Department of Mathematics
PRELIMINARY EXAMINATION

August 10, 2010

8:45–11:45 am

Work all of the following problems, justifying your answers. The problems are weighted evenly.

$\mathbb{N}$ denotes the set of positive integers.

1. Negate each of the following statements in the most informative way possible (i.e., without using the words "no" or "not").
   a. There is an integer x so that $x + y$ is odd for each integer y.
   b. If x is an odd integer, then xy is odd for every integer y.
   c. Given $\varepsilon > 0$, there is $N \in \mathbb{N}$ so that whenever $n > N$, we have $\left|\dfrac{2+n}{1+n} - 1\right| < \varepsilon$.

2. a. State the binomial theorem.
   b. Prove that $(x + y)^5 \equiv x^5 + y^5 \pmod 5$.
   c. Prove by mathematical induction that $(x + y)^{5^n} \equiv x^{5^n} + y^{5^n} \pmod 5$ for every $n \in \mathbb{N}$.

3. Suppose $f: A \to B$ and $g: B \to C$ are functions. Prove or give a counterexample:
   a. If f and g are injective (one-to-one), then $g \circ f$ is injective.
   b. If $g \circ f$ is injective, then f and g are injective.

4. Suppose $f: \mathbb{R} \to \mathbb{R}$.
   a. Give the δ-ε definition of $\lim\limits_{x \to a} f(x) = \ell$.
   b. Determine the limit and *use the definition* to prove your answer:

$$\lim_{x \to 2} \frac{2x + 1}{x^2 + 1} = ?$$

—OVER—

5. Prove that $\int_C (x^3 - y^3)\,dx + (x^3 + y^3)\,dy \geq 0$ for every smooth, simple closed curve $C \subset \mathbb{R}^2$, oriented counterclockwise.

6. Find all the cube roots of $-2 - 2i$.

7. Let $A = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix}$. Prove that A is diagonalizable and find a closed-form expression for $A^k \begin{bmatrix} 1 \\ 7 \end{bmatrix}$, $k \in \mathbb{N}$.

8. Suppose V and W are vector spaces and $T\colon V \to W$ is a linear transformation. Suppose $v_1, \ldots, v_k \in V$. Prove that if $T(v_1), \ldots, T(v_k)$ form a linearly independent set in W, then $v_1, \ldots, v_k$ form a linearly independent set in V.

9. Prove that $\quad x - \dfrac{x^3}{3} \leq \arctan x \leq x \quad$ for all $x \geq 0$.