Norms

By equipping a vector space $\mathcal{S}$ with a norm, we are imbuing it with a sense of **length** and **distance**. Another way to say this is that a norm adds a layer **topological structure** on top of the algebraic structure defining a linear space.

Definition. A **norm** $\|\cdot\|$ on a vector space $\mathcal{S}$ is a mapping

$$\|\cdot\| \;:\; \mathcal{S} \to \mathbb{R}$$

with the following properties for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}$:

1. $\|\boldsymbol{x}\| \geq 0$, and $\quad \|\boldsymbol{x}\| = 0 \;\Leftrightarrow\; \boldsymbol{x} = \boldsymbol{0}$.
2. $\|\boldsymbol{x} + \boldsymbol{y}\| \;\leq\; \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ (triangle inequality)
3. $\|a\boldsymbol{x}\| = |a| \cdot \|\boldsymbol{x}\|$ for any scalar a (homogeneity)

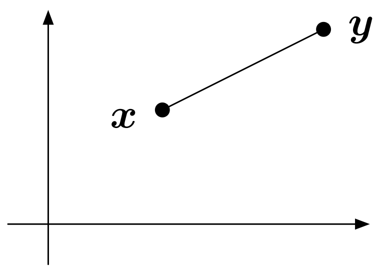
Other related definitions:

- The **length** of $\boldsymbol{x} \in \mathcal{S}$ is simply $\|\boldsymbol{x}\|$
- The **distance** between $\boldsymbol{x}$ and $\boldsymbol{y}$ is $\|\boldsymbol{x} - \boldsymbol{y}\|$
- A linear vector space equipped with a norm is called a **normed linear space**.

Examples:

1. $\mathcal{S} = \mathbb{R}^N$,
$$\|\boldsymbol{x}\|_2 = \left(\sum_{n=1}^{N} |x_n|^2\right)^{1/2}$$
This is called the "ℓ_2 norm" or "standard Euclidean norm"

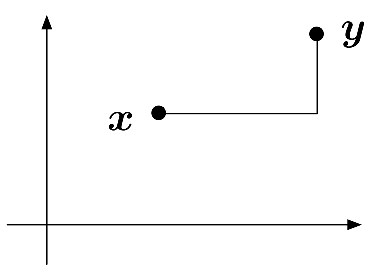
In $\mathbb{R}^2$:



$$\|\boldsymbol{x}-\boldsymbol{y}\|_2 = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

2. $\mathcal{S} = \mathbb{R}^N$
$$\|\boldsymbol{x}\|_1 = \sum_{n=1}^{N} |x_n|$$
This is the "ℓ_1 norm" or "taxicab norm" or "Manhattan norm"

In $\mathbb{R}^2$:



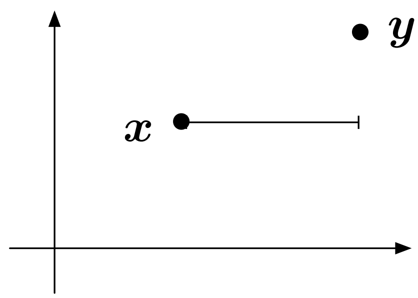
$$\|\boldsymbol{x} - \boldsymbol{y}\|_1 = |x_1 - y_1| + |x_2 - y_2|$$

3. $\mathcal{S} = \mathbb{R}^N$

$$\|\boldsymbol{x}\|_\infty = \max_{n=1,\ldots,N} |x_n|$$

This is the "ℓ_∞ norm" or "Chebyshev norm" or "max norm"

In $\mathbb{R}^2$:



