

# Title

*D. Zack Garza*

# Contents

<b>1</b>	<b>Saturday, November 28: Introduction to <math>\infty</math>-categories</b>	<b>3</b>
----------	--	----------

# 1 | Saturday, November 28: Introduction to $\infty$ -categories

Dealing with size issues: take a Grothendieck Universe  $\mathcal{U}$ : sets whose subsets are closed under all of the usual set operations (small).

## Definition 1.0.1 ( $\infty$ -Category)

An  $\infty$ -category  $\mathcal{C}$  is a (large) simplicial set  $\mathcal{C}$  such that any diagram of the form

$$\begin{array}{ccc} \Lambda_i^n & \xrightarrow{\quad} & \mathcal{C} \\ \downarrow & \nearrow \exists & \\ \Delta_n & & \end{array}$$

admits the indicated lift, where  $\Lambda_i^n$  is an  $i$ -horn (a simplex missing the  $i$ th face) for  $0 < i < n$ .

**Remark 1.0.2:** This is a specialized notion of a Kan complex, and in particular all  $\infty$ -categories are Kan complexes. All inner horns are fillable, i.e. simplicial sets are *inner* Kan complexes. Different to Kan complexes, which include all  $i$ .

## Definition 1.0.3 (Functors between $\infty$ -categories)

A  $\infty$ -functor between two  $\infty$ -categories is a map between simplicial sets.

## Definition 1.0.4 (Nerve of a category)

Given an ordinary category  $\mathcal{C}$ , define the **nerve** of  $\mathcal{C}$  to be the simplicial set given by

$$N(\mathcal{C})_n := \{\text{Functors } F : [n] \rightarrow \mathcal{C}\}$$

where  $[n]$  is the poset category on  $\{1, 2, \dots, n\}$ . So an  $n$ -simplex is a diagram of objects  $X_0, \dots, X_n \in \text{Ob}(\mathcal{C})$  and a sequence of maps. This defines an  $\infty$ -category, and there is a correspondence

$$\{\text{Functors } F : \mathcal{C} \rightarrow \mathcal{D}\} \iff \{\infty\text{-Functors } \hat{F} : N(\mathcal{C}) \rightarrow N(\mathcal{D})\}.$$

Note that taking the nerve of a category preserves the usual categorical structure, since the objects are the 0-simplices and the morphisms are the 1-simplices.

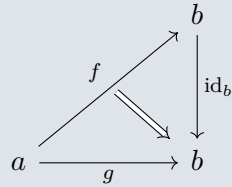
**Remark 1.0.5:** For  $\mathcal{C}$  an  $\infty$ -category, we can define  $\mathcal{C}_0$  to be the “objects” and  $\mathcal{C}_1$  to be the “morphisms”, although we don’t have a good notion of composition yet. There will be boundary map: a 1-simplex has two boundary points, i.e. two objects  $a, b \in \mathcal{C}_0$ , so we can think of this as a map  $f : a \rightarrow b$  where  $a = \partial_1 f, b = \partial_0 f^1$  are the first and second vertices respectively. We’ll also have “degeneracy” maps going up from  $\mathcal{C}_0 \rightarrow \mathcal{C}_1$ , which we should think of as assigning identity morphisms to objects, or conversely that the identity morphism is the degenerate 1-simplex at an

<sup>1</sup>This notation  $\partial_i$  denotes the boundary operator that drops the  $i$ th vertex.

object.

**Definition 1.0.6** (Equivalence of Morphisms)

Given two morphisms  $f, g : a \rightarrow b$  in an  $\infty$ -category, we say  $f \simeq g$  are **equivalent** iff there is a 2-simplex filling in the following diagram:

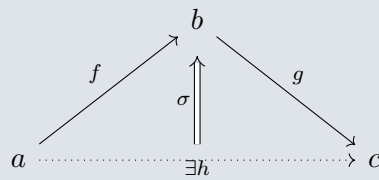


[Link to diagram](#)

**Remark 1.0.7:** This turns out to be an equivalence relation. Note that in an ordinary category, if two morphisms are equivalent then they are already equal.

