Dynamic Simulation

TODOS

- What is dynamic simulation?
- What is the standard approach?
- What are the challenges?
- Enter constraints!

Simulating effects such as incompressibility, inextensibility and joints between atriculated rigid bodies can be achieved in elasticity-based simulations by using high stiffness values. High stiffness values lead to large forces which in turn cause numerical issues in the solver.

We demonstrate these issues based on the example of maintaining a desired distance between two points using a stiff spring [8]. Let $\boldsymbol{x_1}, \boldsymbol{x_2}$ be the positions, $\boldsymbol{v_1}, \boldsymbol{v_2}$ the velocities and $\boldsymbol{a_1}, \boldsymbol{a_2}$ be the accelerations of the two particles. Let $\bar{l}$ be the rest length and $l = \|\boldsymbol{x_1} - \boldsymbol{x_2}\|$ be the current length of the spring with stiffness k. It can be shown that the force that the spring applies at each particle is equal to $\boldsymbol{f_1} = -\boldsymbol{f_2} = \lambda \boldsymbol{u}$, where $\boldsymbol{u} = (\boldsymbol{x_1} - \boldsymbol{x_2})/l$ and $\lambda = -\frac{\delta V}{\delta l} = k(\bar{l} - l)$.

Once the forces, accelerations, velocities and positions are combined into vectors $\boldsymbol{f}, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{x}$, respectively, the motions of the system can be modeled via Newton's Ordinary Differential Equation (ODE) $\boldsymbol{f} = \boldsymbol{M}\boldsymbol{a}$, where $\boldsymbol{M}$ is a $n_d \times n_d$ diagonal matrix and n_d is the total number of independent degrees of freedom for the particles.

This system can be integrated via the symplectic Euler method as follows:

$$\boldsymbol{v_{n+1}} = \boldsymbol{v_n} + h\boldsymbol{a_n}$$
$$\boldsymbol{x_{n+1}} = \boldsymbol{x_n} + h\boldsymbol{v_{n+1}}$$

As the stiffness k of the spring increases, so does the magnitude of the acceleration $\boldsymbol{a}$. Consequently, the integration diverges unless the timestep is prohibitively small. The stability issues are often addressed by switching to an implicit integration scheme, such as the backward Euler method [2]. Replacing current accelerations with future accelerations requires the solution of the following linear system of equations (LSE):

$$(\boldsymbol{M} - h^2\boldsymbol{K})\boldsymbol{v_{n+1}} = \boldsymbol{p} + h\boldsymbol{f}$$

where $\boldsymbol{p} = \boldsymbol{M}\boldsymbol{v_n}$ is the momentum, and $\boldsymbol{K} = \frac{\delta \boldsymbol{f}}{\delta \boldsymbol{x}}$ is the stiffness matrix. Note that $\boldsymbol{K}$ is typically non-singular since elastic forces are invariant under rigid body transforms. When using large stiffness k for springs, the entries of $\boldsymbol{K}$ are large (due to large restorative forces for stiff springs) and dominate the entries of the system matrix $\boldsymbol{H} = \boldsymbol{M} - h^2\boldsymbol{K}$. In these cases, $\boldsymbol{H}$ will be almost non-singular as well, leading to numerical issues and poor convergence for many solvers. Additionally, implicit integration introduces noticable numerical damping [6].

This system results from performing the implicit integration and solving the non-linear system via linearization using the Taylor expansion. Positions can be expressed in terms of velocities and eliminated from the system.

Penalty Forces

In the example above, the energy was derived from Hooke's Law for springs. However, it is also possible to derive energies from geometric displacement functions $\boldsymbol{\phi(x)}$ which vanish in the rest configuration. From the displacement functions, quadratic potential energies of the form $U(\boldsymbol{x}) = \Sigma_i (k/2)\boldsymbol{\phi^2(x)}$, where k is a positive stiffness parameter, are constructed [7]. The potential energy $U(\boldsymbol{x})$ is zero if the displacement function is satisfied, and greater than zero otherwise. The resulting forces are called penalty forces. Using the geometric displacement function $\boldsymbol{\phi}_{\text{spring}}(\boldsymbol{x}) = (\|\boldsymbol{x_i} - \boldsymbol{x_j}\|) - l$ with k_{spring} recovers the behavior of a spring with stiffness k_{spring}. Its displacement function $\boldsymbol{\phi}_{\text{spring}}(\boldsymbol{x})$ is satisfied when the distance of the particles $\boldsymbol{x_i}, \boldsymbol{x_j}$ is equal to a desired rest length l. By constructing different geometric displacement functions, various properties such as the bending angle between triangles and in-plane shearing of triangles can be controlled via the corresponding quadratic energy potentials [2]. Geometric displacement functions with the desired effect are often intuitive and simple to define. However, as the corresponding energy potentials are not physically derived, choosing stiffness parameters that correspond to measurable physical properties of the simulated

material and orchestrating multiple constraints becomes challenging [6, 4]. Additionally, the generated penalty forces do not converge in the limit of infinite stiffess, leading to oscillations unless the timestep is reduced significantly [5].

