Graduate Preliminary Exam, August 12, 2005

Three hours. problems counted equally.

1. Let Z be a set and let $X_1, X_2, \ldots$ be subsets of Z. Prove the formula

$$\left(\bigcap_i X_i\right)^c = \bigcup_i X_i^c \text{ , where } (\cdot)^c \text{ denotes complement.}$$

2. i) Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ and let $a \in \mathbb{R}$. From the definition of the derivative, prove that if f is differentiable at a then f is continuous at a.

ii) Prove the product rule, $(fg)' = f'g + g'f$.

3. For $n = 5, 6$, either give an example of a nonabelian group of order n, or prove that none exists.

4. Find all cube roots of 8 in the complex plane. Write your answers in the form $a + bi$, where a and b are real numbers, and justify your answer.

5. Let C be a circle in the xy-plane, oriented counterclockwise, and not passing through the origin. Prove that $\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ equals 0 if the origin is outside the circle, and 2π if the origin is inside the circle.

6. i) Let $f(x, y)$ be a differentiable function from $\mathbb{R}^2$ to $\mathbb{R}$ and let $g(t) = (x(t), y(t))$ be a differentiable function from $\mathbb{R}$ to $\mathbb{R}^2$. State the chain rule for the derivative of the function $f(g(t))$.

ii) Let $f(x, y)$ be a continuously differentiable function on $\mathbb{R}^2$, and let $P \in \mathbb{R}^2$ be a point. Assume that in a neighborhood of P, the equation $f(x, y) = f(P)$ implicitly defines y as a differentiable function of x, say $y = h(x)$. Show that the tangent line to the graph of $y = h(x)$ at P is perpendicular to the gradient of f at P.

7. Let $a_0, a_1, a_2, \ldots$ be a decreasing sequence of positive real numbers. Prove that for every nonnegative integer m, $0 \le \sum_{i=0}^{m}(-1)^i a_i \le a_0$. (Hint: $\sum_{i=0}^{m+1}(-1)^i a_i = a_0 - \sum_{i=0}^{m}(-1)^i a_{i+1}$.)

8. Find an invertible matrix A and a diagonal matrix B such that $\begin{pmatrix} 8 & 9 \\ -6 & -7 \end{pmatrix} = ABA^{-1}$.

9. i) State the binomial theorem.

ii) Prove that if p is a prime number, then $(x+y)^p \equiv x^p + y^p \mod p$.