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Reading Notes

Reference List

- **Rational Homotopy Theory and Differential Forms** by Griffiths and Morgan
- **Differential Forms in Algebraic Topology** by Bott and Tu
- **Differential Topology** by Hirsch
- **Comprehensive Introduction to Differential Geometry** by Spivak
- **Topology from the Differentiable Viewpoint** by Milnor
- **Topology and Geometry** by Bredon
- **User's Guide to Spectral Sequences** by Mcleary
    - View Here
    - Apparently lots of technical details

General Notes

The standard Serre fibration: $\Omega X \into PX \mapsvia{f} X$ where ΩX is the loop space, PX is the path space, and f is the "evaluation at the endpoint" map. Note that PX is contractible!

Consider a SES $0 \into A \into B \into C \into C$, then look at it as a 2-step filtration of B so $F^0B = B, F^1B = A, F^2B = 0$. The graded pieces are $G_0 = C, G_1 = A$. Can use this to obtain LES from SS.

Homology in the ring-theoretic setting: If R is a Noetherian ring and $I \subseteq R$, then if I can be generated by n elements then $H_I^i(M) = 0$ for any R-module M and $i > n$. Thus to prove I can *not* be generated by n elements, it suffices to find a module M where $H_I^{n+1} \neq 0$.

Griffiths and Morgan

Overall purpose: want to relate C^∞ forms on a manifold to AT invariants. One significant result: given a manifold M, the singular cohomology $H^*(M, \RR)$ is isomorphic to the cohomology of the differential graded algebra of C^∞ forms, $H^*_{DR}(M)$.

> Is this the de Rham cohomology..?

This DGA of smooth forms is actually enough to calculate all of the AT invariants, and can be used to build the Postnikov tower of M ($\tensor \RR$)

One construction is the localization of a CW complex at $\QQ$, this removes torsion and divisibility phenomena. The effect on the Postnikov tower is just then tensoring with $\QQ$.

Things that are homotopy equivalent to CW complexes:

- Manifolds
- Varieties
- Loop spaces of CW complexes
- Eilenberg-MacLane spaces? $K(\pi, n)$.

The Whitehead theorem holds for these: $X \mapsvia{f} Y$ is an homotopy equivalence iff $\pi_*(X) \mapsvia{f_*} \pi_*(Y)$ is an isomorphism.

Recall the weak topology for infinite CW complex: U is open in X iff $U \cap X^n$ is open for every n.

Theorem: Given any $X \mapsvia{f} Y$, we can transform this into an inclusion up to homotopy equivalence. (Just replace Y my the mapping cylinder of f, denoted $M_f \homotopic Y$).

A fibration is anything that satisfies the homotopy lifting property. Examples:

- Locally trivial fiber bundles
- Vector bundles
- Covering spaces

Path spaces are fibrations, loop spaces are contractible.

Homology can be defined with coefficients in any abelian group by tensoring the singular chain groups with \(G\). That is, if we \(H_*(X)\) obtained from

\(\mapsvia{\del_{n+1}} C_n(X) \mapsvia{\del_n} C_{n-1}(X) \mapsvia{\del_{n-1}} C_{n-2}(X) \cdots \mapsvia{\del_1} C_0(X)\)

then we can define \(H_*(X; G)\) via

\(\mapsvia{\del_{n+1}\otimes 1} C_n(X)\otimes G \mapsvia{\del_n\otimes 1} C_{n-1}(X) \otimes G \cdots \mapsvia{\del_1\otimes 1} C_0(X) \otimes G\)

Note that homology has the structure of a graded group, while cohomology has the structure of a graded commutative ring.

Axioms of homology:

- \(X \mapsvia{f} Y\) always induces a map on homology \(H_*(X) \mapsvia{f_*}H_*(Y)\)
- An orientation on \(S^n\) induces an isomorphism \(H_n(S^n) \cong \ZZ\); reversing orientation induces the map \(\ZZ \mapsvia{\phi}\ZZ: \phi(1) = -1\)
- \(Y\subseteq X\) yields the definition of relative homology \(H_*(X, Y)\), and Mayer Vietoris holds; i.e. there is a long exact sequence \(\cdots H_n(Y) \into H_n(X) \into H_n(X, Y) \into H_{n-1}(Y) \cdots\)
- Excision: \(U \subset Y \subset X\) and \(\bar U \subset Y^\circ\) implies \(H_*(X-U, Y-U) \cong H_*(X,Y)\).

Any homology theory satisfying these properties is equivalent to singular homology.

Use notation \([X, Y]\) for homotopy classes of maps \(X\into Y\), then \(\pi_1(X) = [S^1, X]\) and we can define \(\pi_n(X) = [S^n, X]\). Homotopy groups fail excision.

Whitehead theorem: for CW complexes, if \(X \mapsvia{f} Y\) induces \(\pi_n(X) \mapsvia{f_*} \pi_n(Y)\) and \(f_*\) is an isomorphism (and \(Y\) is connected), then \(f\) is a homotopy equivalence. For spaces that aren’t CW complexes, this may fail, and we say \(f\) is a *weak homotopy equivalence* instead.

Hurewicz theorem: the bottom homology and homotopy groups are isomorphic, and homology below the bottom homology is zero.

General note: there are equivalent “relative versions” of most of these theorems.

Spectral Sequence: Page 45.

For any fibration $F \mapsvia{} E \mapsvia{\pi} B$, we get a LES in homotopy $\pi_n(F) \into \pi_n(E) \into \pi_n(B) \mapsvia{\del} \pi_{n-1}(F)$

Basic question: How are the cohomologies of F,E,B related? An easy case is when $E=F\cross B$, but even then $\pi_n(F\cross B) \neq \pi_n(F) \oplus \pi_n(B)$. Need the Kunneth theorem, formula is more complicated.

For CW complexes and a fibration, the relationship is nice - look at the total space of the fibration. It is filtered by increasing n-skeleta, and we use the LES. More general filtrations need a spectral sequence.

Note: use LES as trivial example of spectral sequence! Write out the pages, differentials, etc

The spectral sequence relates the cohomology of *successive pairs* in the filtration to the cohomology of the total space.

Theorem: If B is path-connected and $\pi_1(B, b_0)$ acts trivially on $H^*(F)$, then there are isomorphisms $$ H^n(E^p, E^{p-1}) \cong \prod_{\text{p-cells in B}} H^n(\pi^{-1}e^p, \pi^{-1}\del e^p) \\ \cong C^p(B; H^{n-p}(F)) $$

In other words, for any k, we can think of $H^*(E^p, E^{p-k})$ as a k-th approximation to $H^*(E^p)$.

(Should probably review results about polynomial and exterior algebras. And what does it mean for π_1 to act trivially on a fiber?)

EXAMPLES OF COMPUTATION: Page 54

- ## Cohomology of $\CP^n$ using $S^1 \into S^{2n+1} \into \CP^n$

- Cohomology of the infinite Grassmannian $\lim_n G(k, n)$
    - Answer: $H^*(G(k)) = \ZZ[x_1, x_2, \cdots x_k]$

Postnikov Towers

A decomposition dual to cell decomposition, the atoms of the space are Eilenberg-Maclane spaces $K(\pi, n)$. (Note the spheres are atomic in homology, while the K are atomic in homotopy.)

Gives a way of going back and forth between X and $\pi_*(X)$: defined as a tower of spaces $X_0 \leftarrow X_1 \leftarrow \cdots$

- $X_{i-1} \leftarrow X_i$ is a fibration
- $\pi_k(X_n) = \mathbb{1}[k \leq n]\cdot\pi_k(X) + \mathbb{1}[k > n]\cdot 0$
    - So all lower homotopy groups agree at the n-th spot
- (Probably) $X_i \injects X$

Unique up to homotopy, $X = \lim_n X_n$ (an inverse limit). Essentially constructs X out of $K(\pi_n(X), n)$.

Note: revisit and draw diagrams for Postnikov Towers

Homotopy and homology commute with direct limits.

$(\cdot \tensor \QQ)$ is a right-exact functor, most results in this section are about how terms in exact sequences all become $\QQ$-vector spaces. In particular, $H^*(X; \QQ), H_*(X;\QQ)$ are.

Homotopy theory over $\QQ$ is much easier than over $\ZZ$. Samples results: $$\pi_i(S^{2n-1}) \otimes \QQ = \begin{cases} \QQ & i=2n-1 \\ 0 & \text{otherwise} \end{cases}$$

Then using the fact that $\pi_i(S^{2n-1})$ is always finitely generated, we can conclude $$\pi_i(S^{2n-1}) = \begin{cases} \ZZ & i=2n-1 \\ \text{a finite group} ~G & \text{otherwise} \end{cases}$$ This yields for even n: $$\pi_i(S^{n}) = \begin{cases} \ZZ & i=n \\ \ZZ \oplus G & i=2n-1 \\ H & \text{otherwise} \end{cases}$$ for some finite groups G,H!

Can also obtain Bott Periodicity this way.

Other Reading

Lots of good examples of computations here

Some fibrations

- Hopf: \(S^1 \into S^3 \into S^2\)
- \(S^1 \into S^{2n+1} \into \CP^n\)
- Path space: \(\Omega S^n \into PS^n \into S^n\)

Serre Spectral Sequence Example: For the fibration \(S^1 \into S^3 \into S^2\), the \(E_2\) page: \[\begin{array}{c|ccc} 1 && H^0(S^2, \ZZ) && H^1(S^2, \ZZ) && H^2(S^2, \ZZ) \\ 0 && H^0(S^2, \ZZ) && H^1(S^2, \ZZ) && H^2(S^2, \ZZ) \\ \hline && 0 && 1 && 2 \end{array} \] Which is equal to \[\begin{array}{c|ccc} 1 && H^0(S^2, \ZZ) && 0 && H^2(S^2, \ZZ) \\ 0 && H^0(S^2, \ZZ) && 0 && H^2(S^2, \ZZ) \\ \hline && 0 && 1 && 2 \end{array} \] And \(E_3 = E_\infty\), so \(d_2^{0,1}\) is an isomorphism.

Note: Probably a good starting point for basic calculations? Fill out the missing details for this table.

Challenge: Prove \(\pi_4(S^2) = \frac{\ZZ}{2\ZZ}\)

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Paper on “constructive” algebraic topology J. Rubio, F. Sergeraert / Bull. Sci. math. 126 (2002) 389–412 403

Many constructions in algebraic topology can be organized as solutions of fibration problems.

In particular the classifying space \(BG\) of a topological group \(G\) is the solution for a fibration \(BG \cross_\tau G\) where the fiber space is the given group \(G\), the base space is the classifying space \(BG\) and the product \(BG \cross G\) is twisted in such a way the total space \(BG \cross_\tau G\) is contractible. The same idea where the base space \(X\) is given and the fibre space is unknown leads to the loop space ΩX and the contractible total space $X \cross_\tau \Omega X$.

The handbooks of Algebraic Topology more or less explain the Eilenberg–Moore spectral sequence can be used to “compute” the homology groups of the new objects BG and $ΩX$ if the homology groups of G or X are known. In fact this spectral sequence is in general unable to give you the new homology groups, unless you are in a very special situation. The Serre spectral sequence works in the third situation, when you are looking for the homology groups of a total space $B \cross_\tau F$ if the homology groups of B and F are known; but in general you meet the same difficulties with the higher differentials and the extension problems at abutment.

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if one understood even the stable homotopy groups of spheres very well, one would therefore have a near complete understanding of the group (I assume that $n\neq 4$) of differentiable structures on the n-sphere: see e.g.

The homotopy group $π_{n+k}(S^k)$ is a finite group except

1. for $n=0$ in which case $π_k(S^k)=\ZZ$;
2. $k=2m$ and $n=2m−1$ in which case $π_{4m−1}(S^{2m})≃Z⊕F_m$ for F_m a finite group.

The rough idea of obstruction theory is simple. Suppose we want to construct some kind of function on a CW complex X. We do this by induction: if the function is defined on the k-skeleton X_k, we try to extend it over the $(k + 1)\dash$skeleton X^{k+1}. The obstruction to extending over a $(k + 1)\dash$cell is an element of π_k of something. These obstructions fit together to give a cellular cochain on X with coefficients in this $π_k$. In fact this cochain is a cocycle, so it defines an “obstruction class” in $H_{k+1}(X; π_k(something))$. If this cohomology class is zero, i.e. if there is a cellular $k\dash$cochain $η$ with $0 = δη$, then $η$ prescribes a way to modify our map over the $k\dash$skeleton so that it can be extended over the $(k + 1)\dash$skeleton

The J homomorphism seems to link the framed bordism classes of manifolds to homotopy groups of spheres - for example J takes $\pi_k(SL^n(R))$ to $\pi_{n}^k S^n$.

What is the J homomorphism? Look at the map $$ SO(n) \to \Omega^n S^n\\ A: (\RR^n \to \RR^n) \mapsto A^+ $$

Where we view a matrix as a linear function on $\RR^n$, and take it to its compactification which is a map $S^n\to S^n$. Taking the limit yields a map from $SO^\infty \to QS^0 = \ \Omega^\infty \Sigma^\infty S^0$, and taking π_0 of both sides induces the J homomorphism. The RHS is equal to π_*^s, the stable homotopy groups of spheres. But the homotopy groups of SO were computed by Bott, and have some 8-fold periodicity.

