Math Camp

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Where we're at

- Conditional Probability/Bayes' Rule
- Today: Random Variables
- Probability Mass Functions
- Expectation, Variance
- Famous Discrete Random Variables
- A Brief Introduction to Markov Chains

Random Variable: Intuition

Recall the three parts of our probability model

- Sample Space
- Events
- Probability

Often, we are interested in some function of the sample space

- Number of incumbents who win
- An indicator whether a country defaults on loans (1 if Default, 0 otherwise)
- Number of casualties in a war (rather than all outcomes of casualties)

Random variables: functions defined on the sample space

Definition: Random Variable

Definition

Random Variable: A Random variable X is a function from the sample space to real numbers. In notation,

$$X : \textit{Sample Space} \rightarrow \mathcal{R}$$

- X's domain are all outcomes (Sample Space)
- X's range is the Real line (or some subset of it)
- Because X is defined on outcomes, makes sense to write $p(X)$ (we'll talk about this soon)

Example

Treatment assignment:

- Suppose we have 3 units, flipping fair coin ($\frac{1}{2}$) to assign each unit
- Assign to T =Treatment or C =control
- X = Number of units received treatment

Defining the function:

$$X = \begin{cases} 0 \text{ if } (C, C, C) \\ 1 \text{ if } (T, C, C) \text{ or } (C, T, C) \text{ or } (C, C, T) \\ 2 \text{ if } (T, T, C) \text{ or } (T, C, T) \text{ or } (C, T, T) \\ 3 \text{ if } (T, T, T) \end{cases} .$$

In other words,

$$\begin{aligned} X((C, C, C)) &= 0 \\ X((T, C, C)) &= 1 \\ X((T, C, T)) &= 2 \\ X((T, T, T)) &= 3 \end{aligned}$$

Another Example

X = Number of Calls into congressional office in some period p

- $X(c) = c$

Outcome of Election

- Define v as the proportion of vote the candidate receives
- Define $X = 1$ if $v > 0.50$
- Define $X = 0$ if $v < 0.50$

For example, if $v = 0.48$, then $X(v) = 0$

Big Question: How do we compute P(X=1), P(X=0), etc?

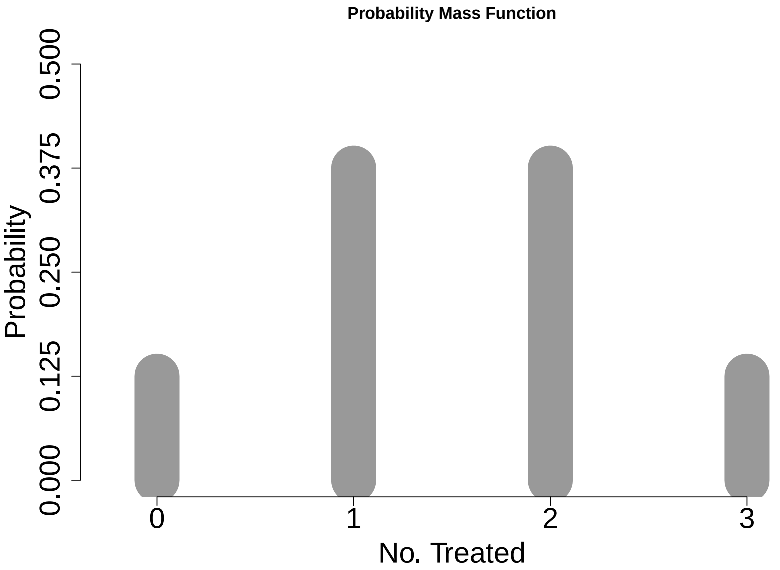
Probability Mass Function: Intuition

Go back to our experiment example–probability comes from probability of outcomes
$P(C, T, C) = P(C)P(T)P(C) = \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{8}$
That's true for all outcomes.

