1}{(z+i)^2} \\
&= \lim_{z\to i} \frac{-2}{(z+i)^3} \\
&= -\frac{2}{(2i)^3} \\
&= \frac{1}{4i}
\end{aligned}$$

$$\implies \int_\gamma f = \frac{2\pi i}{4i} = \pi/2,$$

Exercise 6.3.9 (?)

Use a direct Laurent expansion to show

$$\operatorname*{Res}_{z=0} \frac{1}{z-\sin(z)} = \frac{3!}{5\cdot 4}.$$

Note the necessity: one doesn't know the order of the pole at zero, so it's unclear how many derivatives to take.

Solution:
Expand:

$$\begin{aligned}
\frac{1}{z-\sin(z)} &= z^{-1}\left(1 - z^{-1}\sin(z)\right)^{-1} \\
&= z^{-1}\left(1 - z^{-1}\left(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots\right)\right)^{-1} \\
&= z^{-1}\left(1 - \left(1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \cdots\right)\right)^{-1} \\
&= z^{-1}\left(\frac{1}{3!}z^2 - \frac{1}{5!}z^4 + \cdots\right)^{-1} \\
&= z^{-1}\cdot 3!z^{-2}\left(1 - \frac{1}{5!/3!}z^2 + \cdots\right)^{-1} \\
&= \frac{3!}{z^3}\left(\frac{1}{1-\left(\frac{1}{5\cdot 4}z^2 + \cdots\right)}\right) \\
&= \frac{3!}{z^3}\left(1 + \left(\frac{1}{5\cdot 4}z^2\right) + \left(\frac{1}{5\cdot 4}z^2\right)^2 + \cdots\right) \\
&= 3!z^{-3} + \frac{3!}{5\cdot 4}z^{-1} + O(z)
\end{aligned}$$

.

Exercise 6.3.10 (?)
Compute

$$\operatorname*{Res}_{z=0} \frac{1}{z^2\sin(z)}.$$

Solution:

First expand $(\sin(z))^{-1}$:

$$\begin{aligned}\frac{1}{\sin(z)} &= \left(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots\right)^{-1} \\ &= z^{-1}\left(1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \cdots\right)^{-1} \\ &= z^{-1}\left(1 + \left(\frac{1}{3!}z^2 - \frac{1}{5!}z^4 + \cdots\right) + \left(\frac{1}{3!}z^2 - \cdots\right)^2 + \cdots\right) \\ &= z^{-1}\left(1 + \frac{1}{3!}z^2 \pm O(z^4)\right),\end{aligned}$$

using that $(1-x)^{-1} = 1 + x + x^2 + \cdots$.
Thus

$$\begin{aligned}z^{-2}(\sin(z))^{-1} &= z^{-2} \cdot z^{-1}\left(1 + \frac{1}{3!}z^2 \pm O(z^4)\right) \\ &= z^{-3} + \frac{1}{3!}z^{-1} + O(z).\end{aligned}$$

Exercise 6.3.11 (Keyhole contour and ML estimate)
Compute

$$\int_{[0,\infty]} \frac{\log(x)}{(1+x^2)^2}\,dx.$$

Solution:
Factor $(1+z^2)^2 = (z+i^2(z-i)^2$. Take a keyhole contour similar to the following:

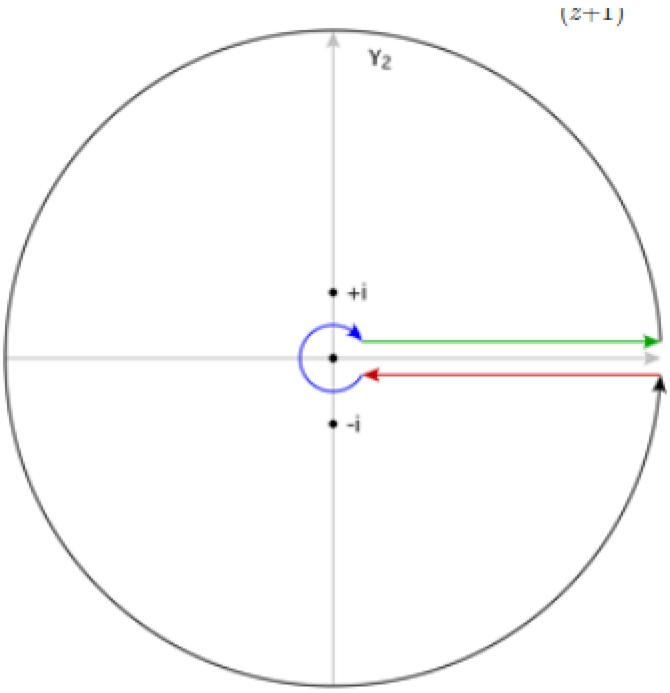


Figure 6: image_2021-06-09-02-11-59

Show that outer radius R and inner radius ρ circles contribute zero in the limit by the ML estimate? Compute the residues by just applying the formula and manually computing derivatives:

$$\begin{aligned}
\operatorname*{Res}_{z=\pm i} f(z) &= \lim_{z\to\pm i} \frac{\partial}{\partial z} \frac{\log^2(z)}{(z\pm i)^2} \\
&= \lim_{z\to\pm i} \frac{2\log(z)(z\pm i)^2 - 2(z\pm i)^2 \log^2(z)}{((z\pm i)^2)^2} \\
&= \frac{2\log(\pm i)(\pm 2i)^2 - 2(\pm 2i)^2 \log^2(\pm i)}{(\pm 2i)^4} \\
&=_? \frac{\pi}{4} \pm \frac{i\pi^2}{16}.
\end{aligned}$$

