Title

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Contents

1 Saturday, November 28: Introduction to ∞-categories

Dealing with size issues: take a Grothendieck Universe $\mathcal{U}$: sets whose subsets are closed under all of the usual set operations (small).

Definition 1.0.1 (∞-Category)
An ∞-category $\mathcal{C}$ is a (large) simplicial set $\mathcal{C}$ such that any diagram of the form

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta_n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

admits the indicated lift, where Λ_i^n is an i-horn (a simplex missing the ith face) for $0 < i < n$.

Remark 1.0.2: This is a specialized notion of a Kan complex, and in particular all ∞-categories are Kan complexes. All inner horns are fillable, i.e. simplicial sets are *inner* Kan complexes. Different to Kan complexes, which include all i.

Definition 1.0.3 (Functors between ∞-categories)
A ∞-functor between two ∞-categories is a map between simplicial sets.

Definition 1.0.4 (Nerve of a category)
Given an ordinary category $\mathcal{C}$, define the **nerve** of $\mathcal{C}$ to be the simplicial set given by

$$N(\mathcal{C})_n \coloneqq \{\text{Functors } F : [n] \to \mathcal{C}\}$$

where $[n]$ is the poset category on $\{1, 2, \cdots, n\}$. So an n-simplex is a diagram of objects $X_0, \cdots, X_n \in \mathrm{Ob}(\mathcal{C})$ and a sequence of maps. This defines an ∞-category, and there is a correspondence

$$\left\{ \text{Functors } F{:}\mathcal{C}{\to}\mathcal{D} \right\} \iff \left\{ \infty\text{-Functors } \widehat{F}{:}N(\mathcal{C}){\to}N(\mathcal{D}) \right\}.$$

Note that taking the nerve of a category preserves the usual categorical structure, since the objects are the 0-simplices and the morphisms are the 1-simplices.

Remark 1.0.5: For $\mathcal{C}$ an ∞-category, we can define $\mathcal{C}_0$ to be the "objects" and $\mathcal{C}_1$ to be the "morphisms", although we don't have a good notion of composition yet. There will be boundary map: a 1-simplex has two boundary points, i.e. two objects $a, b \in \mathcal{C}_0$, so we can think of this as a map $f : a \to b$ where $a = \partial_1 f, b = \partial_0 f$[1] are the first and second vertices respectively. We'll also have "degeneracy" maps going up from $\mathcal{C}_0 \to \mathcal{C}_1$, which we should think of as assigning identity morphisms to objects, or conversely that the identity morphism is the degenerate 1-simplex at an

[1] This notation ∂_i denotes the boundary operator that drops the ith vertex.

object.

Definition 1.0.6 (Equivalence of Morphisms)
Given two morphisms $f, g : a \to b$ in an ∞-category, we say $f \simeq g$ are **equivalent** iff there is a 2-simplex filling in the following diagram:

$$
\begin{tikzcd}
 & b \arrow[d, "\mathrm{id}_b"] \arrow[dl, phantom, "\Searrow"] \\
a \arrow[ur, "f"] \arrow[r, "g"'] & b
\end{tikzcd}
$$

Link to diagram

Remark 1.0.7: This turns out to be an equivalence relation. Note that in an ordinary category, if two morphisms are equivalent then they are already equal.

Definition 1.0.8 (Composition of morphisms)
For 1-simplices $f : a \to b, g : b \to c$, a **composition** of f and g is a 2-simplex σ filling in the following diagram:

$$
\begin{tikzcd}
 & b \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow\sigma"] & \\
a \arrow[ur, "f"] \arrow[rr, dotted, "\exists h"'] & {} & c
\end{tikzcd}
$$

Link to diagram
In this case, $h \coloneqq \partial_1 \sigma$ and we write $h \simeq g \circ f$.

Remark 1.0.9: Note that we're not fixing a choice, but it is well-defined up to the equivalence relation we're using. This is similar to how e.g. coproducts are not baked into the structure of a category, but are instead only well-defined up to canonical isomorphism – and in fact, this characterization is sometimes preferable.

Definition 1.0.10 (Equivalences of objects)
If $f : a \to b$ is a morphism in an ∞-category $\mathcal{C}$, then we say f is an **equivalence** if there exists a morphism $g : b \to a$ such that $\mathrm{id}_a \simeq g \circ f$ and $\mathrm{id}_b \simeq f \circ g$. This is equivalent to finding 2-simplices σ, σ' that fill the following two diagrams:

$$
\begin{tikzcd}
 & b \arrow[d, dashed, "\exists g"] \arrow[dr, "\mathrm{id}_b"] \arrow[dl, phantom, "\Searrow\sigma"] \arrow[dr, phantom, "\sigma'\Swarrow"'] & \\
a \arrow[ur, "f"] \arrow[r, "\mathrm{id}_a"'] & a & b \arrow[l, dashed, "\exists g"]
\end{tikzcd}
$$

Link to diagram

Remark 1.0.11: This is close to what we'd require for an isomorphism in an ordinary category, but we now allow the compositions to only be "weakly equivalent" or homotopic to the identities.

Definition 1.0.12 (Functor Categories)
For $\mathcal{C}, \mathcal{D}$ simplicial sets, we can define a simplicial set $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ whose n-simplices are given by

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D})_n \coloneqq \{\text{Simplicial maps} F : \mathcal{C} \times \Delta^n \to \mathcal{D}\}.$$

Remark 1.0.13: Note that the 0-simplices recover functors if these are ordinary categories. If $\mathcal{D}$ is an ∞-category, then this functor category is again an ∞-category.

Definition 1.0.14 (Morphisms of functors / natural transformations)
A **morphism** in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, say $\eta : F \to G$, is a functor $\eta : C \times \Delta^n \to \mathcal{D}$ such that

$$\eta|_{\mathcal{C}\times\{0\}} = F$$
$$\eta|_{\mathcal{C}\times\{1\}} = G.$$

We call such an η a **natural transformation** from F to G.

Remark 1.0.15: Being an equivalence in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is equivalent to being a pointwise equivalence. I.e., η is an equivalence iff the map η_C given by partially applying an object of $\mathcal{C}$ (i.e. a 1-simplex $\Delta^n \to \mathcal{D}$) is an equivalence in $\mathcal{D}$ for all objects $C \in \mathrm{Ob}(\mathcal{C})$.

Definition 1.0.16 (Equivalences of ∞-categories)
A functor $f : \mathcal{C} \to \mathcal{D}$ of ∞-categories is an **equivalence** iff there exists a functor $g : \mathcal{D} \to \mathcal{C}$ and natural equivalences

$$f \circ g \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$$
$$g \circ f \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}}.$$

If there exists such an equivalence, we will write $\mathcal{C} \simeq \mathcal{D}$.

Remark 1.0.17: For ordinary categories, there is a characteristic property that is much easier to write down in general than an explicit equivalence, namely being essentially surjective and fully faithful.