Math Camp - Homework 3 Solutions

1.

(a) Sketch the graph of a function (any function you like, no need to specify a functional form) that is continuous on [0,3] and has the following properties: an absolute maximum at 0, an absolute minimum at 3, a local minimum at 1 and a local maximum at 2.

(b) Do the same for another function with the following properties: 2 is a critical number, but there is no local minimum and no local maximum.

Solution There are many, many (in fact, uncountably infinitely many) correct answers to this question, but they will all have a few characteristics in common. For the first function, the highest value of the function must be produced by $x = 0$, and the lowest value of the function must be produced by $x = 3$. Furthermore, the graph must change from moving up to moving down at $x = 2$ and from moving down to moving up at $x = 1$. For the second graph, there simply must be a saddle point at $x = 2$ - for $x = 2$ to be a critical point, it must be a local minimum, a local maximum, or a saddle point, but we've specified that there are no local minima and no local maxima - and the graph must not change from increasing to decreasing or vice versa at any point.

2. Find the critical values of these functions:

(a) $f(x) = 5x^{3/2} - 4x$

(b) $s(t) = 3t^4 + 4t^3 - 6t^2$

(c) $f(r) = \frac{r}{r^2+1}$

(d) $h(x) = xlog(x)$

Solutions

(a) $x = \frac{64}{225}$

First, find the derivative of the function.

$$f'(x) = \frac{3}{2}(5)x^{3/2-1} - 4 = \frac{15}{2}x^{1/2} - 4 = \frac{15\sqrt{x}}{2} - 4$$

If we set this derivative equal to 0 and solve, we get the following critical point:

$$\begin{aligned}
\frac{15\sqrt{x}}{2} - 4 &= 0 \\
\frac{15\sqrt{x}}{2} &= 4 \\
15\sqrt{x} &= 8 \\
\sqrt{x} &= \frac{8}{15} \\
x &= \left(\frac{8}{15}\right)^2 \\
x &= \frac{8^2}{15^2} \\
x &= \frac{64}{225}
\end{aligned}$$

(b) The derivative of $s(t)$ requires simple power rule:

$$s'(t) = 4(3)t^{4-1} + 3(4)t^{3-1} - 2(6)t^{2-1} = 12t^3 + 12t^2 - 12t = 12t(t^2 + t - 1)$$

If we set this equal to zero, we immediately see that $t = 0$ is a critical point. However, we cannot "eyeball" if/where $(t^2+t-1) = 0$. For that, we will need to use the quadratic formula. In case you don't remember, the quadratic formula helps us find the roots of a quadratic function – the points at which the function equals 0. Think of a generic quadratic equation, $f(x) = ax^2 + bx + c$, where a, b, c are the coefficients or constants to each term. Then,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In our case with $t^2 + t - 1$, $a = 1, b = 1, c = -1$. Let's plug these into the formula.

$$\begin{aligned}
t &= \frac{-1 \pm \sqrt{1^2 - 4(1)(-1}}{2(1)} \\
&= \frac{-1 \pm \sqrt{1+4}}{2} \\
&= \frac{-1 \pm \sqrt{5}}{2} \\
&= \frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}
\end{aligned}$$

Great. We have three critical points: $t = \frac{-1-\sqrt{5}}{2}, 0, \frac{-1+\sqrt{5}}{2}$.

(c) r $= \pm 1$

We can use the quotient rule to find the derivative.

$$f'(r) = \frac{(r)'(r^2+1) - r(r^2+1)'}{(r^2+1)^2} = \frac{(1)(r^2+1) - r(2r)}{(r^2+1)^2} = \frac{r^2+1-2r^2}{(r^2+1)^2} = \frac{1-r^2}{(r^2+1)^2}$$

To find the critical values, set the numerator of $s'(r)$ equal to 0. (Remember that since we are dealing with a fraction, we only need to find where the numerator equals 0. Also, given the nature of the denominator, we don't have to worry about whether the function is ever undefined.)

