# Title

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# 1 Cotangent Complex and Derived de Rham Cohomology

Reference: MSRI Workshop on Derived AG, Birational Geometry, Moduli Spaces.

Video: `https://www.youtube.com/watch?v=zRPa-VAvl6Q`

## 1.1 Motivation

Basic affine objects in AG: commutative rings, replace with simplicial commutative rings which we'll use as a base diagram. Later: derived stacks and geometric derived stacks.

An evolution of objects. and how we can think about them.

- Algebraic schemes/spaces, e.g. $\mathbb{P}^n$
  - Think of these as etale sheaves of sets (think functor of points), identified as discrete spaces $\mathcal{S}_{\leq 0}$, so every component is contractible and there are no higher homotopy groups
- The Picard stack $\underline{\mathrm{Pic}}_{X/k}$ for $X$ a smooth and proper $k$-scheme, which is an Artin stack (a subclass Deligne-Mumford stacks). Note that this still has automorphisms given by global units on $X$.
  - Think of these as etale sheaves of groupoids $\mathcal{S}_{\leq 1}$.
  - Can take a fundamental groupoid Also note that this is the mapping stack $\underline{\mathrm{Map}}(X, K(\mathbb{G}_m, 1))$.
- $K(\mathbb{G}_m, n)$ a higher stack
  - A sheaf taking values in $n$-truncated spaces, i.e. a space where when basing at any point, there are no homotopy groups above degree $n$.
  - Stack with a single point, where the isotopy is $K(\mathbb{G}_m, n-1)$.

Note that these are all built from affine schemes with a few acceptable moves.

**Example 1.1.1** ***(?):*** We can realize

$$B\mathbb{G}_m = K(\mathbb{G}_m, 1) \cong [\{\mathrm{pt}\}/\mathbb{G}_m]$$

in stack notation. Similarly,

$$K(\mathbb{G}_m, 2) = [\{\mathrm{pt}\}/B\mathbb{G}_m],$$

which is a smooth Artin stack. Mapping into this gives the Picard groupoid of a scheme. It's a higher geometric stack that still has smoothness properties.

**Question 1.1.2:** What does it mean to give a map from a scheme $X$ into a higher stack?

The world of of étale schemes taking values in $\mathcal{S}_{\le n}$ is enriched in topological spaces. There is a topological space

$$M \coloneqq \mathrm{Map}(X, K(\mathbb{G}_m, n))$$

The homotopy groups are

$$\pi_i M = \begin{cases} H^{n-i}_{\text{ét}}(X, \mathbb{G}_m) & 0 \le i \le n \\ 0 & \text{else} \end{cases},$$

so this higher geometric stack that says something about higher cohomology groups. We thus have étale sheaves taking values in higher topological spaces, and has some geometric meaning. They're also built from geometric objects: iterating taking quotients by smooth actions. $K(\mathbb{G}_m, 1)$ is a quotient by a smooth algebraic group, $K(\mathbb{G}_m, 2)$ is now a smooth *Artin stack*, and we can keep going. This is the fundamental process for building geometric higher stacks.

**Remark 1.1.3:** Why derive things? Schemes are equipped with sheaves of commutative rings, so the basic idea is let the sheaves take values in groupoids, stacks, etc. So we can consider replacing the structure sheaf $\mathcal{O}_X$ is itself a sheaf of spaces, and this is the fundamental idea of derived algebraic geometry.

### 1.1.1 Derived Schemes

Consider $\operatorname{Spec} k \otimes^L_{k[x]} k$, a derived tensor product. This is a simplicial commutative ring, and the basic version of an affine derived scheme. This is a complex $C^\cdot$ with homology in degree 0 and 1, where $H_1 = \mathrm{Tor}^1(k \otimes_{k[x]} k)$. So analogously, we'll start with derived schemes and take quotients by smooth groups. In the end, we get derived stacks.

**Example 1.1.4** ***(Fundamental):*** An example is $\mathcal{M}_\varphi$, the moduli of objects in some DG category $\mathcal{C}$.

