Math Camp

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$<$ Course $>$

The Systematic Analysis of Politics

Social Science: systematic analysis of society (Political Science: who gets what, when, and how).

Methodology: Develop and disseminate tools to make inferences about society

- Mathematical models of social world
- Probability and Statistics used across sciences

This class (introduction):

- Math Camp: Develop Tools for Analysis
- Probability theory: systematic model of randomness

Course Goals

First stop in methodology sequence

Big Goal: prepare you to make discoveries about social world

Proximate Goals

1) Mathematical tools to comprehend and use statistical methods
2) Foundation in probability theory/analytic reasoning
3) Practical Computing Tools: `R`
4) Introduction to Logic of Formal Modeling

Course Staff

Me: Justin Grimmer and Avidit Acharya

- Justin: 416 ; Avi 406
- Email: jgrimmer@stanford.edu ; avidit@stanford.edu
- Cell: 617-710-6803
- Office Hours: I'm generally here all the time (9am to 5pm), just stop by [but if you need to see me with 100% probability, schedule a visit];

Extra Info

- Zuhad Hai and Jesse Yoder
- Github for class: `github/justingrimmer/Math18`

Prerequisites

No Formal Prerequisites
BUT

- Successful students will know differential and integral calculus
    1) Limits (intuitive)
    2) Derivatives (tangent lines, differentiation rules)
    3) Integrals (fundamental theorem of calculus/antidifferentiation rules
- We are here to help
    - No mystery to learning math: just hard work
    - Political science increasingly requires math
    - Empirical: calculus and linear algebra
    - Quantitative Methodologist: Real Analysis and Grad level statistics
    - Formal Theory: Real Analysis (through measure theory), Topology

Evaluation

You're not taking this class for a grade $\rightsquigarrow$ that shouldn't matter:

- Math Camp Exam

Grad School Irony Or: How I Learned to Stop Worrying and Love C's

- Grades no longer matter
- Learn as much material as possible
- If you truly only care about learning material, you'll get amazing grades

Homework

Math camp: assigned daily $\rightsquigarrow$ Mechanics of solving problems
Lab Assignment: Twice weekly assignments, help you develop computational and mathematical skills.

Computing/Homeworks

Greatest scientific discovery of 20th Century:

Powerful personal computer (standardize science)

1956: $10,000 megabyte

2015: $<<<$ $ 0.0001 per megabyte

Statistical Computing: R

- R: Scripting language
- Flexible, Cutting Edge Software, great visualization tools and makes learning other programs easier
- More start up costs than STATA, but more payoff

Paper writeup: LaTeX

- Hard to write equations in Word:
- Relatively easy in LaTeX

$$f(x) = \frac{\exp(-\frac{(x-\mu)^2}{2\sigma^2})}{\sqrt{2\pi\sigma^2}}$$

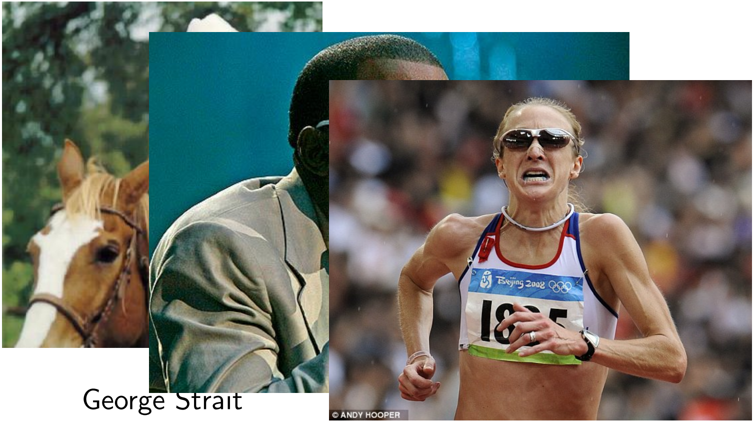
- Tables/Figures/General type/Nice Presentations setting: easier in LaTeX
- If you use start using LaTeX, you'll soon love it

Course Books

1) Simon, Carl and Blume, Lawrence (SB). Mathematics for Economists.
2) Bertsekas, Dimitri P. and Tsitsiklis, John (BT) Introduction to Probability Theory (second edition)

Life in Graduate School/Academy

Three part mixture:

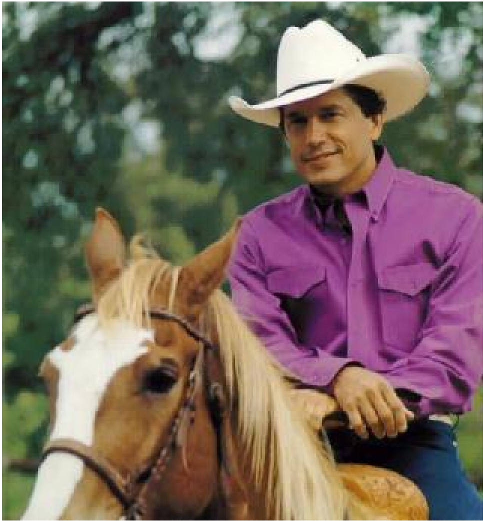


George Strait

Kanye West

Paula Radcliffe

$\frac{1}{3}$ George Strait



- Amarillo By Morning [Terry Stafford 1973, George Strait 1982]
- Ostensibly: song about rodeo cowboys
- Really: song about being academic
- "I ain't got a dime/but what I got is mine/I ain't rich/ but lord I'm free"
- Academics ain't rich (counterfactually)
- But (lord) we're free
- If you're good at methods, you'll be more rich [in expectation] and equally free

$\frac{1}{3}$ Kayne West

- Deal with explicit criticism (part of Hip/Hop culture)
- On masterpiece album My Beautiful Dark Twisted Fantasy
- "Screams from the haters, got a nice ring to it/I guess every superhero needs his theme music"
- Kid Cudi: "These motherf**kers can't fathom the wizadry"
- Academics: intense criticism of ideas
- Very rarely will you be told you're doing a great job
- Self confidence: believe in work

$\frac{1}{3}$ Paula Radcliffe

"It's not a sprint, it's a marathon".

- World class distance running: it is hard
- But not for the obvious reasons
- Marathon: 5:10 minute mile, for 26.2 miles.
- How to train?
  - Old way: get in shape (run far) rely on adrenaline in race
  - Now: races more tactical and agonizing
  - Need to prepare for agony
- Mantra: sustained agony
- Graduate School/Academics: Sustained Agony

$\frac{1}{3}$ Paula Radcliffe

Not crazy to work 30-40 hours on methods alone

- Methods $\rightsquigarrow$ skills use for rest of career
- Methods $\rightsquigarrow$ often takes deep thinking, practice

TAKE BREAKS!

- Regular physical activity $\rightsquigarrow$ improve focus
- Time away from lab $\rightsquigarrow$ more productive when back

$\frac{1}{3}$ Paula Radcliffe

Why work so hard?

- You are all smart Really Smart Mother-in-law brags about you smart
- Everyone entering graduate school at top programs this fall
- Success: work
- Treat grad school like a job
- Who gets ahead? who gets the most work done on the smartest ideas

Preliminaries

What can you learn in a math camp?

1) Introduction to more sophisticated mathematics (notation)
2) Getting acquainted with proof techniques and proofs
3) I'm going to introduce ideas/example problems common in research that will help with your seminar
4) This will not substitute for a richer math background and we won't expect it to

Do not let yourself get lost.

If at. any. point. you have a question please ask !

Smartest people ask the most questions!

Let's get to work

Simple Logical Statements

Sets

A set is a collection of objects.

$$
\begin{aligned}
A &= \{1, 2, 3\} \\
B &= \{4, 5, 6\} \\
C &= \{\text{First year cohort}\} \\
D &= \{\text{Stanford University Faculty}\}
\end{aligned}
$$

Definition

If A is a set, we say that x is an element of A by writing, $x \in A$. If x is not an element of A then, we write $x \notin A$.

- $1 \in \{1, 2, 3\}$
- $4 \in \{4, 5, 6\}$
- Jesse $\notin$ {First year cohort}
- Justin $\in$ {Stanford Faculty}

Why Care?

- Sets are necessary for probability theory
- Defining set is equivalent ot choosing population of interest (usually)

Definition

If A and B are sets, then we say that $A = B$ if, for all $x \in A$ then $x \in B$ and for all $y \in B$ then $y \in A$.

