Math Camp

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Where we're at

- Conditional Probability/Bayes' Rule

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- Today: Random Variables

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- Probability Mass Functions

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- A Brief Introduction to Markov Chains

Random Variable: Intuition

Recall the three parts of our probability model

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Random variables: functions defined on the sample space

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Random Variable: A Random variable X is a function from the sample space to real numbers. In notation,

$$X : Sample\ Space \to \mathcal{R}$$

- *X*'s domain are all outcomes (Sample Space)
- *X*'s range is the Real line (or some subset of it)
- Because *X* is defined on outcomes, makes sense to write $p(X)$ (we'll talk about this soon)

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Treatment assignment:

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Big Question: How do we compute P(X=1), P(X=0), etc?

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$p(X = a) = 0$, for all $a \notin (0, 1, 2, 3)$

Probability Mass Function: Intuition

Probability Mass Function
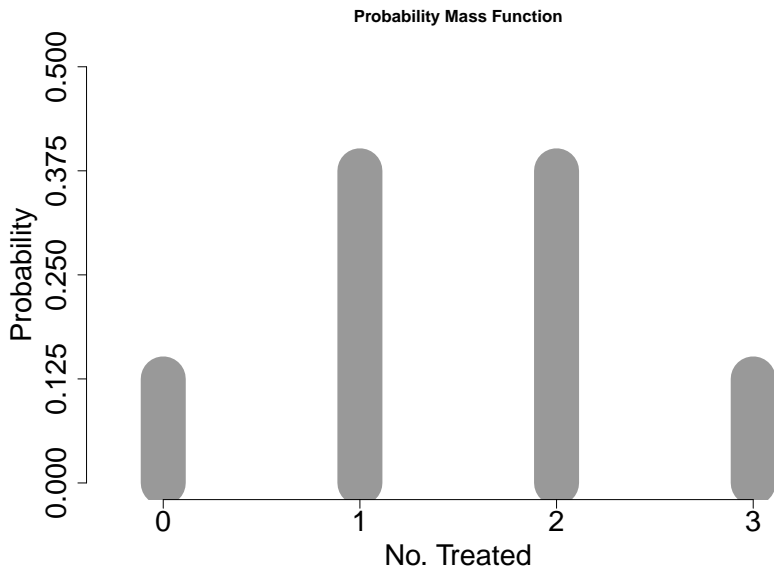
Probability Mass Function: Intuition

Consider outcome of election:

- $X(v) = 1$ if $v > 0.5$ otherwise $X(v) = 0$
- $P(X = 1)$ then is equal to $P(v > 0.5)$

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(Brief aside) Countable: A set is countable if there is a function that can map all its elements to the natural numbers $\{1, 2, 3, 4, \ldots\}$ (one-to-one, injective). If it is onto (from S to all natural numbers, surjective), then we say the set is countably infinite

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Definition

Probability Mass Function: For a discrete random variable X, define the probability mass function $p(x)$ as

$$p(x) \quad = \quad P(X = x)$$

Probability Mass Function: Example 2

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Topic 1 (say, war):

P(afghanistan) = 0.3; P(fire) = 0.0001; P(department) = 0.0001; P(soldier) = 0.2; P(troop) = 0.2; P(war)=0.2997; P(grant)=0.0001

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Topic Models: take a set of documents and estimate topics.

Definition

Cumulative Mass (distribution) Function: For a random variable X, define the cumulative mass function $F(x)$ as,

$$F(x) = P(X \leq x)$$

- Characterizes how probability cumulates as X gets larger
- $F(x) \in [0, 1]$
- $F(x)$ is non-decreasing

