Qual Real Analysis

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1 Preface

Note: linking directly to sections doesn't seem to work yet. Just ctrl-F and search the page for the relevant year.

I'd like to extend my gratitude to Peter Woolfitt for supplying many solutions and checking many proofs of the rest in problem sessions. Many other solutions contain input and ideas from other graduate students and faculty members at UGA, along with questions and answers posted on Math Stack Exchange or Math Overflow.

2 Undergraduate Analysis: Uniform Convergence

2.1 Fall 2018.1

Let $f(x) = \dfrac{1}{x}$. Show that f is uniformly continuous on $(1, \infty)$ but not on $(0, \infty)$.

Concept review omitted.

Strategy omitted.

Solution omitted.

2.2 Fall 2017.1

Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Describe the intervals on which f does and does not converge uniformly.

Concept review omitted.

Strategy omitted.

Solution omitted.

2.3 Spring 2017.4

Let $f(x, y)$ on $[-1, 1]^2$ be defined by

$$f(x, y) = \begin{cases} \dfrac{xy}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Determine if f is integrable.

Concept review omitted.

Solution omitted.

2.4 Fall 2014.1

Let $\{f_n\}$ be a sequence of continuous functions such that $\sum f_n$ converges uniformly.

Prove that $\sum f_n$ is also continuous.

Concept review omitted.

Solution omitted.

2.5 Spring 2015.1

Let (X, d) and (Y, ρ) be metric spaces, $f : X \to Y$, and $x_0 \in X$.

Prove that the following statements are equivalent:

1. For every $\varepsilon > 0 \quad \exists \delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$.
2. The sequence $\{f(x_n)\}_{n=1}^{\infty} \to f(x_0)$ for every sequence $\{x_n\} \to x_0$ in X.

Concept review omitted.

Solution omitted.

2.6 Fall 2014.2

Let I be an index set and $\alpha : I \to (0, \infty)$.

a. Show that

$$\sum_{i\in I} a(i) := \sup_{\substack{J\subset I\\ J \text{ finite}}} \sum_{i\in J} a(i) < \infty \implies I \text{ is countable.}$$

b. Suppose $I = \mathbb{Q}$ and $\sum_{q\in\mathbb{Q}} a(q) < \infty$. Define

$$f(x) := \sum_{\substack{q\in\mathbb{Q}\\ q\leq x}} a(q).$$

Show that f is continuous at $x \iff x \notin \mathbb{Q}$.

Concept review omitted.

Solution omitted.

3 | General Analysis

3.1 Fall 2021.1

Problem 3.1.1 (?)
Let $\{x_n\}_{n-1}^{\infty}$ be a sequence of real numbers such that $x_1 > 0$ and

$$x_{n+1} = 1 - (2 + x_n)^{-1} = \frac{1 + x_n}{2 + x_n}.$$

Prove that the sequence $\{x_n\}$ converges, and find its limit.

Solution omitted.

3.2 Fall 2020.1

Problem 3.2.1 (?)

Show that if x_n is a decreasing sequence of positive real numbers such that $\sum_{n=1}^{\infty} x_n$ converges, then

$$\lim_{n\to\infty} n x_n = 0.$$

Solution omitted.

3.3 Spring 2020.1

Prove that if $f : [0,1] \to \mathbb{R}$ is continuous then

$$\lim_{k\to\infty} \int_0^1 k x^{k-1} f(x)\, dx = f(1).$$

Concept review omitted.

Solution omitted.

3.4 Fall 2019.1

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

a. Prove that if $\lim_{n\to\infty} a_n = 0$, then

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

b. Prove that if $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges, then

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = 0$$

Solution omitted.

3.5 Fall 2018.4

Let $f \in L^1([0,1])$. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)|\sin nx| \, dx = \frac{2}{\pi} \int_0^1 f(x) \, dx$$

Hint: Begin with the case that f is the characteristic function of an interval.

Ask someone to check the last approximation part.

#todo

Solution omitted.

3.6 Fall 2017.4

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

a. Show that $f_n \to 0$ pointwise but not uniformly on $[0,1]$.

b. Show that

$$\lim_{n\to\infty} \int_0^1 n(1-x)^n \sin x \, dx = 0$$

Hint for (a): Consider the maximum of f_n.

Solution omitted.

3.7 Spring 2017.3

Let

$$f_n(x) = ae^{-nax} - be^{-nbx} \quad \text{where } 0 < a < b.$$

Show that

a. $\sum_{n=1}^{\infty} |f_n|$ is not in $L^1([0,\infty), m)$

Hint: $f_n(x)$ has a root x_n.

b.

$$\sum_{n=1}^{\infty} f_n \text{ is in } L^1([0,\infty), m) \quad \text{and} \quad \int_0^\infty \sum_{n=1}^{\infty} f_n(x)\, dm = \ln \frac{b}{a}$$

Not complete.

Add concepts.

Walk through.

Solution omitted.

3.8 Fall 2016.1

Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that f converges to a differentiable function on $(1, \infty)$ and that

$$f'(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n^x}\right)'.$$

Hint:

$$\left(\frac{1}{n^x}\right)' = -\frac{1}{n^x} \ln n$$

Add concepts.

Solution omitted.

3.9 Fall 2016.5

Let $\varphi \in L^\infty(\mathbb{R})$. Show that the following limit exists and satisfies the equality

$$\lim_{n\to\infty} \left(\int_{\mathbb{R}} \frac{|\varphi(x)|^n}{1+x^2}\, dx \right)^{\frac{1}{n}} = \|\varphi\|_\infty.$$

Add concepts.

Solution omitted.

3.10 Fall 2016.6

Let $f, g \in L^2(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)g(x+n)\,dx = 0$$

Rewrite solution.

Concept review omitted.

Solution omitted.

3.11 Spring 2016.1

For $n \in \mathbb{N}$, define

$$e_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad E_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

Show that $e_n < E_n$, and prove Bernoulli's inequality:

$$(1+x)^n \geq 1 + nx \qquad -1 < x < \infty, \; n \in \mathbb{N}.$$

Use this to show the following:

1. The sequence e_n is increasing.
2. The sequence E_n is decreasing.
3. $2 < e_n < E_n < 4$.
4. $\lim_{n\to\infty} e_n = \lim_{n\to\infty} E_n$.

3.12 Fall 2015.1

Define

$$f(x) = c_0 + c_1 x^1 + c_2 x^2 + \ldots + c_n x^n \text{ with } n \text{ even and } c_n > 0.$$

Show that there is a number x_m such that $f(x_m) \leq f(x)$ for all $x \in \mathbb{R}$.

3.13 Spring 2014.2

Let $\{a_n\}$ be a sequence of real numbers such that

$$\{b_n\} \in \ell^2(\mathbb{N}) \implies \sum a_n b_n < \infty.$$

Show that $\sum a_n^2 < \infty$.

Note: Assume a_n, b_n are all non-negative.

Have someone check!

Solution omitted.

4 | Measure Theory: Sets

4.1 Fall 2021.3

Recall that a set $E \subset \mathbb{R}^d$ is measurable if for every $c > 0$ there is an open set $U \subseteq \mathbb{R}^d$ such that $m^*(U \setminus E) < \epsilon$.

a. Prove that if E is measurable then for all $\epsilon > 0$ there exists an elementary set F, such that $m(E \Delta F) < \epsilon$.

   Here $m(E)$ denotes the Lebesgue measure of E, a set F is called elementary if it is a finite union of rectangles and $E \Delta F$ denotes the symmetric difference of the sets E and F.

b. Let $E \subset \mathbb{R}$ be a measurable set, such that $0 < m(E) < \infty$. Use part (a) to show that

$$\lim_{n \to \infty} \int_E \sin(nt)\,dt = 0$$

4.2 Spring 2020.2

Let m_* denote the Lebesgue outer measure on $\mathbb{R}$.

a.. Prove that for every $E \subseteq \mathbb{R}$ there exists a Borel set B containing E such that

$$m_*(B) = m_*(E).$$

b.. Prove that if $E \subseteq \mathbb{R}$ has the property that

$$m_*(A) = m_*(A \bigcap E) + m_*(A \bigcap E^c)$$

for every set $A \subseteq \mathbb{R}$, then there exists a Borel set $B \subseteq \mathbb{R}$ such that $E = B \setminus N$ with $m_*(N) = 0$.

Be sure to address the case when $m_*(E) = \infty$.

Concept review omitted.

Solution omitted.

4.3 Fall 2019.3.

Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) = 1$ and $\{B_n\}_{n=1}^\infty$ be a sequence of $\mathcal{B}$-measurable subsets of X, and

$$B := \left\{ x \in X \;\middle|\; x \in B_n \text{ for infinitely many } n \right\}.$$

a. Argue that B is also a $\mathcal{B}$-measurable subset of X.

b. Prove that if $\displaystyle\sum_{n=1}^\infty \mu(B_n) < \infty$ then $\mu(B) = 0$.

c. Prove that if $\displaystyle\sum_{n=1}^\infty \mu(B_n) = \infty$ **and** the sequence of set complements $\{B_n^c\}_{n=1}^\infty$ satisfies

$$\mu\left(\bigcap_{n=k}^{K} B_n^c\right) = \prod_{n=k}^{K} \left(1 - \mu\left(B_n\right)\right)$$

for all positive integers k and K with $k < K$, then $\mu(B) = 1$.

Hint: Use the fact that $1 - x \leq e^{-x}$ for all x.

Concept review omitted.

Solution omitted.

4.4 Spring 2019.2

Let $\mathcal{B}$ denote the set of all Borel subsets of $\mathbb{R}$ and $\mu : \mathcal{B} \to [0, \infty)$ denote a finite Borel measure on $\mathbb{R}$.

a. Prove that if $\{F_k\}$ is a sequence of Borel sets for which $F_k \supseteq F_{k+1}$ for all k, then

$$\lim_{k\to\infty} \mu(F_k) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right)$$

b. Suppose μ has the property that $\mu(E) = 0$ for every $E \in \mathcal{B}$ with Lebesgue measure $m(E) = 0$. Prove that for every $\epsilon > 0$ there exists $\delta > 0$ so that if $E \in \mathcal{B}$ with $m(E) < \delta$, then $\mu(E) < \varepsilon$.

Concept review omitted.

Strategy omitted.

Solution omitted.

4.5 Fall 2018.2

Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Show that there is a Borel set $B \subset E$ such that $m(E \setminus B) = 0$.

Move this to review notes to clean things up.

What a mess, redo!!

Concept review omitted.

Solution omitted.

4.6 Spring 2018.1

Define

$$E := \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N} \right\}.$$

Prove that $m(E) = 0$.

Concept review omitted.

Solution omitted.

4.7 Fall 2017.2

Let $f(x) = x^2$ and $E \subset [0, \infty) := \mathbb{R}^+$.

1. Show that

$$m^*(E) = 0 \iff m^*(f(E)) = 0.$$

2. Deduce that the map

$$\begin{aligned} \varphi : \mathcal{L}(\mathbb{R}^+) &\to \mathcal{L}(\mathbb{R}^+) \\ E &\mapsto f(E) \end{aligned}$$

is a bijection from the class of Lebesgue measurable sets of $[0, \infty)$ to itself.

