Lesson 01: Geometry of Linear Equations and Basics of Linear Algebra

Learning goals

- ▷ Understand the three "pictures" of a system $Ax = b$: column, row, and matrix view.
- ▷ Distinguish between the cases: unique solution / infinitely many solutions / no solution.
- ▷ Introduce: rank of a matrix, column space $\text{col}(A)$, row space $\text{row}(A)$, nullspace $\text{null}(A)$.
- ▷ First experiments with error, sensitivity, and least-squares solutions.

Quick reference

- ∗ **Column picture**: $Ax = b$ means b is a linear combination of columns of A. If $b \in \text{col}(A)$, the system is consistent.
- ∗ **Row picture**: equations are dot products with rows of A, i.e. hyperplanes.
- ∗ **Matrix picture**: $x \mapsto Ax$ is a linear operator transforming vectors.
- ∗ $\text{rank}(A)$ = number of independent columns = $\dim(\text{col}(A))$.
- ∗ $\text{null}(A) = \{x : Ax = 0\}$: family of solutions = one solution + any element of the nullspace.
- ∗ If no solution exists $\Rightarrow$ use least squares (projection of b onto $\text{col}(A)$).

Exercises A – conceptual

A1 For $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, for which $b = (b_1, b_2)^T$ does the system have a solution?

A2 Find a nonzero vector in the nullspace of A above. How many solutions does $Ax = 0$ have?

A3 If A has size $m \times n$ and rank r, what is $\dim(\text{null}(A))$?

A4 For which relations between m, n, r is the system typically: unique / infinite / LS?

Exercises B – computational

B1 For $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$:

(a) compute $\operatorname{rank}(A)$,

(b) give a basis for $\operatorname{col}(A)$,

(c) describe $\operatorname{null}(A)$.

B2 Decide if $Ax = b$ with $b = (1, 2, 3)^T$ is consistent. If not – give a condition on b for solvability.

B3 Show that if $b \notin \operatorname{col}(A)$, the LS solution satisfies $A^T Ax = A^T b$.

Exercises C – R laboratory

C1 **Three pictures**. In R, check whether b is in $\operatorname{col}(A)$ by comparing $\operatorname{rank}(A)$ and $\operatorname{rank}([A|b])$.

C2 **Rank and nullspace**. Use SVD to find a basis for the nullspace of a chosen matrix.

C3 **Inconsistent system**. Construct A, b with no exact solution. Compute the projection of b onto $\operatorname{col}(A)$ and the LS solution.

C4 **Sensitivity**. Generate an ill-conditioned matrix A, perturb b with noise, and compare errors in x.

C5 **Linear regression**. Compare LS solution via pseudoinverse with `lm()` in R.

Mini-project (home assignment)

- Generate A ($m = 40, n = 3$) of full rank and a hidden x^*.
- Add noise to b with different levels σ.
- For each σ: compute the LS solution, the error $\|x(\sigma) - x^*\|$, and save results to `.csv`.
- Write a short report (3–5 sentences) about the impact of noise and $\kappa(A)$.

Discussion questions

D1 Why does the column picture immediately tell whether the system is consistent?

D2 Does large rank always mean stability? What does $\kappa(A)$ really measure?

D3 When do we prefer the exact solution (if it exists), and when LS?

D4 Why is the LS residual $b - \hat{b}$ orthogonal to $\mathrm{col}(A)$?

D5 Which of today's phenomena will reappear with SVD/NMF/CUR on images?

Suggested grading of homework

- Correctness of calculations: 40%.
- Conclusions and interpretation: 40%.
- Repository/report cleanliness: 20%.