Math Camp

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Where We've Been, Where We're Going

Calculus: Analyze behavior of functions on real line

- Convergence
- Differentiation
- Integration

Linear Algebra

- Data stored in matrices
- Higher dimensional spaces
  - complex world, condition on many factors
  - flood of big data, store in many dimensions
- Linear Algebra:
  - Algebra of matrices
  - Geometry of high dimensional space
  - Calculus (multivariable) in many dimensions

Very important for regression(!!!!)

Points + Vectors

- A point in $\Re^1$
  - 1
  - π
  - e
- An ordered pair in $\Re^2 = \Re \times \Re$
  - $(1, 2)$
  - $(0, 0)$
  - (π, e)
- An ordered triple in $\Re^3 = \Re \times \Re \times \Re$
  - $(3.1, 4.5, 6.11132)$

$\vdots$

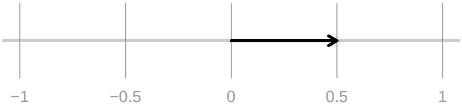
- An ordered n-tuple in $\Re^n = \Re \times \Re \times \ldots \times \Re$
  - $(a_1, a_2, \ldots, a_n)$

Points and Vectors

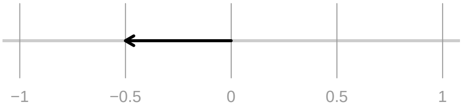
Definition

A point $\boldsymbol{x} \in \Re^n$ is an ordered n-tuple, $(x_1, x_2, \ldots, x_n)$. The vector $\boldsymbol{x} \in \Re^n$ is the arrow pointing from the origin $(0, 0, \ldots, 0)$ to $\boldsymbol{x}$.

One Dimensional Example



One Dimensional Example



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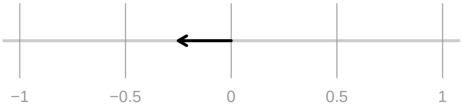
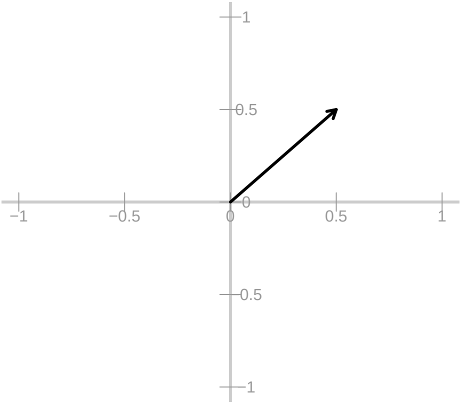


Two Dimensional Example



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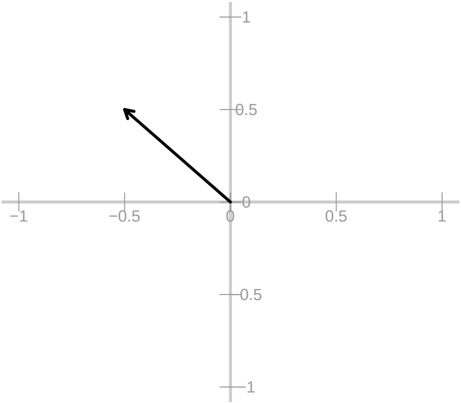
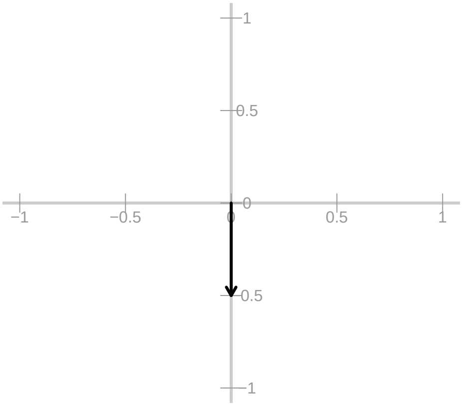


Two Dimensional Example



Three Dimensional Example

- (Latitude, Longitude, Elevation)
- (1, 2, 3)
- (0, 1, 0)

N-Dimensional Example

- Individual campaign donation records

$$\boldsymbol{x} = (1000, 0, 10, 50, 15, 4, 0, 0, 0, \ldots, 2400000000)$$

- Counties have proportion of vote for Obama

$$\boldsymbol{y} = (0.8, 0.5, 0.6, \ldots, 0.2)$$

- Run experiment, assess feeling thermometer of elected official

$$\boldsymbol{t} = (0, 100, 50, 70, 80, \ldots, 100)$$

Arithmetic with Vectors

Definition

Suppose $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *are vectors* $\boldsymbol{u}, \boldsymbol{v} \in \Re^n$*,*

$$\begin{aligned} \boldsymbol{u} &= (u_1, u_2, u_3, \ldots, u_n) \\ \boldsymbol{v} &= (v_1, v_2, v_3, \ldots, v_n) \end{aligned}$$

We will say $\boldsymbol{u} = \boldsymbol{v}$ *if* $u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$
Define the sum of $\boldsymbol{u} + \boldsymbol{v}$ *as*

$$\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \ldots, u_n + v_n)$$

Suppose $k \in \Re$*. We will call* k *a scalar.*
Define $k\boldsymbol{u}$ *as the scalar product*

$$k\boldsymbol{u} = (ku_1, ku_2, \ldots, ku_n)$$

Examples

Suppose:

$$
\begin{aligned}
\boldsymbol{u} &= (1, 2, 3, 4, 5) \\
\boldsymbol{v} &= (1, 1, 1, 1, 1) \\
k &= 2
\end{aligned}
$$

Then,

$$
\begin{aligned}
\boldsymbol{u} + \boldsymbol{v} &= (1+1, 2+1, 3+1, 4+1, 5+1) = (2, 3, 4, 5, 6) \\
k\boldsymbol{u} &= (2 \times 1, 2 \times 2, 2 \times 3, 2 \times 4, 2 \times 5) = (2, 4, 6, 8, 10) \\
k\boldsymbol{v} &= (2 \times 1, 2 \times 1, 2 \times 1, 2 \times 1, 2 \times 1) = (2, 2, 2, 2, 2)
\end{aligned}
$$

Properties of Arithmetic

Challenge Proofs—we can do these!

Theorem

Suppose that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \Re^n$ *and* k *and* l *are scalars.*

a) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$

Properties of Arithmetic

Challenge Proofs—we can do these!

Theorem

Suppose that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \Re^n$ and k and l are scalars.

a) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$

Proof.

$$
\begin{aligned}
\boldsymbol{u} + \boldsymbol{v} &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \\
&= (v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n) \\
&= \boldsymbol{v} + \boldsymbol{u}
\end{aligned}
$$

$\square$

Properties of Arithmetic

Challenge Proofs—we can do these!

Theorem

Suppose that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \Re^n$ and k and l are scalars.

b) $\boldsymbol{u} + \mathbf{0} = \mathbf{0} + \boldsymbol{u} = \boldsymbol{u}$

Properties of Arithmetic

Challenge Proofs—we can do these!