See p.4: http://www.math.toronto.edu/mmica/complex1.pdf

Exercise 6.3.12 (Sinc Function)
Show

$$\int_{(0,\infty)} \frac{\sin(x)}{x}\, dx = \frac{\pi}{2}.$$

Solution:
Take an indented semicircle. Let I be the original integral, then

$$I = \frac{1}{2i} \int_{\mathbb{R}} \frac{e^{iz} - 1}{z} \, dz.$$

Exercise 3.E. [SSh03, 3.14] *Prove that all entire functions that are also injective take the form* $f(z) = az + b$ *with* $a, b \in \mathbb{C}$ *and* $a \neq 0$.

Figure 7: image_2021-05-17-13-33-55

7 Counting Zeros and Poles

7.1 Argument Principle

Definition 7.1.1 (Winding Number)
For $\gamma \subseteq \Omega$ a closed curve not passing through a point z_0, the **winding number of** γ **about** z_0 is defined as

$$n_\gamma(z_0) \coloneqq \frac{1}{2\pi i} \int_\gamma \frac{1}{\xi - z_0} \, d\xi.$$

Theorem 7.1.2 ***(Argument Principle).***
For f meromorphic in γ° with zeros $\{z_j\}$ and poles $\{p_k\}$ repeated with multiplicity where γ does not intersect any zeros or poles, then

$$\Delta_\gamma \arg f(z) \coloneqq \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz = \sum_j n_\gamma(z_j) - \sum_k n_\gamma(p_k) = Z_f - P_f,$$

where Z_f and P_f are the number of zeros and poles respectively enclosed by γ, counted with multiplicity.

Proof (?).
Residue formula applied to $\frac{f'}{f}$? ■

Theorem 1. *1. If f is nonzero and nonsingular at z_0, then $\frac{f'}{f}$ is nonsingular at z_0.*

2. If f has a pole of order n at z_0 then $\frac{f'}{f}$ has a simple pole with residue equal to $-n$ at z_0.

3. If f has a zero of degree n at z_0 then $\frac{f'}{f}$ has a simple pole with residue equal to n at z_0.

Remark 7.1.3: This is useful in numerical computation: if you can compute this integral within an error $E < \pi$ where you know it doesn't contain a pole, you can determine if the contour contains a zero. Canonical example: integrals in rectangles around $\Re(z) = 1/2$ for $\zeta(s)$.

Exercise 7.1.4 (?)
Show that $\partial_{\ln}(fg) = \partial_{\ln} f + \partial_{\ln} g$, and thus

$$\frac{f'(x)}{f(x)} = \frac{g'(x)}{g(x)} + \frac{h'(x)}{h(x)}.$$

7.2 Rouché

Corollary 7.2.1 ***(Rouché's Theorem).***
If f, g are analytic on a domain Ω with finitely many zeros in Ω and $\gamma \subset \Omega$ is a closed curve surrounding each point exactly once, where $|g| < |f|$ on γ, then f and $f + g$ have the same number of zeros.
Alternatively:
Suppose $f = g + h$ with $g \neq 0, \infty$ on γ with $|g| > |h|$ on γ. Then

$$\Delta_\gamma \arg(f) = \Delta_\gamma \arg(h) \quad \text{and } Z_f - P_f = Z_g - P_g.$$

Prove

Corollary 7.2.2 ***(Open Mapping).***
Any holomorphic non-constant map is an open map.

Prove

Corollary 7.2.3 ***(Maximum Modulus).***
If f is holomorphic and nonconstant on an open connected region Ω, then $|f|$ can not attain a maximum on Ω. If Ω is bounded and f is continuous on $\overline{\Omega}$, then $\max_{\overline{\Omega}} |f|$ occurs on $\partial\Omega$.
Conversely, if f attains a local supremum at $z_0 \in \Omega$, then f is constant on Ω.

Prove

Corollary 7.2.4 *(?).*
If f is nonzero on Ω, then f attains a minimum on $\partial\Omega$. This follows from applying the MMP to $1/f$.

7.3 Counting Zeros

Example 7.3.1:

- Take $P(z) = z^4 + 6z + 3$.
- On $|z| < 2$:
  - Set $f(z) = z^4$ and $g(z) = 6z + 3$, then $|g(z)| \leq 6|z| + 3 = 15 < 16 = |f(z)|$.
  - So P has 4 zeros here.
- On $|z| < 1$:
  - Set $f(z) = 6z$ and $g(z) = z^4 + 3$.
  - Check $|g(z)| \leq |z|^4 + 3 = 4 < 6 = |f(z)|$.
  - So P has 1 zero here.

Example 7.3.2:

- Claim: the equation $\alpha z e^z = 1$ where $|\alpha| > e$ has exactly one solution in $\mathbb{D}$.
- Set $f(z) = \alpha z$ and $g(z) = e^{-z}$.
- Estimate at $|z| = 1$ we have $|g| = |e^{-z}| = e^{-\Re(z)} \leq e^1 < |\alpha| = |f(z)|$
- f has one zero at $z_0 = 0$, thus so does $f + g$.

