Math Camp

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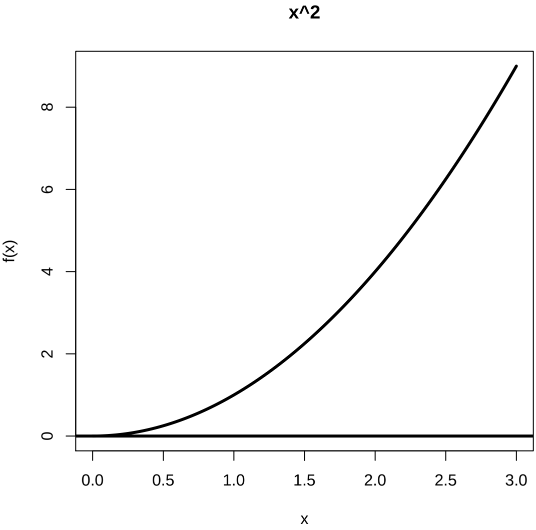
Integration

- Derivatives $\rightsquigarrow$ rates of change
- Integrals$\rightsquigarrow$ area under a curve
- Connection: fundamental theorem of calculus
- Some antiderivative formulas
- Algebra of Integrals
- Improper Integrals
- Monte Carlo principle
- Integrate a lot in probability theory, we'll review more then
- Infinite Series

Area Under a Curve



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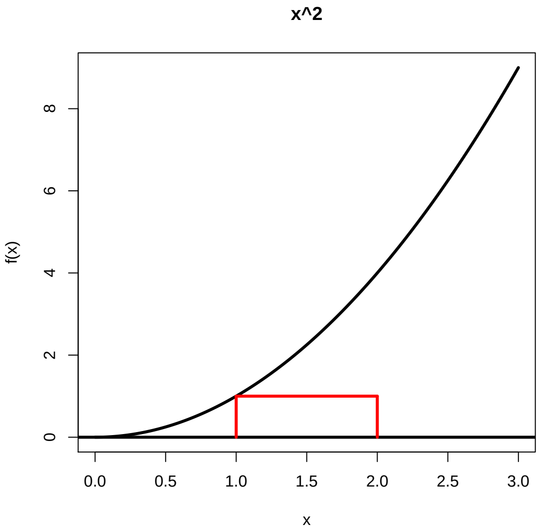

- Approximated area $= \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$

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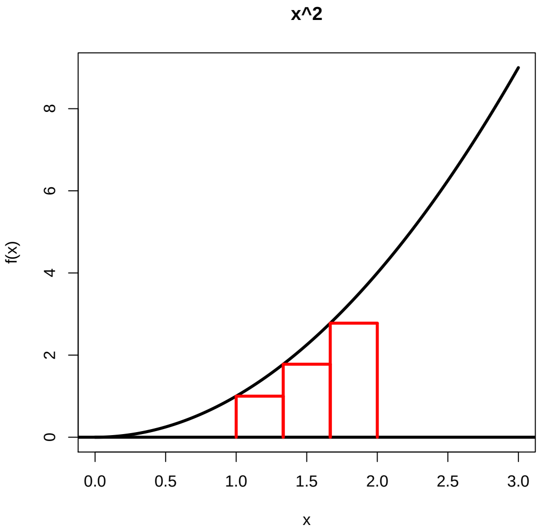

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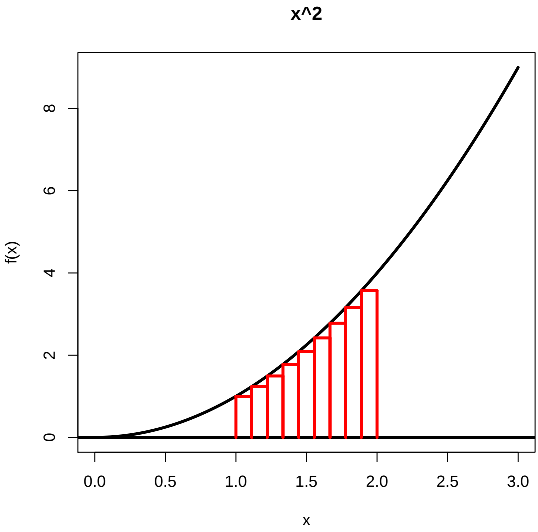

