

Complex Analysis Qualifying Exam Solutions

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1 Week 1

1.1 Integrals and Cauchy's Theorem

1.1.1 5

Show that there is no sequence of polynomials converging uniformly to $f(z) = 1/z$ on S^1 .

Solution

- By Cauchy's integral formula, $\int_{S^1} f = 2\pi i$
- If p_j is any polynomial, then p_j is holomorphic in \mathbb{D} , so $\int_{S^1} p_j = 0$.
- Contradiction: compact sets in \mathbb{C} are bounded, so

$$\left| \int f - \int p_j \right| \leq \int |p_j - f| \leq \int \|p_j - f\|_\infty = \|p_j - f\|_\infty \int_{S^1} 1 dz = \|p_j - f\|_\infty \cdot 2\pi \longrightarrow 0$$

which forces $\int f = \int p_j = 0$.

1.1.2 10

Suppose $f : \mathbb{C} \longrightarrow \mathbb{C}$ is entire and bounded, and use Cauchy's theorem to prove that $f' \equiv 0$ and thus f is constant.

Solution

- Suffices to prove $f' = 0$ because \mathbb{C} is connected (see Stein Ch 1, 3.4)
 - Idea: Fix w_0 , show $f(w) = f(w_0)$ for any $w \neq w_0$
 - Connected = Path connected in \mathbb{C} , so take γ joining w to w_0 .
 - f is a primitive for f' , and $\int_{\gamma} f' = f(w) - f(w_0)$, but $f' = 0$.
- Fix $z_0 \in \mathbb{C}$, let B be the bound for f , so $|f(z)| \leq B$ for all z .
- Apply Cauchy inequalities: if f is holomorphic on $U \supset \bar{D}_R(z_0)$ then setting $\|f\|_C := \sup_{z \in C} |f(z)|$,

$$|f^{(n)}(z_0)| \leq \frac{n! \|f\|_C}{R^n}.$$

- Yields $|f'(z_0)| \leq B/R$
- Take $R \rightarrow \infty$, QED.

1.2 Liouville. The Fundamental Theorem of Algebra, Power Series

1.2.1 1

Suppose f is analytic on $\Omega \supseteq \mathbb{D}$ whose power series $\sum a_n z^n$ has radius of convergence 1.

- Give an example of an f which converges at every point on S^1 .
- Give an example of an f which is analytic at $z = 1$ but $\sum a_n$ diverges.
- Prove that f can not be analytic at every point of S^1 .

Solution:

- Take $\sum \frac{z^n}{n^2}$; then $|z| \leq 1 \implies \left| \frac{z^n}{n^2} \right| \leq \frac{1}{n^2}$ which is summable, so the series converges for $|z| \leq 1$.
- Take $\sum \frac{z^n}{n}$; then $z = 1$ yields the harmonic series, which diverges.
 - For $z \in S^1 \setminus \{1\}$, we have $z = e^{2\pi i t}$ for $0 < t < 2\pi$.
 - So fix t .
 - Toward applying the Dirichlet test, set $a_n = 1/n, b_n = z^n$.
 - Then for all N ,

$$\left| \sum_{n=1}^N b_n \right| = \left| \sum_{n=1}^N \frac{1}{n} z^n \right| = \left| \sum_{n=1}^N z^n \right| = \left| \frac{z - z^{N+1}}{1 - z} \right| \leq \frac{2}{1 - |z|} < \infty.$$

- Thus $\sum a_n b_n < \infty$ and $\sum z^n/n$ converges.
- c. ?

1.2.2 5

Prove the Fundamental Theorem of Algebra: every non-constant polynomial $p(z) = a_n z^n + \dots + a_0 \in \mathbb{C}[x]$ has a root in \mathbb{C} .

Solution:

- Strategy: By contradiction with Liouville's Theorem
- Suppose p is non-constant and has no roots.
- Claim: $1/p(z)$ is a bounded holomorphic function on \mathbb{C} .
 - Holomorphic: clear? Since p has no roots.
 - Bounded: for $z \neq 0$, write

$$\frac{P(z)}{z^n} = a_n + \left(\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right).$$

- The term in parentheses goes to 0 as $|z| \rightarrow \infty$
- Thus there exists an $R > 0$ such that

$$|z| > R \implies \left| \frac{P(z)}{z^n} \right| \geq c := \frac{|a_n|}{2}.$$

- So p is bounded below when $|z| > R$
- Since p is continuous and has no roots in $|z| \leq R$, it is bounded below when $|z| \leq R$.
- Thus p is bounded below on \mathbb{C} and thus $1/p$ is bounded above on \mathbb{C} .
- By Liouville's theorem, $1/p$ is constant and thus p is constant, a contradiction.

1.2.3 6

Find all entire functions f which satisfy the following inequality, and prove the list is complete:

$$|f(z)| \geq z.$$

Solution:

- Suppose f is entire and define $g(z) := \frac{z}{f(z)}$.
- By the inequality, $|g(z)| \leq 1$, so g is bounded.
- g potentially has singularities at the zeros $Z_f := f^{-1}(0)$, but since f is entire, g is holomorphic on $\mathbb{C} \setminus Z_f$.
- Claim: $Z_f \subset \mathbb{C}$ is closed and discrete
 - ???
- Thus the singularities Z_f are isolated
- By Riemann's removable singularity theorem, the singularities Z_f are removable and g has an extension to an entire function \tilde{g} .
- By continuity, we have $|\tilde{g}(z)| \leq 1$ on all of \mathbb{C}
- By Liouville, \tilde{g} is constant, so $\tilde{g}(z) = c_0$ with $|c_0| \leq 1$
- Thus $f(z) = c_0^{-1}z$

Thus all such functions are of the form $f(z) = cz$ for some $c \neq 0 \in \mathbb{C}$.