$$1 - r^2 = 0 \Longrightarrow r^2 = 1 \Longrightarrow r = \pm 1$$

(d) $x = \frac{1}{e}$

The function requires product rule to differentiate.

$$h'(x) = x \cdot (\log x)' + x' \cdot \log x = x \cdot \frac{1}{x} + 1 \cdot \log x = 1 + \log x$$

Now set this derivative equal to zero and solve.

$$\begin{aligned} 1 + \log x &= 0 \\ \log x &= -1 \\ x &= e^{-1} \\ x &= \frac{1}{e} \end{aligned}$$

3. Find the absolute minimum and absolute maximum values of the functions on the given interval:

(a) $f(x) = 3x^2 - 12x + 5, [0, 3]$

(b) $f(t) = t\sqrt{4 - t^2}, [-1, 4]$

(c) $s(x) = x - log(x), [1/2, 2]$

(d) $h(p) = 1 - e^{-p}, [0, 1000]$

Solutions

(a) First, we must identify the critical points; to do this, we must find $f'(x)$. By our rules of derivation, $f'(x) = 6x - 12$. The critical points will then be where $f'(x) = 6x - 12 = 0$. Solving for x then gives $x = 2$. As this is a continuous function over the given interval, the absolute minimum and absolute maximum values must be at critical points or the endpoints of the interval. In this case, the set of candidate values of x is then $\{0, 2, 3\}$.

Evaluating the function at these points gives $f(0) = 5$, $f(2) = -7$, and $f(3) = -4$. So the absolute maximum occurs where $x = 0, f(x) = 5$ and the absolute minimum occurs where $x = 2, f(x) = -7$. It is a useful check of our work to make sure that $x = 2$ gives a local minimum - after all, since $x = 2$ is not one of the endpoints, in order to be the absolute minimum it must be a local minimum as well. To see whether it is a local minimum, we evaluate $f''(x)$ where $x = 2$. $f''(x) = (6x - 12)' = 6$, so $f''(2) = 6 > 0$. Since the second derivative is positive, the first derivative must be increasing at $x = 2$ - in other words, it must be moving from negative to positive - and $x = 2$ is indeed a local minimum.

(b) As before, we must first identify critical points. To find $f'(t)$, we must use both the power rule and the chain rule. The power rule tells us that $f'(t) = t(\sqrt{4 - t^2})' + (t)'\sqrt{4 - t^2}$. By the chain rule, the derivative of $\sqrt{4 - t^2}$ is $\frac{1}{2}(4 - t^2)^{-\frac{1}{2}}(4 - t^2)' = \frac{1}{2}(4 - t^2)^{-\frac{1}{2}}(-2t) = -t(4 - t^2)^{-\frac{1}{2}}$. So the derivative of $f(t)$ is $f'(t) = -t^2(4 - t^2)^{-\frac{1}{2}} + \sqrt{4 - t^2}$. Setting this equal to zero and adding $t^2(4 - t^2)^{-\frac{1}{2}}$ to both sides gives $t^2(4 - t^2)^{-\frac{1}{2}} = \sqrt{(4 - t^2)}$. Multiplying both sides by $\sqrt{4 - t^2}$ then gives $t^2 = 4 - t^2$; solving for t then gives $t = \pm\sqrt{2}$. Of course, $-\sqrt{2}$ is outside of our domain, so we can ignore it and instead only investigate $\sqrt{2}$.

To see something about the behavior of the function at this point, we have to take the second derivative. Omitting the steps involved (it would be good practice to see if you can get the same answer!), $f''(t) = -t^3(4 - t^2)^{-\frac{3}{2}} - 3t(4 - t^2)^{-\frac{1}{2}}$. At $t = \sqrt{2}$, this will be negative, so $t = \sqrt{2}$ produces a local maximum.

