Real Analysis Qualifying Exam Review

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1 Basics

1.1 Table of Notation

Notation	Definition
$\left\lVert f \right\rVert_\infty \coloneqq \sup_{x \in \mathrm{dom}(f)} \lvert f(x) \rvert$	The Sup norm
$\left\lVert f \right\rVert_{L^\infty} \coloneqq \inf \left\{ M \geq 0 \;\middle\vert\; \lvert f(x) \rvert \leq M \text{ for a.e. } x \right\}$	The L^∞ norm
$f_n \overset{n\to\infty}{\to} f$	Convergence of a sequence
$f(x) \overset{\lvert x \rvert\to\infty}{\to} 0$	Vanishing at infinity
$\int_{\lvert x \rvert \geq N} f \overset{N\to\infty}{\to} 0$	Having small tails
$H, \mathcal{H}$	A Hilbert space
X	A topological space

1.2 Useful Techniques

- General advice: try swapping the orders of limits, sums, integrals, etc.

- Limits:
  - Take the $\limsup$ or $\liminf$, which always exist, and aim for an inequality like
$$c \leq \liminf a_n \leq \limsup a_n \leq c.$$
  - $\lim f_n = \limsup f_n = \liminf f_n$ iff the limit exists, so to show some g is a limit, show
$$\limsup f_n \leq g \leq \liminf f_n \qquad (\implies g = \lim f).$$
  - A limit does *not* exist if $\liminf a_n > \limsup a_n$.

- Sequences and Series
  - If f_n has a global maximum (computed using f_n' and the first derivative test) $M_n \to 0$, then $f_n \to 0$ uniformly.
  - For a fixed x, if $f = \sum f_n$ converges *uniformly* on some $B_r(x)$ and each f_n is continuous at x, then f is also continuous at x .

- Equalities
  - Split into upper and lower bounds:
$$a = b \iff a \leq b \text{ and } a \geq b.$$
  - Use an epsilon of room:
$$(\forall \epsilon,\ a < b + \varepsilon) \implies a \leq b.$$
  - Showing something is zero:
$$(\forall \epsilon,\ \|a\| < \varepsilon) \implies a = 0.$$

- Continuity / differentiability: show it holds on $[-M, M]$ for all M to get it to hold on $\mathbb{R}$.

- Simplifications:
  - To show something for a measurable set, show it for bounded/compact/elementary sets/
  - To show something for a function, show it for continuous, bounded, compactly supported, simple, chi functions, L^1, etc
  - Replace a continuous sequence ($\varepsilon \to 0$) with an arbitrary countable sequence ($x_n \to 0$)
  - Intersect with a ball $B_r(\mathbf{0}) \subset \mathbb{R}^n$.

- Integrals
  - Calculus techniques: Taylor series, IVT, MVT, etc.
  - Break up $\mathbb{R}^n = \{|x| \leq 1\} \coprod \{|x| > 1\}$.
    - ♢ Or break integration region into disjoint annuli.

  - Break up into $\{f > g\} \coprod \{f = g\} \coprod \{f < g\}$.
  - Tail estimates!
  - Most of what works for integrals will work for sums.

- Measure theory:

  - Always consider bounded sets, and if E is unbounded write $E = \cup_n B_n(0) \cap E$ and use countable subadditivity or continuity of measure.

  - F_σ sets are Borel, so establish something for Borel sets and use this to extend it to Lebesgue.

  - $s = \inf \{x \in X\} \implies$ for every ε there is an $x \in X$ such that $x \leq s + \varepsilon$.

- Approximate by dense subsets of functions

- Useful facts about compactly supported ($C_c(\mathbb{R})$) continuous functions:

  - Uniformly continuous
  - Bounded almost everywhere

1.3 Definitions

Definition 1.3.1 (Completeness)
A metric space is **complete** if every Cauchy sequence converges.

Fact 1.3.2
If X is complete, then absolutely convergent implies convergent.

Definition 1.3.3 (Continuity and Uniform Continuity)
A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** on a set X iff

$$\forall x_0 \in X, \forall \varepsilon > 0, \quad \exists \delta = \delta(x_0, \varepsilon) > 0 \quad \text{such that} \quad \forall x \in X,\ |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

f is **uniformly continuous** iff

$$\begin{aligned} &\forall \varepsilon \quad \exists \delta(\varepsilon) \text{ such that} \quad \forall x, y, \in X \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \\ \iff &\forall \varepsilon \quad \exists \delta(\varepsilon) \;\Big|\; \quad \forall x, y, \quad |y| < \delta \implies |f(x - y) - f(y)| < \varepsilon. \end{aligned}$$

Remark 1.3.4: The main difference is that δ may depend on x_0 and ε in continuity, but only depends on ε in the uniform version. I.e. once δ is fixed, for continuity one may only range over x, but in uniform continuity one can range over all pairs x, y.

Proposition 1.3.5 ***(Lipschitz implies uniformly continuous).***
If f is Lipschitz on X, then f is uniformly continuous on X.
Supposing that

$$\|f(x) - f(y)\| \leq C\|x - y\|,$$

for a fixed ε take $\delta(\varepsilon) := \varepsilon/C$, then

$$\begin{aligned}\|f(x) - f(y)\| &\leq C\|x - y\| \\ &\leq C\delta \\ &= C\left(\varepsilon/C\right) \\ &= \varepsilon.\end{aligned}$$

Definition 1.3.6 (Nowhere Dense Sets)
A set S is **nowhere dense** iff the closure of S has empty interior iff every interval contains a subinterval that does not intersect S.

Definition 1.3.7 (Meager Sets)
A set is **meager** if it is a *countable* union of nowhere dense sets.

Definition 1.3.8 (F_σ and G_δ Sets)
An F_σ set is a union of closed sets, and a G_δ set is an intersection of opens. [a]

[a] Mnemonic: "F" stands for *ferme*, which is "closed" in French, and σ corresponds to a "sum", i.e. a union.

Definition 1.3.9 (Limsup/Liminf)

$$\begin{aligned}\limsup_n a_n &= \lim_{n\to\infty} \sup_{j\geq n} a_j = \inf_{n\geq 0} \sup_{j\geq n} a_j \\ \liminf_n a_n &= \lim_{n\to\infty} \inf_{j\geq n} a_j = \sup_{n\geq 0} \inf_{j\geq n} a_j.\end{aligned}$$

Definition 1.3.10 (Topological Notions)
Let X be a metric space and A a subset. Let A' denote the limit points of A, and $\overline{A} := A \cup A'$ to be its closure.

- A **neighborhood** of p is an open set U_p containing p.
- An ε-**neighborhood** of p is an open ball $B_r(p) := \left\{q \;\middle|\; d(p,q) < r\right\}$ for some $r > 0$.
- A point $p \in X$ is an **accumulation point** of A iff every neighborhood U_p of p contains a point $q \in Q$
- A point $p \in X$ is a **limit point** of A iff every *punctured* neighborhood $U_p \setminus \{p\}$ contains a point $q \in A$.

- If $p \in A$ and p is not a limit point of A, then p is an **isolated point** of A.
- A is **closed** iff $A' \subset A$, so A contains all of its limit points.
- A point $p \in A$ is **interior** iff there is a neighborhood $U_p \subset A$ that is strictly contained in A.
- A is **open** iff every point of A is interior.
- A is **perfect** iff A is closed and $A \subset A'$, so every point of A is a limit point of A.
- A is **bounded** iff there is a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in A$.
- A is **dense** in X iff every point $x \in X$ is either a point of A, so $x \in A$, or a limit point of A, so $x \in A'$. I.e., $X \subset A \cup A'$.
  - Alternatively, $\overline{A} = X$, so the closure of A is X.

Definition 1.3.11 (Uniform Convergence)

$$(\forall \varepsilon > 0)\,(\exists n_0 = n_0(\varepsilon))\,(\forall x \in S)\,(\forall n > n_0)\,(|f_n(x) - f(x)| < \varepsilon)\,.$$

Negated:[a]

$$(\exists \varepsilon > 0)\,(\forall n_0 = n_0(\varepsilon))\,(\exists x = x(n_0) \in S)\,(\exists n > n_0)\,(|f_n(x) - f(x)| \geq \varepsilon)\,.$$

[a] Slogan: to negate, find a bad x depending on n_0 that are larger than some ε.

Definition 1.3.12 (Pointwise Convergence)
A sequence of functions $\{f_j\}$ is said to **converge pointwise** to f if and only if

$$(\forall \varepsilon > 0)(\forall x \in S)\,(\exists n_0 = n_0(x, \varepsilon))\,(\forall n > n_0)\,(|f_n(x) - f(x)| < \varepsilon)\,.$$

Proposition 1.3.13 ***(Implications of convergence).***
Uniform $\implies$ pointwise $\implies$ almost everywhere $\implies$ (roughly) L^1 convergence. Why these can't be reversed:

- Pointwise but not uniform???: $\dfrac{1}{n}\mathbb{1}_{[0,n]}$
- Almost everywhere but not pointwise???: $n\mathbb{1}_{[0,\frac{1}{n}]}$
- $n\chi_{[n,n+1]}$????

Definition 1.3.14 (Outer Measure)
The **outer measure** of a set is given by

$$m_*(E) \coloneqq \inf_{\substack{\{Q_i\} \rightrightarrows E \\ \text{closed cubes}}} \sum |Q_i|.$$

Definition 1.3.15 (Limsup and Liminf of Sets)

$$\liminf_n E_n := \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n = \left\{x \;\middle|\; x \in E_n \text{ for all but finitely many } n\right\}$$
$$\limsup_n E_n := \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n = \left\{x \;\middle|\; x \in E_n \text{ for infinitely many } n\right\}$$
.

How to derive these definitions:

- For $A := \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n$:
  - $x \in A \iff$ there exists some N such that $x \in \cap_{n \geq N} E_n$, i.e. $x \in E_n$ for all $n \geq N$. So x is in *all* but finitely many n.
- For $B := \bigcap N = 1^{\infty} \bigcup n = N^{\infty} E_n$:
  - $x \in B \iff$ for every N, there exists some $n \geq N$ such that $x \in E_n$. So x is an infinitely many E_n.

Note that $A \subseteq B$ since being in all but finitely many E_n necessarily implies being in infinitely many. This corresponds to $\liminf_n E_n \subseteq \limsup_n E_n$.

Definition 1.3.16 (Lebesgue Measurable Sets)
A subset $E \subseteq \mathbb{R}^n$ is **Lebesgue measurable** iff for every $\varepsilon > 0$ there exists an open set $O \supseteq E$ such that $m_*(O \setminus E) < \varepsilon$. In this case, we define $m(E) := m_*(E)$.

Definition 1.3.17 (L^+: Measurable non-negative functions.)
$f \in L^+$ iff f is measurable and non-negative.

Definition 1.3.18 (Integrability)
A measurable function is **integrable** iff $\|f\|_1 < \infty$.

Definition 1.3.19 (The Infinity Norm)

$$\|f\|_\infty := \inf_{\alpha \geq 0} \left\{\alpha \;\middle|\; m\left\{|f| \geq \alpha\right\} = 0\right\}.$$

Definition 1.3.20 (Essentially Bounded Functions)
A function $f : X \to \mathbb{C}$ is **essentially bounded** iff there exists a real number c such that $\mu(\{|f| > x\}) = 0$, i.e. $\|f\|_\infty < \infty$.

Definition 1.3.21 (L^∞)

$$L^\infty(X) := \left\{ f : X \to \mathbb{C} \;\middle|\; f \text{ is essentially bounded} \right\} := \left\{ f : X \to \mathbb{C} \;\middle|\; \|f\|_\infty < \infty \right\}.$$

Definition 1.3.22 (Dual Norm)
For X a normed vector space and $\Lambda \in X^\vee$,

$$\|\Lambda\|_{X^\vee} := \sup_{\left\{x \in X \;\middle|\; \|x\|_X \leq 1\right\}} |f(x)|.$$

Definition 1.3.23 (Convolution)

$$f * g(x) = \int f(x-y)g(y)dy.$$

Definition 1.3.24 (Fourier Transform)

$$\widehat{f}(\xi) = \int f(x)\, e^{2\pi i x \cdot \xi}\, dx.$$

Definition 1.3.25 (Dilation)

$$\varphi_t(x) = t^{-n}\varphi\left(t^{-1}x\right).$$

Definition 1.3.26 (Approximations to the identity)
For $\varphi \in L^1$, the dilations satisfy $\int \varphi_t = \int \varphi$, and if $\int \varphi = 1$ then φ is an **approximate identity**.

Definition 1.3.27 (Baire Space)
A space X is a **Baire space** if and only if every countable intersections of open, dense sets is still dense.

1.3.1 Functional Analysis

Definition 1.3.28 (Orthonormal sequence)
A countable collection of elements $\{u_i\}$ is **orthonormal** if and only if

1. $\langle u_i,\ u_j \rangle = 0$ for all $j \neq k$ and
2. $\|u_j\|^2 := \langle u_j,\ u_j \rangle = 1$ for all j.

Definition 1.3.29 (Basis of a Hilbert space)
A set $\{u_n\}$ is a **basis** for a Hilbert space $\mathcal{H}$ iff it is dense in $\mathcal{H}$.

Definition 1.3.30 (Completeness of a Hilbert space)
A collection of vectors $\{u_n\} \subset H$ is **complete** iff $\langle x,\ u_n \rangle = 0$ for all $n \iff x = 0$ in H.

Definition 1.3.31 (Dual of a Hilbert space)
The **dual** of a Hilbert space H is defined as

$$H^\vee := \left\{ L : H \to \mathbb{C} \;\middle|\; L \text{ is continuous} \right\}.$$

Definition 1.3.32 (Linear functionals)
A map $L : X \to \mathbb{C}$ is a **linear functional** iff

$$L(\alpha\mathbf{x} + \mathbf{y}) = \alpha L(\mathbf{x}) + L(\mathbf{y})..$$

Definition 1.3.33 (Operator norm)
The **operator norm** of an operator L is defined as

$$\|L\|_{X^\vee} := \sup_{\substack{x \in X \\ \|x\|=1}} |L(x)|.$$

Definition 1.3.34 (Banach Space)
A space is a **Banach space** if and only if it is a complete normed vector space.

Definition 1.3.35 (Hilbert Space)
A **Hilbert space** is an inner product space which is a Banach space under the induced norm.

1.4 Theorems

Theorem 1.4.1 ***(Folland 0.25).***
For $E \subseteq (X, d)$ a metric space, TFAE:

- E is complete and totally bounded.
- E is sequentially compact: Every sequence in E has a subsequence that converges to a point in E.
- E is compact: every open cover has a finite subcover.

Note that E is complete as a metric space with the induced metric iff E is closed in X, and E is bounded iff it is totally bounded.

Theorem 1.4.2 ***(Mean Value Theorem).***
If $f : [a, b] \to \mathbb{R}$ is continuous on a closed interval and differentiable on (a, b), then there exists $\xi \in [a, b]$ such that

$$f(b) - f(a) = f'(\xi)(b - a).$$

Theorem 1.4.3 ***(Lagrange and Cauchy Remainders).***
If f is n times differentiable on a neighborhood of a point p, say $N_\delta(p)$, then for all points x in the deleted neighborhood $N_\delta(p) - \{p\}$, there exists a point ξ strictly between x and p such that

$$\begin{aligned} x \in N_\delta(p) - \{p\} \implies f(x) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(p)}{k!}(x-p)^k + \frac{f^{(n)}(\xi)}{n!}(x-p)^n \\ &= \sum_{k=0}^{n-1} \frac{f^{(k)}(p)}{k!}(x-p)^k + \int_c^x \frac{1}{n!}\frac{\partial^n f}{\partial x^n}(t)(x-t)^n \, dt \end{aligned}$$

Proposition 1.4.4 ***(Sufficient condition for Taylor convergence).***
Given a point c and some $\varepsilon > 0$, if $f \in C^\infty(I)$ and there exists an M such that

$$x \in N_\varepsilon(c) \implies \left| f^{(n)}(x) \right| \leq M^n$$

then the Taylor expansion about c converges on $N_\varepsilon(c)$.

1.4.1 Topology / Sets

Theorem 1.4.5 ***(Heine-Cantor).***
Every continuous function $f : X \to Y$ where X is a compact metric space is uniformly continuous.

