

Title

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1 Reference List

- **Rational Homotopy Theory and Differential Forms** by Griffiths and Morgan
- **Differential Forms in Algebraic Topology** by Bott and Tu
- **Differential Topology** by Hirsch
- **Comprehensive Introduction to Differential Geometry** by Spivak
- **Topology from the Differentiable Viewpoint** by Milnor
- **Topology and Geometry** by Bredon
- **User's Guide to Spectral Sequences** by Mcleary
 - View Here
 - Apparently lots of technical details

2 General Notes

The standard Serre fibration: $\Omega X \rightarrow PX \xrightarrow{f} X$ where ΩX is the loop space, PX is the path space, and f is the “evaluation at the endpoint” map. Note that PX is contractible!

Consider a SES $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow C$, then look at it as a 2-step filtration of B so $F^0 B = B, F^1 B = A, F^2 B = 0$. The graded pieces are $G_0 = C, G_1 = A$. Can use this to obtain LES from SS.

Homology in the ring-theoretic setting: If R is a Noetherian ring and $I \subseteq R$, then if I can be generated by n elements then $H_I^i(M) = 0$ for any R -module M and $i > n$. Thus to prove I can *not* be generated by n elements, it suffices to find a module M where $H_I^{n+1} \neq 0$.

3 Griffiths and Morgan

Overall purpose: want to relate C^∞ forms on a manifold to AT invariants. One significant result: given a manifold M , the singular cohomology $H^*(M, \mathbb{R})$ is isomorphic to the cohomology of the differential graded algebra of C^∞ forms, $H_{DR}^*(M)$.

Is this the de Rham cohomology..?

This DGA of smooth forms is actually enough to calculate all of the AT invariants, and can be used to build the Postnikov tower of M ($\otimes \mathbb{R}$)

One construction is the localization of a CW complex at \mathbb{Q} , this removes torsion and divisibility phenomena. The effect on the Postnikov tower is just then tensoring with \mathbb{Q} .

Things that are homotopy equivalent to CW complexes:

- Manifolds
- Varieties
- Loop spaces of CW complexes
- Eilenberg-MacLane spaces? $K(\pi, n)$.

The Whitehead theorem holds for these: $X \xrightarrow{f} Y$ is an homotopy equivalence iff $\pi_*(X) \xrightarrow{f_*} \pi_*(Y)$ is an isomorphism.

Recall the weak topology for infinite CW complex: U is open in X iff $U \cap X^n$ is open for every n .

Theorem: Given any $X \xrightarrow{f} Y$, we can transform this into an inclusion up to homotopy equivalence. (Just replace Y by the mapping cylinder of f , denoted $M_f \simeq Y$).

A fibration is anything that satisfies the homotopy lifting property. Examples:

- Locally trivial fiber bundles
- Vector bundles
- Covering spaces

Path spaces are fibrations, loop spaces are contractible.

Homology can be defined with coefficients in any abelian group by tensoring the singular chain groups with G . That is, if we $H_*(X)$ obtained from

$$\xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X) \cdots \xrightarrow{\partial_1} C_0(X)$$

then we can define $H_*(X; G)$ via

$$\xrightarrow{\partial_{n+1} \otimes 1} C_n(X) \otimes G \xrightarrow{\partial_n \otimes 1} C_{n-1}(X) \otimes G \cdots \xrightarrow{\partial_1 \otimes 1} C_0(X) \otimes G$$

Note that homology has the structure of a graded group, while cohomology has the structure of a graded commutative ring.

Axioms of homology:

- $X \xrightarrow{f} Y$ always induces a map on homology $H_*(X) \xrightarrow{f_*} H_*(Y)$
- An orientation on S^n induces an isomorphism $H_n(S^n) \cong \mathbb{Z}$; reversing orientation induces the map $\mathbb{Z} \xrightarrow{\phi} \mathbb{Z} : \phi(1) = -1$

- $Y \subseteq X$ yields the definition of relative homology $H_*(X, Y)$, and Mayer Vietoris holds; i.e. there is a long exact sequence $\cdots H_n(Y) \rightarrow H_n(X) \rightarrow H_n(X, Y) \rightarrow H_{n-1}(Y) \cdots$
- Excision: $U \subset Y \subset X$ and $\bar{U} \subset Y^\circ$ implies $H_*(X - U, Y - U) \cong H_*(X, Y)$.

Any homology theory satisfying these properties is equivalent to singular homology.

Use notation $[X, Y]$ for homotopy classes of maps $X \rightarrow Y$, then $\pi_1(X) = [S^1, X]$ and we can define $\pi_n(X) = [S^n, X]$. Homotopy groups fail excision.

Whitehead theorem: for CW complexes, if $X \xrightarrow{f} Y$ induces $\pi_n(X) \xrightarrow{f_*} \pi_n(Y)$ and f_* is an isomorphism (and Y is connected), then f is a homotopy equivalence. For spaces that aren't CW complexes, this may fail, and we say f is a *weak homotopy equivalence* instead.