8 | Conformal Maps

8.1 Linear Fractional Transformations

Definition 8.1.1 (Conformal Map / Biholomorphism)
A map f is **conformal** on Ω iff f is complex-differentiable, $f'(z) \neq 0$ for $z \in \Omega$, and f preserves signed angles (so f is orientation-preserving). Conformal implies holomorphic, and a bijective conformal map has a holomorphic inverse. A bijective conformal map $f : U \to V$ is called a **biholomorphism**, and we say U and V are **biholomorphic**. Self-biholomorphisms of a domain Ω form a group $\underset{\mathbb{C}}{\mathrm{Aut}}(\Omega)$.

Remark 8.1.2: There is an oft-used weaker condition that $f'(z) \neq 0$ for any point. Note that that this condition alone doesn't necessarily imply f is holomorphic, since anti-holomorphic maps may have nonzero derivatives. For example, take $f(z) = \bar{z}$, so $f(x+iy) = x - iy$ – this does not satisfy the Cauchy-Riemann equations.

Remark 8.1.3: A bijective holomorphic map automatically has a holomorphic inverse. This can be weakened: an injective holomorphic map satisfies $f'(z) \neq 0$ and f^{-1} is well-defined on its range and holomorphic.

Definition 8.1.4 (Linear fractional transformation / Mobius transformation)
A map of the following form is a *linear fractional transformation*:

$$T(z) = \frac{az+b}{cz+d},$$

where the denominator is assumed to not be a multiple of the numerator. These have inverses given by

$$T^{-1}(w) = \frac{dw-b}{-cw+a}.$$

Proposition 8.1.5 ***(?).***
Given any three points z_1, z_2, z_3, the following Mobius transformation sends them to $1, 0, \infty$ respectively:

$$f(z) := \frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)}.$$

Such a map is sometimes denoted (z, z_1, z_2, z_3).

Example 8.1.6 ***(?):***

- $(z, i, 1, -1) : \mathbb{D} \to \mathbb{H}$
- $(z, 0, -1, 1) : \mathbb{D} \cap \mathbb{H} \to Q_1$.

Theorem 8.1.7 ***(Cayley Transform).***
The fractional linear transformation given by $F(z) = \dfrac{i-z}{i+z}$ maps $\mathbb{D} \to \mathbb{H}$ with inverse $G(w) = i\dfrac{1-w}{1+w}$.

Theorem 8.1.8 ***(Characterization of conformal maps).***
Conformal maps $\mathbb{D} \to \mathbb{D}$ have the form

$$g(z) = \lambda \frac{1-a}{1-\bar{a}z}, \quad |a| < 1, \quad |\lambda| = 1.$$

Theorem 8.1.9 ***(Riemann Mapping).***
If Ω is simply connected, nonempty, and not $\mathbb{C}$, then for every $z_0 \in \Omega$ there exists a unique

conformal map $F : \Omega \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$.
Thus any two such sets Ω_1, Ω_2 are conformally equivalent.

8.2 By Type

Notation	Definition
$\mathbb{D} := \left\{ z \,\middle|\, \lvert z \rvert \leq 1 \right\}$	The unit disc
$\mathbb{H} := \left\{ x + iy \,\middle|\, y > 0 \right\}$	The upper half-plane
$X_{\frac{1}{2}}$	A "half version of X", see examples
$\mathbb{H}_{\frac{1}{2}}$	The first quadrant
$\mathbb{D}_{\frac{1}{2}}$	The portion of the first quadrant inside the unit disc
$L := \left\{ x + iy \,\middle|\, x \in \mathbb{R},\, 0 < y < \pi \right\}$	The horizontal strip

Remark 8.2.1 ***(Notation):***

Theorem 8.2.2 ***(Classification of Conformal Maps).***
There are 8 major types of conformal maps:

Type/Domains	Formula
Translation	$z \mapsto z + h$
Dilation	$z \mapsto cz$
Rotation	$z \mapsto e^{i\theta}$
Sectors to sectors	$z \mapsto z^n$
$\mathbb{D}_{\frac{1}{2}} \to \mathbb{H}_{\frac{1}{2}}$, the first quadrant	$z \mapsto \dfrac{1+z}{1-z}$
$\mathbb{H} \to S$	$z \mapsto \log(z)$
$\mathbb{D}_{\frac{1}{2}} \to L_{\frac{1}{2}}$	$z \mapsto \log(z)$
$S_{\frac{1}{2}} \to \mathbb{D}_{\frac{1}{2}}$	$z \mapsto e^{iz}$
$\mathbb{D}_{\frac{1}{2}} \to \mathbb{H}$	$z \mapsto \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$
$L_{\frac{1}{2}} \to \mathbb{H}$	$z \mapsto \sin(z)$

Pictures!

