MATH 7200 Notes Algebraic Topology
as taught by Prof. Anna-Marie Bohmann Vanderbilt University Fall Semester 2016

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1. Lecture 1 (2016-08-25)

(Discussion of syllabus omitted.)

The goal of algebraic topology is to study topological spaces via their algebraic invariants. A familiar example from previous courses is the fundamental group, which associates to each topological space X and point $x_0 \in X$ a group $\pi_1(X, x_0)$.

In this course, we will also study homology and cohomology groups (chapters 2-3 of Hatcher's text), which are relatively easy to compute, in addition to higher homotopy groups (chapter 4), which are more powerful but harder to compute.

1.1. Review of Essential Concepts

We want to study topological spaces up to homotopy. (Henceforth all spaces will be assumed topological, and all maps assumed continuous.) We say that two maps $f_0, f_1 : X \to Y$ are ***homotopic***, written $f_0 \simeq f_1$, if there exists a map $H : X \times I \to Y$ such that $H(x, i) = f_i(x)$ for $i = 0, 1$.

A map $f : X \to Y$ is a ***homotopy equivalence*** if there exists a map $g : Y \to X$ such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$. If this occurs, we say that X and Y are ***homotopy equivalent***, or simply ***equivalent*** if no confusion can arise. For example, the letters A and D are homotopy equivalent, but A and B are not. Homotopy equivalence is indeed an equivalence relation — the meaning of studying a space "up to homotopy" is to study its equivalence class under homotopy equivalence.

A space with the homotopy equivalence type of a point is called ***contractible***. For example, $\mathbb{R}^n$ is contractible. (The homotopy is given by multiplication by t.) In general, we won't necessarily write down our homotopies explicitly, as long as we can describe them sufficiently clearly.

A map which is homotopic to the "squish everything to a point" map is called ***nullhomotopic***. Occasionally the word ***essential*** is used to mean "not nullhomotopic."

For the purposes of this class, we will restrict our attention to "nice" spaces, where the word "nice" means "sufficiently well-behaved to make our theorems work." (This typically includes, for example, the Hausdorff property.) Examples include ***manifolds*** and ***CW-complexes***.

A CW-complex is a space that is "inductively constructed" by attaching cells (i.e. disks of various dimensions). For example, by gluing together two points, two segments, and a disk, we obtain a (closed) filled circle. In general, we start with a discrete space (collection of points) X^0 and inductively form a space X^n, called the ***n-skeleton***, from X^{n-1} by attaching n-cells $\{e^n_\alpha\}_{\alpha \in A}$ along maps $\phi_\alpha : S^{n-1}_\alpha \to X^{n-1}$. Explicitly, X^n is the quotient space

$$\frac{\left(\bigsqcup_{\alpha \in A} e^n_\alpha\right) \sqcup X^{n-1}}{x \sim \phi_\alpha(x)}$$

where the denominator indicates that x is identified with $\phi_\alpha(x)$ for all $x \in \partial e^n_\alpha$.

(Some examples of CW-complexes omitted.)

1.2. Operations on Spaces

The product or disjoint union of two CW-complexes is another CW-complex. Moreover, if X is a CW-complex, and $A \subseteq X$ is a ***subcomplex*** (closed subspace which is a union of cells), then we can construct the ***quotient space*** X/A which bears an induced CW-structure. (Everything in A gets crushed to a point, and cells attached to A become attached to that point.)

A less familiar operation is that of ***suspension***. If X is a space, the ***cone on X*** is the space CX given by $(X \times I)/(X \times \{0\} \sim \mathrm{pt})$. (Imagine constructing a "cylinder" $X \times I$ and crushing the bottom face to a point.) The suspension SX is given by $(X \times I)/(X \times \{0\} \sim \mathrm{pt}_1,\ X \times \{1\} \sim \mathrm{pt}_2)$ with both faces crushed in. If $f : X \to Y$ is a map, its ***mapping cone*** M_f is $(CX \sqcup Y)/((x, 1) \sim f(x))$. Think of a witch's hat with the "brim" as Y and the "cone" as CX, with the cone attached to the brim via f.

