Vorlesung: MAT 460 Numerical Differential Equations

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Lecture 3

Topics: Cholesky-factorization and Positive Definiteness

Theorem of Cholesky-factorization

Let $A \in \mathbb{R}^{n\times n}$ be a **symmetric** and **positive definite** matrix.
Then there exist exactly one lower triangular matrix $L \in \mathbb{R}^{n\times n}$ with positive diagonal elements, such that

$$A = L\,L^T.$$

Example:

$$\underbrace{\begin{bmatrix} 25 & -5 & 15 \\ -5 & 10 & -15 \\ 15 & -15 & 29 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & -4 & 2 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 5 & -1 & 3 \\ 0 & 3 & -4 \\ 0 & 0 & 2 \end{bmatrix}}_{L^T}.$$

Note: In contrast to the LU-factorization, the diagonal elements of the Cholesky-factor L are not necessarily 1.

Andre-Louis Cholesky (1875-1918)
published Cholesky-factorization 1924.

(At least) two very different algorithms for the Cholesky-factorization exist.

1. Cholesky-algorithm:

Input: symmetric positive definite matrix $A \in \mathbb{R}^{n\times n}$.

For $j = 1, \ldots, n$

Compute $\ell_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} \ell_{jk}^2}$

For $i = j+1, \ldots, n$

Compute $\ell_{ij} := \left(a_{ij} - \sum_{k=1}^{j-1} \ell_{ik}\ell_{jk}\right)/\ell_{jj}$

Comment: This algorithm computes the elements of $L = [\ell_{ij}] \in \mathbb{R}^{n\times n}$ in the lower trianglular elements (including diagonal elements).

It works by **equating coefficents**.

Example for equating coefficients:

The ansatz

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ 0 & \ell_{22} & \ell_{32} \\ 0 & 0 & \ell_{33} \end{bmatrix}$$

leads to the equations:

$$\begin{aligned} a_{11} &= \ell_{11}^2 \\ a_{21} &= \ell_{11}\,\ell_{21} \\ a_{31} &= \ell_{11}\,\ell_{31} \\ a_{22} &= \ell_{21}^2 + \ell_{22}^2 \\ a_{32} &= \ell_{31}\ell_{21} + \ell_{32}\ell_{22} \\ a_{33} &= \ell_{33}^2 + \ell_{31}^2 + \ell_{32}^2 \end{aligned}$$

reordering gives :

$$\begin{aligned} \ell_{11} &= \sqrt{a_{11}} \\ \ell_{21} &= a_{21}/\ell_{11} \\ \ell_{31} &= a_{31}/\ell_{11} \\ \ell_{22} &= \sqrt{a_{22} - \ell_{21}^2} \\ \ell_{32} &= (a_{32} - \ell_{31}\ell_{21})/\ell_{22} \\ \ell_{33} &= \sqrt{a_{33} - \ell_{31}^2 - \ell_{32}^2} \end{aligned}$$

2. Cholesky-algorithm:

Input: symmetric positive definite matrix $A \in \mathbb{R}^{n\times n}$.

For $j = 1, \ldots, n$

Compute $a_{jj} := \sqrt{a_{jj}}$

For $i = j+1, \ldots, n$

Compute $a_{ij} := a_{ij}/a_{jj}$

For $k = j+1, \ldots, i$

Compute $a_{ik} := a_{ik} - a_{ij}a_{kj}$

Comment: This algorithm overwrites the entries in the lower triangular part of A (including the diagonal elements) with the entries of L.
In other words: After performing the algorithm we have

$$a_{ij} = \ell_{ij} \qquad \text{for } i \geq j.$$

It works by **completing the square**.

More at the end of this lecture.

Cholesky-algorithm, version 2:

Input: symmetric positive definite matrix $A \in \mathbb{R}^{n\times n}$.

For $j = 1, \ldots, n$

Compute $\ell_{jj} := \sqrt{a_{jj}}$

Für $i = j+1, \ldots, n$

Compute $\ell_{ij} := a_{ij}/\ell_{jj}$

For $k = j+1, \ldots, i$

Compute $a_{ik} := a_{ik} - \ell_{ij}\ell_{kj}$

Comment: The only difference between this and the last algorithm is: here the results are saved in the matrix $L = [\ell_{ij}]$ and not in A.

