Math Camp

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Multivariable Calculus

Functions of many variables:

1) Policies may be multidimensional (policy provision and pork buy off)
2) Countries may invest in offensive and defensive resources for fighting wars
3) Ethnicity and resources could affect investment

Today:

0) Determinant
0) Eigenvector/Diagonalization
1) Multivariate functions
2) Partial Derivatives, Gradients, Jacobians, and Hessians
3) Total Derivative, Implicit Differentiation, Implicit Function Theorem
4) Multivariate Integration

Determinant

Suppose we have a square $(n \times n)$ matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

A determinant is a function that assigns a number to square matrices

Determinant

Facts needed to define determinant :

Definition

A permutation of the set of integers $\{1, 2, \ldots, J\}$ *is an arrangement of these integers in some order without omissions or repetition.*

For example, consider $\{1, 2, 3, 4\}$

$\{3, 2, 1, 4\}$

$\{4, 3, 2, 1\}$

If we have J integers then there are $J!$ permutations

Determinant

Definition

An inversion occurs when a larger integer occurs before a smaller integer in a permutation

Even permutation: total inversions are even

Odd permutation: total inversions are odd

Count the inversions

$\{3, 2, 1\}$

$\{1, 2, 3\}$

$\{3, 1, 2\}$

$\{2, 1, 3\}$

$\{1, 3, 2\}$

$\{2, 3, 1\}$

Determinant

Definition

For a square nxn matrix A, we will call an elementary product an n element long product, with no two components coming from the same row or column. We will call a signed elementary product one that multiplies odd permutations of the column numbers by -1.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

There are $n!$ elementary products

Determinant

Definition

Suppose A is an $n \times n$ matrix. Define the determinant function $\det(A)$ to be the sum of signed elementary products from A. Call $\det(A)$ the determinant of A

Suppose A is a 3×3 matrix.

$$\begin{aligned}
\det(A) &= \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\
&= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\
&\quad + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}
\end{aligned}$$

`R Code!`

An Introduction to Eigenvectors, Values, and Diagonalization

Definition

Suppose $\boldsymbol{A}$ is an $N \times N$ matrix and λ is a scalar.
If

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$$

Then $\boldsymbol{x}$ is an eigenvector and λ is the associated eigenvalue

- $\boldsymbol{A}$ stretches the eigenvector $\boldsymbol{x}$
- $\boldsymbol{A}$ stretches $\boldsymbol{x}$ by λ
- To find eigenvectors/values: (`eigen` in `R`)
  - Find λ that solves $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$
  - Find vectors in null space of:

$$(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$$

An Introduction to Eigenvectors, Values, and Diagonalization

Theorem

Suppose $\boldsymbol{A}$ is an invertible $N \times N$ matrix and further suppose that $\boldsymbol{A}$ has N distinct eigenvalues and N linearly independent eigenvectors. Then we can write $\boldsymbol{A}$ as,

$$\boldsymbol{A} = \boldsymbol{W} \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N \end{pmatrix} \boldsymbol{W}^{-1}$$

where $\boldsymbol{W} = (\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_N)$ is an $N \times N$ matrix with the N eigenvectors as column vectors.

Proof:
Note

$$
\begin{aligned}
\boldsymbol{AW} &= \begin{pmatrix} \lambda_1 \boldsymbol{w}_1 & \lambda_2 \boldsymbol{w}_2 & \ldots & \lambda_N \boldsymbol{w}_N \end{pmatrix} \\
&= \boldsymbol{W\Lambda} \\
\boldsymbol{A} &= \boldsymbol{W\Lambda W}^{-1}
\end{aligned}
$$

Examples of Diagonalization

Suppose $\boldsymbol{A}$ is an $N \times N$ invertible matrix with eigenvalues $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ and eigenvectors $\boldsymbol{W}$. Calculate $\boldsymbol{AA} = \boldsymbol{A}^2$

$$
\begin{aligned}
\boldsymbol{AA} &= \boldsymbol{W\Lambda W}^{-1}\boldsymbol{W\Lambda W}^{-1} \\
&= \boldsymbol{W}\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N \end{pmatrix}\boldsymbol{W}^{-1} \\
&= \boldsymbol{W}\begin{pmatrix} \lambda_1^2 & 0 & \ldots & 0 \\ 0 & \lambda_2^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N^2 \end{pmatrix}\boldsymbol{W}^{-1}
\end{aligned}
$$