Note that this function is not actually defined over the entire interval provided - if $t > 2$, then we'd have to take the square root of a negative number. So the *effective* endpoints of the interval, for the purpose of finding the absolute minimum and maximum, are -1 and 2. So our candidates for minimum and maximum are where $t \in \{-1, \sqrt{2}, 2\}$. Plugging in, we get $f(-1) = -\sqrt{3}$, $f(\sqrt{2}) = 2$, and $f(2) = 0$. So the absolute maximum occurs where $t = \sqrt{2}$, and the absolute minimum occurs at $t = -1$.

(c) This one should be less painful than the previous problem. First, we need to find $s'(x)$. Remember, the derivative of the natural log of x is just $\frac{1}{x}$! So $s'(x) = 1 - \frac{1}{x}$. Setting this equal to 0, we get $0 = 1 - \frac{1}{x}$, which means $1 = \frac{1}{x}$, or $x = 1$. Let's check whether this is a minimum or a maximum. $s''(x) = \frac{1}{x^2}$, so $s''(1) = 1 > 0$. Since the second derivative is positive at $x = 1$, $x = 1$ should produce a local minimum. Plugging in $x = 1$ and the endpoints of the interval, we get $s(1/2) = 1.19$, $s(1) = 1$, and $s(2) = 1.31$. $x = 1$ is therefore not only produces a local minimum, but it produces the absolute minimum. The absolute maximum occurs at one of the endpoints, where $x = 2$.

(d) The procedure should be getting familiar by now. First, we find the derivative of $h(p)$, giving us $h'(p) = e^{-p}$. However, we can make an interesting observation when we set this equal to 0, namely that e^{-p} never equals 0! Its limit as p goes to infinity is 0, but it is not 0 for any finite p, let alone one in our interval. So we have no critical points, and the endpoints will give us the absolute minimum and maximum over the interval.

Plugging in, we get $h(0) = 0$ and $h(1000)$ is very close to 1, so over our interval, $p = 0$ produces the absolute minimum and $p = 1000$ produces the absolute maximum.

4. Prove that the function $f(x) = x^5+x^3+x+1$ has no local maximum and no local minimum

Solution: This proof might seem hard to approach, so let's just see what happens when we try to find a local minimum or maximum. First, as usual, we have to find the derivative, and we find that $f'(x) = 5x^4+3x^2+1$. Next, we have to set this equal to zero and solve for x. After looking at the equation $5x^4 + 3x^2 + 1 = 0$, though, we might make an important observation - namely, that this has no solutions! There are two ways we could show this fact. First, we could create a variable $y = x^2$, rewrite the equation as $5y^2+3y+1 = 0$, and then observe that the quadratic equation gives us no solutions. Second, and perhaps more elegantly, we can observe that $x^4 \geq 0$ and $x^2 \geq 0$ for all x. Therefore $5x^4 + 3x^2 + 1 \geq 5(0) + 3(0) + 1 = 1$. So $f'(x) \geq 1 > 0$ for all x. Thus, we see that the derivative never equals zero, and the function has no critical points. But all local maxima and local minima occur at critical points, so the function cannot have a local maximum or local minimum.

5. Does a continuous, differentiable function exist on [0,2] such that f(0) = -1, f(2) = 4, and f'(x) $\leq$ 2 $\forall$ x?
Use the mean value theorem to explain your answer.

Solution:

First we set up the mean value theorem which states that, if a function is continuous and differentiable over some interval, then a c exists such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

We plug in the values given by the problem and find, $f'(c) = \frac{f(2)-f(0)}{2-0} = \frac{4--1}{2} = \frac{5}{2}$.

The problem states that the derivative of the function is less than or equal to 2 over this entire interval, but the Mean Value theorem tell us that that the derivative must equal 2.5 at some point. So by demonstrating this contradiction, we've shown that the earlier values could not have come from a continuous, differentiable function.

6. A friend shows you this graph of a function g(x).

Which of the following could be a graph of g'(x)? For each graph explain why or why not it might be the derivative of g(x).

Solution:

A doesn't work because it is negative and the function we observe is increasing in x. B is constant so this also won't work, the function we observe gets larger at an increasing, not constant rate. C seems to be a plausible candidate because an upward sloping derivative would map to the behavior of the function we observe, that g(x) gets large at an increasing rate. D does not work because it suggests the function would need to be decreasing over some interval and because, when we refer back to g(x), there doesn't seem to be any local minimum, maximum or a saddle point despite the graph in D crossing 0.

