Complex Analysis Qualifying Exam Review

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1 Useful Techniques

1.1 Notation

Notation	Definition
$\mathbb{D} \coloneqq \left\{ z \,\middle\vert\, \|z\| \leq 1 \right\}$	The unit disc
$\mathbb{H} \coloneqq \left\{ x + iy \,\middle\vert\, y > 0 \right\}$	The upper half-plane
$X_{\frac{1}{2}}$	A "half version of X", see examples
$\mathbb{H}_{\frac{1}{2}}$	The first quadrant
$\mathbb{D}_{\frac{1}{2}}$	The portion of the first quadrant inside the unit disc
$S \coloneqq \left\{ x + iy \,\middle\vert\, x \in \mathbb{R},\, 0 < y < \pi \right\}$	The horizonta strip

Remark 1.1.1 ***(Showing a function is constant):*** If you want to show that a function f is constant, try one of the following:

- Write $f = u + iv$ and use Cauchy-Riemann to show $u_x, u_y = 0$, etc.
- Show that f is entire and bounded.

If you additionally want to show f is zero, try one of these:

- Show f is entire, bounded, and $\lim_{z \to \infty} f(z) = 0$.

1.2 Greatest Hits

Things to know well:

- Estimates for derivatives, mean value theorem
- **??**CauchyTheorem]Cauchy's Theorem
- **??**CauchyIntegral]Cauchy's Integral Formula
- **??**CauchyInequality]Cauchy's Inequality
- **??**Morera]Morera's Theorem
- **??**SchwarzReflection]The Schwarz Reflection Principle
- **??**MaximumModulus]Maximum Modulus Principle
- **??**SchwarzLemma]The Schwarz Lemma
- **??**Liouville]Liouville's Theorem

- **??**Casorati]Casorati-Weierstrass Theorem
- **??**Rouche]Rouché's Theorem
- Properties of linear fractional transformations
- Automorphisms of $\mathbb{D}, \mathbb{C}, \mathbb{CP}^1$.

1.3 Basic but Useful Facts

Fact 1.3.1 (Some useful facts about basic complex algebra)

- $z\bar{z} = |z|^2$

$$\Re(z) = \frac{z+\bar{z}}{2} \qquad \Im(z) = \frac{z-\bar{z}}{2i}.$$

- $\operatorname{Arg}(z/w) = \operatorname{Arg}(z) - \operatorname{Arg}(w)$.

- Exponential forms of cosine and sine:

$$\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \qquad \sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right).$$

- Various differentials:

$$\begin{aligned} dz &= dx + i\ dy \\ d\bar{z} &= dx - i\ dy \end{aligned}$$

$$f_z = f_x = f_y/i.$$

- Integral of a complex exponential:

$$\int_0^{2\pi} e^{i\ell x} dx = \begin{cases} 2\pi & \ell = 0 \\ 0 & \text{else} \end{cases}.$$

Fact 1.3.2 (Some useful series)

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$
$$\log(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n} (z-a)^n$$
$$\frac{\partial}{\partial z} \sum_{j=0}^{\infty} a_j z^j = \sum_{j=0}^{\infty} a_{j+1} z^j$$

1.4 Advice

- Consider $1/f(z)$ and $f(1/z)$.

2 Definitions

Definition 2.0.1 (Analytic)
A function $f : \Omega \to \mathbb{C}$ is *analytic* at $z_0 \in \Omega$ iff there exists a power series $g(z) = \sum a_n(z-z_0)^n$ with radius of convergence $R > 0$ and a neighborhood $U \ni z_0$ such that $f(z) = g(z)$ on U.

Definition 2.0.2 (Cauchy-Riemann Equations)

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$
.

Definition 2.0.3 (Entire)
A function that is holomorphic on $\mathbb{C}$ is said to be *entire*.

Definition 2.0.4 (Essential Singularity)
A singularity z_0 is *essential* iff it is neither removable nor a pole.
Equivalently, a Laurent series expansion about z_0 has a principal part with infinitely many terms.

