

# Title

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## 1 Reference List

- **Rational Homotopy Theory and Differential Forms** by Griffiths and Morgan
- **Differential Forms in Algebraic Topology** by Bott and Tu
- **Differential Topology** by Hirsch
- **Comprehensive Introduction to Differential Geometry** by Spivak
- **Topology from the Differentiable Viewpoint** by Milnor
- **Topology and Geometry** by Bredon
- **User's Guide to Spectral Sequences** by Mcleary
  - View Here
  - Apparently lots of technical details

## 2 General Notes

The standard Serre fibration:  $\Omega X \rightarrow PX \xrightarrow{f} X$  where  $\Omega X$  is the loop space,  $PX$  is the path space, and  $f$  is the “evaluation at the endpoint” map. Note that  $PX$  is contractible!

Consider a SES  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow C$ , then look at it as a 2-step filtration of  $B$  so  $F^0 B = B, F^1 B = A, F^2 B = 0$ . The graded pieces are  $G_0 = C, G_1 = A$ . Can use this to obtain LES from SS.

Homology in the ring-theoretic setting: If  $R$  is a Noetherian ring and  $I \subseteq R$ , then if  $I$  can be generated by  $n$  elements then  $H_I^i(M) = 0$  for any  $R$ -module  $M$  and  $i > n$ . Thus to prove  $I$  can *not* be generated by  $n$  elements, it suffices to find a module  $M$  where  $H_I^{n+1} \neq 0$ .

### 3 Griffiths and Morgan

Overall purpose: want to relate  $C^\infty$  forms on a manifold to AT invariants. One significant result: given a manifold  $M$ , the singular cohomology  $H^*(M, \mathbb{R})$  is isomorphic to the cohomology of the differential graded algebra of  $C^\infty$  forms,  $H_{DR}^*(M)$ .

Is this the de Rham cohomology..?

This DGA of smooth forms is actually enough to calculate all of the AT invariants, and can be used to build the Postnikov tower of  $M$  ( $\otimes \mathbb{R}$ )

One construction is the localization of a CW complex at  $\mathbb{Q}$ , this removes torsion and divisibility phenomena. The effect on the Postnikov tower is just then tensoring with  $\mathbb{Q}$ .

Things that are homotopy equivalent to CW complexes:

- Manifolds
- Varieties
- Loop spaces of CW complexes
- Eilenberg-MacLane spaces?  $K(\pi, n)$ .

The Whitehead theorem holds for these:  $X \xrightarrow{f} Y$  is a homotopy equivalence iff  $\pi_*(X) \xrightarrow{f_*} \pi_*(Y)$  is an isomorphism.

Recall the weak topology for infinite CW complex:  $U$  is open in  $X$  iff  $U \cap X^n$  is open for every  $n$ .

Theorem: Given any  $X \xrightarrow{f} Y$ , we can transform this into an inclusion up to homotopy equivalence. (Just replace  $Y$  by the mapping cylinder of  $f$ , denoted  $M_f \simeq Y$ ).

A fibration is anything that satisfies the homotopy lifting property. Examples:

- Locally trivial fiber bundles
- Vector bundles
- Covering spaces

Path spaces are fibrations, loop spaces are contractible.

Homology can be defined with coefficients in any abelian group by tensoring the singular chain groups with  $G$ . That is, if we  $H_*(X)$  obtained from

$$\xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X) \cdots \xrightarrow{\partial_1} C_0(X)$$

then we can define  $H_*(X; G)$  via

$$\xrightarrow{\partial_{n+1} \otimes 1} C_n(X) \otimes G \xrightarrow{\partial_n \otimes 1} C_{n-1}(X) \otimes G \cdots \xrightarrow{\partial_1 \otimes 1} C_0(X) \otimes G$$

Note that homology has the structure of a graded group, while cohomology has the structure of a graded commutative ring.

Axioms of homology:

- $X \xrightarrow{f} Y$  always induces a map on homology  $H_*(X) \xrightarrow{f_*} H_*(Y)$
- An orientation on  $S^n$  induces an isomorphism  $H_n(S^n) \cong \mathbb{Z}$ ; reversing orientation induces the map  $\mathbb{Z} \xrightarrow{\phi} \mathbb{Z} : \phi(1) = -1$

- $Y \subseteq X$  yields the definition of relative homology  $H_*(X, Y)$ , and Mayer Vietoris holds; i.e. there is a long exact sequence  $\cdots H_n(Y) \rightarrow H_n(X) \rightarrow H_n(X, Y) \rightarrow H_{n-1}(Y) \cdots$
- Excision:  $U \subset Y \subset X$  and  $\bar{U} \subset Y^\circ$  implies  $H_*(X - U, Y - U) \cong H_*(X, Y)$ .

Any homology theory satisfying these properties is equivalent to singular homology.

Use notation  $[X, Y]$  for homotopy classes of maps  $X \rightarrow Y$ , then  $\pi_1(X) = [S^1, X]$  and we can define  $\pi_n(X) = [S^n, X]$ . Homotopy groups fail excision.

Whitehead theorem: for CW complexes, if  $X \xrightarrow{f} Y$  induces  $\pi_n(X) \xrightarrow{f_*} \pi_n(Y)$  and  $f_*$  is an isomorphism (and  $Y$  is connected), then  $f$  is a homotopy equivalence. For spaces that aren't CW complexes, this may fail, and we say  $f$  is a *weak homotopy equivalence* instead.

