

Fear factor in a prey–predator system in deterministic and stochastic environment

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ABSTRACT

In this article, a prey–predator model has been studied where apart from direct predation the prey population is affected by the fear induced from predators. The basic reproduction of the prey population is reduced as a cost of fear. Biologically well posedness of the model system has been shown through positivity and boundedness of solutions. Existence criterion and stability analysis of the non-negative equilibrium points have been discussed and necessary conditions for uniform persistence have been derived for corresponding non-autonomous system. Also, sufficient conditions for the existence of positive periodic solution of the non-autonomous system have been established by utilizing the coincidence degree theorem. We further show that the positive periodic solution is global attractor under certain conditions. Next, we construct the stochastic model based upon the deterministic setup by perturbing the intrinsic growth rate of prey and natural mortality rate of predators. It has been shown that the stochastic system admits unique positive global solution initiates from anywhere in the interior of the positive quadrant. The sufficient conditions for extinction, non-persistence and weakly persistence of both the species have been derived along with the stochastic permanence of the system. We have verified our analytical results and make a comparison between the deterministic and stochastic setup by exhaustive numerical simulations.

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1. Introduction

The effects of prey responses to predators on the dynamics of the prey–predator interactions have been researched by several ecologist by taking account of different functional responses and numerical responses [1–3]. Over the past few decades most of the studies have been done considering only the direct impact of predator population on prey species i.e., researchers paid maximum attention on direct killing of the prey species by the predators as this kind of impact is easy to observe in any scenario. But several field data reveal that the indirect impact of predator species on prey species has significant effects on the population dynamics, even in some of the cases this becomes more influential than the direct predation [4–7]. Though only direct killing is observable in nature but all the prey in any habitat respond to the perceived predation risk and as a consequence they exhibit different kinds of anti-predator behaviours like new selection of habitat, foraging behaviours, vigilance and several psychological changes [8–12]. Scared prey usually do not tend to forage much but fear induced by the predators compelled them to adopt mechanisms like starvation [4,5] which actually lessen the birth rate of that prey species and survival of the adults are also affected consequently. Studies reveal that

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the induced fear affect the psychological condition of the juvenile prey and can cause harmful impact on their adult's survival (Example includes, the species like bird responds to the predator's sound and as a cost of fear they leave their nest taking in mind that they are extremely vulnerable. They choose another places where they are invulnerable, only tend to back to their original places if the risk becomes less). Such kind of anti-predator activities can be thought as beneficial for adult's survival but the basic reproduction can be reduced as a long term cost [4]. Also, when some prey become well aware of forthcoming predation risk, the choice of new habitat sometimes leave negative impact on the lifetime reproductive success of individual. The poor choice of habitat selection i.e., the sub-optimal quality of the new territory does not only affect the reproduction but it also influence the survival of the adults [4]. In 2011, the experiment done by Zanette et al. [13] exhibited that there is a 40% reduction in the production of offspring of the song sparrows (*Melospiza melodia*) due to the fear induced by predators. This reduction is due to the anti-predator behaviour that has an effect on birth rate as well as on the offspring survival rates. Thus, theoretical biologist and evolutionary ecologist have realized that not only direct killing or direct impact but also the cost of fear or indirect impact must be incorporated in prey-predator interaction. There are some articles in literature which proposed and analysed mathematical models by incorporating cost of fear [14–16]. Wang et al. [14] proposed and analysed a mathematical model incorporating cost of fear on the birth rate of prey species. They have shown that for a higher birth rate of prey along with a high level of fear, the positive equilibrium of the system is locally asymptotically stable, which indicates that the cost of fear can stabilize the prey-predator system by ruling out oscillations. Wang and Zou [15] in their work exhibited that a high amount of fear can destabilizes the system while large number of predators has a stabilizing effect on the prey-predator interactions. Wang et al. [14] in their study suggested that fear may affect the death rate of prey species also, i.e. induced fear from predators can have significant effects on the intrinsic growth rate of prey species. Later, Elliot et al. [17] reveal that seasonal populations have an intrinsic rate of growth of about 3.6 as compared to 4.2 when fear was incorporated and not incorporated into the model respectively. This study clearly suggests that the intrinsic growth rate of prey species is lowered due to induced fear. Moreover, intra-specific competition in predators cannot be ignored in prey-predator interactions as this play a crucial role in determine the prey-predator dynamics [18–24]. Furthermore, the level of fear may vary due to change in predation pressure, intra-specific competition or resource distribution among the individuals [25]. Motivated by these facts, we made an attempt to study the dynamics of a prey-predator model with intra-specific competition in top predators where the intrinsic growth rate of prey species is affected by the induced fear from predators.

It is to be noted that the parameters related to environment and biology cannot supposed to be constant throughout a time period as they are subject to fluctuate with environmental change; specially the seasonal effects are considered as important selective forces in a periodic environment [26–31]. Several environmental factors like light intensity, temperature, humidity, etc. seasonally varies in a periodic manner and as a consequence affect the basic metabolism parameters. Thus, it is quite justified to consider the system parameters to be periodic as many important factors of interacting populations like mating habit, competition for limiting resources, available food all factors are seasonally forced. Experiment suggests that fear has cross-seasonal effect on reproduction of offspring and survival of adults [32]. Furthermore, the level of fear is duly affected by the seasonal variations [25]. These motivate us to further consider the non-autonomous version of our proposed system where we assume that the system parameters are periodically varying with time t with a fixed period. The main aspect of study of non-autonomous population model is to find the existence of positive periodic solution and to check whether it is globally stable or globally attractive. In this ambience, we obtained sufficient conditions for uniform persistence and existence of global attractive positive periodic solution.

Besides successful study of the deterministic systems, researchers showed interests to extend the deterministic model to a stochastic system as environmental driving forces significantly affect the population growth [33,34]. Since the parameters related to a deterministic system are not capable of capturing the effect of random environmental fluctuation, the deterministic setup has its own limitation in mathematical modelling of ecosystems. May [35] revealed that due to stochastic fluctuations in environmental conditions all the intrinsic parameters involved in ecological models exhibit certain amount of random fluctuations. Researchers have modelled this type of fluctuations by introducing white noise and coloured noise [36]. It is well postulated that when the coloured noise is not strongly correlated, then white noise can be used as an approximation of the coloured noise. Therefore most of the researchers incorporated white noise term to model the ecosystem in order to study how environmental fluctuation affect the dynamics of the interacting populations [37–48]. Motivated by this fact, we extend our deterministic model to a stochastic model by incorporating multiplicative white noise terms. To the best of our knowledge, no work has been done in literature on the stochastic non-autonomous prey-predator system where the prey population is affected by the indirect impact apart from direct predation.

2. Deterministic model

We have considered an ecological system constitutes of a single prey and single predator species. Let $x(t)$ be the density of the prey species, $y(t)$ be the density of predator species respectively at any instant of time t . We assume that the predator species predate the prey species according to Holling type-II functional response (it has been assumed that the predator is limited by its capacity to process food). It is assumed that prey population follows logistic growth in the absence of direct predation and cost of fear. The growth rate of prey can be split up in three parts: (i) intrinsic growth

Table 1

The biological meaning of the related parameters are given below.

Parameters	Description of parameters
r	Intrinsic growth rate of prey species
b	Density dependence coefficient of prey
f	Fear level induced by predator
c_1	Consumption rate of predator
c_2	Conversion efficiency of middle predator
β	Handling time of predator for each prey that is consumed
h	Death rate of predator for intra-specific competition
d	Natural mortality rate of predator

rate (here we have not separated it in birth rate and natural mortality rate of prey), (ii) density dependent death rate due to intra-specific competition, (iii) predation term. This leads to following ODE:

$$\frac{dx}{dt} = rxF(f, y) - bx^2 - \frac{c_1xy}{1 + \beta x},$$

where r is the intrinsic growth rate of prey, b is the prey's density dependence coefficient, $F(f, y)$ is the fear function which accounts for the cost of anti-predator defence of prey due fear induced by predator, where f is the level of fear. Since several field data reveal that effect of fear actually reduce the reproduction process of prey species, so it is quite reasonable to assume that $F(f, y)$ has the following properties:

- (i) $F(0, y) = 1$, if there is no fear of predator, then there will be no loss in basic reproduction of prey.
- (ii) $F(f, 0) = 1$, In the absence of any predator there is no reduction in reproduction of prey due to anti-predator behaviour.
- (iii) $\lim_{f \rightarrow \infty} F(f, y) = 0$, Due to very high level of fear the reproduction of prey species will be zero.
- (iv) $\lim_{y \rightarrow \infty} F(f, y) = 0$, In the presence of very large number of predators, the prey species exhibits large anti-predator behaviours, as a result the reproduction of prey reduced to zero.
- (v) $\frac{\partial F(f, y)}{\partial f} < 0$, the production of prey species decreases if the anti-predator behaviour of prey species increases.
- (vi) $\frac{\partial F(f, y)}{\partial y} < 0$, if the number of predator increases, the production of prey decreases.

Though fear may affect the intra-specific competition of prey species but due to lack of direct experimental evidences we have only considered the case when fear does not affect the strength of intra-specific competition in our model.

We choose here a particular type fear function $F(f, y) = \frac{1}{1+f}$. Though experiment can be done with other type of fear functions such as $F(f, y) = e^{-fy}$, $F(f, y) = \frac{1}{1+f_1y+f_2y^2}$. By considering the fear function as $F(f, y) = \frac{1}{1+f}$ we get the following system of differential equations:

$$\begin{cases} \frac{dx}{dt} = \frac{rx}{1+f} - bx^2 - \frac{c_1xy}{1+\beta x}, \\ \frac{dy}{dt} = -dy - hy^2 + \frac{c_2xy}{1+\beta x}, \end{cases} \quad (1)$$

where $x(0) > 0$, $y(0) > 0$. Here, it is to be noted that all the parameters involved in the model system (1) are positive.

We can have a clear view about the biological interpretations of the associated parameters of the above model from Table 1.

3. Model analysis

3.1. Positivity and boundedness of solutions

We need to show positivity and boundedness of the solutions of the model system (1) to assure that the model system (1) is biologically well-behaved. In this connection, we have the following theorems:

Theorem 3.1. Every solution of the system (1) that initiates in the interior of \mathbb{R}_+^2 remains positive forever.

Proof. From the first equation of (1) we get

$$\frac{dx(t)}{dt} = \frac{rx}{1+f} - bx^2 - \frac{c_1xy}{1+\beta x}.$$

Therefore,

$$x(t) = x(0) \exp \left(\int_0^t \left[\frac{r}{1+f} - bx - \frac{c_1y}{1+\beta x} \right] ds \right) > 0, \text{ for } x(0) > 0.$$

Similarly,

$$y(t) = y(0) \exp \left(\int_0^t \left[-d - hy + \frac{c_2x}{1+\beta x} \right] ds \right) > 0, \text{ for } y(0) > 0.$$

Hence the theorem. \square

Theorem 3.2. The solutions of the system (1) is bounded.

Proof. The proof is a routine work. Hence, we omit this. \square

3.2. Equilibrium points and their existence criterion

The model system (1) has three non-negative equilibrium points:

(i) The trivial equilibrium point $E_1(0, 0)$:

(ii) The predator free equilibrium point $E_2(\frac{r}{b}, 0)$:

(iii) The positive interior equilibrium point $E^*(x^*, y^*)$, where $y^* = \frac{1}{h} \left[-d + \frac{c_2 x^*}{1 + \beta x^*} \right]$. The interior equilibrium point exists if $\frac{c_2 x^*}{1 + \beta x^*} > d$. x^* is the solution of the equation $Ax^4 + Bx^3 + Cx^2 + Dx + E$, where $A_1 = -h^2 b \beta^3$, $A_2 = \beta^2 (h^2 r \beta - h^2 b) - 2h^2 b \beta^2 - (c_2 - d\beta)hbf$, $A_3 = -h^2 b \beta + \beta^2 h^2 r + 2\beta (h^2 r \beta - h^2 b) - (c_2 - d\beta) (hbf + c_1 c_2 f - c_1 d \beta) + d hbf$, $A_4 = 2\beta h^2 r - (c_2 - d_1 \beta) (hc_1 - c_1 d) + (h^2 r \beta - h^2 b) + d (hbf + c_1 c_2 f - c_1 d \beta)$, $A_5 = h^2 + d (hc_1 - c_1 d)$. Now will find sufficient conditions for existence of at least one positive root of the fourth degree equation. Suppose α is a complex root, then its conjugate α^* is also a complex root of the equation. The quadratic with negative discriminant can be written down as $x^2 + px + q = (x - \alpha)(x - \alpha^*) = x^2 - 2\text{Re}(\alpha)x + |\alpha|^2$. We can now factorized the fourth degree equation as below:

$$\begin{aligned} A_1 x^4 + A_2 x^3 + A_3 x^2 + A_4 x + A_5 &= A_1 (x^2 + px + q)(x^2 + mx + n) \\ &= A_1 [x^4 + (p + m)x^3 + (q + pm + n)x^2 + (pn + qm)x + qn], \end{aligned} \quad (2)$$

where we need to determine m and n . By equating the coefficients of like powers of x on the both sides, we obtain $p + m = \frac{A_2}{A_1}$, $q + pm + n = \frac{A_3}{A_1}$, $pn + qm = \frac{A_4}{A_1}$, $qn = \frac{A_5}{A_1}$. Then we have $m = \frac{A_2}{A_1} - p = \frac{A_2}{A_1} + 2\text{Re}(\alpha)$ and $n = \frac{A_5}{qA_1} = \frac{A_5}{|\alpha|^2 A_1}$.

The other two roots of Eq. (2) are $x_3 = \frac{-m + \sqrt{m^2 - 4n}}{2}$ and $x_4 = \frac{-m - \sqrt{m^2 - 4n}}{2}$. Now if the conditions $n > 0$, $m > 0$ hold then we get $x_3 > 0 > x_4$ which guaranteed the existence of at least one positive root of the fourth degree equation and hence the existence of one positive interior equilibrium point is assured.

3.3. Stability analysis

We perform stability analysis of the equilibrium of the model system (1). The results are given in form of theorems as follows:

Theorem 3.3. The trivial equilibrium point $E_1(0, 0)$ is always unstable. The predator equilibrium point $E_2(\frac{r}{b}, 0)$ is locally asymptotically stable (LAS) if $\frac{c_2 r}{b + \beta r} < d$, otherwise it is unstable.

Proof. Since the eigenvalues associated with the Jacobian matrix at E_1 are $r > 0$ and $-d < 0$, so E_0 is a saddle point that is unstable.

From the Jacobian matrix computed at E_2 , it is observed that the eigenvalues are $-r$ and $\frac{c_2 r}{b + \beta r} - d$. Therefore, the axial equilibrium point E_2 is LAS if $\frac{c_2 r}{b + \beta r} < d$. \square

Theorem 3.4. The coexistence equilibrium point $E^*(x^*, y^*)$ is LAS if $\frac{r}{1 + fy^*} - 2bx^* - \frac{c_1 y^*}{(1 + \beta x^*)^2} < 0$.

Proof. The Jacobian matrix calculated at E^* is

$$V(E^*) = \begin{pmatrix} \frac{r}{1 + fy^*} - 2bx^* - \frac{c_1 y^*}{(1 + \beta x^*)^2} & -\frac{rx^* f}{(1 + fy^*)^2} - \frac{c_1 x^*}{1 + \beta x^*} \\ \frac{c_2 y^*}{(1 + \beta x^*)^2} & -hy^* \end{pmatrix}.$$

The trace and determinant of $V(E^*)$ is

$$\begin{aligned} \text{trace} V(E^*) &= \frac{r}{1 + fy^*} - 2bx^* - \frac{c_1 y^*}{(1 + \beta x^*)^2} - hy^*, \\ \det V(E^*) &= - \left(\frac{r}{1 + fy^*} - 2bx^* - \frac{c_1 y^*}{(1 + \beta x^*)^2} \right) hy^* + \left(\frac{rx^* f}{(1 + fy^*)^2} + \frac{c_1 x^*}{1 + \beta x^*} \right) \frac{c_2 y^*}{(1 + \beta x^*)^2}. \end{aligned}$$

Now if the condition $\frac{r}{1 + fy^*} - 2bx^* - \frac{c_1 y^*}{(1 + \beta x^*)^2} < 0$ holds, then $\text{trace} V(E^*) < 0$ and $\det V(E^*) > 0$ and therefore, E^* is LAS.

