

Complex Analysis Problems

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1 Integrals and Cauchy's Theorem

1.1 1

Suppose $f, g : [0, 1] \rightarrow \mathbb{R}$ where f is Riemann integrable and for $x, y \in [0, 1]$,

$$|g(x) - g(y)| \leq |f(x) - f(y)|.$$

Prove that g is Riemann integrable.

1.2 2

State and prove Green's Theorem for rectangles.

Then use it to prove Cauchy's Theorem for functions that are analytic in a rectangle.

1.3 3

Suppose $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of analytic functions on $\mathbb{D}^\circ := \{z \in \mathbb{C} \mid |z| < 1\}$.

Show that if $f_n \rightarrow g$ for some $g : \mathbb{D}^\circ \rightarrow \mathbb{C}$ uniformly on every compact $K \subset \mathbb{D}^\circ$, then g is analytic on \mathbb{D}° .

1.4 4

Suppose $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of entire functions where

- $f_n \rightarrow g$ pointwise for some $g : \mathbb{C} \rightarrow \mathbb{C}$.
- On every line segment in \mathbb{C} , $f_n \rightarrow g$ uniformly.

Show that

- g is entire, and
- $f_n \rightarrow g$ uniformly on every compact subset of \mathbb{C} .

1.5 5

Prove that there is no sequence of polynomials that uniformly converge to $f(z) = \frac{1}{z}$ on S^1 .

1.6 6

Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function that vanishes outside of some finite interval. For each $z \in \mathbb{C}$, define

$$g(z) = \int_{-\infty}^{\infty} f(t)e^{-izt} dt.$$

Show that g is entire.

1.7 7

Suppose $f : \mathbb{C} \rightarrow \mathbb{C}$ is entire and

$$|f(z)| \leq |z|^{\frac{1}{2}} \quad \text{when } |z| > 10.$$

Prove that f is constant.

1.8 8

Let γ be a smooth curve joining two distinct points $a, b \in \mathbb{C}$.

Prove that the function

$$f(z) := \int_{\gamma} \frac{g(w)}{w - z} dw$$

is analytic in $\mathbb{C} \setminus \gamma$.

1.9 9

Suppose that $f : \mathbb{C} \rightarrow \mathbb{C}$ is continuous everywhere and analytic on $\mathbb{C} \setminus \mathbb{R}$ and prove that f is entire.

1.10 10

Prove Liouville's theorem: suppose $f : \mathbb{C} \rightarrow \mathbb{C}$ is entire and bounded. Use Cauchy's formula to prove that $f' \equiv 0$ and hence f is constant.

2 Liouville's Theorem, Power Series**2.1 1**

Suppose f is analytic on a region Ω such that $\mathbb{D} \subseteq \Omega \subseteq \mathbb{C}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series with radius of convergence exactly 1.

- Give an example of such an f that converges at every point of S^1 .
- Given an example of such an f which is analytic at 1 but $\sum_{n=0}^{\infty} a_n$ diverges.
- Prove that f can not be analytic at *every* point of S^1 .

2.2 2

Suppose f is entire and has Taylor series $\sum a_n z^n$ about 0.

- Express a_n as a contour integral along the circle $|z| = R$.
- Apply (a) to show that the above Taylor series converges uniformly on every bounded subset of \mathbb{C} .
- Determine those functions f for which the above Taylor series converges uniformly on all of \mathbb{C} .

2.3 3

Suppose D is a domain and f, g are analytic on D .

Prove that if $fg = 0$ on D , then either $f \equiv 0$ or $g \equiv 0$ on D .

2.4 4

Suppose f is analytic on \mathbb{D}° . Determine with proof which of the following are possible:

- $f\left(\frac{1}{n}\right) = (-1)^n$ for each $n > 1$.
- $f\left(\frac{1}{n}\right) = e^{-n}$ for each even integer $n > 1$ while $f\left(\frac{1}{n}\right) = 0$ for each odd integer $n > 1$.
- $f\left(\frac{1}{n^2}\right) = \frac{1}{n}$ for each integer $n > 1$.
- $f\left(\frac{1}{n}\right) = \frac{n-2}{n-1}$ for each integer $n > 1$.

2.5 5

Prove the Fundamental Theorem of Algebra (using complex analysis).

2.6 6

Find all entire functions that satisfy

$$|f(z)| \geq |z| \quad \forall z \in \mathbb{C}.$$

Prove this list is complete.

2.7 7

Suppose $\sum_{n=0}^{\infty} a_n z^n$ converges for some $z_0 \neq 0$.

- Prove that the series converges absolutely for each z with $|z| < |z_0|$.
- Suppose $0 < r < |z_0|$ and show that the series converges uniformly on $|z| \leq r$.

2.8 8

Suppose f is entire and suppose that for some integer $n \geq 1$,

$$\lim_{z \rightarrow \infty} \frac{f(z)}{z^n} = 0.$$

Prove that f is a polynomial of degree at most $n - 1$.

2.9 9

Find all entire functions satisfying

$$|f(z)| \leq |z|^{\frac{1}{2}} \quad \text{for } |z| > 10.$$

2.10 10

Prove that the following series converges uniformly on the set $\{z \mid \Im(z) < \ln 2\}$:

$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{2^n}.$$

3 Spring 2020 Homework 1**4 Spring 2020 Homework 2**

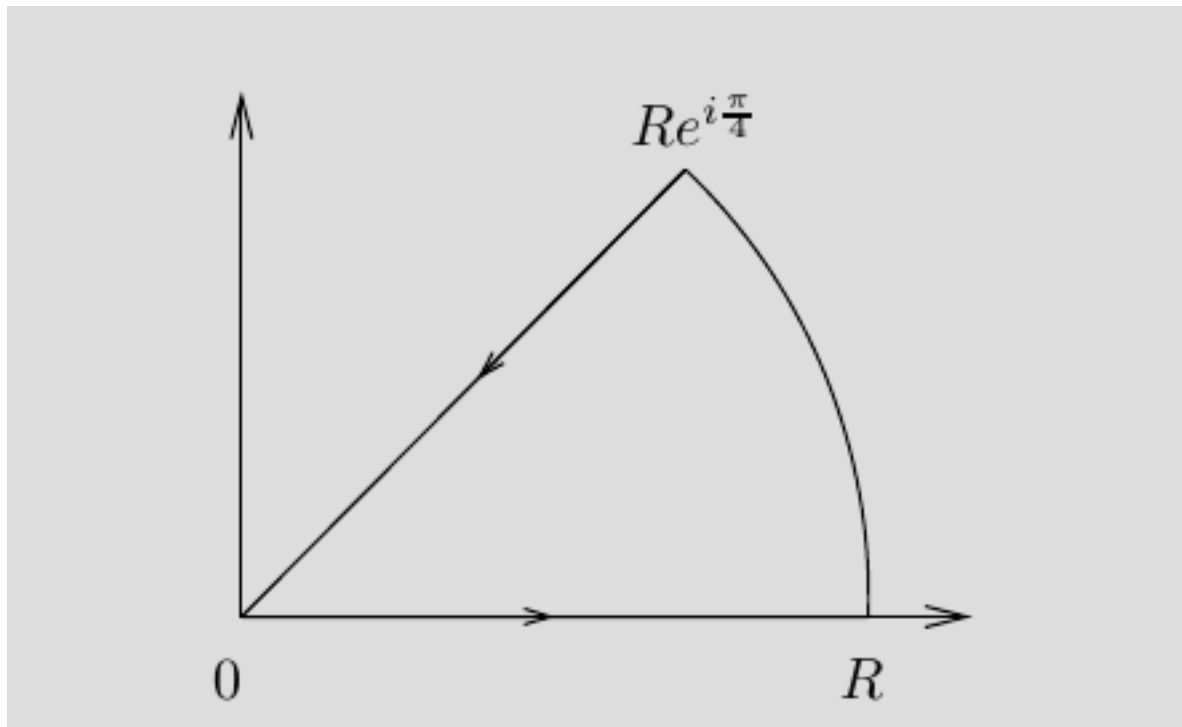
Note on notation: I sometimes use $f_x := \frac{\partial f}{\partial x}$ to denote partial derivatives, and $\partial_z^n f$ as $f^{(n)}(z)$.

