Fall 2004 Preliminary Exam

Instructions: 3 hours, 9 problems counted equally; show your work and justify your answers.

1. Let a be a real number other than 1. Use induction to show that for each positive integer n,

$$\sum_{k=0}^{n-1} a^k = (a^n - 1)/(a - 1).$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let a be a real number. Let S be the statement "If f has a local minimum at a, then $f'(a) = 0$ or f is not differentiable at a".

a) Write the contrapositive of S.
b) Write the converse of S.
c) Which of the statements in (a) and (b) are true? If either statement is false, give a counterexample.

3. Let α be the complex number $-2 + 2\sqrt{3}\,i$. Express α^3 and the two square roots of α in the standard form $a + bi$.

4. Prove that there is no $n \in \mathbb{N}$ so that $(21n - 3)/4$ and $(15n + 2)/4$ are both integers.

5. Let $T : V \to W$ be an injective linear transformation and suppose $v_1, \ldots, v_n$ are linearly independent vectors in V. Prove that the vectors $T(v_1), \ldots, T(v_n)$ are linearly independent in W.

6. Find *all* real numbers x for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n^2+1}}$ converges. [Be sure to justify your analysis.]

7. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$.

a) State the $\epsilon - \delta$ definition of what it means for f to be continuous at a.
b) Prove that if f and g are continuous at 3, then their sum $f + g$ is also continuous at 3.

8. Evaluate the line integral $\oint_C P\,dx + Q\,dy$ where $P = y^2$, $Q = 2x - 3y$, and C is the circle $x^2 + y^2 = 9$.

9. Suppose $f : \mathbb{R} \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are differentiable. Prove that the composite function $f \circ g$ cannot have a differentiable inverse.