

Complex Analysis Problems

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1 Topology and Functions of One Variable (8155a)

1.1 1

Let $x_0 = a, x_1 = b$, and set

$$x_n := \frac{x_{n-1} + x_{n-2}}{2} \quad n \geq 2.$$

Show that $\{x_n\}$ is a Cauchy sequence and find its limit in terms of a and b .

1.2 2

Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $\lim_{x \rightarrow \pm\infty} f(x) = 0$. Prove that f is uniformly continuous.

1.3 3

Give an example of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ that is everywhere differentiable but f' is not continuous at 0.

1.4 4

Suppose $\{g_n\}$ is a uniformly convergent sequence of functions from \mathbb{R} to \mathbb{R} and $f : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly continuous. Prove that the sequence $\{f \circ g_n\}$ is uniformly convergent.

1.5 5

Let f be differentiable on $[a, b]$. Say that f is *uniformly differentiable* iff

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } |x - y| < \delta \implies \left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \varepsilon.$$

Prove that f is uniformly differentiable on $[a, b] \iff f'$ is continuous on $[a, b]$.

1.6 6

Suppose $A, B \subseteq \mathbb{R}^n$ are disjoint and compact. Prove that there exist $a \in A, b \in B$ such that

$$\|a - b\| = \inf \left\{ \|x - y\| \mid x \in A, y \in B \right\}.$$

1.7 7

Suppose $A, B \subseteq \mathbb{R}^n$ are connected and not disjoint. Prove that $A \cup B$ is also connected.

1.8 8

Suppose $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of continuous functions $f_n : [0, 1] \rightarrow \mathbb{R}$ such that

$$f_n(x) \geq f_{n+1}(x) \geq 0 \quad \forall n \in \mathbb{N}, \forall x \in [0, 1].$$

Prove that if $\{f_n\}$ converges pointwise to 0 on $[0, 1]$ then it converges to 0 uniformly on $[0, 1]$.

1.9 9

Show that if $E \subset [0, 1]$ is uncountable, then there is some $t \in \mathbb{R}$ such that $E \cap (-\infty, t)$ and $E \cap (t, \infty)$ are also uncountable.

2 Several Variables (8155h)**2.1 1**

Is the following function continuous, differentiable, continuously differentiable?

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}$$
$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & \text{else.} \end{cases}$$

2.2 2**2.2.1 a**

Complete this definition: “ $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is real-differentiable at a point $p \in \mathbb{R}^n$ iff there exists a linear transformation. . .”

2.2.2 b

Give an example of a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ whose first-order partial derivatives exist everywhere but f is not differentiable at $(0, 0)$.

2.2.3 c

Give an example of a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ which is real-differentiable everywhere but nowhere complex-differentiable.

2.3 3

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

2.3.1 a

Define in terms of linear transformations what it means for f to be differentiable at a point $(a, b) \in \mathbb{R}^2$.

2.3.2 b

State a version of the inverse function theorem in this setting.

2.3.3 c

Identify \mathbb{R}^2 with \mathbb{C} and give a necessary and sufficient condition for a real-differentiable function at (a, b) to be complex differentiable at the point $a + ib$.

2.4 4

Let $f = u + iv$ be complex-differentiable with continuous partial derivatives at a point $z = re^{i\theta}$ with $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

2.5 5

Let $P = (1, 3) \in \mathbb{R}^2$ and define

$$f(s, t) := ps^3 - 6st + t^2.$$

2.5.1 a

State the conclusion of the implicit function theorem concerning $f(s, t) = 0$ when f is considered a function $\mathbb{R}^2 \rightarrow \mathbb{R}$.

2.5.2 b

State the above conclusion when f is considered a function $\mathbb{C}^2 \rightarrow \mathbb{C}$.

2.5.3 c

Use the implicit function theorem for a function $\mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ to prove (b).

There are various approaches: using the definition of the complex derivative, the Cauchy-Riemann equations, considering total derivatives, etc.

2.6 6

Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuously differentiable with $F(0, 0) = 0$ and $\|\nabla F(0, 0)\| < 1$.

Prove that there is some real number $r > 0$ such that $|F(x, y)| < r$ whenever $\|(x, y)\| < r$.

2.7 7

State the most general version of the implicit function theorem for real functions and outline how it can be proved using the inverse function theorem.

3 Conformal Maps (8155c)

Notation: \mathbb{D} is the open unit disc, \mathbb{H} is the open upper half-plane.

3.1 1

Find a conformal map from \mathbb{D} to \mathbb{H} .

3.2 2

Find a conformal map from the strip $\{z \in \mathbb{C} \mid 0 < \Im(z) < 1\}$ to \mathbb{H} .

3.3 3

Find a fractional linear transformation T which maps \mathbb{H} to \mathbb{D} , and explicitly describe the image of the first quadrant under T .

3.4 4

Find a conformal map from $\{z \in \mathbb{C} \mid |z - i| > 1, \Re(z) > 0\}$ to \mathbb{H} .

3.5 5

Find a conformal map from $\left\{z \in \mathbb{C} \mid |z| < 1, \left|z - \frac{1}{2}\right| > \frac{1}{2}\right\}$ to \mathbb{D} .

3.6 6

Find a conformal map from $\{|z - 1| < 2\} \cap \{|z + 1| < 2\}$ to \mathbb{H} .

3.7 7

Let Ω be the region inside the unit circle $|z| = 1$ and outside the circle $\left|z - \frac{1}{4}\right| = \frac{1}{4}$.

Find an injective conformal map from Ω onto some annulus $\{r < |z| < 1\}$ for some constant r .

3.8 8

Let D be the region obtained by deleting the real interval $[0, 1)$ from \mathbb{D} ; find a conformal map from D to \mathbb{D} .

3.9 9

Find a conformal map from $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$ to \mathbb{D} .

3.10 10

Find a conformal map from $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \geq 1\}$ to \mathbb{D} .

3.11 11

Find a bijective conformal map from G to \mathbb{H} , where

$$G := \left\{z \in \mathbb{C} \mid |z - 1| < \sqrt{2}, |z + 1| < \sqrt{2}\right\} \setminus [0, i).$$

3.12 12

Prove that TFAE for a Möbius transformation T given by $T(z) = \frac{az + b}{cz + d}$:

- T maps $\mathbb{R} \cup \{\infty\}$ to itself.
- It is possible to choose a, b, c, d to be real numbers.
- $\overline{T(z)} = T(\bar{z})$ for every $z \in \mathbb{CP}^1$.
- There exist $\alpha \in \mathbb{R}, \beta \in \mathbb{C} \setminus \mathbb{R}$ such that $T(\alpha) = \alpha$ and $T(\bar{\beta}) = \overline{T(\beta)}$.

4 Integrals and Cauchy's Theorem (8155d)

Some interesting problems: 3, 4, 9, 10.

4.1 1

Suppose $f, g : [0, 1] \rightarrow \mathbb{R}$ where f is Riemann integrable and for $x, y \in [0, 1]$,

$$|g(x) - g(y)| \leq |f(x) - f(y)|.$$

Prove that g is Riemann integrable.

4.2 2

State and prove Green's Theorem for rectangles.

Then use it to prove Cauchy's Theorem for functions that are analytic in a rectangle.

4.3 3

Suppose $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of analytic functions on $\mathbb{D}^\circ := \{z \in \mathbb{C} \mid |z| < 1\}$.

Show that if $f_n \rightarrow g$ for some $g : \mathbb{D}^\circ \rightarrow \mathbb{C}$ uniformly on every compact $K \subset \mathbb{D}^\circ$, then g is analytic on \mathbb{D}° .

4.4 4

Suppose $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of entire functions where

- $f_n \rightarrow g$ pointwise for some $g : \mathbb{C} \rightarrow \mathbb{C}$.
- On every line segment in \mathbb{C} , $f_n \rightarrow g$ uniformly.

Show that

- g is entire, and
- $f_n \rightarrow g$ uniformly on every compact subset of \mathbb{C} .

4.5 5

Prove that there is no sequence of polynomials that uniformly converge to $f(z) = \frac{1}{z}$ on S^1 .

4.6 6

Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function that vanishes outside of some finite interval. For each $z \in \mathbb{C}$, define

$$g(z) = \int_{-\infty}^{\infty} f(t)e^{-izt} dt.$$

Show that g is entire.

4.7 7

Suppose $f : \mathbb{C} \rightarrow \mathbb{C}$ is entire and

$$|f(z)| \leq |z|^{\frac{1}{2}} \quad \text{when } |z| > 10.$$

Prove that f is constant.

4.8 8

Let γ be a smooth curve joining two distinct points $a, b \in \mathbb{C}$.

Prove that the function

$$f(z) := \int_{\gamma} \frac{g(w)}{w - z} dw$$

is analytic in $\mathbb{C} \setminus \gamma$.

4.9 9

Suppose that $f : \mathbb{C} \rightarrow \mathbb{C}$ is continuous everywhere and analytic on $\mathbb{C} \setminus \mathbb{R}$ and prove that f is entire.

4.10 10

Prove Liouville's theorem: suppose $f : \mathbb{C} \rightarrow \mathbb{C}$ is entire and bounded. Use Cauchy's formula to prove that $f' \equiv 0$ and hence f is constant.

5 Liouville's Theorem, Power Series (8155e)**5.1 1**

Suppose f is analytic on a region Ω such that $\mathbb{D} \subseteq \Omega \subseteq \mathbb{C}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series with radius of convergence exactly 1.

- Give an example of such an f that converges at every point of S^1 .
- Given an example of such an f which is analytic at 1 but $\sum_{n=0}^{\infty} a_n$ diverges.
- Prove that f can not be analytic at *every* point of S^1 .

