Graduate Preliminary Exam, Spring 2006
(9 problems counted equally, 3 hours)

\# 1. i) State what it means for a sequence $\{a_n\}$ of real numbers to be *Cauchy*.
ii) Prove that every convergent sequence is Cauchy.

\# 2. Prove that for every positive integer n, $n^3 + 2n$ is divisible by 3.

\# 3. Find all complex numbers $z = x + iy$, $(x, y \in \mathbb{R})$, such that $e^z = 2i$.

\# 4. Let $L \subset \mathbb{R}^2$ be the line spanned by the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Let $P \subset \mathbb{R}^2$ be the line defined by the equation $x + 2y = 0$. Find the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(v) = 0$ for $v \in L$ and $T(v) = v$ for $v \in P$.

\# 5. Find an invertible matrix Q so that $Q^{-1} \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} Q$ is diagonal.

\# 6. Let f be a Riemann integrable function on a closed interval $[a, b]$. For $x \in [a, b]$, let $g(x) = \int_a^x f(t)dt$. Prove that g is continuous.

\# 7. Let τ be the arc of the unit circle in the first quadrant, from $(1, 0)$ to $(3/5, 4/5)$. Compute

$$\int_\tau xdy + ydx \ .$$

\# 8. Let $f(x, y) = \frac{x^2 y}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$ and let $f(0, 0) = 0$.
i) Prove that f is continuous at $(0, 0)$.
ii) Prove that f is not differentiable at $(0, 0)$.

\# 9. For each positive integer n, let $f_n(x) = \frac{x}{x+n}$ for $x \in [0, \infty)$. Show that the sequence of functions $\{f_n\}$ converges pointwise on $[0, \infty)$ to the 0-function but does not converge uniformly.