MAT 460 – Numerical Differential Equations – Solutions to Assignment 5 (E,H)

Problem 5

E a,b) The polynomial p, which interpolates the function f at the control points $x-h$, x, $x+h$, is

$$p(t) = f(x-h)\,L_1(t) + f(x)\,L_2(t) + f(x+h)\,L_3(t)$$

with the Lagrange-basis polynomials

$$\begin{aligned}
L_1(t) &= \frac{(t-x)(t-(x+h))}{((x-h)-x)((x-h)-(x+h))} = \frac{t^2-(2x+h)t+(x-h)x}{2h^2}\\
L_2(t) &= \frac{(t-(x-h))(t-(x+h))}{(x-(x-h))(x-(x+h))} = \frac{t^2-2xt+(x-h)(x+h)}{-h^2}\\
L_3(t) &= \frac{(t-(x-h))(t-x)}{((x+h)-(x-h))((x+h)-x)} = \frac{t^2-(2x-h)t+(x-h)x}{2h^2}
\end{aligned}$$

We have

$$\begin{array}{lll}
L_1'(t) = (2t-(2x+h))/(2h^2), & L_1'(x) = -1/(2h), & L_1''(t) = 1/h^2,\\
L_2'(t) = (2t-2x)/(-h^2), & L_2'(x) = 0, & L_2''(t) = -2/h^2,\\
L_3'(t) = (2t-(2x-h))/(2h^2), & L_3'(x) = 1/(2h), & L_3''(t) = 1/h^2.
\end{array}$$

From this we get the approximations

$$\begin{aligned}
f'(x) &\approx p'(x) = f(x-h)\,L_1'(x) + f(x)\,L_2'(x) + f(x+h)\,L_3'(x) = (-f(x+h)+f(x-h))/(2h),\\
f''(x) &\approx p''(x) = f(x-h)\,L_1''(x) + f(x)\,L_2''(x) + f(x+h)\,L_3''(x) = (f(x+h)-2f(x)+f(x-h))/h^2.
\end{aligned}$$

E c) The polynomial p, which interpolates the function f at the positions $x-h$, x, $x+h$, is

$$p(t) = f(x-2h)\,L_1(t) + f(x-h)\,L_2(t) + f(x)\,L_3(t)$$

with the Lagrange-basis polynomials

$$\begin{aligned}
L_1(t) &= \frac{(t-(x-h))(t-x)}{((x-2h)-(x-h))((x-2h)-x)} = \frac{t^2-(2x-h)t+(x-h)x}{2h^2}\\
L_2(t) &= \frac{(t-(x-2h))(t-x)}{((x-h)-(x-2h))((x-h)-x)} = \frac{t^2-(2x-2h)t+(x-2h)x}{-h^2}\\
L_3(t) &= \frac{(t-(x-2h))(t-(x-h))}{(x-(x-2h))(x-(x-h))} = \frac{t^2-(2x-3h)t+(x-2h)(x-h)}{2h^2}\,.
\end{aligned}$$

We obtain

$$\begin{aligned}
L_1'(t) &= (2t-(2x-h))/(2h^2), & L_1'(x) &= 1/(2h),\\
L_2'(t) &= (2t-(2x-2h))/(-h^2), & L_2'(x) &= -2/h,\\
L_3'(t) &= (2t-(2x-3h))/(2h^2), & L_3'(x) &= 3/(2h),
\end{aligned}$$

From this we find the approximation

$$\begin{aligned} f'(x) &\approx p'(x) \\ &= f(x-2h)\,L_1'(x) + f(x-h)\,L_2'(x) + f(x)\,L_3'(x) \\ &= \frac{1}{h}((1/2)f(x-2h) - 2\,f(x-h) + (3/2)\,f(x)). \end{aligned}$$

H) Similarily.

To estimate the error we follow the approach from the lecture slides: Let q be the interpolation polynomial of degree $\leq n$ which interpolates f at the positions $x_1, \ldots, x_{n+1}$. Then

$$f(x) = q(x) + R_{f,q}(x), \qquad R_{f,q}(x) = \frac{f^{(n)}(\xi)}{(n+1)!}(x-x_1)\ldots(x-x_{n+1}) \qquad (*)$$

where $\min\{x_1, \ldots x_{n+1}, x\} \leq \xi \leq \max\{x_1, \ldots x_{n+1}, x\}$.

For E a) we have: the interpolation parabola p interpolates f at the positions $x-h, x, x+h$. The difference function $\phi(t) = f(t) - p(t)$ has these three positions as roots. Between the roots ϕ has local extrema. So we can find some $x_1 \in [x-h, x]$, $x_2 \in [x, x+h]$, such that $0 = \phi'(x_j) = f'(x_j) - p'(x_j)$, and thus the polynomial p' interpolates f' at the positions x_1, x_2. With $(*)$ we have

$$\begin{aligned} f'(x) &= p'(x) + R_{f',p'}(x) \\ &= p'(x) + \frac{(f')''(\xi)}{2}(x-x_1)(x-x_2) \qquad \qquad \xi \in [x-h, x+h] \\ &= \frac{f(x+h) - f(x-h)}{2h} + \frac{f'''(\xi)}{2}(x-x_1)(x-x_2) \end{aligned}$$

An estimate for $R_{f',p'}(x)$ is

$$|R_{f',p'}(x)| = |\frac{f'''(\xi)}{2}(x-x_1)(x-x_2)| \leq \max_{\xi \in [x-h, x+h]} |f'''(\xi)|\, \frac{h^2}{2} = \mathcal{O}(h^2).$$