Proposition 8.2.3 ***(Half-plane to Disc).***

$$\begin{aligned} F : \mathbb{H}^\circ &\rightleftharpoons \mathbb{D}^\circ \\ \left\{ z \;\middle|\; \Im(z) > 0 \right\} &\rightleftharpoons \left\{ w \;\middle|\; |w| < 1 \right\} \\ z &\mapsto \frac{i - z}{i + z} \\ i\left(\frac{1 - w}{1 + w}\right) &\leftarrow\!\shortmid w. \end{aligned}$$

Boundary behavior: This maps $\mathbb{R} \to \partial\mathbb{D}$, where $F(\infty) = -1$, and as $x \in \mathbb{R}$ ranges from $-\infty \to \infty$, $F(x)$ travels from $z = -1$ counter-clockwise through S^1 (starting at $z = -1$ and moving through the lower half first). So this extends to a map $\mathbb{H} \to \mathbb{D}$.

Mnemonic: every $z \in \mathbb{H}$ is closed to i than $-i$.

Remark 8.2.4: Some write a similar map:

$$\begin{aligned} \mathbb{H}^\circ &\to \mathbb{D}^\circ \\ z &\mapsto \frac{z - i}{z + i}. \end{aligned}$$

This is just a composition of the above map with the flip $z \mapsto -z$:

$$-\frac{i - z}{i + z} = \frac{z - i}{i + z} = \frac{z - i}{z + i}.$$

Proposition 8.2.5 ***(Right half-plane to Disc).***

$$\begin{aligned} \mathbb{H}_R &\rightleftharpoons \mathbb{D} \\ \left\{ z \;\middle|\; \Re(z) > 0 \right\} &\rightleftharpoons \left\{ w \;\middle|\; |w| < 1 \right\} \\ z &\mapsto \frac{1 - z}{1 + z} \\ \frac{1 - w}{1 + w} &\leftarrow\!\shortmid w. \end{aligned}$$

Just map the *right* half-plane $\mathbb{H}_R$ to the disc $\mathbb{D}$ by precomposing with a rotation $e^{i\pi/2} = i$:

$$\begin{aligned} &\mathbb{H}_R \to \mathbb{H} \to \mathbb{D} \\ &z \mapsto iz \mapsto \frac{i - (iz)}{i + (iz)} = \frac{i(1 - z)}{i(1 + z)} = \frac{1 - z}{1 + z}. \end{aligned}$$

This can easily be inverted:

$$\begin{aligned} & w = \frac{1 + z}{1 + z} \\ \implies & -(1 - w) + z(w + 1) = 0 \\ \implies & z = \frac{1 - w}{1 + w}. \end{aligned}$$

Boundary behavior: Just a rotated version of $\mathbb{H} \to \mathbb{D}$!

Mnemonic: every $z \in \mathbb{H}_R$ is closed to 1 than -1.

Proposition 8.2.6 ***(Sector to sector).***
For $0 < \alpha < 2$:

$$\begin{gathered} F_\alpha : S^\circ_{\frac{\pi}{\alpha}} \rightleftharpoons S^\circ_\pi = \mathbb{H}^\circ \\ \left\{ z \,\middle|\, 0 < \operatorname{Arg}(z) < \frac{\pi}{\alpha} \right\} \rightleftharpoons \left\{ w \,\middle|\, 0 < \operatorname{Arg}(w) < \pi \right\} \\ z \mapsto z^\alpha \\ w^{\frac{1}{\alpha}} \leftarrow\!\shortmid w. \end{gathered}$$

Note that if you look at the image of $\mathbb{H}$ under $z \mapsto z^\alpha$, you get

$$\left\{ z \,\middle|\, 0 < \operatorname{Arg}(z) < \pi \right\} \rightleftharpoons \{0 < \operatorname{Arg}(w) < \alpha\pi\}$$

.

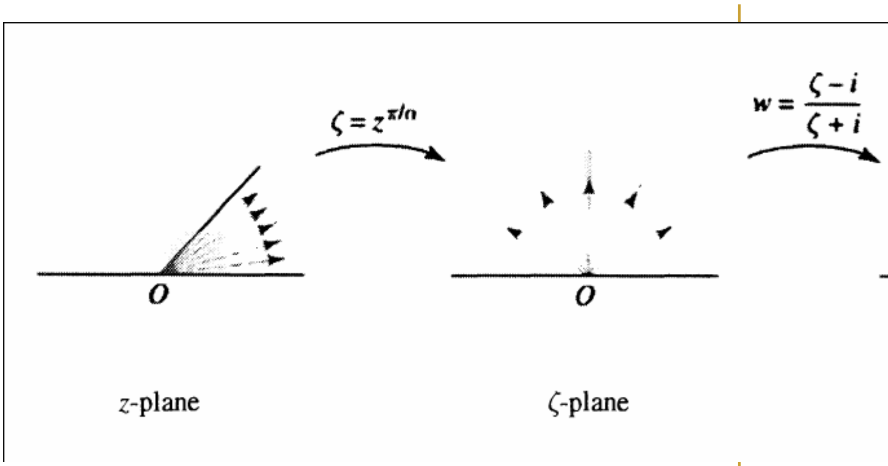
For the inverse, choose a branch cut of log deleting the negative real axis, or more generally fix $0 < \arg w < w^{\frac{1}{\alpha}}$.
Boundary behavior:

- As x travels from $-\infty \to 0$, $F_\alpha(x)$ travels *away* from infinity along the ray $\theta = \alpha\pi$, so $L = \left\{ e^{t\alpha\pi} \,\middle|\, t \in (0, \infty) \right\}$, from $\infty \to 0$.
- As x travels from $0 \to \infty$, $F_\alpha(x)$ travels from $0 \to \infty$ along $\mathbb{R}$.

