MAT 460 – Numerical Differential Equations – Solutions to Assignment 4 (E,H)

Problem 1

E a) Hornerschema for $p(x) = 5x^4 + 3x^3 - 30x^2 + 7x + 8, \quad x_0 = 2$:

5	3	-30	7	8
0	10	26	-8	-2
5	13	-4	-1	6

$\Rightarrow \qquad p(x) = (5x^3 + 13x^2 - 4x - 1)(x-2) + 6$, $p(2) = 6$.

Problem 2

1) General solution: We write the system of equations

$$f_j = p(x_j) = a_3 x_j^3 + a_2 x_j^2 + a_1 x_j + a_0.$$

in matrix-vector form:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}.$$

This matrix is called Vandermonde-matrix.

The Lagrange-representation is

$$p(x) = f_1\, L_1(x) + f_2\, L_2(x) + f_3\, L_3(x) + f_4\, L_4(x)$$

with the Lagrange-basis polynomials

$$\begin{aligned} L_1(x) &= \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} \\ L_2(x) &= \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} \\ L_3(x) &= \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} \\ L_4(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} \end{aligned}$$

Problem 3

E a) From the table of the divided differences

x_j	f_j			
3	6	2	5	-7
1	2	-13	40	
0	15	-133		
-2	281			

$\Rightarrow$ we obtain the interpolation polynomial

$$p(x) = 6 + 2(x-3) + 5(x-3)(x-1) - 7(x-3)(x-1)x\,.$$

E b) The only difference between this problem and problem E a) is that we have one additional interpolation point. So we add entries on the diagonal below the table from E a) (see the lecture slides):

x_j	f_j				
3	6	2	5	-7	$3/2$
1	2	-13	40	-4	
0	15	-133	24		
-2	281	-13			
5	190				

$\Rightarrow$ and obtain the interpolation polynomial

$$p(x) = 6 + 2(x-3) + 5(x-3)(x-1) - 7(x-3)(x-1)x + \frac{3}{2}(x-3)(x-1)x(x+2)\,.$$

E c) From the table of the divided differences

x_j	f_j			
-2	281	-133	40	-7
0	15	-13	5	
1	2	2		
3	6			

$\Rightarrow$ we obtain the interpolation polynomial

$$p(x) = 281 - 133(x+2) + 40(x+2)x - 7(x+2)x(x-1).$$

This polynomial is the same as in E a), in a different notation.

H b)

x_j	f_j				
4	7	-11	-3	6	$5/2$
2	29	1	-33	$-3/2$	
0	27	100	$-63/2$		
-1	-73	$137/2$			
1	64				

$\Rightarrow$ The interpolation polynomial is:

$$p(x) = 7 - 11(x-4) - 3(x-4)(x-2) + 6(x-4)(x-2)x + \frac{5}{2}(x-4)(x-2)x(x+1)\,.$$

To compute the solution of H a) we drop the last term:

$$p(x) = 7 - 11(x-4) - 3(x-4)(x-2) + 6(x-4)(x-2)x\,.$$