Application of largest Lyapunov exponent analysis on the studies of dynamics under external forces

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Abstract

Dynamics of driven dissipative Frenkel-Kontorova model is examined by using largest Lyapunov exponent computational technique. Obtained results show that besides the usual way where behavior of the system in the presence of external forces is studied by analyzing its dynamical response function, the largest Lyapunov exponent analysis can represent a very convenient tool to examine system dynamics. In the dc driven systems, the critical depinning force for particular structure could be estimated by computing the largest Lyapunov exponent. In the dc+ac driven systems, if the substrate potential is the standard sinusoidal one, calculation of the largest Lyapunov exponent offers a more sensitive way to detect the presence of Shapiro steps. When the amplitude of the ac force is varied the behavior of the largest Lyapunov exponent in the pinned regime completely reflects the behavior of Shapiro steps and the critical depinning force, in particular, it represents the mirror image of the amplitude dependence of critical depinning force. This points out an advantage of this technique since by calculating the largest Lyapunov exponent in the pinned regime we can get an insight into the dynamics of the system when driving forces are applied.

Keywords: Frenkel-Kontorova model, Josephson junctions arrays, Shapiro steps, Lyapunov analysis

1. Introduction

The Frenkel - Kontorova (FK) model is widely used to describe systems where competition between length scales determines the ground state energy. Dissipative FK model has been often used as one of the most suitable models for description of different kinds of phenomena in many fields of physics, such as charge or spin density wave systems [1–6], vortex matter [7, 8], Josephson-junction arrays biased by external currents [9–13] and in recent years even superconducting nanowires [14, 15]. The one-dimensional FK model is a simple clas-sical model which describes a chain of particles, usually identical, coupled to their nearest neighbors and subjected to the periodic or quasiperiodic [16, 17] on-site substrate potential. In particular, in the standard FK model, interaction between particles is harmonic, and the on-site substrate potential is sinusoidal. Although very simple, the one-dimensional standard FK model exhibits rich dynamics when it is subjected to external driving forces [18, 19].

The FK model has two distinct types of system ground states, commensurate and incommensurate ones, (in the commensurate grounds states interparticle average distance is a rational number, while for the incommensurate ones it is irrational). If an external dc driving force is applied (the dc driven FK model), there exists a critical threshold value i.e. the critical depinning force F_c which

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separates two different dynamical regimes: the pinned and the sliding regimes. The type of the structure, whether it is commensurate or incommensurate determines depinning transition. In the commensurate structures, particles exhibit the motion with stick and slip intervals or intermittent behavior (Intermittencies type I) [18, 19]. On the other hand, in incommensurate structures, the depinning appears only if the system is above Aubry transition while if it is below, there is no depinning transition, i.e. under any force $F > 0$ particles are in sliding regime [18–20].

In the case of dc+ac driven FK model, the dynamics is characterized by the appearance of the staircase macroscopic response or Shapiro steps in the response function $\bar{v}(\bar{F})$ of the system [20–22]. These steps appear due to interference or dynamical mode-locking of the internal frequency (that comes from the motion of particles over periodic substrate potential) with the frequency of external ac force.

In the case of incommensurate structures, the ac-driven dissipative dynamics exhibits the dynamical Aubry transition [18–20, 22]. The dynamical hull function that describes the driven structure becomes nonanalytical above the transition point and the result of this is the dynamical locking $\bar{v}(\bar{F})$ at certain resonant values. The dynamical mode locking for the commensurate and incommensurate structures, respectively, appear to be one of the universal features of the systems with the competition of time scales in the ac-driven dynamics.

Critical depinning force and the Shapiro steps have been the subjects of our previous investigations [19, 23–26]. The standard way to examine dynamics of a driven system is by analyzing the response function. However, this analysis could be very difficult since in the response function of the systems, Shapiro steps particularly the subharmonic ones are often hardly visible. In realistic systems such as the CDW systems and the systems of Josephson-junction arrays, particularly in the experimental examinations, instead of analyzing the current-voltage characteristics where Shapiro steps could be hardly detected, examination of differential resistance provides more accurate way to determine the presence and properties of the Shapiro steps [1–3, 9]. In general, in studies of ac-driven systems, the main interest is always focused on the existence and robustness (structural stability) of the resonant solutions against the varying of the system parameters, and therefore, an accurate detection of Shapiro steps have always been of great importance for both theoretical and experimental studies.