Proof (?).
Fix $\varepsilon > 0$, we'll find a δ that works for all $x \in X$ uniformly. For every $x \in X$, pick a δ_x neighborhood satisfying the conditions for (assumed) continuity. Take an open cover by $\delta_x/2$ balls, extract a finite subcover, take δ the minimal radius. ■

Proposition 1.4.6 ***(Compact if and only if sequentially compact for metric spaces).***
Metric spaces are compact iff they are sequentially compact, (i.e. every sequence has a convergent subsequence).

Proof (?).
Todo.

Proof.

∎

Proposition 1.4.7 ***(A unit ball that is not compact).***
The unit ball in $C([0,1])$ with the sup norm is not compact.

Proof (?).
Take $f_k(x) = x^n$, which converges to $\chi(x=1)$. The limit is not continuous, so no subsequence can converge.

∎

Proposition 1.4.8 ***(?).***
A *finite* union of nowhere dense is again nowhere dense.

Proposition 1.4.9 ***(Convergent Sums Have Small Tails).***

$$\sum a_n < \infty \implies a_n \to 0 \quad \text{and} \quad \sum_{k=N}^{\infty} a_n \overset{N\to\infty}{\to} 0$$

Theorem 1.4.10 ***(Heine-Borel).***
$X \subseteq \mathbb{R}^n$ is compact $\iff$ X is closed and bounded.

Proposition 1.4.11 ***(Geometric Series).***

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \iff |x| < 1.$$

Corollary 1.4.12 ***(?).***

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = 1.$$

Proposition 1.4.13 ***(?).***
The Cantor set is closed with empty interior.

Proof (?).
Its complement is a union of open intervals, and can't contain an interval since intervals have positive measure and $m(C_n)$ tends to zero.

∎

Corollary 1.4.14 ***(?).***
The Cantor set is nowhere dense.

Proposition 1.4.15 *(?).*
Singleton sets in $\mathbb{R}$ are closed, and thus $\mathbb{Q}$ is an F_σ set.

Theorem 1.4.16 *(Baire).*
$\mathbb{R}$ is a **Baire space** Thus $\mathbb{R}$ can not be written as a countable union of nowhere dense sets.

Lemma 1.4.17 *(?).*
Any nonempty set which is bounded from above (resp. below) has a well-defined supremum (resp. infimum).

1.4.2 Functions

Proposition 1.4.18 *(Existence of Smooth Compactly Supported Functions).*
There exist smooth compactly supported functions, e.g. take

$$f(x) = e^{-\frac{1}{x^2}} \chi_{(0,\infty)}(x).$$

Lemma 1.4.19 *(Function discontinuous on the rationals).*
There is a function discontinuous precisely on $\mathbb{Q}$.

Proof (?).
$f(x) = \frac{1}{n}$ if $x = r_n \in \mathbb{Q}$ is an enumeration of the rationals, and zero otherwise. The limit at every point is 0. ■

Proposition 1.4.20 *(No functions discontinuous on the irrationals).*
There *do not* exist functions that are discontinuous precisely on $\mathbb{R} \setminus \mathbb{Q}$.

Proof (?).
D_f is always an F_σ set, which follows by considering the oscillation ω_f. $\omega_f(x) = 0 \iff f$ is continuous at x, and $D_f = \cup_n A_{\frac{1}{n}}$ where $A_\varepsilon = \{\omega_f \geq \varepsilon\}$ is closed. ■

Proposition 1.4.21 *(Lipschitz $\iff$ differentiable with bounded derivative.).*
A function $f : (a, b) \to \mathbb{R}$ is Lipschitz $\iff$ f is differentiable and f' is bounded. In this case, $|f'(x)| \leq C$, the Lipschitz constant.

1.4.3 Sequences and Series

Proposition 1.4.22 ***(The Cauchy condensation test).***
For $\{a_k\}$ is a non-increasing sequence in $\mathbb{R}$ then

$$\sum_{k\geq 1} a_k < \infty \iff \sum_{k\geq 1} 2^k a_{2^k} < \infty.$$

Proof (?).
Show that

$$\sum a_k \leq \sum 2^k a_{2^k} \leq 2 \sum a_k$$

using

$$\sum a_k = a_0 + a_1 + a_2 + a_3 + \cdots \leq (a_1) + (a_2 + a_2) + a_3 + a_3 + a_3 + a_3 + \cdots$$

where each group with a_k has 2^k terms. ∎

1.5 Uniform Convergence

Proposition 1.5.1 ***(Testing Uniform Convergence: The Sup Norm Test).***
$f_n \to f$ uniformly iff there exists an M_n such that $\|f_n - f\|_\infty \leq M_n \to 0$.

Remark 1.5.2 ***(Negating the Sup Norm test):*** **Negating**: find an x which depends on n for which $\|f_n\|_\infty > \varepsilon$ (negating small tails) or $\|f_n - f_m\| > \varepsilon$ (negating the Cauchy criterion).

Proposition 1.5.3 ***($C(I)$ is complete).***
The space $X = C([0,1])$, continuous functions $f : [0,1] \to \mathbb{R}$, equipped with the norm

$$\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|$$

is a **complete** metric space.

Proof .

1. Let $\{f_k\}$ be Cauchy in X.

2. Define a candidate limit using pointwise convergence:

Fix an x; since

$$|f_k(x) - f_j(x)| \leq \|f_k - f_k\| \to 0$$

the sequence $\{f_k(x)\}$ is Cauchy in $\mathbb{R}$. So define $f(x) := \lim_k f_k(x)$.

3. Show that $\|f_k - f\| \to 0$:

$$|f_k(x) - f_j(x)| < \varepsilon \; \forall x \implies \lim_j |f_k(x) - f_j(x)| < \varepsilon \; \forall x$$

Alternatively, $\|f_k - f\| \leq \|f_k - f_N\| + \|f_N - f_j\|$, where N, j can be chosen large enough to bound each term by $\varepsilon/2$.

4. Show that $f \in X$:

The uniform limit of continuous functions is continuous.

■

Remark 1.5.4: In other cases, you may need to show the limit is bounded, or has bounded derivative, or whatever other conditions define X.

Theorem 1.5.5 ***(Uniform Limit Theorem).***
If $f_n \to f$ pointwise and uniformly with each f_n continuous, then f is continuous. [a]

[a] Slogan: a uniform limit of continuous functions is continuous.

Proof .

- Follows from an $\varepsilon/3$ argument:

$$|F(x) - F(y| \leq |F(x) - F_N(x)| + |F_N(x) - F_N(y)| + |F_N(y) - F(y)| \leq \varepsilon \to 0.$$

  - The first and last $\varepsilon/3$ come from uniform convergence of $F_N \to F$.
  - The middle $\varepsilon/3$ comes from continuity of each F_N.

- So just need to choose N large enough and δ small enough to make all 3 ε bounds hold.

■

Proposition 1.5.6 ***(Uniform Limits Commute with Integrals).***
If $f_n \to f$ uniformly, then $\int f_n = \int f$.

1.5.1 Series

Proposition 1.5.7 ***(p-tests).***
Let n be a fixed dimension and set $B = \left\{x \in \mathbb{R}^n \;\middle|\; \|x\| \leq 1\right\}$.

$$\begin{aligned}
\sum \frac{1}{n^p} < \infty &\iff p > 1 \\
\int_\varepsilon^\infty \frac{1}{x^p} < \infty &\iff p > 1 \\
\int_0^1 \frac{1}{x^p} < \infty &\iff p < 1 \\
\int_B \frac{1}{|x|^p} < \infty &\iff p < n \\
\int_{B^c} \frac{1}{|x|^p} < \infty &\iff p > n
\end{aligned}$$

.

Proposition 1.5.8 ***(Comparison Test).***
If $0 \leq a_n \leq b_n$, then

- $\sum b_n < \infty \implies \sum a_n < \infty$, and
- $\sum a_n = \infty \implies \sum b_n = \infty$.

Proposition 1.5.9 ***(Small Tails for Series of Functions).***
If $\sum f_n$ converges then $f_n \to 0$ uniformly.

Corollary 1.5.10 ***(Term by Term Continuity Theorem).***
If f_n are continuous and $\sum f_n \to f$ converges uniformly, then f is continuous.

Proposition 1.5.11 ***(Weak M-Test).***
If $f_n(x) \leq M_n$ **for a fixed** x where $\sum M_n < \infty$, then the series $f(x) = \sum f_n(x)$ converges.[a]

[a] Note that this is only pointwise convergence of f, whereas the full M-test gives uniform convergence.

Proposition 1.5.12 ***(The Weierstrass M-Test).***

If $\sup_{x \in A} |f_n(x)| \leq M_n$ for each n where $\sum M_n < \infty$, then $\sum_{n=1}^\infty f_n(x)$ converges uniformly and absolutely on A. [a] Conversely, if $\sum f_n$ converges uniformly on A then $\sup_{x \in A} |f_n(x)| \to 0$.

[a] It suffices to show $|f_n(x)| \leq M_n$ for some M_n not depending on x.

Proposition 1.5.13 ***(Cauchy criterion for sums).***
f_n are uniformly Cauchy (so $\|f_n - f_m\|_\infty < \varepsilon$) iff f_n is uniformly convergent.

Derivatives

Theorem 1.5.14 ***(Term by Term Differentiability Theorem).***
If f_n are differentiable, $\sum f_n' \to g$ uniformly, and there exists one point[a] x_0 such that $\sum f_n(x)$ converges, then there exist an f such that $\sum f_n \to f$ uniformly and $f' = g$.[b]

[a] So this implicitly holds if f is the pointwise limit of f_n.
[b] See Abbott theorem 6.4.3, pp 168.

1.6 Commuting Limiting Operations

Proposition 1.6.1 ***(Limits of bounded functions need not be bounded).***

$$\lim_{n\to\infty} \sup_{x\in X} |f_n(x)| \neq \sup_{x\in X} \left| \lim_{n\to\infty} f_n(x) \right|.$$

Proposition 1.6.2 ***(Limits of continuous functions need not be continuous).***

$$\lim_{k\to\infty} \lim_{n\to\infty} f_n(x_k) \neq \lim_{n\to\infty} \lim_{k\to\infty} f_n(x_k).$$

Proposition 1.6.3 ***(Limits of differentiable functions need not be differentiable).***

$$\lim_{n\to\infty} \frac{\partial}{\partial x} f_n \neq \frac{\partial}{\partial n} \left(\lim_{n\to\infty} f_n \right).$$

Note that uniform convergence of f_n and f_n' is sufficient to guarantee that f is differentiable. Even worse: every continuous function is a uniform limit of polynomials by the Weierstrass approximation theorem.

Example 1.6.4 ***(?):*** As a counterexample:

$$f_n(x) := \sqrt{x^2 + \frac{1}{n}} \overset{n\to\infty}{\longrightarrow} f(x) := |x|,$$

and this convergence is even uniform.

Example 1.6.5 ***(?):***

$$f_n(x) := \frac{x}{1 + nx^2}.$$

Then by Calculus, $f_n(x) \leq 1/2\sqrt{n} := M_n$ and $f_n \to 0$ uniformly, so $f' = 0$. But

$$f_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2},$$

and $f_n'(0) \to 1$.

Proposition 1.6.6 *(?)*.

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx \neq \int_a^b \lim_{n\to\infty}(f_n(x))\,dx.$$

1.7 "Almost" Theorems

Theorem 1.7.1 *(Egorov's Theorem)*.
Let $E \subseteq \mathbb{R}^d$ be measurable of positive finite measure with $f_k \to f$ almost everywhere on E. Then for every $\varepsilon > 0$ there is a closed $A_\varepsilon \subseteq E$ with $\mu(E \setminus A_\varepsilon) < \varepsilon$ and $f_k \to f$ uniformly on A_ε.

Proof (of Egorov).

Proof. We may assume without loss of generality that $f_k(x) \to f(x)$ for every $x \in E$. For each pair of non-negative integers n and k, let

$$E_k^n = \{x \in E : |f_j(x) - f(x)| < 1/n, \quad \text{for all } j > k\}.$$

Now fix n and note that $E_k^n \subset E_{k+1}^n$, and $E_k^n \nearrow E$ as k tends to infinity. By Corollary 3.3, we find that there exists k_n such that $m(E - E_{k_n}^n) < 1/2^n$. By construction, we then have

$$|f_j(x) - f(x)| < 1/n \qquad \text{whenever } j > k_n \text{ and } x \in E_{k_n}^n.$$

We choose N so that $\sum_{n=N}^{\infty} 2^{-n} < \epsilon/2$, and let

$$\tilde{A}_\epsilon = \bigcap_{n\geq N} E_{k_n}^n.$$

We first observe that

$$m(E - \tilde{A}_\epsilon) \leq \sum_{n=N}^{\infty} m(E - E_{k_n}^n) < \epsilon/2.$$

Next, if $\delta > 0$, we choose $n \geq N$ such that $1/n < \delta$, and note that $x \in \tilde{A}_\epsilon$ implies $x \in E^n_{k_n}$. We see therefore that $|f_j(x) - f(x)| < \delta$ whenever $j > k_n$. Hence f_k converges uniformly to f on $\tilde{A}_\epsilon$.

Finally, using Theorem 3.4 choose a closed subset $A_\epsilon \subset \tilde{A}_\epsilon$ with $m(\tilde{A}_\epsilon - A_\epsilon) < \epsilon/2$. As a result, we have $m(E - A_\epsilon) < \epsilon$ and the theorem is proved.

■

Theorem 1.7.2 ***(Lusin's Theorem).***
If f is measurable and finite-valued on E with $\mu(E) < \infty$ then for every $\varepsilon > 0$ there exists a closed set F_ε with

$$F_\varepsilon \subset F \qquad\qquad \mu(E - F_\varepsilon) \leq \varepsilon$$

where f restricted to F_ε is continuous.

Note: this means that the separate function $\tilde{f} := f|_{F_\varepsilon}$ is continuous, not that the function f defined on all of E is continuous at points of F_ε.

Proof (of Lusin).

Proof. Let f_n be a sequence of step functions so that $f_n \to f$ a.e. Then we may find sets E_n so that $m(E_n) < 1/2^n$ and f_n is continuous outside E_n. By Egorov's theorem, we may find a set $A_{\epsilon/3}$ on which $f_n \to f$ uniformly and $m(E - A_{\epsilon/3}) \leq \epsilon/3$. Then we consider

$$F' = A_{\epsilon/3} - \bigcup_{n \geq N} E_n$$

for N so large that $\sum_{n \geq N} 1/2^n < \epsilon/3$. Now for every $n \geq N$ the function f_n is continuous on F'; thus f (being the uniform limit of $\{f_n\}$) is also continuous on F'. To finish the proof, we merely need to approximate the set F' by a closed set $F_\epsilon \subset F'$ such that $m(F' - F_\epsilon) < \epsilon/3$.

■

1.8 Slightly Advanced Stuff

Theorem 1.8.1 ***(Weierstrass Approximation).***
If $[a, b] \subset \mathbb{R}$ is a closed interval and f is continuous, then for every $\varepsilon > 0$ there exists a polynomial p_ε such that $\|f - p_\varepsilon\|_{L^\infty([a,b])} \overset{\varepsilon \to 0}{\to} 0$.
Equivalently, polynomials are dense in the Banach space $C([0, 1], \|-\|_\infty)$.

1.9 Examples and Counterexamples

Example 1.9.1 ***(?):*** A series of continuous functions that does *not* converge uniformly but is still continuous:

$$g(x) := \sum \frac{1}{1 + n^2 x}.$$

Take $x = 1/n^2$.

Let all of the following integrals to be over a compact interval $[a, b]$ with $0 \le a < b$.

Questions to ask:

- Where is/isn't f continuous?
- Where is/isn't f differentiable?
- Is f Riemann integrable?

1.9.1 Dirichlet function

$$f(x) = b + (a - b)\ \chi(x \in \mathbb{Q}) = \begin{cases} a, & x \in \mathbb{Q} \\ b, & \text{else} \end{cases}$$

(usually take $a = 1, b = 0$)

- Continuous nowhere
- Discontinuous everywhere
- Not integrable
- Differentiable nowhere

1.9.2 Dirichlet with a Continuous Point

$$f(x) = x\ \chi(\mathbb{Q}) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & \text{else} \end{cases}$$

- Continuous at 0
- Discontinuous at $\mathbb{R} - \{0\}$
- Not integrable
  - $U(f) > \frac{1}{4}$ but $L(f) = 0$.
- Differentiable nowhere

1.9.3 Dirichlet with a Differentiable Point

$$f(x) = x^2\ \chi(\mathbb{Q}) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & \text{else} \end{cases}$$

- Continuous at 0
- Discontinuous at $\mathbb{R} - \{0\}$
- Not integrable
- Differentiable at 0

1.9.4 Dirichlet with Two Functions

$$f(x) = x\ \chi\mathbb{Q} + (-x)\chi(\mathbb{R} - \mathbb{Q}) = \begin{cases} x, & x \in \mathbb{Q} \\ -x, & \text{else} \end{cases}$$

- Continuous at 0
- Discontinuous at $\mathbb{R} - \{0\}$
- Differentiable nowhere.
- Not integrable

Proof (of non-integrability).

