Math Camp

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Questions?

Today:

1) Properties of Expectations
2) Changing Coordinates
3) Moment Generating Functions
4) Inequalities
5) Convergence

Iterated Expectations

Proposition

Suppose X and Y are random variables. Then

$$E[X] = E[E[X|Y]]$$

- Inner Expectation is $E[X|Y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$.
- Outer expectation is over y.

Iterated Expectations

Proof.

$$
\begin{aligned}
E[E[X|Y]] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X|Y}(x|y) f_Y(y) dx dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X|Y}(x|y) f_Y(y) dy dx \\
&= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f(x,y) dy dx \\
&= \int_{-\infty}^{\infty} x f_X(x) dx \\
&= E[X]
\end{aligned}
$$

□

Iterated Expectations

Definition

Suppose Y is a continuous random variable with $Y \in [0, 1]$ and pdf of Y given by

$$f(y) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y^{\alpha_1 - 1}(1 - y)^{\alpha_2 - 1}$$

Then we will say Y is a Beta distribution with parameters α_1 and α_2. Equivalently,

$$Y \sim Beta(\alpha_1, \alpha_2)$$

- Beta is a distribution on proportions
- Beta is a special case of the Dirichlet distribution
- $E[Y] = \frac{\alpha_1}{\alpha_1 + \alpha_2}$

Iterated Expectations

Suppose

$$\begin{aligned}\pi &\sim \text{Beta}(\alpha_1, \alpha_2)\\ Y|\pi, n &\sim \text{Binomial}(n, \pi)\end{aligned}$$

What is $E[Y]$?

$$\begin{aligned}E[Y] &= E[E[Y|\pi]]\\ &= \int_{-\infty}^{\infty} \sum_{j=0}^{N} \binom{N}{j} j p(j|\pi) f(\pi) d\pi\\ &= \int_{-\infty}^{\infty} N\pi f(\pi) d\pi\\ &= N\frac{\alpha_1}{\alpha_1 + \alpha_2}\end{aligned}$$

Change of Coordinates

Proposition

Suppose X is a random variable and $Y = g(X)$, where $g : \Re \rightarrow \Re$ that is a monotonic function.
Define $g^{-1} : \Re \rightarrow \Re$ such that $g^{-1}(g(X)) = X$ and is differentiable. Then,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{\partial g^{-1}(y)}{\partial y} \right| \textit{ if } y = g(x) \textit{ for some } x \\ &= 0 \textit{ otherwise} \end{aligned}$$

Change of Coordinates

Proof.

Suppose $g(\cdot)$ is monotonically increasing (WLOG)

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(g(X) \leq y) \\
&= P(X \leq g^{-1}(y)) \\
&= F_X(g^{-1}(y))
\end{aligned}
$$

Now differentiating to get the pdf

$$
\begin{aligned}
\frac{\partial F_Y(y)}{\partial y} &= \frac{\partial F_X(g^{-1}(y))}{\partial y} \\
&= f_X(g^{-1}(y)) \frac{\partial g^{-1}(y)}{\partial y}
\end{aligned}
$$

Then this is a pdf because $\frac{\partial g^{-1}(Y)}{\partial y} > 0$.

$\square$

Change of Coordinates

Suppose X is a random variable with pdf $f_X(x)$. Suppose $Y = X^n$. Find $f_Y(y)$.
Then $g^{-1}(x) = x^{1/n}$.

$$
\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{\partial g^{-1}(Y)}{\partial y} \right| \\
&= f_X(y^{1/n}) \frac{y^{\frac{1}{n}-1}}{n}
\end{aligned}
$$

We've used this to derive many of the pdfs

- Normal distribution
- Chi-Squared Distribution

Moment Generating Functions

Definition

Suppose X is a random variable with pdf f. Define,

$$E[X^n] \;=\; \int_{-\infty}^{\infty} x^n f(x) dx$$

We will call X^n the n^{th} moment of X

- By this definition $var(X) =$ Second Moment $-$ First Moment^2
- We are assuming that the integral converges