On the next slides we will answer the following questions:

- Why do only positive definite matrices have a Cholesky-factorization?
- What does positive definite actually mean?
- How can we determine definiteness?
- Why is positive definiteness important?
- Where do we find positive definite matrices in applications?

Quadratic forms und Definiteness

The scalar product as matrix product and the term $x^T Ax$.

We know that the scalar product of two vectors $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ is defined as

$$x \cdot y = x_1 y_1 + \ldots + x_n y_n = \sum_{j=1}^{n} x_j y_j.$$

We can write the scalar product as matrix product: First, transpose the vector x, then multiply it with y according to the rules of matrix multiplication:

$$x^T y = [x_1 \ldots x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \sum_{j=1}^{n} x_j y_j = x \cdot y.$$

If we set $y = Ax$ with any matrix $A = [a_{jk}] \in \mathbb{R}^{n \times n}$, we get

$$x^T Ax = x^T y = \sum_{j=1}^{n} x_j y_j = \sum_{j=1}^{n} x_j \left(\sum_{k=1}^{n} a_{jk} x_k \right) = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} x_j x_k.$$

Quadratic Forms

Definition:

A quadratic form in the variables $x_1, \ldots, x_n$ is an expression of the form

$$q_A(x) = x^T Ax = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} x_j x_k,$$

where we can assume that the matrix $A \in \mathbb{R}^{n \times n}$ is symmetric.

Example: a quadratic form in two variables

Let $A = \begin{bmatrix} 2 & -2 \\ 8 & 4 \end{bmatrix}$ (not symmetric). Then

$$\begin{aligned} q_A(x) &= x^T A x \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= 2x_1^2 - 2 x_1 x_2 + 8 x_2 x_1 + 4 x_2^2 \\ &= 2x_1^2 + 3 x_1 x_2 + 3 x_2 x_1 + 4 x_2^2 \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}}_{=:\widetilde{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x^T \widetilde{A} x = q_{\widetilde{A}}(x). \end{aligned}$$

The matrix $\widetilde{A}$ is symmetric.

Example: quadratic forms in 4 variables

Let

$$x_2^2 + 7 x_1 x_2 - x_4^2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^T \begin{bmatrix} 0 & 3.5 & 0 & 0 \\ 3.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + x_4^2 &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ &= x^T I x \qquad I = \text{identity matrix} \\ &= x^T x. \end{aligned}$$

Definitness of quadratic forms

A quadratic form (i. e. the corresponding matrix) is c d

positive definite	if	$x^T A x > 0$ for all $x \neq 0$,
positive semidefinite	if	$x^T A x \geq 0$ for all x,
negative definite	if	$x^T A x < 0$ for all $x \neq 0$,
negativ semidefinite	if	$x^T A x \leq 0$ for all x,
indefinite	otherwise.	

Note: A definite matrix is always semi-definite.

Example: Let A be a diagonal matrix:

$$A = \begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & a_{nn} \end{bmatrix}$$

Then

$$x^T A x = a_{11} x_1^2 + a_{22} x_2^2 + \ldots + a_{nn} x_n^2.$$

A is positive definite, if and only if all diagonal elements are positive.

A is positive semi-definite, if and only if all diagonal elements are non negative.

A is indefinite, if and only if 2 diagonal elements have different signs.

A necessary criteria for positive definiteness

If a symmetric matrix $A = [a_{jk}]$ is positive definite, than all its diagonal elements are positive.

Proof: Let

$$e_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow k\text{-te Zeile}$$

It is easy to see, that $e_k^T A e_k = a_{kk}$, where a_{kk} is the k-th diagonal element of A. If A is positive definite, then

$$a_{kk} = e_k^T A e_k > 0$$

Practical example: The matrix

$$A = \begin{bmatrix} 2 & 3 & 0 & -6 \\ 3 & 7 & 4 & 8 \\ 0 & 4 & -2 & 7 \\ -6 & 8 & 7 & 1 \end{bmatrix}$$

is not positive definite, because $a_{33} = -2 \leq 0$.

