UGA Real Analysis Questions (Fall 2014 – Spring 2021)

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1 Preface

I'd like to extend my gratitude to Peter Woolfitt for supplying many solutions and checking many proofs of the rest in problem sessions. Many other solutions contain input and ideas from other graduate students and faculty members at UGA, along with questions and answers posted on Math Stack Exchange or Math Overflow.

2 Undergraduate Analysis: Uniform Convergence

2.1 Fall 2018 # 1 ✨

Let $f(x) = \frac{1}{x}$. Show that f is uniformly continuous on $(1, \infty)$ but not on $(0, \infty)$.

Relevant concepts omitted.

Solution omitted.

2.2 Fall 2017 # 1 ✨

Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which f does and does not converge uniformly.

Relevant concepts omitted.

Solution omitted.

2.3 Fall 2014 # 1 ✨

Let $\{f_n\}$ be a sequence of continuous functions such that $\sum f_n$ converges uniformly.

Prove that $\sum f_n$ is also continuous.

Relevant concepts omitted.

Solution omitted.

2.4 Spring 2017 # 4 ✨

Let $f(x,y)$ on $[-1,1]^2$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Determine if f is integrable.

Relevant concepts omitted.

Solution omitted.

2.5 Spring 2015 # 1 ✨

Let (X,d) and (Y,ρ) be metric spaces, $f: X \to Y$, and $x_0 \in X$.

Prove that the following statements are equivalent:

1. For every $\varepsilon > 0 \quad \exists \delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$.
2. The sequence $\{f(x_n)\}_{n=1}^{\infty} \to f(x_0)$ for every sequence $\{x_n\} \to x_0$ in X.

Relevant concepts omitted.

Solution omitted.

2.6 Fall 2014 # 2 🚩

Let I be an index set and $\alpha: I \to (0, \infty)$.

a. Show that

$$\sum_{i\in I} a(i) := \sup_{\substack{J\subset I \\ J \text{ finite}}} \sum_{i\in J} a(i) < \infty \implies I \text{ is countable.}$$

b. Suppose $I = \mathbb{Q}$ and $\sum_{q\in\mathbb{Q}} a(q) < \infty$. Define

$$f(x) := \sum_{\substack{q\in\mathbb{Q} \\ q\leq x}} a(q).$$

Show that f is continuous at $x \iff x \notin \mathbb{Q}$.

Stuck on part b

Solution omitted.

2.7 Spring 2014 # 2 ✨

Let $\{a_n\}$ be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that $\sum a_n^2 < \infty$.

Note: Assume a_n, b_n are all non-negative.

Have someone check!

Solution omitted.

3 | General Analysis

3.1 Spring 2020 # 1 ✨

Prove that if $f : [0,1] \to \mathbb{R}$ is continuous then

$$\lim_{k\to\infty} \int_0^1 kx^{k-1} f(x)\, dx = f(1).$$

Relevant concepts omitted.

Solution omitted.

3.2 Fall 2019 # 1 ✨

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

a. Prove that if $\lim_{n\to\infty} a_n = 0$, then

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

b. Prove that if $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges, then

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

Solution omitted.

3.3 Fall 2018 # 4 ✨

Let $f \in L^1([0,1])$. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)|\sin nx|\ dx = \frac{2}{\pi}\int_0^1 f(x)\ dx$$

Hint: Begin with the case that f is the characteristic function of an interval.

Ask someone to check the last approximation part.

Solution omitted.

3.4 Fall 2017 # 4 ✨

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

a. Show that $f_n \to 0$ pointwise but not uniformly on $[0,1]$.

b. Show that

$$\lim_{n\to\infty} \int_0^1 n(1-x)^n \sin x \, dx = 0$$

Hint for (a): Consider the maximum of f_n.

Solution omitted.

3.5 Spring 2017 # 3 🚩

Let

$$f_n(x) = ae^{-nax} - be^{-nbx} \qquad \text{where } 0 < a < b.$$

Show that

a. $\sum_{n=1}^\infty |f_n|$ is not in $L^1([0,\infty), m)$

Hint: $f_n(x)$ has a root x_n.

b.

$$\sum_{n=1}^\infty f_n \text{ is in } L^1([0,\infty), m) \quad \text{and} \quad \int_0^\infty \sum_{n=1}^\infty f_n(x)\, dm = \ln \frac{b}{a}$$

Not complete.

Add concepts.

Walk through.

Solution omitted.

3.6 Fall 2016 # 1 ✨

Define

$$f(x) = \sum_{n=1}^\infty \frac{1}{n^x}.$$

Show that f converges to a differentiable function on $(1, \infty)$ and that

$$f'(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n^x}\right)'.$$

Hint:

$$\left(\frac{1}{n^x}\right)' = -\frac{1}{n^x} \ln n$$

Add concepts.

