

Lecture 7: Logistic Regression

Winter 2018

Kai-Wei Chang

CS @ UCLA

kw+cm146@kwchang.net

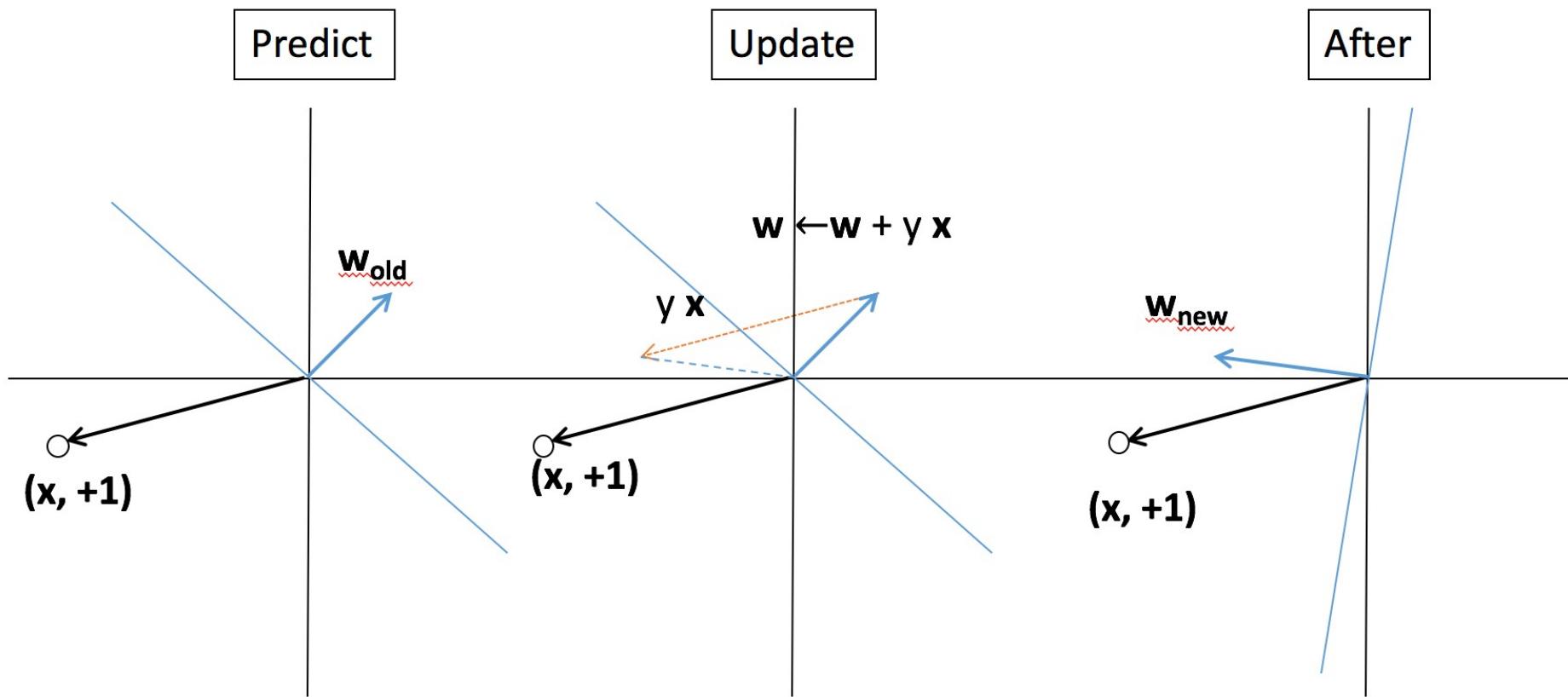
The instructor gratefully acknowledges Dan Roth, Vivek Srikumar, Sriram Sankararaman, Fei Sha, Ameet Talwalkar, Eric Eaton, and Jessica Wu whose slides are heavily used, and the many others who made their course material freely available online.

Homework

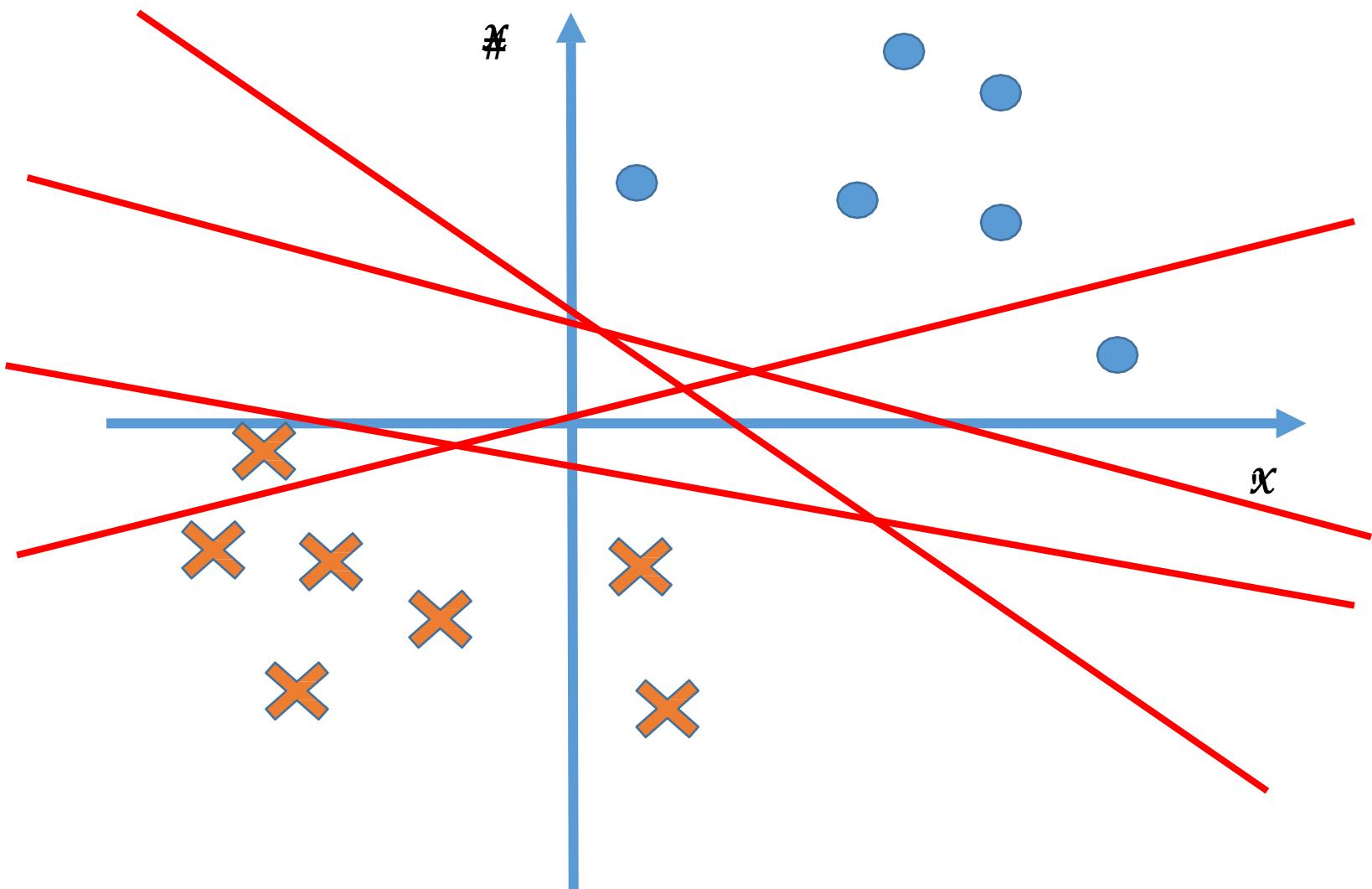
- Hw 1 will be Due next Monday (10/29)
- Midterm: 11/5
- We will have Hw 2 released soon
 - Math Part: Due 10/31
 - Programming part: **11/10**

Recap: Perceptron

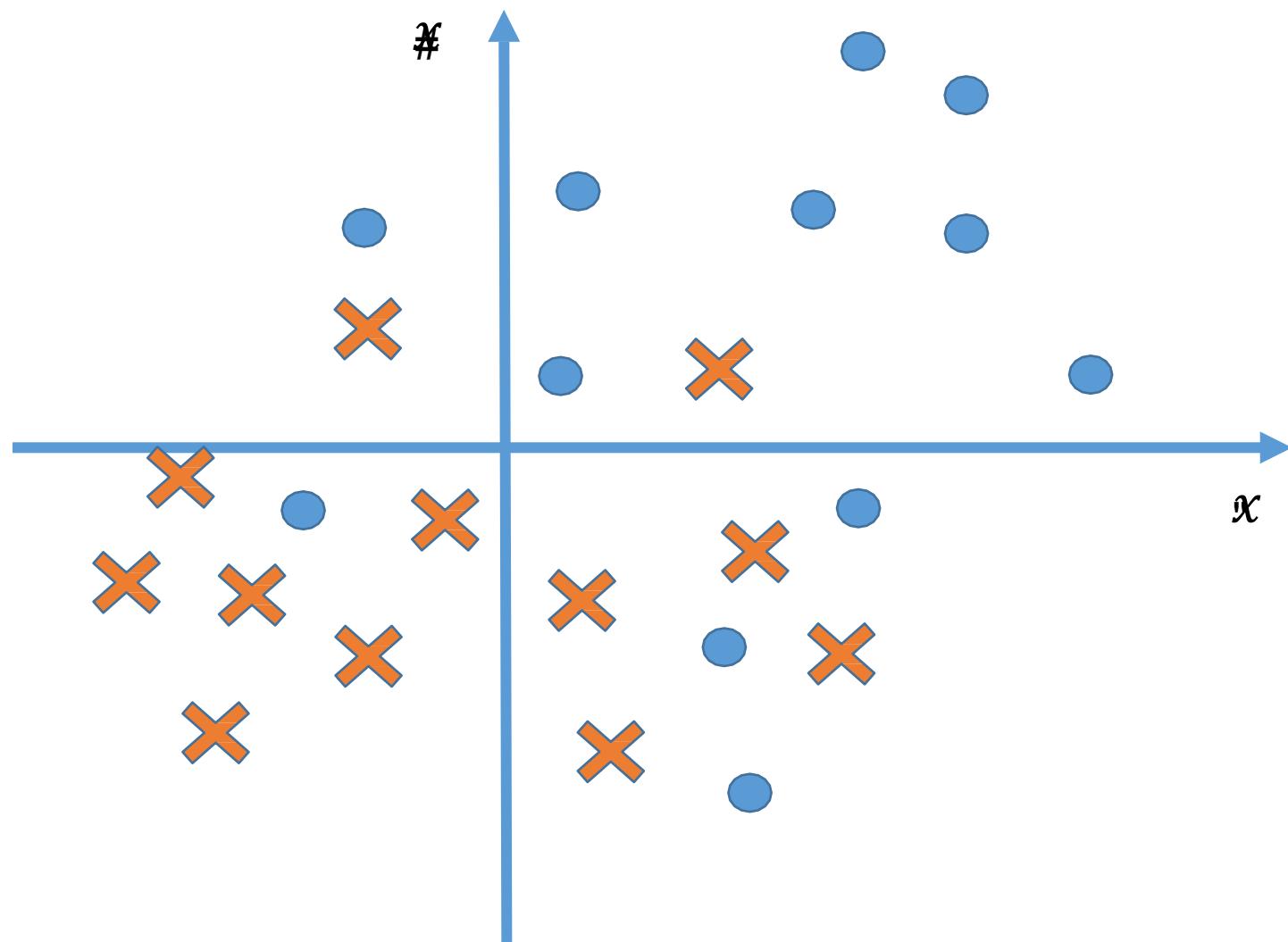
- Learning a linear separator when data is linearly separable.



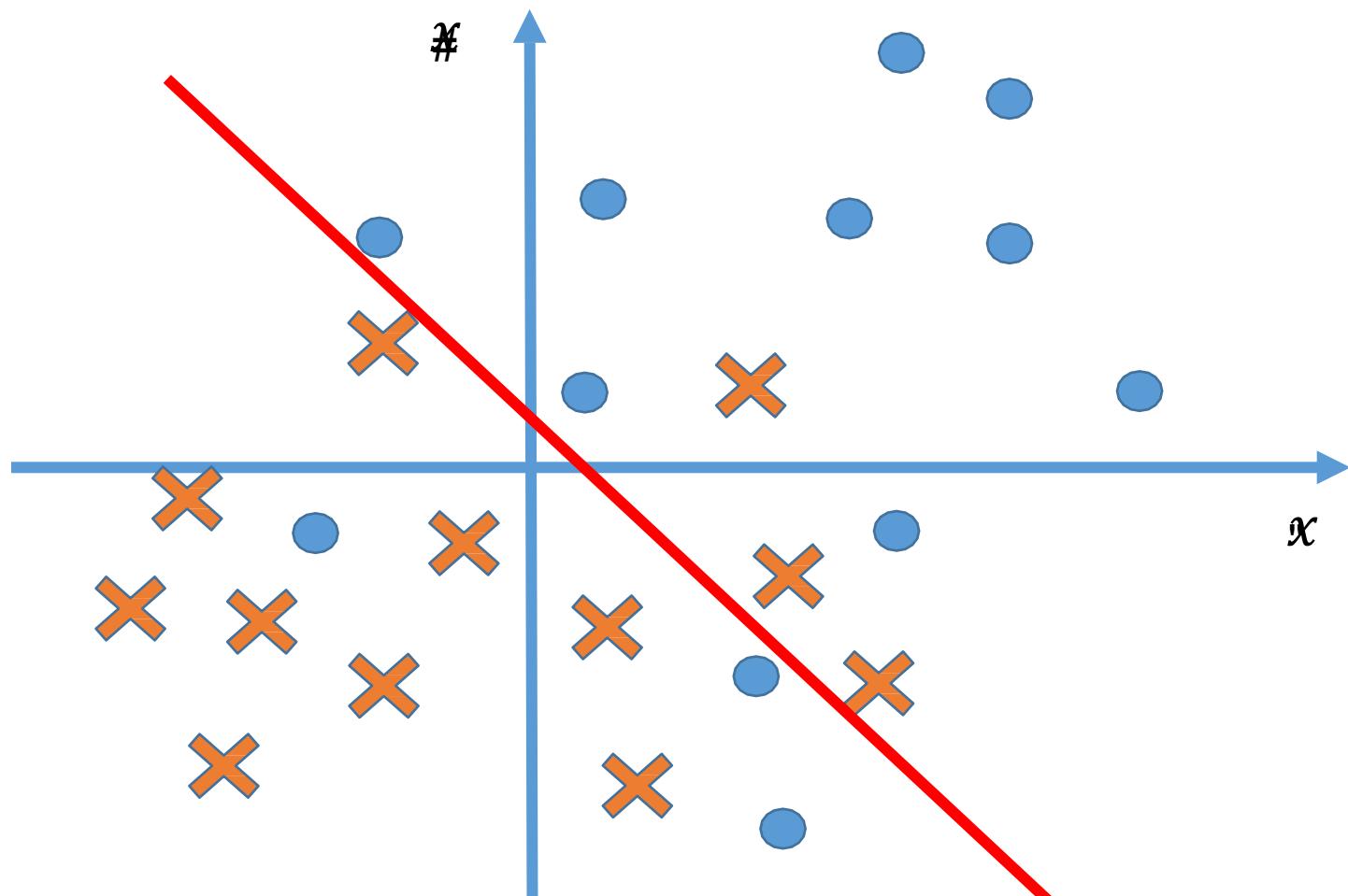
Hypothesis space: linear model



What if data is not linearly separable?

