Title

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Contents

1 Saturday, November 28: Introduction to ∞-categories

Dealing with size issues: take a Grothendieck Universe $\mathcal{U}$: sets whose subsets are closed under all of the usual set operations (small).

Definition 1.0.1 (∞-Category)
An ∞-category $\mathcal{C}$ is a (large) simplicial set $\mathcal{C}$ such that any diagram of the form

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta_n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

admits the indicated lift, where Λ_i^n is an i-horn (a simplex missing the ith face) for $0 < i < n$.

Remark 1.0.2: This is a specialized notion of a Kan complex, and in particular all ∞-categories are Kan complexes. All inner horns are fillable, i.e. simplicial sets are *inner* Kan complexes. Different to Kan complexes, which include all i.

Definition 1.0.3 (Functors between ∞-categories)
A ∞-functor between two ∞-categories is a map between simplicial sets.

Definition 1.0.4 (Nerve of a category)
Given an ordinary category $\mathcal{C}$, define the **nerve** of $\mathcal{C}$ to be the simplicial set given by

$$N(\mathcal{C})_n := \{\text{Functors } F : [n] \to \mathcal{C}\}$$

where $[n]$ is the poset category on $\{1, 2, \cdots, n\}$. So an n-simplex is a diagram of objects $X_0, \cdots, X_n \in \mathrm{Ob}(\mathcal{C})$ and a sequence of maps. This defines an ∞-category, and there is a correspondence

$$\left\{ \text{Functors } F{:}\mathcal{C}{\to}\mathcal{D} \right\} \iff \left\{ \infty\text{-Functors } \widehat{F}{:}N(\mathcal{C}){\to}N(\mathcal{D}) \right\}.$$

Note that taking the nerve of a category preserves the usual categorical structure, since the objects are the 0-simplices and the morphisms are the 1-simplices.

Remark 1.0.5: For $\mathcal{C}$ an ∞-category, we can define $\mathcal{C}_0$ to be the "objects" and $\mathcal{C}_1$ to be the "morphisms", although we don't have a good notion of composition yet. There will be boundary map: a 1-simplex has two boundary points, i.e. two objects $a, b \in \mathcal{C}_0$, so we can think of this as a map $f : a \to b$ where $a = \partial_1 f, b = \partial_0 f$[1] are the first and second vertices respectively. We'll also have "degeneracy" maps going up from $\mathcal{C}_0 \to \mathcal{C}_1$, which we should think of as assigning identity morphisms to objects, or conversely that the identity morphism is the degenerate 1-simplex at an

[1]This notation ∂_i denotes the boundary operator that drops the ith vertex.

object.

Definition 1.0.6 (Equivalence of Morphisms)
Given two morphisms $f, g : a \to b$ in an ∞-category, we say $f \simeq g$ are **equivalent** iff there is a 2-simplex filling in the following diagram:

$$\begin{tikzcd}
& b \arrow[d, "\mathrm{id}_b"] \\
a \arrow[ur, "f"] \arrow[r, "g"'] \arrow[ur, phantom, bend right, "\Downarrow"] & b
\end{tikzcd}$$

Link to diagram

Remark 1.0.7: This turns out to be an equivalence relation. Note that in an ordinary category, if two morphisms are equivalent then they are already equal.

Definition 1.0.8 (Composition of morphisms)
For 1-simplices $f : a \to b, g : b \to c$, a **composition** of f and g is a 2-simplex σ filling in the following diagram:

$$\begin{tikzcd}
& b \arrow[dr, "g"] \\
a \arrow[ur, "f"] \arrow[rr, dotted, "\exists h"'] \arrow[rr, phantom, shift left=4, "\Uparrow\sigma"] & & c
\end{tikzcd}$$

Link to diagram
In this case, $h := \partial_1 \sigma$ and we write $h \simeq g \circ f$.

Remark 1.0.9: Note that we're not fixing a choice, but it is well-defined up to the equivalence relation we're using. This is similar to how e.g. coproducts are not baked into the structure of a category, but are instead only well-defined up to canonical isomorphism – and in fact, this characterization is sometimes preferable.

Definition 1.0.10 (Equivalences of objects)
If $f : a \to b$ is a morphism in an ∞-category $\mathcal{C}$, then we say f is an **equivalence** if there exists a morphism $g : b \to a$ such that $\mathrm{id}_a \simeq g \circ f$ and $\mathrm{id}_b \simeq f \circ g$. This is equivalent to finding 2-simplices σ, σ' that fill the following two diagrams:

$$\begin{tikzcd}
& b \arrow[d, dashed, "\exists g"] \arrow[dr, "\mathrm{id}_b"] & \\
a \arrow[ur, "f"] \arrow[r, "\mathrm{id}_a"'] \arrow[ur, phantom, bend right, "\sigma\,\Downarrow"] & a & b \arrow[l, dashed, "\exists g"] \arrow[ul, phantom, bend left, "\Downarrow\,\sigma'"]
\end{tikzcd}$$

Link to diagram

Remark 1.0.11: This is close to what we'd require for an isomorphism in an ordinary category, but we now allow the compositions to only be "weakly equivalent" or homotopic to the identities.