MAT 460 – Numerical Differential Equations – Solutions to Assignment 3 (E,H)

Problem 1

E) The equation for the line through (x_0, y_0) with slope a is: $y(x) = y_0 + a(x - x_0)$. The intersection x_s of the line with the x-axis is $x_s = x_0 - y_0/a$, because $y(x_s) = 0$.
For the Newton-method we have $y_0 = f(x_0)$ and $a = f'(x_0)$, so $x_1 = x_s = x_0 - f(x_0)/f'(x_0)$. Repeating this gives the iteration method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

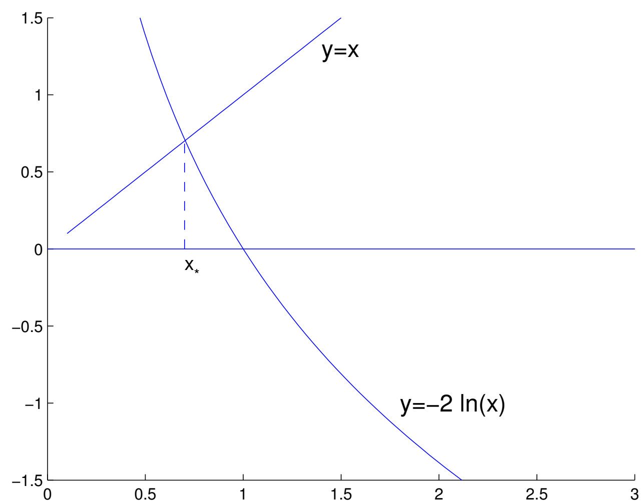
The secant method needs two initial values, x_0 and x_1. The next value is the intersection x_2 between the line through $(x_0, f(x_0))$ and $(x_1, f(x_1))$ (secant) with the x-axis. The secant is the line through $(x_1, f(x_1))$ with the slope $\frac{f(x_1)-f(x_0)}{x_1-x_0}$ so we have

$$x_2 = x_1 - \frac{f(x_1)}{\frac{f(x_1)-f(x_0)}{x_1-x_0}} = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

The rest is explained in the lecture notes.

Problem 2

E) The equation $x = -2\ln(x)$ has exactly one solution x_*, which is in the interval $]0, 1[$, see the Figure.



The derivative of $\phi(x) = -2\ln(x)$ is $\phi'(x) = -2/x$. It is $|\phi'(x)| > 1$ for $x \in]0, 1[$. In particular we have $|\phi'(x_*)| > 1$. So x_* is an repelling fix point and cannot be found by the iteration method $x_{k+1} = \phi(x_k)$. We have

$$x = -2\ln(x) \quad \Leftrightarrow -x/2 = \ln(x) \quad \Leftrightarrow \quad \underbrace{e^{-x/2}}_{=\psi(x)} = x$$

We have $|\psi'(x)| < 1$ for $x \in]0, 1[$. So x_* is a attracting fix point of ψ. The iteration method $x_{k+1} = \psi(x_k)$ converges to $x_* \approx 0.70347$ if we start in the interval $(0, 1)$.

H) The equation $x = \frac{1}{2}x^2 - 12$ has the solutions (fix points of g) $x = 6$ and $x = -4$. The derivative of g at both fix points has the absolute value >1. Both fix points are repelling and cannot be found with the iteration method $x_{k+1} = g(x_k)$.

Problem 3

E) The Newton-method for $f(x) = 0$ with $f(x) = x^2 - a$ is

$$x_{k+1} = x_k - f(x_k)/f'(x_k) = x_k - (x_k^2 - a)/(2x_k) = \underbrace{\frac{1}{2}(x_k + a/x_k)}_{=\phi(x_k)}.$$

The derivative of the function ϕ at the fix point $x_* = \sqrt{a}$ is

$$\phi'(\sqrt{a}) = \frac{1}{2}(1 - a/x^2)|_{x=\sqrt{a}} = 0$$

We conclude that the iteration has order of convergence 2, see lecture slides.

Ha) $f(x) = x^n - a \ \Rightarrow$

$$\phi(x) = x - f(x)/f'(x) = x - (x^n - a)/(n\,x^{n-1}) = \frac{1}{n}((n-1)x + a/x^{n-1}).$$

Hb) We need to check that the iteration method/function

$$\phi(x) = \frac{1}{2}(x + a/x) - (x^2 - a)^2/(8x^2)$$

has the number $\sqrt{a}$ as fix point (which is obviously the case), and that $\phi'(\sqrt{a}) = \phi''(\sqrt{a}) = 0$.

Hc) With $(**)$ we need 4 iterations, with $(*)$ we need 6 iterations to find the value $\sqrt{5} = 2.236067977499790$.

Problem 4

E)

$$\begin{aligned}
\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} &= \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \begin{bmatrix} y_k + 1 & x_k - 1 \\ y_k^2 & 2x_k y_k \end{bmatrix}^{-1} \begin{bmatrix} x_k y_k + x_k - y_k + 1 \\ x_k y_k^2 + 5 \end{bmatrix} \\
&= \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \frac{\begin{bmatrix} 2x_k y_k & -x_k + 1 \\ -y_k^2 & y_k + 1 \end{bmatrix}}{2(y_k + 1)x_k y_k - (x_k - 1)y_k^2} \begin{bmatrix} x_k y_k + x_k - y_k - 1 \\ x_k y_k^2 + 5 \end{bmatrix}.
\end{aligned}$$

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \Rightarrow \qquad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

H)

$$\begin{aligned}
\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} &= \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \begin{bmatrix} 3x_k^2 & 1 \\ 1 & -1/y_k^2 \end{bmatrix}^{-1} \begin{bmatrix} x_k^3 + y_k - 2 \\ x_k + 1/y_k \end{bmatrix} \\
&= \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \frac{\begin{bmatrix} -1/y_k^2 & -1 \\ -1 & 3x_k^2 \end{bmatrix}}{-3x_k^2/y_k^2 - 1} \begin{bmatrix} x_k^3 + y_k - 2 \\ x_k + 1/y_k \end{bmatrix}.
\end{aligned}$$

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \Rightarrow \qquad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 5/2 \end{bmatrix}.$$