If K is a knot in the 3-sphere, then there exists an irr. rep $\pi_1 M \to SU_2$

Let

$$\star : h^{\theta}_{SU_2}(M) + h^{\theta}_{SL_2\mathbb{R}}(M) = h_{\mathbb{R}}(M)$$

Q: Is this also true for $SL_2\mathbb{R}$

If $\sigma_K \neq$ const, then we have irr $SL_2\mathbb{R}$ rep and by $\circledast$, $h^{\theta}_{SU_2} = \frac{1}{2}\sigma_M$, so $h^{\theta}_{SL_2\mathbb{R}} \neq$ const and is thus θ is nonzero somewhere.

$h^{\theta}_{SL_2\mathbb{R}}$ = signed count of irr reps $\pi_1(\cdot) \to SL_2\mathbb{R}$ where

$\rho(u)$ is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (~parabolic)

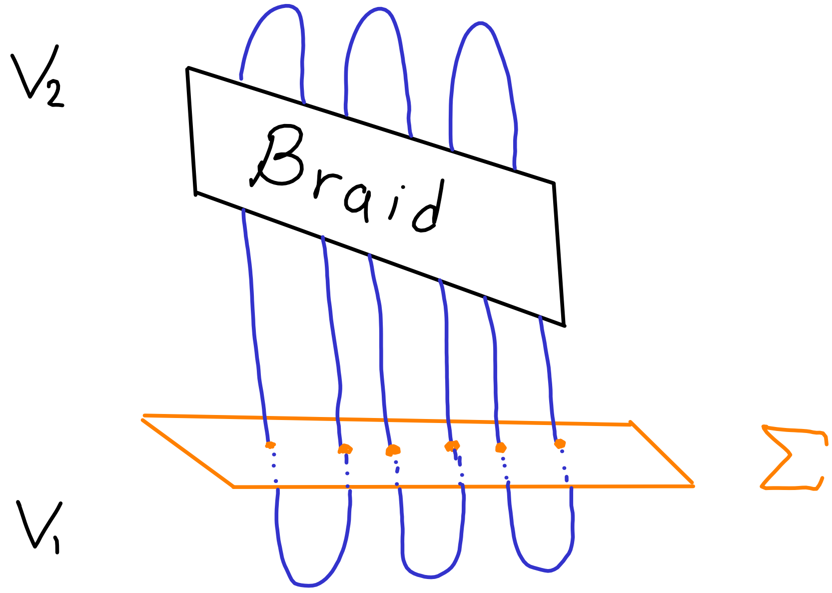
Conj: For K a 2-bridge knot, $\# X^{\circ}_{SL_2\mathbb{R}} \geq \underbrace{\frac{1}{2}\left|\sigma_k\left(\theta = \frac{\pi}{2}\right)\right|}_{\text{Signature}}$

Proved, Gordon 2016. Does it hold for all K?.

Since 2-fold branched cover is a lens space, where π_1 is cyclic and thus has only trivial reps.

How is this invariant defined?

$X_G^{\mathcal{O}} \approx$ character variety



$$\begin{tikzcd}
& \pi_1 V_2 \arrow[dr] & \\
\pi_1 \Sigma \arrow[ur] \arrow[dr] & & \pi_1 M \\
& \pi_1 V_1 \arrow[ur] &
\end{tikzcd}$$

Apply $X_G^{\mathcal{O}_1}$ contravariant

V_i: genus n handlebodies

$SU_2, SL(2,\mathbb{R}) \leq SL(2,\mathbb{C})$

$$\begin{tikzcd}
\text{dim } 4n-6 & X_G^{\mathcal{O}}(V_2) \arrow[dl] & \\
X_G^{\mathcal{O}}(\Sigma) & \text{dim } 2n-3 \arrow[u] \arrow[d] & X_G^{\mathcal{O}}(M) \arrow[ul] \arrow[dl] \\
& X_G^{\mathcal{O}}(V_2) \arrow[ul] &
\end{tikzcd}$$

$\underbrace{\phantom{X_G^{\mathcal{O}}(M)}}_{\text{Compact when } \mathcal{O} \in D_M}$

$\rightarrow \overline{X}^{\mathcal{O}}_{SU_2} \cup \overline{X}^{\mathcal{O}}_{SL(2,\mathbb{R})} \subseteq \overline{X}^{\mathcal{O}}_{SL(2,\mathbb{C})}$

and

$\overline{X}^{\mathcal{O}}_{SU_2} \cap \overline{X}^{\mathcal{O}}_{SL(2,\mathbb{R})} = \underbrace{\overline{X}^{\mathcal{O},\text{ reducible}}_{SL(2,\mathbb{C})}}_{\text{A highly singular subvariety}}$

Smooth mfds, oriented, but not compact

Key idea: blow up along singular locus to produce a Smooth manifolds

$X_{\mathbb{R}}^{\mathcal{O}} \xrightarrow[p]{\text{smooth}} \overline{X}^{\mathcal{O}}_{SU_2} \cup \overline{X}^{\mathcal{O}}_{SL(2,\mathbb{R})}$, $p^{-1} \in C^\infty$, diffeo for reducible reps