

# Real Analysis Qualifying Exam Notes

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## 1 Practice Exam 2 (November 2014)

### 1.1 1: Fubini-Tonelli

#### 1.1.1 a

Carefully state Tonelli's theorem for a nonnegative function  $F(x, t)$  on  $\mathbb{R}^n \times \mathbb{R}$ .

#### 1.1.2 b

Let  $f : \mathbb{R}^n \rightarrow [0, \infty]$  and define

$$\mathcal{A} := \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid 0 \leq t \leq f(x) \right\}.$$

Prove the validity of the following two statements:

1.  $f$  is Lebesgue measurable on  $\mathbb{R}^n \iff \mathcal{A}$  is a Lebesgue measurable subset of  $\mathbb{R}^{n+1}$ .
2. If  $f$  is Lebesgue measurable on  $\mathbb{R}^n$  then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x) dx = \int_0^\infty m\left(\left\{x \in \mathbb{R}^n \mid f(x) \geq t\right\}\right) dt.$$

### 1.2 2: Convolutions and the Fourier Transform

#### 1.2.1 a

Let  $f, g \in L^1(\mathbb{R}^n)$  and give a definition of  $f * g$ .

#### 1.2.2 b

Prove that if  $f, g$  are integrable and bounded, then

$$(f * g)(x) \xrightarrow{|x| \rightarrow \infty} 0.$$

**1.2.3 c****Definition (The Fourier Transform):**

$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi} dx.$$

**Facts:**

- *Riemann-Lebesgue lemma:*  $\widehat{f}$  vanishes at infinity
- $f \in L^1 \implies \widehat{f}$  is bounded and uniformly continuous
- $f, \widehat{f} \in L^1 \implies f$  is bounded, uniformly continuous, and vanishes at infinity
- $f \in L^1$  and  $\widehat{f} = 0$  almost everywhere  $\implies f = 0$  almost everywhere.

**Theorem (Fourier Inversion):**

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

*Proof:* Idea: Fubini-Tonelli doesn't work directly, so introduce a convergence factor, take limits, and use uniqueness of limits.

Use the following facts:

- $f, g \in L^1 \implies \int \widehat{f}g = \int f\widehat{g}$ .
- $g(x) := e^{-\pi|x|^2} \implies \widehat{g}(\xi) = g(\xi)$
- $g_t(x) = t^{-n}g(x/t) = t^{-n}e^{-\pi|x|^2/t^2}$
- $\widehat{g}_t(x) = g(tx) = e^{-\pi t^2|x|^2}$
- $\varphi(\xi) := e^{2\pi i x \cdot \xi} \widehat{g}_t(\xi)$
- $\widehat{\varphi}(\xi) = \mathcal{F}(\widehat{g}_t(\xi - x)) = g_t(x - \xi)$
- $\lim_{t \rightarrow 0} \varphi(\xi) = e^{2\pi i x \cdot \xi}$

Take the modified integral:

$$\begin{aligned}
 I_t(x) &= \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi} e^{-\pi t^2|\xi|^2} \\
 &= \int \widehat{f}(\xi) \varphi(\xi) \\
 &= \int f(\xi) \widehat{\varphi}(\xi) \\
 &= \int f(\xi) \widehat{g}_t(\xi - x) \\
 &= \int f(\xi) g_t(x - \xi) d\xi \\
 &= \int f(y - x) g_t(y) dy \quad (\xi = y - x) \\
 &= (f * g_t) \\
 &\longrightarrow f \text{ in } L^1 \text{ as } t \longrightarrow 0,
 \end{aligned}$$

but we also have

$$\begin{aligned}
 \lim_{t \rightarrow 0} I_t(x) &= \lim_{t \rightarrow 0} \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi} e^{-\pi t^2 |\xi|^2} \\
 &= \lim_{t \rightarrow 0} \int \widehat{f}(\xi) \varphi(\xi) \\
 &=_{DCT} \int \widehat{f}(\xi) \lim_{t \rightarrow 0} \varphi(\xi) \\
 &= \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi}
 \end{aligned}$$

So

$$I_t(x) \longrightarrow \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi} \text{ pointwise and } \|I_t(x) - f(x)\|_1 \longrightarrow 0.$$

So there is a subsequence  $I_{t_n}$  such that  $I_{t_n}(x) \longrightarrow f(x)$  almost everywhere, so  $f(x) = \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi}$  almost everywhere by uniqueness of limits. ■

### 1.3 3: Hilbert Spaces

#### 1.3.1 a

**Theorem (Bessel's Inequality):**

$$\left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2$$

and thus

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

*Proof:* Let  $S_N = \sum_{n=1}^N \langle x, u_n \rangle u_n$

$$\begin{aligned}
\|x - S_N\|^2 &= \langle x - S_N, x - S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \langle x, S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \left\langle x, \sum_{n=1}^N \langle x, u_n \rangle u_n \right\rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^N \langle x, \langle x, u_n \rangle u_n \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^N \overline{\langle x, u_n \rangle} \langle x, u_n \rangle \\
&= \|x\|^2 + \left\| \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|^2 - 2 \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\
&= \|x\|^2 + \sum_{n=1}^N |\langle x, u_n \rangle|^2 - 2 \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\
&= \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2.
\end{aligned}$$

And by continuity of the norm and inner product, we have

$$\begin{aligned}
\lim_{N \rightarrow \infty} \|x - S_N\|^2 &= \lim_{N \rightarrow \infty} \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\
\Rightarrow \left\| x - \lim_{N \rightarrow \infty} S_N \right\|^2 &= \|x\|^2 - \lim_{N \rightarrow \infty} \sum_{n=1}^N |\langle x, u_n \rangle|^2 \\
\Rightarrow \left\| x - \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2.
\end{aligned}$$

Then noting that  $0 \leq \|x - S_N\|^2$ , we have

$$\begin{aligned}
0 &\leq \|x\|^2 - \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \\
\Rightarrow \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 &\leq \|x\|^2 \blacksquare.
\end{aligned}$$

### 1.3.2 b

- Take  $\{a_n\} \in \ell^2$ , then note that  $\sum |a_n|^2 < \infty \Rightarrow$  the tails vanish.
- Define  $x = \lim_{N \rightarrow \infty} S_N$  where  $S_N = \sum_{k=1}^N a_k u_k$

- $\{S_N\}$  is Cauchy and  $H$  is complete, so  $x \in H$ .
- By construction,  $\langle x, u_n \rangle = \left\langle \sum_k a_k u_k, u_n \right\rangle = \sum_k a_k \langle u_k, u_n \rangle = a_n$  since the  $u_k$  are all orthogonal.
- $\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$  by Pythagoras since the  $u_k$  are normal.

**1.3.3 c**

Let  $x$  and  $u_n$  be arbitrary. Then

$$\begin{aligned}
 \left\langle x - \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k, u_n \right\rangle &= \langle x, u_n \rangle - \left\langle \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k, u_n \right\rangle \\
 &= \langle x, u_n \rangle - \sum_{k=1}^{\infty} \langle \langle x, u_k \rangle u_k, u_n \rangle \\
 &= \langle x, u_n \rangle - \sum_{k=1}^{\infty} \langle x, u_k \rangle \langle u_k, u_n \rangle \\
 &= \langle x, u_n \rangle - \langle x, u_n \rangle = 0 \\
 \implies x - \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k &= 0 \quad \text{by completeness.}
 \end{aligned}$$

So

$$x = \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k \implies \|x\|^2 = \sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2. \blacksquare$$

**1.4 4: Lp Spaces**

p-test for integrals:

$$\begin{aligned}
 \int_0^1 x^{-p} < \infty &\iff p < 1 \\
 \int_1^{\infty} x^{-p} < \infty &\iff p > 1.
 \end{aligned}$$

Yields a general technique: break integrals apart at  $x = 1$ .