Hurewicz theorem: the bottom homology and homotopy groups are isomorphic, and homology below the bottom homology is zero.

General note: there are equivalent “relative versions” of most of these theorems.

**Spectral Sequence:** Page 45.

For any fibration  $F \rightarrow E \xrightarrow{\pi} B$ , we get a LES in homotopy  $\pi_n(F) \rightarrow \pi_n(E) \rightarrow \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F)$

Basic question: How are the cohomologies of  $F, E, B$  related? An easy case is when  $E = F \times B$ , but even then  $\pi_n(F \times B) \neq \pi_n(F) \oplus \pi_n(B)$ . Need the Kunneth theorem, formula is more complicated.

For CW complexes and a fibration, the relationship is nice - look at the total space of the fibration. It is filtered by increasing  $n$ -skeleta, and we use the LES. More general filtrations need a spectral sequence.

*Note: use LES as trivial example of spectral sequence! Write out the pages, differentials, etc*

The spectral sequence relates the cohomology of *successive pairs* in the filtration to the cohomology of the total space.

**Theorem:** If  $B$  is path-connected and  $\pi_1(B, b_0)$  acts trivially on  $H^*(F)$ , then there are isomorphisms

$$\begin{aligned} H^n(E^p, E^{p-1}) &\cong \prod_{p\text{-cells in } B} H^n(\pi^{-1}e^p, \pi^{-1}\partial e^p) \\ &\cong C^p(B; H^{n-p}(F)) \end{aligned}$$

In other words, for any  $k$ , we can think of  $H^*(E^p, E^{p-k})$  as a  $k$ -th approximation to  $H^*(E^p)$ .

(Should probably review results about polynomial and exterior algebras. And what does it mean for  $\pi_1$  to act trivially on a fiber?)

**EXAMPLES OF COMPUTATION:** Page 54

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### 3.1 Cohomology of $\mathbb{CP}^n$ using $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$

- Cohomology of the infinite Grassmannian  $\lim_n G(k, n)$ 
  - Answer:  $H^*(G(k)) = \mathbb{Z}[x_1, x_2, \dots, x_k]$

## 3.2 Postnikov Towers

A decomposition dual to cell decomposition, the atoms of the space are Eilenberg-MacLane spaces  $K(\pi, n)$ . (Note the spheres are atomic in homology, while the  $K$  are atomic in homotopy.)

Gives a way of going back and forth between  $X$  and  $\pi_*(X)$ : defined as a tower of spaces  $X_0 \leftarrow X_1 \leftarrow \dots$

- $X_{i-1} \leftarrow X_i$  is a fibration
- $\pi_k(X_n) = \mathbb{Z}[k \leq n] \cdot \pi_k(X) + \mathbb{Z}[k > n] \cdot 0$   
– So all lower homotopy groups agree at the  $n$ -th spot
- (Probably)  $X_i \hookrightarrow X$

Unique up to homotopy,  $X = \lim_n X_n$  (an inverse limit). Essentially constructs  $X$  out of  $K(\pi_n(X), n)$ .

*Note: revisit and draw diagrams for Postnikov Towers*

Homotopy and homology commute with direct limits.

$(\cdot \otimes \mathbb{Q})$  is a right-exact functor, most results in this section are about how terms in exact sequences all become  $\mathbb{Q}$ -vector spaces. In particular,  $H^*(X; \mathbb{Q}), H_*(X; \mathbb{Q})$  are.

Homotopy theory over  $\mathbb{Q}$  is much easier than over  $\mathbb{Z}$ . Samples results:

$$\pi_i(S^{2n-1}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & i = 2n - 1 \\ 0 & \text{otherwise} \end{cases}$$

Then using the fact that  $\pi_i(S^{2n-1})$  is always finitely generated, we can conclude

$$\pi_i(S^{2n-1}) = \begin{cases} \mathbb{Z} & i = 2n - 1 \\ \text{a finite group } G & \text{otherwise} \end{cases}$$

This yields for even  $n$ :

$$\pi_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ \mathbb{Z} \oplus G & i = 2n - 1 \\ H & \text{otherwise} \end{cases}$$

for some finite groups  $G, H$ !

Can also obtain Bott Periodicity this way.

## 3.3 Other Reading

Lots of good examples of computations here

Some fibrations

- Hopf:  $S^1 \rightarrow S^3 \rightarrow S^2$
- $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$
- Path space:  $\Omega S^n \rightarrow PS^n \rightarrow S^n$

Serre Spectral Sequence Example: For the fibration  $S^1 \rightarrow S^3 \rightarrow S^2$ , the  $E_2$  page:

1	$H^0(S^2, \mathbb{Z})$	$H^1(S^2, \mathbb{Z})$	$H^2(S^2, \mathbb{Z})$
0	$H^0(S^2, \mathbb{Z})$	$H^1(S^2, \mathbb{Z})$	$H^2(S^2, \mathbb{Z})$
	0	1	2.

Which is equal to

$1H^0(S^2, \mathbb{Z})$	0	$H^2(S^2, \mathbb{Z})$	
$0H^0(S^2, \mathbb{Z})$	0	$H^2(S^2, \mathbb{Z})$	
	0	1	2.

And  $E_3 = E_\infty$ , so  $d_2^{0,1}$  is an isomorphism.

*Note: Probably a good starting point for basic calculations? Fill out the missing details for this table.*

Challenge: Prove  $\pi_4(S^2) = \frac{\mathbb{Z}}{2\mathbb{Z}}$