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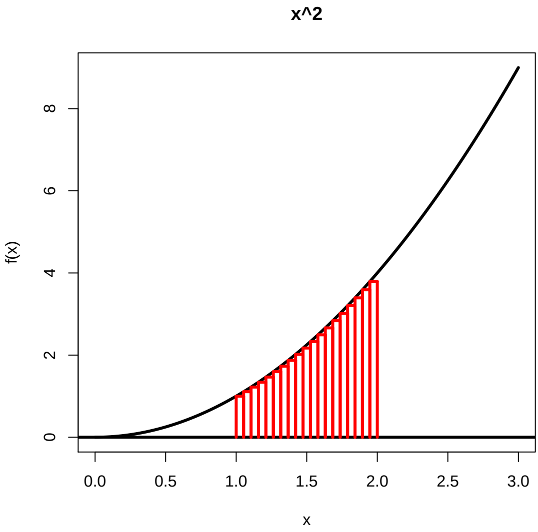

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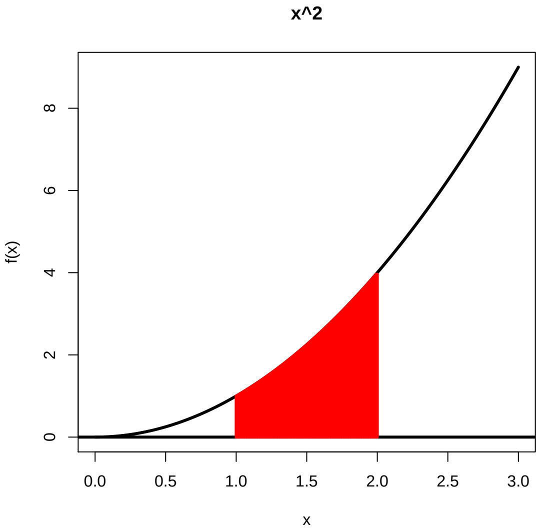

- Approximated area = $\sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$
- As partitions become more refined, they improve
- $\lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) \rightsquigarrow$ Riemann Integral

Definition

Suppose $f : \Re \rightarrow \Re$*. We will define the Riemann Integral as* $\int_a^b f(x)dx$*. If this exists then we say* f *is* ***integrable*** *on* $[a, b]$ *and call* $\int_a^b f(x)dx$ *the* ***integral*** *of f.*

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Theorem

Suppose $f : [a, b] \to \Re$ *is a monotonic function. Then* f *is integrable*

Some Counterexamples

Suppose $f : [0, 1] \rightarrow \frac{1}{x}$

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$$\begin{aligned} f(x) &= 1 \text{ if } x \text{ rational} \\ &= 0 \text{ if } x \text{ irrational} \end{aligned}$$

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$$\begin{aligned} f(x) &= 1 \text{ if } x \text{ rational} \\ &= 0 \text{ if } x \text{ irrational} \end{aligned}$$

Not integrable, because every interval will contain a discontinuous jump

Fundamental Theorem of Calculus

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A deep connection between derivatives and integrals makes integration much easier

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Theorem

Fundamental Theorem of Calculus Suppose $f : [a, b] \rightarrow \Re$ and that f is differentiable on $[a, b]$ and that its derivative, $f^{'}$, is integrable. Then,

$$\int_a^b f^{'}(x)dx = f(b) - f(a)$$

Recipe for Definite Integration

$$\int_a^b f^{'}(x)dx \quad = \quad f(b) - f(a)$$

- Calculate antiderivative
- Evaluate at b
- Evaluate at a

Some Classic Antiderivative Formulas

antiderivative = indefinite integral

$$\begin{aligned}
\int 1 dx &= x + c \\
\int k dx &= kx + c \\
\int x^n dx &= \frac{x^{n+1}}{n+1} + c \\
\int \frac{1}{x} dx &= \log x + c \\
\int e^x dx &= e^x + c \\
\int a^x dx &= \frac{a^x}{\log a} + c
\end{aligned}$$

Uniform Distribution

Suppose $f : \Re \rightarrow \Re$, with

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$$\int_0^{1/2} f(x)dx = \int_0^{1/2} 1dx$$

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&= x|_0^{1/2}
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We will call $f(x) = 1$ the uniform distribution.

