Problem 2 [Variance Reduction Methods for Monte Carlo]

Use a total sample budget of $n = 1000$ to obtain Monte Carlo estimates and sample MC estimate variances for the definite integrals in 2 dimensions (x_1, x_2):

$$(a)\ exp(\sum_{i=1}^{2} 5|x_i - 0.5|)\ \ for\ x_i\ in\ [0, 1]$$

$$(b)\ cos(\pi + \sum_{i=1}^{2} 5x_i)\ \ for\ x_i\ in\ [-1, 1]$$

$$(c)\ |4x - 2| \times |4y - 2|\ \ for\ x, y\ in\ [0, 1]$$

Implement stratification and importance sampling (separately) in the Monte Carlo estimation procedures using the same sample budget $n = 1000$. Compare the 3 different Monte Carlo integral estimates and their sample variances. Discuss the quality of the Monte Carlo estimates from each method.

In [1]:

```
%matplotlib notebook
import numpy as np
import matplotlib.pyplot as plt
SAMPLE_BUDGET = 1000
SAMPLE_TIME = 100
```

In [2]:

```
def rv_gen(my_pdf,maxvalue):
    rv_list = []
    i = 0
    while i < SAMPLE_BUDGET:
        rp = np.array([np.random.rand(),np.random.rand()]).T
        if my_pdf(rp)/maxvalue > np.random.rand():
            i = i + 1
            rv_list.append(rp)
    return rv_list
```

In [3]:

```
def rv_gen_2(my_pdf,maxvalue):
    rv_list = []
    i = 0
    while i < SAMPLE_BUDGET:
        rp = np.array([np.random.rand()*2 - 1,np.random.rand()*2 - 1]).T
        if my_pdf(rp)/maxvalue > np.random.rand():
            i = i + 1
            rv_list.append(rp)
    return rv_list
```

(a) Firstly, we calculate the general MC for (a). Next, we use startified and importance sampling separately to reduce the variance.

In [4]:

```
def my_func1(x):
    if min(x) < 0 or max(x) > 1:
        return 0
    return np.exp(np.sum(5*np.abs(x - 0.5)))
```

We run the MC method for 100 times to get the average mean and variance.

In [5]:

```
a_general_mean_list = []
a_general_var_list = []
for i in range(SAMPLE_TIME):
    x = np.random.rand(SAMPLE_BUDGET,2)
    y = map(lambda t: my_func1(t),x)
    a_general_mean_list.append(np.average(y))
    a_general_var_list.append(np.var(y,ddof = 1))
```