Restrict attention to $\left[\frac{1}{2}, 1\right]$

$$\begin{aligned}
\overline{\int_0^1} f &= \inf\left\{\sum \sup f(x)(x_i - x_{i-1})\right\} \\
\sup f(x) = x_i \implies \sum \sup f(x)(x_i - x_{i-1}) &= \sum x_i(x_i - x_{i-1}) \\
&> \sum \frac{1}{2}(x_i - x_{i-1}) \\
&= \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} \\
\implies \overline{\int_0^1} f &\geq \frac{1}{4}
\end{aligned}$$

and

$$\begin{aligned}
\underline{\int_0^1} f &= \sup\left\{\sum \inf f(x)(x_i - x_{i-1})\right\} \\
\inf f(x) = -x_i \implies \sum \inf f(x)(x_i - x_{i-1}) &= \sum -x_i(x_i - x_{i-1}) \\
&< -\sum \frac{1}{2}(x_i - x_{i-1}) \\
&= -\frac{1}{2}\left(\frac{1}{2}\right) = -\frac{1}{4} \\
\implies \underline{\int_0^1} f &\leq -\frac{1}{4}
\end{aligned}$$

So we have $\underline{\int_0^1} f \lneq 0 \lneq \overline{\int_0^1} f$.

■

1.10 The Thomae function:

$$f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q} \in \mathbb{Q}, \ (p, q) = 1 \\ 0, & \text{else} \end{cases}$$

- Continuous on $\mathbb{R} - \mathbb{Q}$
- Discontinuous on $\mathbb{Q}$
- Integrable with $\int_a^b f = 0$
- Differentiable nowhere

Exercises from Folland:

- Chapter 1: Exercises 3, 7, 10, 12, 14 (with the sets in 3(a) being non-empty) Exercises 15, 17, 18, 19, 22(a), 24, 28 Exercises 26, 30 (also check out 31)

- Chapter 2: Exercises 2, 3, 7, 9 (in 9(c) you can use Exercise 1.29 without proof Exercises 10, 12, 13, 14, 16, 19 Exercises 24, 25, 28(a,b), 33, 34, 35, 38, 41 (note that 24 shows that upper sums are not needed in the definition of integrals, and the extra hypotheses also show that they are not desired either) Exercises 40, 44, 47, 49, 50, 51, 52, 54, 56, 58, 59

- Chapter 3: Exercises 3(b,c), 5, 6, 9, 12, 13, 14, 16, 20, 21, 22

2 | Measure Theory

Fact 2.0.1
Some useful tricks:

- $\mu(A \setminus B) = \mu(A) - \mu(B)$ if $\mu(B) < \infty$
- Write $f = f - f_n + f_n$
- If G is measurable, then there exists an $E \supseteq G$ such $m(G) \leq m(E) + \varepsilon$
- If E is measurable,
  - $E = F_\delta \coprod N$ for N a null set.
  - $E \coprod N = G_\delta$ for N a null set.

2.1 Abstract Measure Theory

Definition 2.1.1 (Measures on measurable spaces)
If $(X, \mathcal{M})$ is a measurable space, then a **measure** is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$.
2. Countable additivity: if $\{E_k\}_{k\geq 1}$ is a countable union of disjoint sets in X, then

$$\mu\left(\coprod_{k\geq 1} E_k\right) = \sum_{k\geq 1} \mu(E_k).$$

If (2) only holds for finitely indexed sums, we say μ is σ**-additive**.

Proposition 2.1.2 ***(Subtraction of Measures).***

$$m(A) = m(B) + m(C) \quad \text{and} \quad m(C) < \infty \implies m(A) - m(C) = m(B).$$

Theorem 2.1.3 ***(Properties of measures).***
Let $(X, \mathcal{M}, \mu)$ be a measure space. Then

1. Monotonicity: $E \subseteq F \implies \mu(E) \leq \mu(F)$.
2. Countable subadditivity: If $tsE_{k k \geq 1}$ is a countable collection,

$$\mu\left(\bigcup_{k \geq 1} E_k\right) \leq \sum_{k \geq 1} \mu(E_k).$$

Proposition 2.1.4 ***(Continuity of Measure).***

$$\begin{aligned} \text{Continuity from below:} \quad & E_n \nearrow E \implies m(E_n) \to m(E) \\ \text{Continuity from above:} \quad & m(E_1) < \infty \text{ and } E_i \searrow E \implies m(E_i) \to m(E). \end{aligned}$$

Mnemonic: $\lim_n \mu(E_n) = \mu(\lim E_n)$.

Proof (sketches).

- From below: break into disjoint annuli $A_2 = E_2 \setminus E_1$,
  - Apply countable disjoint additivity to $E = \coprod A_i$.
- From above: funny step, use $E_1 = (\coprod E_j \setminus E_{j+1}) \coprod (\cap E_j)$.
  - Taking measures yields a telescoping sum, and use countable additivity, then finiteness to subtract.

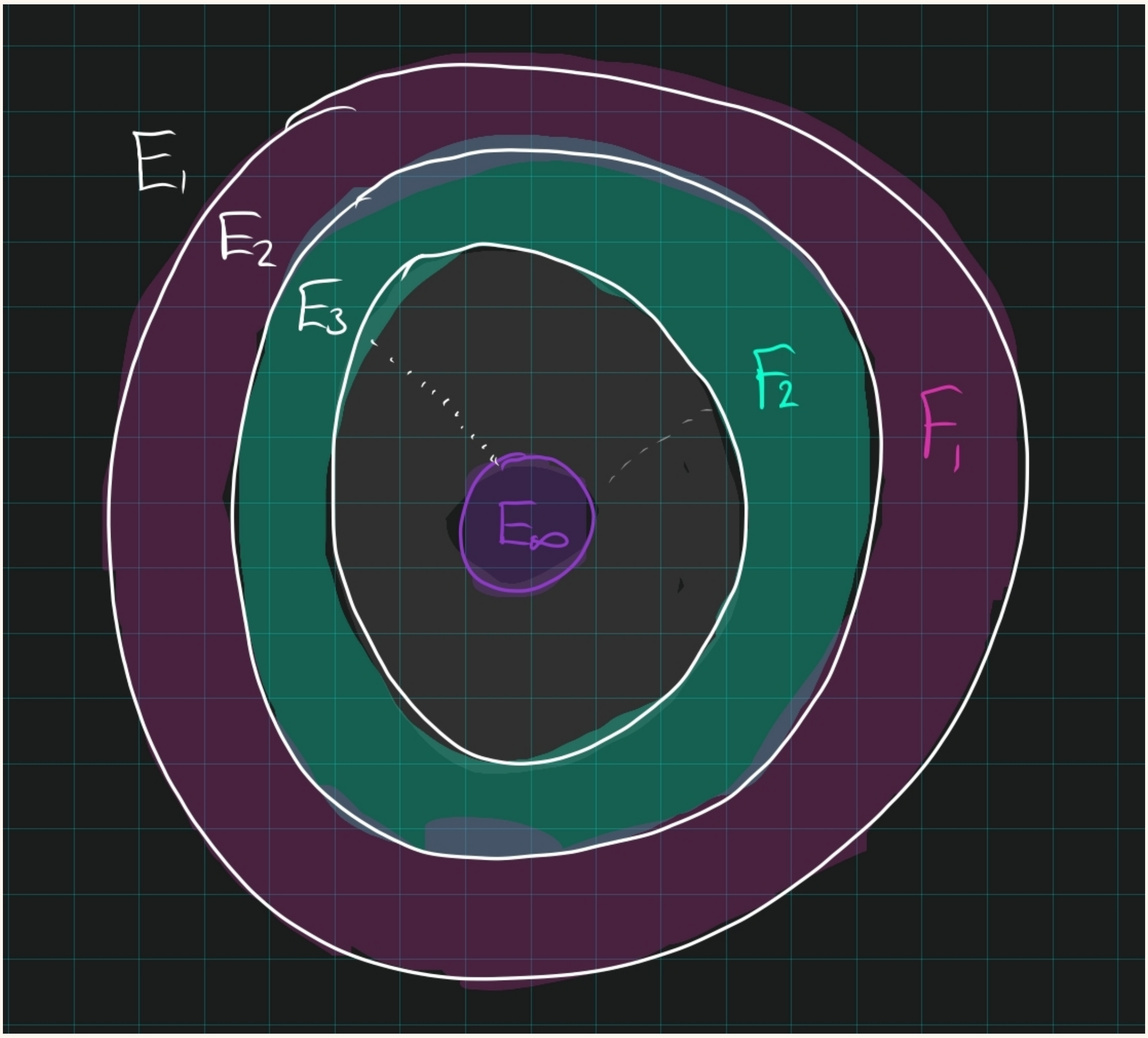

Figure 1: image_2021-05-28-23-29-31

■

Proof (of continuity of measure from below, detailed).
For any measure μ,

$$\mu(F_1) < \infty,\, F_k \searrow F \implies \lim_{k\to\infty} \mu(F_k) = \mu(F),$$

where $F_k \searrow F$ means $F_1 \supseteq F_2 \supseteq \cdots$ with $\cap_{k=1}^{\infty} F_k = F$. - Note that $\mu(F)$ makes sense: each $F_k \in \mathcal{B}$, which is a σ-algebra and closed under countable intersections.

- Take disjoint annuli by setting $E_k \coloneqq F_k \setminus F_{k+1}$

- Funny step: write

$$F_1 = F \coprod \coprod_{k=1}^{\infty} E_k.$$

  - This is because $x \in F_1$ iff x is in every F_k, so in F, **or**
  - $x \notin F_1$ but $x \in F_2$, noting incidentally $x \in F_3, F_4, \cdots$, **or**,
  - $x \notin F_2$ but $x \in F_3$, and so on.

- Now take measures, and note that we get a telescoping sum:

$$\begin{aligned}
\mu(F_1) &= \mu(F) + \sum_{k=1}^{\infty} \mu(E_k) \\
&= \mu(F) + \lim_{N\to\infty} \sum_{k=1}^{N} \mu(E_k) \\
&:= \mu(F) + \lim_{N\to\infty} \sum_{k=1}^{N} \mu(F_k \setminus F_{k+1}) \\
&:= \mu(F) + \lim_{N\to\infty} \sum_{k=1}^{N} \mu(F_k) - \mu(F_{k+1}) && \text{to be justified} \\
&= \mu(F) + \lim_{N\to\infty} [(\mu(F_1) - \mu(F_2)) + (\mu(F_2) - \mu(F_3)) + \cdots \\
&\qquad + (\mu(F_{N-1}) - \mu(F_N)) + (\mu(F_N) - \mu(F_{N+1}))] \\
&= \mu(F) + \lim_{N\to\infty} \mu(F_1) - \mu(F_{N+1}) \\
&= \mu(F) + \mu(F_1) - \lim_{N\to\infty} \mu(F_{N+1}).
\end{aligned}$$

- Justifying the measure subtraction: the general statement is that for any pair of sets $A \subseteq X$, $\mu(X \setminus A) = \mu(X) - \mu(A)$ when $\mu(A) < \infty$:

$$\begin{aligned}
X &= A \coprod (X \setminus A) \\
\implies \mu(X) &= \mu(A) + \mu(X \setminus A) && \text{countable additivity} \\
\implies \mu(X) - \mu(A) &= \mu(X \setminus A) && \text{if } \mu(A) < \infty.
\end{aligned}$$

- Now use that $\mu(F_1) < \infty$ to justify subtracting it from both sides:

$$\begin{aligned}
\mu(F_1) &= \mu(F) + \mu(F_1) - \lim_{N\to\infty} \mu(F_{N+1}) \\
\implies 0 &= \mu(F_1) - \lim_{N\to\infty} \mu(F_{N+1}) \\
\lim_{N\to\infty} \mu(F_{N+1}) &= \mu(F_1).
\end{aligned}$$

- Now use that $\lim_{N\to\infty} \mu(F_{N+1}) = \lim_{N\to\infty} \mu(F_N)$ to conclude.

■

2.2 Outer Measure

Proposition 2.2.1 ***(Properties of Outer Measure).***

1. Monotonicity: $E \subseteq F \implies m_*(E) \leq m_*(F)$.
2. Countable Subadditivity: $m_*(\cup E_i) \leq \sum m_*(E_i)$.
3. Approximation: For all E there exists a $G \supseteq E$ such that $m_*(G) \leq m_*(E) + \varepsilon$.
4. Disjoint[a] Additivity: $m_*(A \coprod B) = m_*(A) + m_*(B)$.

[a] This holds for outer measure **iff** $\mathrm{dist}(A, B) > 0$.

2.3 Measures on $\mathbb{R}^d$

Proposition 2.3.1 ***(Borel Characterization of Measurable Sets).***
If E is Lebesgue measurable, then $E = H \coprod N$ where $H \in F_\sigma$ and N is null.

Proposition 2.3.2 ***(Opens are unions of almost disjoint intervals.).***
Every open subset of $\mathbb{R}$ (resp $\mathbb{R}^n$) can be written as a unique countable union of disjoint (resp. almost disjoint) intervals (resp. cubes).

Theorem 2.3.3 ***(Measurable sets can be approximated by open/closed/compact sets.).***
Suppose E is measurable; then for every $\varepsilon > 0$,

1. There exists an open $O \supset E$ with $m(O \setminus E) < \varepsilon$
2. There exists a closed $F \subset E$ with $m(E \setminus F) < \varepsilon$
3. There exists a compact $K \subset E$ with $m(E \setminus K) < \varepsilon$.

Proof (that measurable sets can be approximated).

- (1): Take $\{Q_i\} \rightrightarrows E$ and set $O = \cup Q_i$.
- (2): Since E^c is measurable, produce $O \supset E^c$ with $m(O \setminus E^c) < \varepsilon$.
  - Set $F = O^c$, so F is closed.
  - Then $F \subset E$ by taking complements of $O \supset E^c$
  - $E \setminus F = O \setminus E^c$ and taking measures yields $m(E \setminus F) < \varepsilon$
- (3): Pick $F \subset E$ with $m(E \setminus F) < \varepsilon/2$.
  - Set $K_n = F \cap \mathbb{D}_n$, a ball of radius n about 0.
  - Then $E \setminus K_n \searrow E \setminus F$
  - Since $m(E) < \infty$, there is an N such that $n \geq N \implies m(E \setminus K_n) < \varepsilon$.

■

Proposition 2.3.4 ***(Translation and Dilation Invariance).***
Lebesgue measure is translation and dilation invariant.

Proof ((Todo) of translation/dilation invariance).
Obvious for cubes; if $Q_i \rightrightarrows E$ then $Q_i + k \rightrightarrows E + k$, etc. ■

Theorem 2.3.5 ***(Non-measurable sets exist).***
There is a non-measurable set $A \subseteq \mathbb{R}$.

Proof (Constructing a non-measurable set).

- Use AOC to choose one representative from every coset of $\mathbb{R}/\mathbb{Q}$ on $[0,1)$, which is countable, and assemble them into a set N
- Enumerate the rationals in $[0,1]$ as q_j, and define $N_j = N + q_j$. These intersect trivially.
- Define $M := \coprod N_j$, then $[0,1) \subseteq M \subseteq [-1,2)$, so the measure must be between 1 and 3.
- By translation invariance, $m(N_j) = m(N)$, and disjoint additivity forces $m(M) = 0$, a contradiction.

■

Proof (of Borel characterization).
For every $\frac{1}{n}$ there exists a closed set $K_n \subset E$ such that $m(E \setminus K_n) \leq \frac{1}{n}$. Take $K = \cup K_n$, wlog $K_n \nearrow K$ so $m(K) = \lim m(K_n) = m(E)$. Take $N := E \setminus K$, then $m(N) = 0$. ■

Proposition 2.3.6 ***(Limsups/infs of measurable sets are measurable.).***
If A_n are all measurable, $\limsup A_n$ and $\liminf A_n$ are measurable.