Solution omitted.

3.7 Fall 2016 # 5 ✨

Let $\varphi \in L^\infty(\mathbb{R})$. Show that the following limit exists and satisfies the equality

$$\lim_{n\to\infty} \left(\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2}\, dx\right)^{\frac{1}{n}} = \|\varphi\|_\infty.$$

Add concepts.

Solution omitted.

3.8 Fall 2016 # 6 ✨

Let $f, g \in L^2(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x) g(x+n)\, dx = 0$$

Rewrite solution.

Relevant concepts omitted.

Solution omitted.

3.9 Spring 2016 # 1

For $n \in \mathbb{N}$, define

$$e_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad E_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

Show that $e_n < E_n$, and prove Bernoulli's inequality:

$$(1+x)^n \geq 1 + nx \qquad\qquad -1 < x < \infty,\ n \in \mathbb{N}.$$

Use this to show the following:

1. The sequence e_n is increasing.
2. The sequence E_n is decreasing.
3. $2 < e_n < E_n < 4$.
4. $\lim\limits_{n\to\infty} e_n = \lim\limits_{n\to\infty} E_n$.

3.10 Fall 2015 # 1

Define

$$f(x) = c_0 + c_1 x^1 + c_2 x^2 + \ldots + c_n x^n \text{ with } n \text{ even and } c_n > 0.$$

Show that there is a number x_m such that $f(x_m) \leq f(x)$ for all $x \in \mathbb{R}$.

3.11 Fall 2020 # 1

Show that if x_n is a decreasing sequence of positive real numbers such that $\sum\limits_{n=1}^{\infty} x_n$ converges, then

$$\lim_{n\to\infty} n x_n = 0.$$

3.12 Fall 2020 # 3

Let f be a non-negative Lebesgue measurable function on $[1, \infty)$.

a. Prove that

$$1 \leq \left(\frac{1}{b-a} \int_a^b f(x)\, dx \right) \left(\frac{1}{b-a} \int_a^b \frac{1}{f(x)}\, dx \right)$$

for any $1 \leq a < b < \infty$.

b. Prove that if f satisfies

$$\int_1^t f(x)\, dx \leq t^2 \log(t)$$

for all $t \in [1, \infty)$, then

$$\int_1^\infty \frac{1}{f(x)\, dx} = \infty.$$

Hint: write

$$\int_1^\infty \frac{1}{f(x)\, dx} = \sum_{k=0}^\infty \int_{2^k}^{2^{k+1}} \frac{1}{f(x)}\, dx.$$

4 | Measure Theory: Sets

4.1 Spring 2020 # 2 ✨

Let m_* denote the Lebesgue outer measure on $\mathbb{R}$.

4.1.1 a.

Prove that for every $E \subseteq \mathbb{R}$ there exists a Borel set B containing E such that

$$m_*(B) = m_*(E).$$

4.1.2 b.

Prove that if $E \subseteq \mathbb{R}$ has the property that

$$m_*(A) = m_*(A \cap E) + m_*(A \cap E^c)$$

for every set $A \subseteq \mathbb{R}$, then there exists a Borel set $B \subseteq \mathbb{R}$ such that $E = B \setminus N$ with $m_*(N) = 0$.

Be sure to address the case when $m_*(E) = \infty$.

Solution omitted.

4.2 Fall 2019 # 3. ✨

Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) = 1$ and $\{B_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{B}$-measurable subsets of X, and

$$B := \left\{ x \in X \;\middle|\; x \in B_n \text{ for infinitely many } n \right\}.$$

a. Argue that B is also a $\mathcal{B}$-measurable subset of X.

b. Prove that if $\displaystyle\sum_{n=1}^{\infty} \mu(B_n) < \infty$ then $\mu(B) = 0$.

c. Prove that if $\displaystyle\sum_{n=1}^{\infty} \mu(B_n) = \infty$ **and** the sequence of set complements $\{B_n^c\}_{n=1}^{\infty}$ satisfies

$$\mu\left(\bigcap_{n=k}^{K} B_n^c\right) = \prod_{n=k}^{K} \left(1 - \mu\left(B_n\right)\right)$$

for all positive integers k and K with $k < K$, then $\mu(B) = 1$.

Hint: Use the fact that $1 - x \leq e^{-x}$ for all x.

Solution omitted.

4.3 Spring 2019 # 2 ✨

Let $\mathcal{B}$ denote the set of all Borel subsets of $\mathbb{R}$ and $\mu : \mathcal{B} \to [0, \infty)$ denote a finite Borel measure on $\mathbb{R}$.