Multivariate Functions

$$\begin{aligned} f(\boldsymbol{x}) &= f(x_1, x_2, \ldots, x_N) \\ &= x_1 + x_2 + \ldots + x_N \\ &= \sum_{i=1}^{N} x_i \end{aligned}$$

Multivariate Functions

Definition

Suppose $f : \Re^n \rightarrow \Re^1$. We will call f a multivariate function. We will commonly write,

$$f(\boldsymbol{x}) \quad = \quad f(x_1, x_2, x_3, \ldots, x_n)$$

- $\Re^n = \Re \underbrace{\times}_{\text{cartesian}} \Re \times \Re \times \ldots \Re$
- The function we consider will take n inputs and output a single number (that lives in $\Re^1$, or the real line)

Example 1

$$f(x_1, x_2) \quad = \quad x_1 + x_2 + x_1 x_2$$

Evaluate at $\boldsymbol{w} = (w_1, w_2) = (1, 2)$

$$\begin{aligned} f(w_1, w_2) &= w_1 + w_2 + w_1 w_2 \\ &= 1 + 2 + 1 \times 2 \\ &= 5 \end{aligned}$$

Preferences for Multidimensional Policy

Recall that in the spatial model, we suppose policy and political actors are located in a space.

Suppose that policy is N dimensional—or $\boldsymbol{x} \in \Re^N$.

Suppose that legislator i's utility is a $U : \Re^N \rightarrow \Re^1$ and is given by,

$$\begin{aligned}
U(\boldsymbol{x}) &= U(x_1, x_2, \ldots, x_N) \\
&= -(x_1 - \mu_1)^2 - (x_2 - \mu_2)^2 - \ldots - (x_N - \mu_N)^2 \\
&= -\sum_{j=1}^{N} (x_j - \mu_j)^2
\end{aligned}$$

Suppose $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_N) = (0, 0, \ldots, 0)$. Evaluate legislator's utility for a policy proposal of $\boldsymbol{m} = (1, 1, \ldots, 1)$.

$$\begin{aligned}
U(\boldsymbol{m}) &= U(1, 1, \ldots, 1) \\
&= -(1-0)^2 - (1-0)^2 - \ldots - (1-0)^2 \\
&= -\sum_{j=1}^{N} 1 = -N
\end{aligned}$$

Regression Models and Randomized Treatments

Often we administer randomized experiments:

The most recent wave of interest began with voter mobilization, and wonder if individual i turns out to vote, Vote_i

- $T = 1$ (treated): voter receives mobilization
- $T = 0$ (control): voter does not receive mobilization

Suppose we find the following regression model, where x_2 is a participant's age:

$$\begin{aligned} f(T, x_2) &= Pr(\text{Vote}_i = 1 | T, x_2) \\ &= \beta_0 + \beta_1 T + \beta_2 x_2 \end{aligned}$$

We can calculate the effect of the experiment as:

$$\begin{aligned} f(T=1, x_2) - f(T=0, x_2) &= \beta_0 + \beta_1 1 + \beta_2 x_2 - (\beta_0 + \beta_1 0 + \beta_2 x_2) \\ &= \beta_0 - \beta_0 + \beta_1(1-0) + \beta_2(x_2 - x_2) \\ &= \beta_1 \end{aligned}$$

Multivariate Derivative

Definition

Suppose $f : X \rightarrow \Re^1$, *where* $X \subset \Re^n$. $f(\boldsymbol{x}) = f(x_1, x_2, \ldots, x_N)$. *If the limit,*

$$\begin{aligned}
\frac{\partial}{\partial x_i} f(\boldsymbol{x}_0) &= \frac{\partial}{\partial x_i} f(x_{01}, x_{02}, \ldots, x_{0i}, x_{0i+1}, \ldots, x_{0N}) \\
&= \lim_{h \rightarrow 0} \frac{f(x_{01}, x_{02}, \ldots, x_{0i} + h, \ldots x_{0N}) - f(x_{01}, x_{02}, \ldots, x_{0i}, \ldots, x_{0N})}{h}
\end{aligned}$$

exists then we call this the partial derivative of f *with respect to* x_i *at the value* $\boldsymbol{x}_0 = (x_{01}, x_{02}, \ldots, x_{0N})$.

