Title

D. Zack Garza

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Contents

1 Undergraduate Analysis: Uniform Convergence

1.1 Fall 2018 # 1 ✨

Let $f(x) = \dfrac{1}{x}$. Show that f is uniformly continuous on $(1, \infty)$ but not on $(0, \infty)$.

> **Solution:**
>
> > **Concepts Used:**
> >
> > - Uniform continuity:
> >
> > $$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \quad \text{such that} \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$
> >
> > - Negating uniform continuity: $\exists \varepsilon > 0$ such that $\forall \delta(\varepsilon)$ there exist x, y such that $|x - y| < \delta$ and $|f(x) - f(y)| > \varepsilon$.
>
> **Claim**: $f(x) = \dfrac{1}{x}$ is uniformly continuous on (c, ∞) for any $c > 0$.
>
> - Note that
>
> $$|x|, |y| > c > 0 \implies |xy| = |x||y| > c^2 \implies \frac{1}{|xy|} < \frac{1}{c^2}.$$
>
> - Letting ε be arbitrary, choose $\delta < \varepsilon c^2$.
>   - Note that δ does not depend on x, y.
> - Then
>
> $$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{x} - \frac{1}{y} \right| \\ &= \frac{|x - y|}{xy} \\ &\leq \frac{\delta}{xy} \\ &< \frac{\delta}{c^2} \\ &< \varepsilon, \end{aligned}$$
>
> which shows uniform continuity.
>
> **Claim**: f is *not* uniformly continuous when $c = 0$.
>
> - Toward a contradiction, let $\varepsilon < 1$.
> - Let $x_n = \dfrac{1}{n}$ for $n \geq 1$.
> - Choose n large enough such that $|x_n - x_{n+1}| = \dfrac{1}{n} - \dfrac{1}{n+1} < \delta$.
>   - Why this can be done: by the archimedean property of $\mathbb{R}$, choose n such that $\dfrac{1}{n} < \varepsilon$.
>   - Then
>
> $$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \leq \frac{1}{n} < \varepsilon \quad \text{since } n + 1 > 1.$$

- Note $f(x_n) = n$ and thus

$$|f(x_{n+1}) - f(x_n)| = (n+1) - n = 1 > \varepsilon,$$

a contradiction.

1.2 Fall 2017 # 1 ✨

Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which f does and does not converge uniformly.

Solution:

> **Concepts Used:**
>
> - $f_N \longrightarrow f$ uniformly $\iff \|f_N - f\|_\infty \longrightarrow 0$.
> - $\displaystyle\sum_{n=0}^{\infty} c_n x^n := \lim_{N \longrightarrow \infty} \sum_{n=0}^{N} c_n x^n$
>   - I.e. an infinite sum is defined as the pointwise limit of its partial sums.
> - If $\displaystyle\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on a set A, then $\displaystyle\sup_{x \in A} |f_n(x)| \longrightarrow 0$.

- Set $f_N(x) = \displaystyle\sum_{n=1}^{N} \frac{x^n}{n!}$.
  - Then by definition, $f_N(x) \longrightarrow f(x)$ pointwise on $\mathbb{R}$.
- For any compact interval $[-M, M]$, we have

$$
\begin{aligned}
\|f_N(x) - f(x)\|_\infty &= \sup_{-M \leq x \leq M} \left| \sum_{n=N+1}^{\infty} \frac{x^n}{n!} \right| \\
&\leq \sup_{-M \leq x \leq M} \sum_{n=N+1}^{\infty} \left| \frac{x^n}{n!} \right| \\
&\leq \sum_{n=N+1}^{\infty} \frac{M^n}{n!} \\
&\leq \sum_{n=0}^{\infty} \frac{M^n}{n!} \quad \text{since all additional terms are positive} \\
&= e^M \\
&< \infty,
\end{aligned}
$$

so $f_N \longrightarrow f$ uniformly on $[-M, M]$ by the M-test.

    Here we've used that e^x is equal to its power series expansion.