In this paper, we will show that dynamics of the dc+ac driven system could be examined in detail and just as successfully by using the largest Lyapunov exponent computation technique. The largest Lyapunov exponent not only reveals the existence of Shapiro steps but also provides an insight into dynamical response of the system and properties of the Shapiro steps. The paper is organized as follows: The model is introduced in Sec. II, simulation results are presented in Sec. III. Finally, Sec. IV concludes the paper.

2. Model

We consider the dissipative (overdamped) dynamics of a series of coupled harmonic oscillators u_l subjected to the standard sinusoidal substrate potential:

$$V(u) = \frac{K}{(2\pi)^2}[1 - \cos(2\pi u)], \qquad (1)$$

The sinusoidal potential is usually written in this form so that K is the pinning strength and the period is 1, $V(u + 1) = V(u)$ [18, 19, 22]. The potential $V(u)$ given in Eq. (1) and the corresponding force $F(u) = -\frac{dV}{du}$ are presented in Figure 1.
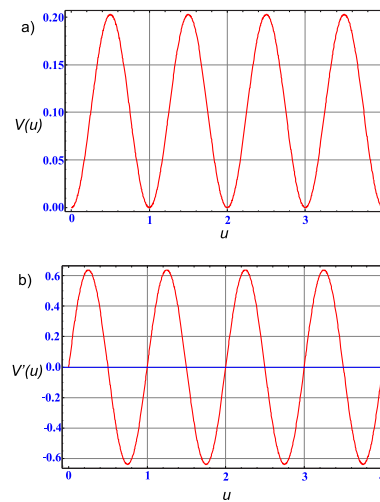

Figure 1: a) Sinusoidal potential described in Eq(1) for $K = 4$; b) Its corresponding force $F(u) = -\frac{dV}{du}$.

The system is driven by dc and ac forces:

$$F(t) = F_{dc} + F_{ac} \cos(2\pi\nu_0 t). \qquad (2)$$

where F_{dc} is the dc force, while F_{ac} and $2\pi\nu_0$ are the amplitude and the circular frequency of the ac force, respectively. Since the ac force $F_{ac} \cos(2\pi\nu_0 t)$ drives the system of particles forward and back during the time of one period $\tau = \frac{1}{\nu_0}$, the average driving force is $\bar{F} = F_{dc}$.

The equations of motion are:

$$\dot{u}_l = u_{l+1} + u_{l-1} - 2u_l + \frac{K}{2\pi} \sin(2\pi u_l) + F(t), \qquad (3)$$

where $l = -\frac{N}{2}, ..., \frac{N}{2}$, u_l is the position of lth particle. This system of equations is normalized in such a way that the parameters which characterize overdamped dynamics are the strength of potential (K) and characteristics of the applied force (F_{ac} and ν_0) [18, 19, 22].

When the system is driven by the forces given in Eq. (2), two different frequency scales appear in the system: the frequency of the external periodic force ν_0 and the characteristic frequency of the motion of particles over the sinusoidal potential driven by the average force F_{dc}. The competition between those frequency scales can result in the appearance of resonance (dynamical mode-locking or Shapiro steps). The steps are called harmonic if the locking appears for integer values of frequency, whereas if it appears for the rational noninteger values of the frequency they are called subharmonic. If $\{u_l(t)\}$ is steady state solution of (3), the transformation

$$\sigma_{i,j,m}\{u_l(t)\} = \{u_{l+i}(t - \frac{m}{\nu_0}) + j\} = \{u'_l(t)\} \qquad (4)$$

produces another steady state solution, where i, j, m are integers.

If solution is invariant under a symmetry operation (4)

$$\sigma_{i,j,m}\{u_l(t)\} = \{u_l(t)\} \qquad (5)$$

then solution of the system (3) is called resonant and its (particle and time) average velocity $\bar{v}$ satisfies the following relation:

$$\bar{v} = \frac{i\omega + j}{m}\nu_0, \qquad (6)$$

$m = 1$ for harmonic, and $m > 1$ for subharmonic steps, while $\omega = \langle u_{l+1} - u_l \rangle$ is the average interparticle distance

also known as the winding number (ω is rational for the commensurate structures and irrational for the incommensurate ones). The triplet i, j, and m is unique only for incommensurate structures [18].

