Qual Complex Analysis

Table of Contents

Contents

1 Preface

I'd like to thank the following individuals for their contributions to this document:

- Edward Azoff, for supplying a problem sheet broken out by topic.
- Mentzelos Melistas, for explaining and documenting many solutions to these questions.
- Jingzhi Tie, for supplying **many** additional problems and solutions.
- Swaroop Hegde for supplying a number of proofs

2 Real Analysis Review

2.1 Tie's Extra Questions: Fall 2015 (Computing area)

Problem 2.1.1 (?)

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic and one-to-one in $|z| < 1$. For $0 < r < 1$, let D_r be the disk $|z| < r$. Show that the area of $f(D_r)$ is finite and is given by

$$S = \pi \sum_{n=1}^{\infty} n|c_n|^2 r^{2n}.$$

Note that in general the area of $f(D_1)$ is infinite.

Solution omitted.

2.2 Tie's Extra Questions: Fall 2015 (Variant)

Problem 2.2.1 (?)

Let $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ be analytic and one-to-one in $r_0 < |z| < R_0$. For $r_0 < r < R < R_0$, let $D(r, R)$ be the annulus $r < |z| < R$. Show that the area of $f(D(r, R))$ is finite and is given by

$$S = \pi \sum_{n=-\infty}^{\infty} n|c_n|^2 (R^{2n} - r^{2n}).$$

Solution omitted.

2.3 Spring 2019.1

Define

$$E(z) = e^x(\cos y + i \sin y).$$

- Show that $E(z)$ is the unique function analytic on $\mathbb{C}$ that satisfies

$$E'(z) = E(z), \quad E(0) = 1.$$

- Conclude from the first part that

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

2.4 Recurrences

Problem 2.4.1 (?)
Let $x_0 = a, x_1 = b$, and set

$$x_n := \frac{x_{n-1} + x_{n-2}}{2} \qquad n \geq 2.$$

Show that $\{x_n\}$ is a Cauchy sequence and find its limit in terms of a and b.

Solution omitted.

2.5 Uniform continuity

Problem 2.5.1 (?)
Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{x \to \pm\infty} f(x) = 0$. Prove that f is uniformly continuous.

Solution omitted.

2.6 Negating uniform continuity

Tie, Fall 2009

Problem 2.6.1 (?)
Show that $f(z) = z^2$ is uniformly continuous in any open disk $|z| < R$, where $R > 0$ is fixed, but it is not uniformly continuous on $\mathbb{C}$.

Solution omitted.

2.7 Non-continuously differentiable

Problem 2.7.1 (?)
Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is everywhere differentiable but f' is not continuous at 0.

Solution omitted.

2.8 Uniformly convergent + uniformly continuous

Problem 2.8.1 (?)
Suppose $\{g_n\}$ is a uniformly convergent sequence of functions from $\mathbb{R}$ to $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous. Prove that the sequence $\{f \circ g_n\}$ is uniformly convergent.

Solution omitted.

2.9 Uniform differentiability

Problem 2.9.1 (?)
Let f be differentiable on $[a,b]$. Say that f is *uniformly differentiable* iff

$$\forall\varepsilon > 0,\ \exists\delta > 0 \text{ such that } \quad |x-y| < \delta \implies \left|\frac{f(x)-f(y)}{x-y} - f'(y)\right| < \varepsilon.$$

Prove that f is uniformly differentiable on $[a,b] \iff f'$ is continuous on $[a,b]$.

Solution omitted.

2.10 Inf distance

Problem 2.10.1 (?)
Suppose $A, B \subseteq \mathbb{R}^n$ are disjoint and compact. Prove that there exist $a \in A, b \in B$ such that

$$\|a-b\| = \inf\left\{\|x-y\| \;\middle|\; x \in A,\ y \in B\right\}.$$

Solution omitted.

2.11 Connectedness

Problem 2.11.1 (?)
Suppose $A, B \subseteq \mathbb{R}^n$ are connected and not disjoint. Prove that $A \cup B$ is also connected.

Solution omitted.

2.12 Pointwise and uniform convergence

Problem 2.12.1 (?)
Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ such that

$$f_n(x) \geq f_{n+1}(x) \geq 0 \quad \forall n \in \mathbb{N},\ \forall x \in [0,1].$$

Prove that if $\{f_n\}$ converges pointwise to 0 on $[0,1]$ then it converges to 0 uniformly on $[0,1]$.

Solution omitted.

2.13

Problem 2.13.1 (?)
Show that if $E \subset [0,1]$ is uncountable, then there is some $t \in \mathbb{R}$ such that $E \cap (-\infty, t)$ and $E \cap (t, \infty)$ are also uncountable.

Solution omitted.

2.14

Problem 2.14.1 (?)
Suppose $f, g : [0,1] \to \mathbb{R}$ where f is Riemann integrable and for $x, y \in [0,1]$,

$$|g(x) - g(y)| \leq |f(x) - f(y)|.$$

Prove that g is Riemann integrable.

Solution omitted.

2.15 Exercises

Problem 2.15.1 (Uniform continuity of x^n)
Show that $f(x) = x^n$ is uniformly continuous on any interval $[-M, M]$.

Solution omitted.

Problem 2.15.2 (?)
Show $f(x) = x^{-n}$ for $n \in \mathbb{Z}_{\geq 0}$ is uniformly continuous on $[0, \infty)$.

Solution omitted.

Problem 2.15.3 (?)
Show that f' bounded implies f is uniformly continuous.

Solution omitted.

Problem 2.15.4 (?)
Show that the Dirichlet function $f(x) = \chi_{I \cap \mathbb{Q}}$ is not Riemann integrable and is everywhere discontinuous.

Solution omitted.

3 | Continuity

3.1 1

Is the following function continuous, differentiable, continuously differentiable?

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & \text{else.} \end{cases}$$

3.2 ?

Show that $f(z) = z^2$ is uniformly continuous in any open disk $|z| < R$, where $R > 0$ is fixed, but it is not uniformly continuous on $\mathbb{C}$.

3.3 6

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable with $F(0, 0) = 0$ and $\|\nabla F(0, 0)\| < 1$.

Prove that there is some real number $r > 0$ such that $|F(x, y)| < r$ whenever $\|(x, y)\| < r$.

3.4 2 Multivariable derivatives

a. Complete this definition: "$f : \mathbb{R}^n \to \mathbb{R}^m$ is real-differentiable a point $p \in \mathbb{R}^n$ iff there exists a linear transformation…"

b. Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ whose first-order partial derivatives exist everywhere but f is not differentiable at $(0, 0)$.

c. Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is real-differentiable everywhere but nowhere complex-differentiable.

4 | Implicit/Inverse Function Theorems

4.1 3

Let $f : \mathbb{R}^2 \to \mathbb{R}$.

a. Define in terms of linear transformations what it means for f to be differentiable at a point $(a, b) \in \mathbb{R}^2$.

b. State a version of the inverse function theorem in this setting.

c. Identify $\mathbb{R}^2$ with $\mathbb{C}$ and give a necessary and sufficient condition for a real-differentiable function at (a, b) to be complex differentiable at the point $a + ib$.

4.2 5

Let $P = (1, 3) \in \mathbb{R}^2$ and define

$$f(s, t) \coloneqq ps^3 - 6st + t^2.$$

a. State the conclusion of the implicit function theorem concerning $f(s, t) = 0$ when f is considered a function $\mathbb{R}^2 \to \mathbb{R}$.

b. State the above conclusion when f is considered a function $\mathbb{C}^2 \to \mathbb{C}$.

c. Use the implicit function theorem for a function $\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ to prove (b).

There are various approaches: using the definition of the complex derivative, the Cauchy-Riemann equations, considering total derivatives, etc.

4.3 7

State the most general version of the implicit function theorem for real functions and outline how it can be proved using the inverse function theorem.

5 Complex Differentiability

5.1 4

Let $f = u + iv$ be complex-differentiable with continuous partial derivatives at a point $z = re^{i\theta}$ with $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

5.2 Tie's Extra Questions: Fall 2016

Let $u(x, y)$ be harmonic and have continuous partial derivatives of order three in an open disc of radius $R > 0$.

a. Let two points $(a, b), (x, y)$ in this disk be given. Show that the following integral is independent of the path in this disk joining these points:

$$v(x, y) = \int_{a,b}^{x,y} \left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right).$$

b. In parts:

- Prove that $u(x, y) + iv(x, y)$ is an analytic function in this disc.
- Prove that $v(x, y)$ is harmonic in this disc.

5.3 Tie's Questions, Spring 2014: Polar Cauchy-Riemann

Let $f = u + iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

5.4 ?

1. Show that the function $u = u(x, y)$ given by

$$u(x,y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx \quad \text{for } n \in \mathbf{N}$$

   is the solution on $D = \{(x, y) \mid x^2 + y^2 < 1\}$ of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,0) = \frac{\sin nx}{n}.$$

2. Show that there exist points $(x, y) \in D$ such that $\limsup_{n\to\infty} |u(x, y)| = \infty$.

6 | Montel

6.1 Convergence of holomorphic functions on line segments

Problem 6.1.1 (?)
Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of entire functions where

- $f_n \to g$ pointwise for some $g : \mathbb{C} \to \mathbb{C}$.
- On every line segment in $\mathbb{C}$, $f_n \to g$ uniformly.

Show that

- g is entire, and
- $f_n \to g$ uniformly on every compact subset of $\mathbb{C}$.

Solution omitted.

6.2 Tie's Extra Questions: Spring 2015

Problem 6.2.1 (?)
Assume $f_n \in H(\Omega)$ is a sequence of holomorphic functions on the region Ω that are uniformly bounded on compact subsets and $f \in H(\Omega)$ is such that the set $\{z \in \Omega : \lim_{n\to\infty} f_n(z) = f(z)\}$ has a limit point in Ω. Show that f_n converges to f uniformly on compact subsets of Ω.

6.3 Spring 2019.7

Problem 6.3.1 (?)
Let $\Omega \subset \mathbb{C}$ be a connected open subset. Let $\{f_n : \Omega \to \mathbb{C}\}_{n=1}^\infty$ be a sequence of holomorphic functions uniformly bounded on compact subsets of Ω. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function such that the set

$$\left\{z \in \Omega \,\middle|\, \lim_{n\to\infty} f_n(z) = f(z)\right\}$$

has a limit point in Ω. Show that f_n converges to f uniformly on compact subsets of Ω.

Solution omitted.

7 Function Convergence

7.1 Fall 2021.4

Problem 7.1.1 (?)
Prove that the sequence $\left(1 + \frac{z}{n}\right)^n$ converges uniformly to e^z on compact subsets of $\mathbb{C}$.

Hint: $e^{n \log w_n} = w_n^n$ and e^z is uniform continuous on compact subsets of $\mathbb{C}$.

Solution omitted.

7.2 Spring 2021.6, Spring 2015, Extras

Problem 7.2.1 (?)
Let $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions on $\mathbb{D}$ and f is also holomorphic on $\mathbb{D}$. Show that the following are equivalent:

- $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$.
- For $0 < r < 1$,

$$\int_{|z|=r} |f_n(z) - f(z)||dz| \overset{n\to\infty}{\longrightarrow} 0.$$

Note: $|\,dz| = |\gamma'(t)|\,dt$ for γ a parameterization of the curve.

Solution omitted.

7.3 Spring 2020 HW 2, SS 2.6.10

Problem 7.3.1 (?)
Can every continuous function on $\overline{\mathbb{D}}$ be uniformly approximated by polynomials in the variable z?

Hint: compare to Weierstrass for the real interval.

Solution omitted.

7.4 Spring 2020 HW 2.5

Problem 7.4.1 (?)
Assume f is continuous in the region $\left\{x + iy \;\middle|\; x \geq x_0,\; 0 \leq y \leq b\right\}$, and the following limit exists independent of y:

$$\lim_{x\to+\infty} f(x+iy) = A.$$

Show that if $\gamma_x := \left\{z = x + it \;\middle|\; 0 \leq t \leq b\right\}$, then

$$\lim_{x\to+\infty} \int_{\gamma_x} f(z)\,dz = iAb.$$

Solution omitted.

7.5 Limiting curve variant

Problem 7.5.1 (?)
Let $0 \leq \alpha \leq 2\pi$ be a fixed angle. Suppose f is continuous on the region $\Omega = \{|z| \geq R, \operatorname{Arg}(z) \in [0, \alpha]\}$ and $\lim_{z \to \infty} z f(z) = A$. Show that

$$\lim_{z \to \infty} \int_{\gamma_R} f(z)\, dz = iA\alpha,$$

where $\gamma_R \coloneqq \{|z| = R, \operatorname{Arg}(z) \in [0, \alpha]\}$ is an arc.

Solution omitted.