- Thus f converges on any bounded interval, since any bounded interval is contained in

> some larger compact interval.
> **Claim**: f does not converge on $\mathbb{R}$.
>
> - If $\displaystyle\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on a set A, then $\displaystyle\sup_{x \in A}|f_n(x)| \longrightarrow 0$.
> - But taking $A = \mathbb{R}$ and $g_n(x) = \dfrac{x^n}{n!}$, we have
>
> $$\sup_{x \in \mathbb{R}}|g_n(x)| = \sup_{x \in \mathbb{R}} \frac{x^n}{n!} = \infty.$$

1.3 Fall 2014 # 1 ✨

Let $\{f_n\}$ be a sequence of continuous functions such that $\sum f_n$ converges uniformly.

Prove that $\sum f_n$ is also continuous.

> **Solution:**
>
> **Claim:** If $F_N \longrightarrow F$ uniformly with each F_N continuous, then F is continuous.
> - Follows from an $\varepsilon/3$ argument:
>
> $$|F(x) - F(y)| \leq |F(x) - F_N(x)| + |F_N(x) - F_N(y)| + |F_N(y) - F(y)| \leq \varepsilon \longrightarrow 0.$$
>
>   - The first and last $\varepsilon/3$ come from uniform convergence of $F_N \longrightarrow F$.
>   - The middle $\varepsilon/3$ comes from continuity of each F_N.
> - Now setting $F_N := \displaystyle\sum_{n=1}^{N} f_n$ yields a finite sum of continuous functions, which is continuous.
> - Each F_N is continuous and $F_N \longrightarrow F$ uniformly, so applying the claim yields the desired result.

1.4 Spring 2017 # 4 🚩

Let $f(x,y)$ on $[-1,1]^2$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2 + y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Determine if f is integrable.

Redo, may just be wrong.

Add concepts.

> **Solution:**

> **Concepts Used:**
>
> - ?
>
> Switching to polar coordinates and integrating over the unit disc $\mathbb{D} \subseteq I^2$, we have
>
> $$\int_{I^2} f \, dA \geq \int_{\mathbb{D}} f \, dA$$
> $$\geq \int_0^{2\pi} \int_0^1 \frac{\cos(\theta)\sin(\theta)}{r^4} \, r \, dr \, d\theta$$
> $$= \int_0^{2\pi} \cos(\theta)\sin(\theta) \int_0^1 \frac{1}{r^3} \, dr \, d\theta$$
> $$= \left(\int_0^1 \frac{1}{r^3} \, dr\right)\left(\int_0^{2\pi} \cos(\theta)\sin(\theta) \, d\theta\right) = \infty,$$
>
> so f is not integrable.

1.5 Spring 2015 # 1 ✨

Let (X, d) and (Y, ρ) be metric spaces, $f : X \longrightarrow Y$, and $x_0 \in X$.

Prove that the following statements are equivalent:

1. For every $\varepsilon > 0$ $\exists \delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$.
2. The sequence $\{f(x_n)\}_{n=1}^\infty \longrightarrow f(x_0)$ for every sequence $\{x_n\} \longrightarrow x_0$ in X.

1.6 Fall 2014 # 2 🚩

Let I be an index set and $\alpha : I \longrightarrow (0, \infty)$.

1. Show that

$$\sum_{i \in I} a(i) := \sup_{\substack{J \subset I \\ J \text{ finite}}} \sum_{i \in J} a(i) < \infty \implies I \text{ is countable.}$$

2. Suppose $I = \mathbb{Q}$ and $\sum_{q \in \mathbb{Q}} a(q) < \infty$. Define

$$f(x) := \sum_{\substack{q \in \mathbb{Q} \\ q \leq x}} a(q).$$

Show that f is continuous at $x \iff x \notin \mathbb{Q}$.

1.7 Spring 2014 # 2 🚩

Let $\{a_n\}$ be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that $\sum a_n^2 < \infty$.

> Note: Assume a_n, b_n are all non-negative.

2 General Analysis

2.1 Spring 2020 # 1 ✨

Prove that if $f : [0, 1] \longrightarrow \mathbb{R}$ is continuous then

$$\lim_{k \longrightarrow \infty} \int_0^1 kx^{k-1} f(x) \, dx = f(1).$$

Solution:

> **Concepts Used:**
>
> - DCT
> - Weierstrass Approximation Theorem

- Suppose p is a polynomial, then

$$
\begin{aligned}
\lim_{k \longrightarrow \infty} \int_0^1 kx^{k-1} p(x) \, dx &= \lim_{k \longrightarrow \infty} \int_0^1 \left(\frac{\partial}{\partial x} x^k \right) p(x) \, dx \\
&= \lim_{k \longrightarrow \infty} \left[x^k p(x) \Big|_0^1 - \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx \right] \quad \text{integrating by parts} \\
&= p(1) - \lim_{k \longrightarrow \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx,
\end{aligned}
$$