The equations of motion (3) have been integrated using the fourth order Runge-Kutta method with periodic boundary conditions. The time step used in the simulation was 0.01τ, where $\tau = \frac{1}{\nu_0}$. The force is varied adiabatically with the step $\Delta F_{dc} = 1 \times 10^{-5}$.

We will focus on the calculation of the largest Lyapunov exponent, which we denote by λ. Detailed explanation of our approach and numerical procedure can be found in Ref. [27]. In further text, we will refer to the quantity of largest Lyapunov exponent as the Lyapunov exponent (LE) for convenience. The LE will be analyzed in different regimes and for different parameters of the systems. It is well known that the LE can indicate the chaotic behavior in dynamical systems, and therefore, the presented studies will give us also an information about the presence or absence of chaos in the driven dissipative one-dimensional FK model.

3. Results

We present a comparative study of system dynamics under external forces using the response function and LE as a function of average driving force.

3.1. dc driven system

First, we will consider the simplest case, the dc driven standard FK model. In Figure 2, the response function $\bar{v}(F_{dc})$ and the Lyapunov exponent $\lambda(F_{dc})$ are presented for the same range of the applied dc force and two different commensurate structures. A clear dynamical transition point between the pinned and the sliding regime can be observed. Naturally, in the case of the response function $\bar{v}(F_{dc})$, this transition happens when the average velocity reaches the non-zero value $\bar{v} \neq 0$ for the lowest F_{dc} in Figure 2 (a) and (b), while in the other case in Figure 2 (c) and (d), the transition is identified by the point where LE becomes zero $\lambda = 0$. Given that the system is dc driven, the LE will keep its zero value in the whole sliding regime. The critical depinning force in the case of $\omega = 1$ (Figure 2 (a)) approximately has the value as $\max(V'(u))$ (see Figure 1 (b)). The system is more stable
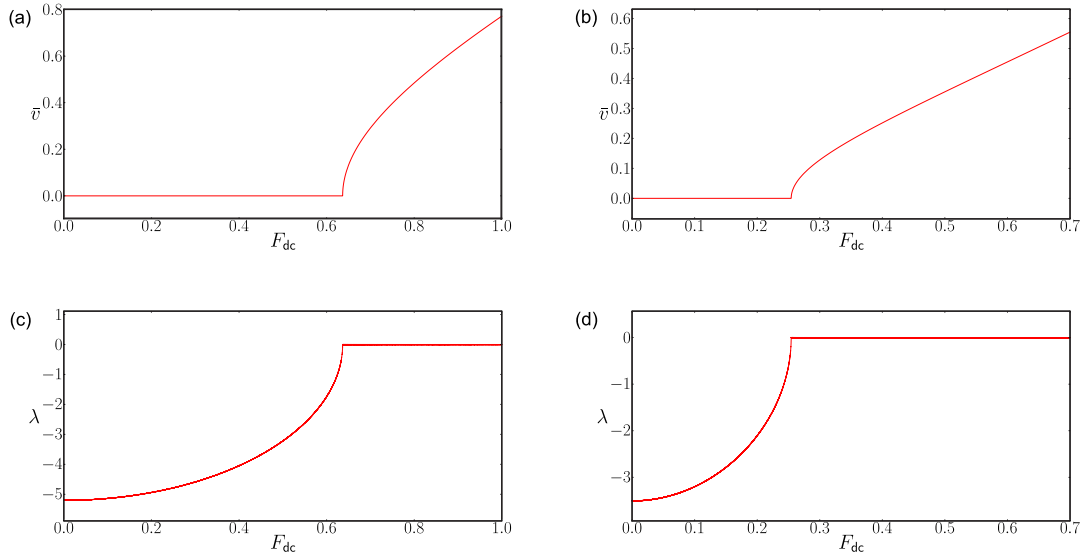
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Figure 2: Average velocity as a function of average driving force $\bar{v}(F_{dc})$ and Lyapunov exponent as a function of average driving force $\lambda(F_{dc})$ for $K = 4$, $F_{ac} = 0$ (dc driven system) and two different commensurate structures $\omega = 1$ in (a) and (c) and $\omega = 1/2$ in (b) and (d).