2. Lecture 2 (2016-08-30)

Recall that the fundamental group $\pi_1(X)$ depends only on low-dimensional information. In fact, for CW-complexes, it is uniquely determined by the 0, 1, and 2-skeleta. (We won't prove this, but it isn't hard to see intuitively.) This means that π_1 cannot distinguish between, say, S^3 and $S^{1,000,000}$.

We would obviously like to study algebraic invariants of topological spaces that take higher-dimensional information into account. One obvious generalization is the ***higher homotopy groups***, where instead of considering homotopy classes of maps $S^1 \to X$, we instead look at homotopy classes of maps $S^n \to X$. This is a fine idea, but these groups turn out to be *very* difficult to compute.

Another solution is ***homology***. *(Motivating example from Hatcher's text omitted. See pg. 99.)*

We first study ***simplicial homology***. Our building blocks in this context are the ***n-simplices*** Δ^n, the n-dimensional analogues of the tetrahedron. We can regard Δ^n as the smallest convex set in $\mathbb{R}^m$ (for some m) containing $n+1$ affinely independent points. In particular, we have the ***standard n-simplex***

$$\Delta^n = \left\{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0 \ \forall i \text{ and } \sum_{i=0}^{n} t_i = 1 \right\}.$$

We will include an ordering on the vertices as part of the information encoded by a simplex. Given an n-simplex Δ^n, we can throw out the ith vertex to obtain an $(n-1)$-simplex, called the ***ith face*** of Δ^n. An ordering of vertices on the face is inherited from the ordering on Δ^n. The union of all faces of Δ^n is the ***boundary*** $\partial\Delta^n$.

Given a space X, a ***Δ-complex structure*** on X is a collection of maps $\{\sigma_\alpha : \Delta^n \to X\}$ (where n depends on α) such that

1. The restriction of each σ_α to $\mathrm{int}(\Delta^n) = \Delta^n \setminus \partial\Delta^n$ is injective, and each point $x \in X$ is in the image of exactly one such restriction.
2. The restriction of each σ_α to a face of Δ^n is another map $\sigma_\beta : \Delta^{n-1} \to X$ in our collection.
3. $A \subseteq X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each α.

This essentially amounts to saying that X is a bunch of simplices glued together along faces. Δ-complex structures are precisely what we need to define simplicial homology.

Definition: Given a space X and a Δ-complex structure on X, let $\Delta_n(X)$ be the free abelian group on the set $\{\sigma_\alpha : \Delta^n \to X\}$ of n-simplices of X. This is the group of finite formal linear combinations of the maps σ_α, thought of as formal symbols. We then define the boundary homomorphisms $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ as follows:

$$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \, \sigma_\alpha \Big|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

Here, the hat means that v_i is omitted. This means that we have a sequence

$$\Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n+1}} \cdots \xrightarrow{\partial_1} \Delta_0(X).$$

Algebraic Interlude: A ***chain complex*** A_* is a sequence of (free) Abelian groups linked by homomorphisms $d_n : A_n \to A_{n-1}$ such that $d^2 = 0$. That is, $d_n \circ d_{n+1} = 0$ for all n. Given a chain complex A_*, its ***homology*** is a graded abelian group $H_*(A)$ such that $H_n(A) = \ker d_n / \operatorname{im} d_{n+1}$. We call elements of A_* ***chains***, elements of $\ker d_n$ ***cycles***, and elements of $\operatorname{im} d_{n+1}$ ***boundaries***. Elements of $H_*(A)$ are called ***homology classes***.

(Standard proof of $\partial^2 = 0$ omitted.)