Inclusions and relationships:

$$\begin{aligned}
 m(X) < \infty &\implies L^\infty \subset L^2 \subset L^1 \\
 \ell^2(\mathbb{Z}) &\subset \ell^1(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z}).
 \end{aligned}$$

**1.4.1 a****Theorem (Holder's Inequality):**

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

*Proof:*

It suffices to show this when  $\|f\|_p = \|g\|_q = 1$ , since

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \iff \int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \leq 1.$$

Using  $AB \leq \frac{1}{p}A^p + \frac{1}{q}B^q$ , we have

$$\int |f| |g| \leq \int \frac{|f|^p}{p} \frac{|g|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1 \blacksquare.$$

**Theorem (Minkowski's Inequality):**

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

*Proof:*

We first note

$$|f + g|^p = |f + g| |f + g|^{p-1} \leq (|f| + |g|) |f + g|^{p-1}.$$

Note that if  $p, q$  are conjugate exponents then

$$\begin{aligned} \frac{1}{q} &= 1 - \frac{1}{p} = \frac{p-1}{p} \\ q &= \frac{p}{p-1}. \end{aligned}$$

Then taking integrals yields

$$\begin{aligned}
\|f + g\|_p^p &= \int |f + g|^p \\
&\leq \int (|f| + |g|) |f + g|^{p-1} \\
&= \int |f| |f + g|^{p-1} + \int |g| |f + g|^{p-1} \\
&= \|f(f + g)^{p-1}\|_1 + \|g(f + g)^{p-1}\|_1 \\
&\leq \|f\|_p \|(f + g)^{p-1}\|_q + \|g\|_p \|(f + g)^{p-1}\|_q \\
&= (\|f\|_p + \|g\|_p) \|(f + g)^{p-1}\|_q \\
&= (\|f\|_p + \|g\|_p) \left( \int |f + g|^{(p-1)q} \right)^{\frac{1}{q}} \\
&= (\|f\|_p + \|g\|_p) \left( \int |f + g|^p \right)^{1 - \frac{1}{p}} \\
&= (\|f\|_p + \|g\|_p) \frac{\int |f + g|^p}{(\int |f + g|^p)^{\frac{1}{p}}} \\
&= (\|f\|_p + \|g\|_p) \frac{\|f + g\|_p^p}{\|f + g\|_p}
\end{aligned}$$

and canceling common terms yields

$$\begin{aligned}
1 &\leq (\|f\|_p + \|g\|_p) \frac{1}{\|f + g\|_p} \\
&\implies \|f + g\|_p \leq \|f\|_p + \|g\|_p \blacksquare.
\end{aligned}$$

### 1.4.2 c

**Definition (Infinity Norm):**

$$\begin{aligned}
L^\infty(X) &= \left\{ f : X \longrightarrow \mathbb{C} \mid \|f\|_\infty < \infty \right\} \\
&\text{where} \\
\|f\|_\infty &= \inf_{\alpha \geq 0} \left\{ \alpha \mid m\{|f| \geq \alpha\} = 0 \right\}.
\end{aligned}$$

**Theorem:**

$$m(X) < \infty \implies \lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty.$$

*Proof:* Let  $M = \|f\|_\infty$ . For any  $L < M$ , let  $S = \{|f| \geq L\}$ . Then  $m(S) > 0$  and



$$\begin{aligned}
\|f\|_p &= \left( \int_X |f|^p \right)^{\frac{1}{p}} \\
&\geq \left( \int_S |f|^p \right)^{\frac{1}{p}} \\
&\geq L m(S)^{\frac{1}{p}} \xrightarrow{p \rightarrow \infty} L \\
&\implies \liminf_p \|f\|_p \geq M.
\end{aligned}$$

We also have

$$\begin{aligned}
\|f\|_p &= \left( \int_X |f|^p \right)^{\frac{1}{p}} \\
&\leq \left( \int_X M^p \right)^{\frac{1}{p}} \\
&= M m(X)^{\frac{1}{p}} \xrightarrow{p \rightarrow \infty} M \\
&\implies \limsup_p \|f\|_p \leq M \blacksquare.
\end{aligned}$$

Note: this doesn't help with this problem at all.

Solution:

$$\begin{aligned}
\int f^p &= \int_{x \leq 1} f^p + \int_{x=1} f^p + \int_{x \geq 1} f^p \\
&= \int_{x \leq 1} f^p + \int_{x=1} 1 + \int_{x \geq 1} f^p \\
&= \int_{x \leq 1} f^p + m(\{f = 1\}) + \int_{x \geq 1} f^p \\
&\xrightarrow{p \rightarrow \infty} 0 + m(\{f = 1\}) + \begin{cases} 0 & m(\{x \geq 1\}) = 0 \\ \infty & m(\{x \geq 1\}) > 0. \end{cases}
\end{aligned}$$

## 1.5 5: Dual Spaces

**Definition:** A map  $L : X \rightarrow \mathbb{C}$  is a linear functional iff

$$L(\alpha \mathbf{x} + \mathbf{y}) = \alpha L(\mathbf{x}) + L(\mathbf{y}).$$

**Theorem (Riesz Representation for Hilbert Spaces):** If  $\Lambda$  is a continuous linear functional on a Hilbert space  $H$ , then there exists a unique  $y \in H$  such that

$$\forall x \in H, \quad \Lambda(x) = \langle x, y \rangle.$$

*Proof:*

- Define  $M := \ker \Lambda$ .
- Then  $M$  is a closed subspace and so  $H = M \oplus M^\perp$
- There is some  $z \in M^\perp$  such that  $\|z\| = 1$ .
- Set  $u := \Lambda(x)z - \Lambda(z)x$
- Check

$$\Lambda(u) = \Lambda(\Lambda(x)z - \Lambda(z)x) = \Lambda(x)\Lambda(z) - \Lambda(z)\Lambda(x) = 0 \implies u \in M$$

- Compute

$$\begin{aligned} 0 &= \langle u, z \rangle \\ &= \langle \Lambda(x)z - \Lambda(z)x, z \rangle \\ &= \langle \Lambda(x)z, z \rangle - \langle \Lambda(z)x, z \rangle \\ &= \Lambda(x)\langle z, z \rangle - \Lambda(z)\langle x, z \rangle \\ &= \Lambda(x)\|z\|^2 - \Lambda(z)\langle x, z \rangle \\ &= \Lambda(x) - \Lambda(z)\langle x, z \rangle \\ &= \Lambda(x) - \langle x, \overline{\Lambda(z)}z \rangle, \end{aligned}$$

- Choose  $y := \overline{\Lambda(z)}z$ .
- Check uniqueness:

$$\begin{aligned} \langle x, y \rangle &= \langle x, y' \rangle \quad \forall x \\ \implies \langle x, y - y' \rangle &= 0 \quad \forall x \\ \implies \langle y - y', y - y' \rangle &= 0 \\ \implies \|y - y'\| &= 0 \\ \implies y - y' &= \mathbf{0} \implies y = y'. \end{aligned}$$

### 1.5.1 b

**Theorem (Continuous iff Bounded):** Let  $L : X \longrightarrow \mathbb{C}$  be a linear functional, then the following are equivalent:

1.  $L$  is continuous
2.  $L$  is continuous at zero
3.  $L$  is bounded, i.e.  $\exists c \geq 0 \mid |L(x)| \leq c\|x\|$  for all  $x \in H$

2  $\implies$  3: Choose  $\delta < 1$  such that

$$\|x\| \leq \delta \implies |L(x)| < 1.$$

Then

$$\begin{aligned} |L(x)| &= \left| L \left( \frac{\|x\|}{\delta} \frac{\delta}{\|x\|} x \right) \right| \\ &= \frac{\|x\|}{\delta} \left| L \left( \delta \frac{x}{\|x\|} \right) \right| \\ &\leq \frac{\|x\|}{\delta} 1, \end{aligned}$$

so we can take  $c = \frac{1}{\delta}$ . ■

3  $\implies$  1:

We have  $|L(x - y)| \leq c\|x - y\|$ , so given  $\varepsilon \geq 0$  simply choose  $\delta = \frac{\varepsilon}{c}$ .