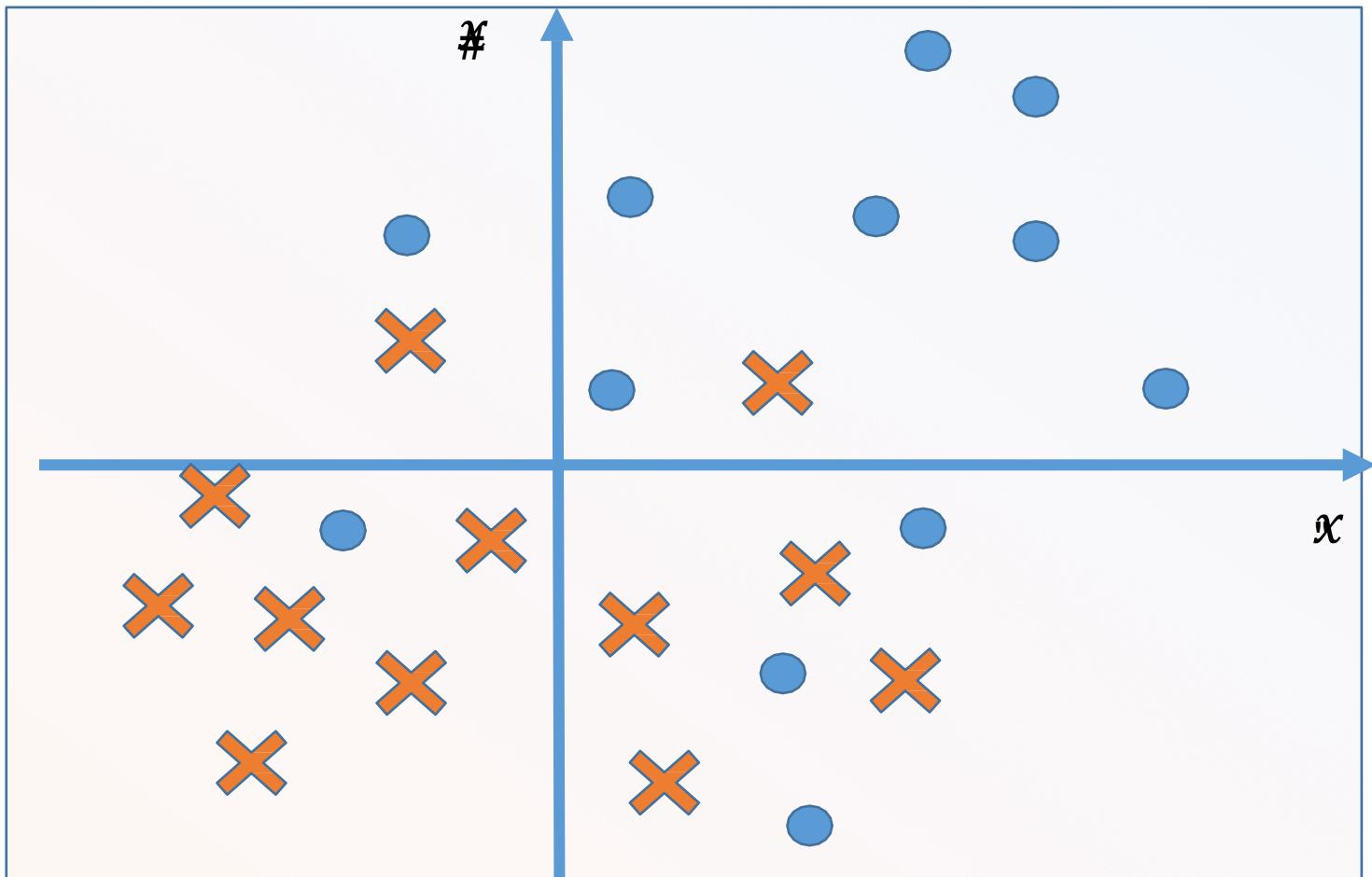


What if data is not linearly separable?



We can allow some mistakes (we will see how to do it later)

What if data is not linearly separable?



We can consider a probabilistic model (today's lecture)

Online Learning

Data instance is provided one at a time

Initialize a model

Loop

 Given one data point (x, y)

 Update the model

Return the final model

Batch Learning v.s. online learning

All the data instances are presented in the training time

Given a training set $\mathcal{D} = \{(x, y)\}$

Training is usually done by minimizing training error
(w/ regularization).

Predict on the test set $\mathcal{D}' = \{(x, y)\}$

Note: We can use online learning
algorithm in the batch setting

Online v.s. Batch

Different learning protocols

The assumption & learning goal may different

E.g., batch learning assumes data are i.i.d (independent and identically distributed) while online learning may provide worst-case bound under adversarial data.

Computation

Space: online learning consider one instance at a time

Convergence rate: some fast converged optimization method requires access to the entire dataset

What you will learn today

- ✓ Logistic Regression
 - ✓ How to model – hypothesis space, model definition
 - ✓ How to predict?
 - ✓ How to train?
- ✓ Maximum Likelihood
- ✓ Gradient descent

Logistic Regression: Setup

- ❖ The setting
 - ❖ Binary classification
 - ❖ Inputs: Feature vectors $x \in R^N$
 - ❖ Labels: $y \in \{-1, +1\}$
- ❖ Training data
 - ❖ $S = \{(x_i, y_i)\}$, m examples
- ❖ Goal
 - ❖ Linear model

Goal: Learn a Linear Classifiers

- ✓ *Linear Threshold Units* (LTUs) classify an example \mathbf{x} using the following classification rule
 - ✓ Output = $\text{sgn}(\mathbf{w}^T \mathbf{x} + b) = \text{sgn}(b + \sum w_i x_i)$
 - ✓ $\mathbf{w}^T \mathbf{x} + b \geq 0$ Predict $y = 1$ Tie is broken arbitrary
 - ✓ $\mathbf{w}^T \mathbf{x} + b < 0$ Predict $y = -1$

b is called the bias term

Recap: A trick to remove the bias term b

$$\begin{aligned} & w^T x + b \\ &= [w^T \ b] \cdot \begin{bmatrix} x \\ 1 \end{bmatrix} \\ &= \tilde{w} \cdot \tilde{x} \end{aligned}$$

$$\begin{aligned} \tilde{w} &= [w^T \ b]^T \\ \tilde{x} &= [x^T \ 1]^T \end{aligned}$$

For simplicity, I may write \tilde{w} and \tilde{x} as w and x when there is no confusion

What is our hypothesis space?

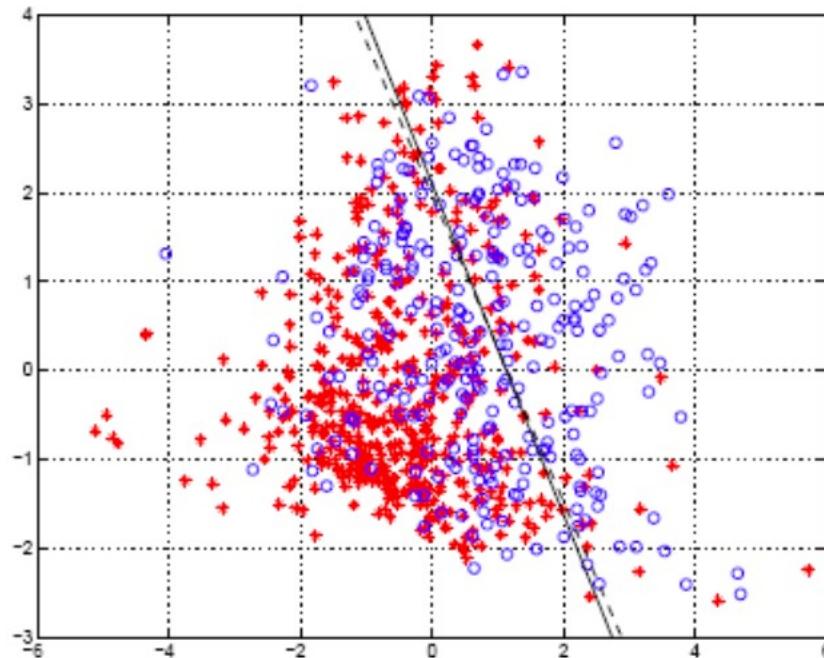
- Formally, we are consider all functions in the following space:

$$H = \{ h \mid h : X \rightarrow Y, h(x) = \text{sgn}(w^T x + b) \}$$

- Learning goal: $y = h(x)$
 - Learn w, b (w and x are n-dimensional vectors)
 - w : weight vector, b : bias

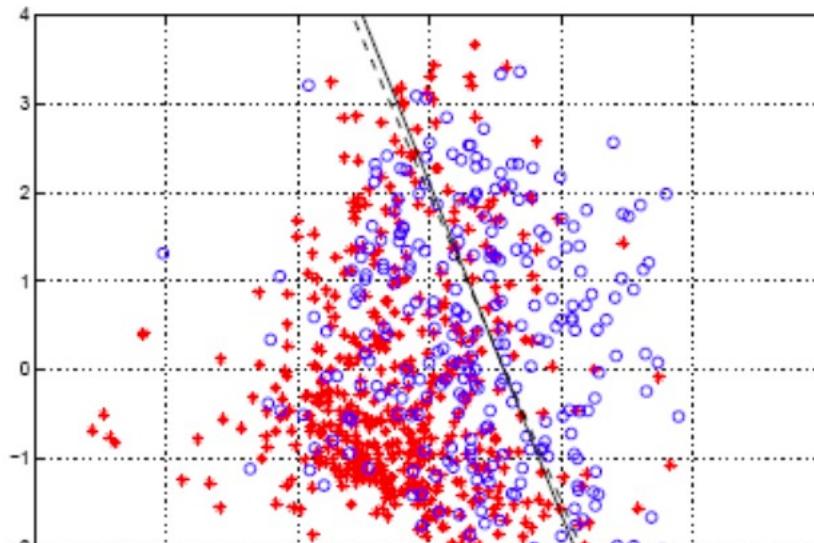
Classification, but...

- ✓ The output y is discrete valued (-1 or 1)
- ✓ Instead of predicting the output, let us try to predict $P(y = 1 | \mathbf{x})$



Classification, but...

- ✓ The output y is discrete valued (-1 or 1)
- ✓ Instead of predicting the output, let us try to predict $P(y = 1 | x)$



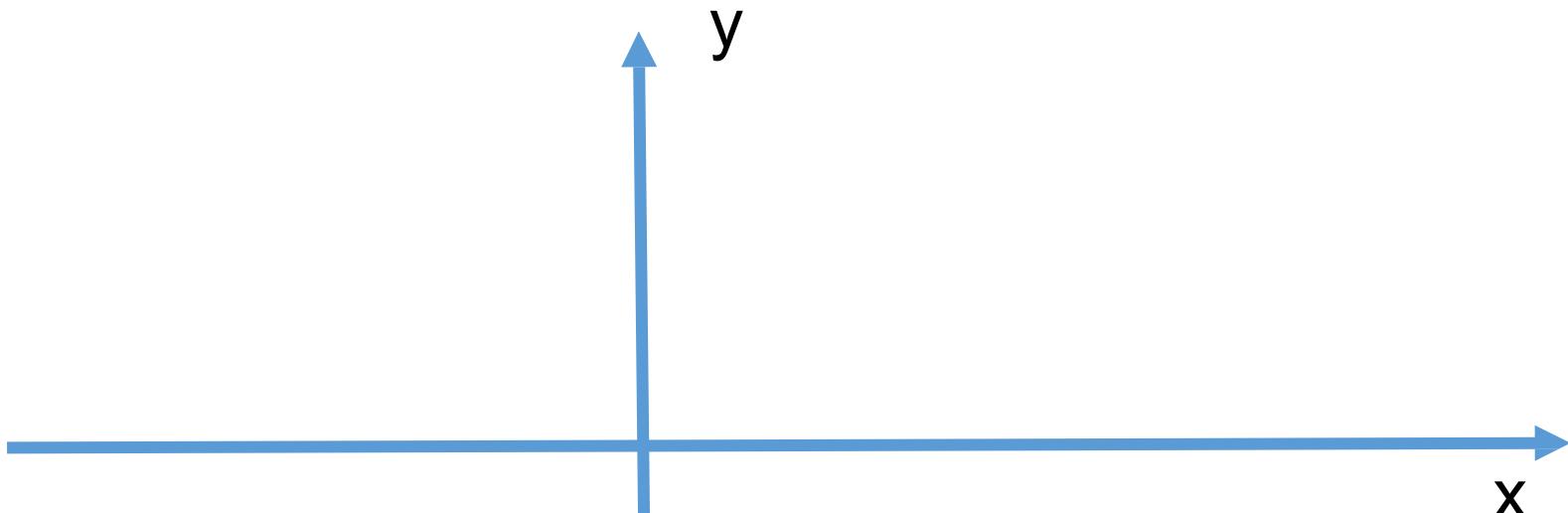
Perceptron does not produce probability estimates

Predict $P(y = 1 | \mathbf{x})$

- ✓ Expand hypothesis space to functions whose output is [0-1]
- ✓ Original problem: $h(x) \in \{-1, 1\}$
- ✓ Modified problem: $h(x) \in [0-1]$
- ✓ Effectively make the problem a regression problem

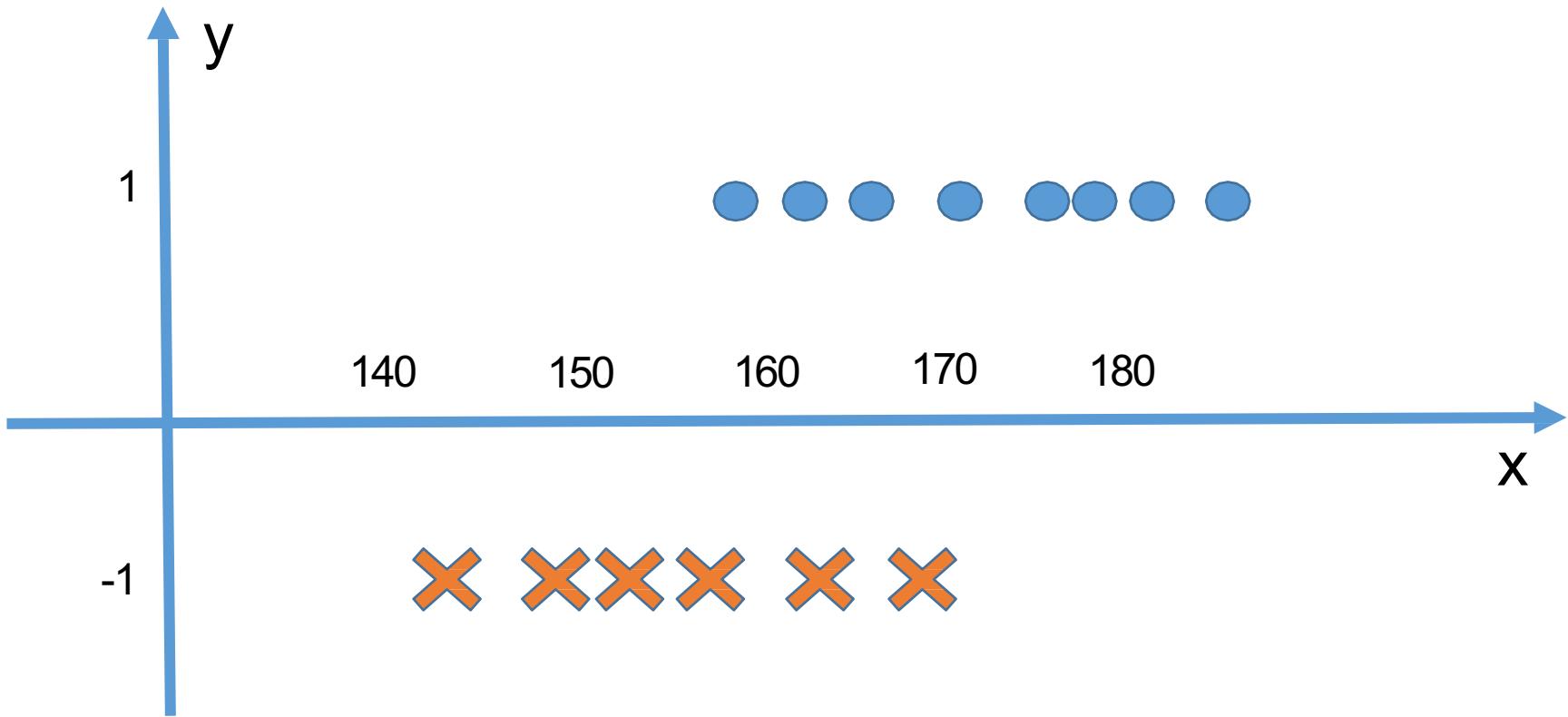
How to make the prediction within [0,1]