Proposition 8.2.7 ***(Sector to Disc).***
The unmotivated formula first:

$$F : S_\alpha \to \mathbb{D}$$

$$\begin{gathered} \left\{ z \,\middle|\, 0 < \operatorname{Arg}(z) < \alpha \right\} \rightleftharpoons \left\{ w \,\middle|\, |w| < 1 \right\} \\ z \rightleftharpoons \frac{z^{\frac{\pi}{\alpha}} - i}{z^{\frac{\pi}{\alpha}} + i}. \end{gathered}$$



Idea: compose some known functions.

$$S_\alpha \to S_\pi = \mathbb{H} \to \mathbb{D}$$
$$z \mapsto z^{\frac{\pi}{\alpha}} \mapsto \left.\frac{z-i}{z+i}\right|_{z=z^{\frac{\pi}{\alpha}}}.$$

Proposition 8.2.8 ***(Upper half-disc to first quadrant).***

$$\left\{z \;\middle|\; |z| < 1,\, \Im(z) > 0\right\} \rightleftharpoons \left\{w \;\middle|\; \Re(w) > 0,\, \Im(w) > 0\right\}$$
$$z \mapsto \frac{1+z}{1-z}$$
$$\frac{w-1}{w+1} \mapsfrom w.$$

- Why this lands in the first quadrant:
  - Use that squares are non-negative and $z = x + iy \in \mathbb{D} \implies x^2 + y^2 < 1$:

$$f(z) = \frac{1 - (x^2 + y^2)}{(1-x)^2 + y^2} + i\frac{2y}{(1-x)^2 + y^2}.$$

- Why the inverse lands in the unit disc:
  - For w in Q1, the distance from w to 1 is smaller than from w to -1.
  - Check that if $w = u + iv$ where $u, v > 0$, the imaginary part of the image is positive:

$$\begin{aligned}\frac{w-1}{w+1} &= \frac{(w-1)\overline{(w+1)}}{|w+1|^2} \\ &= \frac{(u-1+iv)\,(u+1-iv)}{(u+1)^2 + v^2} \\ &= \frac{u^2+v^2+1}{(u+1)^2+v^2} + i\left(\frac{2v}{(u+1)^2+v^2}\right).\end{aligned}$$

Boundary behavior:

- On the upper half circle $\left\{ e^{it} \,\middle|\, t \in (0, \pi) \right\}$, write

$$f(z) = \frac{1 + e^{i\theta}}{1 - e^{i\theta}} = \frac{e^{-i\theta/2} + e^{i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}} = \frac{i}{\tan(\theta/2)},$$

so as t ranges $0 \to \pi$ we have $f(z)$ ranging from $0 \to i\infty$ along the imaginary axis.

- As x ranges from $-1 \to 1$ in $\mathbb{R}$, $f(z)$ ranges from $0 \to \infty$ with $f(0) = 1$.

Proposition 8.2.9 ***(Log: Upper half-plane to horizontal strip).***

$$\begin{aligned} \mathbb{H} &\rightleftharpoons \mathbb{R} \times (0, \pi) \\ \left\{ z \,\middle|\, \Im(z) > 0 \right\} &\rightleftharpoons \left\{ w \,\middle|\, \Im(z) \in (0, \pi) \right\} \\ z &\mapsto \log(z) \\ e^w &\mapsfrom w. \end{aligned}$$

- Why this lands in a strip: use that $\arg(z) \in (0, \pi)$ and $\log(z) = |z| + i \arg(z)$.

Boundary behavior:

- As x travels from $-\infty \to 0$, $F(x)$ travels horizontally from $\infty + i\pi$ to $-\infty + i\pi$.
- As x travels from $o \to \infty$, $F(x)$ travels from $-\infty \to \infty$ in $\mathbb{R}$.

Remark 8.2.10: This extends to a function $\mathbb{C} \setminus \mathbb{R}^{\leq 0} \to \mathbb{R} \times (-\pi, \pi)$. Circles of radius R are mapped to vertical line segments connecting $\ln(R) + i\pi$ to $\ln(R) - i\pi$, and rays are mapped to horizontal lines.

Remark 8.2.11: One can find other specific images of the logarithm:

$$\begin{aligned} \left\{ z \,\middle|\, |z| < 1, \Im(z) > 0 \right\} &\rightleftharpoons \mathbb{R}^{<0} \times (0, \pi) \\ \left\{ z \,\middle|\, |z| > 1, \Im(z) > 0 \right\} &\rightleftharpoons \mathbb{R}^{>0} \times (0, \pi) \end{aligned}$$

.

For the upper half-disc to the negative horizontal half-strip: - As x travels $0 \to 1$ in $\mathbb{R}$, $\log(x)$ travels from $-\infty \to 0$. - As x travels from -1 to 1 along $S^1 \cap \mathbb{H}$, $\log(x)$ travels from $0 \to i\pi$ vertically. - As x travels from $-1 \to 0$, $\log(x)$ travels from $0 + i\pi \to i - \infty + i\pi$ along the top of the strip.