The general MC result for (a) is

In [6]:

```
np.average(a_general_mean_list)
```

Out[6]:

```
19.972359322312212
```

The sample variance is

In [7]:

```
np.mean(a_general_var_list)
```

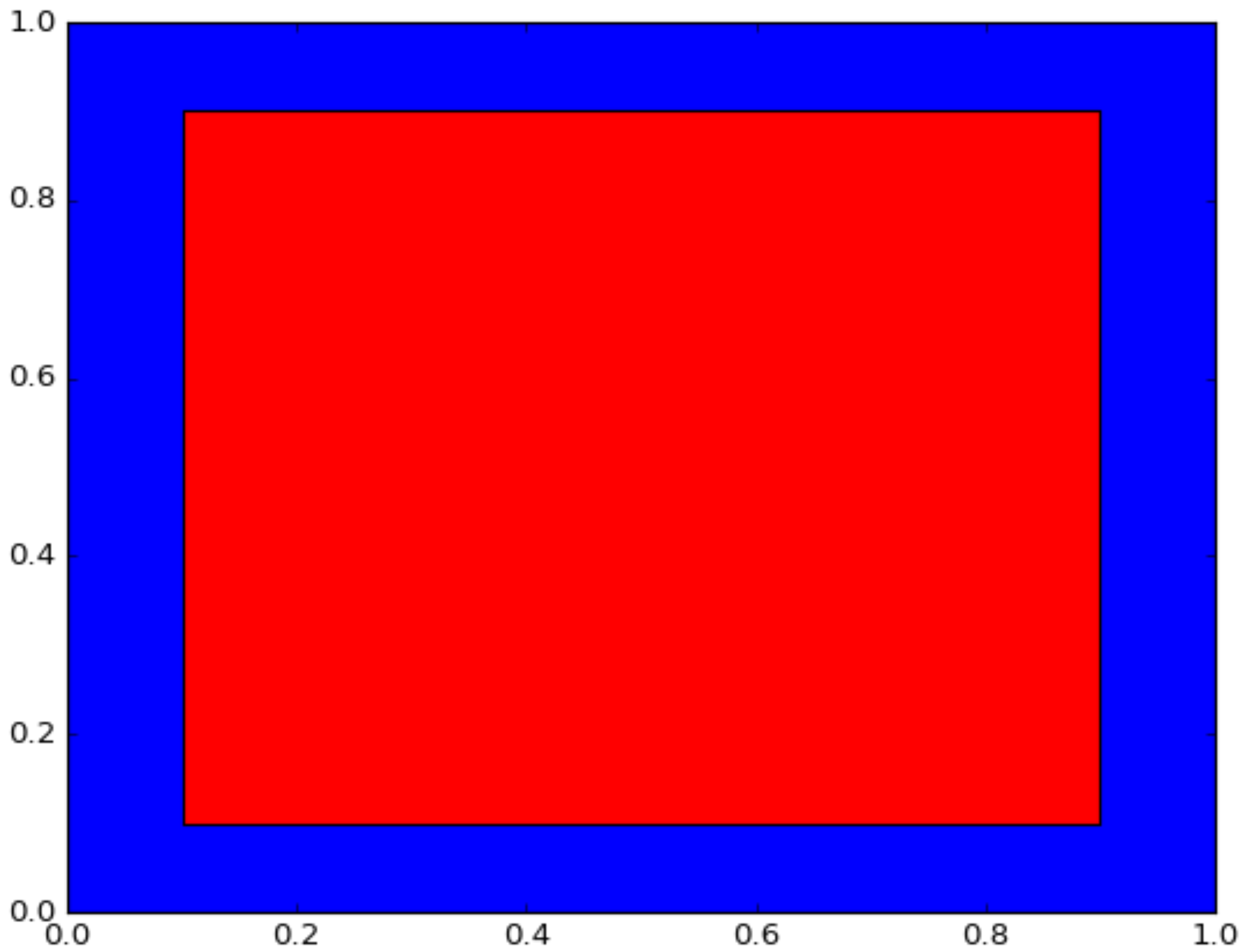
Out[7]:

```
466.32838471488236
```

For stratification sampling, we divide the integral area into 2 subinterval as showing below.

In [8]:

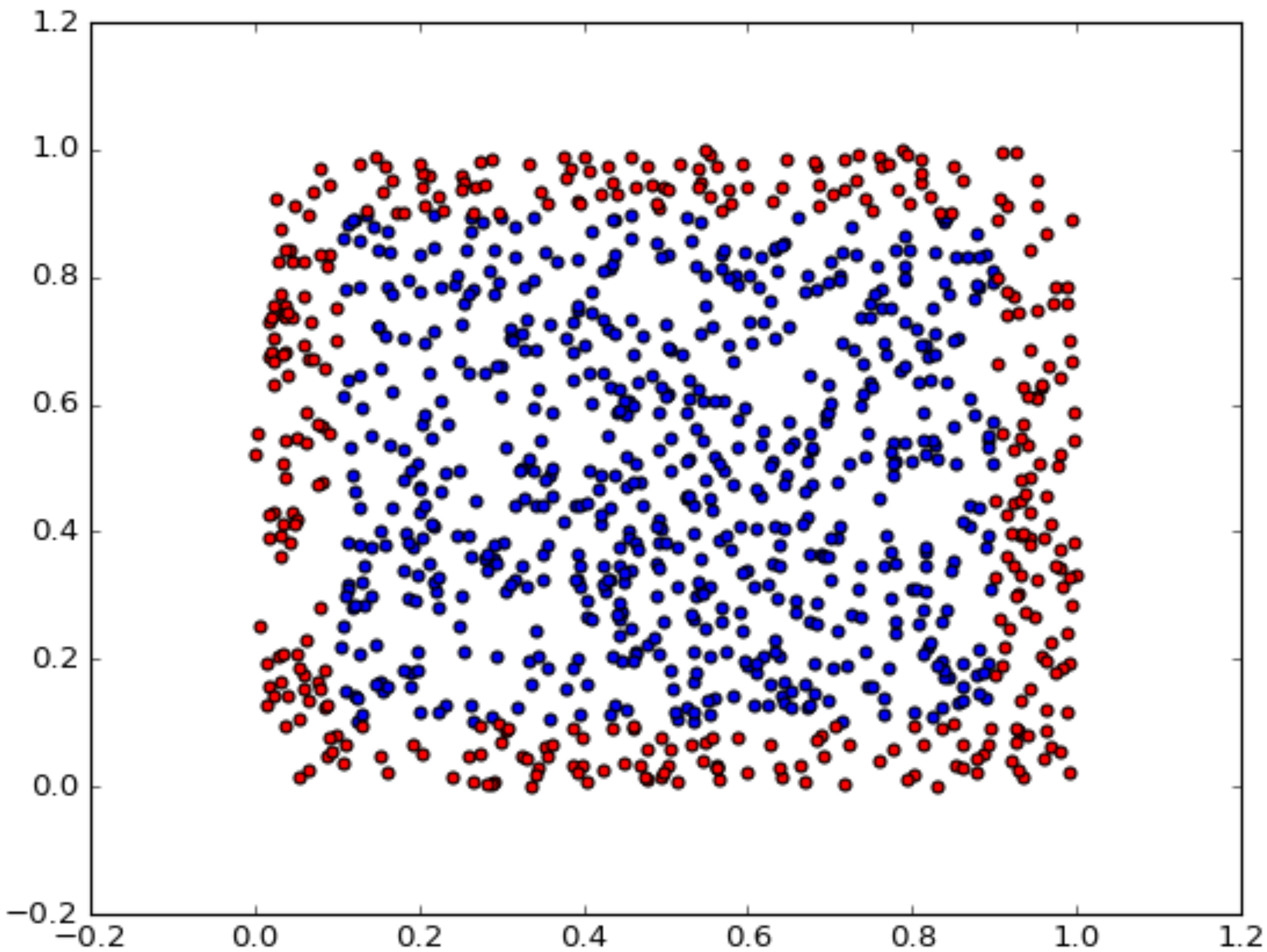
```
T = 0.4
plt.figure()
plt.fill_between([0,1],[0,0],[1,1],facecolor = 'blue')
plt.fill_between([0.5 - T,0.5 + T],[0.5 - T,0.5 - T],[0.5 + T,0.5 + T],facecol
or = 'red')
plt.show()
```



The following figure shows the random number that we used to sample the function.

In [9]:

```
x = np.random.rand(SAMPLE_BUDGET,2)
x1_index = np.logical_and(np.abs(x[:,0] -0.5) < T, np.abs(x[:,1] -0.5) < T)
x1 = x[x1_index]
x2 = x[np.logical_not(x1_index)]
plt.figure()
plt.scatter(x1[:,0],x1[:,1])
plt.hold(True)
plt.scatter(x2[:,0],x2[:,1],c='r')
plt.show()
```



Calculate the mean and variance in each stratified area. Then, we combine them to get the result for the whole area.

In [10]:

```
a_strat_mean_list = []
a_strat_var_list = []
for i in range(SAMPLE_TIME):

        # Gen random number
        x = np.random.rand(SAMPLE_BUDGET,2)
        x1_index = np.logical_and(np.abs(x[:,0] -0.5) < T, np.abs(x[:,1] -0.5)
< T)
        x1 = x[x1_index]
        x2 = x[np.logical_not(x1_index)]
        p = float(sum(x1_index))/SAMPLE_BUDGET

        # Gen samples
        y1 = map(lambda t: my_func1(t), x1)
        y2 = map(lambda t: my_func1(t), x2)

        # Get mean for each subinterval
        m1 = np.average(y1)
        m2 = np.average(y2)

        # Get var for each subinterval
        v1 = np.var(y1,ddof=1)
        v2 = np.var(y2,ddof=1)
        a_strat_mean_list.append(m1*p+m2*(1-p))
        a_strat_var_list.append(v1*p+v2*(1-p))
```

The result of stratified sampling is

In [11]:

```
np.average(a_strat_mean_list)
```

Out[11]:

```
19.939545446804583
```

The average variance of stratified sampling is

In [12]:

```
np.average(a_strat_var_list)
```

Out[12]:

```
297.10033701335038
```

We choose $f_{X_1X_2}(x_1, x_2) = 2\,|x_1 - 0.5| + 2\,|x_2 - 0.5|$, $(x_1, x_2 \in [0, 1])$ for the importance sampling of (a). Since it is close to the function in (a).