### 1.5.2 c

**Definition (Dual Space):**

$$X^\vee := \left\{ L : X \longrightarrow \mathbb{C} \mid L \text{ is continuous} \right\}$$

**Definition (Operator Norm):**

$$\|L\|_{X^\vee} := \sup_{\substack{x \in X \\ \|x\|=1}} |L(x)|$$

**Theorem: (Operator Norm is a Norm)**

*Proof:* The only nontrivial property is the triangle inequality, but

$$\|L_1 + L_2\| = \sup |L_1(x) + L_2(x)| \leq \sup |L_1(x)| + \sup |L_2(x)| = \|L_1\| + \|L_2\|.$$

**Theorem (Completeness in Operator Norm):**

$X^\vee$  equipped with the operator norm is a Banach space.

*Proof:*

- Let  $\{L_n\}$  be Cauchy in  $X^\vee$ .
- Then for all  $x \in X$ ,  $\{L_n(x)\} \subset \mathbb{C}$  is Cauchy and converges to something denoted  $L(x)$ .
- Need to show  $L$  is continuous and  $\|L_n - L\| \longrightarrow 0$ .
- Since  $\{L_n\}$  is Cauchy in  $X^\vee$ , choose  $N$  large enough so that

$$n, m \geq N \implies \|L_n - L_m\| < \varepsilon \implies |L_m(x) - L_n(x)| < \varepsilon \quad \forall x \mid \|x\| = 1.$$

- Take  $n \longrightarrow \infty$  to obtain

$$\begin{aligned} m \geq N &\implies |L_m(x) - L(x)| < \varepsilon \quad \forall x \mid \|x\| = 1 \\ &\implies \|L_m - L\| < \varepsilon \longrightarrow 0. \end{aligned}$$

- 
- Continuity:

$$\begin{aligned}
 |L(x)| &= |L(x) - L_n(x) + L_n(x)| \\
 &\leq |L(x) - L_n(x)| + |L_n(x)| \\
 &\leq \varepsilon \|x\| + c \|x\| \\
 &= (\varepsilon + c) \|x\| \blacksquare.
 \end{aligned}$$

## 2 Exam 2 (2018)

[Link to PDF File](#)

## 3 Exam 2 (2014)

[Link to PDF File](#)

## 4 Qual: Fall 2019

### 4.1 1

See phone photo?

### 4.2 2

DCT?

### 4.3 3

“Follow your nose.”

### 4.4 4

See Problem Set 8.

**Bessel’s Inequality:** For any orthonormal set in a Hilbert space (not necessarily a basis), we have

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

*Proof:*

$$0 \leq \left\| x - \sum_{k=1}^n \langle x, e_k \rangle e_k \right\|^2$$

**Corollary (Parseval’s Identity):** If  $\text{span}\{u_n\}$  is dense in  $\mathcal{H}$ , so it is a basis, then this is an equality.

**Riesz-Fischer:** Let  $U = \{u_n\}_{n=1}^{\infty}$  be an orthonormal set (not necessarily a basis), then

1. There is an isometric surjection

$$\begin{aligned}\mathcal{H} &\longrightarrow \ell^2(\mathbb{N}) \\ \mathbf{x} &\mapsto \{\langle \mathbf{x}, \mathbf{u}_n \rangle\}_{n=1}^{\infty}\end{aligned}$$

i.e. if  $\{a_n\} \in \ell^2(\mathbb{N})$ , so  $\sum |a_n|^2 < \infty$ , then there exists a  $\mathbf{x} \in \mathcal{H}$  such that

$$a_n = \langle \mathbf{x}, \mathbf{u}_n \rangle \quad \forall n.$$

2.  $\mathbf{x}$  can be chosen such that

$$\|\mathbf{x}\|^2 = \sum |a_n|^2$$

Note: the choice of  $\mathbf{x}$  is unique  $\iff \{u_n\}$  is **complete**, i.e.  $\langle \mathbf{x}, \mathbf{u}_n \rangle = 0$  for all  $n$  implies  $\mathbf{x} = \mathbf{0}$ .

*Proof:*

- Given  $\{a_n\}$ , define  $S_N = \sum_{n=1}^N a_n \mathbf{u}_n$ .
- $S_N$  is Cauchy in  $\mathcal{H}$  and so  $S_N \longrightarrow \mathbf{x}$  for some  $\mathbf{x} \in \mathcal{H}$ .
- $\langle x, u_n \rangle = \langle x - S_N, u_n \rangle + \langle S_N, u_n \rangle \longrightarrow a_n$
- By construction,  $\|x - S_N\|^2 = \|x\|^2 - \sum_{n=1}^N |a_n|^2 \longrightarrow 0$ , so  $\|x\|^2 = \sum_{n=1}^{\infty} |a_n|^2$ .

## 4.5 5

See Problem Set 5.

**Heine-Cantor theorem:** Every continuous function on a compact set is uniformly continuous.

Uniform continuity:

$$\begin{aligned} &\forall \varepsilon \quad \exists \delta(\varepsilon) \mid \quad \forall x, y, \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \\ \iff &\forall \varepsilon \quad \exists \delta(\varepsilon) \mid \quad \forall x, y, \quad |y| < \delta \implies |f(x - y) - f(y)| < \varepsilon \end{aligned}$$

Fubini-Tonelli interchange of integrals, where the change of bounds becomes very important.

Continuity in  $L^1$ :

$$\lim_{y \rightarrow 0} \|\tau_y f - f\|_1 = 0.$$

## 5 Basics

**Useful Technique:**  $\lim f_n = \limsup f_n = \liminf f_n$  iff the limit exists, so  $\limsup f_n \leq g \leq \liminf f_n$  implies that  $g = \lim f$ . Similarly, a limit does not exist iff  $\liminf f_n > \limsup f_n$ .

---

**Lemma 5.1** (*Convergent Sums Have Small Tails*).

$$\sum a_n < \infty \implies a_n \longrightarrow 0 \quad \text{and} \quad \sum_{k=N}^{\infty} a_k \xrightarrow{N \rightarrow \infty} 0$$

**Theorem 5.2** (*Heine-Borel*).

$X \subseteq \mathbb{R}^n$  is compact  $\iff X$  is closed and bounded.

**Lemma 5.3** (*Geometric Series*).

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \iff |x| < 1.$$

*Corollary:*  $\sum_{k=0}^{\infty} \frac{1}{2^k} = 1.$

**Definition 5.3.1.**

A set  $S$  is **nowhere dense** iff the closure of  $S$  has empty interior iff every interval contains a subinterval that does not intersect  $S$ .

**Definition 5.3.2.**

A set is **meager** if it is a *countable* union of nowhere dense sets.

Note that a *finite* union of nowhere dense is still nowhere dense.

**Lemma 5.4.**

The Cantor set is closed with empty interior.

*Proof .*

Its complement is a union of open intervals, and can't contain an interval since intervals have positive measure and  $m(C_n)$  tends to zero.

**Corollary:** The Cantor set is nowhere dense. ■

**Definition 5.4.1.**

An  $F_\sigma$  set is a union of closed sets, and a  $G_\delta$  set is an intersection of opens.

*Mnemonic:* “ $F$ ” stands for ferme, which is “closed” in French, and  $\sigma$  corresponds to a “sum”, i.e. a union.

**Lemma 5.5.**

Singleton sets in  $\mathbb{R}$  are closed, and thus  $\mathbb{Q}$  is an  $F_\sigma$  set.

---

**Theorem 5.6 (Baire).**

$\mathbb{R}$  is a **Baire space** (countable intersections of open, dense sets are still dense). Thus  $\mathbb{R}$  can not be written as a countable union of nowhere dense sets.

**Lemma 5.7.**

There is a function discontinuous precisely on  $\mathbb{Q}$ .

*Proof.*

$f(x) = \frac{1}{n}$  if  $x = r_n \in \mathbb{Q}$  is an enumeration of the rationals, and zero otherwise. The limit at every point is 0. ■

**Lemma 5.8.**

There *do not* exist functions that are discontinuous precisely on  $\mathbb{R} \setminus \mathbb{Q}$ .

*Proof.*

$D_f$  is always an  $F_\sigma$  set, which follows by considering the oscillation  $\omega_f$ .  $\omega_f(x) = 0 \iff f$  is continuous at  $x$ , and  $D_f = \bigcup_n A_{\frac{1}{n}}$  where  $A_\varepsilon = \{\omega_f \geq \varepsilon\}$  is closed. ■

**Lemma 5.9.**

Any nonempty set which is bounded from above (resp. below) has a well-defined supremum (resp. infimum).