The image of J was found by Adams in ’66 or so, it is a finite group with order the denominator of some function involving Bernoulli numbers. However, the pattern is more apparent by looking at the p-stems, then the number of connected dots really just depends on the p-adic divisibilty of the horizontal number plus 1. The image of J is just the bottom row in these stem diagrams.

How to read the stem diagrams? Each one is for a fixed p, for example at $p=2$ each dot depicts a factor of 2 and vertical lines denote additive extensions. For example, $\cdot \to \cdot \to \cdot$ vertically would denote $\ZZ_{2^3} = \ZZ_2 \oplus \ZZ_2 \oplus \ZZ_2$, whereas $\cdot \to \cdot$ denotes $\ZZ_2 \oplus \ZZ_2$.

The EHP sequence is good for unstable stuff, not the best for stable - use the Adams spectral sequence instead.

The stable homotopy groups of spheres are isomorphic to the framed cobordism groups of manifolds. The Kervaire invariant has to do with which stable homotopy groups can be represented by exotic spheres. There are several open problems related to differentials and invariants the arise from this SS, e.g. what are the permanent cycles? The Kervaire Invariant Problem was only recently solved by Hopkins-Ravenel-Hill.

The Adams-Novikov SS ends up being cleaner, fewer differentials!

Table of $\pi_{n+k}S^n$: here

Interesting question: $\pi_0(X)$ can be defined for schemes. What are the higher homotopy groups? What do they measure? (More fundamentally, what do higher homotopy groups of spheres measure at all?)

WRT schemes, what are the homotopy spheres? What are the homology spheres, the Eilenberg-MacLane spaces?

Possible Project Outline

- CW Complexes
- Define homotopy
- Define homotopy invariance
- Classification of abelian groups
    - Free and torsion
- Define $\pi_n(X)$
    - Show functoriality
    - Show homotopy invariant
- Whitehead’s Theorem (homotopy and homology versions)
- π_n for $n\geq 2$ is abelian
- Compute $H_* S^n$
- Compute $\pi_k S^1$
- Cellular approximation theorem
- Show $\pi_k S^n = 0$ for $k<n$
- Show π_n only depends on n-skeleton
- Hurewicz theorem
- Show $\pi_n S^n = Z$
- Show $\pi_i S^n = 0$ for $i < n$
- Define fibrations and fiber bundles
- Define suspension and loop spaces
- Show $\Sigma \to \Omega$ adjunction
- Show how to use Σ and Ω to move between π_n using equalities
- Freudenthal Suspension theorem

- The homotopy LES of a fibration
- Hopf Fibration
    - Show $\pi_k S^2 = \pi_k S^3$
    - Show $\pi_3 S^2 = Z$
- Killing off homotopy groups
- SS of a filtration
- Serre SS
    - Compute algebra structure of CP^∞
- Compute $\pi_4 S^2$
- Compute first stable π_k
- Freudenthal Suspension
- Eilenberg-Maclane Spaces
- $H^n (X; G) = [X, K(G, n)]$
- Summary of “easy” results:
    - $\pi_k S^1 = 0, i > 1$
    - $\pi_n S^n = Z$
    - $\pi_3 S^2 = Z$
    - $\pi_k S^2 = \pi_k S^3$
    - $\pi_i(S^n)$ is finite for $i > n$
        - Except for π_{4k-1}
    - See here
- Harder results
    - $\pi_n+1 S^n = Z\delta_2 + Z_2 \delta_{n \geq 3}$
    - $\pi_n+2 S^n = Z_2$
- Exact sequences
- Splitting and extension problem
- Degree of a map to S^n
- Whitehead product and Lie Algebra structure of π_*

In fact, by working the Serre spectral sequence backwards, we can compute the homology of ΩS^n.

Unstable homotopy groups of spheres

Unstable Homotopy Groups of Spheres

Homotopy Groups of SO(n)

Homotopy Groups of SO^n

Useful Higher Homotopy used in Physics

Various higher homotopy groups
π_n are equal for the following spaces:

- SO^3
- $\RP^3$
- S^3
- SU^2

(Maybe these are all diffeomorphic)

Also $\pi_n(\RP^n) = \pi_n(S^n)$

$Sp^4 = SU^2 \cross SU^2$

$J: \pi_k(SO^n) \to \pi_{n+k} S^n$

Homotopy of Infinite Grassmannian

Homotopy of infinite Grassmannian

It is ‘’well-known’’ that the third stable homotopy group of spheres is cyclic of order 24. It is also “well-known” that the quaternionic Hopf map $ν:S^7→S^4$, an S^3-bundle, suspends to a generator of $π_8(S^5)=π^{st}_3$. It is even better known that the complex Hopf map $η:S^3 \to S^2$ suspends to a generator of $π_4(S^3)=π^{st}_1=\ZZ_2$. For this, there is a reasonably elementary argument, see e.g. Bredon, Topology and Geometry, page 465.

As we have seen in this MO question, and also this one, the K3 surface plays an important role in the third stable homotopy group of spheres. It can be viewed as the source of the 24 in this group $π^{st}_3=\ZZ_{24}$. Here is a brief review of how that goes: the stable homotopy groups (in degree n) of spheres are the same as cobordism classes of stably framed manifolds (of dimension n). In dimension 3 the generator is given by $ν=(S^3,Lie)$, the 3-sphere with its Lie group framing (where we think of $S^3⊆H$ as the group of unit quaternions).

The homotopy groups $π_i(S^3)$ for $i\leq 64$ are apparently computed in:

Curtis, Edward B.,Mahowald, Mark, The unstable Adams spectral sequence for S^3, Algebraic topology (Evanston, IL, 1988), 125–162, Contemp. Math., 96, Amer. Math. Soc., Providence, RI, 1989.

The Pontryagin-Thom construction shows that the stable homotopy groups of spheres are the same as the groups of stably framed manifolds up to cobordism.

The best overall summary of results would be Doug Ravenel’s book on the homotopy groups of spheres, and I would also recommend Kochman’s book. Read works of Mark Mahowald for results using the Adams spectral sequence, and Doug Ravenel for the Adams-Novikov spectral sequence. Complete or nearly complete calculations for the homotopy groups of spheres that have been localized at a particular Morava K-theory have been made by Toda, Goerss-Henn-Mahowald-Rezk, and Mark Behrens. If you’re interested in computer calculations of Ext, you should contact Robert Bruner or Christian Nassau. Many others have contributed to the calculation of homotopy groups of spheres and probably deserve to be mentioned (if I omitted someone, it was unintentional).

One of the most recent spectacular advances in algebraic topology was the solution of (most of) the Kervaire invariant 1 problem by Hill, Hopkins and Ravenel about framed manifolds/stable homotopy groups of spheres. They used a tremendous amount of stuff to solve this classical problem: equivariant topology, chromatic homotopy theory, spectral sequences, orthogonal spectra, abstract homotopy theory, …

https://www.youtube.com/watch?v=jie-ww7RBWY

Honors Thesis on Related Stuff, Hopf Invariant 1

Only odd spheres with a unique smooth structure are dimensions 1, 3, 5, and 61

Question 1.2. Let \(M\) be a closed \(n\)-manifold. Suppose \(M\) is homotopy equivalent to \(S^n\). Is \(M\) homeomorphic to \(S^n\)? The answer turns out to be yes for all dimensions.

Question 1.4. Let \(M\) be a closed \(n\)-manifold. Suppose \(M\) is homeomorphic to \(S^n\). Is \(M\) diffeomorphic to \(S^n\)? For \(n= 3\), the answer is yes. It is due to Moise [37] that every closed 3-manifold has a unique smooth structure. In particular, the 3-sphere has a unique smooth structure. **For n= 4, this question is wildly open.** For higher dimensions, Milnor [35] constructed an exotic smooth structure on \(S^7\). Furthermore, Kervaire and Milnor [27] showed that the answer is not true in general for \(n≥5\).

Question 1.6. For which \(n\) does there exist a unique smooth structure on \(S^n\)? Kervaire and Milnor reduced Question 1.5 to a computation of the stable homotopy groups of spheres. In fact, Kervaire and Milnor constructed a group \(Θ_n\), which is the group of h-cobordism classes of homotopy \(n\)-spheres. The group \(Θ_n\) classifies the differential structures on \(S^n\) for \(n≥5\).

Table of n+k homotopy groups of spheres

$K(G, n) \cong \Omega K(G, n+1)$

is adjoint to the map

$\Sigma K(G,n) \to K(G, n+1)$

General Reading Notes

Stacks and orbifolds are concepts from algebraic geometry (respectively geometric topology) that allow us to talk about objects that locally look like the quotient of a smooth object by a group action, in a way that remembers information about the isotropy groups of the action. Such ‘stacky’ objects can behave like smooth objects even if the underlying spaces have singularities. As for spaces, manifolds,and schemes, cohomology theories are important invariants also for stacks and orbifolds, and examples such as ordinary cohomology or K-theory lend themselves to generalization. Special cases of orbifolds are ‘global quotients’, often denoted M/G, for example for a smooth action of a compact Lie group G on a smooth manifold M. In such examples, the orbifold cohomology of M/G is supposed to be the G-equivariant cohomology of M. This suggests away to do orbifold cohomology theories by means of equivariant stable homotopy theory,via suitable G-spectra. However, since the group G is not intrinsic and can vary, one needs equivariant cohomology theories for all groups G, with some compatibility. Global homotopy theory makes this idea precise.

The universal compact lie group is the topological monoid $L(\RR^\infty, \RR^\infty)$ of linear isometric self-embeddings.

What are Quillen equivalences?

What is a Stiefel manifold?

Proof of Hurewicz using spectral sequences: Thm 3.2.2

Survey articles on homotopy

groups of spheres

From Hatcher:

While my Algebraic Topology book and my unfinished book on spectral sequences (referred to in other answers to this question) contain some information about homotopy groups of spheres, they don’t really qualify as a general survey or introduction. One source that fits this bill more closely is Chapter 1 of Doug Ravenel’s “green book” Complex Cobordism and Stable Homotopy Groups of Spheres, from 1986. This introductory chapter starts at a reasonably accessible level, with increasing prerequisites in the later sections of the chapter. More recent surveys ought to exist, although at the moment I can’t recall any. With the recent solution of the Kervaire invariant problem by Hill-Hopkins-Ravenel, this would be a good time for an updated survey.

Connections between homotopy groups of spheres and low-dimensional geometry and topology have traditionally been somewhat limited, with the Hopf bundle being the thing that comes most immediately to mind. A fairly recent connection is Soren Galatius’ theorem that the homology groups of $Aut(F_n)$, the automorphism group of a free group, are isomorphic in a stable range of dimensions to the homology groups of “loop-infinity S-infinity”, the space whose homotopy groups are the stable homotopy groups of spheres.

Other posters have alluded to the Kervaire-Milnor theory (from “Groups of homotopy spheres. I”) which shows how, via Pontryagin-Thom, our knowledge and ignorance about the stable homotopy groups of spheres is reflected in knowledge and ignorance about classification of manifolds. Maybe it’s worth telling this (really beautiful!) story.

In each dimension n, one has a group Θ_n of smooth n-manifolds that are homotopy n-spheres, up to h-cobordism, under connected sum. This has a subgroup bP_{n+1} of boundaries of parallelizable $n+1$-manifolds. Assume n>4, so h-cobordism classes are diffeomorphism classes.

Every homotopy n-sphere S can be shown to have a stable framing. Hence (by P-T) S is a regular fibre of a map $S_{n+k}\to S_k$ for $k\gg0$ whose class in $π_{n+k}(S_k)$ is the obstruction to S (with chosen stable framing) being a framed boundary. Changing the stable framing amounts to adding something in the image of the J-homomorphism $J: π_n(SO(k)) \to π_{n+k}(S_k)$. So we get an injective homomorphism $Θ_n/bP_{n+1}→\coker(J)$ (which is onto e.g. for n odd).

We don't know $\coker(J)$ in high dimensions, so we don't know the order of $Θ_n/bP_{n+1}$. But Serre's finiteness theorem for the stable stems tells us that $Θ_n/bP_{n+1}$ is finite!

The subgroup bP_{n+1} is analyzed via surgery theory and the h-cobordism theorem. There's a nice summary in Lück's Basic introduction to surgery theory.

We have $bP_{odd}=0$. There's a formula for bP_{4p} involving Bernoulli numerators; this comes from a known (thanks to Adams) part of the stable stems, namely, $\im(J)$.

Finally, bP_{4p+2} is at most Z_2. Here S bounds a parallelizable manifold P. We'd like to make this contractible. By framed surgery, we can kill the homotopy groups of P below the middle dimension but the Arf invariant of the pairing on middle-dimensional homology obstructs the final step, that of killing $π_{p+1}$. Say this is non-zero. Can we do better by starting with a different P? Yes, if and only if there's a closed, framed 2_{p+2}-manifold of Kervaire invariant one.

Browder showed that the Kervaire invariant can be one only when $4p+2=2l−2$ for some l, and Hill-Hopkins-Ravenel have shown that $l≤7$. Conclusion: bP_{4p+2} is Z_2 except in dimensions 6, 14, 30, 62, and possibly 126, where it's zero.