This shows that $\Delta_*(X)$ is a chain complex. Its homology $H_*(\Delta_*(X))$ is the ***simplicial homology*** of X. We denote the groups of this homology by $H_n^\Delta(X)$. At this point, we make some observations:

1. It is not at all clear that the simplicial homology groups are invariants of the underlying space X. A priori, it's perfectly possible that two different Δ-complex structures on X might give different simplicial homologies.

2. If X has finitely many n-simplices in its Δ-complex structure, then $H_n^{\Delta}(X)$ is finitely generated.

3. If X is finite-dimensional, then $H_k^{\Delta}(X) = 0$ for $k > n$.

3. Lecture 3 (2016-09-01)

3.1. Singular Homology

A ***singular n-simplex*** in a space X is a map $\sigma : \Delta^n \to X$. Let $C_n(X)$ be the free abelian group on the set of singular n-simplices in X. We get a chain complex

$$\cdots \longrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \longrightarrow \cdots$$

with boundary maps

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \, \sigma\Big|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}.$$

Note that $\sigma\Big|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$ is a map from Δ^{n-1} to X. The same proof as before gives $\partial^2 = 0$.

Definition: The ***nth singular homology group*** of X, denoted by $H_n(X)$, is the nth homology group of this chain complex.

$$H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

One immediately observes that if X is homeomorphic than X', then $H_n(X) \equiv H_n(X')$. However, it's unclear that these groups would ever be finitely generated.

Remark: We can, in fact, regard singular homology as a special case of simplicial homology by building $S(X)$ as a Δ-complex with one n-simplex for each singular n-simplex in X, with each n-simplex attached to its faces as $(n-1)$-simplices in the natural way. This gives a giant Δ-complex $S(X)$ which clearly satisfies $H_n^{\Delta}(S(X)) = H_n(X)$.

3.2. Geometric interpretation of singular homology

In general, a singular n-chain ξ can be written as a sum $\xi = \sum_i \epsilon_i \sigma_i$ with $\epsilon_i = \pm 1$ (and some simplices σ_i possibly repeated.) $\partial \xi$ is then a linear combination of $(n-1)$-simplices with sign ± 1.

(Some things happened here that I don't really understand. See pgs. 108-109 of Hatcher's text. The upshot is that $H_1(X)$ is represented by maps $S^1 \to X$, and $H_2(X)$ is represented by maps of oriented surfaces into X.)

3.3. Properties of singular homology

Proposition: If X decomposes into path components as $X = \bigsqcup_\alpha X_\alpha$, then $H_n(X) = \bigoplus_\alpha H_n(X_\alpha)$.

Proof: Since Δ^n is path-connected, its continuous image is path-connected in X, and hence lies in some X_α. This means that the chain groups decompose as $C_n(X) = \bigoplus_\alpha C_n(X_\alpha)$. The boundary maps respect this decomposition, and hence the homology groups split as desired. **QED**

Proposition: If X is nonempty and path-connected, then $H_0(X) = \mathbb{Z}$. For any nonempty X, $H_0(X)$ is a direct sum of copies of $\mathbb{Z}$, one for each path component.

Proof: By definition, $H_0(X) = C_0(X)/\operatorname{im}\partial_1$. Define a map $\epsilon : C_0(X) \to \mathbb{Z}$ which sends a linear combination of 0-simplices (i.e. points of X) to the sum of its coefficients. This is indeed an abelian group homomorphism, and if X is nonempty, then ϵ is surjective.

We claim that $\ker\epsilon = \operatorname{im}\partial_1$ if X is path-connected. The inclusion $\operatorname{im}\partial_1 \subseteq \ker\epsilon$ is clear from the definition of ∂_1. For the reverse inclusion, suppose we have a 0-chain $\sum_i n_i\sigma_i$ such that $\sum_i n_i = 0$. Pick some basepoint $x_0 \in X$, and for each i, pick a path τ_i from x_0 to x_i. Each τ_i is a singular 1-simplex, whose 0th face is σ_i and whose 1st face is σ_0. It follows that $\partial\tau_i = \sigma_i - \sigma_0$, and hence that $\partial(\sum_i n_i\tau_i) = \sum_i n_i\sigma_i - \sum_i n_i\sigma_0 = \sum_i n_i\sigma_i$. **QED**

Proposition: If X is a point, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) = \mathbb{Z}$.

