The Frobenius-Schur Indicator and the Finite Rotation Groups

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This thesis represents my own work in accordance with University regulations.

/s/ Daniel Penner

Contents

Chapter 1

Representation theory of finite groups

1.1 Basics of linear representations

In this first chapter, we will build the theory of group representations and their characters from the bottom up. In the first section, we present the basic definitions, prove the fundamental decomposition theorem of group representations, and introduce a couple simple examples. In the second section, we will present a broad overview of the theory of group characters, beginning with basic definitions and elementary lemmas, and proceeding to prove the orthogonality relations of irreducible characters, and a theorem which reduces the study of representations to the study of their characters. Along the way we will briefly discuss the group algebra as it relates to this paper, and will conclude with orthogonality and divisibility theorems for character tables, and some simple examples of character tables. Much of the standard theory follows the first half of J.P. Serre's classic book on the subject (see [7]).

1.1.1 Definitions and theory

Let G be a finite group and V be a complex finite-dimensional vector space (note: the infinite-dimensional case will not be mentioned in this paper). A **linear representation** of G in V is a homomorphism $\rho : G \to GL(V)$. Here $GL(V)$ denotes the group of isomorphisms of V, which is isomorphic to the group of invertible $n \times n$ complex matrices, where $n = \dim(V)$, and we call n the **degree** of the representation. More concretely, for all $s, t \in G$ there are complex $n \times n$ matrices $\rho_s, \rho_t, \rho_{st}$ such that $\rho_{st} = \rho_s \rho_t$. In particular, this means that $\rho_1 = I_n$, where $1 \in G$ is the identity element and I_n is the $n \times n$ identity matrix.

Let $\rho : G \to GL(V)$ be a representation of degree n, and $W \subset V$ be a proper subspace (that is, neither the zero space nor the whole space). We say W is **G-invariant** if $\rho_s w \in W$ for all $w \in W$ and all $s \in G$. If W is a G-invariant subspace of V, then for each $s \in G$, the restriction $\rho_s\big|_W$ is an endomorphism of W, and $\rho_{st}\big|_W = \rho_s\big|_W \rho_t\big|_W$ for all $s, t \in G$, so $\rho\big|_W : G \to GL(W)$ is a homomorphism, and thus a linear representation. We call it a **subrepresentation** of ρ. If $V = W_1 \oplus W_2$

for two G-invariant subspaces W_1, W_2 of V, then we say that $\rho = \rho\big|_{W_1} \oplus \rho\big|_{W_2}$, or that ρ is a **direct sum** of its subrepresentations. Finally, we say that two representations $\rho^1 : G \to GL(V_1)$ and $\rho^2 : G \to GL(V_2)$ **isomorphic** if there is a vector space isomorphism $f : V_1 \to V_2$ such that, for all $s \in G$, we have $f \circ \rho^1_s \circ f^{-1} = \rho^2_s$.

The following lemma allows us to decompose a representation into a direct sum of subrepresentations.

Lemma 1.1.1. *Let $\rho : G \to GL(V)$ be a representation, and let $\rho\big|W : G \to GL(W)$ be a subrepresentation to a G-invariant subspace $W \subset V$. Then there is a G-invariant complement $W' \subset V$ such that $V = W \oplus W'$, and thus $\rho = \rho\big|_W \oplus \rho\big|_{W'}$.*

Proof. Let p be a projection of V onto W. Then we construct a new projection onto W by averaging over all the elements of G:

$$p' = \frac{1}{|G|} \sum_{s \in G} \rho_s \cdot p \cdot \rho_{s^{-1}}.$$

Since W is G-invariant, and the $\mathrm{Im}(p) \subset W$, we obtain that $\mathrm{Im}(p') \subset W$ as well. Since p is a projection, it restricts to the identity on W. Therefore if $w \in W$, then for all $s \in G$, since $\rho_{s^{-1}} w \in W$, we have

$$p'w = \rho_s \cdot p \cdot \rho_{s^{-1}} w = \rho_s \cdot \rho_{s^{-1}} w = w.$$

So p' is a projection onto W. Let $W' = \mathrm{Ker}(p')$.

Since the sum in p' runs over all elements of G, we have that for any $s \in G$, $\rho_s \cdot p' \cdot \rho_{s^{-1}} = p'$, and thus $\rho_s \cdot p' = p' \cdot \rho_s$. So if $w' \in W'$, we have $p' \cdot \rho_s w' = \rho_s \cdot p' w' = \rho_s \cdot 0 = 0$. So $\rho_s w' \in W'$, and thus W' is G-invariant. So we have two G-invariant subspaces W, W' of V such that $V = W \oplus W'$, and therefore $\rho = \rho\big|_W \oplus \rho\big|_{W'}$. □

We say that a representation $\rho : G \to GL(V)$ is **irreducible** if V has no proper G-invariant subspaces. The following theorem allows us to focus all our attention on understanding irreducible representations.

Theorem 1.1.2 (Maschke's Theorem)**.** *Every representation $\rho : G \to GL(V)$ is a direct sum of irreducible subrepresentations.*

Proof. We go by induction on $n = \dim(V)$. If $n = 1$, then V has no proper subspaces at all, and thus ρ is irreducible.

Now suppose $n > 1$. If V has no G-invariant subspace, then it is irreducible. Otherwise, let $W \subset V$ be a G-invariant subspace. The preceding lemma gives us that $V = W \oplus W^\perp$, where $W^\perp$ is also G-invariant, and thus $\rho = \rho\big|_W \oplus \rho\big|_{W^\perp}$. But these are both representations of degree strictly less than n, and so they are covered by our inductive hypothesis, and are thus direct sums of irreducible subrepresentations. So ρ is a direct sum of irreducible subrepresentations. □

1.1.2 Examples

To show some of the theory in action, let's examine the representations of S_3, the symmetric group on three letters. Every group has a representation $\rho^1_s = 1$ for all $s \in G$, called the **trivial representation**. This is an example of a degree-one representation, all of which are simply homomorphisms from $G \to \mathbb{C}$.

S_3 has another degree-one representation, given by sending an permutation $s \in S_3$ to its sign, $\mathrm{sgn}(s) \in \{\pm 1\}$. Note that this is a representation since sign of a permutation respects products of permutations, and hence it is a homomorphism $G \to \{\pm 1\}$. This representation appears to be different from the trivial representation (that is, nonisomorphic), but it is not immediately obvious how to prove that there are no satisfactory endomorphisms of $\mathbb{C}$. Their non-isomorphism will follow from basic character theory a bit later. However, as we saw in the proof of Maschke's theorem, both the trivial and sign representations are irreducible, since they have degree one.

Next, since S_3 acts on the set $X = \{1, 2, 3\}$, each $s \in S_3$ defines a permutation $x \mapsto sx$, where $1x = x$ and $s(tx) = (st)x$ for any $t \in S_3$. Let V be a $|X| =$ 3-dimensional vector space with a basis $\{e_x\}_{x \in X}$ indexed by the elements of X. Then the elements of S_3 permute the bases of this vector space, which induces a representation of S_3 as a set of 3×3 permutation matrices. Hence we call this the **permutation representation** of S_3. We can form this type of representation for any group G which acts on a finite set X. It is not clear at the moment whether or not this representation, which has degree three, is irreducible. As before, we defer our curiosity to character theory, which will make everything clear.

A particular and very important permutation representation is that induced by G acting on itself by left-multiplication. We call this the **regular representation** of G. In this case let V be a $|G| = n$-dimensional vector space with basis $\{e_s\}_{s \in G}$ indexed by the elements of G. Then for $s, t \in G$, we let ρ_s send $e_t \mapsto e_{st}$, and this induces a set of $n \times n$ permutation matrices. We can already see that this (degree n) representation is not irreducible, however, since the vector space element $\sum_{s \in G} e_s$ generates a 1-dimensional subspace on which every basis vector e_t acts as the identity (since it simply permutes the terms in the sum), and is thus G-invariant. We will return to the regular representation in the context of characters later.

1.2 Character theory

1.2.1 Basic definitions and lemmas

If $\rho : G \to GL(V)$ is a representation, define a function $\chi_\rho : G \to \mathbb{C}$ as $\chi_\rho(s) = \mathrm{Tr}(\rho_s)$. We call this function the **character** of the representation ρ (we will drop the subscript when the representation is clear from context).

Proposition 1.2.1. *If χ is the character of a degree n representation $\rho : G \to GL(V)$, then if $s, t \in G$, we have the following:*

1. $\chi(1) = n$
2. $\chi(s^{-1}) = \overline{\chi(s)}$
3. $\chi(tst^{-1}) = \chi(s)$

Proof. 1. $\rho_1 = I_n$, the $n \times n$ identity matrix, whose trace is n.

2. Since G is finite, there is some $m > 0$ such that $s^m = 1$. Therefore $\rho_s^m = I_n$, since ρ is a homomorphism. So the eigenvalues λ_i of ρ_s are roots of unity, whose complex conjugates are their multiplicative inverses. Since $\mathrm{Tr}(\rho_s) = \sum_i \lambda_i$, it follows that $\overline{\chi(s)} = \overline{\mathrm{Tr}(\rho_s)} = \mathrm{Tr}(\rho_s^{-1}) = \chi(s^{-1})$.

3. For any square matrices A, B, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. So if $A = \rho_{ts} = \rho_t\rho_s$ and $B = \rho_{t^{-1}}$, we see that $\chi(tst^{-1}) = \mathrm{Tr}(\rho_{tst^{-1}}) = \mathrm{Tr}(\rho_{ts}\rho_{t^{-1}}) = \mathrm{Tr}(\rho_{t^{-1}}\rho_{ts}) = \mathrm{Tr}(\rho_s) = \chi(s)$.

□

Proposition 1.2.2. *If $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ are representations of a group G with characters χ_1 and χ_2, respectively, then the character of the representation $\rho = \rho_1 \oplus \rho_2$ is $\chi = \chi_1 \oplus \chi_2$.*

Proof. Since every matrix ρ_s of the representation ρ is in block diagonal form with upper block $\rho_1(s)$ and lower block $\rho_2(s)$, the trace of ρ_s is simply the sum of the traces of its blocks. □

The following lemma and its corollaries facilitates much of the subsequent theory.

Lemma 1.2.3 (Schur's Lemma)**.** *Let $\rho^1 : G \to GL(V_1)$ and $\rho^2 : G \to GL(V_2)$ be irreducible representations of G, and let $f : V_1 \to V_2$ be a linear transformation for which $\rho_s^2 f = f\rho_s^1$ for all $s \in G$. Then the following statements hold:*

1. *If ρ^1, ρ^2 are not isomorphic representations, $f = 0$.*

2. *If $V_1 \simeq V_2$ and $\rho^1 \simeq \rho^2$, then $f = \lambda I$ for some $\lambda \in \mathbb{C}$.*

Proof. 1. Suppose $f \neq 0$. Let $W_1 = \mathrm{Ker}(f) \subset V_1$, and $W_2 = \mathrm{Im}(f) \subset V_2$. If $w \in W_1$, then for any $s \in G$ we have $0 = \rho_s^2 fw = f\rho_s^1 w$, and hence $\rho_s^1 w \in \mathrm{Ker}(f) = W_1$. So W_1 is G-invariant (since $f \neq 0$ and thus its kernel is nontrivial). Similarly, if $w' = fw \in W_2$, for any $s \in G$ we have $\rho_s^2 w' = \rho_s^2 fw = f\rho_s^1 w \in \mathrm{Im}(f) = W_2$. So W_2 is G-invariant as well. Since ρ^1, ρ^2 are irreducible, this means that either $W_1 = 0$ and $W_2 = V_2$, or that $W_1 = V_1$ and $W_2 = 0$. The latter is impossible since $f \neq 0$, so we have that f is both injective and surjective, and hence an isomorphism of vector spaces. Since $\rho_s^2 f = f\rho_s^1$ for any $s \in G$, ρ^1, ρ^2 are isomorphic representations of G.

2. Suppose $V_1 = V_2 = V$ and $\rho^1 = \rho^2 = \rho$. We know f has at least one eigenvalue, since V is a vector space over an algebraically closed field ($\mathbb{C}$). Denote one such eigenvalue by λ, and note that the linear transformation $g = f - \lambda I$ has nontrivial kernel W. So if $w \in W$ and $s \in G$, we see that $0 = \rho_s gw = g\rho_s w$, so that $\rho_s w \in \mathrm{Ker}(g) = W$, and therefore W is G-invariant. Since V is irreducible, W is trivial or the whole space. But we know it is not trivial. So $W = V$, and therefore $g = 0$, so that $f = \lambda I$.

□

Corollary 1.2.4. *Let $h : V_1 \to V_2$ be a linear map from a vector space V_1 of dimension n to a vector space V_2 of dimension m, and define*

$$h' = \frac{1}{|G|} \sum_{s \in G} \rho_{s^{-1}}^2 h \rho_s^1,$$

where $\rho^i : G \to GL(V_i)$ are representations. Then the following hold:

1. *If ρ^1, ρ^2 are non-isomorphic representations, then $h' = 0$.*

2. *If $V_1 = V_2 = V$ and $\rho^1 = \rho^2 = \rho$, then $h' = (Tr(h)/dim(V))I$, where I is the identity on V.*

3. *Writing $\rho_s^1 = (r_{i_1 j_1}(s))_{i_1, j_1 \in \{1,\ldots,n\}}$ (an $n \times n$ matrix), $\rho_s^2 = (r_{i_2 j_2}(s))_{i_2, j_2 \in \{1,\ldots,m\}}$ (an $m \times m$ matrix), and $h = (h_{i_2 i_1})$ (an $m \times n$ matrix), so that we write $h' = (h'_{i_2 i_1})$ where*

$$h'_{i_2 i_1} = \frac{1}{|G|} \sum_{s \in G, j_1, j_2} r_{i_2 j_2}(s^{-1}) h_{j_2 j_1} r_{j_1 i_1}(s),$$

   then if ρ^1, ρ^2 are non-isomorphic representations, and i_1, i_2, j_1, j_2 are arbitrary indices,

$$\frac{1}{|G|} \sum_{s \in G} r_{i_2 j_2}(s^{-1}) r_{j_1 i_1}(s) = 0.$$

4. *Given the same notation as above, if $V_1 = V_2 = V$ and $\rho^1 = \rho^2 = \rho$, we have that*

$$\frac{1}{|G|} \sum_{s \in G} r_{i_2 j_2}(s^{-1}) r_{j_1 i_1}(s) = \frac{\delta_{i_2 i_1} \delta_{j_2 j_1}}{dim(V)} = \begin{cases} 1/dim(V) & \text{if } i_1 = i_2 \text{ and } j_1 = j_2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

1. Since h is linear, and h' is a linear combination of linear transformations, it is itself linear. Also, it satisfies $\rho_s^2 h' = h' \rho_s^1$ for all $s \in G$, since the sum is symmetric in the elements of G. So by the first result of Schur's Lemma, if ρ^1, ρ^2 are non-isomorphic, then $h' = 0$.

