MATHEMATICS PRELIMINARY EXAM: FALL, 2006

(1) Consider the statement:

"If there exists a purple apple, then all lemons are pink."

(A) Give the negation, the converse, and the contrapositive of the statement above.
(B) Assuming the statement is true, which (ones, if any) of the statements formulated in part (A) must necessarily be true?

(2) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} .$$

(3) Compute the determinant of the matrix

$$B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 0 & 1 \\ 4 & -1 & 1 & 1 \\ -1 & 2 & 1 & 2 \end{bmatrix} .$$

(4) Consider the vector space of polynomials over $\mathbb{Q}$ spanned by

$$p_1(x) = x^2 + x + 1, \quad p_2(x) = x^2 + 2x, \quad p_3(x) = x^2 + 2, \quad p_4(x) = x - 1 .$$

Find the dimension of this vector space.

(5) Let $f(x) = xe^{2x}$. Writing $f^{(n)}(x)$ for the n^{th} derivative of $f(x)$, prove by induction that $f^{(n)}(x) = 2^n xe^{2x} + n2^{n-1}e^{2x}$ for all $n \geq 0$.

(6) (A) Find the Maclaurin series expansion of $f(x) = xe^{2x}$ (that is, the Taylor series expansion of $f(x)$ about $a = 0$).
(B) If $T_5(x)$ is the polynomial consisting of the terms of the Maclaurin series of $f(x)$ through degree 5, and $T_5(x)$ is used to approximate $f(x)$ on the interval $[0, 1/2]$, find a bound for the maximum error $|f(x) - T_5(x)|$ on $[0, 1/2]$.

(7) Using the method of Lagrange Multipliers, find the maximum and minimum values of $f(x, y) = xy$ on the ellipse $x^2 + 4y^2 = 8$.

(8) Let R be the planar region between the circles $x^2 + y^2 = 4$ and $(x - 1)^2 + y^2 = 1$, and lying in the halfplane $y \geq 0$. Let ∂R be its boundary, oriented so that R is on its left. Either using the definition or by applying theorems of Calculus, compute the line integral

$$\int_{\partial R} x\,dy + y\,dx .$$

(9) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous for all x. Using an ε–δ argument, show that $f(x)g(x)$ is continuous for all x.