Title

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Contents

1 | Saturday, November 28: Introduction to ∞-categories

Dealing with size issues: take a Grothendieck Universe $\mathcal{U}$: sets whose subsets are closed under all of the usual set operations (small).

> **Definition 1.0.1** (∞-Category)
> An ∞-category $\mathcal{C}$ is a (large) simplicial set $\mathcal{C}$ such that any diagram of the form
>
> $$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \\ \Delta_n & \exists & \end{array}$$
>
> admits the indicated lift, where Λ_i^n is an i-horn (a simplex missing the ith face) for $0 < i < n$.

Remark 1.0.2: This is a specialized notion of a Kan complex, and in particular all ∞-categories are Kan complexes. All inner horns are fillable, i.e. simplicial sets are *inner* Kan complexes. Different to Kan complexes, which include all i.

> **Definition 1.0.3** (Functors between ∞-categories)
> A ∞-functor between two ∞-categories is a map between simplicial sets.

> **Definition 1.0.4** (Nerve of a category)
> Given an ordinary category $\mathcal{C}$, define the **nerve** of $\mathcal{C}$ to be the simplicial set given by
>
> $$N(\mathcal{C})_n := \{\text{Functors } F : [n] \to \mathcal{C}\}$$
>
> where $[n]$ is the poset category on $\{1, 2, \cdots, n\}$. So an n-simplex is a diagram of objects $X_0, \cdots, X_n \in \mathrm{Ob}(\mathcal{C})$ and a sequence of maps. This defines an ∞-category, and there is a correspondence
>
> $$\left\{ \text{Functors } F : \mathcal{C} \to \mathcal{D} \right\} \iff \left\{ \text{∞-Functors } \widehat{F} : N(\mathcal{C}) \to N(\mathcal{D}) \right\}.$$
>
> Note that taking the nerve of a category preserves the usual categorical structure, since the objects are the 0-simplices and the morphisms are the 1-simplices.

Remark 1.0.5: For $\mathcal{C}$ an ∞-category, we can define $\mathcal{C}_0$ to be the "objects" and $\mathcal{C}_1$ to be the "morphisms", although we don't have a good notion of composition yet. There will be boundary map: a 1-simplex has two boundary points, i.e. two objects $a, b \in \mathcal{C}_0$, so we can think of this as a map $f : a \to b$ where $a = \partial_1 f, b = \partial_0 f$[1] are the first and second vertices respectively. We'll also have "degeneracy" maps going up from $\mathcal{C}_0 \to \mathcal{C}_1$, which we should think of as assigning identity morphisms to objects, or conversely that the identity morphism is the degenerate 1-simplex at an

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[1]This notation ∂_i denotes the boundary operator that drops the ith vertex.

object.

Definition 1.0.6 (Equivalence of Morphisms)
Given two morphisms $f, g : a \to b$ in an ∞-category, we say $f \simeq g$ are **equivalent** iff there is a 2-simplex filling in the following diagram:
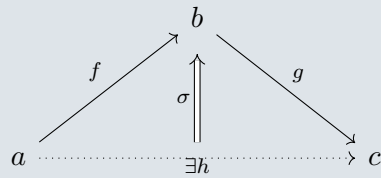

Link to diagram

Remark 1.0.7: This turns out to be an equivalence relation. Note that in an ordinary category, if two morphisms are equivalent then they are already equal.

Definition 1.0.8 (Composition of morphisms)
For 1-simplices $f : a \to b, g : b \to c$, a **composition** of f and g is a 2-simplex σ filling in the following diagram:
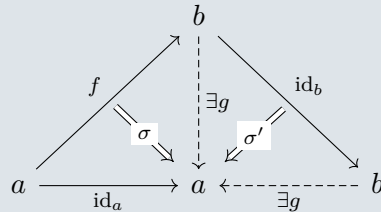

Link to diagram
In this case, $h := \partial_1 \sigma$ and we write $h \simeq g \circ f$.

Remark 1.0.9: Note that we're not fixing a choice, but it is well-defined up to the equivalence relation we're using. This is similar to how e.g. coproducts are not baked into the structure of a category, but are instead only well-defined up to canonical isomorphism – and in fact, this characterization is sometimes preferable.

Definition 1.0.10 (Equivalences of objects)
If $f : a \to b$ is a morphism in an ∞-category $\mathcal{C}$, then we say f is an **equivalence** if there exists a morphism $g : b \to a$ such that $\mathrm{id}_a \simeq g \circ f$ and $\mathrm{id}_b \simeq f \circ g$. This is equivalent to finding 2-simplices σ, σ' that fill the following two diagrams:

Link to diagram

Remark 1.0.11: This is close to what we'd require for an isomorphism in an ordinary category, but we now allow the compositions to only be "weakly equivalent" or homotopic to the identities.

> **Definition 1.0.12** (Functor Categories)
> For $\mathcal{C}, \mathcal{D}$ simplicial sets, we can define a simplicial set $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ whose n-simplices are given by
>
> $$\mathrm{Fun}(\mathcal{C}, \mathcal{D})_n := \{\text{Simplicial maps} F : \mathcal{C} \times \Delta^n \to \mathcal{D}\}.$$

Remark 1.0.13: Note that the 0-simplices recover functors if these are ordinary categories. If $\mathcal{D}$ is an ∞-category, then this functor category is again an ∞-category.

> **Definition 1.0.14** (Morphisms of functors / natural transformations)
> A **morphism** in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, say $\eta : F \to G$, is a functor $\eta : C \times \Delta^n \to \mathcal{D}$ such that
>
> $$\eta|_{\mathcal{C} \times \{0\}} = F$$
> $$\eta|_{\mathcal{C} \times \{1\}} = G.$$
>
> We call such an η a **natural transformation** from F to G.

Remark 1.0.15: Being an equivalence in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is equivalent to being a pointwise equivalence. I.e., η is an equivalence iff the map η_C given by partially applying an object of $\mathcal{C}$ (i.e. a 1-simplex $\Delta^n \to \mathcal{D}$) is an equivalence in $\mathcal{D}$ for all objects $C \in \mathrm{Ob}(\mathcal{C})$.

> **Definition 1.0.16** (Equivalences of ∞-categories)
> A functor $f : \mathcal{C} \to \mathcal{D}$ of ∞-categories is an **equivalence** iff there exists a functor $g : \mathcal{D} \to \mathcal{C}$ and natural equivalences
>
> $$f \circ g \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$$
> $$g \circ f \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}}.$$