Justification for Lie Group Computations Here is a very fundamental way to create interesting Riemannian manifolds: Let G be a semi-simple Lie group, let K be its maximal compact subgroup, let $Γ$ be a discrete subgroup of G, and form G/K. This quotient is called the symmetric space attached to G.

The Riemannian structure comes from an invariant metric on G, and so G acts as isometries on G/K by left translation.

If you now take a discrete subgroup Γ of G, you can form the double quotient $\Gamma\backslash G/K$. These are some of the most celebrated Riemannian manifolds in mathematics. In the case of $SL_2(\RR)$, we know via uniformization that all genus $g\geq 2$ Riemann surfaces can be described in this way. In the case of $SL_2(\CC)$ we get hyperbolic 3-manifolds, from symplectic groups we get moduli spaces of abelian varieties, … .

P.S. I should also note that the study of spaces $\Gamma\backslash G/K$ for certain Γ (so-called congruence subgroups) is one of the basic topics of the Langlands program, and the function theory and cohomology of these spaces (especially their representation-theoretic structure) is conjectured to govern a vast amount of number theory. Trying to understand and work on these conjectures was my own motivation for learning Lie theory.

Cartan’s Generalization of Klein’s Erlangen Program

More info

A Klein geometry consists of a Lie group G together with a Lie subgroup H of G. Together G and H determine a homogeneous space G/H, on which the group G acts by left-translation. Klein’s aim was then to study objects living on the homogeneous space which were congruent by the action of G. A Cartan geometry extends the notion of a Klein geometry by attaching to each point of a manifold a copy of a Klein geometry, and to regard this copy as tangent to the manifold. Thus the geometry of the manifold is infinitesimally identical to that of the Klein geometry, but globally can be quite different. In particular, Cartan geometries no longer have a well-defined action of G on them. However, a Cartan connection supplies a way of connecting the infinitesimal model spaces within the manifold by means of parallel transport.

- Irreducible reps of SO_3 describe the periodic table
- Irreducible reps of SU_2 lead to Dirac equations of the electron
- Standard Model: $SU_3 \cross SU_2 \cross U_1$

$SO(n)$ and its relativistic analog $SO(1,n)$ define symmetries of space time. $SU(n)$ defines fundamental quantum mechanical phase space symmetries. $Sp(n)$ defines symmetries in cases where there is a quaternionic structure.

Misc

Homotopy and Homology Results

- $\pi_1(SL_n(\RR)) = \ZZ\delta_2 + \ZZ_2 \delta_{n\geq 3}$ See Lemma 5.3
- $\pi_1(SO_n(\RR)) = \pi_1(SL_n(\RR))$

Classifying Space

Usually look at this in the context of a topological group G, and denote BG the classifying space of G. It is the quotient of some contractible space EG by a free action of G, so we have something that looks like $G \to EG \to BG$ and $BG = EG/G$.

For a discrete group G, we have $BG = K(G,1)$, so that $\pi_1(BG) = G$ and $\pi_k(BG) = 0$ for $k \neq 1$.

Question: what is a principal bundle? According to Wikipedia, any G-principal bundle is a pullback of $EG \to BG$.

Note that contractibility of EG shows that BG is $K(G, 1)$.

Examples

Note that EG is always a contractible space upon which G acts freely.

We also have $BX \homotopic \Omega X$

- $G \to EG \to BG = EG/G$
- $\ZZ \to \RR \to S^1$
- $\ZZ^n \to \RR^n \to T^n$

- $\ZZ^{\ast n} \to ??? \to \bigvee_n S^1$
- $\ZZ_2 \to S^\infty \to \RP^\infty$
- $\ZZ_n \to S^\infty \to L_n^\infty$
- $S^0 \to S^\infty \to \RP^\infty$
- $S^1 \to S^\infty \to \CP^\infty$
- $S^3 \to S^\infty \to \HP^\infty$
- NOT TRUE: $S^7 \to S^\infty \to \OP^\infty$
- $T^n \to ? \to (\CP^\infty)^n$
- $O_n \to V_n(\RR^\infty) \to Gr_n(\RR^\infty)$
- $GL_n(\RR) \to V_n(\RR^\infty) \to Gr_n(\RR^\infty)$
- $SO_n \to ? \to ?$
- $Gr_n(\RR^\infty) \to ? \to Gr_n(\RR^\infty)$
- $\pi_1(\Sigma_g) \to ? \to \Sigma_g$
- $S_n \to ??? \to \theset{U \subset \RR^\infty,~ |U| = n}$

Note that $V_n(X)$ is the Stiefel manifold of dimension n orthonormal frames in X.

Also, $\pi_1(\Sigma_g) = <\theset{a_i, b_i}_i^n> \mid \prod_i^g [a_i, b_i]>$

A principal G bundle is a locally trivial free G-space with orbit space B. If G is discrete, then a principal G-bundle over X with total space $\tilde X$ is equivalent to a regular covering map with $Aut(\tilde X) = G$. Under some hypothesis, there exists a classifying space BG such that $\theset{ \text{isomorphism classes of G-bundles over X}} \cong [X, BG]$, i.e. bundles of G's over X are equivalent to maps from X into the classifying space, i.e.
$$\hom(X, BG) \cong \theset{G\dash\text{bundles over } X}$$

It is useful to think of BG as a space whose points are copies of G, so the classifying map $X \mapsvia{f} BG$ assigns each $x \in X$ to the fiber above x, which is a G.

There is a standard procedure in homotopy theory for constructing a classifying space for every group. One starts by constructing a 2-complex with the given fundamental group, and then one inductively attaches higher dimensional cells to kill all higher homotopy groups. Each element $c\in \pi_n(X_{n−1})$ is represented by some continuous map $\gamma_c:S^n\to X_{n−1}$ with image in the $n\dash$-skeleton. Let X_n be obtained from $X_{n−1}$ by attaching an $(n+1)\dash$cell along γ_c, for each $c\in π_n(X_{n−1})$.

Conjecture: $B(G \cross H) = BG \cross BH$ Proof outline: $EG \cross EH$ is contractible, and $G \cross H$ acts freely on it with quotient equal to the RHS.

Conjecture: $B(G \ast H) = BG \vee BH$

Unknown: $B(G \tensor H) = BG \tensor BH$

Unknown: $B(G \semidirect_\phi H) = ?$

Paper on Chow Rings

Recent result: Chow Rings computed in 2005 for $BGL_n, BSL_n, BSp_n, BO_n, BSO_n$ Cohomology for classifying spaces of linear algebraic groups (equivalently compact Lie Groups) have an algebraic analog: Chow rings of the classifying spaces. For a finite abelian group, the chow ring is the symmetric algebra on the group of characters.

There is a map from the Chow ring back into cohomology, which in general fails surjectivity and injectivity. Tensoring this map with $\QQ$ creates an isomorphism, though. In this case, both have the ring structure of invariants under the Weyl group in the symmetric algebra of the ring of characters of a maximal torus. (Classical result, Leray and Borel.)

Chow rings have not been computed for PGL_n. Need to know about Chern classes, Euler classes,

A_* known for all O_n and SO_n for n odd in 80s, general result for SO_n 2004. PGL_n case is much harder. Understood for $n=2$, since $PGL_2 \cong SO_3$. Other bits that have been computed: $H^*(BPGL_3, \ZZ_3), H^*(BPGL_n, \ZZ_2)$ for $n = 2 \mod 4$ in 70s/80s, incomplete results for $H^*(BPGL_p, \ZZ_p)$ in 2003.

Term “equivariant” pops up a lot, symplectic forms, schemes, stacks

Further Reading

Characteristic classes are elements of $H^*(BG)$, can be used to define characteristic classes for bundles.

Connected covers can kill higher homotopy?

You can realize any Eilenberg-MacLane space as a classifying space.

Claim: $\pi_{i+k}B^kG = \pi_i G$.

Proof: If G is a topological group, there is a universal principal $G\dash$bundle $EG \to BG$ which induces a LES in homotopy. Since EG is contractible, $\pi_i EG = \pi_{i+1}EG = 0$, so $\pi_{i+1}BG \cong \pi_i G$. When G is an E_2 space, BG is a topological group, and so $\pi_{i+2}(B^2G) = \pi_{i+2}(B(BG)) = \pi_{i+1}(BG) = \pi_i(G)$ and we conclude the result.

Corollary: If G is a discrete group, $B^kG = K(G, n)$. Proof: Then $\pi_0 G = G$ and $\pi_i G = 0$ for $i > 0$, so $\pi_k B^k G = G$.

It’s possible to take classifying spaces of stacks. E.g. there is a stack that classifies principal bundles *with connections*, but it has issues: it is not a presentable stack, i.e. not covered by a manifold, so an associated sheaf is not representable.

Stable homotopy of BG: same sort of techniques as in S^n, break into components.

EG can be constructed as $\bigcup_n G \ast G \ast \cdots \ast G$, where $\ast$ is join of two spaces: the suspension of the smash product. For example, $G = \ZZ_2$ implies $EG = \bigcup_n \ZZ_2 \ast \cdots = \bigcup_n S^{n-1} = S^\infty$.

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Difference between low and high dimensional topology

High-dimensional topology refers to manifolds of dimension 5 and above, or in relative terms, embeddings in codimension 3 and above. Low-dimensional topology is concerned with questions in dimensions up to 4, or embeddings in codimension up to 2.

The distinction is because surgery theory works in dimension 5 and above (in fact, it works topologically in dimension 4, though this is very involved to prove), and thus the behavior of manifolds in dimension 5 and above is controlled algebraically by surgery theory. In dimension 4 and below (topologically, in dimension 3 and below), surgery theory does not work, and other phenomena occur.

The precise reason for the difference at dimension 5 is because the Whitney embedding theorem, the key technical trick which underlies surgery theory, requires 2+1 dimensions.

Low-dimensional topology is strongly geometric, as reflected in the uniformization theorem in 2 dimensions – every surface admits a constant curvature metric; geometrically, it has one of 3 possible geometries: positive curvature/spherical, zero curvature/flat, negative curvature/hyperbolic – and the geometrization theorem in 3 dimensions – every 3-manifold can be cut into pieces, each of which has one of 8 possible geometries.

Characteristic Classes

A characteristic class is a way of associating to each principal bundle on a topological space X a cohomology class of X. The cohomology class measures the extent to which the bundle is “twisted” — particularly, whether it possesses sections or not. In other words, characteristic classes are global invariants which measure the deviation of a local product structure from a global product structure. They are one of the unifying geometric concepts in algebraic topology, differential geometry and algebraic geometry.

Let G be a topological group, and for a topological space X, write $bG(X)$ for the set of isomorphism classes of principal G-bundles. This bG is a contravariant functor from Top (the category of topological spaces and continuous functions) to Set (the category of sets and functions), sending a map f to the pullback operation $f*$.

A characteristic class c of principal G-bundles is then a natural transformation from \(bG\) to a cohomology functor \(H*\), regarded also as a functor to Set.

In other words, a characteristic class associates to any principal \(G\)-bundle \(P → X\) in \(bG(X)\) an element \(c(P)\) in \(H^* (X)\) such that, if \(f : Y → X\) is a continuous map, then \(c(f^* P) = f^* c(P)\). On the left is the class of the pullback of \(P\) to \(Y\); on the right is the image of the class of \(P\) under the induced map in cohomology.

When the theory was put on an organized basis around 1950 (with the definitions reduced to homotopy theory) it became clear that the most fundamental characteristic classes known at that time (the Stiefel–Whitney class, the Chern class, and the Pontryagin classes) were reflections of the classical linear groups and their maximal torus structure.

The prime mechanism then appeared to be this: Given a space \(X\) carrying a vector bundle, that implied in the homotopy category a mapping from \(X\) to a classifying space \(BG\), for the relevant linear group \(G\). For the homotopy theory the relevant information is carried by compact subgroups such as the orthogonal groups and unitary groups of \(G\). Once the cohomology \({\displaystyle H^* (BG)}\) was calculated, once and for all, the contravariance property of cohomology meant that characteristic classes for the bundle would be defined in \({\displaystyle H^* (X)}\) in the same dimensions.

Characteristic numbers solve the oriented and unoriented bordism questions: two manifolds are (respectively oriented or unoriented) cobordant if and only if their characteristic numbers are equal.

Schoenflies Problem

A sharpening of Jordan curve theorem. It states that not only does every simple closed curve in the plane separate the plane into two regions, one (the “inside”) bounded and the other (the “outside”) unbounded; but also that these two regions are homeomorphic to the inside and outside of a standard circle in the plane.

Uniformization Theorem

The uniformization theorem says that every simply connected Riemann surface is conformally equivalent to one of the three Riemann surfaces: the open unit disk, the complex plane, or the Riemann sphere. In particular it implies that every Riemann surface admits a Riemannian metric of constant curvature.

For compact Riemann surfaces, those with universal cover the unit disk are precisely the hyperbolic surfaces of genus greater than 1, all with non-abelian fundamental group; those with universal cover the complex plane are the Riemann surfaces of genus 1, namely the complex tori or elliptic curves with fundamental group \(Z_2\); and those with universal cover the Riemann sphere are those of genus zero, namely the Riemann sphere itself, with trivial fundamental group.