Example 2: Area Under a Line

Suppose $f : \Re \rightarrow \Re$, with

$$f(x) \quad = \quad x$$

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&= \frac{x^2}{2}|_2^t \\
&= \frac{t^2}{2} - \frac{2^2}{2} \\
&= \frac{t^2}{2} - \frac{4}{2} = \frac{t^2}{2} - 2
\end{aligned}$$

Integration Facts

Theorem

If $f_1, f_2 : [a, b] \to \Re$ and f_1, f_2 are integrable on [a,b], then

i) *Consider the interval $[a, b]$ and $c \in [a, b]$. Then,*

$$\begin{aligned}
\int_c^c f_1^{'}(x)dx &= f_1(c) - f_1(c) = 0 \\
\int_a^b f_1^{'}(x)dx &= \int_a^c f_1^{'}(x)dx + \int_c^b f_1^{'}(x)dx \\
&= (f_1(c) - f_1(a) + (f_1(b) - f_1(c)) \\
&= f_1(b) - f_1(a)
\end{aligned}$$

Theorem

If $f_1^{'}, f_2^{'} : [a, b] \rightarrow \Re$ and $f_1^{'}, f_2^{'}$ are integrable on [a,b] and $f_1^{'}$ has antiderivative is f_1 and $f_2^{'}$ has antiderivative f_2, then

ii) *For $c_1, c_2 \in \Re$ then*

$$\int_a^b (c_1 f_1^{'}(x) + c_2 f_2^{'}(x))dx = c_1 \int_a^b f_1^{'}(x)dx + c_2 \int_a^b f_2^{'}(x)dx$$

Challenge Problems

$$\int_0^1 x dx$$
$$\int_0^1 (x^2 + x + 1) dx$$
$$\int_1^2 (\frac{1}{x} + e^x)$$

Let's Prove Taylor Theorem (And Come Up With Intuition Too!)

Theorem

Taylor's Theorem *Suppose $f : \Re \rightarrow \Re$, $f(x)$ is infinitely differentiable function. Then, the taylor expansion of $f(x)$ around a is given by*

$$
\begin{aligned}
f(x) &= f(a) + \frac{f^{'}(a)}{1!}(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 + \frac{f^{'''}(a)}{3!}(x-a)^3 + \ldots \\
f(x) &= \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x-a)^n
\end{aligned}
$$

Zero Order Approximation

$$f(x) = f(a) + \int_a^x f^{'}(t_1)dt_1$$

First order approximation:

Zero Order Approximation

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f(x) &= f(a) + \int_a^x f^{'}(t_1) dt_1 \\
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f(a) + f(x) - f(a) = f(x) &
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$$f(x) = f(a) + \int_a^x f^{'}(a)dt_1 + \int_a^x \int_a^{t_1} f^{''}(t_2)dt_2 dt_1$$

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\int_a^x \int_a^{t_1} f^{''}(t_2)dt_2 dt_1 &= \int_a^x \left(f^{'}(t_1) - f^{'}(a)\right) dt_1 \\
&= f(x) - f(a) - f^{'}(a)(x-a)
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Next step:

$$f(x) = f(a) + \int_a^x f^{'}(a)dt_1 + \int_a^x \int_a^{t_1} f^{''}(a)dt_2 dt_1 + \int_a^x \int_a^{t_1} \int_a^{t_2} f^{'''}(t_3)dt_3 t_2 t_1$$

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Flip bounds on the remainder term and you realize it contains R_k and that the additional term cancels out the new f^{k+1} term.
Can obtain error bounds with computation of remainder. Because expansion around each point is necessarily finite as $k \rightarrow \infty$ remainder goes to zero.