Here, it is to be noted that the Jacobian matrix $V(E^*)$ will have purely imaginary eigenvalues if $\text{trace} V(E^*) = 0$ provided that $\det V(E^*) > 0$. This assured the existence of a Hopf bifurcation point at $f = f_h$ around the interior equilibrium point

E^* . The critical value for the Hopf bifurcation parameter f is $f_h = \frac{1}{y^*} \frac{(r-2bx^*-hy^*)(1+\beta x^*)^2 - c_1 y^*}{(2bx^*+hy^*)(1+\beta x^*)^2 + c_1 y^*}$ for $f = f_h$. The transversality condition for Hopf bifurcation around E^* is given by $\frac{d(\text{trace} V(E^*))}{df} = -\frac{ry^*}{(1+fy^*)^2} \neq 0$ for $f = f_h$. This guaranteed the existence of Hopf bifurcation around E^* for the Hopf bifurcation parameter f . It has been further investigated that the intra-specific competition rate h also acts as Hopf bifurcation parameter. \square

4. Non-autonomous system

Next as we intend to investigate the above mentioned prey–predator system in periodically varying season where all the parameters involved in the system are to be seasonally forced, we get the following non-autonomous system of ordinary differential equations

$$\begin{cases} \frac{dx(t)}{dt} = \frac{r(t)x(t)}{1+f(t)y(t)} - b(t)x^2(t) - \frac{c_1(t)x(t)y(t)}{1+\beta(t)x(t)}, \\ \frac{dy(t)}{dt} = -d(t)y(t) - h(t)y^2(t) + \frac{c_2(t)x(t)y(t)}{1+\beta(t)x(t)}, \end{cases} \quad (3)$$

where $x(0) > 0, y(0) > 0$.

Here all the associated coefficients are assumed to be continuous, bounded and non-negative functions on $\mathbb{R}_+ = (0, +\infty)$. For convenience we have utilized the following notations $g^M = \max g(t)$, $g^L = \min g(t)$, where $g(t)$ is a bounded continuous function defined on $[0, +\infty]$.

We are interested in investigating the persistence of the system (3). In view of this we first state the following Lemma [49].

4.1. Uniform persistence

Lemma 4.1. Consider the following equation

$$z'(t) = z(t)[d_1 - d_2 z(t)] \quad \text{where } d_2 > 0$$

if $d_1 > 0$ then $\lim_{t \rightarrow \infty} z(t) = \frac{d_1}{d_2}$.

Theorem 4.2. If

$$\frac{c_2^L x_0}{1 + \beta^M x_0} > d^M \quad (4)$$

$$\frac{r^L}{1 + f^M \phi_2} > \frac{c_1^M \phi_1}{\beta^L} \quad (5)$$

where

$$\phi_1 = \frac{c_2^M - d^L \beta^L}{\beta^L h^L}$$

$$\phi_2 = \frac{1}{h^M} \left[-d^M + \frac{c_2^L x_0}{1 + \beta^M x_0} \right]$$

then the system (3) is uniformly persistence.

Proof. Since

$$\frac{dx(t)}{dt} \leq x(t)[r^M - b^L x(t)].$$

We consider the following equation,

$$z'(t) = z(t)[r^M - b^L x(t)].$$

Using Lemma 4.1 we have

$$\lim_{t \rightarrow \infty} z(t) = \frac{r^M}{b^L}.$$

By comparison theorem there exist a $T_1 > 0$ such that,

$$\begin{aligned} x(t) &\leq \frac{r^M}{b^L} \quad \forall \quad t \geq T_1 \\ &= \phi_0 \quad (\text{say}). \end{aligned}$$

For the predator species there exist $T_2 > 0$ such that,

$$y(t) \leq \frac{c_2^M - d^L \beta^L}{\beta^L h^L} = \phi_1 \quad \forall \quad t \geq T_2.$$

Now from the second equation of system (3) we have,

$$\frac{dy(t)}{dt} \geq y(t) \left[-d^M + \frac{c_2^L x(t)}{1 + \beta^M x(t)} - h^M y(t) \right].$$

Now, $\frac{d}{dx} \left(\frac{c_2^L x(t)}{1 + \beta^M x(t)} \right) = \frac{c_2^L}{(1 + \beta^M x(t))^2} > 0$. So $\frac{c_2^L x(t)}{1 + \beta^M x(t)}$ is an increasing function of x , so $\frac{c_2^L x(t)}{1 + \beta^M x(t)} \geq \frac{c_2^L x_0}{1 + \beta^M x_0}$ where $x(0) = x_0$. From here we get,

$$\frac{dy(t)}{dt} \geq y(t) \left[-d^M + \frac{c_2^L x_0}{1 + \beta^M x_0} - h^M y(t) \right].$$

So this and the inequality (4) together imply there exist a $T_3 > 0$ such that

$$y(t) \geq \frac{1}{h^M} \left[-d^M + \frac{c_2^L x_0}{1 + \beta^M x_0} \right] = \phi_2 \quad \forall \quad t \geq T_3.$$

Now,

$$\frac{dx(t)}{dt} \geq x(t) \left[\frac{r^L}{1 + f^M \phi_2} - b^M x(t) - \frac{c_1^M \phi_1}{\beta^L} \right]$$

This and the inequality (5) together imply that there exists a $T_3 > 0$ such that

$$x(t) \geq \frac{\frac{r^L}{1 + f^M \phi_2} - \frac{c_1^M \phi_1}{\beta^L}}{b^M} = \phi_3 \quad \forall \quad t \geq T_4.$$

We choose $T = \max T_i$; $i = 1, 2, 3, 4$. For any $t > T$ we have $\phi_3 \leq x(t) \leq \phi_0$ and $\phi_2 \leq y(t) \leq \phi_1$.

Therefore, $D = \left\{ (x(t), y(t)) : \phi_3 \leq x(t) \leq \phi_0, \phi_2 \leq y(t) \leq \phi_1 \right\}$ is invariant set of (3). Hence we conclude that the system (3) is uniformly persistent. \square

4.2. Existence of positive periodic solution

To establish sufficient conditions for existence of positive periodic solution of our non-autonomous system (3), we first briefly stated a few concepts [50–53] which is fundamental for this section. Suppose Θ_1 and Θ_2 be euclidean Banach spaces, $S : \text{Dom} S \subset \Theta_1 \rightarrow \Theta_2$ be a linear operator, $R : \Theta_1 \rightarrow \Theta_2$ be a continuous mapping. The operator S is said to be a Fredholm operator with index zero if $\dim \text{Ker} S = \text{codim} \text{Im} S < \infty$ and $\text{Im} S \in \Theta_2$ is closed. For the operator S to be Fredholm operator with index zero it is necessary that there exist continuous projectors $P : \Theta_1 \rightarrow \Theta_1$ and $Q : \Theta_2 \rightarrow \Theta_2$ such that $\text{Im} P = \text{Ker} S$, $\text{Im} S = \text{Ker} Q = \text{Im}(I - Q)$, then the restriction S_p of S to $\text{Dom} S \cap \text{Ker} P : (I - Q)\Theta_1 \rightarrow \text{Im} S$ is invertible. It is to be noted that the inverse of S_p is denoted by K_p . Now suppose Λ denotes an open bounded subset of the Banach space Θ_1 , the mapping R will be called S -compact on $\bar{\Lambda}$ if $QR(\bar{\Lambda})$ is bounded and $K_p(I - Q)R : \bar{\Lambda} \rightarrow \Theta_1$ is compact. Since $\text{Im} Q$ is isomorphic to $\text{Ker} S$ then there exists an isomorphism $J : \text{Im} Q \rightarrow \text{Ker} S$.

Lemma 4.3 (Continuation Theorem [50]). Let $\Lambda \subset \Theta_1$ be an open bounded set. Let S be a Fredholm mapping of index zero and R be S -compact on $\bar{\Lambda}$. Assume that the following conditions hold

(a) for each $\lambda \in (0, 1)$, $x \in \delta \Lambda \cap \text{Dom} S$, $Sx \neq \lambda Rx$;

(b) for each $x \in \delta \Lambda \cap \text{Ker} S$, $QRx \neq 0$;

(c) $\deg\{QR, \Lambda \cap \text{Ker} S, 0\} \neq 0$.

Then $Sx = Rx$ has at least one solution in $\Lambda \cap \text{Dom} S$.

Lemma 4.4 ([54]). Suppose that a function f is integral, uniformly continuous and non-negative on $[0, \infty)$, then $\lim_{t \rightarrow \infty} f(t) = 0$.

Definition 4.1. Let $x(t)$ is any solution of (3) and $\bar{x}(t)$ is a ω periodic solution of the non-autonomous system satisfying

$$\lim_{t \rightarrow \infty} |\bar{x}(t) - x(t)| = 0,$$

then the solution $\bar{x}(t)$ is said to be globally attractive. For reader's convenience we denote,

$$\bar{f} = \frac{1}{\omega} \int_0^\omega f(t) dt, f_s = \min_{t \in [0, \omega]} f(t), f_H = \max_{t \in [0, \omega]} f(t).$$

Theorem 4.5. The non-autonomous system (3) possess at least one ω positive periodic solution if the conditions stated below hold,

$$\begin{aligned} [E_1] \quad & [c_2(t) - d(t)\beta(t)]_H > 0, \\ [E_2] \quad & [c_2(t)e^{H_1} - d(t)\beta(t)e^{L_1} - d(t)]_S > 0 \end{aligned}$$

Proof. Let the positive solution $(x(t), y(t))$ of the system (3) be arbitrary. By making change of variables we get $\Phi_1(t) = \ln x(t)$, $\Phi_2(t) = \ln y(t)$.

Then Eq. (3) becomes,

$$\begin{cases} \frac{d\Phi_1(t)}{dt} = \frac{r(t)}{1+f(t)e^{\Phi_2(t)}} - be^{\Phi_1(t)} - \frac{c_1(t)e^{\Phi_2(t)}}{1+\beta(t)e^{\Phi_1(t)}} \\ \frac{d\Phi_2(t)}{dt} = -d(t) - h(t)e^{\Phi_2(t)} + \frac{c_2(t)e^{\Phi_1(t)}}{1+\beta(t)e^{\Phi_1(t)}} \end{cases} \quad (6)$$

It is quite obvious, that if the non-autonomous system (6) possess a ω - periodic solution $(\Phi_1^*(t), \Phi_2^*(t))^T$, we can conclude that $Z^* = (x^*(t), y^*(t))^T = (\exp(\Phi_1^*(t)), \exp(\Phi_2^*(t)))^T$ is a positive ω - periodic solution of the system (3). To make the proof complete, it is required to verify that the system (6) has at least one ω - periodic solution.

Set $\Theta_1 = \Theta_2 = (\Phi_1(t), \Phi_2(t))^T \in C(\mathbb{R}, \mathbb{R}^2) : \Phi_i(t + \omega) = \Phi_i, i = 1, 2$. and

$$\|(\Phi_1(t), \Phi_2(t))^T\| = \sum_{i=1}^2 \max_{t \in [0, \omega]} |\Phi_i(t)|,$$

$\|\cdot\|$ denotes here the general Euclidean norm. Here Θ_1 and Θ_2 are both Banach spaces when we endow them with the norm defined above.

Let $S : \text{Dom}S \cap \Theta_1, S(\Phi_1(t), \Phi_2(t))^T = (\frac{d\Phi_1(t)}{dt}, \frac{d\Phi_2(t)}{dt})^T$, where $\text{Dom}S = (\Phi_1(t), \Phi_2(t))^T \in C^1(\mathbb{R}, \mathbb{R}^2)$ and $R : \Theta_1 \rightarrow \Theta_1$,

$$R \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \frac{r(t)}{1+f(t)e^{\Phi_2(t)}} - be^{\Phi_1(t)} - \frac{c_1(t)e^{\Phi_2(t)}}{1+\beta(t)e^{\Phi_1(t)}} \\ -d(t) - h(t)e^{\Phi_2(t)} + \frac{c_2(t)e^{\Phi_1(t)}}{1+\beta(t)e^{\Phi_1(t)}} \end{pmatrix}$$

Now define

$$P \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = Q \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\omega} \int_0^\omega \Phi_1(t) dt \\ \frac{1}{\omega} \int_0^\omega \Phi_2(t) dt \end{pmatrix}, \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \in \Theta_1 = \Theta_2.$$

Obviously, we have

$$\begin{aligned} \text{Ker}S &= \{p | p \in \Theta_1, p = h, h \in \mathbb{R}^2\}, \\ \text{Im}S &= \left\{ q | q \in \Theta_2, \int_0^\omega y(t) dt = 0 \right\}. \end{aligned}$$

Note. We have $\dim \text{Ker}S = 2$. From the first isomorphism theorem, it can be easily proved that $\text{codim} \text{Im}S$, i.e. $\dim \frac{\Theta_2}{\text{Im}S}$ is also 2. Therefore, $\dim \text{Ker}S = \text{codim} \text{Im}S = 3$.

As image of S is closed in Θ_2 , S is a Fredholm operator of index zero. It can easily be shown that P and Q are continuous projectors (Idempotent operators) s.t

$$\text{Im}P = \text{Ker}S, \text{Ker}Q = \text{Im}S = \text{Im}(I - Q).$$

Furthermore, It can be easily verified that the inverse K_P of S_P has the form

$$K_P : \text{Im}S \rightarrow \text{Dom}S \cap \text{Ker}P, K_P(q) = \int_0^t q(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t q(s) ds dt.$$

Consequently, $QR : \Theta_1 \rightarrow \Theta_2$ and $K_P(I - Q)R : \Theta_1 \rightarrow \Theta_1$ lead

$$QRp = \begin{pmatrix} \frac{1}{\omega} \int_0^\omega \left[\frac{r(t)}{1+f(t)e^{\Phi_2(t)}} - be^{\Phi_1(t)} - \frac{c_1(t)e^{\Phi_2(t)}}{1+\beta(t)e^{\Phi_1(t)}} \right] dt \\ \frac{1}{\omega} \int_0^\omega [-d(t) - h(t)e^{\Phi_2(t)} + \frac{c_2(t)e^{\Phi_1(t)}}{1+\beta(t)e^{\Phi_1(t)}}] dt \end{pmatrix},$$

$$K_P(I - Q)Rp = \int_0^t Rp(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t Rp(s) ds dt - \left(\frac{t}{\omega} - \frac{1}{2} \right) \int_0^\omega Rp(s) ds.$$

From the Lebesgue theorem, it follows that $K_P(I - Q)R$ and QR are continuous furthermore, $QR(\bar{\Lambda})$ and $K_P(I - Q)R(\bar{\Lambda})$ both are relatively compact for any open bounded set $\Lambda \subset \Theta_1$. Consequently, R is S - compact on $\bar{\Lambda}$ where $\Lambda \subset \Theta_1$ is any open bounded set.

For applying Lemma 4.4, it is necessary to have a suitable open bounded subset Λ .

From the operator equation $Sx = \lambda Rx$, $\lambda \in (0, 1)$, we get

$$\begin{cases} \frac{d\phi_1}{dt} = \lambda \left[\frac{r(t)}{1+f(t)e^{\phi_2(t)}} - be^{\phi_1(t)} - \frac{c_1(t)e_2^{\phi_1(t)}}{1+\beta(t)e^{\phi_1(t)}} \right], \\ \frac{d\phi_2}{dt} = \lambda \left[-d(t) - h(t)e^{\phi_2(t)} + \frac{c_2(t)e^{\phi_1(t)}}{1+\beta(t)e^{\phi_1(t)}} \right] \end{cases} \quad (7)$$

Suppose that $(\phi_1(t), \phi_2(t))^T \in \Theta_1$ is a solution of (7) for a certain $\lambda \in (0, 1)$. Then from we have from (7)

$$\frac{r(t)}{1+f(t)e^{\phi_2(t)}} = be^{\phi_1(t)} + \frac{c_1(t)e_2^{\phi_1(t)}}{1+\beta(t)e^{\phi_1(t)}}, \quad (8)$$

$$d(t) + h(t)e^{\phi_2(t)} = \frac{c_2(t)e^{\phi_1(t)}}{1+\beta(t)e^{\phi_1(t)}}. \quad (9)$$

Now, from (8) we obtain

$$b(t)e^{\phi_1(t)} \leq r(t) \Rightarrow e^{\phi_1(t)} \leq \frac{r(t)}{b(t)} \Rightarrow \phi_1(t) \leq \ln \left[\frac{r(t)}{b(t)} \right]_H = H_1.$$

From (9)

$$\frac{c_2(t)e^{\phi_1(t)}}{1+\beta(t)e^{\phi_1(t)}} \geq d(t) \Rightarrow (c_2(t) - d(t)\beta(t))e^{\phi_1(t)} \geq d(t) \Rightarrow e^{\phi_1(t)} \geq \frac{[d(t)]_S}{[(c_2(t) - d(t)\beta(t))]_H} = S'.$$

From condition $[E_1]$, $S' > 0$, then we have $\phi_1(t) \geq \ln S' = L_1$.