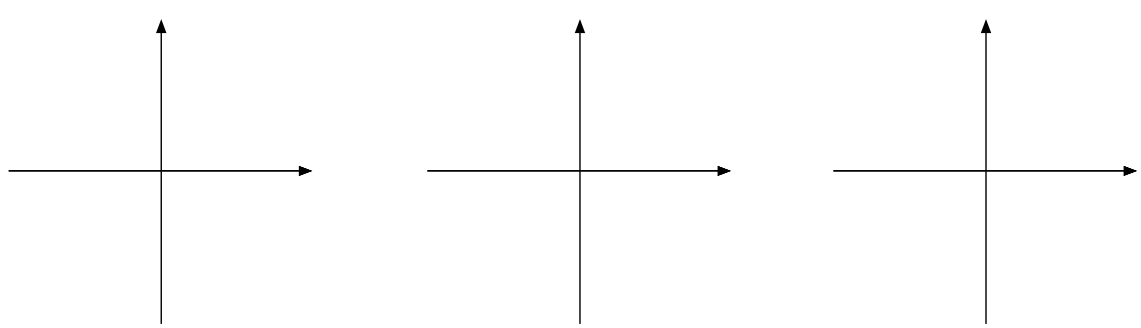
$$\|\boldsymbol{x}-\boldsymbol{y}\|_\infty = \max\left(|x_1 - y_1|,\ |x_2 - y_2|\right)$$

4. $\mathcal{S} = \mathbb{R}^N$

$$\|\boldsymbol{x}\|_p = \left(\sum_{n=1}^N |x_n|^p\right)^{1/p} \qquad \text{for some } 1 \le p < \infty$$

This is the "ℓ_p norm".

Draw the "ℓ_p unit balls" $\boldsymbol{\mathcal{B}}_p = \{\boldsymbol{x} \in \mathbb{R}^2 \ :\ \|\boldsymbol{x}\|_p \le 1\}$



$p = 1$ $\qquad$ $p = 2$ $\qquad$ $p = \infty$

5. The same definitions extend easily to infinite sequences:
$\mathcal{S}$ = sequences $\{x_n\}_{n\in\mathbb{Z}}$ indexed by the integers $n \in \mathbb{Z}$

$$\|\boldsymbol{x}\|_p = \left(\sum_{n=-\infty}^{\infty} |x_n|^p\right)^{1/p}$$

You can verify at home that the set of all sequences that have $\|\boldsymbol{x}\|_p < \infty$ is a (normed) linear space; we call this space ℓ_p.

6. $\mathcal{S}$ = continuous-time signals on the real line

$$\|\boldsymbol{x}\|_2 = \left(\int_{-\infty}^{\infty} |x(t)|^2 \ \mathrm{d}t\right)^{1/2}$$

This is called the L_2 norm[1]. In engineering, when $x(t)$ is a signal varying with time, we often refer to $\|\boldsymbol{x}\|_2^2$ as the **energy** in the signal.

Similarly,

$$\|\boldsymbol{x}\|_p = \left(\int_{-\infty}^{\infty} |x(t)|^p \ \mathrm{d}t\right)^{1/p}$$

and

$$\|\boldsymbol{x}\|_\infty = \sup_{t\in\mathbb{R}} |x(t)|, \quad \text{where sup = "least upper bound"}$$

Note that we are also using $\|\cdot\|_p$ for the discrete version of these norms, but I do not expect this will cause any confusion.

[1]The L is for Lebesgue, the mathematician who formalized the modern theory of integration in the early 1900s.

The set of functions of a continuous variable that have finite L_p norm are a normed linear space; we call this space $L_p(\mathbb{R})$.

7. $\mathcal{S}$ = real-valued functions on an interval $[a,b]$:

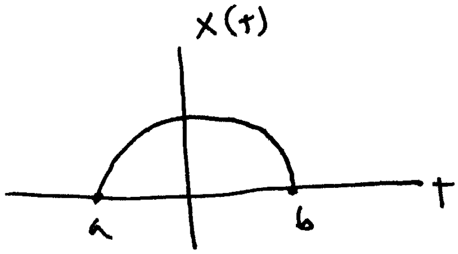
$$\|\boldsymbol{x}\|_p = \left(\int_a^b |x(t)|^p\right)^{1/p}$$
$$\|\boldsymbol{x}\|_\infty = \sup_{t\in[a,b]} |x(t)|$$

