2016 General Preliminary Exam
Complex Analysis

This exam has eight problems. Solve as many of them as you can, in whatever order you like. We do not expect that you will have time to solve them all.

Problem 1. State the principle of analytic continuation for holomorphic functions, and give a specific example of how we used it in our development of complex analysis last semester.

Problem 2. One of the 'miracles' of complex analysis is that a function which is holomorphic (i.e. complex-differentiable once) is infinitely complex-differentiable. Prove this fact.

Problem 3. Compute the following integrals. All contours are oriented counter-clockwise.

(a) $\int_\gamma e^{2z^2}dz$ where γ is the square of side length 2 centered at 0.

(b) $\int_{|z|=5}(\frac{1}{z}-\frac{1}{z^2})^3 dz$.

(c) $\int_{-\infty}^{\infty}\frac{x+1}{(x^2+1)^2}dx$.

Problem 4. Let g be an entire function satisfying the bound

$$|g(z)| \leq A|z|^n \qquad \forall\, z \in \mathbb{C}$$

for some $A > 0$ and integer $n \geq 0$. Prove that g is a polynomial.

Problem 5. (a) Find an explicit conformal map between the following two sets:

$$\Omega_1 = \{z : |z| < 1 \text{ and } 0 < arg(z) < \alpha < 2\pi\} \quad \text{and} \quad \Omega_2 = \{x+iy : x, y > 0\}.$$

(b) Now let Ω_1^* and Ω_2^* be the sets above with one point removed from each. How many conformal maps are there between these two sets? Prove it.

Problem 6. Determine the conformal automorphism group of the punctured disk: $\mathbb{D}^* = \{0 < |z| < 1\}$, and prove your claim.

Problem 7. Prove that no two of $\mathbb{D} = \{z : |z| < 1\}$, $\mathbb{C}$, and the Riemann sphere $\mathbb{S}$ are conformally equivalent. Recall that two sets are conformally equivalent if there is a bijective, holomorphic map with a holomorphic inverse between them.

Problem 8. Let f be a holomorphic function satisfying $f(0) = 0$, $f'(0) = 0$ and $f''(0) = 0$. Prove that there exists some $\epsilon > 0$ such that for all $w \in B_\epsilon(0)$, $f(z) = w$ has at least 3 zeros, counted with multiplicity.

Real Analysis Prelim Exam
07 July 2016 – 13:00 - 15:00.

Instructions: This exam consists of two groups of 4 problems each. (Group B is on the other side of this sheet.) You need to complete at least two problems from each group. However, completing more of the problems can only help you.

Complete at least 2 of the following problems:

Problem A1. Let $f : \mathbb{R}^d \to [0,1]$ be a Lebesgue measurable function, and for each $n = 1, 2, 3, \ldots$, let

$$W_n = \{x \in \mathbb{R}^d : 2^{-n} \le f(x) \le 1 - 2^{-n}\}.$$

Prove that either (i) there is a measurable set $X \subseteq \mathbb{R}^d$ such that $f = 1_X$ almost everywhere, or (ii) $\lambda(W_n) > 0$ for some n.

Problem A2. Let $f_0, f_1, \ldots, f_n, \ldots : \mathbb{R}^d \to \mathbb{R}$ be a sequence of Lebesgue measurable functions. Prove that the set

$$\{x \in \mathbb{R}^d : (f_0(x), f_1(x), \ldots, f_n(x), \ldots) \text{ is convergent}\}$$

is measurable.
[*Hint*: For $q \in \mathbb{Q}$ and $n, k \in \mathbb{N}$, let $C_{q,n,k} = \{x : |f_n(x) - q| \le 2^{-k}\}$.]

Problem A3. Suppose $f : \mathbb{R}^d \to [0, \infty)$ is an absolutely integrable function.
(a) Prove that the function $\mu : \mathcal{M}^d \to [0,1]$ given by $\mu(A) = \int_{\mathbb{R}^d} f \cdot 1_A$ is a countably-additive measure.
(b) Now, let $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n \subseteq \cdots$ be measurable sets, and let $M = \bigcup_{n=0}^{\infty} M_n$.
Prove that the limit $\lim_{n\to\infty} \int_{\mathbb{R}^d} f \cdot 1_{M_n}$ exists.
[*Hint*: You are going to need the Monotone Convergence Theorem.]

Problem A4. Let $f : \mathbb{R}^d \to [0, \infty)$ be a Lebesgue measurable function.
(a) For all $n, k \in \mathbb{N}$, let $W_{n,k} = \{x \in \mathbb{R}^d : k2^{-n} \le f(x) < (k+1)2^{-n}\}$. Use these sets to prove that there is a sequence of simple functions $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots : \mathbb{R}^d \to [0, \infty)$ such that:
- $\varphi_0 = 0$;
- $\varphi_n \to f$ pointwise;
- For all n, for all $x \in \mathbb{R}^d$, $\varphi_n(x) \le \varphi_{n+1}(x) \le f(x)$.

