2017 GRADUATE PRELIMINARY EXAM

All problems are weighted equally. Throughout $\mathbb{R}$ denotes the real numbers and $\mathbb{C}$ the complex numbers.

(1) Negate the following statements in a "non-cheap" way – especially, avoid using the word "not."
   (a) For all real numbers x, there is a real number y such that $|x-y| \geq 2017$.
   (b) The function $f : \mathbb{R} \to \mathbb{R}$ is continuous.

(2) Let $V = \{(x, y, z) \in \mathbb{R}^3 \mid 3x + 4y + 5z = 0\}$.
   (a) Show that V is a linear subspace of $\mathbb{R}^3$.
   (b) Prove or disprove: there is a linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ with kernel equal to V.
   (c) Prove or disprove: there is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with image equal to V.
   (d) Prove or disprove: there is a linear transformation $U : \mathbb{R}^3 \to \mathbb{R}^3$ with kernel and image equal to V.

(3) Show (we suggest by induction) that for all non-negative integers n, we have

$$\int_0^\infty x^n e^{-x} dx = n!$$

(4) For a positive integer n, let I_n denote the $n \times n$ identity matrix.
   (a) Let A be a 2×2 real matrix with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. Show: $A^2 = I_2$.
   (b) Find a 3×3 real matrix whose only eigenvalues (in $\mathbb{C}$) are 1 and -1 such that $A^2 \neq I_3$.
   (c) Let A be an $n \times n$ real *symmetric* matrix whose only eigenvalues (in $\mathbb{C}$) are 1 and -1. Show that $A^2 = I_n$.

(5) Let $\{f_n : [0,1] \to \mathbb{R}\}_{n=1}^\infty$ be a sequence of continuous functions that converges uniformly to 0. Show that the sequence $\int_0^1 f_n(x)dx$ converges to 0.

(6) Use the ϵ, δ definition of the limit to show: $\lim_{x \to 1} \frac{x^2+1}{x} = 2$.

(7) Let $z = f(x, y)$ be a smooth surface. Show that the gradient is perpendicular to the level curves. (Suggestion: let $\gamma(t)$ be a curve contained in a level set of f, and consider the derivative of $f \circ \gamma$.)

(8) Let X and Y be sets and let $f : X \to Y$ and $g : Y \to X$ be functions. We suppose throughout that $g(f(x)) = x$ for all $x \in X$.
   (a) Show that f is injective.
   (b) Show that g is surjective.
   (c) Give an example in which neither f nor g is bijective.