From (9) we obtain

$$\begin{aligned} b(t)e^{\phi_2(t)} &\leq \frac{c_2(t)e^{\phi_1(t)}}{1+\beta(t)e^{\phi_1(t)}} \Rightarrow b(t)e^{\phi_2(t)} \leq c_2(t)e^{\phi_1(t)} \leq c_2(t)e^{H_1} \\ &\Rightarrow e^{\phi_2(t)} \leq \frac{c_2(t)e^{H_1}}{b(t)} \Rightarrow \phi_2(t) \leq \ln \left[\frac{c_2(t)e^{H_1}}{b(t)} \right]_H = H_2. \end{aligned}$$

and

$$\begin{aligned} h(t)e^{\phi_2(t)}[1+\beta(t)e^{\phi_1(t)}] &= c_2(t)e^{\phi_1(t)} - d(t)\beta(t)e^{\phi_1(t)} - d(t) \\ &\Rightarrow e^{\phi_2(t)} \geq \frac{[c_2(t)e^{H_1} - d(t)\beta(t)e^{L_1} - d(t)]_S}{[h(t)]_H} = S'' \end{aligned}$$

From condition $[E_2]$, $S'' > 0$, then we obtain that $\phi_2(t) \geq \ln S'' = L_2$.

So, we have $|\phi_1(t)| \leq H_1$, $|\phi_2(t)| \leq H_2$. Clearly, $H'_i (i = 1, 2)$ are independent of λ . Denote $\tilde{H} = H_1 + H_2 + \epsilon$, where ϵ is large enough s.t each solution $(\phi_1^*(t), \phi_2^*(t))^T$ (if system possess at least one solution) of algebraic system of equations

$$\begin{cases} \frac{\bar{r}}{1+\bar{f}e^{\phi_2}} - \bar{b}e^{\phi_1} - \frac{\bar{c}_1e_2^{\phi_1}}{1+\bar{\beta}e^{\phi_1}} = 0, \\ -\bar{d} - \bar{h}e^{\phi_2} + \frac{\bar{c}_2e^{\phi_1}}{1+\bar{\beta}e^{\phi_1}} = 0 \end{cases} \quad (10)$$

satisfies $\|(\phi_1^*, \phi_2^*)^T\| < \tilde{H}$ provided that the system (10) has at least one solution.

Now, we set $\Lambda = \{(\phi_1(t), \phi_2(t))^T \in \Theta_1 : \|(\phi_1(t), \phi_2(t))^T\| < \tilde{H}\}$. It is clear that the condition (a) of Lemma 4.3 is satisfied. When $(\phi_1(t), \phi_2(t))^T \in \delta\Lambda \cap \text{Ker}S$, $(\phi_1(t), \phi_2(t))^T$ is a constant vector in R^2 with $\|(\phi_1(t), \phi_2(t))^T\| = \tilde{H}$. If the system (10) has at least one solution, then we get

$$QR \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \frac{\bar{r}}{1+\bar{f}e^{\phi_2}} - \bar{b}e^{\phi_1} - \frac{\bar{c}_1e_2^{\phi_1}}{1+\bar{\beta}e^{\phi_1}} \\ -\bar{d} - \bar{h}e^{\phi_2} + \frac{\bar{c}_2e^{\phi_1}}{1+\bar{\beta}e^{\phi_1}} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, the condition (b) in Lemma 4.3 is satisfied.

As we are going to prove the condition (c) in Lemma 4.3, we first define a homomorphism mapping $J : \text{Im}Q \Rightarrow \text{Ker}S$, by $(\phi_1, \phi_2)^T \rightarrow (\phi_1, \phi_2)^T$ and a projector $\psi : \text{Dom}S \times [0, 1] \rightarrow \Theta_1$ by

$$\psi(\phi_1, \phi_2, \mu) = \begin{pmatrix} \frac{\bar{r}}{1+\bar{f}e^{\phi_2}} - \bar{b}e^{\phi_1} \\ -\bar{d} - \bar{h}e^{\phi_2} \end{pmatrix} + \mu \begin{pmatrix} -\frac{\bar{c}_1e_2^{\phi_1}}{1+\bar{\beta}e^{\phi_1}} \\ \frac{\bar{c}_2e^{\phi_1}}{1+\bar{\beta}e^{\phi_1}} \end{pmatrix}.$$

Here $\mu \in [0, 1]$ is a parameter. For $(\phi_1, \phi_2)^T \in \delta\Lambda \cap \text{Ker}S = \delta\Lambda \cap R^2$, we get $\psi(\phi_1, \phi_2, \mu) \neq 0$. We can prove by contradiction. If our claim is not true then we can find a constant vector $(\phi_1, \phi_2)^T$ with $\|(\phi_1(t), \phi_2(t))^T\| = \tilde{H}$ s.t

$\psi(\Phi_1, \Phi_2, \mu) = 0$. But we already showed that $\|(\Phi_1(t), \Phi_2(t))^T\| \leq \tilde{H}$, a contradiction. Hence ψ is homotopic and so from homotopy invariance theorem we have

$$\begin{aligned} \deg(JQR(\Phi_1, \Phi_2)^T, \Lambda \cap \text{Ker}S, (0, 0)^T) &= \deg(\psi(\Phi_1, \Phi_2, 1)^T, \Lambda \cap \text{Ker}S, (0, 0)^T) \\ &= \deg(\psi(\Phi_1, \Phi_2, 0)^T, \Lambda \cap \text{Ker}S, (0, 0)^T) \\ &= \deg\left(\frac{\bar{r}}{1 + \bar{f}e^{\Phi_2}} - \bar{b}e^{\Phi_1}, -\bar{d} - \bar{h}e^{\Phi_2} + \frac{\bar{c}_2 e^{\Phi_1}}{1 + \bar{\beta}e^{\Phi_1}}, \Lambda \cap \text{Ker}S, (0, 0)^T\right) \\ &= \text{sgn det} \begin{pmatrix} -\bar{b}e^{\Phi_1} & -\frac{\bar{r}\bar{f}e^{\Phi_2}}{(1 + \bar{f}e^{\Phi_2})^2} \\ 0 & -\bar{h}e^{\Phi_2} \end{pmatrix} = 1 \neq 0 \end{aligned}$$

Then the condition (c) of Lemma 4.3 is also satisfied. By Lemma 4.3 the non-autonomous system has at least one positive ω -periodic solution on $\Lambda \cap \text{Dom}S$.

This completes the proof. \square

4.3. Global attractivity

Theorem 4.6. If we assume that all the conditions $[E_1]$, $[E_2]$ and the following conditions hold:

$$\begin{aligned} [E_3] \quad &0 < x(0), y(0) < +\infty; \\ [E_4] \quad &\left[\mu_1 b(t) - \mu_1 \frac{c_1(t)e^{H_2}\beta(t)}{1 + \beta(t)e^{L_1}} - \mu_2 c_2(t) - \mu_2 \frac{c_2(t)e^{H_1}\beta(t)}{1 + \beta(t)e^{L_1}} \right]_{t \in [0, \omega]}^S > 0; \\ [E_5] \quad &\left[-\mu_1 \frac{r(t)f(t)}{1 + f(t)e^{L_2}} - \mu_1 c_1(t) + \mu_2 h(t) \right]_{t \in [0, \omega]}^S > 0; \end{aligned}$$

then the system (3) has only one positive ω -periodic solution which is globally attractive.

Proof. From Theorem 4.5, we know that the system (3) has at least one positive ω -periodic solution $(\tilde{x}(t), \tilde{y}(t))$ and also, we have $e^{L_1} \leq \tilde{x}(t) \leq e^{H_1}$, $e^{L_2} \leq \tilde{y}(t) \leq e^{H_2}$. Suppose, $(x(t), y(t))$ is any positive periodic solution of (3).

Let

$$V(t) = \mu_1 |\ln x(t) - \ln \tilde{x}(t)| + \mu_2 |\ln y(t) - \ln \tilde{y}(t)|. \quad (11)$$

By calculating the Dini derivative (right hand derivative) of (11) along the solution of (3) we get

$$D^+ V(t) = \mu_1 \text{sgn}(x(t) - \tilde{x}(t)) \left(\frac{\dot{x}(t)}{x(t)} - \frac{\dot{\tilde{x}}(t)}{\tilde{x}(t)} \right) + \mu_2 \text{sgn}(y(t) - \tilde{y}(t)) \left(\frac{\dot{y}(t)}{y(t)} - \frac{\dot{\tilde{y}}(t)}{\tilde{y}(t)} \right).$$

Now,

$$\mu_1 \text{sgn}(x(t) - \tilde{x}(t)) \left(\frac{\dot{x}(t)}{x(t)} - \frac{\dot{\tilde{x}}(t)}{\tilde{x}(t)} \right) \quad (12)$$

$$\begin{aligned} &= \mu_1 \text{sgn}(x(t) - \tilde{x}(t)) \left[\frac{r(t)}{1 + f(t)y(t)} - b(t)x(t) - \frac{c_1(t)y(t)}{1 + \beta(t)x(t)} - \frac{r(t)}{1 + f(t)\tilde{y}(t)} + b(t)\tilde{x}(t) + \frac{c_1(t)\tilde{y}(t)}{1 + \beta(t)\tilde{x}(t)} \right] \\ &= \mu_1 \text{sgn}(x(t) - \tilde{x}(t)) \left[\frac{-r(t)f(t)(y(t) - \tilde{y}(t))}{(1 + f(t)y(t))(1 + f(t)\tilde{y}(t))} - b(t)(x(t) - \tilde{x}(t)) \right. \\ &\quad \left. - \frac{c_1(t)(y(t) - \tilde{y}(t))}{1 + \beta(t)x(t)} + \frac{c_1(t)\tilde{y}(t)\beta(t)(x(t) - \tilde{x}(t))}{(1 + \beta(t)x(t))(1 + \beta(t)\tilde{x}(t))} \right] \\ &\leq \mu_1 \frac{r(t)f(t) |y(t) - \tilde{y}(t)|}{1 + f(t)\tilde{y}(t)} - \mu_1 b(t) |x(t) - \tilde{x}(t)| + \mu_1 c_1(t) |y(t) - \tilde{y}(t)| + \mu_1 \frac{c_1(t)\tilde{y}(t)\beta(t) |x(t) - \tilde{x}(t)|}{1 + \beta(t)\tilde{x}(t)}, \quad (13) \end{aligned}$$

and

$$\begin{aligned} \mu_2 \text{sgn}(y(t) - \tilde{y}(t)) \left(\frac{\dot{y}(t)}{y(t)} - \frac{\dot{\tilde{y}}(t)}{\tilde{y}(t)} \right) &= \mu_2 \text{sgn}(y(t) - \tilde{y}(t)) \left[-h(t)(y(t) - \tilde{y}(t)) + \frac{c_2(t)x(t)}{1 + \beta(t)x(t)} \right. \\ &\quad \left. - \frac{c_2(t)\tilde{x}(t)}{1 + \beta(t)x(t)} + \frac{c_2(t)\tilde{x}(t)}{1 + \beta(t)x(t)} - \frac{c_2(t)\tilde{x}(t)}{1 + \beta(t)\tilde{x}(t)} \right] \\ &= \mu_2 \text{sgn}(y(t) - \tilde{y}(t)) \left[-h(t)(y(t) - \tilde{y}(t)) + \frac{c_2(t)(x(t) - \tilde{x}(t))}{1 + \beta(t)x(t)} \right] \end{aligned}$$

$$+ \frac{c_2(t)\tilde{x}(t)\beta(t)(\tilde{x}(t) - x(t))}{(1 + \beta(t)x(t))(1 + \beta(t)\tilde{x}(t))} \Big] \leq -\mu_2 h(t) |y(t) - \tilde{y}(t)| + \mu_2 c_2(t) |x(t) - \tilde{x}(t)| + \mu_2 \frac{c_2(t)\tilde{x}(t)\beta(t) |x(t) - \tilde{x}(t)|}{1 + \beta(t)\tilde{x}(t)}. \quad (14)$$

Adding (12) and (14) we obtain

$$D^+V(t) \leq - \left[\mu_1 b(t) - \mu_1 \frac{c_1(t)\tilde{y}(t)\beta(t)}{1 + \beta(t)\tilde{x}(t)} - \mu_2 c_2(t) - \mu_2 \frac{c_2(t)\tilde{x}(t)\beta(t)}{1 + \beta(t)\tilde{x}(t)} \right] |x(t) - \tilde{x}(t)| \\ - \left[-\mu_1 \frac{r(t)f(t)}{1 + f(t)\tilde{y}(t)} - \mu_1 c_1(t) + \mu_2 h(t) \right] |y(t) - \tilde{y}(t)|.$$

Then we have

$$D^+V(t) \leq -\phi_1 |x(t) - \tilde{x}(t)| - \phi_2 |y(t) - \tilde{y}(t)|,$$

where,

$$\phi_1 = \left[\mu_1 b(t) - \mu_1 \frac{c_1(t)e^{H_1}\beta(t)}{1 + \beta(t)e^{L_1}} - \mu_2 c_2(t) - \mu_2 \frac{c_2(t)e^{H_1}\beta(t)}{1 + \beta(t)e^{L_1}} \right], \\ \phi_2 = \left[-\mu_1 \frac{r(t)f(t)}{1 + f(t)e^{L_2}} - \mu_1 c_1(t) + \mu_2 h(t) \right].$$

If the conditions $[E_3] - [E_5]$ hold then clearly $V(t)$ is a non-decreasing on $[0, \infty)$. Now, we integrate the above inequality from 0 to t and observing the condition $[E_3]$, we obtain that

$$V(t) + \phi_1 \int_0^t |x(t) - \tilde{x}(t)| dt + \phi_2 \int_0^t |y(t) - \tilde{y}(t)| dt \leq V(0) < +\infty \quad \forall t > 0$$

From Lemma 4.4, we get

$$\lim_{t \rightarrow \infty} |x(t) - \tilde{x}(t)| = 0,$$

$$\lim_{t \rightarrow \infty} |y(t) - \tilde{y}(t)| = 0.$$

So, we claim that $(\tilde{x}(t), \tilde{y}(t))$ is globally attractive. \square

5. Stochastic model

To capture what environmental fluctuation affect the model system (3) it is necessary to take into account the stochastic differential equation (SDE) model corresponding to the non-autonomous deterministic model (3). Since the demographic stochasticity mainly affects the uncertain growth as well as death of a population, then we construct the SDE by perturbing the growth rate of prey population and natural mortality rate of predator population as (for example see [55,56]): $r(t) \rightarrow r(t) + \sigma_1(t)\frac{dB_1(t)}{dt}$ and $-d(t) \rightarrow -d(t) + \sigma_2(t)\frac{dB_2(t)}{dt}$, we get the following SDE;

$$\begin{cases} dx(t) = x(t) \left[\frac{r(t)}{1+f(t)y(t)} - b(t)x(t) - \frac{c_1(t)y(t)}{1+\beta(t)x(t)} \right] dt + \frac{\sigma_1(t)x(t)}{1+f(t)y(t)} dB_1(t), \\ dy(t) = y(t) \left[-d(t) - h(t)y(t) + \frac{c_2(t)x(t)}{1+\beta(t)x(t)} \right] dt + \sigma_2(t)y(t) dB_2(t), \end{cases} \quad (15)$$

with $x(0) > 0$, $y(0) > 0$,

where $B_i(t)$, $i = 1, 2$ are standard mutually independent Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\sigma_i(t)$, $i = 1, 2$ represent the intensity of the white noise.

To study the dynamics of stochastic system we first briefly go through some basic preliminaries.

5.1. Preliminaries

Before proceed, we here mention some basic notations and lemma which will be needed for further section.

$$F^u = \sup_{t \geq 0} F(t), \quad F^l = \inf_{t \geq 0} F(t), \quad \langle F(t) \rangle = \frac{1}{t} \int_0^t F(t) dt, \quad F^* = \limsup_{t \rightarrow \infty} F(t), \quad F_* = \liminf_{t \rightarrow \infty} F(t),$$

where the function $F(t)$ is continuous and also bounded on $[0, +\infty)$.