4.1 Stein And Shakarchi**4.1.1 2.6.1**

Show that

$$\int_0^\infty \sin(x^2) dx = \int_0^\infty \cos(x^2) dx = \frac{\sqrt{2\pi}}{4}.$$

Hint: integrate e^{-x^2} over the following contour, using the fact that $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$:

**4.1.2 2.6.2**

Show that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Hint: use the fact that this integral equals $\frac{1}{2i} \int_{-\infty}^\infty \frac{e^{ix} - 1}{x} dx$, and integrate around an indented semicircle.

4.1.3 2.6.5

Suppose $f \in C^1_{\mathbb{C}}(\Omega)$ and $T \subset \Omega$ is a triangle with $T^\circ \subset \Omega$. Apply Green's theorem to show that $\int_T f(z) dz = 0$.

Assume that f' is continuous and prove Goursat's theorem.

Hint: Green's theorem states

$$\int_T F dx + G dy = \int_{T^\circ} \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx dy.$$

4.1.4 2.6.6

Suppose that f is holomorphic on a punctured open set $\Omega \setminus \{w_0\}$ and let $T \subset \Omega$ be a triangle containing w_0 . Prove that if f is bounded near w_0 , then $\int_T f(z) dz = 0$.

4.1.5 2.6.7

Suppose $f : \mathbb{D} \rightarrow \mathbb{C}$ is holomorphic and let $d := \sup_{z, w \in \mathbb{D}} |f(z) - f(w)|$ be the diameter of the image of f . Show that $2|f'(0)| \leq d$, and that equality holds iff f is linear, so $f(z) = a_1 z + a_2$.

Hint: $2f'(0) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi) - f(-\xi)}{\xi^2} d\xi$ whenever $0 < r < 1$.

4.1.6 2.6.8

Suppose that f is holomorphic on the strip $S = \{x + iy \mid x \in \mathbb{R}, -1 < y < 1\}$ with $|f(z)| \leq A(1 + |z|)^\nu$ for ν some fixed real number. Show that for all $z \in S$, for each integer $n \geq 0$ there exists an $A_n \geq 0$ such that $|f^{(n)}(x)| \leq A_n(1 + |x|)^\nu$ for all $x \in \mathbb{R}$.

Hint: Use the Cauchy inequalities.

4.1.7 2.6.9

Let $\Omega \subset \mathbb{C}$ be open and bounded and $\varphi : \Omega \rightarrow \Omega$ holomorphic. Prove that if there exists a point $z_0 \in \Omega$ such that $\varphi(z_0) = z_0$ and $\varphi'(z_0) = 1$, then φ is linear.

Hint: assume $z_0 = 0$ (explain why this can be done) and write $\varphi(z) = z + a_n z^n + O(z^{n+1})$ near 0. Let $\varphi_k = \varphi \circ \varphi \circ \dots \circ \varphi$ and prove that $\varphi_k(z) = z + k a_n z^n + O(z^{n+1})$. Apply Cauchy's inequalities and let $k \rightarrow \infty$ to conclude.

4.1.8 2.6.10

Can every continuous function on $\overline{\mathbb{D}}$ be uniformly approximated by polynomials in the variable z ?

Hint: compare to Weierstrass for the real interval.

4.1.9 2.6.13

Suppose f is analytic, defined on all of \mathbb{C} , and for each $z_0 \in \mathbb{C}$ there is at least one coefficient in the expansion $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ is zero. Prove that f is a polynomial.

Hint: use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.

4.1.10 2.6.14

Suppose that f is holomorphic in an open set containing \mathbb{D} except for a pole $z_0 \in \partial\mathbb{D}$. Let $\sum_{n=0}^{\infty} a_n z^n$ be the power series expansion of f in \mathbb{D} , and show that $\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} = z_0$.

4.1.11 2.6.15

Suppose f is continuous and nonvanishing on $\bar{\mathbb{D}}$, and holomorphic in \mathbb{D} . Prove that if $|z| = 1 \implies |f(z)| = 1$, then f is constant.

Hint: Extend f to all of \mathbb{C} by $f(z) = 1/\overline{f(1/\bar{z})}$ for any $|z| > 1$, and argue as in the Schwarz reflection principle.

4.2 Additional Problems**4.2.1 1**

Let $a_n \neq 0$ and show that

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L \implies \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = L.$$

In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

4.2.2 2

Let f be a power series centered at the origin. Prove that f has a power series expansion about any point in its disc of convergence.

4.2.3 3

Prove the following:

- $\sum_n n z^n$ does not converge at any point of S^1
- $\sum_n \frac{z^n}{n^2}$ converges at every point of S^1 .

c. $\sum_n \frac{z^n}{n}$ converges at every point of S^1 except $z = 1$.

4.2.4 4

Without using Cauchy's integral formula, show that if $|a| < r < |b|$, then

$$\int_{\gamma} \frac{dz}{(z - \alpha)(z - \beta)} = \frac{2\pi i}{\alpha - \beta}$$

where γ denotes the circle centered at the origin of radius r with positive orientation.

4.2.5 5

Assume f is continuous in the region $\{x + iy \mid x \geq x_0, 0 \leq y \leq b\}$, and the following limit exists independent of y :

$$\lim_{x \rightarrow +\infty} f(x + iy) = A.$$

Show that if $\gamma_x := \{z = x + it \mid 0 \leq t \leq b\}$, then

$$\lim_{x \rightarrow +\infty} \int_{\gamma_x} f(z) dz = iAb.$$

4.2.6 6

Show by example that there exists a function $f(z)$ that is holomorphic on $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$ and for all $r < 1$,

$$\int_{|z|=r} f(z) dz = 0,$$

but f is not holomorphic at $z = 0$.

4.2.7 7

Let f be analytic on a region R and suppose $f'(z_0) \neq 0$ for some $z_0 \in R$. Show that if C is a circle of sufficiently small radius centered at z_0 , then

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{dz}{f(z) - f(z_0)}.$$

Hint: use the inverse function theorem.

4.2.8 8

Assume two functions $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$ have continuous partial derivatives at (x_0, y_0) . Show that $f := u + iv$ has derivative $f'(z_0)$ at $z_0 = x_0 + iy_0$ if and only if

$$\lim_{r \rightarrow 0} \frac{1}{\pi r^2} \int_{|z - z_0| = r} f(z) dz = 0.$$

4.2.9 9 (Cauchy's Formula for Exterior Regions)

Let γ be a piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2 . Assume f' exists in an open set containing γ and Ω_2 with $\lim_{z \rightarrow \infty} f(z) = A$. Show that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1 \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}.$$

4.2.10 10

Let $f(z)$ be bounded and analytic in \mathbb{C} . Let $a \neq b$ be any fixed complex numbers. Show that the following limit exists:

$$\lim_{R \rightarrow \infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

Use this to show that $f(z)$ must be constant.

4.2.11 11

Suppose $f(z)$ is entire and

$$\lim_{z \rightarrow \infty} \frac{f(z)}{z} = 0.$$

Show that $f(z)$ is a constant.

4.2.12 12

Let f be analytic in a domain D and γ be a closed curve in D . For any $z_0 \in D$ not on γ , show that

$$\int_{\gamma} \frac{f'(z)}{(z-z_0)} dz = \int_{\gamma} \frac{f(z)}{(z-z_0)^2} dz.$$

Give a generalization of this result.