8 Series Convergence

8.1 Fall 2020.2

Problem 8.1.1 (?)
Expand $\dfrac{1}{1 - z^2} + \dfrac{1}{z - 3}$ in a series of the form $\displaystyle\sum_{-\infty}^{\infty} a_n z^n$ so it converges for

- $|z| < 1$,
- $1 < |z| < 3$,
- $|z| > 3$.

Solution omitted.

8.2 Spring 2020 HW 2.2

Problem 8.2.1 (?)
Let f be a power series centered at the origin. Prove that f has a power series expansion about any point in its disc of convergence.

Concept review omitted.

Solution omitted.

8.3 Fall 2015, Spring 2020 HW 2, Ratio Test

Problem 8.3.1 (?)
Let $a_n \neq 0$ and show that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L \implies \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L.$$

In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

8.4 Analytic on circles

Problem 8.4.1 (?)
Suppose f is analytic on a region Ω such that $\mathbb{D} \subseteq \Omega \subseteq \mathbb{C}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series with radius of convergence exactly 1.

a. Give an example of such an f that converges at every point of S^1.

b. Give an example of such an f which is analytic at 1 but $\sum_{n=0}^{\infty} a_n$ diverges.

c. Prove that f can not be analytic at *every* point of S^1.

Solution omitted.

8.5 Spring 2020 HW 2.3: series on the circle

Problem 8.5.1 (?)
Prove the following:

a. $\sum_n nz^n$ does not converge at any point of S^1

b. $\sum_n \frac{z^n}{n^2}$ converges at every point of S^1.

c. $\sum_n \frac{z^n}{n}$ converges at every point of S^1 except $z = 1$.

Concept review omitted.

Solution omitted.

8.6 Uniform convergence of series

Problem 8.6.1 (?)
Suppose $\sum_{n=0}^{\infty} a_n z^n$ converges for some $z_0 \neq 0$.

a. Prove that the series converges absolutely for each z with $|z| < |z|_0$.

b. Suppose $0 < r < |z_0|$ and show that the series converges uniformly on $|z| \leq r$.

8.7 Sine series?

Problem 8.7.1 (?)
Prove that the following series converges uniformly on the set $\left\{z \;\middle|\; \Im(z) < \ln 2\right\}$:

$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{2^n}.$$

Suppose $0 < r < |z_0|$ and show that the series converges uniformly on $|z| \leq r$.

8.8 Fall 2015 Extras

Assume $f(z)$ is analytic in $\mathbb{D}$ and $f(0) = 0$ and is not a rotation (i.e. $f(z) \neq e^{i\theta} z$). Show that $\sum_{n=1}^{\infty} f^n(z)$ converges uniformly to an analytic function on compact subsets of $\mathbb{D}$, where $f^{n+1}(z) = f(f^n(z))$.

9 | Holomorphicity

9.1 Fall 2019.6

Problem 9.1.1 (?)
A holomorphic mapping $f : U \to V$ is a local bijection on U if for every $z \in U$ there exists an open disc $D \subset U$ centered at z so that $f : D \to f(D)$ is a bijection. Prove that a holomorphic map $f : U \to V$ is a local bijection if and only if $f'(z) \neq 0$ for all $z \in U$.

Concept review omitted.

Solution omitted.

9.2 Spring 2020 HW 1.7

Problem 9.2.1 (?)
Prove that $f(z) = |z|^2$ has a derivative at $z = 0$ and nowhere else.

Solution omitted.

9.3 Spring 2020 HW 1.8

Problem 9.3.1 (?)
Let $f(z)$ be analytic in a domain, and prove that f is constant if it satisfies any of the following conditions:

a. $|f(z)|$ is constant.
b. $\Re(f(z))$ is constant.

c. $\arg(f(z))$ is constant.
d. $\overline{f(z)}$ is analytic.

How do you generalize (a) and (b)?

Solution omitted.

Solution omitted.

Solution omitted.

Solution omitted.

9.4 Spring 2020 HW 1.9

Problem 9.4.1 (?)
Prove that if $z \mapsto f(z)$ is analytic, then $z \mapsto \overline{f(\bar{z})}$ is analytic.

Solution omitted.

Solution omitted.

Solution omitted.

9.5 Spring 2020 HW 1.10

Problem 9.5.1 (?)

a. Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

b. Use (a) to show that the logarithm function, defined as

$$\operatorname{Log} z = \log r + i\theta \text{ where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi.$$

is holomorphic on the region $r > 0, -\pi < \theta < \pi$.

Also show that this function is not continuous in $r > 0$.

Solution omitted.

9.6 Fall 2021.1

Problem 9.6.1 (?)
Let $f(z)$ be an analytic function on $|z| < 1$. Prove that $f(z)$ is necessarily a constant if $f(\bar{z})$ is also analytic.

Solution omitted.

9.7 Holomorphic functions form an integral domain

Problem 9.7.1 (?)
Suppose D is a domain and f, g are analytic on D.
Prove that if $fg = 0$ on D, then either $f \equiv 0$ or $g \equiv 0$ on D.

Solution omitted.

9.8 Holomorphic functions with specified values

Problem 9.8.1 (?)
Suppose f is analytic on $\mathbb{D}^\circ$. Determine with proof which of the following are possible:

a. $f\left(\frac{1}{n}\right) = (-1)^n$ for each $n > 1$.

b. $f\left(\frac{1}{n}\right) = e^{-n}$ for each even integer $n > 1$ while $f\left(\frac{1}{n}\right) = 0$ for each odd integer $n > 1$.

c. $f\left(\frac{1}{n^2}\right) = \frac{1}{n}$ for each integer $n > 1$.

d. $f\left(\frac{1}{n}\right) = \frac{n-2}{n-1}$ for each integer $n > 1$.

Solution omitted.

10 | Geometry

10.1 Some Geometry

Let $z_k(k=1,\cdots,n)$ be complex numbers lying on the same side of a straight line passing through the origin. Show that

$$z_1+z_2+\cdots+z_n\neq 0,\quad 1/z_1+1/z_2+\cdots+1/z_n\neq 0$$

Hint: Consider a special situation first.

10.2 Images of circles

Let $f(z)=z+1/z$. Describe the images of both the circle $|z|=r$ of radius $r(r\neq 0)$ and the ray $\arg z=\theta_0$ under f in terms of well known curves.

10.3 Geometric Identities

Prove that $|z_1+z_2|^2+|z_1-z_2|^2=2\left(|z_1|^2+|z_2|^2\right)$ for any two complex numbers z_1,z_2, and explain the geometric meaning of this identity

10.4 Geometric Identities

Use n-th roots of unity (i.e. solutions of $z^n-1=0$) to show that

$$\cos\frac{2\pi}{n}+\cos\frac{4\pi}{n}+\cos\frac{6\pi}{n}+\cdots+\cos\frac{2(n-1)\pi}{n}+=-1 \text{ and}$$
$$\sin\frac{2\pi}{n}+\sin\frac{4\pi}{n}+\sin\frac{6\pi}{n}+\cdots\frac{2(n-1)\pi}{n}=0$$

Hint: If $z^n + c_1 z^{n-1} + \cdots + c_{n-1} z + c_n = 0$ has roots $z_1, z_2, \ldots, z_n$, then

$$\begin{aligned} z_1 + z_2 + \cdots + z_n &= -c_1 \\ z_1 z_2 \cdots z_n &= (-1)^n c_n \text{ (not used)} \end{aligned}$$

10.5 Geometry from equations

Describe each set in the z-plane in (a) and (b) below, where α is a complex number and k is a positive number such that $2|\alpha| < k$.

(a) $|z - \alpha| + |z + \alpha| = k$;

(b) $|z - \alpha| + |z + \alpha| \leq k$.

10.6 Spring 2020.1, Spring 2020 HW 1.4

Problem 10.6.1 (?)

a. Prove that if $c > 0$,

$$|w_1| = c|w_2| \implies \left|w_1 - c^2 w_2\right| = c|w_1 - w_2|.$$

b. Prove that if $c > 0$ and $c \neq 1$, with $z_1 \neq z_2$, then the following equation represents a circle:

$$\left|\frac{z - z_1}{z - z_2}\right| = c.$$

Find its center and radius.

Hint: use part (a)

Solution omitted.

Solution omitted.

10.7 Spring 2020 HW 1.1

Problem 10.7.1 (?)
Geometrically describe the following subsets of $\mathbb{C}$:

a. $|z-1| = 1$
b. $|z-1| = 2|z-2|$
c. $1/z = \bar{z}$
d. $\Re(z) = 3$
e. $\Im(z) = a$ with $a \in \mathbb{R}$.
f. $\Re(z) > a$ with $a \in \mathbb{R}$.
g. $|z-1| < 2|z-2|$

Solution omitted.

10.8 Fixed argument exercise

Exercise 10.8.1 (?)
Fix $a, b \in \mathbb{C}$ and θ, and describe the locus

$$\left\{ z \;\middle|\; \operatorname{Arg}\left(\frac{z-a}{z-b}\right) = \theta \right\}.$$

Solution omitted.

10.9 Fall 2019.2, Spring 2020 HW 1.11

Problem 10.9.1 (?)
Prove that the distinct complex numbers z_1, z_2, z_3 are the vertices of an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

Solution omitted.

11 | Complex Arithmetic

11.1 Sum of Sines

Use de Moivre's theorem (i.e. $\left(e^{i\theta}\right)^n == \cos n\theta + i\sin n\theta$, or $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$) to find the sum

$$\sin x + \sin 2x + \cdots + \sin nx$$

11.2 Solving Equations

Characterize positive integers n such that $(1+i)^n = (1-i)^n$

11.3 Characters

Let n be a natural number. Show that

$$[1/2(-1+\sqrt{3}i)]^n + [1/2(-1-\sqrt{3}i)]^n$$

is equal to 2 if n is a multiple of 3 , and it is equal to -1 otherwise.

11.4 Spring 2019.3 #complex/qual/stuck

Problem 11.4.1 (?)
Let $R > 0$. Suppose f is holomorphic on $\left\{z \,\middle|\, |z| < 3R\right\}$. Let

$$M_R := \sup_{|z|\le R} |f(z)|, \quad N_R := \sup_{|z|\le R} \left|f'(z)\right|$$

a. Estimate M_R in terms of N_R from above.

b. Estimate N_R in terms of M_{2R} from above.

Solution omitted.

11.5 Spring 2021.1

Warning 11.5.1
The question as written on the original qual has several errors. What is below is the correct version of the inequality.

Problem 11.5.1 (?) 1. Let z_1 and z_2 be two complex numbers.

(a) Show that

$$|1-\bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = \left(1-|z_1|^2\right)\left(1-|z_2|^2\right)$$

(b) Show that if $|z_1| < 1$ and $|z_2| < 1$, then $\left|\frac{z_1 - z_2}{1-\bar{z}_1 z_2}\right| < 1$.

(c) Assume that $z_1 \neq z_2$. Show that $\left|\frac{z_1 - z_2}{1-\bar{z}_1 z_2}\right| = 1$ if only if $|z_1| = 1$ or $|z_2| = 1$.

Solution omitted.

11.6 Spring 2020 HW 1.5

Problem 11.6.1 (?) a. Let $z, w \in \mathbb{C}$ with $\bar{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1 \qquad \text{if } |z| < 1,\ |w| < 1$$

with equality when $|z| = 1$ or $|w| = 1$.

b. Prove that for a fixed $w \in \mathbb{D}$, the mapping $F : z \mapsto \frac{w-z}{1-\overline{w}z}$ satisfies

- F maps $\mathbb{D}$ to itself and is holomorphic.
- $F(0) = w$ and $F(w) = 0$.
- $|z| = 1$ implies $|F(z)| = 1$.
- F is a bijection.

Solution omitted.

11.7 Spring 2020 HW 1.2

Problem 11.7.1 (?)
Prove the following inequality, and explain when equality holds:

$$|z-w| \geq ||z|-|w||.$$

Solution omitted.

11.8 Fall 2020.1, Spring 2020 HW 1.6

Problem 11.8.1 (?)
Use nth roots of unity to show that

$$2^{n-1}\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\cdots\sin\left(\frac{(n-1)\pi}{n}\right)=n.$$

Hint:

$$\begin{aligned}1-\cos(2\theta)&=2\sin^2(\theta)\\ 2\sin(2\theta)&=2\sin(\theta)\cos(\theta).\end{aligned}$$

Concept review omitted.

Solution omitted.

Solution omitted.

11.9 Spring 2020 HW 1.5

Problem 11.9.1 (?)

a. Let $z, w \in \mathbb{C}$ with $\bar{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right|<1 \quad \text{if } |z|<1,\ |w|<1$$

with equality when $|z|=1$ or $|w|=1$.

b. Prove that for a fixed $w \in \mathbb{D}$, the mapping $F: z \mapsto \dfrac{w-z}{1-\overline{w}z}$ satisfies

- F maps $\mathbb{D}$ to itself and is holomorphic.
- $F(0) = w$ and $F(w) = 0$.
- $|z| = 1$ implies $|F(z)| = 1$.