5.2 2

Suppose f is entire and has Taylor series $\sum a_n z^n$ about 0.

- Express a_n as a contour integral along the circle $|z| = R$.
- Apply (a) to show that the above Taylor series converges uniformly on every bounded subset of \mathbb{C} .
- Determine those functions f for which the above Taylor series converges uniformly on all of \mathbb{C} .

5.3 3

Suppose D is a domain and f, g are analytic on D .

Prove that if $fg = 0$ on D , then either $f \equiv 0$ or $g \equiv 0$ on D .

5.4 4

Suppose f is analytic on \mathbb{D}° . Determine with proof which of the following are possible:

- a. $f\left(\frac{1}{n}\right) = (-1)^n$ for each $n > 1$.
- b. $f\left(\frac{1}{n}\right) = e^{-n}$ for each even integer $n > 1$ while $f\left(\frac{1}{n}\right) = 0$ for each odd integer $n > 1$.
- c. $f\left(\frac{1}{n^2}\right) = \frac{1}{n}$ for each integer $n > 1$.
- d. $f\left(\frac{1}{n}\right) = \frac{n-2}{n-1}$ for each integer $n > 1$.

5.5 5

Prove the Fundamental Theorem of Algebra (using complex analysis).

5.6 6

Find all entire functions that satisfy

$$|f(z)| \geq |z| \quad \forall z \in \mathbb{C}.$$

Prove this list is complete.

5.7 7

Suppose $\sum_{n=0}^{\infty} a_n z^n$ converges for some $z_0 \neq 0$.

- a. Prove that the series converges absolutely for each z with $|z| < |z_0|$.
- b. Suppose $0 < r < |z_0|$ and show that the series converges uniformly on $|z| \leq r$.

5.8 8

Suppose f is entire and suppose that for some integer $n \geq 1$,

$$\lim_{z \rightarrow \infty} \frac{f(z)}{z^n} = 0.$$

Prove that f is a polynomial of degree at most $n - 1$.

5.9 9

Find all entire functions satisfying

$$|f(z)| \leq |z|^{\frac{1}{2}} \quad \text{for } |z| > 10.$$

5.10 10

Prove that the following series converges uniformly on the set $\{z \mid \Im(z) < \ln 2\}$:

$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{2^n}.$$

6 Laurent Expansions and Singularities (8155f)**6.1 1**

Find the Laurent expansion of

$$f(z) = \frac{z+1}{z(z-1)}$$

about $z = 0$ and $z = 1$ respectively.

6.2 2

Find the Laurent expansions about $z = 0$ of the following functions:

$$\exp\left(\frac{1}{z}\right) \qquad \cos\left(\frac{1}{z}\right).$$

6.3 3

Find the Laurent expansion of

$$f(z) = \frac{z+1}{z(z-1)^2}$$

about $z = 0$ and $z = 1$ respectively.

Hint: recall that power series can be differentiated.

6.4 4

For the following functions, find the Laurent series about 0 and classify their singularities there:

$$\frac{\sin^2(z)}{z} \\ z \exp\left(\frac{1}{z^2}\right) \\ \frac{1}{z(4-z)}.$$

6.5 5

Find all entire functions with have poles at ∞ .

6.6 6

Find all functions on the Riemann sphere that have a simple pole at $z = 2$ and a double pole at $z = \infty$, but are analytic elsewhere.

6.7 7

Let f be entire, and discuss (with proofs and examples) the types of singularities f might have (removable, pole, or essential) at $z = \infty$ in the following cases:

1. f has at most finitely many zeros in \mathbb{C} .
2. f has infinitely many zeros in \mathbb{C} .

6.8 8

Define

$$f(z) = \frac{\pi^2}{\sin^2(\pi z)}$$
$$g(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z - n)^2}.$$

- a. Show that f and g have the same singularities in \mathbb{C} .
- b. Show that f and g have the same singular parts at each of their singularities.
- c. Show that f, g each have period one and approach zero uniformly on $0 \leq x \leq 1$ as $|y| \rightarrow \infty$.
- d. Conclude that $f = g$.

7 Residues (8155g)**7.1 1**

Calculate

$$\int_0^\infty \frac{1}{(1+z)^2(z+9x^2)} dx.$$

7.2 2

Let $a > 0$ and calculate

$$\int_0^\infty \frac{x \sin(x)}{x^2 + a^2} dx.$$

7.3 3

Calculate

$$\int_0^\infty \frac{\sqrt{x}}{(x+1)^2} dx.$$

7.4 4

Calculate

$$\int_0^\infty \frac{\cos(x) - \cos(4x)}{x^2} dx.$$

7.5 5Let $a > 0$ and calculate

$$\int_0^\infty \frac{x^2}{(x^2 + a^2)^2} dx.$$

7.6 6

Calculate

$$\int_0^\infty \frac{\sin(x)}{x} dx.$$

7.7 7

Calculate

$$\int_0^\infty \frac{\sin(x)}{x(x^2 + 1)} dx.$$

7.8 8

Calculate

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2} dx.$$

7.9 9

Calculate

$$\int_{-\infty}^\infty \frac{1+x^2}{1+x^4} dx.$$

7.10 10

Let $a > 0$ and calculate

$$\int_0^\infty \frac{\cos(x)}{(x^2 + a^2)^2} dx.$$

7.11 11

Calculate

$$\int_0^\infty \frac{\sin^3(x)}{x^3} dx.$$

7.12 12

Let $n \in \mathbb{Z}^{\geq 1}$ and $0 < \theta < \pi$ and show that

$$\frac{1}{2\pi i} \int_{|z|=2} \frac{z^n}{1 - 3z \cos(\theta) + z^2} dz = \frac{\sin(n\theta)}{\sin(\theta)}.$$

7.13 13

Suppose $a > b > 0$ and calculate

$$\int_0^{2\pi} \frac{1}{(a + b \cos(\theta))^2} d\theta.$$

8 Rouché's Theorem (8155h)**8.1 1**

Prove that for every $n \in \mathbb{Z}^{\geq 0}$ the following polynomial has no roots in the open unit disc:

$$f_n(x) := \sum_{k=0}^n \frac{z^k}{k!}.$$

Hint: check $n = 1, 2$ directly.

8.2 2

Assume that $|b| < 1$ and show that the following polynomial has exactly two roots (counting multiplicity) in $|z| < 1$:

$$f(z) := z^3 + 3z^2 + bz + b^2.$$

8.3 3

Let $c \in \mathbb{C}$ with $|c| < \frac{1}{3}$. Show that on the open set $\{z \in \mathbb{C} \mid \Re(z) < 1\}$, the function $f(z) := ce^z$ has exactly one fixed point.

8.4 4

How many roots does the following polynomial have in the open disc $|z| < 1$?

$$f(z) = z^7 - 4z^3 - 1.$$

8.5 5

Let $n \in \mathbb{Z}^{\geq 0}$ and show that the equation

$$e^z = az^n$$

has n solutions in the open unit disc if $|a| > e$, and no solutions if $|a| < \frac{1}{e}$.

8.6 6

Let f be analytic in a domain D and fix $z_0 \in D$ with $w_0 := f(z_0)$. Suppose z_0 is a zero of $f(z) - w_0$ with finite multiplicity m . Show that there exists $\delta > 0$ and $\varepsilon > 0$ such that for each w such that $0 < |w - w_0| < \varepsilon$, the equation $f(z) - w = 0$ has exactly m *distinct* solutions inside the disc $|z - z_0| < \delta$.

8.7 7

For $k = 1, 2, \dots, n$, suppose $|a_k| < 1$ and

$$f(z) := \left(\frac{z - a_1}{1 - \bar{a}_1 z} \right) \left(\frac{z - a_2}{1 - \bar{a}_2 z} \right) \cdots \left(\frac{z - a_n}{1 - \bar{a}_n z} \right).$$

Show that $f(z) = b$ has n solutions in $|z| < 1$.

8.8 8

For each $n \in \mathbb{Z}^{\geq 1}$, let

$$P_n(z) = 1 + z + \frac{1}{2!}z^2 + \cdots + \frac{1}{n!}z^n.$$

Show that for sufficiently large n , the polynomial P_n has no zeros in $|z| < 10$, while the polynomial $P_n(z) - 1$ has precisely 3 zeros there.

8.9 9

Prove that

$$\max_{|z|=1} |a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n| \geq 1.$$

Hint: the first part of the problem asks for a statement of Rouché's theorem.

8.10 10

Use Rouché's theorem to prove the Fundamental Theorem of Algebra.

9 Schwarz Lemma and Reflection Principle (8155i)**9.1 1**

Suppose $f : \mathbb{D} \rightarrow \mathbb{D}$ is analytic and admits a continuous extension $\tilde{f} : \bar{\mathbb{D}} \rightarrow \bar{\mathbb{D}}$ such that $|z| = 1 \implies |f(z)| = 1$.

9.1.1 a

Prove that f is a rational function.

9.1.2 b

Suppose that $z = 0$ is the unique zero of f . Show that

$$\exists n \in \mathbb{N}, \lambda \in S^1 \quad \text{such that} \quad f(z) = \lambda z^n.$$

10 Spring 2020 Homework 1**10.1 1**

Geometrically describe the following subsets of \mathbb{C} :

- a. $|z - 1| = 1$
- b. $|z - 1| = 2|z - 2|$
- c. $1/z = \bar{z}$
- d. $\Re(z) = 3$
- e. $\Im(z) = a$ with $a \in \mathbb{R}$.
- f. $\Re(z) > a$ with $a \in \mathbb{R}$.
- g. $|z - 1| < 2|z - 2|$

10.2 2

Prove the following inequality, and explain when equality holds:

$$|z + w| \geq ||z| - |w||.$$

10.3 3

Prove that the following polynomial has its roots outside of the unit circle:

$$p(z) = z^3 + 2z + 4.$$

Hint: What is the maximum value of the modulus of the first two terms if $|z| \leq 1$?