4.2.13 13

Compute

$$\int_{|z|=1} \left(z + \frac{1}{z} \right)^{2n} \frac{dz}{z}$$

and use it to show that

$$\int_0^{2\pi} \cos^{2n}(\theta) d\theta = 2\pi \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \right).$$

5 Spring 2020 Homework 3

5.1 Stein and Shakarchi

5.1.1 3.8.1

Use the following formula to show that the complex zeros of $\sin(\pi z)$ are exactly the integers, and they are each of order 1:

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}.$$

Calculate the residue of $\frac{1}{\sin(\pi z)}$ at $z = n \in \mathbb{Z}$.

5.1.2 3.8.2

Evaluate the integral

$$\int_{\mathbb{R}} \frac{dx}{1+x^4}.$$

What are the poles of $\frac{1}{1+z^4}$?

5.1.3 3.8.4

Show that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}, \quad \text{for all } a > 0.$$

5.1.4 3.8.5

Show that if $\xi \in \mathbb{R}$, then

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{(1+x^2)^2} dx = \frac{\pi}{2} (1 + 2\pi |\xi|) e^{-2\pi |\xi|}.$$

5.1.5 3.8.6

Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \pi.$$

5.1.6 3.8.7

Show that

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos \theta)^2} = \frac{2\pi a}{(a^2 - 1)^{3/2}}, \quad \text{whenever } a > 1.$$

5.1.7 3.8.8

Show that if $a, b \in \mathbb{R}$ with $a > |b|$, then

$$\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

5.1.8 3.8.9

Show that

$$\int_0^1 \log(\sin \pi x) dx = -\log 2.$$

Hint: use the following contour.

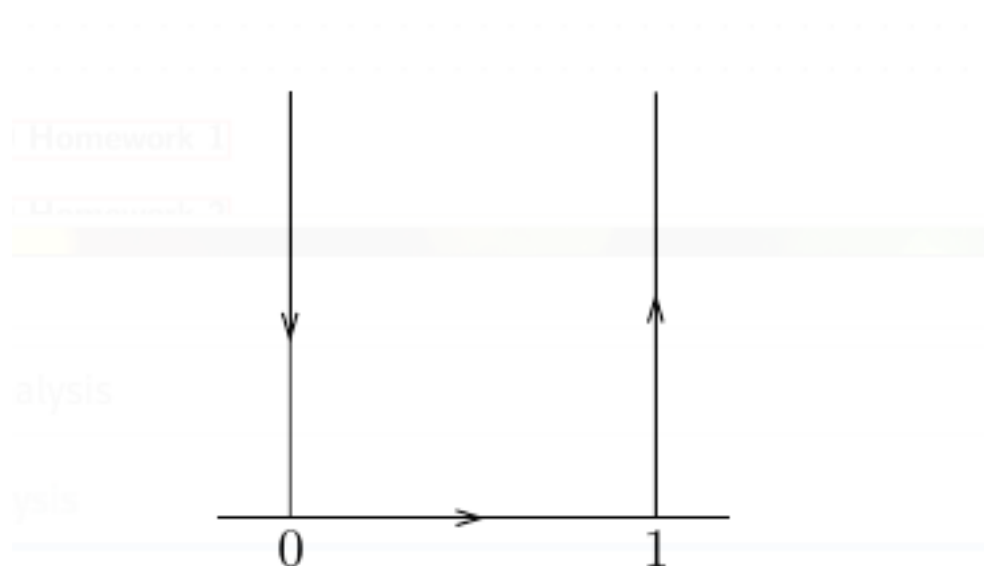


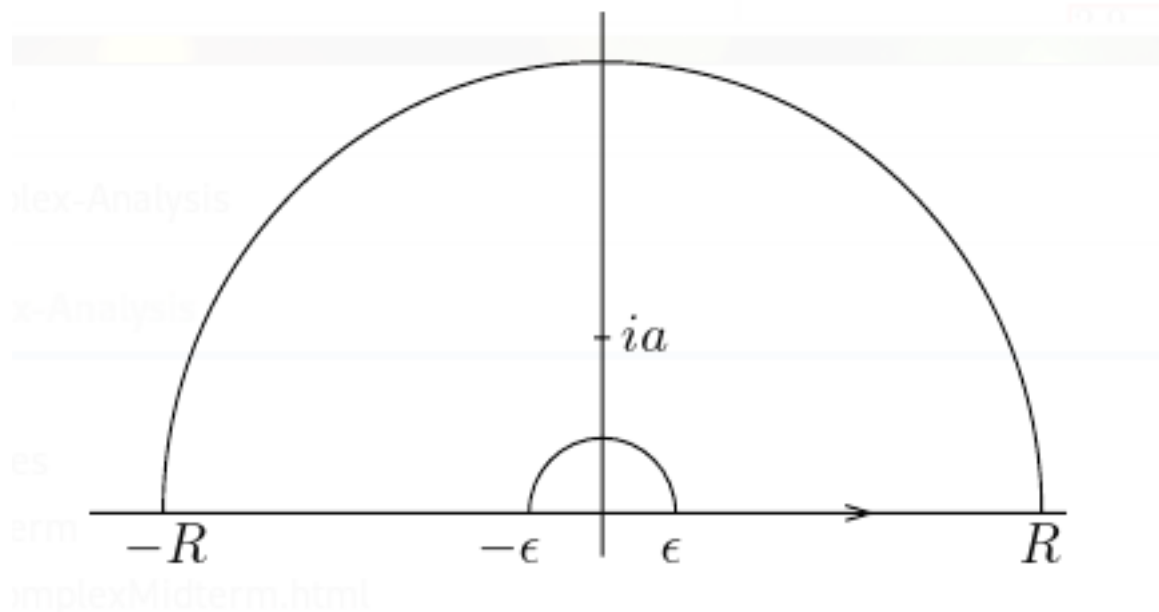
Figure 9. Contour in Exercise 9

5.1.9 3.8.10

Show that if $a > 0$, then

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx = \frac{\pi}{2a} \log a.$$

Hint: use the following contour.

**5.1.10 3.8.14**

Prove that all entire functions that are injective are of the form $f(z) = az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$.

Hint: Apply the Casorati-Weierstrass theorem to $f(1/z)$.

5.1.11 3.8.15

Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

- a. Prove that if f is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \leq AR^k + B$$

for all $R > 0$, some integer $k \geq 0$, and some constants $A, B > 0$, then f is a polynomial of degree $\leq k$.

- b. Show that if f is holomorphic in the unit disc, is bounded, and converges uniformly to zero in the sector $\theta < \arg(z) < \varphi$ as $|z| \rightarrow 0$, then $f \equiv 0$.
- c. Let w_1, \dots, w_n be points on $S^1 \subset \mathbb{C}$. Prove that there exists a point $z \in S^1$ such that the product of the distances from z to the points w_j is at least 1.

Conclude that there exists a point $w \in S^1$ such that the product of the above distances is *exactly* 1.

- d. Show that if the real part of an entire function is bounded, then f is constant.

5.1.12 3.8.17**5.1.13 3.8.19****5.2 Problems From Tie****5.2.1 1**

Prove that if f has two Laurent series expansions,

$$f(z) = \sum c_n(z-a)^n \quad \text{and} \quad f(z) = \sum c'_n(z-a)^n$$

then $c_n = c'_n$.

5.2.2 2

Find Laurent series expansions of

$$\frac{1}{1-z^2} + \frac{1}{3-z}$$

How many such expansions are there? In what domains are each valid?

5.2.3 3

Let P, Q be polynomials with no common zeros. Assume a is a root of Q . Find the principal part of P/Q at $z = a$ in terms of P and Q if a is (1) a simple root, and (2) a double root.

5.2.4 4

Let f be non-constant, analytic in $|z| > 0$, where $f(z_n) = 0$ for infinitely many points z_n with $\lim_{n \rightarrow \infty} z_n = 0$.

Show that $z = 0$ is an essential singularity for f .

Example: $f(z) = \sin(1/z)$.

5.2.5 5

Show that if f is entire and $\lim_{z \rightarrow \infty} f(z) = \infty$, then f is a polynomial.

