

Quantum Error Correction

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"To be an Error and to be Cast out is part of God's Design."

William Blake

- Noise as a longstanding problem in information processing systems
 - e.g., classical computers, modems, CD players, etc.
 - Noise is still a problem in quantum information
- Key idea: to protect a message against noise, *encode* the message by adding redundant information; even if some information is corrupted, redundancy allows us to *decode* and recover the original message

Project Framework

- Goals:
 - to implement various quantum error-correcting codes
 - we chose the 3-qubit, 9-qubit, 7-qubit codes
 - to analyze and compare their performances
 - *when are they effective?*
 - *when should we use error-correcting codes?*
- Tools:
 - Python's Qiskit package
 - IBM's quantum machines

3-Qubit Codes: Classical Inspiration

Classical Error Correction

- Encoding by *repetition codes*:

$$0 \rightarrow 000$$

$$1 \rightarrow 111.$$

- Decoding by *majority voting*:

$$\text{Ex.: } 001 \rightarrow 0.$$

- Analysis: Let p be the probability that a bit is flipped. This method fails when 2 or more bits are flipped, which occurs with probability $3p^2(1-p) + p^3$, so the probability of error is $p_e = 3p^2 - 2p^3$. Then this method is preferred when $p_e < p$, or $p < 1/2$.

Noisy Channels: The Bit Flip Channel

- One model for noise is the *bit flip channel* (analogous to classical channel).
- The bit flip channel flips qubits with probability p and leaves them untouched with probability $1 - p$.
- Equivalent to applying X gate with probability p .
- We protect qubits from this channel with the *bit flip code*.

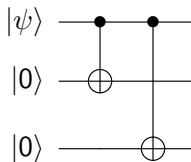
3-Qubit Bit Flip Code: Encoding Logical Bits

- The goal is to correct bit flip errors.
- Encoding:

$$|0\rangle \rightarrow |0_L\rangle \equiv |000\rangle$$

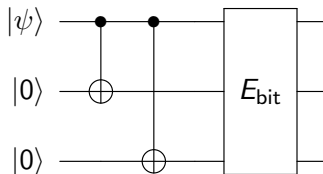
$$|1\rangle \rightarrow |1_L\rangle \equiv |111\rangle.$$

- Encoding circuit for 3-qubit bit flip code:



3-Qubit Bit Flip Code: Detecting Errors

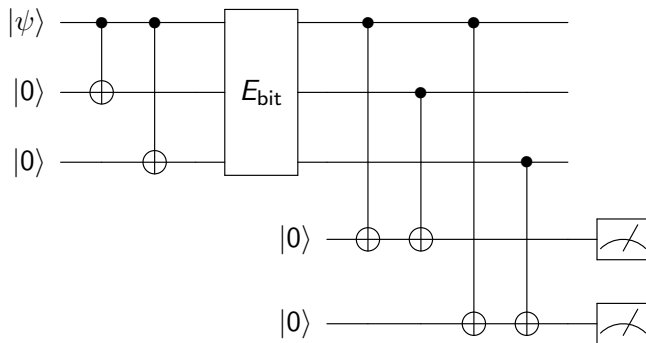
- Suppose there is a bit flip error after encoding:



- Error Detection (or *syndrome diagnosis*):
 - we would like to determine which, if any, of the qubits have been corrupted
 - four error syndromes: no error, bit flip on qubit one, bit flip on qubit two, bit flip on qubit three

3-Qubit Bit Flip Code: Detecting Errors

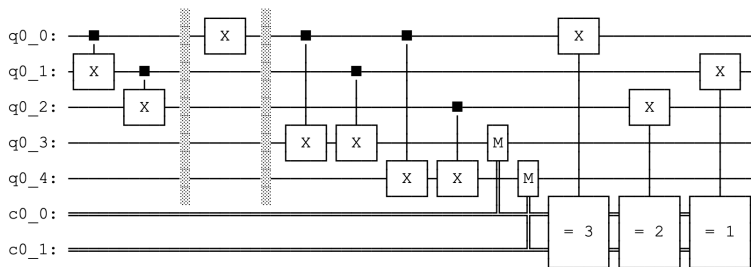
- We can diagnose the syndrome using two ancillary qubits:



- Based on measurement results, we know where the error occurred.