Proposition 8.2.12 ***(Half-discs to half strips).***

$$F : (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^{>0} \to \mathbb{D} \cap \mathbb{H}$$
$$z \mapsto e^{iz}$$
$$\frac{\log(w)}{i}? \leftarrow\!\shortmid w.$$

This uses that $e^{iz} = e^{-\Im(z)} e^{i\Re(z)}$.
Boundary behavior:

Proposition 8.2.13 ***(Half-disc to upper half-plane).***

$$F : ? \rightleftharpoons ?$$
$$z \mapsto -\frac{1}{2}\left(z + z^{-1}\right)$$
.

Proposition 8.2.14 ***(Upper half-plane to vertical half-strip).***

$$? \rightleftharpoons ?$$
$$z \mapsto \sin(z)$$
.

8.3 Schwarz

Theorem 8.3.1 ***(Schwarz Lemma).***
If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic with $f(0) = 0$, then

1. $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$
2. $|f'(0)| \leq 1$.

Moreover, if

- $|f(z_0)| = |z_0|$ for any $z_0 \in \mathbb{D}$, or
- $|f'(0)| = 1$,

then f is a rotation.

Proof (?).
Apply the maximum modulus principle to $f(z)/z$. ■

Exercise 8.3.2 (?)
Show that $\operatorname{Aut}_{\mathbb{C}}(\mathbb{C}) = \left\{ z \mapsto az + b \;\middle|\; a \in \mathbb{C}^\times, b \in \mathbb{C} \right\}$.

Theorem 8.3.3 ***(Biholomorphisms of the disc).***

$$\operatorname{Aut}_{\mathbb{C}}(\mathbb{D}) = \left\{ z \mapsto e^{i\theta} \left(\frac{\alpha - z}{1 - \bar{\alpha} z} \right) \right\}.$$

Proof (?).
Schwarz lemma. ∎

Theorem 8.3.4 ***(?).***

$$\operatorname{Aut}_{\mathbb{C}}(\mathbb{H}) = \left\{ z \mapsto \frac{az + b}{cz + d} \;\middle|\; a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\} \cong \mathrm{PSL}_2(\mathbb{R}).$$

9 Schwarz Reflection

?

10 Schwarz Lemma

?

Montel's theorem

Normal families

Schwarz lemma

Equicontinuity

11 Linear Fractional Transformations

12 Montel's Theorem

13 Unsorted Theorems

Theorem 13.0.1 *(Riemann's Removable Singularity Theorem).*
If f is holomorphic on Ω except possibly at z_0 and f is bounded on $\Omega \setminus \{z_0\}$, then z_0 is a removable singularity.

Theorem 13.0.2 *(Little Picard).*
If $f : \mathbb{C} \to \mathbb{C}$ is entire and nonconstant, then $\mathrm{im}(f)$ is either $\mathbb{C}$ or $\mathbb{C} \setminus \{z_0\}$ for some point z_0.

???

Corollary 13.0.3.
The ring of holomorphic functions on a domain in $\mathbb{C}$ has no zero divisors.

Proof .
???

■

Find the proof!

Morera

Proposition 13.0.4 *(Bounded Complex Analytic Functions form a Banach Space).*
For $\Omega \subseteq \mathbb{C}$, show that $A(\mathbb{C}) := \left\{ f : \Omega \to \mathbb{C} \;\middle|\; f \text{ is bounded} \right\}$ is a Banach space.

Proof .
?

Apply Morera's Theorem and Cauchy's Theorem

■

14 | Proofs of the Fundamental Theorem of Algebra

14.0.1 Argument Principle

Proof (using the argument principle).

- Let $P(z) = a_n z^n + \cdots + a_0$ and $g(z) = P'(z)/P(z)$, note P is holomorphic
- Since $\lim_{|z| \to \infty} P(z) = \infty$, there exist an $R > 0$ such that P has no roots in $\{|z| \geq R\}$.
- Apply the argument principle:

$$N(0) = \frac{1}{2\pi i} \oint_{|\xi|=R} g(\xi)\, d\xi.$$

- Check that $\lim_{|z \to \infty|} z g(z) = n$, so g has a simple pole at ∞
- Then g has a Laurent series $\frac{n}{z} + \frac{c_2}{z^2} + \cdots$
- Integrate term-by-term to get $N(0) = n$.

∎

14.0.2 Rouche's Theorem

Proof (using Rouche's theorem).

- Let $P(z) = a_n z^n + \cdots + a_0$
- Set $f(z) = a_n z^n$ and $g(z) = P(z) - f(z) = a_{n-1} z^{n-1} + \cdots + a_0$, so $f + g = P$.
- Choose $R > \max\left(\frac{|a_{n-1}| + \cdots + |a_0|}{|a_n|}, 1 \right)$, then

$$\begin{aligned}
|g(z)| &\coloneqq |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| \\
&\leq |a_{n-1} z^{n-1}| + \cdots + |a_1 z| + |a_0| \quad \text{by the triangle inequality} \\
&= |a_{n-1}| \cdot |z^{n-1}| + \cdots + |a_1| \cdot |z| + |a_0| \\
&= |a_{n-1}| \cdot R^{n-1} + \cdots + |a_1| R + |a_0| \\
&\leq |a_{n-1}| \cdot R^{n-1} + |a_{n-2}| \cdot R^{n-1} + \cdots + |a_1| \cdot R^{n-1} + |a_0| \cdot R^{n-1} \quad \text{since } R > 1 \implies R^{a+b} \geq R^a \\
&= R^{n-1} \left(|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0| \right) \\
&\leq R^{n-1} \left(|a_n| \cdot R \right) \quad \text{by choice of } R \\
&= R^n |a_n| \\
&= |a_n z^n| \\
&\coloneqq |f(z)|
\end{aligned}$$

- Then $a_n z^n$ has n zeros in $|z| < R$, so $f + g$ also has n zeros.