Rules for Taking Partial Derivatives

Partial Derivative: $\frac{\partial f(\boldsymbol{x})}{\partial x_i}$

- Treat each instance of x_i as a variable that we would differentiate before
- Treat each instance of $\boldsymbol{x}_{-i} = (x_1, x_2, x_3, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ as a constant

Example Partial Derivatives

$$
\begin{aligned}
f(\boldsymbol{x}) &= f(x_1, x_2) \\
&= x_1 + x_2
\end{aligned}
$$

Partial derivative, with respect to x_1 at (x_{01}, x_{02})

$$
\begin{aligned}
\frac{\partial f(x_1, x_2)}{\partial x_1}|_{(x_{01}, x_{02})} &= 1 + 0|_{x_{01}, x_{02}} \\
&= 1
\end{aligned}
$$

Example Partial Derivatives

$$
\begin{aligned}
f(\boldsymbol{x}) &= f(x_1, x_2, x_3) \\
&= x_1^2 \log(x_1) + x_2 x_1 x_3 + x_3^2
\end{aligned}
$$

What is the partial derivative with respect to x_1? x_2? x_3? Evaluated at $\boldsymbol{x}_0 = (x_{01}, x_{02}, x_{03})$.

$$
\begin{aligned}
\frac{\partial f(\boldsymbol{x})}{\partial x_3}|_{\boldsymbol{x}_0} &= x_1 x_2 + 2x_3|_{\boldsymbol{x}_0} \\
&= x_{01} x_{02} + 2x_{03}
\end{aligned}
$$

Rate of Change, Linear Regression

Suppose we regress Approval$_i$ rate for Obama in month i on Employ$_i$ and Gas$_i$. We obtain the following model:

$$\text{Approval}_i = 0.8 - 0.5\text{Employ}_i - 0.25\text{Gas}_i$$

We are modeling $\text{Approval}_i = f(\text{Employ}_i, \text{Gas}_i)$. What is partial derivative with respect to employment?

$$\frac{\partial f(\text{Employ}_i, \text{Gas}_i)}{\partial \text{Employ}_i} = -0.5$$

Gradient

Definition

Suppose $f : X \rightarrow \Re^1$ *with* $X \subset \Re^n$ *is a differentiable function. Define the gradient vector of* f *at* $\boldsymbol{x}_0$, $\nabla f(\boldsymbol{x}_0)$ *as,*

$$\nabla f(\boldsymbol{x}_0) \quad = \quad \left(\frac{\partial f(\boldsymbol{x}_0)}{\partial x_1}, \frac{\partial f(\boldsymbol{x}_0)}{\partial x_2}, \frac{\partial f(\boldsymbol{x}_0)}{\partial x_3}, \ldots, \frac{\partial f(\boldsymbol{x}_0)}{\partial x_n} \right)$$

- The gradient points in the direction that the function is increasing in the fastest direction
- We'll use this to do optimization (both analytic and computational)

Example Gradient Calculation

Suppose

$$
\begin{aligned}
f(\boldsymbol{x}) &= f(x_1, x_2, \ldots, x_n) \\
&= x_1^2 + x_2^2 + \ldots + x_n^2 \\
&= \sum_{i=1}^{n} x_i^2
\end{aligned}
$$

