Question 2

Notice that the utility function might not be strictly increasing. Therefore, we need to modify the notion of feasible allocation.An allocation $(x^i)_{i=1}^m$ is feasible if

$$\sum_{i=1}^{m} x^i \leq \sum_{i=1}^{m} e^i$$

(Sufficiency) Suppose $(\hat{x}^i)_{i=1}^m$ is the solution to the maximization problem. From the second condition we can conclude that this is a feasible allocation. By way of contradiction, suppose that this is not Pareto efficient. Then there must exist some other allocation $(y^i)_{i=1}^m$ such that $u^j(y^i) \geq u^j(\hat{x}^i)$ for all i with at least one inequality. If that is the case, we also have $\sum_{i=1}^m u^j(y^i) \geq \sum_{i=1}^m u^j(\hat{x}^i)$ This contradicts with the assumption that $(x^i)_{i=1}^m$ solves the maximization problem.

(Necessity) Consider an exchange economy $\mathcal{E} = (u_i, e^i)\ i \in \mathcal{I}$. $(\hat{x}^i)_{i=1}^m$ is a Pareto efficient allocation. By way of contradiction, suppose $(\hat{x}^i)_{i=1}^m$ not the solution to the maximization problem. It implies that there exists another feasible allocation $(\tilde{x}^i)_{i=1}^m$ such that $\sum_{i=1}^m u^i(\tilde{x}^i) > \sum_{i=1}^m u^i(\hat{x}^i)$. Therefore, there must be at least one j such that $u^j(\tilde{x}^i) > u^j(\hat{x}^i)$ This violates the definition of Pareto efficient allocation. $\square$

Question 4

Claim: the equilibrium satisfies

$$\begin{aligned} &p_1 = p_2 = p_3 = p^* \\ &x_1^1 = x_2^1 = \frac{1}{2}, \quad x_3^1 = 0 \\ &x_2^2 = x_3^2 = \frac{1}{2}, \quad x_1^2 = 0 \\ &x_1^3 = x_3^3 = \frac{1}{2}, \quad x_2^3 = 0 \end{aligned} \tag{1}$$

With the utility function given in the question, the consumer's optimal demand is to consume the same amount of goods that she values, i.e.,

$$x_1^{1*} = x_2^{1*}, x_2^{2*} = x_3^{2*}, x_1^{3*} = x_3^{3*}$$

The budget constraints yield

$$\begin{aligned} p_1 x_1^{1*} + p_2 x_2^{1*} &= p_1 \\ p_2 x_2^{2*} + p_3 x_3^{2*} &= p_2 \\ p_1 x_1^{3*} + p_3 x_3^{3*} &= p_3 \end{aligned} \tag{2}$$

This gives us

$$\begin{aligned} x_1^{1*} = x_2^{1*} &= \frac{p_1}{p_1 + p_2} \\ x_2^{2*} = x_3^{2*} &= \frac{p_2}{p_2 + p_3} \\ x_1^{3*} = x_3^{3*} &= \frac{p_3}{p_1 + p_3} \end{aligned} \tag{3}$$

Also, market clearing requires that

$$\begin{aligned} x_1^{1*} + x_1^{3*} &= 1 \\ x_2^{2*} + x_1^{1*} &= 1 \\ x_2^{2*} + x_1^{3*} &= 1 \end{aligned} \tag{4}$$

If we normalise p_1 tp $p^* > 0$ and solve for 3 and 4, we will have the result above.

Question 5.3
Recall that function u is strongly increasing means that

$$\forall x, y, x \geq y, x \neq y \Rightarrow u^i(x) > u^i(y)$$

Also recall that excess demand for good k is defined as

$$z_k(p) = \sum_{i=1}^{m} x_k^i - \sum_{i=1}^{m} e_k^i$$

Suppose the price of one good s is non-positive. The consumer's budget constraint is

$$\sum_{k=1}^{s-1} p_k x_k + p_s x_s + \sum_{k=s+1}^{n} p_k x_k \leq \sum_{k=1}^{s-1} p_k e_k + p_s e_s + \sum_{k=s+1}^{n} p_k e_k \tag{5}$$

Suppose $p_s = 0$. The consumer's budget constraint would hold for any $x_s \in \mathbb{R}_+^n$
Also, the utility function is strongly increasing. We could strictly increase the consumer's utility by replacing the original x_s with some unbounded positive demand. The proof is the same is $p_s \leq 0$. □

Question 5.14 (a)
Recall that Theorem 5.5 requires Assumption 5.1:
Assumption 5.1 Each utility function u^i is continuous, strongly increasing and strictly quasi-concave.
However, Cobb-Douglas utility function is not strongly increasing. Consider two bundles $x = (1, 0, 0, ..., 0)$, $y = (1, 1, 1, ..., 1, 0)$ $x \geq y, x \neq y, but u^i(x) = u^i(y)$. Therefore, we cannot apply Theorem 5.5 to conclude that this economy possesses a Walrasian economy.