Theorem

Suppose that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \Re^n$ *and* k *and* l *are scalars.*

b) $\boldsymbol{u} + \mathbf{0} = \mathbf{0} + \boldsymbol{u} = \boldsymbol{u}$

Proof.

$$\begin{aligned} \boldsymbol{u} + \mathbf{0} &= (u_1 + 0, u_2 + 0, \ldots, u_n + 0) \\ &= (0 + u_1, 0 + u_2, \ldots, 0 + u_n) = \mathbf{0} + \boldsymbol{u} \\ &= (u_1, u_2, \ldots, u_n) \\ &= \boldsymbol{u} \end{aligned}$$

□

Properties of Arithmetic

Challenge Proofs—we can do these!

Theorem

Suppose that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \Re^n$ *and* k *and* l *are scalars.*

c) $(l + k)\boldsymbol{u} = l(\boldsymbol{u}) + k(\boldsymbol{u})$

Proof.

How can we show this? $\square$

Challenge Proofs

- Show that $1\boldsymbol{u} = \boldsymbol{u}$
- Show that $\boldsymbol{u} + -1\boldsymbol{u} = \mathbf{0}$

Inner Product

Definition

Suppose $\boldsymbol{u} \in \Re^n$ and $\boldsymbol{v} \in \Re^n$ then define $\boldsymbol{u} \cdot \boldsymbol{v}$,

$$\begin{aligned}
\boldsymbol{u} \cdot \boldsymbol{v} &= u_1 v_1 + u_2 v_2 + \ldots + u_n v_n \\
&= \sum_{i=1}^{N} u_i v_i
\end{aligned}$$

Examples

Suppose $\boldsymbol{u} = (1, 2, 3)$ and $\boldsymbol{v} = (2, 3, 1)$. Then,

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{v} &= 1 \times 2 + 2 \times 3 + 3 \times 1 \\ &= 2 + 6 + 3 \\ &= 11 \end{aligned}$$

Suppose $\boldsymbol{y} = (y_1, y_2, \ldots, y_N)$ and $\mathbf{1} = (1, 1, 1, \ldots, 1)$. Then,

$$\begin{aligned} \boldsymbol{y} \cdot \mathbf{1} &= y_1 + y_2 + \ldots + y_n \\ &= \sum_{i=1}^{n} y_i \end{aligned}$$

R Code

Create a vector in R

R Code

Create a vector in R

```
vec <- c(1, 2, 3, 4, 5)
```

R Code

Create a vector in R

```
vec <- c(1, 2, 3, 4, 5)
vec<- c()
```

R Code

Create a vector in R

```
vec <- c(1, 2, 3, 4, 5)
vec<- c()
vec[1]<- 1
```

R Code

Create a vector in R

```
vec <- c(1, 2, 3, 4, 5)
vec<- c()
vec[1]<- 1
vec[2]<- 2
```

R Code

Create a vector in R

```
vec <- c(1, 2, 3, 4, 5)
vec<- c()
vec[1]<- 1
vec[2]<- 2
vec[3]<- 3
```

R Code

Create a vector in R

```
vec <- c(1, 2, 3, 4, 5)
vec<- c()
vec[1]<- 1
vec[2]<- 2
vec[3]<- 3
vec[4]<- 4
```

R Code

Create a vector in R

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vec <- c(1, 2, 3, 4, 5)
vec<- c()
vec[1]<- 1
vec[2]<- 2
vec[3]<- 3
vec[4]<- 4
vec[5]<- 5
```

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Create a vector in R

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vec <- c(1, 2, 3, 4, 5)
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vec[5]<- 5
x1<- c(2, 2, 3, 2)
```

R Code

Create a vector in R

```
vec <- c(1, 2, 3, 4, 5)
vec<- c()
vec[1]<- 1
vec[2]<- 2
vec[3]<- 3
vec[4]<- 4
vec[5]<- 5
x1<- c(2, 2, 3, 2)
x2<- c(5, 3, 1, 3)
```

R Code

Create a vector in R

```
vec <- c(1, 2, 3, 4, 5)
vec<- c()
vec[1]<- 1
vec[2]<- 2
vec[3]<- 3
vec[4]<- 4
vec[5]<- 5
x1<- c(2, 2, 3, 2)
x2<- c(5, 3, 1, 3)
add <- x1 + x2
```

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add
[1] 7 5 4 5
```

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```
scalar<- 10 *x1
```

R Code

Create a vector in R

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x2<- c(5, 3, 1, 3)
add <- x1 + x2
add
[1] 7 5 4 5
```

```
scalar<- 10 *x1
scalar
[1] 20 20 30 20
```

R Code

Create a vector in R

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vec <- c(1, 2, 3, 4, 5)
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vec[5]<- 5
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x2<- c(5, 3, 1, 3)
add <- x1 + x2
add
[1] 7 5 4 5
```

```
scalar<- 10 *x1
scalar
[1] 20 20 30 20
output<- x1 %*% x2
```

R Code

Create a vector in R

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vec <- c(1, 2, 3, 4, 5)
vec<- c()
vec[1]<- 1
vec[2]<- 2
vec[3]<- 3
vec[4]<- 4
vec[5]<- 5
x1<- c(2, 2, 3, 2)
x2<- c(5, 3, 1, 3)
add <- x1 + x2
add
[1] 7 5 4 5
```
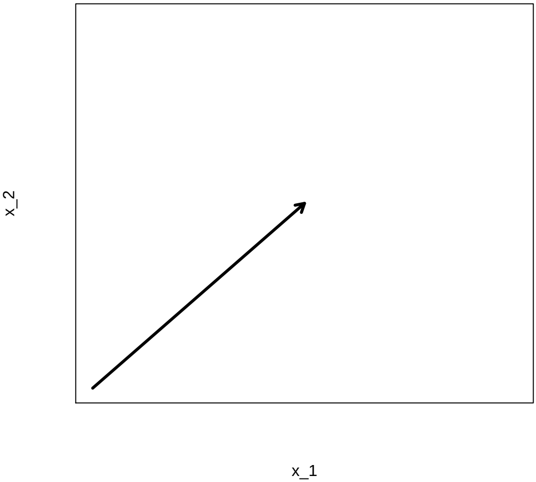