$$\begin{aligned}
p(X = 0) &= P(C, C, C) = \frac{1}{8} \\
p(X = 1) &= P(T, C, C) + P(C, T, C) + P(C, C, T) = \frac{3}{8} \\
p(X = 2) &= P(T, T, C) + P(T, C, T) + P(C, T, T) = \frac{3}{8} \\
p(X = 3) &= P(T, T, T) = \frac{1}{8}
\end{aligned}$$

$p(X = a) = 0$, for all $a \notin (0, 1, 2, 3)$

Probability Mass Function: Intuition



Probability Mass Function: Intuition

Consider outcome of election:

- $X(v) = 1$ if $v > 0.5$ otherwise $X(v) = 0$
- $P(X = 1)$ then is equal to $P(v > 0.5)$

Probability Mass Function

If X is defined on an outcome space that is discrete (countable), we'll call it discrete.

Definition

Probability Mass Function: For a discrete random variable X, define the probability mass function $p(x)$ as

$$p(x) = P(X = x)$$

Probability Mass Function: Example 2

Topics: distinct concepts (war in Afghanistan, national debt, fire department grants)

Mathematically: Probability Mass Function on Words Probability of using word, when discussing a topic

Suppose we have a set of words:

(afghanistan, fire, department, soldier, troop, war, grant)

Topic 1 (say, war):

P(afghanistan) = 0.3; P(fire) = 0.0001; P(department) = 0.0001; P(soldier) = 0.2; P(troop) = 0.2; P(war)=0.2997; P(grant)=0.0001

Topic 2 (say, fire departments):

P(afghanistan) = 0.0001; P(fire) = 0.3; P(department) = 0.2; P(soldier) = 0.0001; P(troop) = 0.0001; P(war)=0.0001; P(grant)=0.2997

Topic Models: take a set of documents and estimate topics.

Definition

Cumulative Mass (distribution) Function: For a random variable X, define the cumulative mass function $F(x)$ as,

$$F(x) = P(X \leq x)$$

- Characterizes how probability cumulates as X gets larger
- $F(x) \in [0, 1]$
- $F(x)$ is non-decreasing

Cumulative Mass Function: Example

Consider the three person experiment. $P(T) = P(C) = 1/2$.
What is $F(2)$?

$$\begin{aligned}
F(2) &= P(X = 0) + P(X = 1) + P(X = 2) \\
&= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} \\
&= \frac{7}{8}
\end{aligned}$$

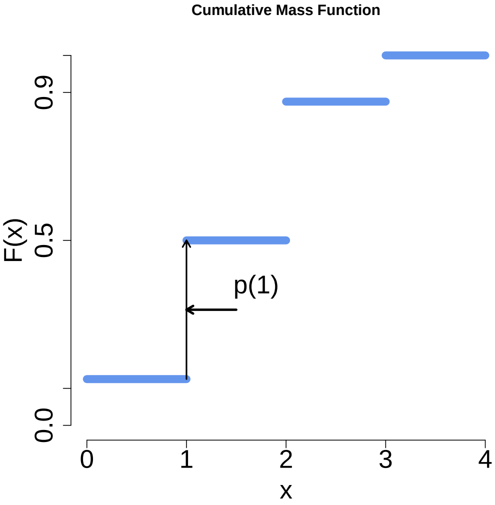
What is $F(2) - F(1)$?

$$\begin{aligned}
F(2) - F(1) &= [P(X = 0) + P(X = 1) + P(X = 2)] \\
&\quad -[P(X = 0) + P(X = 1)] \\
F(2) - F(1) &= P(X = 2)
\end{aligned}$$

Cumulative Mass Function

There is a close relationship between pmf's and cmf's.

Consider Previous example:



Expectation

What can we expect from a trial?
Value of random variable for any outcome
Weighted by the probability of observing that outcome

Definition

Expected Value: define the expected value of a function X as,

$$E[X] = \sum_{x:p(x)>0} xp(x)$$

In words: for all values of x with $p(x)$ greater than zero, take the weighted average of the values

Expectation Example: Simple Experiment

Suppose again X is number of units assigned to treatment, in one of our previous example.
What is $E[X]$?

$$\begin{aligned} E[X] &= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} \\ &= 1.5 \end{aligned}$$

Expectation Example: A Single Person Poll

Suppose that there is a group of N people.