In [13]:

```
def my_pdf1(x):
    if min(x) < 0 or max(x) > 1:
        return 0
    return np.sum(np.abs(x-0.5))*2
```

In [14]:

```
a_imp_mean_list=[]
a_imp_var_list=[]
for i in range(SAMPLE_TIME):
    x = np.array(rv_gen(my_pdf1,2))
    y = map(lambda t: my_func1(t)/my_pdf1(t),x)
    a_imp_mean_list.append(np.average(y))
    a_imp_var_list.append(np.var(y,ddof = 1))
```

The importance sampling result for (a) is

In [15]:

```
np.average(a_imp_mean_list)
```

Out[15]:

```
20.062193758866098
```

The sample variance is

In [16]:

```
np.average(a_imp_var_list)
```

Out[16]:

```
158.36648364465313
```

(b) Let's work for (b). Firstly, we calculate the general MC result.

In [17]:

```
def my_func2(x):
    if min(x) < -1 or max(x) > 1:
        return 0
    return np.cos(np.pi+5*np.sum(x))
```

In [18]:

```
b_general_mean_list = []
b_general_var_list = []
for i in range(SAMPLE_TIME):
    x = np.random.rand(SAMPLE_BUDGET,2)*2 -1
    y = map(lambda t: my_func2(t)*4,x)
    b_general_mean_list.append(np.average(y))
    b_general_var_list.append(np.var(y,ddof = 1))
```