The uniformization theorem is a generalization of the Riemann mapping theorem from proper simply connected open subsets of the plane to arbitrary simply connected Riemann surfaces.

Every Riemann surface is the quotient of a free, proper and holomorphic action of a discrete group on its universal covering and this universal covering is holomorphically isomorphic (one also says: “conformally equivalent” or “biholomorphic”) to one of the following:

1. the Riemann sphere
2. the complex plane
3. the unit disk in the complex plane.

In 3 dimensions, there are 8 geometries, called the eight Thurston geometries. Not every 3-manifold admits a geometry, but Thurston’s geometrization conjecture proved by Grigori Perelman states that every 3-manifold can be cut into pieces that are geometrizable.

Principal Bundles

A principal bundle is a mathematical object that formalizes some of the essential features of the Cartesian product $X \times G$ of a space X with a group G. In the same way as with the Cartesian product, a principal bundle P is equipped with

An action of G on P, analogous to $(x, g)h = (x, gh)$ for a product space. A projection onto X. For a product space, this is just the projection onto the first factor, $(x,g) \mapsto x$.

A principal G-bundle, where G denotes any topological group, is a fiber bundle $\pi:P \to X$ together with a continuous right action $P \times G \to P$ such that G preserves the fibers of P (i.e. if $y \in P_x$ then $yg \in P_x$ for all $g \in G$) and acts freely and transitively on them. This implies that each fiber of the bundle is homeomorphic to the group G itself.

Since the group action preserves the fibers of $\pi:P \to X$ and acts transitively, it follows that the orbits of the G-action are precisely these fibers and the orbit space P/G is homeomorphic to the base space X. Because the action is free, the fibers have the structure of $G\dash$torsors. A G-torsor is a space which is homeomorphic to G but lacks a group structure since there is no preferred choice of an identity element.

One of the most important questions regarding any fiber bundle is whether or not it is trivial, i.e. isomorphic to a product bundle. A principal bundle is trivial if and only if it admits a global cross section.

Examples

The prototypical example of a smooth principal bundle is the frame bundle of a smooth manifold M, often denoted FM or $GL(M)$. Here the fiber over a point $x \in M$ is the set of all frames (i.e. ordered bases) for the tangent space T_xM. The general linear group $GL(n,\RR)$ acts freely and transitively on these frames. These fibers can be glued together in a natural way so as to obtain a principal $GL(n,\RR)$-bundle over M.

A normal (regular) covering space $p:C \to X$ is a principal bundle where the structure group

$$G=\pi _{1}(X)/p_* (\pi_{1}(C))$$

acts on the fibers of p via the monodromy action. In particular, the universal cover of \(X\) is a principal bundle over \(X\) with structure group \(π_1(X)\) (since the universal cover is simply connected and thus \(π_1(C)\) is trivial).

Classification

Any topological group \(G\) admits a classifying space \(BG:\) the quotient by the action of \(G\) of some weakly contractible space \(EG\), i.e. a topological space with vanishing homotopy groups. The classifying space has the property that any \(G\) principal bundle over a paracompact manifold \(B\) is isomorphic to a pullback of the principal bundle \(EG → BG\).

In fact, more is true, as the set of isomorphism classes of principal \(G\) bundles over the base \(B\) identifies with the set of homotopy classes of maps \(B → BG\).

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Preliminaries

Definition: \(n\) connectivity

A space \(X\) is said to be \*(n\dash\)connected* if \(\pi_i X = 0\) for \(1 \leq i \leq n\).

Definition: Weak Homotopy Equivalence

A map \(f: X \to Y\) is called a *weak homotopy equivalence* if the induced maps \(f_i^* : \pi_i(X, x_0) \to \pi_i(Y, f(x_0))\) are isomorphisms for every \(i \geq 0\).

This is a strictly weaker notion than homotopy equivalence - for example, let \(L\) be the long line. Then \(\pi_i(L) = 0\) for all \(i\), but \(L\) is not contractible, and thus \(L \not\sim \pt\). However, the inclusion \(\pt \injects L\) is a weak homotopy equivalence, which can not be a homotopy equivalence.

Any weak homotopy equivalence induces isomorphisms on all integral co/homology groups, and thus co/homology groups with any coefficients by the UCT.

Definition: Cellular Map

If a map $X \mapsvia{f} Y$ satisfies $f(X^{(n)}) \subseteq Y^{(n)}$, then f is said to be a *cellular map*.

Theorem: Cellular Approximation

Any map $X \mapsvia{f} Y$ between CW complexes is homotopic to a cellular map.

Theorem: CW Approximation

For every topological space X, there exists a CW complex Y and a weak homotopy equivalence $f: X \to Y$. Moreover, if X is $n\dash$dimensional, Y may be chosen to be $n\dash$connected and is obtained from X by attaching cells of dimension greater than n.

Theorem: Whitehead

Abbreviated statement: if X, Y are CW complexes, then any map $f: X \to Y$ is a weak homotopy equivalence if and only if it is a homotopy equivalence.

(Note: f must induce maps on all homotopy groups simultaneously.)

Full Statement: If $(X, x_0) \mapsvia{f} (Y, f(x_0))$ such that the induced maps
$$ f_*: \pi_*(X, x_0) \to \pi_*(Y, y_0) \\ [g] \mapsto [f \circ g] $$
are all isomorphisms and Y is connected, then f is a homotopy equivalence.

Theorem: Uniqueness of E-M Spaces

If X is a space with one nontrivial homology group G in degree k, so that X satisfies
$$\pi_i(X) = \cases{G,~i=k\\0,~\text{otherwise}}$$

Then $X \simeq K(G, k)$.

(Note: two spaces with isomorphic homotopy groups may *not* be homotopy-equivalent in general - this is one exception.)

Theorem: Hurewicz

Given a space X, define a family of maps $$ h_k: \pi_k X \to H_k X\\ [f] \mapsto f_*(\mu_k) $$ where $H_k X = \langle \mu_k \rangle$.

If X is $n-1$ connected where $n\geq 2$, then h_k is an isomorphism for all $k \leq n$.

In particular, $\pi_n X \cong H_n X$ as groups.

Theorem: Freudenthal Suspension

If X is an $n\dash$ connected CW complex, then there are maps $\pi_i X \to \pi_{i+1} \Sigma X$ which is an isomorphism for $i\leq 2n$ and a surjection for $i=2n+1$.

Theorem: Homotopy LES for a Fibration

Theorem: Existence of Postnikov Tower

Theorem: Spectral sequence of a Fibration

Theorem: Existence of Whitehead Tower

Fiber Bundles

What is a fiber bundle? Generally speaking, it is similar to a fibration - we require the homotopy lifting property to hold, although it is not necessary that path lifting is unique.
lifting - todo tikz

However, it also satisfies more conditions - in particular, the condition of *local triviality*. This requires that the total space looks like a product locally, although there may some type of global monodromy. Thus with some mild conditions[1], fiber bundles will be instances of fibrations (or alternatively, fibrations are a generalization of fiber bundles, whichever you prefer!)

As with fibrations, we can interpret a fiber bundle as “a family of \(B\)s indexed/parameterized by \(F\)s”, and the general shape data of a fiber bundle is similarly given by

where \(B\) is the base space, \(E\) is the total space, \(\pi: E \to B\) is the projection map, and \(F\) is “the” fiber (in this case, unique up to homeomorphism). Fiber bundles are often described in shorthand by the data \(E \mapsvia{\pi}B\), or occasionally by tuples such as \((E, \pi, B)\).

The local triviality condition is a requirement that the projection \(\pi\) locally factors through the product; that is, for each open set \(U\in B\), there is a homeomorphism \(\varphi\) making this diagram commute:

Fiber bundles may admit right-inverses to the projection map \(s: B\to E\) satisfying \(\pi \circ s = \id_B\), denoted *sections*. Equivalently, for each \(b\in B\), a section is a choice of an element \(e\) in the preimage \(\pi^{-1}(b) \homotopic F\) (i.e. the fiber over \(b\)). Sections are sometimes referred to as *cross-sections* in older literature, due to the fact that a choice of section yields might be schematically represented as such:

foliation diagram
Here, we imagine each fiber as a cross-section or “level set” of the total space, giving rise to a “foliation” of \(E\) by the fibers.[2]

For a given bundle, it is generally possible to choose sections locally, but there may or may not exist globally defined sections. Thus one key question is **when does a fiber bundle admit a global section?**

A bundle is said to be *trivial* if \(E = F \cross B\), and so another important question is **when is a fiber bundle trivial?**

Definition: A fiber bundle in which \(F\) is a \(k\dash\)vector space for some field \(k\) is referred to as a *rank \(n\) vector bundle.* When \(k=\RR, \CC\), they are denoted real/complex vector bundles respectively. A vector bundle of rank \(1\) is often referred to as a *line bundle*.

Example: There are in fact non-trivial fiber bundles. Consider the space E that can appear as the total space in a line bundle over the circle

$$ \RR^1 \to E \to S^1$$

That is, the total spaces that occur when a one-dimensional real vector space (i.e. a real line) is chosen at each point of S^1. One possibility is the trivial bundle $E \cong S^1 \cross \RR \cong S^1 \cross I^\circ \in \text{DiffTop}$, which is an "open cylinder":

cylinder
But another possibility is $E \cong M^\circ \in\text{DiffTop}$, an open Möbius band:

Möbius band
Here we can take the base space B to be the circle through the center of the band; then every open neighborhood U of a point $b\in B$ contains an arc of the center circle crossed with a vertical line segment that misses $\del M$. Thus the local picture looks like $S^1 \cross I^\circ$, while globally $M\not\cong S^1 \cross I^\circ \in \text{Top}$.[3]

So in terms of fiber bundles, we have the following situation $$ \begin{array} &\RR &\to &~~~M &\to &S^1\\ \require{HTML} \style{display: inline-block; transform: rotate(90deg)}{=} &&~~~\require{HTML} \style{display: inline-block; transform: rotate(90deg)}{\neq} &&\require{HTML} \style{display: inline-block; transform: rotate(90deg)}{=}\\ \RR &\to &S^1 \cross I^\circ &\to &S^1 \end{array} $$

and thus M is associated to a nontrivial line bundle over the circle.

Remark: In fact, these are the only two line bundles over S^1. This leads us to a natural question, similar to the group extension question: **given a base B and fiber F, what are the isomorphism classes of fiber bundles over B with fiber F?** In general, we will find that these classes manifest themselves in homology or homotopy. As an example, we have the following result:

Notation: Let $I(F, B)$ denote isomorphism classes of fiber bundles of the form

\(F \to \wait \to B\).

Proposition:

The set of isomorphism classes of smooth line bundles over a space \(B\) satisfies the following isomorphism of abelian groups:

\[I(\RR^1, B) \cong H^1(B; \ZZ_2) \in \text{Ab}\]

in which the RHS is generated by the first Stiefel-Whitney class \(w_1(B)\).

Proof:

Lemma: The structure group of a vector bundle is a general linear group. (Or orthogonal group, by Gram-Schmidt)

Lemma: The classifying space of \(\GL(n, \RR)\) is \(\Gr(n, \RR^\infty)\)

Lemma: \(\Gr(n, \RR^\infty) = \RP^\infty \homotopic K(\ZZ_2, 1)\)

Lemma: For \(G\) an abelian group and \(X\) a CW complex, \([X, K(G, n)] \cong H^n(X; G)\)

The structure group of a vector bundle can be taken to be either the general linear group or the orthogonal group, and the classifying space of both groups are homotopy-equivalent to an infinite real Grassmanian. \[\begin{align} I(\RR^1, B) &= [B, B(\restrictionof{\text{(Sym$~\RR$)}}{\text{Vect}})]\\ &= [B, B(\GL(1, \RR))]\\ &= [B, \Gr(1, \RR^\infty)] \\ &= [B, \RP^\infty] \\ &= [B, K(\ZZ_2, 1)] \\ &= H^1(B; \ZZ_2) \end{align} \] \(\qed\)

This is the general sort of pattern we will find - isomorphism classes of bundles will be represented by homotopy classes of maps into classifying spaces, and for nice enough classifying spaces, these will represent elements in cohomology.

Corollary: There are two isomorphism classes of line bundles over \(S^1\), generated by the Möbius strip, since \(H^1(S^1, \ZZ_2) = \ZZ_2\) (Note: this computation follows from the fact that \(H_1(S^1) = \ZZ\) and an application of both universal coefficient theorems.)

Note: The Stiefel-Whitney class is only a complete invariant of *line* bundles over a space. It is generally an incomplete invariant; for higher dimensions or different types of fibers, other invariants (so-called *characteristic classes*) will be necessary.

Another important piece of data corresponding to a fiber bundle is the *structure group*, which is a subgroup of \(\text{Sym}(F) \in \text{Set}\) and arises from imposing conditions on the structure of the transition functions between local trivial patches. A fiber bundle with structure group \(G\) is referred to as a *\(G\ dash\)bundle*.

Vector Bundles

Definition: A *rank \(n\) vector bundle* is a fiber bundle in which the fibers \(F\) have the structure of a vector space \(k^n\) for some field \(k\); the structure group of such a bundle is a subset of \(\GL(n, k)\).

Note that a vector bundle always has one global section: namely, since every fiber is a vector space, you can canonically choose the 0 element to obtain a global zero section.

