Sample graduate preliminary exam (3 hours, 9 problems counted equally)

1. Define what is meant by an inverse function, and give necessary and sufficient conditions for an inverse function to exist. Find the inverse function of $f(x) = (x+2)/(3x-1)$ for x real, $x \neq 1/3$.

2. Let a be a real number and let I be an open interval containing a. For a function $f : I \to \mathbb{R}$, give the definition of continuity at a. Prove that $f(x) = x^2$ is continuous at $a = 2$.

3. Give the definition for convergence of a sequence of real numbers $\{a_n\}_{n=1,2,\dots}$ to a real number a. Prove that the sequence $a_n = (n^2+2)/(n^2+3n)$ converges to $a = 1$.

4. Work out the Taylor series for $f(x) = \ln x$ around $x = 1$ and use it to approximate $\ln(1.1)$ accurate to 2 decimal places.

5. Evaluate the integral $\displaystyle\int_0^{10\pi} |\sin x| dx$.

6. Define the Jacobian matrix of a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point $a = (a_1, \dots, a_n)$ of $\mathbb{R}^n$. Compute the Jacobian matrix of $f(r,\theta) = (r\cos\theta, r\sin\theta)$ at $a = (0,0)$.

7. Assuming that u, v, and w are linearly independent (in $\mathbb{R}^n$), prove that $2u+v$, $u+v+w$, and $v-w$ are linearly independent.

8. Diagonalize the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

9. Find the 3 cube roots of i and compute $(1+i)^5$.

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