Definition 2.0.5 (Holomorphic)
A function $f : \mathbb{C} \to \mathbb{C}$ is *holomorphic* at z_0 if the following limit converges:

$$\lim_{h \to 0} \frac{1}{h} \left(f(z_0 + h) - f(z_0) \right) \coloneqq f'(z_0).$$

Definition 2.0.6 (Harmonic)
A real function of two variables $u(x, y)$ is *harmonic* iff its Laplacian vanishes:

$$\Delta u \coloneqq \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u = 0.$$

Definition 2.0.7 (Meromorphic)
A function $f : \Omega \to \mathbb{C}$ is *meromorphic* iff there exists a sequence $\{z_n\}$ such that

- $\{z_n\}$ has no limit points in Ω.
- f is holomorphic in $\Omega \setminus \{z_n\}$.
- f has poles at the points $\{z_n\}$.

If f is either holomorphic or has a pole at $z = \infty$ is said to be meromorphic on $\mathbb{CP}^1$.

Definition 2.0.8 (Poles (and associated terminology))
A *pole* z_0 of a meromorphic function $f(z)$ is a zero of $g(z) \coloneqq \dfrac{1}{f(z)}$. If there exists an n such that

$$\lim_{z \to z_0} (z - z_0)^n f(z)$$

is holomorphic and nonzero in a neighborhood of z_0, then the minimal such n is the *order* of the pole. A pole of order 1 is said to be a *simple pole*.
The pole z_0 is *isolated* iff there exists a neighborhood of z_0 containing no other poles of f.

Definition 2.0.9 (Principal Part and Residue)
In a Laurent series $f(z) \coloneqq \sum_{n \in \mathbb{Z}} c_n (z - z_0)^n$, the *principal part of f at z_0* consists of terms with negative degree:

$$P_f(z) \coloneqq \sum_{n=1}^{\infty} c_{-n} (z - z_0)^{-n}.$$

The *residue* of f at z_0 is the coefficient c_{-1}.

Definition 2.0.10 (Removable Singularities)
If z_0 is a singularity of f and there exists a g such that $f(z) = g(z)$ for all z in some deleted neighborhood $U \setminus \{z_0\}$, then z_0 is a *removable singularity* of f.

Definition 2.0.11 (Linear Fractional Transformation)

A map of the following form is a *linear fractional transformation*:

$$T(z) = \frac{az+b}{cz+d},$$

where the denominator is assumed to not be a multiple of the numerator.
These have inverses given by

$$T^{-1}(w) = \frac{dw-b}{-cw+a}.$$

Definition 2.0.12 (Conformal Map / Biholomorphism)
A bijective holomorphic map is a **conformal** (or angle-preserving) map, a.k.a. a **biholomorphism**.
Note that some authors just require the weaker condition that $f'(z) \neq 0$ for any point.

3 Theorems

3.1 Basics

Example 3.1.1 ***(holomorphic vs non-holomorphic):***

- $f(z) = \dfrac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$.
- $f(z) = \bar{z}$ is *not* holomorphic, since $\dfrac{\bar{h}}{h}$ does not converge (but is real differentiable).

Theorem 3.1.2 ***(Green's Theorem).***
If $\Omega \subseteq \mathbb{C}$ is bounded with $\partial\Omega$ piecewise smooth and $f, g \in C^1(\overline{\Omega})$, then

$$\int_{\partial\Omega} f\,dx + g\,dy = \iint_{\Omega} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA.$$

Theorem 3.1.3 ***(Summation by Parts).***
Define the forward difference operator $\Delta f_k = f_{k+1} - f_k$, then

$$\sum_{k=m}^{n} f_k \Delta g_k + \sum_{k=m}^{n-1} g_{k+1} \Delta f_k = f_n g_{n+1} - f_m g_m$$

Note: compare to $\int_a^b f\,dg + \int_a^b g\,df = f(b)g(b) - f(a)g(a)$.

3.2 Holomorphic and Entire Functions

3.2.1 Key Theorems

Theorem 3.2.1 ***(Cauchy's Theorem).***
If f is holomorphic on Ω, then

$$\int_{\partial\Omega} f(z)\,dz = 0.$$

Slogan: closed path integrals of holomorphic functions vanish.

Theorem 3.2.2 ***(Morera's Theorem).***
If f is continuous on a domain Ω and $\displaystyle\int_T f = 0$ for every triangle $T \subset \Omega$, then f is holomorphic.

