Problem 1:

E)

$$A = \begin{bmatrix} 3 & 5 & 1 \\ 6 & 8 & 6 \\ 12 & 26 & -7 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}}_{=L} \underbrace{\begin{bmatrix} 3 & 5 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & 1 \end{bmatrix}}_{=U}, \qquad A * \underbrace{\begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix}}_{x} = \begin{bmatrix} 27 \\ 68 \\ 70 \end{bmatrix}.$$

H)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 13 & 10 & 17 \\ 3 & -4 & 16 & 6 \\ 1 & 27 & -16 & 15 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ 1 & 5 & -3 & 1 \end{bmatrix}}_{=L} \underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & -6 \end{bmatrix}}_{=U}, \qquad A * \underbrace{\begin{bmatrix} 2 \\ 5 \\ -3 \\ 1 \end{bmatrix}}_{x} = \begin{bmatrix} 7 \\ 60 \\ -56 \\ 200 \end{bmatrix}$$

Problem 2

Ea) Let ℓ_{jk} be the coefficients of L, i. e.

$$L = \begin{bmatrix} 1 & 0 & & & & \\ \ell_{21} & 1 & 0 & & & \\ \ell_{31} & \ell_{32} & 1 & 0 & & \\ \vdots & & & \ddots & & \\ \vdots & & & & & \\ \ell_{n1} & \ell_{n2} & \dots & \dots & \ell_{n,n-1} & 1 \end{bmatrix}$$

We need to solve the equations

$$\begin{aligned} x_1 &= b_1 \\ \ell_{21}x_1 + x_2 &= b_2 \\ \ell_{31}x_1 + \ell_{32}x_2 + x_3 &= b_3 \qquad \text{etc.} \end{aligned}$$

Reordering gives

$$\begin{aligned} x_1 &= b_1 \\ x_2 &= b_2 - \ell_{21}x_1 \\ x_3 &= b_3 - \ell_{31}x_1 - \ell_{32}x_2 \qquad \text{etc.} \end{aligned}$$

In general:

$$x_k = b_k - \sum_{j=1}^{k-1} \ell_{kj}x_j.$$

These are $k-1$ multiplications and $k-1$ additions (or subtractions), i. e. $2(k-1)$ flops. The index k runs from 2 to n. In total

$$2 + 4 + 6 + \ldots + 2(n-1) = n(n-1)$$

flops (use the formula from assignment 1).

Eb) To compute one entry in the product matrix AB we need n multiplications and $n-1$ additions. The product matrix has n^2 entries. The total number of flops used to multiply $n \times n$-matrices is then

$$n^2(n + (n-1)) = 2n^3 - n^2.$$

H) For $Ux = b$ we argue as in Ea), only that we have additionally one division in every equation, because the diagonal elements of U are not necessarily 1. The total number of flops is then $n(n-1) + n = n^2$. To compute each component of Ax we need n multiplications and $n-1$ additions. In total we need $n(n + (n-1)) = 2n^2 - n$ flops.

Problem 3
clear.

Problem 4
E) The ansatz

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ 0 & \ell_{22} & \ell_{32} \\ 0 & 0 & \ell_{33} \end{bmatrix}$$

leads to the equations:

$$\begin{aligned} a_{11} &= \ell_{11}^2 \\ a_{21} &= \ell_{11}\,\ell_{21} \\ a_{31} &= \ell_{11}\,\ell_{31} \\ a_{22} &= \ell_{21}^2 + \ell_{22}^2 \\ a_{32} &= \ell_{31}\ell_{21} + \ell_{32}\ell_{22} \\ a_{33} &= \ell_{33}^2 + \ell_{31}^2 + \ell_{32}^2 \end{aligned}$$

Reordering gives:

$$\begin{aligned} \ell_{11} &= \sqrt{a_{11}} \\ \ell_{21} &= a_{21}/\ell_{11} \\ \ell_{31} &= a_{31}/\ell_{11} \\ \ell_{22} &= \sqrt{a_{22} - \ell_{21}^2} \\ \ell_{32} &= (a_{32} - \ell_{31}\ell_{21})/\ell_{22} \\ \ell_{33} &= \sqrt{a_{33} - \ell_{31}^2 - \ell_{32}^2} \end{aligned}$$

Evaluating for $A = \begin{bmatrix} 4 & 10 & -4 \\ 10 & 34 & -7 \\ -4 & -7 & 21 \end{bmatrix}$ gives $L = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 3 & 0 \\ -2 & 1 & 4 \end{bmatrix}$.

H) Similarly:

$$L = \begin{bmatrix} 4 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ -2 & -5 & 2 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}.$$

Problem 5:

E) The idea is to write the quadratic form corresponding to the matrix as sum or difference of squares:

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 25 & 15 \\ 15 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 25x_1^2 + 15x_1x_2 + 15x_2x_1 + 7x_2^2 \\ &= (5x_1)^2 + 2(5x_1)(3x_2) + 7x_2^2 \\ &= (5x_1)^2 + 2(5x_1)(3x_2) + (3x_2)^2 - (3x_2)^2 + 7x_2^2 \qquad \text{(completing the square)} \\ &= (5\,x_1 + 3\,x_2)^2 - 2\,x_2^2 \end{aligned}$$

$\Rightarrow$ matrix is not pos. def. To see this set $x_2 = 1$ and x_1, such that $5\,x_1 + 3\,x_2 = 0$.

Similarly:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 25 & 15 \\ 15 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (5\,x_1 + 3\,x_2)^2 + 3\,x_2^2$$

$\Rightarrow$ matrix is pos. def.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 25 & 15 \\ 15 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (5\,x_1 + 3\,x_2)^2$$

$\Rightarrow$ matrix is not pos. def. but only semi-definite. To see this set $x_1 = -3$, $x_2 = 5$.

H)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (4x_1 + 3x_2)^2$$

$\Rightarrow$ semi-definite.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 16 & -12 \\ -12 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (4x_1 - 3x_2)^2 + x_1^2$$

$\Rightarrow$ pos. definite.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 16 & 20 \\ 20 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (4x_1 + 5x_2)^2 - 25x_2^2$$

$\Rightarrow$ indefinite.

Problem 6:

E) We have for example

$$\begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix}^T \underbrace{\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{=A} \begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}^T \underbrace{\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}}_{A([1,3],[1,3])} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} > 0 \text{ if } A \text{ pos. def. and } [x_1 x_3] \neq [00].$$

H) $x^T A_1 x = (x_1 + x_2)^2 \Rightarrow A_1$ is only semi-definite.
A_2 has A_1 as sub-matrix $\Rightarrow A_2$ not pos. def.
A_3 is pos. def. We can show this using the method of completing the square or by the determinant criteria.
A_4 and A_5 have non positive elements on the diagonal and are therefore not positive definite.