5.2.6 6

- a. Show (without using 3.8.9 in the S&S) that

$$\int_0^{2\pi} \log |1 - e^{i\theta}| d\theta = 0$$

- b. Show that this identity is equivalent to S&S 3.8.9:

$$\int_0^1 \log(\sin(\pi x)) dx = -\log 2.$$

5.2.7 7

Let $0 < a < 4$ and evaluate

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^3} dx$$

5.2.8 8

Prove the fundamental theorem of Algebra using

- Rouche's Theorem.
- The maximum modulus principle.

5.2.9 9

Let f be analytic in a region D and γ a rectifiable curve in D with interior in D . Prove that if $f(z)$ is real for all $z \in \gamma$, then f is constant.

5.2.10 10

For $a > 0$, evaluate

$$\int_0^{\pi/2} \frac{d\theta}{a + \sin^2 \theta}$$

5.2.11 11

Find the number of roots of $p(z) = 4z^4 - 6z + 3$ in $|z| < 1$ and $1 < |z| < 2$ respectively.

5.2.12 12

Prove that $z^4 + 2z^3 - 2z + 10$ has exactly one root in each open quadrant.

5.2.13 13

Prove that for $a > 0$, $z \tan z - a$ has only real roots.

5.2.14 14

Let f be nonzero, analytic on a bounded region Ω and continuous on its closure $\overline{\Omega}$. Show that if $|f(z)| \equiv M$ is constant for $z \in \partial\Omega$, then $f(z) \equiv Me^{i\theta}$ for some real constant θ .

6 Extra Questions from Jingzhi Tie

6.1 Fall 2009

6.1.1 ?

(1) Assume $\displaystyle f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,
$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n};$$

(2) Deduce Liouville's theorem from (1).

6.1.2 ?

Let f be a continuous function in the region
 $D = \{z \text{ such that } |z| > R, 0 \leq \arg z \leq \theta\}$ where $0 < \theta < 2\pi$. If there exists k such that
$$\lim_{|z| \rightarrow \infty} zf(z) = k$$
 for z in the region D .
Show that $\lim_{R' \rightarrow \infty} \int_L f(z) dz = i\theta k$, where L is the part of the circle $|z| = R'$ which lies in the region D .

6.1.3 ?

Suppose that f is an analytic function in the region D which contains the point a . Let

$F(z) = z - a - qf(z), \quad \text{where } q \text{ is a complex parameter}.$

(1) Let $K \subset D$ be a circle with the center at point a and also we assume that $f(z) \neq 0$ for $z \in K$. Prove that the function F has one and only one zero $z=w$ on the closed disc \bar{K} whose boundary is the circle K if $\displaystyle{|q| < \min_{z \in K} \frac{|z-a|}{|f(z)|}}.$

(2) Let $G(z)$ be an analytic function on the disk \bar{K} . Apply the residue theorem to prove that

$$\frac{G(w)}{F'(w)} = \frac{1}{2\pi i} \int_K \frac{G(z)}{F(z)} dz,$$
 where w is the zero from (1).

(3) If $z \in K$, prove that the function

$$\frac{1}{F(z)}$$
 can be represented as a convergent series with respect to q :
$$\frac{1}{F(z)} = \sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z-a)^{n+1}}.$$

6.1.4 ?

Evaluate
$$\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx.$$

6.1.5 ?

Let $f = u + iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

6.1.6 ?

Show that
$$\int_0^{\infty} \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$$
 using complex analysis, $0 < a < n$. Here n is a positive integer.

6.1.7 ?

For $s > 0$, the **gamma function** is defined by
$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt.$$

1. Show that the gamma function is analytic in the half-plane

$\operatorname{Re}(s) > 0$, and is still given there by the integral formula above.

2. Apply the formula in the previous question to show that
- $$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

> Hint: You may need $\Gamma(1-s) = t \int_0^{\infty} e^{-vt} (vt)^{-s} dv$ for $t > 0$.

6.1.8 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree n , then it has n zeros in \mathbb{C} .

6.1.9 ?

Suppose f is entire and there exist $A, R > 0$ and natural number N such that $|f(z)| \geq A|z|^N$ for $|z| \geq R$. Show that

(i) f is a polynomial and

(ii) the degree of f is at least N .

6.1.10 ?

Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be an injective analytic (also called *univalent*) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

6.1.11 ?

Let g be analytic for $|z| \leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

1. Show that g has a unique fixed point in $|z| < 1$.
2. What happens if we replace $|g(z)| < 1$ with $|g(z)| \leq 1$ for $|z| = 1$? Give an example if (a) is not true or give a proof if (a) is still true.
3. What happens if we simply assume that f is analytic for $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z) \not\equiv z$. Can f have more than one fixed point in $|z| < 1$?

> Hint: The map $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$ may be useful.

6.1.12 ?

Find a conformal map from $D = \{z : |z| < 1, |z - 1/2| > 1/2\}$ to the unit disk $\Delta = \{z : |z| < 1\}$.

6.1.13 ?

Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω . Show without using Picard's theorems that $f(z)$ is a constant.

6.1.14 ?

(1) Assume $\displaystyle f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,
$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n};$$

(2) Deduce Liouville's theorem from (1).

6.1.15 ?

Let $f(z)$ be entire and assume that $f(z) \leq M|z|^2$ outside some disk for some constant M . Show that $f(z)$ is a polynomial in z of degree ≤ 2 .

6.1.16 ?

Let $a_n(z)$ be an analytic sequence in a domain D such that

$\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D . Show that $\sum_{n=0}^{\infty} |a'_n(z)|$ converges uniformly on bounded and closed sub-regions of D .

6.1.17 ?

Let $f(z)$ be analytic in an open set Ω except possibly at a point z_0 inside Ω . Show that if $f(z)$ is bounded in near z_0 , then $\displaystyle \int_{\Delta} f(z) dz = 0$ for all triangles Δ in Ω .

6.1.18 ?

Assume f is continuous in the region:

$0 < |z - a| \leq R$, $0 \leq \arg(z - a) \leq \beta_0$ ($0 < \beta_0 \leq 2\pi$) and the limit $\lim_{z \rightarrow a} (z - a)f(z) = A$ exists. Show that

$$\lim_{r \rightarrow 0} \int_{\gamma_r} f(z) dz = iA\beta_0,$$

where $\gamma_r := \{z \mid z = a + re^{it}, 0 \leq t \leq \beta_0\}$.

6.1.19 ?

Show that $f(z) = z^2$ is uniformly continuous in any open disk $|z| < R$, where $R > 0$ is fixed, but it is not uniformly continuous on \mathbb{C} .

6.1.20 ?

(1) Show that the function $u = u(x, y)$ given by

$$u(x, y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx \quad \text{for } n \in \mathbb{N}$$
 is the solution on $D = \{(x, y) \mid x^2 + y^2 < 1\}$ of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = \frac{\sin nx}{n}.$$
 (2) Show that there exist points $(x, y) \in D$ such that

$$\limsup_{n \rightarrow \infty} |u(x, y)| = \infty.$$

6.2 Fall 2011

6.2.1 ?

(1) Assume
$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$
 converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n};$$

(2) Deduce Liouville's theorem from (1).

6.2.2 ?

Let f be a continuous function in the region
 $D = \{z \mid |z| > R, 0 \leq \arg z \leq \theta\} \quad \text{where } 0 \leq \theta \leq 2\pi.$
 If there exists k such that

$$\lim_{z \rightarrow \infty} z f(z) = k$$
 for z in the region D . Show that

$$\lim_{R' \rightarrow \infty} \int_L f(z) dz = i\theta k,$$
 where L is the part of the circle $|z| = R'$ which lies in the region D .

6.2.3 ?