The general result for (b) is

In [19]:

```
np.average(b_general_mean_list)
```

Out[19]:

```
-0.1413668464824451
```

The sample variance is

In [20]:

```
np.average(b_general_var_list)
```

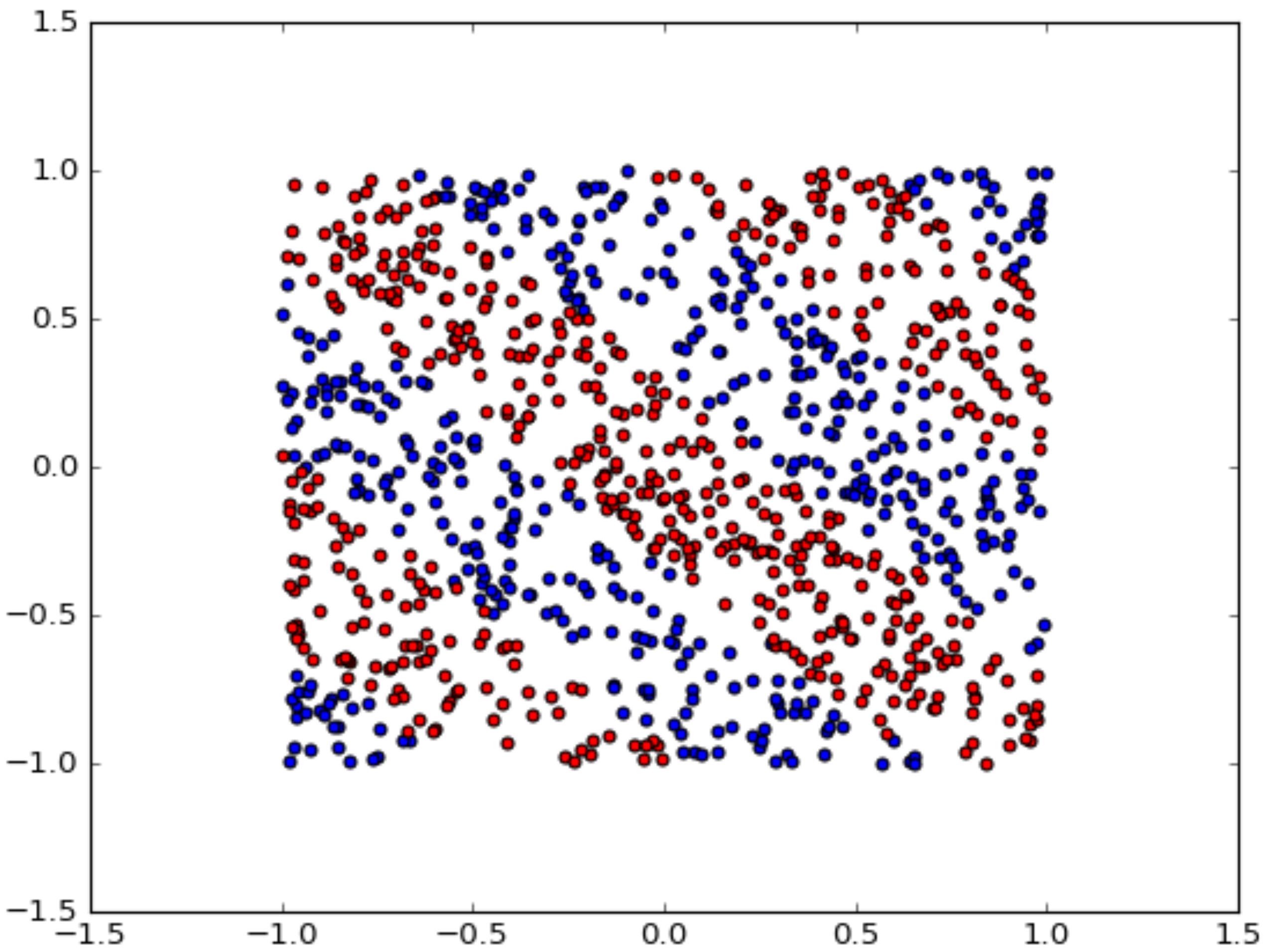
Out[20]:

```
8.0140630861041782
```

For stratified sampling, we divide the area in to 2 subarea. The first area is where the function greater than 0. The rest will be the second area.

In [21]:

```
x = np.random.rand(SAMPLE_BUDGET,2)*2 - 1
x1_index = np.array(map(lambda t: my_func2(t) > 0, x))
x1 = x[x1_index]
x2 = x[np.logical_not(x1_index)]
plt.figure()
plt.scatter(x1[:,0],x1[:,1])
plt.hold(True)
plt.scatter(x2[:,0],x2[:,1],c='r')
plt.show()
```



In [22]:

```
b_strat_mean_list = []
b_strat_var_list = []
for i in range(SAMPLE_TIME):

        # Gen random number
        x = np.random.rand(SAMPLE_BUDGET,2)*2 - 1
        x1_index = np.array(map(lambda t: my_func2(t) > 0, x))
        x1 = x[x1_index]
        x2 = x[np.logical_not(x1_index)]
        p = float(sum(x1_index))/SAMPLE_BUDGET

        # Gen samples
        y1 = map(lambda t: my_func2(t)*4, x1)
        y2 = map(lambda t: my_func2(t)*4, x2)

        # Get mean for each subinterval
        m1 = np.average(y1)
        m2 = np.average(y2)

        # Get var for each subinterval
        v1 = np.var(y1,ddof=1)
        v2 = np.var(y2,ddof=1)

        b_strat_mean_list.append(m1*p+m2*(1-p))
        b_strat_var_list.append(v1*p+v2*(1-p))
```

The stratified sampling MC result for (b) is

In [23]:

```
np.mean(b_strat_mean_list)
```

Out[23]:

```
-0.13472716014905306
```

The variance of the stratified sampling is

In [24]:

```
np.mean(b_strat_var_list)
```

Out[24]:

```
1.5191444564158161
```

Because this function varies between postive and negative, it is hard to find a good pdf that is approximately proportional to it. So we can transform the original problem to an equivalent one. Firstly, we calculate the integral of the following function.

$$1 + cos(\pi + \sum_{i=1}^{2} 5x_i) \quad for\ x_i\ in\ [-1, 1]$$

Next, we subtract 4 from the result to get the original integral. Since the integral of 1 over $[-1, 1]$ is 4.