Walk through.

Solution omitted.

4.8 Spring 2017.1

Let K be the set of numbers in $[0, 1]$ whose decimal expansions do not use the digit 4.

We use the convention that when a decimal number ends with 4 but all other digits are different from 4, we replace the digit 4 *with* $399\cdots$. *For example,* $0.8754 = 0.8753999\cdots$.

Show that K is a compact, nowhere dense set without isolated points, and find the Lebesgue measure $m(K)$.

Concept review omitted.

Solution omitted.

4.9 Spring 2017.2

a. Let μ be a measure on a measurable space $(X, \mathcal{M})$ and f a positive measurable function.

Define a measure λ by

$$\lambda(E) := \int_E f \, d\mu, \quad E \in \mathcal{M}$$

Show that for g any positive measurable function,

$$\int_X g \, d\lambda = \int_X fg \, d\mu$$

b. Let $E \subset \mathbb{R}$ be a measurable set such that

$$\int_E x^2 \, dm = 0.$$

Show that $m(E) = 0$.

Concept review omitted.

Solution omitted.

4.10 Fall 2016.4

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\{E_n\} \subset \mathcal{M}$ satisfies

$$\lim_{n \to \infty} \mu\left(X \backslash E_n\right) = 0.$$

Define

$$G := \left\{ x \in X \;\middle|\; x \in E_n \text{ for only finitely many } n \right\}.$$

Show that $G \in \mathcal{M}$ and $\mu(G) = 0$.

Add concepts.

Solution omitted.

4.11 Spring 2016.3

Let f be Lebesgue measurable on $\mathbb{R}$ and $E \subset \mathbb{R}$ be measurable such that

$$0 < A = \int_E f(x)dx < \infty.$$

Show that for every $0 < t < 1$, there exists a measurable set $E_t \subset E$ such that

$$\int_{E_t} f(x)dx = tA.$$

4.12 Spring 2016.5

Let $(X, \mathcal{M}, \mu)$ be a measure space. For $f \in L^1(\mu)$ and $\lambda > 0$, define

$$\varphi(\lambda) = \mu(\{x \in X | f(x) > \lambda\}) \quad \text{and} \quad \psi(\lambda) = \mu(\{x \in X | f(x) < -\lambda\})$$

Show that φ, ψ are Borel measurable and

$$\int_X |f| \, d\mu = \int_0^\infty [\varphi(\lambda) + \psi(\lambda)] \, d\lambda$$

4.13 Spring 2016.2

Let $0 < \lambda < 1$ and construct a Cantor set C_λ by successively removing middle intervals of length λ.

Prove that $m(C_\lambda) = 0$.

4.14 Fall 2015.2

Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable.

1. Show that there is a sequence of simple functions $s_n(x)$ such that $s_n(x) \to f(x)$ for all $x \in \mathbb{R}$.
2. Show that there is a Borel measurable function g such that $g = f$ almost everywhere.

4.15 Spring 2015.3

Let μ be a finite Borel measure on $\mathbb{R}$ and $E \subset \mathbb{R}$ Borel. Prove that the following statements are equivalent:

1. $\forall \varepsilon > 0$ there exists G open and F closed such that
$$F \subseteq E \subseteq G \quad \text{and} \quad \mu(G \setminus F) < \varepsilon.$$
2. There exists a $V \in G_\delta$ and $H \in F_\sigma$ such that
$$H \subseteq E \subseteq V \quad \text{and} \quad \mu(V \setminus H) = 0$$

4.16 Spring 2014.3

Let $f : \mathbb{R} \to \mathbb{R}$ and suppose

$$\forall x \in \mathbb{R}, \quad f(x) \geq \limsup_{y \to x} f(y)$$

Prove that f is Borel measurable.

4.17 Spring 2014.4

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose f is a measurable function on X. Show that

$$\lim_{n \to \infty} \int_X f^n \, d\mu = \begin{cases} \infty & \text{or} \\ \mu(f^{-1}(1)), & \end{cases}$$

and characterize the collection of functions of each type.

5 Measure Theory: Functions

5.1 Spring 2021.1

Problem 5.1.1 (Spring 2021, 1)
Let $(X, \mathcal{M}, \mu)$ be a measure space and let $E_n \in \mathcal{M}$ be a measurable set for $n \geq 1$. Let

$f_n := \chi_{E_n}$ be the indicator function of the set E and show that

a. $f_n \overset{n\to\infty}{\to} 1$ uniformly $\iff$ there exists $N \in \mathbb{N}$ such that $E_n = X$ for all $n \geq N$.

b. $f_n(x) \overset{n\to\infty}{\to} 1$ for almost every x $\iff$

$$\mu\left(\bigcap_{n\geq 0}\bigcup_{k\geq n}(X \setminus E_k)\right) = 0.$$

Solution omitted.

5.2 Spring 2021.3

Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let $\{f_n\}_{n=1}^{\infty} \subseteq L^1(X, \mu)$. Suppose $f \in L^1(X, \mu)$ such that $f_n(x) \overset{n\to\infty}{\to} f(x)$ for almost every $x \in X$. Prove that for every $\varepsilon > 0$ there exists $M > 0$ and a set $E \subseteq X$ such that $\mu(E) \leq \varepsilon$ and $|f_n(x)| \leq M$ for all $x \in X \setminus E$ and all $n \in \mathbb{N}$.

5.3 Fall 2020.2

a. Let $f : \mathbb{R} \to \mathbb{R}$. Prove that

$$f(x) \leq \liminf_{y\to x} f(y) \text{ for each } x \in \mathbb{R} \iff \left\{x \in \mathbb{R} \;\middle|\; f(x) > a\right\} \text{ is open for all } a \in \mathbb{R}$$

b. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called *lower semi-continuous* iff it satisfies either condition in part (a) above.

Prove that if $\mathcal{F}$ is any family of lower semi-continuous functions, then

$$g(x) = \sup\left\{f(x) \;\middle|\; f \in \mathcal{F}\right\}$$

is Borel measurable.

Note that $\mathcal{F}$ need not be a countable family.

5.4 Fall 2016.2

Let $f, g : [a, b] \to \mathbb{R}$ be measurable with

$$\int_a^b f(x)\, dx = \int_a^b g(x)\, dx.$$

Show that either

1. $f(x) = g(x)$ almost everywhere, or
2. There exists a measurable set $E \subset [a, b]$ such that

$$\int_E f(x)\, dx > \int_E g(x)\, dx$$

Concept review omitted.

Strategy omitted.

Solution omitted.

5.5 Spring 2016.4

Let $E \subset \mathbb{R}$ be measurable with $m(E) < \infty$. Define

$$f(x) = m(E \cap (E + x)).$$

Show that

1. $f \in L^1(\mathbb{R})$.
2. f is uniformly continuous.
3. $\lim_{|x| \to \infty} f(x) = 0$.

Hint:

$$\chi_{E \cap (E+x)}(y) = \chi_E(y)\chi_E(y - x)$$

6 Integrals: Convergence

6.1 Fall 2020.3

Problem 6.1.1 (?)
Let f be a non-negative Lebesgue measurable function on $[1, \infty)$.

a. Prove that

$$1 \leq \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b \frac{1}{f(x)}\,dx\right)$$

for any $1 \leq a < b < \infty$.

b. Prove that if f satisfies

$$\int_1^t f(x)\,dx \leq t^2 \log(t)$$

for all $t \in [1, \infty)$, then

$$\int_1^\infty \frac{1}{f(x)}\,dx = \infty.$$

Hint: write

$$\int_1^\infty \frac{1}{f(x)}\,dx = \sum_{k=0}^{\infty}\int_{2^k}^{2^{k+1}} \frac{1}{f(x)}\,dx.$$

Solution omitted.

6.2 Spring 2021.2

Problem 6.2.1 (?)
Calculate the following limit, justifying each step of your calculation:

$$L := \lim_{n\to\infty}\int_0^n \frac{\cos\left(\frac{x}{n}\right)}{x^2+\cos\left(\frac{x}{n}\right)}\,dx.$$

Solution omitted.

6.3 Spring 2021.5

Problem 6.3.1 (?)
Let $f_n \in L^2([0,1])$ for $n \in \mathbb{N}$, and assume that

- $\|f_n\|_2 \leq n^{\frac{-51}{100}}$ for all $n \in \mathbb{N}$,

- $\widehat{f}_n$ is supported in the interval $[2^n, 2^{n+1}]$, so

$$\widehat{f}_n(\xi) := \int_0^1 f_n(x)e^{2\pi i\xi\cdot x}\,dx = 0 \qquad \text{for } \xi \notin [2^n, 2^{n+1}].$$

Prove that $\sum_{n\in\mathbb{N}} f_n$ converges in the Hilbert space $L^2([0,1])$.

Hint: Plancherel's identity may be helpful.

Warning 6.3.1
Although this mentions Plancherel, probably what is needed is Parseval's identity:

$$\sum_{k\in\mathbb{Z}} \left|\widehat{f}(k)\right|^2 = \int_0^1 |f(x)|^2\,dx.$$

6.4 Fall 2019.2

Prove that

$$\left|\frac{d^n}{dx^n}\frac{\sin x}{x}\right| \leq \frac{1}{n}$$

for all $x \neq 0$ and positive integers n.

Hint: Consider $\int_0^1 \cos(tx)dt$

Solution omitted.

6.5 Spring 2020.5

Compute the following limit and justify your calculations:

$$\lim_{n\to\infty}\int_0^n \left(1+\frac{x^2}{n}\right)^{-(n+1)}\,dx.$$

Not finished, flesh out.

Walk through.

Solution omitted.

6.6 Spring 2019.3

Let $\{f_k\}$ be any sequence of functions in $L^2([0,1])$ satisfying $\|f_k\|_2 \leq M$ for all $k \in \mathbb{N}$.

Prove that if $f_k \to f$ almost everywhere, then $f \in L^2([0,1])$ with $\|f\|_2 \leq M$ and

$$\lim_{k\to\infty} \int_0^1 f_k(x)dx = \int_0^1 f(x)dx$$

Hint: Try using Fatou's Lemma to show that $\|f\|_2 \leq M$ and then try applying Egorov's Theorem.

Solution omitted.

6.7 Fall 2018.6

Compute the following limit and justify your calculations:

$$\lim_{n\to\infty} \int_1^n \frac{dx}{(1+\frac{x}{n})^n \sqrt[n]{x}}$$

Add concepts.

Solution omitted.

6.8 Fall 2018.3

Suppose $f(x)$ and $xf(x)$ are integrable on $\mathbb{R}$. Define F by

$$F(t) := \int_{-\infty}^{\infty} f(x)\cos(xt)dx$$

Show that

$$F'(t) = -\int_{-\infty}^{\infty} xf(x)\sin(xt)dx.$$

Walk through.

Solution omitted.

6.9 Spring 2018.5

Suppose that

- $f_n, f \in L^1$,
- $f_n \to f$ almost everywhere, and
- $\int |f_n| \to \int |f|$.