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More Fundamental Theorem of Calculus

Theorem

Suppose $f^{'} : [a, b] \rightarrow \Re$ *is integrable on [a,b] and suppose that its antiderivative is* $f(x)$*. Define* $F(t) = \int_a^t f^{'}(x)dx$ *for* $a \leq t \leq b$*. Then,* $F^{'}(x_0)$ *is* $f^{'}(x_0)$*.*

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$$\frac{\partial}{\partial t}F(t) = \frac{\partial}{\partial t}(f(t) - f(a))$$

More Fundamental Theorem of Calculus

Theorem

Suppose $f^{'} : [a, b] \rightarrow \Re$ *is integrable on [a,b] and suppose that its antiderivative is* $f(x)$*. Define* $F(t) = \int_a^t f^{'}(x)dx$ *for* $a \leq t \leq b$*. Then,* $F^{'}(x_0)$ *is* $f^{'}(x_0)$*.*

Proof.

intuition $F(t) = \int_a^t f^{'}(x)dx = f(t) - f(a)$.
Now, we want to take the derivative of $F(t)$ and evaluate at x_0 and the derivative of $f(t) - f(a)$ and evaluate at x_0

$$\begin{aligned} \frac{\partial}{\partial t}F(t) &= \frac{\partial}{\partial t}(f(t) - f(a)) \\ F^{'}(t)|_{x_0} &= f^{'}(t)|_{x_0} \end{aligned}$$

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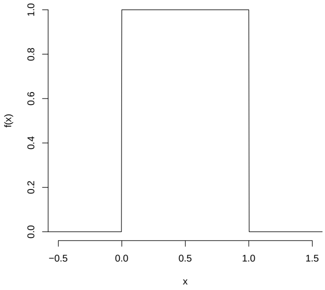
F^{'}(t)|_{x_0} &= f^{'}(t)|_{x_0} \\
F^{'}(x_0) &= f^{'}(x_0)
\end{aligned}$$

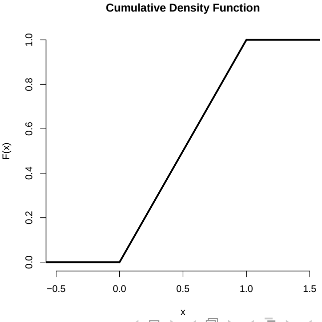
□

Uniform Cumulative Density Function

Suppose that $f^{'} \rightarrow \Re$, $f^{'}(x) = 1$ for $x \in [0, 1]$ and $f^{'}(x) = 0$ otherwise. Define,

$$\begin{aligned} F(t) &= \int_0^t f^{'}(x)dx \\ &= \int_0^t 1dx = x|_0^t \\ &= t \end{aligned}$$





Improper Integrals

Discount rates: valuing the future.
We'll do discrete time with infinite series, we can do them in continuous time with integrals

$$V = \int_0^{\infty} e^{-\delta t} dt$$

- How do we evaluate this integral?
- Improper integrals
- Continuous infinite series

Definition

Definition

Consider $f : [a, \infty) \rightarrow \Re$. If the limit

$$\lim_{t \rightarrow \infty} \int_a^t f(x)dx$$

exists then we will say $\int_a^{\infty} f(x)dx$ converges to L. Otherwise, we say it diverges.

Also apply definition for

- $\int_{-\infty}^{a} f(x)dx = \lim_{t \rightarrow -\infty} \int_t^a f(x)dx$
- $\int_{-\infty}^{\infty} f(x)dx = \lim_{t \rightarrow -\infty} \lim_{y \rightarrow \infty} \int_t^y f(x)dx$.

When do Integrals Converge?

Example 1
$f(x) = 1/x$.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x} dx &= \lim_{t \to \infty} \int_1^t \frac{1}{x} dx \\
&= \lim_{t \to \infty} (\log x)|_1^t \\
&= \lim_{t \to \infty} (\log t) - \lim_{t \to \infty} (\log 1)
\end{aligned}$$

Does not converge

When do Integrals Converge?

