RESTRICTED $\triangle$ via RESTRICTED $\cap$

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Theorem (Kleitman, 1966)

Let $\mathcal{F} \subseteq 2^{[n]}$ be a set family with $|A \triangle B| \leq d\,(\forall A, B \in \mathcal{F})$. Then

$$|\mathcal{F}| \leq \begin{cases} \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{t} & \text{if } d = 2t, \\ 2\left(\binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{t}\right) & \text{if } d = 2t+1. \end{cases}$$

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Tightness: Hamming ball centered at $(\underbrace{0,\ldots,0}_{n})$ or $(\frac{1}{2}, \underbrace{0,\ldots,0}_{n-1})$.

Definition

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- HKP conjectured that the lower bound is asymptotically correct.

Theorem (HOMO, DGLOZ, 2025+)

For homogeneous $D = \{sd, (s+1)d, \ldots, td\}$, *we have*

$$f_D(n) = \begin{cases} (1+o(1)) \cdot \prod_{\ell \in D_{\text{even}}} \frac{2n}{\ell} & \text{if } dst \text{ is even,} \\ (2+o(1)) \cdot \prod_{\ell \in D_{\text{even}}} \frac{2n}{\ell} & \text{if } dst \text{ is odd.} \end{cases}$$

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Theorem (NON-HOMO, DGLOZ, 2025+)

Let $D = \{sd + a, (s+1)d + a, \ldots, td + a\}$ *be non-homogeneous.*

- *If* $D_{\text{even}} = \varnothing$, *then* $f_D(n) = 2$.
- *If* $D_{\text{even}} \neq \varnothing$, *then* $\left\lfloor \frac{2n}{\min(D_{\text{even}})} \right\rfloor \leq f_D(n) \leq n + 2$.

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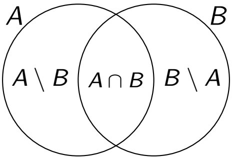

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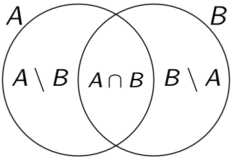

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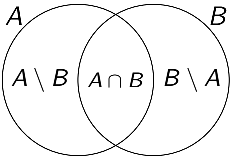

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- Suppose D consists of odd integers. Then

$$(\star) \implies |A \triangle B| + |B \triangle C| + |C \triangle A| = \text{even} \implies f_D(n) \le 2.$$

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- Prove `NON-HOMO` upper bound assuming $D = \{4, 10\}$.
  - Linear `NON-HOMO` lower bounds are trivial.
- Apply our methods to notable binary code problems.
  - Approach binary t-distance set conjecture.

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Consider **uniform** family. The following observation suffices:

$$\mathcal{F} \subseteq \binom{[n]}{3} \text{ (3-}\textbf{uniform}\text{) is } \{0,1\}\text{-}\cap \overset{(\star)}{\Longrightarrow} \{4,6\}\text{-}\triangle. \qquad \blacksquare$$

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- Steiner is of size $\binom{n}{2}/\binom{3}{2}$. It exists iff $\binom{2}{1} \mid \binom{n-1}{1}$ and $\binom{3}{2} \mid \binom{n}{2}$.

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There is $\mathcal{F} \subseteq \binom{[m]}{t}$ of size $|\mathcal{F}| = \left(1 - o(1)\right) \cdot \frac{\binom{m}{t-s+1}}{\binom{t}{t-s+1}}$ such that

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Lemma (Rödl "double" nibble)

There are $\mathcal{F}_1, \mathcal{F}_2 \subseteq \binom{[m]}{t}$, each of size $\big(1 - o(1)\big) \cdot \frac{\binom{m}{t-s+1}}{\binom{t}{t-s+1}}$, such that

- *for every distinct $A, B \in \mathcal{F}_1$ or $\mathcal{F}_2$, we have $|A \cap B| \leq t - s$,*
- *for every $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$, we have $|A \cap B| \leq t - s + 1$.*

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With the help of intersection results, we establish the followings.

