Preliminary Exam, Spring 2005

(3 hours, 8 problems counted equally)

#1. Let $P(x)$ and $Q(x)$ be open sentences containing the variable x, and consider the following statements.

$$A : \forall x, (P(x) \Rightarrow Q(x)) \quad \text{and} \quad B : (\forall x, P(x)) \Rightarrow (\forall x, Q(x))$$

(a) Prove that $A \Rightarrow B$.
(b) Give an example of open sentences $P(x)$ and $Q(x)$ to show that $B \Rightarrow A$ need not be true.

#2. Recall that the Fibonacci numbers are defined by $F_0 = 1, F_1 = 1$, and then

$$F_n = F_{n-1} + F_{n-2} \quad \text{for integers } n \geq 2.$$

Prove that any two successive Fibonacci numbers F_n, F_{n+1} are relatively prime.

#3. Prove that there is exactly one real value of x satisfying $x^3 = 29 - x$.

#4. (a) Give the ϵ, δ definition for continuity of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x_0 \in \mathbb{R}$.
(b) Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that $\{a_n\}$ is a sequence of real numbers with $\lim_{n\to\infty} a_n = L$. Prove that $\lim_{n\to\infty} f(a_n) = f(L)$.

#5. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation satisfying $T(1,1) = (5,3)$ and $T(2,3) = (7,9)$. Find the standard matrix of T.

#6. Suppose A is a 3×3 matrix and $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^3$. Prove the following.
(a) If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent, then $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3$ are linearly dependent.
(b) If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent and $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3$ are linearly dependent, then A is singular.

#7. Find all cube roots of 2-2i and express them in the standard form $a + bi$.

#8. (a) Provide examples to show that the series $\sum_{n=1}^{\infty} a_n^2$ may or may not converge when the series $\sum_{n=1}^{\infty} a_n$ converges conditionally.

(b) Prove that if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then the series $\sum_{n=1}^{\infty} a_n^2$ must converge.

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