Mathematics Preliminary Exam, Fall 2012

Attempt all problems; they are weighted equally. Write $\mathbb{N}$ for the set of positive integers and $\mathbb{R}$ for the set of real numbers.

1. Write the negations of these sentences in as "smooth" a way as possible. (In particular, you may not simply append "It is not the case that ...". Also, you should make explicit any "hidden" quantifiers.)
   (a) There is a real number x such that for every real number y, $|x - y| > 1$.
   (b) A real-valued function that is continuous on a closed interval attains a minimum value on that interval.
   (c) $3n + 1$ is even if and only if $n^2 + 4$ is prime.

2. Let $A = \begin{pmatrix} 7 & -3 \\ 1 & 3 \end{pmatrix}$. Find an invertible matrix P and a diagonal matrix D with $P^{-1}AP = D$. [You should not have to compute P^{-1}.]

3. Let A be an $m \times n$ real matrix. Write A^t for the transpose of A, and $N(A)$ for the nullspace of A. Prove that $N(A^tA) = N(A)$.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(xy) = xf(y) + yf(x)$ for all $x, y \in \mathbb{R}$. Prove that $f(1) = 0$ and that $f(u^n) = nu^{n-1}f(u)$ for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$.

5. Give an ϵ–δ proof that $\displaystyle\lim_{x\to 2} \frac{1}{x^2+1} = \frac{1}{5}$.

6. Let $f : X \to Y$ be a (not necessarily invertible!) function, and $A \subseteq X$.
   (a) Prove that $A \subseteq f^{-1}(f(A))$.
   (b) Prove that if f is injective (one-to-one) then $A = f^{-1}(f(A))$.
   (c) Give an example for which $A \neq f^{-1}(f(A))$.

7. Prove that the line integral $\displaystyle\int_C (x + y^3)\,dx + (e^y + 3xy^2)\,dy$ is path-independent; i.e., it depends only on the endpoints of C.

8. Let $f_n(x) = \dfrac{nx}{n + x}$ for $x \in [0, \infty)$ and $n \in \mathbb{N}$.
   (a) Find a function f such that $\{f_n\}$ converges to f pointwise on $[0, \infty)$.
   (b) Is the convergence uniform on $[0, \infty)$? Justify your answer.

9. Suppose R is a commutative ring (with 1), I is a proper ideal in R, and $a \in R$. Suppose $\langle a\rangle + I = R$. Prove that $a + I$ is a unit (i.e., invertible) in the quotient ring R/I.
   (For half credit: Prove that if a and n are relatively prime integers, then $a + n\mathbb{Z}$ is a unit in $\mathbb{Z}/n\mathbb{Z}$.)