Graduate Preliminary Examination, Spring 2003
[3 hours, problems counted equally, $\mathbb{R}$ denotes the set of real numbers.]

Instructions: Work 7 of the 8 problems.

#1. (a) Define the <u>span</u> of vectors $v_1, ..., v_k$ in $\mathbb{R}^n$.
(b) Prove that if w lies in the span of $v_1, ..., v_k$, then the span of $v_1, ..., v_k, w$ is equal to the span of $v_1, ..., v_k$.

#2. Does there exist a 3×2 complex matrix A such that

$$A^t A = I_2, \text{ and } AA^t = I_3?$$

If so, give an example; if not, prove it. Here I_k is the $k \times k$ identity matrix.

#3. Prove the formula $\sum_{i=1}^n i^2 = \dfrac{n(n+1)(2n+1)}{6}$.

#4. Prove that $x > \ln x$ for all $x > 0$.

#5. Prove that any linear function $L : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous.

#6. Does there exist a divergent series $\sum_{i=0}^{\infty} a_i$ of positive real numbers a_i such that $\sum_{i=0}^{\infty} \sqrt{a_i}$ converges? If so, give an example; if not, prove it.

#7. (a) Let f be a real-valued function defined on $\mathbb{R}^n$. Define <u>differentiability</u> of f at a point p.
(b) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \frac{xy}{x^2 + y^2} \text{ for } (x, y) \neq (0,0), \text{ and } f(0,0) = 0$$

is not differentiable at $(0,0)$.

#8. Consider the line integral $\int_C 2x dx + x^2 y dy$, where C is the boundary of the unit square in the 1st quadrant. Evaluate the line integral (a) directly and (b) by using Green's theorem.

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