Topology Qualifying Exam Workshop

May 2020

Worksheet 3A

Theme: Fundamental groups: Fundamental group, induced homomorphism; free group, group presentation, Tietze's theorem, amalgamated product of groups, Seifert - van Kampen Theorem; cell complex, presentation complex, Classification of surfaces.

Part 1 - 5-10 minutes

Warm-up:

1. Create a cell complex for S^2 and a cell complex for $T = S^1 \times S^1$.
2. State the Seifert-van Kampen Theorem.
3. Briefly describe the classification of compact surfaces without boundary.

Part 2 - 1 hour and 40 minutes

Try these problems:

1. (June 2012) Let X be the space obtained from the torus $T = S^1 \times S^1$ by gluing in a disk D^2 along its boundary $\partial D^2 = S^1$ using the map $\alpha : S^1 \to T$ given by $z \mapsto (z, (1, 0))$ for $z \in S^1$.
   (a) Find $\pi_1(X)$.
   (b) Give a CW complex for X.
2. (May 2017) If $H : X \times [0, 1] \to X$ is a homotopy with $H_0 = H_1 =$ the identity map, show that the map $\gamma : I \to X$ given by $\gamma(t) = H(x_0, t)$ is a loop in X representing an element $g = [\gamma] \in \pi_1(X, x_0)$ which lies in the center of $\pi_1(X, x_0)$, i.e. $gh = hg$ for all $h \in \pi_1(X, x_0)$.
3. (January 2019) Let $U = \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$, and let $W = \mathbb{R}^3 - U$.
   Let $V = \{(x, y, z) : \mathbb{R}^3 : x^2 + y^2 + z^2 = 4\} \cup \{(0, 0, z) \in \mathbb{R}^3 : -2 \leq z \leq 2\}$.
   (a) Describe a deformation retraction of W onto V.
   (b) Use (a) and the Seifert-van Kampen Theorem to compute $\pi_1(W)$.
4. (May 2019) Let A be the annulus $\{re^{i\theta} \in \mathbb{C} : 1 \leq r \leq 2\}$, and let B be the quotient space obtained from A by identifying each point $e^{i\theta}$ on the circle $r = 1$ with the point $2e^{i(\theta+\pi)}$ on the circle $r = 2$. Determine $\pi_1(B)$.

Part 3 - 5-10 minutes

Wrap-up discussion with everyone:

1. Which of these problems did you find easy? Hard?
2. Which topics in the theme do you need to review more?