Show that $\int f_n \to \int f$.

Solution omitted.

6.10 Spring 2018.2

Let

$$f_n(x) := \frac{x}{1+x^n}, \quad x \geq 0.$$

a. Show that this sequence converges pointwise and find its limit. Is the convergence uniform on $[0, \infty)$?

b. Compute

$$\lim_{n\to\infty} \int_0^\infty f_n(x)dx$$

Add concepts.

Solution omitted.

6.11 Fall 2016.3

Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{x\to 0} \int_{\mathbb{R}} |f(y-x) - f(y)|\, dy = 0$$

Missing some stuff.

Solution omitted.

6.12 Fall 2015.3

Problem 6.12.1 (?)
Compute the following limit:

$$\lim_{n\to\infty} \int_1^n \frac{ne^{-x}}{1+nx^2} \sin\left(\frac{x}{n}\right) dx$$

Solution omitted.

6.13 Fall 2015.4

Let $f : [1, \infty) \to \mathbb{R}$ such that $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Show that the following limit exists and satisfies the equality

$$\lim_{x\to\infty} f(x) \leq 1 + \frac{\pi}{4}$$

7 | Integrals: Approximation

7.1 Fall 2021.2

Problem 7.1.1 (?) a. Let $F \subset \mathbb{R}$ be closed, and define

$$\delta_F(y) := \inf_{x\in F} |x-y|.$$

For $y \notin F$, show that

$$\int_F |x-y|^{-2} dx \leq \frac{2}{\delta_F(y)},$$

b. Let $F \subset \mathbb{R}$ be a closed set whose complement has finite measure, i.e. $m(\mathbb{R} \setminus F) < \infty$. Define the function

$$I(x) := \int_{\mathbb{R}} \frac{\delta_F(y)}{|x-y|^2} dy$$

Prove that $I(x) = \infty$ if $x \notin F$, however $I(x) < \infty$ for almost every $x \in F$.

Hint: investigate $\int_F I(x)dx$.

Solution omitted.

Solution omitted.

7.2 Spring 2018.3

Let f be a non-negative measurable function on $[0,1]$.

Show that

$$\lim_{p\to\infty}\left(\int_{[0,1]} f(x)^p dx\right)^{\frac{1}{p}} = \|f\|_\infty.$$

Concept review omitted.

Solution omitted.

7.3 Spring 2018.4

Let $f \in L^2([0,1])$ and suppose

$$\int_{[0,1]} f(x)x^n dx = 0 \text{ for all integers } n \geq 0.$$

Show that $f = 0$ almost everywhere.

Concept review omitted.

Solution omitted.

7.4 Spring 2015.2

Let $f : \mathbb{R} \to \mathbb{C}$ be continuous with period 1. Prove that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(t)dt \quad \forall \alpha \in \mathbb{R}\setminus\mathbb{Q}.$$

Hint: show this first for the functions $f(t) = e^{2\pi ikt}$ for $k \in \mathbb{Z}$.

7.5 Fall 2014.4

Problem 7.5.1 (?)
Let $g \in L^\infty([0,1])$ Prove that

$$\int_{[0,1]} f(x)g(x)\,dx = 0 \quad \text{for all continuous } f : [0,1] \to \mathbb{R} \implies g(x) = 0 \text{ almost everywhere.}$$

Concept review omitted.

Solution omitted.

8 | L^1

8.1 Spring 2021.4

Let f, g be Lebesgue integrable on $\mathbb{R}$ and let $g_n(x) := g(x-n)$. Prove that

$$\lim_{n\to\infty} \|f + g_n\|_1 = \|f\|_1 + \|g\|_1.$$

Concept review omitted.

Solution omitted.

8.2 Fall 2020.4

Problem 8.2.1 (?)
Prove that if $xf(x) \in L^1(\mathbb{R})$, then

$$F(y) := \int f(x)\cos(yx)\,dx$$

defines a C^1 function.

Solution omitted.

8.3 Spring 2020.3

a. Prove that if $g \in L^1(\mathbb{R})$ then

$$\lim_{N \to \infty} \int_{|x| \geq N} |f(x)|\, dx = 0,$$

and demonstrate that it is not necessarily the case that $f(x) \to 0$ as $|x| \to \infty$.

b. Prove that if $f \in L^1([1, \infty])$ and is decreasing, then $\lim_{x \to \infty} f(x) = 0$ and in fact $\lim_{x \to \infty} x f(x) = 0$.

c. If $f : [1, \infty) \to [0, \infty)$ is decreasing with $\lim_{x \to \infty} x f(x) = 0$, does this ensure that $f \in L^1([1, \infty))$?

Concept review omitted.

Solution omitted.

Solution omitted.

Solution omitted.

8.4 Fall 2019.5

a. Show that if f is continuous with compact support on $\mathbb{R}$, then

$$\lim_{y \to 0} \int_{\mathbb{R}} |f(x-y) - f(x)| dx = 0$$

b. Let $f \in L^1(\mathbb{R})$ and for each $h > 0$ let

$$\mathcal{A}_h f(x) := \frac{1}{2h} \int_{|y| \leq h} f(x-y) dy$$

- Prove that $\|\mathcal{A}_h f\|_1 \leq \|f\|_1$ for all $h > 0$.
- Prove that $\mathcal{A}_h f \to f$ in $L^1(\mathbb{R})$ as $h \to 0^+$.

Walk through.

Concept review omitted.

Solution omitted.

Remark 8.4.1: This works for arbitrary $f \in L^1$, using approximation by continuous functions with compact support:

- Choose $g \in C_c^0$ such that $\|f - g\|_1 \to 0$.
- By translation invariance, $\|\tau_h f - \tau_h g\|_1 \to 0$.
- Write

$$\begin{aligned}\|\tau f - f\|_1 &= \|\tau_h f - g + g - \tau_h g + \tau_h g - f\|_1 \\ &\leq \|\tau_h f - \tau_h g\| + \|g - f\| + \|\tau_h g - g\| \\ &\to \|\tau_h g - g\|,\end{aligned}$$

so it suffices to show that $\|\tau_h g - g\| \to 0$.

8.5 Fall 2017.3

Let

$$S = \operatorname*{span}_{\mathbb{C}} \left\{ \chi_{(a,b)} \,\middle|\, a, b \in \mathbb{R} \right\},$$

the complex linear span of characteristic functions of intervals of the form (a, b).

Show that for every $f \in L^1(\mathbb{R})$, there exists a sequence of functions $\{f_n\} \subset S$ such that

$$\lim_{n \to \infty} \|f_n - f\|_1 = 0$$

Concept review omitted.

Solution omitted.

8.6 Spring 2015.4

Problem 8.6.1 (?)
Define

$$f(x, y) := \begin{cases} \dfrac{x^{1/3}}{(1 + xy)^{3/2}} & \text{if } 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Carefully show that $f \in L^1(\mathbb{R}^2)$.

Solution omitted.

8.7 Fall 2014.3

Problem 8.7.1 (?)
Let $f \in L^1(\mathbb{R})$. Show that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } \qquad m(E) < \delta \implies \int_E |f(x)|\, dx < \varepsilon$$

Solution omitted.

Solution omitted.

8.8 Spring 2014.1

Problem 8.8.1 (?)

1. Give an example of a continuous $f \in L^1(\mathbb{R})$ such that $f(x) \not\to 0$ as$|x| \to \infty$.
2. Show that if f is *uniformly* continuous, then

$$\lim_{|x|\to\infty} f(x) = 0.$$

Solution omitted.

9 | Fubini-Tonelli

9.1 Spring 2021.6

Warning 9.1.1
This problem may be much harder than expected. Recommended skip.

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a measurable function and for $x \in \mathbb{R}$ define the set

$$E_x := \left\{ y \in \mathbb{R} \;\middle|\; \mu\left(z \in \mathbb{R} \;\middle|\; f(x,z) = f(x,y) \right) > 0 \right\}.$$

Show that the following set is a measurable subset of $\mathbb{R} \times \mathbb{R}$:

$$E := \bigcup_{x \in \mathbb{R}} \{x\} \times E_x.$$

Hint: consider the measurable function $h(x,y,z) := f(x,y) - f(x,z)$.

9.2 Fall 2021.4

Problem 9.2.1 (?)
Let f be a measurable function on $\mathbb{R}$. Show that the graph of f has measure zero in $\mathbb{R}^2$.

Solution omitted.

9.3 Spring 2020.4

Let $f, g \in L^1(\mathbb{R})$. Argue that $H(x,y) := f(y)g(x-y)$ defines a function in $L^1(\mathbb{R}^2)$ and deduce from this fact that

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x-y)\,dy$$

defines a function in $L^1(\mathbb{R})$ that satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

Strategy omitted.

Concept review omitted.

Solution omitted.

9.4 Spring 2019.4

Let f be a non-negative function on $\mathbb{R}^n$ and $\mathcal{A} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq f(x)\}$.

Prove the validity of the following two statements:

a. f is a Lebesgue measurable function on $\mathbb{R}^n$ $\iff$ $\mathcal{A}$ is a Lebesgue measurable subset of $\mathbb{R}^{n+1}$

b. If f is a Lebesgue measurable function on $\mathbb{R}^n$, then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)dx = \int_0^\infty m\left(\{x \in \mathbb{R}^n : f(x) \geq t\}\right) dt$$

Concept review omitted.

Solution omitted.

9.5 Fall 2018.5

Let $f \geq 0$ be a measurable function on $\mathbb{R}$. Show that

$$\int_{\mathbb{R}} f = \int_0^\infty m(\{x : f(x) > t\})dt$$

Concept review omitted.

Solution omitted.

9.6 Fall 2015.5

Problem 9.6.1 (?)
Let $f, g \in L^1(\mathbb{R})$ be Borel measurable.

- Show that
  - The function

    $$F(x, y) := f(x - y)g(y)$$

    is Borel measurable on $\mathbb{R}^2$, and
  - For almost every $x \in \mathbb{R}$, the function $f(x - y)g(y)$ is integrable with respect to y on $\mathbb{R}$.
- Show that $f * g \in L^1(\mathbb{R})$ and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

Solution omitted.

9.7 Spring 2014.5

Problem 9.7.1 (?)
Let $f, g \in L^1([0,1])$ and for all $x \in [0,1]$ define

$$F(x) := \int_0^x f(y)\,dy \quad \text{and} \quad G(x) := \int_0^x g(y)\,dy.$$

Prove that

$$\int_0^1 F(x)g(x)\,dx = F(1)G(1) - \int_0^1 f(x)G(x)\,dx$$

10 | L^2 and Fourier Analysis

10.1 Fall 2020.5

Problem 10.1.1 (?)
Suppose $\varphi \in L^1(\mathbb{R})$ with

$$\int \varphi(x)\,dx = \alpha.$$

For each $\delta > 0$ and $f \in L^1(\mathbb{R})$, define

$$A_\delta f(x) := \int f(x-y)\delta^{-1}\varphi\left(\delta^{-1}y\right)\,dy.$$

a. Prove that for all $\delta > 0$,

$$\|A_\delta f\|_1 \leq \|\varphi\|_1 \|f\|_1.$$

b. Prove that

$$A_\delta f \to \alpha f \text{ in } L^1(\mathbb{R}) \quad \text{as} \quad \delta \to 0^+.$$

Hint: you may use without proof the fact that for all $f \in L^1(\mathbb{R})$,

$$\lim_{y \to 0} \int_{\mathbb{R}} |f(x-y) - f(x)|\,dx = 0.$$

Remark 10.1.1: See Folland 8.14.

