Quantum Error Correction

Louis Golowich Wenjie Gong Ari Hatzimemos
Dylan Li Dylan Zhou

Physics 160
Harvard University

Final Project Presentation, 13 May 2020

Table of Contents

Introduction

> *"To be an Error and to be Cast out is part of God's Design."*
>
> William Blake

- Noise as a longstanding problem in information processing systems
  - *e.g.*, classical computers, modems, CD players, etc.
  - Noise is still a problem in quantum information
- Key idea: to protect a message against noise, *encode* the message by adding redundant information; even if some information is corrupted, redundancy allows us to *decode* and recover the original message

Project Framework

- Goals:
  - to implement various quantum error-correcting codes
    - we chose the 3-qubit, 9-qubit, 7-qubit codes
  - to analyze and compare their performances
    - *when are they effective?*
    - *when should we use error-correcting codes?*
- Tools:
  - Python's Qiskit package
  - IBM's quantum machines

3-Qubit Codes: Classical Inspiration

Classical Error Correction

- Encoding by *repetition codes*:

$$0 \to 000$$
$$1 \to 111.$$

- Decoding by *majority voting*:

$$\textit{Ex.:} \quad 001 \to 0.$$

- Analysis: Let p be the probability that a bit is flipped. This method fails when 2 or more bits are flipped, which occurs with probability $3p^2(1-p)+p^3$, so the probability of error is $p_e = 3p^2 - 2p^3$. Then this method is preferred when $p_e < p$, or $p < 1/2$.

Noisy Channels: The Bit Flip Channel

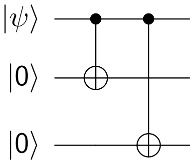
- One model for noise is the *bit flip channel* (analogous to classical channel).
- The bit flip channel flips qubits with probability p and leaves them untouched with probability $1 - p$.
- Equivalent to applying X gate with probability p.
- We protect qubits from this channel with the *bit flip code*.

3-Qubit Bit Flip Code: Encoding Logical Bits

- The goal is to correct bit flip errors.
- Encoding:

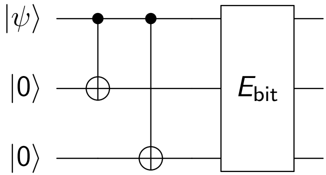
$$
\begin{aligned}
|0\rangle &\rightarrow |0_L\rangle \equiv |000\rangle \\
|1\rangle &\rightarrow |1_L\rangle \equiv |111\rangle \, .
\end{aligned}
$$

- Encoding circuit for 3-qubit bit flip code:



3-Qubit Bit Flip Code: Detecting Errors

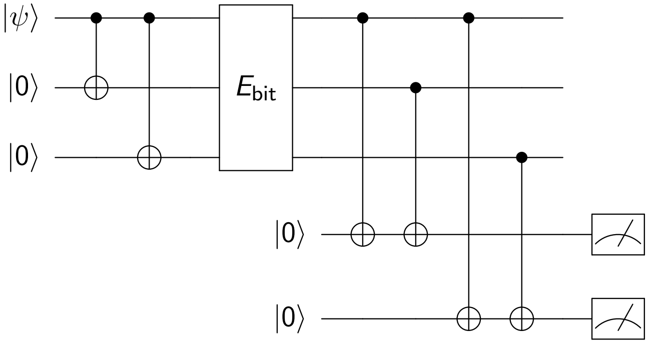
- Suppose there is a bit flip error after encoding:



- Error Detection (or *syndrome diagnosis*):
  - we would like to determine which, if any, of the qubits have been corrupted
  - four error syndromes: no error, bit flip on qubit one, bit flip on qubit two, bit flip on qubit three

3-Qubit Bit Flip Code: Detecting Errors

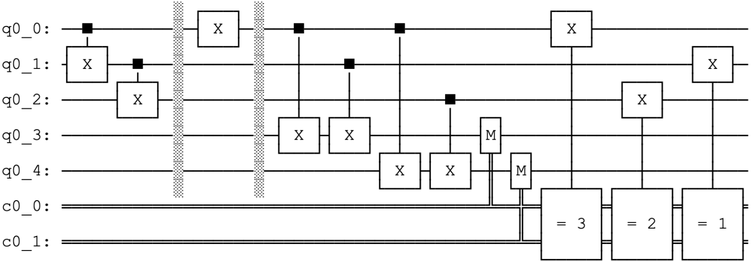
- We can diagnose the syndrome using two ancillary qubits:



- Based on measurement results, we know where the error occured.