Warning:
We can not decide from only looking at the diagonal elements if a matrix is positive definite.

Example: The matrix $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ has positive diagonal elements, but A is not positive.

Explanation: In general

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\,x^2 + 6\,x_1\,x_2 + 2\,x_2^2.$$

If we choose $x_1 = 1$, $x_2 = -1$, then

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 * 1^2 + 6 * 1 * (-1) + 2 * (-1)^2 = -2 < 0.$$

Motivation: Where do we find positive definite matrices in applications?

1. Mechanics

Linearised differential equation of a mechanical system with degrees of freedom $u(t) \in \mathbb{R}^n$:

$$\boxed{M\,\ddot{u}(t) + D\,\dot{u}(t) + S\,u(t) = f(t). \qquad (*)}$$

where

$$\begin{aligned} M &= \text{mass matrix} \\ D &= \text{damping matrix} \\ S &= \text{stiffness matrix} \\ f &= \text{external forces (load)} \end{aligned}$$

Definite quadratic forms:

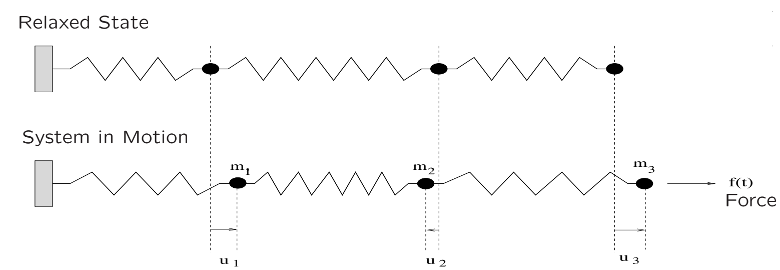
Kinetic energy: $E_{kin} = \frac{1}{2}\,\dot{u}^T M \dot{u} > 0$

Elastic energy: $E_{elast} = \frac{1}{2}\,u^T S u \geq 0$

Energy dissipation: $\dot{u}^T D \dot{u} \geq 0$
(friction)

From $(*)$ we get the energy balance equation: $\frac{d}{dt}(E_{kin} + E_{elast}) = \dot{u}^T f - \dot{u}^T D \dot{u}.$

Practical example: A spring-mass system (here without damping)



Energies: (s_j= spring constants)

$$E_{elast} = \tfrac{s_1}{2}\,u_1^2 + \tfrac{s_2}{2}(u_2-u_1)^2 + \tfrac{s_3}{2}(u_3-u_2)^2 = \frac{1}{2}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^T \begin{bmatrix} s_1+s_2 & -s_2 & 0 \\ -s_2 & s_2+s_3 & -s_3 \\ 0 & -s_3 & s_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$E_{kin} = \tfrac{m_1}{2}\,\dot{u}_1^2 + \tfrac{m_2}{2}\,\dot{u}_2^2 + \tfrac{m_3}{2}\,\dot{u}_3^2 = \frac{1}{2}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix}^T \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix}$$

Equation of motion:

$$\underbrace{\begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix}}_{M} \begin{bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \end{bmatrix} + \underbrace{\begin{bmatrix} s_1+s_2 & -s_2 & 0 \\ -s_2 & s_2+s_3 & -s_3 \\ 0 & -s_3 & s_3 \end{bmatrix}}_{S} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f(t) \end{bmatrix}$$

2. Discretised heat equation (important for projects)

Stationary heat equation in domain Ω:

$$\boxed{-\mathrm{div}(\lambda \nabla T) = f}$$

Ansatz for temperature field:

$$\boxed{T(x) = u_1\,\phi_1(x) + u_2\,\phi_2(x) + \ldots + u_n\,\phi_n(x)}$$

with

$$\begin{aligned} u_k &= \text{temperatures at the vertices,} \\ \phi_k &= \text{hat functions.} \end{aligned}$$

Discretised heat equation:

$$\boxed{S\,u = p}$$

with $u = [u_1 \;\; u_2 \;\; \ldots u_n]^T$ and with the "stiffness matrix"

$$S = [s_{jk}] \in \mathbb{R}^{n\times n}, \qquad s_{jk} = \int_\Omega \lambda\,(\nabla\phi_j)^T \nabla\phi_k$$

We have

$$u^T S u = \int_\Omega \lambda \|\nabla(u_1\phi_1 + u_2\phi_2 + \ldots + u_n\phi_n)\|^2 \geq 0.$$

$\Rightarrow$ S is semi-definite. (Even definite after including the boundary conditions.)