```
scalar<- 10 *x1
scalar
[1] 20 20 30 20
output<- x1 %*% x2
output
[,1]
[1,] 25
```

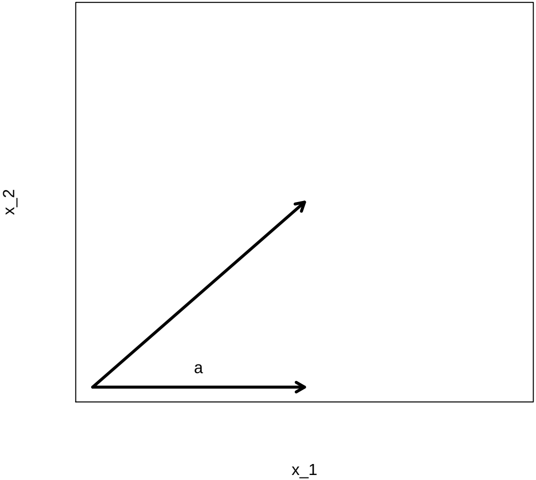
Challenge Problems

- Suppose $\boldsymbol{v} = (1, 4, 1, 4)$ and $\boldsymbol{w} = (4, 1, 4, 1)$. Calculate: $\boldsymbol{v} \cdot \boldsymbol{w}$
- Prove $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$
- Prove $\boldsymbol{v} \cdot \boldsymbol{v} \geq 0$ and $\boldsymbol{v} \cdot \boldsymbol{v} = 0$ if and only if $\boldsymbol{v} = \boldsymbol{0}$.

Vector Length

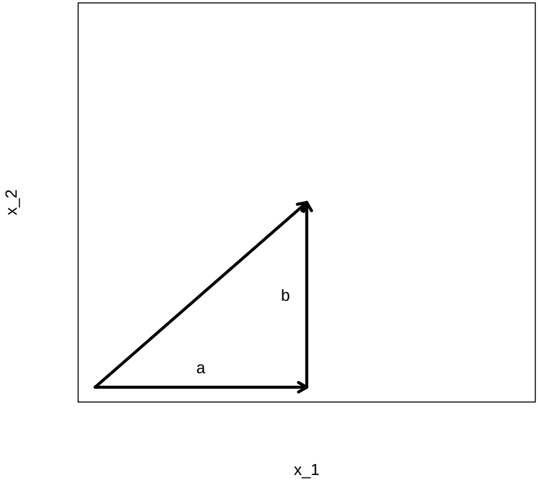


Vector Length



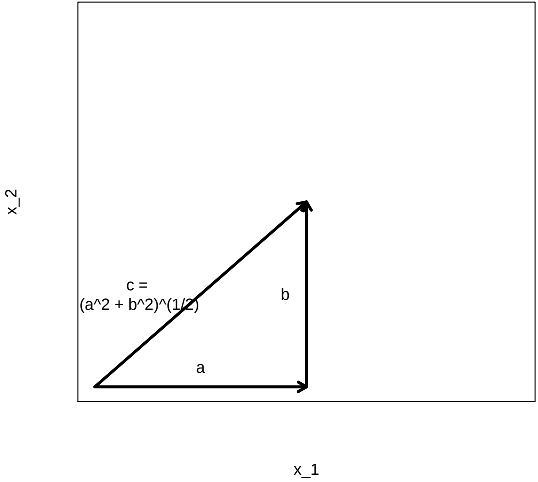
- Pythogorean Theorem: Side with length a

Vector Length



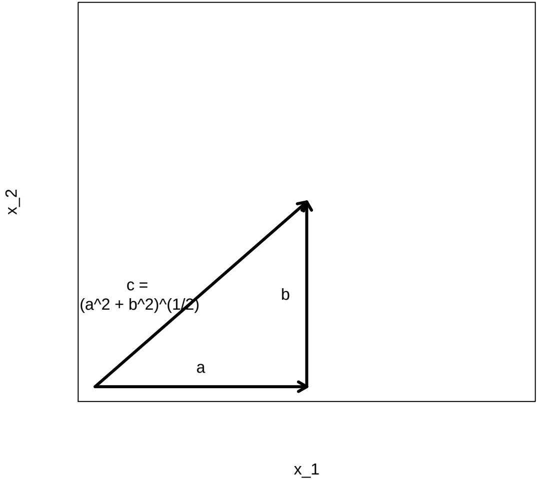
- Pythogorean Theorem: Side with length a
- Side with length b and right triangle

Vector Length



- Pythogorean Theorem: Side with length a
- Side with length b and right triangle
- $c = \sqrt{a^2 + b^2}$

Vector Length



- Pythogorean Theorem: Side with length a
- Side with length b and right triangle
- $c = \sqrt{a^2 + b^2}$
- This is generally true

Vector Length

Definition

Suppose $\boldsymbol{v} \in \Re^n$. *Then, we will define its* *length* *as*

$$\begin{aligned} ||\boldsymbol{v}|| &= (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \\ &= (v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2)^{1/2} \end{aligned}$$

Calculating a Length

Example 1: suppose $\boldsymbol{x} = (1, 1, 1)$.

$$\begin{aligned} ||\boldsymbol{x}|| &= (\boldsymbol{x} \cdot \boldsymbol{x})^{1/2} \\ &= (1 + 1 + 1)^{1/2} \\ &= \sqrt{3} \end{aligned}$$

Example 2: R code for length function