Slogan: if every integral along a triangle vanishes, implies holomorphic.

Theorem 3.2.3 ***(Maximum Modulus).***
If f is holomorphic and nonconstant on an open connected region Ω, then $|f|$ can not attain a maximum on Ω. If Ω is bounded and f is continuous on $\overline{\Omega}$, then $\max_{\overline{\Omega}} |f|$ occurs on $\partial\Omega$. Conversely, if f attains a local supremum at $z_0 \in \Omega$, then f is constant on Ω.

Theorem 3.2.4 ***(Liouville's Theorem).***
If f is entire and bounded, f is constant.

Theorem 3.2.5 ***(Cauchy Integral Formula).***
Suppose f is holomorphic on Ω, then

$$f(z) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(\xi)}{\xi - z}\,d\xi$$

and

$$\frac{\partial^n f}{\partial z^n}(z) = \frac{n!}{2\pi i} \int_{\partial\Omega} \frac{f(\xi)}{(\xi - z)^{n+1}}\,d\xi.$$

Theorem 3.2.6 ***(Cauchy's Inequality).***
For $z_0 \in D_R(z_0) \subset \Omega$, we have

$$\left|f^{(n)}(z_0)\right| \le \frac{n!}{2\pi} \int_0^{2\pi} \frac{\|f\|_\infty}{R^{n+1}} R\,d\theta = \frac{n!\|f\|_\infty}{R^n},$$

where $\|f\|_\infty \coloneqq \sup_{z \in C_R} |f(z)|$.

Slogan 3.2.7
The nth Taylor coefficient of an analytic function is at most $\sup_{|z|=R} |f|/R^n$.

Theorem 3.2.8 ***(Argument Principle).***
For f meromorphic in γ°, if f has no poles and is nonvanishing on γ then

$$\Delta_\gamma \arg f(z) = \int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi(Z_f - P_f),$$

where Z_f and P_f are the number of zeros and poles respectively enclosed by γ, counted with multiplicity.

Theorem 3.2.9 ***(Rouché's Theorem).***
If f, g are analytic on a domain Ω with finitely many zeros in Ω and $\gamma \subset \Omega$ is a closed curve surrounding each point exactly once, where $|g| < |f|$ on γ, then f and $f + g$ have the same number of zeros.
Alternatively:
Suppose $f = g + h$ with $g \neq 0, \infty$ on γ with $|g| > |h|$ on γ. Then

$$\Delta_\gamma \arg(f) = \Delta_\gamma \arg(h) \quad \text{and } Z_f - P_f = Z_g - P_g.$$

Theorem 3.2.10 ***(The Residue Theorem).***
If f is holomorphic on an open set Ω containing a curve γ and its interior γ°, except for finitely many poles $\{z_k\}_{k=1}^N \subset \gamma^\circ$. Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^N \operatorname*{Res}_{z_k} f.$$

Theorem 3.2.11 ***(Cayley Transform).***
The fractional linear transformation given by $F(z) = \dfrac{i - z}{i + z}$ maps $\mathbb{D} \to \mathbb{H}$ with inverse $G(w) = i\dfrac{1 - w}{1 + w}$.

Theorem 3.2.12 ***(Schwarz Lemma).***
If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic with $f(0) = 0$, then

1. $|f(z)| \le |z|$ for all $z \in \mathbb{D}$
2. $|f'(0)| \le 1$.

Moreover, if $|f(z_0)| = |z_0|$ for any $z_0 \in \mathbb{D}$ or $|f'(0)| = 1$, then f is a rotation

Theorem 3.2.13 ***(Mean Value Theorem for Holomorphic Functions).***

$$f(z_0) = \frac{1}{\pi r^2} \iint_{D_r(z_0)} f(z)\,dA.$$

Theorem 3.2.14 ***(Schwarz Reflection).***
If f is continuous and holomorphic on $\mathbb{H}^+$ and real-valued on $\mathbb{R}$, then the extension defined by $F(z) = \overline{f(\overline{z})}$ for $z \in \mathbb{H}^-$ is a well-defined holomorphic function on $\mathbb{C}$.