- Test to determine equality:
    - Take all elements of A, see if in B
    - Take all elements of B, see if in A

Definition

If A and B are sets, then we say that $A \subset B$ is, for all $x \in A$, then $x \in B$.

Difference between definitions?

Theorem

Let A and B be sets. If $A = B$ then $A \subset B$ and $B \subset A$

Proof.

Suppose $A = B$. By definition, if $x \in A$ then $x \in B$. So $A \subset B$. Again, by definition, if $y \in B$ then $y \in A$. So $B \subset A$. □

Theorem

Let A and B be sets. If $A \subset B$ and $B \subset A$ then $A = B$

Proof.

Suppose $A \subset B$ and that $B \subset A$. For all $x \in A$, then $x \in B$. And for all $y \in B$, $y \in A$. Or, every element in A is in B and each element of B is in A. $A = B$. □

Theorem

Let A and B be sets. Then $A = B$ if and only if $A \subset B$ and $B \subset A$.

Proof.

$\Rightarrow$ Suppose $A = B$. By definition, if $x \in A$, $x \in B$. So $A \subset B$. Again, by definition, if $y \in B$ then $y \in A$. So $B \subset A$.
$\Leftarrow$ Suppose $A \subset B$ and that $B \subset A$. For all $x \in A$, then $x \in B$. And for all $y \in B$, $y \in A$. Or, every element in A is in B and each element of B is in A. $A = B$. $\square$

When a proof says if and only if it is showing two things.

- If or that a condition is sufficient
- Only If or that a condition is necessary

Example of sufficient, but not necessary

- If candidate wins the electoral college, then president (can be president through vote of House too)

Example of necessary, but not sufficient

- Only if a candidate is older than 35 can s/he be president (but clearly not sufficient)

Contradiction

- Many ways to prove the same theorem.
- Contradiction: assume theorem is false, show that this leads to logical contradiction
- Indirect proof: setting up proof hardest part

Theorem

Let A and B be sets. Then $A = B$ if and only if $A \subset B$ and $B \subset A$.

Proof.

$\Rightarrow$ Suppose $A = B$. By definition, if $x \in A$, $x \in B$. So $A \subset B$. Again, by definition, if $y \in B$ then $y \in A$. So $B \subset A$.
$\Leftarrow$ Suppose $A \subset B$ and that $B \subset A$. Now, by way of contradiction, suppose that $A \neq B$. $A \neq B$ only if there is $x \in A$ and $x \notin B$ or if $y \in B$ and $y \notin A$. But then, either $A \not\subset B$ or $B \not\subset A$, contradicting our initial assumption. $\square$

Set Builder Notation

- Some famous sets
  - $J = \{1, 2, 3, \ldots\}$
  - $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots, \}$
  - $\Re$ = real numbers (more to come about this)
- Use set builder notation to identify subsets
  - $[a, b] = \{x : x \in \Re \text{ and } a \leq x \leq b\}$
  - $(a, b] = \{x : x \in \Re \text{ and } a < x \leq b\}$
  - $[a, b) = \{x : x \in \Re \text{ and } a \leq x < b\}$
  - $(a, b) = \{x : x \in \Re \text{ and } a < x < b\}$
  - $\emptyset$

Set Operations

We can build new sets with set operations.

Definition

Suppose A and B are sets. Define the Union of sets A and B as the new set that contains all elements in set A or in set B. In notation,

$$\begin{aligned} C &= A \cup B \\ &= \{x : x \in A \text{ or } x \in B\} \end{aligned}$$

- $A = \{1, 2, 3\}, B = \{3, 4, 5\}$, then $C = A \cup B = \{1, 2, 3, 4, 5\}$
- $D = \{\text{First Year Cohort}\}, E = \{\text{Me}\}$, then $F = D \cup E = \{\text{First Year Cohort, ME}\}$

Set Operations

Definition

Suppose A and B are sets. Define the Intersection of sets A and B as the new that contains all elements in set A and set B. In notation,

$$\begin{aligned} C &= A \cap B \\ &= \{x : x \in A \textit{ and } x \in B\} \end{aligned}$$

- $A = \{1, 2, 3\}, B = \{3, 4, 5\}$, then, $C = A \cap B = \{3\}$
- $D = \{\text{First Year Cohort}\}, E = \{\text{Me}\}$, then $F = D \cap E = \emptyset$

Some Facts about Sets (No Venn Diagrams!!!)