Proof: For each n, there is exactly one singular n-simplex in X. This means that all the chain groups are $\mathbb{Z}$, and the boundary maps are given by $\partial_n = 0$ for n odd and $\partial_n = \mathrm{id}$ for n even. **QED**

3.4. Reduced homology groups

Definition: The ***reduced homology groups*** $\tilde{H}_n(X)$ of a nonempty space X are the homology groups of the augmented chain complex

$$\cdots \longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

(Check that this is indeed a chain complex!) Here we have $\tilde{H}_n(X) = H_n(X)$ for $n > 0$, and $H_0(X) \equiv \tilde{H}_0(X) \oplus \mathbb{Z}$ splits, with $\mathbb{Z}$ coming from the path component of the basepoint. One can imagine that the extra $\mathbb{Z}$ in the chain complex comes from the unique map of the (-1)-simplex (i.e. the empty set) into X.

Theorem: If X is path-connected, then there is a map $\pi_1(X) \to H_1(X)$ that realizes $H_1(X)$ as the abelianization of $\pi_1(X)$.

We won't prove this for now, since it would be too much of a detour into homotopy.

4. Lecture 4 (2016-09-06)

4.1. Homotopy invariance of singular homology

Proposition: $f : X \to Y$ (a continuous map) induces $f_* : H_n(X) \to H_n(Y)$ (a homomorphism).

Proof: Define $f_\# : C_n(X) \to C_n(Y)$ to send a singular n-simplex σ in X to the singular n-simplex $f \circ \sigma$ in Y, extending by linearity. We need to show that $f_\#$ is a ***chain map***, that is, $f_\# \partial = \partial f_\#$. (This is equivalent to requiring that a ladder diagram commute.)

(Straightforward verification omitted.) **QED**

Lemma: A chain map between chain complexes induces a homomorphism on homology.

Proof: $f_\#$ takes cycles to cycles. Indeed, if $\partial\alpha = 0$, then $\partial f_\# \alpha = f_\# \partial \alpha = f_\# 0 = 0$. Moreover, $f_\#$ takes boundaries to boundaries, since $f_\# \partial \beta = \partial f_\# \beta$. **QED**

Properties:

1. $(f \circ g)_* = f_* \circ g_*$
2. $(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$

This shows singular homology is a functor.

Theorem: If $f, g : X \to Y$ are homotopic, then they induce the same homomorphism on homology.

Corollary: If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism for all n.

Proof of Theorem: We first need to understand how to decompose $\Delta^n \times I$ into $(n+1)$-simplices. We will omit some combinatorial details (see Hatcher for more), but if we regard $\Delta^n \times I$ as having $[v_0, \ldots, v_n]$ on its bottom and $[w_0, \ldots, w_n]$ on top, then

$$[v_0, \ldots, v_i, w_i, \ldots, w_n]$$

is an $(n+1)$-simplex. The collection of these simplices for all i gives a decomposition of $\Delta^n \times I$ into $(n+1)$-simplices.

Now, given a homotopy $F : X \times I \to Y$ between $f = F_0$ and $g = F_1$, we define "prism operators" $P : C_n(X) \to C_{n+1}(Y)$ as follows:

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i \left. \left[F \circ (\sigma \times \mathrm{id}_I)\right] \right|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]}$$