in the case of $\omega = 1$ than in the case of $\omega = 1/2$, which means that the force required to depin the system is larger. This comes as no surprise given that in the first case, we have one particle, while in the second one, we have two particles per potential well, and competing interactions between the particles makes the system less stable and therefore, easier to depin. Influence of particular config-uration (winding number) on the critical depinning force has been examined in our previous work [28], where it can be seen that in the standard case, the system is the most robust for $\omega = 1$, while as the number of particles per potential well increases (denominator of winding number increases) it becomes less stable. An equivalent obser-vation can be made from the Lyapunov exponent in this context. We can say something about the stability of a particular configuration just by looking at the Lyapunov exponent for $F_{dc} = 0$ when no dc driving force is applied. For the same value of F_{dc}, the LE is more negative in the case $\omega = 1$ where $\lambda \approx -5$ than in the case $\omega = 1/2$ where $\lambda \approx -3.5$. As the LE gets more negative the configuration is going to be more robust to the external disturbances or equivalently the more negative the LE is, the more quickly the trajectory returns to its unperturbed path.

If we consider only the static case (there is no driv-ing force $\bar{F} = 0$), we can examine how different types of structures or potential strength influence the LE. In Figure 3, the Lyapunov exponent as a function of pin-ning strength for two different commensurate structures is presented. As the pinning strength K increases or if the
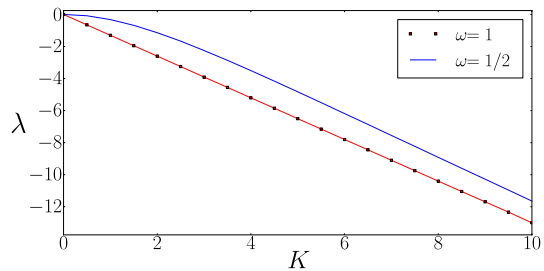

Figure 3: Lyapunov exponent as a function of pinning strength K with-out driving force for (a) $\omega = 1$ and (b) $\omega = 1/2$.

commensurate structure is more stable such as the one for $\omega = 1$, λ becomes more negative. This means that much higher force will have to be applied in order to set the

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particles into collective motion.

3.2. dc+ac driven system

In the case of the dc+ac driven FK model, the dynamics is characterized by the appearance of Shapiro steps. We will examine both the commensurate and incommensurate structures.

In Figure 4, the response function and the corresponding Lyapunov exponent as a function of average driving force for two different commensurate structures are presented. As we can see, in the pinned regime for the same F_{dc}, LE is more negative for more stable structures such as $\omega = 1$. In the sliding regime, negative values of Lyapunov exponent correspond to the Shapiro steps (the region of F_{dc} for which $\lambda < 0$ in Figure 4 (c) and (d) corresponds to the same region of F_{dc} for which steps appear in Figure 4 (a) and (b)).

Since this is the standard FK model, the subharmonic Shapiro steps appear only for noninteger, rational values of ω, while for the integer ones, only harmonic steps exist [29]. As we can see in Figure 4 (a) and (b), for both commensurate structures $\omega = 1$ and $\omega = 1/2$, only harmonic steps are visible which might lead to conclusion that there are no subharmonic steps. However, if we enlarge the region between the first and the second harmonics, results obtained for LE reveal the presence of subharmonic steps for $\omega = 1/2$. In Figure 5, the enlarged regions between the first and second harmonic of the LE in Figure 4 (c) and (d) are presented. For comparison, the corresponding response functions are given in the insets.

In the case of $\omega = 1$, there is no subharmonic mode locking as Figure 5 (a) clearly shows. When ω is noninteger in Figure 5 (b), only the halfinteger step $\frac{3}{2}$ appears in the response function, while on the other hand, the Lyapunov exponent reveals a whole series of subharmonic steps. It was shown in our previous work that those subharmonic steps appear according to Farey rule [27]. In order to better understand the dynamics, let us consider how particles move in the ac+dc driven FK model. The particle motion in the pinning ($F_{dc} < F_c$) and the sliding regime ($F_{dc} > F_c$) is presented in Figure 6 (a) and (b) respectively.

