# Field arithmetic and the complexity of algebraic objects

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*Last updated:* 2021-07-15

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Talk: Danny Krashen

## 1.1 Intro

Missed first 13m

Fix a field $k_0 \in \mathsf{Field}$, we'll consider extensions $k \in \mathsf{Field}_{/k_0}$.

## 1.2 Galois Cohomology

**Definition 1.2.1** (Galois Cohomology)
For $M \in \mathsf{G_k\text{-}Mod}$ for $G_k$ the Galois group of $k \in \mathsf{Field}_{/k_0}$, we can take invariants $M^{G_k}$. The functor $-^{G_k}$ is left-exact, so we define

$$H^*_{\mathrm{Gal}}(G_k; -) \coloneqq \mathbb{R}^*(-)^{G_k}.$$

**Remark 1.2.2:** Note that the tensor product on $\mathsf{G_k\text{-}Mod}$ induces a cup product on $H^*_{\mathrm{Gal}}$. An important example of coefficients is $M = \mu_\ell^{\otimes m}$, where $\mu_\ell^{\otimes 0} \coloneqq \mathbb{Z}/n$. It is known that $H^*_{\mathrm{Gal}}(G_k; \mu^{\otimes 0}) = \mathbb{Z}/n$.

We'll define *symbols*

$$(a_1, \cdots, a_n) \coloneqq (a_1) \smile \cdots \smile (a_n) \in H^*_{\mathrm{Gal}}(k, \mu_\ell^{\otimes n}),$$

which are in fact generators. To remember the $\ell$, we write $(a_1, a_2, \cdots, a_n)_\ell$.

**Remark 1.2.3:** Galois cohomology is a special case of étale cohomology, where for $M \in \mathsf{G_k\text{-}Mod}$,

$$H^n_{\mathrm{Gal}}(G_k; M) = H^n_{\text{ét}}(k; M) = H^n_{\text{ét}}(\operatorname{Spec} k; M).$$

Étale cohomology works for schemes other than just $\operatorname{Spec} k$.

## 1.3 Milnor K-Theory

**Definition 1.3.1** (?)
Given $k \in \mathsf{Field}$, define

$$\mathrm{K}_*^{\mathrm{M}}(k) \coloneqq \bigoplus_{i=1}^{\infty} \mathrm{K}_i^{\mathrm{M}}(k)$$

where

- $\mathrm{K}_0^{\mathrm{M}}(k) = \mathbb{Z}$
- $\mathrm{K}_1^{\mathrm{M}}(k) = k^m$, written additively as elements $\{a\}$ on the left-hand side, so $\{a\} + \{b\} \coloneqq \{ab\}$.
- It's generated by $\mathrm{K}_1^{\mathrm{M}}(k)$, with products written by concatenation:

$$\{a_1, \cdots, a_n\} = \{a_1\}\{a_2\} \cdots \{a_n\}.$$

- The only relations are $\{a, b\} = 0$ when $a + b = 1$, motivated by

$$(a, b)_\ell = 0 \in H^2_{\mathrm{Gal}}(k; \mu_\ell^{\otimes 2}) \iff a + b = 1.$$

- There is a map

$$\begin{aligned} \mathrm{K}_0^{\mathrm{M}}(k) &\to H^*_{\text{ét}}(k; \mu_\ell^{\otimes 0}) \\ \{a\} &\mapsto (a), \end{aligned}$$

and the **Norm-Residue isomorphism** (formerly the **Bloch-Kato conjecture**) states that this is an isomorphism after modding out by $\ell$, i.e.

$$\mathrm{K}_0^{\mathrm{M}}(k)/\ell \xrightarrow{\sim} H^*_{\text{ét}}(k; \mu_\ell^{\otimes 0}).$$