## 6 Uniform Convergence

**Theorem 6.1 (Egorov).**

Let  $E \subseteq \mathbb{R}^n$  be measurable with  $m(E) > 0$  and  $\{f_k : E \rightarrow \mathbb{R}\}$  be measurable functions such that

$$f(x) := \lim_{k \rightarrow \infty} f_k(x) < \infty$$

exists almost everywhere.

Then  $f_k \rightarrow f$  *almost uniformly*, i.e.

$$\forall \varepsilon > 0, \exists F \subseteq E \text{ closed such that } m(E \setminus F) < \varepsilon \text{ and } f_k \xrightarrow{u} f \text{ on } F.$$

**Proposition 6.2.**

The space  $X = C([0, 1])$ , continuous functions  $f : [0, 1] \rightarrow \mathbb{R}$ , equipped with the norm  $\|f\| = \sup_{x \in [0, 1]} |f(x)|$ , is a **complete** metric space.

---

*Proof .*

1. Let  $\{f_k\}$  be Cauchy in  $X$ .
2. Define a candidate limit using pointwise convergence:

Fix an  $x$ ; since

$$|f_k(x) - f_j(x)| \leq \|f_k - f_j\| \longrightarrow 0$$

the sequence  $\{f_k(x)\}$  is Cauchy in  $\mathbb{R}$ . So define  $f(x) := \lim_k f_k(x)$ .

3. Show that  $\|f_k - f\| \longrightarrow 0$ :

$$|f_k(x) - f_j(x)| < \varepsilon \quad \forall x \implies \lim_j |f_k(x) - f_j(x)| < \varepsilon \quad \forall x$$

Alternatively,  $\|f_k - f\| \leq \|f_k - f_N\| + \|f_N - f_j\|$ , where  $N, j$  can be chosen large enough to bound each term by  $\varepsilon/2$ .

4. Show that  $f \in X$ :

The uniform limit of continuous functions is continuous. (Note: in other cases, you may need to show the limit is bounded, or has bounded derivative, or whatever other conditions define  $X$ .)

■

**Lemma 6.3.**

Metric spaces are compact iff they are sequentially compact, (i.e. every sequence has a convergent subsequence).

**Proposition 6.4.**

The unit ball in  $C([0, 1])$  with the sup norm is not compact.

*Proof .*

Take  $f_k(x) = x^n$ , which converges to a dirac delta at 1. The limit is not continuous, so no subsequence can converge.

■

**Lemma 6.5.**

A uniform limit of continuous functions is continuous.

**Lemma 6.6 (Testing Uniform Convergence).**

$f_n \longrightarrow f$  uniformly iff there exists an  $M_n$  such that  $\|f_n - f\|_\infty \leq M_n \longrightarrow 0$ .

**Negating:** find an  $x$  which depends on  $n$  for which the norm is bounded below.

**Useful Technique:** If  $f_n$  has a global maximum (computed using  $f'_n$  and the first derivative test)  $M_n \longrightarrow 0$ , then  $f_n \longrightarrow 0$  uniformly.

**Lemma 6.7 (Baby Commuting Limits with Integrals).**

If  $f_n \longrightarrow f$  uniformly, then  $\int f_n = \int f$ .



---

**Lemma 6.8 (Uniform Convergence and Derivatives).**

If  $f'_n \rightarrow g$  uniformly for some  $g$  and  $f_n \rightarrow f$  pointwise (or at least at one point), then  $g = f'$ .

**Lemma 6.9 (Uniform Convergence of Series).**

If  $f_n(x) \leq M_n$  for a fixed  $x$  where  $\sum M_n < \infty$ , then the series  $f(x) = \sum f_n(x)$  converges pointwise.

**Lemma 6.10.**

If  $\sum f_n$  converges then  $f_n \rightarrow 0$  uniformly.

**Useful Technique:** For a fixed  $x$ , if  $f = \sum f_n$  converges *uniformly* on some  $B_r(x)$  and each  $f_n$  is continuous at  $x$ , then  $f$  is also continuous at  $x$ .

**Lemma 6.11 (M-test for Series).**

If  $|f_n(x)| \leq M_n$  which does not depend on  $x$ , then  $\sum f_n$  converges uniformly.

**Lemma 6.12 (p-tests).**

Let  $n$  be a fixed dimension and set  $B = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ .

$$\begin{aligned}\sum \frac{1}{n^p} < \infty &\iff p > 1 \\ \int_{\varepsilon}^{\infty} \frac{1}{x^p} < \infty &\iff p > 1 \\ \int_0^1 \frac{1}{x^p} < \infty &\iff p < 1 \\ \int_B \frac{1}{|x|^p} < \infty &\iff p < n \\ \int_{B^c} \frac{1}{|x|^p} < \infty &\iff p > n\end{aligned}$$

**Proposition 6.13.**

A function  $f : (a, b) \rightarrow \mathbb{R}$  is Lipschitz  $\iff f$  is differentiable and  $f'$  is bounded. In this case,  $|f'(x)| \leq C$ , the Lipschitz constant.

**Proposition 6.14.**

There exist smooth compactly supported functions, e.g. take

$$f(x) = e^{\frac{-1}{x^2}} \chi_{(0, \infty)}(x).$$

## 7 Measure Theory

**Useful Technique:**  $s = \inf \{x \in X\} \implies$  for every  $\varepsilon$  there is an  $x \in X$  such that  $x \leq s + \varepsilon$ .

**Useful Techniques:** Always consider bounded sets, and if  $E$  is unbounded write  $E = \bigcup_n B_n(0) \cap E$  and use countable subadditivity or continuity of measure.

**Lemma 7.1.**

Every open subset of  $\mathbb{R}$  (resp  $\mathbb{R}^n$ ) can be written as a unique countable union of disjoint (resp. almost disjoint) intervals (resp. cubes).

**Definition 7.1.1.**

The outer measure of a set is given by

$$m_*(E) = \inf_{\substack{\{Q_i\} \rightrightarrows E \\ \text{closed cubes}}} \sum |Q_i|.$$

**Lemma 7.2 (Properties of Outer Measure).**

- Monotonicity:  $E \subseteq F \implies m_*(E) \leq m_*(F)$ .
- Countable Subadditivity:  $m_*(\bigcup E_i) \leq \sum m_*(E_i)$ .
- Approximation: For all  $E$  there exists a  $G \supseteq E$  such that  $m_*(G) \leq m_*(E) + \varepsilon$ .
- Disjoint<sup>a</sup> Additivity:  $m_*(A \amalg B) = m_*(A) + m_*(B)$ .

<sup>a</sup>This holds for outer measure iff  $\text{dist}(A, B) > 0$ .

**Lemma 7.3 (Subtraction of Measure).**

$$m(A) = m(B) + m(C) \quad \text{and} \quad m(C) < \infty \implies m(A) - m(C) = m(B).$$

**Lemma 7.4 (Continuity of Measure).**

$$\begin{aligned} E_i \nearrow E &\implies m(E_i) \longrightarrow m(E) \\ m(E_1) < \infty \text{ and } E_i \searrow E &\implies m(E_i) \longrightarrow m(E). \end{aligned}$$

*Proof.*

1. Break into disjoint annuli  $A_2 = E_2 \setminus E_1$ , etc then apply countable disjoint additivity to  $E = \amalg A_i$ .
2. Use  $E_1 = (\amalg E_j \setminus E_{j+1}) \amalg (\bigcap E_j)$ , taking measures yields a telescoping sum, and use countable disjoint additivity. ■

**Theorem 7.5.**

Suppose  $E$  is measurable; then for every  $\varepsilon > 0$ ,

1. There exists an open  $O \supset E$  with  $m(O \setminus E) < \varepsilon$
2. There exists a closed  $F \subset E$  with  $m(E \setminus F) < \varepsilon$

3. There exists a compact  $K \subset E$  with  $m(E \setminus K) < \varepsilon$ .

*Proof .*

- (1): Take  $\{Q_i\} \rightrightarrows E$  and set  $O = \bigcup Q_i$ .
- (2): Since  $E^c$  is measurable, produce  $O \supset E^c$  with  $m(O \setminus E^c) < \varepsilon$ .
  - Set  $F = O^c$ , so  $F$  is closed.
  - Then  $F \subset E$  by taking complements of  $O \supset E^c$
  - $E \setminus F = O \setminus E^c$  and taking measures yields  $m(E \setminus F) < \varepsilon$
- (3): Pick  $F \subset E$  with  $m(E \setminus F) < \varepsilon/2$ .
  - Set  $K_n = F \cap \mathbb{D}_n$ , a ball of radius  $n$  about 0.
  - Then  $E \setminus K_n \searrow E \setminus F$
  - Since  $m(E) < \infty$ , there is an  $N$  such that  $n \geq N \implies m(E \setminus K_n) < \varepsilon$ .