In [25]:

```
def my_func2_2(x):
    if min(x) < -1 or max(x) > 1:
        return 0
    return 1 + np.cos(np.pi+5*np.sum(x))
```

However, this function is still hard to find an approximate pdf. The best pdf is proportional to the function itself. We choose a quantizatized one of the function as our pdf.

In [26]:

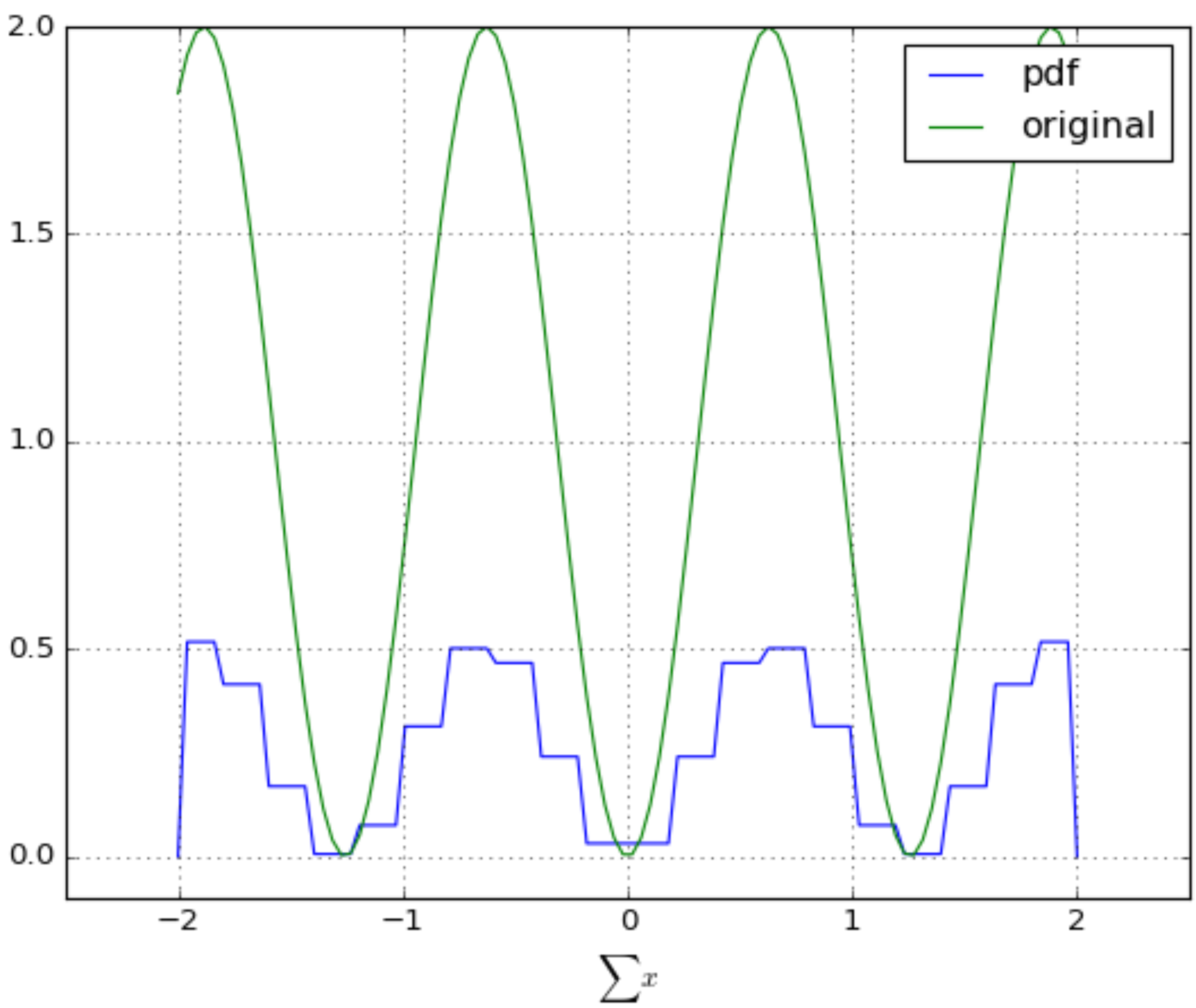
```
NS = 20
xs = np.arange(-2 + 2.0/NS, 2, 4.0/NS)
ys = np.r_[map(lambda t: my_func2_2(np.r_[t/2.0,t/2.0].T),xs)]
ys = np.abs(ys)
NORMAL_FACTOR = sum(2*ys*(4.0 - 4.0/NS - 8.0/NS*np.r_[(NS/2-1):0:-1,0,0:(NS/2)
])/NS)
MAX_VALUE = max(ys)
```