3. Analysis: Lokal extrema

Given: Function $f : \mathbb{R}^n \to \mathbb{R}$.

Taylor expansion about $x \in \mathbb{R}^n$ up to 2nd order:

$$\begin{aligned} f(x+h) &= f(x) + \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(x)\,h_k + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k}(x)\,h_j\,h_k + o(\|h\|^2) \\ &= f(x) + \nabla f(x)^T h + \frac{1}{2} h^T H_f(x)\,h + o(\|h\|^2), \end{aligned}$$

with the Hessian-matrix

$$H_f(x) = \left[\frac{\partial^2 f}{\partial x_j \partial x_k}(x)\right]_{jk}.$$

Necessary for local minimum at x:

$$\nabla f(x) = 0, \qquad H_f(x) \text{ positive semi-definite.}$$

Sufficient for local minimum at x:

$$\nabla f(x) = 0, \qquad H_f(x) \text{ positive definite.}$$

Eigenvalue criteria for definiteness

Remember: $\lambda \in \mathbb{C}$ is called eigenvalue of $A \in \mathbb{R}^{n\times n}$, if $0 \neq v \in \mathbb{R}^n$, such that

$$Av = \lambda\,v.$$

Theorem: Let $A \in \mathbb{R}^{n\times n}$ symmetric. Then:

1. All eigenvalues of A are real.
2. The eigenvectors can be choosen such that they form an orthonormal basis of $\mathbb{R}^n$.
3. The matrix A is positive (semi)definite if and only if all eigenvalues are positive (non negative).

Explanation of the eigenvalue criteria:

Let $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ be an orthonormal basis of eigenvalues:

$$Av_j = \lambda_j\,v_j, \qquad v_j^T v_k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

Every vector $x \in \mathbb{R}^n$ can be written as:

$$x = x_1\,v_1 + x_2\,v_2 + \ldots + x_n\,v_n, \qquad x_j \in \mathbb{R}.$$

Straightforward computation gives: $\quad x^T A x = \lambda_1\,x_1^2 + \lambda_2\,x_2^2 + \ldots + \lambda_n\,x_n^2.$

Check for definiteness by completing the square

A quadratic form in 2 variables can be transformed into a sum or difference of squares by completing the square.

(Method: $a^2 + 2\,a\,b = a^2 + 2\,a\,b + b^2 - b^2 = (a+b)^2 - b^2$.)

Example 1:

$$\begin{aligned} 9\,x_1^2 - 24\,x_1x_2 + 41\,x_2^2 &= (3x_1)^2 + 2(3x_1)(-4x_2) + 41\,x_2^2 \\ &= (3x_1)^2 + 2(3x_1)(-4x_2) + (-4\,x_2)^2 - (-4\,x_2)^2 + 41\,x_2^2 \\ &= (3x_1 - 4x_2)^2 - (-4x_2)^2 + 41\,x_2^2 \\ &= (3x_1 - 4x_2)^2 + (5x_2)^2. \end{aligned}$$

$\Rightarrow$ This form is positive definite.

Example 2:

$$\begin{aligned} 36\,x_1^2 + 4\,x_1\,x_2 &= (6\,x_1)^2 + 2\,(6\,x_1)\,(\frac{1}{3}x_2) \\ &= (6\,x_1)^2 + 2\,(6\,x_1)\,(\frac{1}{3}x_2) + (\frac{1}{3}x_2)^2 - (\frac{1}{3}x_2)^2 \\ &= (6\,x_1 + \frac{1}{3}x_2)^2 - (\frac{1}{3}x_2)^2 \end{aligned}$$

$\Rightarrow$ This form is indefinite.
To see this, choose e. g. $x_2 = 1$ and x_1 such that $6\,x_1 + \frac{1}{3}x_2 = 0$.

