

# Quantum Error Correction

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*"To be an Error and to be Cast out is part of God's Design."*

William Blake

- Noise as a longstanding problem in information processing systems
  - e.g., classical computers, modems, CD players, etc.
  - Noise is still a problem in quantum information
- Key idea: to protect a message against noise, *encode* the message by adding redundant information; even if some information is corrupted, redundancy allows us to *decode* and recover the original message

# Project Framework

- Goals:
  - to implement various quantum error-correcting codes
    - we chose the 3-qubit, 9-qubit, 7-qubit codes
  - to analyze and compare their performances
    - *when are they effective?*
    - *when should we use error-correcting codes?*
- Tools:
  - Python's Qiskit package
  - IBM's quantum machines

# 3-Qubit Codes: Classical Inspiration

## Classical Error Correction

- Encoding by *repetition codes*:

$$0 \rightarrow 000$$

$$1 \rightarrow 111.$$

- Decoding by *majority voting*:

$$\text{Ex.: } 001 \rightarrow 0.$$

- Analysis: Let  $p$  be the probability that a bit is flipped. This method fails when 2 or more bits are flipped, which occurs with probability  $3p^2(1-p) + p^3$ , so the probability of error is  $p_e = 3p^2 - 2p^3$ . Then this method is preferred when  $p_e < p$ , or  $p < 1/2$ .

# Noisy Channels: The Bit Flip Channel

- One model for noise is the *bit flip channel* (analogous to classical channel).
- The bit flip channel flips qubits with probability  $p$  and leaves them untouched with probability  $1 - p$ .
- Equivalent to applying  $X$  gate with probability  $p$ .
- We protect qubits from this channel with the *bit flip code*.

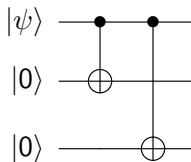
# 3-Qubit Bit Flip Code: Encoding Logical Bits

- The goal is to correct bit flip errors.
- Encoding:

$$|0\rangle \rightarrow |0_L\rangle \equiv |000\rangle$$

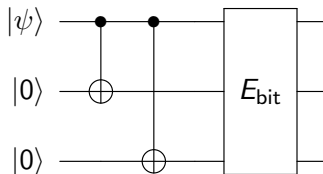
$$|1\rangle \rightarrow |1_L\rangle \equiv |111\rangle.$$

- Encoding circuit for 3-qubit bit flip code:



# 3-Qubit Bit Flip Code: Detecting Errors

- Suppose there is a bit flip error after encoding:

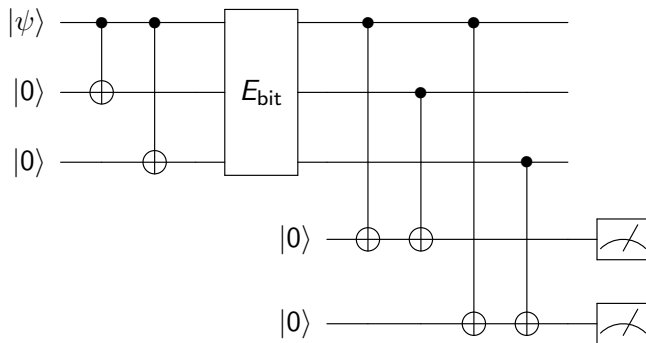


- Error Detection (or *syndrome diagnosis*):
  - we would like to determine which, if any, of the qubits have been corrupted
  - four error syndromes: no error, bit flip on qubit one, bit flip on qubit two, bit flip on qubit three



# 3-Qubit Bit Flip Code: Detecting Errors

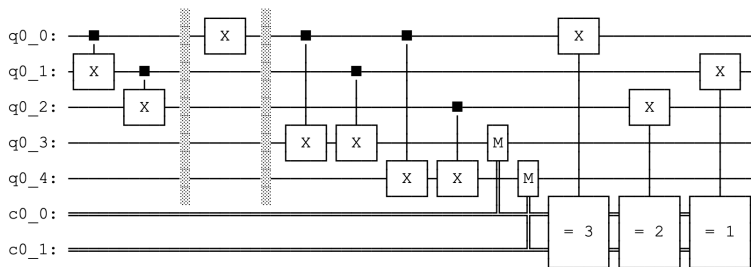
- We can diagnose the syndrome using two ancillary qubits:



- Based on measurement results, we know where the error occurred.

# 3-Qubit Bit Flip Code: Correcting Errors

- Complete circuit for error correction (or *recovery*):



# Analyzing the Bit Flip Code: Simulation

- Let's look at the performance of the 3-qubit bit flip code against bit flip channels of varying error probabilities  $p$ .