Maybe explain the challenges with penalty forces a bit better! Also read [7, 4, 5]. I just skimmed over [7] for now, but want to make sure that I am citing this correctly. The term penalty forces is not used in the paper, I am just following the trail from [6]. [4] is a review that might be intersting to read. [5] would be really interesting to read for once, just to understand why strong penalty forces oscillate. Is this a general problem with penalty forces, or is it an issue with the solver?

Mass Modification

The motion of a particle can be influenced by modifying the inverse mass matrix $\boldsymbol{M^{-1}}$ of the system. For a single particle $\boldsymbol{x_i}$, it is $\boldsymbol{\ddot{x}_i} = \boldsymbol{M_i^{-1}} \boldsymbol{f}$, where $\boldsymbol{M_i^{-1}}$ is the inverse mass matrix for particle i. If, for example, the first diagonal entry of $\boldsymbol{M_i^{-1}}$ is zero, no acceleration in the x-direction is possible. It is possible to construct modified inverse mass matrices $\boldsymbol{W}$ such that the accelerations in the three axes of an arbitraty orthogonal coordinate system can be restricted. The modified inverse mass matrices $\boldsymbol{W}$ can be used in the LSE that results from the implicit integration above. By adding simple velocity and position terms to the system equations the magnitude of the change in velocity in each direction and even the exact position of each constrained particle can be controlled. This approach is called mass modification [2]. In [2], the authors use mass modification to model collision constraints between objects and cloth and other user defined constraints.

Since the velocity and position of each constrained particle is controlled via a single velocity and position term, multiple constraints that affect the same particle have to be handled together. This can lead to constraints which affect arbitrarily many particles. For that reason, self-collisions of cloth are not handled via mass modification in [2]. Instead, penalty forces are used. Additionally, accurately constraining particle positions is only possible for particles whose velocity is constrained as well. The resulting system is unbanded, but sparse, and is solved using a modified version of the conjugate gradient (CG) method.

Constraint-based Dynamics

Hard Constraints

The problem of maintaining hard distance constraints between particles can be formulated as a Differential Algebraic Equation (DAE) [9, 1]. In this framework, the standard ODE $\boldsymbol{f} = \boldsymbol{Ma}$ is handled together with algebraic equations that model the constraints on the system. Distance constraints are typically implemented using holonomic constraints of the form $\boldsymbol{\phi(x)} = 0$. Note that the distance constraint $\boldsymbol{\phi}(x)$ is formulated in terms of the particle positions, whereas the ODE works on accelerations or velocities. Consequently, the constraints need to be differentiated once or twice so that they can be combined with the ODE in terms of velocties or accelerations, respectively. In xPBD, we go the other way! The ODE is tanslated so that it is in terms of positions, so that it can be handled together with the constraints. Is there a reason nobody bothered to do this before? What are the challenges here? Is this exactly what xPBD is? Is there a way to view the simplifications made in terms of the other frameworks?. Using $\boldsymbol{J} = \frac{\delta \boldsymbol{\phi}}{\delta \boldsymbol{x}}$, where $\boldsymbol{J}$ is a $n_c \times n_d$ matrix and n_c is the number of scalar constraints, this leads to the following constraint formulations:

$$\boldsymbol{Jv} = 0$$
$$\boldsymbol{Ja} = \boldsymbol{c(v)}$$

for some $\boldsymbol{c(v)}$. If you check [9], see that $c(v)$ also depends on the positions q. That should be indicated! Additionally, constraint forces are required in order to link the algebraic constraint equations with the ODE describing the motion of the system. It can be shown that the constraint forces $\boldsymbol{f_c}$ applied to the particles have to be in the following form in order to avoid adding linear and angular momentum to the system [1]:

$$\boldsymbol{f_c} = \boldsymbol{J^T \lambda}$$

where the λ are the Lagrange multipliers of the constraints. With external forces $\boldsymbol{f_e}$, the DAE can now be expressed as follows [9]:

$$\begin{pmatrix} \boldsymbol{M} & \boldsymbol{J^T} \\ \boldsymbol{J} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f_e} \\ \boldsymbol{c(v)} \end{pmatrix}$$

Note that the lower block-row of the system drives towards accelerations that satisfy the constraints imposed by $\boldsymbol{\phi(x)}$ (or, stricly speaking, the differentiations thereof) exactly. This is indicated by the lower-right zero block in the system matrix in either formulation. Aren't $\dot{q} = v$ and $\dot{v} = a$ also part of the differential equation? Because $c(v)$ and f_e also depend on q!

