Math Camp - Problem Set 8

The following laws of set algebra will be useful:

Commutative Property
$A \cup B = B \cup A$
$A \cap B = B \cap A$

Associative Law:
$(A \cup B) \cup C = A \cup (B \cup C)$
$(A \cap B) \cap C = A \cap (B \cap C)$

Distributive Law:
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

De Morgan's Law:
$(A \cap B)^c = A^c \cup B^c$

Question 1: Prove that if $A \subset B$ and $C \subset D$, then $A \cup C \subset B \cup D$ and $A \cap C \subset B \cap D$. (*Hint:* This can be proven either directly or by contradiction.)

Question 2: A, B, and D are non-mutually exclusive events contained within a sample space S. Find the simplest form for the following set expressions:

(a) $(A \cap B) \cup (A \cap B^c)$

(b) $(A \cap B) \cap (A \cap B^c)$

(c) $(A \cap B) \cap (A^c \cup B)$

(d) $(A \cap B) \cap (B \cap D)$

(e) $(A^c \cup B^c \cup D^c)^c$

(f) $(A \cup B) \cap (B \cup D)$

Question 3: Are the following statements true or false? Explain. (*Hint:* The inclusion-exclusion principle might be useful.)

(a) If I flip a coin n times, the probability of getting fewer than m heads is equal to the sum of the probability of getting k heads for all integers $0 < k < m$.

(b) Suppose I roll six fair, ordinary dice. Let E_1 be the event in which I rolled exactly one 1, E_2 be the event in which I rolled exactly one 2, E_3 be the event in which I rolled exactly one 3, and so on through E_6. Then:

$$P\left(\bigcup_{i=1}^{6} E_i\right) = P(E_1) + P(E_2) + P(E_3) + P(E_4) + P(E_5) + P(E_6)$$

Question 4: Events A and B are contained within a sample space S. Given that $P(A) = 0.5$, $P(B) = 0.3$ and $P(A \cap B) = 0.1$, find:

(a) $P(A \cup B)$

(b) $P(A \cap B^c)$

(c) $P[(A \cap B^c) \cup (B \cap A^c)]$

(*Hint:* The inclusion-exclusion principle might be useful here, as well.)

Question 5: A political campaign in New Haven, CT. decides to conduct an "experiment" to determine the effectiveness of knocking on a door in turning a resident of that house out to vote. The campaign foolishly denies an offer from a team of political scientists to help them design a protocol for this experiment, and instead directs their two teams of volunteers to each select a random group of the 120 total houses in the district and to go knock on as many of those random doors as they can in the week before the election. The campaign manager directs the teams to count the number of doors on which they knock and to record the names of the residents who live in each house, but neglects to ensure that the two teams select a mutually exclusive set of houses, or to set bounds on how many houses each team chooses.

On election day, the Team 1 members return, and proudly report to the campaign manager that they knocked on 70% of the doors in the electoral district. The Team 2 members return shortly after, and report that they knocked on 40% of the doors in the electoral district. In looking over the names the teams recorded, the campaign manager quickly determines that not only was every house in the district contacted, but some houses were contacted by both teams. (This will make drawing inferences about the effectiveness of door knocking... difficult.)

Use what we have learned about probability to determine how many houses had their doors knocked on by both teams.