Solution omitted.

Solution omitted.

12 | Laurent Expansions

12.1 Tie, Spring 2015:

Let $f(z) = \dfrac{1}{z} + \dfrac{1}{z^2 - 1}$. Find all the Laurent series of f and describe the largest annuli in which these series are valid.

12.2 1

Find the Laurent expansion of

$$f(z) = \frac{z+1}{z(z-1)}$$

about $z = 0$ and $z = 1$ respectively.

Solution omitted.

12.3 2

Find the Laurent expansions about $z = 0$ of the following functions:

$$e^{\frac{1}{z}} \qquad\qquad \cos\left(\frac{1}{z}\right).$$

Solution omitted.

12.4 3

Find the Laurent expansion of

$$f(z) = \frac{z+1}{z(z-1)^2}$$

about $z = 0$ and $z = 1$ respectively.

Hint: recall that power series can be differentiated.

12.5 4

For the following functions, find the Laurent series about 0 and classify their singularities there:

$$\frac{\sin^2(z)}{z}$$
$$z \exp \frac{1}{z^2}$$
$$\frac{1}{z(4-z)}.$$

12.6 Tie's Extra Questions: Fall 2015

Expand the following functions into Laurent series in the indicated regions:

(a) $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$, $2 < |z| < 3$, $3 < |z| < +\infty$.

(b) $f(z) = \sin \dfrac{z}{1-z}$, $0 < |z-1| < +\infty$

12.7 Tie, Fall 2015: Laurent Coefficients

Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Let $f(z) = \sum_{n=1}^{\infty} c_n z^n$ denote the the power series in the open disc. Show that

(1) $c_n \neq 0$ for all large enough n's, and

(2) $\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = z_0$.

12.8 Spring 2020 HW 2, SS 2.6.14

Suppose that f is holomorphic in an open set containing $\mathbb{D}$ except for a pole $z_0 \in \partial\mathbb{D}$. Let $\sum_{n=0}^{\infty} a_n z^n$ be the power series expansion of f in $\mathbb{D}$, and show that $\lim \frac{a_n}{a_{n+1}} = z_0$.

Solution

12.9 2

Suppose f is entire and has Taylor series $\sum a_n z^n$ about 0.

a. Express a_n as a contour integral along the circle $|z| = R$.

b. Apply (a) to show that the above Taylor series converges uniformly on every bounded subset of $\mathbb{C}$.

c. Determine those functions f for which the above Taylor series converges uniformly on all of $\mathbb{C}$.

12.10 Spring 2020 HW 2.4

Without using Cauchy's integral formula, show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{dz}{(z-\alpha)(z-\beta)} = \frac{2\pi i}{\alpha - \beta}$$

where γ denotes the circle centered at the origin of radius r with positive orientation.

Hint: take a Laurent expansion.

12.10.1 Spring 2020 HW 3 # 1

Prove that if f has two Laurent series expansions,

$$f(z) = \sum c_n (z-a)^n \quad \text{and} \quad f(z) = \sum c_n' (z-a)^n$$

then $c_n = c_n'$.

12.10.2 Spring 2020 HW 3 # 2

Find Laurent series expansions of

$$\frac{1}{1-z^2} + \frac{1}{3-z}$$

How many such expansions are there? In what domains are each valid?

13 Singularities

13.1 Spring 2020 HW 3.3

Problem 13.1.1 (?)
Let P, Q be polynomials with no common zeros. Assume a is a root of Q. Find the principal part of P/Q at $z = a$ in terms of P and Q if a is

(1) a simple root, and
(2) a double root.

Solution omitted.

13.2 Spring 2020.4

Problem 13.2.1 (?)
Suppose that f is holomorphic in an open set containing the closed unit disc, except for a simple pole at $z = 1$. Let $f(z) = \sum_{n=1}^{\infty} c_n z^n$ denote the power series in the open unit disc. Show that

$$\lim_{n \to \infty} c_n = -\lim_{z \to 1} (z-1) f(z).$$

Solution omitted.

13.3 Entire functions with poles at ∞

Problem 13.3.1 (?)
Find all entire functions with have poles at ∞.

Solution omitted.

13.4 Functions with specified poles (including at ∞)

Problem 13.4.1 (?)
Find all functions on the Riemann sphere that have a simple pole at $z = 2$ and a double pole at $z = \infty$, but are analytic elsewhere.

Solution omitted.

13.5 Entire functions with singularities at ∞

Problem 13.5.1 (?)
Let f be entire, and discuss (with proofs and examples) the types of singularities f might have (removable, pole, or essential) at $z = \infty$ in the following cases:

1. f has at most finitely many zeros in $\mathbb{C}$.
2. f has infinitely many zeros in $\mathbb{C}$.

Solution omitted.

13.6 Sum formula for $\sin^2$

Problem 13.6.1 (?)
Define

$$f(z) = \frac{\pi^2}{\sin^2(\pi z)}$$
$$g(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}.$$

a. Show that f and g have the same singularities in $\mathbb{C}$.
b. Show that f and g have the same singular parts at each of their singularities.
c. Show that f, g each have period one and approach zero uniformly on $0 \leq x \leq 1$ as $|y| \to \infty$.
d. Conclude that $f = g$.

Solution omitted.

13.7 Spring 2020 HW 3.4, Tie's Extra Questions: Fall 2015

Problem 13.7.1 (?)
Let $f(z)$ be a non-constant analytic function in $|z| > 0$ such that $f(z_n) = 0$ for infinite many points z_n with $\lim_{n\to\infty} z_n = 0$.
Show that $z = 0$ is an essential singularity for $f(z)$.

Hint: an example of such a function is $f(z) = \sin(1/z)$.

Solution omitted.

14 Computing Integrals

14.1 Rational, wedge

14.1.1 Fall 2021.3

Problem 14.1.1 (?)
Suppose $n \geq 2$. Use a wedge of angle $\frac{2\pi}{n}$ to evaluate the integral

$$I = \int_0^\infty \frac{1}{1+x^n} dx$$

Solution omitted.

Solution omitted.

14.1.2 Spring 2020 HW 3, SS 3.8.2

Evaluate the integral

$$\int_{\mathbb{R}} \frac{dx}{1+x^4}.$$

What are the poles of $\dfrac{1}{1+z^4}$?

14.1.3 Spring 2020 HW 3, SS 3.8.6

Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot \pi.$$

14.1.4 Quadratic over quartic

Problem 14.1.2 (?)
Let $a>0$ and calculate

$$\int_0^{\infty} \frac{x^2}{(x^2+a^2)^2}\,dx.$$

14.1.5 Rational function

Problem 14.1.3 (?)
Calculate

$$\int_{-\infty}^{\infty} \frac{1+x^2}{1+x^4}\,dx.$$

14.1.6 Denominator polynomial

Problem 14.1.4 (?)
Calculate

$$\int_0^{\infty} \frac{1}{(1+z)^2(z+9x^2)}\,dx.$$

14.2 Rational, branch cut

14.2.1 Standard example

Problem 14.2.1 (?)
Show that

$$\int_{\mathbb{R}_{\geq 0}} \frac{x^{-s}}{x+1} = \frac{\pi}{\sin(\pi s)}.$$

Solution omitted.

14.2.2 Fall 2019.1

Show that

$$\int_0^\infty \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$$

using complex analysis, $0 < a < n$. Here n is a positive integer.

14.2.3 Spring 2020 HW 3.7

Let $0 < a < 4$ and evaluate

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^3}\, dx$$

14.2.4 Tie's Extra Questions: Fall 2011, Spring 2015

Show that

$$\int_0^\infty \frac{x^{a-1}}{1+x^n}\, dx = \frac{\pi}{n \sin \frac{a\pi}{n}}.$$

using complex analysis, $0 < a < n$. Here n is a positive integer.

14.2.5 Fall 2020.3, Spring 2019.2

Problem 14.2.2 (?)
Let $a \in \mathbb{R}$ with $0 < a < 3$. Evaluate

$$\int_0^\infty \frac{x^{a-1}}{1+x^3} dx.$$

Solution omitted.

14.3 Rational Functions of sin or cos

14.3.1 Cosine in denominator

Problem 14.3.1 (?)
Show

$$\int_0^{2\pi} \frac{1}{a+\cos(\theta)}\, d\theta = \frac{2\pi}{\sqrt{a^2-1}}, \qquad a > 1.$$

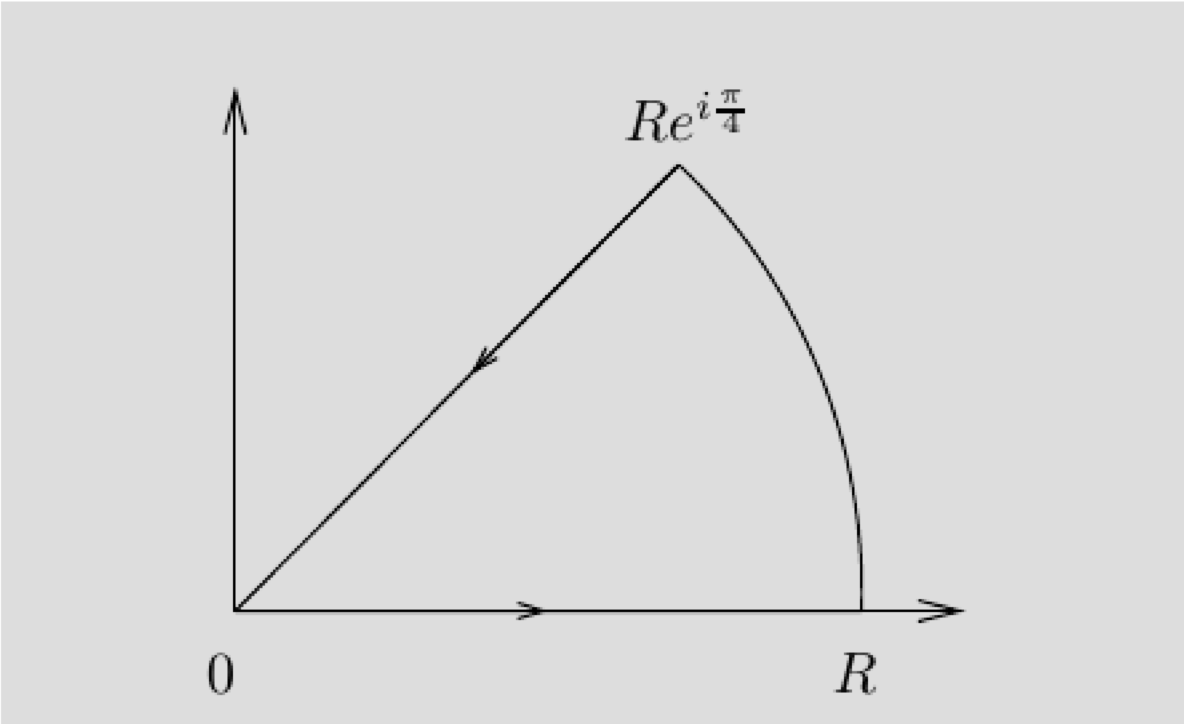
Solution omitted.

14.3.2 Spring 2020 HW 2, SS 2.6.1

Show that

$$\int_0^\infty \sin\left(x^2\right) dx = \int_0^\infty \cos\left(x^2\right) dx = \frac{\sqrt{2\pi}}{4}.$$

Hint: integrate e^{-x^2} over the following contour, using the fact that $\int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}$:



14.3.3 Spring 2020 HW 3, SS 3.8.8

Show that if $a, b \in \mathbb{R}$ with $a > |b|$, then

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

14.3.4 Fresnel

Problem 14.3.2 (?)
Suppose $a > b > 0$ and calculate

$$\int_0^{2\pi} \frac{1}{(a + b\cos(\theta))^2}\, d\theta.$$

14.3.5 Fresnel

Problem 14.3.3 (?)
Let $n \in \mathbb{Z}^{\geq 1}$ and $0 < \theta < \pi$ and show that

$$\frac{1}{2\pi i} \int_{|z|=2} \frac{z^n}{1 - 3z\cos(\theta) + z^2}\, dz = \frac{\sin(n\theta)}{\sin(\theta)}.$$

14.3.6 Spring 2020 HW 3.10

For $a > 0$, evaluate

$$\int_0^{\pi/2} \frac{d\theta}{a + \sin^2\theta}$$

14.3.7 Spring 2020 HW 3, SS 3.8.7

Show that

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos\theta)^2} = \frac{2\pi a}{(a^2 - 1)^{3/2}}, \qquad \text{whenever } a > 1.$$

14.4 Rectangles

14.4.1 Spring 2021.2

Problem 14.4.1 (?)
Let $\xi \in \mathbb{R}$, evaluate

$$\int_{\mathbb{R}} \frac{e^{i\xi x}}{\cosh(x)}\, dx.$$

Solution omitted.