- Suppose $M < N$ people approve of Barack Obama's performance as president
- $N - M$ disapprove of his performance

Define:
Draw one person i, with , $P(\text{Draw } i) = \frac{1}{N}$

$$X = \begin{cases} 1 \text{ if person Approves} \\ 0 \text{ if Disapproves} \end{cases} .$$

E[X]?

$$\begin{aligned} E[X] &= 1 \times P(\text{Approve}) + 0 \times P(\text{Disapprove}) \\ &= 1 \times \frac{M}{N} \end{aligned}$$

Indicator Variables and Probabilities

Proposition

Suppose A is an event. Define random variable I such that $I = 1$ if an outcome in A occurs and $I = 0$ if an outcome in A^c occurs. Then,

$$E[I] = P(A)$$

Proof.

$$\begin{aligned} E[I] &= 1 \times P(A) + 0 \times P(A^c) \\ &= P(A) \end{aligned}$$

$\square$

Functions of Random Variables

We might (or often) apply a function to a random variable $g(X)$. How do we compute $E[g(X)]$?

Proposition

Expected value of a function of a random variable: Suppose X is a discrete random variable that takes on values x_i, $i = \{1, 2, \ldots, \}$, with probabilities $p(x_i)$. If $g : X \rightarrow \mathcal{R}$, then its expected value $E[g(X)]$ is,

$$E[g(X)] \quad = \quad \sum_i g(x_i)p(x_i)$$

Functions of Random Variables

Proof.

Observation $g(X)$ is itself a random variable. Let's say it has unique values y_j $(j = 1, 2, \ldots,)$ So, we know that $E[g(X)] = \sum_j y_j P(g(X) = y_j)$. And we want to show that $\sum_i g(x_i)p(x_i)$ is equal to that.

$$\begin{aligned}
\sum_i g(x_i)p(x_i) &= \sum_j \sum_{i:g(x_i)=y_j} g(x_i)p(x_i) \\
&= \sum_j \sum_{i:g(x_i)=y_j} y_j p(x_i) \\
&= \sum_j y_j \sum_{i:g(x_i)=y_j} p(x_i) \\
&= \sum_j y_j P(g(X) = y_j) \\
&= E[g(X)]
\end{aligned}$$

□

Functions of Random Variables: Example

Let's suppose that X is the number of observations assigned to treatment (from our previous example).
Suppose that $g(X) = X^2$. What is $E[g(X)]$?

$$\begin{aligned}
E[g(X)] = E[X^2] &= 0^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{8} \\
&= 0 + \frac{3}{8} + \frac{12}{8} + \frac{9}{8} \\
&= \frac{24}{8} = 3
\end{aligned}$$

Functions of Random Variables: Corollary

Corollary

Suppose X is a random variable and a and b are constants (not random variables). Then,

$$E[aX + b] = aE[X] + b$$

Proof.

$$\begin{aligned} E[aX+b] &= \sum_{x:p(x)>0} (ax+b)p(x) \\ &= \sum_{x:p(x)>0} axp(x) + \sum_{x:p(x)>0} bp(x) \\ &= a \sum_{x:p(x)>0} xp(x) + b \sum_{x:p(x)>0} p(x) \\ &= aE[X] + b(1) \end{aligned}$$

Variance

Expected value is a measure of central tendency.