```
len.vec<- function(x) {
p1< − sqrt(x% ∗ %x)
return(p1)
}
x <- c(1,1,1)
len.vec(x)
[,1]
[1,] 1.732051
```

Coding Problem

Let's calculate the length of some vectors

- Write a function to assess the length of a vector.
- Use it to calculate the length of:
    - `y<- c(10, 20, 30, 40)`
    - `x<- seq(1, 1000*pi, len=1000)`

Texts in Space

Texts in Space

$$\text{Doc1} \quad = \quad (1, 1, 3, \ldots, 5)$$

Texts in Space

$$
\begin{aligned}
\text{Doc1} &= (1, 1, 3, \ldots, 5) \\
\text{Doc2} &= (2, 0, 0, \ldots, 1)
\end{aligned}
$$

Texts in Space

$$
\begin{aligned}
\text{Doc1} &= (1, 1, 3, \ldots, 5) \\
\text{Doc2} &= (2, 0, 0, \ldots, 1) \\
\textbf{Doc1}, \textbf{Doc2} &\in \Re^{M}
\end{aligned}
$$

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\end{aligned}
$$

Provides many operations that will be useful

Texts in Space

$$\begin{aligned} \text{Doc1} &= (1, 1, 3, \ldots, 5) \\ \text{Doc2} &= (2, 0, 0, \ldots, 1) \\ \textbf{Doc1}, \textbf{Doc2} &\in \Re^M \end{aligned}$$

Provides many operations that will be useful

Inner Product between documents:

Texts in Space

$$\begin{aligned} \text{Doc1} &= (1, 1, 3, \ldots, 5) \\ \text{Doc2} &= (2, 0, 0, \ldots, 1) \\ \textbf{Doc1}, \textbf{Doc2} &\in \Re^{M} \end{aligned}$$

Provides many operations that will be useful
Inner Product between documents:

$$\textbf{Doc1} \cdot \textbf{Doc2} = (1, 1, 3, \ldots, 5)^{'}(2, 0, 0, \ldots, 1)$$

Texts in Space

$$
\begin{aligned}
\text{Doc1} &= (1, 1, 3, \ldots, 5) \\
\text{Doc2} &= (2, 0, 0, \ldots, 1) \\
\textbf{Doc1}, \textbf{Doc2} &\in \Re^{M}
\end{aligned}
$$

Provides many operations that will be useful

Inner Product between documents:

$$
\begin{aligned}
\textbf{Doc1} \cdot \textbf{Doc2} &= (1, 1, 3, \ldots, 5)^{'}(2, 0, 0, \ldots, 1) \\
&= 1 \times 2 + 1 \times 0 + 3 \times 0 + \ldots + 5 \times 1
\end{aligned}
$$

Texts in Space

$$\begin{aligned}
\text{Doc1} &= (1, 1, 3, \ldots, 5) \\
\text{Doc2} &= (2, 0, 0, \ldots, 1) \\
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\end{aligned}$$

Provides many operations that will be useful

Inner Product between documents:

$$\begin{aligned}
\textbf{Doc1} \cdot \textbf{Doc2} &= (1, 1, 3, \ldots, 5)^{'}(2, 0, 0, \ldots, 1) \\
&= 1 \times 2 + 1 \times 0 + 3 \times 0 + \ldots + 5 \times 1 \\
&= 7
\end{aligned}$$

Length of document:

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$$\begin{aligned}||\textbf{Doc1}|| &\equiv \sqrt{\textbf{Doc1} \cdot \textbf{Doc1}} \\ &= \sqrt{(1, 1, 3, \ldots, 5)'(1, 1, 3, \ldots, 5)} \\ &= \sqrt{1^2 + 1^2 + 3^2 + 5^2} \\ &= 6\end{aligned}$$

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&= 6
\end{aligned}$$

Cosine of the angle between documents:

Length of document:

$$
\begin{aligned}
||\mathbf{Doc1}|| &\equiv \sqrt{\mathbf{Doc1} \cdot \mathbf{Doc1}} \\
&= \sqrt{(1, 1, 3, \ldots, 5)'(1, 1, 3, \ldots, 5)} \\
&= \sqrt{1^2 + 1^2 + 3^2 + 5^2} \\
&= 6
\end{aligned}
$$

Cosine of the angle between documents:

$$
\begin{aligned}
\cos\theta &\equiv \left(\frac{\mathbf{Doc1}}{||\mathbf{Doc1}||}\right) \cdot \left(\frac{\mathbf{Doc2}}{||\mathbf{Doc2}||}\right) \\
&= \frac{7}{6 \times 2.24} \\
&= 0.52
\end{aligned}
$$

Measuring Similarity

Documents in space $\rightarrow$ measure similarity/dissimilarity

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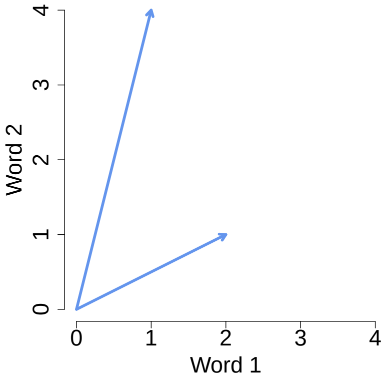
Documents in space $\rightarrow$ measure similarity/dissimilarity

What properties should similarity measure have?

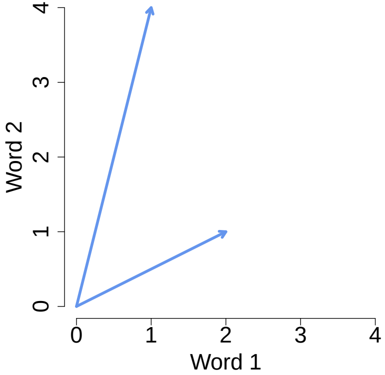
- Maximum: document with itself
- Minimum: documents have no words in common (orthogonal)
- Increasing when more of same words used
- ? $s(a, b) = s(b, a)$.

Measuring Similarity



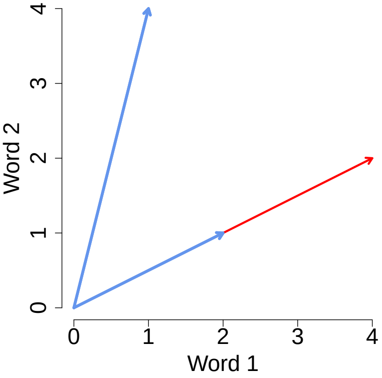
Measure 1: Inner product

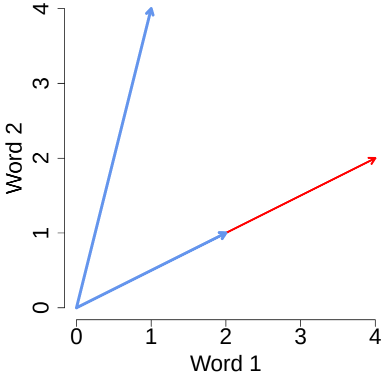
Measuring Similarity



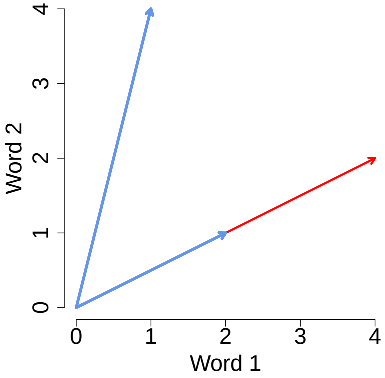
Measure 1: Inner product

$$(2,1)^{'} \cdot (1,4) \quad = \quad 6$$


0
1
2
3
4
Word 2
0
1
2
3
4
Word 1

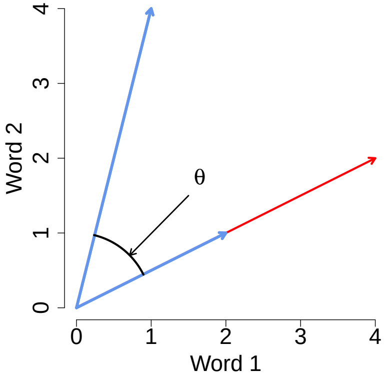


Problem(?): length dependent



Problem(?): length dependent

$$(4, 2)^{'}(1, 4) \quad = \quad 12$$



Problem(?): length dependent

$$
\begin{aligned}
(4,2)^{'}(1,4) &= 12 \\
a \cdot b &= ||a|| \times ||b|| \times \cos\theta
\end{aligned}
$$

Cosine Similarity

$\cos\theta$: removes document length from similarity measure

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$$\cos\theta = \left(\frac{a}{||a||}\right) \cdot \left(\frac{b}{||b||}\right)$$

Cosine Similarity

$\cos\theta$: removes document length from similarity measure

$$\begin{aligned} \cos\theta &= \left(\frac{a}{||a||}\right) \cdot \left(\frac{b}{||b||}\right) \\ \frac{(4,2)}{||(4,2)||} &= (0.89, 0.45) \end{aligned}$$

Cosine Similarity

$\cos\theta$: removes document length from similarity measure

$$
\begin{aligned}
\cos\theta &= \left(\frac{a}{||a||}\right)\cdot\left(\frac{b}{||b||}\right)\\
\frac{(4,2)}{||(4,2)||} &= (0.89, 0.45)\\
\frac{(2,1)}{||(2,1)||} &= (0.89, 0.45)
\end{aligned}
$$

Cosine Similarity

$\cos\theta$: removes document length from similarity measure

$$
\begin{aligned}
\cos\theta &= \left(\frac{a}{||a||}\right) \cdot \left(\frac{b}{||b||}\right) \\
\frac{(4,2)}{||(4,2)||} &= (0.89, 0.45) \\
\frac{(2,1)}{||(2,1)||} &= (0.89, 0.45) \\
\frac{(1,4)}{||(1,4)||} &= (0.24, 0.97)
\end{aligned}
$$

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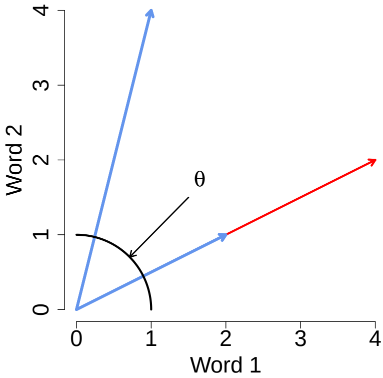
\frac{(1,4)}{||(1,4)||} &= (0.24, 0.97) \\
(0.89, 0.45)^{'}(0.24, 0.97) &= 0.65
\end{aligned}$$

Cosine Similarity



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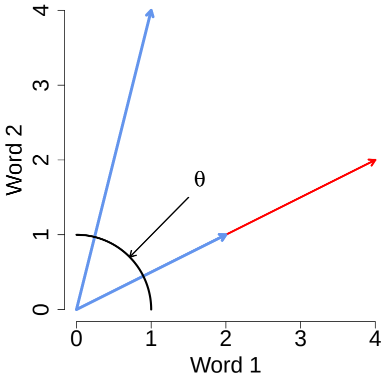

$\cos\theta$: removes document length from similarity measure
Project onto Hypersphere

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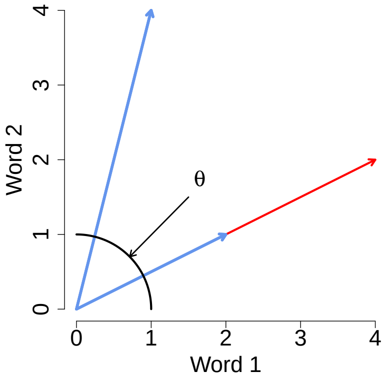
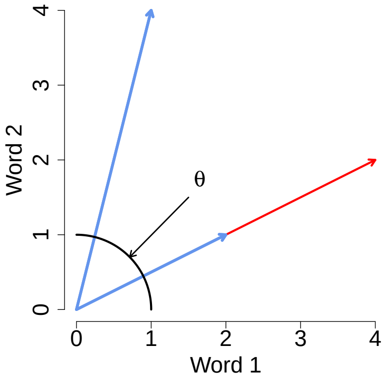


$\cos\theta$: removes document length from similarity measure
Project onto Hypersphere
$\cos\theta \rightarrow$ Inverse distance on Hypersphere

Cosine Similarity



$\cos\theta$: removes document length from similarity measure

Project onto Hypersphere

$\cos\theta \rightarrow$ Inverse distance on Hypersphere

von Mises Fisher distribution : distribution on sphere surface

Matrices

Definition

A Matrix is a rectangular array of numbers

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

If $\boldsymbol{A}$ *has m rows n columns we will say that* $\boldsymbol{A}$ *is an* $m \times n$ *matrix.* *Suppose* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are* $m \times n$ *matrices. Then* $\boldsymbol{X} = \boldsymbol{Y}$ *if* $x_{ij} = y_{ij}$ *for all i and j*

Simple Examples

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

If $\boldsymbol{I}$ is an $n \times n$ matrix we will call an identity matrix.

Simple Examples

$$\boldsymbol{X} \quad = \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}$$

$\boldsymbol{X}$ is an 2×3 matrix

Matrix Algebra

Definition

Suppose $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are* $m \times n$ *matrices. Then define*

$$
\begin{aligned}
\boldsymbol{X} + \boldsymbol{Y} &= \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix} + \begin{pmatrix} y_{11} & y_{12} & \dots & y_{1n} \\ y_{21} & y_{22} & \dots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \dots & y_{mn} \end{pmatrix} \\
&= \begin{pmatrix} x_{11} + y_{11} & x_{12} + y_{12} & \dots & x_{1n} + y_{1n} \\ x_{21} + y_{21} & x_{22} + y_{22} & \dots & x_{2n} + y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} + y_{m1} & x_{m2} + y_{m2} & \dots & x_{mn} + y_{mn} \end{pmatrix}
\end{aligned}
$$

Matrix Algebra

Definition

Suppose $\boldsymbol{X}$ *is an* $m \times n$ *matrix and* $k \in \Re$. *Then,*

$$k\boldsymbol{X} = \begin{pmatrix} kx_{11} & kx_{12} & \dots & kx_{1n} \\ kx_{21} & kx_{22} & \dots & kx_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ kx_{m1} & kx_{m2} & \dots & kx_{mn} \end{pmatrix}$$

Prove theorems about this tonight

R Code

Using matrix command `mat1<- matrix(NA, nrow=3, ncol=2) ## Creating matrix`