10.4 4

- a. Prove that if $c > 0$,

$$|w_1| = c|w_2| \implies |w_1 - c^2 w_2| = c|w_1 - w_2|.$$

- b. Prove that if $c > 0$ and $c \neq 1$, with $z_1 \neq z_2$, then the following equation represents a circle:

$$\left| \frac{z - z_1}{z - z_2} \right| = c.$$

Find its center and radius.

Hint: use part (a)

10.5 5

- a. Let $z, w \in \mathbb{C}$ with $\bar{z}w \neq 1$. Prove that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| < 1 \quad \text{if } |z| < 1, |w| < 1$$

with equality when $|z| = 1$ or $|w| = 1$.

- b. Prove that for a fixed $w \in \mathbb{D}$, the mapping $F : z \mapsto \frac{w - z}{1 - \bar{w}z}$ satisfies
- F maps \mathbb{D} to itself and is holomorphic.
 - $F(0) = w$ and $F(w) = 0$.
 - $|z| = 1$ implies $|F(z)| = 1$.

10.6 6

Use n th roots of unity to show that

$$2^{n-1} \sin\left(\frac{\pi}{n}\right) \sin\left(\frac{2\pi}{n}\right) \cdots \sin\left(\frac{(n-1)\pi}{n}\right) = n.$$

Hint:

$$\begin{aligned} 1 - \cos(2\theta) &= 2\sin^2(\theta) \\ 2\sin(2\theta) &= 2\sin(\theta)\cos(\theta). \end{aligned}$$

10.7 7

Prove that $f(z) = |z|^2$ has a derivative at $z = 0$ and nowhere else.

10.8 8

Let $f(z)$ be analytic in a domain, and prove that f is constant if it satisfies any of the following conditions:

- $|f(z)|$ is constant.
- $\Re(f(z))$ is constant.
- $\arg(f(z))$ is constant.
- $\overline{f(z)}$ is analytic.

How do you generalize (a) and (b)?

10.9 9

Prove that if $z \mapsto f(z)$ is analytic, then $z \mapsto \overline{f(\bar{z})}$ is analytic.

10.10 10

- Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

- Use (a) to show that the logarithm function, defined as

$$\log z = \log r + i\theta \text{ where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi.$$

is holomorphic on the region $r > 0, -\pi < \theta < \pi$.

Also show that this function is not continuous in $r > 0$.

10.11 11

Prove that the distinct complex numbers z_1, z_2, z_3 are the vertices of an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

11 Spring 2020 Homework 2

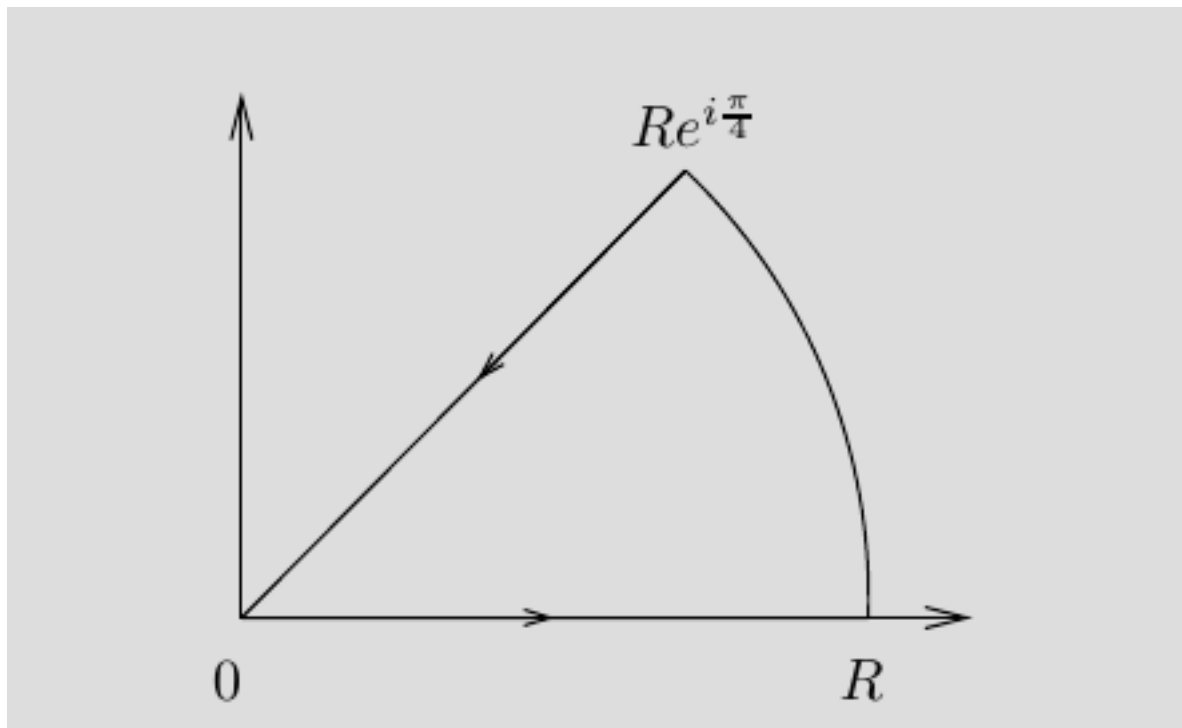
Note on notation: I sometimes use $f_x := \frac{\partial f}{\partial x}$ to denote partial derivatives, and $\partial_z^n f$ as $f^{(n)}(z)$.

11.1 Stein And Shakarchi**11.1.1 2.6.1**

Show that

$$\int_0^\infty \sin(x^2) dx = \int_0^\infty \cos(x^2) dx = \frac{\sqrt{2\pi}}{4}.$$

Hint: integrate e^{-x^2} over the following contour, using the fact that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$:



11.1.2 2.6.2

Show that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Hint: use the fact that this integral equals $\frac{1}{2i} \int_{-\infty}^{\infty} \frac{e^{ix} - 1}{x} dx$, and integrate around an indented semicircle.

11.1.3 2.6.5

Suppose $f \in C_{\mathbb{C}}^1(\Omega)$ and $T \subset \Omega$ is a triangle with $T^\circ \subset \Omega$. Apply Green's theorem to show that $\int_T f(z) dz = 0$.

Assume that f' is continuous and prove Goursat's theorem.

Hint: Green's theorem states

$$\int_T F dx + G dy = \int_{T^\circ} \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx dy.$$

11.1.4 2.6.6

Suppose that f is holomorphic on a punctured open set $\Omega \setminus \{w_0\}$ and let $T \subset \Omega$ be a triangle containing w_0 . Prove that if f is bounded near w_0 , then $\int_T f(z) dz = 0$.

11.1.5 2.6.7

Suppose $f : \mathbb{D} \rightarrow \mathbb{C}$ is holomorphic and let $d := \sup_{z, w \in \mathbb{D}} |f(z) - f(w)|$ be the diameter of the image of f . Show that $2|f'(0)| \leq d$, and that equality holds iff f is linear, so $f(z) = a_1z + a_2$.

Hint: $2f'(0) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi) - f(-\xi)}{\xi^2} d\xi$ whenever $0 < r < 1$.

11.1.6 2.6.8

Suppose that f is holomorphic on the strip $S = \{x + iy \mid x \in \mathbb{R}, -1 < y < 1\}$ with $|f(z)| \leq A(1 + |z|)^\nu$ for ν some fixed real number. Show that for all $z \in S$, for each integer $n \geq 0$ there exists an $A_n \geq 0$ such that $|f^{(n)}(x)| \leq A_n(1 + |x|)^\nu$ for all $x \in \mathbb{R}$.

Hint: Use the Cauchy inequalities.

11.1.7 2.6.9

Let $\Omega \subset \mathbb{C}$ be open and bounded and $\varphi : \Omega \rightarrow \Omega$ holomorphic. Prove that if there exists a point $z_0 \in \Omega$ such that $\varphi(z_0) = z_0$ and $\varphi'(z_0) = 1$, then φ is linear.

Hint: assume $z_0 = 0$ (explain why this can be done) and write $\varphi(z) = z + a_n z^n + O(z^{n+1})$ near 0. Let $\varphi_k = \varphi \circ \varphi \circ \cdots \circ \varphi$ and prove that $\varphi_k(z) = z + k a_n z^n + O(z^{n+1})$. Apply Cauchy's inequalities and let $k \rightarrow \infty$ to conclude.

11.1.8 2.6.10

Can every continuous function on $\overline{\mathbb{D}}$ be uniformly approximated by polynomials in the variable z ?

Hint: compare to Weierstrass for the real interval.

11.1.9 2.6.13

Suppose f is analytic, defined on all of \mathbb{C} , and for each $z_0 \in \mathbb{C}$ there is at least one coefficient in the expansion $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ is zero. Prove that f is a polynomial.

Hint: use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.

11.1.10 2.6.14

Suppose that f is holomorphic in an open set containing \mathbb{D} except for a pole $z_0 \in \partial\mathbb{D}$. Let $\sum_{n=0}^{\infty} a_n z^n$ be the power series expansion of f in \mathbb{D} , and show that $\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} = z_0$.

