Math Camp

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Where are we going?

Probability Theory:

1) Mathematical model of uncertainty
2) Foundation for statistical inference
3) Continues our development of key skills
   - Proofs [precision in thinking, useful for formulating arguments]
   - Statistical computing [basis for much of what you'll do in graduate school]

Model of Probability

Three parts to our probability model

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1) Sample space: set of all things that could happen

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2) Events: subsets of the sample space

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Three parts to our probability model

1) Sample space: set of all things that could happen
2) Events: subsets of the sample space
3) Probability: chance of an event

Sample Spaces: All Things that Can Happen

Definition

The sample space as the set of all things that can occur. We will collect all distinct outcomes into the set S

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Key point: this defines all possible realizations

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  - $E =$ W

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  - Outcome of 2010 election: one event

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Congregational Election Example

- One incumbent:
  - $E = \text{W}$
  - $F = \text{N}$
- Two Incumbents:
  - $E = \{(W, N), (W, W)\}$
  - $F = \{(N, N)\}$
- 435 Incumbents:
  - Outcome of 2010 election: one event
  - All outcomes where Dems retain control of House: one event

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Notation:

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Notation: x is an "element" of a set E:

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Notation: x is an "element" of a set E:
$x \in E$

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- $S = \Re$ and $E = [0, 1]$. What is E^c?

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- What is S^c?

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Suppose $E = W$, $F = N$. Then $E \cap F = \emptyset$ (there is nothing that lies in both sets)

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- Suppose $S = \{(H,H), (H,T), (T,H), (T,T)\}$. $E = \{(H,H)\}$, $F = \{(H,H), (T,H)\}$, and $G = \{(H,T), (T,T)\}$

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  - $E \cap G = \emptyset$

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- Suppose $S = \{H, T\}$. Then $E = H$ and $F = T$, then $E \cap F = \emptyset$
- Suppose $S = \{(H, H), (H, T), (T, H), (T, T)\}$. $E = \{(H, H)\}$, $F = \{(H, H), (T, H)\}$, and $G = \{(H, T), (T, T)\}$
  - $E \cap F = (H, H)$
  - $E \cap G = \emptyset$
  - $F \cap G = \emptyset$
- Suppose $S = \Re_{+}$. $E = \{x : x > 10\}$ and $F = \{x : x < 5\}$. Then $E \cap F = \emptyset$.

Events: Subsets of the Sample Space

Definition

Suppose we have events $E_1, E_2, \ldots, E_N$*.*
Define:

$$\cup_{i=1}^{N} E_i \quad = \quad E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_N$$

$\cup_{i=1}^{N} E_i$ *is the set of outcomes that occur at least once in* $E_1, \ldots, E_N$*.*

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Define:

$$\cap_{i=1}^{N} E_i \quad = \quad E_1 \cap E_2 \cap \ldots \cap E_N$$

$\cap_{i=1}^{N} E_i$ *is the set of outcomes that occur in each* E_i

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    - P is a function
    - Domain: all events E

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1) *For all events E,*
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All probability functions, P, satisfy three axioms:

1) *For all events E,*
   $0 \leq P(E) \leq 1$
2) $P(S) = 1$
3) *For all sequences of mutually exclusive events $E_1, E_2, \ldots, E_N$ (where N can go to infinity)*

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All probability functions, P, satisfy three axioms:

1) *For all events E,*
   $0 \leq P(E) \leq 1$
2) $P(S) = 1$
3) *For all sequences of mutually exclusive events $E_1, E_2, \ldots, E_N$ (where N can go to infinity)*
   $P\left(\cup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} P(E_i)$

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- Suppose we are flipping a pair of fair coins. Then $P(H, H) = 1/4$

Example: Congressional Elections

One candidate example:

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We'll use data to infer these things

Properties of Probability

We can derive intuitive properties of probability theory.

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In words: Probability an outcome in E happens is $1-$ probability an outcome in E doesn't.

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As you add more "outcomes" to a set, it can't reduce the probability.

Examples in R

Simulation: use pseudo-random numbers, computers to gain evidence for claim

Tradeoffs:

Pro Deep understanding of problem, easier than proofs

Con Never as general, can be deceiving if not done carefully (also, never a monte carlo study that shows a new method is wrong)

Walk through `R` code to simulate these two results

To the `R` code!