(1) **FW Th'm** $\implies |\mathcal{F} \setminus \mathcal{F}'| = O(n)$.

(2) **DEF Th'm** $\implies |\mathcal{F}'| \le \frac{n}{3} \cdot \frac{n}{2}$.

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Combining (1) and (2), we obtain

$$|\mathcal{F}| = |\mathcal{F}'| + |\mathcal{F} \setminus \mathcal{F}'| \le \frac{n}{3} \cdot \frac{n}{2} + O(n) = \left(\frac{1}{3} + o(1)\right) \cdot \binom{n}{2}. \qquad \blacksquare$$

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Proof of (1). Recall $\mathcal{F} \setminus \mathcal{F}' = \big\{A \in \mathcal{F} : |A \setminus [6]| < 3\big\}$. Write

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Arbitrarily fix S. For distinct $A, B \in \mathcal{F}_S$,

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WLOG $\varnothing \in \mathcal{F}_S$. Write $\mathcal{F}_S^k \overset{\text{def}}{=} \big\{A \in \mathcal{F}_S : |A| = k\big\}$ $(k = 3, 4, 5)$. Then

$$(\star) \implies \mathcal{F}_S^k \text{ is } \{2\}\text{-}\cap \overset{\textbf{FW}}{\Longrightarrow} |\mathcal{F}_S^k| \le n \overset{\text{sum}}{\Longrightarrow} |\mathcal{F} \setminus \mathcal{F}'| = O(n). \qquad \blacksquare$$

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Suppose $n \geq 2^k k^3$. *If* $\mathcal{F} \subseteq \binom{[n]}{k}$ *is* $\{\ell_1, \ldots, \ell_r\}$-$\cap$, *then* $|\mathcal{F}| \leq \prod_{i=1}^{r} \frac{n-\ell_i}{k-\ell_i}$.

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We see that $\mathcal{F}' \subseteq \binom{[n]}{6}$ is $\{3, 4, 5, 6\}$-$\triangle$. Thus

$$(\star) \Longrightarrow \mathcal{F}' \text{ is } \{3, 4\}\text{-}\cap \overset{\textbf{DEF}}{\Longrightarrow} |\mathcal{F}'| \leq \frac{n-3}{6-3} \cdot \frac{n-4}{6-4} \leq \frac{n}{3} \cdot \frac{n}{2}. \qquad \blacksquare$$

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Theorem (Deza–Erdős–Frankl, 1978)

Suppose $n \geq 2^k k^3$*. If* $\mathcal{F} \subseteq \binom{[n]}{k}$ *is* $\{\ell_1, \ldots, \ell_r\}$*-*$\cap$*, then* $|\mathcal{F}| \leq \prod_{i=1}^{r} \frac{n-\ell_i}{k-\ell_i}$*.*

Moreover, if $|\mathcal{F}| \geq 2^k k^2 n^{r-1}$*, then there exists* $C \in \binom{[n]}{\ell_1}$ *with* $C \subseteq \cap\mathcal{F}$*.*

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The Gram matrix $\big(\langle v_i, v_j\rangle\big)_{m\times m}$ of $v_1, \ldots, v_m$ has rank estimate

$$n \ge \operatorname{rank}_{\mathbb{F}_3} \begin{pmatrix} 1 & -1 & -1 & \cdots & -1 \\ -1 & 1 & -1 & \cdots & -1 \\ -1 & -1 & 1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 1 \end{pmatrix} \ge m - 1 \implies |\mathcal{F}| \le n + 2. \qquad \blacksquare$$

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Conjecture (Folklore, communicated to us by Wei-Hsuan Yu)

If $n \gg t$, then

$$A(\mathcal{H}_n, t) \overset{(1)}{=} A(\mathcal{H}_n, \{2, 4, \ldots, 2t\})$$
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- Barg, Glazyrin, Kao, Lai, Tseng, Yu (2024) settled $t = 2$.

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Let $n \gg t$. If $|D| = t$ and $D \neq \{2, 4, \ldots, 2t\}$ is independent on n, then

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- We modify the HKP linear algebraic method to confirm (2).

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