Suppose that f is an analytic function in the region D which contains the point a . Let

$$F(z) = z - a - qf(z), \quad \text{where } q \text{ is a complex parameter}.$$

(1) Let $K \subset D$ be a circle with the center at

point a and also we assume that $f(z) \neq 0$ for $z \in K$. Prove that the function F has one and only one zero $z=w$ on the closed disc \bar{K} whose boundary is the circle K if $\displaystyle \lim_{|z-a| \rightarrow 0} \frac{|z-a|}{|f(z)|} = \infty$.

(2) Let $G(z)$ be an analytic function on the disk \bar{K} . Apply the residue theorem to prove that
$$\frac{1}{2\pi i} \int_K \frac{G(z)}{F(z)} dz = \sum_{w \in K} \frac{1}{F'(w)}$$
 where w is the zero from (1).

(3) If $z \in K$, prove that the function
$$\frac{1}{F(z)}$$
 can be represented as a convergent series with respect to q :
$$\frac{1}{F(z)} = \sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z-a)^{n+1}}.$$

6.2.4 ?

Evaluate
$$\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx$$
.

6.2.5 ?

Let $f = u + iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

6.2.6 ?

Show that
$$\int_0^{\infty} \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$$
 using complex analysis, $0 < a < n$. Here n is a positive integer.

6.2.7 ?

For $s > 0$, the **gamma function** is defined by
$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt.$$

1. Show that the gamma function is analytic in the half-plane $\operatorname{Re}(s) > 0$, and is still given there by the integral formula above.
2. Apply the formula in the previous question to show that
$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

> Hint: You may need
$$\Gamma(1-s) = \int_0^{\infty} e^{-vt} (vt)^{-s} dv$$
 for $s > 0$.

6.2.8 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If $P_n(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n$ ($a_n \neq 0$) is a polynomial of degree n , then it has n zeros in \mathbb{C} .

6.2.9 ?

Suppose f is entire and there exist $A, R > 0$ and natural number N such that $|f(z)| \geq A|z|^N$ for $|z| \geq R$. Show that (i) f is a polynomial and (ii) the degree of f is at least N .

6.2.10 ?

Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be an injective analytic (also called univalent) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

6.2.11 ?

Let g be analytic for $|z| \leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

- Show that g has a unique fixed point in $|z| < 1$.
- What happens if we replace $|g(z)| < 1$ with $|g(z)| \leq 1$ for $|z| = 1$? Give an example if (a) is not true or give a proof if (a) is still true.
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> Hint: The map

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$$

> may be useful.

6.2.12 ?

Find a conformal map from $D = \{z : |z| < 1, |z - 1/2| > 1/2\}$ to the unit disk $\Delta = \{z : |z| < 1\}$.

6.2.13 ?

Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω . Show without using Picard's theorems that $f(z)$ is a constant.

6.2.14 ?

Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω . Show without using Picard's theorems that $f(z)$ is a constant.

6.2.15 ?

(1) Assume $\displaystyle f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}$$
;

(2) Deduce Liouville's theorem from (1).

6.2.16 ?

Let $f(z)$ be entire and assume that $f(z) \leq M|z|^2$ outside some disk for some constant M . Show that $f(z)$ is a polynomial in z of degree ≤ 2 .

6.2.17 ?

Let $a_n(z)$ be an analytic sequence in a domain D such that $\displaystyle \sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D . Show that $\displaystyle \sum_{n=0}^{\infty} |a'_n(z)|$ converges uniformly on bounded and closed sub-regions of D .

6.2.18 ?

Let $f(z)$ be analytic in an open set Ω except possibly at a point z_0 inside Ω . Show that if $f(z)$ is bounded in near z_0 , then $\displaystyle \int_{\Delta} f(z) dz = 0$ for all triangles Δ in Ω .

6.2.19 ?

Assume f is continuous in the region:
 $0 < |z-a| \leq R$, $0 \leq \arg(z-a) \leq \beta_0$
 $(0 < \beta_0 \leq 2\pi)$ and the limit

$$\lim_{z \rightarrow a} (z-a) f(z) = A$$
 exists. Show that

$$\lim_{r \rightarrow 0} \int_{\gamma_r} f(z) dz = i A \beta_0$$
;
 where

$$\gamma_r = \{ z \mid z = a + r e^{it}, 0 \leq t \leq \beta_0 \}$$
.

6.2.20 ?

Show that $f(z) = z^2$ is uniformly continuous in any open disk $|z| < R$, where $R > 0$ is fixed, but it is not uniformly continuous on \mathbb{C} .

- (1) Show that the function $u = u(x, y)$ given by
$$u(x, y) = \frac{e^{-ny} - e^{-ny}}{2n^2} \sin nx \quad \text{for } n \in \mathbb{N}$$
is the solution on $D = \{(x, y) \mid x^2 + y^2 < 1\}$ of the Cauchy problem for the Laplace equation
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = \frac{\sin nx}{n}.$$
- (2) Show that there exist points $(x, y) \in D$ such that
$$\limsup_{n \rightarrow \infty} |u(x, y)| = \infty.$$

6.3 Spring 2014**6.3.1 ?**

The question provides some insight into Cauchy's theorem. Solve the problem without using the Cauchy theorem.

- Evaluate the integral $\int_{\gamma} z^n dz$ for all integers n . Here γ is any circle centered at the origin with the positive (counterclockwise) orientation.
- Same question as (a), but with γ any circle not containing the origin.
- Show that if $|a| < r < |b|$, then
$$\int_{\gamma} \frac{dz}{(z-a)(z-b)} = \frac{2\pi i}{a-b}.$$
Here γ denotes the circle centered at the origin, of radius r , with the positive orientation.

6.3.2 ?

(1) Assume the infinite series
$$\sum_{n=0}^{\infty} c_n z^n$$
converges in $|z| < R$ and let $f(z)$ be the limit. Show that for $r < R$,
$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n};$$

(2) Deduce Liouville's theorem from (1). Liouville's theorem: If $f(z)$ is entire and bounded, then f is constant.

6.3.3 ?

Let f be a continuous function in the region

$D = \{z \mid |z| > R, 0 \leq \arg z \leq \theta\}$ $\quad \text{where} \quad 0 \leq \theta \leq 2\pi$. If there exists k such that $\lim_{z \rightarrow \infty} zf(z) = k$ for z in the region D . Show that $\lim_{R' \rightarrow \infty} \int_L f(z) dz = i\theta k$, where L is the part of the circle $|z| = R'$ which lies in the region D .

6.3.4 ?

Evaluate $\int_0^\infty \frac{x \sin x}{x^2 + a^2} dx$.

6.3.5 ?

Let $f = u + iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$
6.3.6 ?

Show that $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$ using complex analysis, $0 < a < n$. Here n is a positive integer.

6.3.7 ?

For $s > 0$, the **gamma function** is defined by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt.$$

- Show that the gamma function is analytic in the half-plane $\operatorname{Re}(s) > 0$, and is still given there by the integral formula above.
- Apply the formula in the previous question to show that $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$.

> Hint: You may need $\Gamma(1-s) = \int_0^\infty e^{-vt} (vt)^{-s} dv$ for $t > 0$.

6.3.8 ?

Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If $P_n(z) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + a_n z^n$ ($a_n \neq 0$) is a polynomial of degree n , then it has n zeros in \mathbb{C} .

6.3.9 ?

Suppose f is entire and there exist $A, R > 0$ and natural number N such that $|f(z)| \geq A |z|^N$ for $|z| \geq R$. Show that (i) f is a polynomial and (ii) the degree of f is at least N .

6.3.10 ?

Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be an injective analytic (also called univalent) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

6.3.11 ?

Let g be analytic for $|z| \leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

- Show that g has a unique fixed point in $|z| < 1$.
- What happens if we replace $|g(z)| < 1$ with $|g(z)| \leq 1$ for $|z| = 1$? Give an example if (a) is not true or give a proof if (a) is still true.
- What happens if we simply assume that f is analytic for $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z) \not\equiv z$. Can f have more than one fixed point in $|z| < 1$?

> Hint: The map

$$\psi_{\alpha}(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$$

> may be useful.