- ✓ What does data look like
- ✓ Assume we only have one feature (e.g., predicting gender by height)



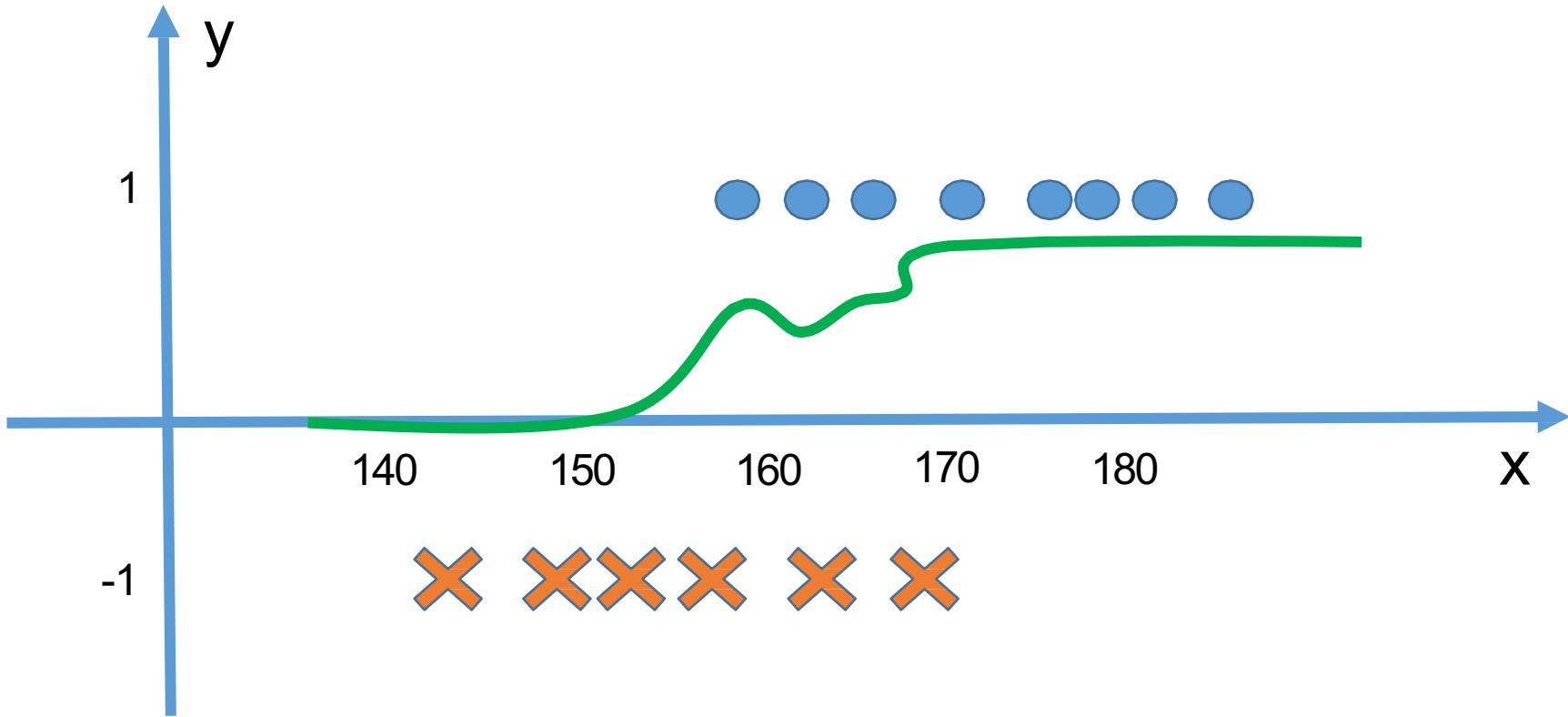
How to make the prediction within $[0,1]$

- ✓ What does data look like
- ✓ Assume we only have one feature



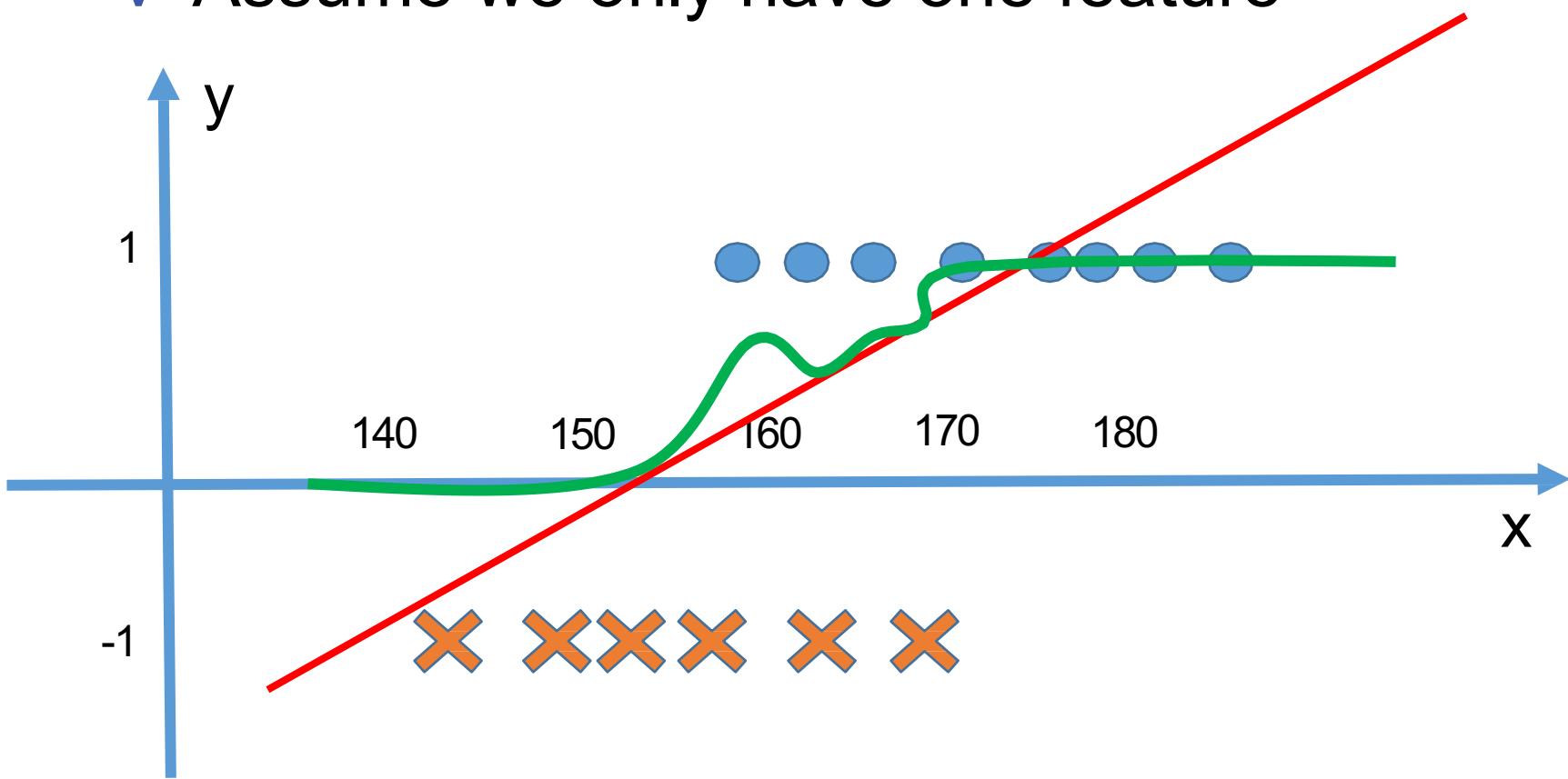
How to make the prediction within $[0,1]$

- ✓ What does $P(y = 1|x)$ look like
- ✓ Assume we only have one feature



How to make the prediction within $[0,1]$

- ✓ What does $y = \mathbf{w}^T \mathbf{x} + b$ look like
- ✓ Assume we only have one feature



How to make the prediction within [0,1]

- What does $y = \mathbf{w}^T \mathbf{x} + b$ look like
- Assume we only have one feature