Proof (That limsups/infs are measurable).
Measurable sets form a sigma algebra, and these are expressed as countable unions/intersections of measurable sets. ■

Theorem 2.3.7 ***(Borel-Cantelli).***
Let $\{E_k\}$ be a countable collection of measurable sets. Then

$$\sum_k m(E_k) < \infty \implies \text{almost every } x \in \mathbb{R} \text{ is in at most finitely many } E_k.$$

Proof (of Borel-Cantelli).

- If $E = \limsup_j E_j$ with $\sum m(E_j) < \infty$ then $m(E) = 0$.
- If E_j are measurable, then $\limsup_j E_j$ is measurable.

- If $\sum_j m(E_j) < \infty$, then $\sum_{j=N}^{\infty} m(E_j) \overset{N\to\infty}{\to} 0$ as the tail of a convergent sequence.
- $E = \limsup_j E_j = \cap_{k=1}^{\infty} \cup_{j=k}^{\infty} E_j \implies E \subseteq \cup_{j=k}^{\infty}$ for all k
- $E \subset \cup_{j=k}^{\infty} \implies m(E) \leq \sum_{j=k}^{\infty} m(E_j) \overset{k\to\infty}{\to} 0.$

■

Proposition 2.3.8 ***(Extending the class of measurable functions.).*** •

Characteristic functions are measurable

- If f_n are measurable, so are $|f_n|, \limsup f_n, \liminf f_n, \lim f_n$,
- Sums and differences of measurable functions are measurable,
- Cones $F(x, y) = f(x)$ are measurable,
- Compositions $f \circ T$ for T a linear transformation are measurable,
- "Convolution-ish" transformations $(x, y) \mapsto f(x - y)$ are measurable

Proof (Convolution).
Take the cone on f to get $F(x, y) = f(x)$, then compose F with the linear transformation $T = [1, -1; 1, 0]$.

■

Definition 2.3.9 (σ-finiteness)
A measure space $(X, \mathcal{M}, \mu)$ is **σ-finite** if X can be written as a union of countably many measurable sets with finite measure.

Proposition 2.3.10 ***(Regularity of measure).***
If $(X, \mathcal{B}, \mu)$ is a Borel measure space where μ is finite on all balls of finite radius, then for any $E \in \mathcal{B}$ and any $\varepsilon > 0$,

- There exists an open set O with $E \subset O$ and $\mu(O \setminus E) < \varepsilon$
- There exists a closed set F with $F \subset E$ and $\mu(E \setminus F) < \varepsilon$.

Problem 2.3.1 (?)
Show that E is measurable iff E is regular in either sense above.

2.4 Exercises

Problem 2.4.1 (?)
Show that if $\sum \mu(E_k) < \infty$ then almost every $x \in X$ is in at most finitely many E_k.

3 Integration

3.1 Unsorted

Definition 3.1.1 (Measurable Function)
A function $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is $(\mathcal{M}_X, \mathcal{M}_Y)$**-measurable** iff $f^{-1}(\mathcal{M}_Y) \subseteq \mathcal{M}_X$. Equivalently, if $\mathcal{E}_Y$ is a generating set for $\mathcal{B}_Y$, $f^{-1}(\mathcal{E}_Y) \subseteq \mathcal{B}_X$.

- An functional on a general measurable space $f : f : (X, \mathcal{M}_X) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$ is **measurable** $\iff$ f is $(\mathcal{M}_X, \mathcal{B}_\mathbb{R})$-measurable.
- A functional $f : \mathbb{R}^d \to \mathbb{R}$ is **Borel measurable** iff f is $(\mathcal{B}_{\mathbb{R}^d}, \mathcal{B}_\mathbb{R})$-measurable.
- A functional $f : \mathbb{R}^d \to \mathbb{R}$ is **Lebesgue measurable** iff f is $(\mathcal{L}_{\mathbb{R}^d}, \mathcal{B}_\mathbb{R})$-measurable.

Using that $\mathcal{B}_\mathbb{R}$ is generated by open/closed rays, it suffices to check any of the following (for all $\alpha \in \mathbb{R}$):

- $f^{-1}(\alpha, \infty) \in \mathcal{M}$
- $f^{-1}[\alpha, \infty) \in \mathcal{M}$
- $f^{-1}(-\infty, \alpha) \in \mathcal{M}$
- $f^{-1}(-\infty, \alpha] \in \mathcal{M}$

Remark 3.1.2: Note that we still require Borel sets in the target for Lebesgue measurability! Taking $(\mathcal{L}_{\mathbb{R}^d}, \mathcal{L}_\mathbb{R})$ functions is too stringent, e.g. this class does not contain continuous functionals.

⚠ **Warning 3.1.3**
If f is $\mathcal{L}$-measurable and h is continuous, it's not necessarily true that $k := f \circ h$ is $\mathcal{L}$-measurable. Standard counterexample: set $g(x) := C(x) + x$ for C the Cantor-Lebesgue function, then $g : [0, 1] \to [0, 2]$ is a homeomorphism. Then $m(g(C)) = 1$ since f is constant on intervals in C^c, so use Vitali's theorem: a set is null iff every subset is measurable. So $g(C)$ contains a non-measurable set A. Define $B := g^{-1}(A)$, then $B \subset C$ and $m(C) = 0$ implies B is measurable and χ_B is a measurable function. But then $k := \chi_B \circ g^{-1}$ is not $\mathcal{L}$-measurable, since $k^{-1}(1) = A$ is a non-measurable set, but χ_B is $\mathcal{L}$-measurable and g^{-1} is continuous.

Proposition 3.1.4 ***(Closure of measurable functions under operations).***
$\mathcal{M}$-measurable functionals are closed under

- Sums
- Products
- Sups/infs
- Limsups/Liminfs
- Limits when they exist, and the limiting function is measurable.
- $\max(f, g)$ and $\min(f, g)$.

Characteristic functions on measurable sets are automatically measurable, since $E \in \mathcal{M} \implies$

$E = \chi_E^{-1}(\{1\})$.

Remark 3.1.5 ***(A common proof technique):***

- Show something holds for indicator functions.
- Show it holds for simple functions by linearity.
- Use $s_k \nearrow f$ and apply MCT to show it holds for f.

Remark 3.1.6 ***(on notation):***

- L^+: measurable functions
- L^1: Lebesgue integrable functions, so $\int |f| < \infty$

Definition 3.1.7 (Simple Function)
A **simple function** $s : \mathbb{C} \to X$ is a finite linear combination of indicator functions of measurable sets, i.e.

$$s(x) = \sum_{j=1}^{n} c_j \chi_{E_j}(x).$$

Definition 3.1.8 (Lebesgue Integral)

$$\int_X f \coloneqq \sup \left\{ \int s(x)\, d\mu \;\middle|\; 0 \le s \le f, s \text{ simple} \right\}.$$

Note that if $s = \sum c_j \chi_{E_j}$ is simple, then

$$\int_X s(x)\, d\mu \coloneqq \sum_{j=1}^{n} c_j \mu(E_j).$$

Remark 3.1.9 ***(Integrals split across disjoint sets):*** A useful fact: for $(X, \mathcal{M})$ a measure space, integrals split across disjoint sets:

$$\int_X f = \int_{X \setminus A} f + \int_A f \qquad \forall A \in \mathcal{M}.$$

Definition 3.1.10 (Essential supremum and infimum, essentially bounded)
An **essential lower bound** b on a function f is any real number such that $S_b \coloneqq \left\{ x \;\middle|\; f(x) < b \right\} = f^{-1}(-\infty, b)$ has measure zero. The **essential infimum** is the supremum of all essential lower bounds, i.e. $\operatorname{ess\,inf} f \coloneqq \sup_b \left\{ b \;\middle|\; \mu S_b = 0 \right\}$. This is the greatest lower bound almost everywhere.
Similarly an **essential upper bound** c is any number such that $S^c \coloneqq f^{-1}(c, \infty)$ has measure zero, and the **essential supremum** is $\operatorname{ess\,sup} f \coloneqq \inf_c \left\{ c \;\middle|\; \mu S^c = 0 \right\}$, which is the least upper bound almost everywhere.

A function is **essentially bounded** if $\|f\|_\infty := \operatorname{ess\,sup} f < \infty$. These are functions which are bounded almost everywhere.

Example 3.1.11 ***(An essentially bounded but not bounded function):*** $f(x) = x\chi_{\mathbb{Q}}(x)$ is essentially bounded but not bounded.

Proposition 3.1.12 ***(L^∞ functions are equivalent to bounded almost-everywhere functions).***
If $f \in L^\infty(X)$, then f is equal to some bounded function g almost everywhere.

Theorem 3.1.13 ***(p-Test for Integrals).***

$$\int_0^1 \frac{1}{x^p} < \infty \iff p < 1$$
$$\int_1^\infty \frac{1}{x^p} < \infty \iff p > 1.$$

Slogan 3.1.14
Large powers of x help us in neighborhoods of infinity and hurt around zero.

Theorem 3.1.15 ***(Monotone Convergence).***
If $f_n : X \to [0, \infty) \in L^+$ and $f_n \nearrow f$ almost everywhere, then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \quad \int f_n \to \int f.$$

Slogan 3.1.16
Measurable, non-negative, increasing pointwise a.e. allows commuting limits and integrals.

Proof (of MCT).

todo

■

Theorem 3.1.17 ***(Dominated Convergence).***
If $f_n \in L^1$ and $f_n \to f$ almost everywhere with $|f_n| \leq g$ for some $g \in L^1$, then $f \in L^1$ and

$$\int |f_n - f| \to 0.$$

As a consequence,

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \quad \int f_n \to \int f < \infty$$

Positivity not *needed.*

Proof (of DCT).

todo

∎

Theorem 3.1.18 ***(Generalized DCT).***
If

- $f_n \in L^1$ with $f_n \to f$ almost everywhere,
- There exist $g_n \in L^1$ with $|f_n| \leq g_n$, $g_n \geq 0$.
- $g_n \to g$ almost everywhere with $g \in L^1$, and
- $\lim \int g_n = \int g$,

then $f \in L^1$ and $\lim \int f_n = \int f < \infty$.

Note that this is the DCT with $|f_n| < |g|$ relaxed to $|f_n| < g_n \to g \in L^1$.

Proof .
Proceed by showing $\limsup \int f_n \leq \int f \leq \liminf \int f_n$:

- $\int f \geq \limsup \int f_n$:

$$\begin{aligned} \int g - \int f = \int (g - f) & \\ &\leq \liminf \int (g_n - f_n) \quad \text{Fatou} \\ &= \lim \int g_n + \liminf \int (-f_n) \\ &= \lim \int g_n - \limsup \int f_n \\ &= \int g - \limsup \int f_n \end{aligned}$$

$$\implies \int f \geq \limsup \int f_n.$$

  - Here we use $g_n - f_n \xrightarrow{n \to \infty} g - f$ with $0 \leq |f_n| - f_n \leq g_n - f_n$, so $g_n - f_n$ are nonnegative (and measurable) and Fatou's lemma applies.

- $\displaystyle\int f \leq \liminf \int f_n$:

$$\begin{aligned}\int g + \int f = \int (g+f) \\ &\leq \liminf \int (g_n + f_n) \\ &= \lim \int g_n + \liminf \int f_n \\ &= \int g + \liminf f_n\end{aligned}$$

$$\int f \leq \liminf \int f_n.$$

  - Here we use that $g_n + f_n \to g + f$ with $0 \leq |f_n| + f_n \leq g_n + f_n$ so Fatou's lemma again applies.

■

Proposition 3.1.19 ***(Convergence in L^1 implies convergence of L^1 norm).***
If $f \in L^1$, then

$$\int |f_n - f| \to 0 \iff \int |f_n| \to \int |f|.$$

Proof .
Let $g_n = |f_n| - |f_n - f|$, then $g_n \to |f|$ and

$$|g_n| = ||f_n| - |f_n - f|| \leq |f_n - (f_n - f)| = |f| \in L^1,$$

so the DCT applies to g_n and

$$\begin{aligned}\|f_n - f\|_1 = \int |f_n - f| + |f_n| - |f_n| = \int |f_n| - g_n \\ &\to_{DCT} \lim \int |f_n| - \int |f|.\end{aligned}$$

■

Theorem 3.1.20 ***(Fatou).***
If f_n is a sequence of nonnegative measurable functions, then

$$\begin{aligned}\liminf_n \int f_n \geq \int \liminf_n f_n \\ \limsup_n \int f_n \leq \int \limsup_n f_n.\end{aligned}$$

Proof (of Fatou).

Prove Fatou

■

Theorem 3.1.21 ***(Tonelli (Non-Negative, Measurable)).***
For $f(x,y)$ **non-negative and measurable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is a **measurable** function
- $F(x) = \displaystyle\int f(x,y)\, dy$ is a **measurable** function,
- For E measurable, the slices $E_x := \left\{ y \;\middle|\; (x,y) \in E \right\}$ are measurable.
- $\displaystyle\int f = \int\int F$, i.e. any iterated integral is equal to the original.

Theorem 3.1.22 ***(Fubini (Integrable)).***
For $f(x,y)$ **integrable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is an **integrable** function
- $F(x) := \displaystyle\int f(x,y)\, dy$ is an **integrable** function,
- For E measurable, the slices $E_x := \left\{ y \;\middle|\; (x,y) \in E \right\}$ are measurable.
- $\displaystyle\int f = \int\int f(x,y)$, i.e. any iterated integral is equal to the original

Theorem 3.1.23 ***(Fubini-Tonelli).***
If any iterated integral is **absolutely integrable**, i.e. $\displaystyle\int\int |f(x,y)| < \infty$, then f is integrable and $\displaystyle\int f$ equals any iterated integral.

Proposition 3.1.24 ***(Measurable Slices).***
Let E be a measurable subset of $\mathbb{R}^n$. Then

- For almost every $x \in \mathbb{R}^{n_1}$, the slice $E_x := \left\{ y \in \mathbb{R}^{n_2} \;\middle|\; (x,y) \in E \right\}$ is measurable in $\mathbb{R}^{n_2}$.
- The function

$$\begin{aligned} F : \mathbb{R}^{n_1} &\to \mathbb{R} \\ x &\mapsto m(E_x) = \int_{\mathbb{R}^{n_2}} \chi_{E_x}\, dy \end{aligned}$$

is measurable and

$$m(E) = \int_{\mathbb{R}^{n_1}} m(E_x)\, dx = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{E_x}\, dy\, dx.$$

Proof (of measurable slices).
$\implies$:

- Let f be measurable on $\mathbb{R}^n$.
- Then the cylinders $F(x,y) = f(x)$ and $G(x,y) = f(y)$ are both measurable on $\mathbb{R}^{n+1}$.
- Write $\mathcal{A} = \{G \leq F\} \cap \{G \geq 0\}$; both are measurable.

$\impliedby$:

- Let A be measurable in $\mathbb{R}^{n+1}$.
- Define $A_x = \left\{y \in \mathbb{R} \;\middle|\; (x,y) \in \mathcal{A}\right\}$, then $m(A_x) = f(x)$.
- By the corollary, A_x is measurable set, $x \mapsto A_x$ is a measurable function, and $m(A) = \displaystyle\int f(x)\, dx$.
- Then explicitly, $f(x) = \chi_A$, which makes f a measurable function.

$\blacksquare$

Proposition 3.1.25 ***(Differentiating Under an Integral).***
If $\left|\dfrac{\partial}{\partial t} f(x,t)\right| \leq g(x) \in L^1$, then letting $F(t) = \displaystyle\int f(x,t)\, dt$,

$$\begin{aligned}\frac{\partial}{\partial t} F(t) &:= \lim_{h \to 0} \int \frac{f(x,t+h) - f(x,t)}{h} dx \\ &\overset{\text{DCT}}{=} \int \frac{\partial}{\partial t} f(x,t)\, dx.\end{aligned}$$

To justify passing the limit, let $h_k \to 0$ be any sequence and define

$$f_k(x,t) = \frac{f(x,t+h_k) - f(x,t)}{h_k},$$

so $f_k \overset{k\to\infty}{\longrightarrow} \dfrac{\partial f}{\partial t}$ pointwise.
Apply the MVT to f_k to get $f_k(x,t) = f_k(\xi,t)$ for some $\xi \in [0,h_k]$, and show that $f_k(\xi,t) \in L_1$.

Proposition 3.1.26 ***(Commuting Sums with Integrals (non-negative)).***
If f_n are non-negative and $\sum \int |f|_n < \infty$, then $\sum \int f_n = \int \sum f_n$.