Solution omitted.

Solution omitted.

10.2 Spring 2020.6

Problem 10.2.1 (?)

a. Show that

$$L^2([0,1]) \subseteq L^1([0,1]) \quad \text{and} \quad \ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z}).$$

b. For $f \in L^1([0,1])$ define

$$\widehat{f}(n) := \int_0^1 f(x) e^{-2\pi i n x}\, dx.$$

Prove that if $f \in L^1([0,1])$ and $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$ then

$$S_N f(x) := \sum_{|n| \leq N} \widehat{f}(n) e^{2\pi i n x}.$$

converges uniformly on $[0,1]$ to a continuous function g such that $g = f$ almost everywhere.

Hint: One approach is to argue that if $f \in L^1([0,1])$ with $\left\{\widehat{f}(n)\right\} \in \ell^1(\mathbb{Z})$ then $f \in L^2([0,1])$.

Concept review omitted.

Solution omitted.

Solution omitted.

Solution omitted.

10.3 Fall 2017.5

Let φ be a compactly supported smooth function that vanishes outside of an interval $[-N, N]$ such that $\displaystyle\int_{\mathbb{R}} \varphi(x)\, dx = 1$.

For $f \in L^1(\mathbb{R})$, define

$$K_j(x) \coloneqq j\varphi(jx), \qquad f * K_j(x) \coloneqq \int_{\mathbb{R}} f(x-y)K_j(y)\,dy$$

and prove the following:

1. Each $f * K_j$ is smooth and compactly supported.

2. $$\lim_{j\to\infty} \|f * K_j - f\|_1 = 0$$

Hint:

$$\lim_{y\to 0} \int_{\mathbb{R}} |f(x-y) - f(x)|dy = 0$$

Add concepts.

Solution omitted.

10.4 Spring 2017.5

Let $f, g \in L^2(\mathbb{R})$. Prove that the formula

$$h(x) \coloneqq \int_{-\infty}^{\infty} f(t)g(x-t)\,dt$$

defines a uniformly continuous function h on $\mathbb{R}$.

10.5 Spring 2015.6

Let $f \in L^1(\mathbb{R})$ and g be a bounded measurable function on $\mathbb{R}$.

1. Show that the convolution $f * g$ is well-defined, bounded, and uniformly continuous on $\mathbb{R}$.
2. Prove that one further assumes that $g \in C^1(\mathbb{R})$ with bounded derivative, then $f * g \in C^1(\mathbb{R})$ and

$$\frac{d}{dx}(f * g) = f * \left(\frac{d}{dx}g\right)$$

10.6 Fall 2014.5

1. Let $f \in C_c^0(\mathbb{R}^n)$, and show

$$\lim_{t \to 0} \int_{\mathbb{R}^n} |f(x+t) - f(x)| \, dx = 0.$$

2. Extend the above result to $f \in L^1(\mathbb{R}^n)$ and show that

$$f \in L^1(\mathbb{R}^n), \quad g \in L^\infty(\mathbb{R}^n) \quad \implies f * g \text{ is bounded and uniformly continuous.}$$

11 | Functional Analysis: General

11.1 Fall 2019.4

Let $\{u_n\}_{n=1}^\infty$ be an orthonormal sequence in a Hilbert space $\mathcal{H}$.

a. Prove that for every $x \in \mathcal{H}$ one has

$$\sum_{n=1}^\infty |\langle x, u_n \rangle|^2 \le \|x\|^2$$

b. Prove that for any sequence $\{a_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$ there exists an element $x \in \mathcal{H}$ such that

$$a_n = \langle x, \ u_n \rangle \text{ for all } n \in \mathbb{N}$$

and

$$\|x\|^2 = \sum_{n=1}^\infty |\langle x, u_n \rangle|^2$$

Concept review omitted.

Solution omitted.

11.2 Spring 2019.5

a. Show that $L^2([0,1]) \subseteq L^1([0,1])$ and argue that $L^2([0,1])$ in fact forms a dense subset of $L^1([0,1])$.

b. Let Λ be a continuous linear functional on $L^1([0,1])$.

Prove the Riesz Representation Theorem for $L^1([0,1])$ by following the steps below:

i. Establish the existence of a function $g \in L^2([0,1])$ which represents Λ in the sense that

$$\Lambda(f) = f(x)g(x)dx \text{ for all } f \in L^2([0,1]).$$

Hint: You may use, without proof, the Riesz Representation Theorem for $L^2([0,1])$.

ii. Argue that the g obtained above must in fact belong to $L^\infty([0,1])$ and represent Λ in the sense that

$$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}dx \quad \text{for all } f \in L^1([0,1])$$

with

$$\|g\|_{L^\infty([0,1])} = \|\Lambda\|_{L^1([0,1])^\vee}$$

Concept review omitted.

Solution omitted.

11.3 Spring 2016.6

Without using the Riesz Representation Theorem, compute

$$\sup\left\{\left|\int_0^1 f(x)e^x dx\right| \;\middle|\; f \in L^2([0,1],m),\ \|f\|_2 \leq 1\right\}$$

11.4 Spring 2015.5

Let $\mathcal{H}$ be a Hilbert space.

1. Let $x \in \mathcal{H}$ and $\{u_n\}_{n=1}^N$ be an orthonormal set. Prove that the best approximation to x in $\mathcal{H}$ by an element in $\operatorname{span}_{\mathbb{C}}\{u_n\}$ is given by

$$\widehat{x} := \sum_{n=1}^N \langle x,\ u_n\rangle u_n.$$

2. Conclude that finite dimensional subspaces of $\mathcal{H}$ are always closed.

11.5 Fall 2015.6

Let $f : [0, 1] \to \mathbb{R}$ be continuous. Show that

$$\sup \left\{ \|fg\|_1 \;\middle|\; g \in L^1[0,1],\ \|g\|_1 \leq 1 \right\} = \|f\|_\infty$$

11.6 Fall 2014.6

Let $1 \leq p, q \leq \infty$ be conjugate exponents, and show that

$$f \in L^p(\mathbb{R}^n) \implies \|f\|_p = \sup_{\|g\|_q = 1} \left| \int f(x)g(x)dx \right|$$

12 | Banach and Hilbert Spaces

12.1 Fall 2021.5

Consider the Hilbert space $\mathcal{H} = L^2([0, 1])$.

a. Prove that of $E \subset \mathcal{H}$ is closed and convex then E contains an element of smallest norm.

> *Hint: Show that if $\|f_n\|_2 \to \min\{f \in E : \|f\|_2\}$ then $\{f_n\}$ is a Cauchy sequence.*

b. Construct a non-empty closed subset $E \subset \mathcal{H}$ which does not contain an element of smallest norm.

12.2 Spring 2019.1

Let $C([0, 1])$ denote the space of all continuous real-valued functions on $[0, 1]$.

a. Prove that $C([0, 1])$ is complete under the uniform norm $\|f\|_u := \sup_{x\in[0,1]} |f(x)|$.

b. Prove that $C([0, 1])$ is not complete under the L^1-norm $\|f\|_1 = \int_0^1 |f(x)|\, dx$.

Add concepts.

Solution omitted.

12.3 Spring 2017.6

Show that the space $C^1([a, b])$ is a Banach space when equipped with the norm

$$\|f\| := \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |f'(x)| \,.$$

Add concepts.

Concept review omitted.

Solution omitted.

12.4 Fall 2017.6

Let X be a complete metric space and define a norm

$$\|f\| := \max\{|f(x)| : x \in X\}.$$

Show that $(C^0(\mathbb{R}), \|-\|)$ (the space of continuous functions $f : X \to \mathbb{R}$) is complete.

Add concepts.

Shouldn't this be a supremum? The max may not exist?

Review and clean up.

Solution omitted.

13 | Extras

Exercise 13.0.1 (?)
Compute the following limits:

- $\lim_{n\to\infty} \sum_{k\geq 1} \frac{1}{k^2} \sin^n(k)$
- $\lim_{n\to\infty} \sum_{k\geq 1} \frac{1}{k} e^{-k/n}$

Solution omitted.

Exercise 13.0.2 (?)
Let $(\Omega, \mathcal{B})$ be a measurable space with a Borel σ-algebra and $\mu_n : \mathcal{B} \to [0, \infty]$ be a σ-additive measure for each n. Show that the following map is again a σ-additive measure on $\mathcal{B}$:

$$\mu(B) \coloneqq \sum_{n\geq 1} \mu_n(B).$$

Solution omitted.

14 | Extra Problems: Measure Theory

14.1 Greatest Hits

- $\star$: Show that for $E \subseteq \mathbb{R}^n$, TFAE:

  1. E is measurable
  2. $E = H \cup Z$ here H is F_σ and Z is null
  3. $E = V \setminus Z'$ where $V \in G_\delta$ and Z' is null.

- $\star$: Show that if $E \subseteq \mathbb{R}^n$ is measurable then $m(E) = \sup \left\{ m(K) \;\middle|\; K \subset E \text{ compact} \right\}$ iff for all $\varepsilon > 0$ there exists a compact $K \subseteq E$ such that $m(K) \geq m(E) - \varepsilon$.

- $\star$: Show that cylinder functions are measurable, i.e. if f is measurable on $\mathbb{R}^s$, then $F(x, y) \coloneqq f(x)$ is measurable on $\mathbb{R}^s \times \mathbb{R}^t$ for any t.

- $\star$: Prove that the Lebesgue integral is translation invariant, i.e. if $\tau_h(x) = x + h$ then

$$\int \tau_h f = \int f.$$

- $\star$: Prove that the Lebesgue integral is dilation invariant, i.e. if $f_\delta(x) = \frac{f(\frac{x}{\delta})}{\delta^n}$ then $\int f_\delta = \int f$.
- $\star$: Prove continuity in L^1, i.e.

$$f \in L^1 \implies \lim_{h\to 0} \int |f(x+h) - f(x)| = 0.$$

- $\star$: Show that

$$f, g \in L^1 \implies f * g \in L^1 \quad \text{and} \quad \|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

- $\star$: Show that if $X \subseteq \mathbb{R}$ with $\mu(X) < \infty$ then

$$\|f\|_p \overset{p\to\infty}{\to} \|f\|_\infty.$$

14.2 Topology

- Show that every compact set is closed and bounded.
- Show that if a subset of a metric space is complete and totally bounded, then it is compact.
- Show that if K is compact and F is closed with K, F disjoint then $\operatorname{dist}(K, F) > 0$.

14.3 Continuity

- Show that a continuous function on a compact set is uniformly continuous.

14.4 Differentiation

- Show that if $f \in C^1(\mathbb{R})$ and both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ exist, then $\lim_{x\to\infty} f'(x)$ must be zero.