## 1.4 Witt Ring

**Remark 1.4.1:** Assume $\operatorname{ch} k \neq 2$, so there is a correspondence between quadratic forms and symmetric bilinear forms given by polarization:

$$\begin{aligned} \text{Quadratic forms} &\rightleftharpoons \text{Symmetric bilinear forms} \\ q_b(x) \coloneqq b(x,x) &\hookleftarrow b(x,y) \\ q &\mapsto b_q(x,y) \coloneqq \frac{1}{2}\left(q(x+y) - q(x) - q(y)\right). \end{aligned}$$

So we'll identify these going forward and write $q$ for an arbitrary symmetric bilinear form or a quadratic form. We say $q$ is **nondegenerate** if there is an induced isomorphism:

$$\begin{aligned} V &\xrightarrow{\sim} V^\vee \\ v &\mapsto b_q(v, -). \end{aligned}$$

*Note that a symmetric bilinear form $q$ on $V$ can be regarded as an element of* $\mathrm{Sym}^2(V^\vee)$.

**Definition 1.4.2** (The Witt Ring)
Let $\mathsf{QuadForm}_{/k}$ be the category of pairs $(V, q)$ with $V \in \mathsf{Vect}_{/k}$ a $k$-vector space and $q \in \mathrm{Sym}^2(V^\vee)$ representing a quadratic form on $V$. The **Witt ring** is generated as a group by isomorphism representing a quadratic form on $V$.

$$W(k) = \frac{\mathbb{Z}\left\langle \left\{ [(V, q)] \in \mathsf{QuadForm}_{/k} \right\} \right\rangle}{\langle q_{\mathrm{hyp}}, (q_1 + q_2) - (q_1 \perp q_2) \rangle} \in \mathsf{AbGrp}.$$

where the **hyperbolic form** is defined as $q_{\mathrm{hyp}}(x, y) = xy$. The ring structure is given by the tensor product (a.k.a. Kronecker product of forms).

**Remark 1.4.3:** Noting that Galois cohomology lives mod $\ell$ for various $\ell$, here $\mathrm{K}_0^{\mathrm{M}}(k)$ lives over $\mathbb{Z}$. So Milnor K-theory relates all of the various mod $\ell$ Galois cohomologies together.

**Definition 1.4.4** (Fundamental ideals and Pfister Forms)
The **fundamental ideal** $I(k) \trianglelefteq W(k)$ is the ideal of even dimensional forms, and set $I^n(k) := (I(k))^n$. There is a map

$$\begin{aligned} \mathrm{K}_n^{\mathrm{M}}(k) &\to I^n(k)/I^{n+1}(k) \\ \{a_1, a_2, \cdots, a_n\} &\mapsto \langle\langle a_1, a_2, \cdots, a_n \rangle\rangle, \end{aligned}$$

which follows from Gram-Schmidt: any form can be diagonalized $q \cong \sum a_i x_i^2$, which we can write as $\langle a_1, a_2, \cdots, a_n \rangle$. We can define the $n$**-fold Pfister forms**

$$\begin{aligned} \langle\langle a \rangle\rangle &:= \langle\langle 1, -a \rangle\rangle \\ \langle\langle a_1, a_2, \cdots, a_n \rangle\rangle &:= \prod_{i=1}^{n} \langle\langle a_i \rangle\rangle. \end{aligned}$$

**Remark 1.4.5:** The **Milnor conjecture** (proved by Voevodsky et al) states that the above map is an isomorphism after modding out by 2, so

$$\mathrm{K}_n^{\mathrm{M}}(k)/2 \xrightarrow{\sim} I^n(k)/I^{n+1}(k).$$

Moreover, the LHS is isomorphic to $H^n(k, \mu_2)$. There are interesting maps going the other way

$$I^n(k) \to I^n(k)/I^{n+1}(k) \xrightarrow{\sim} H^n(k, \mu_2)$$

Upshot: this is surjective – any mod 2 cohomology class comes from a quadratic form, and thus we can reason about cohomology by reasoning about quadratic forms.