In the pinning regime the particles and their relative distance to each other (i.e. first and second particle coordinates u_1 and u_2 respectively) are bounded in Figure 6(a), while on the other hand in the sliding regime in Figure 6

(b) they are not. The clumping of the data points in the second figure actually points at the ac character of the dynamics where the particles move for the first part of the period forward and the second one backward but overall they are moving forward.

In general, in the examination of the properties of Shapiro steps, particular attention has been given to their amplitude dependence.

Numerous theoretical and experimental works done in the charge density wave systems [2, 3, 5] and the systems of Josephson-junction arrays [30] have been dedicated to the amplitude dependence of the Shapiro steps and the critical depinning force. It is well known that Shapiro steps width and the critical depinning force exhibit Bessel like oscillations with the amplitude of the ac force where maxima of one curve corresponds to the minima of the other [23]. Since LE strongly reflects the presence of the Shapiro steps we will further examine how the ac amplitude affects the LE.

In the usual approach, amplitude dependence of critical depinning force and the step width is determined from the response function $\bar{v}(\bar{F})$ of the dc+ac driven system where both forces are applied. On the other hand, in studies of the amplitude dependence of LE, we will consider the pinned state where the dc force is zero $F_{dc} = 0$, and in the computation of the Lyapunov exponent only the range of F_{ac} is explored. The critical depinning force F_c, the size of the first harmonic step ΔF_1 and the corresponding Lyapunov exponent as a function of the ac amplitude are presented in Figure 7 for two different commensurate structures.

In Figure 7 (a) and (b), we have typical Bessel-like oscillations of the critical depinning force F_c and the step size ΔF_1 with the amplitude of ac force. When compared with these results, the amplitude dependence of LE in Figure 7 (c) and (d) shows an interesting property. As we can see, the well known Bessel-like oscillations in Figure 7 (a) and (b) are also reveled in the amplitude dependence of Lyapunov exponent in (c) and (d). *Lyapunov exponent represents the mirror image of the amplitude dependence for critical depinning force.* Considering that the LE is determined in the pinning regime without any need to apply dc force and explore dynamical transitions through a big range of F_{dc} values for various ac amplitudes in order to get response function and extract the general behavior, the above results show that LE computational technique
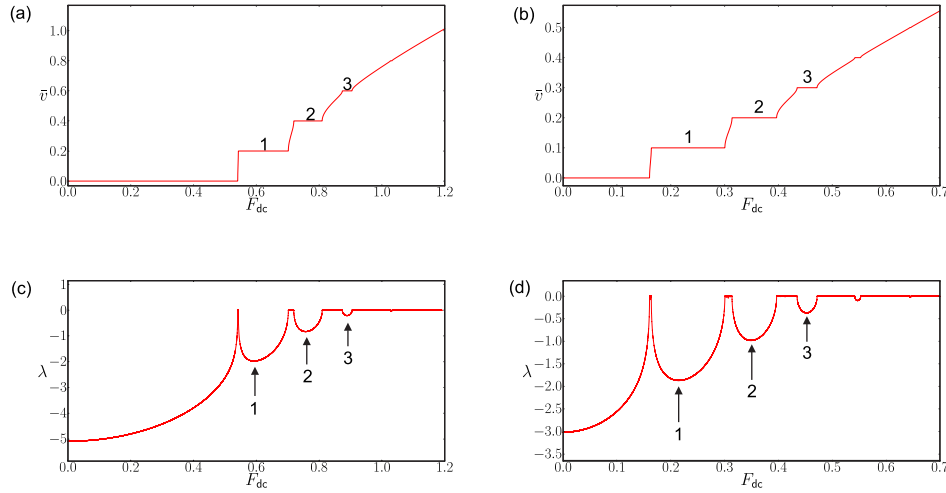
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Figure 4: Average velocity (a) and (b) and the Lyapunov exponent (c) and (d) as a function of average driving force for $K = 4$, $F_{ac} = 0.2$, $v_0 = 0.2$ and two different commensurate structures $\omega = 1$ in (a) and (c) and $\omega = 1/2$ in (b) and (d). Numbers mark first, second and third harmonics. Triplets of integers (i, j, m) in Eq. 6 that determine the first three harmonic steps in case of $\omega = 1$ are $(1, 0, 1)$, $(2, 0, 1)$, $(3, 0, 1)$ respectively and in case $\omega = 1/2$ $(1, 0, 1)$, $(2, 0, 1)$, $(3, 0, 1)$. These triplets are not unique in the case of commensurate structures.

might be a very convenient tool for exploring dynamics of driven FK model. It strongly hints at the Bessel-like nature of the amplitude dependence of critical depinning force and step width without extensive computational effort.