11.1.11 2.6.15

Suppose f is continuous and nonvanishing on $\bar{\mathbb{D}}$, and holomorphic in \mathbb{D} . Prove that if $|z| = 1 \implies |f(z)| = 1$, then f is constant.

Hint: Extend f to all of \mathbb{C} by $f(z) = 1/\overline{f(1/\bar{z})}$ for any $|z| > 1$, and argue as in the Schwarz reflection principle.

11.2 Additional Problems**11.2.1 1**

Let $a_n \neq 0$ and show that

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L \implies \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = L.$$

In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

11.2.2 2

Let f be a power series centered at the origin. Prove that f has a power series expansion about any point in its disc of convergence.

11.2.3 3

Prove the following:

- a. $\sum_n n z^n$ does not converge at any point of S^1
- b. $\sum_n \frac{z^n}{n^2}$ converges at every point of S^1 .
- c. $\sum_n \frac{z^n}{n}$ converges at every point of S^1 except $z = 1$.

11.2.4 4

Without using Cauchy's integral formula, show that if $|a| < r < |b|$, then

$$\int_{\gamma} \frac{dz}{(z - \alpha)(z - \beta)} = \frac{2\pi i}{\alpha - \beta}$$

where γ denotes the circle centered at the origin of radius r with positive orientation.

11.2.5 5

Assume f is continuous in the region $\{x + iy \mid x \geq x_0, 0 \leq y \leq b\}$, and the following limit exists independent of y :

$$\lim_{x \rightarrow +\infty} f(x + iy) = A.$$

Show that if $\gamma_x := \{z = x + it \mid 0 \leq t \leq b\}$, then

$$\lim_{x \rightarrow +\infty} \int_{\gamma_x} f(z) dz = iAb.$$

11.2.6 6

Show by example that there exists a function $f(z)$ that is holomorphic on $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$ and for all $r < 1$,

$$\int_{|z|=r} f(z) dz = 0,$$

but f is not holomorphic at $z = 0$.

11.2.7 7

Let f be analytic on a region R and suppose $f'(z_0) \neq 0$ for some $z_0 \in R$. Show that if C is a circle of sufficiently small radius centered at z_0 , then

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{dz}{f(z) - f(z_0)}.$$

Hint: use the inverse function theorem.

11.2.8 8

Assume two functions $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$ have continuous partial derivatives at (x_0, y_0) . Show that $f := u + iv$ has derivative $f'(z_0)$ at $z_0 = x_0 + iy_0$ if and only if

$$\lim_{r \rightarrow 0} \frac{1}{\pi r^2} \int_{|z-z_0|=r} f(z) dz = 0.$$

11.2.9 9 (Cauchy's Formula for Exterior Regions)

Let γ be a piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2 . Assume f' exists in an open set containing γ and Ω_2 with $\lim_{z \rightarrow \infty} f(z) = A$. Show that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1 \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}.$$

11.2.10 10

Let $f(z)$ be bounded and analytic in \mathbb{C} . Let $a \neq b$ be any fixed complex numbers. Show that the following limit exists:

$$\lim_{R \rightarrow \infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

Use this to show that $f(z)$ must be constant.

11.2.11 11

Suppose $f(z)$ is entire and

$$\lim_{z \rightarrow \infty} \frac{f(z)}{z} = 0.$$

Show that $f(z)$ is a constant.

11.2.12 12

Let f be analytic in a domain D and γ be a closed curve in D . For any $z_0 \in D$ not on γ , show that

$$\int_{\gamma} \frac{f'(z)}{(z-z_0)} dz = \int_{\gamma} \frac{f(z)}{(z-z_0)^2} dz.$$

Give a generalization of this result.

11.2.13 13

Compute

$$\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z}$$

and use it to show that

$$\int_0^{2\pi} \cos^{2n}(\theta) d\theta = 2\pi \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \right).$$

12 Spring 2020 Homework 3**12.1 Stein and Shakarchi****12.1.1 3.8.1**

Use the following formula to show that the complex zeros of $\sin(\pi z)$ are exactly the integers, and they are each of order 1:

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}.$$

Calculate the residue of $\frac{1}{\sin(\pi z)}$ at $z = n \in \mathbb{Z}$.

12.1.2 3.8.2

Evaluate the integral

$$\int_{\mathbb{R}} \frac{dx}{1+x^4}.$$

What are the poles of $\frac{1}{1+z^4}$?

12.1.3 3.8.4

Show that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}, \quad \text{for all } a > 0.$$

12.1.4 3.8.5

Show that if $\xi \in \mathbb{R}$, then

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{(1+x^2)^2} dx = \frac{\pi}{2} (1 + 2\pi |\xi|) e^{-2\pi |\xi|}.$$

12.1.5 3.8.6

Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \pi.$$

12.1.6 3.8.7

Show that

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos \theta)^2} = \frac{2\pi a}{(a^2 - 1)^{3/2}}, \quad \text{whenever } a > 1.$$

12.1.7 3.8.8

Show that if $a, b \in \mathbb{R}$ with $a > |b|$, then

$$\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

12.1.8 3.8.9

Show that

$$\int_0^1 \log(\sin \pi x) dx = -\log 2.$$

Hint: use the following contour.

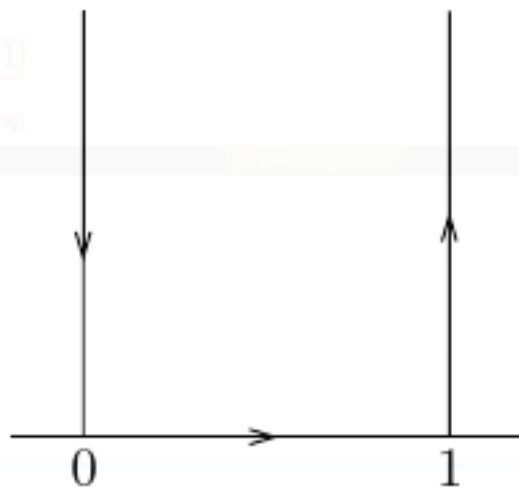


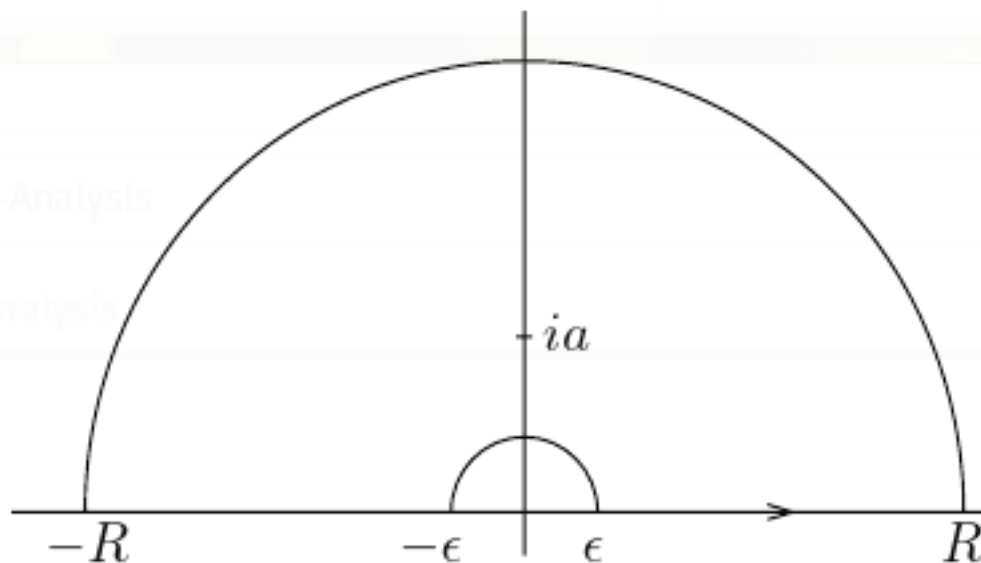
Figure 9. Contour in Exercise 9

12.1.9 3.8.10

Show that if $a > 0$, then

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx = \frac{\pi}{2a} \log a.$$

Hint: use the following contour.



12.1.10 3.8.14

Prove that all entire functions that are injective are of the form $f(z) = az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$.

Hint: Apply the Casorati-Weierstrass theorem to $f(1/z)$.

12.1.11 3.8.15

Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

- a. Prove that if f is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \leq AR^k + B$$

for all $R > 0$, some integer $k \geq 0$, and some constants $A, B > 0$, then f is a polynomial of degree $\leq k$.

- b. Show that if f is holomorphic in the unit disc, is bounded, and converges uniformly to zero in the sector $\theta < \arg(z) < \varphi$ as $|z| \rightarrow 0$, then $f \equiv 0$.
- c. Let w_1, \dots, w_n be points on $S^1 \subset \mathbb{C}$. Prove that there exists a point $z \in S^1$ such that the product of the distances from z to the points w_j is at least 1.

Conclude that there exists a point $w \in S^1$ such that the product of the above distances is *exactly* 1.

- d. Show that if the real part of an entire function is bounded, then f is constant.

12.1.12 3.8.17

Let f be non-constant and holomorphic in an open set containing the closed unit disc.

- a. Show that if $|f(z)| = 1$ whenever $|z| = 1$, then the image of f contains the unit disc.

Hint: Show that $f(z) = w_0$ has a root for every $w_0 \in \mathbb{D}$, for which it suffices to show that $f(z) = 0$ has a root. Conclude using the maximum modulus principle.

- b. If $|f(z)| \geq 1$ whenever $|z| = 1$ and there exists a $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$, then the image of f contains the unit disc.

12.1.13 3.8.19

Prove that maximum principle for harmonic functions, i.e.