Theorem 3.2.15 ***(Classification of Conformal Maps).***
There are 8 major types of conformal maps:

Type/Domains	Formula
Translation/Dilation/Rotation	$z \mapsto e^{i\theta}(cz+h)$
Sectors to sectors	$z \mapsto z^n$
$\mathbb{D}_{\frac{1}{2}} \to \mathbb{H}_{\frac{1}{2}}$, the first quadrant	$z \mapsto \frac{1+z}{1-z}$
$\mathbb{H} \to S$	$z \mapsto \log(z)$
$\mathbb{D}_{\frac{1}{2}} \to S_{\frac{1}{2}}$	$z \mapsto \log(z)$
$S_{\frac{1}{2}} \to \mathbb{D}_{\frac{1}{2}}$	$z \mapsto e^{iz}$
$\mathbb{D}_{\frac{1}{2}} \to \mathbb{H}$	$z \mapsto \frac{1}{2}\left(z+\frac{1}{z}\right)$
$S_{\frac{1}{2}} \to \mathbb{H}$	$z \mapsto \sin(z)$

Theorem 3.2.16 ***(Characterization of conformal maps).***
Conformal maps $\mathbb{D} \to \mathbb{D}$ have the form

$$g(z) = \lambda\frac{1-a}{1-\bar{a}z}, \quad |a|<1, \quad |\lambda|=1.$$

3.2.2 Others

Theorem 3.2.17 ***(Riemann Mapping).***
If Ω is simply connected, nonempty, and not $\mathbb{C}$, then for every $z_0 \in \Omega$ there exists a unique conformal map $F : \Omega \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$.
Thus any two such sets Ω_1, Ω_2 are conformally equivalent.

Theorem 3.2.18 ***(Riemann's Removable Singularity Theorem).***
If f is holomorphic on Ω except possibly at z_0 and f is bounded on $\Omega \setminus \{z_0\}$, then z_0 is a removable singularity.

Proposition 3.2.19 ***(Holomorphic functions have harmonic components).***
If $f(z) = u(x,y) + iv(x,y)$ is holomorphic, then u, v are harmonic.

Proposition 3.2.20 ***(Holomorphic functions are continuous.).***
f is holomorphic at z_0 iff there exists an $a \in \mathbb{C}$ such that

$$f(z_0+h) - f(z_0) - ah = h\psi(h), \quad \psi(h) \overset{h\to 0}{\to} 0.$$

In this case, $a = f'(z_0)$.

Proposition 3.2.21 ***(Cauchy-Riemann implies holomorphic).***
If $f = u + iv$ with $u, v \in C^1(\mathbb{R})$ satisfying the Cauchy-Riemann equations on Ω, then f is holomorphic on Ω and $f'(z) = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) f$.

Proposition 3.2.22 ***(Polar Cauchy-Riemann equations).***

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Proof .

Concepts Used:

- See walkthrough here.
- See problem set 1.

- Take derivative along two paths, along a ray with constant angle θ_0 and along a circular arc of constant radius r_0.
- Then equate real and imaginary parts.

■

Theorem 3.2.23 ***(Open Mapping).***
Any holomorphic non-constant map is an open map.

3.3 Series and Analytic Functions

Proposition 3.3.1 ***(Power Series are Smooth).***
Any power series is smooth and its derivatives can be obtained using term-by-term differentiation.

Proposition 3.3.2 ***(Uniform Convergence of Series).***
A series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly iff

$$\lim_{n\to\infty} \left\| \sum_{k\geq n} f_k \right\|_{\infty} = 0.$$

Theorem 3.3.3 ***(Weierstrass M-Test).***
If $\{f_n\}$ with $f_n : \Omega \to \mathbb{C}$ and there exists a sequence $\{M_n\}$ with $\|f_n\|_\infty \leq M_n$ and $\sum_{n\in\mathbb{N}} M_n < \infty$, then $f(x) \coloneqq \sum_{n\in\mathbb{N}} f_n(x)$ converges absolutely and uniformly on Ω.
Moreover, if the f_n are continuous, by the uniform limit theorem, f is again continuous.

Proposition 3.3.4 ***(Exponential is uniformly convergent in discs).***
$f(z) = e^z$ is uniformly convergent in any disc in $\mathbb{C}$.