14.5 Advanced Limitology

- If f is continuous, is it necessarily the case that f' is continuous?

- If $f_n \to f$, is it necessarily the case that f_n' converges to f' (or at all)?
- Is it true that the sum of differentiable functions is differentiable?
- Is it true that the limit of integrals equals the integral of the limit?
- Is it true that a limit of continuous functions is continuous?
- Show that a subset of a metric space is closed iff it is complete.
- Show that if $m(E) < \infty$ and $f_n \to f$ uniformly, then $\lim \int_E f_n = \int_E f$.

14.6 Uniform Convergence

- Show that a uniform limit of bounded functions is bounded.
- Show that a uniform limit of continuous function is continuous.
  - I.e. if $f_n \to f$ uniformly with each f_n continuous then f is continuous.
- Show that
  - $f_n : [a, b] \to \mathbb{R}$ are continuously differentiable with derivatives f_n'
  - The sequence of derivatives f_n' converges uniformly to some function g
  - There exists *at least one* point x_0 such that $\lim_n f_n(x_0)$ exists,
  - Then $f_n \to f$ uniformly to some differentiable f, and $f' = g$.
- Prove that uniform convergence implies pointwise convergence implies a.e. convergence, but none of the implications may be reversed.
- Show that $\sum \frac{x^n}{n!}$ converges uniformly on any compact subset of $\mathbb{R}$.

14.7 Measure Theory

- Show that continuity of measure from above/below holds for outer measures.

- Show that a countable union of null sets is null.

Measurability

- Show that $f = 0$ a.e. iff $\int_E f = 0$ for every measurable set E.

Integrability

- Show that if f is a measurable function, then $f = 0$ a.e. iff $\int f = 0$.
- Show that a bounded function is Lebesgue integrable iff it is measurable.
- Show that simple functions are dense in L^1.

- Show that step functions are dense in L^1.
- Show that smooth compactly supported functions are dense in L^1.

14.8 Convergence

- Prove Fatou's lemma using the Monotone Convergence Theorem.
- Show that if $\{f_n\}$ is in L^1 and $\sum \int |f_n| < \infty$ then $\sum f_n$ converges to an L^1 function and

$$\int \sum f_n = \sum \int f_n.$$

14.9 Convolution

- Show that if f, g are continuous and compactly supported, then so is $f * g$.
- Show that if $f \in L^1$ and g is bounded, then $f * g$ is bounded and uniformly continuous.
- If f, g are compactly supported, is it necessarily the case that $f * g$ is compactly supported?
- Show that under any of the following assumptions, $f * g$ vanishes at infinity:
  - $f, g \in L^1$ are both bounded.
  - $f, g \in L^1$ with just g bounded.
  - f, g smooth and compactly supported (and in fact $f * g$ is smooth)
  - $f \in L^1$ and g smooth and compactly supported (and in fact $f * g$ is smooth)
- Show that if $f \in L^1$ and g' exists with $\frac{\partial g}{\partial x_i}$ all bounded, then

$$\frac{\partial}{\partial x_i}(f * g) = f * \frac{\partial g}{\partial x_i}$$

14.10 Fourier Analysis

- Show that if $f \in L^1$ then $\widehat{f}$ is bounded and uniformly continuous.
- Is it the case that $f \in L^1$ implies $\widehat{f} \in L^1$?
- Show that if $f, \widehat{f} \in L^1$ then f is bounded, uniformly continuous, and vanishes at infinity.
  - Show that this is not true for arbitrary L^1 functions.
- Show that if $f \in L^1$ and $\widehat{f} = 0$ almost everywhere then $f = 0$ almost everywhere.
  - Prove that $\widehat{f} = \widehat{g}$ implies that $f = g$ a.e.
- Show that if $f, g \in L^1$ then

$$\int \widehat{f} g = \int f \widehat{g}.$$

  - Give an example showing that this fails if g is not bounded.

- Show that if $f \in C^1$ then f is equal to its Fourier *series*.

14.11 Approximate Identities

- Show that if φ is an approximate identity, then

$$\|f * \varphi_t - f\|_1 \overset{t\to 0}{\to} 0.$$

  - Show that if additionally $|\varphi(x)| \leq c(1+|x|)^{-n-\varepsilon}$ for some $c, \varepsilon > 0$, then this converges is almost everywhere.

- Show that is f is bounded and uniformly continuous and φ_t is an approximation to the identity, then $f * \varphi_t$ uniformly converges to f.

L^p Spaces

- Show that if $E \subseteq \mathbb{R}^n$ is measurable with $\mu(E) < \infty$ and $f \in L^p(X)$ then

$$\|f\|_{L^p(X)} \overset{p\to\infty}{\to} \|f\|_\infty.$$

- Is it true that the converse to the DCT holds? I.e. if $\int f_n \to \int f$, is there a $g \in L^p$ such that $f_n < g$ a.e. for every n?
- Prove continuity in L^p: If f is uniformly continuous then for all p,

$$\|\tau_h f - f\|_p \overset{h\to 0}{\to} 0.$$

- Prove the following inclusions of L^p spaces for $m(X) < \infty$:

$$L^\infty(X) \subset L^2(X) \subset L^1(X)$$
$$\ell^2(\mathbb{Z}) \subset \ell^1(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z}).$$

14.12 Unsorted

Proposition 14.12.1 ***(Volumes of Rectangles).***
If $\{R_j\} \rightrightarrows R$ is a covering of R by rectangles,

$$R = \overset{\circ}{\coprod_j} R_j \implies |R| = \sum |R|_j$$
$$R \subseteq \bigcup_j R_j \implies |R| \leq \sum |R|_j.$$

- Show that any disjoint intervals is countable.
- Show that every open $U \subseteq \mathbb{R}$ is a countable union of disjoint open intervals.
- Show that every open $U \subseteq \mathbb{R}^n$ is a countable union of *almost* disjoint closed cubes.
- Show that that Cantor middle-thirds set is compact, totally disconnected, and perfect, with outer measure zero.
- Prove the Borel-Cantelli lemma.

15 | Extra Problems from Problem Sets

15.1 Continuous on compact implies uniformly continuous

Problem 15.1.1 (?)
Show that a continuous function on a compact set is uniformly continuous.

Solution omitted.

15.2 2010 6.1

Problem 15.2.1 (?)
Show that

$$\int_{\mathbb{B}^n} \frac{1}{|x|^p}\, dx < \infty \iff p < n$$

$$\int_{\mathbb{R}^n \setminus \mathbb{B}^n} \frac{1}{|x|^p}\, dx < \infty \iff p > n.$$

Solution omitted.

15.3 2010 6.2

Show that

$$\int_{\mathbb{R}^n} |f| = \int_0^\infty m(A_t)\, dt \qquad A_t := \left\{ x \in \mathbb{R}^n \,\middle|\, |f(x)| > t \right\}.$$

Solution omitted.

15.4 2010 6.5

Suppose $F \subseteq \mathbb{R}$ with $m(F^c) < \infty$ and let $\delta(x) := d(x, F)$ and

$$I_F(x) := \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2}\, dy.$$

a. Show that δ is continuous.

b. Show that if $x \in F^c$ then $I_F(x) = \infty$.

c. Show that $I_F(x) < \infty$ for almost every x

Solution omitted.

15.5 2010 7.1

Let $(X, \mathcal{M}, \mu)$ be a measure space and prove the following properties of $L^\infty(X, \mathcal{M}, \mu)$:

- If f, g are measurable on X then
$$\|fg\|_1 \leq \|f\|_1 \|g\|_\infty.$$
- $\|-\|_\infty$ is a norm on L^∞ making it a Banach space.
- $\|f_n - f\|_\infty \overset{n\to\infty}{\to} 0 \iff$ there exists an $E \in \mathcal{M}$ such that $\mu(X \setminus E) = 0$ and $f_n \to f$ uniformly on E.
- Simple functions are dense in L^∞.

15.6 2010 7.2

Show that for $0 < p < q \leq \infty$, $\|a\|_{\ell^q} \leq \|a\|_{\ell^p}$ over $\mathbb{C}$, where $\|a\|_\infty := \sup\limits_j |a_j|$.

15.7 2010 7.3

Let f, g be non-negative measurable functions on $[0, \infty)$ with

$$A := \int_0^\infty f(y) y^{-1/2}\, dy < \infty$$
$$B := \left(\int_0^\infty |g(y)|\right)^2 dy < \infty.$$

Show that

$$\int_0^\infty \left(\int_0^\infty f(y)\, dy\right) \frac{g(x)}{x}\, dx \leq AB.$$

15.8 2010 7.4

Let $(X, \mathcal{M}, \mu)$ be a measure space and $0 < p < q < \infty$. Prove that if $L^q(X) \subseteq L^p(X)$, then X does not contain sets of arbitrarily large finite measure.

15.9 2010 7.5

Suppose $0 < a < b \leq \infty$, and find examples of functions $f \in L^p((0, \infty))$ if and only if:

- $a < p < b$
- $a \leq p \leq b$
- $p = a$

Hint: consider functions of the following form:

$$f(x) := x^{-\alpha} |\log(x)|^\beta.$$

15.10 2010 7.6

Define

$$F(x) := \left(\frac{\sin(\pi x)}{\pi x}\right)^2$$
$$G(x) := \begin{cases} 1 - |x| & |x| \leq 1 \\ 0 & \text{else.} \end{cases}$$

a. Show that $\widehat{G}(\xi) = F(\xi)$

b. Compute $\widehat{F}$.

c. Give an example of a function $g \notin L^1(\mathbb{R})$ which is the Fourier transform of an L^1 function.

Hint: write $\widehat{G}(\xi) = H(\xi) + H(-\xi)$ *where*

$$H(\xi) := e^{2\pi i \xi} \int_0^1 y e^{2\pi i y \xi}\, dy.$$

15.11 2010 7.7

Show that for each $\epsilon > 0$ the following function is the Fourier transform of an $L^1(\mathbb{R}^n)$ function:

$$F(\xi) := \left(\frac{1}{1 + |\xi|^2}\right)^\epsilon.$$

Hint: show that

$$K_\delta(x) := \delta^{-n/2} e^{\frac{-\pi|x|^2}{\delta}}$$
$$f(x) := \int_0^\infty K_\delta(x) e^{-\pi\delta} \delta^{\epsilon-1}\, d\delta$$
$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1}\, dt$$
$$\implies \widehat{f}(\xi) = \int_0^\infty e^{-\pi\delta|\xi|^2} e^{-\pi\delta} \delta^{\epsilon-1} = \pi^{-s}\Gamma(\epsilon) F(\xi).$$

15.12 2010 7 Challenge 1: Generalized Holder

Suppose that

$$1 \leq p_j \leq \infty, \qquad \sum_{j=1}^{n} \frac{1}{p_j} = \frac{1}{r} \leq 1.$$

Show that if $f_j \in L^{p_j}$ for each $1 \leq j \leq n$, then

$$\prod f_j \in L^r, \qquad \left\| \prod f_j \right\|_r \leq \prod \|f_j\|_{p_j}.$$

15.13 2010 7 Challenge 2: Young's Inequality

Suppose $1 \leq p, q, r \leq \infty$ with

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

Prove that

$$f \in L^p, g \in L^q \implies f * g \in L^r \text{ and } \|f * g\|_r \leq \|f\|_p \|g\|_q.$$

15.14 2010 9.1

Show that the set $\{u_k(j) \coloneqq \delta_{ij}\} \subseteq \ell^2(\mathbb{Z})$ and forms an orthonormal system.