4.3.1 a

Prove that if $\{F_k\}$ is a sequence of Borel sets for which $F_k \supseteq F_{k+1}$ for all k, then

$$\lim_{k \to \infty} \mu\left(F_k\right) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right)$$

4.3.2 b

Suppose μ has the property that $\mu(E) = 0$ for every $E \in \mathcal{B}$ with Lebesgue measure $m(E) = 0$.

Prove that for every $\epsilon > 0$ there exists $\delta > 0$ so that if $E \in \mathcal{B}$ with $m(E) < \delta$, then $\mu(E) < \varepsilon$.

Add concepts.

Solution omitted.

All messed up!

4.4 Fall 2018 # 2 ✨

Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Show that there is a Borel set $B \subset E$ such that $m(E \setminus B) = 0$.

Move this to review notes to clean things up.

Solution omitted.

4.5 Spring 2018 # 1 ✨

Define

$$E := \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N} \right\}.$$

Prove that $m(E) = 0$.

Solution omitted.

4.6 Fall 2017 # 2 ✨

Let $f(x) = x^2$ and $E \subset [0, \infty) := \mathbb{R}^+$.

1. Show that

$$m^*(E) = 0 \iff m^*(f(E)) = 0.$$

2. Deduce that the map

$$\begin{aligned}\varphi : \mathcal{L}(\mathbb{R}^+) &\to \mathcal{L}(\mathbb{R}^+)\\ E &\mapsto f(E)\end{aligned}$$

is a bijection from the class of Lebesgue measurable sets of $[0, \infty)$ to itself.

Walk through.

Solution omitted.

4.7 Spring 2017 # 2 ✨

4.7.1 a

Let μ be a measure on a measurable space $(X, \mathcal{M})$ and f a positive measurable function.

Define a measure λ by

$$\lambda(E) := \int_E f \, d\mu, \quad E \in \mathcal{M}$$

Show that for g any positive measurable function,

$$\int_X g \, d\lambda = \int_X fg \, d\mu$$

4.7.2 b

Let $E \subset \mathbb{R}$ be a measurable set such that

$$\int_E x^2 \, dm = 0.$$

Show that $m(E) = 0$.

Solution omitted.

4.8 Fall 2016 # 4 ✨

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\{E_n\} \subset \mathcal{M}$ satisfies

$$\lim_{n\to\infty} \mu\left(X \setminus E_n\right) = 0.$$

Define

$$G := \left\{x \in X \;\middle|\; x \in E_n \text{ for only finitely many } n\right\}.$$

Show that $G \in \mathcal{M}$ and $\mu(G) = 0$.

Add concepts.

Solution omitted.

4.9 Spring 2016 # 3 🚩

Let f be Lebesgue measurable on $\mathbb{R}$ and $E \subset \mathbb{R}$ be measurable such that

$$0 < A = \int_E f(x) dx < \infty.$$

Show that for every $0 < t < 1$, there exists a measurable set $E_t \subset E$ such that

$$\int_{E_t} f(x) dx = tA.$$

4.10 Spring 2016 # 5 🚩

Let $(X, \mathcal{M}, \mu)$ be a measure space. For $f \in L^1(\mu)$ and $\lambda > 0$, define

$$\varphi(\lambda) = \mu(\{x \in X | f(x) > \lambda\}) \quad \text{and} \quad \psi(\lambda) = \mu(\{x \in X | f(x) < -\lambda\})$$

Show that φ, ψ are Borel measurable and

$$\int_X |f| \, d\mu = \int_0^\infty [\varphi(\lambda) + \psi(\lambda)] \, d\lambda$$

4.11 Fall 2015 # 2

Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable.

1. Show that there is a sequence of simple functions $s_n(x)$ such that $s_n(x) \to f(x)$ for all $x \in \mathbb{R}$.
2. Show that there is a Borel measurable function g such that $g = f$ almost everywhere.

4.12 Spring 2015 # 3

Let μ be a finite Borel measure on $\mathbb{R}$ and $E \subset \mathbb{R}$ Borel. Prove that the following statements are equivalent:

1. $\forall \varepsilon > 0$ there exists G open and F closed such that
$$F \subseteq E \subseteq G \quad \text{and} \quad \mu(G \setminus F) < \varepsilon.$$
2. There exists a $V \in G_\delta$ and $H \in F_\sigma$ such that
$$H \subseteq E \subseteq V \quad \text{and} \quad \mu(V \setminus H) = 0$$

4.13 Spring 2014 # 3

Let $f : \mathbb{R} \to \mathbb{R}$ and suppose
$$\forall x \in \mathbb{R}, \quad f(x) \geq \limsup_{y \to x} f(y)$$
Prove that f is Borel measurable.

4.14 Spring 2014 # 4

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose f is a measurable function on X. Show that
$$\lim_{n \to \infty} \int_X f^n \, d\mu = \begin{cases} \infty & \text{or} \\ \mu(f^{-1}(1)), \end{cases}$$
and characterize the collection of functions of each type.