In [27]:

```
def my_pdf2(x):
    if min(x) <= -1 or max(x) >= 1:
        return 0
    return ys[int(np.floor(sum(x)/(4.0/NS))+NS/2)]/NORMAL_FACTOR
    #return 1
```

In [28]:

```
plt.figure()
xp = np.linspace(-2,2,100)
yp = map(lambda t: my_func2_2(np.r_[t/2.0,t/2.0].T),xp)
ysp = map(lambda t: my_pdf2(np.r_[t/2.0,t/2.0].T),xp)
plt.hold(True)
plt.plot(xp,ysp)
plt.plot(xp,np.abs(yp))
plt.axis([-2.5,2.5,-0.1,2])
plt.legend(['pdf','original'])
plt.xlabel('$\sum x$')
plt.grid()
plt.show()
```



In [29]:

```
b_imp_mean_list=[]
b_imp_var_list = []
for i in range(SAMPLE_TIME):
    x = np.array(rv_gen_2(my_pdf2,MAX_VALUE))
    y = map(lambda t: my_func2_2(t)/my_pdf2(t), x)
    b_imp_mean_list.append(np.average(y))
    b_imp_var_list.append(np.var(y,ddof = 1))
```

The importance sampling result for (b) is

In [30]:

```
np.average(b_imp_mean_list) - 4
```

Out[30]:

```
-0.14902014017738496
```

The sample variance is

In [31]:

```
np.mean(b_imp_var_list)
```

Out[31]:

```
1.4911465089824976
```

(c) Again, we firstly calculate the general MC.

In [32]:

```
def my_func3(x):
    if min(x) < 0 or max(x) > 1:
        return 0
    return np.abs(4*x[0] - 2) * np.abs(4*x[1] - 2)
```

In [33]:

```
PDF2N = - (9*np.pi**6)/10000 + (259*np.pi**4)/3750 - (112*np.pi**2)/75 + 64.0/
7
PDF2MAX = -(((2*7**(0.5)*np.pi**2)/75 - (2*np.pi**2)/15)*((2*7**(0.5)*np.pi**2
)/75 + (2*np.pi**2)/75)*((2*7**(0.5)*np.pi**2)/75 + (8*np.pi**2)/75))

def my_pdf2_1(x):
    if min(x) < -1 or max(x) > 1:
        return 0
    s = sum(x)
    #return (s*s - np.pi**2/100)*(s*s - np.pi**2*9/100)*(s*s - 25/100 * np.pi*
*2)/PDF2N
    return 1
MAX_VALUE = 1
```

In [34]:

```
c_general_mean_list = []
c_general_var_list = []
for i in range(SAMPLE_TIME):
    x = np.random.rand(SAMPLE_BUDGET,2)
    y = map(lambda t: my_func3(t),x)
    c_general_mean_list.append(np.average(y))
    c_general_var_list.append(np.var(y,ddof = 1))
```

The general result for (c) is

In [35]:

```
np.average(c_general_mean_list)
```

Out[35]:

```
1.0007998601123453
```

The sample variance is

In [36]:

```
np.average(c_general_var_list)
```

Out[36]:

```
0.77575139722885256
```

We use the same stratifing schedule in **(a)**.