■

### Lemma 7.6.

Lebesgue measure is translation and dilation invariant.

*Proof .*

Obvious for cubes; if  $Q_i \rightrightarrows E$  then  $Q_i + k \rightrightarrows E + k$ , etc.

■

### Theorem 7.7 (Non-Measurable Sets).

There is a non-measurable set.

*Proof .*

- Use AOC to choose one representative from every coset of  $\mathbb{R}/\mathbb{Q}$  on  $[0, 1]$ , which is countable, and assemble them into a set  $N$
- Enumerate the rationals in  $[0, 1]$  as  $q_j$ , and define  $N_j = N + q_j$ . These intersect trivially.
- Define  $M := \bigsqcup N_j$ , then  $[0, 1] \subseteq M \subseteq [-1, 2]$ , so the measure must be between 1 and 3. By translation invariance,  $m(N_j) = m(N)$ , and disjoint additivity forces  $m(M) = 0$ , a contradiction.

■

### Proposition 7.8 (Borel Characterization of Measurable Sets).

If  $E$  is Lebesgue measurable, then  $E = H \bigsqcup N$  where  $H \in F_\sigma$  and  $N$  is null.

**Useful technique:**  $F_\sigma$  sets are Borel, so establish something for Borel sets and use this to extend it to Lebesgue.

*Proof .*

For every  $\frac{1}{n}$  there exists a closed set  $K_n \subset E$  such that  $m(E \setminus K_n) \leq \frac{1}{n}$ . Take  $K = \bigcup K_n$ , wlog  $K_n \nearrow K$  so  $m(K) = \lim m(K_n) = m(E)$ . Take  $N := E \setminus K$ , then  $m(N) = 0$ .

■

---

**Definition 7.8.1.**

$$\limsup_n A_n := \bigcap_n \bigcup_{j \geq n} A_j = \left\{ x \mid x \in A_n \text{ for inf. many } n \right\}$$
$$\liminf_n A_n := \bigcup_n \bigcap_{j \geq n} A_j = \left\{ x \mid x \in A_n \text{ for all except fin. many } n \right\}$$

**Lemma 7.9.**

If  $A_n$  are all measurable,  $\limsup A_n$  and  $\liminf A_n$  are measurable.

*Proof .*

Measurable sets form a sigma algebra, and these are expressed as countable unions/intersections of measurable sets. ■

**Theorem 7.10 (Borel-Cantelli).**

Let  $\{E_k\}$  be a countable collection of measurable sets. Then

$$\sum_k m(E_k) < \infty \implies \text{almost every } x \in \mathbb{R} \text{ is in at most finitely many } E_k.$$

*Proof .*

- If  $E = \limsup_j E_j$  with  $\sum_j m(E_j) < \infty$  then  $m(E) = 0$ .
- If  $E_j$  are measurable, then  $\limsup_j E_j$  is measurable.
- If  $\sum_j m(E_j) < \infty$ , then  $\sum_{j=N}^{\infty} m(E_j) \xrightarrow{N \rightarrow \infty} 0$  as the tail of a convergent sequence.
- $E = \limsup_j E_j = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j \implies E \subseteq \bigcup_{j=k}^{\infty} E_j$  for all  $k$
- $E \subseteq \bigcup_{j=k}^{\infty} E_j \implies m(E) \leq \sum_{j=k}^{\infty} m(E_j) \xrightarrow{k \rightarrow \infty} 0$ . ■

**Lemma 7.11.**

- Characteristic functions are measurable
- If  $f_n$  are measurable, so are  $|f_n|$ ,  $\limsup f_n$ ,  $\liminf f_n$ ,  $\lim f_n$ ,
- Sums and differences of measurable functions are measurable,
- Cones  $F(x, y) = f(x)$  are measurable,
- Compositions  $f \circ T$  for  $T$  a linear transformation are measurable,
- “Convolution-ish” transformations  $(x, y) \mapsto f(x - y)$  are measurable

---

*Proof (Convolution).*

Take the cone on  $f$  to get  $F(x, y) = f(x)$ , then compose  $F$  with the linear transformation  $T = [1, -1; 1, 0]$ . ■

## 8 Integration

### Definition 8.0.1.

$f \in L^+$  iff  $f$  is measurable and non-negative.

Useful techniques:

- Break integration domain up into disjoint annuli.
- Break real integrals up into  $x < 1$  and  $x > 1$ .

### Definition 8.0.2.

A measurable function is integrable iff  $\|f\|_1 < \infty$ .

Useful facts about  $C_c$  functions:

- Bounded almost everywhere
- Uniformly continuous

### 8.1 Convergence Theorems

#### Theorem 8.1 (*Monotone Convergence*).

If  $f_n \in L^+$  and  $f_n \nearrow f$  a.e., then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \quad \int f_n \longrightarrow \int f.$$

Needs to be positive and increasing.

#### Theorem 8.2 (*Dominated Convergence*).

If  $f_n \in L^1$  and  $f_n \longrightarrow f$  a.e. with  $|f_n| \leq g$  for some  $g \in L^1$ , then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \quad \int f_n \longrightarrow \int f,$$

and more generally,

$$\int |f_n - f| \longrightarrow 0.$$

Positivity *not* needed.

Generalized DCT: can relax  $|f_n| < g$  to  $|f_n| < g_n \longrightarrow g \in L^1$ .

**Lemma 8.3.**

If  $f \in L^1$ , then

$$\int |f_n - f| \longrightarrow 0 \iff \int |f_n| \longrightarrow \int |f|.$$

*Proof.*

Let  $g_n = |f_n| - |f_n - f|$ , then  $g_n \longrightarrow |f|$  and

$$|g_n| = ||f_n| - |f_n - f|| \leq |f_n - (f_n - f)| = |f| \in L^1,$$

so the DCT applies to  $g_n$  and

$$\begin{aligned} \|f_n - f\|_1 &= \int |f_n - f| + |f_n| - |f_n| = \int |f_n| - g_n \\ &\longrightarrow_{DCT} \lim \int |f_n| - \int |f|. \end{aligned}$$

■

**Fatou's Lemma** If  $f_n \in L^+$ , then

$$\begin{aligned} \int \liminf_n f_n &\leq \liminf_n \int f_n \\ \limsup_n \int f_n &\leq \int \limsup_n f_n. \end{aligned}$$

Only need positivity.

**Theorem 8.4 (Tonelli).**

For  $f(x, y)$  **non-negative and measurable**, for almost every  $x \in \mathbb{R}^n$ ,

- $f_x(y)$  is a **measurable** function
- $F(x) = \int f(x, y) dy$  is a **measurable** function,
- For  $E$  measurable, the slices  $E_x := \{y \mid (x, y) \in E\}$  are measurable.
- $\int f = \int \int F$ , i.e. any iterated integral is equal to the original.

**Theorem 8.5 (Fubini).**

For  $f(x, y)$  **integrable**, for almost every  $x \in \mathbb{R}^n$ ,

- $f_x(y)$  is an **integrable** function
- $F(x) := \int f(x, y) dy$  is an **integrable** function,
- For  $E$  measurable, the slices  $E_x := \{y \mid (x, y) \in E\}$  are measurable.
- $\int f = \int \int f(x, y)$ , i.e. any iterated integral is equal to the original

**Theorem 8.6 (Fubini/Tonelli).**

If any iterated integral is **absolutely integrable**, i.e.  $\int \int |f(x, y)| < \infty$ , then  $f$  is integrable and  $\int f$  equals any iterated integral.

