Graduate preliminary examination, Fall 2003

3 hours, 8 problems counted equally

#1. a) Let f be a function defined on an interval (a, b) and let $c \in (a, b)$. Define what it means to say that f is continuous at c.

b) Use this definition to show that $f(x) = 1/x$ is continuous at $x = 1$.

#2. Find a 2×2 matrix P such that $P \begin{pmatrix} 1 & 2 \\ \frac{2}{9} & 1 \end{pmatrix} P^{-1}$ is diagonal.

#3. Show that the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases}$$

is differentiable everywhere in $(-\infty, \infty)$, but that f' is not continuous at 0.

#4. Let M be an $m \times n$ matrix, and let $V = \{v \in \mathbb{R}^n : Mv = 0\}$ and $W = \{M^t y : y \in \mathbb{R}^m\}$.

a) Prove that V and W are vector subspaces of R^n.

b) Prove that $V = \{x \in \mathbb{R}^n : x \cdot w = 0 \text{ for all } w \in W\}$.

#5. Prove by induction that the sum of the cubes of 3 consecutive positive integers is divisible by 9.

#6. a) Define what is meant for an infinite series $\sum_{n=1}^{\infty} b_n$ of real numbers b_n to converge.

b) Let $a_1, a_2, \ldots$ be a sequence of positive real numbers such that

$$a_1 > a_2 > \cdots > a_n > a_{n+1} > \ldots \text{ and } \lim_{n \to \infty} a_n = 0.$$

Prove that the infinite series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

#7. a) Let γ be a path in the plane $\mathbb{R}^2$. Define what is meant by

$$\int_\gamma f(x, y)\, dx + g(x, y)\, dy.$$

b) Compute this line integral in the case where $f(x, y) = 2xy, g(x, y) = x^2 + y^2$ and γ is the straight-line path from $P = (1, 0)$ to $Q = (0, 1)$.

c) Is there another path β from P to Q such that the corresponding line integral takes a different value?

#8. Let A be an $n \times n$ matrix.

a) Define what is meant by the eigenvalues and eigenvectors of A.

b) Show that if v and w are eigenvectors of A corresponding to distinct eigenvalues λ and μ, then v and w are linearly independent.