Now, we introduce Ito's formula for a general n-dimensional stochastic equation as follows [37,57]

$$dX = F(t, X)dt + G(t, X)dB(t) \quad (16)$$

with initial value $X(t_0)$. Here, $F(t, X) = (F_1(t, X), F_2(t, X), \dots, F_n(t, X))$ is a vector function of dimension n , $(G(t, X))_{n \times m}$ is a $n \times m$ matrix function and $B(t) = (B_1(t), B_2(t), \dots, B_m(t))$ is a m -dimensional standard motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Operator L is defined as follows

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^n F_i(t, X) \frac{\partial}{\partial x_i} + \sum_{i,j=1}^n \sum_{k=1}^m G_{ik}(t, X) G_{jk}(t, X) \frac{\partial^2}{\partial x_i \partial x_j}.$$

Let $V(t, X) \in \mathbb{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$. Then

$$LV(t, X) = \frac{\partial V}{\partial t} + \sum_{i=1}^n F_i(t, X) \frac{\partial V}{\partial x_i} + \sum_{i,j=1}^n \sum_{k=1}^m G_{ik}(t, X) G_{jk}(t, X) \frac{\partial^2 V}{\partial x_i \partial x_j}.$$

Thus, we present the Ito's formula as follows:

Lemma 5.1 ([37,57]). Let $X(t)$ satisfies Eq. (16) and a function $V(t, X) \in \mathbb{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$. Then $dV = LV(t, X)dt + V_X G(t, X)dB(t)$, where $V_X(t, X) = \left(\frac{\partial V(t, X)}{\partial x_1}, \frac{\partial V(t, X)}{\partial x_2}, \frac{\partial V(t, X)}{\partial x_3}, \dots, \frac{\partial V(t, X)}{\partial x_n} \right)$.

6. Existence of global positive solutions of the stochastic model

We know, that for any given initial value a SDE will possess a unique global solution if coefficient associated with the SDE satisfy linear growth conditions and also local Lipschitz conditions [57]. It is to be noted that the coefficients of our concerned SDE do not satisfy linear growth conditions but are Lipschitz continuous locally. So it may possible that the solution of stochastic non-autonomous will explode at a finite time. Now in order to show that the solution of our system is global, we will use Lyapunov analysis method.

Theorem 6.1. For any given initial value $(x_0, y_0) \in \mathbb{R}_+^2$ the system (3) will possess a unique solution $(x(t), y(t))$ for all $t \geq 0$ and the solution will remain $\in \mathbb{R}_+$ with probability 1.

Proof. Since, the coefficients associated with the stochastic model (15) is locally Lipschitz continuous then for the system initiates from any value $(x_0, y_0) \in \mathbb{R}_+^2$ will have a unique solution $(x(t), y(t)) \in [0, \tau_e]$, where $\tau_e = \text{explosion time}$.

We have to show that $\tau_e = \infty$.

We choose a non-negative number k_0 be large enough so that $(x_0, y_0) \in [\frac{1}{k_0}, k_0]$. For each integer $k > k_0$, we define the stopping times as $\tau_k = \inf \{t \in [0, \tau_e] : x(t) \notin (\frac{1}{k}, k) \text{ or } y(t) \notin (\frac{1}{k}, k)\}$. Obviously τ_k is increases as $k \rightarrow \infty$. Let

$$\tau_\infty = \lim_{k \rightarrow \infty} \tau_k$$

Hence $\tau_\infty \leq \tau_e$ almost surely.

Now we have to show $\tau_\infty = \infty$. If not, then \exists constants $T > 0$ and $\epsilon \in (0, 1)$ s.t $P\{\tau_k \leq T\} > \epsilon$. Therefore \exists an integer $k_1 > k_0$ s.t $P\{\tau_k \leq T\} > \epsilon \forall k \geq k_1$. We now define a \mathbb{C}^2 -function $V : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$ by $V(x, y) = (x - 1 - \log x) + (y - 1 - \log y)$ which is non-negative.

Using Ito's formula to our model (15), we obtain

$$\begin{aligned} dV(x, y) &= V_x dx + 0.5V_{xx}(dx)^2 + V_y dy + 0.5V_{yy}(dy)^2 \\ &= (x - 1) \left[\frac{r(t)}{1 + f(t)y(t)} - b(t)x(t) - \frac{c_1(t)y(t)}{1 + \beta(t)x(t)} \right] dt \\ &\quad + \frac{\sigma_1^2(t)}{2(1 + f(t)y(t))^2} dt + (y - 1) \left[-d(t) - h(t)y(t) + \frac{c_2(t)x(t)}{1 + \beta(t)x(t)} \right] dt \\ &\quad + \frac{\sigma_2^2(t)}{2} dt + (x - 1) \frac{\sigma_1(t)}{1 + f(t)y(t)} dB_1(t) + (y - 1) \sigma_2(t) dB_2(t) \\ &= LV(x, y)dt + (x - 1) \frac{\sigma_1(t)}{1 + f(t)y(t)} dB_1(t) + (y - 1) \sigma_2(t) dB_2(t), \end{aligned} \quad (17)$$

where

$$\begin{aligned} LV(x, y) &\leq (r^u + b^u)x(t) - \frac{b^l x^2(t)}{1 + f^u y(t)} - \frac{r^l}{1 + f^u y(t)} + \frac{c_1^u}{\beta^l} - h^l y^2(t) + \left(\frac{c_2^u}{\beta^l} + h\right)y(t) + d^u + \frac{(\sigma_1^u)^2}{2(1 + f^l y(t))^2} + \frac{(\sigma_1^u)^2}{2} \\ &\leq G, \end{aligned}$$

where G is a positive number. Now we integrate the both side of the inequality (17) from 0 to $\tau_k \wedge T$ ($\tau_k \wedge T = \min\{\tau_k, T\}$) and taking expectation, we obtain that

$$EV(x(\tau_k \wedge T), y(\tau_k \wedge T)) \leq V(x_0, y_0) + GE(\tau_k \wedge T)$$

$$\leq V(x_0, y_0) + GT$$

Let $\Omega_n = \tau_k \leq T$ for all $k \geq k_1$. Then we have $P(\Omega_k) \geq \epsilon$. Therefore for each $\omega \in \Omega_k$ there exists at least one of $x(\tau_k, \omega)$ or $y(\tau_k, \omega)$ equals to either k or $\frac{1}{k}$.

$$V(x(\tau_k, \omega), y(\tau_k, \omega)) \geq \min \left\{ k - 1 - \log k, \frac{1}{k} - 1 - \log \frac{1}{k} \right\}.$$

Then we have

$$\begin{aligned} V(x_0, y_0) + GT &\geq E[1_{\Omega_k(\omega)} V(x(\tau_k), y(\tau_k))] \\ &\geq \epsilon \min \left\{ \left(k - 1 - \log k \right), \left(\frac{1}{k} - 1 - \log \frac{1}{k} \right) \right\}, \end{aligned}$$

where $1_{\Omega_k(\omega)}$ is the indicator function of Ω_n .

Now when $k \rightarrow \infty$ we have $\infty > V(x_0, y_0) + GT = \infty$ which is a contradiction. Therefore, we have $\tau_\infty = \infty$.

The proof is now complete. \square

7. Stochastic persistence and extinction scenarios

In this current section we motivate to investigate the persistence and extinction criterion of our stochastic system (15). Before going to the main results we want to highlight some severely used definitions [46].

Theorem 7.1. A solution $(x(t), y(t))$ of the system (15) initiating from $(x_0, y_0) \in \mathbb{R}_+^2$ holds the following property $E(y^q) \leq M(q)$,

$$\text{where } M(q) = \max \left\{ q \left(\frac{q}{h^l} \right)^q \left(\frac{\left(\frac{1}{q} - d^l + \frac{c_2^u}{\beta^l} + 0.5(q-1)(\sigma_2^u)^2 \right)}{q+1} \right)^{q+1}, y_0^q \right\}.$$

Proof. Define $V(x, t) = e^t y^q$ for $q \geq 1$. Now, applying Ito's formula to the second equation of (15) and simplifying we obtain,

$$\begin{aligned} dV(x, t) &= qe^t y^q \left[\frac{1}{q} - d(t) - h(t)y(t) + \frac{c_2(t)x(t)}{1 + \beta(t)x(t)} + 0.5(q-1)\sigma_2^2(t) \right] dt + qe^t y^q \sigma_2(t) dB_2(t) \\ &\leq qe^t y^q \left[\frac{1}{q} - d^l - h^l y + \frac{c_2^u}{\beta^l} + 0.5(q-1)(\sigma_2^u)^2 \right] + qe^t y^q \sigma_2(t) dB_2(t). \end{aligned}$$

Integrating both sides from 0 to t and then taking expectation,

$$E(V(x, t)) = y_0^q + q \int_0^t e^s E \left[\frac{1}{q} - d^l - h^l y + \frac{c_2^u}{\beta^l} + 0.5(q-1)(\sigma_2^u)^2 \right] ds.$$

Let $g(y) = \left(\frac{1}{q} - d^l - h^l y + \frac{c_2^u}{\beta^l} + 0.5(q-1)(\sigma_2^u)^2 \right)$. To find the maximum value we first calculate $g'(y)$ and we get

$$\begin{aligned} g'(y) &= qy^{q-1} \left(\frac{1}{q} - d^l - h^l y + \frac{c_2^u}{\beta^l} + 0.5(q-1)(\sigma_2^u)^2 \right) + y^q(-h^l) \\ &= y^{q-1} \left[q \left(\frac{1}{q} - d^l + \frac{c_2^u}{\beta^l} + 0.5(q-1)(\sigma_2^u)^2 \right) - h^l(q+1)y \right]. \end{aligned}$$

From $g'(y) = 0$ we get the critical point as $y = \frac{q \left(\frac{1}{q} - d^l + \frac{c_2^u}{\beta^l} + 0.5(q-1)(\sigma_2^u)^2 \right)}{h^l(q+1)}$. Further, we calculate g'' at the critical point and

notice that the maximum value occurs at the critical point which is $g_{\max} = \left(\frac{q}{h^l} \right)^q \left(\frac{\left(\frac{1}{q} - d^l + \frac{c_2^u}{\beta^l} + 0.5(q-1)(\sigma_2^u)^2 \right)}{q+1} \right)^{q+1}$. So, we

get

$$E(e^t y^q) \leq y_0^q + q \left(\frac{q}{h^l} \right)^q \left(\frac{\left(\frac{1}{q} - d^l + \frac{c_2^u}{\beta^l} + 0.5(q-1)(\sigma_2^u)^2 \right)}{q+1} \right)^{q+1} (e^t - 1).$$

So,

$$E(y^q(t)) \leq \left(y_0^q - \left(\frac{q}{h^l} \right)^q \left(\frac{\left(\frac{1}{q} - d^l + \frac{c_2^u}{\beta^l} + 0.5(q-1)(\sigma_2^u)^2 \right)}{q+1} \right)^{q+1} \right) e^{-t}$$

$$+ q \left(\frac{q}{h^l} \right)^q \left(\frac{\left(\frac{1}{q} - d^l + \frac{c_2^u}{\beta^l} + 0.5(q-1)(\sigma_2^u)^2 \right)}{q+1} \right)^{q+1}.$$

Now, one can notice that at $t = 0$, $E(y^q(t)) \leq y_0^q$. Also when $t \rightarrow \infty$, $E(y^q(t)) \leq q \left(\frac{q}{h^l} \right)^q \left(\frac{\left(\frac{1}{q} - d^l + \frac{c_2^u}{\beta^l} + 0.5(q-1)(\sigma_2^u)^2 \right)}{q+1} \right)^{q+1}$.

$$\text{We take } M(q) = \max \left\{ q \left(\frac{q}{h^l} \right)^q \left(\frac{\left(\frac{1}{q} - d^l + \frac{c_2^u}{\beta^l} + 0.5(q-1)(\sigma_2^u)^2 \right)}{q+1} \right)^{q+1}, y_0^q \right\}.$$

Therefore, we get $E(y^q(t)) \leq M(q)$ for $q \geq 1$. \square

Note. In the similar procedure we can find moment bound for the prey species $x(t)$ also. For conciseness we avoid the proof.

Definition 7.1. 1. We say the species $x(t)$ goes to extinction if $\lim_{t \rightarrow +\infty} x(t) = 0$.

2. We say the population $x(t)$ is non-persistent in mean if $\langle x \rangle^* = 0$.

3. We say the population $x(t)$ is weakly persistent in mean if $\langle x \rangle^* > 0$.

Lemma 7.2. The solution of the system (15) initiates from $(x_0, y_0) \in \mathbb{R}_+^2$ hold the below mentioned properties: $\lim_{t \rightarrow \infty} \sup \frac{\ln x(t)}{t} \leq 0$ and $\lim_{t \rightarrow \infty} \sup \frac{\ln y(t)}{t} \leq 0$ almost surely.

Proof. The proof can be shown here in the similar procedure as Cheng [58] has done. So, we omit this. \square

Lemma 7.3. Now let $x(t) \in C[\Omega \times \mathbb{R}_+, \mathbb{R}_+^0]$, where $\mathbb{R}_+^0 := \{r : r > 0, r \in \mathbb{R}\}$.

(1) If it is possible to find positive constants α_0 , T , $\alpha \geq 0$ such that

$$\ln x(t) \leq \alpha T - \alpha_0 \int_0^t x(s) ds + \sum_{i=1}^n \beta_i B_i(t)$$

for $t \geq T$, where $\beta_i(t)$ ($1 \leq i \leq n$) are continuous bounded functions on \mathbb{R}_+ and $B_i(t)$ are independent standard Brownian motions, then we have: $\langle x \rangle^* \leq \frac{\alpha}{\alpha_0}$ almost surely.

(2) If it is possible to find positive constants α_0 , T , $\alpha \geq 0$ such that,

$$\ln x(t) \geq \alpha T - \alpha_0 \int_0^t x(s) ds + \sum_{i=1}^n \beta_i B_i(t)$$

for $t \geq T$, where $\beta_i(t)$ ($1 \leq i \leq n$) are continuous bounded functions on \mathbb{R}_+ and $B_i(t)$ are independent standard Brownian motions, then we have: $\langle x \rangle^* \geq \frac{\alpha}{\alpha_0}$ almost surely i.e. $\langle x \rangle^* \geq \frac{\alpha}{\alpha_0}$ a.s.

Now, we are going to present the main results of this section.

Now, we apply Ito's formula to (15) we have

$$d \ln x = \left(\frac{r(t)}{1+f(t)y(t)} - b(t)x(t) - \frac{c_1(t)y(t)}{1+\beta(t)x(t)} - \frac{\sigma_1^2(t)}{2(1+f(t)y(t))^2} \right) dt + \frac{\sigma_1(t)}{(1+f(t)y(t))} dB_1(t), \quad (18)$$

$$d \ln y = \left(-d(t) - h(t)y(t) + \frac{c_2(t)x(t)}{1+\beta(t)x(t)} - \frac{\sigma_2^2(t)}{2} \right) dt + \sigma_2(t) dB_2(t). \quad (19)$$

Let $\Upsilon_1(t) = r(t) - \frac{(\sigma_1^u)^2}{2(1+f)y_0^2}$, $\Upsilon_2(t) = -d(t) - \frac{\sigma_2^2(t)}{2}$ and $\Upsilon_3(t) = \frac{r^l}{1+f(t)M(1)} - \frac{\sigma_1^2(t)}{2}$. Then $\langle \Upsilon_2 \rangle^* < 0$. For prey population $x(t)$, we have the following theorem:

Theorem 7.4. (1) If $\langle \Upsilon_1 \rangle^* < 0$, then $x(t)$ goes to extinction almost surely.

(2) If $\langle \Upsilon_1 \rangle^* = 0$, then $x(t)$ is non-persistent in the mean almost surely.

(3) $\langle \Upsilon_3 \rangle^* > 0$, then $x(t)$ is weakly persistent in the mean almost surely.

Proof. (1) For this we follow [46]. We have from (18) that

$$\ln x(t) - \ln x_0 \leq \int_0^t \left[\Upsilon_1(s) - b(s)x(s) - \frac{c_1(s)y(s)}{1+\beta(s)x(s)} \right] ds + \int_0^t \frac{\sigma_1(s)}{1+f(s)y(s)} dB_1(s) \quad (20)$$

Then we get

$$\ln x(t) - \ln x_0 \leq \int_0^t \gamma_1(s) ds + \int_0^t \frac{\sigma_1(s)}{1+f(s)y(s)} dB_1(s).$$

Now, we set $M_1(t) = \int_0^t \frac{\sigma_1(s)}{1+f(s)y(s)} dB_1(s)$, this is a local martingale and the quadratic variation of it is $\langle M_1, M_2 \rangle = \int_0^t \frac{\sigma_1^2(s)}{(1+f(s)y(s))^2} ds \leq \frac{(\sigma_1^u)^2}{(f^l y^l)^2} t$.