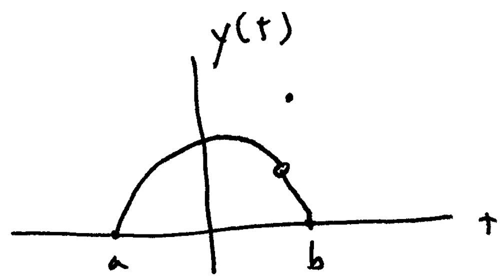
The normed linear space of all signals on the interval $[a,b]$ with finite L_p norm is called $L_p([a,b])$. This definition extends in the obvious way to complex-valued functions as well.

In a normed linear space, we say that

$$\boldsymbol{x} = \boldsymbol{y} \quad \text{if} \quad \|\boldsymbol{x}-\boldsymbol{y}\| = 0.$$

For example, in $L_2([a,b])$, say $y(t) = x(t)$ except at one point





Then

$$\|\boldsymbol{x}-\boldsymbol{y}\|_2 \;=\; \left(\int_a^b |x(t)-y(t)|^2\right)^{1/2} \;=\; 0$$

and so we still say that $\boldsymbol{x} = \boldsymbol{y}$. In general, if $\boldsymbol{x}, \boldsymbol{y} \in L_p$ differ only on a "set of measure zero", then $\boldsymbol{x} = \boldsymbol{y}$.

(A set $\Gamma \subset \mathbb{R}$ has measure zero if

$$\int I_\Gamma(t) \;\, \mathrm{d}t = 0,$$

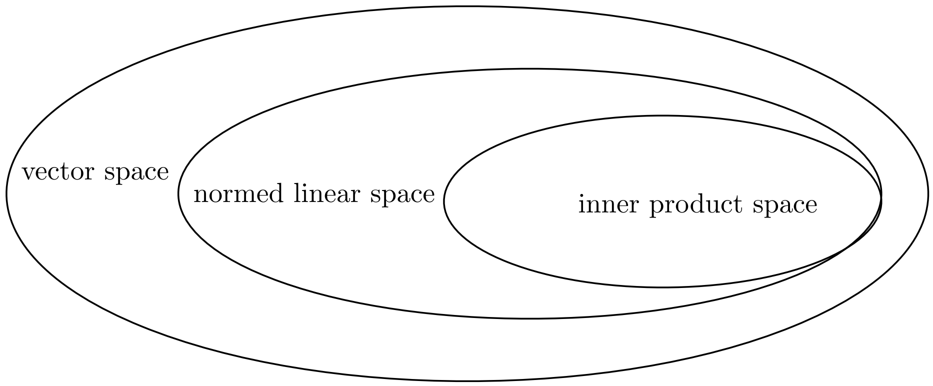
where

$$I_\Gamma(t) = \begin{cases} 1 & t \in \Gamma \\ 0 & t \notin \Gamma \end{cases}$$

is an indicator function.)

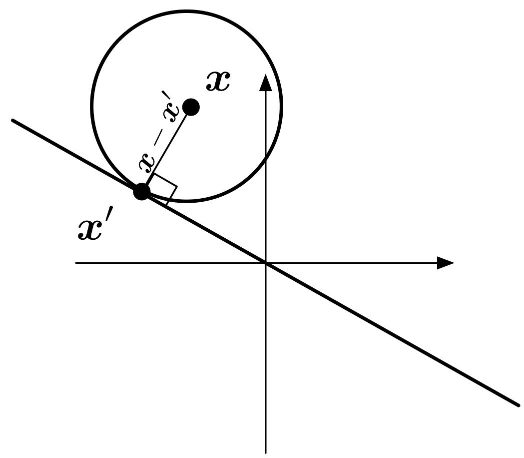
Inner product spaces

Layers of structure:



The abstract definition of an inner product, which we will see very shortly, is simple (and by itself is pretty boring). But it gives us just enough mathematical structure to make sense of many important and fundamental problems.

