Title

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1 | Postnikov Towers

The point of this note is to reword and fill in some details in Akhil Mathew's post

We start with a very general problem: suppose we are given a pointed topological space (X, x_0) and wish to compute the graded group structure of the homotopy groups

$$\pi_*(X) := \bigoplus_{j=1}^{\infty} \pi(X; x_0)$$

Since this construction is functorial, any homeomorphism $f : X \to X'$ will induce a homotopy equivalence $f\_$: an isomorphism $\pi_*(X) \cong \pi_*(X')$ of graded groups, so this provides some invariant of homeomorphism types of spaces. However, we should be careful to note that π_* is not a *complete* invariant of the topological type of X – that is, we can generally construct spaces X' that are homotopy-equivalent to X but not *homeomorphic* to X. They will however provide partial information that will allow us to distinguish spaces that are not even homotopy-equivalent.

Moreover, in this process we will determine what extra information is needed to in fact obtain a *complete* set of homotopy invariants for X. In other words, we hope to answer the question of how much data it takes to uniquely specify $[X]$, the homotopy type of X.

We'll restrict our attention first to the subcategory of CW complexes. By the CW approximation theorem, nice enough spaces X admit a weak equivalence to some CW complex, and since many theorems are much nicer for CW complexes than for arbitrary spaces, we implicitly work in this subcategory from here onward. The very first thing we'll do is replace X with a CW complex $\tilde{X}$ that is homotopy-equivalent to X, and we'll immediately abuse notation by just writing X for $\tilde{X}$ everywhere.

For any group G, suppose for a moment that there existed a space $K(G, n)$ (where n is a natural numbers) satisfying

$$\pi_i(K(G, n)) = \begin{cases} G, & i = n \\ 0, & \text{else} \end{cases}$$

where 0 denotes the trivial group. If such a space existed, this would perhaps be the simplest type of object to work with within homotopy theory – it has exactly one homotopy group concentrated in one dimension. We will show later that not only does such a space exist, and can be constructed in a relatively straightforward manner, but also that it is unique up to homotopy-equivalence, and are thus referred to as *Eilenberg-MacLane spaces*.

2 | The Atoms of Homotopy Theory

For the moment, let's focus on one particular degree on homotopy groups, say $G = \pi_i(X)$. Note that for our original space, if we make the assumption that G is finitely generated, we can appeal

to the classification of (finitely generated) abelian groups and write $G \cong \mathbb{Z}^n \bigoplus_{j=1}^{m} \mathbb{Z}_{k_j}$ for some n and some m. That is, it decomposes as a free group of rank n, along with a sum of finite cyclic groups with orders given by the invariant factors k_j. As an aside, it is perhaps worth mentioning that $\pi_i(X)$ is abelian for $n \geq 2$, so the categorical product and coproduct (here the direct sum) will coincide. This leaves the $i = 1$ case; however, the fundamental group can generally be computed using more elementary methods such as covering space theory and Van Kampen's theorem.

So we've come up with the "atoms" that we might hope to build spaces out of, and we've realized that we can break any given homotopy group of X into free and cyclic groups. The strategy now is to see if we can work out what kind of spaces $K(\mathbb{Z}^n, i)$ and $K(\mathbb{Z}_k, i)$ are, and hope we can prove a theorem that says something like $K(G_1, i) \oplus K(G_2, i) = K(G_1 \oplus G_2, i)$. If we can do this, we now have a method of constructing a space that has the same homotopy groups as X in degree i. If we can then show something like

$$\pi_i(K(G_1, j) \oplus K(G_2, k)) = \begin{cases} G_1, & i = j \\ G_2, & i = k \\ 0, & \text{else} \end{cases}$$

then we will be in business. Explicitly, splitting the homotopy groups of X up by degree and writing

$$\pi_*(X) = \bigoplus_{i=1}^{\infty} \pi_i(X),$$

we could construct a space X' that has the exact same homotopy groups X by taking

$$X' = \bigoplus_{i=1}^{\infty} K(\pi_i(X), i).$$

What does this buy us? It's not quite the case that $X' \simeq X$, since having isomorphic homotopy groups alone is insufficient to guarantee homotopy equivalence. A standard counterexample here are the spaces $S^2 \times \mathbb{RP}^3$ and $S^3 \times \mathbb{RP}^2$; one in fact needs a single map *inducing* all of the isomorphisms on homotopy groups simultaneously for this kind of result to hold.