2. If $V_1 = V_2 = V$ and $\rho^1 = \rho^2 = \rho$, the second result of Schur's Lemma tells us that $h' = \lambda I$ for some $\lambda \in \mathbb{C}$. Hence since $\mathrm{Tr}(\rho_{s^{-1}} h \rho_s) = \mathrm{Tr}(h)$ for any $s \in G$, we have that

$$\mathrm{Tr}(h') = \frac{1}{|G|} \sum_{s \in G} \mathrm{Tr}(\rho_{s^{-1}} h \rho_s) = \frac{1}{|G|} \sum_{s \in G} \mathrm{Tr}(h) = \mathrm{Tr}(h).$$

   But since $h' = \lambda I$, we see that $\lambda = \mathrm{Tr}(h')/\mathrm{dim}(V) = \mathrm{Tr}(h)/\mathrm{dim}(V)$.

3. By the first result of this corollary, if ρ^1, ρ^2 are non-isomorphic, $h' = 0$, and thus $h'_{i_2 i_1} = 0$ for all such indices. Since it is a linear combination of the entries $h_{j_2 j_1}$, and vanishes for any choice of h, we see that the coefficients in this linear combination, $\sum_{s \in G} r_{i_2 j_2}(s^{-1}) r_{j_1 i_1}(s)$ for all $j_1 \in \{1, \ldots, n\}$ and $j_2 \in \{1, \ldots, m\}$, vanish as well. Since this holds for each choice of (i_1, i_2), we obtain the result.

4. The second result of this corollary gives us that if $\rho^1 = \rho^2 = \rho$ and $V_1 = V_2 = V$, then $h' = (\mathrm{Tr}(h)/\mathrm{dim}(V))I$, and thus $h'_{i_2 i_1} = (\mathrm{Tr}(h)/\mathrm{dim}(V))\delta_{i_2 i_1}$, where $\delta_{i_2 i_1} = 1$ if $i_2 = i_1$ and is zero otherwise, for $i_1, i_2 \in \{1, \ldots, n\}$, and we note

that $\mathrm{Tr}(h) = \sum_{j_1,j_2\in\{1,\ldots,n\}} \delta_{j_2j_1}x_{j_2j_1}$, since the off-diagonal entries of h do not contribute to the trace. Therefore:

$$\frac{1}{|G|}\sum_{s\in G,j_1,j_2\in\{1,\ldots,n\}} r_{i_2j_2}(s^{-1})x_{j_2j_1}r_{i_1j_1}(s) = h'_{i_2i_1}$$

$$= \frac{1}{\dim(V)}\sum_{j_1,j_2\in\{1,\ldots,n\}} \delta_{i_2i_1}\delta_{j_2j_1}x_{j_2j_1}.$$

Therefore

$$\frac{\delta_{i_2i_1}\delta_{j_2j_1}}{\dim(V)} = \frac{1}{|G|}\sum_{s\in G} r_{i_2j_2}(s^{-1})r_{i_1j_1}(s)$$

for all choices of $j_1, j_2 \in \{1, \ldots, n\}$. Since this holds for all choices of $i_1, i_2 \in \{1, \ldots, n\}$, we obtain the result.

□

1.2.2 Irreducible characters and the orthogonality relations

We will now use the above-deduced tools to prove a number of surprising facts about **irreducible characters** (that is, the characters of irreducible representations). We will see that most of the interesting information about a given irreducible representation is in a sense stored in its character, and also that the irreducible characters of a group must follow a strict set of guidelines, making them exceedingly pleasant to work with.

First we define a Hermitian product on functions $\varphi, \psi : G \to \mathbb{C}$:

$$(\varphi, \psi) = \frac{1}{|G|}\sum_{s\in G} \varphi(s)\overline{\psi(s)}, \tag{1.1}$$

and a similar symmetric product:

$$\langle \varphi, \psi\rangle = \frac{1}{|G|}\sum_{s\in G} \varphi(s)\psi'(s^{-1}),$$

and note that if we define the function $\psi'(t) = \overline{\psi(t^{-1})}$, then $(\varphi, \psi) = \langle \varphi, \psi\rangle$. Recall that if χ_1, χ_2 are characters of a representation, then $\chi_2(s^{-1}) = \overline{\chi_2(s)}$ (Proposition 1.2.1). Therefore, $(\chi_1, \chi_2) = \langle \chi_1, \chi_2\rangle$, so we can regard these inner products as identical.

Theorem 1.2.5 (Orthogonality Relations)**.** *If χ, χ' are irreducible characters of a group G, then they are orthonormal:*

$$(\chi, \chi') = \begin{cases} 1 & \chi, \chi' \textit{ are characters of isomorphic representations} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let ρ, ρ' be the representations whose characters are χ, χ', respectively. If $\rho_s = (r_{ij}(s))_{i,j\in\{1,\ldots,n\}}$ and $\rho'_s = (r'_{kl}(s))_{k,l\in\{1,\ldots,m\}}$, then

$$(\chi, \chi') = \langle \chi, \chi'\rangle = \sum_{i\in\{1,\ldots,n\}}\sum_{k\in\{1,\ldots,m\}} \langle r_{ii}, r'_{kk}\rangle = \sum_{i,k}\frac{1}{|G|}\sum_{s\in G} r_{ii}(s)r'_k k(s^{-1}).$$

By the third result of Corollary 1.2.4, if ρ, ρ' are non-isomorphic representations, then each term of the outer sum is equal to zero, and hence $(\chi, \chi') = 0$. If ρ, ρ' are isomorphic, then $\chi = \chi'$, so the fourth result of Corollary 1.2.4 reveals the final sum to be

$$(\chi, \chi') = \langle \chi, \chi \rangle = \sum_{i,j \in \{1,\ldots,n\}} \langle r_{ii}, r_{jj} \rangle$$

$$= \sum_{i,j \in \{1,\ldots,n\}} \frac{1}{|G|} \sum_{s \in G} r_{ii}(s) r_{jj}(s^{-1}) = \sum_{i,j \in \{1,\ldots,n\}} \frac{\delta_{ij}}{\dim(V)} = 1.$$

□

The condition of orthonormality of a group's irreducible characters allows us now to see that any group representation has a unique direct sum decomposition into irreducible subrepresentations (to up isomorphism of said subrepresentations):

Theorem 1.2.6. *If $\rho : G \to GL(V)$ is a linear representation with character χ such that*

$$\rho = \rho_1 \oplus \cdots \oplus \rho_n$$

for irreducible representations ρ_i, $i = 1, \ldots, n$, then for any irreducible representation $\rho' : G \to V$ with character χ', if n_i is the number of ρ_i isomorphic to ρ', then $n_i = (\chi, \chi')$.

Proof. By Proposition 1.2.2, $\chi = \sum_{i=1}^n \chi_i$, and therefore $(\chi, \chi') = \sum_{i=1}^n (\chi_i, \chi')$. Theorem 1.2.5 gives us that the i^{th} summand is nonzero if and only if ρ_i is isomorphic to ρ', and so the sum gives us precisely the number n_i of ρ_i's isomorphic to ρ'. □

Corollary 1.2.7. *Two representations are isomorphic if and only if they have the same character.*

Proof. If ρ, ρ' are two representations with the same character $\chi = \sum_{i=1}^n \chi_i$, which decompose as $\rho = \rho_1 \oplus \cdots \oplus \rho_n$ and $\rho' = \rho'_1 \oplus \cdots \oplus \rho'_{n'}$, then if n_{ij} is the number of ρ_i isomorphic to ρ'_j, we have by Theorem 1.2.6 that $n_{ij} = (\chi, \chi'_j)$, which is exactly the number of times χ'_j occurs in the sum decomposition of χ, since ρ, ρ' have the same character. So the two representations have each irreducible subrepresentation occur the same number of times in their direct sum decompositions (up to isomorphism of the subrepresentations), and are therefore isomorphic (note that in Theorem 1.2.6, the numbers n_i are indepdendent of the particular decomposition into irreducible subrepresentations ρ_i). The converse is clear since trace is preserved under isomorphisms. □

With the above results, we can restrict our attention entirely on the characters of representations, since they uniquely determine the isomorphism class of their representations. To this end, we derive some further qualities of irreducible characters that allow us to work freely with them.

Before we proceed, we note a useful criterion for irreducibility of a representation which follows directly from our preceding results.

Proposition 1.2.8. *Let $\rho : G \to GL(V)$ be a representation with character χ. Then (χ, χ) is an integer, and is equal to one if and only if ρ is an irreducible representation of G.*

Proof. By Theorem 1.2.6, $\rho = n_1\rho_1 \oplus \cdots \oplus n_k\rho_k$ for irreducible representaitons ρ_k. By the orthogonality relations, we have that $(\chi, \chi) = \sum_{i=1}^{k} n_k^2$, which is an integer. Furthermore, it equals one precisely if ρ is equal to one copy of one of the irreducible representations, in other words that ρ is irreducible. $\square$

Recall the regular representation ρ^G from section 1.1.2. The following proposition examines its character to see that every irreducible representation of G is contained within it as a subrepresentation, which will in turn yield some of our nice facts about irreducible characters.

Proposition 1.2.9. *Let G be a finite group, and χ^G be the character of its regular representation. Then for $s \in G$,*

$$\chi^G(s) = \begin{cases} |G| & s = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Furthermore, if ρ_i is an irreducible representation of G with character χ_i, then ρ_i is contained in the direct sum decomposition of ρ^G exactly $n_i = \chi_i(1)$ times. That is, we can write

$$\rho^G = n_1\rho_1 \oplus \cdots n_m\rho_m.$$

Proof. The regular representation has degree $|G|$, since the basis $\{e_t\}_{t\in G}$ of the target vector space is indexed by the elements of G. So $\chi^G(1) = |G|$ (Proposition 1.3). Then if $s \neq 1$, since $\rho_s^G e_t = e_{st}$ for any $t \in G$, we see that the diagonal of the matrix ρ_s^G must have all zeros, since it does not fix any of the basis elements e_s. Therefore its trace is zero. The second claim follows from Theorem 1.2.6, since $n_i = (\chi^G, \chi_i) = \chi_i(1)$, using the form for χ^G we just proved. $\square$

Corollary 1.2.10. *If G is a finite group with irreducible characters χ_i, $i = 1, \ldots, n$, then we have the following:*

1. $\sum_{i=1}^{n} \chi_i^2(1) = |G|$
2. *For any $s \in G$ other than the identity,* $\sum_{i=1}^{n} \chi_i(s)\chi_i(1) = 0$.

Proof. By Proposition 1.2.8, for any $s \in G$, we have $\chi^G(s) = \sum_{i=1}^{n} \chi_i(1)\chi_i(s)$. If $s = 1$, then we obtain that

$$|G| = \chi^G(1) = \sum_{i=1}^{n} \chi_i^2(1),$$

and if $s \neq 1$, then we obtain that

$$0 = \chi^G(s) = \sum_{i=1}^{n} \chi_i(1)\chi_i(s).$$

$\square$

We now determine the number of isomorphism classes of irreducible representations for a given group G. Recall that we can partition a group G into **conjugacy classes** $C_1, \ldots, C_m$, which share the property that if $x \in C_i$, then $sxs^{-1} \in C_i$ for any $s \in G$. If a function $f : G \to \mathbb{C}$ is constant on conjugacy classes, then we call it a **class function**. That is, f is a class function if $f(sxs^{-1}) = f(x)$ for all $x, s \in G$. Note that the set $\mathcal{H}_G$ of class functions of G forms a vector space over $\mathbb{C}$, and that the irreducible characters $\chi_1, \ldots, \chi_n$ of G are class functions. We will show that, if we equip $\mathcal{H}_G$ with the Hermitian inner product (1.1), the irreducible characters are an orthonormal basis for $\mathcal{H}_G$.

Lemma 1.2.11. *If $f \in \mathcal{H}_G$ and $\rho : G \to GL(V)$ is an irreducible representation of G with character χ, define $\rho_f : V \to V$ by*

$$\rho_f = \sum_{s \in G} f(s)\rho_s.$$

Then $\rho_f = \lambda I$, where

$$\lambda = \frac{1}{dim(V)} \sum_{s \in G} f(s)\chi(s) = \frac{|G|}{dim(V)}(f, \overline{\chi}).$$

Proof. For any $s \in G$, we see that $\rho_s^{-1}\rho_f\rho_s = \sum_{t \in G} f(t)\rho_{s^{-1}ts} = \sum_{u \in G} f(sus^{-1})\rho_u = \sum_{u \in G} f(u)\rho(u) = \rho_f$, where we substituted $u = s^{-1}ts$. So $\rho_f\rho_s = \rho_s\rho_f$, and we satisfy the conditions for the second statement of Schur's Lemma (Lemma 1.2.3). Therefore $\rho_f = \lambda I$ for some $\lambda \in \mathbb{C}$. But

$$\dim(V)\lambda = \mathrm{Tr}(\lambda I) = \mathrm{Tr}(\rho_f) = \sum_{s \in G} f(s)\mathrm{Tr}(\rho_s) = \sum_{s \in G} f(s)\chi(s).$$

Therefore $\lambda = \frac{1}{\dim(V)} \sum_{s \in G} f(s)\chi(s)$, as desired. ☐

Theorem 1.2.12. *The irreducible characters $\chi_1, \ldots, \chi_n$ are an orthonormal basis of $\mathcal{H}_G$, the space of class functions of G.*

Proof. By Theorem 1.2.5, we have that the irreducible characters are pairwise orthonormal. We proceed to show that they span the space $\mathcal{H}_G$ of class functions of G. Let $W = \mathrm{Span}(\chi_1, \ldots, \chi_n)$. Then $\mathcal{H}_G = W \oplus W^\perp$. Let $f \in W^\perp$, so that $(f, \chi_i) = 0$ for all $i = 1, \ldots, n$. Now for a given representation $\rho : G \to GL(V)$, we can decompose it into a sum of irreducible subrepresentations: $\rho = \rho^1 \oplus \cdots \oplus \rho^n$. Then if $\rho_f^i = \sum_{s \in G} f(s)\rho_s^i$ as in Lemma 1.2.10, we have that $\rho_f^i = \lambda^i I = 0$ (since $\overline{\chi}(s) = \chi(s^{-1})$, and f was assumed to be orthogonal to all the irreducible characters), and therefore $\rho_f = \sum_{s \in G} f(s)\rho_s = 0$ as well.