(b) Prove that there are numbers $c_0, c_1, \ldots, c_p, \ldots$ and a family of measurable sets $M_1, M_2, \ldots, M_p, \ldots$ such that $f = \sum_{p=0}^{\infty} c_p 1_{M_p}$.
[Hint: Notice that $f = \sum_{n=0}^{\infty} (\varphi_{n+1} - \varphi_n)$.]

Complete at least 2 of the following problems:

Problem B1. In a measure space $X = (X, \mathcal{M}, \mu)$, let $f_0, f_1, \ldots, f_n, \ldots$ be a sequence of measurable functions $X \to \mathbb{R}$, and for each n, let $M_n = \{x : |f_n(x)| \geq 2^{-n}\}$.

(a) Define the phrase "$f_n \to 0$ almost uniformly."

(b) Prove that if $\mu(M_n) \leq 2^{-n}$ for all n, then $f_n \to 0$ almost uniformly.

Problem B2. Let $X \subseteq \mathbb{R}^d$ be a Lebesgue measurable set with $\lambda(X) < \infty$, and let $g, f_0, f_1, \ldots, f_n, \ldots$ be Lebesgue measurable functions $\mathbb{R}^d \to \mathbb{R}$ such that $\operatorname{supp}(g) \subseteq X$ and $\operatorname{supp}(f_n) \subseteq X$ for all n. Prove that if $f_n \to g$ in $\mathscr{L}^\infty$-norm, then $\int_{\mathbb{R}^d} f_n \to \int_{\mathbb{R}^d} g$.

Problem B3. In a measure space $X = (X, \mathcal{M}, \mu)$, let $f_0, f_1, \ldots, f_n, \ldots$ be a sequence of measurable functions $X \to \mathbb{R}$.

(a) Define the following two phrases:

- "$f_n \to 0$ in $\mathscr{L}^1$-norm."
- "$f_n \to 0$ in measure."

(b) Prove that if $f_n \to 0$ in $\mathscr{L}^1$-norm, then $f_n \to 0$ in measure.

[*Hint*: "As usual," the sets $M_{n,k} = \{x : |f_n(x)| \geq 2^{-k}\}$ will probably be helpful.]

Problem B4. Let $f : \mathbb{R} \to [0, \infty)$ be a Lebesgue measurable function such that $\operatorname{supp}(f) \subseteq [0, 1]$. Let

$$\begin{aligned} X &= \{x \in [0,1] : f(x) > 1\} \\ Y &= \{x \in [0,1] : f(x) = 1\} \\ Z &= \{x \in [0,1] : f(x) < 1\} \end{aligned}$$

Assuming that $\lim_{n\to\infty} \int_{\mathbb{R}} f^n$ converges in $[0, \infty]$, prove that there are $\alpha, \beta, \gamma \in [0, \infty]$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f^n = \alpha{\cdot}\lambda(X) + \beta{\cdot}\lambda(Y) + \gamma{\cdot}\lambda(Z).$$

[*Hint*: You are going to need the Monotone Convergence Theorem and the Dominated Convergence Theorem.]

Preliminary Examination in Analysis

July, 2015

1. Real Analysis

Solve at least four of the following seven problems. Explain your reasoning and show all work. If you cannot provide a complete solution to a particular problem, a partial solution is welcome, including a discussion about how you think the problem should be approached.

1. Suppose that $(X, \mathcal{A})$ is a measurable space, and f and g are real-valued measurable functions on $(X, \mathcal{A})$. Prove (directly from the definition of measurability) that $f + g$ is measurable.

2. Suppose that $(X, \mathcal{A})$ is a measurable space and suppose that $\mathcal{C} \subset \mathcal{A}$ is an algebra of sets such that $\mathcal{A} = \sigma(\mathcal{C})$. (That is, $\mathcal{A}$ is the smallest $\sigma-$algebra containing $\mathcal{C}$). Suppose that μ and ν are finite measures on $(X, \mathcal{A})$ such that for all sets $C \in \mathcal{C}$, $\mu(C) = \nu(C)$. Prove that for all $A \in \mathcal{A}$, $\mu(A) = \nu(A)$. *Hint: show that the collection consisting of all the sets on which μ and ν agree is a $\sigma-$algebra.*

3. Suppose that $A \subset \mathbb{R}$ is Lebesgue measurable. Prove that the set

$$B = \bigcup_{x \in A} [x - 1, x + 1]$$

   is also Lebesgue measurable.