What about spread? Variance

- For each value, we might measure distance from center
  - Euclidean distance, squared $d(x, E[x])^2 = (x - E[x])^2$
- Then, we might take weighted average of these distances,

$$
\begin{aligned}
E[(X - E[X])^2] &= \sum_{x:p(x)>0} (x - E[X])^2 p(x) \\
&= \sum_{x:p(x)>0} \left(x^2 p(x)\right) - \\
&\quad 2E[X] \sum_{x:p(x)>0} (xp(x)) + E[X]^2 \sum_{x:p(x)>0} p(x) \\
&= E[X^2] - 2E[X]^2 + E[X]^2 \\
&= E[X^2] - E[X]^2 \\
&= \text{Var}(X)
\end{aligned}
$$

Variance

Definition

The variance of a random variable X, $var(X)$, is

$$\begin{aligned} var(X) &= E[(X - E[X])^2] \\ &= E[X^2] - E[X]^2 \end{aligned}$$

- We will define the standard deviation of X, $\text{sd}(X) = \sqrt{\text{var}(X)}$
- $\text{var}(X) \geq 0$.

Variance Calculation

Continue the three person experiment, with $P(T) = P(C) = 1/2$.
What is $\text{Var}(X)$?
We have two components to our variance calculation:

$$\begin{aligned}
E[X^2] &= 3 \\
E[X]^2 &= 1.5^2 = 2.25 \\
\text{Var}(X) &= E[X^2] - E[X]^2 \\
&= 3 - 2.25 = 0.75
\end{aligned}$$

Variance Corollary

Corollary

$Var(aX + b) = a^2 Var(X)$

Proof.

Define $Y = aX + b$. Now, we know that $Var(Y) = E[(Y - E[Y])^2]$. Let's substitute and use our other corollary

$$
\begin{aligned}
Var(Y) &= E[(aX + b - aE[X] - b)^2] \\
&= E[(a^2X^2 - 2a^2XE[X] + a^2E[X]^2)] \\
&= a^2E[X^2] - 2a^2E[X]^2 + a^2E[X]^2 \\
&= a^2(E[X^2] - E[X]^2) \\
&= a^2Var(X)
\end{aligned}
$$

□

Famous Distributions

- Bernoulli
- Binomial
- Multinomial
- Poisson

Models of how world works.

Bernoulli Random Variable

Definition

Suppose X is a random variable, with $X \in \{0, 1\}$ and $P(X = 1) = \pi$. Then we will say that X is Bernoulli random variable,

$$p(k) \quad = \quad \pi^k (1 - \pi)^{1-k}$$

for $k \in \{0, 1\}$ and $p(k) = 0$ otherwise.
We will (equivalently) say that

$$Y \quad \sim \quad Bernoulli(\pi)$$

Bernoulli Random Variable

Suppose we flip a fair coin and $Y = 1$ if the outcome is Heads .

$$
\begin{aligned}
Y & \sim \text{Bernoulli}(1/2) \\
p(1) & = (1/2)^1(1-1/2)^{1-1} = 1/2 \\
p(0) & = (1/2)^0(1-1/2)^{1-0} = (1-1/2)
\end{aligned}
$$

Bernoulli Random Variable Moments

Suppose $Y \sim \text{Bernoulli}(\pi)$

$$
\begin{aligned}
E[Y] &= 1 \times P(Y=1) + 0 \times P(Y=0) \\
&= \pi + 0(1-\pi) = \pi \\
\text{var}(Y) &= E[Y^2] - E[Y]^2 \\
E[Y^2] &= 1^2 P(Y=1) + 0^2 P(Y=0) \\
&= \pi \\
\text{var}(Y) &= \pi - \pi^2 \\
&= \pi(1-\pi)
\end{aligned}
$$

$E[Y] = \pi$
$\text{var}(Y) = \pi(1-\pi)$ What is the maximum variance?