## 1.5 Motivic Cohomology

**Remark 1.5.1: Motivic cohomology** relates the various mod $\ell$ cohomologies together much like $K_*^M$, but additionally relates different twists. In particular, it relates various $H_{\text{ét}}^i(k; \mu_\ell^{\otimes j})$, where Milnor K-theory interprets this "diagonally" when $i = j$. This works by constructing **motivic complexes**

$$\mathbb{Z}(m) \in \mathsf{Ch}(\underset{\text{pre}}{\mathsf{Shsm}}\mathsf{Sch}_{/k}),$$

which are complexes of presheaves on smooth $k$-schemes, usually considered in the Zariski, étale, or Nisnevich topologies.

**Remark 1.5.2: Zariski hypercohomology** is defined as

$$\mathbb{H}^n(X; \mathbb{Z}(m)) = H^{n,m}(X; \mathbb{Z}) = H_{\text{mot}}^n(X; \mathbb{Z}(m)) \qquad \text{for } X \coloneqq \operatorname{Spec} k.$$

These relate to Galois cohomology in the following ways:

- There is a quasi-isomorphism $\mu_\ell^{\otimes m} \xrightarrow{\sim W} \mathbb{Z}/\ell(n)$ in the étale topology.
- There is an isomorphism $H_{\text{zar}}^n(k, \mathbb{Z}(n)) \xrightarrow{\sim} K_n^M(k)$.
- Bloch-Kato identifies $H_{\text{zar}}^*(X; \mathbb{Z}/\ell(n)) \xrightarrow{\sim} H_{\text{ét}}^n(X; \mathbb{Z}/\ell(n))$.

## 1.6 Dimension

**Remark 1.6.1:** There are a number of competing notions for the "dimension" of a field.

**Definition 1.6.2** (Dimension of a field)
If $k$ is finitely generated over either a prime field or an algebraically closed field, we say

$$\dim(k) = \begin{cases} [k : k_0]_{\text{tr}} & k_0 = \bar{k}_0 \\ [k : k_0]_{\text{tr}} + 1 & k_0 \text{ finite} \\ [k : k_0]_{\text{tr}} + 2 & k_0 = \mathbb{Q}. \end{cases}$$

**Definition 1.6.3** (Cohomological dimension)
We define its **cohomological dimension** $\operatorname{cohdim}(k)$, which is at most $n$ if $H^n(G_k; M) = 0$ for all $m > n$ and $M$ torsion,

$$\operatorname{cohdim}(k) \coloneqq \min \left\{ n \;\middle|\; \operatorname{cohdim}(k) \leq n \right\}.$$

Equivalently, $\operatorname{cohdim}(k) = n \iff$ there exists a torsion $M$ with $H^n(G_k; M) \neq 0$ and $H^m(G_k; M) = 0$ for all $m > n$.

**Remark 1.6.4:** $\operatorname{cohdim}(k) = \dim(k)$ if $k$ is finitely generated or a finite extension of $k_0 = \bar{k}_0$, or if $k$ is finitely generated over $\mathbb{Q}$ and has no real orderings. So if $k$ has orderings, $\operatorname{cohdim}(k) = \infty$.

**Definition 1.6.5** (Diophantine Dimension)
We say $k$ is $C_n$ if for $d > 0$ and $m > d^n$, then every homogeneous polynomials of degree $d$ in $m$ variables has a nontrivial root.

$$\operatorname{ddim}(k) := \min \left\{ n \;\middle|\; k \text{ is } C_n \right\}.$$

**Example 1.6.6*(?)*:** If $k$ is finitely generated or finite over $k_0 = \bar{k}_0$, then

$$\operatorname{ddim}(k) = \dim(k) = \operatorname{cohdim}(k).$$

**Definition 1.6.7** ($T_n$-rank)
We say $k$ is $T_n$ if for every $d_1, d_2, \cdots, d_r > 0$ and every system of polynomial equations $f_1 = \cdots = f_r = 0$ with $\deg f_i = d_i$ in $m$ variables, with $m > \sum d_i^n$. Then the **$T_n$-rank** is defined as

$$T_n\text{- rank}(k) := \min \left\{ n \;\middle|\; k \text{ is } T_n \right\}.$$

**Question 1.6.8**
Note that $T_n \implies C_n$, so $T_n\text{- rank}(k) \geq \operatorname{ddim}(k)$, when are they equal? This is likely unknown.

