Title

D. Zack Garza

Sunday 20th September, 2020

Contents

1 | Definitions: Algebraic Topology

- Acyclic
- Alexander duality
- Basis
  - For an R-module M, a basis B is a linearly independent generating set.
- Boundary
- Boundary of a manifold
  - Points $x \in M^n$ defined by

$$\partial M = \{x \in M : H_n(M, M - \{x\}; \mathbb{Z}) = 0\}$$

- Cap Product
  - Denoting $\Delta^p \xrightarrow{\sigma} X \in C_p(X; G)$, a map that sends pairs (p-chains, q-cochains) to $(p-q)$-chains $\Delta^{p-q} \to X$ by

$$\begin{aligned} H_p(X; R) \times H^q(X; R) &\xrightarrow{\frown} H_{p-q}(X; R) \\ \sigma \frown \psi &= \psi(F_0^q(\sigma)) F_q^p(\sigma) \end{aligned}$$

  where F_i^j is the face operator, which acts on a simplicial map σ by restriction to the face spanned by $[v_i \dots v_j]$, i.e. $F_i^j(\sigma) = \sigma|_{[v_i \dots v_j]}$.
- Cellular Homology
- CW Cell
  - An n-cell of X, say e^n, is the image of a map $\Phi : B^n \to X$. That is, $e^n = \Phi(B^n)$. Attaching an n-cell to X is equivalent to forming the space $B^n \coprod_f X$ where $f : \partial B^n \to X$.

* A 0-cell is a point.
* A 1-cell is an interval $[-1,1] = B^1 \subset \mathbb{R}^1$. Attaching requires a map from $S^0 = \{-1,+1\} \to X$
* A 2-cell is a solid disk $B^2 \subset \mathbb{R}^2$ in the plane. Attaching requires a map $S^1 \to X$.
* A 3-cell is a solid ball $B^3 \subset \mathbb{R}^3$. Attaching requires a map from the sphere $S^2 \to X$.

- Cellular Map
  - A map $X \xrightarrow{f} Y$ is said to be cellular if $f(X^{(n)}) \subseteq Y^{(n)}$ where $X^{(n)}$ denotes the n- skeleton.
- Chain
  - An element $c \in C_p(X;R)$ can be represented as the singular p simplex $\Delta^p \to X$.
- Chain Homotopy
  - Given two maps between chain complexes $(C_*, \partial_C) \xrightarrow{f,\ g} (D_*, \partial_D)$, a chain homotopy is a family $h_i : C_i \to B_{i+1}$ satisfying

$$f_i - g_i = \partial_{B,i-1} \circ h_n + h_{i+1} \circ \partial_{A,i}$$

.

$$\begin{tikzcd}
\cdots \arrow[dr, "h_{n-2}"] & A_{n-1} \arrow[l, "d_{A,n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dr, "h_{n-1}"] & A_n \arrow[l, "d_{A,n}"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dr, "h_n"] & A_{n+1} \arrow[l, "d_{A,n+1}"'] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] \arrow[dr, "h_{n+1}"] & \cdots \arrow[l, "d_{A,n+2}"'] \\
\cdots & B_{n-1} \arrow[l, "d_{B,n-1}"'] & B_n \arrow[l, "d_{B,n}"'] & B_{n+1} \arrow[l, "d_{B,n+1}"'] & \cdots \arrow[l, "d_{B,n+2}"']
\end{tikzcd}$$

- Chain Map
  - A map between chain complexes $(C_*, \partial_C) \xrightarrow{f} (D_*, \partial_D)$ is a chain map iff each component $C_i \xrightarrow{f_i} D_i$ satisfies

$$f_{i-1} \circ \partial_{C,i} = \partial_{D,i} \circ f_i$$

(i.e this forms a commuting ladder)

$$\begin{tikzcd}
\cdots & A_{n-1} \arrow[l, "d_{A,n-1}"'] \arrow[d, "f_{n-1}"'] & A_n \arrow[l, "d_{A,n}"'] \arrow[d, "f_n"'] & A_{n+1} \arrow[l, "d_{A,n+1}"'] \arrow[d, "f_{n+1}"'] & \cdots \arrow[l, "d_{A,n+2}"'] \\
\cdots & B_{n-1} \arrow[l, "d_{B,n-1}"'] & B_n \arrow[l, "d_{B,n}"'] & B_{n+1} \arrow[l, "d_{B,n+1}"'] & \cdots \arrow[l, "d_{B,n+2}"']
\end{tikzcd}$$

- Closed manifold

  - A manifold that is compact, with or without boundary.

- Coboundary
- Cochain
  - An cochain $c \in C^p(X; R)$ is a map $c \in \hom(C_p(X; R), R)$ on chains.
- Cocycle

Definition 1.0.1 (Constant Map)**.**
A *constant map* $f : X \to Y$ iff $f(X) = y_0$ for some $y_0 \in Y$, i.e. for every $x \in X$ the output value $f(x) = y_0$ is the same.