- a. If u is a non-constant real-valued harmonic function in a region Ω , then u can not attain a maximum or a minimum in Ω .
- b. Suppose Ω is a region with compact closure $\bar{\Omega}$. If u is harmonic in Ω and continuous in $\bar{\Omega}$, then

$$\sup_{z \in \Omega} |u(z)| \leq \sup_{z \in \bar{\Omega} - \Omega} |u(z)|.$$

Hint: to prove (a), assume u attains a local maximum at z_0 . Let f be holomorphic near z_0 with $\Re(f) = u$, and show that f is not an open map. Then (a) implies (b).

12.2 Problems From Tie**12.2.1 1**

Problem Prove that if f has two Laurent series expansions,

$$f(z) = \sum c_n(z-a)^n \quad \text{and} \quad f(z) = \sum c'_n(z-a)^n$$

then $c_n = c'_n$.

12.2.2 2

Problem Find Laurent series expansions of

$$\frac{1}{1-z^2} + \frac{1}{3-z}$$

How many such expansions are there? In what domains are each valid?

12.2.3 3

Problem Let P, Q be polynomials with no common zeros. Assume a is a root of Q . Find the principal part of P/Q at $z = a$ in terms of P and Q if a is (1) a simple root, and (2) a double root.

12.2.4 4

Problem Let f be non-constant, analytic in $|z| > 0$, where $f(z_n) = 0$ for infinitely many points z_n with $\lim_{n \rightarrow \infty} z_n = 0$.

Show that $z = 0$ is an essential singularity for f .

Example: $f(z) = \sin(1/z)$.

12.2.5 5

Problem Show that if f is entire and $\lim_{z \rightarrow \infty} f(z) = \infty$, then f is a polynomial.

12.2.6 6

Problem

- a. Show (without using 3.8.9 in the S&S) that

$$\int_0^{2\pi} \log |1 - e^{i\theta}| d\theta = 0$$

- b. Show that this identity is equivalent to S&S 3.8.9:

$$\int_0^1 \log(\sin(\pi x)) dx = -\log 2.$$

12.2.7 7

Problem Let $0 < a < 4$ and evaluate

$$\int_0^\infty \frac{x^{a-1}}{1+x^3} dx$$

12.2.8 8

Problem Prove the fundamental theorem of Algebra using

- Rouche's Theorem.
- The maximum modulus principle.

12.2.9 9

Problem Let f be analytic in a region D and γ a rectifiable curve in D with interior in D .

Prove that if $f(z)$ is real for all $z \in \gamma$, then f is constant.

12.2.10 10

Problem For $a > 0$, evaluate

$$\int_0^{\pi/2} \frac{d\theta}{a + \sin^2 \theta}$$

12.2.11 11

Problem Find the number of roots of $p(z) = 4z^4 - 6z + 3$ in $|z| < 1$ and $1 < |z| < 2$ respectively.

12.2.12 12

Problem Prove that $z^4 + 2z^3 - 2z + 10$ has exactly one root in each open quadrant.

12.2.13 13

Problem Prove that for $a > 0$, $z \tan z - a$ has only real roots.

12.2.14 14

Problem Let f be nonzero, analytic on a bounded region Ω and continuous on its closure $\bar{\Omega}$.

Show that if $|f(z)| \equiv M$ is constant for $z \in \partial\Omega$, then $f(z) \equiv Me^{i\theta}$ for some real constant θ .

13 Extra Questions from Jingzhi Tie**13.1 Fall 2009****13.1.1 ?**

(1) Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

(2) Deduce Liouville's theorem from (1).

13.1.2 ?

Let f be a continuous function in the region

$$D = \{z \mid |z| > R, 0 \leq \arg z \leq \theta\} \quad \text{where } 1 \leq \theta \leq 2\pi.$$

If there exists k such that $\lim_{z \rightarrow \infty} z f(z) = k$ for z in the region D . Show that

$$\lim_{R' \rightarrow \infty} \int_L f(z) dz = i\theta k,$$

where L is the part of the circle $|z| = R'$ which lies in the region D .

13.1.3 ?

Suppose that f is an analytic function in the region D which contains the point a . Let

$$F(z) = z - a - qf(z), \quad \text{where } q \text{ is a complex parameter.}$$

- (1) Let $K \subset D$ be a circle with the center at point a and also we assume that $f(z) \neq 0$ for $z \in K$. Prove that the function F has one and only one zero $z = w$ on the closed disc \bar{K} whose boundary is the circle K if $|q| < \min_{z \in K} \frac{|z - a|}{|f(z)|}$.
- (2) Let $G(z)$ be an analytic function on the disk \bar{K} . Apply the residue theorem to prove that $\frac{G(w)}{F'(w)} = \frac{1}{2\pi i} \int_K \frac{G(z)}{F(z)} dz$, where w is the zero from (1).
- (3) If $z \in K$, prove that the function $\frac{1}{F(z)}$ can be represented as a convergent series with respect to q : $\frac{1}{F(z)} = \sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z - a)^{n+1}}$.

13.1.4 ?

Evaluate

$$\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx.$$

13.1.5 ?

Let $f = u + iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

13.1.6 ?

Show that $\int_0^{\infty} \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$ using complex analysis, $0 < a < n$. Here n is a positive integer.

13.1.7 ?

For $s > 0$, the **gamma function** is defined by $\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt$.

1. Show that the gamma function is analytic in the half-plane $\Re(s) > 0$, and is still given there by the integral formula above.
2. Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need $\Gamma(1-s) = t \int_0^{\infty} e^{-vt} (vt)^{-s} dv$ for $t > 0$.

13.1.8 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n \quad (a_n \neq 0)$$

is a polynomial of degree n , then it has n zeros in \mathbb{C} .

13.1.9 ?

Suppose f is entire and there exist $A, R > 0$ and natural number N such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that (i) f is a polynomial and (ii) the degree of f is at least N .

13.1.10 ?

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an injective analytic (also called *univalent*) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

13.1.11 ?

Let g be analytic for $|z| \leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

1. Show that g has a unique fixed point in $|z| < 1$.
2. What happens if we replace $|g(z)| < 1$ with $|g(z)| \leq 1$ for $|z| = 1$? Give an example if (a) is not true or give an proof if (a) is still true.
3. What happens if we simply assume that f is analytic for $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z) \neq z$. Can f have more than one fixed point in $|z| < 1$?

Hint: The map $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$ may be useful.

13.1.12 ?

Find a conformal map from $D = \{z : |z| < 1, |z - 1/2| > 1/2\}$ to the unit disk $\Delta = \{z : |z| < 1\}$.

13.1.13 ?

Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω . Show without using Picard's theorems that $f(z)$ is a constant.

13.1.14 ?

- (1) Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

(2) Deduce Liouville's theorem from (1).

13.1.15 ?

Let $f(z)$ be entire and assume that $f(z) \leq M|z|^2$ outside some disk for some constant M . Show that $f(z)$ is a polynomial in z of degree ≤ 2 .

13.1.16 ?

Let $a_n(z)$ be an analytic sequence in a domain D such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D . Show that $\sum_{n=0}^{\infty} |a'_n(z)|$ converges uniformly on bounded and closed sub-regions of D .

13.1.17 ?

Let $f(z)$ be analytic in an open set Ω except possibly at a point z_0 inside Ω . Show that if $f(z)$ is bounded in near z_0 , then $\int_{\Delta} f(z)dz = 0$ for all triangles Δ in Ω .

13.1.18 ?

Assume f is continuous in the region: $0 < |z - a| \leq R$, $0 \leq \arg(z - a) \leq \beta_0$ ($0 < \beta_0 \leq 2\pi$) and the limit $\lim_{z \rightarrow a} (z - a)f(z) = A$ exists. Show that

$$\lim_{r \rightarrow 0} \int_{\gamma_r} f(z)dz = iA\beta_0,$$

where $\gamma_r := \{z \mid z = a + re^{it}, 0 \leq t \leq \beta_0\}$.

13.1.19 ?

Show that $f(z) = z^2$ is uniformly continuous in any open disk $|z| < R$, where $R > 0$ is fixed, but it is not uniformly continuous on \mathbb{C} .

13.1.20 ?

(1) Show that the function $u = u(x, y)$ given by

$$u(x, y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx \quad \text{for } n \in \mathbf{N}$$

is the solution on $D = \{(x, y) \mid x^2 + y^2 < 1\}$ of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = \frac{\sin nx}{n}.$$

(2) Show that there exist points $(x, y) \in D$ such that $\limsup_{n \rightarrow \infty} |u(x, y)| = \infty$.

13.2 Fall 2011**13.2.1 ?**

- (1) Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

- (2) Deduce Liouville's theorem from (1).

13.2.2 ?

Let f be a continuous function in the region

$$D = \{z \mid |z| > R, 0 \leq \arg z \leq \theta\} \quad \text{where} \quad 0 \leq \theta \leq 2\pi.$$

If there exists k such that $\lim_{z \rightarrow \infty} zf(z) = k$ for z in the region D . Show that

$$\lim_{R' \rightarrow \infty} \int_L f(z) dz = i\theta k,$$

where L is the part of the circle $|z| = R'$ which lies in the region D .

13.2.3 ?

Suppose that f is an analytic function in the region D which contains the point a . Let

$$F(z) = z - a - qf(z), \quad \text{where} \quad q \text{ is a complex parameter.}$$

- (1) Let $K \subset D$ be a circle with the center at point a and also we assume that $f(z) \neq 0$ for $z \in K$. Prove that the function F has one and only one zero $z = w$ on the closed disc \bar{K} whose boundary is the circle K if $|q| < \min_{z \in K} \frac{|z - a|}{|f(z)|}$.
- (2) Let $G(z)$ be an analytic function on the disk \bar{K} . Apply the residue theorem to prove that $\frac{G(w)}{F'(w)} = \frac{1}{2\pi i} \int_K \frac{G(z)}{F(z)} dz$, where w is the zero from (1).
- (3) If $z \in K$, prove that the function $\frac{1}{F(z)}$ can be represented as a convergent series with respect to q : $\frac{1}{F(z)} = \sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z - a)^{n+1}}$.

