Problem 4

System parameters and initial state:

l1 = 15 cm

l2 = 15 cm

l3 = 2 cm

θ1 = 0.93 radians (based on 3/4/5 right triangle)

θ2 = -1.86 radians

θ3 = 0.93 radians

End effector traces a circle of r = 20 cm in 30 seconds

Link 3 is always horizontal

X, Y, and θ of end effector:

X = $l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3)$

Y = $l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3)$

θ = $\theta_1 + \theta_2 + \theta_3$

Create end effector position vector and link angles vector

X vector =

$$\begin{bmatrix} l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ \theta_1 + \theta_2 + \theta_3 \end{bmatrix}$$

θ vector =

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

Create jacobian matrix from X and θ vectors:

J =

$$\begin{bmatrix} -l_1 \sin(\theta_1) - l_2 \sin(\theta_1+\theta_2) - l_3 \sin(\theta_1+\theta_2+\theta_3) & -l_2 \sin(\theta_1+\theta_2) - l_3 \sin(\theta_1+\theta_2+\theta_3) & -l_3 \sin(\theta_1+\theta_2+\theta_3) \\ l_1 \cos(\theta_1) + l_2 \cos(\theta_1+\theta_2) + l_3 \cos(\theta_1+\theta_2+\theta_3) & l_2 \cos(\theta_1+\theta_2) + l_3 \cos(\theta_1+\theta_2+\theta_3) & l_3 \cos(\theta_1+\theta_2+\theta_3) \\ 1 & 1 & 1 \end{bmatrix}$$

Take inverse of Jacobian:

J^{-1} =

$$\begin{bmatrix} \frac{\cos(\theta_1+\theta_2)}{-l_1 \sin(\theta_1)\cos(\theta_1+\theta_2)+l_1\sin(\theta_1+\theta_2)\cos(\theta_1)} & \frac{\sin(\theta_1+\theta_2)}{-l_1 \sin(\theta_1)\cos(\theta_1+\theta_2)+l_1\sin(\theta_1+\theta_2)\cos(\theta_1)} & \frac{-l_3\sin(\theta_1+\theta_2)\cos(\theta_1+\theta_2+\theta_3)+l_3\sin(\theta_1+\theta_2+\theta_3)\cos(\theta_1+\theta_2)}{-l_1 \sin(\theta_1)\cos(\theta_1+\theta_2)+l_1\sin(\theta_1+\theta_2)\cos(\theta_1)} \\ \frac{-l_1\cos(\theta_1)-l_2\cos(\theta_1+\theta_2)}{-l_1 l_2 \sin(\theta_1)\cos(\theta_1+\theta_2)+l_1 l_2\sin(\theta_1+\theta_2)\cos(\theta_1)} & \frac{-l_1\sin(\theta_1)-l_2\sin(\theta_1+\theta_2)}{-l_1 l_2 \sin(\theta_1)\cos(\theta_1+\theta_2)+l_1 l_2\sin(\theta_1+\theta_2)\cos(\theta_1)} & \frac{l_1 l_3\sin(\theta_1)\cos(\theta_1+\theta_2+\theta_3)-l_1 l_3\sin(\theta_1+\theta_2+\theta_3)\cos(\theta_1)+l_2 l_3\sin(\theta_1+\theta_2)\cos(\theta_1+\theta_2+\theta_3)-l_2 l_3\sin(\theta_1+\theta_2+\theta_3)\cos(\theta_1+\theta_2)}{-l_1 l_2 \sin(\theta_1)\cos(\theta_1+\theta_2)+l_1 l_2\sin(\theta_1+\theta_2)\cos(\theta_1)} \\ -\frac{\cos(\theta_1)}{l_2\sin(\theta_1)\cos(\theta_1+\theta_2)-l_2\sin(\theta_1+\theta_2)\cos(\theta_1)} & -\frac{\sin(\theta_1)}{l_2\sin(\theta_1)\cos(\theta_1+\theta_2)-l_2\sin(\theta_1+\theta_2)\cos(\theta_1)} & \frac{l_2\sin(\theta_1)\cos(\theta_1+\theta_2)-l_2\sin(\theta_1+\theta_2)\cos(\theta_1)+l_3\sin(\theta_1)\cos(\theta_1+\theta_2+\theta_3)-l_3\sin(\theta_1+\theta_2+\theta_3)\cos(\theta_1)}{l_2\sin(\theta_1)\cos(\theta_1+\theta_2)-l_2\sin(\theta_1+\theta_2)\cos(\theta_1)} \end{bmatrix}$$