- Thus it suffices to show that

$$\lim_{k \longrightarrow \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx = 0.$$

- Integrating by parts a second time yields

$$
\begin{aligned}
\lim_{k \longrightarrow \infty} \int_0^1 x^k \left(\frac{\partial}{\partial x} p(x) \right) dx &= \lim_{k \longrightarrow \infty} \frac{x^{k+1}}{k+1} p'(x) \Big|_0^1 - \int_0^1 \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \\
&= - \lim_{k \longrightarrow \infty} \int_0^1 \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \\
&= - \int_0^1 \lim_{k \longrightarrow \infty} \frac{x^{k+1}}{k+1} \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \quad \text{by DCT} \\
&= - \int_0^1 0 \left(\frac{\partial^2}{\partial x^2} p(x) \right) dx \\
&= 0.
\end{aligned}
$$

  - The DCT can be applied here because f'' is continuous and $[0, 1]$ is compact, so f''

is bounded on $[0, 1]$ by a constant M and

$$\int_0^1 \left| x^k f''(x) \right| \le \int_0^1 1 \cdot M = M < \infty.$$

- Now use the Weierstrass approximation theorem:
  - If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous, then for every $\varepsilon > 0$ there exists a polynomial $p_\varepsilon(x)$ such that $\|f - p_\varepsilon\|_\infty < \varepsilon$.
- Thus

$$\left| \int_0^1 kx^{k-1} p_\varepsilon(x)\, dx - \int_0^1 kx^{k-1} f(x)\, dx \right| = \left| \int_0^1 kx^{k-1} (p_\varepsilon(x) - f(x))\, dx \right|$$

$$\le \left| \int_0^1 kx^{k-1} \|p_\varepsilon - f\|_\infty\, dx \right|$$

$$= \|p_\varepsilon - f\|_\infty \cdot \left| \int_0^1 kx^{k-1}\, dx \right|$$

$$= \|p_\varepsilon - f\|_\infty \cdot x^k \Big|_0^1$$

$$= \|p_\varepsilon - f\|_\infty \overset{\varepsilon \longrightarrow 0}{\longrightarrow} 0$$

  and the integrals are equal.
- By the first argument,

$$\int_0^1 kx^{k-1} p_\varepsilon(x)\, dx = p_\varepsilon(1) \text{ for each } \varepsilon$$

- Since uniform convergence implies pointwise convergence, $p_\varepsilon(1) \overset{\varepsilon \longrightarrow 0}{\longrightarrow} f(1)$.

2.2 Fall 2019 # 1 ✨

Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers.

2.2.1 a

Prove that if $\lim\limits_{n \longrightarrow \infty} a_n = 0$, then

$$\lim_{n \to \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

2.2.2 b

Prove that if $\sum\limits_{n=1}^\infty \dfrac{a_n}{n}$ converges, then

$$\lim_{n \to \infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

Solution:

> **Concepts Used:**
>
> - Cesaro mean/summation.
> - Break series apart into pieces that can be handled separately.

2.2.3 a

Prove a stronger result:

$$a_k \longrightarrow S \implies S_N := \frac{1}{N} \sum_{k=1}^{N} a_k \longrightarrow S.$$

> Idea: once N is large enough, $a_k \approx S$, and all smaller terms will die off as $N \longrightarrow \infty$. See this MSE answer.

- Use convergence $a_k \longrightarrow S$: choose M large enough such that

$$k \geq M+1 \implies |a_k - S| < \varepsilon.$$

Then

$$\left| \left(\frac{1}{N} \sum_{k=1}^{N} a_k \right) - S \right| = \frac{1}{N} \left| \left(\sum_{k=1}^{N} a_k \right) - NS \right|$$

$$= \frac{1}{N} \left| \left(\sum_{k=1}^{N} a_k \right) - \sum_{k=1}^{N} S \right|$$

$$= \frac{1}{N} \left| \sum_{k=1}^{N} (a_k - S) \right|$$

$$\leq \frac{1}{N} \sum_{k=1}^{N} |a_k - S|$$

$$= \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} |a_k - S|$$

$$\leq \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + \sum_{k=M+1}^{N} \frac{\varepsilon}{2}$$