Then $\nabla f(\boldsymbol{x}^*)$ is

$$
\nabla f(\boldsymbol{x}^*) = (2x_1^*, 2x_2^*, \ldots, 2x_n^*)
$$

So if $\boldsymbol{x}^* = (3, 3, \ldots, 3)$ then

$$
\begin{aligned}
\nabla f(\boldsymbol{x}^*) &= (2 * 3, 2 * 3, \ldots, 2 * 3) \\
&= (6, 6, \ldots, 6)
\end{aligned}
$$

Second Partial Derivative

Definition

Suppose $f : X \rightarrow \Re$ *where* $X \subset \Re^n$ *and suppose that* $\frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_i}$ *exists. Then we define,*

$$\frac{\partial^2 f(\boldsymbol{x})}{\partial x_j \partial x_i} \equiv \frac{\partial}{\partial x_j}\left(\frac{\partial f(\boldsymbol{x})}{\partial x_i}\right)$$

- Second derivative could be with respect to x_i or with some other variable x_j
- Nagging question: does order matter?

Second Partial Derivative: Order Doesn't Matter

Theorem

Young's Theorem *Let* $f: X \to \Re$ *with* $X \subset \Re^n$ *be a twice differentiable function on all of* X*. Then for any* i, j, *at all* $\boldsymbol{x}^* \in X$,

$$\frac{\partial^2}{\partial x_i \partial x_j} f(\boldsymbol{x}^*) = \frac{\partial^2}{\partial x_j \partial x_i} f(\boldsymbol{x}^*)$$

Second Order Partial Derivates

$$f(\boldsymbol{x}) = x_1^2 x_2^2$$

Then,

$$\begin{aligned}
\frac{\partial^2}{\partial x_1 \partial x_1} f(\boldsymbol{x}) &= 2x_2^2 \\
\frac{\partial^2}{\partial x_1 \partial x_2} f(\boldsymbol{x}) &= 4x_1 x_2 \\
\frac{\partial^2}{\partial x_2 \partial x_2} f(\boldsymbol{x}) &= 2x_1^2
\end{aligned}$$

Hessians

Definition

Suppose $f : X \to \Re^1$, $X \subset \Re^n$, with f a twice differentiable function. We will define the Hessian matrix as the matrix of second derivatives at $\boldsymbol{x}^* \in X$,

$$\boldsymbol{H}(f)(\boldsymbol{x}^*) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(\boldsymbol{x}^*) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\boldsymbol{x}^*) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\boldsymbol{x}^*) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\boldsymbol{x}^*) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(\boldsymbol{x}^*) & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\boldsymbol{x}^*) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\boldsymbol{x}^*) & \frac{\partial^2 f}{\partial x_n \partial x_2}(\boldsymbol{x}^*) & \dots & \frac{\partial^2 f}{\partial x_n \partial x_n}(\boldsymbol{x}^*) \end{pmatrix}$$

- Hessians are symmetric
- They describe curvature of a function (think, how bended)
- Will be the basis for second derivative test for multivariate optimization

An Example

Suppose $f : \Re^3 \rightarrow \Re$, with

$$f(x_1, x_2, x_3) \quad = \quad x_1^2 x_2^2 x_3^2$$

$$\nabla f(\boldsymbol{x}) \quad = \quad (2x_1 x_2^2 x_3^2, 2x_1^2 x_2 x_3^2, 2x_1^2 x_2^2, x_3)$$

$$\boldsymbol{H}(f)(\boldsymbol{x}) \quad = \quad \begin{pmatrix} 2x_2^2 x_3^2 & 4x_1 x_2 x_3^2 & 4x_1 x_2^2 x_3 \\ 4x_1 x_2 x_3^2 & 2x_1^2 x_3^2 & 4x_1^2 x_2 x_3 \\ 4x_1 x_2^2 x_3 & 4x_1^2 x_2 x_3 & 2x_1^2 x_2^2 \end{pmatrix}$$

Functions with Multidimensional Codomains

Definition

Suppose $f : \Re^m \rightarrow \Re^n$. We will call f a multivariate function. We will commonly write,

$$f(\boldsymbol{x}) \quad = \quad \begin{pmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \\ \vdots \\ f_n(\boldsymbol{x}) \end{pmatrix}$$

