Graduate preliminary exam, Fall 2002
[3 hours, problems counted equally, $\mathbb{R}$ denotes the set of real numbers.]
Instructions: Work 7 of the 8 problems.

1. (a) Suppose that for each positive integer m, we have a set S_m of real numbers, a real number α_m, and a real function f_m. Formulate the negation of this statement: "There exists a positive integer m such that for every $x \in S_m$, $x \geq \alpha_m$ and $f_m(x) = 0$."
(b) The following statement is not valid: "For any positive integer m, if T is a set of positive integers such that (1) $m \in T$ and (2) $n \in T$ implies $n+1 \in T$, then $T = \{\text{positive integers } n : n \geq m\}$." Explain and correct the flaw.
2. For a function $f : \mathbb{R} \to \mathbb{R}$, give the definition of <u>continuity</u> of f at a point a. For functions f and g from $\mathbb{R}$ to $\mathbb{R}$, prove that if f is continuous at a and g is continuous at $b = f(a)$, then $g \circ f$ is continuous at a. [The composition $g \circ f$ is defined by $(g \circ f)(x) = g(f(x))$ for $x \in \mathbb{R}$.]
3. Define what it means for a series $\sum_{n=1}^{\infty} a_n$ (of real numbers a_n) to <u>converge</u> to a real number S. Prove that the series $\sum_{n=1}^{\infty} 1/10^n$ converges.
4. Determine the 2×2 matrix A such that $A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}5\\-6\end{bmatrix}$ and $A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$. Prove that the matrix A is not diagonalizable.
5. Let $v_1, \ldots, v_k$ be vectors in $\mathbb{R}^n$. Define the <u>span</u> $V = < v_1, \ldots, v_k >$ of the vectors. Prove that there is a subset of the vectors $v_1, \ldots, v_k$ that forms a basis for V.
6. For the cubic polynomial $x^3 - 3x + 2$, use the cubic formula described below to find the root 1. [The roots of this cubic are 1 (with multiplicity 2) and -2. You do not have to prove the validity of the cubic formula.]
For a cubic polynomial $x^3 + px + q$, the roots can all be found by the following formula, carried out in the complex number system. Let s be a square root of $q^2/4 + p^3/27$, set $A = -q/2 + s$, and then let c be a cube root of A. Then $c - p/3c$ is a root of $x^3 + px + q$.
7. Let k and n be positive integers with $k \leq n$. Give the definitions for the <u>permutations</u> and for the <u>combinations</u> of k elements from an n-element set, and state formulas for the numbers of these. Derive a formula for the number of one-to-one functions from a k-element set to an n-element set.
8. (a) State the Cauchy integral formula and use it to evaluate the complex line integral: $\oint_\gamma z^2/(z-i)\, dz$, where γ is a circle centered at i and oriented counterclockwise.
(b) Use a parametrization of the path γ to express the line integral explicitly in terms of real integrals, with limits of integration. [You do not have to evaluate these real integrals.]