3-Qubit Bit Flip Code: Correcting Errors

- Complete circuit for error correction (or *recovery*):



Analyzing the Bit Flip Code: Simulation

- Let's look at the performance of the 3-qubit bit flip code against bit flip channels of varying error probabilities p.

Analyzing the Bit Flip Code: Simulation

- Let's look at the performance of the 3-qubit bit flip code against bit flip channels of varying error probabilities p.
- Setup:
  1. encode a single qubit in state $|0\rangle$ into a logical state $|0_L\rangle = |000\rangle$
  2. create a bit flip channel which adds X gates with probability p
  3. run error correcting code
  4. measure final state

Analyzing the Bit Flip Code: Simulation

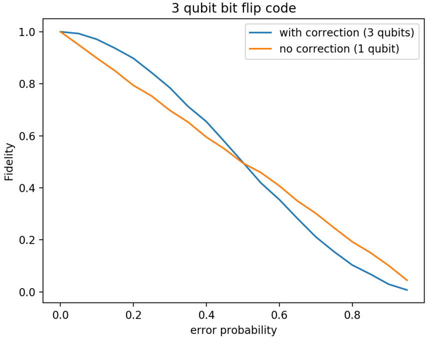
- Let's look at the performance of the 3-qubit bit flip code against bit flip channels of varying error probabilities p.
- Setup:
  1. encode a single qubit in state $|0\rangle$ into a logical state $|0_L\rangle = |000\rangle$
  2. create a bit flip channel which adds X gates with probability p
  3. run error correcting code
  4. measure final state
- We can calculate the accuracy of the error correcting code for a given p by repeating many times and taking the number of times we measure a correct final state $|000\rangle$ and dividing by the total number of trials.

Analyzing the Bit Flip Code: Simulation

- Let's look at the performance of the 3-qubit bit flip code against bit flip channels of varying error probabilities p.
- Setup:
  1. encode a single qubit in state $|0\rangle$ into a logical state $|0_L\rangle = |000\rangle$
  2. create a bit flip channel which adds X gates with probability p
  3. run error correcting code
  4. measure final state
- We can calculate the accuracy of the error correcting code for a given p by repeating many times and taking the number of times we measure a correct final state $|000\rangle$ and dividing by the total number of trials.
- We can compare this to the accuracy of a single qubit (without encoding or error correction) that goes through a bit flip channel with the same p to see when error correction is effective.

Analyzing the Bit Flip Code: Simulation

- Ran tests on Qiskit's simulator
- Probability p ranging from 0 to 1; 10000 trials for each p



- Observe crossover point at $p = 0.5$.
- For $p < 0.5$, error correcting code performs better than a single qubit with no correction.

Noisy Channels: Phase Flip Channel

- Another quantum channel is the *phase flip* error model.
- With probability p the relative phase of states $|0\rangle$ and $|1\rangle$ is flipped, with probability $1 - p$ it is left alone.
- Equivalent to applying Z operator with probability p.
- We fight this channel with the *phase flip code*.

3-Qubit Phase Flip Code

- No classical analog, but it is easy to turn the phase flip channel into a bit flip channel.
- Use x-basis for encoding:

$$|0\rangle \to |0_L\rangle \equiv |+++\rangle$$
$$|1\rangle \to |1_L\rangle \equiv |---\rangle .$$

- Phase flip Z acts as bit flip for this encoding!

The Shor Code

- Can we protect against *arbitrary* errors?