Definition 1.0.2 (Colimit)**.**
For a directed system $(X_i, f_{ij}$, the *colimit* is an object X with a sequence of projections $\pi_i : X \to X_i$ such that for any Y mapping into the system, the following diagram commutes:

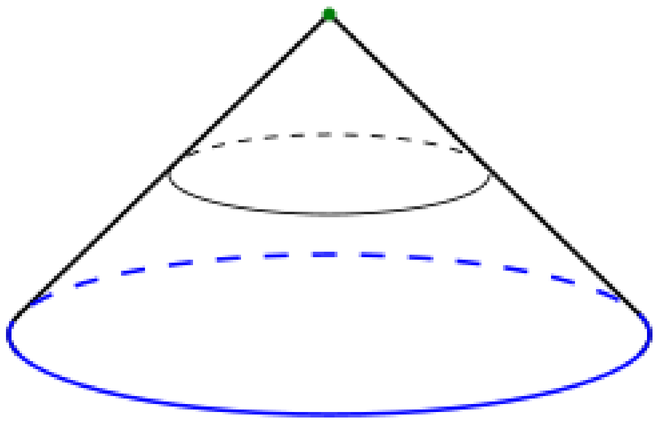
$$\begin{tikzcd}
 & & Y \arrow[d, dashed, "\exists!"] \arrow[ddl, "\psi_j"'] \arrow[ddr, "\psi_i"] & & \\
 & & X \arrow[dl, "\pi_j"] \arrow[dr, "\pi_i"'] & & \\
\cdots \arrow[r] & X_j \arrow[rr, "f_{ij}"] & & X_i \arrow[r] & \cdots
\end{tikzcd}$$

Example 1.1.

- Products
- Pullbacks
- Inverse/Projective limits
- The p-adic integers $\mathbb{Z}_p$.

- Compact
  - A space X is compact iff every open cover of X has a finite subcover.
- Cone
  - For a space X, defined as

$$CX = \frac{X \times I}{X \times \{0\}}.$$



Example: The cone on the circle CS^1

Note that the cone embeds X in a contractible space CX.

- Contractible
  - A space is contractible if its identity map is nullhomotopic.
- Contractible
- Coproduct
- Covering Space
- Cup Product
  - A map taking pairs (p-cocycles, q-cocycles) to $(p+q)$-cocyles by

$$H^p(X;R) \times H^q(X;R) \xrightarrow{\smile} H^{p+q}(X;R)$$
$$(a \cup b)(\sigma) = a(\sigma \circ I_0^p)\ b(\sigma \circ I_p^{p+q})$$

    where $\Delta^{p+q} \xrightarrow{\sigma} X$ is a singular $p+q$ simplex and

$$I_i^j : [i, \cdots, j] \hookrightarrow \Delta^{p+q}$$

    is an embedding of the $(j-i)$-simplex into a $(p+q)$-simplex. On a manifold, the cup product is Poincare dual to the intersection of submanifolds.
  - Applications
    * $T^2 \not\simeq S^2 \vee S^1 \vee S^1$.

      Proof

- CW Complex
- Cycle
- Deck Transformation
- Deformation
- Deformation Retract
  - A map r in $A \underset{r}{\overset{\iota}{\rightleftarrows}} X$ that is a retraction (so $r \circ \iota = \mathrm{id}_A$) **that also satisfies** $\iota \circ r \simeq \mathrm{id}_X$.

Note that this is equality in one direction, but only homotopy equivalence in the other.

  - Equivalently, a map $F : I \times X \to X$ such that

- $F_0(x) = \mathrm{id}_X$
- $F_t(x)\Big|_A = \mathrm{id}_A$
- $F_1(X) = A$

- Degree of a Map
- Derived Functor
  - For a functor T and an R-module A, a *left derived functor* (L_nT) is defined as $h_n(TP_A)$, where P_A is a projective resolution of A.
- Dimension of a manifold
  - For $x \in M$, the only nonvanishing homology group $H_i(M, M - \{x\}\,; \mathbb{Z})$
- Direct Limit
- Direct Product
- Direct Sum
- Eilenberg-MacLane Space
- Euler Characteristic
- Exact Functor
  - A functor T is *right exact* if a short exact sequence

    $$0 \to A \to B \to C \to 0$$

    yields an exact sequence

    $$\dots TA \to TB \to TC \to 0,$$

    and is *left exact* if it yields

    $$0 \to TA \to TB \to TC \to \dots$$

    Thus a functor is exact iff it is both left and right exact, yielding

    $$0 \to TA \to TB \to TC \to 0$$

  - Examples:
    - $\cdot \otimes_R \cdot$ is a right exact bifunctor.
- Exact Sequence
- Excision
- Ext Group
- Flat
  - An R-module is flat if $A \otimes_R \cdot$ is an exact functor.
- Free and Properly Discontinuous
- Free module

- A -module M with a basis $S = \{s_i\}$ of generating elements. Every such module is the image of a unique map $\mathcal{F}(S) = R^S \twoheadrightarrow M$, and if $M = \left\langle S \;\middle|\; \mathcal{R} \right\rangle$ for some set of relations $\mathcal{R}$, then $M \cong R^S/\mathcal{R}$.