**Remark 1.6.9:** There is a famous example of a field $k$ with $\operatorname{cohdim}(k) = 1$ but $\operatorname{ddim}(k) = \infty$.

**Question 1.6.10**
Is it true that $\operatorname{ddim}(k) \geq \operatorname{cohdim}(k)$? Serre showed that this holds when cohdim is replaced by $\operatorname{cohdim}_2$, the 2-primary part – does this hold for all $p$? These are both open.

Why would one expect this to be true?

**Remark 1.6.11:** A recent result: $\operatorname{cohdim}_p$ grows at most linearly in ddim, with slope not 1 but rather $\approx \log_2 p$. These questions say that if an equation has enough variables then there is a solution, but why should this be reflected in cohomology? To show this bound, one would want to show that given some $\alpha \in H^*(k)$, there exists a polynomial $f_\alpha$ where if $f_\alpha$ has a root and $\alpha = 0$ in homology. In special cases, we were able to come up with such polynomials. When $\alpha$ is a symbol, this is closely related to *norm varieties* which have a point iff $\alpha$ is split. One might optimistically hope these are described as hypersurfaces, from which answers to the above would follow, but they turn out to not have such a concrete realization.

## 1.7 Structural Problems in Galois Cohomology

**Remark 1.7.1:** Here we'll describe the problems we need help with! Perhaps insight from motivic cohomology will lend insight to them. We'll write $H^i(k) := H^i(k; \mu_\ell^{\otimes j})$.

### 1.7.1 Period-Index Problems

**Definition 1.7.2** (An extension splitting a cohomology class)
If $\alpha \in H^i(k)$, we say $L_{/k}$ **splits** $\alpha$ if

$$\alpha|_L = 0 \in H^i(L).$$

**Definition 1.7.3** (?)
We define the **index**

$$\operatorname{ind} \alpha := \gcd \left\{ [L : k] \;\middle|\; L_{/k} \text{ finite and splits } \alpha \right\}.$$

and the **period** of $\alpha$ as its (group-theoretic) order $H^i(k)$. Note that $\operatorname{per} \alpha \leq \ell$.

**Remark 1.7.4:** One can show that $\operatorname{per} \alpha \mid \operatorname{ind} \alpha$, and $\operatorname{ind} \alpha \mid (\operatorname{per} \alpha)^m$ for some $m$.

**Question 1.7.5**
For a fixed $k$ and $i, j, \ell$, which is the minimum $m$ such that

$$\operatorname{ind} \alpha \mid (\operatorname{per} \alpha)^m?$$

Alternatively, what is the minimum $m$ such that $\operatorname{ind} \alpha \mid \ell^m$?

**Conjecture 1.7.6.**
If $\operatorname{ddim}(k) = n$ (or $\dim(k) = n$ since $k$ is finitely generated) with $\alpha \in H^2(k, \mu_\ell)$, then

$$\operatorname{ind} \alpha \mid (\operatorname{per} \alpha)^{n-1}.$$

**Remark 1.7.7:** Even in this case, no known bound is known for $k = \mathbb{Q}(t)$, for any choice of $\ell$. How complicated can the cohomology class be? The rough idea is that for $H^i(k)$ with $i$ near $\dim k$, this should have a small index and if $i = \dim k$ then $\operatorname{per} k = \operatorname{ind} k$.

**Remark 1.7.8:** We know per = ind for any number field for classes in $H^2(\operatorname{Spec} k; \mu_N)$, with or without roots.

### 1.7.2 Symbol Length Problem

**Remark 1.7.9:** We know $H^n(k, \mu_\ell^{\otimes n})$ is generated by symbols $(a_1, a_2, \cdots, a_n)$. We can use symbol length to measure complexity, leading to the following:

**Question 1.7.10**
Given $k, n$, what is the minimal number $m$ such that every $\alpha \in H^n(k)$ is a sum of no more than $m$ symbols. I.e. how easy is it to write $\alpha$?