We claim that $\partial P = g_\# - f_\# - P\partial$. Intuitively, we're just saying that "the boundary of a prism consists of the top plus the bottom plus the sides," with some signs thrown in for good measure. Checking this is a straightforward calculation:

$$\partial P(\sigma) = \sum_{j\le i}(-1)^i(-1)^j\ [F\circ(\sigma\times \mathrm{id}_I)]\Big|_{[v_0,\dots,\hat v_j,\dots,v_i,w_i,\dots,w_n]}$$
$$+\sum_{j\ge i}(-1)^i(-1)^{j+1}\ [F\circ(\sigma\times \mathrm{id}_I)]\Big|_{[v_0,\dots,v_i,w_i,\dots,\hat w_j,\dots,w_n]}$$

All of the $i=j$ terms cancel, except for two: for $i=j=0$, we get

$$[F\circ(\sigma\times \mathrm{id}_I)]\Big|_{[\hat v_0,w_0,\dots,w_n]} = g_\#(\sigma),$$

and for $i=j=n$, we get

$$-\ [F\circ(\sigma\times \mathrm{id}_I)]\Big|_{[v_0,\dots,v_n,\hat w_n]} = -f_\#(\sigma).$$

The rest of the terms give precisely $P(\partial\sigma)$.

We now claim that this proves the theorem! If $\alpha\in C_n(X)$ is a cycle, then

$$g_\#(\alpha)-f_\#(\alpha)=\partial P(\alpha).$$

Since $g_\#(\alpha)$ and $f_\#(\alpha)$ differ by a boundary, they represent the same homology class.

Remark: The identity $\partial P+P\partial = g_\# - f_\#$ says that P is a "chain homotopy" between $g_\#$ and $f_\#$.

Remark: The exact same proof can be applied to reduced homologies, giving the same result.

4.2. Excision

When A is a subspace of X, the homologies $H_*(X)$, $H_*(A)$, and $H_*(X/A)$ are related by *exact sequences*. A sequence of Abelian groups

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$$

is said to be ***exact*** if $\ker\alpha_n = \operatorname{im}\alpha_{n+1}$ for all n. This means that the chain complex has trivial homology. For example,

- If $0\longrightarrow A\xrightarrow{\alpha} B$ is exact, then α is injective.
- If $A\xrightarrow{\alpha} B\longrightarrow 0$ is exact, then α is surjective.
- If $0\longrightarrow A\xrightarrow{\alpha} B\longrightarrow 0$ is exact, then α is an isomorphism.
- If $0\longrightarrow A\xrightarrow{\alpha} B\xrightarrow{\beta} C\longrightarrow 0$ is exact, then α is injective, β is surjective, and $\ker\beta=\operatorname{im}\alpha$, so $C=B/\operatorname{im}\alpha$. Sequences of this form are called ***short exact sequences***.

Theorem: If X is a space and A is a nonempty closed subspace that is a deformation retract of some neighborhood in X, then there is an exact sequence

$$\cdots \to \widetilde{H}_n(A) \to \widetilde{H}_n(X) \to \widetilde{H}_n(X/A) \to \widetilde{H}_{n-1}(A) \to \widetilde{H}_{n-1}(X) \to \cdots \to \widetilde{H}_0(X/A) \to 0.$$

Here, the maps $\widetilde{H}_n(A) \to \widetilde{H}_n(X)$ are induced by the inclusion map, and the maps $\widetilde{H}_n(X) \to \widetilde{H}_n(X/A)$ are induced by the quotient map. The maps $\widetilde{H}_n(X/A) \to \widetilde{H}_{n-1}(A)$ are non-obvious and will be constructed during the proof.

Remark: Pairs of spaces (X, A) satisfying these hypotheses are called "good pairs." (Really!)

This theorem can be used to find, for example, the homology groups of S^n.

5. Lecture 05 (2016-09-08)

(Missed some material on relative homology and the long exact homology sequence.)

5.1. Excision

Theorem: Given subspaces $Z \subseteq A \subseteq X$ such that the closure of Z is contained in the interior of A, the inclusion of pairs $(X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X - Z, A - Z) \cong H_n(X, A)$ for all n.

Equivalently, for subspaces $A, B \subseteq X$ whose interiors cover X, then $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \cong H_n(X, A)$ for all n. To translate between these formulations, set $B = X - Z$ or $Z = X - B$.