4. Let $E = [0,1] \setminus \mathbb{Q}$. Construct a closed set $F \subset E$ whose Lebesgue measure exceeds $\frac{3}{4}$.

5. Suppose that $(X, \mathcal{A}, \mu)$ is a measure space and $\{f_n\}$ and f are integrable real-valued functions on $(X, \mathcal{A}, \mu)$. Show that if $f_n(x) \to f(x)$ a.e. and $\int_X |f_n| \, d\mu \to \int_X |f| \, d\mu$, then $\int_X |f_n - f| \, d\mu \to 0$.

2. Complex Analysis

Solve at least four of the following eight problems. Explain your reasoning and show all work. If you cannot provide a complete solution to a particular problem, a partial solution is welcome, including a discussion about how you think the problem should be approached.

1. Classify as removable, pole or essential the isolated singularities of the following functions at the indicated point z_0. For poles, state the order of the pole.

   1. $f(z) = \frac{(e^z-1)^3}{z^6}$, $z_0 = 0$
   2. $g(z) = \frac{\sin z}{z^2-\pi^2}$, $z_0 = \pi$
   3. $h(z) = \frac{\cos \frac{1}{z}}{z^3}$, $z_0 = 0$

2. Show that $x^2 + iy^2$ is differentiable at all points on the line $y = x$, but that f is nowhere analytic.

3. Suppose that f and g are both analytic in a domain D whose closure is compact in $\mathbb{C}$. Show that $|f(z)|+|g(z)|$ takes its maximum on the boundary. *Hint: Consider $f(z)e^{i\alpha} + g(z)e^{i\beta}$ for appropriate α and β.*

4. Calculate the following integrals. (The paths are understood to be traversed counter-clockwise).

   1. $$\int_{|z-i|=10} \left(z+\frac{1}{z}\right)^4 dz$$

   2. $$\int_{|z|=4} \left(\frac{\sin z}{z}\right)^4 dz$$

   3. $$\int_{|z|=\frac{1}{2}} \frac{e^{\sin z}}{z^3 \cos z} dz$$

Real Analysis Preliminary Examination
July, 2014

Please note: The two parts of this exam (Real and Complex) are given equal weight.

1. Definitions and statements
Please answer both of the following.

Problem 1. State one of Littlewood's three principles...

(a) ... in words.

(b) ... in a precise mathematical formulation.

Problem 2. State any three (3) the following theorems or definitions. After each definition, give a quick (no justification needed!) example of such an object. After each theorem, give a short (two sentences at most!) indication of how this theorem was used in our class, or can be used in general. (These should not be long – just enough to give an indication that you know how things fit into the general picture.)

(a) Monotone Convergence Theorem.

(b) Radon-Nikodym Theorem.

(c) The Hahn Decomposition Theorem.

(d) The definition of *unimodular* for a locally compact Hausdorff topological group. (Define unimodular only – not topological group.)

(e) The definition of a *regular content.*

2. Problems
*Answer as many of the following questions as you are able to. You are not expected to complete all of the problems. Provide justification for your answers. Partial solutions **will** be credited, so please show us all your productive work on a problem.*

Problem 3. Prove that μ is σ-finite if and only if there exists a function f in $L^1(\mu)$ such that $f(x) > 0$ for all $x \in X$.

Problem 4. Let $X = Y = \mathbb{R}$, endowed with Borel measures m_X and m_Y, respectively. Show that any open subset of $X \times Y = \mathbb{R}^2$ is measurable for the product measure $m_X \times m_Y$, i.e. that $m_X \times m_Y$ is also a Borel measure.

Problem 5. Let (X, μ) be a measure space. A sequence (f_n) of measurable real-valued functions is said to *converge in measure* to f if given $\epsilon > 0$, there is an integer N and a measurable set B with $\mu(B) < \epsilon$ such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n \geq N$ and all $x \notin B$.

(a) Show that if (f_n) converges in measure to f, then there is a subsequence (f_{n_k}) which converges to f μ-almost everywhere.

(b) Suppose that $\mu(X) < \infty$. Prove that if $f_n \to f$ μ-almost everywhere, then (f_n) converges to f in measure.

Problem 6. Suppose that $\mu(\Omega) = 1$ and that f and g are positive functions on Ω such that $fg \geq 1$. Prove that

$$\int_\Omega f d\mu \int_\Omega g d\mu \geq 1.$$

Problem 7. Construct a Borel set $E \subset \mathbb{R}$ such that

$$0 < m(E \cap I) < m(I)$$

for every finite, non-empty, non-degenerate (i.e. not equal to a single point) interval I.

Problem 8. Given that the Radon-Nikodym Theorem is true for a finite measure space, prove that it is true for a σ-finite measure space.