Proof .
- Idea: MCT.
- Let $F_N = \sum^N f_n$ be a finite partial sum;
- Then there are simple functions $\varphi_n \nearrow f_n$
- So $\sum^N \varphi_n \nearrow F_N$ and MCT applies

$\blacksquare$

Theorem 3.1.27 *(Commuting Sums with Integrals (integrable)).*
If $\{f_n\}$ integrable with either $\sum \int |f_n| < \infty$ or $\int \sum |f_n| < \infty$, then

$$\int \sum f_n = \sum \int f_n.$$

Proof .

- By Tonelli, if $f_n(x) \geq 0$ for all n, taking the counting measure allows interchanging the order of "integration".
- By Fubini on $|f_n|$, if either "iterated integral" is finite then the result follows.

■

Proposition 3.1.28 *(?).*
If $f_k \in L^1$ and $\sum \|f_k\|_1 < \infty$ then $\sum f_k$ converges almost everywhere and in L^1.

Proof (?).
Define $F_N = \sum^N f_k$ and $F = \lim_N F_N$, then $\|F_N\|_1 \leq \sum^N \|f_k\| < \infty$ so $F \in L^1$ and $\|F_N - F\|_1 \to 0$ so the sum converges in L^1. Almost everywhere convergence: ?

■

3.2 Examples of (Non)Integrable Functions

Example 3.2.1 *(Examples of integrable functions):*

- $\displaystyle\int \frac{1}{1+x^2} = \arctan(x) \stackrel{x\to\infty}{\to} \pi/2 < \infty$
- Any bounded function (or continuous on a compact set, by EVT)
- $\displaystyle\int_0^1 \frac{1}{\sqrt{x}} < \infty$
- $\displaystyle\int_0^1 \frac{1}{x^{1-\varepsilon}} < \infty$
- $\displaystyle\int_1^\infty \frac{1}{x^{1+\varepsilon}} < \infty$

Example 3.2.2 *(Examples of non-integrable functions):*

- $\displaystyle\int_0^1 \frac{1}{x} = \infty.$

- $\int_1^\infty \frac{1}{x} = \infty.$
- $\int_1^\infty \frac{1}{\sqrt{x}} = \infty$
- $\int_1^\infty \frac{1}{x^{1-\varepsilon}} = \infty$
- $\int_0^1 \frac{1}{x^{1+\varepsilon}} = \infty$

3.3 L^1 Facts

Proposition 3.3.1 *(Zero in L^1 iff zero almost everywhere).*
For $f \in L^+$,

$$\int f = 0 \iff f \equiv 0 \text{ almost everywhere.}$$

Proof .

- Obvious for simple functions:
  - If $f(x) = \sum_{j=1}^n c_j \chi_{E_j}$, then $\int f = 0$ iff for each j, either $c_j = 0$ or $m(E_j) = 0$.
  - Since nonzero c_j correspond to sets where $f \neq 0$, this says $m(\{f \neq 0\}) = 0$.
- $\Longleftarrow$:
  - If $f = 0$ almost everywhere and $\varphi \nearrow f$, then $\varphi = 0$ almost everywhere since $\varphi(x) \leq f(x)$ -Then

$$\int f = \sup_{\varphi \leq f} \int \varphi = \sup_{\varphi \leq f} 0 = 0.$$

- $\implies$:
  - Instead show negating "$f = 0$ almost everywhere" implies $\int f \neq 0$.
  - Write $\{f \neq 0\} = \cup_{n \in \mathbb{N}} S_n$ where $S_n := \left\{ x \;\middle|\; f(x) \geq \frac{1}{n} \right\}$.
  - Since "not $f = 0$ almost everywhere", there exists an n such that $m(S_n) > 0$.
  - Then

$$0 < \frac{1}{n}\chi_{E_n} \leq f \implies 0 < \int \frac{1}{n}\chi_{E_n} \leq \int f.$$

■

Proposition 3.3.2 *(Translation Invariance).*

The Lebesgue integral is translation invariant, i.e.

$$\int f(x)\ dx = \int f(x+h)\ dx \qquad \text{for any} h.$$

Proof .

- Let $E \subseteq X$; for characteristic functions,

$$\int_X \chi_E(x+h) = \int_X \chi_{E+h}(x) = m(E+h) = m(E) = \int_X \chi_E(x)$$

by translation invariance of measure.
- So this also holds for simple functions by linearity.
- For $f \in L^+$, choose $\varphi_n \nearrow f$ so $\displaystyle\int \varphi_n \to \int f$.
- Similarly, $\tau_h \varphi_n \nearrow \tau_h f$ so $\displaystyle\int \tau_h f \to \int f$
- Finally $\left\{\displaystyle\int \tau_h \varphi\right\} = \left\{\displaystyle\int \varphi\right\}$ by step 1, and the suprema are equal by uniqueness of limits.

■

Proposition 3.3.3 ***(Integrals distribute over disjoint sets).***
If $X \subseteq A \cup B$, then $\displaystyle\int_X f \leq \int_A f + \int_{A^c} f$ with equality iff $X = A \coprod B$.

Proposition 3.3.4 ***(Uniformly continuous L^1 functions vanish at infinity.).***
If $f \in L^1$ and f is uniformly continuous, then $f(x) \overset{|x|\to\infty}{\to} 0$.

Warning 3.3.5
This doesn't hold for general L^1 functions, take any train of triangles with height 1 and summable areas.

Theorem 3.3.6 ***(Small Tails in L^1).***
If $f \in L^1$, then for every ε there exists a radius R such that if $A = B_R(0)^c$, then $\displaystyle\int_A |f| < \varepsilon$.

Proof .

- Approximate with compactly supported functions.
- Take $g \overset{L_1}{\to} f$ with $g \in C_c$
- Then choose N large enough so that $g = 0$ on $E := B_N(0)$
- Then

$$\int_E |f| \leq \int_E |f-g| + \int_E |g|.$$

■

Proposition 3.3.7 *(L^1 functions are absolutely continuous.).*
$m(E) \to 0 \implies \int_E f \to 0.$

Proof (?).
Approximate with compactly supported functions. Take $g \xrightarrow{L_1} f$, then $g \leq M$ so $\int_E f \leq \int_E f - g + \int_E g \to 0 + M \cdot m(E) \to 0.$ ∎

Proposition 3.3.8 *(L^1 functions are finite almost everywhere.).*
If $f \in L^1$, then $m(\{f(x) = \infty\}) = 0$.

Proof (?).
Idea: Split up domain Let $A = \{f(x) = \infty\}$, then $\infty > \int f = \int_A f + \int_{A^c} f = \infty \cdot m(A) + \int_{A^c} f \implies m(X) = 0.$ ∎

Theorem 3.3.9 *(Continuity in L^1).*

$$\|\tau_h f - f\|_1 \overset{h \to 0}{\to} 0$$

Proof .
Approximate with compactly supported functions. Take $g \xrightarrow{L_1} f$ with $g \in C_c$.

$$\begin{aligned}
\int f(x+h) - f(x) &\leq \int f(x+h) - g(x+h) + \int g(x+h) - g(x) + \int g(x) - f(x) \\
&\overset{?\to?}{\to} 2\varepsilon + \int g(x+h) - g(x) \\
&= \int_K g(x+h) - g(x) + \int_{K^c} g(x+h) - g(x) \\
&\overset{??}{\to} 0,
\end{aligned}$$

which follows because we can enlarge the support of g to K where the integrand is zero on K^c, then apply uniform continuity on K. ∎

Proposition 3.3.10 *(Integration by parts, special case).*

$$\begin{aligned}
F(x) := \int_0^x f(y)dy \quad &\text{and} \quad G(x) := \int_0^x g(y)dy \\
\implies \int_0^1 F(x)g(x)dx &= F(1)G(1) - \int_0^1 f(x)G(x)dx.
\end{aligned}$$

Proof (?).
Fubini-Tonelli, and sketch region to change integration bounds. ■

Theorem 3.3.11 ***(Lebesgue Density).***

$$A_h(f)(x) := \frac{1}{2h}\int_{x-h}^{x+h} f(y)dy \implies \|A_h(f) - f\| \overset{h\to 0}{\to} 0.$$

Proof (?).
Fubini-Tonelli, and sketch region to change integration bounds, and continuity in L^1. ■

3.4 Lp Facts

Proposition 3.4.1 ***(Dense subspaces of $L^2(I)$).***
The following are dense subspaces of $L^2([0,1])$:

- Simple functions
- Step functions
- $C_0([0,1])$
- Smoothly differentiable functions $C_0^\infty([0,1])$
- Smooth compactly supported functions C_c^∞

Theorem 3.4.2 ***(?).***

$$m(X) < \infty \implies \lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

Proof (?).
Let $M = \|f\|_\infty$.

- For any $L < M$, let $S = \{|f| \geq L\}$.
- Then $m(S) > 0$ and

$$
\begin{aligned}
\|f\|_p &= \left(\int_X |f|^p\right)^{\frac{1}{p}} \\
&\geq \left(\int_S |f|^p\right)^{\frac{1}{p}} \\
&\geq L\ m(S)^{\frac{1}{p}} \overset{p\to\infty}{\to} L \\
&\implies \liminf_p \|f\|_p \geq M.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\|f\|_p &= \left(\int_X |f|^p\right)^{\frac{1}{p}} \\
&\leq \left(\int_X M^p\right)^{\frac{1}{p}} \\
&= M\ m(X)^{\frac{1}{p}} \xrightarrow{p\to\infty} M \\
&\implies \limsup_p \|f\|_p \leq M.
\end{aligned}
$$

■

Theorem 3.4.3 ***(Duals for L^p spaces).***
For $1 \leq p < \infty$, $(L^p)^\vee \cong L^q$.

Proof ($p = 1$ case).
?

■

todo

Proof ($p = 2$ case).
Use Riesz Representation for Hilbert spaces.

■

Proposition 3.4.4 ***(L^1 is not quite the dual of L^∞.).***
$L^1 \subset (L^\infty)^\vee$, since the isometric mapping is always injective, but *never* surjective.

3.5 Counterexamples

Proposition 3.5.1 ***(a.e. convergence never implies L^p convergence).***
Sequences $f_k \overset{a.e.}{\to} f$ but $f_k \overset{L^p}{\not\to} f$:

- For $1 \leq p < \infty$: The skateboard to infinity, $f_k = \chi_{[k,k+1]}$.

Then $f_k \xrightarrow{a.e.} 0$ but $\|f_k\|_p = 1$ for all k.

Converges pointwise and a.e., but not uniformly and not in norm.

- For $p = \infty$: The sliding boxes $f_k = k \cdot \chi_{[0, \frac{1}{k}]}$.

  Then similarly $f_k \xrightarrow{a.e.} 0$, but $\|f_k\|_p = 1$ and $\|f_k\|_\infty = k \to \infty$

Converges a.e., but not uniformly, not pointwise, and not in norm.

Proposition 3.5.2 ***(The four big counterexamples in convergence).***

1. Uniform: $f_n \rightrightarrows f : \forall \varepsilon\ \exists N \mathrel{\Big|} n \geq N \implies |f_N(x) - f(x)| < \varepsilon \quad \forall x$.
2. Pointwise: $f_n(x) \to f(x)$ for all x. (This is just a sequence of numbers)
3. Almost Everywhere: $f_n(x) \to f(x)$ for almost all x.
4. Norm: $\|f_n - f\|_1 = \displaystyle\int |f_n(x) - f(x)| \to 0$.

We have $1 \implies 2 \implies 3$, and in general no implication can be reversed, but (**warning**) none of $1, 2, 3$ imply 4 or vice versa.

- $f_n = (1/n)\chi_{(0,n)}$. This converges uniformly to 0, but the integral is identically 1. So this satisfies 1,2,3 and not 4.

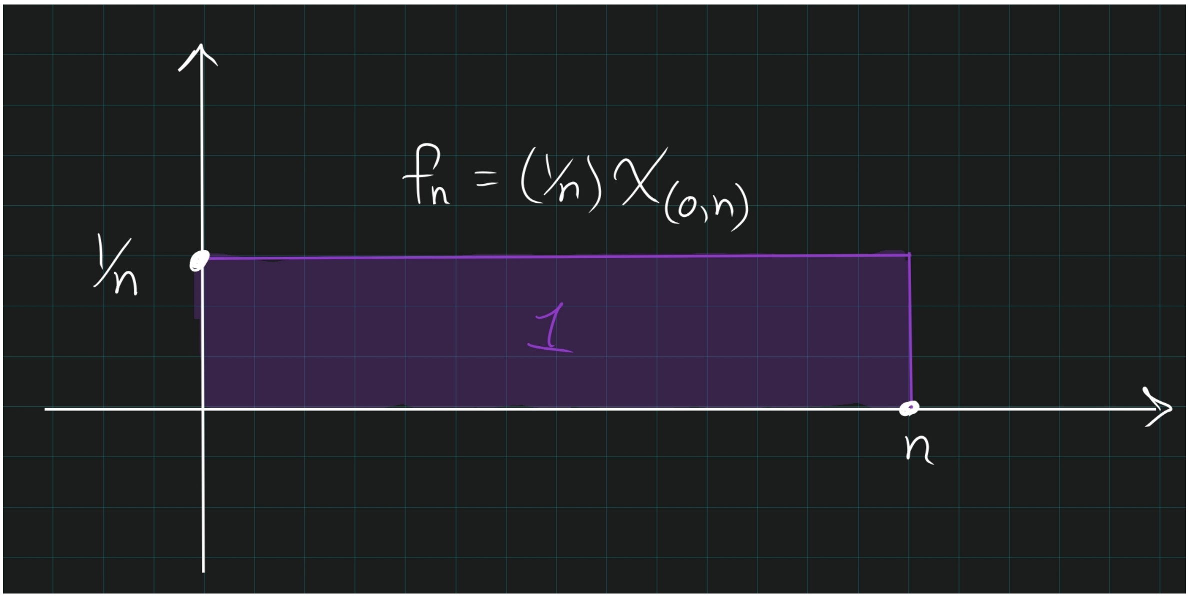

Figure 2: image_2021-05-21-16-38-30

- $f_n = \chi_{(n,n+1)}$ (skateboard to infinity). This satisfies 2,3 but not 1, 4.

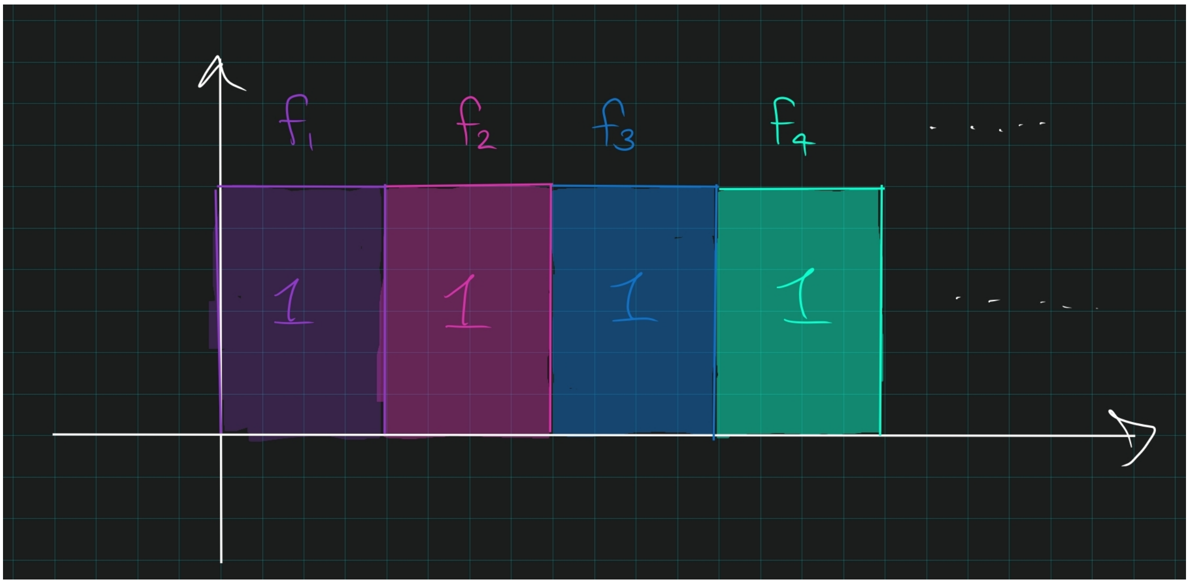

Figure 3: image_2021-05-21-16-42-08

- $f_n = n\chi_{(0,\frac{1}{n})}$. This satisfies 3 but not 1,2,4.