In particular, applying this to the regular representation ρ^G of G gives us that $0 = \rho_f e_1 = \sum_{s \in G} f(s)\rho_s e_1 = \sum_{s \in G} f(s)e_s$, so that the coefficients $f(t)$ in the linear combination are zero. Therefore $f = 0$, so that $W^\perp = \{0\}$, and finally $\mathcal{H}_G = W$. So the span of the irreducible characters generates the whole space of class functions. ☐

Corollary 1.2.13. *The number of irreducible representations of G is equal to the number of conjugacy classes of G.*

Proof. By Theorem 1.2.11, the space $\mathcal{H}_G$ of class functions of G has dimension equal to n, the number of irreducible characters of G. On the other hand, since class functions are constant on each of the conjugacy classes $C_1, \ldots, C_m$ of G, we see that the space of class functions has dimension m, since we can represent each class function by an m-vector of its values on the m conjugacy classes. So $n = \dim(\mathcal{H}_G) = m$. $\square$

1.2.3 The group algebra

In this section, we discuss an alternate and equivalent way to view group representations, which will assist us in deriving some facts about character tables in the following section, and also enrich our analysis in the next chapter.

If G is a finite group and K is a commutative ring, we define the **group algebra** of G over K to the set $K[G]$ of linear combinations $x = \sum_{s \in G} a_s s$, for $a_s \in K$, where if $x' = \sum_{s \in G} b_s s \in K[G]$ then $x + x' = \sum_{s \in G}(a_s + b_s)s$ and $xx' = \sum_{s,t \in G}(b_t a_{t^{-1}s})s$, so that addition and multiplication are directly induced by that in K and in G. We see that it is an algebra, since it is a ring as well as a (finitely-generated) K-module with basis G (therefore of dimension $|G|$).

For now, let us take $K = \mathbb{C}$. Since $\mathbb{C}[G]$ is a ring, we can define modules on it. In particular, if we have a linear representation $\rho : G \to GL(V)$, where V is a complex vector space, then for any $x = \sum_{s \in G} a_s s \in \mathbb{C}[G]$ and $v \in V$, we can let $xv = \sum_{s \in G} a_s \rho_s v$, and see that this definition of scalar multiplication defines a $\mathbb{C}[G]$-module over V. We see that this module V is finitely-generated, since if we choose a basis $\beta = \{v_1, \ldots, v_n\}$ of the original V (when thought of as a complex vector space) every element of (the $\mathbb{C}[G]$-module) V is a linear combination of the basis elements v_i by elements of the group algebra $\mathbb{C}[G]$.

On the other hand, if we're given an arbitrary finitely-generated $\mathbb{C}[G]$-module V, let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V, and note that for any $s, t \in G$, we have that $[st]_\beta = [s]_\beta [t]_\beta$, and therefore that $[s]_\beta [s^{-1}]_\beta = [ss^{-1}]_\beta = [1]_\beta$, the $n \times n$ identity matrix. This shows that the matrix of s under the basis β, for every $s \in G$, is invertible, so that by choosing this basis we have fixed a linear representation of G of degree $\dim V$.

Therefore, we see that we can speak equivalently of finitely-generated $\mathbb{C}[G]$-modules and linear representations of G in $\mathbb{C}$-vector spaces. In fact, all of the theory developed thus far can be expressed in the language of $\mathbb{C}[G]$-modules, and though we will not reprove most things (see [4] for a thorough treatment of the general theory), we will make use of this language when it provides clarity or an additional perspective.

We call an algebra A **semisimple** if every left A-module V is semisimple, which simply means that V can be written as a finite direct sum of irreducible submodules $V = V_1 \oplus \cdots \oplus V_n$ (in alignment with our definitions for representations, a module is irreducible if it has no proper nonzero submodules). The following lemma is essentially a generalization of Maschke's theorem (Theorem 1.1.2) in the language of $K[G]$-modules.

Lemma 1.2.14. *If K has characteristic zero, then the group algebra $K[G]$ is semisimple.*

Proof. The proof is essentially the same as our proof of Lemma 1.1.1 (which facilitated our original version of Maschke's theorem): if W is a submodule of a $K[G]$-module V, then let p be a projection of V onto W which is K-linear. Then we can create a new projection

$$p' = \frac{1}{|G|} \sum_{s \in G} \rho_s \cdot p \cdot \rho_{s^{-1}}$$

by averaging the conjugated projection p under all the elements of G. This projection is $K[G]$ linear, and therefore we can factorize V into a direct sum of W and the kernel of p'. Using induction on dimension, as we did for Maschke's theorem, we see that every $K[G]$-module is semisimple, or in other words that $K[G]$ is a semisimple algebra. $\square$

Now, if we have any representations $\rho^1 : G \to GL(V_1)$ and $\rho^2 : G \to GL(V_2)$ over a field K, we let the space of **intertwining operators** be defined as

$$\mathrm{Hom}_G(V_1, V_2) = \{f \in \mathrm{Hom}_K(V, W) : \rho^1_s \circ f = f \circ \rho^2_s, s \in G\}.$$

In the case where $V_1 = V_2$ and $\rho^1 = \rho^2$ we write $\mathrm{Hom}_G(V, V) = \mathrm{End}_G(V)$, the set of G-covariant endomorphisms of V. Equivalently, for modules M, N over a group algebra $K[G]$, we can define $\mathrm{Hom}_G(M, N) = \{f \in \mathrm{Hom}_K(M, N) : \rho^1_s f = f \rho^2_s, s \in G\}$, and $\mathrm{End}_G(M)$ is defined similarly.

A ring K is a **division ring** (sometimes called a **skew field**) if every element of K has a multiplicative inverse. Note that it differs from a field only in that it needn't be a commutative ring (but clearly every field is also a division ring).

Now note that the statement of Schur's lemma is equivalent to saying that if ρ^1, ρ^2 are irreducible representations, then $f \in \mathrm{Hom}_G(V_1, V_2)$ is either 0 or an isomorphism. It follows immediately from the invertibility of every nonzero intertwining operator that $\mathrm{End}_G(V)$ is a division ring if our chosen $\rho : G \to GL(V)$ is irreducible.

We take for granted the following corollary of the structure theorem of semisimple algebras, which decomposes semisimple algebras into a product of matrix rings over the endomorphism rings of the components of the semisimple algebra:

Theorem 1.2.15 (Artin-Wedderburn)**.** *If $K[G] \simeq V_1 \oplus \cdots \oplus V_n$ is a semisimple group algebra over a division ring K, then $K[G]$ can be decomposed into a product of matrix rings $\mathbb{M}_n(K_i)$, where the matrices are $n \times n$ and their entries come from the division rings $K_i = End_G(V_i)$:*

$$A = \prod_{i=1}^{N} \mathbb{M}_{n_i}(K_i).$$

In particular, we might be interested in the case $K = \mathbb{C}$, which corresponds to the theory of finite group representations we have developed. First note that since $\mathbb{C}[G]$ is an algebra, we can consider it as a $|G|$-dimensional module over itself. Then we can index a basis $\beta = \{e_s\}_{s \in G}$ of $\mathbb{C}[G]$ by the elements of G, and see that the set of $|G| \times |G|$ matrices of the elements of G under this basis are is in fact the regular representation. So in the correspondence we developed earlier between finitely-generated $\mathbb{C}[G]$-modules and representations of G over $\mathbb{C}$, this tells us that $\mathbb{C}[G]$ (as a module over itself) corresponds to the regular representation ρ^G. Now note the following fact.

Proposition 1.2.16. *$\mathbb{C}$ is the only division ring of finite degree over $\mathbb{C}$.*

Proof. Let V be a division ring of finite dimension over $\mathbb{C}$, and let $v \in V$. Then there is some complex polynomial in v with coefficients in $\mathbb{C}$ which evaluates to zero. But since $\mathbb{C}$ is algebraically closed, this polynomial factors completely, and therefore one of its degree-one factors $(v - z_i)$ is equal to zero for some $z_i \in \mathbb{C}$. Therefore $v \in \mathbb{C}$, so that $V = \mathbb{C}$. □

Therefore $\mathbb{C}[G]$ decomposes into a finite product of matrix rings

$$\mathbb{C}[G] = \prod_{i=1}^{N} \mathbb{M}_{n_i}(\mathbb{C}).$$

Furthermore, the number N of factors in the decomposition is the number of distinct irreducible representations of G over $\mathbb{C}$, and the dimension n_i is equal to the dimension of the i^{th} irreducible representation ρ_i of G. To see this, we let $\rho^G = n_1\rho_1 \oplus \cdots n_N\rho_N$ be the decomposition of the regular representation of G into irreducible subrepresentations, where $\rho_i : G \to GL(V_i)$. The ring of endomorphisms of each V_i is isomorphic to $\mathbb{M}_{m_i}(\mathbb{C})$, where $m_i = \dim V_i$, and so the representations ρ_i, by linearly extending them and taking them all together as a homomorphism from $\mathbb{C}[G]$ to the Cartesian product of the endomorphism rings of the V_i gives us the same factorization for $\mathbb{C}[G]$ as above, except we know that N is the number of irreducible representations, and n_i is the degree of the irreducible representation ρ_i. Note that this factorization of the group algebra is directly analogous to the decomposition of the regular representation of G into the direct sum of all irreducible representations of G, as seen in Proposition 1.2.8.

In chapter two, we will consider the somewhat more intricate case in which we take $K = \mathbb{R}$. Since our generalized Maschke's theorem, Lemma 1.2.13, holds for any commutative ring of zero characteristic, we see that $\mathbb{R}[G]$ is semisimple, and therefore that Theorem 1.2.14 allows us to deconstruct this group algebra into a product of matrix rings over division rings of finite degree over $\mathbb{R}$. We will see later that there are only three such division rings, and that this decomposition provides us with new information about the nature of a group's irreducible representations.

1.2.4 Character tables: some more theory

Following our result in Corollary 1.2.12 that the number of irreducible characters is equal to the number of conjugacy classes, we introduce the notion of a **character table**, a square matrix $\mathcal{C}^G$ whose ij^{th} entry $\mathcal{C}^G_{ij}$ is the value that the irreducible character χ_i takes on the conjugacy class C_j of G. As Corollary 1.2.7 reduced our study of representations of G to the study of its irreducible characters, our task has been reduced to filling out the character table, and investigating what sort of information we can deduce about G from knowing its values.

To the first end, we have already found a number of useful tools. By Corollary 1.2.9, the squares of the degrees of the representations sum to $|G|$. By Theorem 1.2.5, the rows of $\mathcal{C}^G$ are pairwise orthonormal. We will now proceed to show that the columns of the character table are pairwise orthogonal, and that the degrees of the representations divide $|G|$.

Proposition 1.2.17. *The columns of $\mathcal{C}^G$ are pairwise orthogonal.*

Proof. Consider a modified character table T where we multiply each entry in column k (the values of the characters on conjugacy class k) by $\sqrt{|C_k|/|G|}$. So if $g_k \in C_k$ is a representative element of each conjugacy class $k = 1, \ldots, n$, then we see that from our orthogonality relations on the rows,

$$\delta_{ij} = \frac{1}{|G|} \sum_{s \in G} \chi_i(s) \overline{\chi_j(s)} = \frac{1}{|G|} \sum_{k=1}^{n} |C_k| \chi_i(g_k) \overline{\chi_j(g_k)} = \sum_{k=1}^{n} T_{ik} \overline{T_{jk}}.$$

This shows that the rows of T are pairwise orthonormal under the standard Hermitian dot product $\mathbb{C}^n$. It follows, then, that $T^*T = I = TT^*$, i.e. that T is a unitary matrix and thus its columns are also pairwise orthonormal. This means that for any two conjugacy classes C_i, C_j, we have

$$\delta_{ij} = \sum_{k=1}^{n} T_{ki} \overline{T_{kj}} = \sum_{k=1}^{n} \frac{\sqrt{|C_i||C_j|} \chi_k(g_i) \overline{\chi_k(g_j)}}{|G|},$$

and therefore that

$$\sum_{k=1}^{n} \chi_k(g_i) \overline{\chi_k(g_j)} = \frac{|G|}{\sqrt{|C_i||C_j|}} \delta_{ij}.$$

□

We will now prove that the degree of an irreducible representation divides the size of the group (adapting the proofs from [7, chapter 6] and [4, chapter 22]). This will require some facts about the center of the group algebra, as well as some elementary facts about algebraic integers. A complex number is an **algebraic integer** if it is the root of a polynomial with integer coefficients.

Lemma 1.2.18. *We have the following facts about algebraic integers.*

1. *The set of algebraic integers forms a subring of $\mathbb{C}$.*

2. *If χ is a character of a finite group G, then for every element $s \in G$, the number $\chi(s)$ is an algebraic number.*

3. *If a number is both an algebraic integer and a rational number, then it is an integer.*

Proof. 1. We will first show that a complex number x is an algebraic integer if and only if the subring $\mathbb{Z}[x]$ of $\mathbb{C}$ consisting of polynomials in x with integer coefficients is a finitely-generated $\mathbb{Z}$-module.

Now, for any two algebraic integers x, y, we know that the rings $\mathbb{Z}[x], \mathbb{Z}[y]$ are finitely-generated as $\mathbb{Z}$-modules. But then clearly the tensor product $\mathbb{Z}[x] \otimes \mathbb{Z}[y]$ is finitely-generated, since it can be generated by the tensor products of the generators of its factors. Therefore the ring $\mathbb{Z}[x, y] = \mathbb{Z}[x][y]$ is also finitely-generated, since we can map the tensor product into $\mathbb{C}$ surjective onto this ring by sending its generators to the products of the powers of x, y. But then every element $z \in \mathbb{Z}[x, y]$ is in a finitely-generated $\mathbb{Z}$-module contained in $\mathbb{C}$ which itself contains both $\mathbb{Z}[z]$. Since $\mathbb{Z}$ is a noetherian ring (i.e. every ideal is finitely-generated - in the case of $\mathbb{Z}$, each ideal can be generated by one number), this is equivalent to saying that $\mathbb{Z}[z]$ is a finitely-generated $\mathbb{Z}$-module. By our first result here, this means that z is an algebraic integer. Since $\mathbb{Z}[x, y]$ contains $x + y$ as well as the product xy, this shows that the algebraic integers form a ring.