Example: Winning a War

Suppose country 1 is engaged in a conflict and can either win or lose.
Define $Y = 1$ if the country wins and $Y = 0$ otherwise.
Then,

$$Y \sim \text{Bernoulli}(\pi)$$

Suppose country 1 is deciding whether to fight a war.
Engaging in the war will cost the country c.
If they win, country 1 receives B.
What is 1's expected utility from fighting a war?

$$\begin{aligned}
E[U(\text{war})] &= (\text{Utility}|\text{win}) \times P(\text{win}) + (\text{Utility}|\text{lose}) \times P(\text{lose}) \\
&= (B - c)P(Y = 1) + (-c)P(Y = 0) \\
&= B \times p(Y = 1) - c(P(Y = 1) + P(Y = 0)) \\
&= B \times \pi - c
\end{aligned}$$

Binomial Random Variable

- A model to count the number of successes across N trials
    - Assume the Bernoulli trials are independent
    - Each Bernoulli trial i is

$$Y_i \sim \text{Bernoulli}(\pi)$$

    Independent and identically distributed.
- $Z =$ number of successful trials
- Derive probability mass function $P(Z = M) = p(M)$
- One way to obtain M successful trials:

$$P(Y_1 = 1, Y_2 = 0, Y_3 = 1, \ldots, Y_N = 1)$$

$$\begin{aligned}
&= P(Y_1 = 1)P(Y_2 = 0)\cdots P(Y_N = 1) \\
&= \underbrace{P(Y_1 = 1)P(Y_3 = 1)\cdots P(Y_M = 1)}_{M} \times \underbrace{P(Y_2 = 0)\cdots P(Y_N = 0)}_{N-M} \\
&= \underbrace{\pi\pi\cdots\pi}_{M} \times \underbrace{(1-\pi)(1-\pi)\cdots(1-\pi)}_{N-M} \\
&= \pi^M(1-\pi)^{N-M}
\end{aligned}$$

Are we done? No

- This is just one instance of M successes
- How many total instances?
  - N total trials
  - We want to select M
- $\binom{N}{M} = \frac{N!}{(N-M)!M!}$

Then,

$$P(Z = M) \quad = \quad p(M) = \binom{N}{M}\pi^M(1-\pi)^{N-M}$$

Definition

Suppose Y is a random variable that counts the number of successes in N independent and identically distributed Bernoulli trials. Then Y is a Binomial random variable,

$$p(k) = \binom{N}{k}\pi^k(1-\pi)^{N-k}$$

for $k \in \{0, 1, 2, \ldots, N\}$ and $p(k) = 0$ otherwise.
Equivalently,

$$Y \sim Binomial(N, \pi)$$

Binomial Example

Recall our experiment example:
$P(T) = P(C) = 1/2$.
$Z =$ number of units assigned to treatment

$$
\begin{aligned}
Z & \sim \text{Binomial}(1/2) \\
p(0) & = \binom{3}{0}(1/2)^0(1-1/2)^{3-0} = 1 \times \frac{1}{8} \\
p(1) & = \binom{3}{1}(1/2)^1(1-1/2)^{2} = 3 \times \frac{1}{8} \\
p(2) & = \binom{3}{2}(1/2)^2(1-1/2)^{1} = 3 \times \frac{1}{8} \\
p(3) & = \binom{3}{3}(1/2)^3(1-1/2)^{0} = 1 \times \frac{1}{8}
\end{aligned}
$$

Binomial Random Variable Moments

$Z = \sum_{i=1}^{N} Y_i$ where $Y_i \sim \text{Bernoulli}(\pi)$

$$
\begin{aligned}
E[Z] &= E[Y_1 + Y_2 + Y_3 + \ldots + Y_N] \\
&= \sum_{i=1}^{N} E[Y_i] \\
&= N\pi \\
\text{var}(Z) &= \sum_{i=1}^{N} \text{var}(Y_i) \\
&= N\pi(1-\pi)
\end{aligned}
$$

$E[Z] = N\pi$
$\text{var}(Z) = N\pi(1-\pi)$

Voter Turnout

Suppose we have a set N voters, with iid turnout decisions $Y_i \sim \text{Bernoulli}(\pi)$
What is the probability that at least M voters turnout?

