MAT 460 – Numerical Differential Equations – Solutions to Assignment 2 (E)

Problem 1: Let $x = [x_1 \;\; x_2 \;\; \ldots \;\; x_n]^\top \in \mathbb{R}^n$. We need to show that $x^\top Ax > 0$ if $x \neq 0$. To prove this we consider a polynomial with coefficients x_j:

$$p_x(t) = x_1 + x_2\,t + x_3\,t^2 + \ldots + x_n\,t^{n-1} = \sum_{j=1}^{n} x_j\,t^{j-1}, \qquad t \in \mathbb{R}.$$

The square of p_x is

$$p_x^2(t) \;=\; p_x(t)\,p_x(t) \;=\; \left(\sum_{j=1}^{n} x_j\,t^{j-1}\right)\left(\sum_{k=1}^{n} x_k\,t^{k-1}\right) \;=\; \sum_{j=1}^{n}\sum_{k=1}^{n} t^{j+k-2} x_j x_k.$$

If $x \neq 0$, then $p_x(t)$ is not the zero polynomial. So $p_x^2(t) > 0$ for all $t \in \; R$ with possibly finitely many exceptions. We have

$$0 < \int_0^1 p_x(t)^2\,dt = \int_0^1 \sum_{j=1}^{n}\sum_{k=1}^{n} t^{j+k-2} x_j x_k\,dt = \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\int_0^1 t^{j+k-2}\,dt\right) x_j x_k = \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{1}{j+k-1}\,x_j x_k = x^\top Ax.$$

Problem 2: Let

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}.$$

Then

$$\begin{aligned}
\|Ax\|_1 &= \|y\|_1 \\
&= |y_1| + |y_2| \\
&= |ax_1 + bx_2| + |cx_1 + dx_2| \\
&\leq |a|\,|x_1| + |b|\,|x_2| + |c|\,|x_1| + |d|\,|x_2| \\
&= (|a| + |c|)|x_1| + (|b| + |d|)\,|x_2| \\
&= \leq \max\{\underbrace{|a| + |c|}_{C_1(A)}, \underbrace{|b| + |d|}_{C_2(A)}\}\,(\underbrace{|x_1| + |x_2|}_{\|x\|_1}).
\end{aligned}$$

$\Rightarrow$

$$\frac{\|Ax\|_1}{\|x\|_1} \leq \max\{C_1(A), C_2(A)\} \qquad \text{for all } x \neq 0. \qquad (*)$$

$\Rightarrow$

$$\|A\|_1 \leq \max\{C_1(A), C_2(A)\}$$

We need to find x such that we have equality in $(*)$.
Let e. g. $C_1(A) \geq C_2(A)$, then $C_1(A) = \max\{C_1(A), C_2(A)\}$. Set $x = [1 \;\; 0]^T$. Then $\frac{\|Ax\|_1}{\|x\|_1} = C_1(A)$. Combined with $(*)$ this gives $\|A\|_1 = C_1(A)$.

Problem 3: Ea) Lets first consider the 2-norm:

$$A = \begin{bmatrix} 4 & -13 \\ 8 & -1 \end{bmatrix} \qquad \Rightarrow \qquad A^T A = \begin{bmatrix} 4 & 8 \\ -13 & -1 \end{bmatrix}\begin{bmatrix} 4 & -13 \\ 8 & -1 \end{bmatrix} = \begin{bmatrix} 80 & -60 \\ -60 & 170 \end{bmatrix}.$$

We compute the eigenvalues of A^TA:
$\mathrm{trace}(A^TA) = 80 + 170 = 250$, $\det(A^TA) = 80 * 170 - (-60) * (-60) = 13600 - 3600 = 10000$.

$$\lambda_{1,2} = \frac{\mathrm{trace}(A^TA)}{2} \pm \sqrt{\frac{\mathrm{trace}(A^TA)^2}{4} - \det(A^TA)} = 125 \pm \sqrt{15625 - 1000} = 125 \pm 75 = \begin{cases} 200 \\ 50 \end{cases}$$

The singular values of A are the square roots of the eigenvalues of A^TA:

$$\sigma_{1,2} = \begin{cases} \sqrt{200} = 10\sqrt{2}, \\ \sqrt{50} = 5\sqrt{2}. \end{cases}$$

$\Rightarrow$

$$\begin{aligned} \|A\|_2 &= \max\{\sigma_1, \sigma_2\} = 10\sqrt{2}, \\ \|A^{-1}\|_2 &= 1/\min\{\sigma_1, \sigma_2\} = 1/(5\sqrt{2}), \\ \mathrm{cond}_2(A) &= \|A\|_2\, \|A^{-1}\|_2 = 2. \end{aligned}$$

Now lets consider the other norms: We have

$$\det(A) = 4 * (-1) - 8 * (-13) = 100, \qquad A^{-1} = \frac{1}{100}\begin{bmatrix} -1 & 13 \\ -8 & 4 \end{bmatrix} = \begin{bmatrix} -0.01 & 0.13 \\ -.08 & 0.04 \end{bmatrix}.$$