- Free Product
- Free product with amalgamation
- Fundamental Class
  - For a connected, closed, orientable manifold, $[M]$ is a generator of $H_n(M;\mathbb{Z}) = \mathbb{Z}$.
- Fundamental classes
- Fundamental Group
- Generating Set
  - $S = \{s_i\}$ is a generating set for an R- module M iff

$$x \in M \implies x = \sum r_i s_i$$

    for some coefficients $r_i \in R$ (where this sum may be infinite).
- Gluing Along a Map
- Group Ring
- Homologous
- Homotopic
- Homotopy
- Homotopy Class
- Homotopy Equivalence
- Homotopy Extension Property
- Homotopy Groups
- Homotopy Lifting Property
- Injection
  - A map ι with a **left** inverse f satisfying $f \circ \iota = \mathrm{id}$
- Intersection Pairing For a manifold M, a map on homology defined by

$$\begin{aligned} H_{\widehat{i}}M \otimes H_{\widehat{j}}M &\to H_{\widehat{i+j}}X \\ \alpha \otimes \beta &\mapsto \langle \alpha, \beta \rangle \end{aligned}$$

  obtained by conjugating the cup product with Poincare Duality, i.e.

$$\langle \alpha, \beta \rangle = [M] \frown ([\alpha]^\vee \smile [\beta]^\vee)$$

Then, if $[A], [B]$ are transversely intersecting submanifolds representing α, β, then

$$\langle \alpha, \beta \rangle = [A \cap B]$$

.

If $\widehat{i} = j$ then $\langle \alpha, \beta \rangle \in H_0 M = \mathbb{Z}$ is the signed number of intersection points.

- Inverse Limit
- Intersection Pairing
  - The pairing obtained from dualizing Poincare Duality to obtain

    $$\mathrm{F}(H_i M) \otimes \mathrm{F}(H_{n-i} M) \to \mathbb{Z}$$

    Computed as an oriented intersection number between two homology classes (perturbed to be transverse).
- Intersection Form
  - The nondegenerate bilinear form cohomology induced by the Kronecker Pairing:

    $$I : H^k(M_n) \times H^{n-k}(M^n) \to \mathbb{Z}$$

    where $n = 2k$.
    * When k is odd, I is skew-symmetric and thus a *symplectic form.*
    * When k is even (and thus $n \equiv 0 \mod 4$) this is a symmetric form.
    * Satisfies $I(x, y) = (-1)^{k(n-k)} I(y, x)$
- Kronecker Pairing
  - A map pairing a chain with a cochain, given by

    $$\begin{aligned} H^n(X; R) \times H_n(X; R) &\to R \\ ([\psi, \alpha]) &\mapsto \psi(\alpha) \end{aligned}$$

    which is a nondegenerate bilinear form.
- Kronecker Product
- Lefschetz duality
- Lefschetz Number
- Lens Space
- Local Degree
  - At a point $x \in V \subset M$, a generator of $H_n(V, V - \{x\})$. The degree of a map $S^n \to S^n$ is the sum of its local degrees.
- Local Orientation
- Limit
- Linear Independence

- A generating S for a module M is linearly independent if $\sum r_i s_i = 0_M \implies \forall i,\ r_i = 0$ where $s_i \in S, r_i \in R$.

- Local homology
  - $H_n(X, X - A; \mathbb{Z})$ is the local homology at A, also denoted $H_n(X \mid A)$
- Local Homology
- Local orientation of a manifold
  - At a point $x \in M^n$, a choice of a generator μ_x of $H_n(M, M - \{x\}) = \mathbb{Z}$.
- Long exact sequence
- Loop Space
- Manifold
  - An n-manifold is a Hausdorff space in which each neighborhood has an open neighborhood homeomorphic to $\mathbb{R}^n$.
- Manifold with boundary
  - A manifold in which open neighborhoods may be isomorphic to either $\mathbb{R}^n$ or a half-space $\left\{\mathbf{x} \in \mathbb{R}^n \mid x_i > 0\right\}$.
- Mapping Cone
- Mapping Cylinder
- Mapping Path Space
- Mayer-Vietoris Sequence
- Monodromy
- Moore Space
- N-cell
- N-connected

Definition 1.0.3 (Nullhomotopic)**.**
A map $X \xrightarrow{f} Y$ is *nullhomotopic* if it is homotopic to a constant map $X \xrightarrow{g} \{y_0\}$; that is, there exists a homotopy

$$\begin{aligned} F : X \times I \to Y \\ F|_{X\times\{0\}} = f \quad F(x,0) = f(x) \\ F|_{X\times\{1\}} = g \quad F(x,1) = g(x) = y_0 \end{aligned}$$
.