14.4.2 Spring 2020 HW 3, SS 3.8.9

Show that

$$\int_0^1 \log(\sin \pi x) dx = -\log 2.$$

Hint: use the following contour.

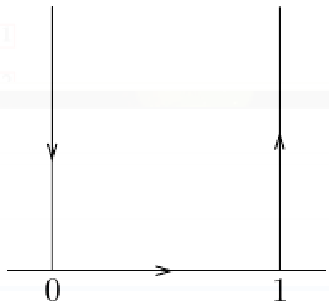


Figure 9. Contour in Exercise 9

14.5 Branch Cuts

14.5.1 Tie's Extra Questions: Spring 2015

Compute the following integrals:

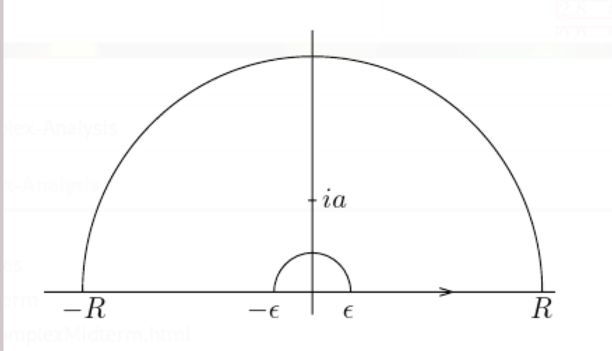
- $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^n}\,dx$, $0 < a < n$
- $\displaystyle\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx$

14.5.2 Spring 2020 HW 3, SS 3.8.10

Show that if $a > 0$, then

$$\int_0^\infty \frac{\log x}{x^2+a^2}dx = \frac{\pi}{2a}\log a.$$

Hint: use the following contour.



14.5.3 Spring 2020.2

Problem 14.5.1 (?)
Compute the following integral carefully justifying each step:

$$\int_0^\infty \frac{\log x}{1+x^3}.$$

14.5.4 Square root in numerator

Problem 14.5.2 (?)
Calculate

$$\int_0^\infty \frac{\sqrt{x}}{(x+1)^2}\,dx.$$

14.5.5 Square root

Problem 14.5.3 (?)
Calculate

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx.$$

14.6 Trigonometric transforms

14.6.1 Spring 2020 HW 3, SS 3.8.4

Show that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}, \qquad \text{for all } a > 0.$$

14.6.2 Spring 2020 HW 2, 2.6.2

Show that

$$\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Hint: use the fact that this integral eexercises $\frac{1}{2i} \int_{-\infty}^{\infty} \frac{e^{ix} - 1}{x} dx$, *and integrate around an indented semicircle.*

14.6.3 Spring 2020 HW 3, SS 3.8.5

Show that if $\xi \in \mathbb{R}$, then

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{(1 + x^2)^2} dx = \frac{\pi}{2} (1 + 2\pi |\xi|) e^{-2\pi |\xi|}.$$

14.6.4 $\sin$ in numerator

Problem 14.6.1 (?)
Let $a > 0$ and calculate

$$\int_{0}^{\infty} \frac{x \sin(x)}{x^2 + a^2} \, dx.$$

14.6.5 sin in numerator

Problem 14.6.2 (?)
Calculate

$$\int_0^\infty \frac{\sin(x)}{x(x^2+1)}\,dx.$$

14.6.6 sinc

Problem 14.6.3 (?)
Calculate

$$\int_0^\infty \frac{\sin(x)}{x}\,dx.$$

14.6.7 cos in numerator

Problem 14.6.4 (?)
Let $a > 0$ and calculate

$$\int_0^\infty \frac{\cos(x)}{(x^2+a^2)^2}\,dx.$$

14.6.8 sin in numerator

Problem 14.6.5 (?)
Calculate

$$\int_0^\infty \frac{\sin^3(x)}{x^3}\,dx.$$

14.6.9 sin in numerator

Problem 14.6.6 (?)
Evaluate
$$\int_0^\infty \frac{x\sin x}{x^2+a^2}\,dx.$$

14.6.10 Tie's Extra Questions: Fall 2009

Evaluate
$$\int_0^\infty \frac{x\sin x}{x^2+a^2}\,dx.$$

14.6.11 Cosine over quadratic

Problem 14.6.7 (?)
Show that
$$\int_0^\infty \frac{\cos(x)}{x^2+b^2}\,dx = \frac{\pi e^{-b}}{2b}.$$

Solution omitted.

14.6.12 Tie's Extra Questions: Fall 2016

Compute the integral $\displaystyle\int_{-\infty}^\infty \frac{e^{-2\pi i x\xi}}{\cosh \pi x} dx$ where $\cosh z = \dfrac{e^z+e^{-z}}{2}$.

14.6.13 Tie's Extra Questions: Fall 2015

Prove by *justifying all steps* that for all $\xi \in \mathbb{C}$ we have $e^{-\pi\xi^2} = \displaystyle\int_{-\infty}^\infty e^{-\pi x^2} e^{2\pi i x\xi} dx$.

Hint: You may use that fact in Example 1 on p. 42 of the textbook without proof, i.e., you may assume the above is true for real values of ξ.

14.6.14 Multiple cosines in numerator

Problem 14.6.8 (?)
Calculate

$$\int_0^\infty \frac{\cos(x) - \cos(4x)}{x^2}\, dx.$$

14.6.15 Tie's Extra Questions: Fall 2011

Evaluate $\displaystyle\int_0^\infty \frac{x \sin x}{x^2 + a^2}\, dx$.

14.7 Unsorted

14.7.1 Spring 2020 HW 3.6

a. Show (without using 3.8.9 in the S&S) that

$$\int_0^{2\pi} \log\left|1 - e^{i\theta}\right|\, d\theta = 0$$

b. Show that this identity is equivalent to S&S 3.8.9:

$$\int_0^1 \log(\sin(\pi x))\, dx = -\log 2.$$

14.7.2 Tie's Extra Questions: Spring 2015

Compute the following integrals.

(i) $\displaystyle\int_0^\infty \frac{1}{(1+x^n)^2}\, dx,\ n \geq 1$

(ii) $\displaystyle\int_0^\infty \frac{\cos x}{(x^2+a^2)^2}\, dx,\ a \in \mathbb{R}$

(iii) $\displaystyle\int_0^\pi \frac{1}{a + \sin\theta}\, d\theta,\ a > 1$

(iv) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{a + \sin^2\theta},\ a > 0.$

(v) $\displaystyle\int_{|z|=2} \frac{1}{(z^5-1)(z-3)}\,dz$

(vi) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin \pi a}{\cosh \pi x + \cos \pi a} e^{-ix\xi}\,dx,\ 0<a<1,\ \xi \in \mathbb{R}$

(vii) $\displaystyle\int_{|z|=1} \cot^2 z\,dz.$

14.8 Conceptual

14.8.1 Spring 2020 HW 3, SS 3.8.1

Use the following formula to show that the complex zeros of $\sin(\pi z)$ are exactly the integers, and they are each of order 1:

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}.$$

Calculate the residue of $\dfrac{1}{\sin(\pi z)}$ at $z = n \in \mathbb{Z}$.

14.8.2 Zeros using residue theorem

Problem 14.8.1 (?)
Suppose that f is an analytic function in the region D which contains the point a. Let

$$F(z) = z - a - qf(z), \quad \text{where} \quad q \text{ is a complex parameter.}$$

1. Let $K \subset D$ be a circle with the center at point a and also we assume that $f(z) \neq 0$ for $z \in K$. Prove that the function F has one and only one zero $z = w$ on the closed disc $\overline{K}$ whose boundary is the circle K if
$$|q| < \min_{z \in K} \frac{|z-a|}{|f(z)|}..$$
2. Let $G(z)$ be an analytic function on the disk $\overline{K}$. Apply the residue theorem to prove that
$$\frac{G(w)}{F'(w)} = \frac{1}{2\pi i}\int_K \frac{G(z)}{F(z)}dz,$$
where w is the zero from (1).

14.8.3 Tie's Extra Questions: Fall 2009

Suppose that f is an analytic function in the region D which contains the point a. Let

$$F(z) = z - a - qf(z), \quad \text{where } q \text{ is a complex parameter.}$$

(1) Let $K \subset D$ be a circle with the center at point a and also we assume that $f(z) \neq 0$ for $z \in K$. Prove that the function F has one and only one zero $z = w$ on the closed disc $\overline{K}$ whose boundary is the circle K if $|q| < \min_{z\in K} \dfrac{|z-a|}{|f(z)|}$.

(2) Let $G(z)$ be an analytic function on the disk $\overline{K}$. Apply the residue theorem to prove that $\dfrac{G(w)}{F'(w)} = \dfrac{1}{2\pi i}\displaystyle\int_K \dfrac{G(z)}{F(z)}dz$, where w is the zero from (1).

(3) If $z \in K$, prove that the function $\dfrac{1}{F(z)}$ can be represented as a convergent series with respect to q: $\dfrac{1}{F(z)} = \displaystyle\sum_{n=0}^{\infty} \dfrac{(qf(z))^n}{(z-a)^{n+1}}$.

14.8.4 Tie's Extra Questions: Spring 2015

Problem 14.8.2 (?)
Let $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha}z}$ with $|\alpha| < 1$ and $\mathbb{D} = \{z : \ |z| < 1\}$. Prove that

a.

$$\frac{1}{\pi}\iint_{\mathbb{D}} |\psi'_\alpha|^2 dxdy = 1.$$

b.

$$\frac{1}{\pi}\iint_{\mathbb{D}} |\psi'_\alpha| dxdy = \frac{1-|\alpha|^2}{|\alpha|^2}\log\frac{1}{1-|\alpha|^2}.$$

Solution omitted.

15 | Cauchy's Theorem

15.1 Entire and O of polynomial implies polynomial

Problem 15.1.1 (?)
Let $f(z)$ be entire and assume that $|f(z)| \leq M|z|^2$ outside of some disk for some constant M. Show that $f(z)$ is a polynomial in z of degree ≤ 2.

Solution omitted.

15.2 Uniform sequence implies uniform derivatives

Problem 15.2.1 (?)
Let $a_n(z)$ be an analytic sequence in a domain D such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D. Show that $\sum_{n=0}^{\infty} |a_n'(z)|$ converges uniformly on bounded and closed sub-regions of D.

15.3 Tie's Extra Questions: Spring 2014

Problem 15.3.1 (?)
The question provides some insight into Cauchy's theorem. Solve the problem without using the Cauchy theorem.

1. Evaluate the integral $\int_\gamma z^n dz$ for all integers n. Here γ is any circle centered at the origin with the positive (counterclockwise) orientation.
2. Same question as (a), but with γ any circle not containing the origin.
3. Show that if $|a| < r < |b|$, then $\int_\gamma \frac{dz}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$. Here γ denotes the circle centered at the origin, of radius r, with the positive orientation.

Solution omitted.

15.4 Fall 2019.3, Spring 2020 HW 2.9 (Cauchy's Formula for Exterior Regions)

Problem 15.4.1 (?)
Let γ be a piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2. Assume f' exists in an open set containing γ and Ω_2 with $\lim_{z\to\infty} f(z) = A$. Show that

$$F(z) := \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1 \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}.$$

NOTE (DZG): I think there is a typo in this question....probably this should equal $f(z)$ for $z \in \Omega_1$, which is Cauchy's formula...

Solution omitted.

15.5 Tie's Extra Questions: Fall 2009 (Proving Cauchy using Green's)

Problem 15.5.1 (?)
State and prove Green's Theorem for rectangles. Use this to prove Cauchy's Theorem for functions that are analytic in a rectangle.

Problem 15.5.2 (Variant)
Suppose $f \in C^1_{\mathbb{C}}(\Omega)$ and $T \subset \Omega$ is a triangle with $T^\circ \subset \Omega$.

- Apply Green's theorem to show that $\int_T f(z)\, dz = 0$.
- Assume that f' is continuous and prove Goursat's theorem.

Hint: Green's theorem states

$$\int_T F dx + G dy = \int_{T^\circ} \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right) dx dy.$$

Solution omitted.

15.6 No polynomials converging uniformly to $1/z$

Problem 15.6.1 (?)
Prove that there is no sequence of polynomials that uniformly converge to $f(z) = \frac{1}{z}$ on S^1.

Solution omitted.