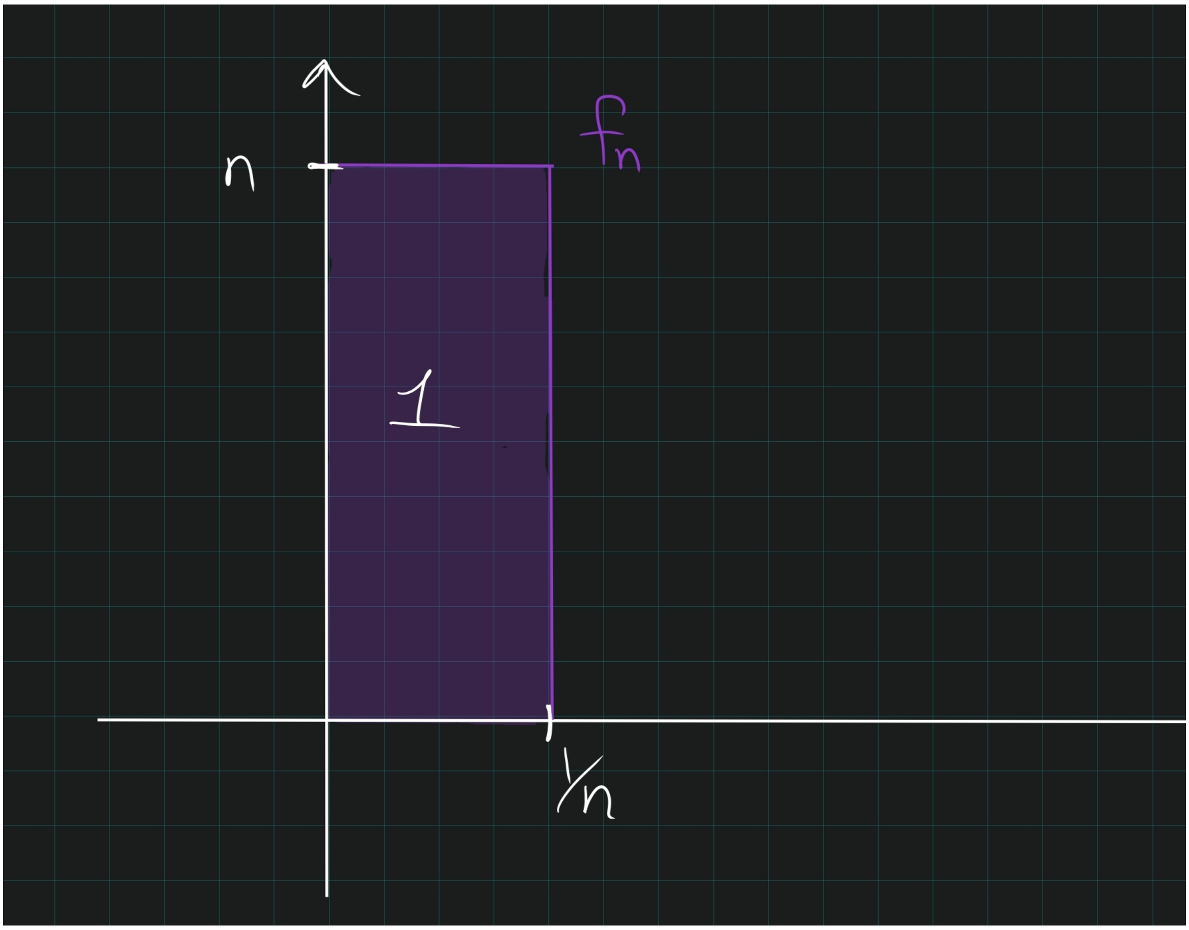

Figure 4: image_2021-05-21-16-54-38

- f_n : one can construct a sequence where $f_n \to 0$ in L^1 but is not 1,2, or 3. The construction:
  - Break I into 2 intervals, let f_1 be the indicator on the first half, f_2 the indicator on the second.
  - Break I into $2^2 = 4$ intervals, like f_3 be the indicator on the first quarter, f_4 on the second, etc.
  - Break I into 2^k intervals and cyclic through k indicator functions.

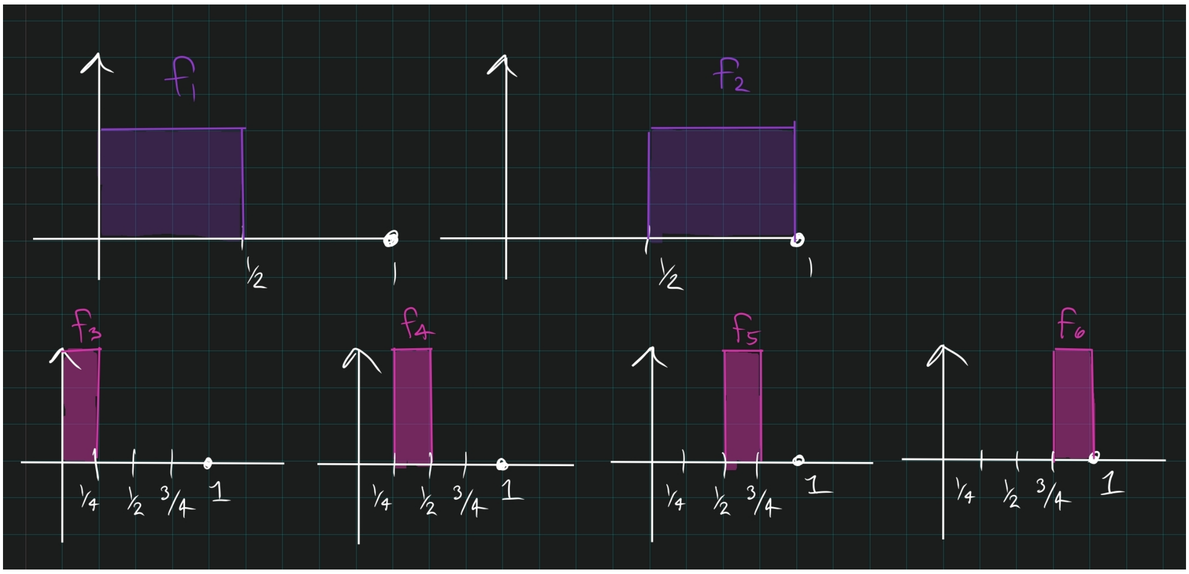

Figure 5: image_2021-05-21-16-49-09

- Then $\int f_n = 1/2^n \to 0$, but $f_n \not\to 0$ pointwise since for every x, there are infinitely many n for which $f_n(x) = 0$ and infinitely many for which $f_n(x) = 1$.

Proposition 3.5.3 ***(Functional analytic properties of L^1 and L^2).***
For any measure space $(X, \mathcal{M}, \mu)$,

- $L^1(X)$ is Banach space.
- $L^2(X)$ is a (possibly non-separable) Hilbert space.

4 Fourier Transform and Convolution

4.1 The Fourier Transform

Proposition 4.1.1 ***(?).***
If $\widehat{f} = \widehat{g}$ then $f = g$ almost everywhere.

Proposition 4.1.2 ***(Riemann-Lebesgue: Fourier transforms have small tails.).***

$$f \in L^1 \implies \widehat{f}(\xi) \to 0 \text{ as } |\xi| \to \infty,$$

if $f \in L^1$, then $\widehat{f}$ is continuous and bounded.

Proof (?).

- Boundedness:

$$\left|\widehat{f}(\xi)\right| \leq \int |f| \cdot \left|e^{2\pi i x\cdot\xi}\right| = \|f\|_1.$$

- Continuity:
- $\left|\widehat{f}(\xi_n) - \widehat{f}(\xi)\right|$
- Apply DCT to show $a \overset{n\to\infty}{\to} 0$.

■

Theorem 4.1.3 *(Fourier Inversion)*.

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(x) e^{2\pi i x\cdot\xi} d\xi.$$

Warning 4.1.4
Fubini-Tonelli does not work here!

Proof (?).
Idea: Fubini-Tonelli doesn't work directly, so introduce a convergence factor, take limits, and use uniqueness of limits.

- Take the modified integral:

$$\begin{aligned}
I_t(x) &= \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}\ e^{-\pi t^2|\xi|^2} \\
&= \int \widehat{f}(\xi)\varphi(\xi) \\
&= \int f(\xi)\widehat{\varphi}(\xi) \\
&= \int f(\xi)\widehat{\widehat{g}}(\xi - x) \\
&= \int f(\xi) g_t(x-\xi)\ d\xi \\
&= \int f(y-x) g_t(y)\ dy \quad (\xi = y - x) \\
&= (f * g_t) \\
&\to f \text{ in } L^1 \text{ as } t \to 0.
\end{aligned}$$

- We also have

$$\begin{aligned}\lim_{t\to 0} I_t(x) &= \lim_{t\to 0} \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}\ e^{-\pi t^2|\xi|^2} \\ &= \lim_{t\to 0} \int \widehat{f}(\xi)\varphi(\xi) \\ &=_{DCT} \int \widehat{f}(\xi) \lim_{t\to 0} \varphi(\xi) \\ &= \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}\end{aligned}$$

.

- So

$$I_t(x) \to \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi} \text{ pointwise and } \|I_t(x) - f(x)\|_1 \to 0.$$

- So there is a subsequence I_{t_n} such that $I_{t_n}(x) \to f(x)$ almost everywhere
- Thus $f(x) = \int \widehat{f}(\xi)\ e^{2\pi i x\cdot\xi}$ almost everywhere by uniqueness of limits.

■

Proposition 4.1.5 ***(Eigenfunction of the Fourier transform).***

$$g(x) := e^{-\pi|t|^2} \implies \widehat{g}(\xi) = g(\xi) \quad \text{and} \quad \widehat{g}_t(x) = g(tx) = e^{-\pi t^2|x|^2}.$$

4.2 Approximate Identities

Example 4.2.1 ***(of an approximation to the identity.):***

$$\varphi(x) := e^{-\pi x^2}.$$

Theorem 4.2.2 ***(Convolving against an approximate identity converges in L^1.).***

$$\|f * \varphi_t - f\|_1 \overset{t\to 0}{\to} 0.$$

Proof (?).

$$\begin{aligned}
\|f - f * \varphi_t\|_1 &= \int f(x) - \int f(x-y)\varphi_t(y)\ dydx \\
&= \int f(x) \int \varphi_t(y)\ dy - \int f(x-y)\varphi_t(y)\ dydx \\
&= \int \int \varphi_t(y)[f(x) - f(x-y)]\ dydx \\
&=_{FT} \int \int \varphi_t(y)[f(x) - f(x-y)]\ dxdy \\
&= \int \varphi_t(y) \int f(x) - f(x-y)\ dxdy \\
&= \int \varphi_t(y) \|f - \tau_y f\|_1 dy \\
&= \int_{y<\delta} \varphi_t(y)\|f - \tau_y f\|_1 dy + \int_{y\geq\delta} \varphi_t(y)\|f - \tau_y f\|_1 dy \\
&\leq \int_{y<\delta} \varphi_t(y)\varepsilon + \int_{y\geq\delta} \varphi_t(y)\left(\|f\|_1 + \|\tau_y f\|_1\right) dy \quad \text{by continuity in } L^1 \\
&\leq \varepsilon + 2\|f\|_1 \int_{y\geq\delta} \varphi_t(y) dy \\
&\leq \varepsilon + 2\|f\|_1 \cdot \varepsilon \quad \text{since } \varphi_t \text{ has small tails} \\
&\overset{\varepsilon\to 0}{\to} 0.
\end{aligned}$$

■

Theorem 4.2.3 ***(Convolutions vanish at infinity).***

$$f, g \in L^1 \text{ and bounded} \implies \lim_{|x|\to\infty} (f * g)(x) = 0.$$

Proof (?).

- Choose $M \geq f, g$.
- By small tails, choose N such that $\int_{B_N^c} |f|, \int_{B_n^c} |g| < \varepsilon$
- Note
$$|f * g| \leq \int |f(x-y)|\ |g(y)|\ dy := I.$$
- Use $|x| \leq |x-y| + |y|$, take $|x| \geq 2N$ so either
$$|x-y| \geq N \implies I \leq \int_{\{x-y\geq N\}} |f(x-y)|M\ dy \leq \varepsilon M \to 0$$
then
$$|y| \geq N \implies I \leq \int_{\{y\geq N\}} M|g(y)|\ dy \leq M\varepsilon \to 0.$$

■

Proposition 4.2.4 ***(Corollary of Young's inequality).***
Take $q = 1$ in Young's inequality to obtain

$$\|f * g\|_p \leq \|f\|p\|g\|1.$$

Proposition 4.2.5 ***(L^1 is closed under convolution.).***
If $f, g \in L^1$ then $f * g \in L^1$.

5 Functional Analysis

5.1 Theorems

Fact 5.1.1 (Pythagoras)

$$\langle v,\ w \rangle = 0 \implies \|v + w\| = \|v\| + \|w\|.$$

Theorem 5.1.2 ***(Bessel's Inequality).***
For any orthonormal set $\{u_n\} \subseteq \mathcal{H}$ a Hilbert space (not necessarily a basis),

$$\left\| x - \sum_{n=1}^{N} \langle x, u_n \rangle\, u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n \rangle|^2$$

and thus

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2.$$

Proof (of Bessel's inequality).

- Let $S_N = \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n$

$$
\begin{aligned}
\|x - S_N\|^2 &= \langle x - S_n,\ x - S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\langle x,\ S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\left\langle x,\ \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n \right\rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^{N} \langle x,\ \langle x,\ u_n \rangle u_n \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^{N} \overline{\langle x,\ u_n \rangle} \langle x,\ u_n \rangle \\
&= \|x\|^2 + \left\| \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n \right\|^2 - 2 \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 - 2 \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
&= \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2.
\end{aligned}
$$

- By continuity of the norm and inner product, we have

$$
\begin{aligned}
\lim_{N\to\infty} \|x - S_N\|^2 &= \lim_{N\to\infty} \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
\implies \left\| x - \lim_{N\to\infty} S_N \right\|^2 &= \|x\|^2 - \lim_{N\to\infty} \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
\implies \left\| x - \sum_{n=1}^{\infty} \langle x,\ u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2.
\end{aligned}
$$

- Then noting that $0 \le \|x - S_N\|^2$,

$$
\begin{aligned}
0 &\le \|x\|^2 - \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2 \\
\implies \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2 &\le \|x\|^2 \blacksquare.
\end{aligned}
$$

$\blacksquare$

Theorem 5.1.3 ***(Riesz Representation for Hilbert Spaces).***
If Λ is a continuous linear functional on a Hilbert space H, then there exists a unique $y \in H$

such that

$$\forall x \in H, \quad \Lambda(x) = \langle x,\ y \rangle.$$

Proof (?).

- Define $M := \ker \Lambda$.
- Then M is a closed subspace and so $H = M \oplus M^{\perp}$
- There is some $z \in M^{\perp}$ such that $\|z\| = 1$.
- Set $u := \Lambda(x)z - \Lambda(z)x$
- Check

$$\Lambda(u) = \Lambda(\Lambda(x)z - \Lambda(z)x) = \Lambda(x)\Lambda(z) - \Lambda(z)\Lambda(x) = 0 \implies u \in M$$

- Compute

$$\begin{aligned}
0 &= \langle u,\ z \rangle \\
&= \langle \Lambda(x)z - \Lambda(z)x,\ z \rangle \\
&= \langle \Lambda(x)z,\ z \rangle - \langle \Lambda(z)x,\ z \rangle \\
&= \Lambda(x)\langle z,\ z \rangle - \Lambda(z)\langle x,\ z \rangle \\
&= \Lambda(x)\|z\|^2 - \Lambda(z)\langle x,\ z \rangle \\
&= \Lambda(x) - \Lambda(z)\langle x,\ z \rangle \\
&= \Lambda(x) - \left\langle x,\ \overline{\Lambda(z)}z \right\rangle,
\end{aligned}$$

- Choose $y := \overline{\Lambda(z)}z$.
- Check uniqueness:

$$\begin{aligned}
\langle x,\ y \rangle &= \langle x,\ y' \rangle \quad \forall x \\
\implies \langle x,\ y - y' \rangle &= 0 \quad \forall x \\
\implies \langle y - y',\ y - y' \rangle &= 0 \\
\implies \|y - y'\| &= 0 \\
\implies y - y' &= \mathbf{0} \implies y = y'.
\end{aligned}$$

■

Theorem 5.1.4 ***(Functionals are continuous if and only if bounded).***
Let $L : X \to \mathbb{C}$ be a linear functional, then the following are equivalent:

1. L is continuous
2. L is continuous at zero

3. L is bounded, i.e. $\exists c \geq 0$ such that $|L(x)| \leq c\|x\|$ for all $x \in H$

Proof (?).
$2 \implies 3$: Choose $\delta < 1$ such that

$$\|x\| \leq \delta \implies |L(x)| < 1.$$

Then

$$\begin{aligned} |L(x)| &= \left| L\left(\frac{\|x\|}{\delta} \frac{\delta}{\|x\|} x \right) \right| \\ &= \frac{\|x\|}{\delta} \left| L\left(\delta \frac{x}{\|x\|} \right) \right| \\ &\leq \frac{\|x\|}{\delta} 1, \end{aligned}$$

so we can take $c = \frac{1}{\delta}$. ■
$3 \implies 1$:
We have $|L(x-y)| \leq c\|x-y\|$, so given $\varepsilon \geq 0$ simply choose $\delta = \frac{\varepsilon}{c}$.
■

Theorem 5.1.5 ***(The operator norm is a norm).***
If H is a Hilbert space, then $(H^\vee, \|-\|_{\mathrm{op}})$ is a normed space.