Proposition: A rank \(n\) vector bundle is trivial iff it admits \(k\) linearly independent global sections.

Example: The tangent bundle of a manifold is an \(\RR\)-vector bundle. Let \(M^n\) be an \(n\dash\)dimensional manifold. For any point \(x\in M\), the tangent space \(T_xM\) exists, and so we can define \[TM = \coprod_{x\in M} T_xM = \theset{(x, t) \mid x\in M, t \in T_xM} \]

Then \(TM\) is a manifold of dimension \(2n\) and there is a corresponding fiber bundle \[\RR^n \to TM \mapsvia{\pi} M \]

given by a natural projection \(\pi:(x, t) \mapsto x\)

Example A circle bundle is a fiber bundle in which the fiber is isomorphic to \(S^1\) as a topological group. Consider circle bundles over a circle, which are of the form \[S^1 \to E \mapsvia{\pi} S^1 \]

There is a trivial bundle, when $E = S^1 \cross S^1 = T^2$, the torus:
torus bundle

There is also a nontrivial bundle, $E = K$, the Klein bottle:
Klein bottle

As in the earlier example involving the Möbius strip, since K is nonorientable, $T^2 \not\cong K$ and there are thus at least two distinct bundles of this type.

Remark: A section of the tangent bundle TM is equivalent to a *vector field* on M.

Definition: If the tangent bundle of a manifold is trivial, the manifold is said to be *parallelizable*.

Proposition: The circle S^1 is parallelizable.

Proof Let $M = S^1$, then there is a rank 1 vector bundle $$\RR \to TM \to M$$ and since $TM = S^1 \cross \RR$ (why?), we find that S^1 is parallelizable. $\qed$

Proposition: The sphere S^2 is not parallelizable.

Proof: Let $M = S^2$, which is associated to the rank 2 vector bundle $$\RR^2 \to TM \to M$$

Then TM is trivial iff there are 2 independent global sections. Since there is a zero section, a second independent section must be everywhere-nonzero - however, this would be a nowhere vanishing vector field on S^2, which by the Hairy Ball theorem does not exist.

Alternate proof: such a vector field would allow a homotopy between the identity and the antipodal map on S^2, contradiction by basic homotopy theory.$\qed$

Classifying Spaces

Definition: A *principal \(G\dash\)bundle* is a fiber bundle \(F \to E \to B\) in which for each fiber \(\pi^{-1}(b)\definedas F_b\), satisfying the condition that \(G\) acts freely and transitively on \(F_b\). In other words, there is a continuous group action \(\actson: E\cross G \to E\) such that for every \(f \in F_b\) and \(g\in G\), we have \(g\actson f \in F_b\) and \(g\actson f \neq f\).

Example: A covering space \(\hat X \mapsvia{p} X\) yields a principal \(\pi_1(X)\dash\)bundle.

Remark: A consequence of this is that each \(F_b \cong G \in \text{TopGrp}\) (which may also be taken as the definition). Furthermore, each \(F_b\) is then a *homogeneous space*, i.e. a space with a transitive group action \(G\actson F_b\) making \(F_b \cong G/G_x\).

Remark: Although each fiber \(F_b\) is isomorphic to \(G\), there is no preferred identity element in \(F_b\). Locally, one can form a local section by choosing some \(e\in F_b\) to serve as the identity, but the fibers can only be given a global group structure iff the bundle is trivial. This property is expressed by saying \(F_b\) is a *\(G\dash\)torsor*.

Remark: Every fiber bundle \(F\to E\to B\) is a principal \(\Aut(F)\dash\)fiber bundle. Also, in local trivializations, the transition functions are elements of \(G\).

Proposition: A principal bundle is trivial iff it admits a global section. Thus all principal vector bundles are trivial, since the zero section always exists.

Definition: A principal bundle \(F \to E \mapsvia{\pi} B\) is *universal* iff \(E\) is weakly contractible, i.e. if \(E\) has the homotopy type of a point.

Definition: Given a topological group \(G\), a *classifying space*, denoted \(BG\), is the base space of a universal principal \(G\dash\)bundle \[G \to EG \mapsvia{\pi} BG \] making \(BG\) a quotient of the contractible space \(EG\) by a \(G\dash\)action. We shall refer to this as *the classifying bundle*.

Classifying spaces satisfy the property that any other principal \(G\dash\)bundle over a space \(X\) is isomorphic to a pullback of the classifying bundle along a map \(X \to BG\).

Let \(I(G, X)\) denote the set of isomorphism classes of principal \(G\dash\)bundles over a base space \(X\), then \[I(G, X) \cong [X, BG]_{\text{hoTop}} \]

So in other words, isomorphism classes of principal \(G\dash\)bundles over a base \(X\) are equivalent to homotopy classes of maps from \(X\) into the classifying space of \(G\).

Proposition: Grassmannians are classifying spaces for vector bundles. That is, there is a bijective correspondence:

\[[X, \Gr(n, \RR)] \cong \theset{\text{isomorphism classes of rank $n$ $\RR\dash$vector bundles over $X$}} \]

It is also the case that every such vector bundle is a pullback of the principal bundle \[\GL(n, \RR) \to V_n(\RR^\infty) \to \Gr(n, \RR) \]

Representing \(K(G, n)\) Geometrically

(Summary of full material found here)

Starting with a low-dimensional example, using the fact that \(K(\ZZ, 2) = \CP^\infty = \varinjlim \CP^n\).

We want to make this into an abelian topological group, so let \[X = \theset{\CC \to \CC \mid f~\text{is rational}} \]

Note that as a vector space, this is isomorphic to \(\CC^\infty\), and there is a way to topologize \(X\) such that this is a homeomorphism as well.

Then let \[\hat X = \theset{f \in X \mid f~\text{is nonzero}} / \theset{f = \lambda f \mid \lambda \in \CC^*} \]

which is the set of rational, nonzero, complex functions, modulo multiplication by constants. This is the “projectivization” of X, and is isomorphic to $\CP^\infty$ as abelian topological groups.

Then every function $f\in \hat X$ has (say) n zeros and m poles, which corresponds to $n+m$ points in $\CC \cup \infty = \CP^1 = S^2$. If we attach an integer to every point, we can encode the difference between zeros and poles while simultaneously encoding their orders. So we then define

$$ Y_n = \theseset{\theseset{(z_i, k_i)}_{i=1}^n \mid \sum_{i=1}^n k_i = 0}\subset \CP^1 \cross \ZZ $$

in which each element is a collections of n points on the Riemann sphere, each with an integer attached, such that these integers sum to zero.

Then any rational complex function $f: \CC \to \CC$ with a total of n combined zeros and poles can be uniquely recovered as some $y\in Y_n$, so we have

$$ \hat X \cong \union_{n=1}^\infty Y_n $$

In other words, we perform the following procedure: each rational function has a finite number of zeros and poles, so we imagine these as points on the Riemann sphere, each with an integer attached such that the sum of these integers equals zero.

Then, we imagine wiggling these functions continuously - this induces movement of the points on the sphere. When any two points coincide, they “coalesce” and their corresponding integer labels are added together.

We thus envision points in $K(\ZZ, 2) = CP^\infty$ as collections of “particle-antiparticle” swarms on the S^2:

swarm 3
There are two immediate generalizations:

- First, replace S^2 with S^n to obtain a corresponding picture for $K(\ZZ, n)$
- Then, replace $\ZZ$ with an arbitrary abelian group. We then require that the aggregate word formed by all of the points multiples to the identity, and that coalescing corresponds to multiplication of elements.

group swarm
This yields a geometric picture for \(K(G, n)\) for any abelian group \(G\). \(\qed\)

Why study characteristic classes?

> A characteristic class is a way of associating to each principal bundle X a cohomology class of X. The cohomology class measures the extent the bundle is “twisted” — and whether it possesses sections.

> Characteristic numbers solve the oriented and unoriented bordism questions: two manifolds are (respectively oriented or unoriented) cobordant if and only if their characteristic numbers are equal.

> When the theory was put on an organised basis around 1950 (with the definitions reduced to homotopy theory) it became clear that the most fundamental characteristic classes known at that time (the Stiefel–Whitney class, the Chern class, and the Pontryagin classes) were reflections of the classical linear groups and their maximal torus structure.

Why study cobordism?

> Cobordism is a much coarser equivalence relation than diffeomorphism or homeomorphism of manifolds, and is significantly easier to study and compute. It is not possible to classify manifolds up to diffeomorphism or homeomorphism in dimensions ≥ 4 – because the word problem for groups cannot be solved – but it is possible to classify manifolds up to cobordism.

> It performed an important role, historically speaking, in developments in topology in the 1950s and early 1960s, in particular in the Hirzebruch–Riemann–Roch theorem, and in the first proofs of the Atiyah–Singer index theorem.

> Every vector bundle theory (real, complex etc.) has an extraordinary

> cohomology theory called K-theory

Why study K-theory?

> Examples of results gleaned from the K-theory approach include the Grothendieck–Riemann–Roch theorem, Bott periodicity, the Atiyah-Singer index theorem, and the Adams operations.

> It played a major role in the second proof of the Index Theorem (circa 1962).

> Already in 1955, Jean-Pierre Serre had used the analogy of vector bundles with projective modules to formulate Serre’s conjecture, which states that every finitely generated projective module over a polynomial ring is free;

> Geometrically, finitely generated projective modules correspond to vector bundles over affine space, and free modules to trivial vector bundles. Affine space is topologically contractible, so It admits no non-trivial topological vector bundles. It also admits no non-trivial holomorphic vector bundles. Jean-Pierre Serre remarked that the corresponding question was not known for algebraic vector bundles: “It is not known whether there exist projective A-modules of finite type which are not free.” Here A is a polynomial ring over a field, that is, A = $k[x_1, ..., x_n]$. Theorem: every finitely generated projective module over a polynomial ring is free.

Note: another example of the use of the correspondence between isomorphism classes and cohomology >the Picard group of a ringed space X, denoted by Pic(X), is the group of isomorphism classes of invertible sheaves (or line bundles) on X. Alternatively, the Picard group can be defined as the sheaf cohomology group $H^{1} (X,{\mathcal {O}}_{X}^{*})$.

> The chromatic view-point, which studies stable homotopy theory via its relationships to the moduli of formal groups, and related topics such as topological modular forms, use a sizable amount of (fairly abstract) algebraic geometry. And Lurie’s work on derived algebraic geometry was motivated in part by establishing foundations adequate to the task of defining equivariant forms of TMF.

> And overall, read classical papers by Adams, Pontryagin, Quillen, Serre, Sullivan, Thom…John Francis has a list of classical papers for the Kan seminar on his homepage.

- Seiberg-Witten theory
- Contact 3-manifolds
- Hyperbolic 3-manifolds
- Symplectic 4-manifolds
- Why do Eilenberg-MacLane spaces have complicated higher cohomology?
    - Dually, why do spheres have higher complicated homotopy?
- What is a Postnikov tower? Why is it useful?

Recent Developments in Algebraic Topology

Big Theorems: - Brouwer Fixed Point - Division Algebras over \(\RR\) - Borsuk-Ulam - Invariance of domain - Aatiyah-Singer Index Theorem - \(K\dash\)theory and bordism - Brown Representability Spectrum

Opinion from MO: modern Topology starts with spectra.

Stuff to Review: - Characteristic Classes (see Milnor) - Stiefel-Whitney class - Mapping cylinder/cone - Chern class - Euler class (or Euler bundle..? Find in old email) - Homotopy limit/colimit - The \(J\)-homomorphism - See beginning of Hopkins talk - \(p\dash\)adic stuff - Linearly independent sections - Normal bundle - Tubular neighborhoods - Resolutions - Suspension - Colimits - Smash product - Picard group (e.g. of \(L_{K(n)}S^O\)) - Regular representations

Stuff to Look Into - Kervaire Invariant 1 Problem - Equivariant topology - Chromatic homotopy theory - Orthogonal spectra - Model categories - The cobordism hypothesis - Milnor’s conjecture - Vandiver’s conjecture - \(\infty\dash\)categories - Classifying orbifolds - \(K\dash\)theory - Deformation theory - Topological modular forms - Riemann-Roch - Localizations - Completions - Steenrod Algebra - Tate Twist - Formal groups - Local fields - Nishida’s Theorem: - The \(S^1\) spectrum is some kind of “thickening” of \(\ZZ\)? - Hodge Conjecture - Floer homology - Fukaya categories - Symplectic geometry - Lagrangian submanifolds - Morava E-Theory

Contact Geometry

> Contact geometry also has applications to low-dimensional topology; for example, it has been used by Kronheimer and Mrowka to prove the property P conjecture, by Michael Hutchings to define an invariant of smooth three-manifolds, and by Lenhard Ng to define invariants of knots. It was also used by Yakov Eliashberg to derive a topological characterization of Stein manifolds of dimension at least six.

Reading Notes

> Computing various homological invariants of associative algebras (such as Tor and Ext of various modules, Hochschild (co)homology, cyclic homology etc.) has been an active research topic in ring theory for many years. More recently (about 15 years ago), ring theorists became interested in associative algebras up to homotopy, or A_∞-algebras, as a recipe to produce meaningful “higher structures” on classical objects like Yoneda Ext-algebras.