15.7 Eventually sublinear implies constant

Problem 15.7.1 (?)
Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire and

$$|f(z)| \leq |z|^{\frac{1}{2}} \qquad \text{when } |z| > 10.$$

Prove that f is constant.

Solution omitted.

15.8 The Cauchy pole function is holomorphic

Problem 15.8.1 (?)
Let γ be a smooth curve joining two distinct points $a, b \in \mathbb{C}$.
Prove that the function

$$f(z) := \int_\gamma \frac{g(w)}{w - z}\, dw$$

is analytic in $\mathbb{C} \setminus \gamma$.

Solution omitted.

15.9 Schwarz reflection proof

Problem 15.9.1 (?)
Suppose that $f : \mathbb{C} \to \mathbb{C}$ is continuous everywhere and analytic on $\mathbb{C} \setminus \mathbb{R}$ and prove that f is entire.

Solution omitted.

15.10 Prove Liouville

Problem 15.10.1 (?)
Prove Liouville's theorem: suppose $f : \mathbb{C} \to \mathbb{C}$ is entire and bounded. Use Cauchy's formula to prove that $f' \equiv 0$ and hence f is constant.

Solution omitted.

15.11 Tie's Extra Questions Fall 2009 (Fractional residue formula)

Problem 15.11.1 (?)
Assume f is continuous in the region:

$$0 < |z-a| \leq R, \quad 0 \leq \operatorname{Arg}(z-a) \leq \beta_0 \qquad \beta_0 \in (0, 2\pi].$$

and the following limit exists:

$$\lim_{z \to a} (z-a) f(z) = A.$$

Show that

$$\lim_{r \to 0} \int_{\gamma_r} f(z) dz = iA\beta_0 \ ,$$

where

$$\gamma_r := \{ z \mid z = a + re^{it}, \ 0 \leq t \leq \beta_0 \}..$$

Problem 15.11.2 (Alternative version)

Let f be a continuous function in the region

$$D = \{z \mid |z| > R, 0 \le \arg z \le \theta\} \quad \text{where} \quad 1 \le \theta \le 2\pi.$$

If there exists k such that $\lim_{z \to \infty} zf(z) = k$ for z in the region D. Show that

$$\lim_{R' \to \infty} \int_L f(z)dz = i\theta k,$$

where L is the part of the circle $|z| = R'$ which lies in the region D.

Solution omitted.

15.12 Spring 2020 HW 2, 2.6.7

Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and let $d := \sup_{z,w \in \mathbb{D}} |f(z) - f(w)|$ be the diameter of the image of f. Show that $2|f'(0)| \le d$, and that equality holds iff f is linear, so $f(z) = a_1 z + a_2$.

Hint:

$$2f'(0) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi) - f(-\xi)}{\xi^2} \, d\xi$$

whenever $0 < r < 1$.

15.13 Spring 2020 HW 2, 2.6.8

Suppose that f is holomorphic on the strip $S = \left\{x + iy \mid x \in \mathbb{R}, \; -1 < y < 1\right\}$ with $|f(z)| \le A\,(1 + |z|)^\nu$ for ν some fixed real number. Show that for all $z \in S$, for each integer $n \ge 0$ there exists an $A_n \ge 0$ such that $\left|f^{(n)}(x)\right| \le A_n(1 + |x|)^\nu$ for all $x \in \mathbb{R}$.

Hint: Use the Cauchy inequalities.

15.14 Spring 2020 HW 2, 2.6.9

Let $\Omega \subset \mathbb{C}$ be open and bounded and $\varphi : \Omega \to \Omega$ holomorphic. Prove that if there exists a point $z_0 \in \Omega$ such that $\varphi(z_0) = z_0$ and $\varphi'(z_0) = 1$, then φ is linear.

Hint: assume $z_0 = 0$ (explain why this can be done) and write $\varphi(z) = z + a_n z^n + O(z^{n+1})$ near 0. Let $\varphi_k = \varphi \circ \varphi \circ \cdots \circ \varphi$ and prove that $\varphi_k(z) = z + k a_n z^n + O(z^{n+1})$. Apply Cauchy's inequalities and let $k \to \infty$ to conclude.

15.15 Spring 2020 HW 2, 6

Show by example that there exists a function $f(z)$ that is holomorphic on $\left\{ z \in \mathbb{C} \;\middle|\; 0 < |z| < 1 \right\}$ and for all $r < 1$,

$$\int_{|z|=r} f(z)\, dz = 0,$$

but f is not holomorphic at $z = 0$.

15.16 Spring 2020 HW 2, 7

Let f be analytic on a region R and suppose $f'(z_0) \neq 0$ for some $z_0 \in R$. Show that if C is a circle of sufficiently small radius centered at z_0, then

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{dz}{f(z) - f(z_0)}.$$

Hint: use the inverse function theorem.

15.17 Spring 2020 HW 2, 8

Assume two functions $u, b : \mathbb{R}^2 \to \mathbb{R}$ have continuous partial derivatives at (x_0, y_0). Show that $f := u + iv$ has derivative $f'(z_0)$ at $z_0 = x_0 + iy_0$ if and only if

$$\lim_{r \to 0} \frac{1}{\pi r^2} \int_{|z - z_0| = r} f(z) dz = 0.$$

15.18 Spring 2020 HW 2, 10

Let $f(z)$ be bounded and analytic in $\mathbb{C}$. Let $a \neq b$ be any fixed complex numbers. Show that the following limit exists:

$$\lim_{R\to\infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}\, dz.$$

Use this to show that $f(z)$ must be constant.

15.19 Spring 2020 HW 2, 11

Suppose $f(z)$ is entire and

$$\lim_{z\to\infty} \frac{f(z)}{z} = 0.$$

Show that $f(z)$ is a constant.

15.20 Spring 2020 HW 2, 12

Let f be analytic in a domain D and γ be a closed curve in D. For any $z_0 \in D$ not on γ, show that

$$\int_\gamma \frac{f'(z)}{(z-z_0)} dz = \int_\gamma \frac{f(z)}{(z-z_0)^2} dz.$$

Give a generalization of this result.

15.21 Spring 2020 HW 2, 13

Compute

$$\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z}$$

and use it to show that

$$\int_0^{2\pi} \cos^{2n}(\theta)\, d\theta = 2\pi \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right).$$

16 | Maximum Modulus

16.1 Spring 2020 HW 3.8

Problem 16.1.1 (?)
Prove the fundamental theorem of Algebra using the maximum modulus principle.

16.2 Spring 2020.7

Problem 16.2.1 (?)
Let f be analytic on a bounded domain D, and assume also that f that is continuous and nowhere zero on the closure $\overline{D}$.
Show that if $|f(z)| = M$ (a constant) for z on the boundary of D, then $f(z) = e^{i\theta} M$ for z in D, where θ is a real constant.

Solution omitted.

16.3 Fall 2020.6

Problem 16.3.1 (?)
Suppose that U is a bounded, open and simply connected domain in $\mathbb{C}$ and that $f(z)$ is a complex-valued non-constant continuous function on $\overline{U}$ whose restriction to U is holomorphic.

- Prove the maximum modulus principle by showing that if $z_0 \in U$, then

$$|f(z_0)| < \sup\{|f(z)| : z \in \partial U\}.$$

- Show furthermore that if $|f(z)|$ is constant on ∂U, then $f(z)$ has a zero in U (i.e., there exists $z_0 \in U$ for which $f(z_0) = 0$).

Solution omitted.

16.4 Spring 2020 HW 3, SS 3.8.15

Problem 16.4.1 (?)
Use the Cauchy inequalities or the maximum modulus principle to solve the following problems:

a. Prove that if f is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \leq AR^k + B$$

for all $R > 0$, some integer $k \geq 0$, and some constants $A, B > 0$, then f is a polynomial of degree $\leq k$.

b. Show that if f is holomorphic in the unit disc, is bounded, and converges uniformly to zero in the sector $\theta < \arg(z) < \varphi$ as $|z| \to 1$, then $f \equiv 0$.

c. Let $w_1, \cdots w_n$ be points on $S^1 \subset \mathbb{C}$. Prove that there exists a point $z \in S^1$ such that the product of the distances from z to the points w_j is at least 1. Conclude that there exists a point $w \in S^1$ such that the product of the above distances is *exactly* 1.

d. Show that if the real part of an entire function is bounded, then f is constant.

Solution omitted.

Solution omitted.

Solution omitted.

Solution omitted.

Solution omitted.

16.5 Spring 2020 HW 3, 3.8.17

Let f be non-constant and holomorphic in an open set containing the closed unit disc.

a. Show that if $|f(z)| = 1$ whenever $|z| = 1$, then the image of f contains the unit disc.

Hint: Show that $f(z) = w_0$ has a root for every $w_0 \in \mathbb{D}$, for which it suffices to show that $f(z) = 0$ has a root. Conclude using the maximum modulus principle.

b. If $|f(z)| \geq 1$ whenever $|z| = 1$ and there exists a $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$, then the image of f contains the unit disc.

16.6 Spring 2020 HW 3, 3.8.19

Prove that maximum principle for harmonic functions, i.e.

a. If u is a non-constant real-valued harmonic function in a region Ω, then u can not attain a maximum or a minimum in Ω.

b. Suppose Ω is a region with compact closure $\overline{\Omega}$. If u is harmonic in Ω and continuous in $\overline{\Omega}$, then

$$\sup_{z \in \Omega} |u(z)| \leq \sup_{z \in \overline{\Omega} - \Omega} |u(z)|.$$

Hint: to prove (a), assume u attains a local maximum at z_0. Let f be holomorphic near z_0 with $\Re(f) = u$, and show that f is not an open map. Then (a) implies (b).

16.7 Spring 2020 HW 3.9

Let f be analytic in a region D and γ a rectifiable curve in D with interior in D.

Prove that if $f(z)$ is real for all $z \in \gamma$, then f is constant.

16.8 Spring 2020 HW 3.14

Let f be nonzero, analytic on a bounded region Ω and continuous on its closure $\overline{\Omega}$.

Show that if $|f(z)| \equiv M$ is constant for $z \in \partial\Omega$, then $f(z) \equiv Me^{i\theta}$ for some real constant θ.

16.9 Tie's Extra Questions: Spring 2015

Let $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ with $|\alpha| < 1$ and $\mathbb{D} = \{z : \ |z| < 1\}$. Prove that

- $\displaystyle \frac{1}{\pi} \iint_{\mathbb{D}} |\psi'_\alpha|^2 dx dy = 1.$

- $\dfrac{1}{\pi}\iint_{\mathbb{D}}|\psi'_\alpha|dxdy = \dfrac{1-|\alpha|^2}{|\alpha|^2}\log\dfrac{1}{1-|\alpha|^2}$.

16.10 Tie's Extra Questions: Spring 2015

Let Ω be a simply connected open set and let γ be a simple closed contour in Ω and enclosing a bounded region U anticlockwise. Let $f:\ \Omega \to \mathbb{C}$ be a holomorphic function and $|f(z)| \leq M$ for all $z \in \gamma$. Prove that $|f(z)| \leq M$ for all $z \in U$.

16.11 Tie's Extra Questions: Fall 2015

Assume $f(z)$ is analytic in region D and Γ is a rectifiable curve in D with interior in D. Prove that if $f(z)$ is real for all $z \in \Gamma$, then $f(z)$ is a constant.

17 | Liouville's Theorem

17.1 Spring 2020.3, Extras Fall 2009

Problem 17.1.1 (?)

- Assume $f(z) = \sum_{n=0}^\infty c_n z^n$ converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2 d\theta = \sum_{n=0}^\infty |c_n|^2 r^{2n}$$

- Deduce Liouville's theorem from (a).

Solution omitted.

17.2 FTA via Liouville

Problem 17.2.1 (?)
Prove the Fundamental Theorem of Algebra (using complex analysis).

Solution omitted.

17.3 Entire functions satisfying an inequality

Problem 17.3.1 (?)
Find all entire functions that satisfy

$$|f(z)| \geq |z| \quad \forall z \in \mathbb{C}.$$

Prove this list is complete.

Concept review omitted.

Solution omitted.

17.4 Entire functions with an asymptotic bound

Problem 17.4.1 (?)
Find all entire functions satisfying

$$|f(z)| \leq |z|^{\frac{1}{2}} \quad \text{for } |z| > 10.$$

Solution omitted.

17.5 Tie's Extra Questions: Fall 2009

Problem 17.5.1 (?)
Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω. Show without using Picard's theorems that $f(z)$ is a constant.

Solution omitted.

17.6 Tie's Extra Questions: Fall 2015

Problem 17.6.1 (?)
Let $f(z)$ be bounded and analytic in $\mathbb{C}$. Let $a \neq b$ be any fixed complex numbers. Show that the following limit exists:

$$\lim_{R\to\infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

Use this to show that $f(z)$ must be a constant (Liouville's theorem).