In the case of incommensurate structures, the winding number is irrational. We will therefore consider the inverse golden mean $\omega = \frac{\sqrt{5}-1}{2}$ [20], which is best approximated by the ratios of two successive Fibonacci numbers, but optimally excluding the first few members of the sequence. We therefore, use the winding number $\omega = \frac{89}{144}$ in our analysis. In Figure 8, the response functions for the set of parameters $K = 4$, $v_0 = 0.2$, $\omega = \frac{89}{144}$ are presented for different values of amplitude of the ac force F_{ac}, where the insets represents the LE for the same interval of F_{dc} (the values of parameters are chosen so that system is above Aubry transition [20, 31] since for incommensurate structure below Aubry transition no mode locking is possible [22]).

As the ac amplitude starts to increase from zero, the steps start to develop, changing in their size and number as the ac amplitude changes. The unique minimal triplets of integers (i, j, m) in Eq. 6 that determine the first

three harmonic steps in Figure 8 are $(-3, 2, 1)$, $(2, -1, 1)$, $(-1, 1, 1)$ [31]. If we analyze only the response function we might think that the large harmonic steps are usually the only one that appear (some very small subharmonic steps might appear at the high amplitudes). However, if instead, we calculate the corresponding Lyapunov exponent for the same set of parameters and intervals of force we can detect the presence of resonances which have been invisible on the response function. These results in Figure 8 clearly show the advantages of LE computational technique.

Though the LE analysis is applied here to investigate the properties of Shapiro steps it also gives us a possibility to investigate weather the model exibits chaotic behavior. Presence of chaos has been studied in the Josephson junction systems where even on-step positive Lyapunov exponent was found [13, 32]. It is interesting that in our studies of the *overdamped* dc+ac driven FK model, even in the case of incommensurate structures and the large values of ac amplitude F_{ac}, chaos has never been detected. The main reasons for the absence of chaos are the over-damped character of the model and the Middleton's no-passing rule [22, 33–35]. The rule states that dynamics
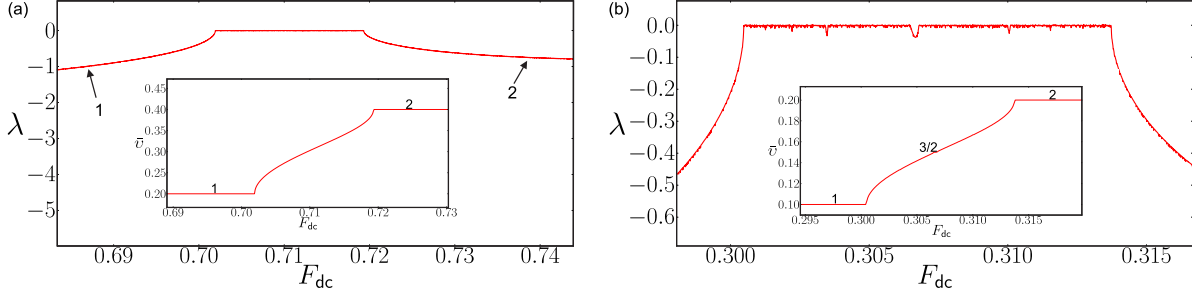
Figure 5: Lyapunov exponent as a function of average driving force between first and second harmonics for $K = 4$, $F_{ac} = 0.2$, $v_0 = 0.2$ and two different commensurate structures $\omega = 1$ at (a) and $\omega = 1/2$ at (b). The insets show the corresponding average velocities.
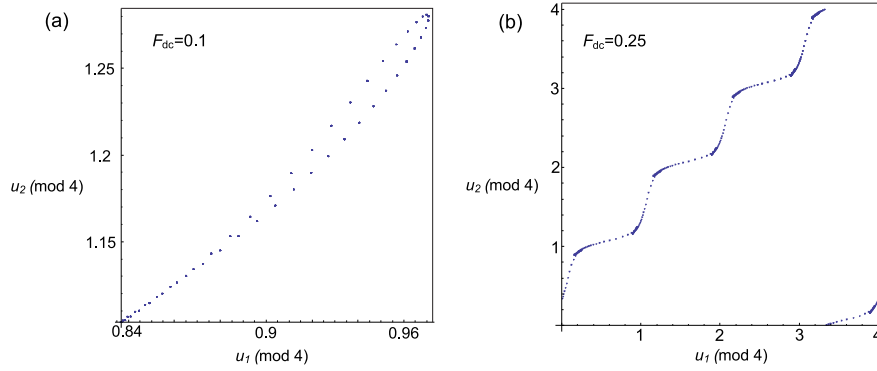