6.3.12 ?

Find a conformal map from $D = \{z : |z| < 1, |z - 1/2| > 1/2\}$ to the unit disk $\Delta = \{z : |z| < 1\}$.

6.4 Fall 2015**6.4.1 ?**

Let $a_n \neq 0$ and assume that $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L$. Show that $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$.
 $\rho_n^{\frac{1}{n}} = L$. In particular, this shows that when applicable, the ratio test can be used to calculate the radius of

convergence of a power series.

6.4.2 ?

(a) Let z, w be complex numbers, such that $|z| \neq 1$.

Prove that

$|\frac{w - z}{1 - \bar{w}z}| < 1$ if $|z| < 1$ and $|w| < 1$, and also that

$|\frac{w - z}{1 - \bar{w}z}| = 1$ if $|z| = 1$ or $|w| = 1$.

(b) Prove that for fixed w in the unit disk \mathbb{D} , the mapping $F: z \mapsto \frac{w - z}{1 - \bar{w}z}$ satisfies the following conditions:

(i) F maps \mathbb{D} to itself and is holomorphic.

(ii) F interchanges 0 and w , namely, $F(0) = w$ and $F(w) = 0$.

(iii) $|F(z)| = 1$ if $|z| = 1$.

(iv) $F: \mathbb{D} \rightarrow \mathbb{D}$ is bijective.

> Hint: Calculate $F \circ F$.

6.4.3 ?

Use n -th roots of unity (i.e. solutions of $z^n - 1 = 0$) to show that

$$2^{n-1} \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \cdots \sin \frac{(n-1)\pi}{n} = n$$

> Hint: $1 - \cos 2\theta = 2 \sin^2 \theta$; $\sin 2\theta = 2 \sin \theta \cos \theta$.

(a) Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

(b) Use these equations to show that the logarithm function

defined by $\log z = \log r + i\theta$

where $z = r e^{i\theta}$ with $-\pi < \theta < \pi$

is a holomorphic function in the region

$r > 0$, $-\pi < \theta < \pi$. Also show that $\log z$ defined

above is not continuous in $r > 0$.

6.4.4 ?

Assume f is continuous in the region:
 $x \geq x_0$, $0 \leq y \leq b$ and the limit

$$\lim_{x \rightarrow +\infty} f(x + iy) = A$$
 exists uniformly with respect to y (independent of y). Show that

$$\lim_{x \rightarrow +\infty} \int_{\gamma_x} f(z) dz = iA b$$
 ; , ;
 where $\gamma_x := \{ z \mid z = x + it, 0 \leq t \leq b \}$.

6.4.5 ?

(Cauchy's formula for "exterior" region) Let γ be piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2 . Assume $f'(z)$ exists in an open set containing γ and Ω_2 and $\lim_{z \rightarrow \infty} f(z) = A$. Show that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

6.4.6 ?

Let $f(z)$ be bounded and analytic in \mathbb{C} . Let $a \neq b$ be any fixed complex numbers. Show that the following limit exists

$$\lim_{R \rightarrow \infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

 Use this to show that $f(z)$ must be a constant (Liouville's theorem).

6.4.7 ?

Prove by *justifying all steps* that for all $\xi \in \mathbb{C}$ we have

$$e^{-\pi \xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x \xi} dx ; .$$

> Hint: You may use that fact in Example 1 on p. 42 of the textbook without proof, i.e., you may assume the above is true for real values of ξ .

6.4.8 ?

Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Let

$$\int_{|z|=1} f(z) dz$$

$f(z) = \sum_{n=1}^{\infty} a_n z^n$
 $f(z) = \sum_{n=1}^{\infty} c_n z^n$ denote the power series in the open disc. Show that (1) $c_n \neq 0$ for all large enough n 's, and (2)
$$\lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}} = z_0.$$

6.4.9 ?

Let $f(z)$ be a non-constant analytic function in $|z| > 0$ such that $f(z_n) = 0$ for infinite many points z_n with $\lim_{n \rightarrow \infty} z_n = 0$. Show that $z=0$ is an essential singularity for $f(z)$. (An example of such a function is $f(z) = \sin(1/z)$.)

6.4.10 ?

Let f be entire and suppose that $\lim_{z \rightarrow \infty} f(z) = \infty$. Show that f is a polynomial.

6.4.11 ?

Expand the following functions into Laurent series in the indicated regions:

(a)

$$f(z) = \frac{z^2 - 1}{(z+2)(z+3)}, \quad 2 < |z| < 3,$$
$$3 < |z| < +\infty.$$

(b)

$$f(z) = \sin \frac{z}{1-z}, \quad 0 < |z-1| < +\infty$$

6.4.12 ?

Assume $f(z)$ is analytic in region D and Γ is a rectifiable curve in D with interior in D . Prove that if $f(z)$ is real for all $z \in \Gamma$, then $f(z)$ is a constant.

6.4.13 ?

Find the number of roots of $z^4 - 6z + 3 = 0$ in $|z| < 1$ and $1 < |z| < 2$ respectively.

6.4.14 ?

Prove that $z^4 + 2z^3 - 2z + 10 = 0$ has exactly one root in each open quadrant.

6.4.15 ?

(1) Let $f(z) \in H(\mathbb{D})$, $\operatorname{Re}(f(z)) > 0$, $f(0) = a > 0$. Show that $|\frac{f(z)-a}{f(z)+a}| \leq |z|$, $|f'(0)| \leq 2a$.

(2) Show that the above is still true if $\operatorname{Re}(f(z)) > 0$ is replaced with $\operatorname{Re}(f(z)) \geq 0$.

6.4.16 ?

Assume $f(z)$ is analytic in \mathbb{D} and $f(0)=0$ and is not a rotation (i.e. $f(z) \neq e^{i\theta}z$). Show that $\sum_{n=1}^{\infty} f^n(z)$ converges uniformly to an analytic function on compact subsets of \mathbb{D} , where $f^{n+1}(z) = f(f^n(z))$.

6.4.17 ?

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic and one-to-one in $|z| < 1$. For $0 < r < 1$, let D_r be the disk $|z| < r$. Show that the area of $f(D_r)$ is finite and is given by $S = \pi \sum_{n=1}^{\infty} n |c_n|^2 r^{2n}$. (Note that in general the area of $f(D_1)$ is infinite.)

6.4.18 ?

Let $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ be analytic and one-to-one in $r_0 < |z| < R_0$. For $r_0 < r < R_0$, let $D(r, R)$ be the annulus $r < |z| < R$. Show that the area of $f(D(r, R))$ is finite and is given by $S = \pi \sum_{n=-\infty}^{\infty} n |c_n|^2 (R^{2n} - r^{2n})$.

6.5 Spring 2015**6.5.1 ?**

Let $a_n(z)$ be an analytic sequence in a domain D such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D . Show that $\sum_{n=0}^{\infty} |a'_n(z)|$ converges uniformly on bounded and closed sub-regions of D .

6.5.2 ?

Let f_n, f be analytic functions on the unit disk \mathbb{D} . Show that the following are equivalent.

(i) $f_n(z)$ converges to $f(z)$ uniformly on compact subsets in \mathbb{D} .

(ii) $\int_{|z|=r} |f_n(z) - f(z)| \, |dz|$ converges to 0 if $0 < r < 1$.

6.5.3 ?

Let f and g be non-zero analytic functions on a region Ω . Assume $|f(z)| = |g(z)|$ for all $z \in \Omega$. Show that $f(z) = e^{i\theta} g(z)$ in Ω for some $0 \leq \theta < 2\pi$.

6.5.4 ?

