Problem 1

E) Write out the equation for the line through the point $(x_0, y_0) \in \mathbb{R}^2$ with slope $a \in \mathbb{R}$ and compute the intersection of the line with the x-axis. Use this to derive the following iteration-methods: Newton-method, secant-method, and Regula Falsi.

H) **First Programming Assignment**

Write a function `regulafalsi(f,a,b,tol)`, which finds the root of a function `f` in the interval `[a,b]` with the Regula Falsi. The iteration shall be repeated as long as the difference between two consecutive iteration values is greater than `tol` (see lecture slides). Then use your Regula-Falsi function to find, with ten digits accuracy, the smallest positive root of the function $f(x) = 1 + \cos(x)\cosh(x)$.

Hints:

- First, plot f to find a good initial value.
- The Matlab and Python command to find the absolute value is `abs`.
- In Matlab, to pass a function as input parameter to another function, write `@` before the function name. For example: To find the fixpoint of $x = \cos(x)$ in the interval $[1, 2]$ with accuracy 0.01 call your function with `regulafalsi(@cos,1,2,0.01)`.

Problem 2 **2 Points**

E) The equation $x = -2\ln(x)$ has exactly one solution x_*. This solution lies in the interval $(0, 1)$. You can not find it using the fixpoint iteration-method $x_{k+1} = -2\ln(x_k)$. (Why not?) Find an iteration-method which converges to x_*.

H) Show that the function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = \frac{1}{2}x^2 - 12$ has exactly two fixpoints. Are they attracting or repelling fixpoints? Can you find them with the iteration-method $x_{k+1} = g(x_k)$?

Problem 3 **6 Points**

Let $a > 0$.

E) Show that the Heron-method

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right), \qquad x_0 > 0 \qquad (*)$$

is the Newton-method to compute $\sqrt{a}$. Show, that this method has quadratic order of convergence.

H a) Show that the iteration-method

$$x_{k+1} = \frac{1}{n}\left((n-1)\,x_k + \frac{a}{x_k^{n-1}}\right), \qquad x_0 > 0$$

is the Newton-method to compute $\sqrt[n]{a}$, assuming the function $f(x)$ is given by $f(x) = x^n - a$.

H b) Show that the following iteration-method to compute $\sqrt{a}$ has order of convergence 3.

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right) - \frac{(x_k^2 - a)^2}{8\,x_k^3}, \qquad x_0 > 0. \qquad (**)$$

H c) Using the method (∗∗), how many iterations are necessary to find the number $\sqrt{5}$ with 15 digits accuracy, if the initial value is $x_0 = 5$? How many iterations are necessary for the same problem, when the Newton-method (∗) is used. (For the computations use Python, Matlab, or a calculator.)

Problem 4 **2 Points**

Write out the Newton-method to find the root of f and find the next iterated value $[x_1\ y_1]^T$ after the start vector $[x_0\ y_0]^T = [1\ 1]^T$.

E) $$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy + x - y - 1 \\ xy^2 + 5 \end{bmatrix}$$

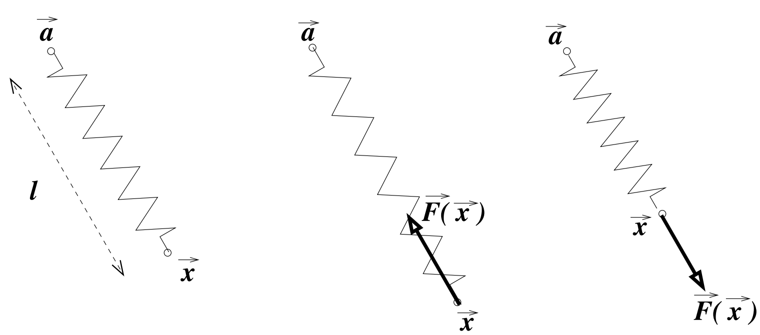
H) $$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^3 + y - 2 \\ x + \frac{1}{y} \end{bmatrix}.$$

Second Programming Assignment

Consider a spring in $\mathbb{R}^2$ which is pivot-mounted at one of its endpoints $\vec{a} \in \mathbb{R}^2$. The spring constant is $s > 0$ and the relaxed length of the spring is $\ell > 0$. Let $\vec{F}(\vec{x})$ be the force the spring exerts on its other endpoint $\vec{x}$. According to Hooke's law we have

$$\vec{F}(\vec{x}) \quad = \quad -s\left(\|\vec{x} - \vec{a}\|_2 - \ell\right)\frac{\vec{x} - \vec{a}}{\|\vec{x} - \vec{a}\|_2} = s\left(\frac{\ell}{\|\vec{x} - \vec{a}\|_2} - 1\right)(\vec{x} - \vec{a}).$$