**Corollary 8.7 (Measurable Slices).**

Let  $E$  be a measurable subset of  $\mathbb{R}^n$ . Then

- For almost every  $x \in \mathbb{R}^{n_1}$ , the slice  $E_x := \{y \in \mathbb{R}^{n_2} \mid (x, y) \in E\}$  is measurable in  $\mathbb{R}^{n_2}$ .
- The function

$$F : \mathbb{R}^{n_1} \longrightarrow \mathbb{R}$$

$$x \mapsto m(E_x) = \int_{\mathbb{R}^{n_2}} \chi_{E_x} dy$$

is measurable and

$$m(E) = \int_{\mathbb{R}^{n_1}} m(E_x) dx = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{E_x} dy dx$$

*Proof (Measurable Slices).*

$\implies :$

- Let  $f$  be measurable on  $\mathbb{R}^n$ .
- Then the cylinders  $F(x, y) = f(x)$  and  $G(x, y) = f(y)$  are both measurable on  $\mathbb{R}^{n+1}$ .
- Write  $\mathcal{A} = \{G \leq F\} \cap \{G \geq 0\}$ ; both are measurable.

$\impliedby :$

- Let  $A$  be measurable in  $\mathbb{R}^{n+1}$ .
- Define  $A_x = \{y \in \mathbb{R} \mid (x, y) \in A\}$ , then  $m(A_x) = f(x)$ .
- By the corollary,  $A_x$  is measurable set,  $x \mapsto m(A_x)$  is a measurable function, and  $m(A) = \int f(x) dx$ .
- Then explicitly,  $f(x) = \chi_A$ , which makes  $f$  a measurable function. ■

**Proposition 8.8 (Differentiating Under an Integral).**

If  $\left| \frac{\partial}{\partial t} f(x, t) \right| \leq g(x) \in L^1$ , then letting  $F(t) = \int f(x, t) dt$ ,

$$\frac{\partial}{\partial t} F(t) := \lim_{h \rightarrow 0} \int \frac{f(x, t+h) - f(x, t)}{h} dx$$

$$\stackrel{\text{DCT}}{=} \int \frac{\partial}{\partial t} f(x, t) dx.$$

To justify passing the limit, let  $h_k \rightarrow 0$  be any sequence and define

$$f_k(x, t) = \frac{f(x, t + h_k) - f(x, t)}{h_k},$$

so  $f_k \xrightarrow{\text{pointwise}} \frac{\partial}{\partial t} f$ .

Apply the MVT to  $f_k$  to get  $f_k(x, t) = f_k(\xi, t)$  for some  $\xi \in [0, h_k]$ , and show that  $f_k(\xi, t) \in L_1$ .

**Proposition 8.9 (Swapping Sum and Integral).**

If  $f_n$  are non-negative and  $\sum \int |f|_n < \infty$ , then  $\sum \int f_n = \int \sum f_n$ .

*Proof.*

MCT. Let  $F_N = \sum_{n=1}^N f_n$  be a finite partial sum; then there are simple functions  $\varphi_n \nearrow f_n$  and so  $\sum_{n=1}^N \varphi_n \nearrow F_N$ , so apply MCT. ■

**Lemma 8.10.**

If  $f_k \in L^1$  and  $\sum \|f_k\|_1 < \infty$  then  $\sum f_k$  converges almost everywhere and in  $L^1$ .

*Proof.*

Define  $F_N = \sum_{k=1}^N f_k$  and  $F = \lim_N F_N$ , then  $\|F_N\|_1 \leq \sum_{k=1}^N \|f_k\|_1 < \infty$  so  $F \in L^1$  and  $\|F_N - F\|_1 \rightarrow 0$  so the sum converges in  $L^1$ . Almost everywhere convergence: ? ■

## 8.2 $L^1$ Facts

**Lemma 8.11 (Translation Invariance).**

The Lebesgue integral is translation invariant, i.e.  $\int f(x) dx = \int f(x+h) dx$  for any  $h$ .

*Proof.*

- For characteristic functions,  $\int_E f(x+h) = \int_{E+h} f(x) = m(E+h) = m(E) = \int_E f$  by translation invariance of measure.
- So this also holds for simple functions by linearity
- For  $f \in L^+$ , choose  $\varphi_n \nearrow f$  so  $\int \varphi_n \rightarrow \int f$ .
- Similarly,  $\tau_h \varphi_n \nearrow \tau_h f$  so  $\int \tau_h f \rightarrow \int f$
- Finally  $\left\{ \int \tau_h \varphi \right\} = \left\{ \int \varphi \right\}$  by step 1, and the suprema are equal by uniqueness of limits. ■

**Lemma 8.12 (Integrals Distribute Over Disjoint Sets).**

If  $X \subseteq A \cup B$ , then  $\int_X f \leq \int_A f + \int_{A^c} f$  with equality iff  $X = A \amalg B$ .



**Lemma 8.13 (Unif Cts  $L^1$  Functions Decay Rapidly).**

If  $f \in L^1$  and  $f$  is uniformly continuous, then  $f(x) \xrightarrow{|x| \rightarrow \infty} 0$ .

Doesn't hold for general  $L^1$  functions, take any train of triangles with height 1 and summable areas.

**Lemma 8.14 ( $L^1$  Functions Have Small Tails).**

If  $f \in L^1$ , then for every  $\varepsilon$  there exists a radius  $R$  such that if  $A = B_R(0)^c$ , then  $\int_A |f| < \varepsilon$ .

*Proof.*

Approximate with compactly supported functions. Take  $g \xrightarrow{L^1} f$  with  $g \in C_c$ , then choose  $N$  large enough so that  $g = 0$  on  $E := B_N(0)^c$ , then  $\int_E |f| \leq \int_E |f - g| + \int_E |g|$ . ■

**Lemma 8.15 ( $L^1$  Functions Have Absolutely Continuity).**

$m(E) \rightarrow 0 \implies \int_E f \rightarrow 0$ .

*Proof.*

Approximate with compactly supported functions. Take  $g \xrightarrow{L^1} f$ , then  $g \leq M$  so  $\int_E f \leq \int_E f - g + \int_E g \rightarrow 0 + M \cdot m(E) \rightarrow 0$ . ■

**Lemma 8.16 ( $L^1$  Functions Are Finite a.e.).**

If  $f \in L^1$ , then  $m(\{f(x) = \infty\}) = 0$ .

*Proof.*

Idea: Split up domain Let  $A = \{f(x) = \infty\}$ , then  $\infty > \int f = \int_A f + \int_{A^c} f = \infty \cdot m(A) + \int_{A^c} f \implies m(A) = 0$ . ■

**Proposition 8.17 (Continuity in  $L^1$ ).**

$$\|\tau_h f - f\|_1 \xrightarrow{h \rightarrow 0} 0$$

*Proof.*

Approximate with compactly supported functions. Take  $g \xrightarrow{L^1} f$  with  $g \in C_c$ .

$$\begin{aligned} \int f(x+h) - f(x) &\leq \\ \int f(x+h) - g(x+h) + \int g(x+h) - g(x) + \int g(x) - f(x) & \\ \longrightarrow 2\varepsilon + \int g(x+h) - g(x) & \\ = \int_K g(x+h) - g(x) + \int_{K^c} g(x+h) - g(x) &\longrightarrow 0, \end{aligned}$$

which follows because we can enlarge the support of  $g$  to  $K$  where the integrand is zero on  $K^c$ , then apply uniform continuity on  $K$ . ■

**Proposition 8.18** (*Integration by Parts, Special Case*).

$$\begin{aligned} F(x) &:= \int_0^x f(y)dy \quad \text{and} \quad G(x) := \int_0^x g(y)dy \\ \implies \int_0^1 F(x)g(x)dx &= F(1)G(1) - \int_0^1 f(x)G(x)dx. \end{aligned}$$

*Proof.*

Fubini-Tonelli, and sketch region to change integration bounds. ■

**Theorem 8.19** (*Lebesgue Density*).

$$A_h(f)(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(y)dy \implies \|A_h(f) - f\| \xrightarrow{h \rightarrow 0} 0.$$

*Proof.*

Fubini-Tonelli, and sketch region to change integration bounds, and continuity in  $L^1$ . ■

### 8.3 $L^p$ Spaces

**Lemma 8.20.**

The following are dense subspaces of  $L^2([0, 1])$ :

- Simple functions
- Step functions
- $C_0([0, 1])$
- Smoothly differentiable functions  $C_0^\infty([0, 1])$
- Smooth compactly supported functions  $C_c^\infty$

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**Theorem 8.21 (Dual  $L^p$  Spaces).**

For  $p \neq \infty$ ,  $(L^p)^\vee \cong L^q$ .