Hurewicz theorem: the bottom homology and homotopy groups are isomorphic, and homology below the bottom homology is zero.

General note: there are equivalent “relative versions” of most of these theorems.

Spectral Sequence: Page 45.

For any fibration $F \rightarrow E \xrightarrow{\pi} B$, we get a LES in homotopy $\pi_n(F) \rightarrow \pi_n(E) \rightarrow \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F)$

Basic question: How are the cohomologies of F, E, B related? An easy case is when $E = F \times B$, but even then $\pi_n(F \times B) \neq \pi_n(F) \oplus \pi_n(B)$. Need the Kunneth theorem, formula is more complicated.

For CW complexes and a fibration, the relationship is nice - look at the total space of the fibration. It is filtered by increasing n -skeleta, and we use the LES. More general filtrations need a spectral sequence.

Note: use LES as trivial example of spectral sequence! Write out the pages, differentials, etc

The spectral sequence relates the cohomology of *successive pairs* in the filtration to the cohomology of the total space.

Theorem: If B is path-connected and $\pi_1(B, b_0)$ acts trivially on $H^*(F)$, then there are isomorphisms

$$\begin{aligned} H^n(E^p, E^{p-1}) &\cong \prod_{p\text{-cells in } B} H^n(\pi^{-1}e^p, \pi^{-1}\partial e^p) \\ &\cong C^p(B; H^{n-p}(F)) \end{aligned}$$

In other words, for any k , we can think of $H^*(E^p, E^{p-k})$ as a k -th approximation to $H^*(E^p)$.

(Should probably review results about polynomial and exterior algebras. And what does it mean for π_1 to act trivially on a fiber?)

EXAMPLES OF COMPUTATION: Page 54

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3.1 Cohomology of \mathbb{CP}^n using $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$

- Cohomology of the infinite Grassmannian $\lim_n G(k, n)$
 - Answer: $H^*(G(k)) = \mathbb{Z}[x_1, x_2, \dots, x_k]$

3.2 Postnikov Towers

A decomposition dual to cell decomposition, the atoms of the space are Eilenberg-MacLane spaces $K(\pi, n)$. (Note the spheres are atomic in homology, while the K are atomic in homotopy.)

Gives a way of going back and forth between X and $\pi_*(X)$: defined as a tower of spaces $X_0 \leftarrow X_1 \leftarrow \dots$

- $X_{i-1} \leftarrow X_i$ is a fibration
- $\pi_k(X_n) = \mathbb{Z}[k \leq n] \cdot \pi_k(X) + \mathbb{Z}[k > n] \cdot 0$
 - So all lower homotopy groups agree at the n -th spot
- (Probably) $X_i \hookrightarrow X$

Unique up to homotopy, $X = \lim_n X_n$ (an inverse limit). Essentially constructs X out of $K(\pi_n(X), n)$.

Note: revisit and draw diagrams for Postnikov Towers

Homotopy and homology commute with direct limits.

$(\cdot \otimes \mathbb{Q})$ is a right-exact functor, most results in this section are about how terms in exact sequences all become \mathbb{Q} -vector spaces. In particular, $H^*(X; \mathbb{Q}), H_*(X; \mathbb{Q})$ are.

Homotopy theory over \mathbb{Q} is much easier than over \mathbb{Z} . Samples results:

$$\pi_i(S^{2n-1}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & i = 2n - 1 \\ 0 & \text{otherwise} \end{cases}$$

Then using the fact that $\pi_i(S^{2n-1})$ is always finitely generated, we can conclude

$$\pi_i(S^{2n-1}) = \begin{cases} \mathbb{Z} & i = 2n - 1 \\ \text{a finite group } G & \text{otherwise} \end{cases}$$

This yields for even n :

$$\pi_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ \mathbb{Z} \oplus G & i = 2n - 1 \\ H & \text{otherwise} \end{cases}$$

for some finite groups G, H !

Can also obtain Bott Periodicity this way.

3.3 Other Reading

Lots of good examples of computations here

Some fibrations

- Hopf: $S^1 \rightarrow S^3 \rightarrow S^2$
- $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$
- Path space: $\Omega S^n \rightarrow PS^n \rightarrow S^n$

Serre Spectral Sequence Example: For the fibration $S^1 \rightarrow S^3 \rightarrow S^2$, the E_2 page:

Which is equal to

$$\begin{array}{ccc}
 1H^0(S^2, \mathbb{Z}) & 0 & H^2(S^2, \mathbb{Z}) \\
 0H^0(S^2, \mathbb{Z}) & 0 & H^2(S^2, \mathbb{Z}) \\
 \hline
 & 0 & 1 & 2.
 \end{array}$$

And $E_3 = E_\infty$, so $d_2^{0,1}$ is an isomorphism.

Note: Probably a good starting point for basic calculations? Fill out the missing details for this table.

Challenge: Prove $\pi_4(S^2) = \frac{\mathbb{Z}}{2\mathbb{Z}}$