Consider the following motivating example in the plane $\mathbb{R}^2$. Let $\mathcal{T}$ be a one dimensional subspace (i.e. a line through the origin). Now suppose we are given a vector $\boldsymbol{x}$. What is the closest point $\boldsymbol{x}'$ in $\mathcal{T}$ to $\boldsymbol{x}$?



The salient feature of this point $\boldsymbol{x}'$ is that

$$\boldsymbol{x} - \boldsymbol{x}' \perp \boldsymbol{v} \quad \text{for all } \boldsymbol{v} \in \mathcal{T}.$$

So all we need to define this optimality property is the notion of **orthogonality** which follows immediately from defining an inner product. More on this later ...

Definition: An **inner product** on a (real- or complex-valued) vector space $\mathcal{S}$ is a mapping

$$\langle \cdot, \cdot \rangle \; : \; \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{C}$$

that obeys[2]

1. $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \overline{\langle \boldsymbol{y}, \boldsymbol{x} \rangle}$

2. For any $a, b \in \mathbb{C}$

$$\langle a\boldsymbol{x} + b\boldsymbol{y}, \boldsymbol{z} \rangle \; = \; a\langle \boldsymbol{x}, \boldsymbol{z} \rangle + b\langle \boldsymbol{y}, \boldsymbol{z} \rangle$$

[2] We are using $\bar{a}$ to denote the complex conjugate of a scalar a, and $\boldsymbol{x}^{\mathrm{H}}$ to denote the conjugate transpose of a vector $\boldsymbol{x}$.

3. $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \geq 0 \quad \text{and} \quad \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \Leftrightarrow \boldsymbol{x} = \mathbf{0}$

Standard Examples:

1. $\mathcal{S} = \mathbb{R}^N$,
$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle \;=\; \sum_{n=1}^{N} x_n y_n \;=\; \boldsymbol{y}^{\mathrm{T}} \boldsymbol{x}$$

2. $\mathcal{S} = \mathbb{C}^N$,
$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle \;=\; \sum_{n=1}^{N} x_n \,\overline{y_n} \;=\; \boldsymbol{y}^{\mathrm{H}} \boldsymbol{x}$$

3. $\mathcal{S} = L_2([a, b])$,
$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle \;=\; \int_a^b x(t)\overline{y(t)} \; \mathrm{d}t$$

Slightly less standard examples:

1. $\mathcal{S} = \mathbb{R}^{M \times N}$ (the set of $M \times N$ matrices with real entries)
$$\langle \boldsymbol{X}, \boldsymbol{Y} \rangle = \operatorname{trace}(\boldsymbol{Y}^{\mathrm{T}} \boldsymbol{X}) = \sum_{m=1}^{M} \sum_{n=1}^{N} X_{m,n} Y_{m,n}$$
(Recall that $\operatorname{trace}(\boldsymbol{X})$ is the sum of the entries on the diagonal of $\boldsymbol{X}$.) This is called the *trace inner product* or *Frobenius inner product* or *Hilbert-Schmidt inner product*.

2. $\mathcal{S}$ = zero-mean Gaussian random variables with finite variance,

$$\langle X, Y \rangle = \mathrm{E}[XY]$$

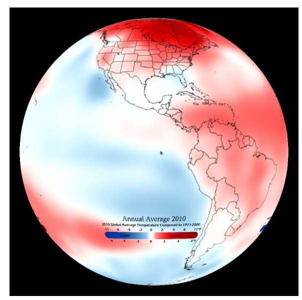
3. $\mathcal{S}$ = differentiable real-valued functions on $\mathbb{R}$,

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \int_{-\infty}^{\infty} x(t)y(t)\ \mathrm{d}t + \int_{-\infty}^{\infty} x'(t)y'(t)\,\mathrm{d}t,$$

   where $x'(t)$ is the derivative of $x(t)$. This is called a *Sobolev inner product.*

4. $\mathcal{S}$ = real-valued functions $x(\theta, \phi)$ on the sphere in $3D$

   Difference in average temperature, 2010 vs. 1971-2000



   A natural (and valid) inner product is

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} x(\theta, \phi)\, y(\theta, \phi) \sin\theta\ \ \mathrm{d}\phi\ \ \mathrm{d}\theta$$

   If we think of θ as being latitude and ϕ as longitude, the $\sin\theta$ can be interpreted as a weight for the size of the "circle" of equal latitude (these get smaller as you go towards the poles).