Solution omitted.

18 | Polynomials

18.1 Big O Estimates

18.1.1 Tie's Extra Questions: Fall 2011, Fall 2009 (Polynomial upper bound, $d = 2$)

Problem 18.1.1 (?)
Let $f(z)$ be entire and assume that $f(z) \leq M|z|^2$ outside some disk for some constant M. Show that $f(z)$ is a polynomial in z of degree ≤ 2.

Solution omitted.

18.1.2 Tie's Extra Questions: Spring 2015, Fall 2016 (Polynomial upper bound, d arbitrary)

Problem 18.1.2 (?) a. Let Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Assume the existence of a non-negative integer m, and of positive constants L and R, such that for all z with $|z| > R$ the inequality

$$|f(z)| \leq L|z|^m$$

holds. Prove that f is a polynomial of degree $\leq m$.

b. Let $f: \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose that there exists a real number M such that for all $z \in \mathbb{C}, \Re(f) \leq M$. Prove that f must be a constant.

Solution omitted.

18.1.3 Asymptotic to z^n

Problem 18.1.3 (?)
Suppose f is entire and suppose that for some integer $n \geq 1$,

$$\lim_{z \to \infty} \frac{f(z)}{z^n} = 0.$$

Prove that f is a polynomial of degree at most $n-1$.

Solution omitted.

18.1.4 Spring 2021.3, Tie's Extra Questions: Spring 2014, Fall 2009 (Polynomial lower bound, d arbitrary)

Problem 18.1.4 (?)
Suppose f is entire and there exist $A, R > 0$ and natural number N such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that

- f is a polynomial and
- the degree of f is at least N.

Solution omitted.

Solution omitted.

18.2 Misc

18.2.1 Spring 2021.4

Problem 18.2.1 (?)
Let $f = u + iv$ be an entire function such that $\Re(f(x+iy))$ is polynomial in x and y. Show that $f(z)$ is polynomial in z.

Solution omitted.

18.2.2 Spring 2019.4 (Eventually bounded implies rational)

Problem 18.2.2 (?)
Let f be a meromorphic function on the complex plane with the property that $|f(z)| \leq M$ for all $|z| > R$, for some constants $M > 0, R > 0$.
Prove that $f(z)$ is a rational function, i.e., there exist polynomials p, q so that $f = \frac{p}{q}$.

Solution omitted.

18.2.3 Spring 2020 HW 3.5, Tie's Extra Questions: Fall 2015

Problem 18.2.3 (?)
Let f be entire and suppose that $\lim_{z \to \infty} f(z) = \infty$. Show that f is a polynomial.

Solution omitted.

18.2.4 Spring 2020 HW 2, SS 2.6.13

Problem 18.2.4 (?)
Suppose f is analytic, defined on all of $\mathbb{C}$, and for each $z_0 \in \mathbb{C}$ there is at least one coefficient in the expansion $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ is zero. Prove that f is a polynomial.

Hint: use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.

Solution omitted.

19 Rouché's Theorem

19.1 Standard Applications

19.1.1 Tie's Extra Questions: Fall 2009, Fall 2011, Spring 2014 (FTA)

Problem 19.1.1 (?)
Use Rouche's theorem to prove the Fundamental Theorem of Algebra.

Solution omitted.

Solution omitted.

19.1.2 Tie's Extra Questions: Fall 2015 (Standard polynomial)

Problem 19.1.2 (?)
Find the number of roots of $z^4 - 6z + 3 = 0$ in $|z| < 1$ and $1 < |z| < 2$ respectively.

Solution omitted.

19.1.3 Tie's Extra Questions: Fall 2016 (Standard polynomial)

Problem 19.1.3 (?)
Prove that all the roots of the complex polynomial

$$f(z) = z^7 - 5z^3 + 12 = 0$$

lie between the circles $|z| = 1$ and $|z| = 2$.

Solution omitted.

19.1.4 Spring 2020 HW 3.11 (Standard polynomial)

Problem 19.1.4 (?)
Find the number of roots of $p(z) = z^4 - 6z + 3$ in $|z| < 1$ and $1 < |z| < 2$ respectively.

Note: the original problem used $4z^4 - 6z + 3$, but I don't think it's possible to use Rouché on that at all!

Solution omitted.

19.1.5 Standard polynomial

Problem 19.1.5 (?)
How many roots does the following polynomial have in the open disc $|z| < 1$?

$$f(z) = z^7 - 4z^3 - 1.$$

Solution omitted.

19.1.6 Spring 2020 HW 1.3 (Standard polynomial)

Problem 19.1.6 (?)
Prove that the following polynomial has its roots outside of the unit circle:

$$p(z) = z^3 + 2z + 4.$$

Hint: What is the maximum value of the modulus of the first two terms if $|z| \leq 1$?

Solution omitted.

19.1.7 Polynomials with parameters

Problem 19.1.7 (?)
Assume that $|b| < 1$ and show that the following polynomial has exactly two roots (counting multiplicity) in $|z| < 1$:

$$f(z) := z^3 + 3z^2 + bz + b^2.$$

Solution omitted.

19.1.8 Tie's Extra Questions: Spring 2015 (Power series)

Problem 19.1.8 (?)
Let $0 < r < 1$. Show that polynomials $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ have no zeros in $|z| < r$ for all sufficiently large n's.

Solution omitted.

19.2 Exponentials

19.2.1 UMN Fall 2009 (Solutions as zeros)

Problem 19.2.1 (?)
Find the number of solutions to the following equation on $|z| < 1$:

$$6z^3 + 1 = -e^z.$$

Solution omitted.

19.2.2 UMN Spring 2009 (Checking the equality case)

Problem 19.2.2 (?)
Find the number of roots on $|z| \leq 1$ of

$$f(z) = z^6 + 4z^2 e^{z+1} - 3.$$

Solution omitted.

19.2.3 Right half-plane estimate

Problem 19.2.3 (?)
Find the number of zeros z with $\Re(z) > 0$ for the following function:

$$f(z) \coloneqq z^3 - z + 1.$$

Solution omitted.

19.2.4 Zeros of e^z

Problem 19.2.4 (?)
Prove that for every $n \in \mathbb{Z}^{\geq 0}$ the following polynomial has no roots in the open unit disc:

$$f_n(z) \coloneqq \sum_{k=0}^{n} \frac{z^k}{k!}.$$

Hint: check $n = 1, 2$ directly.

Solution omitted.

19.2.5 More e^z

Problem 19.2.5 (?)
Let $n \in \mathbb{Z}^{\geq 0}$ and show that the equation

$$e^z = az^n$$

has n solutions in the open unit disc if $|a| > e$, and no solutions if $|a| < \frac{1}{e}$.

Solution omitted.

19.2.6 Zeros of partial sums of exponential

Problem 19.2.6 (?)
For each $n \in \mathbb{Z}^{\geq 1}$, let

$$P_n(z) = 1 + z + \frac{1}{2!}z^2 + \cdots + \frac{1}{n!}z^n.$$

Show that for sufficiently large n, the polynomial P_n has no zeros in $|z| < 10$, while the polynomial $P_n(z) - 1$ has precisely 3 zeros there.

Solution omitted.

19.3 Working for the estimate

19.3.1 Max of a polynomial on S^1

Problem 19.3.1
Prove that

$$\max_{|z|=1} \left| a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n \right| \geq 1.$$

Hint: the first part of the problem asks for a statement of Rouche's theorem.

Solution omitted.

19.3.2 Fixed points

Problem 19.3.2 (?)
Let $c \in \mathbb{C}$ with $|c| < \frac{1}{3}$. Show that on the open set $\left\{ z \in \mathbb{C} \;\middle|\; \Re(z) < 1 \right\}$, the function $f(z) := ce^z$ has exactly one fixed point.

Solution omitted.

19.3.3 $z \sin(z) = 1$

Problem 19.3.3 (?)
Show that $z \sin(z) = a$ has only real solutions.

Solution omitted.

#stuck

19.3.4 Spring 2020 HW 3.13 #stuck

Problem 19.3.4 (?)
Prove that for $a > 0$, $z \tan z - a$ has only real roots.

19.3.5 UMN Spring 2011 (Constant coefficient trick)

Problem 19.3.5 (?)
Let $a \in \mathbb{C}$ and $n \geq 2$. Show that the following polynomial has one root in $|z| \leq 2$:

$$f(z) = az^n + z + 1.$$

Solution omitted.

20 | Argument Principle

20.1 Spring 2020 HW 3.12, Tie's Extra Questions Fall 2015 (Root counting with argument principle)

Problem 20.1.1 (?)
Prove that $f(z) = z^4 + 2z^3 - 2z + 10$ has exactly one root in each open quadrant.

Solution omitted.

Solution omitted.

20.1.1 n-to-one functions

Problem 20.1.2 (?)
Let f be analytic in a domain D and fix $z_0 \in D$ with $w_0 := f(z_0)$. Suppose z_0 is a zero of $f(z) - w_0$ with finite multiplicity m. Show that there exists $\delta > 0$ and $\varepsilon > 0$ such that for each w such that $0 < |w - w_0| < \varepsilon$, the equation $f(z) - w = 0$ has exactly m *distinct* solutions inside the disc $|z - z_0| < \delta$.

Solution omitted.

20.1.2 Blaschke products are n to one

Problem 20.1.3 (?)
For $k = 1, 2, \cdots, n$, suppose $|a_k| < 1$ and

$$f(z) := \left(\frac{z - a_1}{1 - \bar{a}_1 z}\right)\left(\frac{z - a_2}{1 - \bar{a}_2 z}\right)\cdots\left(\frac{z - a_n}{1 - \bar{a}_n z}\right).$$

Show that $f(z) = b$ has n solutions in $|z| < 1$.

Solution omitted.

21 | Morera

21.1 Uniform limit theorem

Problem 21.1.1 (?)
Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of analytic functions on $\mathbb{D} := \left\{z \in \mathbb{C} \;\middle|\; |z| < 1\right\}$.
Show that if $f_n \to g$ for some $g : \mathbb{D} \to \mathbb{C}$ uniformly on every compact $K \subset \mathbb{D}$, then g is analytic on $\mathbb{D}$.

Solution omitted.

21.2 Fourier transforms are entire

Problem 21.2.1 (?)
Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that vanishes outside of some finite interval. For each $z \in \mathbb{C}$, define

$$g(z) = \int_{-\infty}^{\infty} f(t)e^{-izt}\, dt.$$

Show that g is entire.

Solution omitted.

21.3 Tie's Extra Questions: Fall 2009, Fall 2011

Problem 21.3.1 (?)
Let $f(z)$ be analytic in an open set Ω except possibly at a point z_0 inside Ω. Show that if $f(z)$ is bounded in near z_0, then $\int_\Delta f(z)dz = 0$ for all triangles Δ in Ω.

Solution omitted.

21.4 Fall 2021.2

Problem 21.4.1 (?)
Let $\gamma(t)$ be a piecewise smooth curve in $\mathbb{C}, t \in [0, 1]$. Let $F(w)$ be a continuous function on γ. Show that $f(z)$ defined by

$$f(z) := \int_\gamma \frac{F(w)}{w - z} dw$$

is analytic on the complement of the curve γ.

Solution omitted.

Solution omitted.

21.5 Spring 2020 HW 2, SS 2.6.6

Problem 21.5.1 (?)
Suppose that f is holomorphic on a punctured open set $\Omega \setminus \{w_0\}$ and let $T \subset \Omega$ be a triangle containing w_0. Prove that if f is bounded near w_0, then $\int_T f(z)\, dz = 0$.

Solution omitted.

See also conformal map exercises.

21.6 Classifying conformal maps

Problem 21.6.1 (?)
Define

$$G := \left\{ z \in \mathbb{C} \;\middle|\; \Re(z) > 0,\ |z-1| > 1 \right\}.$$

Find all of the injective conformal maps $G \to \mathbb{D}$. These may be expressed as compositions of maps, but explain why this list is complete.

Solution omitted.

22 Half-planes, discs, strips

22.1 Tie's Extra Questions: Spring 2015 (Good Practice)

Problem 22.1.1 (?)
Find a conformal map

1. from $\{z : |z - 1/2| > 1/2, \mathrm{Re}(z) > 0\}$ to $\mathbb{H}$
2. from $\{z : |z - 1/2| > 1/2, |z| < 1\}$ to $\mathbb{D}$
3. from the intersection of the disk $|z + i| < \sqrt{2}$ with $\mathbb{H}$ to $\mathbb{D}$.
4. from $\mathbb{D} \setminus [a, 1)$ to $\mathbb{D} \setminus [0, 1)$ $(0 < a < 1)$.

Short solution possible using Blaschke factors.

5. from $\{z : |z| < 1, \operatorname{Re}(z) > 0\} \setminus (0, 1/2]$ to $\mathbb{H}$.