Moment Generating Functions

Proposition

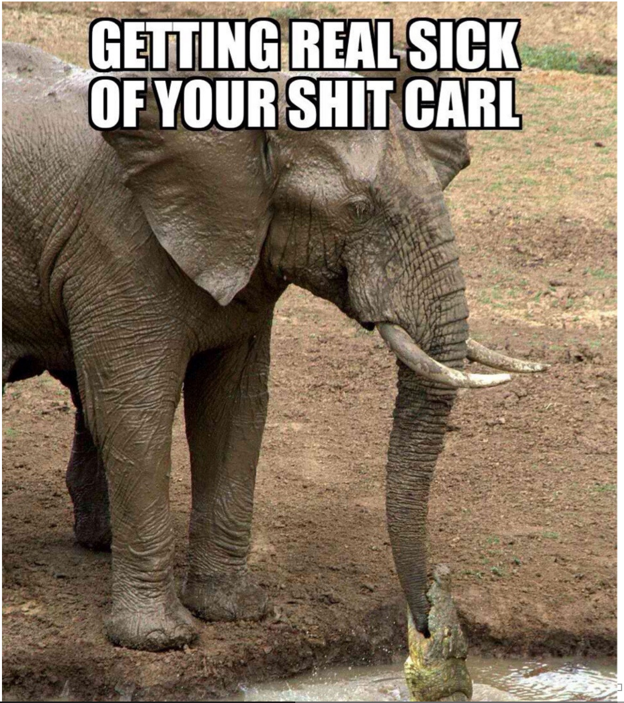
Suppose X is a random variable with pdf $f(x)$. Call $M(t) = E[e^{tX}]$,

$$
\begin{aligned}
M(t) &= E[e^{tX}] \\
&= \int_{-\infty}^{\infty} e^{tx} f(x) dx
\end{aligned}
$$

We will call $M(t)$ the moment generating function, because:

$$
\frac{\partial^n M(t)}{\partial^n t}|_0 = E[X^n]
$$

(Assuming that we can interchange derivative and integral)


GETTING REAL SICK
OF YOUR SHIT CARL

Moment Generating Functions

Proof.

Recall the Taylor Expansion of e^{tX} at 0,

$$e^{tX} = 1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \ldots$$

Then,

$$E[e^{tX}] = 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E[X^3] + \ldots$$

Differentiate once:

$$\begin{aligned} \frac{\partial M(t)}{\partial t} &= 0 + E[X] + \frac{2t}{2!}E[X^2] + \ldots \\ M^{'}(0) &= 0 + E[X] + 0 + 0 \ldots \end{aligned}$$

$\square$

Proof.

Differentiate n times

$$\begin{aligned}\frac{\partial^n M(t)}{\partial^n t} &= 0 + 0 + 0 + \ldots + \frac{n \times n-1 \times \ldots 2 \times t^0 E[X^n]}{n!} + \frac{n! t E[X^{n+1}]}{(n+1)!} + \ldots \\ &= \frac{n! E[X^n]}{n!} + \frac{n! t E[X^{n+1}]}{(n+1)!} + \ldots\end{aligned}$$

Evaluated at 0, yields $M^n(0) = E[X^n]$

□

- If two random variables, X and Y have the same moment generating functions, then $F_X(x) = F_Y(x)$ for almost all x.

The Moments of the Normal Distribution

Suppose $Z \sim N(0,1)$.

$$E[e^{tX}] \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx}e^{-x^2/2}dx$$

$tx - \frac{1}{2}x^2 = -\frac{1}{2}\left((x-t)^2 - t^2\right)$

$$\begin{aligned} E[e^{tX}] &= \frac{1}{\sqrt{2\pi}}e^{\frac{t^2}{2}}\int_{-\infty}^{\infty} e^{-(x-t)^2/2}dx \\ &= e^{\frac{t^2}{2}} \end{aligned}$$

Extracting Moments of the Normal Distribution

$$
\begin{aligned}
M^{'}(0) &= E[X] = e^{t^2/2}t|_0 = 0 \\
M^{''}(0) &= E[X^2] = e^{t^2/2}(t^2+1)|_0 = 1 \\
M^{'''}(0) &= E[X^3] = e^{t^2/2}t(t^2+3)|_0 = 0 \\
M^{''''}(0) &= E[X^4] = e^{t^2/2}(t^4+6t^2+3)|_0 = 3 \\
M^{5}(0) &= E[X^5] = e^{t^2/2}t(t^4+10t^2+15)|_0 = 0 \\
M^{6}(0) &= E[X^6] = e^{t^2/2}(t^6+15t^4+45t^2+15)|_0 = 15
\end{aligned}
$$

Proposition

Suppose X_i are a sequence of independent random variables. Define

$$Y = \sum_{i=1}^{N} X_i$$

Then

$$M_Y(t) = \prod_{i=1}^{N} M_{X_i}(t)$$

Proof.

$$
\begin{aligned}
M_Y(t) &= E[e^{tY}] \\
&= E[e^{t\sum_{i=1}^{N} X_i}] \\
&= E[e^{tX_1 + tX_2 + \ldots tX_N}] \\
&= E[e^{tX_1}]E[e^{tX_2}] \ldots E[e^{tX_N}] \text{ (by independence)} \\
&= \prod_{i=1}^{N} E[e^{tX_i}]
\end{aligned}
$$

□

Inequalities and Limit Theorems

Limit Theorems

- What happens when we consider a long sequence of random variables ?
- What can we reasonably infer from data?
    - Laws of large numbers: averages of random variables converge on expected value?
    - Central Limit Theorems: sum of random variables have normal distribution?
- We'll focus on intuition for both, but we'll prove some stuff too.