**Remark 1.7.11:** We'd like a bound in terms of $\operatorname{ddim}(k)$ and $\dim(k)$. One can construct fields needing arbitrarily long symbols, but perhaps for finite dimensional fields, one feels there should be a bound. Danny feels that there may not be such a bound once $n \geq 4$.

**Remark 1.7.12:** What's known: for number fields (or global fields, i.e. a reasonable notion of dimension with $\dim k = 2$) which lie over finitely generated or prime fields and have a primitive $\ell$th root of unity, we know every class in $H^2$ can be written as exactly one symbol.

**Remark 1.7.13:** A result of Malgri (?): assuming we have roots of unity, if $\ell = p^t$, then for $H^2$ one needs at most $t(p^{\operatorname{ddim}(k)-1} - 1)$ symbols. If $\operatorname{ddim}(k) < \infty$ this yields a bound, and conjecturally this shouldn't depend on ???

For higher degree cohomology, we know almost nothing except for special cases of $H^4$ for "3-dimensional" $p$-adic curves.

**Remark 1.7.14:** If one can bound the symbol length, one can uniformly write down a generic element in cohomology as a sum of at most $n$ symbols. The inability to be able to write down a general form of a cohomology class for a given field is what makes this difficult – they have "complexity" that isn't necessarily bounded in a known way.

# 2 | Tuesday, July 13

**Remark 2.0.1:** Fix a $k_0 \in \mathsf{Field}$.

**Outline**

- Arithmetic problems: consider "complexity" of cohomology or algebraic structures (Witt group, symbol length, index of classes).
  - Examples were $\operatorname{ddim}, \operatorname{cohdim}$, the period-index problem, the period-symbol length problem, which we saw last time.
- Algebraic structure problems: describe (algebraic) structural features of the class of all field extensions $k \in \mathsf{Field}_{/k_0}$.

Today we'll describe a way to connect these using a notion of *essential dimension*. Computing this is difficult in general, but finding lower/upper bounds can be tractable. We'll get upper bounds from *canonical dimensions*, and lower bounds from cohomological invariants.

**Remark 2.0.2:** For a particularly concrete problem, consider

$$\alpha \in H^2(k; \mu_\ell) \subseteq H^2(k; \mathbb{G}_m)[\ell] := \mathrm{Br}(k)[\ell],$$

i.e. this is a subgroup of the $\ell$-torsion of the **Brauer group**. Suppose we know

$$\operatorname{ind} \alpha := \gcd \left\{ [L : k] \;\middle|\; \alpha_L = 0 \right\} = \min \left\{ [L : K] \;\middle|\; \alpha_L = 0 \right\},$$

where the last equality holds in the special case of $\mathrm{Br}(k)$. If $k$ contains a primitive $\ell$th root of unity, we can identify $\mu_\ell = \mathbb{Z}/\ell = \mu_\ell^{\otimes 2}$, and thus identify

$$H^2(k; \mu_\ell) = H^2(k; \mu_\ell^{\otimes 2}) = K_2^M(k)/\ell.$$

So we can write $\alpha = \alpha_1 + \cdots + \alpha_r$ as a sum of symbols with $\alpha_i = (b_i, c_i)_\ell$ with $b_i, c_i \in k^\times$.

**Question 2.0.3**
How big does $n$ have to be?

**Remark 2.0.4:** It follows from "the literature" (after stringing several results together) that there almost exists an absolute bounds depending only on $\ell, n$ but not $k$. However, we do not know what this bound actually is. There are some known cases:

- $\ell = n = 2, 3$: $r \leq 1$, so only one symbol is needed.
- $\ell = n = 4$: probably $r \leq 4$.
- $\ell = 2, n = 4$: $r \leq 2$, a classical results on central simple algebras.
- $\ell = 2, n = 8 : r \leq 4$

**Remark 2.0.5:** It turns out that if $k$ contains a field $k_0$ with $\operatorname{ddim} k_0 < \infty$, one can produce an explicit bound. Given some $\alpha \in H^2(k; \mu_\ell)$ we can find some $k_0 \subseteq L \subseteq k$ with $L$ finitely generated over $k_0$ and $[L : k_0]_{\mathrm{tr}}$ depending only on the period $\ell$ and index $n$, such that $\alpha \in \operatorname{im}\left( H^2(L; \mu) \to H^2(k; \mu) \right)$.

