MAT 460 – Numerical Differential Equations – Solutions to Assignment 6 (E,H)

The iterative method defined by the Butcher-table

$$\begin{array}{c|cc} 0 & & \\ c & a & \\ \hline & b_1 & b_2 \end{array}$$

is

$$y_{j+1} = y_j + h(b_1 k_1 + b_2 k_2), \qquad k_1 = f(t_j, y_j), \quad k_2 = f(t_j + c\,h, y_j + h\,a\,k_1).$$

This can be written in one line:

$$y_{j+1} = y_j + h\,(b_1\, f(t_j, y_j) + b_2\, f(t_j + c\,h, y_j + h\,a\, f(t_j, y_j))\,).$$

For $b_1 = b_2 = 1/2$, $c = 1$, $a = 1$ this is the method of Heun.
For $b_1 = 0$ $b_2 = 1$, $c = 1/2$, $a = 1/2$ this is the method of Collatz (see lecture slides).
The values y_{j+1} and k_2 depend also on the time step size h:

$$y_{j+1}(h) = y_j + h\,(b_1\, k_1 + b_2\, k_2(h)) \qquad (*)$$

To show that a method has order p, we need to show that the Taylor-series of y_{j+1} about $h = 0$ coincides with the Taylor-series of the exact solution to the initial value problem

$$y'(t) = f(t, y(t)), \qquad y(t_j) = y_j \qquad (**)$$

up to order p. Here we have $p = 2$. So we need to compute Taylor-series up to order 2. From $(**)$ it follows that:

$$y''(t) = \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t))\, y'(t)$$

Substituting $t = t_j$ gives

$$y''(t_j) = \frac{\partial f}{\partial t}(t_j, y_j) + \frac{\partial f}{\partial y}(t_j, y_j)\, \underbrace{f(t_j, y_j)}_{=k_1}$$

The Taylor-series of the exact solution is then

$$\begin{aligned} y(t_j + h) &= y(t_j) + y'(t_j)\, h + y''(t_j) \frac{h^2}{2} + \mathcal{O}(h^3) \\ &= y_j + k_1\, h + \left(\frac{\partial f}{\partial t}(t_j, y_j) + \frac{\partial f}{\partial y}(t_j, y_j)\, k_1 \right) \frac{h^2}{2} + \mathcal{O}(h^3). \qquad (***) \end{aligned}$$

From $(*)$ we compute

$$y'_{j+1}(h) = (b_1\, k_1 + b_2\, k_2(h)) + h\, b_2\, k'_2(h), \qquad y''_{j+1}(h) = 2\, b_2\, k'_2(h) + h\, b_2\, k''_2(h)$$

We have $k_2(0) = k_1$ and

$$k'_2(h) = \frac{d}{dh} f(t_j + c\,h, y_j + h\,a\,k_1) = \frac{\partial f}{\partial t}(t_j + c\,h, y_j + h\,a\,k_1)\, c + \frac{\partial f}{\partial y}(t_j + c\,h, y_j + h\,a\,k_1)\, a\, k_1.$$

(Fortunately we don't have to compute $k_2''(h)$, because this term does not appear in the Taylor-series about $h = 0$.) We get:

$$\begin{aligned} y_{j+1}(h) &= y_j + y_{j+1}'(0)\,h + y_{j+1}''(0)\,\frac{h^2}{2} + \mathcal{O}(h^3) \\ &= y_j + (b_1 + b_2)\,k_1\,h + 2\,b_2\left(\frac{\partial f}{\partial t}(t_j, y_j)\,c + \frac{\partial f}{\partial y}(t_j, y_j)\,a\,k_1\right)\frac{h^2}{2} + \mathcal{O}(h^3). \qquad (****) \end{aligned}$$

The Taylor-series $(***)$ and $(****)$ are equal, when $b_1 + b_2 = 1$, and $2\,b_2\,c = 1 = 2\,b_2\,a$.