Weak Law of Large Numbers

Proof plan:

- Markov's Inequality
- Chebyshev's Inequality
- Weak Law of Large Numbers

Markov's Inequality

Proposition

Suppose X is a random variable that takes on non-negative values. Then, for all $a > 0$,

$$P(X \geq a) \quad \leq \quad \frac{E[X]}{a}$$

Markov's Inequality

Proof.

For $a > 0$,

$$\begin{aligned} E[X] &= \int_0^\infty xf(x)dx \\ &= \int_0^a xf(x)dx + \int_a^\infty xf(x)dx \end{aligned}$$

Because $X \geq 0$,

$$E[X] \geq \int_a^\infty xf(x)dx \geq \int_a^\infty af(x)dx = aP(X \geq a)$$

$$\frac{E[X]}{a} \geq P(X \geq a)$$

□

Chebyshev's Inequality

Proposition

If X is a random variable with mean μ and variance σ^2, then, for any value $k > 0$,

$$P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

Chebyshev's Inequality

Proof.

Define the random variable

$$Y = (X - \mu)^2$$

Where $\mu = E[X]$.
Then we know Y is a non-negative random variable. Set $a = k^2$.
Applying the inequality:

$$\begin{aligned} P(Y \geq k^2) &\leq \frac{E[Y]}{k^2} \\ P((X-\mu)^2 \geq k^2) &\leq \frac{E[(X-\mu)^2]}{k^2} \\ P((X-\mu)^2 \geq k^2) &\leq \frac{\sigma^2}{k^2} \end{aligned}$$

□

Chebyshev's Inequality

Further we know that,

$$(X - \mu)^2 \geq k^2$$

Implies that

$$|X - \mu| \geq k$$

Thus, we have shown

$$P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

Sequence of Random Variables

Sequence of Independent and Identically, Distributed Random variables.

- Sequence: $X_1, X_2, \ldots, X_n, \ldots$
- Think of a sequence as sampled data:
  - Suppose we are drawing a sample of N observations
  - Each observation will be a random variable, say X_i
  - With realization x_i

Mean/Variance of Sample Mean

Proposition

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Let $\bar{X}_n$ be the sample mean. Then $E[\bar{X}_n] = \mu$ and $var(\bar{X}_n) = \frac{\sigma^2}{n}$

Proof.

$$
\begin{aligned}
E[\bar{X}_n] &= \frac{1}{n}\sum_{i=1}^{n} E[X_i] \\
&= \frac{1}{n} n\mu = \mu
\end{aligned}
$$

□

Mean/Variance of Sample Mean

$$
\begin{aligned}
\text{var}(\bar{X}_n) &= \frac{1}{n^2}\text{var}(\sum_{i=1}^{n} X_i) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\text{var}(X_i) \\
&= \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}
\end{aligned}
$$

Weak Law of Large Numbers

Proposition

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution with mean μ and $Var(X_i) = \sigma^2$. Then, for all $\epsilon > 0$,

$$P\left\{\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| \geq \epsilon\right\} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Weak Law of Large Numbers

Proof.

From our previous proposition

$$\frac{\mathrm{E}[X_1 + X_2 + \cdots + X_n]}{n} = \frac{\sum_{i=1}^{n} E[X_i]}{n} = \mu$$

Further,

$$\mathrm{E}[(\frac{\sum_{i=1}^{n} X_i - \mu}{n})^2] = \frac{\mathrm{Var}(X_1 + X_2 + \cdots + X_n)}{n^2} = \frac{\sum_{i=1}^{n} \mathrm{Var}(X_i)}{n^2} = \frac{\sigma^2}{n}$$

Apply Chebyshev's Inequality:

$$P\left\{\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| \geq \epsilon\right\} \leq \frac{\sigma^2}{n\epsilon^2}$$

$\square$

Suppose $X_1, X_2, \ldots$ are iid normal distributions,

$$X_i \sim \text{Normal}(0, 10)$$

$$P\left\{\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| \geq 0.1\right\} \text{ as } n \rightarrow \infty$$