Cumulative Mass Function: Example

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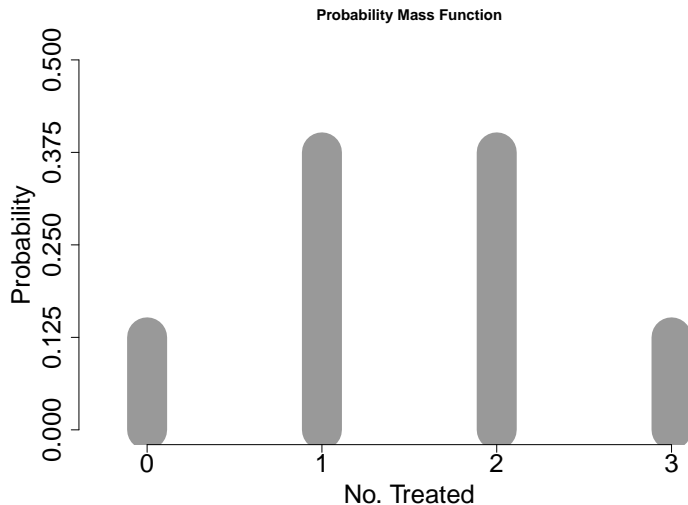Consider the three person experiment. $P(T) = P(C) = 1/2$.
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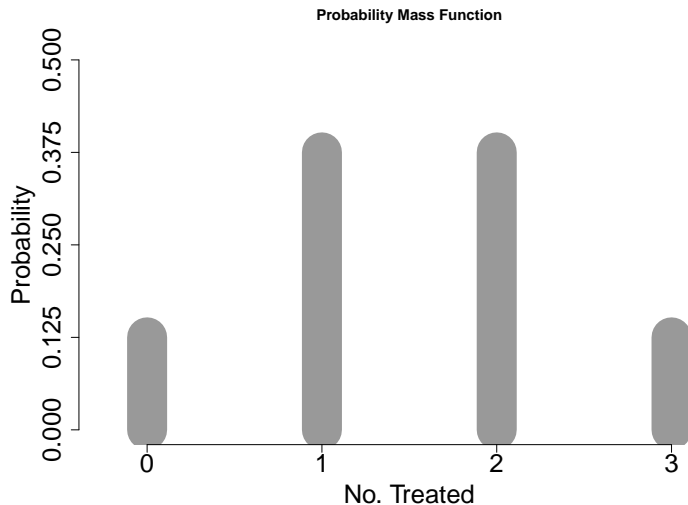$$

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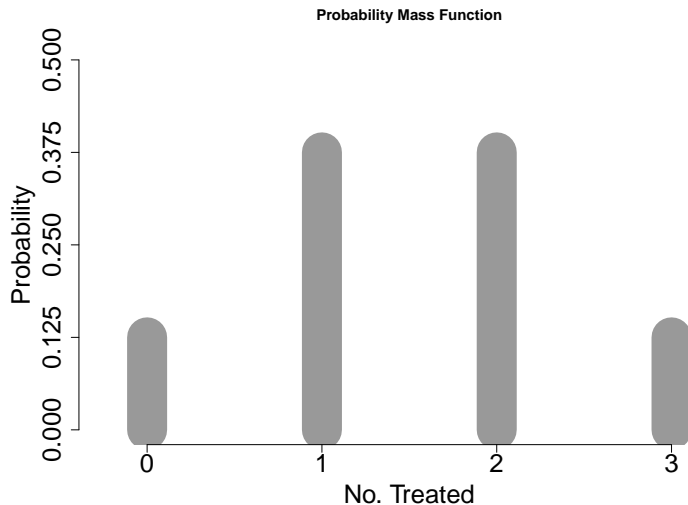
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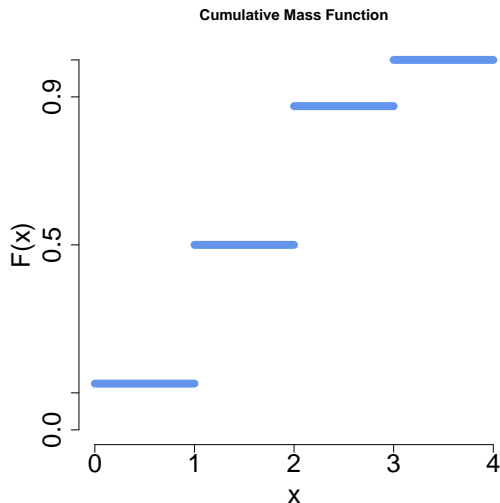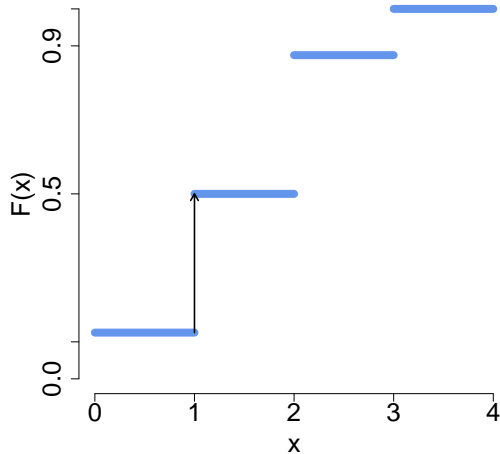
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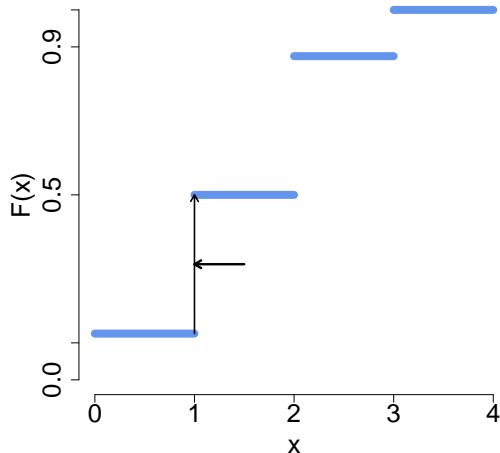# Cumulative Mass Function