Probability cannot be larger than 1

1

140

150

160

170

180

x

-1

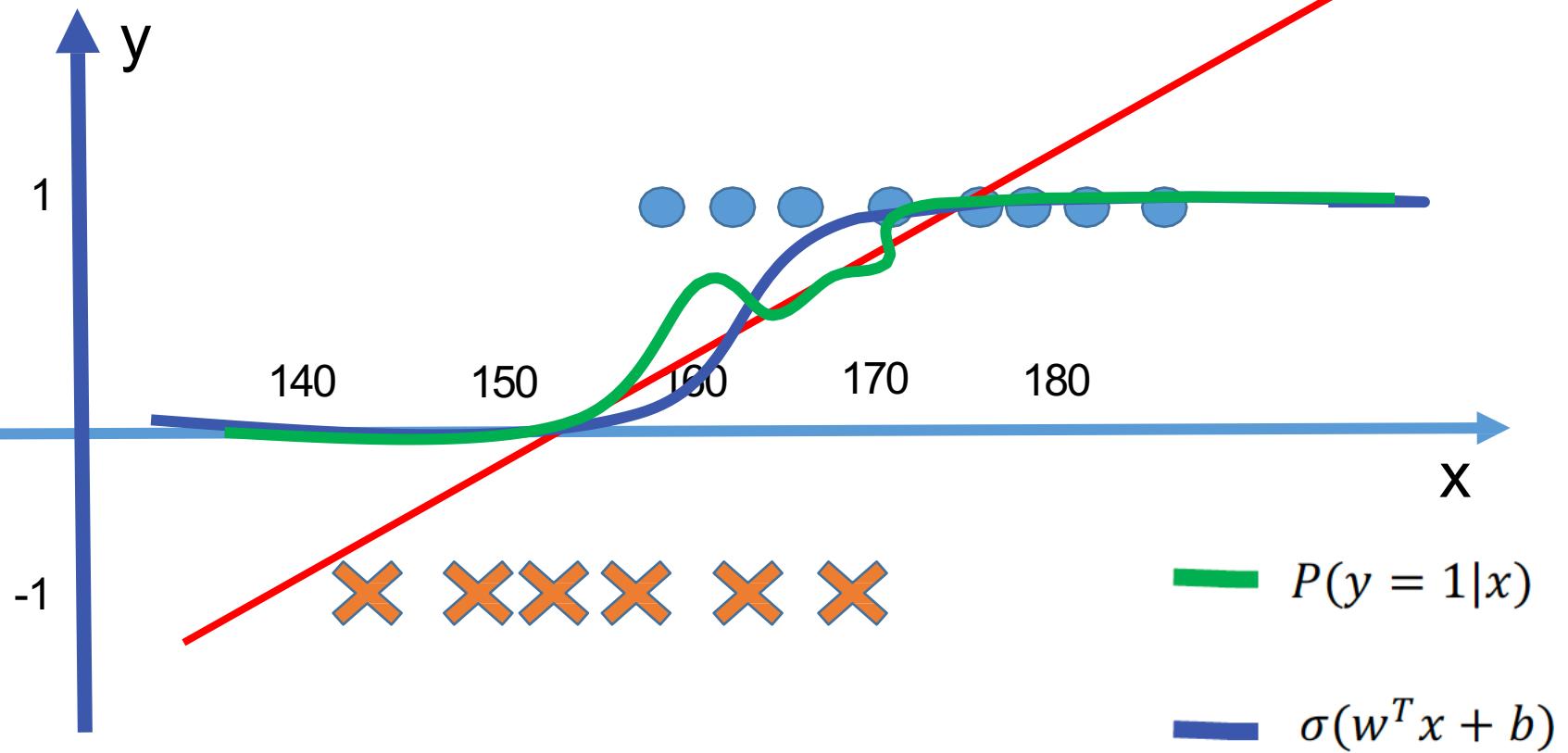
Probability cannot be negative

Define a linear model to fit the curve

- Define a transformation function such that

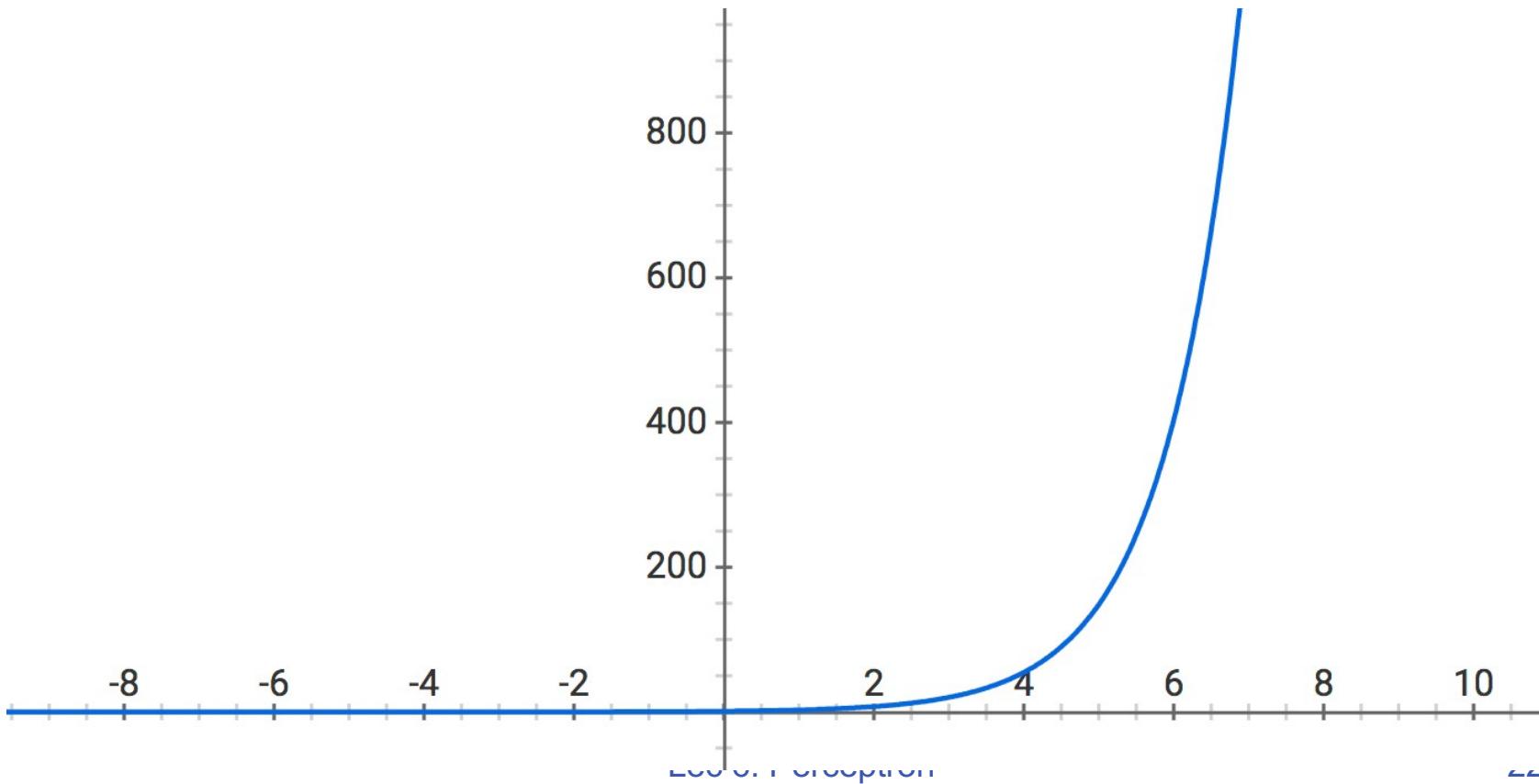
$$\sigma(w^T x + b) = P(y = 1|x)$$

$w^T x + b$



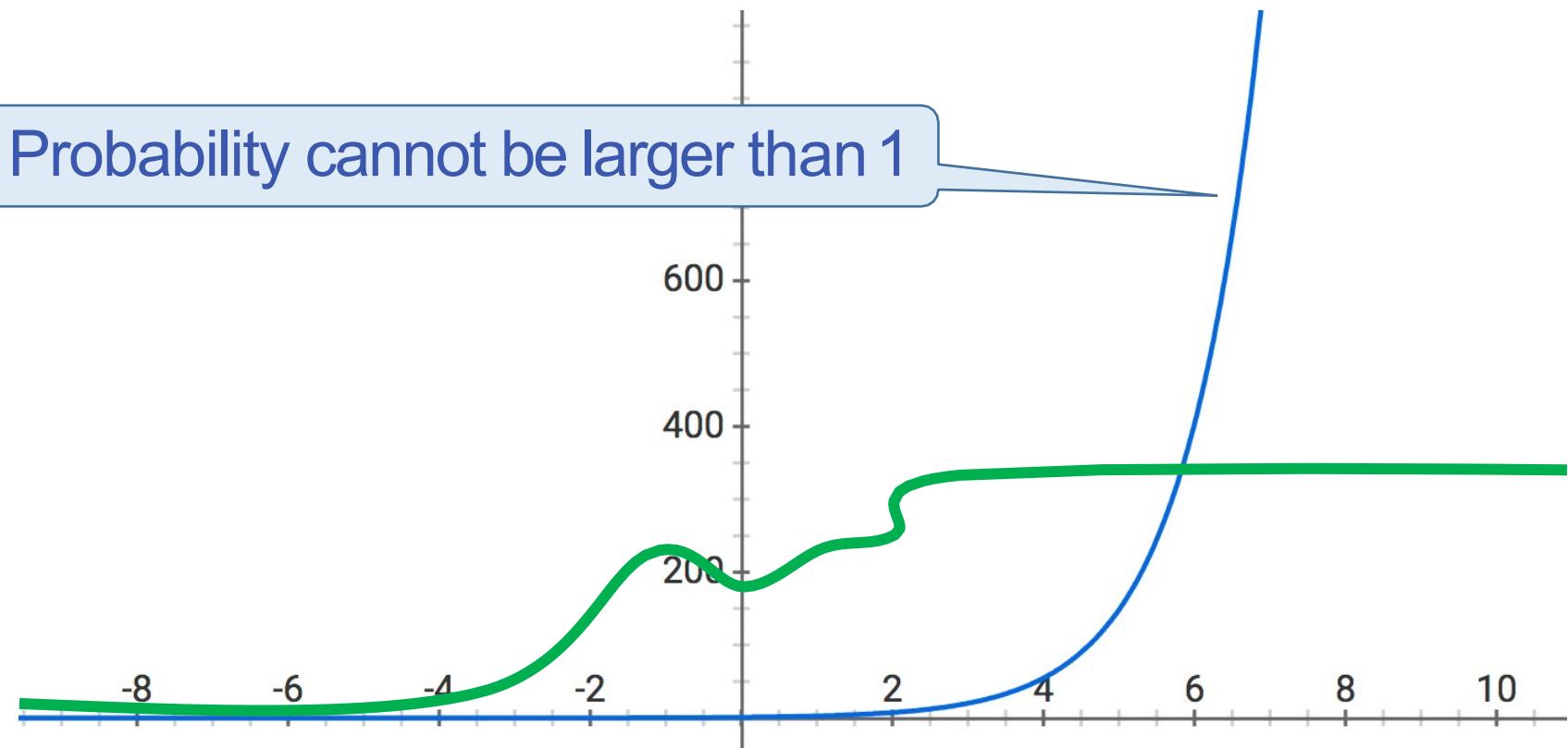
How can we design such a transformation function?

- ✓ Idea 1: function always output positive value
 - ✓ $\exp(\cdot)$ is always positive



How can we design such a transformation function?

- ❖ Idea 1: function always output positive value
 - ❖ $\exp(\cdot)$ is always positive
 - ❖ $\exp(w^T x + b)$ always return positive value



How can we design such a transformation function?

- ↙ Idea 2: normalize the value such that it is less than 1
 - ❖ $\exp(w^T x + b) \in (0, \infty)$ grows very fast, so we need to use \exp to normalize itself
 - ❖ Let's use
- $$\sigma(w^T x + b) = \frac{\exp(w^T x + b)}{1 + \exp(w^T x + b)}$$
- ❖ When $w^T x + b \rightarrow \infty$,
 $\sigma(w^T x + b) \rightarrow 1$
 - ❖ When $w^T x + b \rightarrow -\infty$,
 $\sigma(w^T x + b) \rightarrow 0$

The Sigmoid function

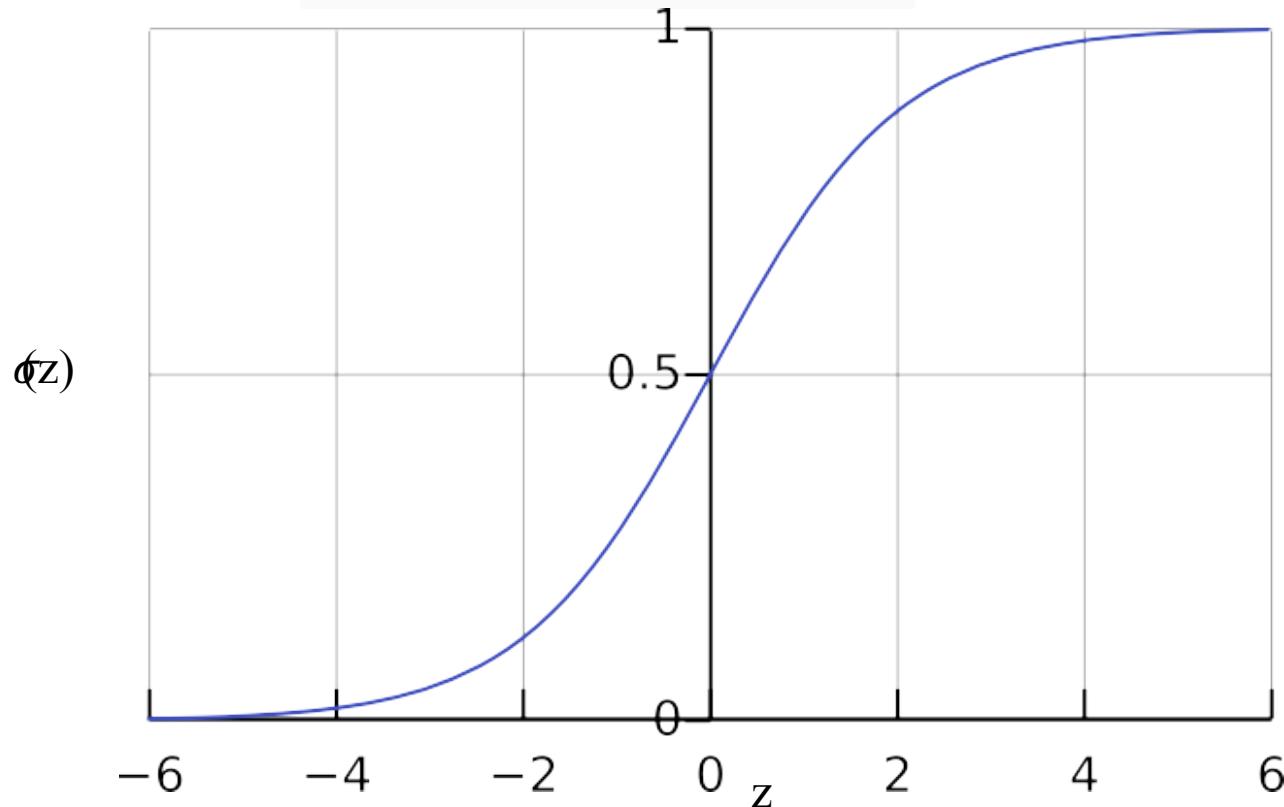
The $\sigma(z)$ function is called sigmoid function
(or logistic function)

$$\begin{aligned}\sigma(z) &= \frac{\exp(z)}{1 + \exp(z)} \\ &= \frac{1}{1 + \exp(-z)}\end{aligned}$$

What is the domain and the range of the sigmoid function?

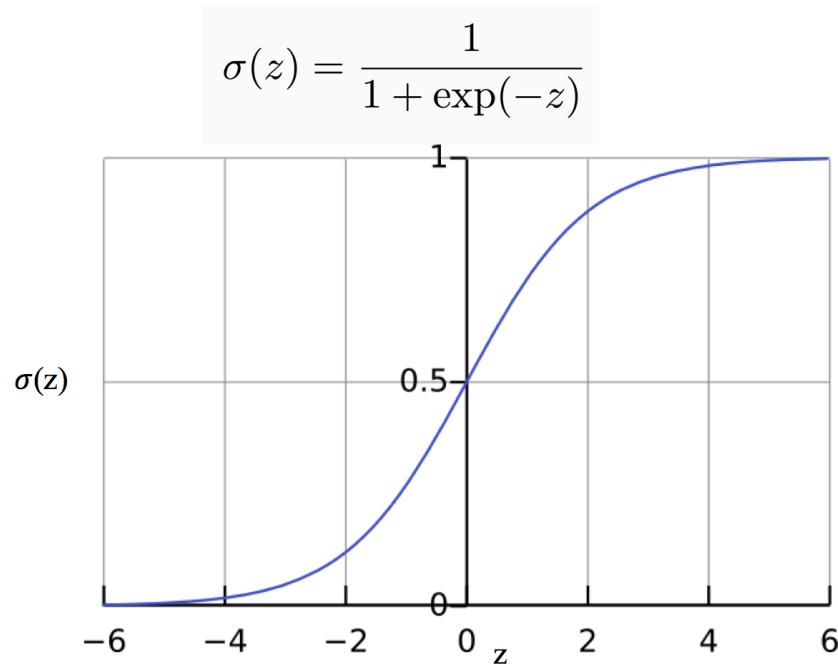
The Sigmoid function

$$\sigma(z) = \frac{1}{1 + \exp(-z)}$$



logistic function or sigmoid function

- ❖ When $z \rightarrow \infty$ what is $\sigma(z)$?
- ❖ When $z \rightarrow -\infty$ what is $\sigma(z)$?
- ❖ When $z = 0$ what is $\sigma(z)$?



The properties of Sigmoid function

$$\sigma(z) = \frac{1}{1 + \exp(-z)}$$

Exercise

What is its derivative with respect to z?

$$\frac{d\sigma}{dz} = \frac{d}{dz} \frac{1}{1 + \exp(-z)}$$

The Sigmoid function

$$\sigma(z) = \frac{1}{1 + \exp(-z)}$$

What is its derivative with respect to z ?

$$\begin{aligned}\frac{d\sigma}{dz} &= \frac{d}{dz} \frac{1}{1 + \exp(-z)} \\ &= \frac{1}{(1 + \exp(-z))^2} \cdot \exp(-z) \\ &= \left(1 - \frac{1}{1 + \exp(-z)}\right) \cdot \frac{1}{1 + \exp(-z)} \\ &= \sigma(z)(1 - \sigma(z)).\end{aligned}$$

Nice computational properties

The hypothesis space for logistic regression

All functions of the form

Let's ignore b from now by
using the trick in page 11

$$h_{\mathbf{w}}(\mathbf{x}) = \sigma(\mathbf{w}^T \mathbf{x}) = \frac{1}{1 + \exp(-\mathbf{w}^T \mathbf{x})}$$

That is, a linear function, composed with a
sigmoid function (the **logistic function**)

$$\sigma(z) = \frac{1}{1 + \exp(-z)}$$

We want to find $h_Q(x)$ such that

$$h_w(x) \approx P(y = 1 | x)$$

Check point (Modeling)

- ❖ What is the goal of logistic regression?
 - ❖ Model $P(y = 1|x)$
- ❖ What is the hypothesis space?

$$H = \{ h \mid h : X \rightarrow P(Y \mid X), h(x) = \sigma(w^T x + b) \}$$

$$\sigma(z) = \frac{1}{1 + \exp(-z)}$$

Understanding a ML model

- ✓ What are the input / output
 - ✓ The input/output of data may be different from your model
 - ✓ E.g., feature selection may remove some features from data
- ✓ What is the hypothesis space?
- ✓ What is the loss function
(criteria of model fitting)
- ✓ How to make prediction
- ✓ How to learn the model
- ✓ Properties / Guarantees of the model
- ✓ Pros/Cons/Connections to other models

Understanding a ML model

- ✓ What are the input / output
 - ✓ The input/output of data may be different from your model
 - ✓ E.g., feature selection may remove some features from data
- ✓ What is the hypothesis space?
- ✓ What is the loss function
(criteria of model fitting)
- ✓ How to make prediction
- ✓ How to learn the model
- ✓ Properties / Guarantees of the model
- ✓ Pros/Cons/Connections to other models

modeling

Understanding a ML model

- ✓ What are the input / output
 - ✓ The input/output of data may be different from your model
 - ✓ E.g., feature selection may remove some features from data
 - ✓ What is the hypothesis space?
 - ✓ What is the loss function
(criteria of model fitting)
 - ✓ How to make prediction
 - ✓ How to learn the model
- algorithm
- ✓ Properties / Guarantees of the model
 - ✓ Pros/Cons/Connections to other models

Understanding a ML model

- ✓ What are the input / output
 - ✓ The input/output of data may be different from your model
 - ✓ E.g., feature selection may remove some features from data
- ✓ What is the hypothesis space?
- ✓ What is the loss function
 - (criteria of model fitting)
- ✓ How to make prediction
- ✓ How to learn the model
- ✓ Properties / Guarantees of the model
- ✓ Pros/Cons/Connections to other models **analysis**

Logistic regression

- ✓ What are the input / output
- ✓ What is the hypothesis space?
- ✓ What is the loss function
(criteria of model fitting)
- ✓ How to make prediction
- ✓ How to learn the model
- ✓ Properties / Guarantees of the model
- ✓ Pros/Cons/Connections to other models

Logistic regression

- ✓ What are the input / output
- ✓ What is the hypothesis space?
- ✓ What is the loss function
(criteria of model fitting)
- ✓ How to make prediction
- ✓ How to learn the model
- ✓ Properties / Guarantees of the model
- ✓ Pros/Cons/Connections to other models

Predicting probabilities

According to the logistic regression model,
we have

$$P(y = 1 | \mathbf{x}; \mathbf{w}) = \sigma(\mathbf{w}^T \mathbf{x}) = \frac{1}{1 + \exp(-\mathbf{w}^T \mathbf{x})}$$

$$P(y = -1 | \mathbf{x}; \mathbf{w}) = 1 - \sigma(\mathbf{w}^T \mathbf{x}) = \frac{\exp(-\mathbf{w}^T \mathbf{x})}{1 + \exp(-\mathbf{w}^T \mathbf{x})}$$

