Math Camp - Homework 2 Solutions

1. Given that:

$\lim_{x \to a} f(x) = -3$ $\qquad$ $\lim_{x \to a} g(x) = 0$ $\qquad$ $\lim_{x \to a} h(x) = 8$

find the following limits. If the limit doesn't exist, explain why.

(a) $\lim_{x \to a}[f(x) + h(x)]$
$-3 + 8 = 5$

(b) $\lim_{x \to a}[f(x)]^2$
$(-3)^2 = 9$

(c) $\lim_{x \to a} \frac{f(x)}{h(x)}$
$\frac{-3}{8}$

(d) $\lim_{x \to a} \frac{g(x)}{f(x)}$
$\frac{0}{-3} = 0$

2. Find the following limits:

(a) The key here is to factor the initial expression

$$\begin{aligned}
\lim_{x \to -4} \frac{(x+4)(x+1)}{(x+4)(x-1)} &= \lim_{x \to -4} \frac{x+1}{x-1} \\
&= \frac{\lim_{x \to -4}(x+1)}{\lim_{x \to -4}(x-1)} \\
&= \frac{-3}{-5} \\
&= \frac{3}{5}
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x \to 4^-} \sqrt{16 - x^2} &= \lim_{x \to 4^-} \sqrt{(4+x)(4-x)} \\
&= \lim_{x \to 4^-} \sqrt{4+x}\sqrt{4-x} \\
&= \lim_{x \to 4^-} \sqrt{4+x} \cdot \lim_{x \to 4^-} \sqrt{4-x} \\
&= \sqrt{8} * \sqrt{0} \\
&= 0
\end{aligned}$$

A critical aspect of this limit, which allows for it to exist, is that it is a left-hand limit.

(c)

$$\begin{aligned}
\lim_{x\to -1} \frac{x-2}{x^2+4x-3} &= \frac{\lim_{x\to -1}(x-2)}{\lim_{x\to -1}(x^2+4x-3)} \\
&= \frac{-1-2}{(-1)^2+4(-1)-3} \\
&= \frac{-3}{-6} \\
&= \frac{1}{2}
\end{aligned}$$

(d)

$$\begin{aligned}
\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} &= \lim_{x\to -4} \frac{\frac{x+4}{4x}}{4+x} \\
&= \lim_{x\to -4} \frac{4+x}{4x}\frac{1}{4+x} \\
&= \lim_{x\to -4} \frac{1}{4x} \\
&= \frac{1}{4(-4)} \\
&= -\frac{1}{16}
\end{aligned}$$

Alternatively, we can use L'Hôpital's Rule:

$$\begin{aligned}
\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} &= \lim_{x\to -4} \frac{-\frac{1}{x^2}}{1} \\
&= \lim_{x\to -4} (-\frac{1}{x^2}) \\
&= -\frac{1}{16}
\end{aligned}$$

3. Differentiate the following using the rules we have discussed (chain rule, product rule, etc.)

(a)

$$\begin{aligned}
f(x) &= 4x^3+2x^2+5x+11 \\
f'(x) &= 12x^2+4x+5
\end{aligned}$$

Derivative of a constant is zero

(b)

$$\begin{aligned} y &= \sqrt{30} \\ y' &= 0 \end{aligned}$$

(c) Need to apply the chain rule to the second term, first by bringing down the exponent and then by taking the derivative of sin(x)

$$\begin{aligned} y &= 2^3 + \sin^3 x \\ y' &= 0 + 3sin^2(x)cos(x) \\ &= 3sin^2(x)cos(x) \end{aligned}$$

(d)

$$\begin{aligned} h(t) &= \log(9t+1) \\ h'(t) &= \frac{1}{9t+1} * 9 \end{aligned}$$

(e)

$$\begin{aligned} g(x) &= x^3 \cos 11x \\ g'(x) &= 3x^2cos(11x) - 11x^3sin(11x) \end{aligned}$$