Example Functions

Suppose we have some policy $\boldsymbol{x} \in \Re^M$. Suppose we have N legislators where legislator i has utility

$$U_i(\boldsymbol{x}) \;=\; \sum_{j=1}^{M} -(x_j - \mu_{ij})^2$$

We can describe the utility of all legislators to the proposal as

$$f(\boldsymbol{x}) \;=\; \begin{pmatrix} \sum_{j=1}^{M} -(x_j - \mu_{1j})^2 \\ \sum_{j=1}^{M} -(x_j - \mu_{2j})^2 \\ \vdots \\ \sum_{j=1}^{M} -(x_j - \mu_{Nj})^2 \end{pmatrix}$$

Jacobian

Definition

Suppose $f : X \rightarrow \Re^n$, *where* $X \subset \Re^m$, *with* f *a differentiable function. Define the Jacobian of* f *at* $\boldsymbol{x}$ *as*

$$\boldsymbol{J}(f)(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \ldots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \ldots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{x_1} & \frac{\partial f_n}{x_2} & \ldots & \frac{\partial f_n}{x_m} \end{pmatrix}$$

Example of Jacobian

$$f(r, \theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

$$\boldsymbol{J}(f)(r, \theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

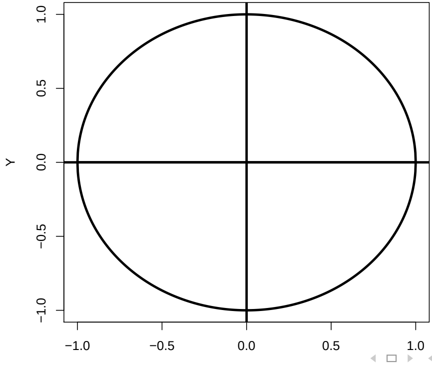
Implicit Functions and Differentiation

We have defined functions explicitly

$$Y = f(x)$$

We might also have an implicit function:

$$1 = x^2 + y^2$$



Implicit Function Theorem (From Avi Acharya's Notes)

Definition

Suppose $X \subset \Re^m$ *and* $Y \subset \Re$*. Let* $f : X \cup Y \rightarrow \Re$ *be a differentiable function (with continuous partial derivatives). Let* $(\boldsymbol{x}^*, y^*) \in X \cup Y$ *such that*

$$\begin{aligned} \frac{\partial f(\boldsymbol{x}^*, y^*)}{\partial y} &\neq 0 \\ f(\boldsymbol{x}^*, y^*) &= 0 \end{aligned}$$

Then there exists $B \subset \Re^n$ *such that there is a differentiable function* $g : B \rightarrow \Re$ *such that* $x^* \in B$ *then* $g(x^*) = y^*$ *and* $f(x, g(x)) = 0$*. The derivative of* g *for* $x \in B$ *is given by*

$$\frac{\partial g}{\partial x_j} = -\frac{\frac{\partial f}{\partial x_j}}{\frac{\partial f}{\partial y}}$$

Example 1: Implicit Function Theorem

Suppose that the equation is

$$\begin{aligned} 1 &= x^2 + y^2 \\ 0 &= x^2 + y^2 - 1 \end{aligned}$$

$$\begin{aligned} y &= \sqrt{1 - x^2} \text{ if y>0} \\ y &= -\sqrt{1 - x^2} \text{ if y} < 0 \end{aligned}$$

Example 1: Implicit Function Theorem

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= 2x \\
\frac{\partial f}{\partial y} &= 2y = 2\sqrt{1 - x^2} \text{ if y} > 0 \\
\frac{\partial f}{\partial y} &= 2y = -2\sqrt{1 - x^2} \text{ if y} < 0 \\
\frac{\partial g(x)}{\partial x}\Big|_{x_0} &= -\frac{\partial f / \partial x}{\partial f / \partial y} \\
&= -\frac{2x_0}{2y} = -\frac{x_0}{\sqrt{1 - x_0^2}} \text{ if y} > 0 \\
&= -\frac{2x_0}{2y} = \frac{x_0}{\sqrt{1 - x_0^2}} \text{ if y} < 0
\end{aligned}
$$