Figure 6: Particle motion for $F_{ac} = 0.2$, $v_0 = 0.2$, in pinning $F_{dc} = 0.1$ (a), and sliding $F_{dc} = 0.25$ (b) regimes.

preserve partial order relation among different configurations, i.e. if $\{u_l(t_0)\} < \{\tilde{u}_l(t_0)\}$ then $\{u_l(t)\} < \{\tilde{u}_l(t)\}$ and it is satisfied in one dimensional dc+ac driven FK model in dissipative limit. This rule implies that structures do not evolve toward less regularity than they already possess, and that the dissipative character of the dynamics smooths out any spatial complexity caused by the time evolution.

4. Conclusion

In this paper, dynamics of the driven overdamped FK model is analyzed by using Lyapunov exponent computational technique. The obtained results have shown that it was often sufficient to calculate the largest Lyapunov exponent in order to get an insight into the system behavior.

In the dc driven systems, just by looking at the Lyapunov exponent when no driving force is applied we can estimate how the system will respond to the external force, the more negative Lyapunov exponent is, the more robust system will be. In the dc+ac driven systems, calculation of LE represents a very convenient tool to detect presence of any dynamical mode-locking or Shapiro steps. Dependence of the LE on the ac amplitude obtained in the pinned regime represents a mirror image of the amplitude dependance of the critical depinning force. This clearly shows the advantages of the LE computational technique, since we can work from the pinned state and without actually driving the system get an insight into dynamics and the properties of Shapiro steps which appear when dc+ac forces are applied.
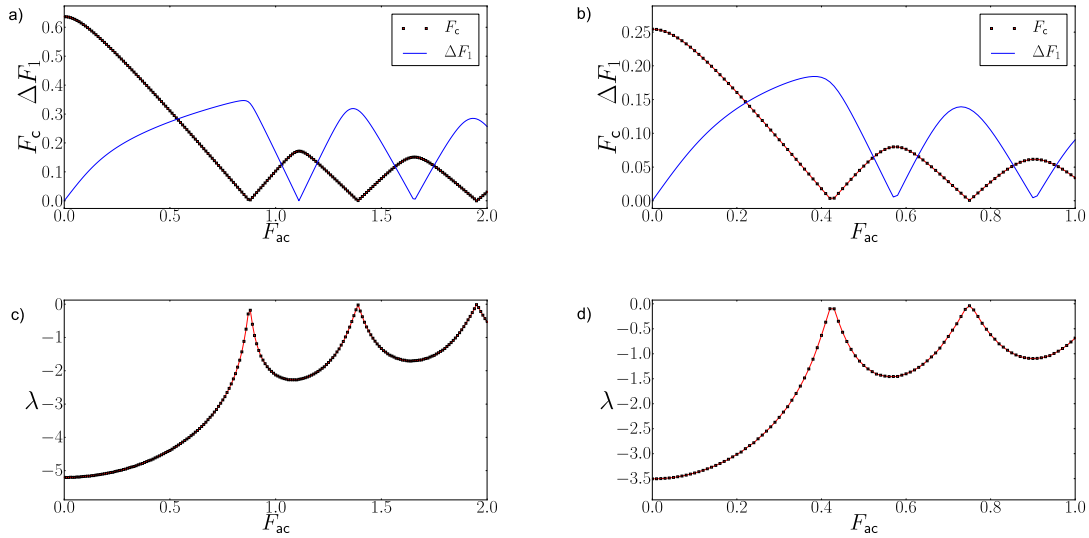
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Figure 7: (a),(b) Critical force F_c and harmonic step width ΔF_1 as a function of ac amplitude F_{ac} for $K = 4$, $v_0 = 0.2$ for $\omega = 1$ and $\omega = 1/2$ respectively. Whereas (c),(d) represent Lyapunov exponent λ for $F_{dc} = 0$ as a function of ac amplitude for $K = 4$, $v_0 = 0.2$ for $\omega = 1$ and $\omega = 1/2$ respectively.