2. First note that if χ is the character of a representation ρ, then $\chi(s)$ is the sum of the eigenvalues $\lambda_1, \ldots, \lambda_k$ of the matrix ρ_s. Let m_s be the degree of s in G; that is, the least integer such that $s^m = 1$. Then since λ_i is an eigenvalue of ρ_s, we have that $\lambda_i^{m_s}$ is an eigenvalue of $\rho_s^{m_s} = I$, the identity matrix. So $\lambda_i^{m_s} = 1$, and therefore λ_i is a root of unity. So the value $\chi(s)$ is a sum of roots of unity. Clearly the roots of unity are algebraic integers, since they are roots of polynomials of the form $x^p - 1$. But since the algebraic integers form a ring, a sum of roots of unity is also an algebraic integer, and therefore $\chi(s)$ is an algebraic integer.

3. Let $x \in \mathbb{Q}$ be an algebraic integer. Suppose that $x \notin \mathbb{Z}$. Then we can write x as a reduced fraction $x = p/q$, where p and q are relatively prime integers. Since x is an algebraic number we have some $a_0, \ldots, a_{n-1} \in \mathbb{Z}$ such that $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$, or, mutiplying through by q^n, that

$$p^n + a_{n-1}qp^{n-1} + \cdots + a_1 q^{n-1} p + a_0 q^n = 0.$$

But since $p \neq 0$ (we assumed $x \notin \mathbb{Z}$), this means that q divides p^n, contradicting that they are relatively prime. So $x \in \mathbb{Z}$.

□

Lemma 1.2.19. *If $\chi : G \to \mathbb{C}$ is an irreducible character of a finite group G with conjugacy classes $C_1, \ldots, C_m$, and $g_i \in C_i$ for each $i = 1, \ldots, m$, then*

$$\lambda_i = |C_i| \frac{\chi(g_i)}{\chi(1)}$$

is an algebraic integer.

Proof. For each conjugacy class C_i, let $\sigma_i = \sum_{s \in C_i} s$, an element of the group algebra $\mathbb{C}[G]$. Then for the irreducible $\mathbb{C}[G]$-module U (which we know is a submodule of $\mathbb{C}[G]$ by our discussion of the regular representation in section 1.2.3) with character χ (that is, the module corresponding to the representation whose character is χ), we have that $u\sigma_i = \lambda_i u$ for every element $u \in U$.

To see why, we first note that σ_i is in the center of the group algebra $\mathbb{C}[G]$ for each i (since conjugation by any group element fixes the element σ_i). Then for any element $t \in \mathbb{C}[G]$, $ut\sigma_i = u\sigma_i t$, and therefore $f : U \to U$ defined by $f(u) = u\sigma_i$ is a homomorphism such that $\rho_s f = f\rho_s$ for any $s \in G$. Therefore Schur's lemma tells us that it is a multiple of the identity transformation. Thus $u\sigma_i = \eta u$ for some $\eta \in \mathbb{C}$. But picking a basis of U and taking its trace gives us that $\sum_{s\in C_i} \chi(s) = \eta\chi(1)$, so that $\eta = |C_i|\chi(g_i)/\chi(1) = \lambda$, as desired.

In general, if we enumerate the elements of G as $s_1, \ldots, s_n$, and we have an element $r = \sum_{i=1}^{n} a_i s_i \in \mathbb{C}[G]$ for integers a_i, then its product with the group elements s_i, $s_i r = \sum_{j=1}^{n} b_{ij} s_j$ with $b_{ij} = a_l$, where$s_l = s_i^{-1} s_j$, is an integer.

Our elements σ_k are of this form, where $b_{ij} = 1$ when $s_i^{-1} s_j \in C_k$, and $b_{ij} = 0$ otherwise. Therefore we see that the statement that $u\sigma_k = \lambda u$ implies that u is an eigenvector of eigenvalue λ of the matrix (b_{ij}) of the b_{ij} for which $s_i\sigma_k = \sum_{j=1}^{n} b_{ij} s_j$, which are integers. So λ is an eigenvalue of an integer matrix, i.e. a root of its characteristic polynomial, which has all integer coefficients (since its coefficients are products and sums of the matrix entries). Therefore λ is an algebraic integer. □

Proposition 1.2.20. *Let $\chi : G \to \mathbb{C}$ be an irreducible character of a finite group G. Then $\chi(1)$ divides $|G|$.*

Proof. Enumerate the conjugacy classes of G as $C_1, \ldots, C_m$, and let $g_i \in C_i$ for each $i = 1, \ldots, m$ (that is, any element of the conjugacy class C_i). Then since the irreducible characters are orthonormal and the conjugacy classes partition G, we have that

$$|G| = \sum_{i=1}^{m} |C_i|\chi(g_i)\overline{\chi(g_i)},$$

and therefore that

$$\frac{|G|}{\chi(1)} = \sum_{i=1}^{m} \frac{|C_i|\chi(g_i)\overline{\chi(g_i)}}{\chi(1)}.$$

Since $|C_i|\chi(g_i)/\chi(1)$ (Lemma 1.2.18) and $\overline{\chi(g_i)} = \chi(g_i^{-1})$ (Lemma 1.2.17(2)) are algebraic integers for $i = 1, \ldots, m$, the whole sum is an algebraic integer, since the algebraic integers form a ring (Lemma 1.2.17(1)). Therefore $|G|/\chi(1)$ is an algebraic integer. Since it is also a rational number, it is an integer (Lemma 1.2.17(3)). So $\chi(1)$ divides $|G|$. □

1.2.5 Some examples of character tables

To get started, we return to the example of $S_3 \simeq D_3$, the smallest non-abelian group, which we began to explore in section 1.1.2. Let $\chi_1 : S_3 \to \{1\}$ be the character of the trivial representation. Recall the sign representation $\rho_{sgn} : S \to \{\pm 1\}$, which sends a permutation to its sign. We can now conclude that it is not isomorphic to the trivial representation, since their characters differ on the conjugacy classes of odd permutations (that is, sign -1). Furthermore, ρ_{sgn} is irreducible since it has degree one. Denote its character be χ_2. So we have obtained two irreducible characters of degree one. S_3 has three conjugacy classes: $C_1 = \{id\}$, $C_2 = \{(123), (132)\}$, and $C_3 = \{(12), (13), (23)\}$. By Corollary 1.2.12, we know that S_3 has three irreducible representations. By Corollary 1.2.9, $\sum_{i=1}^{3} \chi_i^2(1) = |G| = 6$, and therefore $\chi_3^2(1) = 4$, so that $\chi_3(1) = 2$, and we obtain that S_3 has one irreducible representation of degree two. Using Proposition 1.2.13, we can obtain the entries in the third row (corresponding to χ_3) by noting that the second and third columns are both orthogonal to the first. Therefore, we have obtained the full character table:

S_3	C_1	C_2	C_3
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

Next we will find the character tables of D_4, the dihedral group of 8 elements, and the quaternion group Q_8, the smallest non-abelian groups larger than S_3. First, $D_4 = \langle x, y | x^4 = y^2 = yxyx = 1\rangle$ has five conjugacy classes: $C_1 = \{1\}$, $C_2 = \{x^2\}$, $C_3 = \{y, x^2y\}$, $C_4 = \{x, x^3\}$, and $C_5 = \{xy, x^3y\}$, and therefore by Corollary 1.2.12, we know it has five irreducible representations, where we let $\chi_1 = \mathrm{Tr}(\rho_1)$ be the trivial representation as usual. By Corollary 1.2.9, $\sum_{i=1}^{5} \chi_i^2(1) = |G| = 8$, so that $\sum_{i=2}^{5} \chi_i^2(1) = 7$. The only way to add four squares to make seven is to have $\chi_2(1) = \chi_3(1) = \chi_4(1) = 1$, and $\chi_5(1) = 2$. So we have filled out the first row and first column of the table.

Note that the elements of C_2, C_3, C_5 have order 2, and those of C_4 have order 4. Therefore their values under the representations must have orders dividing those orders. So our degree one representations must map $x \mapsto \{1, -1, i, -i\}$, since these are the fourth roots of unity, and $y \mapsto \{\pm 1\}$, since these are the only complex numbers of order 2. If $x \mapsto 1$, then if $y \mapsto -1$, we obtain a very simple representation ρ_2 which is not isomorphic to ρ_1, since its character values differ. So we have filled out the second row as

D_4	C_1	C_2	C_3	C_4	C_5
χ_2	1	1	-1	1	-1.

Now if $\rho_3(x) = -1$, then $\rho_3(x^2) = 1$. If $\rho_3(y) = 1$, we obtain another representation whose character sends C_2, C_3 to 1, and C_4, C_5 to -1. Note that this is not isomorphic to ρ_1, ρ_2, since their characters differ:

D_4	C_1	C_2	C_3	C_4	C_5
χ_3	1	1	1	-1	-1.

If $\rho_4(x) = -1$ and $\rho_4(y) = -1$, we obtain yet another representation whose character sends C_2, C_5 to 1 and C_3, C_4 to -1:

D_4	C_1	C_2	C_3	C_4	C_5
χ_4	1	1	-1	-1	1.

Next if $\rho(x) = \pm i$ for some degree one representation ρ, then we would have that $\rho(x^2) = \rho(x)^2 = -1$, which is impossible since if $\rho(y) = 1$ then we would have $1 = \chi(y) = \chi(x^2y) \neq \chi(x^2)\chi(y) = -1$, and if $\rho(y) = -1$ then we would similarly have $-1 = \chi(y) = \chi(x^2y) \neq \chi(x^2)\chi(y) = 1$. So this situation is impossible, confirming that we have found all the right characters of degree one:

D_4	C_1	C_2	C_3	C_4	C_5
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2				

This leaves our degree two character χ_5, whose values can be found by column orthogonality to the first column using Proposition 1.2.13. By comparison with column one, we easily see that $\chi_5(C_2) = -2$, and $\chi_5(C_j) = 0$ for $j = 3, 4, 5$:

D_4	C_1	C_2	C_3	C_4	C_5
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

1.2.6 Motivation: the character table of the quaternion group

In this section, we compute the character table of the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, where $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, and $ki = j$. We see that Q_8 has five conjugacy classes: $C_1 = \{1\}$, $C_2 = \{-1\}$, $C_3 = \{\pm i\}$, $C_4 = \{\pm j\}$, and $C_5 = \{\pm k\}$, and thus five distinct irreducible characters, with χ_1 again the trivial character. Applying Corollary 1.2.9 exactly as with D_4, we obtain that there are four degree one characters $\chi_{1,\dots,4}$, and one degree two character χ_5. Since $\pm i, \pm j, \pm k$ have order 4, they must all be mapped into $\{\pm 1, \pm i\}$ by the degree one representations, and this choice will determine the character (since $i^2 = j^2 = k^2 = -1$). If $\chi(i) = i$ (resp. $\chi(i) = -i$), then since $i(-i) = 1$, we must have $\chi(-i) = -i$ (resp. $\chi(-i) = i$). But this is impossible since characters are class functions. So we must have $\chi(i) \in \{\pm 1\}$ for all degree one characters (and the same holds for $\chi(j), \chi(k)$, by symmetry). If $\chi(i) = 1$, then $\chi(-1) = \chi(i^2) = 1$ as well, and to avoid being the trivial character let us put $\chi(j) = -1$. Then we have $\chi(k) = \chi(ij) = \chi(i)\chi(j) = -1$. So if one of the i, j, k are sent to one, the other two are sent to -1 (again by symmetry). But this determines three distinct non-trivial degree one characters:

Q_8	C_1	C_2	C_3	C_4	C_5
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2				

Note that we cannot have $\chi(i) = \chi(j) = \chi(k) = -1$, since then $\chi(-1) = \chi(ijk) = -1$, which implies that $-1 = \chi(i) \neq \chi(-i) = \chi(-1)\chi(i) = 1$, which is a contradiction since characters are class functions. This confirms that our degree one characters are correct. At this stage, we can fill in the bottom row by column orthogonality:

Q_8	C_1	C_2	C_3	C_4	C_5
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Note that this character table is identical to the one we obtained for D_4 at the end of the previous chapter. However, it is clear that D_4 and Q_8 are non-isomorphic groups: D_4 has three elements of order two (x^2, y, x^2y), while Q_8 only has one (-1; the rest of the non-identity elements have order 4). Thus this example shows us that while character tables do uniquely determine the isomorphism classes of a group's irreducible representations (Corollary 1.2.7), they are not in general strong enough to distinguish between isomorphism classes of groups.

We see that any representation of either D_4 or Q_8 constructed from their degree one representations will be the same, since the degree one characters are themselves representations. This reduces the representation-theoretic distinction between these two groups to the difference between their irreducible degree two representations. Just how different are they? The answer to this question will lead us to explore further how much information we can obtain about a representation simply by knowing its character.

Chapter 2

Real representations and the Frobenius-Schur indicator

In this chapter, we will further examine the depth of information one can derive about a group and its representations from its irreducible characters. As we saw in the previous chapter, representations of a group are isomorphic precisely if their characters are equal, but that in general non-isomorphic groups can have identical sets of characters. We will see that there are other ways to distinguish groups by their representations that may not carry over to the groups' characters. In particular, we can decide, given a group's character, whether it is possible to find a representation of the group over the real numbers which takes this character. Interestingly, we will also see that this criterion can be computed simply by looking at the groups' character tables and conjugacy class structure.

2.1 Real, complex, and quaternionic representations

Let $\rho : G \to GL(V)$ be a representation of a finite group G in a complex vector space V. One way to further classify ρ is to determine whether there is a basis of V such that, for all $s \in G$, the matrix ρ_s under this basis has only real-valued entries. If this is the case, we call ρ a **real representation**. Note that if ρ is a real representation, then its character χ is certainly real-valued, since trace of a linear transformation is independent of the choice of basis, and therefore if there is some real-valued matrix $A = \rho_s$, then $\chi_s = \mathrm{Tr}(A) \in \mathbb{R}$. If ρ has a complex-valued character, then we call ρ a **complex representation**. Clearly a complex-valued character cannot be the trace of a matrix with real-valued entries, so our definition is consistent (i.e. no representation can be both real and complex).