Proof .
Apply the estimate

$$|e^z| \leq \sum \frac{|z|^n}{n!} = e^{|z|}.$$

Now by the M-test,

$$|z| \leq R < \infty \implies \left|\sum \frac{z^n}{n!}\right| \leq e^R < \infty.$$

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Proposition 3.3.5 ***(Checking radius of convergence).***
For a power series $f(z) = \sum a_n z^n$, define R by

$$\frac{1}{R} \coloneqq \limsup |a_n|^{\frac{1}{n}}.$$

Then f converges absolutely on $|z| < R$ and diverges on $|z| > R$.

3.4 Others

Theorem 3.4.1 ***(Casorati-Weierstrass).***
If f is holomorphic on $\Omega \setminus \{z_0\}$ where z_0 is an essential singularity, then for every $V \subset \Omega \setminus \{z_0\}$, $f(V)$ is dense in $\mathbb{C}$.

The image of a disc punctured at an essential singularity is dense in $\mathbb{C}$.

Theorem 3.4.2 ***(Little Picard).***
If $f : \mathbb{C} \to \mathbb{C}$ is entire and nonconstant, then $\operatorname{im}(f)$ is either $\mathbb{C}$ or $\mathbb{C} \setminus \{z_0\}$ for some point z_0.

???

Theorem 3.4.3 ***(Continuation Principle / Identity Theorem).***
If f is holomorphic on a bounded connected domain Ω and there exists a sequence $\{z_i\}$ with a limit point in Ω such that $f(z_i) = 0$, then $f \equiv 0$ on Ω.

Two functions agreeing on a set with a limit point are equal on a domain.

Corollary 3.4.4.
The ring of holomorphic functions on a domain in $\mathbb{C}$ has no zero divisors.

Proof .
???

■

Find the proof!

Proposition 3.4.5 ***(Injectivity Relates to Derivatives).***
If z_0 is a zero of f' of order n, then f is $(n+1)$-to-one in a neighborhood of z_0.

Proof .
?

■

Proposition 3.4.6 ***(Bounded Complex Analytic Functions form a Banach Space).***
For $\Omega \subseteq \mathbb{C}$, show that $A(\mathbb{C}) \coloneqq \left\{ f : \Omega \to \mathbb{C} \;\middle|\; f \text{ is bounded} \right\}$ is a Banach space.

Proof .
?

Apply Morera's Theorem and Cauchy's Theorem

■

4 Residues

Remark 4.0.1: Check: do you need residues? You may be able to just compute an integral

- Directly by parameterization:

$$\int_\gamma f = \int_a^b f(z(t))\, z'(t) \qquad \text{for } z(t) \text{ a parameterization of } \gamma,$$

- Finding a primitive F,
- Writing $z = z_0 + re^{i\theta}$

Proof (of Cauchy's inequality).

- Given $z_0 \in \Omega$, pick the largest disc $D_R(z_0) \subset \Omega$ and let $C_R = \partial D_R$.
- Then apply the integral formula.

■

Proposition 4.0.2 ***(Residues for simple poles (order 1)).***
If z_0 is a simple pole of f, then

$$\operatorname*{Res}_{z_0} f = \lim_{z \to z_0} (z - z_0) f(z).$$

Example 4.0.3 ***(Residue of a simple pole (order 1)):*** Let $f(z) = \dfrac{1}{1+z^2}$, then $\operatorname{Res}(i, f) = \dfrac{1}{2i}$.

Proposition 4.0.4 ***(For higher order poles).***
If f has a pole z_0 of order n, then

$$\operatorname*{Res}_{z=z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{\partial}{\partial z}\right)^{n-1} (z - z_0)^n f(z).$$

5 Conformal Maps

Fact 5.0.1
A bijective holomorphic map automatically has a holomorphic inverse. This can be weakened: an injective holomorphic map satisfies $f'(z) \neq 0$ and f^{-1} is well-defined on its range and holomorphic.

