Title

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Contents

1 Saturday, November 28: Introduction to ∞-categories

Dealing with size issues: take a Grothendieck Universe $\mathcal{U}$: sets whose subsets are closed under all of the usual set operations (small).

Definition 1.0.1 (∞-Category)
An ∞-category $\mathcal{C}$ is a (large) simplicial set $\mathcal{C}$ such that any diagram of the form

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta_n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

admits the indicated lift, where Λ_i^n is an i-horn (a simplex missing the ith face) for $0 < i < n$.

Remark 1.0.2: All inner horns are fillable, i.e. simplicial sets are *inner* Kan complexes. Different to Kan complexes, which include all i.

Definition 1.0.3 (Functors between ∞-categories)
A ∞-functor between two ∞-categories is a map between simplicial sets.

Definition 1.0.4 (Nerve of a category)
Given an ordinary category $\mathcal{C}$, define the **nerve** of $\mathcal{C}$ to be the simplicial set given by

$$N(\mathcal{C})_n \coloneqq \{\text{Functors } F : [n] \to \mathcal{C}\}$$

where $[n]$ is the poset category on $\{1, 2, \cdots, n\}$. So an n-simplex is a diagram of objects $X_0, \cdots, X_n \in \text{Ob}(\mathcal{C})$ and a sequence of maps. This defines an ∞-category, and there is a correspondence

$$\left\{ \text{Functors } F{:}\mathcal{C}{\to}\mathcal{D} \right\} \iff \left\{ \infty\text{-Functors } \widehat{F}{:}N(\mathcal{C}){\to}N(\mathcal{D}) \right\}.$$

Note that taking the nerve of a category preserves the usual categorical structure, since the objects are the 0-simplices and the morphisms are the 1-simplices.