1) $A \cap B = B \cap A$
2) $A \cup B = B \cup A$
3) $(A \cap B) \cap C = A \cap (B \cap C)$
4) $(A \cup B) \cup C = A \cup (B \cup C)$
5) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
6) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Break into groups, derive for the remaining facts

Ordered Pair

You've seen an ordered pair before,

$$(a, b)$$

Definition

Suppose we have two sets, A and B. Define the Cartesian product of A and B, $A \times B$ as the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. In other words,

$$A \times B \quad = \quad \{(a, b) : a \in A \text{ and } b \in B\}$$

Example:
$A = \{1, 2\}$ and $B = \{3, 4\}$, then,
$A \times B = \{(1, 3); (1, 4); (2, 3); (2, 4)\}$

Function

Start with general and move to specific— (abstract just takes time to get acquainted)

Definition

A relation is a set of ordered pairs. A function F is a relation such that,

$$(x, y) \in F \quad ; \quad (x, z) \in F \Rightarrow y = z$$

We will commonly write a function as $F(x)$, where $x \in$ Domain F and $F(x) \in$ Codomain F. It is common to see people write,

$$F : A \to B$$

where A is domain and B is codomain

Examples

- $F(x) = x$
- $F(x) = x^2$
- $F(x) = \sqrt{x}$

R Computing Language

- We're going to use R throughout the course
- R as calculator :

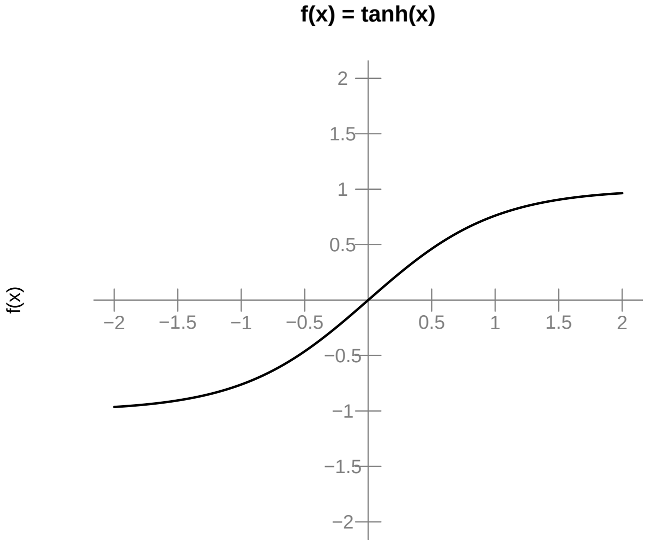
```
> 1 + 1
[1] 2
> 'Hello World'
[1] ''Hello World"
```

- `object<- 2 ## assign numbers to objects`
- R has functions defined, we can define them to objects as well

```
first.func<- function(x) {
out<- 2*x
return(out) }
```

```
first.func(2)
[1] 4
```

Plotting Functions




```
x<- seq(-2, 2,
len=1000)
tanhx<- tanh(x)
plot(tanhx~x)
```

Exponents, Logarithms, and All That

$$
\begin{aligned}
f(x) &= 2^x \\
g(x) &= e^x
\end{aligned}
$$

Some rules of exponents remember a could equal e

$$
\begin{aligned}
a^x \times a^y &= a^{x+y} \\
(a^x)^y &= a^{x \times y} \\
\frac{a^x}{a^y} &= a^{x-y} \\
\frac{1}{a^x} &= a^{-x} \\
a^x \times b^x &= (a \times b)^x \\
a^0 &= 1 \\
a^1 &= a \\
1^x &= 1
\end{aligned}
$$

Exponents, Logarithms, and All That

Logaritm log is a class of functions.

- $\log_e z =$ what number x solves $e^x = z$.
- We'll call $\log_e$ natural logarithm. And we'll assume $\log_e = \log$
- $\log e = 1$ (because $e^1 = e$)
- $\log_{10} 1000 = 3$ (because $10^3 = 1000$)

Some rules of logarithms

- $\log(a \times b) = \log(a) + \log(b)$ (!!!!!!)
- $\log(\frac{a}{b}) = \log(a) - \log(b)$
- $\log(a^b) = b \log(a)$
- $\log(1) = 0$
- $\log(e) = 1$

Properties of Functions

Two important properties of functions

Definition

A function $f : A \rightarrow B$ is 1-1 (one-to-one, or injective) if for all $y \in A$ and $z \in A$ in Domain, $f(y) = f(z)$ implies $y = z$. In other words, preserves distinctiveness.