```
mat1[1,1]<- 1
mat1[1,2]<- 2
mat1[2,1]<- 1
mat1[2,2]<- 4
mat1[3,1]<- exp(1)
mat1[3,2]<- 4
```

R Code

Using rbind

```
r1<- c(1, 2)
r2<- c(1, 4)
r3<- c(exp(1) , 4)
mat1<- rbind(r1, r2, r3)
```

R Code

Using cbind

```
c1<- c(1, 1, exp(1) )
c2<- c(2, 4, 4)
```

R Code

```
dim(mat1)
[1] 3 2
mat1 + mat1
[,1] [,2]
[1,] 2.000000 4
[2,] 2.000000 8
[3,] 5.436564 8
```

R Code

What if the matrices are of different dimension

```
mat1<- matrix(1, nrow=3, ncol=2)
mat2<- matrix(2, nrow=10, ncol=3)
mat1 + mat2
Error in mat1 + mat2 :  non-conformable arrays
```

Matrix Transpose

We will use matrix transpose to flip the dimensionality of a matrix

Matrix Transpose

We will use matrix transpose to flip the dimensionality of a matrix

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix}$$

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$$\boldsymbol{X}^{'} = \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{pmatrix}$$

Matrix Transpose

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$$
\begin{aligned}
\boldsymbol{X} &= \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \\
\boldsymbol{X}^{'} &= \begin{pmatrix} x_{11} & x_{21} & & \\ x_{12} & x_{22} & & \\ \vdots & \vdots & & \\ x_{1n} & x_{2n} & & \end{pmatrix}
\end{aligned}
$$

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$$

$$
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$$

If $\boldsymbol{X}$ is an $m \times n$ then $\boldsymbol{X}^{'}$ is $n \times m$.

Matrix Transpose

We will use matrix transpose to flip the dimensionality of a matrix

$$
\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \ldots & x_{mn} \end{pmatrix}
$$

$$
\boldsymbol{X}^{'} = \begin{pmatrix} x_{11} & x_{21} & \ldots & x_{m1} \\ x_{12} & x_{22} & \ldots & x_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \ldots & x_{mn} \end{pmatrix}
$$

If $\boldsymbol{X}$ is an $m \times n$ then $\boldsymbol{X}^{'}$ is $n \times m$.
If $\boldsymbol{X} = \boldsymbol{X}^{'}$ then we say $\boldsymbol{X}$ is symmetric.

Matrix Transpose

Example 1: $\boldsymbol{X} = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ then $\boldsymbol{X}^{'} = \begin{pmatrix} 4 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}$

In `R`