$$= \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + (N - M)\frac{\varepsilon}{2}$$

$$\overset{\varepsilon \longrightarrow 0}{\longrightarrow} \frac{1}{N} \sum_{k=1}^{M} |a_k - S| + 0$$

$$\overset{N \longrightarrow \infty}{\longrightarrow} 0 + 0.$$

> Note: M is fixed, so the last sum is some constant c, and $c/N \longrightarrow 0$ as $N \longrightarrow \infty$ for any constant. To be more careful, choose M first to get $\varepsilon/2$ for the tail, then choose $N(M) > M$ for the remaining truncated part of the sum.

2.2.4 b

- Define

$$\Gamma_n := \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

- $\Gamma_1 = \sum_{k=1}^{n} \frac{a_k}{k}$ is the original series and each Γ_n is a tail of Γ_1, so by assumption $\Gamma_n \overset{n \rightarrow \infty}{\Longrightarrow} 0$.
- Compute

$$\frac{1}{n} \sum_{k=1}^{n} a_k = \frac{1}{n}(\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \mathbf{\Gamma_{n+1}})$$

.

- This comes from consider the following summation:

$\Gamma_1:$	a_1	$+\dfrac{a_2}{2}$	$+\dfrac{a_3}{3}$	$+\cdots$		
$\Gamma_2:$		$\dfrac{a_2}{2}$	$+\dfrac{a_3}{3}$	$+\cdots$		
$\Gamma_3:$			$\dfrac{a_3}{3}$	$+\cdots$		
$\displaystyle\sum_{i=1}^{n}\Gamma_i:$	a_1	$+a_2$	$+a_3$	$+\cdots \quad a_n$	$+\dfrac{a_{n+1}}{n+1}$	$+\cdots$

- Use part (a): since $\Gamma_n \overset{n \rightarrow \infty}{\Longrightarrow} 0$, we have $\dfrac{1}{n} \sum_{k=1}^{n} \Gamma_k \overset{n \rightarrow \infty}{\Longrightarrow} 0$.

- Also a minor check: $\Gamma_n \longrightarrow 0 \implies \dfrac{1}{n}\Gamma_n \longrightarrow 0$.
- Then

$$\frac{1}{n} \sum_{k=1}^{n} a_k = \frac{1}{n}(\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n - \mathbf{\Gamma_{n+1}})$$

$$= \left(\frac{1}{n} \sum_{k=0}^{n} \Gamma_k \right) - \left(\frac{1}{n}\Gamma_{n+1} \right)$$

$$\overset{n \rightarrow \infty}{\Longrightarrow} 0.$$

2.3 Fall 2018 # 4 ✨

Let $f \in L^1([0,1])$. Prove that

$$\lim_{n \longrightarrow \infty} \int_0^1 f(x)|\sin nx| \, dx = \frac{2}{\pi} \int_0^1 f(x) \, dx$$

> Hint: Begin with the case that f is the characteristic function of an interval.

Add concepts.

Solution:

> **Concepts Used:**
>
> - ?

Case of characteristic function

- First suppose $f(x) = \chi_{[0,1]}(x)$.

- Note that $\sin(nx)$ has a period of $2\pi/n$, and thus $\left\lfloor \dfrac{n}{2\pi} \right\rfloor$ full periods in $[0,1]$.

- Taking the absolute value yields a new function with half the period, so a period of π/n and $\lfloor \pi/n \rfloor$ full periods in $[0,1]$.

- We can compute the integral over one full period (which is independent of *which* period is chosen), and since $\sin(x)$ is positive and agrees with $|\sin(nx)|$ on the first period, we have

$$
\begin{aligned}
\int_{\text{One Period}} |\sin(nx)| \, dx &= \int_0^{\pi/n} \sin(nx) \, dx \\
&= \frac{1}{n} \int_0^\pi \sin(u) \, du \quad u = nx \\
&= \frac{1}{n} - \cos(u) \, \Big|_0^\pi \\
&= \frac{2}{n}.
\end{aligned}
$$