4.15 Spring 2017 # 1 ✨

Let K be the set of numbers in $[0,1]$ whose decimal expansions do not use the digit 4.

We use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 with $399\cdots$. For example, $0.8754 = 0.8753999\cdots$.

Show that K is a compact, nowhere dense set without isolated points, and find the Lebesgue measure $m(K)$.

Solution omitted.

4.16 Spring 2016 # 2 🚩

Let $0 < \lambda < 1$ and construct a Cantor set C_λ by successively removing middle intervals of length λ.

Prove that $m(C_\lambda) = 0$.

5 Measure Theory: Functions

5.1 Fall 2016 # 2 ✨

Let $f, g : [a, b] \to \mathbb{R}$ be measurable with

$$\int_a^b f(x)\, dx = \int_a^b g(x)\, dx.$$

Show that either

1. $f(x) = g(x)$ almost everywhere, or
2. There exists a measurable set $E \subset [a, b]$ such that

$$]\int_E f(x)\, dx > \int_E g(x)\, dx$$

Add concepts.

Solution omitted.

5.2 Spring 2016 # 4

Let $E \subset \mathbb{R}$ be measurable with $m(E) < \infty$. Define

$$f(x) = m(E \cap (E + x)).$$

Show that

1. $f \in L^1(\mathbb{R})$.
2. f is uniformly continuous.
3. $\lim_{|x| \to \infty} f(x) = 0$.

Hint:

$$\chi_{E \cap (E+x)}(y) = \chi_E(y)\chi_E(y - x)$$

5.3 Spring 2021 # 1

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $E_n \in \mathcal{M}$ be a measurable set for $n \geq 1$. Let $f_n := \chi_{E_n}$ be the indicator function of the set E and show that

a. $f_n \overset{n\to\infty}{\rightrightarrows} 1$ uniformly $\iff$ there exists $N \in |NN$ such that $E_n = X$ for all $n \geq N$.

b. $f_n(x) \overset{n\to\infty}{\to} 1$ for almost every $x \iff$

$$\mu\left(\bigcap_{n \geq 0} \bigcup_{k \geq n} (X \setminus E_k) \right) = 0.$$

5.4 Spring 2021 # 3

Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let $\{f_n\}_{n=1}^{\infty} \subseteq L^1(X, \mu)$. Suppose $f \in L^1(X, \mu)$ such that $f_n(x) \overset{n\to\infty}{\to} f(x)$ for almost every $x \in X$. Prove that for every $\varepsilon > 0$ there exists $M > 0$ and a set $E \subseteq X$ such that $\mu(E) \leq \varepsilon$ and $|f_n(x)| \leq M$ for all $x \in X \setminus E$ and all $n \in \mathbb{N}$.

5.5 Fall 2020 # 2

a. Let $f : \mathbb{R} \to \mathbb{R}$. Prove that

$$f(x) \le \liminf_{y \to x} f(y) \text{ for each } x \in \mathbb{R} \iff \left\{x \in \mathbb{R} \;\middle|\; f(x) > a\right\} \text{ is open for all } a \in \mathbb{R}$$

b. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called *lower semi-continuous* iff it satisfies either condition in part (a) above.

Prove that if $\mathcal{F}$ is an y family of lower semi-continuous functions, then

$$g(x) = \sup\left\{f(x) \;\middle|\; f \in \mathcal{F}\right\}$$

is Borel measurable.

Note that $\mathcal{F}$ need not be a countable family.

6 | Integrals: Convergence

6.1 Fall 2019 # 2 ✨

Prove that

$$\left|\frac{d^n}{dx^n}\frac{\sin x}{x}\right| \le \frac{1}{n}$$

for all $x \neq 0$ and positive integers n.

Hint: Consider $\int_0^1 \cos(tx)dt$

Solution omitted.

6.2 Spring 2020 # 5 ✨

Compute the following limit and justify your calculations:

$$\lim_{n \to \infty} \int_0^n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx.$$

Not finished, flesh out.

Walk through.

Solution omitted.

6.3 Spring 2019 # 3 ✨

Let $\{f_k\}$ be any sequence of functions in $L^2([0,1])$ satisfying $\|f_k\|_2 \leq M$ for all $k \in \mathbb{N}$.

Prove that if $f_k \to f$ almost everywhere, then $f \in L^2([0,1])$ with $\|f\|_2 \leq M$ and

$$\lim_{k\to\infty} \int_0^1 f_k(x)dx = \int_0^1 f(x)dx$$

Hint: Try using Fatou's Lemma to show that $\|f\|_2 \leq M$ and then try applying Egorov's Theorem.

Solution omitted.