15.15 2010 9.2

Consider $L^2([0,1])$ and define

$$\begin{aligned} e_0(x) &= 1 \\ e_1(x) &= \sqrt{3}(2x-1). \end{aligned}$$

a. Show that $\{e_0, e_1\}$ is an orthonormal system.

b. Show that the polynomial $p(x)$ where $\deg(p) = 1$ which is closest to $f(x) = x^2$ in $L^2([0,1])$ is given by

$$h(x) = x - \frac{1}{6}.$$

Compute $\|f-g\|_2$.

15.16 2010 9.3

Let $E \subseteq H$ a Hilbert space.

a. Show that $E \perp\subseteq H$ is a closed subspace.

b. Show that $(E^{\perp})^{\perp} = \mathrm{cl}_H(E)$.

15.17 2010 9.5b

Let $f \in L^1((0,2\pi))$.

i. Show that for an $\epsilon > 0$ one can write $f = g + h$ where $g \in L^2((0,2\pi))$ and $\|H\|_1 < \epsilon$.

15.18 2010 9.6

Prove that every closed convex $K \subset H$ a Hilbert space has a unique element of minimal norm.

15.19 2010 9 Challenge

Let U be a unitary operator on H a Hilbert space, let $M := \left\{x \in H \;\middle|\; Ux = x\right\}$, let P be the orthogonal projection onto M, and define

$$S_N := \frac{1}{N}\sum_{n=0}^{N-1} U^n.$$

Show that for all $x \in H$,

$$\|S_N x - Px\|_H \overset{N\to\infty}{\to} 0.$$

15.20 2010 10.1

Let ν, μ be signed measures, and show that

$$\nu \perp \mu \text{ and } \nu \ll |\mu| \implies \nu = 0.$$

15.21 2010 10.2

Let $f \in L^1(\mathbb{R}^n)$ with $f \neq 0$.

a. Prove that there exists a $c > 0$ such that

$$Hf(x) \geq \frac{c}{(1+|x|)^n}.$$

15.22 2010 10.3

Consider the function

$$f(x) := \begin{cases} \dfrac{1}{|x| \left(\log\left(\frac{1}{x}\right)\right)^2} & |x| \leq \dfrac{1}{2} \\ 0 & \text{else.} \end{cases}$$

a. Show that $f \in L^1(\mathbb{R})$.

b. Show that there exists a $c > 0$ such that for all $|x| \leq 1/2$,

$$Hf(x) \geq \frac{c}{|x| \log\left(\frac{1}{|x|}\right)}.$$

Conclude that Hf is not locally integrable.

15.23 2010 10.4

Let $f \in L^1(\mathbb{R})$ and let $\mathcal{U} := \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, y > 0\right\}$ denote the upper half plane. For $(x,y) \in \mathcal{U}$ define

$$u(x,y) := f * P_y(x) \qquad \text{where } P_y(x) := \frac{1}{\pi}\left(\frac{y}{t^2+y^2}\right).$$

a. Prove that there exists a constant C independent of f such that for all $x \in \mathbb{R}$,

$$\sup_{y>0} |u(x, y)| \leq C \cdot Hf(x).$$

Hint: write the following and try to estimate each term:

$$u(x, y) = \int_{|t|<y} f(x-t) P_y(t)\, dt + \sum_{k=0}^{\infty} \int_{A_k} f(x-t) P_y(t)\, dt \qquad A_k := \left\{ 2^k y \leq |t| < 2^{k+1} y \right\}.$$

b. Following the proof of the Lebesgue differentiation theorem, show that for $f \in L^1(\mathbb{R})$ and for almost every $x \in \mathbb{R}$,

$$u(x, y) \overset{y \to 0}{\to} f(x).$$

16 | Midterm Exam 2 (December 2014)

16.1 Fall 2014 Midterm 1.1

Note: (a) is a repeat.

- Let $\Lambda \in L^2(X)^\vee$.
  - Show that $M := \left\{ f \in L^2(X) \;\middle|\; \Lambda(f) = 0 \right\} \subseteq L^2(X)$ is a closed subspace, and $L^2(X) = M \oplus M \perp$.
  - Prove that there exists a unique $g \in L^2(X)$ such that $\Lambda(f) = \int_X g \bar{f}$.

16.2 Fall 2014 Midterm 1.2

a. In parts:

- Given a definition of $L^\infty(\mathbb{R}^n)$.
- Verify that $\|-\|_\infty$ defines a norm on $L^\infty(\mathbb{R}^n)$.
- Carefully proved that $(L^\infty(\mathbb{R}^n), \|-\|_\infty)$ is a Banach space.

b. Prove that for any measurable $f : \mathbb{R}^n \to \mathbb{C}$,

$$L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \quad \text{and} \quad \|f\|_2 \leq \|f\|_1^{\frac{1}{2}} \cdot \|f\|_\infty^{\frac{1}{2}}.$$

16.3 Fall 2014 Midterm 1.3

a. Prove that if $f, g : \mathbb{R}^n \to \mathbb{C}$ is both measurable then $F(x, y) := f(x)$ and $h(x, y) := f(x-y)g(y)$ is measurable on $\mathbb{R}^n \times \mathbb{R}^n$.

b. Show that if $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, then $f * g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is well defined, and carefully show that it satisfies the following properties:

$$\|f * g\|_\infty \leq \|g\|_1 \|f\|_\infty \|f * g\|_1 \qquad \leq \|g\|_1 \|f\|_1 \|f * g\|_2 \leq \|g\|_1 \|f\|_2.$$

Hint: first show $|f * g|^2 \leq \|g\|_1 \left(|f|^2 * |g|\right)$.

16.4 Fall 2014 Midterm 1.4

Note: (a) is a repeat.

Let $f : [0, 1] \to \mathbb{R}$ be continuous, and prove the Weierstrass approximation theorem: for any $\varepsilon > 0$ there exists a polynomial P such that $\|f - P\|_\infty < \varepsilon$.

17 | Midterm Exam 1 (October 2018)

17.1 Fall 2018 Midterm 1.1

Let $E \subseteq \mathbb{R}^n$ be bounded. Prove the following are equivalent:

1. For any $\epsilon > 0$ there exists and open set G and a closed set F such that

$$F \subseteq E \subseteq G \qquad m(G \setminus F) < \epsilon.$$

2. There exists a G_δ set V and an F_σ set H such that

$$m(V \setminus H) = 0.$$

17.2 Fall 2018 Midterm 1.2

Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of extended real-valued Lebesgue measurable functions.

a. Prove that $\sup_k f_k$ is a Lebesgue measurable function.

b. Prove that if $\lim_{k\to\infty} f_k(x)$ exists for every $x \in \mathbb{R}^n$ then $\lim_{k\to\infty} f_k$ is also a measurable function.

17.3 Fall 2018 Midterm 1.3

a. Prove that if $E \subseteq \mathbb{R}^n$ is a Lebesgue measurable set, then for any $h \in \mathbb{R}$ the set

$$E + h := \left\{ x + h \;\middle|\; x \in E \right\}$$

is also Lebesgue measurable and satisfies $m(E + h) = m(E)$.

b. Prove that if f is a non-negative measurable function on $\mathbb{R}^n$ and $h \in \mathbb{R}^n$ then the function

$$\tau_h d(x) := f(x - h)$$

is a non-negative measurable function and

$$\int f(x)\, dx = \int f(x - h)\, dx.$$

17.4 Fall 2018 Midterm 1.4

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function.

a. Prove that for all $\alpha > 0$,

$$A_\alpha := \left\{ x \in \mathbb{R}^n \;\middle|\; |f(x)| > \alpha \right\} \implies m(A_\alpha) \leq \frac{1}{\alpha} \int |f(x)|\, dx.$$

b. Prove that

$$\int |f(x)|\, dx = 0 \iff f = 0 \text{ almost everywhere.}$$

17.5 Fall 2018 Midterm 1.5

Let $\{f_k\}_{k=1}^{\infty} \subseteq L^2([0,1])$ be a sequence which *converges in* L^1 to a function f.

a. Prove that $f \in L^1([0,1])$.

b. Give an example illustrating that f_k may not converge to f almost everywhere.

c. Prove that $\{f_k\}$ must contain a subsequence that converges to f almost everywhere.

18 | Midterm Exam 2 (November 2018)

18.1 Fall 2018 Midterm 2.1

Let $f, g \in L^1([0,1])$, define $F(x) = \int_0^x f(y)\,dy$ and $G(x) = \int_0^x g(y)\,dy$, and show

$$\int_0^1 F(x)g(x)\,dx = F(1)G(1) - \int_0^1 f(x)G(x)\,dx.$$

18.2 Fall 2018 Midterm 2.2

Let $\varphi \in L^1(\mathbb{R}^n)$ such that $\int \varphi = 1$ and define $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$. Show that if f is bounded and uniformly continuous then $f * \varphi_t \overset{t\to 0}{\to} f$ uniformly.

18.3 Fall 2018 Midterm 2.3

Let $g \in L^\infty([0,1])$.

a. Prove

$$\|g\|_{L^p([0,1])} \overset{p\to\infty}{\to} \|g\|_{L^\infty([0,1])}.$$

b. Prove that the map

$$\begin{aligned}\Lambda_g : L^1([0,1]) &\to \mathbb{C} \\ f &\mapsto \int_0^1 fg\end{aligned}$$

defines an element of $L^1([0,1])^\vee$ with $\|\Lambda_g\|_{L^1([0,1])^\vee} = \|g\|_{L^\infty([0,1])}$.

18.4 Fall 2018 Midterm 2.4

See section 20.3

19 | Practice Exam (November 2014)

19.1 Fall 2018 Practice Midterm 1.1

Let $m_*(E)$ denote the Lebesgue outer measure of a set $E \subseteq \mathbb{R}^n$.

a. Prove using the definition of Lebesgue outer measure that

$$m\left(\bigcup_{j=1}^\infty E_j\right) \leq \sum_{j=1}^\infty m_*(E_j).$$

b. Prove that for any $E \subseteq \mathbb{R}^n$ and any $\epsilon > 0$ there exists an open set G with $E \subseteq G$ and

$$m_*(E) \leq m_*(G) \leq m_*(E) + \epsilon.$$

19.2 Fall 2018 Practice Midterm 1.2

a. See section 17.1

b. Let f_k be a sequence of extended real-valued Lebesgue measurable function.

   i. Prove that $\inf_k f_k, \sup_k f_k$ are both Lebesgue measurable function.

Hint: argue that

$$\left\{x \;\middle|\; \inf_k f_k(x) < a\right\} = \bigcup_k \left\{x \;\middle|\; f_k(x) < a\right\}.$$

ii. Carefully state Fatou's Lemma and deduce the Monotone Converge Theorem from it.