From the strong law of large number for martingales [57], we can derive that

$$\lim_{t \rightarrow \infty} \frac{M_1(t)}{t} = 0 \text{ a.s.} \quad (21)$$

Then

$$\frac{\ln x(t) - \ln x_0}{t} \leq \frac{1}{t} \int_0^t \gamma_1(s) ds + \frac{M_1(t)}{t} \quad (22)$$

By taking limit superior on both side of (22), we get

$$\limsup_{t \rightarrow \infty} \frac{\ln x(t)}{t} \leq \langle \gamma_1 \rangle^* < 0.$$

So, $\lim_{t \rightarrow \infty} x(t) = 0$.

(2) From Eq. (20) we get

$$\frac{\ln x(t) - \ln x_0}{t} \leq \langle \gamma_1 \rangle - b^l \langle x(t) \rangle + \frac{M_1(t)}{t} \quad (23)$$

By the specific property of limit superior, we can easily notice that for an arbitrarily given and sufficient small $\epsilon > 0$, there exist a $T > 0$ such that $\langle \gamma_1 \rangle \leq \langle \gamma_1 \rangle^* + \frac{\epsilon}{2}$ for all $t \geq T$. Also from (21), it indicates that $\frac{M_1(t)}{t} \leq \frac{\epsilon}{2}$ for all $t \geq T$.

Now, by substituting the above inequality in (23) we have

$$\ln x(t) - \ln x_0 \leq (\langle \gamma_1 \rangle^* + \epsilon)t - b^l \int_0^t x(s) ds.$$

Now, we assume that $\langle \gamma \rangle^* = 0$, then we have

$$\ln \frac{x(t)}{x_0} \leq \epsilon t - b^l x_0 \int_0^t \frac{x(s)}{x_0} ds.$$

As $b^l > 0$ we have from Lemma 7.3

$$\langle x(t) \rangle^* \leq \frac{\epsilon}{b^l}. \quad (24)$$

By arbitrariness of ϵ , as the solution of (15) is non-negative so we must have $\langle x(t) \rangle^* = 0$. From Definition 7.1, it follows that the prey species $x(t)$ is non-persistent in the mean $x(t)$ a.s.

(3) We need only to show that, there exist a constant $m_1 > 0$ such that for any solution $(x(t), y(t))$ of (15) with initial value $(x_0, y_0) \in \mathbb{R}_+^2$,

$$\langle x(t) \rangle^* \geq m_1 > 0$$

almost surely.

If not, then for arbitrary $\epsilon_1 > 0$ there exist a solution $(\tilde{x}(t), \tilde{y}(t))$ with positive initial value $x_0 > 0$ and $y_0 > 0$ such that, $P\{\langle \tilde{x}(t) \rangle^* < \epsilon_1\} > 0$.

Now, let ϵ_1 be sufficiently small so that

$$\begin{cases} \langle \gamma_3 \rangle^* - b^u \epsilon_1 > 0, \\ \langle \gamma_2 \rangle^* + c_2^u \epsilon_1^2 < 0. \end{cases} \quad (25)$$

From (19),

$$\frac{\ln \tilde{y}(t) - \ln y_0}{t} \leq \langle \gamma_2 \rangle - h^l \langle \tilde{y}(t) \rangle + c_2^u \langle \tilde{x}(t) \rangle + \frac{M_2(t)}{t}, \quad (26)$$

where $M_2(t) = \int_0^t \sigma_2(s) dB_2(s)$ and

$$\lim_{t \rightarrow \infty} \frac{M_2(t)}{t} = 0. \quad (27)$$

From (26) we get $[t^{-1} \ln \tilde{y}(t)]^* = \langle \gamma_2 \rangle^* + c_2^u \epsilon_1 < 0$.

Therefore,

$$\lim_{t \rightarrow \infty} \tilde{y}(t) = 0. \quad (28)$$

From (20),

$$\frac{\ln x(t) - \ln x_0}{t} \geq \langle \gamma_3 \rangle - b^u \langle \tilde{x}(t) \rangle - c_1^u \langle \tilde{y} \rangle + \frac{M_1(t)}{t}.$$

Taking limit superior to the both side of the above inequality and using (21), (25) and (28), we get

$$[t^{-1} \ln \tilde{x}(t)]^* \geq \langle \gamma_3 \rangle^* - b^u \epsilon_1 > 0.$$

We have shown that $P\{[t^{-1} \ln \tilde{x}(t)]^* > 0\} > 0$ which contradicts Lemma 7.2.

Hence, $\langle x(t) \rangle^* > 0$, the prey population $x(t)$ is weakly persistent in the mean almost surely. \square

In case of predator species we have the following theorem;

Theorem 7.5. (1) If $b_* \langle \gamma_2 \rangle^* + c_2^* \langle \gamma_1 \rangle^* < 0$, then $y(t)$ goes to extinction almost surely.

(2) If $b_* \langle \gamma_2 \rangle^* + c_2^* \langle \gamma_1 \rangle^* = 0$, then $y(t)$ is non-persistent in the mean almost surely.

(3) $\langle \gamma_2 \rangle^* + \left(\frac{c_2(t) \tilde{x}(t)}{1 + \beta \tilde{x}(t)} \right)^* > 0$, then $y(t)$ is weakly persistent in the mean almost surely.

Proof. If $\langle \gamma_1 \rangle^* \leq 0$ then from previous theorem it follows that $\langle x(t) \rangle^* = 0$.

Now by using (19),

$$\frac{\ln y(t) - \ln y_0}{t} \leq \langle \gamma_2 \rangle + c_2^u \langle x(t) \rangle + \frac{M_2(t)}{t}.$$

So, we get $[t^{-1} \ln y(t)]^* \leq \langle \gamma_2 \rangle^* < 0$, then $\lim_{t \rightarrow \infty} y(t) = 0$.

Now, if $\langle \gamma_1 \rangle^* > 0$, from the property of superior limit, interior limit and from (27) that for arbitrarily chosen small there exists a $T > 0$ such that

$$\frac{\ln x(t) - \ln x_0}{t} \leq \langle \gamma_1 \rangle^* + \frac{\epsilon}{2} - b_* \langle x(t) \rangle + \frac{\epsilon}{2} \quad \forall t > T.$$

Applying Lemma 7.3 and arbitrariness of ϵ we have

$$\langle x(t) \rangle^* \leq \frac{\langle \gamma_1 \rangle^*}{r_*}. \quad (29)$$

Now by (19) we get

$$[t^{-1} \ln y(t)]^* \leq \langle \gamma_2 \rangle^* + c_2^* \frac{\langle \gamma_1 \rangle^*}{r_*}. \quad (30)$$

Then

$$[t^{-1} \ln y(t)]^* \leq \frac{b_* \langle \gamma_2 \rangle^* + c_2^* \langle \gamma_1 \rangle^*}{b_*} < 0.$$

That is $\lim_{t \rightarrow \infty} y(t) = 0$ almost surely.

(2) In the previous case we have shown that $\langle \gamma_1 \rangle^* \leq 0$ then $\lim_{t \rightarrow \infty} y(t) = 0$. i.e. $\langle y(t) \rangle^* = 0$.

Now, we have to show that $\langle y(t) \rangle^* = 0$ when $\langle \gamma_1 \rangle^* > 0$.

If not then $\langle y(t) \rangle^* > 0$, following Lemma 7.3 and by (30) we have

$$0 = [t^{-1} \ln y(t)]^* \leq \langle \gamma_2 \rangle^* + c_2^* \langle x(t) \rangle^* \quad (31)$$

for any arbitrary $\epsilon > 0$ we will get a $T > 0$ such that

$$\begin{aligned} \langle \gamma_2 \rangle &< \langle \gamma_2 \rangle^* + \frac{\epsilon}{3}, \\ \langle c_2(t) x(t) \rangle &\leq c_2^* \langle x(t) \rangle^* + \frac{\epsilon}{3} \quad \forall t > T, \\ \frac{M_2(t)}{t} &< \frac{\epsilon}{3}. \end{aligned}$$

Putting in (19) we get

$$\frac{\ln y(t) - \ln y_0}{t} \leq \langle \gamma_2 \rangle^* + \epsilon + c_2^* \langle x(t) \rangle^* - h_* \langle y(t) \rangle \quad \forall t > T.$$

Now by Lemma 7.3 and (31),

$$\langle y(t) \rangle^* \leq \frac{\langle \gamma_2 \rangle^* + \epsilon + c_2^* \langle x(t) \rangle^*}{h_*}.$$

Now, as ϵ is arbitrary and from (29) we get

$$\langle y(t) \rangle^* \leq \frac{\langle \gamma_2 \rangle^* b_* + c_2^* \langle \gamma_1 \rangle^*}{b_* h_*} = 0.$$

which is a contradiction.

Therefore $\langle y(t) \rangle^* = 0$ almost surely.

(3) We just only to show that $\langle y(t) \rangle^* > 0$ almost surely.

If it is not true then for any arbitrary $\epsilon_2 > 0$, a solution $(\tilde{x}(t), \tilde{y}(t))$ of the system (15) will exist with positive initial value $(x_0, y_0) \in \mathbb{R}_+^2$ such that $P(\tilde{y}(t))^* < \epsilon_2$.

Let ϵ_2 be sufficiently small so that

$$\langle \gamma_2 \rangle^* + \left\langle \frac{c_2(t)\tilde{x}(t)}{1 + \beta\tilde{x}(t)} \right\rangle^* > h^u + \frac{c_1^u c_2^u}{(\beta^l)^2 \alpha^l} + \frac{c_2^u}{\beta^l} \langle \zeta(t) \rangle^*, \quad (32)$$

where $\zeta(t)$ is defined later.

From (19)

$$\frac{\ln \tilde{y}(t) - \ln y_0}{t} = \langle \gamma_2 \rangle + \left\langle \frac{c_2(t)\tilde{x}(t)}{1 + \beta\tilde{x}(t)} \right\rangle - \langle h(t)\tilde{y} \rangle + \frac{M_2(t)}{t} + \left\langle \frac{c_2(t)\tilde{x}(t)}{1 + \beta\tilde{x}(t)} - \frac{c_2(t)\tilde{x}(t)}{1 + \beta\tilde{x}(t)} \right\rangle,$$

where $(\tilde{x}(t), \tilde{y}(t))$ is the solution of the following model system

$$\begin{cases} d\tilde{x}(t) = \tilde{x}(t) \left[r(t) - b(t)\tilde{x}(t) \right] dt + \frac{\sigma_1(t)\tilde{x}(t)}{1+f(t)\tilde{y}(t)} dB_1(t), \\ d\tilde{y}(t) = \tilde{y}(t) \left[\frac{c_2}{\beta} - h(t)\tilde{y} \right] dt + \sigma_2(t)\tilde{y}(t) dB_2(t), \end{cases} \quad (33)$$

with initial value $x_0 > 0$ and $y_0 > 0$.

$\tilde{x}(t) \leq \tilde{x}(t), \tilde{y}(t) \leq \tilde{y}(t)$ a.s for $t \in [0, \infty)$, as

$$\begin{aligned} \frac{c_2(t)\tilde{x}(t)}{1 + \beta(t)\tilde{x}(t)} - \frac{c_2(t)\tilde{x}(t)}{1 + \beta(t)\tilde{x}(t)} &= \frac{c_2(t)(\tilde{x}(t)) - \tilde{x}(t)}{(1 + \beta(t)\tilde{x}(t))(1 + \beta(t)\tilde{x}(t))}, \\ &= -\frac{c_2(t)(\tilde{x}(t)) - \tilde{x}(t)}{(1 + \beta(t)\tilde{x}(t))(1 + \beta(t)\tilde{x}(t))} \\ &\geq -\frac{c_2(t)(\tilde{x}(t) - \tilde{x}(t))}{\beta(t)}. \end{aligned}$$

After calculation we get

$$\frac{\ln \tilde{y}(t) - \ln y_0}{t} = \langle \gamma_2 \rangle + \left\langle \frac{c_2(t)\tilde{x}(t)}{1 + \beta\tilde{x}(t)} \right\rangle - \langle h(t)\tilde{y} \rangle + \frac{M_2(t)}{t} - \left\langle \frac{c_2^u(\tilde{x}(t) - \tilde{x}(t))}{\beta^l} \right\rangle. \quad (34)$$

Let us now consider the Lyapunov function $V_2(t) = |\ln \tilde{x}(t) - \tilde{x}(t)|$. Obviously, V_2 is positive function on \mathbb{R}_+ . After using Ito's formula, and by (33) and (34) we get

$$\begin{aligned} d^+ V_2(t) &= \left[\left(r(t) - \frac{\sigma_1^2(t)}{2(1+f(t)\tilde{y}(t))^2} - b(t)\tilde{x}(t) \right) dt + \frac{\sigma_1(t)}{1+f(t)\tilde{y}(t)} dB_1(t) \right] \\ &\quad - \left[\left(r(t) - \frac{\sigma_1^2(t)}{2(1+f(t)\tilde{y}(t))^2} - b(t)\tilde{x}(t) - \frac{c_1(t)\tilde{y}(t)}{1 + \beta(t)\tilde{x}(t)} \right) dt + \frac{\sigma_1(t)}{1+f(t)\tilde{y}(t)} dB_1(t) \right] \\ &\leq \left[\left(r(t) - \frac{\sigma_1^2(t)}{2(1+f(t)\tilde{y}(t))^2} - b(t)\tilde{x}(t) \right) dt + \sigma_1^u dB_1(t) \right] \\ &\quad - \left[\left(r(t) - \frac{\sigma_1^2(t)}{2(1+f(t)\tilde{y}(t))^2} - b(t)\tilde{x}(t) - \frac{c_1(t)\tilde{y}(t)}{1 + \beta(t)\tilde{x}(t)} \right) dt + \frac{\sigma_1^l}{1+f^u\tilde{y}^u} dB_1(t) \right] \\ &\leq \left[\frac{c_1^u \tilde{y}(t)}{\beta^l} - b^l (\tilde{x}(t) - \tilde{x}(t)) + \frac{\sigma_1^2(t)}{2(1+f(t)\tilde{y}(t))^2} - \frac{\sigma_1^2(t)}{2(1+f(t)\tilde{y}(t))^2} \right] dt \\ &\quad + \left(\sigma_1^u - \frac{\sigma_1^l}{1+f^u\tilde{y}^u} \right) dB_1(t). \end{aligned} \quad (35)$$

Now integrating the above inequality from 0 to t and then dividing by t , we have

$$\frac{V_2(t) - V_2(0)}{t} \leq \frac{c_1^u}{\beta^l} \langle \tilde{y}(t) \rangle - b^l \langle \tilde{x}(t) - \tilde{x}(t) \rangle + \langle \zeta(t) \rangle, \quad (36)$$

where $\zeta(t) = \frac{\sigma_1^2(t)}{2(1+f(t)\tilde{y}(t))^2} - \frac{\sigma_1^2(t)}{2(1+f(t)\tilde{y}(t))^2} \geq 0$ a.s because $\tilde{y}(t) \leq \tilde{y}(t)$ a.s for $t \in [0, \infty)$.

Now, $\frac{V_2(t)}{t} \geq 0$, $V_2(0) = 0$ yields

$$\langle \tilde{x}(t) - \tilde{x}(t) \rangle \leq \frac{c_1^u}{b^l \beta^l} \langle \tilde{y}(t) \rangle + \langle \zeta(t) \rangle. \quad (37)$$

Using the above inequality and from (34) we get

$$\begin{aligned} \frac{\ln \tilde{y}(t) - \ln y_0}{t} &\geq \langle \gamma_2 \rangle + \left\langle \frac{c_2(t) \tilde{x}(t)}{1 + \beta \tilde{x}(t)} \right\rangle - \langle h(t) \tilde{y} \rangle + \frac{M_2(t)}{t} \\ &\quad - \frac{c_1^u c_2^u}{(\beta^l)^2 \alpha^l} \langle \tilde{y}(t) \rangle - \frac{c_2^u}{\beta^l} \langle \zeta(t) \rangle. \end{aligned} \quad (38)$$

taking limit superior and by (32),

$$[t^{-1} \ln \tilde{y}(t)]^* \geq \langle \gamma_2 \rangle^* + \left\langle \frac{c_2(t) \tilde{x}(t)}{1 + \beta \tilde{x}(t)} \right\rangle^* - \left(h^u + \frac{c_1^u c_2^u}{(\beta^l)^2 \alpha^l} \right) \epsilon_2 - \frac{c_2^u}{\beta^l} \langle \zeta(t) \rangle^* > 0,$$

which contradicts Lemma 7.2. Therefore, $\langle y(t) \rangle^* > 0$ almost surely. This implies that $y(t)$ is weakly persistent in the mean almost surely. \square

8. Stochastic permanence

Definition 8.1. Let $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ be the solution of a stochastic population model with initial value $x(0) \in \mathbb{R}_+^n$. If for any $\epsilon \in (0, 1)$ there exist a positive $\delta = \delta(\epsilon)$ and $\chi = \chi(\epsilon)$ s.t

$$\lim_{t \rightarrow \infty} \inf P\{|x(t)| = \sqrt{\sum_{i=1}^n x_i^2(t)} \geq \delta\} \geq 1 - \epsilon, \quad \lim_{t \rightarrow \infty} \inf P\{|x(t)| \leq \chi\} \geq 1 - \epsilon,$$

then the stochastic population model is said to be stochastically permanent.