$$P(k \geq M) = \sum_{k=M}^{N} \binom{N}{k} \pi^k (1-\pi)^{N-k}$$

`R Code!`

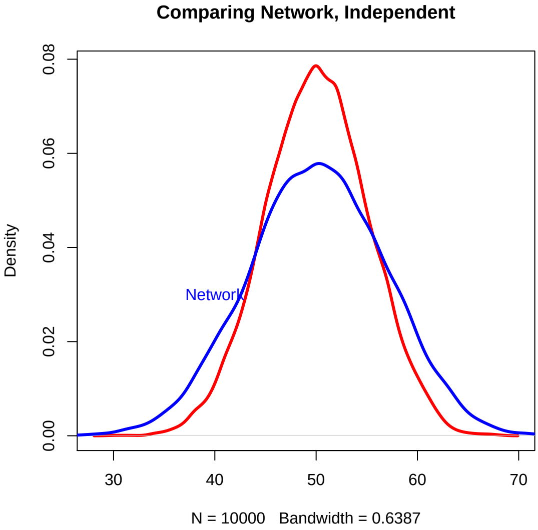
Voter Turnout, with Spillovers

Suppose we have the same set of N voters.
Now, $N/2$ are leaders, who turnout with probability $(1/2)$
But, $N/2$ are followers, whose turnout depends on a specific leader
Suppose follower i depends on only one leader j (and each follower has their own leader)

$$\begin{aligned} Y_i &\sim \text{Bernoulli}(0.9) \text{ if } j \text{ votes} \\ Y_i &\sim \text{Bernoulli}(0.1) \text{ if } j \text{ does not} \end{aligned}$$

Let Z be the number of voters who turnout.

Voter Turnout, with Spillovers



Trials with More than Two Outcomes

Definition

Suppose we observe a trial, which might result in J outcomes.
And that $P(outcome \ = i) = \pi_i$
$\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_J)$ *where* $Y_j = 1$ *if outcome j occurred and 0 otherwise.*
Then $\boldsymbol{Y}$ *follows a* *multinomial* *distribution, with*

$$p(\boldsymbol{y}) \quad = \quad \pi_1^{y_1} \pi_2^{y_2} \ldots \pi_k^{y_J}$$

if $\sum_{i=1}^{J} y_i = 1$ *and the pmf is* 0 *otherwise.*
Equivalently, we'll write

$$\begin{aligned} \boldsymbol{Y} & \sim Multinomial(1, \boldsymbol{\pi}) \\ \boldsymbol{Y} & \sim Categorical(\boldsymbol{\pi}) \end{aligned}$$

Multinomial Properties + Notes

Computer scientists: commonly call Multinomial$(1, \boldsymbol{\pi})$ Discrete$(\boldsymbol{\pi})$.

$$
\begin{aligned}
E[X_i] &= N\pi_i \\
\text{var}(X_i) &= N\pi_i(1 - \pi_i)
\end{aligned}
$$

Investigate Further in Homework!

Counting the Number of Events

Often interested in counting number of events that occur:

1) Number of wars started
2) Number of speeches made
3) Number of bribes offered
4) Number of people waiting for license

Generally referred to as event counts

Stochastic processes: a course provide introduction to many processes (Queing Theory)

Poisson Distribution

Definition

Suppose X is a random variable that takes on values $X \in \{0, 1, 2, \ldots, \}$ and that $P(X = k) = p(k)$ is,

$$p(k) \quad = \quad e^{-\lambda}\frac{\lambda^k}{k!}$$

for $k \in \{0, 1, \ldots, \}$ and 0 otherwise. Then we will say that X follows a Poisson distribution with rate parameter λ.

$$X \quad \sim \quad Poisson(\lambda)$$

Example: Poisson Distribution

Suppose the number of threats a president makes in a term is given by $X \sim \text{Poisson}(5)$. What is the probability the president will make ten or more threats?