For E 1) and E 2) see the lecture slides.
E 3) For the solution to the ODE $y'(t) = f(t, y(t))$ we write

$$y(t+h) = y(t) + \int_t^{t+h} y'(\tau)\,d\tau \;=\; y(t) + \int_t^{t+h} f(\tau, y(\tau))\,d\tau.$$

The main idea of Adams-Bashforth methods is to replace the integrand $f(\tau, y(\tau))$, by the interpolation polynomial $p(\tau)$ to the interpolation points $(t_j, f(t_j, y_j)), (t_{j-1}, f(t_{j-1}, y_{j-1}))\dots(t_{j-s}, f(t_{j-s}, y_{j-s}))$. For $s = 2$ we have

$$\begin{aligned} y(t_j + h) &= y(t_j) + \int_{t_j}^{t_j+h} f(\tau, y(\tau))\,d\tau \\ &\approx y_j + \int_t^{t+h} p(\tau)\,d\tau \\ &= y_j + \int_{t_j}^{t_j+h} f(t_j, y_j)\,L_1(\tau) + f(t_{j-1}, y_{j-1})\,L_2(\tau) + f(t_{j-2}, y_{j-2})\,L_3(\tau))\,d\tau \\ &= y_j + f(t_j, y_j)\underbrace{\int_{t_j}^{t_j+h} L_1(\tau)\,d\tau}_{=(23/12)h} + f(t_{j-1}, y_{j-1})\underbrace{\int_{t_j}^{t_j+h} L_2(\tau)\,d\tau}_{=-(16/12)h} + f(t_{j-2}, y_{j-2})\underbrace{\int_{t_j}^{t_j+h} L_3(\tau)\,d\tau}_{=(5/12)h}. \end{aligned}$$

Here L_1, L_2, L_3 are the corresponding Langrange-basis polynomials.

Adams-Moulton methods are derived similarly, with the difference that here we also use the (unkown) interpolation point $(t_{j+1}, f(y_{j+1}))$ in the interpolation polynomial.

The BDF-method we obtain if we replace the derivative in the ODE $y'(t) = f(t, y(t))$ by a backward difference quotient (see last homework) and substitute $f(t, y(t))$ by the (yet unknown) value $f(t_{j+1}, y_{j+1})$.

H 1 a) Euler step: **(2 Points)**

$$y(3.1) \approx y_e = y(3) + 0.1 * 5 * (3 + y(3)^2) = -1 + 0.5 * 4 = 1.$$

H 1 b) Collatz step: **(2 Points)**

$$y(3.2) \approx y_c = y(3) + 0.2 * 5 * (3.1 + y_e^2) = -1 + 4.1 = 3.1.$$

H 2) General Runge-Kutta-method:

$$y_{j+1} = y_j + h \sum_{i=1}^{s} b_i\, k_i, \qquad k_i = f\left(t_j + c_i\, h,\ y_j + h \sum_{l=1}^{s} a_{il} k_l\right), \quad i = 1, \ldots, s$$

where $h = t_{j+1} - t_j$ is the time step size. The parameter c_i, b_i, a_{il} are given in the Butcher-table (see lecture slides). Parameter, which are not given in the table, are zero. For the 3/8-rule we have

$$y_{j+1} = y_j + h\,(\frac{1}{8}\,k_1 + \frac{3}{8}\,k_2 + \frac{3}{8}\,k_3 + \frac{1}{8}\,k_4),$$

where

$$\begin{aligned}
k_1 &= f(t_j, y_j) \\
k_2 &= f(t_j + \frac{1}{3}h,\ y_j + \frac{h}{3}k_1) \\
k_3 &= f(t_j + \frac{2}{3}h,\ y_j + h(-\frac{1}{3}k_1 + k_2)) \\
k_4 &= f(t_j + h,\ y_j + h(k_1 - k_2 + k_3)).
\end{aligned}$$