Example 2
$f(x) = \frac{1}{x^2}$

$$
\begin{aligned}
\int_1^{\infty} \frac{1}{x^2} dx &= \lim_{t \to \infty} \int_1^t \frac{1}{x^2} dx \\
&= \lim_{t \to \infty} -\frac{1}{x}|_1^t \\
&= \lim_{t \to \infty} -\frac{1}{t} + \frac{1}{1} \\
&= 0 + 1
\end{aligned}
$$

Substitution (slides borrowed from math.hmc.edu)

Sometimes, antidifferentiating is hard

$$\int (x^2 - 1)^4 2x dx$$

But we can use substitution to simplify. Suspend disbelief and set:

$$\begin{aligned} u &= x^2 - 1 \\ du &= 2x dx \end{aligned}$$

Rewriting the original,

$$\begin{aligned} \int (x^2 - 1)^4 (2x dx) &= u^4 du \\ &= \frac{u^5}{5} + c \\ &= \frac{(x^2-1)^5}{5} + c \end{aligned}$$

Substitution Rule (slides borrowed from math.hmc.edu)

Just chain rule in reverse. We know that the antiderivative of

$$\int f(g(x))g^{'}(x)dx = F(g(x))$$

So, with substitution rule, we look for ways to set up chain rule

Substitution Rule (slides borrowed from math.hmc.edu)

$$\int -e^{-x} dx$$

$$\begin{aligned} u &= -x \\ du &= -dx \\ \int e^u du &= e^u + c \\ &= e^{-x} + c \end{aligned}$$

Substitution Rule (slides borrowed from math.hmc.edu)

We can also multiply by 1 (creatively) to set up substitution rule

$$
\begin{aligned}
\int e^{-2x} dx &= -\frac{1}{2}\int -2e^{-2x} dx \\
u &= -2x \\
du &= -2dx \\
-\frac{1}{2}\int e^{u} du &= -\frac{1}{2}e^{u} + c \\
&= -\frac{1}{2}e^{-2x} + c
\end{aligned}
$$

Example: Exponential Distribution

Suppose $f : [0, \infty) \rightarrow \Re$, with $f(x) = e^{-x}$. Evaluate

$$\begin{aligned}
\int_0^{\infty} e^{-x} dx &= \lim_{t \rightarrow \infty} \int_0^{t} e^{-x} dx \\
&= \lim_{t \rightarrow \infty} -e^{-x}|_0^t \\
&= \lim_{t \rightarrow \infty} -e^{-t} + 1 \\
&= 0 + 1
\end{aligned}$$

We will call $f(x) = e^{-x}$ the exponential distribution

Integration by Parts

Consider:

$$\int x \cos(x) dx$$

That is hard to integrate.
Instead we'll use Integration by parts

Integration by Parts

Define:

$$g(x) = u(x)v(x)$$

Integration by Parts

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Let's differentiate $g(x)$

Integration by Parts

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$$g(x) \quad = \quad u(x)v(x)$$

Let's differentiate $g(x)$

$$g^{'}(x) \quad = \quad u^{'}(x)v(x) + v^{'}(x)u(x)$$

Integration by Parts

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$$\int g^{'}(x)dx = \int u^{'}(x)v(x)dx + \int u(x)v^{'}(x)dx$$

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$$\begin{aligned}\int g^{'}(x)dx &= \int u^{'}(x)v(x)dx + \int u(x)v^{'}(x)dx \\ u(x)v(x) + c - \int u^{'}(x)v(x)dx &= \int u(x)v^{'}(x)dx\end{aligned}$$

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\int g^{'}(x)dx &= \int u^{'}(x)v(x)dx + \int u(x)v^{'}(x)dx \\
u(x)v(x) + c - \int u^{'}(x)v(x)dx &= \int u(x)v^{'}(x)dx \\
uv - \int vdu &= \int udv
\end{aligned}$$