Proof (?).
The only nontrivial property is the triangle inequality, but

$$\|L_1 + L_2\|_{\mathrm{op}} = \sup |L_1(x) + L_2(x)| \leq \sup |L_1(x)| + |\sup L_2(x)| = \|L_1\|_{\mathrm{op}} + \|L_2\|_{\mathrm{op}}.$$

■

Theorem 5.1.6 ***(The operator norm on $X^\vee$ yields a Banach space).***
If X is a normed vector space, then $(X^\vee, \|-\|_{\mathrm{op}})$ is a Banach space.

Proof (?).

- Let $\{L_n\}$ be Cauchy in $X^\vee$.
- Then for all $x \in C$, $\{L_n(x)\} \subset \mathbb{C}$ is Cauchy and converges to something denoted $L(x)$.
- Need to show L is continuous and $\|L_n - L\| \to 0$.
- Since $\{L_n\}$ is Cauchy in $X^\vee$, choose N large enough so that

$$n, m \geq N \implies \|L_n - L_m\| < \varepsilon \implies |L_m(x) - L_n(x)| < \varepsilon \quad \forall x \;\Big|\; \|x\| = 1.$$

- Take $n \to \infty$ to obtain

$$\begin{aligned} m \geq N &\implies |L_m(x) - L(x)| < \varepsilon \quad \forall x \;\Big|\; \|x\| = 1 \\ &\implies \|L_m - L\| < \varepsilon \to 0. \end{aligned}$$

- Continuity:

$$\begin{aligned} |L(x)| &= |L(x) - L_n(x) + L_n(x)| \\ &\leq |L(x) - L_n(x)| + |L_n(x)| \\ &\leq \varepsilon\|x\| + c\|x\| \\ &= (\varepsilon + c)\|x\|\blacksquare. \end{aligned}$$

$\blacksquare$

Theorem 5.1.7 ***(Riesz-Fischer).***
Let $U = \{u_n\}_{n=1}^{\infty}$ be an orthonormal set (not necessarily a basis), then

1. There is an isometric surjection

$$\begin{aligned} &\mathcal{H} \to \ell^2(\mathbb{N}) \\ &\mathbf{x} \mapsto \{\langle \mathbf{x},\ \mathbf{u}_n \rangle\}_{n=1}^{\infty} \end{aligned}$$

i.e. if $\{a_n\} \in \ell^2(\mathbb{N})$, so $\sum |a_n|^2 < \infty$, then there exists a $\mathbf{x} \in \mathcal{H}$ such that

$$a_n = \langle \mathbf{x},\ \mathbf{u}_n \rangle \quad \forall n.$$

2. $\mathbf{x}$ can be chosen such that

$$\|\mathbf{x}\|^2 = \sum |a_n|^2$$

Note: the choice of $\mathbf{x}$ *is unique* $\iff$ $\{u_n\}$ *is* ***complete****, i.e.* $\langle \mathbf{x},\ \mathbf{u}_n \rangle = 0$ *for all* n *implies* $\mathbf{x} = \mathbf{0}$.

Proof (?).

- Given $\{a_n\}$, define $S_N = \sum^{N} a_n \mathbf{u}_n$.
- S_N is Cauchy in $\mathcal{H}$ and so $S_N \to \mathbf{x}$ for some $\mathbf{x} \in \mathcal{H}$.
- $\langle x,\ u_n \rangle = \langle x - S_N,\ u_n \rangle + \langle S_N,\ u_n \rangle \to a_n$
- By construction, $\|x - S_N\|^2 = \|x\|^2 - \sum^{N} |a_n|^2 \to 0$, so $\|x\|^2 = \sum^{\infty} |a_n|^2$.

$\blacksquare$

6 | Extra Problems: Measure Theory

6.1 Greatest Hits

- $\star$: Show that for $E \subseteq \mathbb{R}^n$, TFAE:

  1. E is measurable
  2. $E = H \cup Z$ here H is F_σ and Z is null
  3. $E = V \setminus Z'$ where $V \in G_\delta$ and Z' is null.

- $\star$: Show that if $E \subseteq \mathbb{R}^n$ is measurable then $m(E) = \sup \left\{ m(K) \mid K \subset E \text{ compact} \right\}$ iff for all $\varepsilon > 0$ there exists a compact $K \subseteq E$ such that $m(K) \geq m(E) - \varepsilon$.

- $\star$: Show that cylinder functions are measurable, i.e. if f is measurable on $\mathbb{R}^s$, then $F(x, y) := f(x)$ is measurable on $\mathbb{R}^s \times \mathbb{R}^t$ for any t.

- $\star$: Prove that the Lebesgue integral is translation invariant, i.e. if $\tau_h(x) = x + h$ then $\displaystyle\int \tau_h f = \int f$.

- $\star$: Prove that the Lebesgue integral is dilation invariant, i.e. if $f_\delta(x) = \dfrac{f(\frac{x}{\delta})}{\delta^n}$ then $\displaystyle\int f_\delta = \int f$.

- $\star$: Prove continuity in L^1, i.e.

$$f \in L^1 \implies \lim_{h \to 0} \int |f(x+h) - f(x)| = 0.$$

- $\star$: Show that

$$f, g \in L^1 \implies f * g \in L^1 \quad \text{and} \quad \|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

- $\star$: Show that if $X \subseteq \mathbb{R}$ with $\mu(X) < \infty$ then

$$\|f\|_p \overset{p \to \infty}{\to} \|f\|_\infty.$$

6.2 By Topic

6.2.1 Topology

- Show that every compact set is closed and bounded.

- Show that if a subset of a metric space is complete and totally bounded, then it is compact.
- Show that if K is compact and F is closed with K, F disjoint then $\operatorname{dist}(K, F) > 0$.

6.2.2 Continuity

- Show that a continuous function on a compact set is uniformly continuous.

6.2.3 Differentiation

- Show that if $f \in C^1(\mathbb{R})$ and both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ exist, then $\lim_{x\to\infty} f'(x)$ must be zero.

6.2.4 Advanced Limitology

- If f is continuous, is it necessarily the case that f' is continuous?
- If $f_n \to f$, is it necessarily the case that f_n' converges to f' (or at all)?
- Is it true that the sum of differentiable functions is differentiable?
- Is it true that the limit of integrals equals the integral of the limit?
- Is it true that a limit of continuous functions is continuous?
- Show that a subset of a metric space is closed iff it is complete.
- Show that if $m(E) < \infty$ and $f_n \to f$ uniformly, then $\lim \int_E f_n = \int_E f$.

Uniform Convergence

- Show that a uniform limit of bounded functions is bounded.
- Show that a uniform limit of continuous function is continuous.
  - I.e. if $f_n \to f$ uniformly with each f_n continuous then f is continuous.
- Show that
  - $f_n : [a, b] \to \mathbb{R}$ are continuously differentiable with derivatives f_n'
  - The sequence of derivatives f_n' converges uniformly to some function g
  - There exists *at least one* point x_0 such that $\lim_n f_n(x_0)$ exists,
  - Then $f_n \to f$ uniformly to some differentiable f, and $f' = g$.
- Prove that uniform convergence implies pointwise convergence implies a.e. convergence, but none of the implications may be reversed.
- Show that $\sum \frac{x^n}{n!}$ converges uniformly on any compact subset of $\mathbb{R}$.

Measure Theory

- Show that continuity of measure from above/below holds for outer measures.

- Show that a countable union of null sets is null.

Measurability

- Show that $f = 0$ a.e. iff $\int_E f = 0$ for every measurable set E.

Integrability

- Show that if f is a measurable function, then $f = 0$ a.e. iff $\int f = 0$.
- Show that a bounded function is Lebesgue integrable iff it is measurable.
- Show that simple functions are dense in L^1.
- Show that step functions are dense in L^1.
- Show that smooth compactly supported functions are dense in L^1.

Convergence

- Prove Fatou's lemma using the Monotone Convergence Theorem.
- Show that if $\{f_n\}$ is in L^1 and $\sum \int |f_n| < \infty$ then $\sum f_n$ converges to an L^1 function and
$$\int \sum f_n = \sum \int f_n.$$

Convolution

- Show that if f, g are continuous and compactly supported, then so is $f * g$.
- Show that if $f \in L^1$ and g is bounded, then $f * g$ is bounded and uniformly continuous.
- If f, g are compactly supported, is it necessarily the case that $f * g$ is compactly supported?
- Show that under any of the following assumptions, $f * g$ vanishes at infinity:
  - $f, g \in L^1$ are both bounded.
  - $f, g \in L^1$ with just g bounded.
  - f, g smooth and compactly supported (and in fact $f * g$ is smooth)
  - $f \in L^1$ and g smooth and compactly supported (and in fact $f * g$ is smooth)
- Show that if $f \in L^1$ and g' exists with $\dfrac{\partial g}{\partial x_i}$ all bounded, then
$$\frac{\partial}{\partial x_i}(f * g) = f * \frac{\partial g}{\partial x_i}$$

Fourier Analysis

- Show that if $f \in L^1$ then $\widehat{f}$ is bounded and uniformly continuous.
- Is it the case that $f \in L^1$ implies $\widehat{f} \in L^1$?
- Show that if $f, \widehat{f} \in L^1$ then f is bounded, uniformly continuous, and vanishes at infinity.
  - Show that this is not true for arbitrary L^1 functions.

- Show that if $f \in L^1$ and $\widehat{f} = 0$ almost everywhere then $f = 0$ almost everywhere.
  - Prove that $\widehat{f} = \widehat{g}$ implies that $f = g$ a.e.
- Show that if $f, g \in L^1$ then

$$\int \widehat{f} g = \int f \widehat{g}.$$

  - Give an example showing that this fails if g is not bounded.
- Show that if $f \in C^1$ then f is equal to its Fourier *series*.

Approximate Identities

- Show that if φ is an approximate identity, then

$$\|f * \varphi_t - f\|_1 \overset{t \to 0}{\to} 0.$$

  - Show that if additionally $|\varphi(x)| \leq c(1 + |x|)^{-n-\varepsilon}$ for some $c, \varepsilon > 0$, then this converges is almost everywhere.
- Show that is f is bounded and uniformly continuous and φ_t is an approximation to the identity, then $f * \varphi_t$ uniformly converges to f.

L^p Spaces

- Show that if $E \subseteq \mathbb{R}^n$ is measurable with $\mu(E) < \infty$ and $f \in L^p(X)$ then

$$\|f\|_{L^p(X)} \overset{p \to \infty}{\to} \|f\|_\infty.$$

- Is it true that the converse to the DCT holds? I.e. if $\int f_n \to \int f$, is there a $g \in L^p$ such that $f_n < g$ a.e. for every n?
- Prove continuity in L^p: If f is uniformly continuous then for all p,

$$\|\tau_h f - f\|_p \overset{h \to 0}{\to} 0.$$

- Prove the following inclusions of L^p spaces for $m(X) < \infty$:

$$L^\infty(X) \subset L^2(X) \subset L^1(X)$$
$$\ell^2(\mathbb{Z}) \subset \ell^1(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z}).$$

6.2.5 Unsorted

Proposition 6.2.1 ***(Volumes of Rectangles).***

If $\{R_j\} \rightrightarrows R$ is a covering of R by rectangles,

$$R = \coprod_j^{\circ} R_j \implies |R| = \sum |R|_j$$
$$R \subseteq \bigcup_j R_j \implies |R| \leq \sum |R|_j.$$

- Show that any disjoint intervals is countable.
- Show that every open $U \subseteq \mathbb{R}$ is a countable union of disjoint open intervals.
- Show that every open $U \subseteq \mathbb{R}^n$ is a countable union of *almost* disjoint closed cubes.
- Show that that Cantor middle-thirds set is compact, totally disconnected, and perfect, with outer measure zero.
- Prove the Borel-Cantelli lemma.

6.3 Rectangles

6.4 Outer Measure

6.5 Lebesgue Measurable Sets

6.6 Lebesgue Measurable Functions

7 Extra Problems from Problem Sets

7.1 2010 6.1

Show that

$$\int_{\mathbb{B}^n} \frac{1}{|x|^p} \, dx < \infty \iff p < n$$
$$\int_{\mathbb{R}^n \setminus \mathbb{B}^n} \frac{1}{|x|^p} \, dx < \infty \iff p > n.$$

7.2 2010 6.2

Show that

$$\int_{\mathbb{R}^n} |f| = \int_0^\infty m(A_t)\, dt \qquad\qquad A_t := \left\{ x \in \mathbb{R}^n \;\middle|\; |f(x)| > t \right\}.$$

7.3 2010 6.5

Suppose $F \subseteq \mathbb{R}$ with $m(F^c) < \infty$ and let $\delta(x) := d(x, F)$ and

$$I_F(x) := \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2}\, dy.$$

a. Show that δ is continuous.

b. Show that if $x \in F^c$ then $I_F(x) = \infty$.

c. Show that $I_F(x) < \infty$ for almost every x

7.4 2010 7.1

Let $(X, \mathcal{M}, \mu)$ be a measure space and prove the following properties of $L^\infty(X, \mathcal{M}, \mu)$:

- If f, g are measurable on X then

$$\|fg\|_1 \leq \|f\|_1 \|g\|_\infty.$$

- $\|-\|_\infty$ is a norm on L^∞ making it a Banach space.
- $\|f_n - f\|_\infty \overset{n\to\infty}{\to} 0 \iff$ there exists an $E \in \mathcal{M}$ such that $\mu(X \setminus E) = 0$ and $f_n \to f$ uniformly on E.
- Simple functions are dense in L^∞.

7.5 2010 7.2

Show that for $0 < p < q \leq \infty$, $\|a\|_{\ell^q} \leq \|a\|_{\ell^p}$ over $\mathbb{C}$, where $\|a\|_\infty := \sup_j |a_j|$.

7.6 2010 7.3

Let f, g be non-negative measurable functions on $[0, \infty)$ with

$$A := \int_0^\infty f(y) y^{-1/2}\, dy < \infty$$
$$B := \left(\int_0^\infty |g(y)| \right)^2 dy < \infty.$$

Show that

$$\int_0^\infty \left(\int_0^\infty f(y)\, dy \right) \frac{g(x)}{x}\, dx \leq AB.$$

7.7 2010 7.4

Let $(X, \mathcal{M}, \mu)$ be a measure space and $0 < p < q < \infty$. Prove that if $L^q(X) \subseteq L^p(X)$, then X does not contain sets of arbitrarily large finite measure.

7.8 2010 7.5

Suppose $0 < a < b \leq \infty$, and find examples of functions $f \in L^p((0, \infty))$ if and only if:

- $a < p < b$
- $a \leq p \leq b$
- $p = a$

Hint: consider functions of the following form:

$$f(x) := x^{-\alpha} |\log(x)|^\beta.$$

7.9 2010 7.6

Define

$$F(x) := \left(\frac{\sin(\pi x)}{\pi x}\right)^2$$
$$G(x) := \begin{cases} 1-|x| & |x| \le 1 \\ 0 & \text{else.} \end{cases}$$

a. Show that $\widehat{G}(\xi) = F(\xi)$

b. Compute $\widehat{F}$.

c. Give an example of a function $g \notin L^1(\mathbb{R})$ which is the Fourier transform of an L^1 function.