Implicit Function Theorem: Frequently Asked Questions

- Q: What's the deal with the implicit function theorem failing?
- A: Consider our proposed solution

$$\begin{aligned} y &= \sqrt{1-x^2} \\ \frac{\partial y}{\partial x} &= -\frac{x}{\sqrt{1-x^2}} \end{aligned}$$

As $x \rightarrow 1$ or $x \rightarrow -1$ this derivative diverges
The intuition from the Implicit Function Theorem is that any function $g(x) = y$ there would need an "infinite" slope.

Implicit Function Theorems: Frequently Asked Questions

- Q: What's the deal with the following equation?:

$$\frac{\partial g(x)}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

- A: Consider, first, the following example:

$$\begin{aligned}
0 &= f(x,y) \\
0 &= x^2 - y \\
\frac{\partial y}{\partial x} &= 2x \\
\frac{\partial f(x,y)/\partial x}{\partial f(x,y)/\partial y} &= \frac{2x}{-1} = -\frac{\partial y}{\partial x}
\end{aligned}$$

In this example, the negative sign is "moving things to the other side". In general, the negative sign will capture that we want to measure the compensatory behavior of the function: how y moves in response to some x_i along a level curve

Example 2: Implicit Function Theorem (From Jim Fearon)

Suppose there n individuals, each individual i earns pre-tax income $y_i > 0$.
Total income $Y = \sum_{i=1}^{n} y_i$
Per capita income: $\bar{y} = Y/n$
Individuals pay a proportional tax $t \in (0, 1)$
Suppose:

$$U_i(t, y_i) \quad = \quad y_i(1 - t^2) + t\bar{y}$$

Example 2: Implicit Function Theorem (From Jim Fearon)

An individual's optimal tax rate is:

$$\begin{aligned}
\frac{\partial U_i(t, y_i)}{\partial t} &= -2y_i t + \bar{y} \\
0 &= -2y_i t^* + \bar{y} \\
\frac{\bar{y}}{2y_i} &= t_i^*
\end{aligned}$$

Checking the second derivative:

$$\frac{\partial U_i(t, y_i)}{\partial^2 t} = -2y_i$$

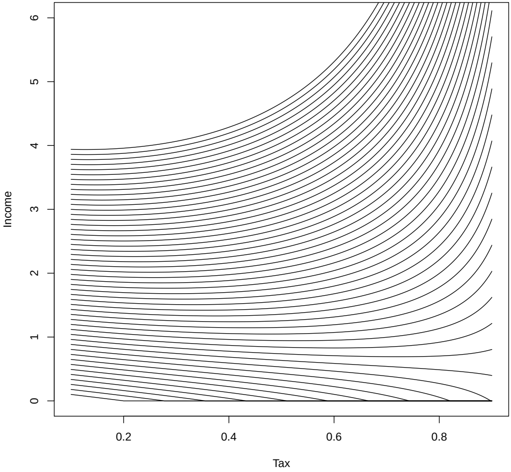
Example 2: Implicit Function Theorem (From Jim Fearon)

If we set utility equal to some constant a, it defines an implicit function
Define Marginal rate of Substitution as

$$\text{MRS} \quad = \quad -\frac{\partial U(t, y_i)/\partial t}{\partial U(t, y_i/\partial y_i} = \frac{\partial Y(t)}{\partial t}$$

$$\begin{aligned} \partial U(t, y_i)/\partial t &= -2y_i t + \bar{y} \\ \partial U(t, y_i/\partial y_i &= (1 - t^2) \\ \text{MRS} &= \frac{2y_i t - \bar{y}}{1 - t^2} \end{aligned}$$

Example 2: Implicit Function Theorem (From Jim Fearon)



Multivariate Integration

Suppose we have a function $f : X \rightarrow \Re^1$,
with $X \subset \Re^2$.
We will integrate a function over an area.
Area under function.
Suppose that area, A, is in 2-dimensions

- $A = \{x, y : x \in [0, 1], y \in [0, 1]\}$
- $A = \{x, y : x^2 + y^2 \leq 1\}$
- $A = \{x, y : x < y, x, y \in (0, 2)\}$

How do calculate the area under the function over these regions?