Integration by Parts

Integration by Parts

$$\int x \cos(x) dx$$

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$$u = x$$

Integration by Parts

$$\int x \cos(x) dx$$

$$\begin{aligned} u &= x \\ du &= 1 \end{aligned}$$

Integration by Parts

$$\int x \cos(x) dx$$

$$\begin{aligned} u &= x \\ du &= 1 \\ dv &= \cos(x) \end{aligned}$$

Integration by Parts

$$\int x \cos(x) dx$$

$$\begin{aligned}
u &= x \\
du &= 1 \\
dv &= \cos(x) \\
v &= \sin(x)
\end{aligned}$$

Integration by Parts

$$\int x\cos(x)dx$$

$$\begin{aligned} u &= x \\ du &= 1 \\ dv &= \cos(x) \\ v &= \sin(x) \end{aligned}$$

$$\int x\cos(x)dx = x\sin(x) - \int \sin(x)1dx$$

Integration by Parts

$$\int x \cos(x) dx$$

$$\begin{aligned} u &= x \\ du &= 1 \\ dv &= \cos(x) \\ v &= \sin(x) \end{aligned}$$

$$\begin{aligned} \int x \cos(x) dx &= x \sin(x) - \int \sin(x) 1 dx \\ &= x \sin(x) + \cos(x) \end{aligned}$$

Integration by Parts

Challenge:

$$\int \exp(x)\cos(x)dx$$

$$\int \log(x)dx$$

$$\int \arctan(x)dx$$

Integration by Parts

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`Wolfram Alpha (briefly)`

Monte Carlo and Integration (via Jackman)

Suppose that we want to compute some integral $\int_{-\infty}^{\infty} x f(x) dx$, but $f(x)$ is complicated.

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Suppose that we want to compute some integral $\int_{-\infty}^{\infty} x f(x) dx$, but $f(x)$ is complicated.

$$f(x) = \frac{\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{\sqrt{2\pi}}$$

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$d_1, d_2, d_3, ..., d_T$, then we can approximate this with:

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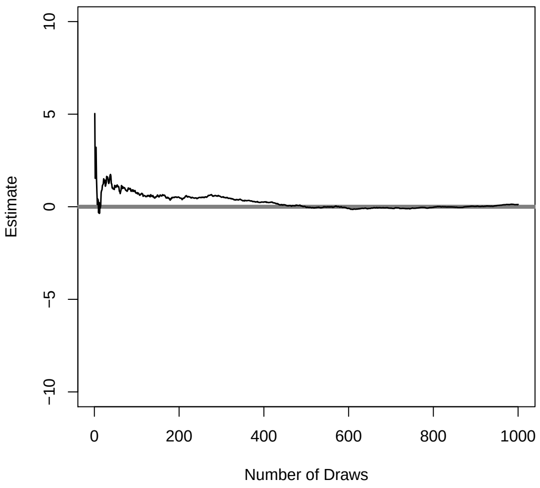
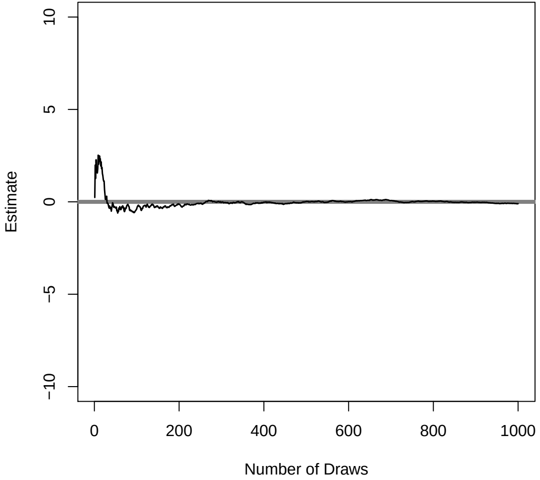
as $T \rightarrow \infty$, Expected value $\rightarrow \int_{-\infty}^{\infty} xf(x)dx$