We can ask if the converse is true: if we have a real-valued character χ of some finite group G, is it necessarily the character of a real representation ρ? The answer, as we shall see, is negative; there are groups with real-valued characters whose representation cannot be taken as real-valued. If ρ has real-valued character but is not a real representation, we say that it is a **quaternionic representation**, for reasons that will be made clear shortly. In particular, the theory we develop below will give us the distinction we were looking for at the end of the previous chapter: the degree two irreducible representation of D_4 is real, while that of Q_8 is quaternionic.

In the following section we will prove a simple computational criterion for classifying complex group representations as real, complex, or quaternionic, which solely depends on the group's characters. Afterwards, we will see how this classification informs the relationship between the decomposition of the complex group algebra $\mathbb{C}[G]$ into irreducible sub-modules (i.e. the decomposition of the regular representation into irreducible subrepresentations), and the corresponding decomposition of the real group algebra $\mathbb{R}[G]$ into irreducible submodules. Finally, we will work through some simple examples to illustrate the theory.

2.2 The Frobenius-Schur indicator

In this section, we will prove the following theorem, which allows us to classify a complex character as real, complex, or quaternionic by a simple computation. The proof vaguely follows that by Serre in [7, chapter 13.2], with certain details adapted from [3]. First, if $\chi : G \to \mathbb{C}$ is a character of a complex representation $\rho : G \to GL(V)$, we define the **Frobenius-Schur indicator** of χ to be

$$\iota(\chi) = \frac{1}{|G|} \sum_{s \in G} \chi(s^2).$$

Theorem 2.2.1 (Frobenius-Schur)**.** *Let $\chi : G \to \mathbb{C}$ be a character of a complex irreducible representation $\rho : G \to GL(V)$. Then the Frobenius-Schur indicator $\iota(\chi)$ satisfies the following statements:*

1. *$\iota(\chi) = 0$ if and only if ρ is a complex representation.*
2. *$\iota(\chi) = 1$ if and only if ρ is a real representation.*
3. *$\iota(\chi) = -1$ if and only if ρ is a quaternionic representation.*

2.2.1 Relation to G-invariant bilinear forms

Recall that if V is a complex, finite-dimensional vector space, then $B : V \times V \to \mathbb{C}$ is a **bilinear form** if B is linear in each coordinate separately. We say that B is **symmetric** if $B(v, w) = B(w, v)$ for all $v, w \in V$, and that B is **alternating** or **skew-symmetric** if $B(v, w) = -B(w, v)$ for all $v, w \in V$ (or equivalently, if $B(v, v) = 0$ for every $v \in V$). Given a representation $\rho : G \to GL(V)$ of a finite group G over V, we say that B is G**-invariant** if for every $s \in G$ we have $B(\rho_s v, \rho_s w) = B(v, w)$. Finally, we say that B is **non-degenerate** if for each $v \in V$, we have that if $B(v, w) = 0$ for all $w \in V$, then $v = 0$.

Recall that for any vector space V over $\mathbb{C}$ (or over any field), there is a dual space we denote as V^*, the space of linear maps $f : V \to \mathbb{C}$. For a representation $\rho : G \to GL(V)$, we can define its dual representation $\rho^* : G \to GL(V^*)$ over the dual space of V by setting $\rho^*_s \varphi = \varphi \rho_{s^{-1}}$, for $\varphi \in V^*$ and $s \in G$. Therefore the character χ^* of ρ^* can be computed by the character χ of ρ: $\chi^*(s) = \chi(s^{-1}) = \overline{\chi(s)}$.

Therefore $\chi^* = \overline{\chi}$, so that $\rho_s^* = \overline{\rho_{s^{-1}}}^T$, the conjugate transpose. It follows then that $\chi^*(s) = \overline{\chi(s)}$.

The following facts will allow us to reduce the classification of a representation ρ as complex, real, or quaternionic to determining of the existence of certain types of bilinear forms on the representation vector space V. Afterwards, we will see that the existence of such bilinear forms is determined by the indicator function defined at the beginning of the section.

Lemma 2.2.2. *Let $\chi : G \to \mathbb{C}$ be a character of a complex irreducible representation $\rho : G \to GL(V)$. Then χ is real-valued if and only if V has a G-invariant non-degenerate bilinear form B. Furthermore, if this holds, B is unique up to multiplication by a scalar; in other words, the space of such forms is one-dimensional.*

Proof. First, note that χ is real-valued on all of G if and only if $\chi = \overline{\chi} = \chi^*$ on all of G, where χ^* is the character of the dual representation ρ^*. Since two representations of G are isomorphic precisely when their characters are equal (Corollary 1.2.7), we conclude that χ is real-valued if and only if ρ an equivalent representation to its dual; that is, if and only if there is a vector space isomorphism $f : V \to V^*$ such that for every $s \in G$, $f \circ \rho_s = \rho_s^* \circ f$.

But then we can define a bilinear form $B : V \times V \to \mathbb{C}$ as $B(v, w) = f_v(w)$, which is G-invariant since $B(\rho_s v, \rho_s w) = f_{\rho_s v}(\rho_s w) = \rho_s^* \circ f_v \circ \rho_s w = f_v \circ \rho_{s^{-1}}^* \circ \rho_s w = f_v(w) = B(v, w)$. Furthermore, B is non-degenerate since Schur's Lemma gives us that f is an isomorphism (and is in fact multiplication by a scalar), and f being injective is equivalent to B being non-degenerate.

Conversely, given a G-invariant non-degenerate bilinear form $B(v, w)$, we can set $f : V \to V^*$ so that $f_v(w) = B(v, w)$. Since B is non-degenerate, f_v is the zero map if and only if $v = 0$. Hence f is injective, and since $\dim V = \dim V^*$, it is an isomorphism. Also, $(\rho_s^* \circ f_v)(\rho_{s^{-1}} w) = B(\rho_s v, \rho_s w) = B(v, w) = f_v(w)$, so that f is an isomorphism of representations. Therefore, $\rho \simeq \rho^*$ if and only if there is a G-invariant non-degenerate bilinear form on V.

Furthermore, let $B' : V \times V \to \mathbb{C}$ be another non-degenerate G-invariant bilinear form with corresponding isomorphism $g : V \to V^*$, where g commutes with ρ. Then $g^{-1} \circ f$ is an automorphism of V which commutes with ρ. By Schur's lemma, $g^{-1} \circ f = \lambda I$ for some $\lambda \in \mathbb{C}$, which means that $f = \lambda g$, and therefore that B, B' are identical up to scalar multiplication. This gives us that B is essentially unique, that that the space of G-invariant non-degenerate bilinear forms has dimension 1. $\square$

Proposition 2.2.3. *Let $\chi : G \to \mathbb{C}$ be a character of a complex irreducible representation $\rho : G \to GL(V)$. Then the following statements hold:*

1. *ρ is a complex representation if and only if V has no G-invariant non-degenerate bilinear forms.*

2. *ρ is a real representation if and only if V has a (unique, up to scalar multiplication) G-invariant non-degenerate symmetric bilinear form.*

3. *ρ is a quaternionic representation if and only if V has a (unique, up to scalar multiplication) G-invariant non-degenerate alternating bilinear form.*

Proof. Since ρ is a complex representation if and only if at least one value of χ is non-real, we have by Lemma 2.2.2 that ρ is a complex representation if and only if V has no G-invariant non-degenerate bilinear form. This establishes our first claim.

Now suppose χ is real-valued, and let $B : V \times V \to \mathbb{C}$ be the G-invariant non-degenerate bilinear form. We claim that B is either symmetric or alternating. To do this, we see that we can write $B(v, w) = B^+(v, w) + B^-(v, w)$, where $B^+(v, w) = \frac{1}{2}(B(v, w) + B(w, v))$ and $B^-(v, w) = \frac{1}{2}(B(v, w) - B(w, v))$. Note that B^+ is symmetric, and B^- is alternating, and that both are G-invariant and non-degenerate since they are linear combinations of such forms. Recall from the proof of Lemma 2.2.3 that there is a bijection between G-invariant non-degenerate bilinear forms B on V and isomorphisms $f : V \to V^*$ such that $f \circ \rho_s = \rho_s^* \circ f$ for all $s \in G$. By Schur's Lemma (Lemma 1.2.3), such an f is a multiple of the identity map. As we defined the bijection in the proof of Lemma 2.2.2, we see that all such bilinear forms are multiples of each other. Since B^+, B^- are G-invariant, non-degenerate bilinear maps on V, we have that $B^+ = \lambda B^-$ for some $\lambda \in \mathbb{C}$. But this is only possible if either $B^+ = 0$ or $B^- = 0$, since one is symmetric and the other is alternating (and not both, since then $B = 0$, contradicting its nondegeneracy). So we have either $B = B^+$ or $B = B^-$.

I claim that $B = B^+$ if and only if ρ is real. First suppose that $\rho : G \to GL(V)$ is a real representation. Then there is a basis under which the matrices ρ_s have real entries. This means that there is a G-invariant $\mathbb{R}$-subspace $V_0 \subset V$ such that $V = \mathbb{C} \otimes_{\mathbb{R}} V_0$. In particular, we see that $\dim_{\mathbb{R}}(V) = 2\dim_{\mathbb{R}}(V_0)$, so that if we have an $\mathbb{R}$-basis $\{x_1, \ldots, x_n\}$ of V_0, we can extend it to an $\mathbb{R}$-basis $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ of V by letting $y_i = ix_i$.

Then we can construct a G-invariant symmetric non-degenerate bilinear form as follows. Let $(,)$ be a positive-definite symmetric inner product on V_0, and define $B_0 = \frac{1}{|G|}\sum_{s \in G}(\rho_s v, \rho_s, w)$. Then clearly B_0 is positive-definite, symmetric, and G-invariant (since the sum is simply permuted when we act on it by G). From here we simply extend B_0 to V as follows. Define $B : V \times V \to \mathbb{C}$ as $B_0(a + ib, c + id) = (B_0(a, c) - B_0(b, d)) + i(B_0(a, d) + B_0(b, c))$. B inherits symmetry and G-invariance from B_0. To see that B is non-degenerate, we note that if we have some $a + ib$ for which $B(a + ib, c) = 0$ for all $c \in V_0$, then by linear independence in V we have $0 = B(a + ib, c) = B(a, c) + iB(b, c)$, so that $B(a, c) = B(b, c) = 0$ for all $c \in V_0$, and therefore $a = b = 0$, so that $a + ib = 0$. So B is non-degenerate. Therefore there exists a G-invariant symmetric non-degenerate bilinear form on V, and since we saw that B is unique up to scalar multiplication, we have that $B = B^+$.

Conversely, suppose that $B = B^+$; that is, there exists a G-invariant non-degenerate symmetric bilinear form on V. As before we note that we can take it to be an isomorphism $B : V \to V^*$ sending $v \mapsto B_v(w) = B(v, w)$. Then let $(,)$ be a positive-definite Hermitian inner product on V, which we take to be G-invariant as well (we can simply average a non-G-invariant one to obtain it, as before). Similarly, we can let $B' : V \to V^*$ be the isomorphism defined by $v \mapsto (v, w)$ (taking the inner product as a function of w with fixed v). Then let $T = B'^{-1} \circ B$ be an $\mathbb{R}$-automorphism of V, where we are thinking of V as a $2n$-dimensional $\mathbb{R}$-vector space as before (n being its dimension as a complex one). Then $T(v)$ is the element of V for which $B(v, w) = (f(v), w)$ for all $w \in V$. Furthermore, we see that

$(T(\rho_s v), w) = B(\rho_s v, w) = B(v, \rho_{s^{-1}}, w) = (T(v), \rho_{s^{-1}} w) = (\rho_s T(v), w)$, so that $\rho_s \circ T = T \circ \rho_s$. Since this latter property holds for T^2 as well and ρ is an irreducible representation, Schur's lemma tells us that $T^2 = \lambda I$ for some $\lambda \in \mathbb{C}$. But since $(,)$ is Hermitian and B is symmetric, we have that

$$(\lambda v, w) = (T^2(v), w) = B(T(v), w) = B(w, T(v))$$

$$= (T(w), T(v)) = \overline{(T(v), T(w))} = (v, T^2(w)) = (v, \lambda w),$$

and therefore that $\lambda \in \mathbb{R}$. Then since $(T(v), w) = \overline{v, T(w)}$, we have then that $(T(v), T(v)) = \overline{(v, T^2(w))} = \lambda(v, v)$, so that λ is positive, because $(,)$ is positive-definite. Rescaling so that $\lambda = 1$, we have that $T^2 = I$, and therefore can only have eigenvalues ± 1.

Now let $V^{\pm} = \{v \in V : T(v) = \pm v\}$ be the eigenspaces corresponding to the eigenvalues ± 1. Note now that

$$(T(iv), w) = B(iv, w) = B(v, iw) = (T(v), iw) = (-iT(v), w),$$

so that $T(iv) = -iT(v)$, and therefore if $v \in V^-$, then $T(iv) = -iT(v) = v$, so that $iv \in V^+$. But then if $v \in V^+$, we have $T(\rho_s v) = \rho_s T(v) = \rho_s v$, so that V^+ is G-invariant. Therefore V has a real G-invariant subspace of dimension n (over $\mathbb{R}$ - where n is the dimension of V over $\mathbb{C}$; this can be seen since multiplication by i swaps V^+ with V^-). This shows that ρ is a real representation.

It follows then that, if there exists such a B at all, $B = B^-$ if and only if ρ is non-real and non-complex, i.e. ρ is quaternionic. $\square$

2.2.2 The symmetric and alternating squares

Before we can prove Theorem 2.2.1, we need to take a short detour to discuss tensor products of linear representations and the decomposition of their characters.