Suppose we want to guarantee that we have at most a 0.01 probability of being more than 0.1 away from the true μ. How big do we need n?

$$\begin{aligned} 0.01 &= \frac{10}{n(0.1^2)} \\ n &= \frac{1000}{0.01} \\ n &= 100,000 \end{aligned}$$

Sequences and Convergence

Sequence (refresher):

$$\{a_i\}_{i=1}^{\infty} \quad = \quad \{a_1, a_2, a_3, \ldots, a_n, \ldots, \}$$

Definition

We say that the sequence $\{a_i\}_{i=1}^{\infty}$ *converges to real number* A *if for each* $\epsilon > 0$ *there is a positive integer* N *such that for* $n \geq N$, $|a_n - A| < \epsilon$

Sequences and Convergence

Sequence of functions:

$$\{f_i\}_{i=1}^{\infty} \quad = \quad \{f_1, f_2, f_3, \ldots, f_n, \ldots,\}$$

Definition

Suppose $f_i : X \rightarrow \Re$ for all i. Then $\{f_i\}_{i=1}^{\infty}$ converges pointwise to f if, for all $x \in X$ and $\epsilon > 0$, there is an N such that for all $n \geq N$,

$$|f_n(x) - f(x)| < \epsilon$$

This is as strong of a statement as we're likely to make in statistics

Convergence Definitions

Define $\widehat{\theta}_n$ to be estimator for θ based on n observations.
Sequence of estimators: increasing sample size

$$\left\{\widehat{\theta}_i\right\}_{i=1}^{n} = \left\{\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3, \ldots, \widehat{\theta}_n\right\}$$

Question: What can we say about $\left\{\widehat{\theta}_i\right\}_{i=1}^{n}$ as $n \rightarrow \infty$?

- What is the probability $\widehat{\theta}_n$ differs from θ?
- What is the probability $\left\{\widehat{\theta}_i\right\}_{i=1}^{n}$ converges to θ?
- What is sampling distribution of $\widehat{\theta}_n$ as $n \rightarrow \infty$?

Convergence in Probability

Definition

We will say the sequence $\widehat{\theta}_n$ converges in probability to θ (perhaps a non-degenerate RV) if,

$$\lim_{n\to\infty} Prob(|\widehat{\theta}_n - \theta| > \epsilon) = 0$$

For any $\epsilon > 0$

- ϵ is a tolerance parameter: how much error around θ?
- In the limit, convergence in probability implies sampling distribution collapses on a spike at θ
- $\left\{\widehat{\theta}_i\right\}$ need not actually converge to θ, only P($|\theta_n$ - $\theta| > \epsilon) = 0$

Example (Cassella and Burger)

Suppose $S \sim$ Uniform(0,1). Define $X(s) = s$.
Suppose X_n is define as follows:

$$\begin{array}{rcl} X_1(s) = s + I(s \in [0,1]) & , & X_2(s) = s + I(s \in [0,1/2]) \\ X_3(s) = s + I(s \in [1/2,1]) & , & X_4(s) = s + I(s \in [0,1/3]) \\ X_5(s) = s + I(s \in [1/3,2/3]) & , & X_6(s) = s + I(s \in [2/3,1]) \end{array}$$

Does $X_n(s)$ pointwise converge to $X(s)$?
Does $X_n(s)$ converge in probability to $X(s)$?

$$P(|X_n - X| > \epsilon) \quad = \quad P(s \in [l_n, u_n])$$

Length of $[l_n, u_n] \rightarrow 0 \Rightarrow P(s \in [L_n, U_n]) = 0$

Almost Sure Convergence

Definition

We will say the sequence $\widehat{\theta}_n$ converges almost surely to θ if,

$$Prob(\lim_{n \to \infty} |\widehat{\theta}_n - \theta| > \epsilon) = 0$$

- Stronger: says that sequence converges to θ (almost everywhere))
- Think about definition of random variable: $\widehat{\theta}_n$ is a function from sample space to real line.
- Almost sure says that, for all outcomes (s) in sample space (S) $s \in S$,

$$\widehat{\theta}_n(s) \quad \rightarrow \quad \theta(s)$$

Except for a subset $\mathcal{N} \subset S$ such that $P(\mathcal{N}) = 0$.