19.3 Fall 2018 Practice Midterm 1.3

a. Prove that if $f, g \in L^+(\mathbb{R})$ then

$$\int (f+g) = \int f + \int g.$$

Extend this to establish that if $\{f_k\} \subseteq L^+(\mathbb{R}^n)$ then

$$\int \sum_k f_k = \sum_k \int f_k.$$

b. Let $\{E_j\}_{j\in\mathbb{N}} \subseteq \mathcal{M}(\mathbb{R}^n)$ with $E_j \nearrow E$. Use the countable additivity of μ_f on $\mathcal{M}(\mathbb{R}^n)$ established above to show that

$$\mu_f(E) = \lim_{j\to\infty} \mu_f(E_j).$$

19.4 Fall 2018 Practice Midterm 1.4

a. Show that $f \in L^1(\mathbb{R}^n) \implies |f(x)| < \infty$ almost everywhere.

b. Show that if $\{f_k\} \subseteq L^1(\mathbb{R}^n)$ with $\sum \|f_k\|_1 < \infty$ then $\sum f_k$ converges almost everywhere and in L^1.

c. Use the Dominated Convergence Theorem to evaluate

$$\lim_{t\to 0} \int_0^1 \frac{e^{tx^2} - 1}{t}\, dx.$$

20 | Practice Exam (November 2014)

20.1 Fall 2018 Practice Midterm 2.1

a. Carefully state Tonelli's theorem for a nonnegative function $F(x,t)$ on $\mathbb{R}^n \times \mathbb{R}$.

b. Let $f : \mathbb{R}^n \to [0,\infty]$ and define

$$\mathcal{A} := \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; 0 \le t \le f(x) \right\}.$$

Prove the validity of the following two statements:

1. f is Lebesgue measurable on $\mathbb{R}^n$ $\iff$ $\mathcal{A}$ is a Lebesgue measurable subset of $\mathbb{R}^{n+1}$.
2. If f is Lebesgue measurable on $\mathbb{R}^n$ then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)dx = \int_0^\infty m\left(\left\{ x \in \mathbb{R}^n \;\middle|\; f(x) \ge t \right\} \right) dt.$$

20.2 Fall 2018 Practice Midterm 2.2

a. Let $f, g \in L^1(\mathbb{R}^n)$ and give a definition of $f * g$.

b. Prove that if f, g are integrable and bounded, then

$$(f * g)(x) \overset{|x| \to \infty}{\to} 0.$$

c. In parts:

1. Define the *Fourier transform* of an integrable function f on $\mathbb{R}^n$.
2. Give an outline of the proof of the Fourier inversion formula.
3. Give an example of a function $f \in L^1(\mathbb{R}^n)$ such that $\widehat{f}$ is not in $L^1(\mathbb{R}^n)$.

20.3 Fall 2018 Practice Midterm 2.3

Let $\{u_n\}_{n=1}^\infty$ be an orthonormal sequence in a Hilbert space H.

a. Let $x \in H$ and verify that

$$\left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|_H^2 = \|x\|_H^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 .$$

for any $N \in \mathbb{N}$ and deduce that

$$\sum_{n=1}^\infty |\langle x, u_n \rangle|^2 \le \|x\|_H^2.$$

b. Let $\{a_n\}_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ and prove that there exists an $x \in H$ such that $a_n = \langle x,\ u_n\rangle$ for all $n \in \mathbb{N}$, and moreover x may be chosen such that

$$\|x\|_H = \left(\sum_{n\in\mathbb{N}} |a_n|^2\right)^{\frac{1}{2}}.$$

c. Prove that if $\{u_n\}$ is *complete*, Bessel's inequality becomes an equality.

Solution omitted.

Solution omitted.

20.4 Fall 2018 Practice Midterm 2.4

a. Prove Holder's inequality: let $f \in L^p, g \in L^q$ with p, q conjugate, and show that

$$\|fg\|_p \leq \|f\|_p \cdot \|g\|_q.$$

b. Prove Minkowski's Inequality:

$$1 \leq p < \infty \implies \|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Conclude that if $f, g \in L^p(\mathbb{R}^n)$ then so is $f+g$.

c. Let $X = [0,1] \subset \mathbb{R}$.

1. Give a definition of the Banach space $L^\infty(X)$ of essentially bounded functions of X.
2. Let f be non-negative and measurable on X, prove that

$$\int_X f(x)^p \, dx \overset{p\to\infty}{\to} \begin{cases} \infty & \text{or} \\ m\left(\left\{f^{-1}(1)\right\}\right) \end{cases},$$

and characterize the functions of each type

Solution omitted.

20.5 Fall 2018 Practice Midterm 2.5

Let X be a normed vector space.

a. Give the definition of what it means for a map $L : X \to \mathbb{C}$ to be a *linear functional*.

b. Define what it means for L to be *bounded* and show L is bounded $\iff$ L is continuous.

c. Prove that $(X^\vee, \|-\|_{\text{op}})$ is a Banach space.

DZG: this comes from some tex file that I found when studying for quals, so is definitely not my own content! I've just copied it here for extra practice.

21 | May 2016 Qual

21.1 May 2016, 1

Consider the function $f(x) = \dfrac{x}{1-x^2}$, $x \in (0,1)$.

1. By using the $\epsilon - \delta$ definition of the limit only, prove that f is continuous on $(0,1)$. (Note: You are not allowed to trivialize the problem by using properties of limits).

2. Is f uniformly continuous on $(0,1)$? Justify your answer.

Proof omitted.

Proof omitted.

21.2 (May 2016, 2)

Let $\{a_k\}_{k=1}^\infty$ be a bounded sequence of real numbers and E given by:

$$E := \left\{ s \in \mathbb{R} : \text{ the set } \{k \in \mathbb{N} : \, a_k \geq s\} \text{ has at most finitely many elements} \right\}.$$

Prove that $\limsup_{k\to\infty} a_k = \inf E$.

Proof omitted.

21.3 (May 2016, 3)

Assume (X, d) is a compact metric space.

1. Prove that X is both complete and separable.
2. Suppose $\{x_k\}_{k=1}^{\infty} \subseteq X$ is a sequence such that the series $\sum_{k=1}^{\infty} d(x_k, x_{k+1})$ converges. Prove that the sequence $\{x_k\}_{k=1}^{\infty}$ converges in X.

21.4 (May 2016, 4)

Suppose that $f\colon [0,2] \to \mathbb{R}$ is continuous on $[0,2]$, differentiable on $(0,2)$, and such that $f(0) = f(2) = 0$, $f(c) = 1$ for some $c \in (0,2)$. Prove that there exists $x \in (0,2)$ such that $|f'(x)| > 1$.

Proof omitted.

21.5 (May 2016, 5)

Let $f_n(x) = n^{\beta} x(1-x^2)^n$, $x \in [0,1]$, $n \in \mathbb{N}$.

1. Prove that $\{f_n\}_{n=1}^{\infty}$ converges pointwise on $[0,1]$ for every $\beta \in \mathbb{R}$.
2. Show that the convergence in part (a) is uniform for all $\beta < \frac{1}{2}$, but not uniform for any $\beta \geq \frac{1}{2}$.

21.6 (May 2016, 6)

1. Suppose $f\colon [-1,1] \to \mathbb{R}$ is a bounded function that is continuous at 0. Let $\alpha(x) = -1$ for $x \in [-1,0]$ and $\alpha(x) = 1$ for $x \in (0,1]$. Prove that $f \in \mathcal{R}(\alpha)[-1,1]$, i.e., f is Riemann integrable with respect to α on $[-1,1]$, and $\int_{-1}^{1} f\,d\alpha = 2f(0)$.
2. Let $g\colon [0,1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 g(x)x^{3k+2}\,dx = 0$ for all $k = 0,1,2,\ldots$. Prove that $g(x) = 0$ for all $x \in [0,1]$.

Proof omitted.

Proof omitted.

22 | Metric Spaces and Topology

22.1 (May 2019, 1)

Let (M, d_M), (N, d_N) be metric spaces. Define $d_{M\times N}\colon (M\times N)\times(M\times N)\to\mathbb{R}$ by

$$d_{M\times N}((x_1, y_1), (x_2, y_2)) := d_M(x_1, x_2) + d_N(y_1, y_2).$$

1. Prove that $(M \times N, d_{M\times N})$ is a metric space.
2. Let $S \subseteq M$ and $T \subseteq N$ be compact sets in (M, d_M) and (N, d_N), respectively. Prove that $S \times T$ is a compact set in $(M \times N, d_{M\times N})$.

Proof omitted.

Proof omitted.

22.2 (June 2003, 1b,c)

(b) Show by example that the union of infinitely many compact subsets of a metric space need not be compact. (c) If (X, d) is a metric space and $K \subset X$ is compact, define $d(x_0, K) = \inf_{y\in K} d(x_0, y)$. Prove that there exists a point $y_0 \in K$ such that $d(x_0, K) = d(x_0, y_0)$.

22.3 (January 2009, 4a)

Consider the metric space $(\mathbb{Q}, d)$ where $\mathbb{Q}$ denotes the rational numbers and $d(x, y) = |x - y|$. Let $E = \{x \in \mathbb{Q} : x > 0,\, 2 < x^2 < 3\}$. Is E closed and bounded in $\mathbb{Q}$? Is E compact in $\mathbb{Q}$?

22.4 (January 2011 3a)

Let (X, d) be a metric space, $K \subset X$ be compact, and $F \subset X$ be closed. If $K \cap F = \emptyset$, prove that there exists an $\epsilon > 0$ so that $d(k, f) \geq \epsilon$ for all $k \in K$ and $f \in F$.

Proof omitted.

22.5 5?

Let (X, d) be an unbounded and connected metric space. Prove that for each $x_0 \in X$, the set $\{x \in X : d(x, x_0) = r\}$ is nonempty.

23 | Sequences and Series

23.1 (June 2013 1a)

Let $a_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)$. Prove that $\lim_{n\to\infty} a_n = 1/2$.

23.2 (January 2014 2)

1. Produce sequences $\{a_n\}$, $\{b_n\}$ of positive real numbers such that
$$\liminf_{n\to\infty}(a_n b_n) > \left(\liminf_{n\to\infty} a_n\right)\left(\liminf_{n\to\infty} b_n\right).$$
2. If $\{a_n\}$, $\{b_n\}$ are sequences of positive real numbers and $\{a_n\}$ converges, prove that
$$\liminf_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\liminf_{n\to\infty} b_n\right).$$

23.3 (May 2011 4a)

Determine the values of $x \in \mathbb{R}$ for which $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{1 + n|x|^n}$ converges, justifying your answer carefully.

23.4 (June 2005 3b)

If the series $\sum_{n=0}^{\infty} a_n$ converges conditionally, show that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is 1.

23.5 (January 2011 5)

Suppose $\{a_n\}$ is a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$ and $\sum a_n$ diverges. Prove that for all $x > 0$ there exist integers $n(1) < n(2) < \ldots$ such that $\sum_{k=1}^{\infty} a_{n(k)} = x$.