Definition 2.2.4. *Let V_1, V_2 be vector spaces with bases $\{e_{i_1}\}, \{e_{i_2}\}$, respectively. Then we can define their* **tensor product** *$V = V_1 \otimes V_2$ to be the set of pairs $v_1 \otimes v_2$, for $v_1 \in V_1$ and $v_2 \in V_2$, such that*

1. *The product $v_1 \otimes v_2$ is linear in both arguments.*
2. *The set of products $\{e_{i_1} \otimes e_{i_2}\}$ forms a basis for the space V, and therefore dim V = dim $V_1 \cdot$ dim V_2.*

Accordingly, given two linear representations $\rho^1 : G \to GL(V_1)$ and $\rho^2 : G \to GL(V_2)$, we can define their tensor product $\rho : G \to GL(V_1 \otimes V_2)$ by setting $\rho_s(v_1 \otimes v_2) = \rho^1_s(v_1) \otimes \rho^2_s(v_2)$ for each $s \in G$. In this case we write that $\rho = \rho^1 \otimes \rho^2$. Note that the character χ of ρ is equal to the product of the characters χ_1 and χ_2 of ρ^1, ρ^2, respectively: $\chi = \chi_1 \cdot \chi_2$. To see this we simply note that entries of the matrix representation of ρ_s are the entrywise products of the entries of ρ^1_s, ρ^2_s, so that the trace χ is the sum of all products of the diagonal entries of the factor matrices.

Given a representation $\rho : G \to GL(V)$ of degree n with character χ, we are interested in the tensor product space $V \otimes V$, with corresponding representation

$\rho \otimes \rho$ and character χ^2. Let $\{e_i\}_{i=1}^n$ be a basis of V, and define $\theta : V \otimes V \to V \otimes V$ to be the automorphism which switches the basis elements in the coordinates: $\theta(e_i \otimes e_j) = e_j \otimes e_i$ for each pair (i, j). Since each element in $V \times V$ can be written $x \otimes y = (\sum_{i=1}^n a_i e_i) \otimes (\sum_{j=1}^n b_j e_j) = \sum_{(i,j)} a_i b_j (e_i \otimes e_j)$, we see that $\theta(x \otimes y) = \sum_{(i,j)} a_i b_j (e_j \otimes e_i) = y \otimes x$.

Since $\theta^2 = I$, the identity transformation, we obtain that the eigenvalues of θ are ± 1. Let $\mathrm{Sym}^2(V)$ denote the set of elements fixed by θ (the eigenspace of the eigenvalue 1), and let $\mathrm{Alt}^2(V)$ denote the set elements $v \in V \otimes V$ such that $\theta(v) = -v$ (the eigenspace of eigenvalue -1). We refer to these as the **symmetric square** and the **alternating square**, respectively. Clearly, these two subsets are disjoint. Furthermore, the set of elements $\beta_1 = \{(e_i + e_j) \otimes (e_j + e_i)\{_{i \leq j}$ is a linearly independent subset of $\mathrm{Sym}^2(V)$, while the set $\beta_2 = \{(e_i - e_j) \otimes (e_j - e_i)\}_{i<j}$ is a linearly independent subset of $\mathrm{Alt}^2(V)$. Note that $|\beta_1| = \frac{n(n+1)}{2}$ and $|\beta_2| = \frac{n(n-1)}{2}$, so that $|\beta_1| + |\beta_2| = n$. This shows us that β_1 and β_2 are bases of the symmetric and alternating squares, respectively, and therefore that

$$V \otimes V = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V).$$

Note that since representations on $V \otimes V$ are linear in each coordinate as well, that the symmetric and alternating squares are in fact G-invariant subspaces of $V \otimes V$, so that we can restrict a representation $\rho : G \to GL(V \times V)$ to representations $\rho_\sigma^2 : G \to GL(\mathrm{Sym}^2(V))$ and $\rho_\alpha^2 : G \to GL(\mathrm{Alt}^2(V))$. The following lemma decomposes their characters.

Lemma 2.2.5. *Let $\rho : G \to GL(V)$ be a representation with character χ. Let χ_σ^2 and χ_α^2 be the characters of the symmetric and alternating squares of V, respectively. Then if $s \in G$, we have*

$$\chi_\sigma^2(s) = \frac{1}{2}(\chi(s)^2 + \chi(s^2)),$$
$$\chi_\alpha^2(s) = \frac{1}{2}(\chi(s)^2 - \chi(s^2)).$$

Proof. First recall that we can construct an inner product (v_1, v_2) on V which is G-invariant (in the sense that $(\rho_t v_1, \rho_t v_2) = (v_1, v_2)$ for each $t \in G$, $v_1, v_2 \in V$), and therefore we can pick an orthonormal basis $\{e_i\}_{i=1}^n$ for V of eigenvectors of ρ_s, for our given (and fixed) $s \in G$. Therefore for each $i = 1, \ldots, n$, there is some eigenvalue $\lambda_i \in \mathbb{C}$ such that $\rho_s e_i = \lambda_i e_i$. Therefore $\chi(s) = \sum_i \lambda_i$, and since $\rho_{s^2} = (\rho_s)^2$, the eigenvalues of ρ_{s^2} are $\lambda_1^2, \ldots, \lambda_n^2$, so that $\chi(s^2) = \sum_i \lambda_i^2$.

Next we see that the eigenvalues of ρ_σ^2 and ρ_α^2, the symmetric and alternating squares of ρ, are the products $\lambda_i \lambda_j$:

$$(\rho_s \otimes \rho_s)((e_i \pm e_j) \otimes (e_j \pm e_i)) = \lambda_i \lambda_j ((e_i \pm e_j) \otimes (e_j \pm e_i)),$$

so that

$$\chi_\sigma^2 = \sum_{i \leq j} \lambda_i \lambda_j = \sum_{i=1}^n \lambda_i^2 + \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}\left(\left(\sum_{i=1}^n \lambda_i\right)^2 + \sum_{i=1}^n \lambda_i^2\right) = \frac{1}{2}(\chi(s)^2 + \chi(s^2))$$

$$\chi_\alpha^2 = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}\left(\left(\sum_{i=1}^n \lambda_i\right)^2 - \sum_{i=1}^n \lambda_i^2\right) = \frac{1}{2}(\chi(s)^2 - \chi(s^2)).$$

$\square$

Note that Lemma 2.2.3 implies that $\chi^2 = \chi_\sigma^2(s) + \chi_\alpha^2(s)$, which we already knew, since $V \otimes V = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V)$, and thus the character χ^2 of $\rho \otimes \rho = \rho_\sigma^2 \oplus \rho_\alpha^2$ can be written as the sum of the characters of its direct summands (Proposition 1.2.2).

2.2.3 Bringing it all together

We now explicate the relationship between the symmetric and alternating squares of a representation into a vector space V and the space of bilinear forms on V.

First, note that there is a bijective correspondence between non-degenerate bilinear forms $B : V \times V \to \mathbb{C}$ and linear functionals $f : V \otimes V \to \mathbb{C}$ of the tensor product: since tensor products were defined such that they are bilinear, these are in fact the same structures. In other words, there is a bijection between such bilinear forms and elements of the dual space $(V \otimes V)^* \simeq V^* \otimes V^*$. Furthermore, since the space S_B of bilinear forms $B : V \times V \to \mathbb{C}$ decomposes as $S_B = S_\alpha \oplus S_\sigma$, where S_α is the subspace of alternating forms and S_σ the space of symmetric forms; and similarly $V \otimes V = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V)$ as seen before; we note that the bijection $S_B \sim V^* \otimes V^*$ respects this sum: it restricts to a bijection between the space of symmetric (resp., alternating) bilinear forms and elements of $\mathrm{Sym}^2(V^*)$ (resp., $\mathrm{Alt}^2(V^*)$).

In particular, the space of non-degenerate symmetric (resp., alternating) G-invariant bilinear forms on V is in bijective correspondence with the elements of $\mathrm{Sym}^2(V^*)$ (resp., $\mathrm{Alt}^2(V^*)$).

Now, with all the pieces in place, we can prove our main theorem about the Frobenius-Schur indicator.

Proof of Theorem 2.2.1. First, recall from Lemma 2.2.5 that if χ_σ^2 and χ_α^2 are the characters of the symmetric and alternating squares, respectively, then $\chi_\sigma^2(s) = \frac{1}{2}(\chi(s)^2 + \chi(s^2))$, and that $\chi_\alpha^2(s) = \frac{1}{2}(\chi(s)^2 - \chi(s^2))$. Note that we can decompose the Frobenius-Schur indicator as follows:

$$\iota(\chi) = \frac{1}{|G|}\sum_{s \in G} \chi(s^2) = \frac{1}{|G|}\sum_{s \in G} \chi_\sigma^2(s) - \chi_\alpha^2(s) = (\chi_1, \chi_\sigma^2 - \chi_\alpha^2) = (\chi_1, \chi_\sigma^2) - (\chi_1, \chi_\alpha^2).$$

Furthermore, we note that $(\chi_1, \chi) = (\chi_1, \chi^*)$ for any character χ with dual character χ^*, since $\chi^*(s) = \chi(s^{-1})$, and in the inner product we sum over all $s \in G$. Now, we have seen that the dimension d_σ (respectively, d_α) of the space of G-invariant non-degenerate symmetric (resp., alternating) bilinear forms on V is equal to the number of times the trivial representation occurs in the decomposition of the dual of the symmetric square ρ_σ^2 (resp., alternating square ρ_α^2). But then we can rewrite the indicator function as

$$\iota(\chi) = d_\sigma - d_\alpha.$$

By Lemma 2.2.3, we have three distinct cases:

1. ρ is a complex representation; the space of G-invariant non-degenerate bilinear forms has dimension zero; $d_\sigma = d_\alpha = 0$; and thus $\iota(\chi) = 0$.

2. ρ is a real representation; the space of G-invariant non-degenerate symmetric bilinear forms has dimension one, and the space of G-invariant non-degenerate alternating bilinear forms has dimension zero; $d_\sigma = 1$, $d_\alpha = 0$; and thus $\iota(\chi) = 1$.

3. ρ is a quaternionic representation; the space of G-invariant non-degenerate symmetric bilinear forms has dimension zero, and the space of G-invariant non-degenerate alternating bilinear forms has dimension one; $d_\sigma = 0$, $d_\alpha = 1$; and thus $\iota(\chi) = -1$.

$\square$

2.3 Decomposing the group algebra $\mathbb{R}[G]$

In section 1.2.3, we defined the group algebra of a finite group G over a commutative ring K, and noted that if we take $K = \mathbb{C}$, then there is a bijective correspondence between finitely-generated modules over the algebra $\mathbb{C}[G]$ and representations of G in $\mathbb{C}$-vector spaces. In discovering this correspondence, we did not use any particular properties of $\mathbb{C}$ that do not also hold of $\mathbb{R}$, and so we see that there is also a bijective correspondence between finitely-generated modules over the real group algebra $\mathbb{R}[G]$ and representations of G in $\mathbb{R}$-vector spaces. Furthermore, since $\mathbb{R}$ has characteristic zero, Lemma 1.2.13 tells us that the algebra $\mathbb{R}[G]$ is semisimple, which, when combined with Theorem 1.2.14 (the Artin-Wedderburn decomposition theorem), implies that $\mathbb{R}[G]$ can be factorized into a finite product of matrix rings with coefficients in division rings of finite degree of $\mathbb{R}$. The following result (the slick, elementary proof proceeds according to [5]) corresponds to Proposition 1.2.15, in which we classified the division rings of finite degree over $\mathbb{C}$.

Theorem 2.3.1 (Frobenius' Theorem)**.** *Let D be a division ring of finite degree over $\mathbb{R}$. Then D is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or the ring $\mathbb{H}$ of quaternions.*

Proof. Suppose that D is not isomorphic to $\mathbb{R}$. Then there is some element $x \in D \backslash \mathbb{R}$. Let $\mathbb{R}_x$ be the two-dimensional $\mathbb{R}$-vector space with basis $\{1, x\}$. First we note that $\mathbb{R}_x$ is a commutative subset of D, of degree two over $\mathbb{R}$, and therefore is isomorphic to $\mathbb{C}$. Furthermore, it contains all the elements of D which commute with x, and it has maximal degree over $\mathbb{R}$ among such subsets (this phrase makes sense since we took D to be finite-degree over $\mathbb{R}$ in the first place).

To see this, let F be a commutative subspace of D containing $\mathbb{R}_x$ such that it has maximal degree (over $\mathbb{R}$) of all such spaces. Then if some $x' \in D$ commutes with the elements of F, the vector space $\{a + rx' : a \in F, r \in \mathbb{R}\}$ is commutative, so that it is actually equal to F (by our maximality assumption). Therefore $x' \in F$, and F is maximal as a commutative subset of D. Then for any $f \in F$, the set of elements which commute with f is equal to the set which commutes with f^{-1}, so that $f^{-1} \in F$ and therefore F is a field. Since it is commutative and contains $\mathbb{R}_x$, it is isomorphic to $\mathbb{C}$ and therefore equal to $\mathbb{R}_x$.

Therefore we can let i denote some element of D such that $i^2 = -1$ (it exists since $\mathbb{R}_x \subset D$ is isomorphic to $\mathbb{C}$). Then D becomes a $\mathbb{C}$-vector space (not a commutative one) with left scalar multiplication. Let $g : D \to D$ be the linear transformation defined by $g(x) = xi$. Then $g^2(x) = -x$, so that the eigenvalues of g can only be $\pm i$ (note that they needn't both be eigenvalues necessarily). Then if we let $D^\pm = \{x \in D : xi = \pm xi\}$ be the eigenspaces of g in D corresponding of to $\pm i$, then since for any $x \in D$ we can write $x = \frac{1}{2}(x - ixi) + \frac{1}{2}(x + ixi)$, where $x \mp ixi \in D^\pm$, and clearly $D^+ \cap D^- = \emptyset$, $D = D^+ \oplus D^-$.

Then since D^+ is a maximal set which commutes with some element $i \in D \setminus \mathbb{R}$, we see that, as before, $D^+ \simeq \mathbb{C}$. Therefore if $D^- = 0$, we have that $D = D^+ \simeq \mathbb{C}$. Otherwise, suppose that there is some nonzero $y \in D^-$. Then for any other $z \in D^-$ we have $yzi = -yiz = iyz$, so that $yz \in D^+$. Then if $h : D^- \to D^+$ is given by $h(x) = xy$, we see that h^{-1} is defined by right-multiplication by y^{-1} (recall that D is a division ring), and therefore $D^- \simeq D^+ \simeq \mathbb{C}$, and in particular both have dimension one as $\mathbb{C}$-vector spaces.