There is a close relationship between pmf's and cmf's.
Consider Previous example:

Cumulative Mass Function

Expectation

What can we expect from a trial?

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Expected Value: define the expected value of a function X as,

$$E[X] = \sum_{x:p(x)>0} xp(x)$$

In words: for all values of x with p(x) greater than zero, take the weighted average of the values

Expectation Example: Simple Experiment

Suppose again X is number of units assigned to treatment, in one of our previous example.

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&= 1.5
\end{aligned}
$$

Expectation Example: A Single Person Poll

Suppose that there is a group of N people.

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Indicator Variables and Probabilities

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□

Functions of Random Variables

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$$E[g(X)] = \sum_i g(x_i) p(x_i)$$

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&= \sum_j y_j P(g(X) = y_j) \\
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\end{aligned}
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$\square$

Functions of Random Variables: Example

Let's suppose that X is the number of observations assigned to treatment (from our previous example).

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&= 0 + \frac{3}{8} + \frac{12}{8} + \frac{9}{8} \\
&= \frac{24}{8} = 3
\end{aligned}
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Functions of Random Variables: Corollary

Corollary

Suppose X is a random variable and a and b are constants (not random variables). Then,

$$E[aX + b] = aE[X] + b$$

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&= aE[X] + b(1)
\end{aligned}
$$

Variance

Expected value is a measure of central tendency.

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&= E[X^2] - 2E[X]^2 + E[X]^2 \\
&= E[X^2] - E[X]^2 \\
&= \text{Var}(X)
\end{aligned}
$$

Variance

Definition

The variance of a random variable X, $var(X)$, is

$$
\begin{aligned}
var(X) &= E[(X - E[X])^2] \\
&= E[X^2] - E[X]^2
\end{aligned}
$$

- We will define the standard deviation of X, $sd(X) = \sqrt{var(X)}$
- $var(X) \geq 0$.

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Continue the three person experiment, with $P(T) = P(C) = 1/2$.

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E[X^2] &= 3 \\
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E[X^2] &= 3 \\
E[X]^2 &= 1.5^2 = 2.25 \\
Var(X) &= E[X^2] - E[X]^2 \\
&= 3 - 2.25 = 0.75
\end{aligned}
$$

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\begin{aligned}
Var(Y) &= E[(aX + b - aE[X] - b)^2] \\
&= E[(a^2 X^2 - 2a^2 X E[X] + a^2 E[X]^2)]
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&= a^2 (E[X^2] - E[X]^2)
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$$

□

Famous Distributions

- Bernoulli
- Binomial
- Multinomial
- Poisson

Models of how world works.

Bernoulli Random Variable

Definition

Suppose X is a random variable, with $X \in \{0, 1\}$ and $P(X = 1) = \pi$.
Then we will say that X is Bernoulli random variable,

$$p(k) \;=\; \pi^k (1 - \pi)^{1-k}$$

for $k \in \{0, 1\}$ and $p(k) = 0$ otherwise.
We will (equivalently) say that

$$Y \;\sim\; Bernoulli(\pi)$$

Bernoulli Random Variable

Suppose we flip a fair coin and $Y = 1$ if the outcome is Heads .

$$
\begin{aligned}
Y &\sim \text{Bernoulli}(1/2) \\
p(1) &= (1/2)^1(1 - 1/2)^{1-1} = 1/2 \\
p(0) &= (1/2)^0(1 - 1/2)^{1-0} = (1 - 1/2)
\end{aligned}
$$

Bernoulli Random Variable Moments

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$E[Y] = \pi$

$\text{var}(Y) = \pi(1-\pi)$ What is the maximum variance?

Example: Winning a War

Suppose country 1 is engaged in a conflict and can either win or lose.