A linear vector space equipped with an inner product is called an **inner product space**.

Induced norms

A valid inner product induces a valid norm by

$$\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}.$$

(Check this on your own as an exercise.)

It is not hard to see that in $\mathbb{R}^N/\mathbb{C}^N$, the standard inner product induces the ℓ_2 norm.

Properties of induced norms

In addition to the triangle inequality,

$$\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|,$$

induced norms obey some very handy inequalities (note that these are not necessarily true for norms in general, only for norms induced by an inner product):

1. **Cauchy-Schwarz Inequality**

   $$|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \;\leq\; \|\boldsymbol{x}\| \, \|\boldsymbol{y}\|$$

   Equality is achieved above when (and only when) $\boldsymbol{x}$ and $\boldsymbol{y}$ are **colinear**:

   $$\exists\, a \in \mathbb{C} \quad \text{such that} \quad \boldsymbol{y} = a\boldsymbol{x}.$$

2. **Pythagorean Theorem**

   $$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0 \;\Rightarrow\; \|\boldsymbol{x} + \boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2$$

The left-hand side above also implies that $\|\boldsymbol{x}-\boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2$.

3. **Parallelogram Law**

$$\|\boldsymbol{x}+\boldsymbol{y}\|^2 + \|\boldsymbol{x}-\boldsymbol{y}\|^2 = 2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2.$$

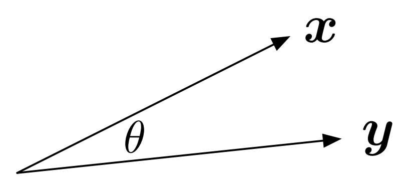
   You can prove this by expanding $\|\boldsymbol{x}+\boldsymbol{y}\|^2 = \langle \boldsymbol{x}+\boldsymbol{y}, \boldsymbol{x}+\boldsymbol{y}\rangle$ and similarly for $\|\boldsymbol{x}-\boldsymbol{y}\|^2$.

4. **Polarization Identity**

$$\operatorname{Re}\{\langle \boldsymbol{x}, \boldsymbol{y}\rangle\} = \frac{\|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{x}-\boldsymbol{y}\|^2}{4}$$

Angles between vectors

In $\mathbb{R}^2$ (and $\mathbb{R}^3$), we are very familiar with the geometrical notion of an angle between two vectors.



We have

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \|\boldsymbol{x}\|\,\|\boldsymbol{y}\|\,\cos\theta$$

Notice that this relationship depends only on norms and inner products. Therefore, we can extend the definition to any inner product space.

Definition: The **angle** between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in an inner product space is

$$\cos\theta = \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{\|\boldsymbol{x}\|\,\|\boldsymbol{y}\|},$$

where the norm is the one induced by the inner product.

Definition: Vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in an inner product space are **orthogonal** to one another if

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0.$$

Example: (Weighted inner product)

$$\mathcal{S} = \mathbb{R}^2, \quad \langle \boldsymbol{x}, \boldsymbol{y} \rangle_Q = \boldsymbol{y}^{\mathrm{T}} \boldsymbol{Q} \boldsymbol{x}, \quad \text{where} \quad \boldsymbol{Q} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

so $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 4x_1y_1 + x_2y_2$. What is the norm induced by this inner product? Draw the unit ball $\mathcal{B}_Q = \{\boldsymbol{x} \in \mathbb{R}^2 \ : \ \|\boldsymbol{x}\|_Q \leq 1\}$.

Find a vector which is orthogonal to $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ under $\langle \cdot, \cdot \rangle_Q$.