```
mat1<- matrix(c(1, 2, 3), nrow=3, ncol=2)
mat2<- t(mat1)
dim(mat1)
3 2
dim(mat2)
2 3
```

Matrix Multiplication

How do we multiply matrices?

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Because we want to use matrix multiplication to solve equations we won't use an intuitive definition

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Suppose we have two matrices

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Suppose we have two matrices

$\boldsymbol{X} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\boldsymbol{Y} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Matrix Multiplication

How do we multiply matrices?

Because we want to use matrix multiplication to solve equations we won't use an intuitive definition

Suppose we have two matrices

$\boldsymbol{X} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\boldsymbol{Y} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

We will create a new matrix $\boldsymbol{A}$ by matrix multiplication:

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$$\boldsymbol{A} = \boldsymbol{X}\boldsymbol{Y}$$

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We will create a new matrix $\boldsymbol{A}$ by matrix multiplication:

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{X}\boldsymbol{Y} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \end{aligned}$$

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$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{X}\boldsymbol{Y} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 \times 1 + 1 \times 3 & \\ & \end{pmatrix} \end{aligned}$$

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$$
\begin{aligned}
\boldsymbol{A} &= \boldsymbol{X}\boldsymbol{Y} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 1 \times 1 + 1 \times 3 & 1 \times 2 + 1 \times 4 \\ 1 \times 1 + 1 \times 3 & \\ & \end{pmatrix}
\end{aligned}
$$

Matrix Multiplication

How do we multiply matrices?

Because we want to use matrix multiplication to solve equations we won't use an intuitive definition

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$$
\begin{aligned}
\boldsymbol{A} &= \boldsymbol{X}\boldsymbol{Y} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 1 \times 1 + 1 \times 3 & 1 \times 2 + 1 \times 4 \\ 1 \times 1 + 1 \times 3 & 1 \times 2 + 1 \times 4 \\ & \end{pmatrix}
\end{aligned}
$$

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Because we want to use matrix multiplication to solve equations we won't use an intuitive definition

Suppose we have two matrices

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We will create a new matrix $\boldsymbol{A}$ by matrix multiplication:

$$\begin{aligned}
\boldsymbol{A} &= \boldsymbol{X}\boldsymbol{Y} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 1 \times 1 + 1 \times 3 & 1 \times 2 + 1 \times 4 \\ 1 \times 1 + 1 \times 3 & 1 \times 2 + 1 \times 4 \\ \end{pmatrix} \\
&= \begin{pmatrix} 4 & 6 \\ 4 & 6 \end{pmatrix}
\end{aligned}$$

Matrix Multiplication

Definition

Suppose $\boldsymbol{X}$ *is an* $m \times n$ *matrix and* $\boldsymbol{Y}$ *is an* $n \times k$ *matrix. Then define the matrix* $\boldsymbol{A}$ *an* $m \times k$ *matrix that obtains from multiplying* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *as,*

$$\begin{aligned}
\boldsymbol{A} &= \boldsymbol{X}\boldsymbol{Y} \\
&= \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1k} \\ y_{21} & y_{22} & \cdots & y_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nk} \end{pmatrix} \\
&= \begin{pmatrix} x_{11}y_{11} + x_{12}y_{21} + \ldots + x_{1n}y_{n1} & \cdots & x_{11}y_{1k} + x_{12}y_{2k} + \ldots + x_{1n}y_{nk} \\ \vdots & \ddots & \vdots \\ x_{m1}y_{11} + x_{m2}y_{21} + \ldots + x_{mn}y_{n1} & \cdots & x_{m1}y_{11} + x_{m2}y_{12} + \ldots + x_{mn}y_{nk} \end{pmatrix}
\end{aligned}$$

Definition

Suppose $\boldsymbol{X}$ is an $m \times n$ matrix and $\boldsymbol{Y}$ is an $n \times k$ matrix. Write the row vectors of $\boldsymbol{X} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_m \end{pmatrix}$ *and $\boldsymbol{Y}$ as column vector* $\boldsymbol{Y} = \begin{pmatrix} \boldsymbol{y}_1 & \boldsymbol{y}_2 & \dots & \boldsymbol{y}_k \end{pmatrix}$.

Then the $m \times k$ matrix $\boldsymbol{A} = \boldsymbol{X}\boldsymbol{Y}$ can be written as

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{x}_1 \cdot \boldsymbol{y}_1 & \boldsymbol{x}_1 \cdot \boldsymbol{y}_2 & \dots & \boldsymbol{x}_1 \cdot \boldsymbol{y}_k \\ \boldsymbol{x}_2 \cdot \boldsymbol{y}_1 & \boldsymbol{x}_2 \cdot \boldsymbol{y}_2 & \dots & \boldsymbol{x}_2 \cdot \boldsymbol{y}_k \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{x}_m \cdot \boldsymbol{y}_1 & \boldsymbol{x}_m \cdot \boldsymbol{y}_2 & \dots & \boldsymbol{x}_m \cdot \boldsymbol{y}_k \end{pmatrix}$$

Matrix Multiplication

Let's work on an example together!

$$\boldsymbol{X} = \begin{pmatrix} 1 & 4 & 5 \\ 10 & 2 & 3 \end{pmatrix}$$

$$\boldsymbol{Y} = \begin{pmatrix} 2 & 3 \\ 1 & 5 \\ 3 & 5 \end{pmatrix}$$ What is $\boldsymbol{X}\boldsymbol{Y}$?

Matrix Multiplication

Let's work on an example together!

$$\boldsymbol{X} = \begin{pmatrix} 1 & 4 & 5 \\ 10 & 2 & 3 \end{pmatrix}$$

$$\boldsymbol{Y} = \begin{pmatrix} 2 & 3 \\ 1 & 5 \\ 3 & 5 \end{pmatrix}$$ What is $\boldsymbol{XY}$?

Not all matrices can be multiplied.

Matrix $\boldsymbol{AB}$ exists only if the number of columns in $\boldsymbol{A}$ = number of rows in $\boldsymbol{B}$. If $\boldsymbol{AB}$ exists we will say the matrices are conformable

Matrix Multiplication with a Vector

Suppose $\boldsymbol{X} = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 1 & 5 & 1 & 2 \\ 3 & 5 & 3 & 4 \end{pmatrix}$ a 3×4 matrix and that $\boldsymbol{v} = \begin{pmatrix} 3 \\ 3 \\ 4 \\ 10 \end{pmatrix}$ a 4×1 matrix (or a column vector) what is
$\boldsymbol{X}\boldsymbol{v}$?
What is $\boldsymbol{X}^{'}\boldsymbol{v}$?

Algebraic Properties

Suppose $\boldsymbol{X}$ is an $m \times n$ matrix and $\boldsymbol{Y}$ is an $n \times k$ matrix. Suppose that $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$ as the identity matrix and that $k \in Re$.

- $\boldsymbol{X}\boldsymbol{Y} \neq \boldsymbol{Y}\boldsymbol{X}$ in general !!!! (but it could)
- $\boldsymbol{X}\boldsymbol{I} = \boldsymbol{X}$ (let's talk it out!)
- $(\boldsymbol{X}^{'})^{'} = \boldsymbol{X}$
- $(\boldsymbol{X}\boldsymbol{Y})^{'} = \boldsymbol{Y}^{'}\boldsymbol{X}^{'}$
- $(k\boldsymbol{X})^{'} = k\boldsymbol{X}^{'}$
- $(\boldsymbol{X} + \boldsymbol{Y})^{'} = \boldsymbol{X}^{'} + \boldsymbol{Y}^{'}$

Examples, Implenting in R

R and matrix multiplication

```
X<- matrix(NA, nrow=2, ncol=3)
Y<- matrix(NA, nrow=3, ncol=2)
X[1,]<- c(1, 4, 5)
X[2,]<- c(10, 2, 3)
Y[1,]<- c(2, 3)
Y[2,]<- c(1, 5)
Y[3,]<- c(3, 5)