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Definition

Suppose Y is a random variable that counts the number of successes in N independent and identically distributed Bernoulli trials. Then Y is a Binomial *random variable,*

$$p(k) \;=\; \binom{N}{k} \pi^k (1-\pi)^{1-k}$$

for $k \in \{0, 1, 2, \ldots, N\}$ and $p(k) = 0$ otherwise.
Equivalently,

$$Y \;\sim\; Binomial(N, \pi)$$

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Recall our experiment example:

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$E[Z] = N\pi$
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Voter Turnout

Suppose we have a set N voters, with iid turnout decisions
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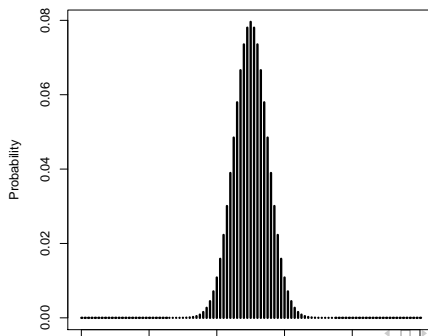$$P(k \geq M) = \sum_{k=M}^{N} \binom{N}{k} \pi^k (1-\pi)^{N-k}$$

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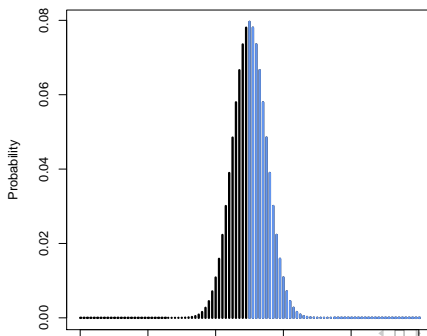$$P(k \geq M) = \sum_{k=M}^{N} \binom{N}{k} \pi^k (1-\pi)^{N-k}$$

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R Code!

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$$Y_i \sim \text{Bernoulli}(0.9) \text{ if } j \text{ votes}$$

Voter Turnout, with Spillovers

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Now, $N/2$ are leaders, who turnout with probability $(1/2)$

But, $N/2$ are followers, whose turnout depends on a specific leader

Suppose follower i depends on only one leader j (and each follower has their own leader)

$$Y_i \sim \text{Bernoulli}(0.9) \text{ if } j \text{ votes}$$
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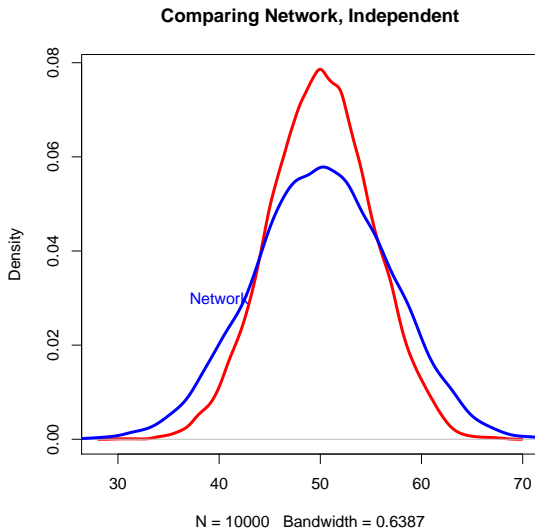
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Voter Turnout, with Spillovers

Comparing Network, Independent

Network

N = 10000   Bandwidth = 0.6387

Trials with More than Two Outcomes

Definition

Suppose we observe a trial, which might result in J outcomes.
And that $P(\text{outcome} = i) = \pi_i$
$\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_J)$ where $Y_j = 1$ if outcome j occurred and 0 otherwise.
Then $\boldsymbol{Y}$ follows a multinomial distribution, with

$$p(\boldsymbol{y}) \;\;=\;\; \pi_1^{y_1} \pi_2^{y_2} \ldots \pi_k^{y_k}$$

if $\sum_{i=1}^{k} y_i = 1$ and the pmf is 0 otherwise.
Equivalently, we'll write

$$\boldsymbol{Y} \;\;\sim\;\; \text{Multnomial}(1, \boldsymbol{\pi})$$
$$\boldsymbol{Y} \;\;\sim\;\; \text{Categorial}(\boldsymbol{\pi})$$

Multinomial Properties + Notes

Computer scientists: commonly call Multinomial$(1, \boldsymbol{\pi})$ Discrete$(\boldsymbol{\pi})$.

$$
\begin{aligned}
E[X_i] &= N\pi_i \\
\text{var}(X_i) &= N\pi_i(1 - \pi_i)
\end{aligned}
$$

Investigate Further in Homework!