(Note: Many variations on this problem are possible including more general rearrangements. You may also wish to show that if $\sum a_n$ converges conditionally then given any $x \in \mathbb{R}$ there is a rearrangement of $\{b_n\}$ of $\{a_n\}$ such that $\sum b_n = r$. See Rudin Thm. 3.54 for a further generalization.)

23.6 (June 2008 # 4b)

Assume $\beta > 0$, $a_n > 0$, $n = 1, 2, \ldots$, and the series $\sum a_n$ is divergent. Show that $\sum \frac{a_n}{\beta + a_n}$ is also divergent.

24 Continuity of Functions

25 Differential Calculus

25.1 (June 2005 1a)

Use the definition of the derivative to prove that if f and g are differentiable at x, then fg is differentiable at x.

25.2 (January 2006 2b)

Assume that f is differentiable at a. Evaluate

$$\lim_{x\to a} \frac{a^n f(x) - x^n f(a)}{x-a}, \quad n \in \mathbb{N}.$$

25.3 (June 2007 3a)

Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable, that $f(x) \le g(x)$ for all $x \in \mathbb{R}$, and that $f(x_0) = g(x_0)$ for some x_0. Prove that $f'(x_0) = g'(x_0)$.

25.4 (June 2008 3a)

Prove that if f' exists and is bounded on $(a, b]$, then $\lim\limits_{x\to a^+} f(x)$ exists.

25.5 (January 2012 4b, extended)

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f' \in C(\mathbb{R})$. Assume that there are $a, b \in \mathbb{R}$ with $\lim\limits_{x\to\infty} f(x) = a$ and $\lim\limits_{x\to\infty} f'(x) = b$. Prove that $b = 0$. Then, find a counterexample to show that the assumption $\lim\limits_{x\to\infty} f'(x)$ exists is necessary.

25.6 (June 2012 1a)

Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 0$. Prove that f is differentiable at $x = 0$ if and only if there is a function $g : \mathbb{R} \to \mathbb{R}$ which is continuous at $x = 0$ and satisfies $f(x) = xg(x)$ for all $x \in \mathbb{R}$.

26 | Integral Calculus

26.1 (January 2006 4b)

Suppose that f is continuous and $f(x) \geq 0$ on $[0, 1]$. If $f(0) > 0$, prove that $\int_0^1 f(x)dx > 0$.

26.2 (June 2005 1b)

Use the definition of the Riemann integral to prove that if f is bounded on $[a, b]$ and is continuous everywhere except for finitely many points in (a, b), then $f \in \mathscr{R}$ on $[a, b]$.

26.3 (January 2010 5)

Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, $f \geq 0$ on $[a, b]$, and put $M = \sup\{f(x) : x \in [a, b]\}$. Prove that

$$\lim_{p\to\infty} \left(\int_a^b f(x)^p \, dx \right)^{1/p} = M.$$

26.4 (January 2009 4b)

Let f be a continuous real-valued function on $[0, 1]$. Prove that there exists at least one point $\xi \in [0, 1]$ such that $\int_0^1 x^4 f(x) \, dx = \frac{1}{5} f(\xi)$.

Proof omitted.

26.5 (June 2009 5b)

Let φ be a real-valued function defined on $[0,1]$ such that φ, φ', and φ'' are continuous on $[0,1]$. Prove that

$$\int_0^1 \cos x \frac{x\varphi'(x) - \varphi(x) + \varphi(0)}{x^2}\, dx < \frac{3}{2}||\varphi''||_\infty,$$

where $||\varphi''||_\infty = \sup_{[0,1]} |\varphi''(x)|$. Note that $3/2$ may not be the smallest possible constant.

27 | Sequences and Series of Functions

27.1 (June 2010 6a)

Let $f : [0,1] \to \mathbb{R}$ be continuous with $f(0) \neq f(1)$ and define $f_n(x) = f(x^n)$. Prove that f_n does not converge uniformly on $[0,1]$.

27.2 (January 2008 5a)

Let $f_n(x) = \dfrac{x}{1+nx^2}$ for $n \in \mathbb{N}$. Let $\mathcal{F} := \{f_n : n = 1, 2, 3, \ldots\}$ and $[a,b]$ be any compact subset of $\mathbb{R}$. Is $\mathcal{F}$ equicontinuous? Justify your answer.

27.3 (January 2005 4, June 2010 6b)

If $f : [0,1] \to \mathbb{R}$ is continuous, prove that

$$\lim_{n\to\infty} \int_0^1 f(x^n)\, dx = f(0).$$

27.4 (January 2020 4a)

Let $M < \infty$ and $\mathcal{F} \subseteq C[a,b]$. Assume that each $f \in \mathcal{F}$ is differentiable on (a,b) and satisfies $|f(a)| \leq M$ and $|f'(x)| \leq M$ for all $x \in (a,b)$. Prove that $\mathcal{F}$ is equicontinuous on $[a,b]$.

27.5 (June 2005 5)

Suppose that $f \in C([0,1])$ and that $\int_0^1 f(x)x^n\,dx = 0$ for all $n = 99, 100, 101, \ldots$. Show that $f \equiv 0$.

Note: Many variations on this problem exist. See June 2012 6b and others.

27.6 (January 2005 3b)

Suppose $f_n : [0,1] \to \mathbb{R}$ are continuous functions converging uniformly to $f : [0,1] \to \mathbb{R}$. Either prove that $\lim_{n\to\infty} \int_{1/n}^1 f_n(x)\,dx = \int_0^1 f(x)\,dx$ or give a counterexample.

28 | Miscellaneous Topics

Bounded Variation

28.1 (January 2018)

Let $f\colon [a,b] \to \mathbb{R}$. Suppose $f \in \mathrm{BV}[a,b]$. Prove f is the difference of two increasing functions.

28.2 (January 2007, 6a)

Let f be a function of bounded variation on $[a,b]$. Furthermore, assume that for some $c > 0$, $|f(x)| \geq c$ on $[a,b]$. Show that $g(x) = 1/f(x)$ is of bounded variation on $[a,b]$.

28.3 (January 2017, 2a)

Define $f\colon [0,1] \to [-1,1]$ by

$$f(x) := \begin{cases} x \sin\left(\frac{1}{x}\right) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

Determine, with justification, whether f is if bounded variation on the interval $[0,1]$.

28.4 (January 2020, 6a)

Let $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ and a strictly increasing sequence $\{x_n\}_{n=1}^{\infty} \subseteq (0,1)$ be given. Assume that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and define $\alpha\colon [0,1] \to \mathbb{R}$ by

$$\alpha(x) := \begin{cases} a_n & x = x_n \\ 0 & \text{otherwise} \end{cases}.$$

Prove or disprove: α has bounded variation on $[0,1]$.

Metric Spaces and Topology

1. Find an example of a metric space X and a subset $E \subseteq X$ such that E is closed and bounded but not compact.

28.5 (May 2017 6)

Let (X,d) be a metric space. A function $f\colon X \to \mathbb{R}$ is said to be lower semi-continuous (l.s.c) if $f^{-1}(a,\infty) = \{x \in X : f(x) > a\}$ is open in X for every $a \in \mathbb{R}$. Analogously, f is upper semi-continuous (u.s.c) if $f^{-1}(-\infty,b) = \{x \in X : f(x) < b\}$ is open in X for every $b \in \mathbb{R}$.

1. Prove that a function $f\colon X \to \mathbb{R}$ is continuous if and only if f is both l.s.c. and u.s.c.

2. Prove that f is lower semi-continuous if and only if $\liminf_{n\to\infty} f(x_n) \geq f(x)$ whenever $\{x_n\}_{n=1}^{\infty} \subseteq X$ such that $x_n \to x$ in X.

28.6 (January 2017 3)

Let (X,d) be a compact metric space. Suppose that $f_n \colon X \to [0,\infty)$ is a sequence of continuous functions with $f_n(x) \geq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in X$, and such that $f_n \to 0$ pointwise on X. Prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on X.

29 | Integral Calculus

29.1 1.

(June 2014 1)Define $\alpha \colon [-1,1] \to \mathbb{R}$ by

$$\alpha(x) := \begin{cases} -1 & x \in [-1,0] \\ 1 & x \in (0,1]. \end{cases}$$

Let $f \colon [-1,1] \to \mathbb{R}$ be a function that is uniformly bounded on $[-1,1]$ and continuous at $x=0$, but not necessarily continuous for $x \neq 0$. Prove that f is Riemann-Stieltjes integrable with respect to α over $[-1,1]$ and that

$$\int_{-1}^{1} f(x)d\alpha(x) = 2f(0).$$

29.2 (June 2017 2)

Prove : $f \in \mathcal{R}(\alpha)$ on $[a,b]$ if and only if for any $a < c < b$, $f \in \mathcal{R}(\alpha)$ on $[a,c]$ and on $[c,b]$. In addition, if either condition holds, then we have that

$$\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha.$$

29.3 (Spring 2017 7)

Prove that if $f \in \mathcal{R}$ on $[a,b]$ and $\alpha \in C^1[a,b]$, then the Riemann integral $\displaystyle\int_a^b f(x)\alpha'(x)dx$ exists and

$$\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx.$$

30 | Sequences and Series (and of Functions)

30.1 (January 2006 1)

Let the power series series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ have radii of convergence R_1 and R_2, respectively.

30.2 ?

If $R_1 \neq R_2$, prove that the radius of convergence, R, of the power series $\sum_{n=0}^{\infty}(a_n+b_n)x^n$ is $\min\{R_1, R_2\}$. What can be said about R when $R_1 = R_2$?

30.3 ?

Prove that the radius of convergence, R, of $\sum_{n=0}^{\infty} a_n b_n x^n$ satisfies $R \geq R_1 R_2$. Show by means of example that this inequality can be strict.

30.4 ?

Show that the infinite series $\sum_{n=0}^{\infty} x^n 2^{-nx}$ converges uniformly on $[0, B]$ for any $B > 0$. Does this series converge uniformly on $[0, \infty)$?

30.5 (January 2006 4a)

Let

$$f_n(x) = \begin{cases} \frac{1}{n} & x \in (\frac{1}{2^{n+1}}, \frac{1}{2^n}] \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\sum_{n=1}^{\infty} f_n$ does not satisfy the Weierstrass M-test but that it nevertheless converges uniformly on $\mathbb{R}$.

30.6 ?

Let $f_n \colon [0,1) \to \mathbb{R}$ be the function defined by

$$f_n(x) := \sum_{k=1}^{n} \frac{x^k}{1+x^k}.$$

1. Prove that f_n converges to a function $f \colon [0,1) \to \mathbb{R}$.
2. Prove that for every $0 < a < 1$ the convergence is uniform on $[0,a]$.
3. Prove that f is differentiable on $(0,1)$.

January 2019 Qualifying Exam

1. Suppose that $f : [0,1] \to \mathbb{R}$ is differentiable and $f(0) = 0$. Assume that there is a $k > 0$ such that

$$|f'(x)| \leq k|f(x)|$$

for all $x \in [0,1]$. Prove that $f(x) = 0$ for all $x \in [0,1]$.

Proof omitted.