*Proof (p=1).*

?

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Proof (p=2) > Use Riesz Representation for Hilbert spaces.

For the  $p = \infty$  case:  $L^1 \subset (L^\infty)^\vee$ , since the isometric mapping is always injective, but *never* surjective. So this containment is always proper (requires Hahn-Banach Theorem).

## 9 Fourier Transform and Convolution

### 9.1 The Fourier Transform

**Definition 9.0.1** (Convolution).

$$f * g(x) = \int f(x-y)g(y)dy.$$

**Definition 9.0.2** (The Fourier Transform).

$$\widehat{f}(\xi) = \int f(x) e^{2\pi i x \cdot \xi} dx.$$

**Lemma 9.1.**

If  $\widehat{f} = \widehat{g}$  then  $f = g$  almost everywhere.

**Lemma 9.2 (Riemann-Lebesgue: Fourier transforms have small tails).**

$$f \in L^1 \implies \widehat{f}(\xi) \rightarrow 0 \text{ as } |\xi| \rightarrow \infty.$$

**Lemma 9.3.**

If  $f \in L^1$ , then  $\widehat{f}$  is continuous and bounded.

*Proof .*

- Boundedness:

$$|\widehat{f}(\xi)| \leq \int |f| \cdot |e^{2\pi i x \cdot \xi}| = \|f\|_1.$$

- Continuity:

– Apply DCT to show  $|\widehat{f}(\xi_n) - \widehat{f}(\xi)| \xrightarrow{n \rightarrow \infty} 0$ .

■

## 9.2 Approximate Identities

**Definition 9.3.1** (Dilation).

$$\varphi_t(x) = t^{-n} \varphi(t^{-1}x).$$

**Definition 9.3.2** (Approximation to the Identity).

For  $\varphi \in L^1$ , the dilations satisfy  $\int \varphi_t = \int \varphi$ , and if  $\int \varphi = 1$  then  $\varphi$  is an *approximate identity*.

Example:  $\varphi(x) = e^{-\pi x^2}$

**Theorem 9.4** (*Convolution Against Approximate Identities Converge in  $L^1$* ).

$$\|f * \varphi_t - f\|_1 \xrightarrow{t \rightarrow 0} 0.$$

*Proof .*

$$\begin{aligned} \|f - f * \varphi_t\|_1 &= \int f(x) - \int f(x-y)\varphi_t(y) \, dy dx \\ &= \int f(x) \int \varphi_t(y) \, dy - \int f(x-y)\varphi_t(y) \, dy dx \\ &= \int \int \varphi_t(y)[f(x) - f(x-y)] \, dy dx \\ &=_{FT} \int \int \varphi_t(y)[f(x) - f(x-y)] \, dx dy \\ &= \int \varphi_t(y) \int f(x) - f(x-y) \, dx dy \\ &= \int \varphi_t(y) \|f - \tau_y f\|_1 dy \\ &= \int_{y < \delta} \varphi_t(y) \|f - \tau_y f\|_1 dy + \int_{y \geq \delta} \varphi_t(y) \|f - \tau_y f\|_1 dy \\ &\leq \int_{y < \delta} \varphi_t(y) \varepsilon + \int_{y \geq \delta} \varphi_t(y) (\|f\|_1 + \|\tau_y f\|_1) dy \quad \text{by continuity in } L^1 \\ &\leq \varepsilon + 2\|f\|_1 \int_{y \geq \delta} \varphi_t(y) dy \\ &\leq \varepsilon + 2\|f\|_1 \cdot \varepsilon \quad \text{since } \varphi_t \text{ has small tails} \\ &\xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

■

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**Theorem 9.5 (Convolutions Vanish at Infinity).**

$$f, g \in L^1 \text{ and bounded} \implies \lim_{|x| \rightarrow \infty} (f * g)(x) = 0.$$

*Proof.*

- Choose  $M \geq f, g$ .
- By small tails, choose  $N$  such that  $\int_{B_N^c} |f|, \int_{B_N^c} |g| < \varepsilon$
- Note

$$|f * g| \leq \int |f(x - y)| |g(y)| dy := I.$$

- Use  $|x| \leq |x - y| + |y|$ , take  $|x| \geq 2N$  so either

$$|x - y| \geq N \implies I \leq \int_{\{|x-y| \geq N\}} |f(x - y)| M dy \leq \varepsilon M \rightarrow 0$$

then

$$|y| \geq N \implies I \leq \int_{\{|y| \geq N\}} M |g(y)| dy \leq M \varepsilon \rightarrow 0.$$

■

Proposition (Young's Inequality?) :

$$\frac{1}{r} := \frac{1}{p} + \frac{1}{q} - 1 \implies \|f * g\|_r \leq \|f\|_p \|g\|_q.$$

**Corollary 9.6.**

Take  $q = 1$  to obtain

$$\|f * g\|_p \leq \|f\|_p \|g\|_1.$$

**Corollary 9.7.**

If  $f, g \in L^1$  then  $f * g \in L^1$ .

## 10 Extra Problems

Integration

- Show that if  $f \in C^1(\mathbb{R})$  and  $\lim_{x \rightarrow \infty} f(x), f'(x)$  exist, then  $\lim_{x \rightarrow \infty} f'(x) = 0$ .

Basics

- If  $f$  is continuous, is it necessarily the case that  $f'$  is continuous?

- 
- If  $f_n \rightarrow f$ , is it necessarily the case that  $f'_n$  converges to  $f'$  (or at all)?
  - Is it true that the sum of differentiable functions is differentiable?
  - Is it true that the limit of integrals equals the integral of the limit?
  - Is it true that a limit of continuous functions is continuous?
  - Prove that uniform convergence implies pointwise convergence implies a.e. convergence, but none of the implications may be reversed.
  - Show that if  $K$  is compact and  $F$  is closed with  $K, F$  disjoint then  $\text{dist}(K, F) > 0$ .
  - Show that if  $f_n \rightarrow f$  uniformly with each  $f_n$  continuous then  $f$  is continuous.
  - Show that a subset of a metric space is closed iff it is complete.
  - Show that if a subset of a metric space is complete and totally bounded, then it is compact.
  - Show that every compact set is closed and bounded.
  - Show that a uniform limit of bounded functions is bounded.
  - Show that a uniform limit of continuous function is continuous.
  - Show that if  $f_n \rightarrow f$  pointwise,  $f'_n \rightarrow g$  uniformly for some  $f, g$ , then  $f$  is differentiable and  $g = f'$ .

### Measure Theory

- $\star$ : Show that for  $E \subseteq \mathbb{R}^n$ , TFAE:
  1.  $E$  is measurable
  2.  $E = H \cup Z$  here  $H$  is  $F_\sigma$  and  $Z$  is null
  3.  $E = V \setminus Z'$  where  $V \in G_\delta$  and  $Z'$  is null.
- Show that continuity of measure from above/below holds for outer measures.
- $\star$ : Show that if  $E \subseteq \mathbb{R}^n$  is measurable then  $m(E) = \sup_{K \subseteq E \text{ compact}} m(K)$  iff for all  $\varepsilon > 0$  there exists a compact  $K \subseteq E$  such that  $m(K) \geq m(E) - \varepsilon$ .
- Show that a countable union of null sets is null.

### Continuity

- Show that a continuous function on a compact set is uniformly continuous.