4.2. *Three different combination rules were used.* We then tried to identify the rules used to combine individual drug predictions into a combination score. Letting P() indicate probability of sensitivity, the rules used are:

$$\begin{aligned} P(TFAC) &= P(T) + P(F) + P(A) + P(C) - P(T)P(F)P(A)P(C), \\ P(TET) &= P(ET) = \max[P(E), P(T)], \text{ and} \end{aligned}$$

5 1

Inclusion/Exclusion

Proposition

Suppose $E_1, E_2, \ldots, E_n$ *are events. Then*

$$\begin{aligned} P(E_1 \cup E_2 \cup \cdots \cup E_n) &= \sum_{i=1}^{N} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} \cap E_{i_2}) + \cdots \\ &\quad + (-1)^{r+1} \sum_{i_1 < i_2 < \cdots < i_r} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}) \\ &\quad + \cdots + (-1)^{n+1} P(E_1 \cap E_2 \cap \cdots E_n) \end{aligned}$$

Proof: Version 1, Intuition

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\text{count} &= -\sum_{i=1}^{m} \binom{m}{i}(-1)^{i}
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$$\begin{aligned} 0 = (-1+1)^m &= \sum_{i=0}^{m} \binom{m}{i}(-1)^i \\ 0 &= 1 + \sum_{i=1}^{m} \binom{m}{i}(-1)^i \end{aligned}$$

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1 &= \text{count}
\end{aligned}
$$

Inclusion/Exclusion

Corollary

Suppose E_1 and E_2 are events. Then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

`R Code!`

Proposition

Consider events E_1 and E_2. Then

$$P(E_1 \cap E_2) \quad = \quad P(E_1) - P(E_1 \cap E_2^c)$$

Proof.

$\square$

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Proof.

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P(E_1 \cap E_2) &= P(E_1) - P(E_1 \cap E_2^c)
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$$P(E_{k+1}) - P(\cup_{i=1}^{k} E_i \cap E_{k+1}) \leq P(E_{k+1})$$

Proof Continued

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Proof Continued

$$P(\cup_{i=1}^{k} E_i) \leq \sum_{i=1}^{k} P(E_i)$$

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Suprising Probability Facts

Formalized Probabilistic Reasoning: helps us to avoid silly reasoning

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- "Group A has a higher rate of some behavior, therefore most of the behavior is from group A" $\rightsquigarrow$ confuses two different problems (explain more tomorrow)
- "This is a low probability event, therefore god designed it" $\rightsquigarrow$ (1) Even if we stipulate to a low probability event, intelligent design is an assumption (2) Low probability obviously doesn't imply divine intervention. Take 100 balls and let them sort into an undetermined bins. You'll get a result, but probability of that result $= 1/(10^{29} \times$ Number of Atoms in Universe)

Suprising Probability Facts

Formalized Probabilistic Reasoning: helps us to avoid silly reasoning

- "What are the odds" $\rightsquigarrow$ not great, but neither are all the other non-pattens that are missed
- "There is no way a candidate has a 80% chance of winning, the forecasted vote share is only 55%" $\rightsquigarrow$ confuses different events
- "Group A has a higher rate of some behavior, therefore most of the behavior is from group A" $\rightsquigarrow$ confuses two different problems (explain more tomorrow)
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Easy Problems

Surprising Probability Facts:Birthday Problem

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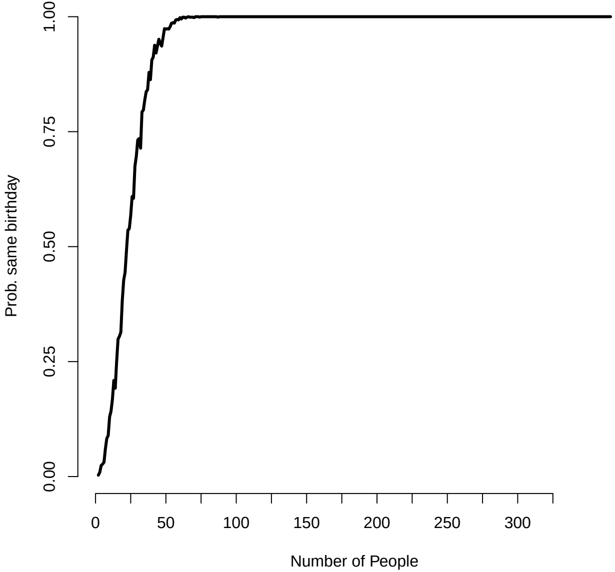
- Examine via simulation


Prob. same birthday
0.00
0.25
0.50
0.75
1.00
0
50
100
150
200
250
300
Number of People

Surprising Probability Facts: the E-Harmony Problem

Curse of dimensionality and on-line dating:

eHarmony matches you based on compatibility in the most important areas of life - like values, character, intellect, sense of humor, and 25 other dimensions.

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\end{aligned}
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1 in 536,870,912 people

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Across many "variables" (events) agreement is harder

Probability Theory

- Today: Introducing probability model
- Conditional probability, Bayes' rule, and independence