5.1 Plane to Disc

$$\varphi : \mathbb{H} \to \mathbb{D}$$
$$\varphi(z) = \frac{z-i}{z+i} \qquad f^{-1}(z) = i\left(\frac{1+w}{1-w}\right).$$

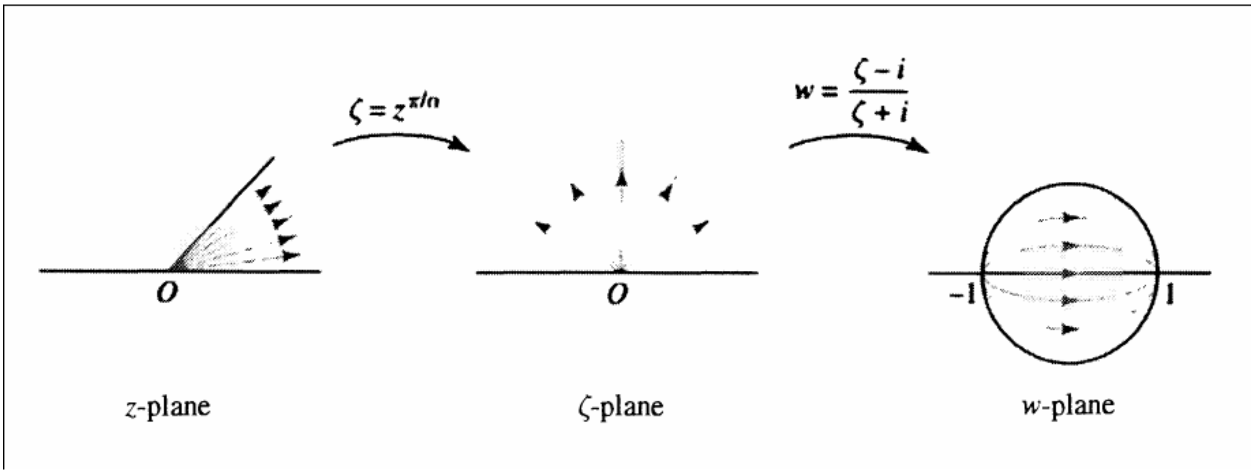
5.2 Sector to Disc

For $S_\alpha \coloneqq \left\{ z \in \mathbb{C} \,\middle|\, 0 < \arg(z) < \alpha \right\}$ an open sector for α some angle, first map the sector to the half-plane:

$$\begin{aligned} g &: S_\alpha \to \mathbb{H} \\ g(z) &= z^{\frac{\pi}{\alpha}}. \end{aligned}$$

Then compose with a map $\mathbb{H} \to \mathbb{D}$:

$$\begin{aligned} f &: S_\alpha \to \mathbb{D} \\ f(z) &= (\varphi \circ g)(z) = \frac{z^{\frac{\pi}{\alpha}} - i}{z^{\frac{\pi}{\alpha}} + i}. \end{aligned}$$



5.3 Strip to Disc

- Map to horizontal strip by rotation $z \mapsto \lambda z$.
- Map horizontal strip to sector by $z \mapsto e^z$
- Map sector to $\mathbb{H}$ by $z \mapsto z^{\frac{\pi}{\alpha}}$.
- Map $\mathbb{H} \to \mathbb{D}$.

6 Schwarz Reflection

Remark 6.0.1: $\mathbb{H}^+, \mathbb{H}^-$ can be replaced with any region symmetric about a line segment $L \subseteq \mathbb{R}$.

7 Zeros and Poles

7.1 Counting Zeros

Example 7.1.1: • Take $P(z) = z^4 + 6z + 3$.

- On $|z| < 2$:
  - Set $f(z) = z^4$ and $g(z) = 6z + 3$, then $|g(z)| \leq 6|z| + 3 = 15 < 16 = |f(z)|$.
  - So P has 4 zeros here.
- On $|z| < 1$:
  - Set $f(z) = 6z$ and $g(z) = z^4 + 3$.
  - Check $|g(z)| \leq |z|^4 + 3 = 4 < 6 = |f(z)|$.
  - So P has 1 zero here.

Example 7.1.2: • Claim: the equation $\alpha z e^z = 1$ where $|\alpha| > e$ has exactly one solution in $\mathbb{D}$.

- Set $f(z) = \alpha z$ and $g(z) = e^{-z}$.
- Estimate at $|z| = 1$ we have $|g| = |e^{-z}| = e^{-\Re(z)} \leq e^1 < |\alpha| = |f(z)|$
- f has one zero at $z_0 = 0$, thus so does $f + g$.