(f)

$$\begin{aligned} f(x) &= \log(x^2e^x) \\ f'(x) &= \frac{1}{x^2e^x} * (2xe^x + e^x x^2) \\ &= \frac{2xe^x + e^x x^2}{x^2e^x} \\ &= \frac{2}{x} + 1 \end{aligned}$$

(g)

$$
\begin{aligned}
h(y) &= \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3) \\
&= \frac{y}{y^2} + \frac{5y^3}{y^2} - \frac{3y}{y^4} - \frac{15y^3}{y^4} \\
&= \frac{1}{y} + 5y - \frac{3}{y^3} - \frac{15}{y} \\
&= 5y - \frac{14}{y} - \frac{3}{y^3} \\
h'(y) &= 5 + \frac{14}{y^2} + \frac{9}{y^4}
\end{aligned}
$$

(h)

$$
\begin{aligned}
g(t) &= \frac{3t-1}{2t+1} \\
g'(t) &= \frac{(3)(2t+1) - (3t-1)(2)}{(2t+1)^2} \\
&= \frac{5}{(2t+1)^2}
\end{aligned}
$$

4. Differentiate the following using both the product and quotient rules:

(a) $f(x) = \dfrac{x^2 - 2x}{x^4 + 6}$

First let's use the quotient rule:

$$
\begin{aligned}
h(x) &= \frac{f(x)}{g(x)} \\
f(x) &= x^2 - 2x \\
g(x) &= x^4 + 6 \\
f'(x) &= 2x - 2 \\
g'(x) &= 4x^3 \\
h'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\
&= \frac{(2x-2)(x^4+6) - (x^2-2x)(4x^3)}{(x^4+6)^2} \\
&= \frac{2x^5 + 12x - 2x^4 - 12 - 4x^5 + 8x^4}{(x^4+6)^2} \\
&= \frac{-2x^5 + 6x^4 + 12x - 12}{(x^4+6)^2}
\end{aligned}
$$

Now we can do the same thing with the product rule:

$$\begin{aligned}
j(x) &= k(x)m(x) \\
k(x) &= x^2 - 2x \\
m(x) &= (x^4+6)^{-1} \\
k'(x) &= 2x - 2 \\
m'(x) &= -(x^4+6)^{-2}(4x^3) = -\frac{4x^3}{(x^4+6)^2} \\
j'(x) &= k(x)m'(x) + k'(x)m(x) \\
&= (x^2-2x)(-\frac{4x^3}{(x^4+6)^2}) + (2x-2)(x^4+6)^{-1} \\
&= -\frac{(x^2-2x)(4x^3)}{(x^4+6)^2} + \frac{2x-2}{x^4+6} \\
&= -\frac{4x^5-8x^4}{x^4+6)^2} + \frac{2x-2}{x^4+6} \\
&= -\frac{4x^5-8x^4}{x^4+6)^2} + \frac{2x-2}{x^4+6}\frac{x^4+6}{x^4+6} \\
&= -\frac{4x^5-8x^4}{(x^4+6)^2} + \frac{2x^5+12x-2x^4-12}{(x^4+6)^2} \\
&= \frac{2x^5+12x-2x^4-12-4x^5-8x^4}{(x^4+6)^2} \\
&= \frac{-2x^5+6x^4+12x-12}{(x^4+6)^2}
\end{aligned}$$

The quotient rule is simply a derivation of the product rule combined with the chain rule:

$$\begin{aligned}
h(x) &= \frac{f(x)}{g(x)} \\
&= f(x)g(x)^{-1}
\end{aligned}$$

Apply product and chain rules:

$$\begin{aligned}
h'(x) &= f'(x)g(x)^{-1} + f(x)(-1)g(x)^{-2}g'(x) \\
&= f'(x)g(x)g(x)^{-2} - f(x)g(x)^{-2}g'(x) \\
&= [f'(x)g(x) - f(x)g'(x)]g(x)^{-2} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}
\end{aligned}$$

which is the quotient rule.