Predicting probabilities

According to the logistic regression model,
we have

$$P(y = 1 | \mathbf{x}; \mathbf{w}) = \sigma(\mathbf{w}^T \mathbf{x}) = \frac{1}{1 + \exp(-\mathbf{w}^T \mathbf{x})}$$

$$P(y = -1 | \mathbf{x}; \mathbf{w}) = 1 - \sigma(\mathbf{w}^T \mathbf{x}) = \frac{\exp(-\mathbf{w}^T \mathbf{x})}{1 + \exp(-\mathbf{w}^T \mathbf{x})}$$

$$\frac{1}{1 + \exp(\mathbf{w}^T \mathbf{x})}$$



Predicting probabilities

According to the logistic regression model,
we have

$$P(y = 1 | \mathbf{x}; \mathbf{w}) = \sigma(\mathbf{w}^T \mathbf{x}) = \frac{1}{1 + \exp(-\mathbf{w}^T \mathbf{x})}$$

$$P(y = -1 | \mathbf{x}; \mathbf{w}) = 1 - \sigma(\mathbf{w}^T \mathbf{x}) = \frac{1}{1 + \exp(\mathbf{w}^T \mathbf{x})}$$

Predicting probabilities

According to the logistic regression model,
we have

$$P(y = 1 | \mathbf{x}; \mathbf{w}) = \sigma(\mathbf{w}^T \mathbf{x}) = \frac{1}{1 + \exp(-\mathbf{w}^T \mathbf{x})}$$

$$P(y = -1 | \mathbf{x}; \mathbf{w}) = 1 - \sigma(\mathbf{w}^T \mathbf{x}) = \frac{1}{1 + \exp(\mathbf{w}^T \mathbf{x})}$$

Or equivalently

$$P(y | \mathbf{x}; \mathbf{w}) = \frac{1}{1 + \exp(-y \mathbf{w}^T \mathbf{x})}$$

Predicting a label

$$P(y = 1 | \mathbf{x}; \mathbf{w}) = \sigma(\mathbf{w}^T \mathbf{x}) = \frac{1}{1 + \exp(-\mathbf{w}^T \mathbf{x})}$$

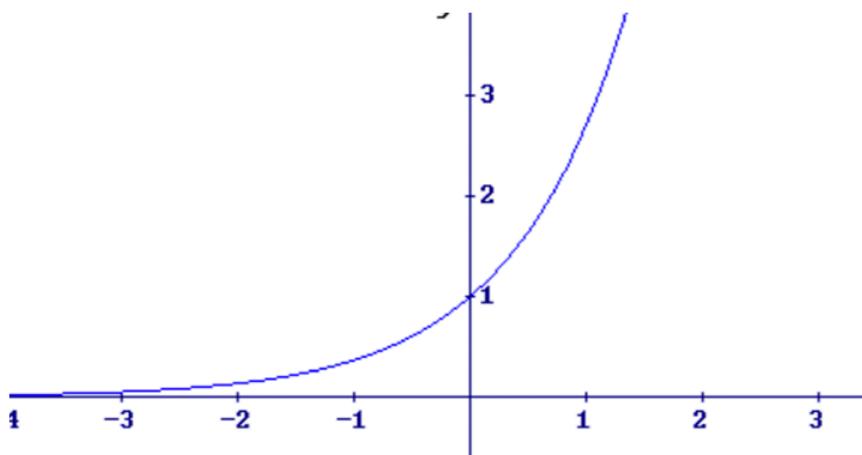
- ✓ Compute $P(y = 1 | \mathbf{x}; \mathbf{w})$
- ✓ If this is greater than half, predict 1 else predict -1
 - ✓ What does this correspond to in terms of $\mathbf{w}^T \mathbf{x}$?

Decision Boundary

- ❖ Predict $y=1$ if $P(y=1|x, w) \geq P(y=-1|x, w)$

- ❖ When does this happen?

- ❖ $\frac{1}{1+\exp(-w^T x)} \geq 0.5$
 $\Rightarrow 1 + \exp(-w^T x) \leq 2$
 $\Rightarrow \exp(-w^T x) \leq 1$
 $\Rightarrow w^T x \geq 0$



DecisionType equation here. boundary

$$P(y = 1 | \mathbf{x}, \mathbf{w}) = \frac{e^{\mathbf{w}^T \mathbf{x}}}{1 + e^{\mathbf{w}^T \mathbf{x}}} = \frac{1}{1 + e^{-\mathbf{w}^T \mathbf{x}}}$$

$$P(y = -1 | \mathbf{x}, \mathbf{w}) = \frac{1}{1 + e^{\mathbf{w}^T \mathbf{x}}}$$

$$\log \frac{P(Y = 1 | x)}{P(Y = -1 | x)} = \mathbf{w}^T \mathbf{x}$$

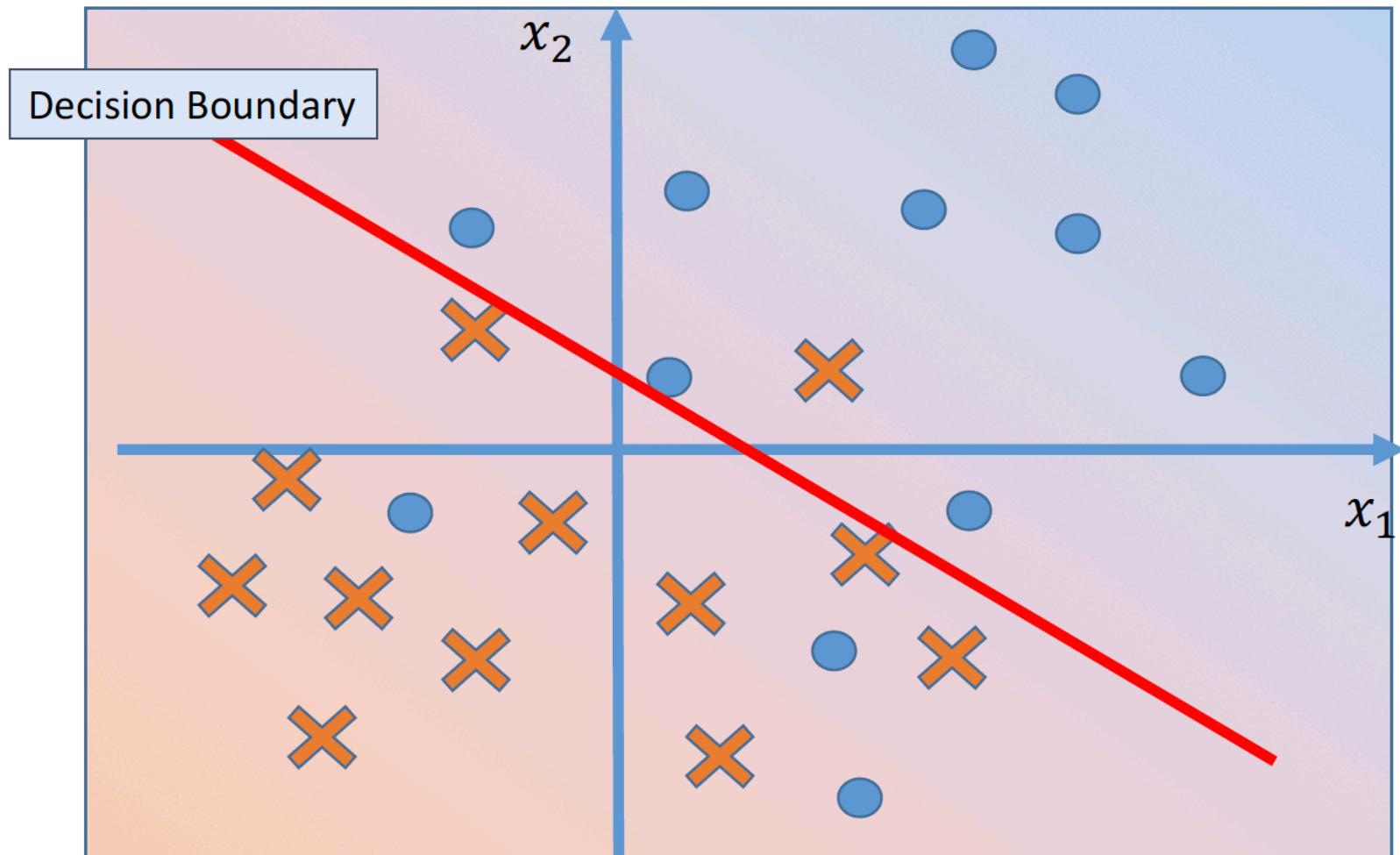
$$\log(z) \geq 0 \quad \text{iff } z \geq 1$$

❖ The decision boundary?

$$\mathbf{w}^T \mathbf{x} = 0$$

❖ $\sigma(\mathbf{w}^T \mathbf{x})$ is non-linear but the decision boundary is linear!

Prediction by logistic regression



A probabilistic model of modeling $\sigma(w^T x + b) = P(y = 1|x)$

How to train a logistic regression model?

Logistic Regression: Setup

❖ The setting

- ❖ Binary classification
- ❖ Inputs: Feature vectors $x \in R^N$
- ❖ Labels: $y \in \{-1, +1\}$

❖ Training data

- ❖ $S = \{(\mathbf{x}_i, y_i)\}$, m examples

❖ Hypothesis space

$$H = \{ h \mid h : X \rightarrow P(Y \mid X), h(x) = \sigma(w^T x + b) \}$$

$$\sigma(z) = \frac{1}{1 + \exp(-z)}$$

Logistic Regression: Setup

- ❖ Training data

- ❖ $S = \{(x_i, y_i)\}$, m examples

- ❖ Hypothesis space

$$H = \{ h \mid h : X \rightarrow Y, h(x) = \sigma(w^T x + b) \}$$

$$\sigma(z) = \frac{1}{1 + \exp(-z)}$$

- ❖ Learning Goal:

- ❖ Find an $h \in H$, such that $h(x) \approx P(y = 1 \mid x)$

- ❖ How to?

Maximum Likelihood

Which bag of words more likely generate:

aDaaa



Maximum Likelihood

Which bag of words more likely generate:

aDaaa

$$0.7 \times 0.1 \times 0.7 \times 0.7 \times 0.7 \\ = 2.401 \times 10^{-2}$$



$$0.2 \times 0.1 \times 0.2 \times 0.2 \times 0.2 \\ = 1.6 \times 10^{-4}$$



Maximum Likelihood

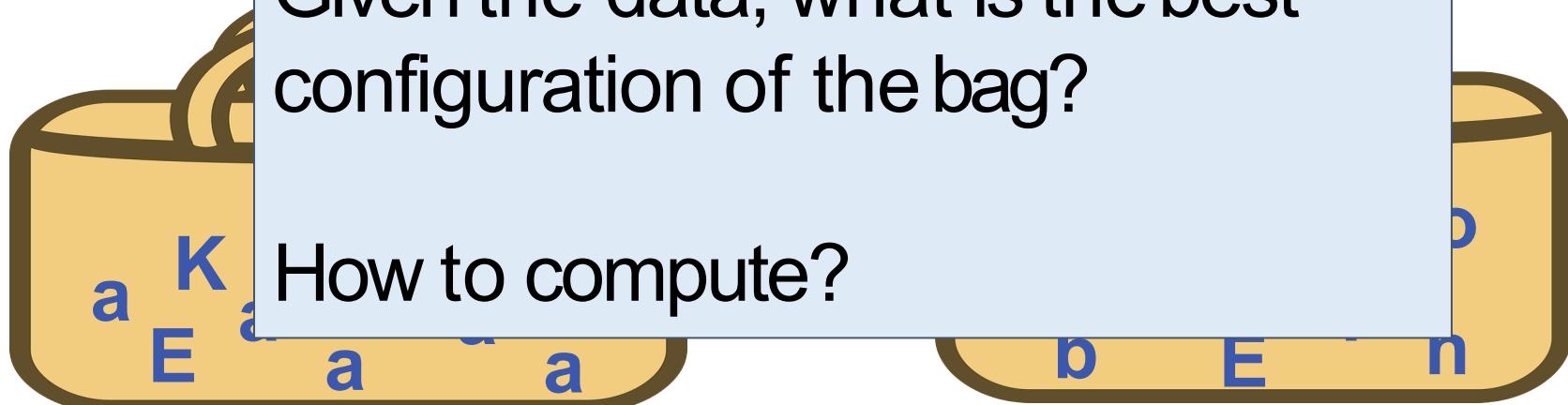
Which bag of words more likely generate:

aDaaa

$$0.7 \times 0.1 \times 0.7 \times 0.7 \times 0.7 = 2.401 \times 0.2 \times 0.1 \times 0.2 \times 0.2$$

Given the data, what is the best configuration of the bag?

How to compute?



Drawing color cards from the envelope

- ❖ Let say we have several cards in the envelope
- ❖ Assume



$$P(\text{card} = \text{yellow}) = \theta$$

$$P(\text{card} = \text{purple}) = 1 - \theta$$

Drawing color cards from the envelope

- ❖ Sample with replacement n times
- ❖ k times we get yellow card, and n-k times, we get purple card
- ❖ The joint probability (likelihood)
$$\theta^k(1 - \theta)^{n-k}$$
- ❖ What is the best θ making the joint probability maximal?

Drawing color cards from the envelope

❖ Solving $\max_{\theta} \theta^k (1 - \theta)^{n-k}$

❖ Equivalently, we can solve

$$\max_{\theta} \log(\theta^k (1 - \theta)^{n-k})$$

$$\max_{\theta} k \log \theta + (n - k) \log(1 - \theta)$$

❖ At the optimum,

$$\frac{d(k \log \theta + (n - k) \log(1 - \theta))}{d\theta} = 0$$

The usual trick: Convert products to sums by taking log

Recall that this works only because log is an increasing function and the maximizer will not change

Drawing color cards from the envelope

$$\frac{d(k \log \theta + (n-k) \log(1-\theta))}{d\theta} = 0$$

$$\Rightarrow \frac{k}{\theta} - \frac{n-k}{1-\theta} = 0$$

For this simple problem, we have a closed-form solution.
We are not always lucky like this

$$\Rightarrow \theta(n - k) = (1 - \theta)k$$

$$\Rightarrow \theta = \frac{k}{n}$$



Back to Logistic regression

- ❖ Training data

- ❖ $S = \{(x_i, y_i)\}$, m examples

- ❖ Hypothesis space

$$H = \{ h \mid h : X \rightarrow Y, h(x) = \sigma(w^T x + b) \}$$

$$\sigma(z) = \frac{1}{1 + \exp(-z)}$$

- ❖ Learning Goal:

- ❖ Find an $h \in H$, such that $h(x) \approx P(y = 1 \mid x)$
 - ❖ How to? Maximum Likelihood Estimator

Maximum likelihood estimator for logistic regression

$$\operatorname{argmax}_{\mathbf{w}} P(S|\mathbf{w}) = \operatorname{argmax}_{\mathbf{w}} \prod_{i=1}^m P(y_i|\mathbf{x}_i, \mathbf{w})$$

Maximum likelihood estimator for logistic regression

$$\operatorname{argmax}_{\mathbf{w}} P(S|\mathbf{w}) = \operatorname{argmax}_{\mathbf{w}} \prod_{i=1}^m P(y_i|\mathbf{x}_i, \mathbf{w})$$

Equivalent to solving

$$\max_{\mathbf{w}} \sum_i^m \log P(y_i|\mathbf{x}_i, \mathbf{w})$$

Maximum likelihood estimator for logistic regression

$$\operatorname{argmax}_{\mathbf{w}} P(S|\mathbf{w}) = \operatorname{argmax}_{\mathbf{w}} \prod_{i=1}^m P(y_i|\mathbf{x}_i, \mathbf{w})$$

$$\max_{\mathbf{w}} \sum_i^m \log P(y_i|\mathbf{x}_i, \mathbf{w})$$

By definition, we know that

$$P(y|\mathbf{w}, \mathbf{x}) = \sigma(y_i \mathbf{w}^T \mathbf{x}_i) = \frac{1}{1 + \exp(-y_i \mathbf{w}^T \mathbf{x}_i)}$$