And from this it follows

$$\begin{aligned} \|A\|_\infty &= \text{row sum norm} = \max\{4 + 13, 8 + 1\} = 17 \\ \|A^{-1}\|_\infty &= \text{row sum norm} = \max\{0.01 + 0.13, 0.08 + 0.04\} = 0.14 \\ \mathrm{cond}_\infty(A) &= 17 * 0.14 = 2.38. \\ \|A\|_1 &= \text{column sum norm} = \max\{4 + 8, 13 + 1\} = 14 \\ \|A^{-1}\|_1 &= \text{column sum norm} = \max\{0.01 + 0.08, 0.13 + 0.04\} = 0.17 \\ \mathrm{cond}_1(A) &= 14 * 0.17 = 2.38. \end{aligned}$$

Eb) Next we compute the 2-norm:

$$A = \begin{bmatrix} 0 & -2 & 0 \\ 5 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \Rightarrow \qquad A^\top A = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

The singular values are the square roots of the eigenvalues of A^TA, so $\sigma_1(A) = 5, \sigma_2(A) = 3, \sigma_3(A) = 2$, $\|A\|_2 = \sigma_1(A) = 5$, $\|A^{-1}\|_2 = 1/\sigma_3(A) = 1/2$, $\mathrm{cond}_2(A) = \|A\|_2\|A^{-1}\|_2 = \sigma_1(A)/\sigma_3(A) = 5/2$.

Now the other norms:

$$A^{-1} = \begin{bmatrix} 0 & 1/5 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

Therefore:

$$\begin{aligned} \|A\|_\infty &= \text{row sum norm} = 5, \\ \|A^{-1}\|_\infty &= \text{row sum norm} = 1/2, \\ \mathrm{cond}_\infty(A) &= 5/2, \\ \|A\|_1 &= \text{column sum norm} = 5, \\ \|A^{-1}\|_1 &= \text{column sum norm} = 1/2, \\ \mathrm{cond}_1(A) &= 5/2. \end{aligned}$$

Ec) All the quantities we want to compute have the value 1 for the identity matrix.

Problem 4:

The value of the force applied by the spring k to its endpoints is

$$F_k = \text{stiffness} * \text{stretching of spring} = s_k * (u_k - u_{k-1}),$$

where $u_0 = 0$ (the first spring is fixed at the top). Let $1 \leq k \leq n-1$. Three forces act on the mass k: The force from the spring above which pulls the mass up, the force from the spring below which pulls the mass down, and the gravitational force f_k. The spring force which pulls up is equal to the sum of the two forces which pull down, since the masses are not in motion (by assumption). We get the following equations.

$$s_k(u_k - u_{k-1}) = s_{k+1}\,(u_{k+1} - u_k) + f_k, \qquad k = 2, \ldots n-1.$$

Reordered:

$$-s_k\, u_{k-1} + (s_k + s_{k+1})\, u_k - s_{k+1}\, u_{k+1} = f_k, \qquad k = 2, \ldots n-1.$$

These are the rows in the middle of the matrix in the linear equation system on assignment 2. For $k = 1$ we set $u_0 = 0$:

$$(s_1 + s_2)\, u_1 - s_2\, u_2 = f_1.$$

For $k = n$ we have no spring pulling down, so we set $s_{n+1} = 0$ in the general formula. This gives:

$$-s_n u_{n-1} + s_n\, u_n = f_n.$$

Which completes all equations.

We want to show that the stiffness matrix is positive definite.The stiffness matrix S can be split into n matrices S_j, $j = 1, \ldots, n$ where each of the matrices has only one entry s_j. We have

$$S = S_1 + S_2 + \ldots + S_n,$$

where

$$S_1 = \operatorname{diag}(s_1, 0, \ldots, 0) \in \mathbb{R}^{n \times n},$$

and for $j \geq 2$,

$$S_j = \operatorname{diag}(\underbrace{0, \ldots, 0}_{j-2 \text{ zeros}}, \begin{bmatrix} s_j & -s_j \\ s_j & s_j \end{bmatrix}, 0 \ldots, 0) \in \mathbb{R}^{n \times n}$$

We have

$$u^\top S_1 u \;=\; s_1\, u_1^2$$

and for $j \geq 2$,

$$\begin{aligned} u^\top S_j u &= \begin{bmatrix} u_{j-1} & u_j \end{bmatrix} \begin{bmatrix} s_j & -s_j \\ -s_j & s_j \end{bmatrix} \begin{bmatrix} u_{j-1} \\ u_j \end{bmatrix} \\ &= s_j\, u_{j-1}^2 - 2 s_j\, u_{j-1} u_j + s_j\, u_j^2 \\ &= s_j (u_{j-1} - u_j)^2. \end{aligned}$$

Using this the potential (elastic) energy is,

$$E(u) \;=\; \frac{1}{2} u^\top S u = \frac{1}{2} \left(\sum_{j=1}^{n} u^\top S_j u \right) = \frac{1}{2} \left(s_1\, u_1^2 + \sum_{j=2}^{n} s_j (u_{j-1} - u_j)^2 \right) \geq 0.$$

Therefore the form $E(u)$ is positive semidefinite. Definiteness we can show as follows: If $E(u) = 0$, then all squares are zero. This implies $u_1 = 0$, $u_2 = u_1$, $u_3 = u_2$, $\ldots$ $u_n = u_{n-1}$. So $u = 0$.