■

14.0.3 Liouville's Theorem

Proof (using Liouville's theorem).

- Suppose p is nonconstant and has no roots, then $\frac{1}{p}$ is entire. We will show it is also bounded and thus constant, a contradiction.
- Write $p(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right)$
- Outside a disc:
  - Note that $p(z) \overset{z\to\infty}{\to} \infty$. so there exists an R large enough such that $|p(z)| \geq \frac{1}{A}$ for any fixed chosen constant A.
  - Then $|1/p(z)| \leq A$ outside of $|z| > R$, i.e. $1/p(z)$ is bounded there.
- Inside a disc:
  - p is continuous with no roots and thus must be bounded below on $|z| < R$.
  - p is entire and thus continuous, and since $\overline{D}_r(0)$ is a compact set, p achieves a min A there
  - Set $C := \min(A, B)$, then $|p(z)| \geq C$ on all of $\mathbb{C}$ and thus $|1/p(z)| \leq C$ everywhere.
  - So $1/p(z)$ is bounded an entire and thus constant by Liouville's theorem – but this forces p to be constant. $\lightning$

■

14.0.4 Open Mapping Theorem

Proof (using the Open Mapping theorem).

- p induces a continuous map $\mathbb{CP}^1 \to \mathbb{CP}^1$
- The continuous image of compact space is compact;
- Since the codomain is Hausdorff space, the image is closed.
- p is holomorphic and non-constant, so by the Open Mapping Theorem, the image is open.
- Thus the image is clopen in $\mathbb{CP}^1$.
- The image is nonempty, since $p(1) = \sum a_i \in \mathbb{C}$
- $\mathbb{CP}^1$ is connected
- But the only nonempty clopen subset of a connected space is the entire space.
- So p is surjective, and $p^{-1}(0)$ is nonempty.
- So p has a root.

■

14.0.5 Generalized Liouville

Theorem 14.0.1 ***(Generalized Liouville).***
If X is a compact complex manifold, any holomorphic $f : X \to \mathbb{C}$ is constant.

Lemma 14.0.2 ***(?).***
If $f : X \to Y$ is a nonconstant holomorphic map between Riemann surfaces with X compact, then

- f must be surjective,
- Y must be compact,
- $f^{-1}(q)$ is finite for all $q \in Y$,
- The branch and ramification loci consist of finitely many points.

Proof (of FTA, using Generalized Liouville).
Given a nonconstant $p \in \mathbb{C}[x]$, regard it as a function $p : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ by extending so that $p(\infty) = \infty$. Since p is nonconstant, by the lemma p is surjective, so there exists some $x \neq \infty$ in $\mathbb{P}^1(\mathbb{C})$ with $p(x) = 0$.

■

15 | Appendix

15.1 Misc Basic Algebra

Fact 15.1.1 (Standard forms of conic sections)

- Circle: $x^2 + y^2 = r^2$
- Ellipse: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$
- Hyperbola: $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$
  - Rectangular Hyperbola: $xy = \frac{c^2}{2}$.
- Parabola: $-4ax + y^2 = 0$.

Mnemonic: Write $f(x, y) = Ax^2 + Bxy + Cy^2 + \cdots$, then consider the discriminant $\Delta = B^2 - 4AC$:

- $\Delta < 0 \iff$ ellipse
  - $\Delta < 0$ and $A = C, B = 0 \iff$ circle
- $\Delta = 0 \iff$ parabola
- $\Delta > 0 \iff$ hyperbola

Fact 15.1.2 (Completing the square)

$$x^2 - bx = (x-s)^2 - s^2 \quad \text{where} s = \frac{b}{2}$$
$$x^2 + bx = (x+s)^2 - s^2 \quad \text{where} s = \frac{b}{2}.$$

Fact 15.1.3
The sum of the interior angles of an n-gon is $(n-2)\pi$, where each angle is $\frac{n-2}{n}\pi$.

Definition 15.1.4 (The Dirichlet Problem)
Given a bounded piecewise continuous function $u : S^1 \to \mathbb{R}$, is there a unique extension to a continuous harmonic function $\tilde{u} : \mathbb{D} \to \mathbb{R}$?

Remark 15.1.5: More generally, this is a boundary value problem for a region where the *values* of the function on the boundary are given. Compare to prescribing conditions on the normal vector on the boundary, which would be a Neumann BVP. Why these show up: a harmonic function on a simply connected region has a harmonic conjugate, and solutions of BVPs are always analytic functions with harmonic real/imaginary parts.