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  - 3 run error correcting code
  - 4 measure final state

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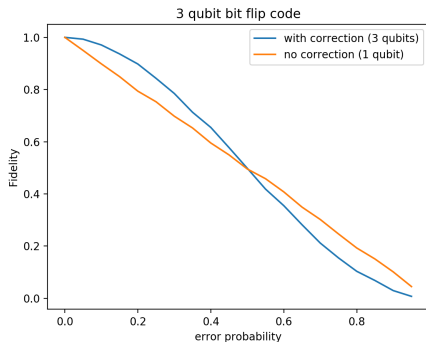
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- We can calculate the accuracy of the error correcting code for a given  $p$  by repeating many times and taking the number of times we measure a correct final state  $|000\rangle$  and dividing by the total number of trials.
- We can compare this to the accuracy of a single qubit (without encoding or error correction) that goes through a bit flip channel with the same  $p$  to see when error correction is effective.

# Analyzing the Bit Flip Code: Simulation

- Ran tests on Qiskit's simulator
- Probability  $p$  ranging from 0 to 1; 10000 trials for each  $p$



- Observe crossover point at  $p = 0.5$ .
- For  $p < 0.5$ , error correcting code performs better than a single qubit with no correction.

# Noisy Channels: Phase Flip Channel

- Another quantum channel is the *phase flip* error model.
- With probability  $p$  the relative phase of states  $|0\rangle$  and  $|1\rangle$  is flipped, with probability  $1 - p$  it is left alone.
- Equivalent to applying  $Z$  operator with probability  $p$ .
- We fight this channel with the *phase flip code*.



# 3-Qubit Phase Flip Code

- No classical analog, but it is easy to turn the phase flip channel into a bit flip channel.
- Use  $x$ -basis for encoding:

$$\begin{aligned}|0\rangle &\rightarrow |0_L\rangle \equiv |+++ \rangle \\ |1\rangle &\rightarrow |1_L\rangle \equiv |-- - \rangle .\end{aligned}$$

- Phase flip  $Z$  acts as bit flip for this encoding!

# The Shor Code

- Can we protect against *arbitrary* errors?

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- Yes!  $\longrightarrow$  The *Shor code*

# The Shor Code: Encoding

- By combining the 3-qubit phase flip and bit flip codes, the Shor code protects against arbitrary errors.
- First encode the qubit using the phase flip code:

$$|0\rangle \rightarrow |+++ \rangle, \quad |1\rangle \rightarrow |-- - \rangle.$$

- Then encode each of those qubits with the bit flip code:

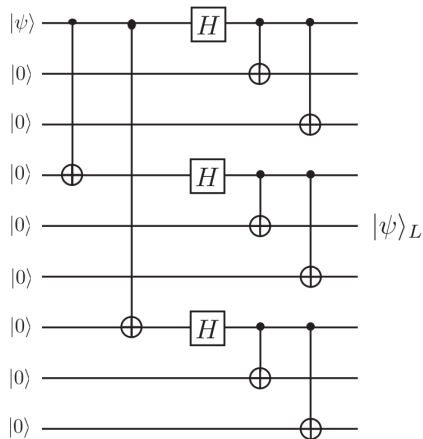
$$|+\rangle \rightarrow (|000\rangle + |111\rangle)/\sqrt{2}, \quad |-\rangle \rightarrow (|000\rangle - |111\rangle)/\sqrt{2}.$$

- The result is a 9-qubit code with codewords

$$\begin{aligned} |0\rangle \rightarrow |0_L\rangle &\equiv \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}} \\ |1\rangle \rightarrow |1_L\rangle &\equiv \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}. \end{aligned}$$

# The Shor 9-Qubit Code: Encoding

Encoding circuit for 9-qubit code:



# The Shor 9-Qubit Code: Correcting Errors

## Bit Flip Error Correction

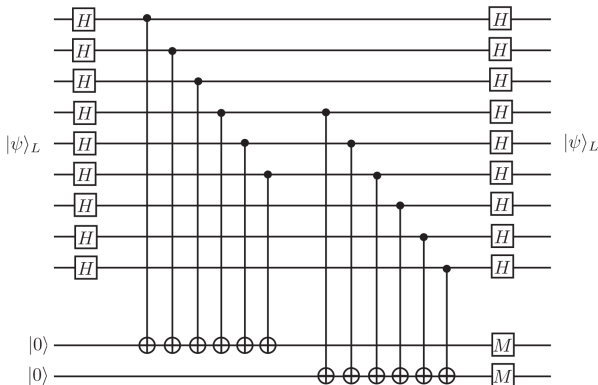
- On each block of three (i.e. qubits 0-2, 3-5, and 6-8), the 3-qubit circuit is utilized to correct for bit flips.

## Phase Flip Error Correction

- The phase of the first two blocks of three (qubits 0-2 and 3-5) and the second two blocks of three (qubits 3-5 and 6-8) are compared to correct for phase flips.
- The phase correction necessitates two ancillary qubits. Thus, we need 8 ancilla: 6 for bit flip correction, and 2 for phase flip correction.