Multivariate Integration

Definition

Suppose $f : X \rightarrow \Re$ where $X \subset \Re^n$. We will say that f is integrable over $A \subset X$ if we are able to calculate its area with refined partitions of A and we will write the integral $I = \int_A f(\boldsymbol{x}) d\boldsymbol{A}$

That's horribly abstract. There is an extremely helpful theorem that makes this manageable.

Theorem

Fubini's Theorem Suppose $A = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$ and that $f : A \rightarrow \Re$ is integrable. Then

$$\int_A f(\boldsymbol{x}) d\boldsymbol{A} = \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \ldots \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(\boldsymbol{x}) dx_1 dx_2 \ldots dx_{n-1} dx_n$$

Multivariate Integration Recipe

$$\int_A f(\boldsymbol{x}) d\boldsymbol{A} \;=\; \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \ldots \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(\boldsymbol{x}) dx_1 dx_2 \ldots dx_{n-1} dx_n$$

1) Start with the inside integral x_1 is the variable, everything else a constant
2) Work inside to out, iterating
3) At the last step, we should arrive at a number

Intuition: Three Dimensional Jello Molds, a discussion

Multivariate Uniform Distribution

Suppose $f : [0,1] \times [0,1] \rightarrow \Re$ and $f(x_1, x_2) = 1$ for all $x_1, x_2 \in [0,1] \times [0,1]$. What is $\int_0^1 \int_0^1 f(x) dx_1 dx_2$?

$$
\begin{aligned}
\int_0^1 \int_0^1 f(x) dx_1 dx_2 &= \int_0^1 \int_0^1 1 dx_1 dx_2 \\
&= \int_0^1 x_1 |_0^1 dx_2 \\
&= \int_0^1 (1 - 0) dx_2 \\
&= \int_0^1 1 dx_2 \\
&= x_2 |_0^1 \\
&= 1
\end{aligned}
$$

Example 2

Suppose $f : [a_1, b_1] \times [a_2, b_2] \rightarrow \Re$ is given by

$$f(x_1, x_2) = x_1 x_2$$

Find $\int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2) dx_1 dx_2$

$$\begin{aligned}
\int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2) dx_1 dx_2 &= \int_{a_2}^{b_2} \int_{a_1}^{b_1} x_2 x_1 dx_1 dx_2 \\
&= \int_{a_2}^{b_2} \frac{x_1^2}{2} x_2 |_{a_1}^{b_1} dx_2 \\
&= \frac{b_1^2 - a_1^2}{2} \int_{a_2}^{b_2} x_2 dx_2 \\
&= \frac{b_1^2 - a_1^2}{2} \left(\frac{x_2^2}{2} |_{a_2}^{b_2} \right) \\
&= \frac{b_1^2 - a_1^2}{2} \frac{b_2^2 - a_2^2}{2}
\end{aligned}$$

Example 3: Exponential Distributions

Suppose $f : \Re^2_{+} \rightarrow \Re$ and that

$$f(x_1, x_2) = 2\exp(-x_1)\exp(-2x_2)$$

Find:

$$\begin{aligned}
\int_0^{\infty}\int_0^{\infty} f(x_1, x_2) &= 2\int_0^{\infty}\int_0^{\infty}\exp(-x_1)\exp(-2x_2)dx_1 dx_2 \\
&= 2\int_0^{\infty}\exp(-x_1)dx_1\int_0^{\infty}\exp(-2x_2)dx_2 \\
&= 2(-\exp(-x)|_0^{\infty})(-\frac{1}{2}\exp(-2x_2)|_0^{\infty}) \\
&= 2\left[(-\lim_{x_1\rightarrow\infty}\exp(-x_1) + 1)(-\frac{1}{2}\lim_{x_2\rightarrow\infty}\exp(-2x_2) + \frac{1}{2})\right] \\
&= 2[\frac{1}{2}] \\
&= 1
\end{aligned}$$