What about if figure 3 was the graph of g(x)? Which of the graphs might potentially be the derivative of g(x) then?

Again, A doesn't work because it is negative and the function we observe is increasing in x. B seems to be plausible as the derivative, since g(x) appears to increase at a constant rate, its derivative should be flat and greater than 0. C won't work because the slop of g(x) is constant and does not increase in x. D doesn't work, again because it suggests the function would need to be decreasing at some point over the interval we observe.

7.

Suppose we were interested in learning about how years of schooling affect the probability that a person turns out to vote. To simplify things, let's say we just have one observation of each variable. Let Y be our single observation of the dependent variable (whether or not a person turns out to vote) and X be our single observation of the independent variable, (the number of years of education that same person has). We believe that the process used to generate our data takes the following form:

$$Y = \beta X + \epsilon$$

where ϵ is an error term. We include this error term because we think random occurrences in the world will make our model produce estimates that are slightly wrong sometimes, but we believe that on average, this model accurately relates X to Y. We observe the values of X and Y, but what about β? How do we know the value of β that best approximates this relationship, i.e., what's the slope of this line?

There are different criteria we could use, but a popular choice is the method of least squares. In this process, we solve for the value of β that minimizes the sum of squared errors in our data. Using the tools of minimization we've been practicing, find the value of β that minimizes this quantity. (Hint: In this case there is only one observation, so the sum of squared errors is equal to the single error squared.)

Solution:

First we must identify the quantity we are trying to minimize, the sum of squared errors. Using some simple algebra, we get:

$$\begin{aligned} Y &= \beta X + \epsilon \\ Y - \beta X &= \epsilon \\ (Y - \beta X)^2 &= \epsilon^2 \end{aligned}$$

If we have many data points, there would be a summation sign in front of both sides of this equation, but since there is only one data point here, we have the quantity of interest. We want to find the value of β that minimizes $(Y - \beta X)^2$, the squared error. Let's call this function $f(\beta)$.

The first step to minimizing a function is finding its first derivative with respect to the variable of interest, which is β in this case. For that we will need the chain rule, since this is a nested function. Define $f(\beta)$ as $h(g(\beta))$, where $h(u) = u^2$ and $g(u) = (Y - uX)$ for any u. So, by the chain rule,

$$\begin{aligned} f'(\beta) &= h'(g(\beta)) * g'(\beta) \\ &= 2(Y - \beta X) * -X \\ &= -2X(Y - \beta X) \end{aligned}$$

With our first derivative in hand, we need to solve for β^*, the value of β at which $f'(\beta) = 0$. Again, some algebra:

$$
\begin{aligned}
f'(\beta) = -2X(Y - \beta^* X) &= 0 \\
-2XY + 2X^2\beta^* &= 0 \\
-XY + X^2\beta^* &= 0 \\
X^2\beta^* &= XY \\
\beta^* &= \frac{XY}{X^2} \\
\beta^* &= \frac{Y}{X}
\end{aligned}
$$

Is this a minimum or a maximum? Remember, the first derivative will equal 0 in both cases. To find out, we perform the second derivative test.

$$
\begin{aligned}
f''(\beta) &= \frac{df'(\beta)}{d\beta} \\
&= \frac{d}{d\beta} - 2X(Y - \beta X) \\
&= \frac{d}{d\beta} - 2XY + 2X^2\beta \\
&= 2X^2
\end{aligned}
$$

We know that the quantity $2X^2 > 0$, so the second derivative is positive, so we have found a minimum. We have set no bounds on the domain, and there is only one minimum (i.e. only one solution for β^*), so we are done. $\frac{Y}{X}$ is the value of β that minimizes the squared error. For any other value of β, the squared distance between the true outcome Y, and the predicted outcome, βX, would be larger, which means our model would not fit as well.