Suppose f is analytic in an open set containing the unit disc \mathbb{D} and $|f(z)| = 1$ when $|z| = 1$. Show that either $f(z) = e^{i\theta}$ for some $\theta \in \mathbb{R}$ or there are finite number of $z_k \in \mathbb{D}$, $k \leq n$ and $\theta \in \mathbb{R}$ such that

$$f(z) = e^{i\theta} \prod_{k=1}^n \frac{z - z_k}{1 - \bar{z}_k z}, \quad z \in \mathbb{D}.$$

> Also cf. Stein et al, 1.4.7, 3.8.17

6.5.5 ?

(1) Let $p(z)$ be a polynomial, $R > 0$ any positive number, and $m \geq 1$ an integer. Let $M_R = \sup_{|z|=R} |z^m p(z) - 1|$. Show that $M_R > 1$.

(2) Let $m \geq 1$ be an integer and $K = \{z \in \mathbb{C} : r \leq |z| \leq R\}$ where $0 < r < R$. Show (i) using (1) as well as, (ii) without using (1) that there exists a positive number $\varepsilon_0 > 0$ such that for each polynomial $p(z)$,

$$\sup_{|z| \in K} |p(z) - z^{-m}| \geq \varepsilon_0.$$

6.5.6 ?

Let $f(z) = \frac{1}{z} + \frac{1}{z^2 - 1}$. Find all the Laurent series of f and describe the largest annuli in which these series are valid.

6.5.7 ?

Suppose f is entire and there exist $A, R > 0$ and natural number N such that $|f(z)| \leq A |z|^N$ for $|z| \geq R$. Show that (i)

f is a polynomial and (ii) the degree of f is at most N .

6.5.8 ?

Suppose f is entire and there exist $A, R > 0$ and natural number N such that $|f(z)| \geq A |z|^N$ for $|z| \geq R$. Show that (i) f is a polynomial and (ii) the degree of f is at least N .

6.5.9 ?

(1) Explicitly write down an example of a non-zero analytic function in $|z| < 1$ which has infinitely zeros in $|z| < 1$.

(2) Why does not the phenomenon in (1) contradict the uniqueness theorem?

6.5.10 ?

(1) Assume u is harmonic on open set Ω and z_n is a sequence in Ω such that $u(z_n) = 0$ and $\lim z_n \in \Omega$. Prove or disprove that u is identically zero. What if Ω is a region?

(2) Assume u is harmonic on open set Ω and $u(z) = 0$ on a disc in Ω . Prove or disprove that u is identically zero. What if Ω is a region?

(3) Formulate and prove a Schwarz reflection principle for harmonic functions

> cf. Theorem 5.6 on p.60 of Stein et al.

> Hint: Verify the mean value property for your new function obtained by Schwarz reflection principle.

6.5.11 ?

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s < r$, there exists a constant $c > 0$ such that $\|f\|_{(\infty, s)} \leq c \|f\|_{(1, r)}$, where $\|f\|_{(\infty, s)} = \sup_{z \in D_s(z_0)} |f(z)|$ and $\|f\|_{(1, r)} = \int_{D_r(z_0)} |f(z)| dx dy$.

> Note: Exercise 3.8.20 on p.107 in Stein et al is a straightforward consequence of this stronger result using the integral form of the Cauchy-Schwarz inequality in real analysis.

6.5.12 ?

(1) Let f be analytic in $\Omega: 0 < |z-a| < r$ except at a sequence of poles $a_n \in \Omega$ with $\lim_{n \rightarrow \infty} a_n = a$. Show that for any $w \in \mathbb{C}$, there exists a sequence $z_n \in \Omega$ such that $\lim_{n \rightarrow \infty} f(z_n) = w$.

(2) Explain the similarity and difference between the above assertion and the Weierstrass-Casorati theorem.

6.5.13 ?

Compute the following integrals.

(i) $\int_0^\infty \frac{1}{(1+x^n)^2} dx$,
 $n \geq 1$ (ii)
 $\int_0^\infty \frac{\cos x}{(x^2+a^2)^2} dx$,
 $a \in \mathbb{R}$ (iii)
 $\int_0^\pi \frac{1}{a + \sin \theta} d\theta$,
 $a > 1$

(iv) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{a + \sin^2 \theta}$, $a > 0$. (v)
 $\int_{|z|=2} \frac{1}{(z^5-1)(z-3)} dz$ (v)
 $\int_{-\infty}^\infty \frac{\sin \pi a}{\cosh \pi x + \cos \pi a} e^{-ix} dx$,
 $0 < a < 1$, $x \in \mathbb{R}$ (vi)
 $\int_{|z|=1} \cot^2 z dz$.

6.5.14 ?

Compute the following integrals.

(i) $\int_0^\infty \frac{\sin x}{x} dx$ (ii)
 $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$ (iii)
 $\int_0^\infty \frac{x^{a-1}}{(1+x)^2} dx$,
 $0 < a < 2$

(i) $\int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx$,
 $a, b > 0$ (ii)
 $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx$,
 $0 < a < n$

(iii) $\int_0^\infty \frac{\log x}{1+x^n} dx$,
 $n \geq 2$ (iv)
 $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$ (v)
 $\int_0^\pi \log|1 - a \sin \theta| d\theta$,

$a \in \mathbb{C}$

6.5.15 ?

Let $0 < r < 1$. Show that polynomials

$P_n(z) = 1 + 2z + 3z^2 + \cdots + n z^{n-1}$ have no zeros in $|z| < r$ for all sufficiently large n 's.

6.5.16 ?

Let f be an analytic function on a region Ω . Show that f is a constant if there is a simple closed curve γ in Ω such that its image $f(\gamma)$ is contained in the real axis.

6.5.17 ?

(1) Show that $\frac{\pi^2}{\sin^2 \pi z}$ and $g(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ have the same principal part at each integer point.

(2) Show that

$h(z) = \frac{\pi^2}{\sin^2 \pi z} - g(z)$ is bounded

on \mathbb{C} and conclude that

$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} + \dots$

6.5.18 ?

Let $f(z)$ be an analytic function on

$\mathbb{C} \setminus \{z_0\}$, where z_0 is a fixed point.

Assume that $f(z)$ is bijective from

$\mathbb{C} \setminus \{z_0\}$ onto its image, and that $f(z)$

is bounded outside $D_r(z_0)$, where r is some fixed positive number. Show that there exist $a, b, c, d \in \mathbb{C}$ with

$ad - bc \neq 0$, $c \neq 0$ such that

$f(z) = \frac{az + b}{cz + d}$.

6.5.19 ?

Assume $f(z)$ is analytic in $D: |z| < 1$ and $f(0) = 0$ and

is not a rotation (i.e. $f(z) \neq e^{i\theta} z$). Show that

$\sum_{n=1}^{\infty} f^n(z)$ converges uniformly to an

analytic function on compact subsets of D , where

$f^{n+1}(z) = f(f^n(z))$.

6.5.20 ?

Let f be a non-constant analytic function on D with

$f(\mathbb{D}) \subseteq \mathbb{D}$. Use $\psi_a(f(z))$ (where $a=f(0)$, $\displaystyle \psi_a(z) = \frac{a-z}{1-\bar{a}z}$) to prove that
$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}.$$

6.5.21 ?

Find a conformal map

1. from $\{z: |z - 1/2| > 1/2, \text{Re}(z) > 0\}$ to \mathbb{H}
2. from $\{z: |z - 1/2| > 1/2, |z| < 1\}$ to \mathbb{D}
3. from the intersection of the disk $|z + i| < \sqrt{2}$ with \mathbb{H} to \mathbb{D} .
4. from $\mathbb{D} \setminus [a, 1)$ to $\mathbb{D} \setminus [0, 1)$ ($0 < a < 1$). [Short solution possible using Blaschke factor]
5. from $\{z: |z| < 1, \text{Re}(z) > 0\} \setminus (0, 1/2]$ to \mathbb{H} .

6.5.22 ?

Let C and C' be two circles and let $z_1 \in C$, $z_2 \notin C$, $z'_1 \in C'$, $z'_2 \notin C'$. Show that there is a unique fractional linear transformation f with $f(C) = C'$ and $f(z_1) = z'_1$, $f(z_2) = z'_2$.