Technical Details: Completeness, convergence, and Hilbert Spaces

As we can see from the properties above, inner product spaces have almost all of the geometrical properties of the familiar Euclidean space $\mathbb{R}^3$ (or more generally $\mathbb{R}^N$). In fact, in the coming sections, we will see that any space with an inner product defined (which comes with its induced norm) is directly analogous to Euclidean space.

To make all of this work nicely in infinite dimensions, we need a technical condition on $\mathcal{S}$ called **completeness**. Roughly, this means that there are no points "missing" from the space. More precisely, if we have an infinite sequence of vectors in $\mathcal{S}$ that get closer and closer to one another, then they converge to something in $\mathcal{S}$. Even more precisely, we call an inner product space complete if every Cauchy sequence is a convergent sequence; that is, for every sequence $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots \in \mathcal{S}$ for which

$$\lim_{\min(m,n)\to\infty} \|\boldsymbol{x}_m - \boldsymbol{x}_n\| = 0, \quad \text{will also have} \quad \lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x}^* \in \mathcal{S},$$

where $\|\cdot\|$ is the norm induced by the inner product.

An example of a space which is **not** complete is continuous, bounded functions on $[0,1]$. It is easy to come up with a sequence of functions which are all continuous but converge to a discontinuous function.

All finite dimensional inner product spaces are complete, as is $L_2([a,b])$ and $L_2(\mathbb{R})$ (and in fact, every example of an inner product space we have given above). The former is a basic result in mathematical analysis, the latter is a result from the modern theory of Lebesgue integration. Determining whether or not a space is complete is far outside the scope of this course; it is enough for us to know what this concept means.

An inner product space which is also complete is called a **Hilbert space**. As all of the inner product spaces we will encounter in this course are complete, we will use this descriptor from now on. The Wikipedia pages on these topics are actually pretty good:
http://en.wikipedia.org/wiki/Hilbert_space
http://en.wikipedia.org/wiki/Complete_metric_space

The point of asking that the space be complete is that it gives us confidence in writing expressions like

$$x(t) = \sum_{n=1}^{\infty} \alpha_n \psi_n(t).$$

What is on the left is a sum of an infinite number of terms; the equality above means that as we include more and more terms in this sum, it converges to something which we call $x(t)$. There are different ways we might define convergence, depending on how much of a role we want the order of terms to play in the result. But we say that $\sum_{n=1}^{\infty} \alpha_n \psi_n(t)$, where the $\psi_n(t)$ are in a Hilbert space $\mathcal{S}$, is convergent if there is an $x(t)$ such that

$$\left\| x(t) - \sum_{n=1}^{N} \alpha_n \psi_n(t) \right\| \to 0 \quad \text{as} \quad N \to \infty.$$

If $\mathcal{S}$ is complete, we know that $x(t)$ will also be in $\mathcal{S}$.

Finally, we note that this notion of completeness only depends on having a norm defined. A complete normed linear space (where the norm is not necessarily induced by an inner product) is called a **Banach space**.

Technical Details: Proofs of properties of induced norms

1. **Cauchy-Schwarz**
   Set
   $$\boldsymbol{z} = \boldsymbol{x} - \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{\|\boldsymbol{y}\|^2}\, \boldsymbol{y},$$
   and notice that $\langle \boldsymbol{z}, \boldsymbol{y} \rangle = 0$, since
   $$\langle \boldsymbol{z}, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle - \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{\|\boldsymbol{y}\|^2}\, \|\boldsymbol{y}\|^2 = 0.$$
   We can write $\boldsymbol{x}$ in terms of $\boldsymbol{y}$ and $\boldsymbol{z}$ as
   $$\boldsymbol{x} = \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{\|\boldsymbol{y}\|^2}\, \boldsymbol{y} + \boldsymbol{z},$$
   and since $\boldsymbol{y} \perp \boldsymbol{z}$,
   $$\begin{aligned} \|\boldsymbol{x}\|^2 &= \frac{|\langle \boldsymbol{x}, \boldsymbol{y} \rangle|^2}{\|\boldsymbol{y}\|^4} \|\boldsymbol{y}\|^2 + \|\boldsymbol{z}\|^2 \\ &= \frac{|\langle \boldsymbol{x}, \boldsymbol{y} \rangle|^2}{\|\boldsymbol{y}\|^2} + \|\boldsymbol{z}\|^2. \end{aligned}$$
   Thus
   $$|\langle \boldsymbol{x}, \boldsymbol{y} \rangle|^2 \;=\; \|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^2 - \|\boldsymbol{z}\|^2 \|\boldsymbol{y}\|^2 \;\leq\; \|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^2.$$
   We have equality above if and only if $\boldsymbol{z} = \boldsymbol{0}$. If $\boldsymbol{z} = \boldsymbol{0}$, then $\boldsymbol{x}$ is co-linear with $\boldsymbol{y}$, as
   $$\boldsymbol{x} = \alpha \boldsymbol{y}, \quad \text{with} \;\; \alpha = \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{\|\boldsymbol{y}\|^2}.$$