6.4 Fall 2018 # 6 ✨

Compute the following limit and justify your calculations:

$$\lim_{n\to\infty} \int_1^n \frac{dx}{(1+\frac{x}{n})^n \sqrt[n]{x}}$$

Add concepts.

Solution omitted.

6.5 Fall 2018 # 3 ✨

Suppose $f(x)$ and $xf(x)$ are integrable on $\mathbb{R}$. Define F by

$$F(t) := \int_{-\infty}^{\infty} f(x)\cos(xt)dx$$

Show that

$$F'(t) = -\int_{-\infty}^{\infty} xf(x)\sin(xt)dx.$$

Walk through.

Solution omitted.

6.6 Spring 2018 # 5 ✨

Suppose that

- $f_n, f \in L^1$,
- $f_n \to f$ almost everywhere, and
- $\int |f_n| \to \int |f|$.

Show that $\int f_n \to \int f$.

Solution omitted.

6.7 Spring 2018 # 2 ✨

Let

$$f_n(x) := \frac{x}{1+x^n}, \quad x \geq 0.$$

a. Show that this sequence converges pointwise and find its limit. Is the convergence uniform on $[0, \infty)$?

b. Compute

$$\lim_{n\to\infty} \int_0^\infty f_n(x)dx$$

Add concepts.

Solution omitted.

6.8 Fall 2016 # 3 ✨

Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{x \to 0} \int_{\mathbb{R}} |f(y-x) - f(y)|\, dy = 0$$

Missing some stuff.

Solution omitted.

6.9 Fall 2015 # 3 🚩

Compute the following limit:

$$\lim_{n \to \infty} \int_1^n \frac{ne^{-x}}{1+nx^2} \sin\left(\frac{x}{n}\right) dx$$

6.10 Fall 2015 # 4 🚩

Let $f : [1, \infty) \to \mathbb{R}$ such that $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Show that the following limit exists and satisfies the equality

$$\lim_{x \to \infty} f(x) \leq 1 + \frac{\pi}{4}$$

6.11 Spring 2021 # 2

Calculate the following limit, justifying each step of your calculation:

$$L := \lim_{n \to \infty} \int_0^n \frac{\cos\left(\frac{x}{n}\right)}{x^2 + \cos\left(\frac{x}{n}\right)} dx.$$

6.12 Spring 2021 # 5

Let $f_n \in L^2([0,1])$ for $n \in \mathbb{N}$, and assume that

- $\|f_n\|_2 \leq n^{\frac{-51}{100}}$ for all $n \in \mathbb{N}$,
- $hatf_n$ is supported in the interval $[2^n, 2^{n+1}]$, so

$$\widehat{f_n}(\xi) \coloneqq \int_0^1 f_n(x) e^{2\pi i \xi \cdot x}\, dx = 0 \qquad\qquad \text{for } \xi \notin [2^n, 2^{n+1}].$$

Prove that $\sum_{n\in\mathbb{N}} f_n$ converges in the Hilbert space $L^2([0,1])$.

Hint: Plancherel's identity may be helpful.

7 Integrals: Approximation

7.1 Spring 2018 # 3 ✨

Let f be a non-negative measurable function on $[0,1]$.

Show that

$$\lim_{p\to\infty} \left(\int_{[0,1]} f(x)^p dx \right)^{\frac{1}{p}} = \|f\|_\infty .$$

Solution omitted.

7.2 Spring 2018 # 4 ✨

Let $f \in L^2([0,1])$ and suppose

$$\int_{[0,1]} f(x) x^n dx = 0 \text{ for all integers } n \geq 0.$$

Show that $f = 0$ almost everywhere.

Solution omitted.

7.3 Spring 2015 # 2 🚩

Let $f : \mathbb{R} \to \mathbb{C}$ be continuous with period 1. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(t)dt \quad \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

Hint: show this first for the functions $f(t) = e^{2\pi ikt}$ for $k \in \mathbb{Z}$.

7.4 Fall 2014 # 4 🚩

Let $g \in L^\infty([0,1])$ Prove that

$$\int_{[0,1]} f(x)g(x)\,dx = 0 \quad \text{for all continuous } f : [0,1] \to \mathbb{R} \implies g(x) = 0 \text{ almost everywhere.}$$

8 | L^1

8.1 Spring 2020 # 3 ✨

a. Prove that if $g \in L^1(\mathbb{R})$ then

$$\lim_{N\to\infty} \int_{|x|\geq N} |f(x)|\,dx = 0,$$

and demonstrate that it is not necessarily the case that $f(x) \to 0$ as $|x| \to \infty$.

b. Prove that if $f \in L^1([1,\infty])$ and is decreasing, then $\lim_{x\to\infty} f(x) = 0$ and in fact $\lim_{x\to\infty} xf(x) = 0$.

c. If $f : [1,\infty) \to [0,\infty)$ is decreasing with $\lim_{x\to\infty} xf(x) = 0$, does this ensure that $f \in L^1([1,\infty))$?