Next note that, as before, $\mathbb{R}_y$ is isomorphic to $\mathbb{C}$ and thus $y^2 \in \mathbb{R}_y$. But since $y^2 = h(y) \in D^+ \simeq \mathbb{C}$, $y^2 \in D^+ \cap \mathbb{R}_y = \mathbb{R}$. Furthermore, since y is a square root of y^2, and $y \notin \mathbb{R} \subset D^+$, y^2 cannot have two real square roots, and hence $y^2 < 0$. Therefore we can define some $j \in D^-$ for which $j^2 = -1$ (in particular, it is a positive multiple of y). Finally let $k = ij \in D^-$, and note that j, k are linearly independent over $\mathbb{R}$. Therefore $\{j, k\}$ is a basis for D^- over $\mathbb{R}$, and $\{1, i\}$ is a basis for D^+ over $\mathbb{R}$, so that $\{1, i, j, k\}$ is a basis for D over $\mathbb{R}$. Furthermore, $k^2 = ijij = -i^2j^2 = -1$, and $ijk = k^2$, so that $i^2 = j^2 = k^2 = ijk = -1$, the defining relations of the quaternions. So $D \simeq \mathbb{H}$, the quaternions. $\square$

Corollary 2.3.2. *If G is a finite group, then the group algebra of G over $\mathbb{R}$ factorizes into a product of matrix rings as follows:*

$$\mathbb{R}[G] = \prod_{j=1}^{N_1} \mathbb{M}_{n_j}(\mathbb{R}) \prod_{k=1}^{N_2} \mathbb{M}_{n_k}(\mathbb{C}) \prod_{l=1}^{N_3} \mathbb{M}_{n_l}(\mathbb{H}).$$

Proof. This follows directly from Frobenius' theorem (2.3.1) and the Artin-Wedderburn decomposition theorem (Theorem 1.2.14). $\square$

In the previous section, we inquried as to whether irreducible representations over the complex numbers could be realized over the reals. In this section, we dig deeper, and see that when we transform an irreducible representation over the complex numbers into a representation over the real numbers, we can obtain the irreducible representations over the reals, and furthermore determine their character and their corresponding simple component of the real group algebra $\mathbb{R}[G]$.

First we note that if $\rho : G \to GL(V)$ is an irreducible representation over the complex numbers with character χ, where V has dimension n and basis $\{x_1, \ldots, x_n\}$, then we can transform it into a $2n$-dimensional vector space over $\mathbb{R}$ by letting $y_i = ix_i$ for $i = 1, \ldots, n$, and taking the $\mathbb{R}$-basis $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. This turns ρ into a $2n$-degree representation $\rho^{\mathbb{R}}$ over the real numbers with character $\chi + \overline{\chi}$ (since we multiply the basis elements by i in the "non-real" half of the vector space).

The question, then, is whether or not ρ remains irreducible. We take note of the following criterion first. If χ_0 is the character of a representation $\rho^0 : G \to GL(V_0)$

over the real numbers, then the inner product (χ_0, χ_0) is equal to the number of irreducible representations contained in the decomposition of ρ^0 over the complex numbers. Since every G-invariant endomorphism of V_0 is a representation of G over V_0, we see that the dimension over $\mathbb{R}$ of the endomorphism ring $\mathrm{End}_G(V_0)$ is equal to (χ_0, χ_0). This will allow us to determine the simple component of the real group algebra $\mathbb{R}[G]$ which corresponds to representation ρ.

There are three scenarios:

1. If ρ is a real representation, then we know that there is a basis of V under which ρ has all real entries and still has character χ, and therefore the representation of character $\chi + \overline{\chi} = 2\chi$ is reducible over $\mathbb{R}$, and is in fact a sum of two copies of $\rho^{\mathbb{R}}$. Therefore it corresponds to one of the simple components $\mathbb{M}_{n_i}(\mathbb{R})$ in the decomposition of $\mathbb{R}[G]$.

2. If ρ is a complex representation, then $\chi \neq \overline{\chi}$, and therefore this ρ corresponds to an irreducible real representation of degree $2n$ and character $\chi + \overline{\chi}$. Note that since $(\chi+\overline{\chi}, \chi+\overline{\chi}) = 2$, we see that its endomorphism ring has dimension 2 over $\mathbb{R}$ and is thus equal to $\mathbb{C}$. Therefore it corresponds to one of the simple components $\mathbb{M}_{n_i}(\mathbb{C})$ in the decomposition of $\mathbb{R}[G]$.

3. If ρ is a quaternionic representation, then $\chi = \overline{\chi}$, and so ρ corresponds to a real representation of degree $2n$ and character 2χ, which is not reducible over the real numbers. This is because, if it were, we would have a realization of ρ over the real numbers, contradicting that it is quaternionic. Furthermore, $(2\chi, 2\chi) = 4$, and therefore the endomorphism ring of the representation has dimension 4 over $\mathbb{R}$ and is thus equal to $\mathbb{H}$. So it corresponds to one of the simple components $\mathbb{M}_{n_i}(\mathbb{H})$ in the decomposition of $\mathbb{R}[G]$.

2.4 Some simple examples

First, we return to the groups whose character tables we have computed.

As a simplest example, we compute the Frobenius-Schur indicator of the degree two representation of S_3. Recall that its character is given by:

S_3	C_1	C_2	C_3
χ_3	2	-1	0

where $C_1 = \{id\}$, $C_2 = \{(123), (132)\}$ and $C_3 = \{(12), (13), (23)\}$. Then since $(123)^2 = (132)$, and $(12)^2 = (13)^2 = (23)^2 = id$, we have that

$$\iota(\chi_3) = \frac{1}{|G|} \sum_{s \in G} \chi_3(s^2) = \frac{1}{6}(4\chi(C_1) + 2\chi_3(C_2)) = \frac{1}{6}(8 - 2) = 1.$$

Therefore χ_3 is the character of a real representation.

Now, we return to the identical character tables of D_4 and Q_8 to see that their degree two representations are indeed not the same. Recall that the degree two character of D_4 is given by

D_4	C_1	C_2	C_3	C_4	C_5
χ_5	2	-2	0	0	0

where $C_1 = \{1\}$, $C_2 = \{x^2\}$, $C_3 = \{y, x^2y\}$, $C_4 = \{x, x^3\}$ and $C_5 = \{xy, x^3y\}$.

Note that the elements of C_4 square to x^2; and that those of C_1, C_2, C_3, and C_5 square to 1. Then we can compute the indicator:

$$\iota(\chi_5) = \frac{1}{|G|}\sum_{s\in G}\chi_5(s^2) = \frac{1}{8}(6\chi_5(C_1) + 2\chi_5(C_2)) = 1.$$

Therefore χ_5 is also the character of a real representation.

Now recall that the degree two character of Q_8 is

Q_8	C_1	C_2	C_3	C_4	C_5
χ_5	2	-2	0	0	0

where $C_1 = \{1\}$, $C_2 = \{-1\}$, $C_3 = \{\pm i\}$, $C_4 = \{\pm j\}$, and $C_5 = \{\pm k\}$. In this case the elements of C_3, C_4, and C_5 square to $-1 \in C_2$, and those of C_1 and C_2 square to $1 \in C_1$. Then

$$\iota(\chi_5) = \frac{1}{|G|}\sum_{s\in G}\chi_5(s) = \frac{1}{8}(2\chi_5(C_1) + 6\chi_5(C_2)) = -1.$$

Indeed, the degree two character of Q_8 is quaternionic; it cannot be realized over $\mathbb{R}$. In the final chapter, we will examine the finite subgroups of the special orthogonal group SO_3 for more examples of character tables and Frobenius-Schur indicators. We will be led to compute the character tables of the symmetric group S_4, and the alternating groups A_4 and A_5, which have more high-degree irreducible representations than those seen above. However, we will see that they for the most part turn out to be realizable over $\mathbb{R}$. As it turns out, quaternionic representations are somewhat rare. For an example of a slightly larger group which has a quaternionic representation, we note that the special linear group $SL_2(\mathbb{F}_3)$ of 2×2 matrices of determinant one with entries in the finite field $\mathbb{F}_3$ has a degree two quaternionic irreducible representation.

Chapter 3

The finite subgroups of SO_3

In this section, we examine as a source of examples the finite subgroups of the special orthogonal group SO_3, defined either as the space of 3×3 matrices of determinant 1, or as the space of rotations in three dimensions. The classification of these subgroups is standard, and not particularly relevant to the contents of this paper. We defer to [1, chapter 6] or [2, chapter 3] for the details of the classification proof. We will take the following theorem as given:

Theorem 3.0.1 (Classification of the finite subgroup sof SO_3)**.** *Every finite subgroup of the special orthogonal group SO_3 is isomorphic to one of the following:*

1. *A cyclic group C_n of order n*
2. *A dihedral group D_n of order $2n$*
3. *The tetrahedral group T, defined as the group of symmetries of a regular tetrahedron inscribed in the unit sphere. The group T is isomorphic to the alternating group A_4.*
4. *The octahedral group O of symmetries of a regular octahedron (cube) inscribed in the unit sphere. It is isomorphic to the symmetric group S_4.*
5. *The icosahedral group I of symmetries of a regular icosahedron inscribed in the unit sphere. It is isomorphic to the alternating group A_5.*

In the remaining sections, we will determine the character tables of each class of groups listed above, and determine the Frobenius-Schur indicators of their representations of degree greater than one (if any).

3.1 The cyclic groups C_n

Since $C_n = \langle x \rangle$ is abelian, it has n distinct conjugacy classes, and thus n nonisomorphic irreducible representations (since the sum of the squares of their dimensions must equal n, they all have degree one). Since every element of C_n has order dividing n and all representations are one-dimensional, the values $\chi_i(x^j)$ must be powers of $\omega = e^{2\pi i/n}$, the primitive n^{th} root of unity. Of course, the value of a character on all of C_n is determined by its value on a generator x, which we can send to any of the n distinct powers of ω (the n^{th} power, which is 1, accounts for the trivial character). Taking powers of $\chi_i(x)$ generates the rest of the the values, since the characters are one-dimensional, and thus homomorphisms. From this, we know we needn't bother calculating the Frobenius-Schur indicators of the characters, since unless the generator x is sent to 1 (the trivial character) or to -1 (which will necessarily determine a real representation that takes on 1 on even powers of the generator and -1 on odd powers), it must be sent to a non-real number (as all other powers of roots of unity are), and hence the representations are complex.

Note that, in any case, a character of degree one is not interesting from the perspective of real, complex, and quaternionic representations, since if the value of the character is always real, the representation is necessarily real, anyway (the character is itself the representation).

3.2 The dihedral groups D_n

We write $D_n = \langle x, y | x^n = y^2 = xyxy = 1\rangle$ in its usual presentation. First, we note that the cases in which n is even are different than those in which n is odd, as can be seen by the difference in conjugacy class structures.

If n is odd, then D_n has $\frac{n-1}{2} + 2$ conjugacy classes: the class of the identity; the single class (of size n) containing the elements $x^k y$ for all $k = 1, \ldots, n$; and the $(n-1)/2$ classes which each contain $\{x^k, x^{-k}\}$ for each $1 \leq k \leq \lfloor n/2 \rfloor$. In this case, we can easily identify two irreducible (degree one) characters as the trivial one χ_1 and the representation χ_2 sending $x \mapsto 1$ and $y \mapsto -1$. Since this is a homomorphism and is distinct from the trivial representation, we see that it is irreducible.

Contrarily, if n is even, then D_n has $\frac{n}{2} + 3$ conjugacy classes: the class of the identity; the class of elements $x^k y$ for k even; the class of elements $x^k y$ for k odd; and the classes containing $\{x^k, x^{-k}\}$ for each $1 \leq k \leq \frac{n}{2} - 1$. Here we can easily identify four degree one representations: χ_1, the trivial character; χ_2 which sends $x \mapsto 1$ and $y \mapsto -1$; χ_3 which sends $x \mapsto -1$ and $y \mapsto 1$; and χ_4 which sends $x \mapsto -1$ and $y \mapsto -1$. Again, these are all distinct homomorphisms from $D_n \to \mathbb{C}$, and are thus irreducible representations.

To obtain the remaining representations, in either case, we can construct the necessary amount of distinct two-dimensional representations as follows: let $\gamma_n = e^{2\pi i/n}$. Then we can define

$$\rho_x^k = \begin{bmatrix} \gamma_n^k & 0 \\ 0 & \gamma_n^{-k} \end{bmatrix}, \text{ and } \rho_y^k = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

so that $(\rho_y^k)^2 = I$, $(\rho_x^k)^n = I$, and $\rho_y^k \rho_x^k \rho_y^k \rho_x^k = I$, so that ρ^k is a representation of D_n for each $k = 1, \ldots, \lfloor (n-1)/2 \rfloor$. We see that they are distinct since their characters take different values on x for each k. To see that they are irreducible, we simply take the inner product of their characters with themselves, and see that it comes out to one.

In the case of n odd, this gives us our remaining $(n-1)/2$ representations; and in the case of n even, this gives us our remaining $\frac{n}{2} - 1$ representations, and therefore we have found all the irreducible characters of D_n for all n. Note that our two-dimensional characters are all real, as defined, but the representations we gave are not. So in this case computing the Frobenius-Schur indicators wouldn't be a bad idea.

First note that every element of the form $x^k y$ has order 2, and that if n is even, $x^{n/2}$ does as well. So if n is odd, there are $n+1$ elements whose square is the identity; and if n is even, there are $n+2$. For the other elements, if n is odd, then squaring is an automorphism of the subgroup $\langle x \rangle$ (since n and 2 are relatively prime), and so each conjugacy class $\{x^k, x^{-k}\}$ contains two squares. Therefore the Frobenius-Schur indicator, where χ_k is the character of ρ^k as defined earlier, is

$$\iota(\chi_k) = \frac{1}{|D_n|} \sum_{s \in D_n} \chi_k(s) = \frac{1}{2n}(2(n+1) - 2) = 1,$$

since the sum of the characters of the remaining conjugacy classes add to -2 (since the character is orthogonal to the trivial character). Thus all the irreducible representations of D_n for n odd are real representations.