13.2.4 ?

Evaluate $\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx$.

13.2.5 ?

Let $f = u + iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

13.2.6 ?

Show that $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$ using complex analysis, $0 < a < n$. Here n is a positive integer.

13.2.7 ?

For $s > 0$, the **gamma function** is defined by $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$.

1. Show that the gamma function is analytic in the half-plane $\Re(s) > 0$, and is still given there by the integral formula above.
2. Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need $\Gamma(1-s) = t \int_0^\infty e^{-vt} (vt)^{-s} dv$ for $t > 0$.

13.2.8 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree n , then it has n zeros in \mathbb{C} .

13.2.9 ?

Suppose f is entire and there exist $A, R > 0$ and natural number N such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that (i) f is a polynomial and (ii) the degree of f is at least N .

13.2.10 ?

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an injective analytic (also called univalent) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

13.2.11 ?

Let g be analytic for $|z| \leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

- Show that g has a unique fixed point in $|z| < 1$.
- What happens if we replace $|g(z)| < 1$ with $|g(z)| \leq 1$ for $|z| = 1$? Give an example if (a) is not true or give an proof if (a) is still true.
- What happens if we simply assume that f is analytic for $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z) \neq z$. Can f have more than one fixed point in $|z| < 1$?

Hint: The map $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$ may be useful.

13.2.12 ?

Find a conformal map from $D = \{z : |z| < 1, |z - 1/2| > 1/2\}$ to the unit disk $\Delta = \{z : |z| < 1\}$.

13.2.13 ?

Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω . Show without using Picard's theorems that $f(z)$ is a constant.

13.2.14 ?

Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω . Show without using Picard's theorems that $f(z)$ is a constant.

13.2.15 ?

- (1) Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

- (2) Deduce Liouville's theorem from (1).

13.2.16 ?

Let $f(z)$ be entire and assume that $f(z) \leq M|z|^2$ outside some disk for some constant M . Show that $f(z)$ is a polynomial in z of degree ≤ 2 .

13.2.17 ?

Let $a_n(z)$ be an analytic sequence in a domain D such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D . Show that $\sum_{n=0}^{\infty} |a'_n(z)|$ converges uniformly on bounded and closed sub-regions of D .

13.2.18 ?

Let $f(z)$ be analytic in an open set Ω except possibly at a point z_0 inside Ω . Show that if $f(z)$ is bounded in near z_0 , then $\int_{\Delta} f(z)dz = 0$ for all triangles Δ in Ω .

13.2.19 ?

Assume f is continuous in the region: $0 < |z - a| \leq R$, $0 \leq \arg(z - a) \leq \beta_0$ ($0 < \beta_0 \leq 2\pi$) and the limit $\lim_{z \rightarrow a} (z - a)f(z) = A$ exists. Show that

$$\lim_{r \rightarrow 0} \int_{\gamma_r} f(z)dz = iA\beta_0,$$

where $\gamma_r := \{z \mid z = a + re^{it}, 0 \leq t \leq \beta_0\}$.

13.2.20 ?

Show that $f(z) = z^2$ is uniformly continuous in any open disk $|z| < R$, where $R > 0$ is fixed, but it is not uniformly continuous on \mathbb{C} .

(1) Show that the function $u = u(x, y)$ given by

$$u(x, y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx \quad \text{for } n \in \mathbf{N}$$

is the solution on $D = \{(x, y) \mid x^2 + y^2 < 1\}$ of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = \frac{\sin nx}{n}.$$

(2) Show that there exist points $(x, y) \in D$ such that $\limsup_{n \rightarrow \infty} |u(x, y)| = \infty$.

13.3 Spring 2014**13.3.1 ?**

The question provides some insight into Cauchy's theorem. Solve the problem without using the Cauchy theorem.

1. Evaluate the integral $\int_{\gamma} z^n dz$ for all integers n . Here γ is any circle centered at the origin with the positive (counterclockwise) orientation.

2. Same question as (a), but with γ any circle not containing the origin.
3. Show that if $|a| < r < |b|$, then $\int_{\gamma} \frac{dz}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$. Here γ denotes the circle centered at the origin, of radius r , with the positive orientation.

13.3.2 ?

- (1) Assume the infinite series $\sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$ and let $f(z)$ be the limit. Show that for $r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

- (2) Deduce Liouville's theorem from (1). Liouville's theorem: If $f(z)$ is entire and bounded, then f is constant.

13.3.3 ?

Let f be a continuous function in the region

$$D = \{z \mid |z| > R, 0 \leq \arg Z \leq \theta\} \quad \text{where} \quad 0 \leq \theta \leq 2\pi.$$

If there exists k such that $\lim_{z \rightarrow \infty} z f(z) = k$ for z in the region D . Show that

$$\lim_{R' \rightarrow \infty} \int_L f(z) dz = i\theta k,$$

where L is the part of the circle $|z| = R'$ which lies in the region D .

13.3.4 ?

Evaluate $\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx$.

13.3.5 ?

Let $f = u + iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

13.3.6 ?

Show that $\int_0^{\infty} \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$ using complex analysis, $0 < a < n$. Here n is a positive integer.

13.3.7 ?

For $s > 0$, the **gamma function** is defined by $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$.

- Show that the gamma function is analytic in the half-plane $\Re(s) > 0$, and is still given there by the integral formula above.
- Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need $\Gamma(1-s) = t \int_0^\infty e^{-vt} (vt)^{-s} dv$ for $t > 0$.

13.3.8 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree n , then it has n zeros in \mathbb{C} .

13.3.9 ?

Suppose f is entire and there exist $A, R > 0$ and natural number N such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that (i) f is a polynomial and (ii) the degree of f is at least N .

13.3.10 ?

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an injective analytic (also called univalent) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

13.3.11 ?

Let g be analytic for $|z| \leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

- Show that g has a unique fixed point in $|z| < 1$.
- What happens if we replace $|g(z)| < 1$ with $|g(z)| \leq 1$ for $|z| = 1$? Give an example if (a) is not true or give a proof if (a) is still true.
- What happens if we simply assume that f is analytic for $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z) \neq z$. Can f have more than one fixed point in $|z| < 1$?

Hint: The map $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$ may be useful.

13.3.12 ?

Find a conformal map from $D = \{z : |z| < 1, |z - 1/2| > 1/2\}$ to the unit disk $\Delta = \{z : |z| < 1\}$.

13.4 Fall 2015**13.4.1 ?**

Let $a_n \neq 0$ and assume that $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L$. Show that $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$. In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

13.4.2 ?

(a) Let z, w be complex numbers, such that $\bar{z}w \neq 1$. Prove that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

(b) Prove that for fixed w in the unit disk \mathbb{D} , the mapping

$$F : z \mapsto \frac{w - z}{1 - \bar{w}z}$$

satisfies the following conditions:

- (c) F maps \mathbb{D} to itself and is holomorphic.
- (ii) F interchanges 0 and w , namely, $F(0) = w$ and $F(w) = 0$.
- (iii) $|F(z)| = 1$ if $|z| = 1$.
- (iv) $F : \mathbb{D} \mapsto \mathbb{D}$ is bijective.

Hint: Calculate $F \circ F$.

13.4.3 ?

Use n -th roots of unity (i.e. solutions of $z^n - 1 = 0$) to show that

$$2^{n-1} \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \cdots \sin \frac{(n-1)\pi}{n} = n.$$

Hint: $1 - \cos 2\theta = 2 \sin^2 \theta$, $\sin 2\theta = 2 \sin \theta \cos \theta$.

(a) Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

(b) Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \quad \text{where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

is a holomorphic function in the region $r > 0$, $-\pi < \theta < \pi$. Also show that $\log z$ defined above is not continuous in $r > 0$.

13.4.4 ?

Assume f is continuous in the region: $x \geq x_0$, $0 \leq y \leq b$ and the limit

$$\lim_{x \rightarrow +\infty} f(x + iy) = A$$

exists uniformly with respect to y (independent of y). Show that

$$\lim_{x \rightarrow +\infty} \int_{\gamma_x} f(z) dz = iAb,$$

where $\gamma_x := \{z \mid z = x + it, 0 \leq t \leq b\}$.

13.4.5 ?

(Cauchy's formula for "exterior" region) Let γ be piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2 . Assume $f'(z)$ exists in an open set containing γ and Ω_2 and $\lim_{z \rightarrow \infty} f(z) = A$. Show that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

13.4.6 ?

Let $f(z)$ be bounded and analytic in \mathbb{C} . Let $a \neq b$ be any fixed complex numbers. Show that the following limit exists

$$\lim_{R \rightarrow \infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

Use this to show that $f(z)$ must be a constant (Liouville's theorem).

13.4.7 ?

Prove by *justifying all steps* that for all $\xi \in \mathbb{C}$ we have $e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x \xi} dx$.

Hint: You may use that fact in Example 1 on p. 42 of the textbook without proof, i.e., you may assume the above is true for real values of ξ .

13.4.8 ?

Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Let $\sum c_n z^n$ denote the power series in the open disc. Show that (1) $c_n \neq 0$ for all large enough n 's, and (2) $\lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}} = z_0$.

13.4.9 ?