8 Linear Fractional Transformations

9 Appendix: Proofs of the Fundamental Theorem of Algebra

9.0.1 Fundamental Theorem of Algebra: Argument Principle

- Let $P(z) = a_n z^n + \cdots + a_0$ and $g(z) = P'(z)/P(z)$, note P is holomorphic
- Since $\lim_{|z| \to \infty} P(z) = \infty$, there exist an $R > 0$ such that P has no roots in $\{|z| \geq R\}$.
- Apply the argument principle:

$$N(0) = \frac{1}{2\pi i} \oint_{|\xi|=R} g(\xi)\, d\xi.$$

- Check that $\lim_{|z \to \infty|} z g(z) = n$, so g has a simple pole at ∞
- Then g has a Laurent series $\frac{n}{z} + \frac{c_2}{z^2} + \cdots$
- Integrate term-by-term to get $N(0) = n$.

9.0.2 Fundamental Theorem of Algebra: Rouche's Theorem

- Let $P(z) = a_n z^n + \cdots + a_0$
- Set $f(z) = a_n z^n$ and $g(z) = P(z) - f(z) = a_{n-1}z^{n-1} + \cdots + a_0$, so $f + g = P$.
- Choose $R > \max\left(\dfrac{|a_{n-1}| + \cdots + |a_0|}{|a_n|}, 1\right)$, then

$$\begin{aligned}
|g(z)| &\coloneqq |a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| \\
&\leq |a_{n-1}z^{n-1}| + \cdots + |a_1 z| + |a_0| \quad \text{by the triangle inequality} \\
&= |a_{n-1}| \cdot |z^{n-1}| + \cdots + |a_1| \cdot |z| + |a_0| \\
&= |a_{n-1}| \cdot R^{n-1} + \cdots + |a_1| R + |a_0| \\
&\leq |a_{n-1}| \cdot R^{n-1} + |a_{n-2}| \cdot R^{n-1} + \cdots + |a_1| \cdot R^{n-1} + |a_0| \cdot R^{n-1} \quad \text{since } R > 1 \implies R^{a+b} \geq R^a \\
&= R^{n-1} \left(|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0|\right) \\
&\leq R^{n-1} \left(|a_n| \cdot R\right) \quad \text{by choice of } R \\
&= R^n |a_n| \\
&= |a_n z^n| \\
&\coloneqq |f(z)|
\end{aligned}$$

- Then $a_n z^n$ has n zeros in $|z| < R$, so $f + g$ also has n zeros.

9.0.3 Fundamental Theorem of Algebra: Liouville's Theorem

- Suppose p is nonconstant and has no roots, then $\dfrac{1}{p}$ is entire. We will show it is also bounded and thus constant, a contradiction.
- Write $p(z) = z^n \left(a_n + \dfrac{a_{n-1}}{z} + \cdots + \dfrac{a_0}{z^n}\right)$
- Outside a disc:
  - Note that $p(z) \overset{z\to\infty}{\to} \infty$. so there exists an R large enough such that $|p(z)| \geq \dfrac{1}{A}$ for any fixed chosen constant A.
  - Then $|1/p(z)| \leq A$ outside of $|z| > R$, i.e. $1/p(z)$ is bounded there.
- Inside a disc:
  - p is continuous with no roots and thus must be bounded below on $|z| < R$.
  - p is entire and thus continuous, and since $\overline{D}_r(0)$ is a compact set, p achieves a min A there
  - Set $C \coloneqq \min(A, B)$, then $|p(z)| \geq C$ on all of $\mathbb{C}$ and thus $|1/p(z)| \leq C$ everywhere.
  - So $1/p(z)$ is bounded an entire and thus constant by Liouville's theorem – but this forces p to be constant. ※

9.0.4 Fundamental Theorem of Algebra: Open Mapping Theorem

- p induces a continuous map $\mathbb{CP}^1 \to \mathbb{CP}^1$
- The continuous image of compact space is compact;
- Since the codomain is Hausdorff space, the image is closed.
- p is holomorphic and non-constant, so by the Open Mapping Theorem, the image is open.
- Thus the image is clopen in $\mathbb{CP}^1$.
- The image is nonempty, since $p(1) = \sum a_i \in \mathbb{C}$
- $\mathbb{CP}^1$ is connected
- But the only nonempty clopen subset of a connected space is the entire space.
- So p is surjective, and $p^{-1}(0)$ is nonempty.
- So p has a root.