Theorem 8.1. If

$$2(\max\{\sigma_1^u, \sigma_2^u\})^2 < \min\left\{\frac{r^l}{1 + f^u M(1)} - \frac{c_1^u}{\beta^l}, \frac{c_2^l}{\beta^u} - d^u\right\}, \quad (39)$$

then the system (15) is stochastically permanent.

Proof. At first we need to show that for any $\epsilon > 0$ there exists a constant $\delta > 0$ such that

$$P_*\{|x(t)| \geq \delta\} \geq 1 - \epsilon. \quad (40)$$

Let the condition hold

$$2(\phi + 1)(\max\{\sigma_1^u, \sigma_2^u\})^2 < \min\left\{\frac{r^l}{1 + f^u M(1)} - \frac{c_1^u}{\beta^l}, \frac{c_2^l}{\beta^u} - d^u\right\}. \quad (41)$$

By (39) there exists a constant $p > 0$ that obeys

$$\min\left\{\frac{r^l}{1 + f^u M(1)} - \frac{c_1^u}{\beta^l}, \frac{c_2^l}{\beta^u} - d^u - 2(\phi + 1)(\max\{\sigma_1^u, \sigma_2^u\})^2 - p\right\} > 0. \quad (42)$$

Let $V(x, y) = x + y$ then by Ito's formula

$$\begin{aligned} dV(x, y) &= \left\{x(t) \left(\frac{r(t)}{1 + f(t)y(t)} - b(t)x(t) - \frac{c_1(t)y(t)}{1 + \beta(t)x(t)} \right) + \frac{c_2(t)x(t)y(t)}{1 + \beta(t)x(t)} \right\} dt \\ &\quad + y(t)(-d(t) - h(t)y(t)) dt + \frac{\sigma_1(t)x(t)}{1 + f(t)y(t)} dB_1(t) + \sigma_2(t)y(t) dB_2(t). \end{aligned}$$

Now, let $Q(x, y) = \frac{1}{V(x, y)}$. By applying Ito's formula we get

$$\begin{aligned} dQ(X) &= \left[-Q^2(X) \left(x(t) \left(\frac{r(t)}{1 + f(t)y(t)} - b(t)x(t) - \frac{c_1(t)y(t)}{1 + \beta(t)x(t)} \right) + \frac{c_2(t)x(t)y(t)}{1 + \beta(t)x(t)} \right) \right] dt \\ &\quad - Q^2(X)y(t)(-d(t) - h(t)y(t)) dt + Q^3(X) \left(\frac{\sigma_1^2(t)x^2(t)}{(1 + f(t)y(t))^2} + y^2(t)\sigma_2^2(t) \right) dt \end{aligned}$$

$$\begin{aligned}
& - Q^2(X) \left(\frac{\sigma_1(t)x(t)}{1+f(t)y(t)} dB_1(t) + \sigma_2(t)y(t) dB_2(t) \right) \\
& = LQ(X)dt - Q^2(X) \left(\frac{\sigma_1(t)x(t)}{1+f(t)y(t)} dB_1(t) + \sigma_2(t)y(t) dB_2(t) \right).
\end{aligned}$$

Now, we choose a positive constant ϕ so that it obeys (39). Then

$$\begin{aligned}
& L(1+Q(X))^\phi \\
& = \phi(1+Q(X))^{\phi-1} LQ(X) + \frac{1}{2} \phi(\phi-1)(1+Q(X))^{\phi-2} Q^4(X) \left(\frac{\sigma_1^2(t)x^2(t)}{(1+f(t)y(t))^2} + \sigma_2^2(t)y^2(t) \right).
\end{aligned}$$

So, we choose a sufficiently small positive constant p such that it obeys (41).

Now, we take $M(X) = e^{pt} (1+Q(X))^\phi$. Then

$$\begin{aligned}
LM(X) &= pe^{pt}(1+Q(X))^\phi + e^{pt}L(1+Q(X))^\phi \\
&= e^{pt}(1+Q(X))^{\phi-2} \left[p(1+Q(X))^2 + \phi(1+Q(X))LQ(X) + \frac{1}{2} \phi(\phi-1)Q^4(X) \left(\frac{\sigma_1^2(t)x^2(t)}{(1+f(t)y(t))^2} + \sigma_2^2(t)y^2(t) \right) \right] \\
&= e^{pt}(1+Q(X))^{\phi-2} \left[p(1+Q(X))^2 - \phi Q^2(X)x(t) \left(\frac{r(t)}{1+f(t)y(t)} - b(t)x(t) - \frac{c_1(t)y(t)}{1+\beta(t)x(t)} \right) - \phi Q^2(X)y(t) \right. \\
&\quad \left(-d(t) - h(t)y(t) + \frac{c_2(t)x(t)}{1+\beta(t)x(t)} \right) \\
&\quad - \phi Q^3(X) \left(x(t) \left(\frac{r(t)}{1+f(t)y(t)} - b(t)x(t) - \frac{c_1(t)y(t)}{1+\beta(t)x(t)} \right) + y(t) \frac{c_2(t)x(t)}{1+\beta(t)x(t)} \right) \\
&\quad - \phi Q^3(X)y(t) (-d(t) - h(t)y(t)) + \phi Q^3(X) \left(\frac{\sigma_1^2(t)x^2(t)}{(1+f(t)y(t))^2} + \sigma_2^2(t)y^2(t) \right) \\
&\quad \left. + \frac{\phi(\phi+1)}{2} Q^4(X) \left(\frac{\sigma_1^2(t)x^2(t)}{(1+f(t)y(t))^2} + \sigma_2^2(t)y^2(t) \right) \right]. \tag{43}
\end{aligned}$$

It is obvious that

$$\phi Q^3(X) \left(\frac{\sigma_1^2(t)x^2(t)}{(1+f(t)y(t))^2} + \sigma_2^2(t)y^2(t) \right) \leq \phi Q^3(X) (\sigma_1^2(t)x^2(t) + \sigma_2^2(t)y^2(t)) \leq \phi Q(X) (2 \max\{\sigma_1^u, \sigma_2^u\})^2,$$

and

$$\frac{\phi(\phi+1)}{2} Q^4(X) \left(\frac{\sigma_1^2(t)x^2(t)}{(1+f(t)y(t))^2} + \sigma_2^2(t)y^2(t) \right) \leq \frac{\phi(\phi+1)^2}{Q} (X) (2 \max\{\sigma_1^u, \sigma_2^u\})^2.$$

Hence,

$$\begin{aligned}
& LQ(X) \\
& \leq e^{pt}(1+Q(X))^{\phi-2} \left[p + \phi \max\{b^u, h^u\} + \left(2p - \phi \min \left\{ \frac{r^l}{1+f^u M(1)} - \frac{c_1^u}{\beta^l}, \frac{c_2^l}{\beta^u} - d^u \right\} \right. \right. \\
& \quad \left. \left. \phi \max\{b^u, h^u\} + 4\phi(\max\{\sigma_1^u, \sigma_2^u\})^2 \right) Q(X) + \left(p - \phi \min \left\{ \frac{r^l}{1+f^u M(1)} - \frac{c_1^u}{\beta^l}, \frac{c_2^l}{\beta^u} - d^u \right\} \right. \right. \\
& \quad \left. \left. + \frac{\phi(\phi+1)}{2} (2 \max\{\sigma_1^u, \sigma_2^u\})^2 \right) Q^2(X) \right].
\end{aligned}$$

By (41) there exists constant $c > 0$ such that, $LQ(X) \leq ce^{pt}$. Therefore,

$$E[e^{pt}(1+Q(X))^\phi] \leq (1+Q(0))^\phi + \frac{c(e^{pt}-1)}{p}.$$

Therefore,

$$\begin{aligned}
\limsup_{t \rightarrow \infty} E \left[\frac{1}{|x(t)|^\phi} \right] &\leq 2^\phi \limsup_{t \rightarrow \infty} EQ^\phi(X) \\
&\leq 2^\phi \frac{c}{p} = B.
\end{aligned}$$

Then for any $\epsilon > 0$ letting $\delta = (\frac{\epsilon}{B})^{\frac{1}{\phi}}$ and using Chebyshev's inequality we obtain

$$P\{|X(t)| < \delta\} = P\{|X(t)|^{-\phi} > \delta^{-\phi}\}$$

$$\begin{aligned} &\leq \frac{E[|X(t)|^{-\phi}]}{\delta^{-\phi}} \\ &= \delta^\phi E[|X(t)|^{-\phi}]. \end{aligned}$$

So, we get $P_*\{|X(t)| \geq \delta\} \geq 1 - \epsilon$.

Now, we have to prove that for arbitrary $\epsilon > 0$ there exists a positive constant χ such that, $P_*\{|x(t)| \leq \chi\} \geq 1 - \epsilon$.

For this purpose, let us define $U(X) = x^u + y^u$, where $0 < u < 1$, $X = (x, y) \in \mathbb{R}_2^+$.

Applying Ito's formula we have

$$\begin{aligned} dU((t)) = & ux^u \left(\frac{r(t)}{1+f(t)y(t)} - b(t)x(t) - \frac{c_1(t)y(t)}{1+\beta(t)x(t)} + \frac{u-1}{2} \frac{\sigma_1^2(t)}{(1+f(t)y(t))^2} \right) dt \\ & + uy^u \left(-d(t) - h(t)y(t) + \frac{c_2(t)x(t)}{1+\beta(t)x(t)} + \frac{u-1}{2} \sigma_2^2(t) \right) dt + ux^u \frac{\sigma_1(t)}{1+f(t)y(t)} dB_1(t) + uy^u \sigma_2(t) dB_2(t). \end{aligned} \quad (44)$$

Let z_0 be sufficiently large so that X_0 lies within the interval $[\frac{1}{z_0}, z_0]$. For each integer $z \geq z_0$ we define the stopping time, $\tau_z = \inf\{t \geq 0 : X(t) \notin (\frac{1}{z}, z)\}$. So obviously τ_z increases as $z \rightarrow \infty$.

Therefore,

$$\begin{aligned} &E[\exp\{t \wedge \tau_z\} X^u(t \wedge \tau_z)] - X^u(0) \\ &\leq uE \int_0^{t \wedge \tau_z} \exp s \cdot x^u(s) \left(1 + u \left(\frac{r(s)}{1+f(s)y(s)} - b(s)x(s) - \frac{1-u}{2} \frac{\sigma_1^2(s)}{(1+f(s)y(s))^2} \right) \right) ds \\ &+ uE \int_0^{t \wedge \tau_z} \exp s \cdot y^u(s) \left(1 + u \left(-d(s) - h(s)y(s) + \frac{c_2}{\beta} - \frac{1-u}{2} \sigma_2^2(s) \right) \right) ds \\ &\leq E \int_0^{t \wedge \tau_z} (\alpha_1 + \alpha_2) \exp s ds \\ &\leq (\alpha_1 + \alpha_2)(\exp t - 1). \end{aligned}$$

Here α_1 and α_2 are positive constants.

Taking $z \rightarrow \infty$ we obtain

$$\exp t \cdot E[X^u(t)] \leq X^u(0) + (\alpha_1 + \alpha_2)(\exp t - 1).$$

From the above equation we obtain

$$\limsup_{t \rightarrow \infty} E[X^u(t)] \leq \alpha_1 + \alpha_2.$$

So for any $\epsilon > 0$, let us choose $\chi = \frac{(\alpha_1 + \alpha_2)^{\frac{1}{u}}}{\epsilon^{\frac{1}{u}}}$ and by Chebyshev's inequality we obtain

$$P\{|X(t)| > \chi\} = P\{|X(t)|^u > \chi^u\} \leq \frac{E[|X(t)|^u] - \chi^u}{\chi^u}.$$

That is, $P_*\{|X(t)| > \chi\} \leq \frac{E[|X(t)|^u] - \chi^u}{\chi^u} \leq \epsilon$.

Consequently,

$$P_*\{|X(t)| \leq \chi\} \geq 1 - \epsilon. \quad (45)$$

The proof is now complete. \square

9. Numerical simulation with biological implications

In this section, computer based simulations have been performed to validate our analytical findings obtained in previous sections. The numerical simulations and figures have been done using 'Matlab 2014a' and 'Wolfram Mathematica 9'. For convenience we have exhibited all the numerical results both for deterministic system and stochastic system separately.

9.1. Deterministic system

We have studied the deterministic system considering both the autonomous and non-autonomous versions due to biological importance. The numerical findings have been presented as follows:

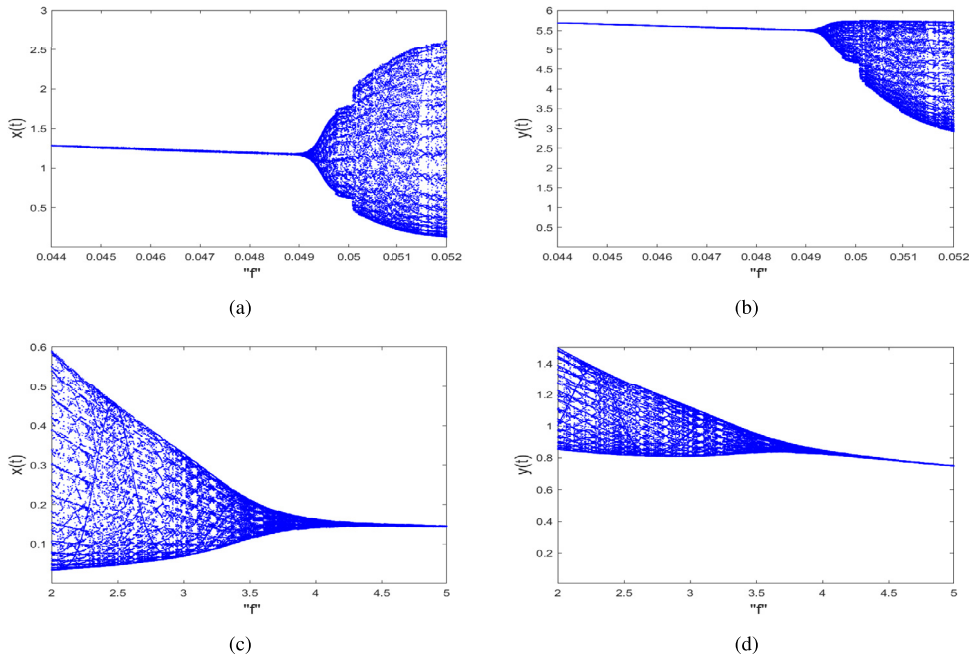


Fig. 1. Bifurcation diagrams of the model system (1) for the bifurcation parameter f . (a) and (b) represent that the system (1) switches its stability when f crosses the threshold value 0.049. (c) and (d) exhibit that the system (1) switches its stability again when f exceeds the other threshold value 4.2 i.e. two consecutive Hopf bifurcations occur for two different values of f .

9.1.1. Autonomous version

For the autonomous version of our model system (1) we have considered the parameters values as $r = 2$, $f = 0.04$, $b = 0.5$, $c_1 = 0.6$, $\beta = 2$, $d = 0.03$, $h = 0.02$, $c_2 = 0.4$. For these set of parameters values the model system (1) has three non-negative equilibrium points i.e. the trivial equilibrium point $E_1(0, 0)$, the axial equilibrium point $E_2(4, 0)$ and the coexistence equilibrium point $E^*(1.377, 5.838)$. The axial equilibrium point $E_2(4, 0)$ is unstable since $\frac{c_2 r}{b + \beta r} = 0.177 > d = 0.03$ i.e., the condition of LAS of the axial equilibrium point in Theorem 3.3 is violated in this case. We are now mainly concern about dynamical behaviour of the system (1) around the positive interior equilibrium point $E^*(1.377, 5.838)$. It is observed that the fear level (f) acts as Hopf bifurcation parameter in system dynamics. Interestingly, the system (1) switches its stability twice i.e. two consecutive Hopf bifurcations occur for two different threshold values of f . For the above set of parameters values the system remain stable up to the threshold value 0.049 of f , then becomes unstable (see Figs. 1(a) and 1(b)). If we increase the level of fear f then the system exhibits periodic coexistence around the positive interior equilibrium point up to the value 4.2 of f , then again a Hopf bifurcation occurs and the coexistence equilibrium point E^* becomes stable (see Figs. 1(c) and 1(d)).