Maximum likelihood estimator for logistic regression

$$\operatorname{argmax}_{\mathbf{w}} P(S|\mathbf{w}) = \operatorname{argmax}_{\mathbf{w}} \prod_{i=1}^m P(y_i|\mathbf{x}_i, \mathbf{w})$$

$$\max_{\mathbf{w}} \sum_i^m \log P(y_i|\mathbf{x}_i, \mathbf{w})$$

Equivalent to solving

$$\max_{\mathbf{w}} \sum_i^m -\log(1 + \exp(-y_i \mathbf{w}^T \mathbf{x}_i))$$

$$P(y|\mathbf{w}, \mathbf{x}) = \frac{1}{1 + \exp(-y \mathbf{w}^T \mathbf{x})}$$

Maximum likelihood estimator for logistic regression

$$\operatorname{argmax}_{\mathbf{w}} P(S|\mathbf{w}) = \operatorname{argmax}_{\mathbf{w}} \prod_{i=1}^m P(y_i|\mathbf{x}_i, \mathbf{w})$$

$$\max_{\mathbf{w}} \sum_i^m \log P(y_i|\mathbf{x}_i, \mathbf{w})$$

Equivalent to solving

$$\max_{\mathbf{w}} \sum_i^m -\log(1 + \exp(-y_i \mathbf{w}^T \mathbf{x}_i))$$

This is somehow ugly



Minimizing Negative Log-likelihood

- ❖ It is convenient to work with its negation termed **negative log likelihood**

$$\max_{\mathbf{w}} \sum_i^m -\log(1 + \exp(-y_i \mathbf{w}^T \mathbf{x}_i))$$

$$\min_{\mathbf{w}} \sum_i^m \log(1 + \exp(-y_i \mathbf{w}^T \mathbf{x}_i))$$



Alternative Formulation

- ❖ Sometimes, we define
 - ❖ Inputs: Feature vectors $x \in R^N$
 - ❖ Labels: $y \in \{0, 1\}$ (or $y \in [0, 1]$)
- ❖ Training data
 - ❖ $S = \{(x_i, y_i)\}$, m examples
- ❖ Hypothesis space

$$H = \{ h \mid h : X \rightarrow Y, h(x) = \sigma(w^T x + b) \}$$

$$\sigma(z) = \frac{1}{1 + \exp(-z)}$$

Alternative representation

$$P(y|\mathbf{w}, \mathbf{x}) = \sigma(\mathbf{w}^T \mathbf{x}) = \frac{1}{1 + \exp(-y - \mathbf{w}^T \mathbf{x}_i)}$$

Maximum Likelihood estimator:

$$\operatorname{argmax}_{\mathbf{w}} P(S|\mathbf{w}) = \operatorname{argmax}_{\mathbf{w}} \prod_{i=1}^m P(y_i|\mathbf{x}_i, \mathbf{w})$$

$$P(y_i|\mathbf{x}_i, \mathbf{w}) = \begin{cases} \sigma(\mathbf{w}^T \mathbf{x}) & \text{if } y_i = 1 \\ 1 - \sigma(\mathbf{w}^T \mathbf{x}) & \text{if } y_i = 0 \end{cases}$$

Alternative representation

$$P(y|\mathbf{w}, \mathbf{x}) = \sigma(\mathbf{w}^T \mathbf{x}) = \frac{1}{1 + \exp(-y \mathbf{w}^T \mathbf{x}_i)}$$

Maximum Likelihood estimator:

$$\underset{\mathbf{w}}{\operatorname{argmax}} P(S|\mathbf{w}) = \underset{\mathbf{w}}{\operatorname{argmax}} \prod_{i=1}^m P(y_i|\mathbf{x}_i, \mathbf{w})$$

Equivalent to solving

$$P(y|\mathbf{x}_i, \mathbf{w}) = \begin{cases} \sigma(\mathbf{w}^T \mathbf{x}) & \text{if } y = 1 \\ 1 - \sigma(\mathbf{w}^T \mathbf{x}) & \text{if } y = 0 \end{cases}$$

$$\underset{\mathbf{w}}{\operatorname{argmax}} \prod_{i=1}^m P(y = 1|\mathbf{x}_i, \mathbf{w})^{y_i} P(y = 0|\mathbf{x}_i, \mathbf{w})^{1-y_i}$$

Alternative representation

$$P(y|\mathbf{w}, \mathbf{x}) = \sigma(\mathbf{w}^T \mathbf{x}) = \frac{1}{1 + \exp(-y \mathbf{w}^T \mathbf{x}_i)}$$

Maximum Log-Likelihood estimator:

$$\operatorname{argmax}_{\mathbf{w}} \prod_{i=1}^m P(y=1|\mathbf{x}_i, \mathbf{w})^{y_i} P(y=0|\mathbf{x}_i, \mathbf{w})^{1-y_i}$$

Equivalent to solving

$$P(y|\mathbf{x}_i, \mathbf{w}) = \begin{cases} \sigma(\mathbf{w}^T \mathbf{x}) & \text{if } y = 1 \\ 1 - \sigma(\mathbf{w}^T \mathbf{x}) & \text{if } y = 0 \end{cases}$$

$$\operatorname{argmax}_{\mathbf{w}} \sum_{i=1}^m y_i \log \sigma(\mathbf{w}^T \mathbf{x}) + (1 - y_i) \log(1 - \sigma(\mathbf{w}^T \mathbf{x}))$$

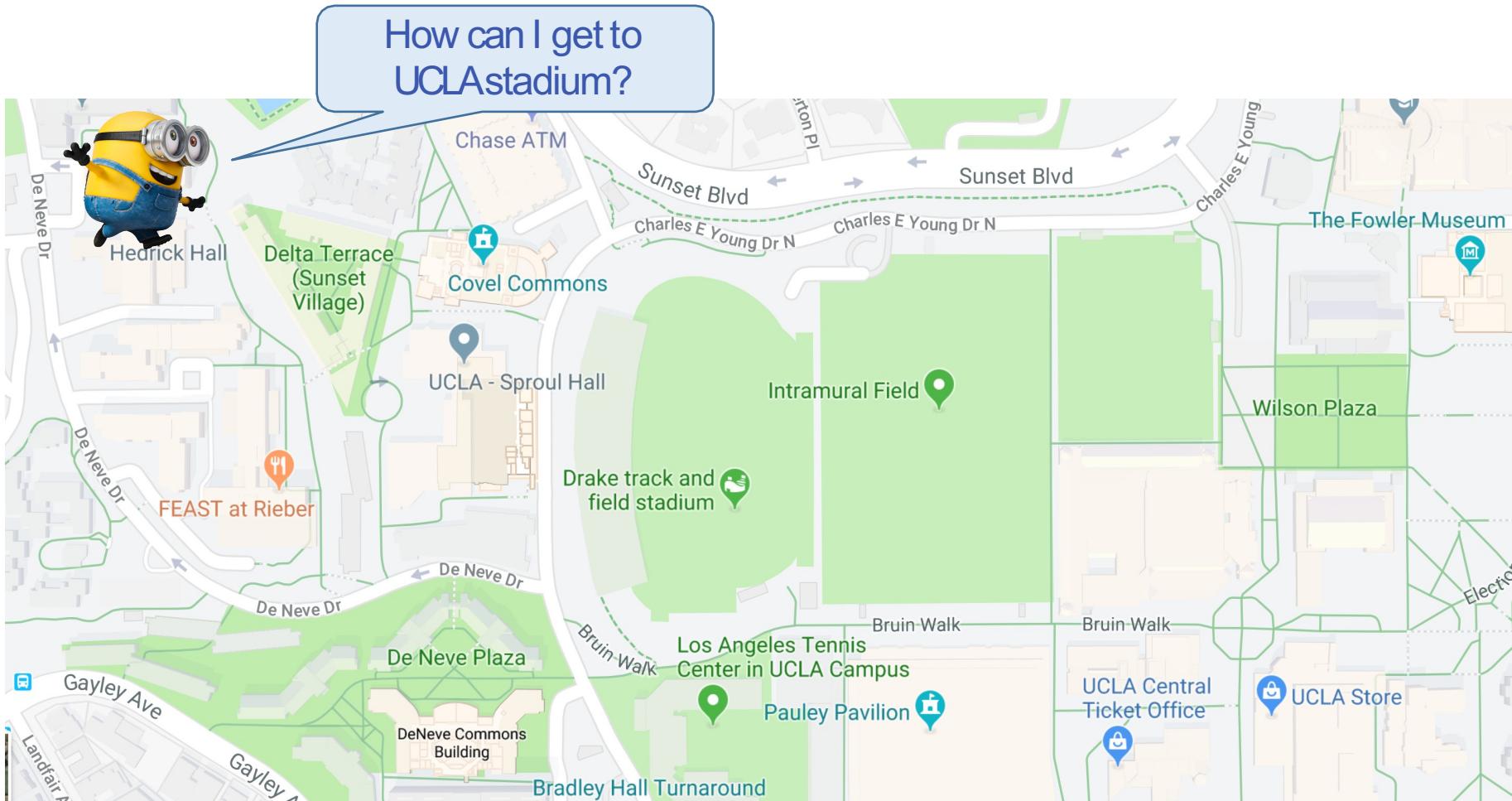
How to solve this?

$$\min_{\mathbf{w}} \sum_i^m \log(1 + \exp(-y_i \mathbf{w}^T \mathbf{x}_i))$$

- ✓ There is no closed-form solution
- ✓ One way to solve it is by gradient descent