The Shor Code

- Can we protect against *arbitrary* errors?
- Yes! $\longrightarrow$ The *Shor code*

The Shor Code: Encoding

- By combining the 3-qubit phase flip and bit flip codes, the Shor code protects against arbitrary errors.
- The result is a 9-qubit code with codewords

$$|0\rangle \rightarrow |0_L\rangle \equiv \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \rightarrow |1_L\rangle \equiv \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}.$$

7-Qubit Code

Encodes 1 logical qubit using 7 physical qubits:

$$|\bar{0}\rangle = \frac{|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle}{\sqrt{8}}$$

$$|\bar{1}\rangle = \frac{|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle}{\sqrt{8}}$$

$$H^{\otimes 7}|\bar{0}\rangle = \frac{|\bar{0}\rangle + |\bar{1}\rangle}{\sqrt{2}}$$

$$H^{\otimes 7}|\bar{1}\rangle = \frac{|\bar{0}\rangle - |\bar{1}\rangle}{\sqrt{2}}$$

7-Qubit Code

$$|\bar{0}\rangle = \frac{|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle}{\sqrt{8}}$$

$$|\bar{1}\rangle = \frac{|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle}{\sqrt{8}}$$

$$H^{\otimes 7}|\bar{0}\rangle = \frac{|\bar{0}\rangle + |\bar{1}\rangle}{\sqrt{2}}$$

$$H^{\otimes 7}|\bar{1}\rangle = \frac{|\bar{0}\rangle - |\bar{1}\rangle}{\sqrt{2}}$$

- Of the 16 bit strings above, any two differ by ≥ 3 bits
- Intuition: therefore a single bit flip can be recovered
  - X error flips bit in $|\bar{0}\rangle, |\bar{1}\rangle$
  - Z error flips bit in $H^{\otimes 7}|\bar{0}\rangle, H^{\otimes 7}|\bar{1}\rangle$

Example recovery for X error in qubit 3

$$|\bar{0}\rangle = \frac{|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle}{\sqrt{8}}$$

$$X^{(3)}\,|\bar{0}\rangle = \frac{|0010000\rangle + |1000101\rangle + |0100011\rangle + |1110110\rangle + |0011111\rangle + |1001010\rangle + |0101100\rangle + |1111001\rangle}{\sqrt{8}}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \\ \\ \\ \\ \\ \\ \\ \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = |3\rangle \ \text{(in binary)}$$

Example recovery for X error in qubit 3

$$|\bar{0}\rangle = \frac{|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle}{\sqrt{8}}$$

$$X^{(3)}\,|\bar{0}\rangle = \frac{|0010000\rangle + |1000101\rangle + |0100011\rangle + |1110110\rangle + |0011111\rangle + |1001010\rangle + |0101100\rangle + |1111001\rangle}{\sqrt{8}}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \\ \\ \\ \\ \\ \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = |3\rangle \text{ (in binary)}$$

Example recovery for X error in qubit 3

$$|\bar{0}\rangle = \frac{|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle}{\sqrt{8}}$$

$$X^{(3)}|\bar{0}\rangle = \frac{|0010000\rangle + |1000101\rangle + |0100011\rangle + |1110110\rangle + |0011111\rangle + |1001010\rangle + |0101100\rangle + |1111001\rangle}{\sqrt{8}}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \\ \\ \\ \\ \\ \\ \\ \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = |3\rangle \text{ (in binary)}$$

Example recovery for X error

- In fact:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} X^{(i)} \left|\bar{0}\right\rangle = |i\rangle \ \text{(in binary) for all } i = 1, \dots, 7$$

- Let H be matrix above. To recover from single X error, apply map

$$|v\rangle \otimes |0\rangle_A \mapsto |v\rangle \otimes |Hv\rangle_A$$

  and measure subsystem A. Result will be index i of bit flip, in binary!
- Also works for logical state 1, and for phase flips.

7-qubit code: Why does it work?