**Definition 1.0.8** (Composition of morphisms)

For 1-simplices  $f : a \rightarrow b, g : b \rightarrow c$ , a **composition** of  $f$  and  $g$  is a 2-simplex  $\sigma$  filling in the following diagram:



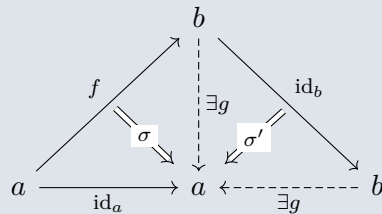
[Link to diagram](#)

In this case,  $h := \partial_1 \sigma$  and we write  $h \simeq g \circ f$ .

**Remark 1.0.9:** Note that we're not fixing a choice, but it is well-defined up to the equivalence relation we're using. This is similar to how e.g. coproducts are not baked into the structure of a category, but are instead only well-defined up to canonical isomorphism – and in fact, this characterization is sometimes preferable.

**Definition 1.0.10** (Equivalences of objects)

If  $f : a \rightarrow b$  is a morphism in an  $\infty$ -category  $\mathcal{C}$ , then we say  $f$  is an **equivalence** if there exists a morphism  $g : b \rightarrow a$  such that  $\text{id}_a \simeq g \circ f$  and  $\text{id}_b \simeq f \circ g$ . This is equivalent to finding 2-simplices  $\sigma, \sigma'$  that fill the following two diagrams:



[Link to diagram](#)

**Remark 1.0.11:** This is close to what we'd require for an isomorphism in an ordinary category, but we now allow the compositions to only be “weakly equivalent” or homotopic to the identities.

**Definition 1.0.12** (Functor Categories)

For  $\mathcal{C}, \mathcal{D}$  simplicial sets, we can define a simplicial set  $\text{Fun}(\mathcal{C}, \mathcal{D})$  whose  $n$ -simplices are given by

$$\text{Fun}(\mathcal{C}, \mathcal{D})_n := \{\text{Simplicial maps } F : \mathcal{C} \times \Delta^n \rightarrow \mathcal{D}\}.$$

**Remark 1.0.13:** Note that the 0-simplices recover functors if these are ordinary categories. If  $\mathcal{D}$  is an  $\infty$ -category, then this functor category is again an  $\infty$ -category. 


**Definition 1.0.14** (Morphisms of functors / natural transformations)

A **morphism** in  $\text{Fun}(\mathcal{C}, \mathcal{D})$ , say  $\eta : F \rightarrow G$ , is a functor  $\eta : \mathcal{C} \times \Delta^n \rightarrow \mathcal{D}$  such that

$$\eta|_{\mathcal{C} \times \{0\}} = F$$

$$\eta|_{\mathcal{C} \times \{1\}} = G.$$

We call such an  $\eta$  a **natural transformation** from  $F$  to  $G$ .

**Remark 1.0.15:** Being an equivalence in  $\text{Fun}(\mathcal{C}, \mathcal{D})$  is equivalent to being a pointwise equivalence. I.e.,  $\eta$  is an equivalence iff the map  $\eta_C$  given by partially applying an object of  $\mathcal{C}$  (i.e. a 1-simplex  $\Delta^n \rightarrow \mathcal{D}$ ) is an equivalence in  $\mathcal{D}$  for all objects  $C \in \text{Ob}(\mathcal{C})$ . 

**Definition 1.0.16** (Equivalences of  $\infty$ -categories)

A functor  $f : \mathcal{C} \rightarrow \mathcal{D}$  of  $\infty$ -categories is an **equivalence** iff there exists a functor  $g : \mathcal{D} \rightarrow \mathcal{C}$  and natural equivalences

$$f \circ g \xrightarrow{\sim} \text{id}_{\mathcal{D}}$$

$$g \circ f \xrightarrow{\sim} \text{id}_{\mathcal{C}}.$$

If there exists such an equivalence, we will write  $\mathcal{C} \simeq \mathcal{D}$ .

**Remark 1.0.17:** For ordinary categories, there is a characteristic property that is much easier to write down in general than an explicit equivalence, namely being essentially surjective and fully faithful. 