**Slogan 2.0.6**
Central simple algebras of a given period and index have finite essential dimension.

An important property is that

$$\operatorname{ddim} L \leq \operatorname{ddim} k_0 + [L : k_0]_{\mathrm{tr}}.$$

Recall that we can bound the symbol length in $H^2(k; \mu_\ell)$ in terms of $\operatorname{ddim} L$. The idea is that we can bound the transcendence degree in terms of $\ell, n$. This bound can be made very explicit, although it's not tight: for $\ell = 2, n = 8$, it's $2^{17 + \operatorname{ddim} k_0} - 1$. This is an improvement over $k_0 = \mathbb{Q}$ though, where it's known there's a bound but it can't be written down. The lower bound is *very* low: it is hard to show a symbol can not be written with very few symbols.

## 2.1 Pfister Form

**Remark 2.1.1:** Recall $W(k)$, whose elements are isometry classes of nondegenerate quadratic forms with addition given by perpendicular sum and the Kronecker product. There is a hyperbolic form $xy$, or $x^2 - y^2$ in $\operatorname{ch} k \neq 2$, which we can write as $\langle 1, -1 \rangle$, and a fundamental ideal of even-dimensional forms $\langle 1, -a \rangle = \langle\langle a \rangle\rangle$. We write

$$\langle\langle a_1, a_2, \cdots, a_n \rangle\rangle \coloneqq \langle\langle a_1 \rangle\rangle \langle\langle a_2 \rangle\rangle \cdots \langle\langle a_n \rangle\rangle \in I^n(k),$$

which in fact generate $I^n(k)$.

**Question 2.1.2**
Given $q \in I^n(k)$ of dimension $d$, we know we can write $q \sim q_1 \perp \cdots \perp q_r$ where $q_i$ are $n$-fold Pfister forms. How many are needed? Is this number even bounded?

**Theorem 2.1.3** ***((Vishik)).***
If $d < 2^n + 2^{n-1}$ then $r$ is bounded by some small number.

**Remark 2.1.4:** For $d \geq 2^n + 2^{n-1}$, it's not so clear, although it is bounded when $n \geq 3$. Why is $n \leq 3$ easy and $n \geq 4$ hard?