Another important observation is that not only the fear level f but the intra-specific competition rate h among the predators also acts as Hopf bifurcation parameter. It is observed that if the intra-specific competition rate h among the top predators increases while the values of other parameters fixed as before; the system (1) goes to Hopf bifurcation. In this case, the system remains unstable up to the threshold value 0.032 of h and the populations exhibit stable coexistence when h exceeds the threshold value 0.032 of h (see Figs. 2(a) and 2(b)).

9.1.2. Non-autonomous version

To capture the seasonal effect we consider the rate parameters to be sinusoidal function. We set the parameters values as $r(t) = .8 + .1 \sin t$, $f(t) = 5 + .5 \sin t$, $b(t) = 0.22 + 0.1 \sin t$, $c_1(t) = 0.6 + 0.1 \sin t$, $\beta(t) = 4 + 0.05 \sin t$, $d(t) = 0.06 + 0.01 \sin t$, $h(t) = 0.02 + 0.01 \sin t$, $c_2(t) = 0.6 + 0.1 \sin t$. One can observe that the condition $[E_1]$, that is $[c_2(t) - d(t)\beta(t)]_H = 0.2165 > 0$ is satisfied and also the condition $[E_2]$ that is $[c_2(t)e^{H_1} - d(t)\beta(t)e^{L_1} - d(t)]_L = 3.6111 > 0$ also hold. Hence, we can claim that system (3) possess a positive periodic solution starting from the positive quadrant and the solution will persist in future time also. Biologically, both the species coexist in the oscillatory mode. We have shown it numerically in Fig. 3, Fig. 3 also reveals the fact after certain time different trajectories initiates from different values of the both the species come very much close to each other which actually indicate the fact that the positive periodic solution of the non-autonomous system (3) is a global attractor or in other word globally asymptotically stable.

To show uniform persistence of our system (3) we choose $r(t) = 3.8 + .1 \sin t$, $f(t) = 2 + .1 \sin t$, $h(t) = 0.05 + 0.01 \sin t$ then we notice that $c_2^M = 0.7 > 0.1975 = d^L \beta^L$ also $\frac{c_2^L x_0}{1 + \beta^M x_0} = 0.0763 > 0.07 = d^M$, and $\frac{r^L}{1 + f^M \phi_2} = 2.5631 > 2.2544 =$

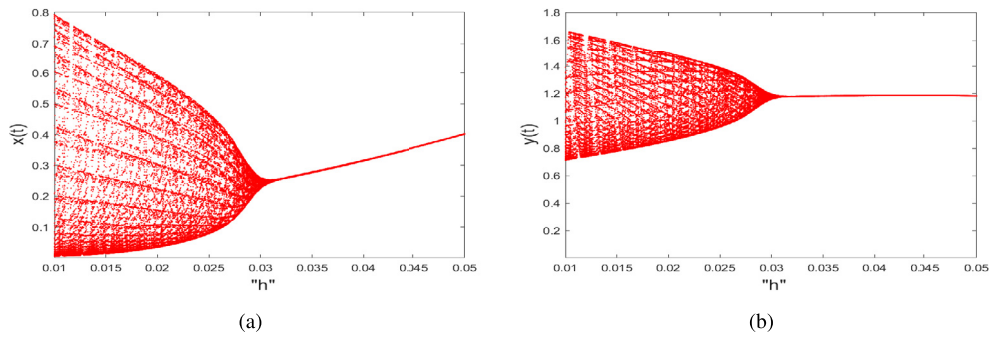


Fig. 2. Bifurcation diagram of the model system (1) for the bifurcation parameter h . (a) and (b) represent that the system (1) becomes stable from its unstable position when h crosses the threshold value 0.032.

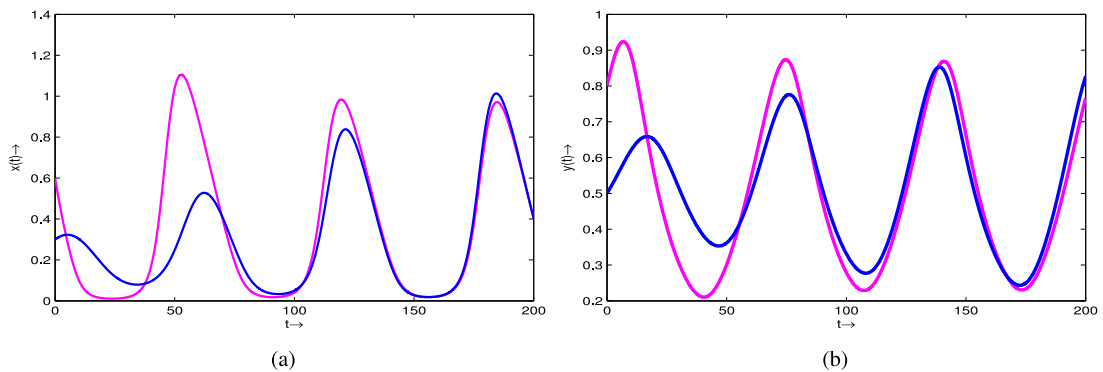


Fig. 3. Coexistence of globally asymptotically stable positive periodic solution. (a) and (b) represent the time evolution of the prey and predator species respectively for a particular set of parameters values.

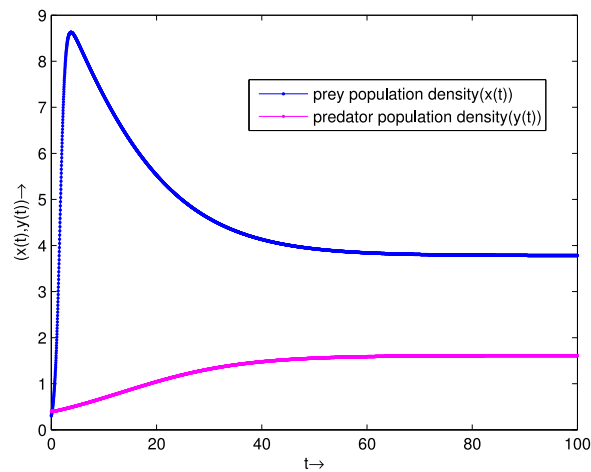


Fig. 4. Time evolution of prey-predator species shows that system is persistent.

$\frac{c_1^M \phi_1}{\beta_L}$. Then all the conditions of uniform persistence of the deterministic system (3) hold here, the corresponding figure (Fig. 4) also exhibits the same.

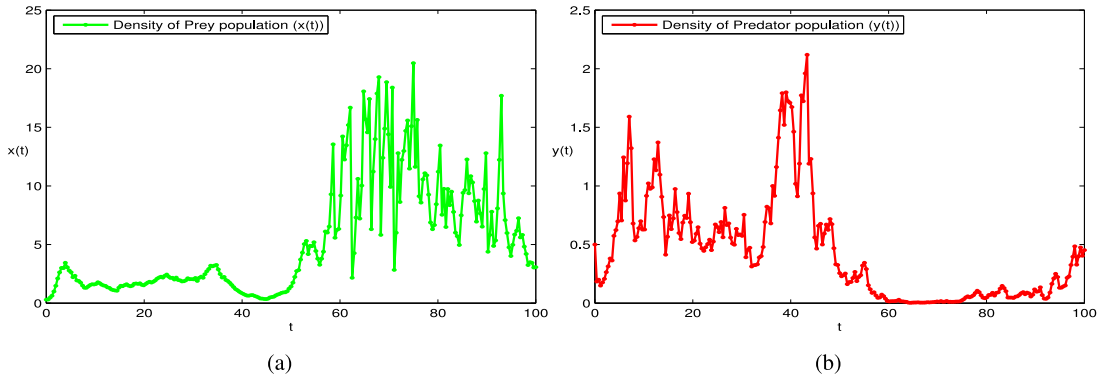


Fig. 5. Computer simulation for the stochastic system (15) shows weakly persistence in mean of both the species. (a) and (b) represent the trajectories for the prey and predator populations respectively of the stochastic system (15).

9.2. Stochastic non-autonomous system

According to the Milstein's method in Hingham [59] we consider the following discretized equations corresponding to the stochastic system (23):

$$\begin{aligned}
 x_{k+1} &= x_k + x_k \left[\frac{r(k\Delta t)}{1 + f(\Delta t)y_k} - b(\Delta t)x_k - \frac{c_1(\Delta t)y_k}{1 + \beta(\Delta t)x_k} \right] \Delta t \\
 &\quad + \sigma_1(\Delta t) \frac{x_k}{1 + f(\Delta t)y_k} \sqrt{\Delta t} \xi_k + \frac{\sigma_1^2(k\Delta t)}{2} \frac{x_k}{1 + f(\Delta t)y_k} (\xi_k^2 - 1) \Delta t \\
 y_{k+1} &= y_k + y_k \left[-d(\Delta t) - h(\Delta t)y_k + \frac{c_2(\Delta t)x_k}{1 + \beta(\Delta t)x_k} \right] \Delta t + \sigma_2(k\Delta t) y_k \sqrt{\Delta t} \eta_k + \frac{\sigma_2^2(k\Delta t)}{2} y_k (\eta_k^2 - 1) \Delta t
 \end{aligned}$$

where ξ_k and η_k ($k = 1, 2, \dots, n$) are independent Gaussian random variables $N(0,1)$.

Simulation for Weakly persistence:

We have chosen the parameters values as $r(t) = 3.8 + .1 \sin t$, $f(t) = 2 + 0.1 \sin t$, $b(t) = 0.21 + 0.1 \sin t$, $c_1(t) = 0.6 + 0.1 \sin t$, $\beta(t) = 4 + 0.05 \sin t$, $d(t) = 0.06 + 0.01 \sin t$, $h(t) = 0.05 + 0.01 \sin t$, $c_2(t) = 0.6 + 0.1 \sin t$; we further choose $\frac{\sigma_1^2}{2} = 0.09 + 0.02 \sin t$, $\frac{\sigma_2^2}{2} = 0.07 + 0.02 \sin t$. We verified the condition of weakly persistence of prey of Theorem 7.4 i.e. $\langle \gamma_3 \rangle^* = 0.0309 > 0$ hold here. Condition of weakly persistence of predator species of Theorem 7.5 also holds good. Fig. 5 depicts the fact.

Remark. We notice that in the absence of environmental noise for the same set of parameters values the deterministic system is uniformly persistent (see Fig. 4) whether both the species of the stochastic system become weakly persistent depending upon the intensity of the noise (see Fig. 5).

Simulation for Permanence:

We have set the parameters values as, $r(t) = 3.8 + .1 \sin t$; $f(t) = 2 + 0.5 \sin t$; $b(t) = 0.21 + 0.1 \sin t$; $c_1(t) = 0.6 + 0.1 \sin t$; $\beta(t) = 4 + 0.05 \sin t$; $d(t) = 0.06 + 0.01 \sin t$; $h(t) = 0.05 + 0.01 \sin t$; $c_2(t) = 0.6 + 0.1 \sin t$; we choose $\frac{\sigma_1^2}{2} = 0.005 + 0.005 \sin t$, $\frac{\sigma_2^2}{2} = 0.005 + 0.005 \sin t$. Now, we get $2(\max\{\sigma_1^u, \sigma_2^u\})^2 = 0.04$ and also $\min\{\frac{r^l}{1+f^u M(1)} - \frac{c_1^u}{\beta^l}, \frac{c_2^l}{\beta^u} - d^u\} = 0.0534$, so the condition for stochastic permanence in Theorem 8.1 is satisfied. We validate the result by the corresponding figures (see Figs. 6(a) and 6(b)).

Remark. In the deterministic counterpart for the same set of parameters values we observed that the system is uniformly persistent in the absence of environmental noise (see Fig. 4). Here for the stochastic system when the intensity of noise is very low we observe that system becomes stochastically permanent and strongly persist for any future time (see Fig. 6).

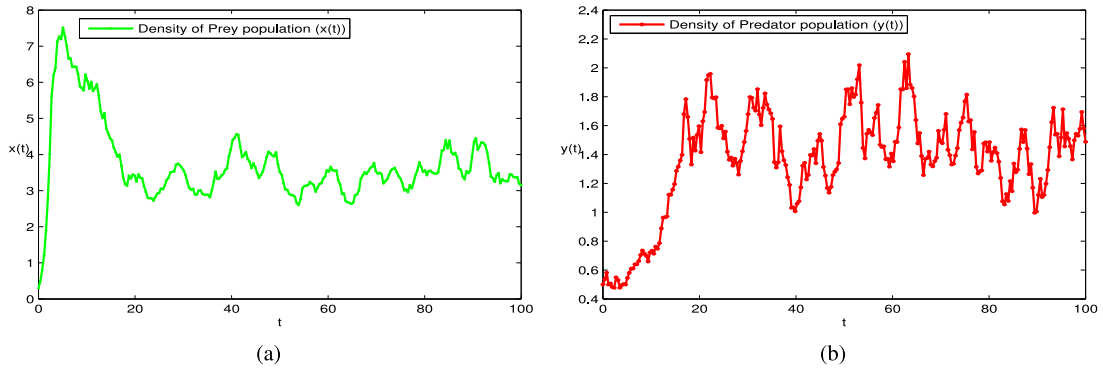


Fig. 6. Time evolution of prey–predator species shows that system (15) is stochastically permanent. (a) and (b) represent the trajectories for the prey and predator populations respectively of the stochastic system (15).

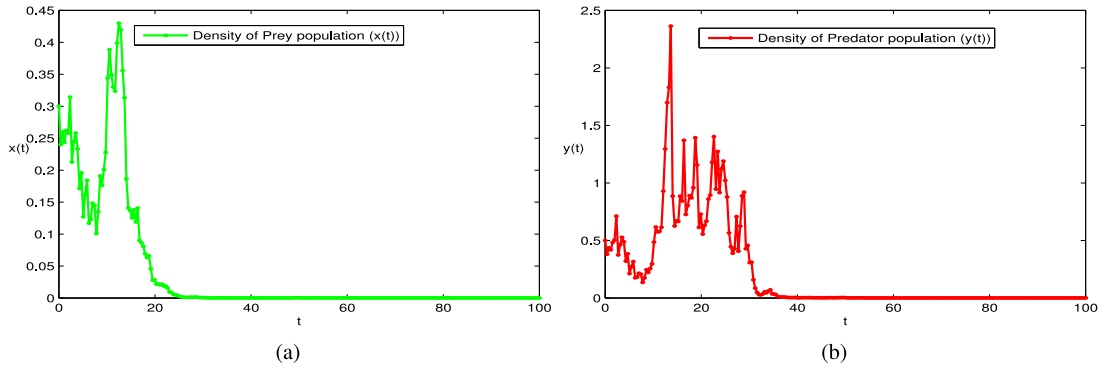


Fig. 7. (a) and (b) represent time evolution of the prey and predator populations respectively of the system (15) when the intensity of noise is high. These figures clearly depict that high intensity of noise lead both the species of the system to extinction.

Simulation for Extinction:

First we choose the parameters values as $r(t) = .2 + .01 \sin t$, $f(t) = 1 + 0.1 \sin t$, $b(t) = 0.21 + 0.1 \sin t$, $c_1(t) = 0.6 + 0.1 \sin t$, $\beta(t) = 4 + 0.05 \sin t$, $d(t) = 0.06 + 0.01 \sin t$, $h(t) = 0.05 + 0.01 \sin t$, $c_2(t) = 0.6 + 0.1 \sin t$. In Fig. 3 we choose $\frac{\sigma_1^2}{2} = 1.1 + 0.02 \sin t$, $\frac{\sigma_2^2}{2} = 0.4 + 0.02 \sin t$. We get $\langle \gamma_1 \rangle^* = -0.55861 < 0$ and also $b_* \langle \gamma_2 \rangle^* + c_2^* \langle \gamma_1 \rangle^* = -0.4661627 < 0$. So both the conditions of extinction of the prey–predator species of Theorems 7.4 and 7.5 are satisfied. The corresponding figures (see Fig. 7) also depicts the fact.