### Measurability

- Show that  $f = 0$  a.e. iff  $\int_E f = 0$  for every measurable set  $E$ .
- $\star$ : Show that cylinder functions are measurable, i.e. if  $f$  is measurable on  $\mathbb{R}^s$ , then  $F(x, y) := f(x)$  is measurable on  $\mathbb{R}^s \times \mathbb{R}^t$  for any  $t$ .
- Show that if  $f$  is a measurable function, then  $f = 0$  a.e. iff  $\int f = 0$ .

### Integrability

- $\star$ : Prove that the Lebesgue integral is translation invariant, i.e. if  $\tau_h(x) = x + h$  then  $\int \tau_h f = \int f$ .
- $\star$ : Prove that the Lebesgue integral is dilation invariant, i.e. if  $f_\delta(x) = \frac{f(\frac{x}{\delta})}{\delta^n}$  then  $\int f_\delta = \int f$ .
- $\star$ : Prove continuity in  $L^1$ , i.e.

$$f \in L^1 \implies \lim_{h \rightarrow 0} \int |f(x+h) - f(x)| = 0.$$

- 
- Show that a bounded function is Lebesgue integrable iff it is measurable.
  - Show that simple functions are dense in  $L^1$ .
  - Show that step functions are dense in  $L^1$ .
  - Show that smooth compactly supported functions are dense in  $L^1$ .

#### Convergence

- Prove Fatou's lemma using the Monotone Convergence Theorem.
- Show that if  $\{f_n\}$  is in  $L^1$  and  $\sum \int |f_n| < \infty$  then  $\sum f_n$  convergence to an  $L^1$  function and  $\int \sum f_n = \sum \int f_n$ .

#### Fourier Analysis

- Show that if  $f \in L^1$  then  $\hat{f}$  is bounded and uniformly continuous.
- Is it the case that  $f \in L^1$  implies  $\hat{f} \in L^1$ ?
- Show that if  $f, \hat{f} \in L^1$  then  $f$  is bounded, continuous, and vanishes at infinity.
  - Show that this is not true for arbitrary  $L^1$  functions.
- Show that if  $f \in L^1$  and  $\hat{f} = 0$  a.e. then  $f = 0$  a.e.
  - Prove that  $\hat{f} = \hat{g}$  implies that  $f = g$  a.e.
- Show that if  $f, g \in L^1$  then  $\int \hat{f}g = \int f\hat{g}$ .
  - Give an example showing that this fails if  $g$  is not bounded.
- Show that if  $f \in C^1$  then  $f$  is equal to its Fourier series.

#### Convolution

- Show that  $f, g \in L^1 \implies f * g \in L^1$  and  $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ .
- Show that  $f \in L^1, g \leq M \implies f * g \leq M'$  and is uniformly continuous.
- Show that if  $f, g \in L^1$  with  $f \leq M, g \leq M'$ , then  $f * g \xrightarrow{x \rightarrow \infty} 0$ .
- Show that if  $f \in L^1$  and  $g'$  exists with  $\frac{\partial g}{\partial x_i}$  all bounded, then  $\frac{\partial}{\partial x_i}(f * g) = f * \frac{\partial g}{\partial x_i}$ .
- Show that if  $f, g$  are smooth and compactly supported then  $f * g$  is smooth and  $f * g \xrightarrow{x \rightarrow \infty} 0$ .
- $\star$ : show that if  $f, g \in L^1$ , then  $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ .
- Is it the case that  $f, g \in C_c$  implies that  $f * g \in C_c$ ?
- Show that if  $f \in L^1$  and  $g \in C_c^\infty$  then  $f * g$  is smooth and  $f * g$  vanishes at infinity.
- Show that if  $f, g \in L^1$  and  $g$  is bounded, then  $\lim_{|x| \rightarrow \infty} (f * g)(x) = 0$ .

#### Fourier Transforms

##### Approximate Identities

- Show that if  $\varphi$  is an approximate identity, then  $\|f * \varphi_t - f\|_1 \xrightarrow{t \rightarrow 0} 0$ .
  - Show that if additionally  $|\varphi(x)| \leq c(1 + |x|)^{-n-\varepsilon}$  for some  $c, \varepsilon > 0$ , then this converges is almost everywhere.
- Show that if  $f$  is bounded and uniformly continuous and  $\varphi_t$  is an approximation to the identity, then  $f * \varphi_t$  uniformly converges to  $f$ .

##### $L_p$

- Show that if  $E \subseteq \mathbb{R}^n$  is measurable with  $\mu(E) < \infty$  and  $f \in L^p(X)$  then  $\|f\|_{L^p(X)} \xrightarrow{p \rightarrow \infty} \|f\|_\infty$ .

- Is it true that the converse to the DCT holds? I.e. if  $\int f_n \rightarrow \int f$ , is there a  $g \in L^p$  such that  $f_n < g$  a.e. for every  $n$ ?
- Prove continuity in  $L^p$ : If  $f$  is uniformly continuous then  $\|\tau_h f - f\|_p \rightarrow 0$  as  $h \rightarrow 0$  for all  $p$ .

## 11 Inequalities and Equalities

**Proposition 11.1 (Reverse Triangle Inequality).**

$$|\|x\| - \|y\|| \leq \|x - y\|.$$

**Proposition 11.2 (Chebyshev's Inequality).**

$$\mu(\{x : |f(x)| > \alpha\}) \leq \left(\frac{\|f\|_p}{\alpha}\right)^p.$$

**Proposition 11.3 (Holder's Inequality (when surjective)).**

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \|fg\|_1 \leq \|f\|_p \|g\|_q.$$

*Application:* For finite measure spaces,

$$1 \leq p < q \leq \infty \implies L^q \subset L^p \quad (\text{and } \ell^p \subset \ell^q).$$

*Proof (Holder's Inequality).*

Fix  $p, q$ , let  $r = \frac{q}{p}$  and  $s = \frac{r}{r-1}$  so  $r^{-1} + s^{-1} = 1$ . Then let  $h = |f|^p$ :

■

$$\|f\|_p^p = \|h \cdot 1\|_1 \leq \|1\|_s \|h\|_r = \mu(X)^{\frac{1}{s}} \|f\|_q^{\frac{q}{r}} \implies \|f\|_p \leq \mu(X)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q.$$

Note: doesn't work for  $\ell_p$  spaces, but just note that  $\sum |x_n| < \infty \implies x_n < 1$  for large enough  $n$ , and thus  $p < q \implies |x_n|^q \leq |x_n|^p$ .

**Proposition 11.4 (Cauchy-Schwarz Inequality).**



$$|\langle f, g \rangle| = \|fg\|_1 \leq \|f\|_2 \|g\|_2 \quad \text{with equality} \iff f = \lambda g.$$

Note: Relates inner product to norm, and only happens to relate norms in  $L^1$ .

*Proof .*  
?



**Proposition 11.5 (Minkowski's Inequality:).**

$$1 \leq p < \infty \implies \|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Note: does not handle  $p = \infty$  case. Use to prove  $L^p$  is a normed space.

**Proposition 11.6 (Young's Inequality\*).**

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \implies \|f * g\|_r \leq \|f\|_p \|g\|_q.$$

]

**Application:** Some useful specific cases:

$$\begin{aligned} \|f * g\|_1 &\leq \|f\|_1 \|g\|_1 \\ \|f * g\|_p &\leq \|f\|_1 \|g\|_p, \\ \|f * g\|_\infty &\leq \|f\|_2 \|g\|_2 \\ \|f * g\|_\infty &\leq \|f\|_p \|g\|_q. \end{aligned}$$

**Proposition 11.7 (? Inequality).**

$$(a + b)^p \leq 2^p(a^p + b^p).$$

**Proposition 11.8 (Bezel's Inequality:).**

For  $x \in H$  a Hilbert space and  $\{e_k\}$  an orthonormal sequence,

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

Note: this does not need to be a basis.

**Proposition 11.9 (*Parseval's Identity*).**

Equality in Bessel's inequality, attained when  $\{e_k\}$  is a *basis*, i.e. it is complete, i.e. the span of its closure is all of  $H$ .

**11.1 Less Explicitly Used Inequalities****Proposition 11.10 (*AM-GM Inequality*).**

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

**Proposition 11.11 (*Jensen's Inequality*).**

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$