$$
\begin{aligned}
P(X \geq 10) &= e^{-\lambda} \sum_{k=10}^{\infty} \frac{5^k}{k!} \\
&= 1 - P(X < 10)
\end{aligned}
$$

`R code!`

Poisson Distribution

Properties:

1) It is a probability distribution.
   Recall the Taylor expansion of e^x

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \\
e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} &= e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2!} + \ldots) \\
&= e^{-\lambda}(e^{\lambda}) = 1
\end{aligned}
$$

Poisson Distribution

Properties:

2) $E[X] = \lambda$

$$
\begin{aligned}
E[X] &= e^{-\lambda}\sum_{k=0}^{\infty} k\frac{\lambda^k}{k!} \\
&= e^{-\lambda}\lambda\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}
\end{aligned}
$$

Define $j = k - 1$, then

$$
\begin{aligned}
E[X] &= e^{-\lambda}\lambda\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} \\
&= e^{-\lambda}\lambda e^{\lambda} \\
&= \lambda
\end{aligned}
$$

Poisson Distribution

Properties:

3) $\text{var}(X) = \lambda$

$$
\begin{aligned}
E[X^2] &= \sum_{k=0}^{\infty} \frac{k^2 e^{-\lambda} \lambda^k}{k!} \\
&= \lambda e^{-\lambda} \left(\sum_{k=1}^{\infty} \frac{k \lambda^{k-1}}{(k-1)!} \right)
\end{aligned}
$$

Let $j = k - 1$,

$$
\begin{aligned}
E[X^2] &= \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{(j+1)\lambda^j}{j!} \\
&= \lambda e^{-\lambda} \left(\sum_{j=0}^{\infty} \frac{(j)\lambda^j}{j!} + \sum_{j=0}^{\infty} \frac{(1)\lambda^j}{j!} \right) \\
&= \lambda e^{-\lambda} (\lambda e^{\lambda} + e^{\lambda})
\end{aligned}
$$

Poisson Distribution

Properties

3) $\text{var}(X) = \lambda$

$$
\begin{aligned}
E[X^2] &= \lambda e^{-\lambda}(\lambda e^{\lambda} + e^{\lambda}) \\
&= \lambda(\lambda + 1)
\end{aligned}
$$

$\text{var}(X) = E[X^2] - E[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

Very useful distribution, with strong assumptions. We'll explore in homework!

Often interested in how processes evolve over time

- Given voting history, probability of voting in the future
- Given history of candidate support, probability of future support
- Given prior conflicts, probability of future war
- Given previous words in a sentence, probability of next word

Potentially complex history

Stochastic Process

Definition

Suppose we have a sequence of random variables $\{X\}_{i=0}^{M} = X_0, X_1, X_2, \ldots, X_M$ *that take on the countable values of* S. *We will call* $\{X\}_{i=0}^{M}$ *a stochastic process with state space* S.

If index gives time, then we might condition on history to obtain probability

$$\text{PMF } X_t, \text{ given history} \quad = \quad P(X_t | X_{t-1}, X_{t-2}, \ldots, X_1, X_0)$$

Still Complex

Markov Chain

Definition

Suppose we have a stochastic process $\{X\}_{i=0}^{M}$ *with countable state space* S*. Then* $\{X\}_{i=0}^{M}$ *is a markov chain if:*

$$P(X_t|X_{t-1}, X_{t-2}, \ldots, X_1, X_0) \quad = \quad P(X_t|X_{t-1})$$

A Markov chain's future depends only on its current state

Transition Matrix

Habitual turnout?

$$\boldsymbol{T} \quad = \quad \begin{pmatrix} & \text{Vote}_t & \text{Not Vote}_t \\ \text{Vote}_{t-1} & 0.8 & 0.2 \\ \text{Not Vote}_{t-1} & 0.3 & 0.7 \end{pmatrix}$$

- Suppose someone starts as a voter—what is their behavior after
- 1 iteration?
- 2 interations?
- The long run?

`R Code!`

Tomorrow: Continuous Random Variables!