Proof idea: Show that $H_n(X)$ is actually giben by chains with "small images" in X.

Let $\mathcal{U} = \{U_j\}$ be a collection of sets whose interiors cover X. Let $C_n^{\mathcal{U}}(X)$ be the subgroup of $C_n(X)$ consisting of chains $\sum n_i\sigma_i$ such that the image of each σ_i is contained in one of the sets $U_j \in \mathcal{U}$. Note that the boundary maps $\partial_n : C_n(X) \to C_{n-1}(X)$ take $C_n^{\mathcal{U}}(X)$ to $C_{n-1}^{\mathcal{U}}(X)$, so we can consider the homology $H_n^{\mathcal{U}}(X)$ of this chain complex.

Proposition: The inclusion $i : C_n^{\mathcal{U}}(X) \to C_n(X)$ is a chain homotopy equivalence. That is, there is a map $p : C_n(X) \to C_n^{\mathcal{U}}(X)$ such that ip and pi are chain-homotopic to the identity. Hence, $H_n^{\mathcal{U}}(X) \cong H_n(X)$.

Summary of Proof:

1. We first need to define *barycentric subdivision*, a method of dividing simplices into smaller simplices.
2. We then create a subdivision operator on *linear chains*, a restricted type of chain. We omit some details for now, but if $Y \subseteq \mathbb{R}^m$ is a subset of Euclidean space, we want to consider the collection $LC_n(Y)$ of linear maps $\Delta^n \to Y$. The subdivision operator $S : LC_n(Y) \to LC_n(Y)$ should then break linear chains into smaller linear chains in such a way that $\partial S = S\partial$, i.e., such that S is a chain map. We also need to show that there's a chain homotopy $T : LC_n(Y) \to LC_{n+1}(Y)$ with $\partial T + T\partial = \mathrm{id} - S$.

3. We now pass to general chains. If $\sigma \in C_n(X)$, define $S(\sigma) = \sigma_\# S(\Delta^n)$, where if $\sigma : \Delta^n \to X$, then $\sigma_\# : C_n(\Delta^n) \to C_n(X)$. We then define $T : C_n(X) \to C_{n+1}(X)$ by $T(\sigma) = \sigma_\# T(\Delta^n)$.

4. Iterate this construction. The amount of subdividing we need depends on the chain we started with.

6. Lecture 06 (2016-09-13)

6.1. Barycentric Subdivision

Definition: The barycentric subdivision of Δ^0 (a point) is simply Δ^0. The barycentric subdivision of Δ^1 (a line segment) is two line segments (drop a vertex in the middle). To subdivide Δ^n, subdivide each of its faces Δ^{n-1} inductively, and drop a new vertex in its barycenter. Then connect each division of each face to the new vertex to subdivide Δ^n into $(n+1)!$ pieces.

(This is difficult to describe textually. See Hatcher's text for pictures.)

Fact: The diameter of each simplex in the barycentric subdivision of $[v_0, \dots, v_n]$ is at most $\frac{n}{n-1} \operatorname{diam}([v_0, \dots, v_n])$. Thus, by repeatedly subdividing, we can make these diameters arbitrarily small.

6.2. Linear Chains

Definition: Let Y be a subset of Euclidean space. We define $LC_n(Y)$ as the free Abelian group generated by all linear maps $\Delta^n \to Y$.

We now obtain a chain complex

$$\cdots \longrightarrow LC_n(Y) \longrightarrow LC_{n-1}(Y) \longrightarrow \cdots \longrightarrow LC_1(Y) \longrightarrow LC_0(Y) \longrightarrow LC_{-1}(Y)$$

where, as before, $LC_{-1}(Y)$ consists of linear maps from the empty simplex (i.e. $[\varnothing]$) to Y.