In [9], the DAE is approached by eliminating the λ from the system entirely and constructing an ODE in terms of positions and velocities. In [8], the authors suggest applying implicit integration schemes to the system directly by constructing the following Karush-Kuhn-Tucker (KKT) equation system:

$$\begin{pmatrix} \boldsymbol{M} & \boldsymbol{-J^T} \\ \boldsymbol{J} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\mu} \end{pmatrix} = \begin{pmatrix} \boldsymbol{p} + h\boldsymbol{f_e} \\ 0 \end{pmatrix}$$

If internal forces are taken into account, the upper-left matrix $\boldsymbol{M}$ is replaced by the matrix $\boldsymbol{H}$ from above.

Reverse-engineering how the authors arrived at this system is quite enlightening. Start out from the equations of motion [9]

$$\boldsymbol{\dot{v} = M^{-1}(f - J^T)\lambda}$$

and perform implicit integration:

$$\begin{aligned} \boldsymbol{v_{n+1}} &= \boldsymbol{v_n} + h\boldsymbol{M^{-1}}(\boldsymbol{f_e(x_{n+1})} - \boldsymbol{J^T(x_{n+1})\lambda}) \\ \boldsymbol{Mv_{n+1}} &= \boldsymbol{p} + h\boldsymbol{f_e(x_{n+1})} - h\boldsymbol{J^T(x_{n+1})\lambda} \\ \boldsymbol{Mv_{n+1}} + h\boldsymbol{J^T(x_{n+1})\lambda} &= \boldsymbol{p} + h\boldsymbol{f_e(x_{n+1})} \\ \boldsymbol{Mv_{n+1}} + \boldsymbol{J^T(x_{n+1})\mu} &= \boldsymbol{p} + h\boldsymbol{f_e(x_{n+1})} \end{aligned}$$

If we assume that $\boldsymbol{f_e}$ and the constraint gradients $\boldsymbol{J}$ are constant across the time step, we arrive at the formulation from the paper. For the external forces, which are usually only comprised of gravitational forces, this is not a big deal. Is this what authors mean when they say that the constraints are effectively linearized during one solve, e.g. second page of [3]?

Note that the system matrix is sparse, which can be exploited by sparse-matrix solvers in order to solve the system efficiently [1]. Alternatively, the Schur complement can be constructed since the mass matrix in the upper left block is invertible. This leads to a smaller, albeit less sparse system [8]:

$$\boldsymbol{JM^{-1}J^T\mu = -JM^{-1}(p + hf_e)}$$

If the constraints are not redundant, $\boldsymbol{JM^{-1}J^T}$ is non-singular and symmetric positive definite [1], which are desirable properties for many solvers. According to [6], the common approaches for linearizing the constraint forces and stabilizing the constraints $\boldsymbol{\phi(x)} = 0$ are notoriously unstable. Additionally, instabilities in the traverse direction of the constraints occur when the tensile force with respect to particle masses is large when using hard constraints [8].

Compliant Constraints

- It is well know that kinematic constraints of the form $\boldsymbol{\phi(x) = 0}$ are the physical lmit of strong potential forces of the form $\frac{k}{2}\boldsymbol{\phi^2(x)}$ for very large k.
- Till now: Penalty forces or hard constraints
- Basically: We still want to restrict the motion, but leave some wiggle room.
- Start from hard constraints, but try to find a way to express constraints as the limit of strong potentials, i.e. simply using a very high stiffness. This is easier to do by using the inverse of stiffness, called compliance. Why can't we just express the problem in terms of stiffness? Where would that fail and why?
- Benefits (Figure out why this is true!):

  - almost entire range of parameters from zero to infinity allowed
  - removing numerical problems which occur when constraints are degenerate or overdefined
  - stabilize the linear equation solving process by making the matrices strongly positive definite
    * If you look that the systems that we have constructed earlier, the lower right block matrix is zero. This means that the entire system matrix is indefinite from my understanding (Check whether this is actually true!). Adding the compliance matrix, which is strictly positive definite, makes the entire system matrix positive definite, which is a nice property for most solvers

- From a constraint, define the constraint potential in terms of the compliance. Fill in the equations
- In order to remove the large stiffness parameters (Remove from where exactly? Is this just removal from $\boldsymbol{K}$), introduce $\boldsymbol{\lambda}$ (Show the equations).
- Show the resulting system!
- In the limit of infinite stiffness, hard constraints are recovered.
- Don't use intuitive geometric displacement functions, instead derive geometric displacement functions from elasticity theory
- The approximations for the symplectic Euler solver do not make sense to me, in particular the time-average for lamba. No idea where the term $\beta\dot{\phi}$ comes from. The mean of a function f over and interval (a, b) is defined as $\frac{1}{b-a}\int_a^b f(x)dx$

References

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