3-Qubit Bit Flip Code: Correcting Errors

- Complete circuit for error correction (or *recovery*):



Analyzing the Bit Flip Code: Simulation

- Let's look at the performance of the 3-qubit bit flip code against bit flip channels of varying error probabilities p .

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 - 2 create a bit flip channel which adds X gates with probability p
 - 3 run error correcting code
 - 4 measure final state

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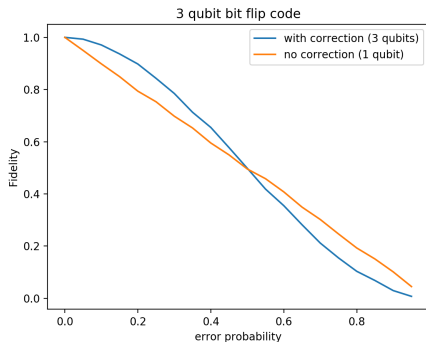
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- We can calculate the accuracy of the error correcting code for a given p by repeating many times and taking the number of times we measure a correct final state $|000\rangle$ and dividing by the total number of trials.
- We can compare this to the accuracy of a single qubit (without encoding or error correction) that goes through a bit flip channel with the same p to see when error correction is effective.

Analyzing the Bit Flip Code: Simulation

- Ran tests on Qiskit's simulator
- Probability p ranging from 0 to 1; 10000 trials for each p



- Observe crossover point at $p = 0.5$.
- For $p < 0.5$, error correcting code performs better than a single qubit with no correction.

Noisy Channels: Phase Flip Channel

- Another quantum channel is the *phase flip* error model.
- With probability p the relative phase of states $|0\rangle$ and $|1\rangle$ is flipped, with probability $1 - p$ it is left alone.
- Equivalent to applying Z operator with probability p .
- We fight this channel with the *phase flip code*.

3-Qubit Phase Flip Code

- No classical analog, but it is easy to turn the phase flip channel into a bit flip channel.
- Use x -basis for encoding:

$$\begin{aligned}|0\rangle &\rightarrow |0_L\rangle \equiv |+++ \rangle \\ |1\rangle &\rightarrow |1_L\rangle \equiv |-- - \rangle .\end{aligned}$$

- Phase flip Z acts as bit flip for this encoding!

The Shor Code

- Can we protect against *arbitrary* errors?
- Yes! \longrightarrow The *Shor code*
- The Shor code combines 3-qubit phase flip and bit flip codes.

7-Qubit Code

Encodes 1 logical qubit using 7 physical qubits:

$$|\bar{0}\rangle = \frac{|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle}{\sqrt{8}}$$

$$|\bar{1}\rangle = \frac{|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle}{\sqrt{8}}$$

$$H^{\otimes 7} |\bar{0}\rangle = \frac{|\bar{0}\rangle + |\bar{1}\rangle}{\sqrt{2}}$$

$$H^{\otimes 7} |\bar{1}\rangle = \frac{|\bar{0}\rangle - |\bar{1}\rangle}{\sqrt{2}}$$

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- Of the 16 bit strings above, any two differ by ≥ 3 bits
- Intuition: therefore a single bit flip can be recovered
 - X error flips bit in $|\bar{0}\rangle, |\bar{1}\rangle$
 - Z error flips bit in $H^{\otimes 7} |\bar{0}\rangle, H^{\otimes 7} |\bar{1}\rangle$

Example recovery for X error in qubit 3

$$|\bar{0}\rangle = \frac{|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle}{\sqrt{8}}$$

$$X^{(3)} |\bar{0}\rangle = \frac{|00\textcolor{red}{1}0000\rangle + |10\textcolor{red}{0}0101\rangle + |01\textcolor{red}{0}0011\rangle + |11\textcolor{red}{1}0110\rangle + |00\textcolor{red}{1}1111\rangle + |10\textcolor{red}{0}1010\rangle + |01\textcolor{red}{0}1100\rangle + |11\textcolor{red}{1}1001\rangle}{\sqrt{8}}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = |3\rangle \text{ (in binary)}$$

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Example recovery for X error

- In fact:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} X^{(i)} |\bar{0}\rangle = |i\rangle \text{ (in binary) for all } i = 1, \dots, 7$$

- Let H be matrix above. To recover from single X error, apply map

$$|v\rangle \otimes |0\rangle_A \mapsto |v\rangle \otimes |Hv\rangle_A$$

and measure subsystem A . Result will be index i of bit flip, in binary!