# The Shor 9-Qubit Code: Correcting Phase Errors

- The phase correction circuit, shown below, converts the qubits from the x-basis to the z-basis and checks parity of each block of two.



# The Shor 9-Qubit Code: Correcting Phase Errors

- The following corrections are performed depending on the measured ancilla for phase flip correction:

10  $\rightarrow \sigma_z$  on block 1

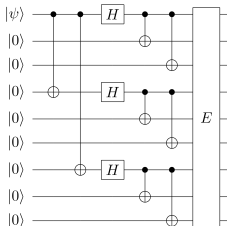
01  $\rightarrow \sigma_z$  on block 2

11  $\rightarrow \sigma_z$  on blocks 1 and 2.



# The Shor 9-Qubit Code: Error Correction Methodology

- We only consider error that occurs between the encoding step and the correcting step, thus simulating a memory error.



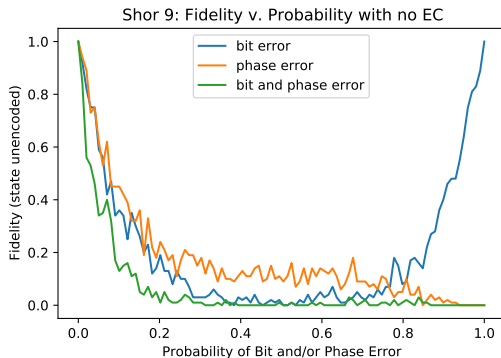
- Specifically, we consider a complete phase flip and/or bit flip (i.e.  $X$  or  $Z$ ) that occurs independently on each of the 9 physical qubits with probability  $p$ .
- After the error, we measure the ancilla and apply the appropriate error correcting steps. Finally, we run the encoding circuit in reverse and measure the output to determine fidelity.

# The Shor 9-Qubit Code: Simulation Performance with No Error Correction

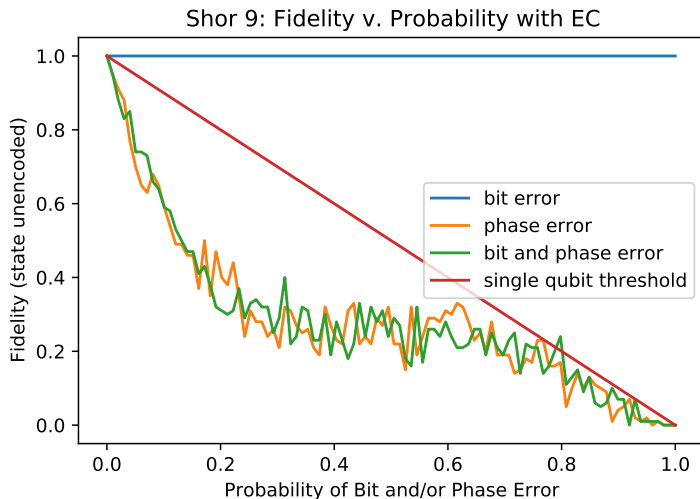
- Initial state:

$$|0\rangle \rightarrow |0_L\rangle \equiv \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle).$$

- Fidelity of un-encoded state measured against  $|000000000\rangle$ .



# The Shor 9-Qubit Code: Simulation Performance with Error Correction



# 7-Qubit Code

Encodes 1 logical qubit using 7 physical qubits:

$$|\bar{0}\rangle = \frac{|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle}{\sqrt{8}}$$

$$|\bar{1}\rangle = \frac{|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle}{\sqrt{8}}$$

$$H^{\otimes 7} |\bar{0}\rangle = \frac{|\bar{0}\rangle + |\bar{1}\rangle}{\sqrt{2}}$$

$$H^{\otimes 7} |\bar{1}\rangle = \frac{|\bar{0}\rangle - |\bar{1}\rangle}{\sqrt{2}}$$

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$$H^{\otimes 7} |\bar{1}\rangle = \frac{|\bar{0}\rangle - |\bar{1}\rangle}{\sqrt{2}}$$

- Of the 16 bit strings above, any two differ by  $\geq 3$  bits
- Intuition: therefore a single bit flip can be recovered
  - $X$  error flips bit in  $|\bar{0}\rangle, |\bar{1}\rangle$
  - $Z$  error flips bit in  $H^{\otimes 7} |\bar{0}\rangle, H^{\otimes 7} |\bar{1}\rangle$

# Example recovery for $X$ error in qubit 3

$$|\bar{0}\rangle = \frac{|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle}{\sqrt{8}}$$

$$X^{(3)} |\bar{0}\rangle = \frac{|00\textcolor{red}{1}0000\rangle + |10\textcolor{red}{0}0101\rangle + |01\textcolor{red}{0}0011\rangle + |11\textcolor{red}{1}0110\rangle + |00\textcolor{red}{1}1111\rangle + |10\textcolor{red}{0}1010\rangle + |01\textcolor{red}{0}1100\rangle + |11\textcolor{red}{1}1001\rangle}{\sqrt{8}}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = |3\rangle \text{ (in binary)}$$