Conversely, if $\boldsymbol{x} = \alpha\boldsymbol{y}$ for some $\alpha \in \mathbb{C}$, then

$$\boldsymbol{z} = \alpha\boldsymbol{y} - \frac{\alpha\langle\boldsymbol{y},\boldsymbol{y}\rangle}{\|\boldsymbol{y}\|^2}\boldsymbol{y} = \mathbf{0}.$$

2. **Pythagorean Theorem**

$$\begin{aligned}\|\boldsymbol{x}+\boldsymbol{y}\|^2 = \langle\boldsymbol{x}+\boldsymbol{y},\boldsymbol{x}+\boldsymbol{y}\rangle &= \langle\boldsymbol{x},\boldsymbol{x}\rangle + \langle\boldsymbol{x},\boldsymbol{y}\rangle + \langle\boldsymbol{y},\boldsymbol{x}\rangle + \langle\boldsymbol{y},\boldsymbol{y}\rangle \\ &= \|\boldsymbol{x}\|^2 + \langle\boldsymbol{x},\boldsymbol{y}\rangle + \overline{\langle\boldsymbol{x},\boldsymbol{y}\rangle} + \|\boldsymbol{y}\|^2 \\ &= \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 \qquad (\text{since } \langle\boldsymbol{x},\boldsymbol{y}\rangle = 0)\end{aligned}$$

3. **Parallelogram Law**. As above, we have

$$\begin{aligned}\|\boldsymbol{x}+\boldsymbol{y}\|^2 &= \|\boldsymbol{x}\|^2 + \langle\boldsymbol{x},\boldsymbol{y}\rangle + \overline{\langle\boldsymbol{x},\boldsymbol{y}\rangle} + \|\boldsymbol{y}\|^2 \\ &= \|\boldsymbol{x}\|^2 + 2\,\mathrm{Re}\left\{\langle\boldsymbol{x},\boldsymbol{y}\rangle\right\} + \|\boldsymbol{y}\|^2, \\ \|\boldsymbol{x}-\boldsymbol{y}\|^2 &= \|\boldsymbol{x}\|^2 - \langle\boldsymbol{x},\boldsymbol{y}\rangle - \overline{\langle\boldsymbol{x},\boldsymbol{y}\rangle} + \|\boldsymbol{y}\|^2 \\ &= \|\boldsymbol{x}\|^2 - 2\,\mathrm{Re}\left\{\langle\boldsymbol{x},\boldsymbol{y}\rangle\right\} + \|\boldsymbol{y}\|^2,\end{aligned}$$

and so

$$\|\boldsymbol{x}+\boldsymbol{y}\|^2 + \|\boldsymbol{x}-\boldsymbol{y}\|^2 = 2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2.$$

4. **Polarization Identity**. Using the expansions above, we quickly see that

$$\|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{x}-\boldsymbol{y}\|^2 = 4\,\mathrm{Re}\left\{\langle\boldsymbol{x},\boldsymbol{y}\rangle\right\}.$$