- Then break the integral up into integrals over periods $P_1, P_2, \cdots, P_N$ where $N := \lfloor n/\pi \rfloor$:

$$
\begin{aligned}
\int_0^1 |\sin(nx)| \, dx &= \left(\sum_{j=1}^N \int_{P_j} |\sin(nx)| \, dx \right) + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)| \, dx \\
&= \left(\sum_{j=1}^N \frac{2}{n} \right) + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)| \, dx \\
&= N \left(\frac{2}{n} \right) + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)| \, dx \\
&:= \left\lfloor \frac{n}{\pi} \right\rfloor \frac{2}{n} + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)| \, dx \\
&= \frac{2}{\pi} + \int_{N\lfloor \pi/n \rfloor}^1 |\sin(nx)| \, dx \\
&:= \frac{2}{\pi} + R(n)
\end{aligned}
$$

so it suffices to show that $R(n) \overset{n \to \infty}{\Longrightarrow} 0$.

> Need to justify removing floor function and cancellation.

- Showing this: ????????????

> No clue how to show this.

General case

> Not sure. Approximate f by simple functions...?

2.4 Fall 2017 # 4 ✨

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

1. Show that $f_n \longrightarrow 0$ pointwise but not uniformly on $[0,1]$.

> Hint: Consider the maximum of f_n.

2.

$$\lim_{n \to \infty} \int_0^1 n(1-x)^n \sin x \, dx = 0$$

> Add concepts.

> Walk through.

> **Solution:**
>
> > **Concepts Used:**
> >
> > - ?

2.4.1 a

Let $G(x) = \sum_{n=1}^{\infty} nx(1-x)^n$. Applying the ratio test, we have

$$\left| \frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n} \right| = \frac{n+1}{n} |1-x| \overset{n \to \infty}{\longrightarrow} |1-x| < 1 \iff 0 \leq x \leq 2,$$

and in particular, this series converges on $[0,2]$. Thus its terms go to zero, and $nx(1-x)^n \longrightarrow 0$ on $[0,1] \subset [0,2]$.

To see that the convergence is not uniform, let $x_n = \dfrac{1}{n}$ and $\varepsilon > \dfrac{1}{e}$, then

$$\sup_{x \in [0,1]} |nx(1-x)^n - 0| \geq |nx_n(1-x_n)^n| = \left| \left(1 - \frac{1}{n}\right)^n \right| \overset{n \to \infty}{\longrightarrow} e^{-1} > \varepsilon.$$

2.4.2 b

> Note: could use the first part with $\sin(x) \leq x$, but then integral ends up more complicated.

Noting that $\sin(x) \leq 1$, we have We have

$$\left| \int_0^1 n(1-x)^n \sin(x) \right| \leq \int_0^1 |n(1-x)^n \sin(x)|$$

$$\leq \int_0^1 |n(1-x)^n|$$

$$= n \int_0^1 (1-x)^n$$

$$= -\frac{n(1-x)^{n+1}}{n+1}$$

$$\overset{n\longrightarrow\infty}{\Longrightarrow} 0.$$

2.5 Spring 2017 # 3 ✨

Let

$$f_n(x) = ae^{-nax} - be^{-nbx} \quad \text{where } 0 < a < b.$$

Show that

a. $\displaystyle\sum_{n=1}^{\infty} |f_n|$ is not in $L^1([0,\infty), m)$

> Hint: $f_n(x)$ has a root x_n.

b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0,\infty), m) \quad \text{and} \quad \int_0^\infty \sum_{n=1}^{\infty} f_n(x)\, dm = \ln \frac{b}{a}$$

Not complete.

Add concepts.

Walk through.

Solution:

> **Concepts Used:**
>
> - ?

2.5.1 a

Letting $x_n := \dfrac{1}{n}$, we have

$$\sum_{k=1}^{\infty} |f_k(x)| \geq |f_n(x_n)| = \left| ae^{-ax} - be^{-bx} \right| := M.$$

In particular, $\sup_x |f_n(x)| \not\to 0$, so the terms do not go to zero and the sum can not converge.

2.5.2 b

?

2.6 Fall 2016 # 1 ✨

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that f converges to a differentiable function on $(1, \infty)$ and that

$$f'(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n^x} \right)'.$$

Hint:

$$\left(\frac{1}{n^x} \right)' = -\frac{1}{n^x} \ln n$$

Add concepts.

Solution:

Concepts Used:

- ?

- Set $f_N(x) := \sum_{n=1}^{N} n^{-x}$, so $f(x) = \lim_{N \to \infty} f_N(x)$.