Solution omitted.

22.2 Tie's Extra Questions: Fall 2016 (Half-strip)

Problem 22.2.1 (?)
Find the conformal map that takes the upper half-plane conformally onto the half-strip

$$\left\{ w = x + iy \;\middle|\; -\pi/2 < x < \pi/2,\, y > 0 \right\}.$$

Solution omitted.

23 | Lunes, Bigons

23.1 Fall 2019.5, Tie's extra questions: Fall 2009, Fall 2011, Spring 2014, Spring 2015

Problem 23.1.1 (?)
Find a conformal map from

$$D = \left\{ z \in \mathbb{C} \;\middle|\; |z| < 1 \text{ and } \left| z - \frac{1}{2} \right| > \frac{1}{2} \right\}$$

to the unit disk $\Delta = \{z : |z| < 1\}$.

Solution omitted.

23.2 Fall 2021.7

Problem 23.2.1 (?)
Find a conformal map from the intersection of $|z-1|<2$ and $|z+1|<2$ to the upper half plane.

Warning 23.2.1
DZG: I'm 90% sure this is meant to be $|z-1|, |z+1| < \sqrt{2}$ or $|z-1|^2, |z+1|^2 < 2$. Otherwise computing the argument of the resulting lines is tricky...

Solution omitted.

23.3 Spring 2020.5, Spring 2019.6

Problem 23.3.1 (Spring 2020.5)
Find a conformal map that maps the region

$$R = \left\{ z \;\middle|\; \Re(z) > 0, \left| z - \frac{1}{2} \right| > \frac{1}{2} \right\}$$

to the upper half plane.

Problem 23.3.2 (Spring 2019.6)
Find a conformal map from

$$\left\{ z \;\middle|\; |z-1/2| > 1/2, \Re(z) > 0 \right\}$$

to $\mathbb{H}$.

Solution omitted.

23.4 UMN Spring 2009

Problem 23.4.1 (Lune, one intersection)
Find a conformal map from the region bounded by $\left|z - \frac{i}{2}\right| = \frac{1}{2}$ and $|z-i| = 1$ to $\mathbb{D}$.

Solution omitted.

24 | Joukowski Maps, Blaschke Factors, Slits

24.1 Spring 2021.7 (Slit)

Problem 24.1.1 (?)
Let R be the intersection of the right half-plane and the outside of the circle $\left|z-\frac{1}{2}\right|=\frac{1}{2}$ with the line segment $[1,2]$ removed, i.e.

$$R=\left\{z\in\mathbb{C}\;\middle|\;\Re(z)>0,\;\left|z-\frac{1}{2}\right|>\frac{1}{2}\right\}\setminus\left\{z:=x+iy\;\middle|\;1\leq x\leq 2,\;y=0\right\}.$$

Find a conformal map from R to $\mathbb{H}$ the upper half-plane.

Concept review omitted.

Solution omitted.

24.2 Exercises (Lune)

Problem 24.2.1 (?)
Let $\lambda=\frac{1}{2}\left(1+i\sqrt{3}\right)$ and find a map

$$R:=\{|z-\lambda|<1\}\cap\left\{\left|z-\bar{\lambda}\right|<1\right\}\longrightarrow\mathbb{D}.$$

Solution omitted.

24.3 Fall 2020.5, Spring 2019.6 (Joukowski)

Problem 24.3.1 (?)
Consider the function $f(z)=\frac{1}{2}\left(z+\frac{1}{z}\right)$ for $z\in\mathbb{C}\setminus\{0\}$. Let $\mathbb{D}$ denote the open unit disc.

a. Show that f is one-to-one on the punctured disc $\mathbb{D}\setminus\{0\}$. What is the image of the circle $|z|=r$ under this map when $0<r<1$?

b. Show that f is one-to-one on the domain $\mathbb{C}\setminus\mathbb{D}$. What is the image of this domain under this map?

c. Show that there exists a map $g : \mathbb{C}\backslash[-1,1] \to \mathbb{D}\backslash\{0\}$ such that $(g \circ f)(z) = z$ for all $z \in \mathbb{D}\backslash\{0\}$. Describe the map g by an explicit formula.

Solution omitted.

24.4 Tie's Extra Questions: Spring 2015 (Joukowski)

Problem 24.4.1 (?)
Prove that $f(z) = -\frac{1}{2}\left(z + \frac{1}{z}\right)$ is a conformal map from the half disc

$$\{z = x + iy : \ |z| < 1, \ y > 0\}$$

to $\mathbb{H} := \{z = x + iy : \ y > 0\}$.

Solution omitted.

24.5 UMN Spring 2008

Problem 24.5.1 (?)
Define $A := \{\Re(z) > 0, \Im(z) > 0\}$. Find a conformal equivalence $\Delta \cap A \to A$.

Solution omitted.

25 Linear Fractional Transformations

25.1 Tie's Extra Questions: Spring 2015

Problem 25.1.1 (?)
Let C and C' be two circles and let $z_1 \in C$, $z_2 \notin C$, $z_1' \in C'$, $z_2' \notin C'$. Show that there is a unique fractional linear transformation f with $f(C) = C'$ and $f(z_1) = z_1'$, $f(z_2) = z_2'$.

Solution omitted.

25.2 UMN Fall 2012

Problem 25.2.1 (?)
Suppose f is holomorphic on Δ^* and $\Re(f) \geq 0$. Show that f has a removable singularity at $z = 0$.

Solution omitted.

25.3 UMN Fall 2009

Problem 25.3.1 (?)
Suppose f is entire and $f(\mathbb{C}) \subseteq \mathbb{H}$. Show that f must be constant.

Solution omitted.

26 Schwarz Lemma

26.1 Fall 2020.4 (Schwarz double root) #stuck

Problem 26.1.1 (?)
Let $\mathbb{D} := \{z : |z| < 1\}$ denote the open unit disk. Suppose that $f(z) : \mathbb{D} \to \mathbb{D}$ is holomorphic, and that there exists $a \in \mathbb{D}\backslash\{0\}$ such that $f(a) = f(-a) = 0$.

- Prove that $|f(0)| \leq |a|^2$.
- What can you conclude when $|f(0)| = |a|^2$?

Solution omitted.

26.2 Fall 2021.5

Problem 26.2.1 (?)
Assume f is an entire function such that $|f(z)| = 1$ on $|z| = 1$. Prove that $f(z) = e^{i\theta} z^n$, where θ is a real number and n a non-negative integer.

Suggestion: First use the maximum and minimum modulus theorem to show

$$f(z) = e^{i\theta} \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k} z}$$

if f has zeros.

Solution omitted.

26.3 Fall 2021.6 (Schwarz manipulation)

Problem 26.3.1 (?)
Show that if $f : D(0, R) \to \mathbb{C}$ is holomorphic, with $|f(z)| \leq M$ for some $M > 0$, then

$$\left| \frac{f(z) - f(0)}{M^2 - \overline{f(0)} f(z)} \right| \leq \frac{|z|}{MR}.$$

Concept review omitted.

Solution omitted.

26.4 Scaling Schwarz

Problem 26.4.1 (?)
Let $\overline{B}(a, r)$ denote the closed disc of radius r about $a \in \mathbb{C}$. Let f be holomorphic on an open set containing $\overline{B}(a, r)$ and let

$$M := \sup_{z \in \overline{B}(a,r)} |f(z)|.$$

Prove that

$$z \in \overline{B}\left(a, \frac{r}{2}\right), z \neq a, \qquad \frac{|f(z) - f(a)|}{|z - a|} \leq \frac{2M}{r}.$$

Solution omitted.

26.5 Bounding derivatives

Problem 26.5.1 (?)
Suppose $f : \mathbb{D} \to \mathbb{H}$ is analytic and satisfies $f(0) = 2$. Find a sharp upper bound for $|f'(0)|$, and prove it is sharp by example.

Concept review omitted.

Solution omitted.

26.6 Schwarz for higher order zeros

Problem 26.6.1 (?)
Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic, has a single zero of order k at $z = 0$, and satisfies $\lim_{|z|\to 1} |f(z)| = 1$. Give with proof a formula for $f(z)$.

Solution omitted.

26.7 Schwarz with an injective function

Problem 26.7.1 (?)
Suppose $f, g : \mathbb{D} \to \Omega$ are holomorphic with f injective and $f(0) = g(0)$.
Show that

$$\forall\, 0 < r < 1, \qquad g\left(\{|z| < r\}\right) \subseteq f\left(\{|z| < r\}\right).$$

The first part of this problem asks for a statement of the Schwarz lemma.

Solution omitted.

26.8 Reflection principle

Problem 26.8.1 (?)
Let $S \coloneqq \left\{ z \in \mathbb{D} \;\middle|\; \Im(z) \geq 0 \right\}$. Suppose $f : S \to \mathbb{C}$ is continuous on S, real on $S \cap \mathbb{R}$, and holomorphic on S°.
Prove that f is the restriction of a holomorphic function on $\mathbb{D}$.

Solution omitted.

27 | Blaschke Factors

27.1 Spring 2019.5, Spring 2021.5 (Blaschke contraction)

Problem 27.1.1 (?)
Let f be a holomorphic map of the open unit disc $\mathbb{D}$ to itself. Show that for any $z, w \in \mathbb{D}$,

$$\left| \frac{f(w) - f(z)}{1 - \overline{f(w)} f(z)} \right| \leq \left| \frac{w - z}{1 - \overline{w} z} \right|.$$

Show that this inequality is strict for $z \neq w$ except when f is a linear fractional transformation from $\mathbb{D}$ to itself.

Concept review omitted.

Solution omitted.

27.2 Schwarz-Pick derivative

Problem 27.2.1 (?)
Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic. Prove that

$$\forall a \in \mathbb{D}, \qquad \frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2}.$$

Solution omitted.

27.3 Schwarz and Blaschke products

Problem 27.3.1 (?)
Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic and admits a continuous extension $\tilde{f} : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ such that $|z| = 1 \implies |f(z)| = 1$.

a. Prove that f is a rational function.

b. Suppose that $z = 0$ is the unique zero of f. Show that

$$\exists n \in \mathbb{N}, \lambda \in S^1 \quad \text{suchthat} \quad f(z) = \lambda z^n.$$

c. Suppose that $a_1, \cdots, a_n \in \mathbb{D}$ are the zeros of f and prove that

$$\exists \lambda \in S^1 \quad \text{suchthat} \quad f(z) = \lambda \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}.$$

Solution omitted.

27.3.1 Tie's Extra Questions: Fall 2009

Problem 27.3.2 (?)
Let g be analytic for $|z| \leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

1. Show that g has a unique fixed point in $|z| < 1$.
2. What happens if we replace $|g(z)| < 1$ with $|g(z)| \leq 1$ for $|z| = 1$? Give an example if (a) is not true or give an proof if (a) is still true.
3. What happens if we simply assume that f is analytic for $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z) \not\equiv z$. Can f have more than one fixed point in $|z| < 1$?

Hint: The map $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ may be useful.

Solution omitted.

Solution omitted.

Solution omitted.

27.3.2 Tie's Extra Questions: Fall 2015 (Blaschke factor properties) #complex/exercises/completed

Problem 27.3.3 (?) a. Let z, w be complex numbers, such that $\bar{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left|\frac{w-z}{1-\overline{w}z}\right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

b. Prove that for fixed w in the unit disk $\mathbb{D}$, the mapping

$$F : z \mapsto \frac{w-z}{1-\overline{w}z}$$

satisfies the following conditions:

- F maps $\mathbb{D}$ to itself and is holomorphic.
- F interchanges 0 and w, namely, $F(0) = w$ and $F(w) = 0$.
- $|F(z)| = 1$ if $|z| = 1$.
- $F : \mathbb{D} \mapsto \mathbb{D}$ is bijective.

Hint: Calculate $F \circ F$.

27.4 Tie's Extra Questions: Spring 2015

Problem 27.4.1 (?)
Suppose f is analytic in an open set containing the unit disc $\mathbb{D}$ and $|f(z)| = 1$ when $|z|$=1. Show that either $f(z) = e^{i\theta}$ for some $\theta \in \mathbb{R}$ or there are finite number of $z_k \in \mathbb{D}$, $k \leq n$ and $\theta \in \mathbb{R}$ such that

$$f(z) = e^{i\theta} \prod_{k=1}^{n} \frac{z - z_k}{1 - \bar{z}_k z} ..$$

Also cf. Stein et al, 1.4.7, 3.8.17

27.5 Tie's Extra Questions: Spring 2015 (Equality of modulus)

> *Problem* 27.5.1 (?)
> Let f and g be non-zero analytic functions on a region Ω. Assume $|f(z)| = |g(z)|$ for all z in Ω. Show that $f(z) = e^{i\theta} g(z)$ in Ω for some $0 \le \theta < 2\pi$.