> This offers two different perspectives on associative algebras: homological invariants are “Abelian” (i. e. arise when one works with additive categories, e.g. chain complexes of modules over a ring), while homotopical invariants are “non-Abelian” (i. e. arise from non-additive categories, like the category of all differential graded associative algebras). However, these two perspectives are closely related, and it is often possible to recover homological information from the homotopical one, and the other way round. For experts in homotopical algebra on a larger scale (beyond the associative ring theory), this philosophy is already present in works of Stasheff and Hinich on homotopy algebras.

Hochschild Homology

Reference: Chapter 9 of Weibel

> The Hochschild homology of an \(R\dash R\dash\)bimodule reflects some ring-theoretic stuff. For instance, \(H_1(R,R)\) of a \(k\dash\)algebra \(R\) is the module of differentials \(\Omega_R/k\). If \(Q\subseteq R\) then there is an algebraic decomposition of this homology that is analogous to the Hodge decomposition in complex manifold theory.

> Taking the above example further if \(k\) is a ring and \(X\) a simplicial set, the cyclic homology (Hochschild homology taking into account a cyclic action on the corresponding simplicial set) of the simplicial module \(k[X]\) is the same as the \(S_1\dash\)equivariant homology of the geometric realization of \(X\) with coefficients in \(k\).

> K-theory is like homology on rings. Moreover, algebraic topology is clearly interested in vector bundles; on a nice space \(X\) the category of rank \(n\) \(\RR\dash\)vector bundles on \(X\) is equivalent to the category of rank \(n\) finitely generated projective modules over the continuous functions \(C(X,\RR)\). The group \(K_0(C(X))\) is the Grothendieck group of the isomorphism classes of finitely generated projectives.

> We look first at vector bundles over a space \(X\). A vector bundle assigns a vector space to every point of \(X\). This is done in a continuous fashion. The \(k\dash\)dimensional vector bundles over \(X\) are equivalent to the homotopy classes of maps from \(X\) to a fixed space \(BO_k, [X, BO_k]\), [Ste51]. So, as is the case with many geometric problems, the classification of isomorphism classes of \(k\dash\)dimensional vector bundles is reduced to the computation of homotopy classes of maps. Furthermore, it is clear that studying \(BO_k\) is very useful for this problem. It comes about by a standard construction which builds a classifying space, \(BG\), for any group \(G\).

Amazing result: all \(n\dash\)dimensional manifolds, \(M_n\) , immerse in \(R_{2n}-\alpha(n)\) where \(\alpha(n)\) is the number of ones in the binary expansion of \(n\).

What is a foliation? What is an orbifold?

Terms to look up: - Foliation - Orbifold - Sectional curvature - Of a Riemannian manifold - “Type” of a quadratic form - Isotropy group - Uniformization - Haar measure - Hodge decomposition - Equivariant

Review covering spaces, sheets, regular coverings.

How to study Riemannian geometry? (Without much of the associated analysis.)

Write up a quick summary of classification in manifolds for low dimensions. Uniformization, etc

General Reading

Sheaves for Homotopy Theorists

The algebraic K-groups of a scheme X are defined by producing a spectrum K(X) and setting \(K^n(X) \definedas π_{-n}K(X)\).

> Be warned that K-theorists and geometers write \(K_n(X)\) for our \(K^{−n}(X)\) – we will stick to homotopy-theoretic notation, however.

K can be defined so that it actually gives a contravariant functor (Schemes) → (Spectra), and so the groups \(K^n(X)\) do indeed start to look like a cohomology theory. For X a topological space, the analogue of K(X) turns out to be the mapping spectrum \(bu^X\), where bu is the spectrum representing complex connective K-theory (so that \(bu\) is the connective cover of the spectrum \(BU\)).

Thus, the functor K can be thought of as a substitute for the spectrum bu – it is sort of a device for storing all the same information that bu provides, but without an honest ‘space’ to house it in.

Most of the important results about algebraic K-theory arise from the study of the spectrum K(X), rather than that of the disembodied abelian groups \(K^n(X)\). For example, if the scheme X is covered by two open sets U and V , one wants a Mayer-Vietoris sequence \[\cdots \rightarrow K ^ { n } (X) \rightarrow K ^ { n } (

U) \oplus K ^ { n } (V) \rightarrow K ^ { n } (U \cap V) \rightarrow K ^ { n + 1 } (X) \rightarrow \cdots \]

This follows formally once one proves the stronger result that \[\begin{CD} \mathcal{K}(C) @>>> \mathcal{K}(U) \\ @VVV @VVV \\ \mathcal{K}(V) @>>> \mathcal{K}(U\intersect V) \end{CD} \] is a homotopy pullback diagram.

Our first guess is motivated by the hypothesis that a contravariant functor \(E\) : (Schemes) → (Spectra), in analogy to the functor \(K\), should look like a cohomology theory. A weak equivalence of schemes \(X \to Y\) would be expected to have the property that \(E(Y) → E(X)\) is a weak equivalence of spectra for every such \(E\).

Believing that this gives the ‘correct’ definition of weak equivalence reduces to believing that the whole homotopy theory of schemes is captured in the two basic facts that \(\mathrm{hocolim}~U_\wait \longrightarrow X\) and \(X \cross I \to X\) are weak equivalences. The surprising answer is yes. If we use this definition in the case of reasonable topological spaces (like CW-complexes) we do recover the usual notion of weak equivalence. It says that the homotopy-theory of topological spaces can indeed be ‘generated’ by these two fundamental properties.

The problem is that Zariski open covers don’t tell the whole story about the homotopy-type of schemes. The first indications of this date back to Grothendieck’s work on etale cohomology: looking at only Zariski covers makes the spectrum of a field look contractible, whereas in real life they behave more like \(K(π, 1)\) (for example, they have nontrivial covering spaces).

We begin with the observation that algebraic topologists, despite what we are usually led to believe, are really not very interested in the category of topological spaces. Of course there are certain subcategories, like the category of manifolds, in which they are highly interested—but the notion of ‘topological space’ is much too broad, admitting a host of pathological objects for which the machinery of algebraic topology is simply not intended. On the other hand, categories such as topological manifolds end up being much too small for many purposes. The industrious homotopy-theorist finds himself wanting to glue manifolds together, quotient out by subspaces, divide out by group actions—in short, take various colimits—all of which have the disadvantage of perhaps producing something which is no longer a manifold. Gluing manifolds together might lead one to study some category of cell complexes, but of course these categories end up not being closed under colimits either. Thus, what the algebraic topologist desires is a setting in which he can study the category of manifolds (or finite complexes, if he is daring), have all small colimits at his disposal, and yet does not have to worry about the pathology that goes along with objects as diverse as topological spaces.

Reading: Mark Hovey's List of Open Problems in Algebraic Topology

> Note: a vector bundle is a locally free sheaf. A scheme is a generalization of a ring, in the same way that a manifold is a generalization of R^n

To look up and learn more about: ## Specific Problems/Conjectures {#specific-problemsconjectures } (and concepts therein)

- The Weil Conjectures
    - What are the actual statements? What is their importance within AG?
    - What were Deligne's contributions?
- The Ravenel conjectures
    - Particularly Mike Hopkins' contributions
- The generating hypothesis in stable homotopy theory
- Elliptic cohomology
    - Hopkins' viewpoint on elliptic spectra
- The etale fundamental group
- Topological modular forms
- Index theory
- Chromatic homotopy theory
- The chromatic spectral sequence
- The chromatic splitting conjecture
    - Is this similar to fracture theorems in stable homotopy..?
- Bousfield localization
- The telescope conjecture
- Loop spaces
    - Particularly infinite loop spaces, and why they're important
- Weyl groups
- The Kervaire invariant 1 problem

- The Adams Spectral Sequence
- **Morava \(K\) and \(E\) theory**
- The Morava stabilizer groups
- Lubin-Tate spaces
- **Spectra**
    - Really spell out the details of how they’re constructed (find/pick one of the nicer definitions) and their duality with homology theories
    - Extract some of the motivating examples of computations with spectra from my textbook annotations
    - Especially try to understand the bordism spectrum, \(MO\)? And \(MU\)
        - Want to know \(K(MU), TC(MU)\), etc
    - And ring spectra
- Hopf Algebras
    - Same story, extract and understand the motivating examples
    - Also piece together theory of e.g. comodules over them
    - Also Hopf Algebroids
- Localization
    - See some actual computations of computing “at a prime” or “away from a prime”
- Gerbes
- Simplicial sets
    - Find some results that actually work/compute with these
- **\(K\dash\)theory**
- The Novikov conjecture
- Metrics and curvature on manifolds
- Understand the \(H\) notation, i.e. \(H\ZZ\)
- \(A_\infty\)
- \(H\dash\)spaces
- Motivate and understand phantom maps
- Nilpotence
- Profinite groups
- Quasi-coherent sheaves over a scheme
- Moduli stacks

- Formal group laws
- Symmetric monoidal categories
- Quillen equivalence
- Model categories
    - And cofibrant generation
- **The Steenrod Algebra**
- Cohomology operations
- Thom spectrum
    - And Thom space
- Tate cohomology
- Get a more precise understanding of \(n\dash\)categories
- The Milnor conjecture
- The Grothendieck-Witt Ring
- The chromatic redshift conjecture
- Algebraic K Theory
    - And how topological cyclic homology is related and more computable
- Formal Groups
    - Where they come up, what they're used for, why important
- Mapping class group: orientation-preserving diffeos / id component
- Perfect fields
- Spf: Formal spectrum
- Morava \(E\dash\)theory and \(K\dash\)theory
    - \(E\) behaves like a local ring and \(K\) its residue fields
- Brauer groups
- Massey products
- Descent

General Theory

- Algebras
    - Precise descriptions of polynomial, exterior, and symmetric algebras
    - Differential graded and Lie algebras
- Homotopy and derived categories

    - Sort out weak equivalences vs homotopy equivalences vs quasi-isomorphisms etc
- Limits, colimits
- Pushouts, pullbacks
- Adjunction
- Equivalence of categories
    - Need to state this precisely!
- \(E_n\) rings
- Sard’s Theorem
- Generalized Poincare
- Langlands and Geometric Langlands
- L functions for elliptic curves
- The Frobenius
- Classification of surfaces
- Complex structures, almost-complex structures
- Trivial, canonical, universal bundles
- Exceptional divisor
- Blowup
- Hodge numbers
- Jacobian of a curve
- Abelian varieties
- Schemes
- Stacks
- Coarse moduli space
    - Somehow a nicer version of a stack where torsion in the automorphism groups aren’t messing things up
- Closed points of a scheme
- Smooth projective curves
- Chow rings
- Teichmüller space: space of complex structures

Notes

> Flipping roles, generalized cohomology theories can be studied in their own right. They come from a category called the stable homotopy category (which

> is much like a derived category of chain complexes), and each of them can be determined by a certain amount of data involving cohomology operations. Much of this data can be recovered by looking at how the generalized cohomology theory behaves on certain spaces (projective spaces and classifying spaces being the canonical examples).

> After a lot of hard work (with some of the bigger names including Adams, Milnor, and Quillen, though I am leaving a lot of important names out) it is discovered, starting from almost pure calculation, that the stable homotopy category has a connection to the category of 1-dimensional formal groups, via the study of characteristic classes. Each generalized cohomology theory determines some amount of formal group data.

> via things like BP-theory and the Adams-Novikov spectral sequence, leads to better qualitative understanding of the stable homotopy category, new guesses about what phenomena can occur (e.g. the Ravenel conjectures), new techniques which are computationally useful, and new theorems (e.g. the solution of most of the Ravenel conjectures).

> Take a look at Machlachlan and Reid’s book “The Arithmetic of Hyperbolic 3-Manifolds”.

> Since finite volume hyperbolic structures are unique whenever an n-manifold (n≥3) has them, any invariants of the hyperbolic structure are invariants of the manifold. Hyperbolic manifolds are K(π,1)-spaces, so they’re not just diffeo/homeomorphism invariants, but invariants of the homotopy-type.

> I think the whole field of anabelian geometry fits the bill, even if it’s perhaps more focused on going the other way around (i.e. applying homotopy theory to number theory). Anabelian geometry is a ‘program’ launched by Grothendieck in his famous Esquisse d’un Programme, and is all about translating arithmetic geometric problems to problems in homotopy theory.

> As an example of a specific instance of the anabelian philosophy, we have Grothendieck’s celebrated ‘section conjecture’, which states (in one form) that for a ‘nice’ curve X over a number field F, the rational points are in bijection with the sections of the exact sequence \[1 \rightarrow \pi_1(X_{\bar{F}})

> \rightarrow \pi_1(X) \rightarrow G_F \rightarrow 1 \] where GF is the absolute Galois group of F and π1 is the algebraic (etale) fundamental group. In case the curve is over the complex numbers, the etale π1 is the profinite completion of the regular fundamental group, so there is a very close connection to the classical stuff of Hatcher. The conjecture is still a wide open problem, but any proof would mean you could check something of number theoretic interest (existence of rational points on curves) by studying maps between certain generalized homotopy groups!