A<- X%*%Y
> A
[,1] [,2]
[1,] 21 48
[2,] 31 55
```

Matrix Inversion

Big topic: suppose $\boldsymbol{X}$ is an $n \times n$ matrix. We want to find the matrix X^{-1} such that

$$\boldsymbol{X}^{-1}\boldsymbol{X} \;=\; \boldsymbol{X}\boldsymbol{X}^{-1} = \boldsymbol{I}$$

where $\boldsymbol{I}$ is the $n \times n$ identity matrix.
Why?

- Regression
- Solving systems of equations
- Will provide intuition about "colinearity", "fixed effects", "treatment designs" and what we can learn as social scientists

Calculate $\rightsquigarrow$ Properties of Inverses $\rightsquigarrow$ when do inverses exist $\rightsquigarrow$ Application to regression analysis

Some Motivating Examples

Consider the following equations:

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 0 \\
0x_1 + 0x_2 + x_3 &= 5
\end{aligned}
$$

Some Motivating Examples

Consider the following equations:

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 0 \\
x_1 + x_2 + 0x_3 &= 0 \\
0x_1 + x_2 + x_3 &= 0 \\
x_1 + 0x_2 + x_3 &= 0
\end{aligned}
$$

Some Motivating Examples

Consider the following equations:

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 0 \\
0x_1 + 5x_2 + 0x_3 &= 5 \\
0x_1 + 0x_2 + 3x_3 &= 6
\end{aligned}
$$

Some Motivating Examples

Consider the following equations:

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 0 \\
0x_1 + 5x_2 + 0x_3 &= 5 \\
0x_1 + 0x_2 + 3x_3 &= 6
\end{aligned}
$$

$$
\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}
$$

Some Motivating Examples

Consider the following equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 0x_1 + 5x_2 + 0x_3 &= 5 \\ 0x_1 + 0x_2 + 3x_3 &= 6 \end{aligned}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\boldsymbol{x} = (x_1, x_2, x_3)$$

Some Motivating Examples

Consider the following equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 0x_1 + 5x_2 + 0x_3 &= 5 \\ 0x_1 + 0x_2 + 3x_3 &= 6 \end{aligned}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$
$$\boldsymbol{x} = (x_1, x_2, x_3)$$
$$\boldsymbol{b} = (0, 5, 6)$$

Some Motivating Examples

Consider the following equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 0x_1 + 5x_2 + 0x_3 &= 5 \\ 0x_1 + 0x_2 + 3x_3 &= 6 \end{aligned}$$

$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
$\boldsymbol{x} = (x_1, x_2, x_3)$
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The system of equations are now,

Some Motivating Examples

Consider the following equations:

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$\boldsymbol{x} = (x_1, x_2, x_3)$
$\boldsymbol{b} = (0, 5, 6)$
The system of equations are now,

$$\boldsymbol{Ax} = \boldsymbol{b}$$

Some Motivating Examples

Consider the following equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 0x_1 + 5x_2 + 0x_3 &= 5 \\ 0x_1 + 0x_2 + 3x_3 &= 6 \end{aligned}$$

$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
$\boldsymbol{x} = (x_1, x_2, x_3)$
$\boldsymbol{b} = (0, 5, 6)$
The system of equations are now,

$$\boldsymbol{Ax} = \boldsymbol{b}$$

$\boldsymbol{A}^{-1}$ exists if and only if $\boldsymbol{Ax} = \boldsymbol{b}$ has only one solution.

Matrix Inversion, Definition

Definition

Suppose $\boldsymbol{X}$ is an $n \times n$ matrix. We will call $\boldsymbol{X}^{-1}$ the inverse of $\boldsymbol{X}$ if

$$\boldsymbol{X}^{-1}\boldsymbol{X} \;=\; \boldsymbol{X}\boldsymbol{X}^{-1} = \boldsymbol{I}$$

If $\boldsymbol{X}^{-1}$ exists then $\boldsymbol{X}$ is invertible. If $\boldsymbol{X}^{-1}$ does not exist, then we will say $\boldsymbol{X}$ is singular.

Matrix Inversion

You'll never invert a matrix by hand.

We're going to use R