So the naive hope of writing X as a product of simple spaces won't work. As it turns out, what we *can* get is a way to write X in a way that's often referred to as a "twisted" product, which is formalized in the notion of a (Serre) fibration.

3 | Twisted Products

It's perhaps worth spelling out exactly what "twisted product" means here, since it's often glossed over. Suppose we have a fibration $F \hookrightarrow E \xrightarrow{p} B$ which is in fact a fiber bundle – note that this is a strictly stronger condition, requiring the fibration to have *local trivialization*. This amounts to

asking that for every open neighborhood $U \in B$, we have $p^{-1}(B) = F \times B$ up to homeomorphism. This is of course strictly weaker than requiring $E \cong F \times B$ globally, which would be denoted a trivial bundle; there may in fact be monodromy in the total space E that topologically distinguishes it from this product. A primary example of this phenomenon is the Mobius bundle,

$$I \hookrightarrow M \twoheadrightarrow S^1$$

which is a line bundle over the circle. Note that locally we do have $M \cong S^1 \times I$, but since M is not homeomorphic to a cylinder, this produces a nontrivial bundle. There is instead a global "twist", arising from the fact M is nonorientable and parallel-transporting a small arc in M once around the core circle is not the identity map. By a somewhat non-precise abuse of notation, one might instead write something like $M \cong S^1 \rtimes_\varphi I$, indicating that this is a nontrivial bundle. One could further ask, given a base B and a fiber F, how many distinct nontrivial bundles $F \to E \to B$ exist, which is a bit far afield for us now but leads to the rich and fruitful theory classifying spaces.

The moral of the story here is that we can generalize a product of spaces $A \times B \times C \times \cdots$ as a "twisted product" $A \rtimes_{\varphi_1} B \rtimes_{\varphi_1} C \rtimes_{\varphi_1} \cdots$ by constructing an interlocking series of fibrations. Luckily for us, our next step towards solving our original problem will involve the construction of Postnikov towers, which explicitly encode how this interlocking works.
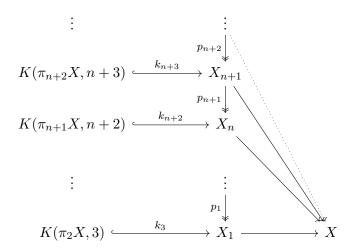
4 | **Postnikov Towers**

First I want to define exactly what a Postnikov tower is and how we can explicitly build them using CW complexes, since this is something that is treated differently among different sources.

The data is a Postnikov tower is a sequence of spaces $\{X_i\}$ equipped with maps $p_n : X_n \to X_{n-1}$, where each p_n is a fibration, and

$$\pi_i(X_n) = \begin{cases} \pi_i(X), & i \leq n \\ 0, & i > n. \end{cases}$$

such that the following diagram commutes:

$$\vdots \qquad\qquad \vdots$$

$$K(\pi_{n+2}X, n+3) \xhookrightarrow{\;k_{n+3}\;} X_{n+1}$$

$$\downarrow p_{n+2}$$

$$K(\pi_{n+1}X, n+2) \xhookrightarrow{\;k_{n+2}\;} X_n$$

$$\downarrow p_{n+1}$$

$$\vdots \qquad\qquad \vdots$$

$$\downarrow p_1$$

$$K(\pi_2 X, 3) \xhookrightarrow{\;k_3\;} X_1 \longrightarrow X$$

Note that we have fibrations

$$K(\pi_{i+1}(X), i+2) \to X_i \to X_{i-1}$$

for all $i > 1$, and so using our earlier notation we can suggestively write $X_i \cong K(\pi_{i+1}(X) \rtimes X_{i-1}$. It also turns out there is a weak equivalence $X \to \varprojlim_i X_i$, and so carrying this out inductively allows us to write

$$X \simeq K(\pi_1(X), 2) \rtimes K(\pi_2(X), 3) \rtimes K(\pi_3(X), 4) \rtimes \cdots$$

yielding our "twisted product".

This spaces can be constructed by a relatively easy process. If X already has the homotopy type of a CW complex, first let $X_i = X^{(i)}$, the i-skeleton of X. Since every finitely-generated group has a presentation, we can write $\pi_{i+1}(X) = \langle a, b, c, \cdots \rangle / \langle R_1, R_2, \cdots \rangle$ where $a, b, c, \cdots$ are generators and R_i are relations.