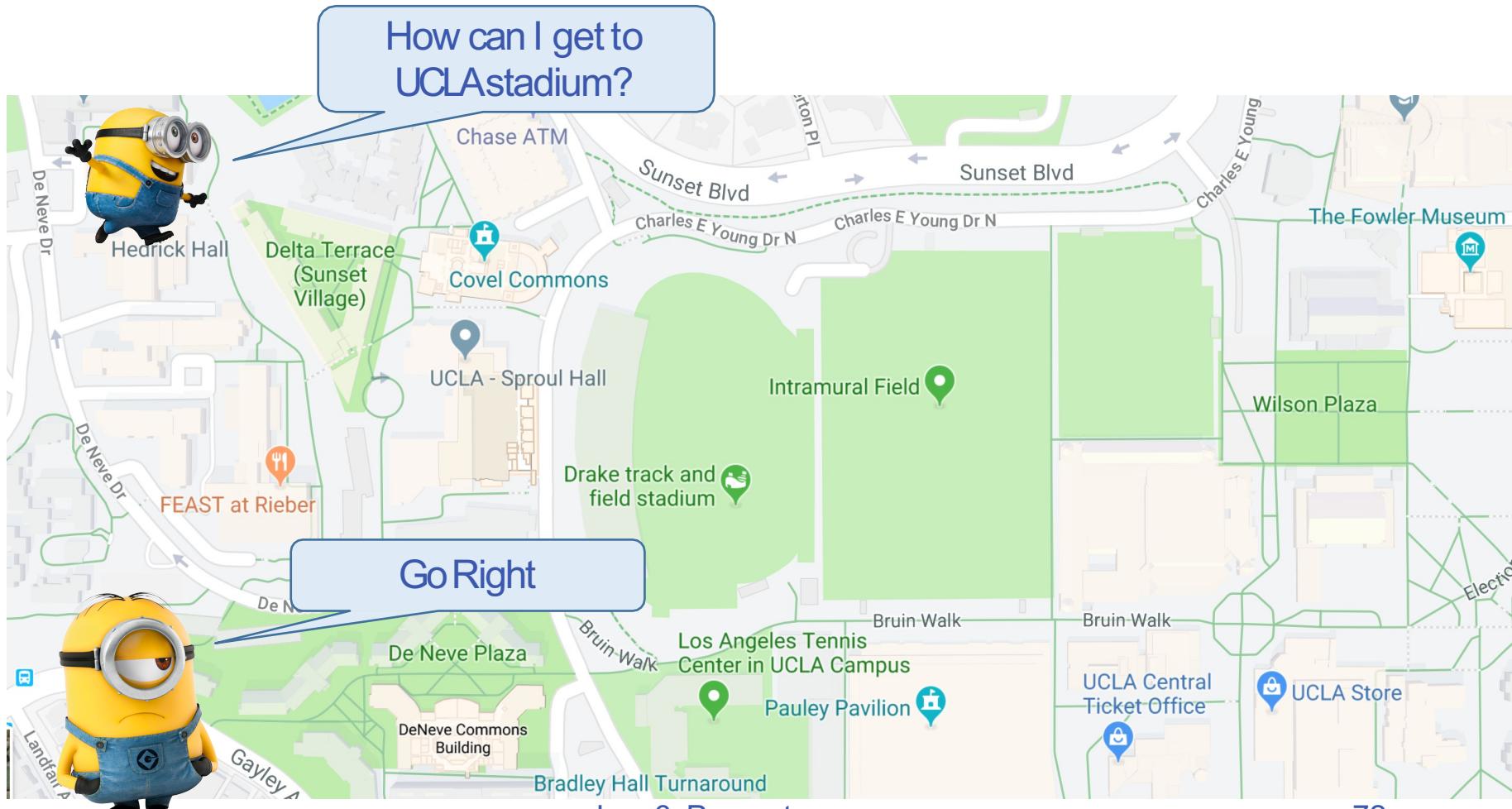
Intuition

✓ Asking direction



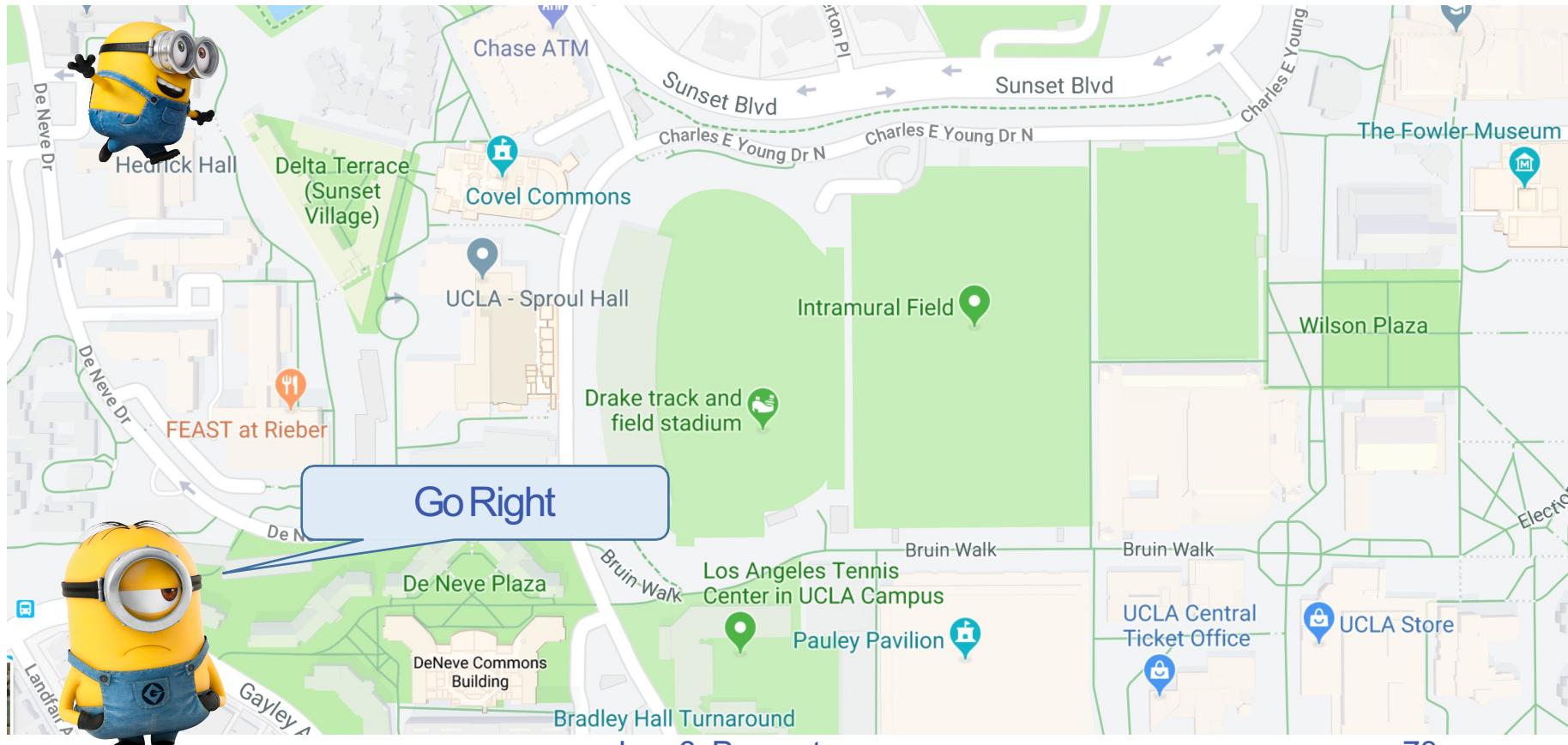
Intuition

✓ Asking direction



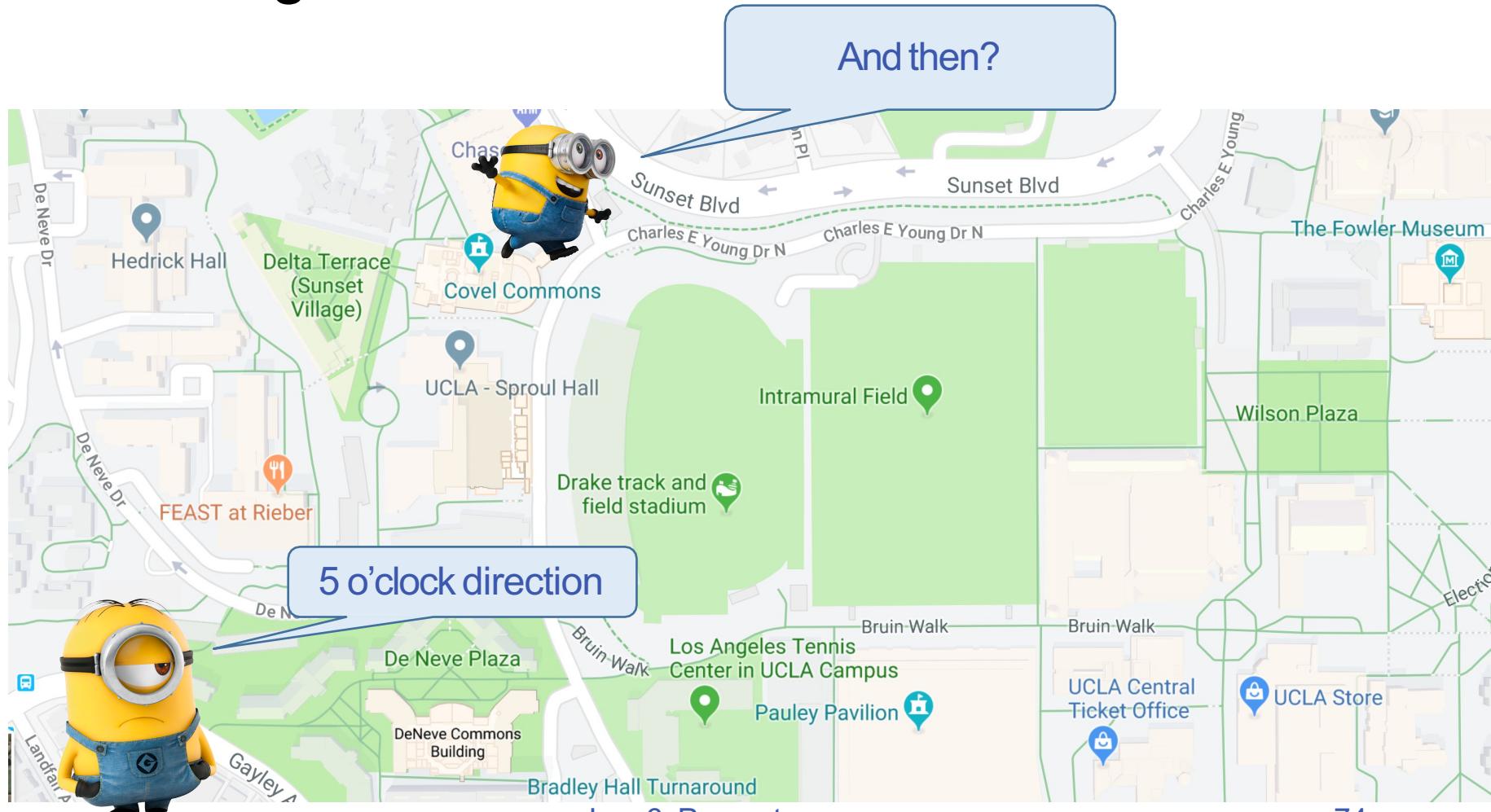
Intuition

✓ Asking direction



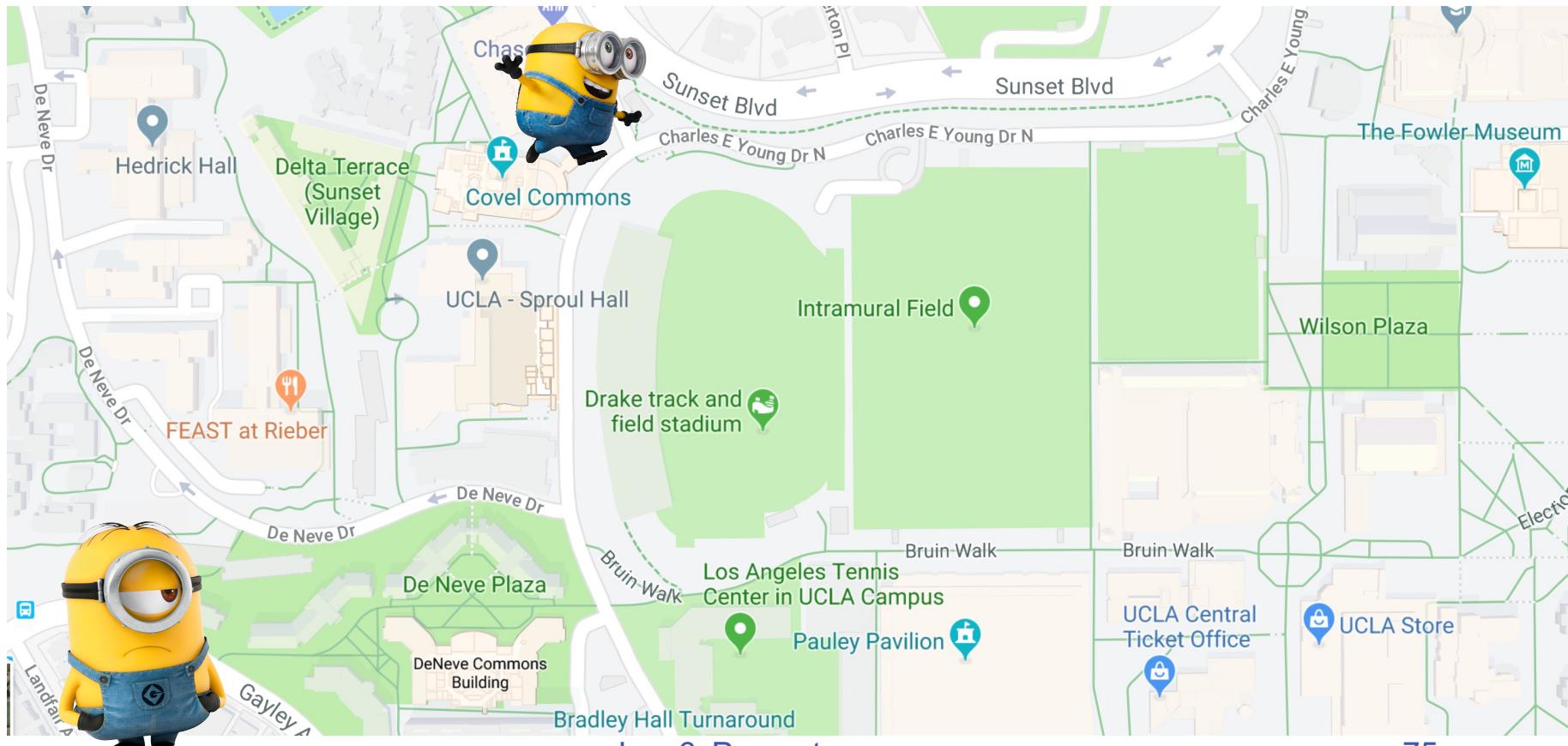
Intuition

✓ Asking direction



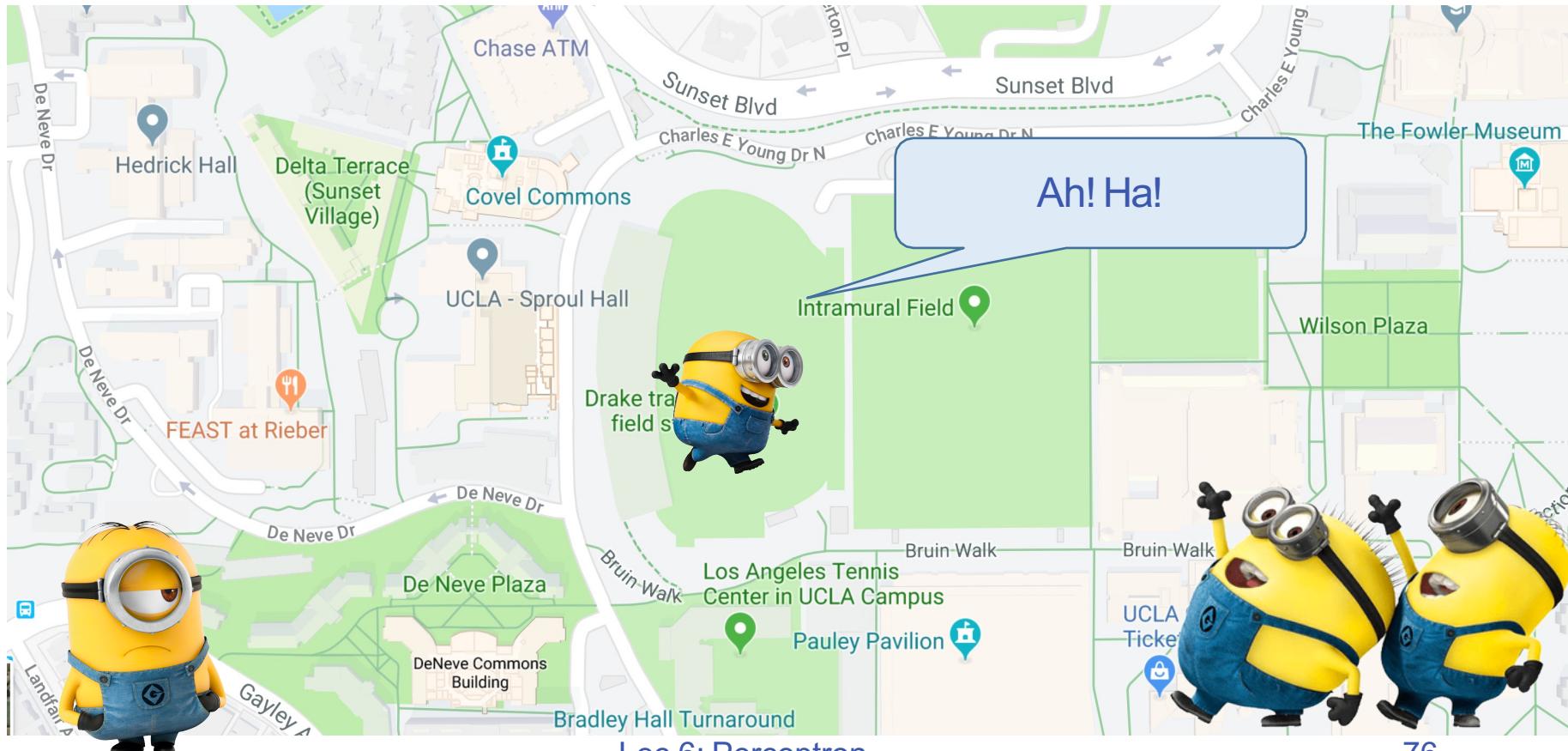
Intuition

▼ Asking direction



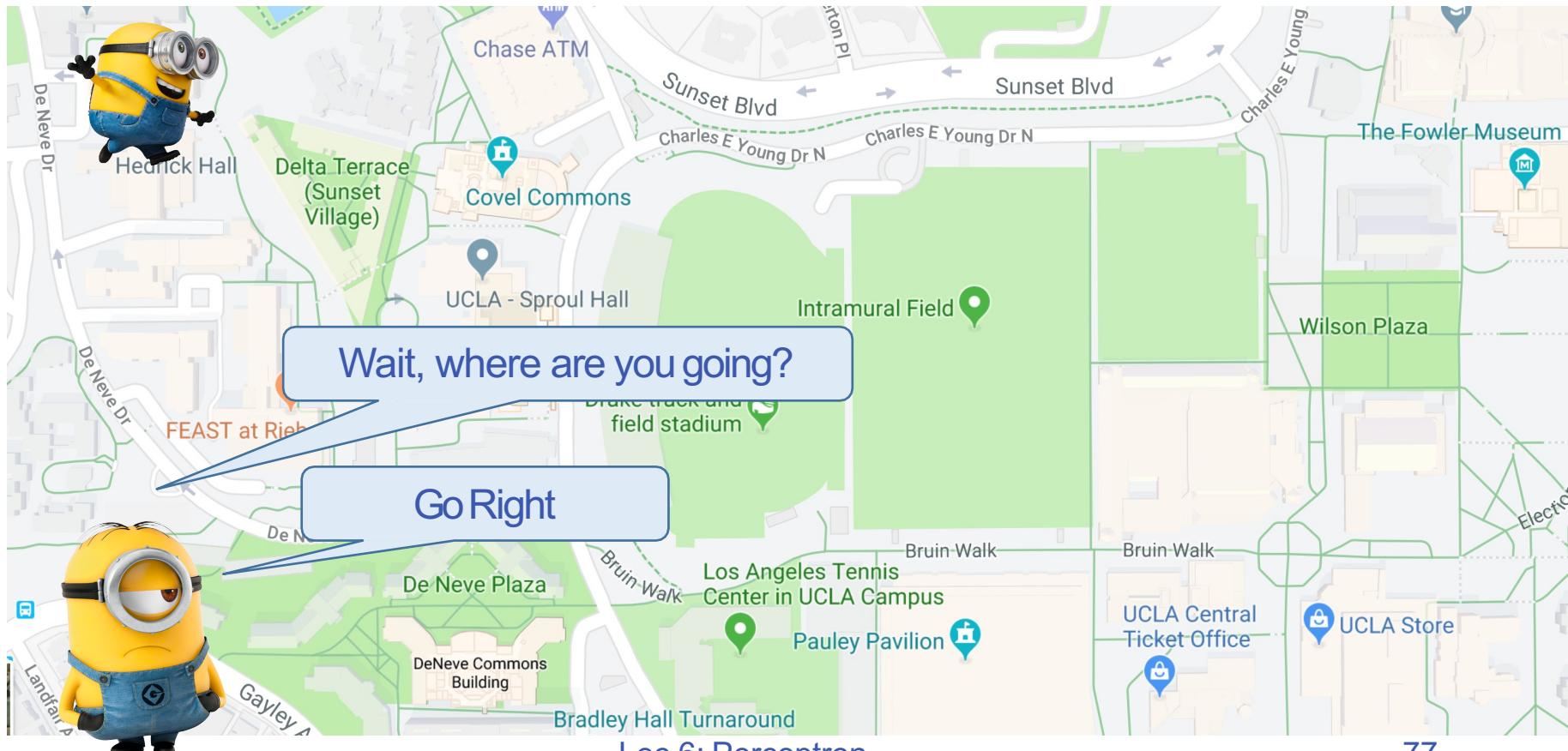
Intuition

✓ Asking direction



Intuition

- What may go wrong? Incorrect Step-size

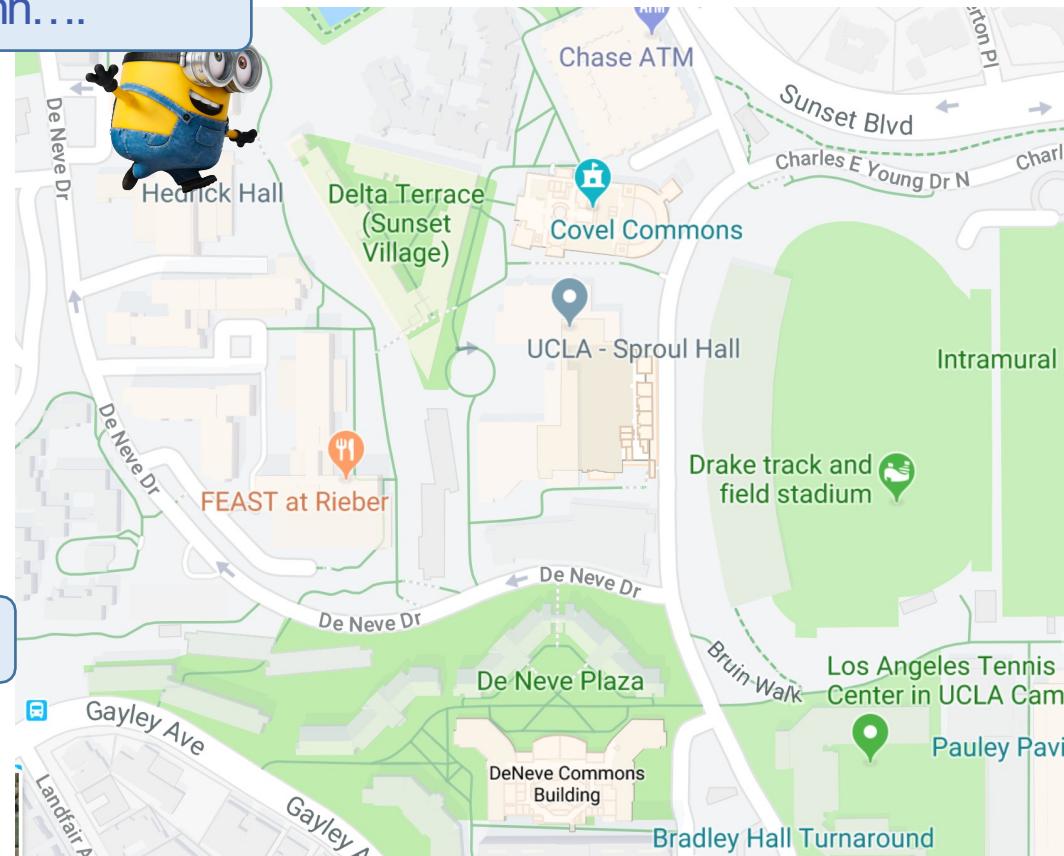


Intuition

- What may go wrong? Incorrect Direction

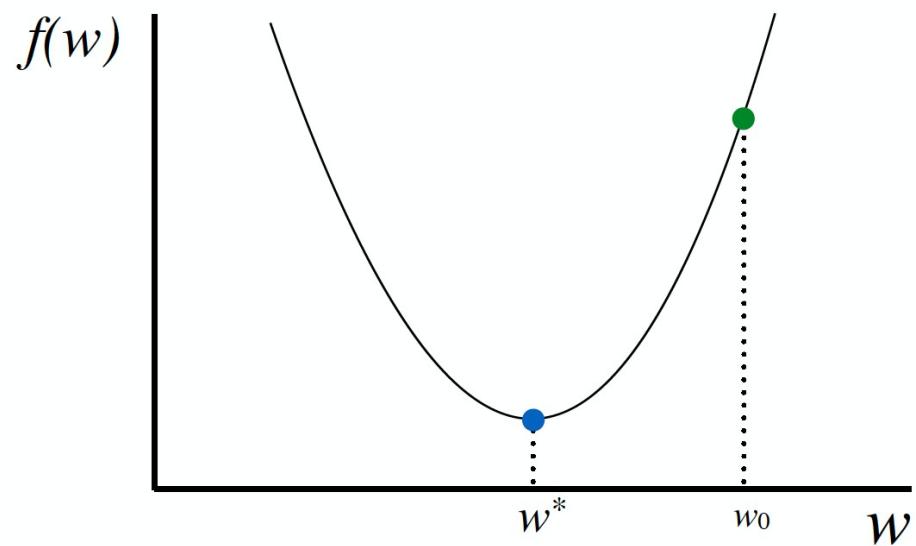


AHHhhhhhhh....



Gradient descent

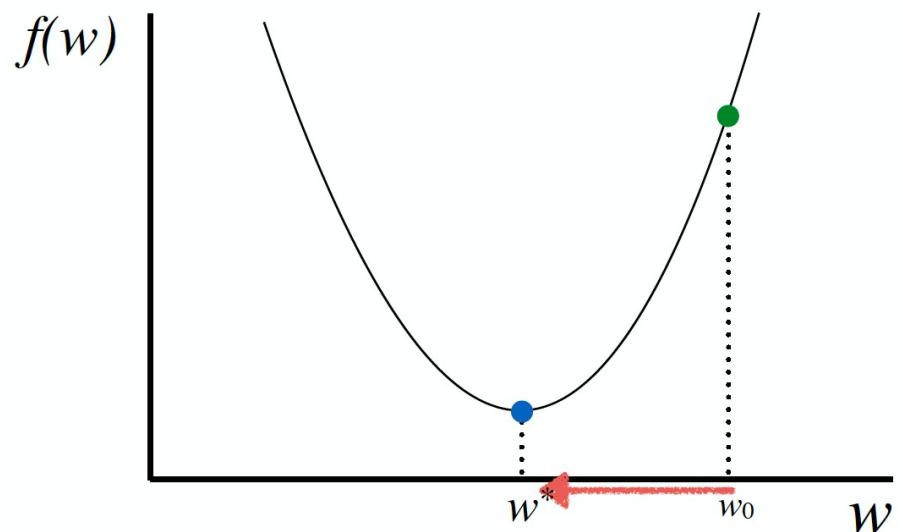
Start at a random point



Gradient descent

Start at a random point

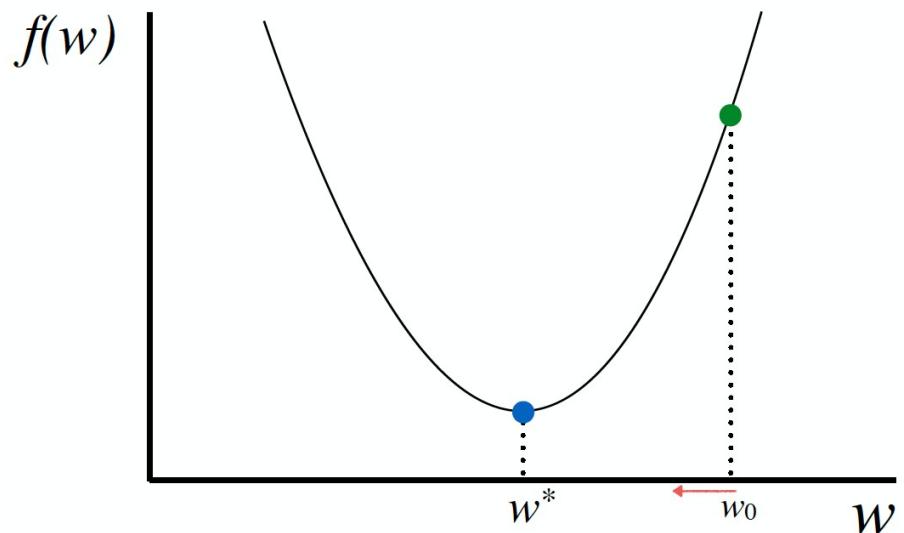
Determine a descent direction



Gradient descent

Start at a random point

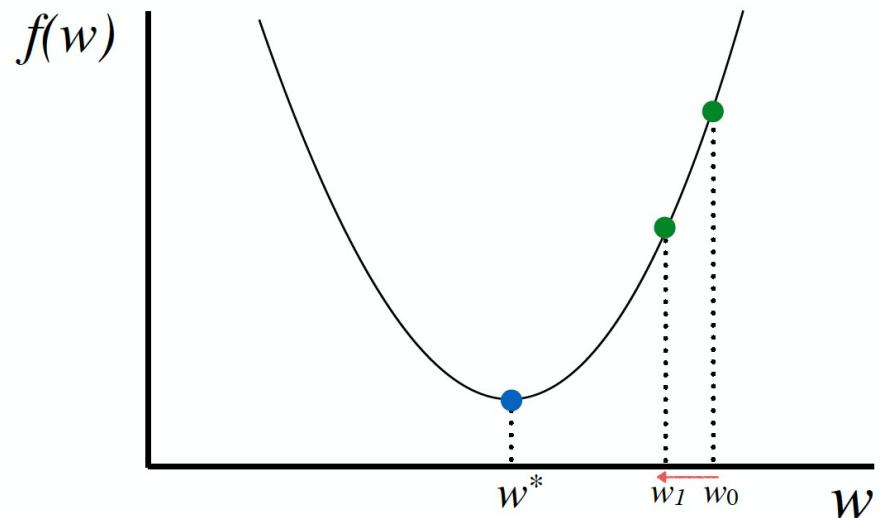
Determine a descent direction
Choose a step size



Gradient descent

