Notes on Mathematics

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List of Definitions

List of Theorems

Chapter 1

Set Theory

Definition: Empty Set, $\varnothing$

The ***empty set***, denoted by $\varnothing$, is the set that contains no elements at all.

In other words, $x \notin \varnothing$ for all x.

Definition: Subset, $A \subseteq B$

Let A and B be sets. We say that A is a ***subset*** of B, denoted by $A \subseteq B$, if every element of A is also an element of B, i.e., for all $a \in A$ we have $a \in B$.

Definition: Superset, $A \supseteq B$

Let A and B be sets. We say that A is a ***superset*** of B, denoted by $A \supseteq B$, if B is a subset of A.

Note that the empty set is a subset of every set, and that every set is a superset of the empty set.

Definition: Set Equality, $A = B$, $A \neq B$

Let A and B be sets. We say that A and B are ***equal***, denoted by $A = B$, if $A \subseteq B$ and $B \subseteq A$. If this is not the case, then we write $A \neq B$ to denote that A and B are not equal.

Definition: Proper Subset, $A \subsetneq B$

Let A and B be sets. We say that A is a ***proper subset*** of B, denoted by $\subsetneq$, if $A \subseteq B$ and $A \neq B$.

Definition: Proper Superset, $A \supsetneq B$

Let A and B be sets. We say that A is a ***proper superset*** of B, denoted by $\supsetneq$, if $A \supseteq B$ and $A \neq B$.

Definition: Power Set, $\mathcal{P}(A)$, 2^A

Let A be a set. The ***power set*** of A, denoted by $\mathcal{P}(A)$ or 2^A, is the set containing all subsets of A.

Note that $\mathcal{P}(A)$ contains both the empty set $\varnothing$ and the set A itself.

Definition: Cartesian Product, $A \times B$

Let A and B be sets. The ***Cartesian product*** of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) consisting of an element $a \in A$ followed by an element $b \in B$.

$$A \times B \coloneqq \{(a, b) : a \in A \text{ and } b \in B\}$$

Definition: Relation, Binary Relation, $a \mathrel{R} b$, $a \not\mathrel{R} b$

Let A and B be sets. A ***relation*** between A and B, also known as a ***binary relation*** between A and B, is a subset $R \subseteq A \times B$ of their Cartesian product.

Given $a \in A$ and $b \in B$, we write $a \mathrel{R} b$ to denote that $(a, b) \in R$. If this is the case, then we say that R *relates* a to b. Similarly, we write $a \not\mathrel{R} b$ to denote that $(a, b) \notin R$.

If $A = B$, then instead of saying that R is a relation *between* A and A, we simply say that R is a binary relation *on* A.

For example, the less-than relation $<$ is a binary relation on the set $\mathbb{N}$ of natural numbers. We write $3 < 4$ to indicate that the ordered pair $(3, 4) \in \mathbb{N} \times \mathbb{N}$ is an element of the relation $<$.

Definition: Function, Domain, Codomain, Maps, $f : A \to B$, $f(x)$

Let A and B be sets. A ***function*** from A to B is a binary relation $f \subseteq A \times B$ between A and B that has the following property: for each $x \in A$, there exists a unique $y \in B$ such that $x \mathrel{f} y$. We call the set A the ***domain*** of the function f, and we call B the ***codomain*** of f.

We write $f : A \to B$ to denote that f is a function from A to B, and given $x \in A$, we write $f(x)$ to denote the unique element of B that is related to x by f. We say that f ***maps*** x to $f(x)$.

Definition: Injective Function, Injection

Let A and B be sets. A function $f : A \to B$ is ***injective***, or an ***injection***, if no two distinct elements of A are mapped to the same element of B by f.

$$\forall x, y \in A : x \neq y \implies f(x) \neq f(y)$$

Definition: Identity Function, id_A

Let A be a set. The ***identity function*** on A is the function $\mathrm{id}_A : A \to A$ defined by $\mathrm{id}_A(a) \coloneqq a$ for all $a \in A$.

Definition: Cartesian Product, $\prod_{i\in I} A_i$

Let I be an index set, and let $\{A_i\}_{i\in I}$ be an indexed family of sets. The ***Cartesian product*** of $\{A_i\}_{i\in I}$, denoted by $\prod_{i\in I} A_i$, is the set of all functions $f : I \to \bigcup_{i\in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$.

$$\prod_{i\in I} A_i \coloneqq \left\{ f : I \to \bigcup_{i\in I} A_i \;\middle|\; \forall i \in I : f(i) \in A_i \right\}$$

Chapter 2

Topology

Definition: Topology

Let X be a set. A ***topology*** on X

Definition: Metric, Distance Function

Let X be a set. A ***metric*** on X, also known as a ***distance function*** on X, is a function $d : X \times X \to \mathbb{R}$ that satisfies the following requirements:

- ***Positive-definiteness***: $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.
- ***Symmetry***: $d(x, y) = d(y, x)$ for all $x, y \in X$.
- ***Triangle inequality***: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Definition: Metric Space, Underlying Set, Point

A ***metric space*** is an ordered pair (X, d) consisting of a set X, called the ***underlying set*** of the metric space, and a metric d on X. The elements of X are called the ***points*** of the metric space (X, d).