2×2 matrices in general (for $a \neq 0$):

$$\begin{aligned}
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \underbrace{\begin{bmatrix} a & b \\ b & c \end{bmatrix}}_{=A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= a\,x_1^2 + 2b\,x_1x_2 + c\,x_2^2 \\
&= a\left(x_1^2 + 2\frac{b}{a}x_1x_2\right) + c\,x_2^2 \\
&= a\left(x_1^2 + 2\frac{b}{a}x_1x_2 + \left(\frac{b}{a}\right)^2 x_2^2\right) \\
&\quad -\frac{b^2}{a}x_2^2 + c\,x_2^2 \\
&= a\left(x_1 + \frac{b}{a}x_2\right)^2 + \left(c - \frac{b^2}{a}\right)x_2^2 \\
&= a\left(x_1 + \frac{b}{a}x_2\right)^2 + \frac{1}{a}\underbrace{\left(ac - b^2\right)}_{=\det(A)}x_2^2
\end{aligned}$$

Conclusion: A is positive definite if and only if

$$a > 0 \quad \text{and} \quad \det(A) > 0.$$

The determinant criteria for positive definiteness

The criteria from the last slide for positive definiteness of a 2×2 matrix can be generalized.

Theorem: A symmetric matrix $A = [a_{jk}] \in \mathbb{R}^{n\times n}$ is positive definite if and only if the determinants of all upper left sub-matrices are positive, i. e.

$$\det[a_{11}] = a_{11} > 0, \quad \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > 0, \quad \det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} > 0, \quad \ldots \quad \det(A) > 0.$$

Relation between completing the square and Cholesky-factorization

Example:

$$\begin{aligned}
q_A(x) = x^TA\,x &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} 25 & -5 & 15 \\ -5 & 10 & -15 \\ 15 & -15 & 29 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= 25x_1^2 - 10x_1x_2 + 30x_1x_3 + 10x_2^2 - 30x_2x_3 + 29x_3^2
\end{aligned}$$

Completing the square we get:

$$\begin{aligned}
q_A(x) &= (5x_1)^2 + 2(5x_1)(-x_2 + 3x_3) + (-\mathbf{x_2} + \mathbf{3x_3})^2 - (-\mathbf{x_2} + \mathbf{3x_3})^2 \\
&\quad + 10x_2^2 - 30x_2x_3 + 29x_3^2 \\
&= (5x_1 - x_2 + 3x_3)^2 + \text{ quad. form in } x_2, x_3 \\
&\ \ \vdots \\
&= (5x_1 - x_2 + 3x_3)^2 + (3x_2 - 4x_3)^2 + (2x_3)^2 \\
&= y_1^2 + y_2^2 + y_3^2 = y^Ty
\end{aligned}$$

with

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5x_1 - x_2 + 3x_3 \\ 3x_2 - 4x_3 \\ 2x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 & -1 & 3 \\ 0 & 3 & -4 \\ 0 & 0 & 2 \end{bmatrix}}_{=:L^T} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In summary: $x^TA\,x = y^Ty = (L^Tx)^T(L^Tx) = x^TLL^Tx. \quad \Rightarrow \quad A = LL^T.$

Why do only positive definite matrices have a Cholesky-factorization?

In general: Let $B \in \mathbb{R}^{n\times n}$ not singular (invertible).
Then the matrix $A = B^TB$ is positive definite.

Explanation:

Let $x \in \mathbb{R}^n$, $x \neq 0$. Choose $y = Bx$. Then also $y \neq 0$, becaues B is not singular. And further:

$$x^TAx = x^TB^TBx = (Bx)^TBx = y^Ty = y_1^2 + y_2^2 + \ldots + y_n^2 > 0.$$

$\Rightarrow$ A is positive definite.

Conclusion: If $A = LL^T = (L^T)^T\underbrace{(L^T)}_{B}$ and $\det(L) \neq 0$, then A is positive definite.

Cholesky-factorization with MATLAB

`U=chol(A)`	gives back a upper triangular matrix `U`, such that `U'*U=A`.
`L=chol(A,'lower')`	gives back a lower triangular matrix `L`, such that `L*L'=A`.

If `A` is not symmetric and not positive definite, an error message is returned.