Solution omitted.

8.2 Fall 2019 # 5. ✨

8.2.1 a

Show that if f is continuous with compact support on $\mathbb{R}$, then

$$\lim_{y \to 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dx = 0$$

8.2.2 b

Let $f \in L^1(\mathbb{R})$ and for each $h > 0$ let

$$\mathcal{A}_h f(x) := \frac{1}{2h} \int_{|y| \leq h} f(x-y) dy$$

i. Prove that $\|\mathcal{A}_h f\|_1 \leq \|f\|_1$ for all $h > 0$.

ii. Prove that $\mathcal{A}_h f \to f$ in $L^1(\mathbb{R})$ as $h \to 0^+$.

Walk through.

Solution omitted.

8.3 Fall 2017 # 3 ✨

Let

$$S = \operatorname{span}_{\mathbb{C}} \left\{ \chi_{(a,b)} \;\middle|\; a, b \in \mathbb{R} \right\},$$

the complex linear span of characteristic functions of intervals of the form (a, b).

Show that for every $f \in L^1(\mathbb{R})$, there exists a sequence of functions $\{f_n\} \subset S$ such that

$$\lim_{n \to \infty} \|f_n - f\|_1 = 0$$

Walk through.

Solution omitted.

8.4 Spring 2015 # 4

Define

$$f(x, y) := \begin{cases} \dfrac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Carefully show that $f \in L^1(\mathbb{R}^2)$.

8.5 Fall 2014 # 3

Let $f \in L^1(\mathbb{R})$. Show that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } \qquad m(E) < \delta \implies \int_E |f(x)|\, dx < \varepsilon$$

8.6 Spring 2014 # 1

1. Give an example of a continuous $f \in L^1(\mathbb{R})$ such that $f(x) \not\to 0$ as$|x| \to \infty$.
2. Show that if f is *uniformly* continuous, then

$$\lim_{|x| \to \infty} f(x) = 0.$$

8.7 Spring 2021 # 4

Let f, g be Lebesgue integrable on $\mathbb{R}$ and let $g_n(x) \coloneqq g(x-n)$. Prove that

$$\lim_{n \to \infty} \|f + g_n\|_1 = \|f\|_1 + \|g\|_1.$$

Relevant concepts omitted.

Solution omitted.

8.8 Fall 2020 # 4

Prove that if $xf(x) \in L^1(\mathbb{R})$, then

$$F(y) := \int f(x)\cos(yx)\, dx$$

defines a C^1 function.

9 | Fubini-Tonelli

9.1 Spring 2020 # 4 ✨

Let $f, g \in L^1(\mathbb{R})$. Argue that $H(x, y) := f(y)g(x-y)$ defines a function in $L^1(\mathbb{R}^2)$ and deduce from this fact that

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x-y)\, dy$$

defines a function in $L^1(\mathbb{R})$ that satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

Hint/strategy omitted.

Relevant concepts omitted.

Solution omitted.

9.2 Spring 2019 # 4 ✨

Let f be a non-negative function on $\mathbb{R}^n$ and $\mathcal{A} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq f(x)\}$.

Prove the validity of the following two statements:

a. f is a Lebesgue measurable function on $\mathbb{R}^n$ $\iff$ $\mathcal{A}$ is a Lebesgue measurable subset of $\mathbb{R}^{n+1}$

b. If f is a Lebesgue measurable function on $\mathbb{R}^n$, then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x) dx = \int_0^\infty m\left(\{x \in \mathbb{R}^n : f(x) \geq t\}\right) dt$$

Add concepts.

Relevant concepts omitted.

Solution omitted.

9.3 Fall 2018 # 5 ✨

Let $f \geq 0$ be a measurable function on $\mathbb{R}$. Show that

$$\int_{\mathbb{R}} f = \int_0^\infty m(\{x : f(x) > t\})dt$$

:::{.concept} - Claim: If $E \subseteq \mathbb{R}^a \times \mathbb{R}^b$ is a measurable set, then for almost every $y \in \mathbb{R}^b$, the slice E^y is measurable.

$$m(E) = \int_{\mathbb{R}^b} m(E^y)\,dy.$$

- Set $g = \chi_E$, which is non-negative and measurable, so apply Tonelli. - Conclude that $g^y = \chi_{E^y}$ is measurable, the function $y \mapsto \int g^y(x)\,dx$ is measurable, and $\int\int g^y(x)\,dx\,dy = \int g$. - But $\int g = m(E)$ and $\int\int g^y(x)\,dx\,dy = \int m(E^y)\,dy$. :::

Solution omitted.