- Orientable manifold
  - A manifold for which an orientation exists, see "Orientation of a Manifold".
- Orientation Cover

- For any manifold M, a two sheeted orientable covering space $\tilde{M}_o$. M is orientable iff $\tilde{M}$ is disconnected. Constructed as

$$\tilde{M} = \coprod_{x \in M} \left\{ \mu_x \;\middle|\; \mu_x \text{ is a local orientation} \right\}$$

- Orientation of a manifold
  - A family of $\{\mu_x\}_{x \in M}$ with local consistency: if $x, y \in U$ then μ_x, μ_y are related via a propagation.
    * Formally, a function

$$\begin{aligned} M^n \to \coprod_{x \in M} H(X \;\middle|\; \{x\}) \\ x \mapsto \mu_x \end{aligned}$$

      such that $\forall x \exists N_x$ in which $\forall y \in N_x$, the preimage of each μ_y under the map $H_n(M \;\middle|\; N_x) \twoheadrightarrow H_n(M \;\middle|\; y)$ is a single generator μ_{N_x}.
  - TFAE:
    * M is orientable.
    * The map $W : (M, x) \to \mathbb{Z}_2$ is trivial.
    * $\tilde{M}_o = M \coprod \mathbb{Z}_2$ (two sheets).
    * $\tilde{M}_o$ is disconnected
    * The projection $\tilde{M}_o \to M$ admits a section.
- Oriented manifold
- Path
- Path Lifting Property
- Perfect Pairing
  - A pairing alone is an R-bilinear module map, or equivalently a map out of a tensor product since $p : M \otimes_R N \to L$ can be partially applied to yield $\varphi : M \to L^N = \hom_R(N, L)$. A pairing is **perfect** when φ is an isomorphism.
    * Example: $\det_M : k^2 \times k^2 \to k$
- Poincare Duality
  - For a closed, orientable n-manifold, following map $[M] \frown \cdot$ is an isomorphism:

$$\begin{aligned} D : H^k(M; R) \to H_{n-k}(M; R) \\ D(\alpha) = [M] \frown \alpha \end{aligned}$$

- Projective Resolution
- Properly Discontinuous
- Pullback
- Pushout
- Quasi-isomorphism

- R-orientability
- Relative boundaries
- Relative cycles
- Relative homotopy groups
- Retraction
  - A map r in $A \underset{r}{\overset{\iota}{\rightleftarrows}} X$ satisfying

$$r \circ \iota = \mathrm{id}_A.$$

    Equivalently $X \twoheadrightarrow_r A$ and $r|_A = \mathrm{id}_A$. If X retracts onto A, then i_* is injective.
- Short exact sequence
- Simplicial Complex
- Simplicial Map
  - For a map

$$K \xrightarrow{f} L$$

    between simplicial complexes, f is a simplicial map if for any set of vertices $\{v_i\}$ spanning a simplex in K, the set $\{f(v_i)\}$ are the vertices of a simplex in L.
- Simply Connected
- Singular Chain

$$x \in C_n(x) \implies X = \sum_i n_i \sigma_i = \sum_i n_i (\Delta^n \xrightarrow{\sigma_i} X)$$

- Singular Cochain

$$x \in C^n(x) \implies X = \sum_i n_i \psi_i = \sum_i n_i (\sigma_i \xrightarrow{\psi_i} X)$$

- Singular Homology
- Smash Product
- Surjection
  - A map π with a **right** inverse f satisfying

$$\pi \circ f = \mathrm{id}$$

- Suspension Compact represented as $\Sigma X = CX \coprod_{\mathrm{id}_X} CX$, two cones on X glued along X. Explicitly given by

$$\Sigma X = \frac{X \times I}{(X \times \{0\}) \cup (X \times \{1\}) \cup (\{x_0\} \times I)}$$

- Tor Group
  - For an R-module
    $$\mathrm{Tor}_R^n(\,\cdot\,, B) = L_n(\,\cdot\, \otimes_R B)$$
    where L_n denotes the nth left derived functor.
- Universal Cover
- Universal Coefficient Theorem for Cohomology
- Universal Coefficient Theorem for Change of Coefficient Ring
- Weak Homotopy Equivalence
- Weak Topology
- Wedge Product