Quantum Error Correction

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Introduction

"To be an Error and to be Cast out is part of God's Design."

William Blake

- Noise as a longstanding problem in information processing systems
  - *e.g.*, classical computers, modems, CD players, etc.
  - Noise is still a problem in quantum information
- Key idea: to protect a message against noise, *encode* the message by adding redundant information; even if some information is corrupted, redundancy allows us to *decode* and recover the original message

Project Framework

- Goals:
  - to implement various quantum error-correcting codes
    - we chose the 3-qubit, 9-qubit, 7-qubit codes
  - to analyze and compare their performances
    - *when are they effective?*
    - *when should we use error-correcting codes?*
- Tools:
  - Python's Qiskit package
  - IBM's quantum machines

3-Qubit Codes: A Review

Classical Inspiration

- Encoding by *repetition codes*:

$$0 \to 000$$
$$1 \to 111.$$

- Decoding by *majority voting*:

$$\textit{Ex.:}\quad 001 \to 0.$$

- Analysis: Let p be the probability that a bit is flipped. This method fails when 2 or more bits are flipped, which occurs with probability $3p^2(1-p)+p^3$, so the probability of error is $p_e = 3p^2 - 2p^3$. Then this method is preferred when $p_e < p$, or $p < 1/2$.

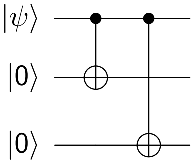
3-Qubit Codes: A Review

The Quantum Version: 3-Qubit Bit Flip Code

- The goal is to correct bit flip errors.
- Encoding:

$$
\begin{aligned}
|0\rangle &\rightarrow |0_L\rangle \equiv |000\rangle \\
|1\rangle &\rightarrow |1_L\rangle \equiv |111\rangle \,.
\end{aligned}
$$

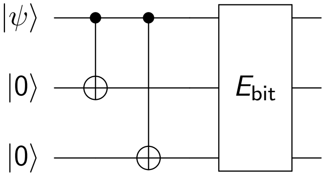
- Encoding circuit for 3-qubit bit flip code:



3-Qubit Codes: A Review

The Quantum Version: 3-Qubit Bit Flip Code

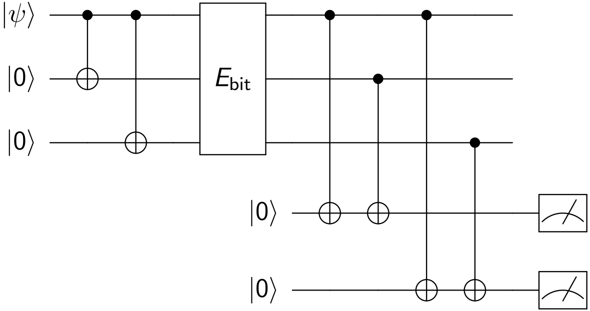
- Suppose there is a bit flip error after encoding:



- Error Detection (or *syndrome diagnosis*):
  - we would like to determine which, if any, of the qubits have been corrupted
  - four error syndromes: no error, bit flip on qubit one, bit flip on qubit two, bit flip on qubit three

The Quantum Version: 3-Qubit Bit Flip Code

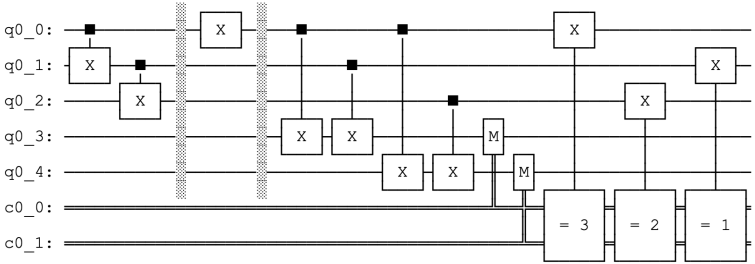
- Error Detection (or *syndrome diagnosis*):
  - we can diagnose the syndrome using two ancillary qubits:



  - Based on measurement results, we know where the error occured.

The Quantum Version: 3-Qubit Bit Flip Code

- Error Correction (or *recovery*):



The Shor Code

7-Qubit Code

Encodes 1 logical qubit using 7 physical qubits:

$$|\bar{0}\rangle = \frac{|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle}{\sqrt{8}}$$

$$|\bar{1}\rangle = \frac{|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle}{\sqrt{8}}$$

$$H^{\otimes 7}|\bar{0}\rangle = \frac{|\bar{0}\rangle + |\bar{1}\rangle}{\sqrt{2}}$$

$$H^{\otimes 7}|\bar{1}\rangle = \frac{|\bar{0}\rangle - |\bar{1}\rangle}{\sqrt{2}}$$

7-Qubit Code

$$|\overline{0}\rangle = \frac{|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle}{\sqrt{8}}$$

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$$H^{\otimes 7}|\overline{0}\rangle = \frac{|\overline{0}\rangle + |\overline{1}\rangle}{\sqrt{2}}$$

$$H^{\otimes 7}|\overline{1}\rangle = \frac{|\overline{0}\rangle - |\overline{1}\rangle}{\sqrt{2}}$$

- Of the 16 bit strings above, any two differ by ≥ 3 bits
- Intuition: therefore a single bit flip can be recovered
  - X error flips bit in $|\overline{0}\rangle, |\overline{1}\rangle$
  - Z error flips bit in $H^{\otimes 7}|\overline{0}\rangle, H^{\otimes 7}|\overline{1}\rangle$

Example recovery for X error in qubit 3

$$|\bar{0}\rangle = \frac{|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle}{\sqrt{8}}$$

$$X^{(3)}\,|\bar{0}\rangle = \frac{|0010000\rangle + |1000101\rangle + |0100011\rangle + |1110110\rangle + |0011111\rangle + |1001010\rangle + |0101100\rangle + |1111001\rangle}{\sqrt{8}}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \left(\vphantom{\begin{matrix} 1\\1\\1\\1\\1 \end{matrix}} \quad \right) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = |3\rangle \text{ (in binary)}$$

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Example recovery for X error

- In fact:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} X^{(i)} |\bar{0}\rangle = |i\rangle \text{ (in binary) for all } i = 1, \ldots, 7$$

- Let H be matrix above. To recover from single X error, apply map

$$|v\rangle \otimes |0\rangle_A \mapsto |v\rangle \otimes |Hv\rangle_A$$

  and measure subsystem A. Result will be index i of bit flip, in binary!
- Also works for logical state 1, and for phase flips.

7-qubit code: Why does it work?

- The kernel of the matrix

$$H = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \in \mathbb{F}_2^{3\times 7}$$

  consists of the 16 bit strings defining $\left|\overline{0}\right\rangle, \left|\overline{1}\right\rangle$
- A bit flip at position i of a vector v adds the ith row of H to Hv (basic linear algebra)
- The ith row of H is i in binary
- Same reasoning for phase flips = bit flips in rotated basis