6.5.23 ?

Assume $f_n \in H(\Omega)$ is a sequence of holomorphic functions on the region Ω that are uniformly bounded on compact subsets and $f \in H(\Omega)$ is such that the set
$$\{z \in \Omega: \lim_{n \rightarrow \infty} f_n(z) = f(z)\}$$
 has a limit point in Ω . Show that f_n converges to f uniformly on compact subsets of Ω .

6.5.24 ?

Let

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$$
with $|\alpha| < 1$ and $\mathbb{D} = \{z: |z| < 1\}$. Prove that

-
$$\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_\alpha|^{-2} dx dy = 1.$$

- $\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_{\alpha}| dx dy = \frac{1-|\alpha|^2}{\log \frac{1}{1-|\alpha|^2}}$.

6.5.25 ?

Prove that

$f(z) = -\frac{1}{2} \left(z + \frac{1}{z} \right)$ is a conformal map from half disc $\{z = x + iy : |z| < 1, y > 0\}$ to upper half plane $\mathbb{H} = \{z = x + iy : y > 0\}$.

6.5.26 ?

Let Ω be a simply connected open set and let γ be a simple closed contour in Ω and enclosing a bounded region U anticlockwise. Let $f: \Omega \rightarrow \mathbb{C}$ be a holomorphic function and $|f(z)| \leq M$ for all $z \in \gamma$. Prove that $|f(z)| \leq M$ for all $z \in U$.

6.5.27 ?

Compute the following integrals. (i)

$$\int_0^{\infty} \frac{x^{a-1}}{1+x^n} dx, \quad 0 < a < n$$

(ii)

$$\int_0^{\infty} \frac{\log x}{(1+x^2)^2} dx$$

6.5.28 ?

Let $0 < r < 1$. Show that polynomials

$P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ have no zeros in $|z| < r$ for all sufficiently large n 's.

6.5.29 ?

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s < r$, there exists a constant $c > 0$ such that

$$\|f\|_{(\infty, s)} \leq c \|f\|_{(1, r)}, \quad \text{where}$$

$$\|f\|_{(\infty, s)} = \sup_{z \in D_s(z_0)} |f(z)|$$

$$\text{and } \|f\|_{(1, r)} = \int_{D_r(z_0)} |f(z)| dx dy.$$

6.5.30 ?

Let $\psi_{\alpha}(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$ with $|\alpha| < 1$ and $\mathbb{D} = \{z : |z| < 1\}$. Prove that

- $\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_{\alpha}|^2 dx dy = 1$.

-
$$\frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_{\alpha}| dx dy = \frac{1-|\alpha|^2}{\log \frac{1}{1-|\alpha|^2}}.$$

Prove that
$$f(z) = -\frac{1}{2} \left(z + \frac{1}{z} \right)$$
 is a conformal map from half disc $\{z = x + iy : |z| < 1, y > 0\}$ to upper half plane $\mathbb{H} = \{z = x + iy : y > 0\}$.

6.5.31 ?

Let Ω be a simply connected open set and let γ be a simple closed contour in Ω and enclosing a bounded region U anticlockwise. Let $f: \Omega \rightarrow \mathbb{C}$ be a holomorphic function and $|f(z)| \leq M$ for all $z \in \gamma$. Prove that $|f(z)| \leq M$ for all $z \in U$.

6.5.32 ?

Compute the following integrals. (i)
$$\int_0^{\infty} \frac{x^{a-1}}{1+x^n} dx, \quad 0 < a < n$$
 (ii)
$$\int_0^{\infty} \frac{\log x}{(1+x^2)^2} dx$$

6.5.33 ?

Let $0 < r < 1$. Show that polynomials $P_n(z) = 1 + 2z + 3z^2 + \dots + n z^{n-1}$ have no zeros in $|z| < r$ for all sufficiently large n 's.

6.5.34 ?

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s < r$, there exists a constant $c > 0$ such that
$$\|f\|_{(\infty, s)} \leq c \|f\|_{(1, r)},$$
 where
$$\|f\|_{(\infty, s)} = \sup_{z \in D_s(z_0)} |f(z)|$$
 and
$$\|f\|_{(1, r)} = \int_{D_r(z_0)} |f(z)| dx dy.$$

6.6 Fall 2016

6.6.1 ?

Let $u(x, y)$ be harmonic and have continuous partial derivatives of order three in an open disc of radius $R > 0$.

(a) Let two points $(a, b), (x, y)$ in this disk be given. Show that the following integral is independent of the path in this disk joining these points:

$$v(x, y) = \int_{[a, b]^{x, y}} \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right).$$

(b) \hfill

(i) Prove that $u(x,y)+i v(x,y)$ is an analytic function in this disc.

(ii) Prove that $v(x,y)$ is harmonic in this disc.

6.6.2 ?

(a) $f(z) = u(x,y) + i v(x,y)$ be analytic in a domain $D \subset \mathbb{C}$. Let $z_0 = (x_0, y_0)$ be a point in D which is in the intersection of the curves $u(x,y) = c_1$ and $v(x,y) = c_2$, where c_1 and c_2 are constants. Suppose that $f'(z_0) \neq 0$. Prove that the lines tangent to these curves at z_0 are perpendicular.

(b) Let $f(z) = z^2$ be defined in \mathbb{C} .

(i) Describe the level curves of $\operatorname{Re}\{f\}$ and of $\operatorname{Im}\{f\}$.

(ii) What are the angles of intersections between the level curves $\operatorname{Re}\{f\} = 0$ and $\operatorname{Im}\{f\}$? Is your answer in agreement with part a) of this question?

6.6.3 ?

(a) $f: D \rightarrow \mathbb{C}$ be a continuous function, where $D \subset \mathbb{C}$ is a domain. Let $\alpha: [a,b] \rightarrow D$ be a smooth curve. Give a precise definition of the *complex line integral* $\int_{\alpha} f.$

(b) Assume that there exists a constant M such that $|f(\tau)| \leq M$ for all $\tau \in \operatorname{Image}(\alpha)$. Prove that $\left| \int_{\alpha} f \right| \leq M \times \operatorname{length}(\alpha).$

(c) Let C_R be the circle $|z| = R$, described in the counterclockwise direction, where $R > 1$. Provide an upper bound for $\left| \int_{C_R} \frac{\log\{z\}}{z^2} dz \right|$, which depends [only] on R and other constants.

6.6.4 ?

(a) Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be an entire function. Assume the existence of a non-negative integer m , and of positive constants L and R , such that for all z with $|z| > R$ the inequality $|f(z)| \leq L |z|^m$ holds. Prove that f is a

polynomial of degree $\leq m$.

(b) Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be an entire function. Suppose that there exists a real number M such that for all $z \in \mathbb{C}$ $\operatorname{Re}(f(z)) \leq M$. Prove that f must be a constant.

6.6.5 ?

Prove that all the roots of the complex polynomial $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z|=1$ and $|z|=2$.

6.6.6 ?

(a) Let F be an analytic function inside and on a simple closed curve C , except for a pole of order $m \geq 1$ at $z=a$ inside C . Prove that

$$\frac{1}{2\pi i} \oint_C F(\tau) d\tau = \lim_{\tau \rightarrow a} \frac{d^{m-1}}{d\tau^{m-1}} \left((\tau-a)^m F(\tau) \right).$$

(b) Evaluate $\oint_C \frac{e^{\tau}}{(\tau^2 + \pi^2)^2} d\tau$ where C is the circle $|z|=4$.

6.6.7 ?

Find the conformal map that takes the upper half-plane conformally onto the half-strip $\{w = x+iy: -\pi/2 < x < \pi/2, y > 0\}$.

6.6.8 ?

Compute the integral $\int_{-\infty}^{\infty} \frac{e^{-2\pi i x} dx}{\cosh \pi x}$ where $\cosh z = \frac{e^z + e^{-z}}{2}$.