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Cyclic Branched Covers and the L-Space Conjecture

Let W^3 be closed, connected, oriented. TFAE?.

- CTF: W admits a co-oriented taut foliation
- NLS: W is not an L-space
- LO: $\pi_1 W$ is left orderable

Main sources of 3-manifolds:

Surgery, or branched covers

Showing not LO is amenable to computation, but showing LO is undecidable

One way: exhibit $\rho : \pi_1 W \longrightarrow \mathrm{Homeo}^+(\mathbb{R})$

$\hookrightarrow$ Difficult in general

Cyclic branched covers

$$\Sigma_n(K) \longrightarrow S^3$$

K a prime knot

Let $\mathcal{L}(K) = \{n \geq 2 \mid \Sigma_n K \text{ is an L-space}\}$

1) $= \emptyset$ for $K = P(-3,5,5)$ pretzel

2) $= \{2,3,\dots,m\}$ for $K = T(2,3)$

3) $= \mathbb{N} - \{0,1\}$ for $K =$ figure 8

Conjecture: Case 2 iff $\pi_1(S^3 \setminus K)$ bi-orderable

<u>What's Known</u>

$$\omega \in \begin{cases} \text{excellent} = \text{CTF} + \text{LO} \\ \quad \sqcup \\ \text{Total L-Space} = \text{L-Space} - \text{LO} \end{cases}$$

Equivalent to L-Space Conjecture

<u>Thm</u>: $\Sigma_n T(p,q)$ is a total L-space iff

π_1 is finite iff

(p,q,n) is a Platonic triple iff

$$(p,q,n) \in \left\{ \begin{matrix} (2,3,\ 2 \leq n \leq 5), & (3,5,2) \\ (2,5,\ 2 \leq n \leq 3), & \\ (2,\ k \geq 7,\ 2), & \\ (3,4,2), & \end{matrix} \right\}$$

<u>Conjecture</u>: K a satellite knot $\Rightarrow \Sigma_n K$ is a satellite knot. (Open)

<u>Hyperbolic Knots</u>

① K is alternating $\Rightarrow \Sigma_2 K$ is an L-Space, not LO $\Rightarrow$ is a total L-space

Some experimental results

$\sim$265k knots w/ Σ_2 hyperbolic, ≤ 15 crossings

73% L-Spaces

$\geq$73% CTF

$\geq$44% not LTO

2-bridge knots (Nice examples, $\pi_1 \leq \langle \alpha, \beta \rangle$)

Alternating, $\Sigma_2 K$ is a total L-Space

$\Sigma_n K$ is LO for $n \gg 0$

- when K is (p,q) with $p \equiv 3 \bmod 4$

Parabolic

meridian $\mapsto \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$

- $\sigma(K) \neq 0 \Rightarrow \exists$ a "real p-rep", rep $\to SL(2, \mathbb{R})$

High enough branched covers yield LOs

General hyperbolic knots

If the trace field of K has a real place then $\Sigma_n K$ is LO for $n \gg 0$

(Smooth point of character variety)

Look at fractional Dehn twist coef.

Thm: $b \in B^n$ pseudo-Anisov $\Rightarrow \Sigma_{2k} b$ is excellent ($|o(b)| \geq 2$)

Thm: K a fibred hyperbolic knot in a $\mathbb{Z}HS$ (genus(fibre) $= g$) & monodromy $\Sigma_n K$ is excellent for $n|c(h)| \geq 1$

Conjecture: K an L-Space knot and $\exists n$ s.t. $\Sigma_n K$ is LO then K is a certain torus knot