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 \end{aligned}$$

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## Example recovery for $X$ error

- In fact:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} X^{(i)} |\bar{0}\rangle = |i\rangle \text{ (in binary) for all } i = 1, \dots, 7$$

- Let  $H$  be matrix above. To recover from single  $X$  error, apply map

$$|v\rangle \otimes |0\rangle_A \mapsto |v\rangle \otimes |Hv\rangle_A$$

and measure subsystem  $A$ . Result will be index  $i$  of bit flip, in binary!

- Also works for logical state 1, and for phase flips.

# 7-qubit code: Why does it work?

- The kernel of the matrix

$$H = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \in \mathbb{F}_2^{3 \times 7}$$

consists of the 16 bit strings defining  $|\bar{0}\rangle, |\bar{1}\rangle$

- A bit flip at position  $i$  of a vector  $v$  adds the  $i$ th row of  $H$  to  $Hv$  (basic linear algebra)
- The  $i$ th row of  $H$  is  $i$  in binary
- Same reasoning for phase flips = bit flips in rotated basis

# 7-qubit code: Initialization

$$|\bar{0}\rangle = \frac{|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle}{\sqrt{8}}$$

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$|\psi\rangle \rightarrow$

$|0\rangle \rightarrow$

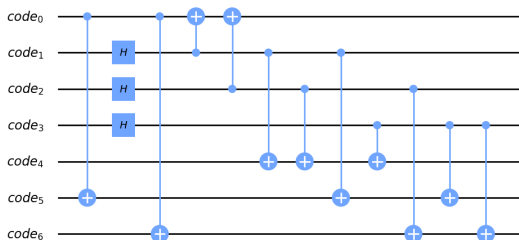
$|0\rangle \rightarrow$

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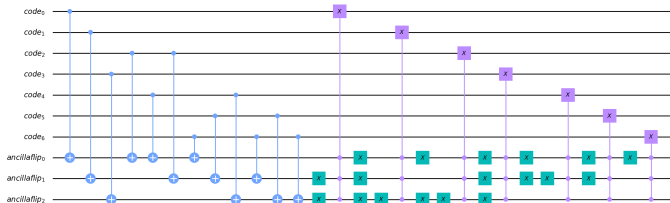
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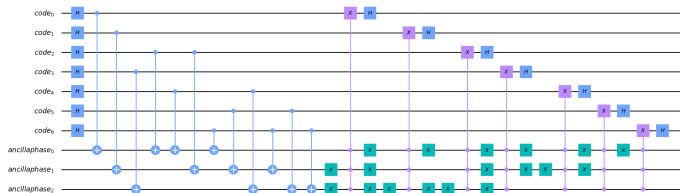
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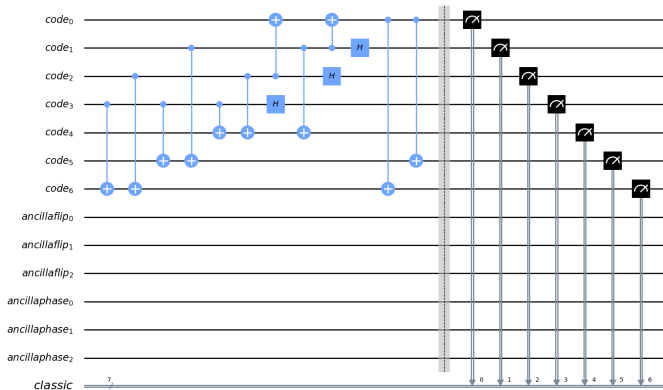
# 7-qubit code: Flip correction



# 7-qubit code: Phase correction



# 7-qubit code: Measurement



# 7-qubit code: Fidelity of X Gate under Depolarization

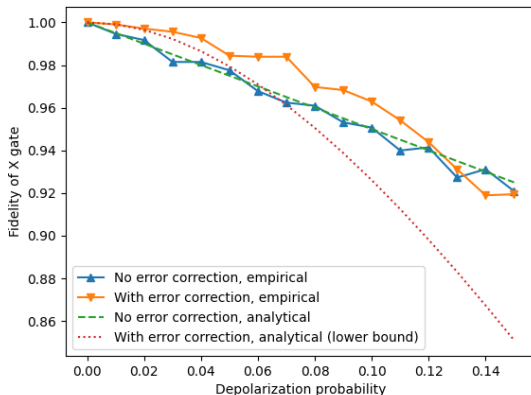
Encode  $|\bar{0}\rangle$

X

Correct flip

Correct phase

Decode, measure



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X

Correct flip

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