**Remark 2.1.5:** Consider the following objects:

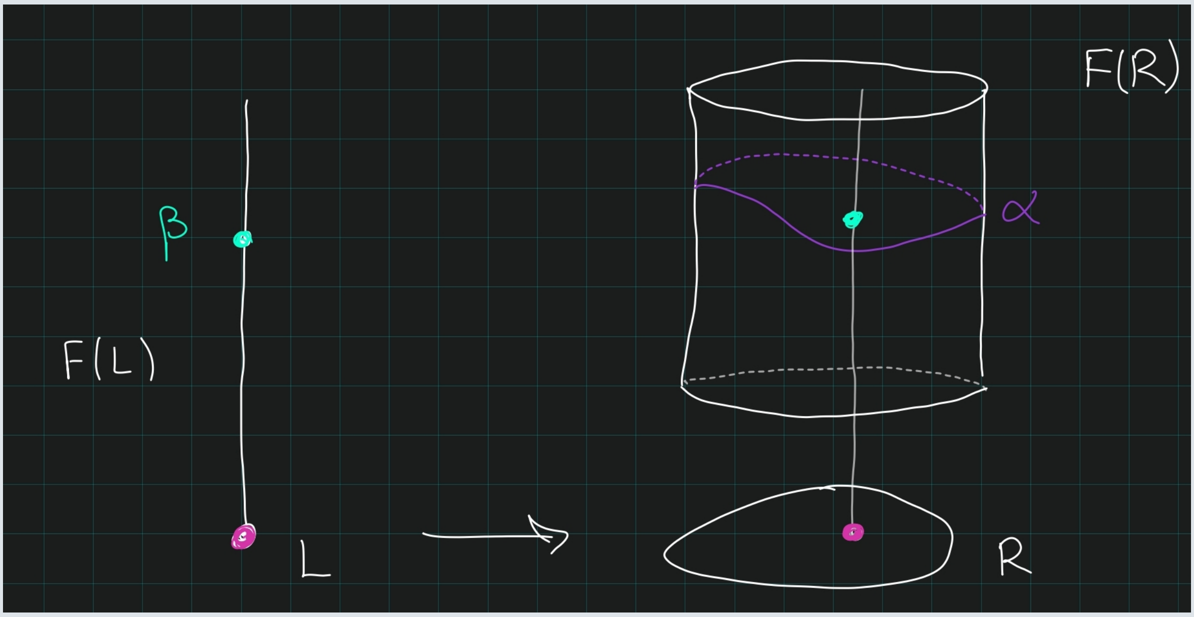
- $H^2(k; \mu)$
- $\operatorname{Br}(k)$
- $W(k)$
- $I^n(k)$
- $q \in I^n(k)$ with $\dim q = d$

These can all be viewed as functors $\mathsf{Field}_{/k_0} \to \mathsf{Set}$ taking field extensions to sets.

**Definition 2.1.6** (Essential dimension of a functor)
Given a functor $f$ and $\alpha \in F(k)$, define

$$\operatorname{essdim}(\alpha) = \min \left\{ [L : k_0]_{\mathrm{tr}} \;\middle|\; \alpha \in \operatorname{im}(F(L) \to F(k)) \right\}$$
$$\operatorname{essdim}(F) = \min \left\{ \operatorname{essdim}(\alpha) \;\middle|\; \alpha \in F(k)\ \forall k_{/k_0} \right\}.$$

**Definition 2.1.7** (Versal)
Given a functor $F : \mathsf{Alg}_{/k_0} \to \mathsf{Set}$, we say $\alpha \in F(R)$ is **versal** if for every $\beta \in F(K)$, for any $k_{/k_0}$, there exists a morphism $R \to k$ such that $\beta$ is the image of $\alpha$ under $F(R) \to F(k)$.




**Observation 2.1.8**
If there exists a versal $\alpha \in F(R)$ then krulldim $R \geq \operatorname{essdim}(F)$, so the essential dimension is bounded above by the Krull dimension.

**Example 2.1.9** ***(?):*** Let $F(k)$ be the set of quadratic forms of dimension $n$ over $k$, then $\operatorname{essdim} F = n$. Every such $q$ can be diagonalized to yields $q \simeq \langle a_1, a_2, \cdots, a_n \rangle$ which is defined over $L := k_0(a_1, a_2, \cdots, a_n)$. Alternatively,

$$q = \langle x_1, x_2, \cdots, x_n \rangle / k_0[x_1^{\pm 1}, x_2^{\pm 1}, \cdots, x_n^{\pm 1}]$$

is versal. Thus every such quadratic form comes from "specializing".

Considering now the fundamental ideals, the Milnor conjectures yield an isomorphism $I^n / I^{n+1} \cong H^n(k; \mu_2)$, so there is a SES

$$1 \to I^{n+1} \to I^n \xrightarrow{e_n} H^n(k; \mu_2) \to 1.$$

Thus a quadratic form $q$ of dimension $d$ in $I^{n+1}$ is equivalent to $q \in I^n$ such that $e_n(q) = 0$.