10 | Appendix

10.1 Misc Basic Algebra

Fact 10.1.1 (Standard forms of conic sections)

- Circle: $x^2 + y^2 = r^2$
- Ellipse: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$
- Hyperbola: $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$
  - Rectangular Hyperbola: $xy = \frac{c^2}{2}$.
- Parabola: $-4ax + y^2 = 0$.

Mnemonic: Write $f(x,y) = Ax^2 + Bxy + Cy^2 + \cdots$, then consider the discriminant $\Delta = B^2 - 4AC$:

- $\Delta < 0 \iff$ ellipse
  - $\Delta < 0$ and $A = C, B = 0 \iff$ circle
- $\Delta = 0 \iff$ parabola
- $\Delta > 0 \iff$ hyperbola

Fact 10.1.2 (Completing the square)

$$x^2 - bx = (x-s)^2 - s^2 \quad \text{where} s = \frac{b}{2}$$
$$x^2 + bx = (x+s)^2 - s^2 \quad \text{where} s = \frac{b}{2}.$$

Fact 10.1.3
The sum of the interior angles of an n-gon is $(n-2)\pi$, where each angle is $\frac{n-2}{n}\pi$.

Basics

- Show that $\frac{1}{z}\sum_{k=1}^{\infty}\frac{z^k}{k}$ converges on $S^1 \setminus \{1\}$ using summation by parts.
- Show that any power series is continuous on its domain of convergence.
- Show that a uniform limit of continuous functions is continuous.

??

- Show that if f is holomorphic on $\mathbb{D}$ then f has a power series expansion that converges uniformly on every compact $K \subset \mathbb{D}$.
- Show that any holomorphic function f can be uniformly approximated by polynomials.
- Show that if f is holomorphic on a connected region Ω and $f' \equiv 0$ on Ω, then f is constant on Ω.
- Show that if $|f| = 0$ on $\partial\Omega$ then either f is constant or f has a zero in Ω.
- Show that if $\{f_n\}$ is a sequence of holomorphic functions converging uniformly to a function f on every compact subset of Ω, then f is holomorphic on Ω and $\left\{f_n'\right\}$ converges uniformly to f' on every such compact subset.
- Show that if each f_n is holomorphic on Ω and $F \coloneqq \sum f_n$ converges uniformly on every compact subset of Ω, then F is holomorphic.
- Show that if f is once complex differentiable at each point of Ω, then f is holomorphic.

11 | Draft of Problem Book

- Prove the triangle inequality

- Prove the reverse triangle inequality
- Show that $\sum z^{k-1}/k$ converges for all $z \in S^1$ except $z = 1$.
- What is an example of a noncontinuous limit of continuous functions?
- Show that the uniform limit of continuous functions is continuous.
- Show that f is holomorphic if and only if $\bar{\partial} f = 0$.
- Show $n^{\frac{1}{n}} \xrightarrow{n\to\infty} 1$.
- Show that if f is holomorphic with $f' = 0$ on Ω then f is constant.
- Show that holomorphic implies analytic.
- Use Cauchy's inequality to prove Liouville's theorem

Problem 11.0.1 (?)
What is a pair of conformal equivalences between $\mathbb{H}$ and $\mathbb{D}$?

Solution:

$$\begin{aligned} F : HH &\to \mathbb{D} \\ z &\mapsto \frac{i-z}{i+z} \end{aligned}$$

$$\begin{aligned} G : \mathbb{D} &\to \mathbb{H} \\ w &\mapsto i\frac{1-w}{1+w}. \end{aligned}$$

Mnemonic: any point in $\mathbb{H}$ is closer to i than $-i$, so $|F(z)| < 1$.

- Maps $\mathbb{R} \to S^1 \setminus \{-1\}$.

Problem 11.0.2 (?)
What is conformal equivalence $\mathbb{H} \rightleftharpoons S \coloneqq \left\{ w \in \mathbb{C} \;\middle|\; 0 < \arg(w) < \alpha\pi \right\}$?

Solution:

$$f(z) = z^{\alpha}.$$