If n is even, then first recall that there are $n+2$ elements which square to the identity. The remaining $n-2$ elements are the $n-1$ powers of x excluding $x^{n/2}$. Now since $x^{2j} = x^{2k}$ if and only if $k = j \bmod n/2$, we see that each even power of x has two square roots. Since n is even, the conjugacy classes $\{x^k, x^{-k}\}$ are either even or odd classes. So each even class contains four squares, except that of $x^{n/2}$ (which contains two), and each odd one contains none. Therefore the indicator is computed as follows:

$$\iota(\chi_k) = \frac{1}{|D_n|} \sum_{s \in D_n} \chi_k(s) = \frac{1}{2n} \left(2(n+2) - 2(\gamma_n^{n/2} + \gamma_n^{-n/2}) + 4 \sum_{k=1}^{\frac{n}{2}-1} (\gamma_n^{k/2} + \gamma_n^{-k/2}) \right)$$

$$= \frac{1}{2n}(2n + 2 - 2 + 0) = 1.$$

As in the odd case, we see that D_n has all real irreducible representations for even n, as well.

3.3 The tetrahedral group T

To find the character table of the tetrahedral group, we enumerate the vertices of a regular tetrahedron as v_1, v_2, v_3, v_4, and note that T simply permutes the set V of these vertices with orientation-preserving (that is, even) permutations. So T is isomorphic to A_4, and we can instead find its character table. Its elements are the identity, the eight three-cycles, and the three double-transpositions. A little fiddling reveals that the double-transpositions form a single conjugacy class $\mathcal{C}_2$, and that the three-cycles split into two conjugacy classes in the alternating group A_4: $\mathcal{C}_3 : (123), (134), (142), (243)$, and $\mathcal{C}_4 : (132), (143), (124), (234)$ (the identity is conjugacy class $\mathcal{C}_1$). So by the main theorem of characters, there are four nonisomorphic irreducible characters of T, where χ_1 is the trivial character. The main theorem also gives us that $\sum_{i=1}^4 \chi_i(1)^2 = |T| = 12$, so that $\sum_{i=2}^4 \chi_i(1)^2 = 11$, for which the only solution is $1^2 + 1^2 + 3^2 = 11$. So χ_2, χ_3 are one-dimensional, and χ_4 is three-dimensional.

If ω is a primitive third root of unity, we see that for χ_2 to be non-trivial, it must take on either ω or ω^2 on conjugacy classes $\mathcal{C}_3$ and $\mathcal{C}_4$, since otherwise its value on both would be 1 (since these conjugacy classes have elements of order 3, and so the order of their character value on the one-dimensional characters must have order dividing 3), and thus the value on $\mathcal{C}_2$ would also be 1, since we can obtain double-transpositions as products of certain three-cycles. So say that $\chi_2(\mathcal{C}_3) = \omega$, and $\chi_2(\mathcal{C}_4) = \omega^2$, which implies that $\chi_2(\mathcal{C}_2) = 1$, since again its elements can be obtained as products of those from the other two classes. By the same argument, χ_3 must take the values ω^2 and ω on the final two classes, and since it is distinct it must reverse them. This fills out the first three rows of the character table:

T	1	(12)(34)	(123)	(132)
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3			

Finally we can use orthogonality of columns to fill out the rest: since the first and second column are orthogonal, $1+1+1+3\chi_4(\mathcal{C}_2) = 0$, so that $\chi_4(\mathcal{C}_2) = -1$. Now for the third and fourth columns, we have that if $i = 3, 4$, then $1+(\omega+\omega^2)+3\chi_4(\mathcal{C}_i) = 0$, and hence $\chi_4(\mathcal{C}_i) = 0$. So the complete table is as below.

T	1	(12)(34)	(123)	(132)
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

Now, we calculcate the Frobenius-Schur indicator of the three-dimensional irreducible character χ_4 of T. We see that

$$\iota(\chi_4) = \frac{1}{|G|} \sum_{g \in G} \chi_4(g^2) = \frac{1}{12}(3 + 3 + 3 + 3) = 1.$$

Hence χ_4 is realizable as a representation over $\mathbb{R}$. As in our subsequent examples, this is completely natural, since the tetrahedral group is in fact a finite subgroup of SO_3, the space of 3×3 orthogonal (and thus real-valued) matrices of determinant one.

3.4 The octahedral group O

Recall that the octahedral group O is isomorphic to the symmetric group S_4. Its elements are the identity, the eight three-cycles, the three double-transpositions, the six four-cycles, and the six transpositions. In the symmetric group, conjugacy class is determined by cycle type, so we have obtained our five conjugacy classes: $\mathcal{C}_1$ of the identity; $\mathcal{C}_2$ of the transpositions; $\mathcal{C}_3$ of the double-transpositions; $\mathcal{C}_4$ of the three-cycles; and $\mathcal{C}_5$ of the four-cycles. Therefore, there are five irreducible characters, with χ_1 being the trivial character. We must have that $\sum_{i=2}^{5} \chi_i(1) = |O| - 1 = 23$. If any of the squares were equal to 16, this could not add up; hence they must all be at most 9. But we must have at least two equal to 9 for it to add to 23, so we have that $\chi_2(1) = 1$, $\chi_3(1) = 2$, and $\chi_4(1) = \chi_5(1) = 3$. To obtain a non-trivial degree-one representation, we need look no further than the sign map, which sends a permutation to its sign. The transpositions and four-cycles are odd, and hence are sent to -1 (those are $\mathcal{C}_2$ and $\mathcal{C}_5$); while the three-cycles and double-transpositions are sent to 1 ($\mathcal{C}_3$ and $\mathcal{C}_4$). So we have filled out the first two rows:

O	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2				
χ_4	3				
χ_5	3				

To fill out another row, we consider the permutation representation of O operating on the set of pairs of opposite vertices of the octahedron inscribed in the unit sphere. Recall that the transpositions $\mathcal{C}_2$ correspond to the rotations of angle π about the line through opposite edges; the double-transpositions $\mathcal{C}_3$ correspond to the rotations of angle π about lines going through the center of opposite faces; the three-cycles $\mathcal{C}_4$ correspond to the rotations of angle $2\pi/3$ about the line through opposite pairs of vertices; and the four-cycles $\mathcal{C}_5$ correspond to the to the rotations of angle $\pi/2$ about the same lines as $\mathcal{C}_2$.

We see that $\mathcal{C}_1$ fixes all four pairs; $\mathcal{C}_2$ fixes the two pairs adjacent to the edges through which the axis of rotation goes; $\mathcal{C}_3$ and $\mathcal{C}_5$ do not fix any pairs; and $\mathcal{C}_4$ fixes the pair of vertices through which the axis of rotation goes. So we have a representation χ_A of character

O	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
χ_A	4	2	0	1	0

Now note that if we subtract the trivial character from χ_A, we obtain a character with values as follows:

O	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
$\chi_A - \chi_1$	3	1	-1	0	-1

This character is irreducible, since its inner product with itself is equal to one. Call it χ_4. Then we have obtained the fourth row of our character table:

O	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2				
χ_4	3	1	-1	0	-1
χ_5	3				

Since the product of two characters is a character (in particular, its representation is the tensor product of the two component representations, as we saw in section 2.2); we take the product $\chi_2\chi_4$, and see that its character is equal to

O	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
$\chi_4\chi_2$	3	-1	-1	0	1

In fact, this character is also irreducible, which we can see by taking its inner product with itself. We let $\chi_5 = \chi_4\chi_2$, and see that we have filled out most of the table:

O	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2				
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

Now we can obtain χ_3 by column orthogonality. It is easily seen this way that $\chi_3(\mathcal{C}_2) = \chi_3(\mathcal{C}_5) = 0$, that $\chi_3(\mathcal{C}_3) = 2$, and that $\chi_3(\mathcal{C}_4) = -1$. This completes the character table:

O	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

Now we may determine whether the degree three and two representations of O are real or quaternionic by computing their Frobenius-Schur indicators.

We see that the elements of $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_4$ square to the identity, so that we have $s^2 \in \mathcal{C}_1$ for 10 elements $g \in O$. The elements of $\mathcal{C}_5$ square to other elements of $\mathcal{C}_5$, so there are 8 elements $s^2 \in \mathcal{C}_5$. Finally, the elements of $\mathcal{C}_3$ square to those in $\mathcal{C}_2$, so there are 6 elements $s^2 \in \mathcal{C}_2$. From this we obtain that:

$$\frac{1}{|G|}\sum_{s\in G}\chi_3(s^2) = \frac{1}{24}(10\times 2 + 6\times 2 + 8\times -1) = 1.$$

$$\frac{1}{|G|}\sum_{s\in G}\chi_4(s^2) = \frac{1}{24}(10\times 3 + 6\times -1 + 8\times 0) = 1,$$

$$\frac{1}{|G|}\sum_{s\in G}\chi_5(s^2) = \frac{1}{24}(10\times 3 + 6\times -1 + 8\times 0) = 1,$$

so that indeed all three multi-degree representations of O can be realized over the real numbers. In fact this can be inferred at least in the case of the degree three representations, since O, as a finite subgroup of SO_3, has a representation as three-dimensional orthogonal matrices of determinant one; and we obtained the other degree three character by taking its tensor product with the sign representation. However, in the case of the degree two character, we do not have any obvious corresponding representation over the reals, and so this computation is at least somewhat enlightening.

3.5 The icosahedral group I

To find the character table of $I \simeq A_5$, first we determine the conjugacy classes of A_5. Now, the conjugacy classes of the symmetric group S_5 are equivalent to partitioning the group by cycle type. In general, for an alternating group $A_n \subset S_n$, each conjugacy class (that is, cycle type) that remains in A_n - the even ones - either remains an intact conjugacy class, or splits in two. In general, it doesn't split if its cycle decomposition contains either an even cycle or two cycles of the same length. In the case of A_5, the five-cycles split, while the double-transpositions and three-cycles do not. So we have five conjugacy classes: $\mathcal{C}_1$ of the identity (size 1); $\mathcal{C}_2$ of the double-transpositions (size 15); $\mathcal{C}_3$ of the three-cycles (size 20); $\mathcal{C}_4$ of the five-cycles generated by (12345) (size 12); and $\mathcal{C}_5$ of the five-cycles generated by (13524) (size 12). Therefore we have five irreducible representations, with χ_1 again the trivial representation. This means that $\sum_{i=2}^{5} \chi_i^2(1) = 59$, which we can easily see is only possible if the values are 5, 4, 3 and 3.

Following [1], we use geometric aspects of the rotation groups to fill out our table. We obtain one three-dimensional representation as the set of orthogonal matrices corresponding to the rotations in the icosahedral group I. The trace of a rotation matrix which rotates by angle θ is $1 + 2\cos\theta$, since we can pick a basis under which the rotation matrix takes a block form with a 1 in one corner, and the block $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ in the other. Since these angles are π for $\mathcal{C}_2$, we get that $\chi_2(\mathcal{C}_2) = -1$; for $\mathcal{C}_3$, $\theta = 2\pi/3$, so $\chi_2(\mathcal{C}_3) = 0$; for $\mathcal{C}_4$, $\theta = 2\pi/5$ and $\chi_2(\mathcal{C}_4) = \alpha = 1 + 2\cos(2\pi/5)$; similarly for $\mathcal{C}_5$, $\theta = 4\pi/5$ so that $\chi_2(\mathcal{C}_5) = \beta = 1 + 2\cos(4\pi/5)$.

Next, we consider the action of $I \simeq A_5$ on $\{1, \ldots, 5\}$; a permutation representation of dimension 5. The character χ_P at each group element is equal to the number of indices fixed by that element. We see, then, that $\chi_P(\mathcal{C}_1) = 5$; $\chi_P(\mathcal{C}_2) = 1$; $\chi_P(\mathcal{C}_3) = 2$; $\chi_P(\mathcal{C}_4) = \chi_P(\mathcal{C}_5) = 0$. We know that the trivial representation appears in the decomposition of every permutation representation, so we take the inner product $(\chi_P - \chi_1, \chi_P - \chi_1) = 1$, and see that $\chi_4 = \chi_P - \chi_1$ is irreducible. So we have filled out three rows of the character table:

I	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
χ_1	1	1	1	1	1
χ_2	3	-1	0	α	β
χ_3	3				
χ_4	4	0	1	-1	-1
χ_5	5				

Now recall that I is the group of symmetries of a regular dodecahedron inscribed in the unit circle, and consider the operation of I on the pairs of opposite faces of this dodecahedron. This gives us a degree six character with values $\chi_D(\mathcal{C}_1) = 6$; $\chi_D(\mathcal{C}_2) = 2$; $\chi_D(\mathcal{C}_3) = 0$; $\chi_D(\mathcal{C}_4) = \chi_D(\mathcal{C}_5) = 1$. This is because the edge rotations of angle π fix two pairs of faces (those on either side of the edges); the vertex rotatiions fix no faces; and the face rotations of both angles $2\pi/5$ and $4\pi/5$ only fix the faces they rotate about. We know, as before, that the trivial representation appears in this character's decomposition. So we compute the inner product $(\chi_D - \chi_1, \chi_D - \chi_1) = 1$, and therefore $\chi_5 = \chi_D - \chi_1$ is irreducible. This fills out the bottom row of our character table. We can fill out the third row by column orthogonality. Hence we have the completed character table of $I \simeq A_5$:

I	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
χ_1	1	1	1	1	1
χ_2	3	-1	0	α	β
χ_3	3	-1	0	β	α
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

where again $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Since the elements of $\mathcal{C}_1$ and $\mathcal{C}_2$ square to the identity; the elements of $\mathcal{C}_3$ square to other elements of $\mathcal{C}_3$, and the elements of $\mathcal{C}_4$ square to those in $\mathcal{C}_5$ (and vice versa), we can calculate the indicators of the non-trivial characters χ:

$$\iota(\chi) = \frac{1}{|G|}\sum_{s\in G}\chi(s^2) = \frac{1}{60}(16\chi(\mathcal{C}_1) + 20\chi(\mathcal{C}_3) + 12(\chi(\mathcal{C}_4) + \chi(\mathcal{C}_5))).$$

Therefore

$$\begin{aligned}\iota(\chi_2) = \iota(\chi_3) &= \frac{1}{60}(48 + 0 + 12) = 1,\\ \iota(\chi_4) &= \frac{1}{60}(64 + 20 - 24) = 1,\\ \iota(\chi_5) &= \frac{1}{60}(80 - 20) = 1.\end{aligned}$$

As before, we see that all non-trivial irreducible representations of $I \simeq A_5$ are indeed real.

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