- The kernel of the matrix

$$H = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \in \mathbb{F}_2^{3\times 7}$$

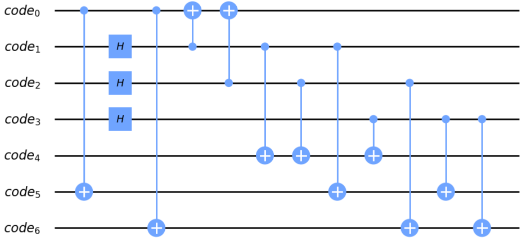
  consists of the 16 bit strings defining $\left|\overline{0}\right\rangle, \left|\overline{1}\right\rangle$
- A bit flip at position i of a vector v adds the ith row of H to Hv (basic linear algebra)
- The ith row of H is i in binary
- Same reasoning for phase flips = bit flips in rotated basis

7-qubit code: Initialization

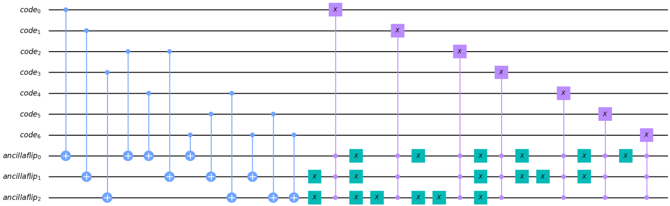
$$|\bar{0}\rangle = \frac{|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle}{\sqrt{8}}$$

$$|\bar{1}\rangle = \frac{|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle}{\sqrt{8}}$$

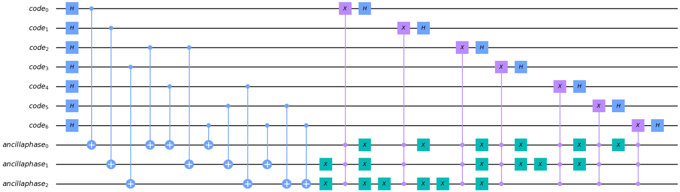
$|\psi\rangle \rightarrow$
$|0\rangle \rightarrow$
$|0\rangle \rightarrow$
$|0\rangle \rightarrow$
$|0\rangle \rightarrow$
$|0\rangle \rightarrow$
$|0\rangle \rightarrow$



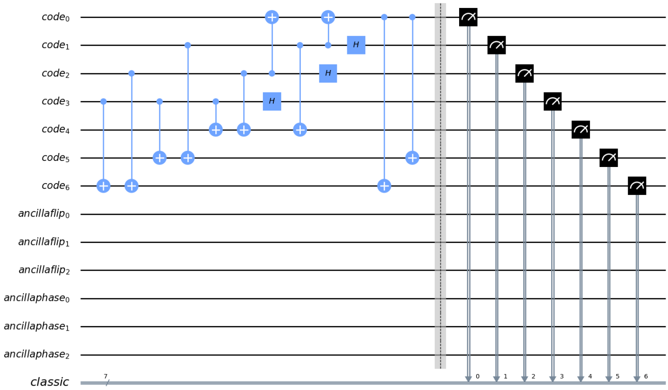
7-qubit code: Flip correction



7-qubit code: Phase correction

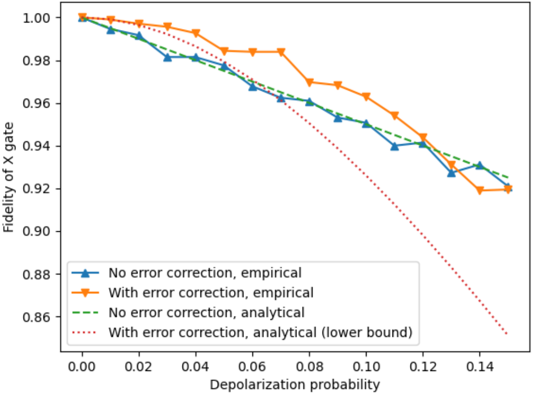


7-qubit code: Measurement



7-qubit code: Fidelity of X Gate under Depolarization

Encode $\lvert\overline{0}\rangle$	X	Correct flip	Correct phase	Decode, measure



7-qubit code: Fidelity of X Gate under Depolarization

Encode $|\bar{0}\rangle$ | X | Correct flip | Correct phase | Decode, measure