Let $f(z)$ be a non-constant analytic function in $|z| > 0$ such that $f(z_n) = 0$ for infinite many points z_n with $\lim_{n \rightarrow \infty} z_n = 0$. Show that $z = 0$ is an essential singularity for $f(z)$. (An example of such a function is $f(z) = \sin(1/z)$.)

13.4.10 ?

Let f be entire and suppose that $\lim_{z \rightarrow \infty} f(z) = \infty$. Show that f is a polynomial.

13.4.11 ?

Expand the following functions into Laurent series in the indicated regions:

(a) $f(z) = \frac{z^2 - 1}{(z + 2)(z + 3)}, \quad 2 < |z| < 3, \quad 3 < |z| < +\infty.$

(b) $f(z) = \sin \frac{z}{1 - z}, \quad 0 < |z - 1| < +\infty$

13.4.12 ?

Assume $f(z)$ is analytic in region D and Γ is a rectifiable curve in D with interior in D . Prove that if $f(z)$ is real for all $z \in \Gamma$, then $f(z)$ is a constant.

13.4.13 ?

Find the number of roots of $z^4 - 6z + 3 = 0$ in $|z| < 1$ and $1 < |z| < 2$ respectively.

13.4.14 ?

Prove that $z^4 + 2z^3 - 2z + 10 = 0$ has exactly one root in each open quadrant.

13.4.15 ?

(1) Let $f(z) \in H(\mathbb{D})$, $\operatorname{Re}(f(z)) > 0$, $f(0) = a > 0$. Show that

$$\left| \frac{f(z) - a}{f(z) + a} \right| \leq |z|, \quad |f'(0)| \leq 2a.$$

(2) Show that the above is still true if $\operatorname{Re}(f(z)) > 0$ is replaced with $\operatorname{Re}(f(z)) \geq 0$.

13.4.16 ?

Assume $f(z)$ is analytic in \mathbb{D} and $f(0) = 0$ and is not a rotation (i.e. $f(z) \neq e^{i\theta}z$). Show that $\sum_{n=1}^{\infty} f^n(z)$ converges uniformly to an analytic function on compact subsets of \mathbb{D} , where $f^{n+1}(z) = f(f^n(z))$.

13.4.17 ?

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic and one-to-one in $|z| < 1$. For $0 < r < 1$, let D_r be the disk $|z| < r$. Show that the area of $f(D_r)$ is finite and is given by

$$S = \pi \sum_{n=1}^{\infty} n |c_n|^2 r^{2n}.$$

(Note that in general the area of $f(D_1)$ is infinite.)

13.4.18 ?

Let $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ be analytic and one-to-one in $r_0 < |z| < R_0$. For $r_0 < r < R < R_0$, let $D(r, R)$ be the annulus $r < |z| < R$. Show that the area of $f(D(r, R))$ is finite and is given by

$$S = \pi \sum_{n=-\infty}^{\infty} n |c_n|^2 (R^{2n} - r^{2n}).$$

13.5 Spring 2015**13.5.1 ?**

Let $a_n(z)$ be an analytic sequence in a domain D such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D . Show that $\sum_{n=0}^{\infty} |a'_n(z)|$ converges uniformly on bounded and closed sub-regions of D .

13.5.2 ?

Let f_n, f be analytic functions on the unit disk \mathbb{D} . Show that the following are equivalent.

- (i) $f_n(z)$ converges to $f(z)$ uniformly on compact subsets in \mathbb{D} .
- (ii) $\int_{|z|=r} |f_n(z) - f(z)| |dz|$ converges to 0 if $0 < r < 1$.

13.5.3 ?

Let f and g be non-zero analytic functions on a region Ω . Assume $|f(z)| = |g(z)|$ for all z in Ω . Show that $f(z) = e^{i\theta}g(z)$ in Ω for some $0 \leq \theta < 2\pi$.

13.5.4 ?

Suppose f is analytic in an open set containing the unit disc \mathbb{D} and $|f(z)| = 1$ when $|z|=1$. Show that either $f(z) = e^{i\theta}$ for some $\theta \in \mathbb{R}$ or there are finite number of $z_k \in \mathbb{D}$, $k \leq n$ and $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} \prod_{k=1}^n \frac{z - z_k}{1 - \bar{z}_k z}$.

Also cf. Stein et al, 1.4.7, 3.8.17

13.5.5 ?

- (1) Let $p(z)$ be a polynomial, $R > 0$ any positive number, and $m \geq 1$ an integer. Let $M_R = \sup\{|z^m p(z) - 1| : |z| = R\}$. Show that $M_R > 1$.
- (2) Let $m \geq 1$ be an integer and $K = \{z \in \mathbb{C} : r \leq |z| \leq R\}$ where $r < R$. Show (i) using (1) as well as, (ii) without using (1) that there exists a positive number $\varepsilon_0 > 0$ such that for each polynomial $p(z)$,

$$\sup\{|p(z) - z^{-m}| : z \in K\} \geq \varepsilon_0.$$

13.5.6 ?

Let $f(z) = \frac{1}{z} + \frac{1}{z^2 - 1}$. Find all the Laurent series of f and describe the largest annuli in which these series are valid.

13.5.7 ?

Suppose f is entire and there exist $A, R > 0$ and natural number N such that $|f(z)| \leq A|z|^N$ for $|z| \geq R$. Show that (i) f is a polynomial and (ii) the degree of f is at most N .

13.5.8 ?

Suppose f is entire and there exist $A, R > 0$ and natural number N such that $|f(z)| \geq A|z|^N$ for $|z| \geq R$. Show that (i) f is a polynomial and (ii) the degree of f is at least N .

13.5.9 ?

- (1) Explicitly write down an example of a non-zero analytic function in $|z| < 1$ which has infinitely zeros in $|z| < 1$.
- (2) Why does not the phenomenon in (1) contradict the uniqueness theorem?

13.5.10 ?

- (1) Assume u is harmonic on open set O and z_n is a sequence in O such that $u(z_n) = 0$ and $\lim z_n \in O$. Prove or disprove that u is identically zero. What if O is a region?
- (2) Assume u is harmonic on open set O and $u(z) = 0$ on a disc in O . Prove or disprove that u is identically zero. What if O is a region?

- (3) Formulate and prove a Schwarz reflection principle for harmonic functions

cf. Theorem 5.6 on p.60 of Stein et al.

Hint: Verify the mean value property for your new function obtained by Schwarz reflection principle.

13.5.11 ?

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s < r$, there exists a constant $c > 0$ such that

$$\|f\|_{(\infty,s)} \leq c \|f\|_{(1,r)},$$

where $\|f\|_{(\infty,s)} = \sup_{z \in D_s(z_0)} |f(z)|$ and $\|f\|_{(1,r)} = \int_{D_r(z_0)} |f(z)| dx dy$.

Note: Exercise 3.8.20 on p.107 in Stein et al is a straightforward consequence of this stronger result using the integral form of the Cauchy-Schwarz inequality in real analysis.

13.5.12 ?

- (1) Let f be analytic in $\Omega : 0 < |z - a| < r$ except at a sequence of poles $a_n \in \Omega$ with $\lim_{n \rightarrow \infty} a_n = a$. Show that for any $w \in \mathbb{C}$, there exists a sequence $z_n \in \Omega$ such that $\lim_{n \rightarrow \infty} f(z_n) = w$.
- (2) Explain the similarity and difference between the above assertion and the Weierstrass-Casorati theorem.

13.5.13 ?

Compute the following integrals.

$$\begin{aligned} & i \int_0^\infty \frac{1}{(1+x^n)^2} dx, n \geq 1 \quad (\text{ii}) \int_0^\infty \frac{\cos x}{(x^2+a^2)^2} dx, a \in \mathbb{R} \quad (\text{iii}) \int_0^\pi \frac{1}{a+\sin \theta} d\theta, a > 1 \\ & iv \int_0^{\frac{\pi}{2}} \frac{d\theta}{a+\sin^2 \theta}, a > 0. \quad (\text{v}) \int_{|z|=2} \frac{1}{(z^5-1)(z-3)} dz \quad (\text{vi}) \int_{-\infty}^\infty \frac{\sin \pi a}{\cosh \pi x + \cos \pi a} e^{-ix\xi} dx, 0 < a < 1, \\ & \xi \in \mathbb{R} \quad (\text{vii}) \int_{|z|=1} \cot^2 z dz. \end{aligned}$$

13.5.14 ?

Compute the following integrals.

$$\begin{aligned} & i \int_0^\infty \frac{\sin x}{x} dx \quad (\text{ii}) \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx \quad (\text{iii}) \int_0^\infty \frac{x^{a-1}}{(1+x)^2} dx, 0 < a < 2 \\ & i \int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx, a, b > 0 \quad (\text{ii}) \int_0^\infty \frac{x^{a-1}}{1+x^n} dx, 0 < a < n \\ & iii \int_0^\infty \frac{\log x}{1+x^n} dx, n \geq 2 \quad (\text{iv}) \int_0^\infty \frac{\log x}{(1+x^2)^2} dx \quad (\text{v}) \int_0^\pi \log |1 - a \sin \theta| d\theta, a \in \mathbb{C} \end{aligned}$$

13.5.15 ?

Let $0 < r < 1$. Show that polynomials $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ have no zeros in $|z| < r$ for all sufficiently large n 's.

13.5.16 ?

Let f be an analytic function on a region Ω . Show that f is a constant if there is a simple closed curve γ in Ω such that its image $f(\gamma)$ is contained in the real axis.

13.5.17 ?

(1) Show that $\frac{\pi^2}{\sin^2 \pi z}$ and $g(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ have the same principal part at each integer point.