9.4 Fall 2015 # 5 🚩

Let $f, g \in L^1(\mathbb{R})$ be Borel measurable.

1. Show that

- The function

$$F(x, y) := f(x - y)g(y)$$

  is Borel measurable on $\mathbb{R}^2$, and
- For almost every $y \in \mathbb{R}$,

$$F_y(x) := f(x - y)g(y)$$

  is integrable with respect to y.

2. Show that $f * g \in L^1(\mathbb{R})$ and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

9.5 Spring 2014 # 5 🚩

Let $f, g \in L^1([0,1])$ and for all $x \in [0,1]$ define

$$F(x) := \int_0^x f(y)\,dy \quad \text{and} \quad G(x) := \int_0^x g(y)\,dy.$$

Prove that

$$\int_0^1 F(x)g(x)\,dx = F(1)G(1) - \int_0^1 f(x)G(x)\,dx$$

9.6 Spring 2021 # 6

⚠ **Warning 9.6.1**
This problem may be much harder than expected. Recommended skip.

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a measurable function and for $x \in \mathbb{R}$ define the set

$$E_x := \left\{ y \in \mathbb{R} \;\middle|\; \mu\left(z \in \mathbb{R} \;\middle|\; f(x,z) = f(x,y) \right) > 0 \right\}.$$

Show that the following set is a measurable subset of $\mathbb{R} \times \mathbb{R}$:

$$E := \bigcup_{x \in \mathbb{R}} \{x\} \times E_x.$$

Hint: consider the measurable function $h(x,y,z) := f(x,y) - f(x,z)$.

10 | L^2 and Fourier Analysis

10.1 Spring 2020 # 6 ✨

10.1.1 a

Show that

$$L^2([0,1]) \subseteq L^1([0,1]) \quad \text{and} \quad \ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z}).$$

10.1.2 b

For $f \in L^1([0,1])$ define

$$\widehat{f}(n) := \int_0^1 f(x) e^{-2\pi i n x}\, dx.$$

Prove that if $f \in L^1([0,1])$ and $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$ then

$$S_N f(x) := \sum_{|n| \leq N} \widehat{f}(n) e^{2\pi i n x}.$$

converges uniformly on $[0,1]$ to a continuous function g such that $g = f$ almost everywhere.

Hint: One approach is to argue that if $f \in L^1([0,1])$ with $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$ then $f \in L^2([0,1])$.

Solution omitted.

10.2 Fall 2017 # 5 ✨

Let φ be a compactly supported smooth function that vanishes outside of an interval $[-N, N]$ such that $\int_{\mathbb{R}} \varphi(x)\, dx = 1$.

For $f \in L^1(\mathbb{R})$, define

$$K_j(x) := j\varphi(jx), \qquad f * K_j(x) := \int_{\mathbb{R}} f(x-y) K_j(y)\, dy$$

and prove the following:

1. Each $f * K_j$ is smooth and compactly supported.

2. $$\lim_{j \to \infty} \|f * K_j - f\|_1 = 0$$

Hint:

$$\lim_{y \to 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dy = 0$$

Add concepts.

Solution omitted.

10.3 Spring 2017 # 5

Let $f, g \in L^2(\mathbb{R})$. Prove that the formula

$$h(x) := \int_{-\infty}^{\infty} f(t) g(x-t)\, dt$$

defines a uniformly continuous function h on $\mathbb{R}$.

10.4 Spring 2015 # 6

Let $f \in L^1(\mathbb{R})$ and g be a bounded measurable function on $\mathbb{R}$.

1. Show that the convolution $f * g$ is well-defined, bounded, and uniformly continuous on $\mathbb{R}$.
2. Prove that one further assumes that $g \in C^1(\mathbb{R})$ with bounded derivative, then $f * g \in C^1(\mathbb{R})$ and

$$\frac{d}{dx}(f * g) = f * \left(\frac{d}{dx} g\right)$$

10.5 Fall 2014 # 5

1. Let $f \in C_c^0(\mathbb{R}^n)$, and show

$$\lim_{t \to 0} \int_{\mathbb{R}^n} |f(x+t) - f(x)|\, dx = 0.$$

2. Extend the above result to $f \in L^1(\mathbb{R}^n)$ and show that

$$f \in L^1(\mathbb{R}^n), \quad g \in L^\infty(\mathbb{R}^n) \quad \implies f * g \text{ is bounded and uniformly continuous.}$$

10.6 Fall 2020 # 5

Suppose $\varphi \in L^1(\mathbb{R})$ with

$$\int \varphi(x)\, dx = \alpha.$$

For each $\delta > 0$ and $f \in L^1(\mathbb{R})$, define

$$A_\delta f(x) := \int f(x-y) \delta^{-1} \varphi\left(\delta^{-1} y\right)\, dy.$$

a. Prove that for all $\delta > 0$,

$$\|A_\delta f\|_1 \leq \|\varphi\|_1 \|f\|_1.$$

b. Prove that

$$A_\delta f \to \alpha f \text{ in } L^1(\mathbb{R}) \quad \text{as} \quad \delta \to 0^+.$$

Hint: you may use without proof the fact that for all $f \in L^1(\mathbb{R})$,

$$\lim_{y \to 0} \int_{\mathbb{R}} |f(x-y) - f(x)|\, dx = 0.$$

11 | Functional Analysis: General

11.1 Fall 2019 # 4 ✨

Let $\{u_n\}_{n=1}^\infty$ be an orthonormal sequence in a Hilbert space $\mathcal{H}$.