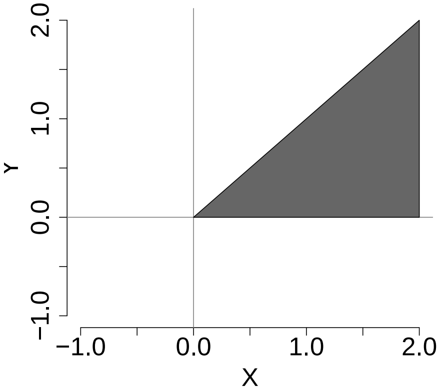
Challenge Problems

1) Find $\int_0^1 \int_0^1 x_1 + x_2 dx_1 dx_2$
2) Demonstrate that

$$\int_0^b \int_0^a x_1 - 3x_2 dx_1 dx_2 = \int_0^a \int_0^b x_1 - 3x_2 dx_2 dx_1$$

More Complicated Bounds of Integration

So far, we have integrated over rectangles. But often, we are interested in more complicated regions



How do we do this?

Example 4: More Complicated Regions

Suppose $f : [0,1] \times [0,1] \rightarrow \Re$, $f(x_1, x_2) = x_1 + x_2$. Find area of function where $x_1 < x_2$.

Trick: we need to determine bound. If $x_1 < x_2$, x_1 can take on any value from 0 to x_2

$$\begin{aligned}
\iint_{x_1 < x_2} f(\boldsymbol{x}) &= \int_0^1 \int_0^{x_2} x_1 + x_2 dx_1 dx_2 \\
&= \int_0^1 x_2 x_1 |_0^{x_2} dx_2 + \int_0^1 \frac{x_1^2}{2} |_0^{x_2} \\
&= \int_0^1 x_2^2 dx_2 + \int_0^1 \frac{x_2^2}{2} \\
&= \frac{x_2^3}{3} |_0^1 + \frac{x_2^3}{6} |_0^1 \\
&= \frac{1}{3} + \frac{1}{6} \\
&= \frac{3}{6} = \frac{1}{2}
\end{aligned}$$

Consider the same function and let's switch the bounds.

$$
\begin{aligned}
\iint_{x_1<x_2} f(\boldsymbol{x}) &= \int_0^1 \int_{x_1}^1 x_1 + x_2 dx_2 dx_1 \\
&= \int_0^1 x_1 x_2|_{x_1}^1 + \int_0^1 \frac{x_2^2}{2}|_{x_1}^1 dx_1 \\
&= \int_0^1 x_1 - x_1^2 + \int_0^1 \frac{1}{2} - \frac{x_1^2}{2} dx_1 \\
&= \frac{x_1^2}{2}|_0^1 - \frac{x_1^3}{3}|_0^1 + \frac{x_1}{2}|_0^1 - \frac{x_1^3}{6}|_0^1 \\
&= \frac{1}{2} - \frac{1}{3} + \frac{1}{2} - \frac{1}{6} \\
&= 1 - \frac{3}{6} \\
&= \frac{1}{2}
\end{aligned}
$$

Example 5: More Complicated Regions

Suppose $f[0,1] \times [0,1] \rightarrow \Re$, $f(x_1, x_2) = 1$. What is the area of $x_1 + x_2 < 1$? Where is $x_1 + x_2 < 1$? Where, $x_1 < 1 - x_2$

$$
\begin{aligned}
\iint_{x_1 + x_2 < 1} f(\boldsymbol{x}) d\boldsymbol{x} &= \int_0^1 \int_0^{1-x_2} 1 dx_1 x_2 \\
&= \int_0^1 x_1|_0^{1-x_2} dx_2 \\
&= \int_0^1 (1 - x_2) dx_2 \\
&= x_2|_0^1 - \frac{x_2^2}{2}|_0^1 \\
&= 1 - (\frac{1}{2}) \\
&= \frac{1}{2}
\end{aligned}
$$