Preliminary Exam, Spring 2004
(3 hours, 8 problems counted equally)

1. Suppose $f, g : A \to A$ are functions with $f \circ g$ injective.

   a) Prove that g must be injective.

   b) Give an example to show that f need not be injective.

2. a) Give the definition for a function $L : \mathbb{R}^n \to \mathbb{R}^m$ to be a linear transformation.

   b) Prove that the kernel of a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ is a subspace of $\mathbb{R}^n$.

3. a) Define what it means for vectors $v_1, ..., v_n$ in a vector space V to be linearly independent.

   b) Suppose that vectors $v_1, ..., v_n$ in a vector space V span V and that no proper subset of $\{v_1, ..., v_n\}$ spans V. Prove that $v_1, ..., v_n$ are linearly independent.

4. Diagonalize the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$ and use the diagonal form of A to compute A^2.

5. Suppose L is a real number and $f, g : \mathbb{R} \to \mathbb{R}$. Prove that if $\lim_{x \to 0} g(x) = L$ and f is continuous at L, then $\lim_{x \to 0} f(g(x))$ also exists.

6. Define a sequence (x_n) by $x_1 = 1$ and $x_{n+1} = \frac{1}{4}x_n^2 + 1$ for $n \in \mathbb{N}$.

   a) Give an inductive proof that $x_n \le 2$ for all $n \in \mathbb{N}$.

   b) Prove that $x_{n+1} \ge x_n$ for each $n \in \mathbb{N}$. [Hint: Consider $x_{n+1} - x_n$.]

7. Suppose f is a continuous function satisfying the equation $f(x) = 5 + \int_0^x 3f(t)dt$ for all real x. Argue that f must be differentiable and then find all such function(s) explicitly.

8. Provide examples of the following. [No justification is required.]

   a) a true implication whose converse is false,

   b) a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at 7, but not differentiable there,

   c) a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is continuous in each variable separately, but is not continuous at $(0, 0)$,

   d) a bounded sequence which is not Cauchy,

   e) a (real) power series whose domain of convergence is the closed interval $[0, 2]$.