11.1.1 a

Prove that for every $x \in \mathcal{H}$ one has

$$\sum_{n=1}^\infty |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

11.1.2 b

Prove that for any sequence $\{a_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$ there exists an element $x \in \mathcal{H}$ such that

$$a_n = \langle x,\ u_n \rangle \text{ for all } n \in \mathbb{N}$$

and

$$\|x\|^2 = \sum_{n=1}^\infty |\langle x, u_n \rangle|^2$$

Solution omitted.

11.2 Spring 2019 # 5 ✨

11.2.1 a

Show that $L^2([0,1]) \subseteq L^1([0,1])$ and argue that $L^2([0,1])$ in fact forms a dense subset of $L^1([0,1])$.

11.2.2 b

Let Λ be a continuous linear functional on $L^1([0,1])$.

Prove the Riesz Representation Theorem for $L^1([0,1])$ by following the steps below:

i. Establish the existence of a function $g \in L^2([0,1])$ which represents Λ in the sense that

$$\Lambda(f) = f(x)g(x)dx \text{ for all } f \in L^2([0,1]).$$

> *Hint: You may use, without proof, the Riesz Representation Theorem for $L^2([0,1])$.*

ii. Argue that the g obtained above must in fact belong to $L^\infty([0,1])$ and represent Λ in the sense that

$$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}dx \quad \text{for all } f \in L^1([0,1])$$

with

$$\|g\|_{L^\infty([0,1])} = \|\Lambda\|_{L^1([0,1])^{\vee}}$$

Solution omitted.

11.3 Spring 2016 # 6 🚩

Without using the Riesz Representation Theorem, compute

$$\sup\left\{ \left| \int_0^1 f(x)e^x dx \right| \;\middle|\; f \in L^2([0,1],m), \; \|f\|_2 \le 1 \right\}$$

11.4 Spring 2015 # 5 🚩

Let $\mathcal{H}$ be a Hilbert space.

1. Let $x \in \mathcal{H}$ and $\{u_n\}_{n=1}^N$ be an orthonormal set. Prove that the best approximation to x in $\mathcal{H}$ by an element in $\operatorname{span}_{\mathbb{C}}\{u_n\}$ is given by

$$\widehat{x} := \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n.$$

2. Conclude that finite dimensional subspaces of $\mathcal{H}$ are always closed.

11.5 Fall 2015 # 6 🚩

Let $f : [0,1] \to \mathbb{R}$ be continuous. Show that

$$\sup \left\{ \|fg\|_1 \ \middle|\ g \in L^1[0,1],\ \|g\|_1 \leq 1 \right\} = \|f\|_\infty$$

11.6 Fall 2014 # 6 🚩

Let $1 \leq p, q \leq \infty$ be conjugate exponents, and show that

$$f \in L^p(\mathbb{R}^n) \implies \|f\|_p = \sup_{\|g\|_q = 1} \left| \int f(x) g(x) dx \right|$$

12 | Functional Analysis: Banach Spaces

12.1 Spring 2019 # 1 ✨

Let $C([0,1])$ denote the space of all continuous real-valued functions on $[0,1]$.

a. Prove that $C([0,1])$ is complete under the uniform norm $\|f\|_u := \sup_{x \in [0,1]} |f(x)|$.

b. Prove that $C([0,1])$ is not complete under the L^1-norm $\|f\|_1 = \int_0^1 |f(x)|\, dx$.

Add concepts.

Solution omitted.

12.2 Spring 2017 # 6 ✨

Show that the space $C^1([a,b])$ is a Banach space when equipped with the norm

$$\|f\| := \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |f'(x)|.$$

Add concepts.

Solution omitted.

12.3 Fall 2017 # 6 ✨

Let X be a complete metric space and define a norm

$$\|f\| := \max\{|f(x)| : x \in X\}.$$

Show that $(C^0(\mathbb{R}), \|-\|)$ (the space of continuous functions $f : X \to \mathbb{R}$) is complete.

Add concepts.

Shouldn't this be a supremum? The max may not exist?

Review and clean up.

Solution omitted.