Neat trick from algebraic geometry: For a stack \(\mathcal M =X/G\) where \(X\) is a \(\CC\dash\)variety and \(G\) is a finite group, then \[H^\wait(\mathcal M; \QQ) \cong \left(H^\wait(X; \QQ)\right)^G \] where the RHS denotes the taking the \(G\dash\) invariant part. Seems to only work over \(\QQ\). The quotient is scheme-theoretic. The actual definite involves equivariant cohomology.

Prototypical example of a moduli space: \(\Gr_\CC(n, k)\). Apparently fundamental classes exist for closed subvarieties? Maybe just closed subvarieties of a moduli space.

You can take the homotopy groups of a cohomology theory.

AG/NT Stuff

There seems to be some interest from topologists and geometers around the **Novikov conjecture**. On the analytic side, **Sarnak's conjecture** seems to have some buzz as well, as does the **Artin conjecture**.

Surface bundles and monodromy representations sound interesting.

I should really get a better grip on etale cohomology and the etale fundamental group.

I should review conference notes and get a list going on the common structures and spaces that are popping up. \(K3\) surfaces, Kahler manifolds, toric varieties, Calibi-Yau manifolds, etc.

Review the deRham-Witt complex and crystalline cohomology, get at least a surface-level understanding of \(L\) functions.

Topology Stuff

Defining THH:

- Take \(A \in \mathcal C\), where \(\mathcal C\) is a “nice” monoidal category, and \(A\) is an algebra object in it. We’ll call the monoidal operation \(\tensor\).
- We’ll make a simplicial object \(THH_\wait(A)\):
    - \(THH_n(A) = A^{\tensor n+1}\). If it’s to be simplicial, need to specify the face/degeneracy maps:
    - Face maps: collapse by cyclic multiplication
    - Degeneracy maps: use the unit of \(A\), can replace any tensor symbol with it. Have a unit map that goes from the unit to \(A\), so somehow this gets you “up” one level (?)
- Now take its geometric realization \(\abs{THH_\wait(A)}\)
    - In general, take \(\mathrm{hocolim}_\Delta THH_\wait(A)\)

Look at the HKR theorem again, identifies \(HH\) with derivations.

When proving stuff about algebras – try with polynomial algebras first, essentially the simplest case.

Simplest coalgebra case: divided power algebras (also a Hopf algebra)

The “dual” of geometric realization is totalization.

Look at Bousfield-Kan spectral sequence

Random Reading

A Fredholm operator \(D\) on a Banach space is one such that the index \(\dim\ker D - \dim\coker D\) makes sense.

The point of this note is to reword and fill in some details in Akhil Mathew's post # The Setup {#the-setup }

We start with a very general problem: suppose we are given a topological space \(X\) and wish to compute \(\pi_*(X)\). Although the homotopy groups themselves are not a complete invariant of \(X\) – that is, we can generally construct spaces that are homotopy-equivalent to \(X\) but not homeomorphic – this will get us partially there. Moreover, in this process we will determine what extra information is needed to in fact obtain a *complete* set of invariants for \(X\).

The very first thing we'll do is replace \(X\) with a CW complex that is homotopy-equivalent to \(X\). By the CW approximation theorem, we can do this for any space \(X\), and since many theorems are much nicer for CW complexes than arbitrary spaces, we implicitly work in this subcategory from here on.

For any group \(G\), suppose for a moment that there existed a space \(K(G,n)\) (where \(n\) is a natural numbers) satisfying \[\pi_i(K(G,n)) = \begin{cases} G, & i = n \\ 0, & \text{else} \end{cases} \] where \(0\) denotes the trivial group. If such a space existed, this would perhaps be the simplest type of object to work with within homotopy theory – it has exactly one homotopy group concentrated in one dimension. We will show later that not only does such a space exist, and can be constructed in a relatively straightforward manner, but also that it is unique up to homotopy-equivalence, and are thus referred to as *Eilenberg-MacLane spaces*.

The Atoms of Homotopy Theory

For the moment, let's focus on one particular degree on homotopy groups, say \(G = \pi_i(X)\). Note that for our original space, if we make the assumption that \(G\) is finitely generated, we can appeal to the classification of (finitely generated) abelian groups and write \(G \cong \ZZ^n \bigoplus_{j=1}^m \ZZ_{k_j}\) for some \(n\) and some \(m\). That is, it decomposes as a free group of rank \(n\), along with a sum of finite cyclic groups with orders given by the invariant factors \(k_j\). As an aside, it is perhaps worth mentioning that \(\pi_i(X)\) is abelian for \(n\geq 2\), so the categorical product and coproduct (here the direct sum) will coincide. This leaves the \(i=1\) case; however, the fundamental group can generally be computed using more elementary methods such as covering space theory and Van Kampen’s theorem.

So we’ve come up with the “atoms” that we might hope to build spaces out of, and we’ve realized that we can break any given homotopy group of \(X\) into free and cyclic groups. The strategy now is to see if we can work out what kind of spaces \(K(\ZZ^n, i)\) and \(K(\ZZ_k, i)\) are, and hope we can prove a theorem that says something like \(K(G_1, i) \oplus K(G_2, i) =K(G_1 \oplus G_2, i)\). If we can do this, we now have a method of constructing a space that has the same homotopy groups as \(X\) in degree \(i\). If we can then show something like \[\pi_i(K(G_1, j) \oplus K(G_2, k)) = \begin{cases} G_1, & i = j \\ G_2, & i = k \\ 0, & \text{else} \end{cases} \] then we will be in business. Explicitly, splitting the homotopy groups of \(X\) up by degree and writing \[\pi_*(X) = \bigoplus_{i=1}^\infty \pi_i(X),\] we could construct a space \(X'\) that has the exact same homotopy groups \(X\) by taking \[X' = \bigoplus_{i=1}^\infty K(\pi_i(X), i).\]

What does this buy us? It’s not quite the case that \(X' \homotopic X\), since having isomorphic homotopy groups alone is insufficient to guarantee homotopy equivalence. A standard counterexample here are the spaces \(S^2\cross \RP^3\) and \(S^3 \cross \RP^2\); one in fact needs a single map *inducing* all of the isomorphisms on homotopy groups simultaneously for this kind of result to hold.

So the naive hope of writing \(X\) as a product of simple spaces won’t work. As it turns out, what we *can* get is a way to write \(X\) in a way that’s often referred to as a “twisted” product, which is formalized in the notion of a (Serre) fibration.

Twisted Products

It’s perhaps worth spelling out exactly what “twisted product” means here, since it’s often glossed over. Suppose we have a fibration \(F \injects E \mapsvia{p} B\) which is in fact a fiber bundle – note that this is a strictly stronger condition, requiring the fibration to have *local trivialization*. This amounts to asking that for every open neighborhood \(U \in B\), we have \(p^{-1}(B) = F \cross B\) up to homeomorphism. This is of course strictly weaker than requiring \(E\cong F\cross B\) globally, which would be denoted a trivial bundle; there may in fact be monodromy in the total space \(E\) that topologically distinguishes it from this product. A primary example of this phenomenon is the Mobius bundle, \[I \injects M \surjects S^1 \]

which is a line bundle over the circle. Note that locally we do have \(M \cong S^1 \cross I\), but since \(M\) is not homeomorphic to a cylinder, this produces a nontrivial bundle. There is instead a global “twist”, arising from the fact \(M\) is nonorientable and parallel-transporting a small arc in \(M\) once around the core circle is not the identity map. By a somewhat non-precise abuse of notation, one might instead write something like \(M \cong S^1 \semidirect_\varphi I\), indicating that this is a nontrivial bundle. One could further ask, given a base \(B\) and a fiber \(F\), how many distinct nontrivial bundles \(F \to E \to B\) exist, which is a bit far afield for us now but leads to the rich and fruitful theory classifying spaces.

The moral of the story here is that we can generalize a product of spaces \(A \times B \times C \times \cdots\) as a “twisted product” \(A \semidirect_{\varphi_1} B \semidirect_{\varphi_1} C \semidirect_{\varphi_1} \cdots\) by constructing an interlocking series of fibrations. Luckily for us, our next step towards solving our original problem will involve the construction of Postnikov towers, which explicitly encode how this interlocking works.

Postnikov Towers

First I want to define exactly what a Postnikov tower is and how we can explicitly build them using CW complexes, since this is something that is treated differently among different sources.

The data is a Postnikov tower is a sequence of spaces \(\theset{X_i}\) equipped with maps \(p_n: X_n \to X_{n-1}\), where each \(p_n\) is a fibration, and \[\pi_i(X_n) = \begin{cases} \pi_i(X), & i \leq n \\ 0, & i > n. \end{cases} \] such that the following diagram commutes:

\[\begin{tikzcd} \vdots & & \vdots \arrow[d, "p_{n+2}"', two heads] \arrow[rrddd, dotted] & & \\ {K(\pi_{n+2}X, n+3)} \arrow[rr, "k_{n+3}", hook] & & X_{n+1} \arrow[d, "p_{n+1}"', two heads] \arrow[rrddd] & & \\ {K(\pi_{n+1}X, n+2)}

\arrow[rr, "k_{n+2}", hook] & & X_n \arrow[rrdd] & & \\ \vdots & & \vdots \arrow[d, "p_1"', two heads] & & \\ {K(\pi_2X, 3)} \arrow[rr, "k_3", hook] & & X_1 \arrow[rr] & & X \end{tikzcd} \] Note that we have fibrations \[K(\pi_{i+1}(X), i+2) \to X_i \to X_{i-1}\] for all \(i > 1\), and so using our earlier notation we can suggestively write \(X_i \cong K(\pi_{i+1}(X) \semidirect X_{i-1}\). It also turns out there is a weak equivalence \(X \to \lim_i X_i\), and so carrying this out inductively allows us to write \[X \homotopic K(\pi_{1}(X),2) \semidirect K(\pi_{2}(X),3) \semidirect K(\pi_{3}(X),4) \semidirect \cdots \]

yielding our “twisted product”.

This spaces can be constructed by a relatively easy process. If \(X\) already has the homotopy type of a \(CW\) complex, first let \(X_i = X^{(i)}\), the \(i\dash\)skeleton of \(X\). Since every group has a presentation, we can write \(\pi_{i+1}(X) = \generators{a, b, c, \cdots} / \generators{R_1, R_2, \cdots}\) where \(a,b,c,\cdots\) are generators and \(R_i\) are relations.

Behrens Hopkins Hill A homotopy n-sphere is a smooth n-manifold which is homotopy equivalent to \(S^n\). Kervaire and Milnor defined Θn to be the group of homotopy spheres up to h-cobordism (where the group operation is given by connect sum). By the h-cobordism theorem [Sma62] (n > 4) and Perelman’s proof of the Poincare conjecture [Per02], [Per03a], [Per03b] (n = 3), for n 6= 4, Θn = 0 if and only if \(S^n\) has a unique differentiable structure (i.e. there are no exotic spheres of dimension n).

We wish to consider the following question: For which n is Θn = 0?

The general belief is that there should be finitely many such n, and these n should be concentrated in relatively low dimensions.

The chromatic tower. Fix a prime p. The chromatic tower of a spectrum X is the tower of Bousfield localizations \[X \rightarrow \cdots \rightarrow X _ { E (n) } \rightarrow X _ { E (n - 1) } \rightarrow \cdots \rightarrow X _ { E (0) } \] where E(n) is the nth Johnson-Wilson spectrum (E(0) = HQ, by convention) with \[E (n) _ { * } = \mathbb { Z } _ { (p) } \left[v _ { 1 } , \dots , v _ { n - 1 } , v _ { n } ^ { \pm } \right] \] The fibers of the chromatic tower \[M _ { n } X \rightarrow X _ { E (n) } \rightarrow X _ { E (n - 1) } \] are called the monochromatic layers. The spectral sequence associated to the chromatic tower is the chromatic spectral sequence \[E _ { 1 } ^ { n , * } = \pi _ { * } M _ { n } X \Rightarrow \pi _ { * } X _ { (p) } \]

Let Mell denote the Deligne-Mumford stack of elliptic curves (over Spec(Z)). For a commutative ring R, the groupoid of R-points of Mell is the groupoid of elliptic curves over R. This stack carries a line bundle ω where for an elliptic curve C, the fiber of ω over C is given by \[ωC = T^∗_e C,\] the tangent space of C at its basepoint e. The stack \(M_{ell}\) admits a compactification \(\overline{M}_{ell}\) whose R points are generalized elliptic curves. The space of integral modular forms of weight k is defined to be the sections \[H ^ { 0 } \left(\overline { \mathcal { M } } _ { e l l } , \omega ^ { \otimes k } \right) \] Motivated by the definition of integral modular forms and this descent spectral sequence in the case of U = Mell , the spectrum Tmf is defined to be the global sections \[\mathrm { Tmf } : = \mathcal { O } ^ { t o p } \left(\overline { \mathcal { M } } _ { e l l } \right) \]

What we want to represent is \(SO(3, \RR)\), where a real representation of a group \(G\) is a map \(G \to \GL(n, \RR)\) for some \(n\). More generally, we can define a \(V\dash\)representation as a map \(G \to \Aut(V)\), where since \(V\) lives in the category of vector spaces, this amounts to requiring that \(G\) maps to a linear map. Equivalently we can ask for a \(V\dash\)linear map \(G\cross V \to V\) (i.e. a group action on \(V\)).

So the goal is to construct a representation \(f: SO(3, \RR)) \to \GL(n, \RR)\) for some \(n\).