Example 15.1.6*(Dirichlet problem on the strip):* See section 27, example 1 in Brown and Churchill. On the strip $(x, y) \in (0, \pi) \times (0, \infty)$, set up the BVP for temperature on a thin plate with no sinks/sources:

$$\Delta T = 0 \qquad T(0, y) = 0,\ T(\pi, y) = 0\ \forall y$$
$$T(x, 0) = \sin(x) \qquad T(x, y) \overset{y \to \infty}{\longrightarrow} 0.$$

Then the following function is harmonic on $\mathbb{R}^2$ and satisfies that Dirichlet problem:

$$T(x, y) = e^{-y} \sin(x) = \Re(-ie^{iz}) = \Im(e^{iz}).$$

Definition 15.1.7 (Logarithmic Derivative)
The **logarithmic derivative** of f is $(\ln f)' = f'/f$.

Remark 15.1.8: Why this is useful: deriving the argument principle. If f has a pole of order n at z_0, then write $f(z) = (z - z_0)^{-n} g(z)$ with g analytic in a neighborhood of z_0. Then a direct computation of the derivatives will show

$$(\ln f)' := \frac{f'(z)}{f(z)} = -\frac{n}{z - z_0} + \frac{g'(z)}{g(z)}.$$

Thus

$$\frac{1}{2\pi i} \int_\gamma (\ln f)' = -n,$$

for γ a small circle about z_0. A similar argument for z_0 a **zero** of f yields

$$\frac{1}{2\pi i} \int_\gamma h = +n.$$

Exercise 15.1.9 (?)
Show that there is no continuous square root function defined on all of $\mathbb{C}$.

Solution:
Suppose $f(z)^2 = z$. Then f is a section to the covering map

$$\begin{aligned} p : \mathbb{C}^\times &\to \mathbb{C}^\times \\ z &\mapsto z^2, \end{aligned}$$

so $p \circ f = \mathrm{id}$. Using $\pi_1(\mathbb{C}^\times) = \mathbb{Z}$, the induced maps are $p_*(1) = 2$ and $f_*(1) = n$ for some $n \in \mathbb{Z}$. But then $p_* \circ f_*$ is multiplication by $2n$, contradicting $p_* \circ f_* = \mathrm{id}$ by functoriality.

Theorem 15.1.10 ***(Uniformization).***
Every Riemann surface S is the quotient of a free proper holomorphic action of a group G on the universal cover $\tilde{S}$ of S, so $S \cong \tilde{S}/G$ is a biholomorphism. Moreover, $\tilde{S}$ is biholomorphic to either

- $\mathbb{CP}^1$
- $\mathbb{C}$
- $\mathbb{D}$

Basics

- Show that $\displaystyle\frac{1}{z}\sum_{k=1}^{\infty}\frac{z^k}{k}$ converges on $S^1 \setminus \{1\}$ using summation by parts.
- Show that any power series is continuous on its domain of convergence.
- Show that a uniform limit of continuous functions is continuous.

??

- Show that if f is holomorphic on $\mathbb{D}$ then f has a power series expansion that converges uniformly on every compact $K \subset \mathbb{D}$.

- Show that any holomorphic function f can be uniformly approximated by polynomials.
- Show that if f is holomorphic on a connected region Ω and $f' \equiv 0$ on Ω, then f is constant on Ω.
- Show that if $|f| = 0$ on $\partial\Omega$ then either f is constant or f has a zero in Ω.
- Show that if $\{f_n\}$ is a sequence of holomorphic functions converging uniformly to a function f on every compact subset of Ω, then f is holomorphic on Ω and $\{f_n'\}$ converges uniformly to f' on every such compact subset.
- Show that if each f_n is holomorphic on Ω and $F := \sum f_n$ converges uniformly on every compact subset of Ω, then F is holomorphic.
- Show that if f is once complex differentiable at each point of Ω, then f is holomorphic.

16 Draft of Problem Book

- Prove the triangle inequality
- Prove the reverse triangle inequality
- Show that $\sum z^{k-1}/k$ converges for all $z \in S^1$ except $z = 1$.
- What is an example of a noncontinuous limit of continuous functions?
- Show that the uniform limit of continuous functions is continuous.
- Show that f is holomorphic if and only if $\bar{\partial} f = 0$.
- Show $n^{\frac{1}{n}} \overset{n\to\infty}{\to} 1$.
- Show that if f is holomorphic with $f' = 0$ on Ω then f is constant.
- Show that holomorphic implies analytic.
- Use Cauchy's inequality to prove Liouville's theorem

Problem 16.0.1 (?)
What is a pair of conformal equivalences between $\mathbb{H}$ and $\mathbb{D}$?

Solution:

$$
\begin{aligned}
F : HH &\to \mathbb{D} \\
z &\mapsto \frac{i-z}{i+z}
\end{aligned}
$$

$$
\begin{aligned}
G : \mathbb{D} &\to \mathbb{H} \\
w &\mapsto i\frac{1-w}{1+w}.
\end{aligned}
$$

Mnemonic: any point in $\mathbb{H}$ is closer to i than $-i$, so $|F(z)| < 1$.

- Maps $\mathbb{R} \to S^1 \setminus \{-1\}$.

Problem 16.0.2 (?)

What is conformal equivalence $\mathbb{H} \rightleftharpoons S := \left\{ w \in \mathbb{C} \;\middle|\; 0 < \arg(w) < \alpha\pi \right\}$?

Solution:

$$
f(z) = z^{\alpha}.
$$