- If an interchange of limits is justified, we have

$$
\frac{\partial}{\partial x} \lim_{N \to \infty} \sum_{n=1}^{N} n^{-x} = \lim_{h \to 0} \lim_{N \to \infty} \frac{1}{h} \left[\left(\sum_{n=1}^{N} n^{-x} \right) - \left(\sum_{n=1}^{N} n^{-(x+h)} \right) \right]
$$

$$
\overset{?}{=} \lim_{N \to \infty} \lim_{h \to 0} \frac{1}{h} \left[\left(\sum_{n=1}^{N} n^{-x} \right) - \left(\sum_{n=1}^{N} n^{-(x+h)} \right) \right]
$$

$$
= \lim_{N \to \infty} \lim_{h \to 0} \frac{1}{h} \left[\sum_{n=1}^{N} n^{-x} - n^{-(x+h)} \right] \quad (1)
$$

$$
= \lim_{N \to \infty} \sum_{n=1}^{N} \lim_{h \to 0} \frac{1}{h} \left[n^{-x} - n^{-(x+h)} \right] \quad \text{since this is a finite sum}
$$

$$
:= \lim_{N \to \infty} \sum_{n=1}^{N} \frac{\partial}{\partial x} \left(\frac{1}{n^x} \right)
$$

$$
= \lim_{N \to \infty} \sum_{n=1}^{N} -\frac{\ln(n)}{n^x},
$$

where the combining of sums in (1) is valid because $\sum n^{-x}$ is absolutely convergent for $x > 1$ by the p-test.
- Thus it suffices to justify the interchange of limits and show that the last sum converges on $(1, \infty)$.
- Claim: $\sum n^{-x} \ln(n)$ converges.
  - Use the fact that for any fixed $\varepsilon > 0$,

$$
\lim_{n \to \infty} \frac{\ln(n)}{n^\varepsilon} \overset{L.H.}{=} \lim_{n \to \infty} \frac{1/n}{\varepsilon n^{\varepsilon - 1}} = \lim_{n \to \infty} \frac{1}{\varepsilon n^\varepsilon} = 0,
$$

  - This implies that for a fixed $\varepsilon > 0$ and for any constant $c > 0$ there exists an N large enough such that $n \geq N$ implies $\ln(n)/n^\varepsilon < c$, i.e. $\ln(n) < cn^\varepsilon$.
  - Taking $c = 1$, we have $n \geq N \implies \ln(n) < n^\varepsilon$
  - We thus break up the sum:

$$
\sum_{n \in \mathbb{N}} \frac{\ln(n)}{n^x} = \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{\ln(n)}{n^x}
$$

$$
\leq \sum_{n=1}^{N-1} \frac{\ln(n)}{n^x} + \sum_{n=N}^{\infty} \frac{n^\varepsilon}{n^x}
$$

$$
:= C_\varepsilon + \sum_{n=N}^{\infty} \frac{n^\varepsilon}{n^x} \quad \text{with } C_\varepsilon < \infty \text{ a constant}
$$

$$
= C_\varepsilon + \sum_{n=N}^{\infty} \frac{1}{n^{x-\varepsilon}},
$$

  where the last term converges by the p-test if $x - \varepsilon > 1$.
  - But ε can depend on x, and if $x \in (1, \infty)$ is fixed we can choose $\varepsilon < |x - 1|$ to ensure this.
- Claim: the interchange of limits is justified.
  - ?

2.7 Fall 2016 # 5 ✨

Let $\varphi \in L^\infty(\mathbb{R})$. Show that the following limit exists and satisfies the equality

$$\lim_{n \longrightarrow \infty} \left(\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1 + x^2} \, dx \right)^{\frac{1}{n}} = \|\varphi\|_\infty.$$

> Walk through.

> Add concepts.

Solution:

> **Concepts Used:**
>
> - ?