Definition: Given $b \in Y$, define $b : LC_n(Y) \to LC_{n+1}(Y)$ by

$$b([w_0, \dots, w_n]) = [b, w_0, \dots, w_n]$$

where the RHS is obtained from the LHS by linear interpolation. (This is why we work with $LC_n(Y)$ instead of $C_n(Y)$; linearity of the simplices guarantees that this interpolation can be done uniquely.) We call these maps ***cone operators*** because they construct the "cone" connecting the given simplex $[w_0, \dots, w_n]$ to the new point b.

It is a simple computation to check that

$$\partial b[w_0, \dots, w_n] = [w_0, \dots, w_n] - b\partial[w_0, \dots, w_n].$$

This means that b is a chain homotopy from id to the zero map, since $\partial b + b\partial = \operatorname{id} - 0$.

We now define ***subdivision homomorphisms*** $S : LC_n(Y) \to LC_n(Y)$ by induction. If $\lambda : \Delta^n \to Y$, and b_λ is the image of the barycenter of Δ^n in Y, then

$$S(\lambda) = b_\lambda S(\partial\lambda).$$

We start this induction with $S[\varnothing] = [\varnothing]$, so that $S = \mathrm{id}$ on $n = -1$ and $n = 0$.

Observe that

$$\begin{aligned}\partial S\lambda &= \partial b_\lambda(S\partial\lambda)\\ &= S\partial\lambda - b_\lambda\partial(S\partial\lambda)\\ &= S\partial\lambda - b_\lambda S\partial\partial\lambda.\end{aligned}$$

This shows that $\partial S = S\partial$. We now want to build a chain homotopy $T : LC_n(Y) \to LC_{n+1}(Y)$ between S and id. Again, we work inductively: define $T = 0$ for $n = -1$, and

$$T\lambda = b_\lambda(\lambda - T\partial\lambda)$$

for $n \geq 0$. We want to check that $\partial T + T\partial = \mathrm{id} - S$ inductively.

$$\begin{aligned}\partial T\lambda &= \partial b_\lambda(\lambda - T\partial\lambda)\\ &= \lambda - T\partial\lambda - b_\lambda\partial(\lambda - T\partial\lambda)\\ &= \lambda - T\partial\lambda - b_\lambda(S\partial\lambda + T\partial\partial\lambda)\\ &= \lambda - T\partial\lambda - S\lambda\end{aligned}$$

This is the desired result.

6.3. General Chains

Now, we want to generalize to spaces X which are not subsets of Euclidean space. The key observation here is that Δ^n is a subset of Euclidean space, so if we have a map $\Delta^n \to X$, we can subdivide Δ^n and take compositions to obtain a subdivision of a singular n-simplex in X. We define $S : C_n(X) \to C_n(X)$ by

$$S\sigma = \sigma_\# S(\Delta^n).$$

(Technically, we should write $S(\mathrm{id}_{\Delta^n})$ instead of $S(\Delta^n)$, but this abuse of notation should not cause any confusion.) This means that $S\sigma$ is a signed sum of restrictions of σ to n-simplices in the subdivision of Δ^n. It is not hard to see that S is a chain map:

$$\begin{aligned}
\partial S\sigma &= \partial\sigma_{\#}S\Delta^n \\
&= \sigma_{\#}\partial S\Delta^n \\
&= \sigma_{\#}S\partial\Delta^n \\
&= \sigma_{\#}S\sum_{i=0}^{n}(-1)^i\Delta_i^n \\
&= \sum_{i=0}^{n}(-1)^i\sigma_{\#}S\Delta_i^n \\
&= \sum_{i=0}^{n}(-1)^i S\left(\sigma\Big|_{\Delta_i^n}\right) \\
&= S\left(\sum_{i=0}^{n}(-1)^i\ \sigma\Big|_{\Delta_i^n}\right) \\
&= S\partial\sigma
\end{aligned}$$

where we have written Δ_i^n for the ith face of Δ^n. We do the same thing with T, defining $T : C_n(X) \to C_{n+1}(X)$ by

$$T\sigma = \sigma_{\#}T(\Delta^n).$$

A similar computation shows that $\partial T + T\partial = \mathrm{id} - S$.

