Department of Mathematics
PRELIMINARY EXAMINATION

August 10, 2007

2:30–5:00 pm

Work all of the following problems, justifying your answers. The problems are weighted evenly.

$\mathbb{R}$ denotes the set of real numbers and $\mathcal{P}_n$ denotes the vector space of (real) polynomials of degree $\leq n$. As usual, $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ is the binomial coefficient.

1. a. Suppose A and B are false statements. Is the statement

$$(A \implies B) \implies (A \vee B)$$

true or false? ("$\vee$" denotes "or.")

b. Give the negation of the statement

If all blockoids are split and some blockoid is nontrivial, then there is a short blockoid.

2. Let k be a nonnegative integer. Prove by mathematical induction that for all $n \geq k$ we have

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}.$$

3. Suppose A, B, and C are sets, and $f\colon B \to C$ and $g\colon A \to B$ are functions.

a. Prove that if f and g are surjective (onto), then so is $f \circ g$.

b. Prove or give a counterexample: If g is not surjective, then $f \circ g$ is not surjective.

4. a. Complete the δ-ε definition of a limit: Given a function $f\colon \mathbb{R} \to \mathbb{R}$ and $a, L \in \mathbb{R}$,

$$\lim_{x \to a} f(x) = L \qquad \text{means} \quad \ldots$$

b. Using the definition, prove that

$$\lim_{x \to 1} \frac{1}{1+x^2} = \frac{1}{2}.$$

5. Let Δ be the triangle in $\mathbb{R}^2$ with vertices $(0,1)$, $(2,0)$, and $(2,1)$, traversed counterclockwise. Evaluate the line integral

$$\int_{\Delta} y\,dx + 2x\,dy.$$

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6. Suppose V and W are vector spaces and $T\colon V \to W$ is a linear transformation. Suppose that $T(v) = 0$ only when $v = 0$. Prove that if $\{v_1, \dots, v_k\}$ is a linearly independent set of vectors in V, then $\{T(v_1), \dots, T(v_k)\}$ is a linearly independent set of vectors in W.

7. Give an example (without proof) of each of the following:

   a. An integrable function $f\colon \mathbb{R} \to \mathbb{R}$ so that the function $F(x) = \displaystyle\int_0^x f(t)\,dt$ is differentiable everywhere but at $x = 1$.

   b. A sequence $\{a_n\}$ of real numbers such that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges and $\displaystyle\sum_{n=1}^{\infty} a_n^2$ diverges.

   c. A basis for the subspace of $\mathcal{P}_3$ spanned by x^2+x+1, x^3-x+2, x^3+x^2+3, and $-x^3+x^2+2x+1$.