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The presented results could be important not only for driven dissipative systems such as charge- or spin-density wave conductors and systems of Josephson junctions which are closely related to the dissipative FK model, but for many areas of physics [19, 22]. Possibility to estimate the response of the system to the applied force and to get a glimpse into dynamics just by analyzing one characteristics of the system such as Lyapunov exponent when the system is static certainly represents an advantage which could help both theoretical and experimental studies of dynamical systems in general.

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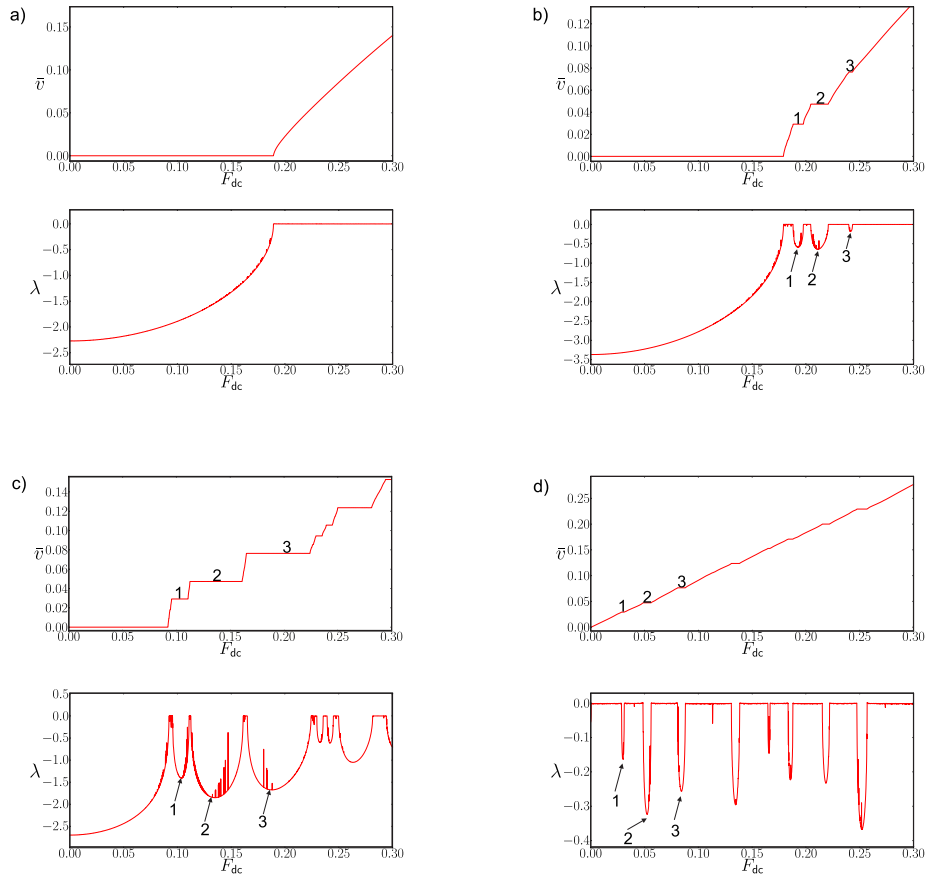
Figure 8: Average velocity and largest Lyapunov exponent as a function of average driving force for incommensurate structure $\omega = \frac{89}{144}$, $K = 4$, $v_0 = 0.2$ and for different values of the ac amplitude (a) $F_{\text{ac}} = 0$, (b) $F_{\text{ac}} = 0.05$, (c) $F_{\text{ac}} = 0.2$ and (d) $F_{\text{ac}} = 1$. Number mark first, second and third harmonics. The unique minimal triplets of integers (i, j, m) in Eq. 6 that determine the first three harmonic steps are $(-3, 2, 1)$, $(2, -1, 1)$, $(-1, 1, 1)$ respectively [31].