Example (Cassella and Burger)

Suppose $S \sim$ Uniform(0,1).
Suppose X_n is define as follows:

$$\begin{aligned} X_1(s) = s + I(s \in [0,1]) \quad &, \quad X_2(s) = s + I(s \in [0, 1/2]) \\ X_3(s) = s + I(s \in [1/2, 1]) \quad &, \quad X_4(s) = s + I(s \in [0, 1/3]) \\ X_5(s) = s + I(s \in [1/3, 2/3]) \quad &, \quad X_6(s) = s + I(s \in [2/3, 1]) \end{aligned}$$

Does $X_n(s)$ converge almost surely to $X(s) = s$?
No!: the sequence doesn't converge for each s
For each value of s the sequence varies between s and $s + 1$ infinitely often

Convergence in Distribution

We've talked about $\widehat{\theta}_n$'s sampling distribution converging to a normal distribution.

This is convergence in distribution

Definition

$\widehat{\theta}_n$, with cdf $F_n(x)$, converges in distribution to random variable Y with cdf $F(x)$ if

$$\lim_{n \to \infty} |F_n(x) - F(x)| = 0$$

For all $x \in \Re$ where $F(x)$ is continuous.

- Weakest form of convergence almost sure $\rightarrow$ probability $\rightarrow$ distribution
- Says that cdfs are equal, says nothing about convergence of underlying RV
- Useful for justifying use of some sampling distributions

Convergence in Distribution $\not\Rightarrow$ Convergence in Probability

Define $X \sim N(0,1)$ and each $X_n = -X$. Then:
$X_n \sim N(0,1)$ for all n so X_n trivially converges to X. But,

$$\begin{aligned} P(|X_n - X| > \epsilon) &= P(|X + X| > \epsilon) \\ &= P(|2X| > \epsilon) \\ &= P(|X| > \epsilon/2) \not\rightsquigarrow 0 \end{aligned}$$

Central Limit Theorem

Proposition

Let $X_1, X_2, \ldots$ be a sequence of independent random variables with mean μ and variance σ^2. Let X_i have a cdf $P(X_i \leq x) = F(x)$ and moment generating function $M(t) = E[e^{tX_i}]$. Let $S_n = \sum_{i=1}^{n} X_i$. Then

$$\lim_{n\rightarrow\infty} P\left(\frac{S_n - \mu n}{\sigma\sqrt{n}} \leq x\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left(-\frac{z^2}{2}\right) dz$$

Proof plan:

1) Rely on Fact that convergence of MGFs$\rightsquigarrow$ convergence in CDFs
2) Show that MGFs, in limit, converge on normal MGF

Proposition

Let F_n be a sequence of cumulative distribution functions with the corresponding moment generating functions M_n. F be a cdf with the moment generating functions M. If $\lim_{n\rightarrow\infty} M_n(t) \rightarrow M(t)$ for all t in some interval, then $F_n(x) \rightsquigarrow F(x)$ for all x (when F is continuous).

Proposition

Suppose $\lim_{n\rightarrow\infty} a_n \rightarrow a$, then

$$\lim_{n\rightarrow\infty}\left(1 + \frac{a_n}{n}\right)^n = e^a$$

Proposition

Suppose $M(t)$ is a moment generating function some random variable X. Then $M(0) = 1$.

Proof of Central Limit Theorem (Courtsey of Swarthmore Notes)

Proof. Suppose $X_1, \ldots, X_n$ are iid variables with $E[X] = 0$, variance σ_x^2, Moment Generating Function (MGF) $M_x(t)$.
Let $S_n = \sum_{i=1}^n X_i$ and $Z_n = \frac{S_n}{\sigma_x \sqrt{n}}$.
$M_{S_n} = (M_x(t))^n$ and $M_{Z_n}(t) = \left(M_x\left(\frac{t}{\sigma_x\sqrt{n}}\right)\right)^n$
Using Taylor's Theorem we can write

$$M_x(s) \quad = \quad M_x(0) + sM_x^{'}(0) + \frac{1}{2}s^2 M_x^{''}(0) + e_s$$

$e_s/s^2 \rightarrow 0$ as $s \rightarrow 0$.

$$M_x(s) = M_x(0) + sM_x^{'}(0) + \frac{1}{2}s^2 M_x^{''}(0) + e_s$$

Filling in the values we have

$$M_x(s) = 1 + 0 + \frac{\sigma_x^2}{2}s^2 + \underbrace{e_s}_{\text{Goes to zero}}$$

Set $s = \frac{t}{\sigma_x \sqrt{n}}$ $\lim_{n \to \infty} s \to 0$. Then

$$\begin{aligned} M_{Z_n}(t) &= \left(1 + \frac{\sigma_x^2}{2}\left(\frac{t}{\sigma_x\sqrt{n}}\right)^2\right)^n \\ &= \left(1 + \frac{t^2/2}{n}\right)^n \\ \lim_{n\to\infty} M_{Z_n}(t) &= e^{\frac{t^2}{2}} \end{aligned}$$