- Also works for logical state 1, and for phase flips.

7-qubit code: Why does it work?

- The kernel of the matrix

$$H = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \in \mathbb{F}_2^{3 \times 7}$$

consists of the 16 bit strings defining $|\bar{0}\rangle, |\bar{1}\rangle$

- A bit flip at position i of a vector v adds the i th row of H to Hv (basic linear algebra)
- The i th row of H is i in binary
- Same reasoning for phase flips = bit flips in rotated basis

7-qubit code: Initialization

$$|\bar{0}\rangle = \frac{|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle}{\sqrt{8}}$$

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$|\psi\rangle \rightarrow$

$|0\rangle \rightarrow$

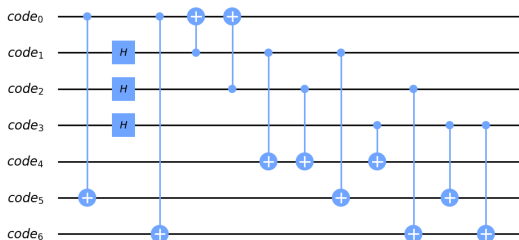
$|0\rangle \rightarrow$

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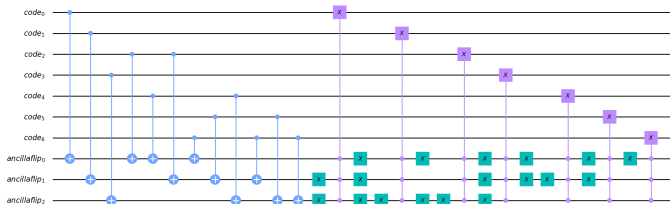
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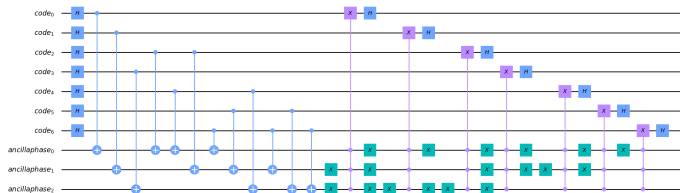
$|0\rangle \rightarrow$



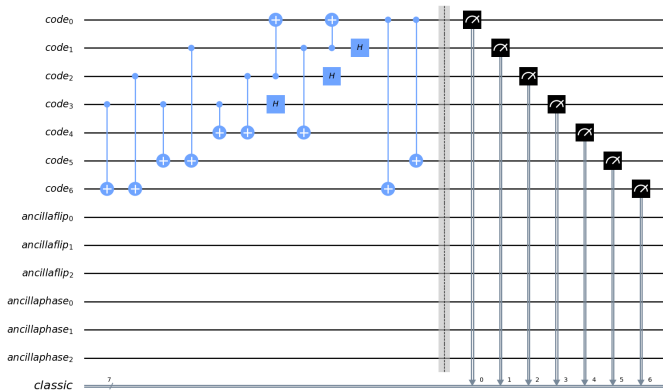
7-qubit code: Flip correction



7-qubit code: Phase correction



7-qubit code: Measurement



7-qubit code: Fidelity of X Gate under Depolarization

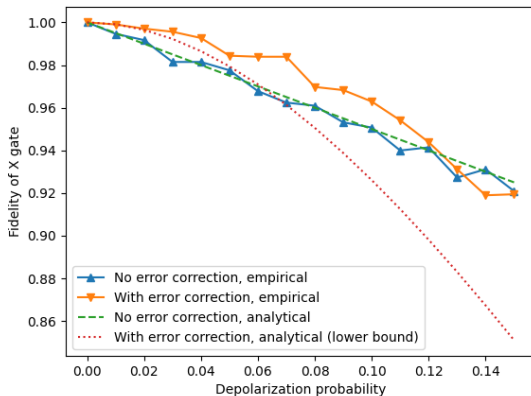
Encode $|\bar{0}\rangle$

X

Correct flip

Correct phase

Decode, measure



7-qubit code: Fidelity of X Gate under Depolarization

Encode $|\bar{0}\rangle$

X

Correct flip

Correct phase

Decode, measure