Remark. When environmental fluctuation that is noise intensity is very high then both the species go to extinction after a certain time.

Now we consider some other cases. We take the parameters values as $r(t) = .2 + .1 \sin t$, $f(t) = 2 + .1 \sin t$, $b(t) = 0.22 + 0.1 \sin t$, $c_1(t) = 0.6 + 0.1 \sin t$, $\beta(t) = 4 + 0.05 \sin t$, $d(t) = 0.06 + 0.01 \sin t$, $h(t) = 0.02 + 0.01 \sin t$, $c_2(t) = 0.6 + 0.1 \sin t$. We consider the stochastic system (15) with the intensity of the noise as $\frac{\sigma_1^2}{2} = 0.21 + 0.02 \sin t$, $\frac{\sigma_2^2}{2} = 1.5 + 0.02 \sin t$. The conditions for weakly persistence of prey and extinction for predator species are satisfied. We exhibit the fact by Fig. 8.

Remark. Due to the intensity of the environmental noise in a prey–predator system one population may be permanently washed out from the system whether the other population survives and persist for all future time.

Effect of fear:

How the induced fear by predator affect the whole system in a periodically varying season and also in the presence of environmental noise is an area of utmost interest. For these purpose, we set the following set of parameters values as, $r(t) = 5 + .5 \sin t$, $b(t) = 0.2 + 0.1 \sin t$, $c_1(t) = 0.7 + 0.1 \sin t$, $\beta(t) = 4 + 0.06 \sin t$, $d(t) = 0.02 + 0.01 \sin t$, $h(t) = 0.04 + 0.01 \sin t$, $c_2(t) = 0.6 + 0.1 \sin t$. We also choose when the noise intensity is low as $\frac{\sigma_1^2}{2} = 0.02 + 0.01 \sin t$ and $\frac{\sigma_2^2}{2} = 0.02 + 0.01 \sin t$. In the absence of fear ($f = 0$), we have observed that both the species strongly persist in the

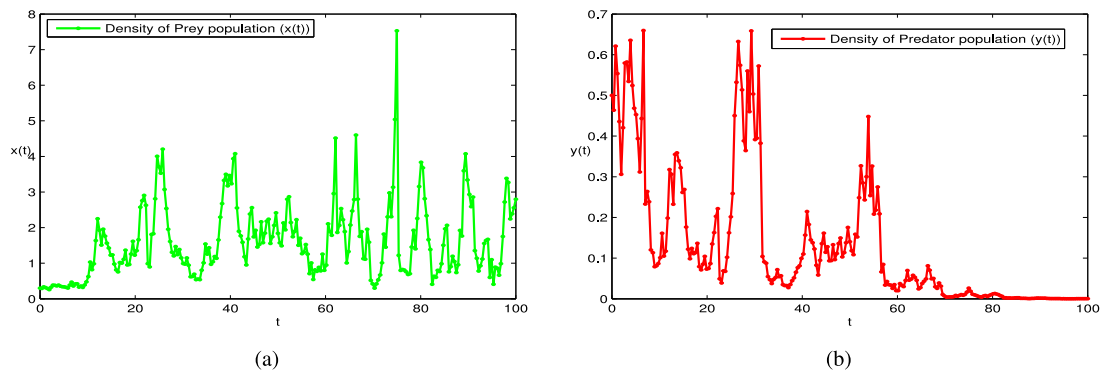


Fig. 8. Time evolution of the stochastic prey–predator system (15). (a) represents that the prey species persist, (b) exhibits the fact that the predator species goes to extinction.

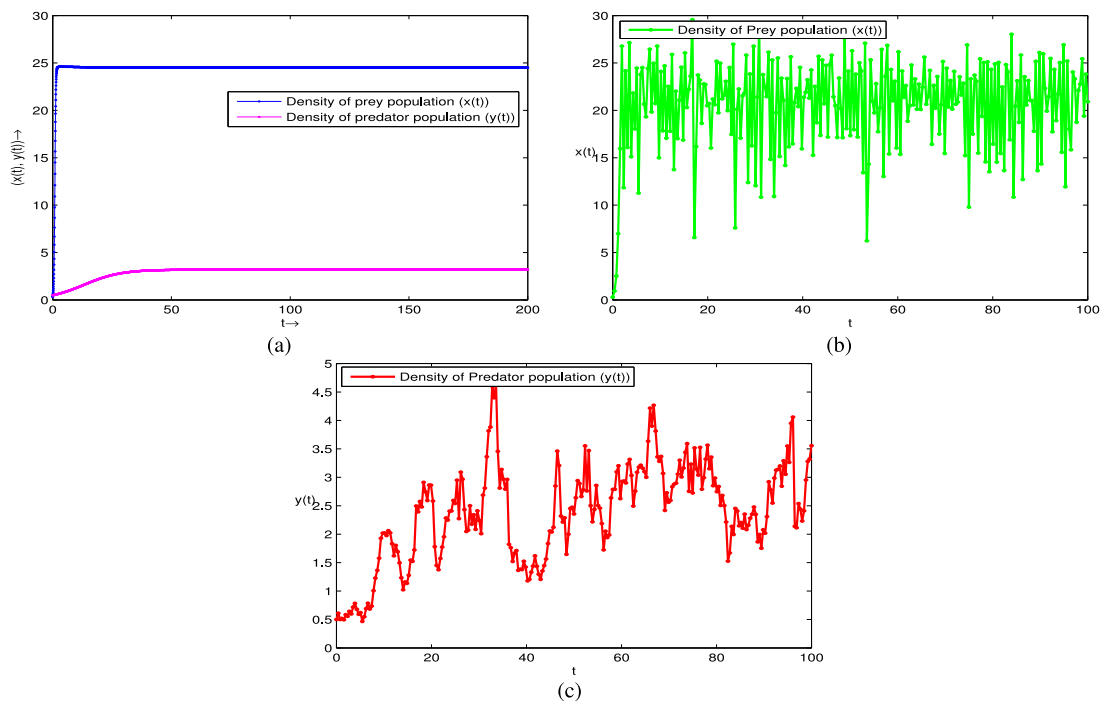


Fig. 9. Trajectories of the prey–predator species of the underlying systems (3) and (15) when there is no fear. (a), (b) and (c) exhibit that in the absence of fear both the prey–predator species of our underlying systems (3) and (15) strongly persist.

system and the prey species has comparatively a higher growth rate. The corresponding figure (see Fig. 9) also validates this fact.

Now we consider the case when the level of fear is low ($f(t) = 1 + 0.5 \sin t$), all other parameters values and strength of noise have been kept fixed as above. In this case we noticed that when we consider the deterministic system (3) the prey and predator both species become uniformly persistent and exhibit stable behaviour (see Fig. 10(a)) and for the stochastic system (15) with added environmental noise both the species become stochastically permanent (see Fig. 10(b) and Fig. 10(c)). The corresponding conditions are also satisfied here.

Now we increase the level of fear by $f(t) = 8 + 0.5 \sin t$, we kept all other parameters values and the intensity of noise fixed as above. In this case we observed that the deterministic system (3) have periodic coexistence (see Fig. 11) and both the species of the corresponding stochastic system (15) are fluctuating randomly. Here to be noticed that the rapidity of fluctuation depends upon the noise intensity.

Remark. In the absence of fear both the prey and predator species exhibits strong persistence and prey population maintains a higher growth rate, when the fear level is low then the dynamics of the systems remains almost same but the

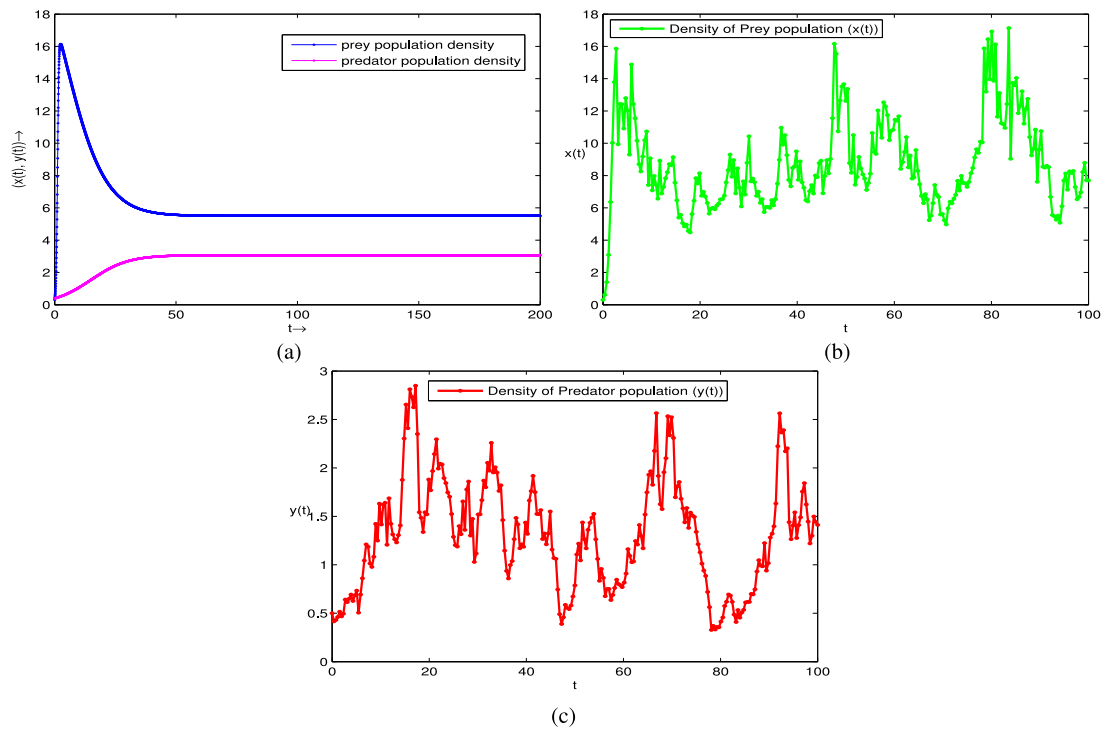


Fig. 10. (a) represents that when the fear level is low then the deterministic system (3) exhibits stable behaviour. (b) and (c) represent that both the species becomes stochastically permanent of the system (15).

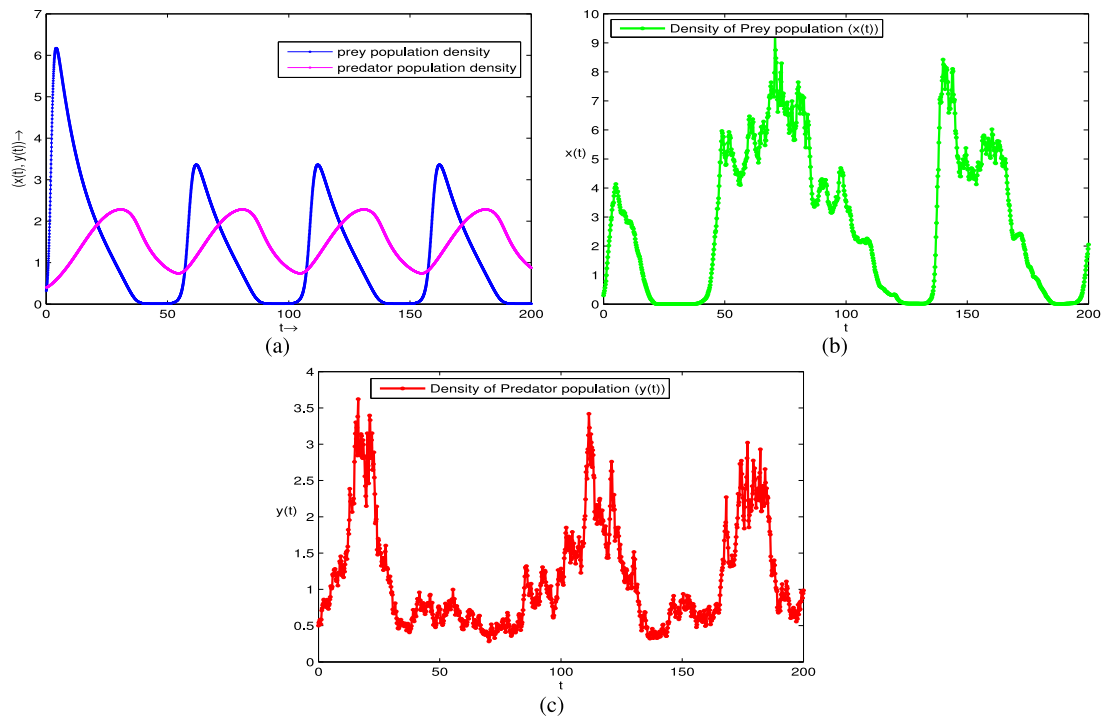


Fig. 11. (a) exhibits that when the fear level is high the deterministic system (3) has periodic coexistence. (b) and (c) depict the facts that both the species fluctuate randomly of the stochastic system (15) for high level of fear.

growth rate of prey population deteriorates and for a high level of fear the dynamics of the system in both of deterministic and stochastic environment changed drastically and the populations exhibits periodic co-existence.

10. Conclusion

Recent field studies suggest that the intrinsic growth rate of prey species in an ecological system is affected by the fear induced from predators [17]. Moreover, the level of fear varies by several factors like mating habits, food resources, intra-specific competition etc [25]. In this view, in our present work at first we proposed a two species prey–predator model incorporating fear function due to induced fear from predators where the predators are engaged in intra-specific competition. The well-posedness of the model along with the existence criterion of the non-negative equilibrium points and their stability have been discussed. We observed that both the fear level (f) and intra-specific competition rate (h) among top predators play a vital role in the dynamics of the autonomous system. It is found that two consecutive Hopf bifurcations occur for different level of fear (see Fig. 1). The intra-specific competition rate h also leads the system to Hopf bifurcation; engagement of more predators in intra-specific competition enhances the stable coexistence of all the species (see Fig. 2).

Next, we consider non-autonomous version of the model system incorporating seasonal variation of the intrinsic parameters of the system. Here, we derived the conditions of uniform persistence and it is found that the level of fear is duly involved in the persistence of the species. If fear level is very high then the uniform persistence of the system will be lost. We have also derived sufficient conditions under which the non-autonomous system possess a unique globally attractive positive periodic solution. It is an interesting finding through numerical simulation that in the case of periodic co-existence of both the species, the predator species follows almost the same pattern of periodicity with that of prey species (see Fig. 3). Also, fear does not affect the periodic co-existence of the system when the intrinsic growth rate is low. But when the intrinsic growth rate is high then a higher level of fear will force the interacting species for periodic coexistence (see Fig. 11(a)).

Finally, we extended our model into a stochastic model by perturbing the intrinsic growth rate of prey species and natural mortality rate of predator species with uncorrelated white noise terms. We have shown that the solution of the stochastic system will not explode at a finite time and without any parametric restriction the system will possess a unique global solution starting from any interior of the positive quadrant. We also established sufficient conditions for extinction, persistence and stochastic permanence of both the species in view of parametric restrictions and noise intensity. It is observed that when the intensity of noise is very low the system becomes stochastically permanent (see Fig. 6). If the strength of the noise becomes little bit higher then unlike from its deterministic counterpart both the species of the stochastic system become weakly persistence (see Figs. 4 and 5). If the intensity of the environmental noise is very high then both the species extinct after a certain time (see Fig. 7). Thus it may be concluded that environmental noise significantly affects the dynamics of the system; especially it can lead the species for its extinction.

The level of fear also has also significant effects in the dynamics of the system. Under a moderate intensity of noise with higher value of intrinsic growth rate, it is found that in the absence of fear both the species of the stochastic system as well as the deterministic system strongly persist (see Fig. 9), whereas with high level of fear the species exhibit periodic coexistence (see Fig. 11). From Fig. 11 it is clear that the prey species goes to minimum level in a periodic manner as a cost of fear. When prey is in minimum level then predators also deteriorate due to lack of food and then prey species again grows up due to less number of predators and the system shows periodic behaviour. This kind of periodic oscillation is observed in grasshoppers–spiders prey–predator system. Field experiment revealed that the more number of predators induced physiological changes in the scared grasshoppers that reduces the basic reproduction of grasshoppers but in a predator free region or when the spiders are in less numbers then due to absence of fear they grow naturally [60]. Studies also revealed that fear of wolves apart from their direct killing of prey (elk) have significant effect on their ecosystem. Due to induced fear the elks change their habit to feed them in a more safer area, as a consequence wolves cannot get sufficient amount of food and grow less [61]. This type of periodic behaviour is depicted in Fig. 11.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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