For E b) we could follow a similar approach but this would not result in the optimal error estimate. Difference formulas can also be derived directly from Taylor series, which is here the better way. To derive the difference formula for the second derivative, we assume that f is 4-times continuously differentiable and write out the Taylor-series

$$\begin{aligned} f(x+h) &= f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{6}h^3 + \frac{f^{(4)}(\xi_1)}{24}h^4, \qquad \xi_1 \in [x, x+h], \\ f(x-h) &= f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{6}h^3 + \frac{f^{(4)}(\xi_2)}{24}h^4, \qquad \xi_2 \in [x-h, x]. \end{aligned}$$

Adding both equations and dividing the result by h^2 gives

$$\frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x) + \underbrace{\frac{f^{(4)}(\xi_1) + f^{(4)}(\xi_2)}{24}\,h^2}_{R(h)}.$$

So the remainder is of order $R(h) = \mathcal{O}(h^2)$. We can further simplify the representation of $R(h)$. We have

$$\min\{f^{(4)}(\xi_1), f^{(4)}(\xi_2)\} \leq \frac{f^{(4)}(\xi_1) + f^{(4)}(\xi_2)}{2} \leq \max\{f^{(4)}(\xi_1), f^{(4)}(\xi_2)\},$$

and with the mean value theorem some $\xi \in [\xi_1, \xi_2]$ exist, such that

$$f^{(4)}(\xi) = \frac{f^{(4)}(\xi_1) + f^{(4)}(\xi_2)}{2}.$$

This gives

$$R(h) = \frac{f^{(4)}(\xi)}{12} h^2.$$

Problem 6

E) Let

$$I(f) := \int_a^b f(x)\,dx, \qquad Q(f) := (b-a)\left(\gamma_1\, f(a) + \gamma_2\, f(\frac{a+b}{2}) + \gamma_3\, f(b)\right)$$

We have to find the weights $\gamma_1, \gamma_2, \gamma_3$ such that

$$I(f) = Q(f) \text{ for all polynomials } f(x) = a_2\, x^2 + a_1\, x + a_0, \quad a_k \in \mathbb{R}. \qquad (*)$$

$I(f)$ and $Q(f)$ depend linearly on f, i. e. for two integrable functions f_1, f_2 and two scalars $\lambda_1, \lambda_2 \in \mathbb{R}$ we have

$$\begin{aligned} I(\lambda_1\, f_1 + \lambda_2\, f_2) &= \lambda_1\, I(f_1) + \lambda_2\, I(f_2), \\ Q(\lambda_1\, f_1 + \lambda_2\, f_2) &= \lambda_1\, Q(f_1) + \lambda_2\, Q(f_2). \end{aligned}$$

If we assume that the weights γ_j satisfy

$$I(1) = Q(1), \quad I(x) = Q(x), \quad I(x^2) = Q(x^2), \qquad (**)$$

then $(*)$ follows from linearity. $(**)$ is a system of linear equations:

$$\left.\begin{array}{rcl} Q(1) &=& I(1) \\ Q(x) &=& I(x) \\ Q(x^2) &=& I(x^2) \end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{rcl} \gamma_1 + \gamma_2 + \gamma_3 &=& \frac{b-a}{b-a} = 1 \\ \gamma_1\, a + \gamma_2 \frac{a+b}{2} + \gamma_3\, b &=& \frac{\frac{b^2}{2} - \frac{a^2}{2}}{b-a} = \frac{1}{2}(a+b) \\ \gamma_1\, a^2 + \gamma_2 \left(\frac{a+b}{2}\right)^2 + \gamma_3\, b^2 &=& \frac{\frac{b^3}{3} - \frac{a^3}{3}}{b-a} = \frac{1}{3}(a^2 + ab + b^2). \end{array}\right.$$

In matrix form we have

$$\begin{bmatrix} 1 & 1 & 1 \\ a & \frac{a+b}{2} & b \\ a^2 & \left(\frac{a+b}{2}\right)^2 & b^2 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2}(a+b) \\ \frac{1}{3}(a^2 + ab + b^2) \end{bmatrix}$$

from which we obtain the solution $\gamma_1 = \gamma_3 = \frac{1}{6}$, $\gamma_2 = \frac{2}{3}$ (check), so we finally obtain

$$Q(x) = (b-a)\left(\frac{1}{6}\, f(a) + \frac{2}{3}\, f(\frac{a+b}{2}) + \frac{1}{6}\, f(b)\right).$$

This is the Simpson-rule (see lecture slides).

H a) We have to show that $Q(x^3) = I(x^3) = \frac{b^4 - a^4}{4}$.

H b) Let $I(f) = \int_0^1 f(x)\,dx$ and let $Q(x) = \gamma_1 f(0) + \gamma_2 f(1/3)$. The following must hold true.

$$\begin{aligned} 1 = I(1) &= Q(1) = \gamma_1 + \gamma_2, \\ \frac{1}{2} = I(x) &= Q(x) = \gamma_1 * 0 + \gamma_2 * \frac{1}{3}. \end{aligned}$$

We obtain $\gamma_2 = 3/2$, $\gamma_1 = -\frac{1}{2}$, so we obtain $Q(f) = -\frac{1}{2} f(0) + \frac{3}{2} f(1/3)$.

We see that $Q(x^2) = (3/2) * (1/9) = 1/6 \neq 1/3 = I(x^2)$. So this quadrature formula is not exact for polynomials of degree 2.