Graduate preliminary examination, Spring 2007

3 hours, 8 problems counted equally

1) Write the following statement in symbolic form, and then give (in symbolic form and in English) its contrapositive and its negation:

"If all birds can swim or some fish can fly, then no whales can walk"

2) Using mathematical induction, show that for each positive integer n,

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1} n^2 \ = \ (-1)^{n-1} n(n+1)/2$$

(Hint: in the induction step, consider separately the cases where n is even or odd.)

3) a) Give the $\varepsilon - \delta$ definition of the one-sided limit $\lim_{x \to a^+} f(x) = L$.

b) Using the definition and basic properties of $\ln(x)$, show that $\lim_{x \to 0^+} 1/\ln(x) = 0$.

4) a) Find the Taylor expansion of $f(x) = \ln(2+x)$ about the origin.

b) Find the radius of convergence R of that series;

c) Use Taylor's theorem to show that the series converges to $f(x)$ on $[0, R/2]$.

5) Let γ be the closed path which goes counterclockwise around the circle $C(0,2) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 2^2\}$ from the point $(2,0)$ to the point $(0,-2)$, then goes to the origin along the y-axis, and then back to $(2,0)$ along the x-axis.

Compute $\int_\gamma 2x dy + y dx$ in one of two ways:

a) Directly, using the definition of a line integral; or

b) By using Green's theorem or the general Stokes' theorem.

6) Give an example (proof not required) of each of the following:

a) A function $f : \mathbb{R} \to \mathbb{R}$ which satisfies $f(x) = 0$ for all $x \leq 0$, but which is nonzero for $x > 0$ and has derivatives of all orders for all x.

b) An infinite-dimensional vector space over the field $\mathbb{Z}/2\mathbb{Z}$.

c) A matrix in $M_2(\mathbb{C})$ which is not diagonalizable .

d) A power series $\sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[[z]]$ with radius of convergence 0 .

7) Let $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -5 \\ 2 & -1 & 1 \end{bmatrix} \in M_3(\mathbb{R})$, and let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation $T_A(\vec{x}) = A\vec{x}$. Find $\dim_{\mathbb{R}}(Ker(A))$, and use this to determine $\dim_{\mathbb{R}}(\mathrm{Im}(A))$.

8) a) State the Spectral Theorem (over $\mathbb{R}$).

b) Let $A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$. Find an orthogonal matrix P for which $P^{-1}AP$ is diagonal.