6.4. Iteration

We now define

$$D_m = \sum_{i=0}^{m-1} TS^i$$

where S^i is the i-fold composition of S. We claim that D_m is a chain homotopy between id and S^m, since

$$\begin{aligned}\partial D_m + D_m\partial &= \sum_{i=0}^{m-1} \partial T S^i + T S^i \partial \\ &= \sum_{i=0}^{m-1} \partial T S^i + T \partial S^i \\ &= \sum_{i=0}^{m-1} (\partial T + T\partial) S^i \\ &= \sum_{i=0}^{m-1} (\mathrm{id} - S) S^i \\ &= \sum_{i=0}^{m-1} S^i - S^{i+1} \\ &= S^0 - S^m = \mathrm{id} - S^m.\end{aligned}$$

We now claim that for any n-simplex $\sigma : \Delta^n \to X$, there exists m such that $S^m(\sigma)$ lies in $C_n^{\mathcal{U}}(X)$. Indeed, we can always force the diameter of simplices in $S^m(\Delta^n)$ smaller than the Lebesgue number of the cover $\sigma^{-1}(\mathcal{U})$ by taking m sufficiently large. Henceforth, we will write $m(\sigma)$ to denote the smallest m for which this holds.

Define $D : C_n(X) \to C_{n+1}(X)$ by $D\sigma = D_{m(\sigma)}\sigma$. We want a chain map $\rho : C_n(X) \to C_n(X)$ whose image lies in $C_n^{\mathcal{U}}(X)$ satisfying $\partial D + D\partial = \mathrm{id} - \rho$. Well, simply define $\rho = \mathrm{id} - \partial D - D\partial$. A simple computation shows that $\partial\rho = \rho\partial$. We need only check that $\rho\sigma$ lies in $C_n^{\mathcal{U}}(X)$:

$$\begin{aligned}\rho(\sigma) &= \sigma - \partial D\sigma - D\partial\sigma \\ &= \sigma - \partial D_{m(\sigma)}\sigma - D\partial\sigma \\ &= S^{m(\sigma)}\sigma + D_{m(\sigma)}\partial\sigma - D\partial\sigma\end{aligned}$$

Now, $S^{m(\sigma)}\sigma$ definitely lies in $C_n^{\mathcal{U}}(X)$. Moreover, $D_{m(\sigma)}\partial\sigma - D\partial\sigma$ is a linear combination of terms of the form $D_{m(\sigma)}(\sigma_j) - D_{m(\sigma_j)}(\sigma_j)$ where σ_j is the jth face of σ. Since $m(\sigma_j) \leq m(\sigma)$, we see that all these terms lie in $C_n^{\mathcal{U}}(X)$.

Thus, we can view ρ as a map $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$. We see that $\partial D + D\partial = \mathrm{id} - i\rho$ and $\rho i = \mathrm{id}$, as desired. **QED**

Proof of Excision: Use the preceding proposition with $\mathcal{U} = \{A, B\}$. (All that work, and we're only using a two-element cover!) Abusing notation, we denote $C_n^{\mathcal{U}}(X) = C_n(A + B)$. Observe that D and ρ take chains in A to chains in A. Thus, we obtain maps

$$C_n(A+B)/C_n(A) \overset{\rho}{\longrightarrow} C_n(X)/C_n(A)$$

satisfying some properties. (In particular, D is still a chain homotopy equivalence.) Now, $H_n(B, A \cap B)$ is the homology of $C_n(B)/C_n(A \cap B) = C_n(A +$

$B)/C_n(A)$. But the homology of $C_n(X)/C_n(A)$ is precisely $H_n(X, A)$. Hence, $H_n(B, A \cap B) \cong H_n(X, A)$. **QED**