Let L be the LHS and R be the RHS.
Claim: $L \leq R$. - Since $|\varphi| \leq \|\varphi\|_\infty$ a.e., we can write

$$L^{\frac{1}{n}} := \int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1 + x^2}$$
$$\leq \int_{\mathbb{R}} \frac{\|\varphi\|_\infty^n}{1 + x^2}$$
$$= \|\varphi\|_\infty^n \int_{\mathbb{R}} \frac{1}{1 + x^2}$$
$$= \|\varphi\|_\infty^n \arctan(x) \Big|_{-\infty}^{\infty}$$
$$= \|\varphi\|_\infty^n \left(\frac{\pi}{2} - \frac{-\pi}{2} \right)$$
$$= \pi \|\varphi\|_\infty^n$$

$$\implies L^{\frac{1}{n}} \leq \sqrt[n]{\pi \|\varphi\|_\infty^n}$$
$$\implies L \leq \pi^{\frac{1}{n}} \|\varphi\|_\infty$$
$$\overset{n \longrightarrow \infty}{\longrightarrow} \|\varphi\|_\infty,$$

where we've used the fact that $c^{\frac{1}{n}} \overset{n \longrightarrow \infty}{\longrightarrow} 1$ for any constant c.

> Actually true? Need conditions?

Claim: $R \leq L$.
- We will show that $R \leq L + \varepsilon$ for every $\varepsilon > 0$.
- Set

$$S_\varepsilon := \left\{ x \in \mathbb{R}^n \ \middle| \ |\varphi(x)| \geq \|\varphi\|_\infty - \varepsilon \right\}.$$

- Then we have

$$\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2}\, dx \geq \int_{S_\varepsilon} \frac{|\varphi(x)|^n}{1+x^2}\, dx \quad S_\varepsilon \subset \mathbb{R}$$

$$\geq \int_{S_\varepsilon} \frac{(\|\varphi\|_\infty - \varepsilon)^n}{1+x^2}\, dx \qquad \text{by definition of } S_\varepsilon$$

$$= (\|\varphi\|_\infty - \varepsilon)^n \int_{S_\varepsilon} \frac{1}{1+x^2}\, dx$$

$$= (\|\varphi\|_\infty - \varepsilon)^n C_\varepsilon \qquad \text{where } C_\varepsilon \text{ is some constant}$$

$$\implies \left(\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2}\, dx \right)^{\frac{1}{n}} \geq (\|\varphi\|_\infty - \varepsilon) C_\varepsilon^{\frac{1}{n}}$$

$$\overset{n \to \infty}{\longrightarrow} (\|\varphi\|_\infty - \varepsilon) \cdot 1$$

$$\overset{\varepsilon \to 0}{\longrightarrow} \|\varphi\|_\infty,$$

where we've again used the fact that $c^{\frac{1}{n}} \longrightarrow 1$ for any constant.

2.8 Fall 2016 # 6 ✨

Let $f, g \in L^2(\mathbb{R})$. Show that

$$\lim_{n \to \infty} \int_{\mathbb{R}} f(x) g(x+n)\, dx = 0$$

Add concepts.

Solution:

Concepts Used:

- ?

- Use the fact that L^p has small tails: if $h \in L^2(\mathbb{R})$, then for any $\varepsilon > 0$,

$$\forall \varepsilon, \ \exists N \in \mathbb{N} \quad \text{such that} \quad \int_{|x| \geq N} |h(x)|^2\, dx < \varepsilon.$$

How to prove small tails in L^p?

- So choose n large enough so the tails of both f and g are smaller than ε.
- Apply Cauchy-Schwarz:

$$\left| \int_{\mathbb{R}} f(x) g(x+n)\, dx \right| \leq \int_{\mathbb{R}} |f(x) g(x+n)|\, dx$$

$$\leq \int_{\mathbb{R}} \cdot$$

2.9 Spring 2016 # 1 ▶

For $n \in \mathbb{N}$, define

$$]e_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad E_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

Show that $e_n < E_n$, and prove Bernoulli's inequality:

$$(1+x)^n \geq 1 + nx \text{ for } -1 < x < \infty \text{ and } n \in \mathbb{N}$$

Use this to show the following:

1. The sequence e_n is increasing.
2. The sequence E_n is decreasing.
3. $2 < e_n < E_n < 4$.
4. $\lim\limits_{n \longrightarrow \infty} e_n = \lim\limits_{n \longrightarrow \infty} E_n$.

2.10 Fall 2015 # 1 ▶

Define

$$f(x) = c_0 + c_1 x^1 + c_2 x^2 + \ldots + c_n x^n \text{ with } n \text{ even and } c_n > 0.$$

Show that there is a number x_m such that $f(x_m) \leq f(x)$ for all $x \in \mathbb{R}$.