Topology Qual Workshop Day 4: Counterexamples

Warm-up Problems:

- For a set $A \subset X$, $\overline{A}$ is defined to be the intersection of all closed sets containing A. Using this definition, show: $x \in \overline{A}$ if and only if every open set U containing x intersects A.

- Let $f : X \to Y$ be a continuous, bijective, open map. Show that f is a homeomorphism. [Open map can be replaced with closed map and the same result holds.]

- Provide an example of a topological space where open sets are compact. Then find an infinite set with a Hausdorff topology where open sets are compact.

(1) (June '09 # 2A) For a topological space X and $y \in X$, the *path component* P_y of X containing y is the largest path-connected subset with $y \in P_y \subseteq X$.

   (a) Show that this concept is well defined (that is, show that every point y is contained in a largest path-connected subset).

   (b) Give an example of a space and a point $y \in X$ so that P_y is neither an open nor a closed subset of X.

(2) (June '08 # A1) Let A be a subset of a topological space X, and let B be a subset of a topological space Y. Let $X \times Y$ be the product space, and let $\text{int}_Z(C)$ denote the interior of the set C in the space Z. Prove or give a counterexample to $\text{int}_{X\times Y}(A \times B) = \text{int}_X(A) \times \text{int}_Y(B)$.

(3) (Jan '04 # A3) Let (X, τ) be a Hausdorff space and let $\tau' = \{U \subseteq X : X \backslash U \subseteq X \text{ is compact}\} \cup \{\emptyset\}$. Show that τ' is a topology on X and is coarser than τ. Show that, in general, they need not be equal.

(4) Let X be a compact space. If $X = A \cup B$ with both A and B Hausdorff, must X be Hausdorff?

Tips: The following are common places to look for counterexamples:

- Flea and Comb (Differentiates Connected and Path Connected)
- Comb (Sans the flea) (Differentiates Path Connected and Locally Path Connected)
- Topologists Sine Curve (Differentiates Connected and Path Connected)
- 2 or 3 point sets (Especially useful if you assume your space must be compact)
- Any space with the Discrete Topology
- Any space with the Indiscrete Topology

Topology Qual Workshop Day 4: Assorted Problems

(1) (June '10 # A1) Let X be a topological space and let $f, g : X \to \mathbb{R}$ be continuous functions.

(a) Show that the set $L = \{p \in X : f(p) \leq g(p)\}$ is a closed subset of X.

(b) Show that the function $h : X \to \mathbb{R}$ given by $h = \min\{f(p), g(p)\}$ is continuous.

(2) (June '08 # A3)

(a) Prove that a map $f : X \to Y$ between topological spaces is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.

(b) Prove that if f is continuous and $f(\overline{A})$ closed for some $A \subseteq X$, then $f(\overline{A}) = \overline{f(A)}$

(3) (Jan '08 # B8) Suppose that X is an arbitrary topological space and Y is a compact space. Consider the projection map $\pi : X \times Y \to X$ defined by $\pi(x, y) = x$. Prove that if $X \times Y$ has the product topology, then π is a closed map.

(4) (June '04 # A2) Let X be the unit sphere in $\mathbb{R}^3$ and define an equivalence relation on X by

$$(x, y, z) \sim (x', y', z') \Leftrightarrow z = z'$$

Let $Z = X/\sim$ be the quotient space under this equivalence relation, with the quotient topology. Show that Z is homeomorphic to the interval $[-1, 1]$.