Counting the Number of Events

Often interested in counting number of events that occur:

1) Number of wars started

2) Number of speeches made

3) Number of bribes offered

4) Number of people waiting for license

Generally referred to as event counts

Stochastic processes: a course provide introduction to many processes (Queing Theory)

Poisson Distribution

Definition

*Suppose X is a random variable that takes on values $X \in \{0, 1, 2, \ldots, \}$
and that $P(X = k) = p(k)$ is,*

$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

*for $k \in \{0, 1, \ldots, \}$ and 0 otherwise. Then we will say that X follows a
Poisson distribution with rate parameter λ.*

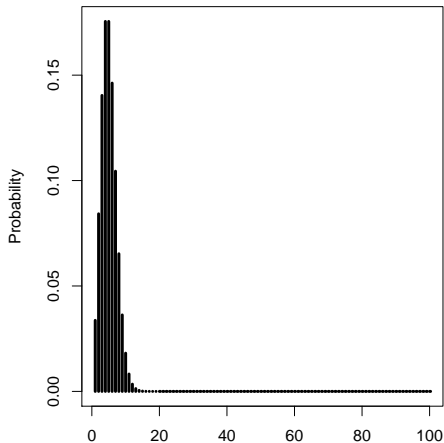$$X \sim Poisson(\lambda)$$

Example: Poisson Distribution

Suppose the number of threats a president makes in a term is given by
$X \sim \text{Poisson}(5)$.

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more threats?

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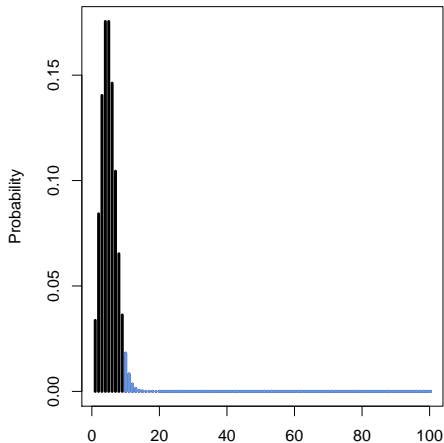$X \sim \text{Poisson}(5)$. What is the probability the president will make ten or
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R code!

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Properties:

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$$e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2!} + \ldots)$$

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\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \\
e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} &= e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2!} + \ldots) \\
&= e^{-\lambda}(e^{\lambda}) = 1
\end{aligned}
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Poisson Distribution

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$$E[X^2] \;=\; \sum_{k=0}^{\infty} \frac{k^2 e^{-\lambda}\lambda^k}{k!}$$

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E[X^2] &= \sum_{k=0}^{\infty} \frac{k^2 e^{-\lambda} \lambda^k}{k!} \\
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Very useful distribution, with strong assumptions. We'll explore in homework!

Often interested in how processes evolve over time

- Given voting history, probability of voting in the future

- Given history of candidate support, probability of future support

- Given prior conflicts, probability of future war

- Given previous words in a sentence, probability of next word

Potentially complex history

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Potentially complex history

Stochastic Process

Definition

Suppose we have a sequence of random variables
$\{X\}_{i=0}^{M} = X_0, X_1, X_2, \ldots, X_M$ *that take on the countable values of S. We will call $\{X\}_{i=0}^{M}$ a stochastic process with state space S.*

If index gives time, then we might condition on history to obtain probability

$$\text{PMF } X_t, \text{ given history } = P(X_t | X_{t-1}, X_{t-2}, \ldots, X_1, X_0)$$

Still Complex

Markov Chain

Definition

Suppose we have a stochastic process $\{X\}_{i=0}^{M}$ with countable state space S. Then $\{X\}_{i=0}^{M}$ is a markov chain if:

$$P(X_t|X_{t-1}, X_{t-2}, \ldots, X_1, X_0) = P(X_t|X_{t-1})$$

A Markov chain's future depends only on its current state

Transition Matrix

Habitual turnout?

$$\boldsymbol{T} \;=\; \begin{pmatrix} & \text{Vote}_t & \text{Not Vote}_t \\ \text{Vote}_{t-1} & 0.8 & 0.2 \\ \text{Not Vote}_{t-1} & 0.3 & 0.7 \end{pmatrix}$$

- Suppose someone starts as a voter—what is their behavior after
- 1 iteration?
- 2 interations?
- The long run?

R Code!

Tomorrow: Continuous Random Variables!