Explanation: The force vector $\vec{F}(\vec{x})$ points along the line between the endpoints of the spring, its absolute value is the spring constant times the difference between relaxed length and stretched length of the spring, i.e. $\|\vec{F}(\vec{x})\|_2 = s\,\left|\|\vec{x} - \vec{a}\|_2 - \ell\right|$. When the spring is stretched the force acts in the direction of $\vec{a}$. When the spring is compressed the force acts in the opposite direction, away from the point $\vec{a}$. See the sketch (left: relaxed spring, middle: stretched spring, right: compressed spring).



The Jacobi matrix of the force function $\vec{F}$ is given by

$$\vec{F}'(\vec{x}) = s\left(\left(\frac{\ell}{\|\vec{x} - \vec{a}\|_2} - 1\right)I - \ell\,\frac{(\vec{x} - \vec{a})(\vec{x} - \vec{a})^\top}{\|\vec{x} - \vec{a}\|_2^3}\right).$$

Notation: I is the identity matrix. $\vec{x} - \vec{a}$ is a column vector, so $(\vec{x} - \vec{a})^\top$ so is a row vector (transposed). The product $(\vec{x} - \vec{a})(\vec{x} - \vec{a})^\top$ (matrix multiplication) is a quadratic matrix.

Now, lets consider a mass m, whose center is at the position $\vec{x} \in \mathbb{R}^2$, which is attached to two springs with spring constants $s_1, s_2 > 0$ (see the Figure). The springs are fixed at the points $\vec{a}_1$ and $\vec{a}_2$. The relaxed lengths of the springs are ℓ_1 and ℓ_2. Three forces act on the mass: the spring forces

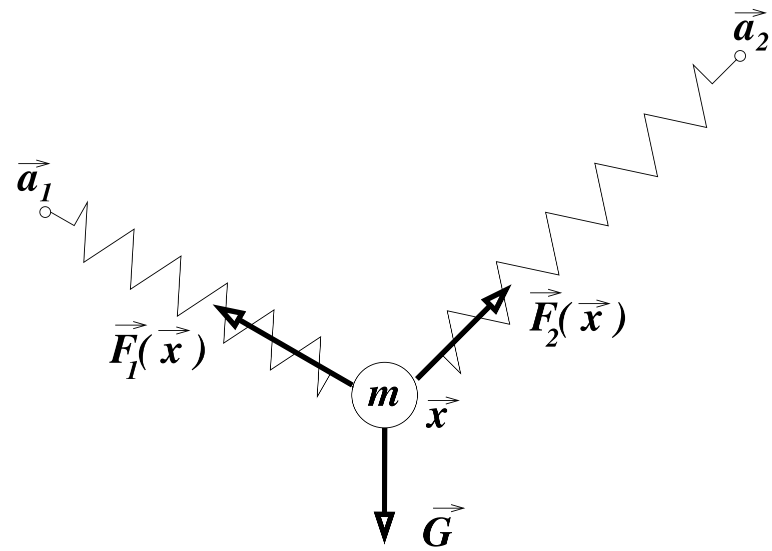
$$\vec{F}_k(\vec{x}) = -s_k\left(|\vec{x} - \vec{a}_k| - \ell_k\right)\frac{\vec{x} - \vec{a}_k}{|\vec{x} - \vec{a}_k|} = s_k\left(\frac{\ell_k}{|\vec{x} - \vec{a}_k|} - 1\right)(\vec{x} - \vec{a}_k), \qquad k = 1, 2$$

and the gravitational force

$$\vec{G} = \begin{bmatrix} 0 \\ -g\,m \end{bmatrix}, \qquad g = 9.81\,m/s^2 \text{ (gravitational constant)}.$$

So the total force acting on the mass at the point $\vec{x}$ is given by

$$\vec{F}_{total}(\vec{x}) = \vec{F}_1(\vec{x}) + \vec{F}_2(\vec{x}) + \vec{G}.$$



Task: Compute, using the the Newton-method, the equilibrium position $\vec{x}$ of the mass m, i.e. find the position vector $\vec{x}$ which satisfies $\vec{F}_{total}(\vec{x}) = \vec{0}$ for the given data.

$$\vec{a}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \vec{a}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad s_1 = s_2 = 10, \qquad \ell_1 = \ell_1 = 2, \qquad m = 1.$$

As initial condition for the Newton-method, use $\vec{x}_0 = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$ and (in a second run of your program) $\vec{x}_0 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$. How can you explain that the results are different?