(2) Show that $h(z) = \frac{\pi^2}{\sin^2 \pi z} - g(z)$ is bounded on \mathbb{C} and conclude that $\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$.

13.5.18 ?

Let $f(z)$ be an analytic function on $\mathbb{C} \setminus \{z_0\}$, where z_0 is a fixed point. Assume that $f(z)$ is bijective from $\mathbb{C} \setminus \{z_0\}$ onto its image, and that $f(z)$ is bounded outside $D_r(z_0)$, where r is some fixed positive number. Show that there exist $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$, $c \neq 0$ such that $f(z) = \frac{az+b}{cz+d}$.

13.5.19 ?

Assume $f(z)$ is analytic in $\mathbb{D} : |z| < 1$ and $f(0) = 0$ and is not a rotation (i.e. $f(z) \neq e^{i\theta}z$). Show that $\sum_{n=1}^{\infty} f^n(z)$ converges uniformly to an analytic function on compact subsets of \mathbb{D} , where $f^{n+1}(z) = f(f^n(z))$.

13.5.20 ?

Let f be a non-constant analytic function on \mathbb{D} with $f(\mathbb{D}) \subseteq \mathbb{D}$. Use $\psi_a(f(z))$ (where $a = f(0)$, $\psi_a(z) = \frac{a-z}{1-\bar{a}z}$) to prove that $\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}$.

13.5.21 ?

Find a conformal map

1. from $\{z : |z - 1/2| > 1/2, \operatorname{Re}(z) > 0\}$ to \mathbb{H}
2. from $\{z : |z - 1/2| > 1/2, |z| < 1\}$ to \mathbb{D}
3. from the intersection of the disk $|z + i| < \sqrt{2}$ with \mathbb{H} to \mathbb{D} .

4. from $\mathbb{D} \setminus [a, 1)$ to $\mathbb{D} \setminus [0, 1)$ ($0 < a < 1$).

Short solution possible using Blaschke factor

5. from $\{z : |z| < 1, \operatorname{Re}(z) > 0\} \setminus (0, 1/2]$ to \mathbb{H} .

13.5.22 ?

Let C and C' be two circles and let $z_1 \in C$, $z_2 \notin C$, $z'_1 \in C'$, $z'_2 \notin C'$. Show that there is a unique fractional linear transformation f with $f(C) = C'$ and $f(z_1) = z'_1$, $f(z_2) = z'_2$.

13.5.23 ?

Assume $f_n \in H(\Omega)$ is a sequence of holomorphic functions on the region Ω that are uniformly bounded on compact subsets and $f \in H(\Omega)$ is such that the set $\{z \in \Omega : \lim_{n \rightarrow \infty} f_n(z) = f(z)\}$ has a limit point in Ω . Show that f_n converges to f uniformly on compact subsets of Ω .

13.5.24 ?

Let $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$ with $|\alpha| < 1$ and $\mathbb{D} = \{z : |z| < 1\}$. Prove that

- $\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_\alpha|^2 dx dy = 1.$
- $\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_\alpha| dx dy = \frac{1 - |\alpha|^2}{|\alpha|^2} \log \frac{1}{1 - |\alpha|^2}.$

13.5.25 ?

Prove that $f(z) = -\frac{1}{2} \left(z + \frac{1}{z} \right)$ is a conformal map from half disc $\{z = x + iy : |z| < 1, y > 0\}$ to upper half plane $\mathbb{H} = \{z = x + iy : y > 0\}$.

13.5.26 ?

Let Ω be a simply connected open set and let γ be a simple closed contour in Ω and enclosing a bounded region U anticlockwise. Let $f : \Omega \rightarrow \mathbb{C}$ be a holomorphic function and $|f(z)| \leq M$ for all $z \in \gamma$. Prove that $|f(z)| \leq M$ for all $z \in U$.

13.5.27 ?

Compute the following integrals. (i) $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx$, $0 < a < n$ (ii) $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$

13.5.28 ?

Let $0 < r < 1$. Show that polynomials $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ have no zeros in $|z| < r$ for all sufficiently large n 's.

13.5.29 ?

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s < r$, there exists a constant $c > 0$ such that

$$\|f\|_{(\infty,s)} \leq c\|f\|_{(1,r)},$$

where $\|f\|_{(\infty,s)} = \sup_{z \in D_s(z_0)} |f(z)|$ and $\|f\|_{(1,r)} = \int_{D_r(z_0)} |f(z)| dx dy$.

13.5.30 ?

Let $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$ with $|\alpha| < 1$ and $\mathbb{D} = \{z : |z| < 1\}$. Prove that

- $\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_\alpha|^2 dx dy = 1$.
- $\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_\alpha| dx dy = \frac{1 - |\alpha|^2}{|\alpha|^2} \log \frac{1}{1 - |\alpha|^2}$.

Prove that $f(z) = -\frac{1}{2} \left(z + \frac{1}{z} \right)$ is a conformal map from half disc $\{z = x + iy : |z| < 1, y > 0\}$ to upper half plane $\mathbb{H} = \{z = x + iy : y > 0\}$.

13.5.31 ?

Let Ω be a simply connected open set and let γ be a simple closed contour in Ω and enclosing a bounded region U anticlockwise. Let $f : \Omega \rightarrow \mathbb{C}$ be a holomorphic function and $|f(z)| \leq M$ for all $z \in \gamma$. Prove that $|f(z)| \leq M$ for all $z \in U$.

13.5.32 ?

Compute the following integrals. (i) $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx$, $0 < a < n$ (ii) $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$

13.5.33 ?

Let $0 < r < 1$. Show that polynomials $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ have no zeros in $|z| < r$ for all sufficiently large n 's.

13.5.34 ?

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s < r$, there exists a constant $c > 0$ such that

$$\|f\|_{(\infty,s)} \leq c\|f\|_{(1,r)},$$

where $\|f\|_{(\infty,s)} = \sup_{z \in D_s(z_0)} |f(z)|$ and $\|f\|_{(1,r)} = \int_{D_r(z_0)} |f(z)| dx dy$.

13.6 Fall 2016**13.6.1 ?**

Let $u(x, y)$ be harmonic and have continuous partial derivatives of order three in an open disc of radius $R > 0$.

- (a) Let two points $(a, b), (x, y)$ in this disk be given. Show that the following integral is independent of the path in this disk joining these points:

$$v(x, y) = \int_{a,b}^{x,y} \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right).$$

(b)

- (i) Prove that $u(x, y) + iv(x, y)$ is an analytic function in this disc.
(ii) Prove that $v(x, y)$ is harmonic in this disc.

13.6.2 ?

- (a) $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain $D \subset \mathbb{C}$. Let $z_0 = (x_0, y_0)$ be a point in D which is in the intersection of the curves $u(x, y) = c_1$ and $v(x, y) = c_2$, where c_1 and c_2 are constants. Suppose that $f'(z_0) \neq 0$. Prove that the lines tangent to these curves at z_0 are perpendicular.
- (b) Let $f(z) = z^2$ be defined in \mathbb{C} .
- (c) Describe the level curves of $\operatorname{Re}(f)$ and of $\operatorname{Im}(f)$.
- (ii) What are the angles of intersections between the level curves $\operatorname{Re}(f) = 0$ and $\operatorname{Im}(f)$? Is your answer in agreement with part a) of this question?

13.6.3 ?

- (a) $f : D \rightarrow \mathbb{C}$ be a continuous function, where $D \subset \mathbb{C}$ is a domain. Let $\alpha : [a, b] \rightarrow D$ be a smooth curve. Give a precise definition of the *complex line integral*

$$\int_{\alpha} f.$$

- (b) Assume that there exists a constant M such that $|f(\tau)| \leq M$ for all $\tau \in \operatorname{Image}(\alpha)$. Prove that

$$\left| \int_{\alpha} f \right| \leq M \times \operatorname{length}(\alpha).$$

- (c) Let C_R be the circle $|z| = R$, described in the counterclockwise direction, where $R > 1$. Provide an upper bound for $\left| \int_{C_R} \frac{\log(z)}{z^2} \right|$, which depends only on R and other constants.

13.6.4 ?

- (a) Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function. Assume the existence of a non-negative integer m , and of positive constants L and R , such that for all z with $|z| > R$ the inequality

$$|f(z)| \leq L|z|^m$$

holds. Prove that f is a polynomial of degree $\leq m$.

- (b) Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function. Suppose that there exists a real number M such that for all $z \in \mathbb{C}$

$$\operatorname{Re}(f) \leq M.$$

Prove that f must be a constant.

13.6.5 ?

Prove that all the roots of the complex polynomial

$$z^7 - 5z^3 + 12 = 0$$

lie between the circles $|z| = 1$ and $|z| = 2$.

13.6.6 ?

- (a) Let F be an analytic function inside and on a simple closed curve C , except for a pole of order $m \geq 1$ at $z = a$ inside C . Prove that

$$\frac{1}{2\pi i} \oint_C F(\tau) d\tau = \lim_{\tau \rightarrow a} \frac{d^{m-1}}{d\tau^{m-1}} ((\tau - a)^m F(\tau)).$$

- (b) Evaluate

$$\oint_C \frac{e^\tau}{(\tau^2 + \pi^2)^2} d\tau$$

where C is the circle $|z| = 4$.

13.6.7 ?

Find the conformal map that takes the upper half-plane conformally onto the half-strip $\{w = x + iy : -\pi/2 < x < \pi/2, y > 0\}$.

13.6.8 ?

Compute the integral $\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{\cosh \pi x} dx$ where $\cosh z = \frac{e^z + e^{-z}}{2}$.