Question 5.14 (b)
Theorem 5.3 Suppose $\mathbf{z} : \mathbb{R}_{++}^n \to \mathbb{R}^n$ satisfies the following three conditions:

1. z(·) is continuous on $\mathbb{R}_{++}^n$;
2. $p \cdot z(p) = 0$ for all $p \cdot 0$;
3. If p^m is a sequence of price vectors in $\mathbb{R}_{++}^n$ converging to $\overline{p} = 0$, and $\overline{p}_k = 0$ for some good k, then for some good k' with $\overline{p}_{k'} = 0$, the associated sequence of excess demands in the market for good k', $z_{k'}(p^m)$, is unbounded above.

Then there is a price vector $p \gg 0$ such that $z(p) = 0$.

(1) The consumer's problem is:

$$\begin{aligned} \max_{\mathbf{x}\in\mathbb{R}^n_+} \quad & x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \\ \text{s.t.} \quad & \sum_{k=1}^{n} p_k x_k \leq \sum_{k=1}^{n} p_k e_k \end{aligned} \tag{6}$$

To solve the maximization problem, write the Lagrangian:

$$\mathbb{L} = u^i(x) - \lambda_i(px - pe)$$

The first order condition is

$$\begin{aligned} \frac{\alpha_k^i u^i(x)}{x_k^i} &= \lambda_i p_k, \quad \forall k \\ \Rightarrow \frac{p_k}{p_j} &= \frac{\alpha_k^i}{\alpha_j^i}\frac{x_j^i}{x_k^i}, \quad \forall k, j \\ \Rightarrow x_k^i &= \frac{\alpha_k^i}{\alpha_j^i}\frac{p_j}{p_k} x_j^i \\ \Rightarrow \sum_{k=1}^{n} p_k x_k^i = \sum_{k=1}^{n} \frac{\alpha_k^i}{\alpha_j^i} p_j x_j^i &= \sum_{k=1}^{n} p_k e_k^i \\ \Rightarrow x_k^i &= \frac{\alpha_k^i}{p_k}\left(\sum_{j=1}^{n} p_j e_j\right) \end{aligned}$$

The last equation follows from $\sum_{k=1}^{n} \alpha_k^i = 1$.
The excess demand function is

$$\begin{aligned} z_k(\mathbf{p}) &:= \sum_{i\in\mathcal{I}} x_k^i - \sum_{i\in\mathcal{I}} e_k^i \\ &= \sum_{i\in\mathcal{I}} \frac{\alpha_k^i}{p_k}\left(\sum_{j=1}^{n} p_j e_j\right) - \sum_{i\in\mathcal{I}} e_k^i \end{aligned} \tag{7}$$

Notice that x_j^i are continuous, so the aggregate excess demand function is also continuous.

(2) Using 7, we have

$$
\begin{aligned}
\mathbf{p}\cdot\mathbf{z}(\mathbf{p}) &= \sum_{k=1}^{n}\left\{\sum_{i\in\mathcal{I}}\alpha_k\left(\sum_{j=1}^{n}p_je_j\right)-p_ke_k^i\right\} \\
&= \sum_{i\in\mathcal{I}}\sum_{k=1}^{n}\left\{\alpha_k\left(\sum_{j=1}^{n}p_je_j\right)-p_ke_k^i\right\} \\
&= \sum_{i\in\mathcal{I}}\left(\sum_{k=1}^{n}\alpha_k\sum_{j=1}^{n}p_je_j\right)-\sum_{i\in\mathcal{I}}\sum_{k=1}^{n}p_ke_k^i \\
&= \sum_{i\in\mathcal{I}}\left(\sum_{k=1}^{n}p_ke_k^i-\sum_{k=1}^{n}p_ke_k^i\right) \\
&= 0
\end{aligned}
\tag{8}
$$

Condition 2 is satisfied.
(3) Suppose $\tilde{\mathbf{p}}$ is the price vector that satisfies $\tilde{\mathbf{p}} \to \overline{p} \neq 0$ and $\tilde{p}_k = 0$ for some k.
From 7 we know that the excess demand for good k is related to $\frac{\sum_{j=1}^{n}\tilde{p}_je_j}{\tilde{p}_k}$. $\sum_{j=1}^{n}\tilde{p}_je_j$ is strictly positive, while $\tilde{p}_k \to 0$.
Therefore, the demand is unbounded from above. Correspondingly, $z_k(\tilde{p}$ is also unbounded from above. Condition 3 is satisfied. We could apply Theorem 5.3 to conclude that a Walrasian equilibrium exists. □