In [37]:

```
c_strat_mean_list = []
c_strat_var_list = []
for i in range(SAMPLE_TIME):

        # Gen random number
        x = np.random.rand(SAMPLE_BUDGET,2)
        x1_index = np.logical_and(np.abs(x[:,0] -0.5) < T, np.abs(x[:,1] -0.5)
< T)
        x1 = x[x1_index]
        x2 = x[np.logical_not(x1_index)]
        p = float(sum(x1_index))/SAMPLE_BUDGET

        # Gen samples
        y1 = map(lambda t: my_func3(t), x1)
        y2 = map(lambda t: my_func3(t), x2)

        # Get mean for each subinterval
        m1 = np.average(y1)
        m2 = np.average(y2)

        # Get var for each subinterval
        v1 = np.var(y1,ddof=1)
        v2 = np.var(y2,ddof=1)
        c_strat_mean_list.append(m1*p+m2*(1-p))
        c_strat_var_list.append(v1*p+v2*(1-p))
```

The stratified sampling result for **(c)** is

In [38]:

```
np.mean(c_strat_mean_list)
```

Out[38]:

```
0.99763951790286565
```

The variance of (c) for stratified sampling is

In [39]:

```
np.mean(c_strat_var_list)
```

Out[39]:

```
0.54310101648507181
```

We choose $f(x_1, x_2) = 4\,|x_1 - 0.5|$ for the importance sampling.

In [40]:

```
def my_pdf3(x):
    if min(x) < 0 or max(x) > 1:
        return 0
    return 4 * np.abs(x[0]-0.5)
```

In [41]:

```
c_imp_mean_list=[]
c_imp_var_list=[]
for i in range(SAMPLE_TIME):
    x = np.array(rv_gen(my_pdf3,2))
    y = map(lambda t: my_func3(t)/my_pdf3(t),x)
    zx = np.logical_or(x < 0,x > 1)
    zy = np.logical_or(zx[:,0],zx[:,1])
    y = y*np.logical_not(zy)
    c_imp_mean_list.append(np.average(y))
    c_imp_var_list.append(np.var(y,ddof = 1))
```

The importance sampling result for (c) is

In [42]:

```
np.average(c_imp_mean_list)
```

Out[42]:

```
1.0013596121812709
```

The sample variance is

In [43]:

```
np.mean(c_imp_var_list)
```

Out[43]:

```
0.33147887572116125
```

In [46]:

```
print("(%c) general mean: %.4f \t stratified mean: %.4f \t importance mean: %.4f"%('a',np.mean(a_general_mean_list),np.mean(a_strat_mean_list),np.mean(a_imp_mean_list)))
print("(%c) general mean: %.4f \t stratified mean: %.4f \t importance mean: %.4f"%('b',np.mean(b_general_mean_list),np.mean(b_strat_mean_list),np.mean(b_imp_mean_list)-4))
print("(%c) general mean: %.4f \t stratified mean: %.4f \t importance mean: %.4f"%('c',np.mean(c_general_mean_list),np.mean(c_strat_mean_list),np.mean(c_imp_mean_list)))
```

```
(a) general mean: 19.9724       stratified mean: 19.9395        i
mportance mean: 20.0622
(b) general mean: -0.1414       stratified mean: -0.1347        i
mportance mean: -0.1490
(c) general mean: 1.0008        stratified mean: 0.9976         i
mportance mean: 1.0014
```

In [45]:

```
print("(%c) general var: %.4f \t stratified var: %.4f \t importance var: %.4f"%('a',np.mean(a_general_var_list),np.mean(a_strat_var_list),np.mean(a_imp_var_list)))
print("(%c) general var: %.4f \t stratified var: %.4f \t importance var: %.4f"%('b',np.mean(b_general_var_list),np.mean(b_strat_var_list),np.mean(b_imp_var_list)))
print("(%c) general var: %.4f \t stratified var: %.4f \t importance var: %.4f"%('c',np.mean(c_general_var_list),np.mean(c_strat_var_list),np.mean(c_imp_var_list)))
```

```
(a) general var: 466.3284       stratified var: 297.1003        i
mportance var: 158.3665
(b) general var: 8.0141         stratified var: 1.5191          i
mportance var: 1.4911
(c) general var: 0.7758         stratified var: 0.5431          i
mportance var: 0.3315
```

We can see that both method reduce the variance. The importance sampling is better than the stratified sampling in the ability of reducing the variance. However, it is hard to find a pdf for (b) that approximately proportional to the function. In this case, the importance sampling may not as easy as stratified sampling for implementation.