## 1.2 Simplicial Rings

We need to agree on what the local affine modules will look like. For our purposes, they'll be simplicial commutative rings. Consider the derived category $\mathcal{C} \coloneqq D(\mathbb{Z})_{\ge 0}$ and its connective[^def:connective] objects, which are chain complexes $C.$ where $H_{<0}(C.) = 0$. There is a **derived tensor product** $\otimes^L$ which makes $\mathcal{C}$ into a symmetric monoidal category, . Basic idea: we want to look at commutative algebra objects in this symmetric monoidal category $(D(\mathbb{Z}), \otimes^L)$.[1] Note that of working in a symmetric monoidal abelian category, we will be looking at connective chain complexes, and simplicial rings are one way of studying commutative algebra objects here.

[1] This is a familiar move: people in the 60s knew one could do AG in some ambient symmetric monoidal abelian category.

[^def:connective] **Connective** means $H_{<0} = 0$.

We have some choices for making sense of DAG:

- $E_\infty$-ring spectra
- Simplicial commutative rings, Note that this is what we will choose.
- Over $\mathbb{Q}$, $\mathbb{Q}$-commutative DGAs.

**Definition 1.2.1** (Simplicial Commutative Ring)
Let $\Delta$ denote the *simplex category*, the category of non-empty finite ordered sets with order-preserving maps. We have the following situation:

$$\begin{tikzcd}
{[0]} \arrow[r, shift left=2] \arrow[r, leftrightarrow] \arrow[r, shift right=2] & {[1]} \arrow[r, shift left=4] \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, shift right=4] & {[2]} & \cdots \\
\\
\{0\} & \{0 \to 1\} & \{0 \to 1 \to 2\} & \cdots
\end{tikzcd}$$

The arrows going up are **face maps** (or **coface maps**), and the others are **degeneracy** maps. If $\mathcal{C}$ is a category, then $s\mathcal{C} \coloneqq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ is the category of simplicial objects of $\mathcal{C}$.

**Example 1.2.2** ***(of simplicial categories):*** sSets $\simeq$ Top: this is not an equivalence of categories, but rather they equivalent homotopy theories (theory up to weak equivalence). There are notions of weak equivalence (isomorphism on $\pi_0$, and for each choice of basepoint, an isomorphism on all $\pi_{\geq 1}$ on each side.

Here there is an $n$-simplex on the LHS (sSets),

$$\Delta^n = \hom_\Delta(\cdot, [n])$$

and on the RHS we have

$$\Delta^n_{\mathrm{Top}} \coloneqq \left\{ [x_0, \cdots, x_n] \in \mathbb{R}^n \;\middle|\; x_i \geq 0, \sum x_i = 1 \right\}$$

If you make a functor $\Delta^n \rightleftharpoons \Delta^n_{\mathrm{Top}}$, then by Yoneda the presheaf category $\mathrm{Presh}(\Delta) \coloneqq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$ is generated by representable objects. Everything on in sSets is generated by taking colimits of the $\Delta^n$, so we can make some assignment and extend by colimits to get a functor sSets $\to$ Top. We have a notion of weak equivalence for Top, and so the notion of weak equivalence on sSets is just given by pullback along the functor sSets $\to$ Top, and this induces an equivalence of homotopy theories.

The functor back Top $\to$ sSets is the **singular complex construction**. Considering $\Delta^{\cdot}_{\mathrm{Top}}$, this is a cosimplicial object in Top.

**Remark 1.2.3:** Top will denote that 1-category, while $\mathcal{T}$op will be its full $\infty$-category.

So we have a natural cosimplicial object in Top, so $\mathrm{Sing}(X) \coloneqq \mathrm{Hom}_{\mathrm{Top}}(\Delta^{\cdot}_{\mathrm{Top}}, X)$ is a simplicial object in sSets. As in singular homology, we can get a simplicial abelian group by taking the free abelian group $\mathbb{Z}[\mathrm{Sing}(X)]$. Note that this is just composing functors $\Delta^{\mathrm{op}} \to \mathrm{Set}$ and $\mathrm{Set} \to \mathbb{Z}\text{-}\mathrm{mod}$. We can use this to create a chain complex $C.(\mathbb{Z}[\mathrm{Sing}(X)])$, and as expected. $H_i(C.) \cong H_i^{\mathrm{Sing}}(X, \mathbb{Z})$, the singular homology.