

Title

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1 | Saturday, November 28: Introduction to ∞ -categories

Dealing with size issues: take a Grothendieck Universe \mathcal{U} : sets whose subsets are closed under all of the usual set operations (small).

Definition 1.0.1 (∞ -Category)

An ∞ -category \mathcal{C} is a (large) simplicial set \mathcal{C} such that any diagram of the form

$$\begin{array}{ccc} \Lambda_i^n & \xrightarrow{\quad} & \mathcal{C} \\ \downarrow & \nearrow \exists & \\ \Delta_n & & \end{array}$$

admits the indicated lift, where Λ_i^n is an i -horn (a simplex missing the i th face) for $0 < i < n$.

Remark 1.0.2: This is a specialized notion of a Kan complex, and in particular all ∞ -categories are Kan complexes. All inner horns are fillable, i.e. simplicial sets are *inner* Kan complexes. Different to Kan complexes, which include all i .

Definition 1.0.3 (Functors between ∞ -categories)

A ∞ -functor between two ∞ -categories is a map between simplicial sets.

Definition 1.0.4 (Nerve of a category)

Given an ordinary category \mathcal{C} , define the **nerve** of \mathcal{C} to be the simplicial set given by

$$N(\mathcal{C})_n := \{\text{Functors } F : [n] \rightarrow \mathcal{C}\}$$

where $[n]$ is the poset category on $\{1, 2, \dots, n\}$. So an n -simplex is a diagram of objects $X_0, \dots, X_n \in \text{Ob}(\mathcal{C})$ and a sequence of maps. This defines an ∞ -category, and there is a correspondence

$$\{\text{Functors } F : \mathcal{C} \rightarrow \mathcal{D}\} \iff \{\infty\text{-Functors } \hat{F} : N(\mathcal{C}) \rightarrow N(\mathcal{D})\}.$$

Note that taking the nerve of a category preserves the usual categorical structure, since the objects are the 0-simplices and the morphisms are the 1-simplices.

Remark 1.0.5: For \mathcal{C} an ∞ -category, we can define \mathcal{C}_0 to be the “objects” and \mathcal{C}_1 to be the “morphisms”, although we don’t have a good notion of composition yet. There will be boundary map: a 1-simplex has two boundary points, i.e. two objects $a, b \in \mathcal{C}_0$, so we can think of this as a map $f : a \rightarrow b$ where $a = \partial_1 f, b = \partial_0 f^1$ are the first and second vertices respectively. We’ll also have “degeneracy” maps going up from $\mathcal{C}_0 \rightarrow \mathcal{C}_1$, which we should think of as assigning identity morphisms to objects, or conversely that the identity morphism is the degenerate 1-simplex at an

¹This notation ∂_i denotes the boundary operator that drops the i th vertex.

object.

Definition 1.0.6 (Equivalence of Morphisms)

Given two morphisms $f, g : a \rightarrow b$ in an ∞ -category, we say $f \simeq g$ are **equivalent** iff there is a 2-simplex filling in the following diagram:



[Link to diagram](#)

Remark 1.0.7: This turns out to be an equivalence relation. Note that in an ordinary category, if two morphisms are equivalent then they are already equal.

Definition 1.0.8 (Composition of morphisms)

For 1-simplices $f : a \rightarrow b, g : b \rightarrow c$, a **composition** of f and g is a 2-simplex σ filling in the following diagram:



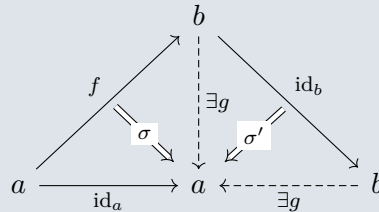
[Link to diagram](#)

In this case, $h := \partial_1 \sigma$ and we write $h \simeq g \circ f$.

Remark 1.0.9: Note that we're not fixing a choice, but it is well-defined up to the equivalence relation we're using. This is similar to how e.g. coproducts are not baked into the structure of a category, but are instead only well-defined up to canonical isomorphism – and in fact, this characterization is sometimes preferable.

Definition 1.0.10 (Equivalences of objects)

If $f : a \rightarrow b$ is a morphism in an ∞ -category \mathcal{C} , then we say f is an **equivalence** if there exists a morphism $g : b \rightarrow a$ such that $\text{id}_a \simeq g \circ f$ and $\text{id}_b \simeq f \circ g$. This is equivalent to finding 2-simplices σ, σ' that fill the following two diagrams:



[Link to diagram](#)

Remark 1.0.11: This is close to what we'd require for an isomorphism in an ordinary category, but we now allow the compositions to only be “weakly equivalent” or homotopic to the identities.