Suppose we are working with Euler angles \[E \definedas \theset{ (\phi, \theta, \psi) \suchthat \phi,\psi \in [0, 2\pi], \theta \in [0,\pi]} / \sim\]

where \(2\pi \phi \sim \phi, 2\pi\psi \sim \psi, \pi\psi\sim\psi\).

We can then cook up a representation \(g: E \to \GL(n, \RR)\) by writing down appropriate rotation matrices in the parameters \(\phi, \theta, \psi\).

\[R = \left[\begin{array} { c c c } { 1 } & { 0 } & { 0 } \\ { 0 } & { \cos \alpha } & { - \sin \alpha } \\ { 0 } & { \sin \alpha } & { \cos \alpha } \end{array} \right] \left[\begin{array} { c c c } { \cos \beta } & { 0 } & { \sin \beta } \\ { 0 } & { 1 } & { 0 } \\ { - \sin \beta } & { 0 } & { \cos \beta } \end{array} \right] \left[\begin{array}

{ c c c } { \cos \gamma } & { - \sin \gamma } & { 0 } \\ { \sin \gamma } & { \cos \gamma } & { 0 } \\ { 0 } & { 0 } & { 1 } \end{array} \right] = = \left[\begin{array} { c c c } { 0 } & { 0 } & { 1 } \\ { \sin (\alpha + \gamma) } & { \cos (\alpha + \gamma) } & { 0 } \\ { - \cos (\alpha + \gamma) } & { \sin (\alpha + \gamma) } & { 0 } \end{array} \right] \]

Note that each angle lives in a real interval with the endpoints identified, which is topologically a circle. So, up to homeomorphism, we have \(E \cong S^1 \cross S^1 \cross S^1 = (S^1)^3 = T^3\), a real 3-torus. The question then becomes whether or not the representation we are looking for, say \(SO(3, \RR) \to \GL(n, \RR)\), factors through the representation \(E\ to \GL(n, \RR)\) that we have cooked up. This amounts to asking if there is a map \(h: \SO^3 \to T^3\) that makes the following diagram commute:

\[\begin{tikzcd} T^3 \arrow[rrdd, "g"] & & \\ & & \\ {SO(3, \mathbb{R})} \arrow[rr, "f"] \arrow[dd, "h", dashed] & & {\mathrm{Gl}(n, \mathbb{R})} \end{tikzcd} \]

In other words, can we factor the complicated representation \(f\) that we want through a simpler representation \(g\)? In order for this to be a fully faithful representation, this amounts to asking if we can replace \(SO(3, \RR)\) by \(E\) up to isomorphism in some category.

In particular, we should ask that this diagram preserves all of the structure we care about. Since we are viewing everything in sight as a smooth topological group, we should ask that all maps be continuous group homomorphisms.

In particular, we would want \(h\) to be a homeomorphism – however, something has already gone awry, because this is not the case. Perhaps the easiest way to see this is by considering fundamental groups. Topologically, we have \(\SO(3, \RR) \cong \RP^3\) and in particular \(\pi_1(\RP^3) \cong \ZZ/2\ZZ\). However, \(\pi_1(T^3) \cong \ZZ^3\), and so these spaces are not homeomorphic.

We might instead ask that \(h\) be a covering map, so that \(T^3\) covers \(\SO(3, \RR)\) in some sufficiently nice way instead. It turns out that such a surjective map can be constructed, but from the above discussion, we know that injectivity will have to fail, and it is this phenomenon which introduces the singularities that result in gimbal lock. When writing out such a map, this singular points will be to coordinates at which the map will be less than full rank, and fail to be a local homeomorphism in neighborhoods of these points.

To explicitly construct such a map, we’ll identify $SO(3, \RR)$ with $\RP^3$ and look for a map $h': T^3 \to \RP^3$.

Nix that, we’ll actually want to look at all possible covers of $\RP^3$. One such covering space is S^3, where the covering map is given by identifying antipodal points.

Definition [Hatcher]: For a topological space X, a covering space of X is a space $\tilde X$ and a map $p: \tilde X \to X$ such that for each point $x\in X$, there exists an open neighborhood $U_x\subet X$ of x such that $p^{-1}i(U_x) = \coprod \tilde{U}_x$ such that each $\tilde{U}_x$ is homeomorphic to U_x, where the homeomorphism is realized by p.

One fact we can use here is that if p is a covering map, then the induced map on fundamental groups $p_*: \tilde X \to X$ is an injection. Since we are taking $X = \RP^3$ here and we know $\pi_1(\RP^3) = \ZZ/2\ZZ$, we can narrow our search a bit by asking what groups G can fit into an injective group homomorphism $G \to \ZZ/2\ZZ$. This narrows our choices for $\tilde X$ considerably, since this forces $\pi_1(\tilde X) = \ZZ/2\ZZ\text{ or }0$. This immediately excludes T^3 from being a possibility, since $\pi_1(T^3) = \ZZ^3$ and an infinite group can’t be injected into a finite group.

We can in fact instead appeal to the classification of covering spaces, which says that there is a Galois correspondence between covering spaces of X and subgroups of $\pi_1(X)$. Since $\ZZ/2\ZZ$ has exactly two subgroups, 0 and $\ZZ/2\ZZ$, this leaves us with exactly two choices. The identity map $\RP^2 \to \RP^2$ yields the covering space corresponding to $\ZZ/2\ZZ$, and the remaining choice can be realized by taking $S^3 \mapsvia{p} \RP^2$ where $p(\vector{x}) = [x_1, x_2, x_3]$ in homogeneous coordinates. Topologically, noting that $$p(-\vector x) = [-x_1, -x_2, -x_3] = [x_1, x_2, x_3],$$

which follows from the identity $[x,y, \cdots] = [\lambda x, \lambda y, \cdots]$ for any scalar λ in projective spaces, we find that this map is realized by taking points on the sphere and identifying them with their antipodes.

Since \(\pi_1(S^3) = 0\), the induced map \(p_*(\pi_1(S^3)) \to \pi_1(\RP^3)\) will be the zero map, and thus \(p_*(\pi_1(S^3)) = 0\) and we obtain the second possible covering space. This can also be seen from the fact that \(S^3\) is simply connected, so \(\pi_1(S^3) = 0\), and thus \(S^3\) is the universal cover of \(\RP^3\).

Of particular importance here is the fact that the only possibilities for covers were \(S^3\) or \(\RP^3\) itself. We know that \(\RP^3 \not\cong T^3\) by looking at fundamental groups; we can also find that \(S^3 \not\cong T^3\) by looking at \(\pi_3\). In particular, \(\pi_1(S^3) = 0\) while \(\pi_1(T^3) = \ZZ^3\). So there can not exist a covering map \(T^3 \to \RP^3\), which means that any such map we construct must fail the local homeomorphic mapping condition in a neighborhood of at least one point.

So let's attempt to define such a map and see what goes wrong. Since \(T^3 = S^1 \cross S^1 \cross S^1\), let a coordinate on the torus be given by \((\theta_1, \theta_2, \theta_3)\) where each \(\theta_i \in [0, \2pi)\). Inspired by how nicely the previous map \(S^3 \to \RP^3\) worked, let us first define \[p: \T^3 \to \RP^3 \\ (\theta_1, \theta_2, \theta_3) \mapsto [\theta_1, \theta_2, \theta_3]\] Since we hope to represent all elements of \(SO(3,\RR)\) in this way, we want a surjective map.

The most prestigious journals may be Annals of Math., Invent. Math., and the Journal of the AMS. Others of very high quality include Duke Math. J., Publ. Math. IHES, Annales de l'ENS, Math. Annalen, Compositio Math., J. reine und angew. Math., Internat. Math. Research Notices, and (though I may be biased) Algebra & Number Theory.

Participate in research groups: If you are a new graduate student in your department, I recommend that you contact your department chair, coordinator, advisor, and graduate studies chair to ask them about any available research groups to join so you can participate in the group's research publications and seminar's talks.

At the end of the talk, you should try to answer the questions: What question(s) is the speaker trying to answer? Why should we care about them? What flavor of results has the speaker proved? Do I have a small example of the phenomenon under discussion? You can even scribble down these questions at the start of the talk, and jot down answers to them during the talk.

Try to extract three words from the talk (no matter how tangentially related to the subject at hand) that you want to know the definition of.

Try to ask one question at as many seminars as possible, either during the talk, or privately afterwards. The act of trying to formulating an interesting question (for you, not the speaker!) is a worthwhile exercise, and can focus the mind.

Courses and reading courses. During the first year, you will be taking many courses, some required. In the Winter and Spring terms, try to set up a reading course with other graduate students in an area that you are interested in knowing more about.

However, you should be able to answer the following questions at the end of any seminar: “What is the major theorem or question that the speaker is approaching?” “Why is he or she interested in this question or theorem?”

Summer research. Do a research project with your potential advisor over the summer. At least once every two weeks, write down what you have learned (from papers, books, conversations) or discovered (through your hard and persistent work). Keep this in a place that you will be able to refer to later

Ideally, you should have a preprint out by spring of Year n-1. If you do, or you are close, then begin talking to the people that you’ve met at conferences about speaking at their schools’ research seminars.

What if you are interested in liberal arts jobs? If you think it is a possibility that you will want to take a liberal arts job or teaching job after you graduate, then attend the Joint Math Meetings and begin meeting people from the types of schools you are considering a job at.

1. Co-author a paper with someone who has more experience. Approach a professor who is working on an interesting project and offer your services in return for a junior authorship. He’ll appreciate the help and will give you lots of good comments on the paper because his name will be on it.

Advisory Relationship

You should have a discussion with your advisor very early on about your career goals. By this, I mean, you should talk openly about whether or not you are planning to pursue a tenure-track research position, a teaching position at a SLAC (small, liberal arts college), an “alt-ac” career in industry, or any other goal.

Show up to your advisory meetings with a clear agenda, starting with things that are most time-sensitive. Your advisors are busy and may not have the time to be as involved as they want to be in your life. Make it easy for them. Same with emails that you are sending out. If it’s longer than two paragraphs: stop and delete the whole thing. Set up a meeting instead.

Other people’s successes are not your failures. Try as much as possible not to compete with your peers. Don’t undermine them, don’t put them down, don’t be upset when they succeed. Foster a support network, based on care and trust. Help them and they will help you. Share materials with each other, read each other’s papers, study together. They will be your colleagues, now and forever. Success in academia hinges upon long-term collaborative interactions.

If you have debilitating stage fright or presentation anxiety, I would recommend talking to your doctor about whether you’re a good candidate for beta blockers, such as propranolol.

- Every lecture should make only one main point.
  The German philosopher G. W. F. Hegel wrote that any philosopher who uses the word “and” too often cannot be a good philosopher. I think he was right, at least insofar as lecturing goes.
- Every lecture should state one main point and repeat it over and over, like a theme with variations
  As I looked through his Collected Papers, however, another picture emerged. The editors had gone out of their way to publish every little scrap Riesz had ever published. It was clear that Riesz’s publications were few. What is more surprising is that the papers had been published several times. Riesz would publish the first rough version of an idea in some obscure Hungarian journal. A few years later he would send a

series of notes to the French Academy’s Comptes Rendus in which the same material was further elaborated. A few more years would pass, and he would publish the definitive paper, either in French or in English. Adam Koranyi, who took courses with Frederick Riesz, told me that Riesz would lecture on the same subject year after year while meditating on the definitive version to be written. No wonder the final version was perfect.

- Every Mathematician Has Only a Few Tricks
- Use the Feynman Method
  Richard Feynman was fond of giving the following advice on how to be a genius. You have to keep a dozen of your favorite problems constantly present in your mind, although by and large they will lay in a dormant state. Every time you hear or read a new trick or a new result, test it against each of your twelve problems to see whether it helps. Every once in a while there will be a hit,
- Give Lavish Acknowledgments
- Write Informative Introductions

Nowadays reading a mathematics paper from top to bottom is a rare event. If we wish our paper to be read, we had better provide our prospective readers with strong motivation to do so. A lengthy introduction, summarizing the history of the subject, giving everybody his due, and perhaps enticingly outlining the content of the paper in a discursive manner, will go some of the way towards getting us a couple of readers.

- \(p\dash\)completions
- The limit definitions of pullbacks/pushouts and products/coproducts
- The action of \(\pi_1 \actson \pi_n\)
- The unit of an adjunction
- Monad, and its uses within Topology
- A statement of the Yoneda lemma
- Homotopy equivalence vs weakly homotopy equivalent vs quasi-isomorphisms
- What are the subobjects and quotients of a category?
- Kan extensions
- Kan complex

- Bousfield localization
- Loop and Path spaces
- Try to write a composition series as an exact sequence
- Add lens spaces to examples
- Do some examples of natural transformations and the Yoneda lemma
- Look at branched surfaces, foliations, orbifolds
- Sutured manifolds
- Dehn twists
- Train tracks

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[1] A fiber bundle \(E \to B\) is a fibration when \(B\) is paracompact.↩

[2] When \(E\) is in fact a product \(F\cross B\), this actually is a foliation in the technical sense.↩

[3] Due to the fact that, for example, \(M\) is nonorientable and orientability distinguishes topological spaces up to homeomorphism.↩