Estimate
10
5
0
−5
−10
0
200
400
600
800
1000
Number of Draws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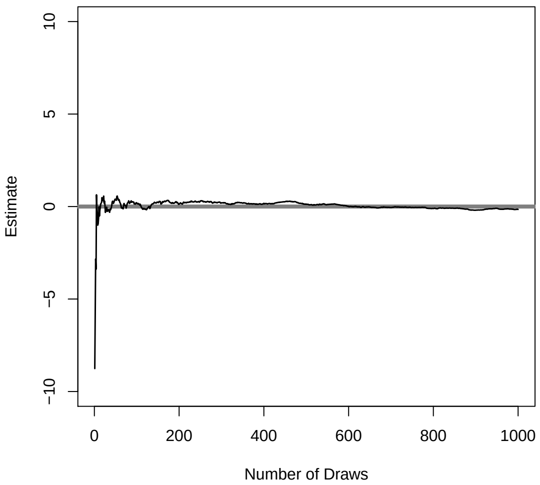

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`R code for quantiles! MonteCarlo.R`

Infinite Series

- Interactions are often repeated
  - Countries: Fight now or fight later
  - Congress: Caro, LBJ, and the Southern Strategy
  - FDA: Do I approve this drug?
  - Bargain: Do I make a deal now, or wait?
- General idea :
  - Actions have continuation value:
  - Value in the present time
  - Stream of benefits in the future
- Infinite Series to model

Formal definition $\rightsquigarrow$ Heuristics $\rightsquigarrow$ example problem (from JF)

Infinite Series

Definition

An infinite series is a pair $(\{a_n\}_{n=1}^{\infty}, \{S_n\}_{n=1}^{\infty})$ *where* $\{a_n\}_{n=1}^{\infty}$ *is a sequence and* $S_n = \sum_{k=1}^{n} a_k$.

Definition

The infinite series $(\{a_n\}_{n=1}^{\infty}, \{S_n\}_{n=1}^{\infty})$ *converges if the sequence* $\{S_n\}_{n=1}^{\infty}$ *converges to* S. *We'll write this as,*

$$\sum_{n=1}^{\infty} a_n = S$$

Infinite Series

- Example 1

Infinite Series

- Example 1
    - $a_n = \{0, 1, 0, 1, 0, 1, \ldots, \}$

Infinite Series

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    - $a_n = \{0, 1, 0, 1, 0, 1, \ldots, \}$
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- Example 2

Infinite Series

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  - $a_n = \left\{ \frac{1}{n(n+1)} \right\}_{n=1}^{\infty}$

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  - We know that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

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$$S_m = \sum_{i=1}^{m} a_i$$

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$$
\begin{aligned}
S_m &= \sum_{i=1}^{m} a_i \\
&= \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \ldots + \frac{1}{(m)(m+1)}
\end{aligned}
$$

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S_m &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots \left(\frac{1}{m} - \frac{1}{m+1}\right)
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So S_m converges on 1. (the sequence S_m converges, just like we prove other sequence convergence)

How Do We Assess Convergence?

Theorem

If $\{S_n\}_{n=1}^{\infty}$ *converges then* $\{a_n\}_{n=1}^{\infty}$ *is converges to zero*

- Necessary!
- But not sufficient

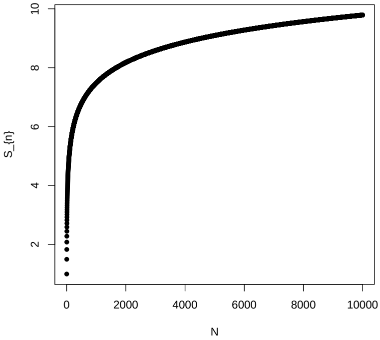
Infinite Series Convergence

Example 1:

- $a_n = \frac{1}{n}$. S_n,

$$S_n \quad = \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

Does this converge?



Infinite Series Convergence

Suppose $n = 2^k$

Infinite Series Convergence

Suppose $n = 2^k$

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2^k}$$

Infinite Series Convergence

Suppose $n = 2^k$

$$
\begin{aligned}
S_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2^k} \\
&= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^{k-1}+1} + \ldots + \frac{1}{2^k}\right)
\end{aligned}
$$

Infinite Series Convergence

Suppose $n = 2^k$

$$
\begin{aligned}
S_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2^k} \\
&= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^{k-1}+1} + \ldots + \frac{1}{2^k}\right) \\
&> 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \ldots + 2^{k-1}\left(\frac{1}{2^k}\right) = 1 + \frac{k}{2}
\end{aligned}
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Infinite Series Convergence

Suppose $n = 2^k$

$$
\begin{aligned}
S_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2^k} \\
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\end{aligned}
$$

We know that $1 + \frac{k}{2}$ does not converge.