- $f(x) = x$
- $f(x) = x^2$

Definition

A function $f : A \rightarrow B$ is onto (surjective) if for all $b \in B$ there exists ($\exists$) $a \in A$ such that $f(a) = b$.

- $f : \{\ldots, -2, -1, 0, 1, 2, \ldots\} \rightarrow \{0, 1, 2, \ldots\}$ and $f(x) = |x|$. onto, but not 1-1.
- $f : R \rightarrow R$ $f(x) = x$. Onto and 1-1, bijective

Composite Functions

Definition

Suppose $f : A \rightarrow B$ and $g : B \rightarrow C$. Then, define,

$$g \circ f = g(f(x))$$

- $f(x) = x$, $g(x) = x^2$. Then $g \circ f = x^2$.
- $f(x) = \sqrt{x}$, $g(x) = e^x$. Then $g \circ f = e^{\sqrt{x}}$.
- $f(x) = sin(x)$, $g(x) = |x|$. Then $g \circ f = |sin(x)|$.

Inverse Function

Definition

Suppose a function f is 1-1. Then we'll define f^{-1} as its inverse if,

$$f^{-1}(f(x)) = x$$

Why do we need 1-1?

Induction

Well Ordering Principle Every non-empty set J has a smallest number

Theorem

If $P(n)$ is a statement containing the variable n such that

i. *$P(1)$ is a true statement, and*

ii. *for each $k \in 1, 2, 3, 4, \ldots, n, \ldots$ if $P(k)$ is true then $P(k+1)$ is true*

then $P(n)$ is true for all $n \in 1, 2, 3, 4, \ldots, n, \ldots$

Induction and Contradiction

We'll use contradiction and well ordering to prove that induction works.

Proof.

Suppose $P(n)$ is some statement about the variable n and that

i. $P(1)$ is true

ii. If $P(k)$ is true then $P(k+1)$ is true.

Now suppose, by way of contradiction that there exists N such that $P(N)$ is false. This implies that

$$S = \{x : P(x) \text{ is not true } \}$$

By well ordering principle, there is smallest member of S, call it n_0. By i. we know that $n_0 > 1$. Further, because n_0 is smallest member of S, then $P(n_0)$ is false, but $P(n_0 - 1)$ is true. But now we have a problem, because if $P(n_0 - 1)$ is true, then $P(n_0)$ is also true. This implies that there is no smallest element of S. CONTRADICTION $\square$

Summing N numbers

Induction is a useful proof technique.

Theorem

$\sum_{i=1}^{N} i = 1 + 2 + 3 + 4 + \ldots + N = \frac{N(N+1)}{2}$

Two conditions to show:

i. $\sum_{i=1}^{1} i = 1$ and $\frac{1(1+1)}{2} = 1$

Summing N numbers

ii. Suppose true N. Then, for $N+1$ we have,

$$\begin{aligned}
\sum_{i=1}^{N+1} i &= \sum_{i=1}^{N} i + (N+1) \\
&= \frac{N(N+1)}{2} + \frac{2(N+1)}{2} \\
&= \frac{(N+1)(N+2)}{2} \\
&= \frac{(N+1)((N+1)+1)}{2}
\end{aligned}$$

Conditions of induction met. Therefore, proof complete

Very Simple R Code

Finite, Countable, and Uncountable

Three sizes of sets

1) A set, X is finite if there is a bijective function from $\{1, 2, 3, \ldots, n\}$ to X.
2) A set X is countably infinite if there is a bijective function from $\{1, 2, 3, 4, \ldots, \}$ to X.
3) A set X is uncountably infinite if it is not countable

The Real numbers are uncountably infinite

Recap

We've covered a lot.
PLEASE don't worry—we're here to help!

1) Sets + Operations
2) Functions
3) Contradiction, Induction, and direct proofs

Tomorrow:

- Convergence of sequences
- Limits
- Continuity
- Derivatives