Start at a random point

Determine a descent direction
Choose a step size
Update



Gradient descent

Start at a random point

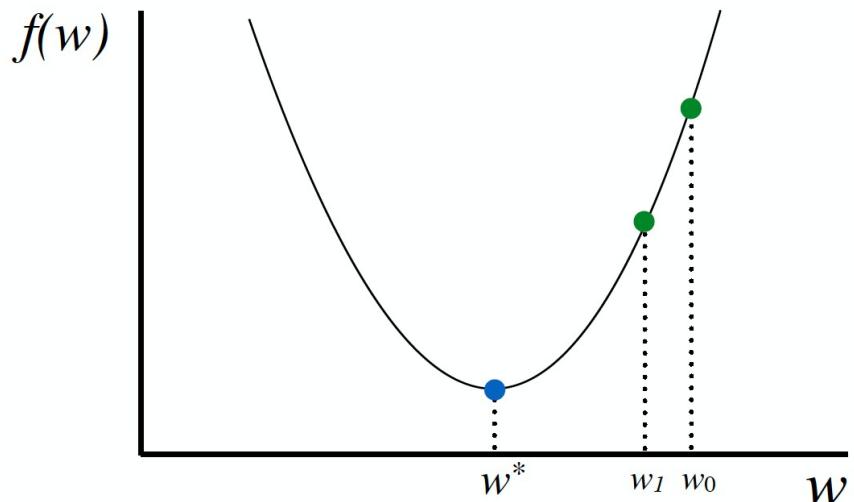
Repeat

Determine a descent direction

Choose a step size

Update

Until stopping criterion is satisfied



Gradient descent

Start at a random point

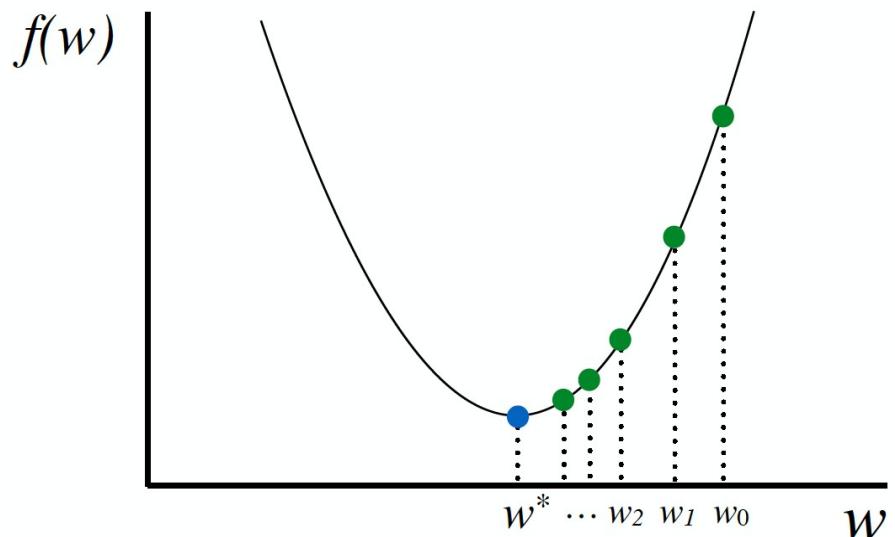
Repeat

Determine a descent direction

Choose a step size

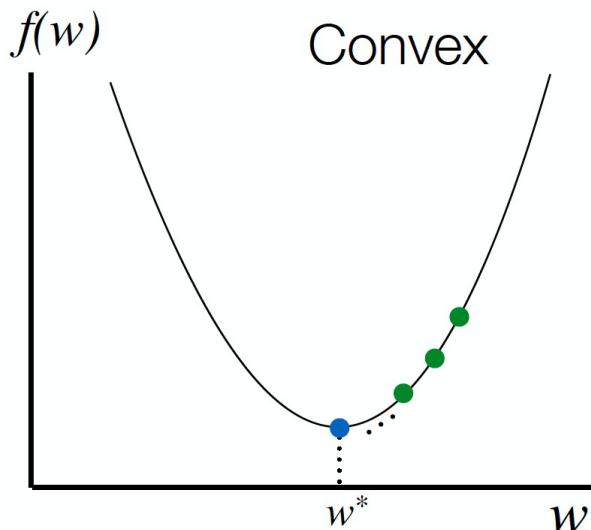
Update

Until stopping criterion is satisfied

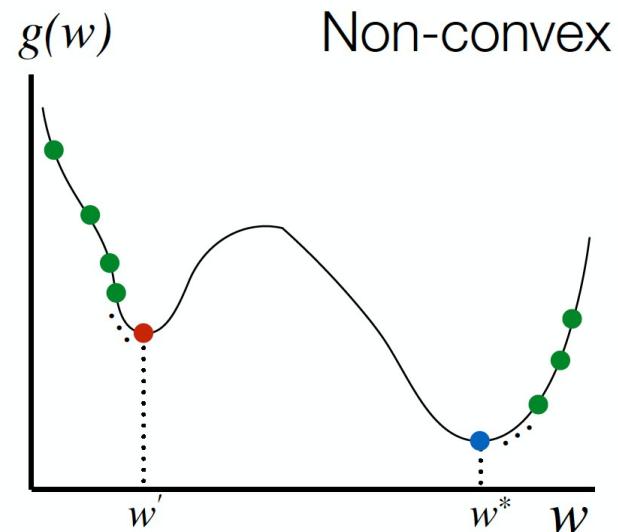


Where will we converge?

- ✓ If function is convex, it converges to the global optimum (need proper choice of step-size)



Any local minimum is a global minimum



Multiple local minima may exist

Exercise

❖ Let $\sigma(z) = \frac{1}{1+\exp(-z)}$, show

$$\sigma(-z) = 1 - \sigma(z)$$

❖ What is the gradient of

$$\underset{\mathbf{w}}{\operatorname{argmax}} \sum_{i=1}^m y_i \log \sigma(\mathbf{w}^T \mathbf{x}) + (1 - y_i) \log(1 - \sigma(\mathbf{w}^T \mathbf{x}))$$