Solution omitted.

28 Fixed Points

28.1 Fall 2020.7

> *Problem* 28.1.1 (?)
> Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = 0$. Let $n \ge 1$, and define the function $f_n(z)$ to be the n-th composition of f with itself; more precisely, let
>
> $$f_1(z) := f(z), f_2(z) := f(f(z)), \text{ in general } f_n(z) := f\left(f_{n-1}(z)\right).$$
>
> Suppose that for each $z \in \mathbb{D}$, $\lim_{n\to\infty} f_n(z)$ exists and equals to $g(z)$. Prove that either $g(z) \equiv 0$ or $g(z) = z$ for all $z \in D$.

Solution omitted.

29 Open Mapping, Riemann Mapping, Casorati-Weierstrass

29.1 Spring 2020.6 (Prove the open mapping theorem)

> *Problem* 29.1.1 (?)
> Prove the open mapping theorem for holomorphic functions: If f is a non-constant holomorphic function on an open set U in $\mathbb{C}$, then $f(U)$ is also an open set.

Solution omitted.

29.2 Fall 2019.4, Spring 2020 HW 3 SS 3.8.14, Tie's Extras Fall 2009, Problem Sheet (Entire univalent functions are linear)

Problem 29.2.1 (Entire univalent functions are affine/linear)
Let $f : \mathbb{C} \to \mathbb{C}$ be an injective analytic (also called univalent) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

Hint: Apply the Casorati-Weierstrass theorem to $f(1/z)$.

Solution omitted.

Solution omitted.

29.3 Tie's Extra Questions: Spring 2015

Problem 29.3.1 (?)

1. Let f be analytic in $\Omega : 0 < |z - a| < r$ except at a sequence of poles $a_n \in \Omega$ with $\lim_{n\to\infty} a_n = a$. Show that for any $w \in \mathbb{C}$, there exists a sequence $z_n \in \Omega$ such that $\lim_{n\to\infty} f(z_n) = w$.
2. Explain the similarity and difference between the above assertion and the Weierstrass-Casorati theorem.

DZG: I think it's also necessary to state that $z_n \to a$.

Solution omitted.

29.4 Dense images #stuck

Problem 29.4.1 (?)
Suppose $f : \mathbb{H} \cup \mathbb{R} \to \mathbb{C}$ satisfies the following:

- $f(i) = i$
- f is continuous
- f is analytic on $\mathbb{H}$
- $f(z) \in \mathbb{R} \iff z \in \mathbb{R}$.

Show that $f(\mathbb{H})$ is a dense subset of $\mathbb{H}$.

Solution omitted.

29.5 Tie's Extra Questions: Spring 2015

Problem 29.5.1 (?)
Let $f(z)$ be an analytic function on $\mathbb{C}\backslash\{z_0\}$, where z_0 is a fixed point. Assume that $f(z)$ is bijective from $\mathbb{C}\backslash\{z_0\}$ onto its image, and that $f(z)$ is bounded outside $D_r(z_0)$, where r is some fixed positive number. Show that there exist $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$, $c \neq 0$ such that $f(z) = \dfrac{az+b}{cz+d}$.

30 | Schwarz Reflection

30.1 Tie's Extra Questions: Spring 2015 (Reflection for harmonic functions)

Problem 30.1.1 (?) (1) Assume u is harmonic on open set O and z_n is a sequence in O such that $u(z_n) = 0$ and $\lim z_n \in O$. Prove or disprove that u is identically zero. What if O is a region?

(2) Assume u is harmonic on open set O and $u(z) = 0$ on a disc in O. Prove or disprove that u is identically zero. What if O is a region?

(3) Formulate and prove a Schwarz reflection principle for harmonic functions

cf. Theorem 5.6 on p.60 of Stein et al.

Hint: Verify the mean value property for your new function obtained by Schwarz reflection principle.

Solution omitted.

30.2 Reflection for the disc

Problem 30.2.1 (?)
a. State the standard Schwarz reflection principle involving reflection across the real axis.

b. Give a linear fractional transformation T mapping $\mathbb{D}$ to $\mathbb{H}$. Let $g(z) = \bar{z}$, and show

$$(T^{-1} \circ g \circ T)(z) = 1/\bar{z}.$$

c. Suppose that f is holomorphic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, and real on S^1. Show that f must be constant.

Solution omitted.

30.3 Spring 2020 HW 2, SS 2.6.15 (Constant on boundary and nonvanishing implies constant, using Schwarz)

Problem 30.3.1 (?)
Suppose f is continuous and nonvanishing on $\overline{\mathbb{D}}$, and holomorphic in $\mathbb{D}$. Prove that if $|z| = 1 \implies |f(z)| = 1$, then f is constant.

Hint: Extend f to all of $\mathbb{C}$ by $f(z) = 1/\overline{f(1/\bar{z})}$ for any $|z| > 1$, and argue as in the Schwarz reflection principle.

Solution omitted.

31 Unsorted

31.1 Tie's Extra Questions: Fall 2015

1. Let $f(z) \in H(\mathbb{D})$, $\mathrm{Re}(f(z)) > 0$ and $f(0) = a > 0$. Show that

$$\left| \frac{f(z) - a}{f(z) + a} \right| \leq |z|, \quad |f'(0)| \leq 2a.$$

2. Show that the above is still true if $\mathrm{Re}(f(z)) > 0$ is replaced with $\mathrm{Re}(f(z)) \geq 0$.

31.2 Tie's Extra Questions: Spring 2015

(1) Let $p(z)$ be a polynomial, $R > 0$ any positive number, and $m \geq 1$ an integer. Let $M_R = \sup\{|z^m p(z) - 1| : |z| = R\}$. Show that $M_R > 1$.

(2) Let $m \geq 1$ be an integer and $K = \{z \in \mathbb{C} : r \leq |z| \leq R\}$ where $r < R$. Show (i) using (1) as well as, (ii) without using (1) that there exists a positive number $\varepsilon_0 > 0$ such that for each polynomial $p(z)$,

$$\sup\{|p(z) - z^{-m}| : z \in K\} \geq \varepsilon_0\,.$$

31.3 Tie's Extra Questions: Spring 2015

(1) Explicitly write down an example of a non-zero analytic function in $|z| < 1$ which has infinitely zeros in $|z| < 1$.

(2) Why does not the phenomenon in (1) contradict the uniqueness theorem?

31.4 Tie's Extra Questions: Spring 2015

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s < r$, there exists a constant $c > 0$ such that

$$||f||_{(\infty,s)} \leq c||f||_{(1,r)},$$

where $|f||_{(\infty,s)} = \sup_{z\in D_s(z_0)}|f(z)|$ and $||f||_{(1,r)} = \displaystyle\int_{D_r(z_0)} |f(z)|dxdy$.

Note: Exercise 3.8.20 on p.107 in Stein et al is a straightforward consequence of this stronger result using the integral form of the Cauchy-Schwarz inequality in real analysis.

31.5 Tie's Extra Questions: Spring 2015

Let f be an analytic function on a region Ω. Show that f is a constant if there is a simple closed curve γ in Ω such that its image $f(\gamma)$ is contained in the real axis.

31.6 Tie's Extra Questions: Spring 2015

(1) Show that $\dfrac{\pi^2}{\sin^2 \pi z}$ and $g(z) = \displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ have the same principal part at each integer point.

(2) Show that $h(z) = \dfrac{\pi^2}{\sin^2 \pi z} - g(z)$ is bounded on $\mathbb{C}$ and conclude that $\dfrac{\pi^2}{\sin^2 \pi z} = \displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$.

31.7 Tie's Extra Questions: Spring 2015

Assume $f(z)$ is analytic in $\mathbb{D} : |z| < 1$ and $f(0) = 0$ and is not a rotation (i.e. $f(z) \neq e^{i\theta} z$). Show that $\displaystyle\sum_{n=1}^{\infty} f^n(z)$ converges uniformly to an analytic function on compact subsets of $\mathbb{D}$, where $f^{n+1}(z) = f(f^n(z))$.

31.8 Tie's Extra Questions: Spring 2015

Let f be a non-constant analytic function on $\mathbb{D}$ with $f(\mathbb{D}) \subseteq \mathbb{D}$. Use $\psi_a(f(z))$ (where $a = f(0)$, $\psi_a(z) = \dfrac{a-z}{1-\bar{a}z}$) to prove that $\dfrac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \dfrac{|f(0)| + |z|}{1 - |f(0)||z|}$.

31.9 Tie's Extra Questions: Spring 2015

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s < r$, there exists a constant $c > 0$ such that

$$\|f\|_{(\infty,s)} \leq c\|f\|_{(1,r)},$$

where $\|f\|_{(\infty,s)} = \sup_{z \in D_s(z_0)} |f(z)|$ and $\|f\|_{(1,r)} = \displaystyle\int_{D_r(z_0)} |f(z)| dx dy$.

31.10 Tie's Extra Questions: Spring 2015

Let Ω be a simply connected open set and let γ be a simple closed contour in Ω and enclosing a bounded region U anticlockwise. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function and $|f(z)| \leq M$ for all

$z \in \gamma$. Prove that $|f(z)| \leq M$ for all $z \in U$.

31.11 Tie's Extra Questions: Spring 2015

Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s < r$, there exists a constant $c > 0$ such that

$$\|f\|_{(\infty,s)} \leq c\|f\|_{(1,r)},$$

where $\|f\|_{(\infty,s)} = \sup_{z\in D_s(z_0)}|f(z)|$ and $\|f\|_{(1,r)} = \int_{D_r(z_0)} |f(z)|dxdy$.

31.12 Tie's Extra Questions: Fall 2016

a. $f(z) = u(x,y) + iv(x,y)$ be analytic in a domain $D \subset \mathbb{C}$. Let $z_0 = (x_0, y_0)$ be a point in D which is in the intersection of the curves $u(x,y) = c_1$ and $v(x,y) = c_2$, where c_1 and c_2 are constants. Suppose that $f'(z_0) \neq 0$. Prove that the lines tangent to these curves at z_0 are perpendicular.

b. Let $f(z) = z^2$ be defined in $\mathbb{C}$.

- Describe the level curves of $\mathrm{Re}(f)$ and of $\mathrm{Im}(f)$.

- What are the angles of intersections between the level curves $\mathrm{Re}(f) = 0$ and $\mathrm{Im}(f)$? Is your answer in agreement with part a) of this question?

31.13 Tie's Extra Questions: Fall 2016

a. $f : D \to \mathbb{C}$ be a continuous function, where $D \subset \mathbb{C}$ is a domain.Let $\alpha : [a,b] \to D$ be a smooth curve. Give a precise definition of the *complex line integral*

$$\int_\alpha f.$$

b. Assume that there exists a constant M such that $|f(\tau)| \leq M$ for all $\tau \in \mathrm{Image}(\alpha)$. Prove that

$$|\int_\alpha f| \leq M \times \mathrm{length}(\alpha).$$

c. Let C_R be the circle $|z| = R$, described in the counterclockwise direction, where $R > 1$. Provide an upper bound for $|\int_{C_R} \frac{\log(z)}{z^2}|$, which depends *only* on R and other constants.

31.14 Tie's Extra Questions: Fall 2016

a. Let F be an analytic function inside and on a simple closed curve C, except for a pole of order $m \geq 1$ at $z = a$ inside C. Prove that

$$\frac{1}{2\pi i} \oint_C F(\tau) d\tau = \lim_{\tau \to a} \frac{d^{m-1}}{d\tau^{m-1}} ((\tau - a)^m F(\tau))).$$

b. Evaluate

$$\oint_C \frac{e^\tau}{(\tau^2 + \pi^2)^2} d\tau$$

where C is the circle $|z| = 4$.

31.15 Tie's Extra Questions: Spring 2014, Fall 2009, Fall 2011

For $s > 0$, the **gamma function** is defined by $\Gamma(s) = \displaystyle\int_0^\infty e^{-t} t^{s-1} dt$.

- Show that the gamma function is analytic in the half-plane $\Re(s) > 0$, and is still given there by the integral formula above.
- Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need $\Gamma(1-s) = t \displaystyle\int_0^\infty e^{-vt}(vt)^{-s} dv$ *for* $t > 0$.

31.15.1 Tie's Extra Questions: Fall 2011

Problem 31.15.1 (?)

- Show that the function $u = u(x, y)$ given by

$$u(x,y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx \quad \text{for } n \in \mathbf{N}$$

is the solution on $D = \{(x, y) \ |x^2 + y^2 < 1\}$ of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,0) = \frac{\sin nx}{n}.$$

- Show that there exist points $(x,y) \in D$ such that $\limsup_{n\to\infty} |u(x,y)| = \infty$.