Definition: Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is ***continuous*** at a point $x_0 \in X$ if, for every $\epsilon > 0$, there exists $\delta > 0$, such that for all $x \in X$, if $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \epsilon$.

Definition: Neighborhood

Let (X, T) be a topological space, and let $x \in X$. A subset $A \subseteq X$ is a ***neighborhood*** of x if there exists an open set $U \in T$ such that $x \in U$ and $U \subseteq A$.

Chapter 3

Group Theory

Definition: Group

A ***group*** is an algebraic structure $\langle G; 1, {}^{-1}, \cdot \rangle$ consisting of:

- a set G, called the ***underlying set***;
- a distinguished element $1 \in G$, called the ***identity element***;
- a unary operation ${}^{-1} : G \to G$, written as $x \mapsto x^{-1}$, called ***inversion***;
- a binary operation $\cdot : G \times G \to G$, written as $(x, y) \mapsto x \cdot y$, called the ***group operation*** or ***group product***;

satisfying the following requirements:

- ***Associative property***: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in G$.
- ***Identity property***: $1 \cdot x = x \cdot 1 = x$ for all $x \in G$.
- ***Inverse property***: $x \cdot x^{-1} = x^{-1} \cdot x = 1$ for all $x \in G$.

Definition: Abelian Group

An ***abelian group*** is a group $\langle G; 1, {}^{-1}, \cdot \rangle$ that satisfies the following additional requirement:

- ***Commutative property***: $x \cdot y = y \cdot x$ for all $x, y \in G$.

Chapter 4

Ring Theory

In this chapter, we introduce a new class of algebraic structures, called rngs and rings, whose study is collectively called ***ring theory***. Rngs and rings are more complicated than groups because their definition involves not one, but two binary operations.

Definition: Rng

A ***rng*** (pronounced as "*rung*") is an algebraic structure $\langle R; 0, -, +, \cdot \rangle$ consisting of:

- a set R, called the ***underlying set***;
- a distinguished element $0 \in R$, called the ***zero element***;
- a unary operation $- : R \to R$, written as $x \mapsto -x$, called ***negation***;
- a binary operation $+ : R \times R \to R$, written as $(x, y) \mapsto x + y$, called ***addition***;
- a binary operation $\cdot : R \times R \to R$, written as $(x, y) \mapsto x \cdot y$, called ***multiplication***;

satisfying the following requirements:

- ***Additive structure***: $\langle R; 0, -, + \rangle$ is an abelian group.
- ***Associativity***: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in R$.
- ***Left distributivity***: $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ for all $x, y, z \in R$.
- ***Right distributivity***: $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ for all $x, y, z \in R$.

The key ingredient in the definition of a rng is ***distributivity***, which establishes a link between two different binary operations. We begin our study of rngs by proving a simple (but important) result to demonstrate the utility of the distributive property.

Multiplying by Zero Yields Zero

Theorem: Let $\langle R; 0, -, +, \cdot \rangle$ be a rng. For any $x \in R$, we have $0 \cdot x = x \cdot 0 = 0$.

Proof: Let $x \in R$ be given. Because 0 is the identity element of the abelian group $\langle R; 0, -, + \rangle$, we have $0 = 0 + 0$. Using left distributivity, it follows that $0 \cdot x = (0 + 0) \cdot x = (0 \cdot x) + (0 \cdot x)$, and by canceling one copy of $0 \cdot x$ on both sides, we conclude that $0 \cdot x = 0$. We similarly apply right distributivity to the expression $x \cdot 0 = x \cdot (0 + 0) = (x \cdot 0) + (x \cdot 0)$ to conclude that $x \cdot 0 = 0$.

Definition: Zero Rng, Trivial Rng

The ***zero rng*** or ***trivial rng*** is the rng $\langle \{0\}; 0, -, +, \cdot \rangle$ whose underlying set is a singleton containing the distinguished element 0, and whose operations are defined by $-0 = 0 + 0 = 0 \cdot 0 = 0$.

Definition: Nonzero Rng, Nontrivial Rng

A rng is ***nonzero*** or ***nontrivial*** if its underlying set contains more than one element.

Definition: Ring

A ***ring*** is an algebraic structure $\langle R; 0, 1, -, +, \cdot \rangle$ consisting of:

- a set R, called the ***underlying set***;
- a distinguished element $0 \in R$, called the ***zero element***;
- a distinguished element $1 \in R$, called the ***identity element***;
- a unary operation $- : R \to R$, written as $x \mapsto -x$, called ***negation***;
- a binary operation $+ : R \times R \to R$, written as $(x, y) \mapsto x + y$, called ***addition***;
- a binary operation $\cdot : R \times R \to R$, written as $(x, y) \mapsto x \cdot y$, called ***multiplication***;

satisfying the following requirements:

- ***Rng structure***: $\langle R; 0, -, +, \cdot \rangle$ is a rng.
- ***Identity***: $1 \cdot x = x \cdot 1 = x$ for all $x \in R$.

Definition: Commutative Ring

A ***commutative ring*** is a ring R that satisfies the following additional requirement:

- ***Commutativity***: $x \cdot y = y \cdot x$ for all $x, y \in R$.