Infinite Series Convergence

Suppose $n = 2^k$

$$
\begin{aligned}
S_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2^k} \\
&= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^{k-1}+1} + \ldots + \frac{1}{2^k}\right) \\
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\end{aligned}
$$

We know that $1 + \frac{k}{2}$ does not converge.
And we know that $S_n > 1 + \frac{k}{2} \rightsquigarrow$ does not converge (!!)

Infinite Series Convergence

Suppose $n = 2^k$

$$
\begin{aligned}
S_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2^k} \\
&= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^{k-1}+1} + \ldots + \frac{1}{2^k}\right) \\
&> 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \ldots + 2^{k-1}\left(\frac{1}{2^k}\right) = 1 + \frac{k}{2}
\end{aligned}
$$

We know that $1 + \frac{k}{2}$ does not converge.
And we know that $S_n > 1 + \frac{k}{2} \rightsquigarrow$ does not converge (!!)

Theorem

$\sum_{k=1}^{\infty} \frac{1}{k^p}$ *converges if and only if* $p > 1$.

Geometric Series and Discount Rates

Definition

A geometric series is an infinite series such that $a_n = cr^n$ *and that* $S_n = \sum_{k=0}^{n} cr^k = c + cr + cr^2 + cr^3 + \ldots cr^n$

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$$S_n = \sum_{k=0}^{n} cr^k$$

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$$(1 - r)S_n = (1 - r)\sum_{k=0}^{n} cr^k$$

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$$
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(1-r)S_n &= (1-r)\sum_{k=0}^{n} cr^k \\
(1-r)\sum_{k=0}^{n} cr^k &= c + cr + cr^2 + \ldots + cr^n -
\end{aligned}
$$

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& \quad (cr + cr^2 + cr^3 + \ldots cr^n + cr^{n+1})
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If $|r| < 1$, then the geometric series converges to $\frac{c}{1-r}$. If $|r| > 1$, the geometric series diverges

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& \quad (cr + cr^2 + cr^3 + \ldots cr^n + cr^{n+1}) \\
&= c - cr^{n+1} \\
S_n &= c\left(\frac{1-r^{n+1}}{1-r}\right)
\end{aligned}
$$

$$\begin{aligned} S_n &= c\left(\frac{1 - r^{n+1}}{1 - r}\right) \\ &= c\left(\frac{1}{1-r}\right) - c\left(\frac{r^{n+1}}{1-r}\right) \end{aligned}$$

$c\left(\frac{r^{n+1}}{1-r}\right)$ converges if and only if $|r| < 1$.

Discount Rates and IR (Fearon, Part 2)

Suppose states are choosing between attacking another country to obtain a short time gain, or cooperating for peace

	C	D
C	20,20	10,25
D	25,10	15,15

Grim-trigger: cooperate, until defect. Then defect forever

Suppose states discount future $\delta \in [0, 1]$.

$$\begin{aligned}
V(C) &= 20 + \delta 20 + \delta^2 20 + \delta^3 20 + \ldots \\
&= \frac{20}{1-\delta} \\
V(D) &= 25 + \delta 15 + \delta^2 15 + \delta^3 15 \\
&= 25 + \delta \frac{15}{1-\delta}
\end{aligned}$$

When Will States Cooperate? (Fearon, Part 2)

$$
\begin{aligned}
V(C) &> V(D) \\
\frac{20}{1-\delta} &> 25 + \delta \frac{15}{1-\delta} \\
\frac{1}{1-\delta}(20 - \delta 15) &> 25 \\
(20 - \delta 15) &> 25(1-\delta) \\
10\delta &> 5 \\
\delta &> \frac{1}{2}
\end{aligned}
$$

Linear Algebra Tuesday!