Hint: write $\widehat{G}(\xi) = H(\xi) + H(-\xi)$ *where*

$$H(\xi) := e^{2\pi i\xi} \int_0^1 y e^{2\pi i y\xi}\, dy.$$

7.10 2010 7.7

Show that for each $\epsilon > 0$ the following function is the Fourier transform of an $L^1(\mathbb{R}^n)$ function:

$$F(\xi) := \left(\frac{1}{1+|\xi|^2}\right)^{\epsilon}.$$

Hint: show that

$$\begin{aligned} K_\delta(x) &:= \delta^{-n/2} e^{\frac{-\pi|x|^2}{\delta}} \\ f(x) &:= \int_0^\infty K_\delta(x) e^{-\pi\delta} \delta^{\epsilon-1}\, d\delta \\ \Gamma(s) &:= \int_0^\infty e^{-t} t^{s-1}\, dt \\ \implies \widehat{f}(\xi) &= \int_0^\infty e^{-\pi\delta|\xi|^2} e^{-\pi\delta} \delta^{\epsilon-1} = \pi^{-s}\Gamma(\epsilon)F(\xi). \end{aligned}$$

7.11 2010 7 Challenge 1: Generalized Holder

Suppose that

$$1 \le p_j \le \infty, \qquad \sum_{j=1}^{n} \frac{1}{p_j} = \frac{1}{r} \le 1.$$

Show that if $f_j \in L^{p_j}$ for each $1 \le j \le n$, then

$$\prod f_j \in L^r, \qquad \left\|\prod f_j\right\|_r \le \prod \|f_j\|_{p_j}.$$

7.12 2010 7 Challenge 2: Young's Inequality

Suppose $1 \le p, q, r \le \infty$ with

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

Prove that

$$f \in L^p, g \in L^q \implies f * g \in L^r \text{ and } \|f * g\|_r \le \|f\|_p \|g\|_q.$$

7.13 2010 9.1

Show that the set $\{u_k(j) \coloneqq \delta_{ij}\} \subseteq \ell^2(\mathbb{Z})$ and forms an orthonormal system.

7.14 2010 9.2

Consider $L^2([0,1])$ and define

$$\begin{aligned} e_0(x) &= 1 \\ e_1(x) &= \sqrt{3}(2x - 1). \end{aligned}$$

a. Show that $\{e_0, e_1\}$ is an orthonormal system.

b. Show that the polynomial $p(x)$ where $\deg(p) = 1$ which is closest to $f(x) = x^2$ in $L^2([0,1])$ is given by

$$h(x) = x - \frac{1}{6}.$$

Compute $\|f - g\|_2$.

7.15 2010 9.3

Let $E \subseteq H$ a Hilbert space.

a. Show that $E \perp\subseteq H$ is a closed subspace.

b. Show that $(E^\perp)^\perp = \mathrm{cl}_H(E)$.

7.16 2010 9.5b

Let $f \in L^1((0, 2\pi))$.

i. Show that for an $\epsilon > 0$ one can write $f = g + h$ where $g \in L^2((0, 2\pi))$ and $\|H\|_1 < \epsilon$.

7.17 2010 9.6

Prove that every closed convex $K \subset H$ a Hilbert space has a unique element of minimal norm.

7.18 2010 9 Challenge

Let U be a unitary operator on H a Hilbert space, let $M := \left\{x \in H \,\middle|\, Ux = x\right\}$, let P be the orthogonal projection onto M, and define

$$S_N := \frac{1}{N} \sum_{n=0}^{N-1} U^n.$$

Show that for all $x \in H$,

$$\|S_N x - Px\|_H \overset{N\to\infty}{\to} 0.$$

7.19 2010 10.1

Let ν, μ be signed measures, and show that

$$\nu \perp \mu \text{ and } \nu \ll |\mu| \implies \nu = 0.$$

7.20 2010 10.2

Let $f \in L^1(\mathbb{R}^n)$ with $f \neq 0$.

a. Prove that there exists a $c > 0$ such that

$$Hf(x) \geq \frac{c}{(1+|x|)^n}.$$

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Consider the function

$$f(x) := \begin{cases} \dfrac{1}{|x|\left(\log\left(\frac{1}{x}\right)\right)^2} & |x| \leq \dfrac{1}{2} \\ 0 & \text{else.} \end{cases}$$

a. Show that $f \in L^1(\mathbb{R})$.

b. Show that there exists a $c > 0$ such that for all $|x| \leq 1/2$,

$$Hf(x) \geq \frac{c}{|x| \log\left(\frac{1}{|x|}\right)}.$$

Conclude that Hf is not locally integrable.

7.22 2010 10.4

Let $f \in L^1(\mathbb{R})$ and let $\mathcal{U} := \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; y > 0\right\}$ denote the upper half plane. For $(x,y) \in \mathcal{U}$ define

$$u(x,y) := f * P_y(x) \qquad \text{where } P_y(x) := \frac{1}{\pi}\left(\frac{y}{t^2+y^2}\right).$$

a. Prove that there exists a constant C independent of f such that for all $x \in \mathbb{R}$,

$$\sup_{y>0} |u(x,y)| \leq C \cdot Hf(x).$$

Hint: write the following and try to estimate each term:

$$u(x,y) = \int_{|t|<y} f(x-t)P_y(t)\,dt + \sum_{k=0}^{\infty} \int_{A_k} f(x-t)P_y(t)\,dt \quad A_k := \left\{2^k y \leq |t| < 2^{k+1} y\right\}.$$

b. Following the proof of the Lebesgue differentiation theorem, show that for $f \in L^1(\mathbb{R})$ and for almost every $x \in \mathbb{R}$,

$$u(x,y) \overset{y \to 0}{\to} f(x).$$

8 | Common Inequalities

8.1 The GOATs

Proposition 8.1.1 ***(Cauchy-Schwarz Inequality).***

$$|\langle f,\ g \rangle| = \leq \|f\|_2 \|g\|_2 \qquad \text{with equality} \iff f = \lambda g.$$

Remark 8.1.2 ***(Different forms of CS):*** In general, Cauchy-Schwarz relates inner product to norm, and only happens to relate norms in L^1. Some other useful forms:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$
$$\left|\int_{\mathbb{R}^n} f(x)\overline{g(x)}dx\right|^2 \leq \int_{\mathbb{R}^n} |f(x)|^2 dx \int_{\mathbb{R}^n} |g(x)|^2 dx.$$

Proposition 8.1.3 ***(Reverse Triangle Inequality).***

$$|\|x\| - \|y\|| \leq \|x-y\|.$$

Proposition 8.1.4 ***(Holder's Inequality).***

$$\frac{1}{p}+\frac{1}{q}=1 \implies \|fg\|_1 \le \|f\|_p\|g\|_q.$$

With integrals:

$$\int_X |fg| \le \left(\int_X |f|^p\right)^{\frac{1}{p}} \left(\int_X |f|^q\right)^{\frac{1}{q}}.$$

Proof (of Holder's inequality).
It suffices to show this when $\|f\|_p = \|g\|_q = 1$, since

$$\|fg\|_1 \le \|f\|_p\|f\|_q \iff \int \frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_q} \le 1.$$

Using $AB \le \frac{1}{p}A^p + \frac{1}{q}B^q$, we have

$$\int |f||g| \le \int \frac{|f|^p}{p}\frac{|g|^q}{q} = \frac{1}{p}+\frac{1}{q} = 1.$$

■

Example 8.1.5 ***(Application of Holder's inequality: containment of L^p spaces):*** For finite measure spaces,

$$1 \le p < q \le \infty \implies L^q \subset L^p \quad (\text{ and } \ell^p \subset \ell^q).$$

Proof (of containment of L^p spaces).
Fix p, q, let $r = \frac{q}{p}$ and $s = \frac{r}{r-1}$ so $r^{-1}+s^{-1}=1$. Then let $h = |f|^p$:

$$\|f\|_p^p = \|h \cdot 1\|_1 \le \|1\|_s \|h\|_r = \mu(X)^{\frac{1}{s}}\|f\|_q^{\frac{q}{r}} \implies \|f\|_p \le \mu(X)^{\frac{1}{p}-\frac{1}{q}}\|f\|_q.$$

Note: doesn't work for ℓ_p spaces, but just note that $\sum |x_n| < \infty \implies x_n < 1$ for large enough n, and thus $p < q \implies |x_n|^q \le |x_n|^q$.

■

Proposition 8.1.6 ***(Bessel's Inequality).***
For $x \in H$ a Hilbert space and $\{e_k\}$ an orthonormal sequence,

$$\sum_{k=1}^{\infty} \|\langle x,\ e_k\rangle\|^2 \le \|x\|^2.$$

Note that this does not need to be a basis.

Proposition 8.1.7 ***(Parseval's Identity).***
Equality in Bessel's inequality, attained when $\{e_k\}$ is a *basis*, i.e. it is complete, i.e. the span of its closure is all of H. This states that if $\{e_k\}$ is an orthonormal basis for H, then

$$\sum_{k\geq 0} |\langle x,\ e_k\rangle|^2 = \|x\|_H^2.$$

Remark 8.1.8: This appears in several other forms:

$$\frac{1}{2\pi}\int_{(-\pi,\pi)} |f|^2 = \sum_{k\in\mathbb{Z}} |c_k|^2 \qquad\qquad c_k := \frac{1}{2\pi}\int_{(-\pi,\pi)} f(x)e^{-ikx}\,dx.$$

Proposition 8.1.9 ***(Plancherel).***

$$\|f\|_{L^2}^2 = \left\|\widehat{f}\right\|_{L^2}$$
$$\int_{\mathbb{R}^d} |f|^2 = \int_{\mathbb{R}^d} \left|\widehat{f}\right|^2.$$

8.2 Less common

Proposition 8.2.1 ***(Markov/Chebyshev's Inequality).***
The most often used form here:

$$\mu\left(f^{-1}\left((\alpha,\infty)\right)\right) := \mu\left(\left\{x\in X \;\middle|\; |f(x)| > \alpha\right\}\right) \leq \frac{1}{\alpha}\|f\|_1 := \frac{1}{\alpha}\int_X |f|.$$

Proof: let S_α be the set appearing, then $\alpha\mu(S_\alpha)$ is the sum of areas of certain boxes below the graph of f. Interpret $\displaystyle\int_X f$ as the total area under the graph to make the inequality obvious.

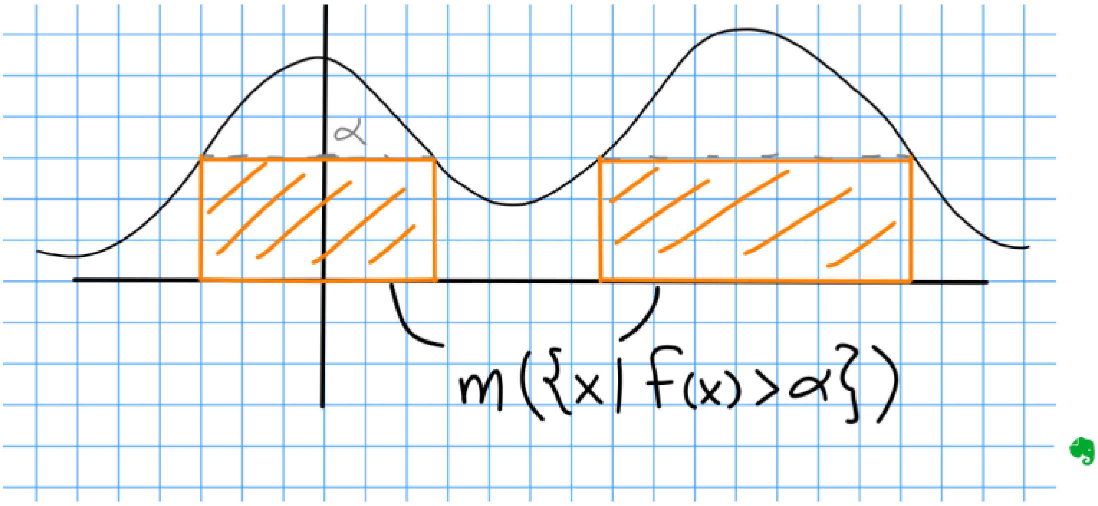

Figure 6: image_2021-06-02-22-59-46

The probability interpretation: $\mathbb{P}(X \geq \alpha) \leq \frac{1}{\alpha}\mathbb{E}(X)$.
The more general version:

$$\mu\left(f^{-1}\left((\alpha, \infty)\right)\right) := \mu\left(\left\{x \in X \;\middle|\; |f(x)| > \alpha\right\}\right) \leq \frac{1}{\alpha^p}\|f\|_p^p := \frac{1}{\alpha^p}\int_X |f|^p.$$

Proof:

$$\|f\|_p^p = \int |f|^p \geq \int_{S_\alpha} |f|^p \geq \alpha^p \int_{S_\alpha} 1 = \alpha^p \mu(S_\alpha).$$

Proposition 8.2.2 ***(Minkowski's Inequality).***

$$1 \leq p < \infty \implies \|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Remark 8.2.3: This does not handle $p = \infty$ case. Use to prove L^p is a normed space.

Proof (of Minkowski's inequality).

- We first note

$$|f+g|^p = |f+g||f+g|^{p-1} \leq \left(|f| + |g|\right)|f+g|^{p-1}.$$

- Note that if p, q are conjugate exponents then

$$\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$$
$$q = \frac{p}{p-1}.$$

- Then taking integrals yields

$$
\begin{aligned}
\|f+g\|_p^p &= \int |f+g|^p \\
&\leq \int (|f|+|g|)\,|f+g|^{p-1} \\
&= \int |f||f+g|^{p-1} + \int |g||f+g|^{p-1} \\
&= \left\| f(f+g)^{p-1} \right\|_1 + \left\| g(f+g)^{p-1} \right\|_1 \\
&\leq \|f\|_p \left\| (f+g)^{p-1}) \right\|_q + \|g\|_p \left\| (f+g)^{p-1}) \right\|_q \\
&= \left(\|f\|_p + \|g\|_p \right) \left\| (f+g)^{p-1}) \right\|_q \\
&= \left(\|f\|_p + \|g\|_p \right) \left(\int |f+g|^{(p-1)q} \right)^{\frac{1}{q}} \\
&= \left(\|f\|_p + \|g\|_p \right) \left(\int |f+g|^{p} \right)^{1-\frac{1}{p}} \\
&= \left(\|f\|_p + \|g\|_p \right) \frac{\int |f+g|^p}{\left(\int |f+g|^p \right)^{\frac{1}{p}}} \\
&= \left(\|f\|_p + \|g\|_p \right) \frac{\|f+g\|_p^p}{\|f+g\|_p}.
\end{aligned}
$$

- Cancelling common terms yields

$$
\begin{aligned}
1 &\leq \left(\|f\|_p + \|g\|_p \right) \frac{1}{\|f+g\|_p} \\
&\implies \|f+g\|_p \leq \|f\|_p + \|g\|_p.
\end{aligned}
$$

■

Proposition 8.2.4 ***(Young's Inequality).***

$$
\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \implies \|f * g\|_r \leq \|f\|_p \|g\|_q
$$

Remark 8.2.5 ***(some useful special cases):***

$$
\begin{aligned}
\|f * g\|_1 &\leq \|f\|_1 \|g\|_1 \\
\|f * g\|_p &\leq \|f\|_1 \|g\| p, \\
\|f * g\|_\infty &\leq \|f\|_2 \|g\|_2 \\
\|f * g\|_\infty &\leq \|f\|_p \|g\|_q.
\end{aligned}
$$

8.3 Inequalities that appear in proofs

Proposition 8.3.1 ***(AM-GM Inequality).***

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Proposition 8.3.2 ***(Jensen's Inequality).***

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Proposition 8.3.3 ***(Young's Product Inequality).***

$$AB \leq \frac{A^p}{p} + \frac{B^q}{q}.$$

Proposition 8.3.4 ***(?).***

$$(a+b)^p \leq 2^{p-1}(a^p + b^p).$$

Proposition 8.3.5 ***(Bernoulli's Inequality).***

$$(1+x)^n \geq 1 + nx \quad x \geq -1, \text{ or } n \in 2\mathbb{Z} \text{ and } \forall x.$$

As a consequence,

$$1 - x \leq e^{-x}.$$

Proposition 8.3.6 ***(Exponential Inequality).***

$$\forall t \in \mathbb{R}, \quad 1 + t \leq e^t.$$

Proof .

- It's an equality when $t = 0$.
- $\frac{\partial}{\partial t} 1 + t < \frac{\partial t}{\partial e}^t \iff t < 0$

∎

Proposition 8.3.7 ***(Young's Convolution Inequality).***

$$\frac{1}{r} := \frac{1}{p} + \frac{1}{q} - 1 \implies \|f * g\|_r \leq \|f\|_p \|g\| q.$$