```
X<- matrix(NA, nrow=3, ncol=3)
X[1,]<- c(2, 3, 4)
X[2,]<- c(3, 1, 3)
X[3,]<- c(2, 4, 2)
X.inv<- solve(X)
> X.inv
[,1] [,2] [,3]
[1,] -0.5 0.5 0.25
[2,] 0.0 -0.2 0.30
[3,] 0.5 -0.1 -0.35
X.inv%*%X
[,1] [,2] [,3]
[1,] 1 0.000000e+00 -2.220446e-16
[2,] 0 1.000000e+00 0.000000e+00
[3,] 0 -2.220446e-16 1.000000e+00
```

Matrix Inversion

1) Calculate Inverses
2) Properties of Inverses

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Theorem

The inverse of matrix $\boldsymbol{X}$, $\boldsymbol{X}^{-1}$, *is unique*

Matrix Inversion

1) Calculate Inverses
2) Properties of Inverses

Theorem

The inverse of matrix $\boldsymbol{X}$, $\boldsymbol{X}^{-1}$, is unique

Proof.

By way of contradiction, suppose not. Then there are at least two matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ such that $\boldsymbol{AX} = \boldsymbol{I}$ and $\boldsymbol{CX} = \boldsymbol{I}$

This implies that,

$$\begin{aligned} \boldsymbol{AXC} &= (\boldsymbol{AX})C \\ &= \boldsymbol{IC} \\ &= \boldsymbol{C} \end{aligned}$$

Matrix Inversion

But it also implies that

$$\begin{aligned} \boldsymbol{AXC} &= \boldsymbol{A}(\boldsymbol{XC}) \\ &= \boldsymbol{A}(\boldsymbol{I}) \\ &= \boldsymbol{A} \end{aligned}$$

So $\boldsymbol{C} = \boldsymbol{AXC} = \boldsymbol{A}$ or $\boldsymbol{C} = \boldsymbol{A}$ but this contradicts our assumption that there are two unique inverses.

Matrix Inversion

Theorem

Suppose $\boldsymbol{A}$ *has inverse* $\boldsymbol{A}^{-1}$ *and* $\boldsymbol{B}$ *has inverse* $\boldsymbol{B}^{-1}$. *Then,*

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

Matrix Inversion

Theorem

Suppose $\boldsymbol{A}$ has inverse $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}$ has inverse $\boldsymbol{B}^{-1}$. Then,

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

Proof.

We need to show that $(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1})(\boldsymbol{AB}) = (\boldsymbol{AB})(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}) = \boldsymbol{I}$.

$$\begin{aligned}
(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1})(\boldsymbol{AB}) &= \boldsymbol{B}^{-1}(\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{B} \\
&= \boldsymbol{B}^{-1}\boldsymbol{I}\boldsymbol{B} \\
&= \boldsymbol{B}^{-1}\boldsymbol{B} \\
&= \boldsymbol{I}
\end{aligned}$$

Matrix Inversion

$$\begin{aligned}
(\boldsymbol{AB})(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}) &= \boldsymbol{A}(\boldsymbol{BB}^{-1})\boldsymbol{A}^{-1} \\
&= \boldsymbol{AIA}^{-1} \\
&= \boldsymbol{AA}^{-1} \\
&= \boldsymbol{I}
\end{aligned}$$

So $\boldsymbol{AB}$ is invertible and $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

Challenge Inversion Proofs

- Show that $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$.
- Show that $(k\boldsymbol{A})^{-1} = \frac{1}{k}\boldsymbol{A}^{-1}$

Matrix Inversion

1) How to Calculate an Inverse
2) Inversion properties
3) When do inverses exist?

Linear Independence: not repeated information in matrix will be the key (for both inversion and regressions)

Matrix Inversion: Existence

Definition

Suppose we have a set of vectors $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$
And consider the system of equations

$$k_1 \boldsymbol{v}_1 + k_2 \boldsymbol{v}_2 + \ldots + k_r \boldsymbol{v}_r \quad = \quad \boldsymbol{0}$$

If the only solution is $k_1 = 0, k_2 = 0, k_3 = 0, \ldots, k_r = 0$ *then we say that the set is linearly independent. If there are other solutions, then the set is linearly dependent.*

Matrix Inversion: Existence

Consider $\boldsymbol{v}_1 = (1,0,0)$, $\boldsymbol{v}_2 = (0,1,0)$, $\boldsymbol{v}_3 = (0,0,1)$
Can we write this as a combination of other vectors?

Matrix Inversion: Existence

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Consider $\boldsymbol{v}_1 = (1, 0, 0)$, $\boldsymbol{v}_2 = (0, 1, 0)$, $\boldsymbol{v}_3 = (0, 0, 1)$, $\boldsymbol{v}_4 = (1, 2, 3)$.
Can we write this as a combination of other vectors?

Matrix Inversion: Existence

Consider $\boldsymbol{v}_1 = (1, 0, 0)$, $\boldsymbol{v}_2 = (0, 1, 0)$, $\boldsymbol{v}_3 = (0, 0, 1)$
Can we write this as a combination of other vectors? no!
Consider $\boldsymbol{v}_1 = (1, 0, 0)$, $\boldsymbol{v}_2 = (0, 1, 0)$, $\boldsymbol{v}_3 = (0, 0, 1)$, $\boldsymbol{v}_4 = (1, 2, 3)$.
Can we write this as a combination of other vectors?

$$\boldsymbol{v}_4 = \boldsymbol{v}_1 + 2\boldsymbol{v}_2 + 3\boldsymbol{v}_3$$

Matrix Inversion: Existence

Theorem

Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_K \in \Re^n$. *If* $K > n$ *then the set is linearly dependent*

Matrix Inversion: Existence

Theorem

Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_K \in \Re^n$. If $K > n$ then the set is linearly dependent

- $\boldsymbol{v}_1 = (v_{11}, v_{21}, \ldots, v_{n1})$

Matrix Inversion: Existence

Theorem

Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_K \in \Re^n$. *If* $K > n$ *then the set is linearly dependent*

- $\boldsymbol{v}_1 = (v_{11}, v_{21}, \ldots, v_{n1})$
- Says that if there are more vectors in the set than elements in each vector, one must be linearly dependent

Matrix Inversion: Existence

We care because of the following theorem

Theorem

Suppose $\boldsymbol{X}$ *is an* $n \times n$ *matrix. Recall we can write this matrix as* $\begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_n \end{pmatrix}$.

Then $\boldsymbol{X}$ *has an inverse if and only if* $S = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\}$ *is linearly independent*

If this is true, we say $\boldsymbol{X}$ has full rank

Linear Regression

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In methods classes you learn about linear regression. For each i (individual) we observe covariates $x_{i1}, x_{i2}, \ldots, x_{ik}$ and independent variable Y_i. Then,

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Linear Regression

In methods classes you learn about linear regression. For each i (individual) we observe covariates $x_{i1}, x_{i2}, \ldots, x_{ik}$ and independent variable Y_i. Then,

$$
\begin{aligned}
Y_1 &= \beta_0 + \beta_1 x_{11} + \beta_2 x_{12} + \ldots + \beta_k x_{1k} \\
Y_2 &= \beta_0 + \beta_1 x_{21} + \beta_2 x_{22} + \ldots + \beta_k x_{2k} \\
\vdots \quad &\vdots \quad \vdots \\
Y_i &= \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} \\
\vdots \quad &\vdots \quad \vdots \\
Y_n &= \beta_0 + \beta_1 x_{n1} + \beta_2 x_{n2} + \ldots + \beta_k x_{nk}
\end{aligned}
$$

Linear Regression

- Define $\boldsymbol{x}_i = (1, x_{i1}, x_{i2}, \ldots, x_{ik})$
- Define $\boldsymbol{X} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_n \end{pmatrix}$
- Define $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_k)$
- Define $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)$.

Then we can write

$$\boldsymbol{Y} \quad = \quad \boldsymbol{X}\beta$$

Linear Regression

$$\begin{aligned}
\boldsymbol{Y} &= \boldsymbol{X}\boldsymbol{\beta} \\
\boldsymbol{X}^{'}\boldsymbol{Y} &= \boldsymbol{X}^{'}\boldsymbol{X}\boldsymbol{\beta} \\
(\boldsymbol{X}^{'}\boldsymbol{X})^{-1}\boldsymbol{X}^{'}\boldsymbol{Y} &= (\boldsymbol{X}^{'}\boldsymbol{X})^{-1}\boldsymbol{X}^{'}\boldsymbol{X}\boldsymbol{\beta} \\
(\boldsymbol{X}^{'}\boldsymbol{X})^{-1}\boldsymbol{X}^{'}\boldsymbol{Y} &= \boldsymbol{\beta}
\end{aligned}$$

Big question: is $(\boldsymbol{X}^{'}\boldsymbol{X})^{-1}$ invertible?
We'll investigate in homework!

An Introduction to Eigenvectors, Values, and Diagonalization

Definition

Suppose $\boldsymbol{A}$ *is an* $N \times N$ *matrix and* λ *is a scalar.*
If

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$$

Then $\boldsymbol{x}$ *is an eigenvector and* λ *is the associated eigenvalue*

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- $\boldsymbol{A}$ stretches the eigenvector $\boldsymbol{x}$

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- $\boldsymbol{A}$ stretches $\boldsymbol{x}$ by λ

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- $\boldsymbol{A}$ stretches $\boldsymbol{x}$ by λ
- To find eigenvectors/values: (`eigen` in `R`)
  - Find λ that solves $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$
  - Find vectors in null space of:

$$(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$$

Definition

Suppose A is an $N \times N$ matrix and A has N linearly independent eigenvectors. Then, we can write A as

$$\boldsymbol{A} = \boldsymbol{W}^{'} \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N \end{pmatrix} \boldsymbol{W}$$

Where $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the eigenvalues and $\boldsymbol{w}$ is a matrix of the eigenvectors.

Definition

Suppose $\boldsymbol{X}$ *is an* $N \times J$ *matrix. Then* $\boldsymbol{X}$ *can be written as:*

$$\boldsymbol{X} = \underbrace{\boldsymbol{U}}_{N\times N}\underbrace{\boldsymbol{S}}_{N\times J}\underbrace{\boldsymbol{V}^{'}}_{J\times J}$$

Where:

$$\begin{aligned} \boldsymbol{U}^{'}\boldsymbol{U} &= \boldsymbol{I}_N \\ \boldsymbol{V}^{'}\boldsymbol{V} &= \boldsymbol{V}\boldsymbol{V}^{'} = \boldsymbol{I}_J \end{aligned}$$

$\boldsymbol{S}$ *contains* $\min(N, J)$ *singular values,* $\sqrt{\lambda_j} \geq 0$ *down the diagonal and then 0's for the remaining entries*