## 2.2 Canonical Dimension

**Definition 2.2.1** (Canonical Dimension)
This is a generalization of essdim. Letting $k_{/k_0}$, suppose $F : \mathsf{Field}_{/k} \to \mathsf{Set}_+$ is a functor now

from extensions of $k$ (not $k_0$) into pointed sets. For $\alpha \in F(k)$, define a new functor

$$\check{F}_\alpha(L) := \begin{cases} \emptyset & \alpha_L \neq \mathrm{pt} \\ \{\mathrm{pt}\} & \alpha_L = \mathrm{pt}, \end{cases}$$

and define the **canonical dimension**

$$\mathrm{candim}(\alpha) = \mathrm{essdim}(\check{F}(\alpha)).$$

**Remark 2.2.2:** This measures how many parameters are needed to trivialize/split $\alpha$. To have $\mathrm{candim}(\alpha) \leq r$ means that if $\alpha = \mathrm{pt}$ means the following: if $\alpha_L = \mathrm{pt}$ then there exists an $E$ with $k \subseteq E \subseteq L$ with $[E : k]_{\mathrm{tr}} \leq r$ such that $\alpha_E = \mathrm{pt}$.

**Definition 2.2.3** (Generic splitting scheme)
Given $F$ as above and $\alpha \in F(k)$, we say an $X \in \mathsf{Sch}_{/k}$ is a **generic splitting scheme** for $\alpha$ if

$$\alpha_L = 0 \iff X(L) \neq \emptyset.$$

**Remark 2.2.4:** So this encodes triviality of $\alpha$ into polynomial equations.

**Example 2.2.5** ***(?):*** If $X$ is a generic splitting scheme for $\alpha$ finite type over $L$ implies $\mathrm{candim}(\alpha) \leq \dim X$.

**Question 2.2.6**
Does there exists a finite type generic splitting scheme for cohomology classes in $H^i(k; \mu_\ell^{\otimes j})$?

**Remark 2.2.7:** We do know this in special cases:

- $i = 1$: Yes, these are etale algebras, so finite schemes over $k$.
- $i = 2$: Yes, Danny shows these exist for all twists.
  - $j = 1$: Classical, these are Severi-Brauer varieties.
- For symbols, $i = 3, j = 2, \ell$ a prime: see Merkurjev-Suslin
- For symbols, $i = 4, j = 3, \ell = 3$: see Albert algebras
- For symbols, $\ell$ prime: this can be done up to prime-to-$\ell$ extensions, see Rost's "Norm Varieties". Related to Bloch-Kato conjecture.
- For symbols, $\ell = 2$: see Pfister quadrics.

**Remark 2.2.8:** Upshot: if there exists generic splitting schemes for classes in $H^i(k; \mu_2)$ for $i \geq 3$, one could bound Pfister numbers and thus essdim. Write $\mathcal{I}_d^n(k)$ to be the set of quadratic forms of dimension $d$ in $I^n$, then $\mathrm{essdim}(\mathcal{I}_d^n) < \infty$ would imply that if $q \in \mathcal{I}_d^n(k)$ for $k \supseteq k_0$ then $q$ would be defined over some $L_{/k_0}$ with $[L : k_0]_{\mathrm{tr}} < \infty$.

If we knew that $\mathrm{ddim}\, k_0 < \infty$, e.g. if $k_0$ contains a finite field, this yields a bound on $\mathrm{ddim}\, L$ and thus on $\mathrm{cohdim}\, L$. If there is a versal element in $\alpha \in \mathcal{I}_d^n$, then $\alpha$ needs some finite number $m$ of

Pfister forms to be written. Everything else is a specialization of $\alpha$, so the length $m$ will almost give an upper bound.

**Warning 2.2.9**

This may seem like a correct argument, but it is not! A problem arises where you may have denominators – specialization can get worse, but only a finite number of times, which is how the actual argument goes.

**Remark 2.2.10:** If you knew the essential dimensions were finite with some given bound, and some general period-index conjecture were known, these would give bounds on symbol length in $H^i(L; \mu_2)$. There's an argument pushing things into higher powers of the fundamental ideal, thus higher degree cohomology, which disappear at some point and yield a bound. Motives enter the picture in terms of the tools used to attack these problems.