Determine x and y input

Polar coordinates:

Circle of radius 20, therefore $r = 20$

Circle completes revolution of 2π radians in 30 seconds, therefore $\phi = (\pi t/15)$ where t is time in seconds

Convert polar coordinates to cartesian:

$x = rcos(\phi) = 20cos(\pi t/15)$

$y = rsin(\phi) = 20sin(\pi t/15)$

Take derivative to get $\dot{x}$ and $\dot{y}$

$\dot{x} = -\frac{4\pi \sin\left(\frac{\pi t}{15}\right)}{3}$ $\dot{y} = \frac{4\pi \cos\left(\frac{\pi t}{15}\right)}{3}$

Create velocity vector $\dot{x}$, $\dot{y}$, and θdot (θdot = 0 because link 3 is always horizontal, thus end effector angle to initial reference frame is always zero)

Velocity vector $\dot{\mathbf{x}}$= $\begin{bmatrix} -\frac{4\pi \sin\left(\frac{\pi t}{15}\right)}{3} \\ \frac{4\pi \cos\left(\frac{\pi t}{15}\right)}{3} \\ 0 \end{bmatrix}$

Multiply inverse jacobian by veclocity vector to get angle velocities vector
Angle velocities vector **θdot** =

$$\begin{bmatrix} -\frac{4\pi \sin\left(\frac{\pi t}{15}\right)\cos\left(\theta_1+\theta_2\right)}{3(-l_1\sin\left(\theta_1\right)\cos\left(\theta_1+\theta_2\right)+l_1\sin\left(\theta_1+\theta_2\right)\cos\left(\theta_1\right))} + \frac{4\pi \sin\left(\theta_1+\theta_2\right)\cos\left(\frac{\pi t}{15}\right)}{3(-l_1\sin\left(\theta_1\right)\cos\left(\theta_1+\theta_2\right)+l_1\sin\left(\theta_1+\theta_2\right)\cos\left(\theta_1\right))} \\ \frac{4\pi(-l_1\sin\left(\theta_1\right)-l_2\sin\left(\theta_1+\theta_2\right))\cos\left(\frac{\pi t}{15}\right)}{3(-l_1l_2\sin\left(\theta_1\right)\cos\left(\theta_1+\theta_2\right)+l_1l_2\sin\left(\theta_1+\theta_2\right)\cos\left(\theta_1\right))} - \frac{4\pi(-l_1\cos\left(\theta_1\right)-l_2\cos\left(\theta_1+\theta_2\right))\sin\left(\frac{\pi t}{15}\right)}{3(-l_1l_2\sin\left(\theta_1\right)\cos\left(\theta_1+\theta_2\right)+l_1l_2\sin\left(\theta_1+\theta_2\right)\cos\left(\theta_1\right))} \\ -\frac{4\pi \sin\left(\theta_1\right)\cos\left(\frac{\pi t}{15}\right)}{3(l_2\sin\left(\theta_1\right)\cos\left(\theta_1+\theta_2\right)-l_2\sin\left(\theta_1+\theta_2\right)\cos\left(\theta_1\right))} + \frac{4\pi \sin\left(\frac{\pi t}{15}\right)\cos\left(\theta_1\right)}{3(l_2\sin\left(\theta_1\right)\cos\left(\theta_1+\theta_2\right)-l_2\sin\left(\theta_1+\theta_2\right)\cos\left(\theta_1\right))} \end{bmatrix}$$

Where the elements correspond to θdot1, θdot2, and θdot3, respectively.

Initial angle velocity values can be calculated from initial link angles, link lengths, and t = 0.
Angle values over time can be calculated from
$\theta[i] \ = \ \theta[i-1] \ + \ \theta dot[i-1](\Delta t)$ where Δt is the chosen time increment.

Angle velocities over time can be calculated by substituting time[i], $\theta 1[i]$, and $\theta 2[i]$ into the the θdot equations.