

Math Camp

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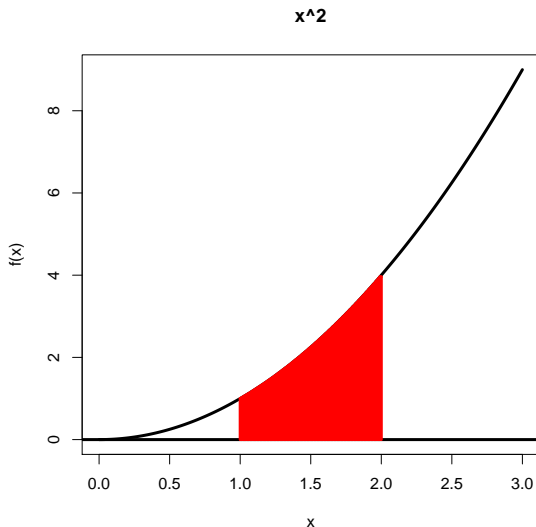
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Integration

- **Derivatives** \rightsquigarrow rates of change
- **Integrals** \rightsquigarrow area under a curve
- **Connection**: fundamental theorem of calculus
- Some **antiderivative** formulas
- Algebra of Integrals
- Improper Integrals
- Monte Carlo principle
- **Integrate a lot in probability theory**, we'll review more then
- Infinite Series

Area Under a Curve



- Approximated area = $\sum_{i=1}^n f(x_{i-1})(x_i - x_{i-1})$
- As partitions become more refined, they improve
- $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_{i-1})(x_i - x_{i-1}) \rightsquigarrow$ **Riemann Integral**

Definition

Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$. We will define the Riemann Integral as $\int_a^b f(x)dx$. If this exists then we say f is **integrable** on $[a, b]$ and call $\int_a^b f(x)dx$ the **integral** of f .

Theorem

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function. Then f is integrable

Theorem

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is a monotonic function. Then f is integrable

Some Counterexamples

Suppose $f : [0, 1] \rightarrow \frac{1}{x}$

$$\int_0^1 \frac{1}{x} dx$$

Then $\frac{1}{x}$ is not integrable on $[a, b]$ because the area that the integral would represent is infinite.

$$\begin{aligned} f(x) &= 1 \text{ if } x \text{ rational} \\ &= 0 \text{ if } x \text{ irrational} \end{aligned}$$

Not integrable, because every interval will contain a discontinuous jump

Fundamental Theorem of Calculus

A **deep** connection between derivatives and integrals makes integration much easier

Theorem

***Fundamental Theorem of Calculus** Suppose $f : [a, b] \rightarrow \mathbb{R}$ and that f is differentiable on $[a, b]$ and that its derivative, f' , is integrable. Then,*

$$\int_a^b f'(x) dx = f(b) - f(a)$$

Recipe for Definite Integration

$$\int_a^b f'(x) dx = f(b) - f(a)$$

- Calculate **antiderivative**
- Evaluate at b
- Evaluate at a

Some Classic Antiderivative Formulas

antiderivative = indefinite integral

$$\int 1 dx = x + c$$

$$\int k dx = kx + c$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\int \frac{1}{x} dx = \log x + c$$

$$\int e^x dx = e^x + c$$

$$\int a^x dx = \frac{a^x}{\log a} + c$$

Uniform Distribution

Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$, with

$$f(x) = 1 \text{ if } x \in [0, 1]$$

$$f(x) = 0 \text{ otherwise}$$

What is the area under $f(x)$ from $[0, 1/2]$?

$$\begin{aligned} \int_0^{1/2} f(x) dx &= \int_0^{1/2} 1 dx \\ &= x \Big|_0^{1/2} \\ &= (1/2) - (0) \\ &= 1/2 \end{aligned}$$

We will call $f(x) = 1$ the **uniform distribution**.

Example 2: Area Under a Line

Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$, with

$$f(x) = x$$

Evaluate the $\int_2^t f(x)dx$

$$\begin{aligned}\int_2^t f(x)dx &= \int_2^t xdx \\ &= \frac{x^2}{2} \Big|_2^t \\ &= \frac{t^2}{2} - \frac{2^2}{2} \\ &= \frac{t^2}{2} - \frac{4}{2} = \frac{t^2}{2} - 2\end{aligned}$$

Integration Facts

Theorem

If $f_1, f_2 : [a, b] \rightarrow \mathbb{R}$ and f_1, f_2 are integrable on $[a, b]$, then

i) Consider the interval $[a, b]$ and $c \in [a, b]$. Then,

$$\int_c^c f_1'(x) dx = f_1(c) - f_1(c) = 0$$

$$\begin{aligned} \int_a^b f_1'(x) dx &= \int_a^c f_1'(x) dx + \int_c^b f_1'(x) dx \\ &= (f_1(c) - f_1(a)) + (f_1(b) - f_1(c)) \\ &= f_1(b) - f_1(a) \end{aligned}$$

Theorem

If $f_1', f_2' : [a, b] \rightarrow \mathbb{R}$ and f_1', f_2' are integrable on $[a, b]$ and f_1' has antiderivative f_1 and f_2' has antiderivative f_2 , then

ii) For $c_1, c_2 \in \mathbb{R}$ then

$$\int_a^b (c_1 f_1'(x) + c_2 f_2'(x)) dx = c_1 \int_a^b f_1'(x) dx + c_2 \int_a^b f_2'(x) dx$$

Challenge Problems

$$\int_0^1 x dx$$

$$\int_0^1 (x^2 + x + 1) dx$$

$$\int_1^2 \left(\frac{1}{x} + e^x\right)$$

Let's Prove Taylor Theorem (And Come Up With Intuition Too!)

Theorem

Taylor's Theorem Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x)$ is infinitely differentiable function. Then, the taylor expansion of $f(x)$ around a is given by

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

Zero Order Approximation

$$f(x) = f(a) + \int_a^x f'(t_1) dt_1$$

$$\int_a^x f'(t_1) dt_1 = f(x) - f(a)$$

$$f(a) + f(x) - f(a) = f(x)$$

First order approximation:

$$f(x) = f(a) + \int_a^x f'(a) dt_1 + \int_a^x \int_a^{t_1} f''(t_2) dt_2 dt_1$$

$$\int_a^x f'(a) dt_1 = f'(a)(x - a)$$

$$\begin{aligned} \int_a^x \int_a^{t_1} f''(t_2) dt_2 dt_1 &= \int_a^x \left(f'(t_1) - f'(a) \right) dt_1 \\ &= f(x) - f(a) - f'(a)(x - a) \\ &= f(a) + f'(x - a) + f(x) - f(a) - f'(a)(x - a) \\ &= f(x) \end{aligned}$$

Next step:

$$f(x) = f(a) + \int_a^x f'(a) dt_1 + \int_a^x \int_a^{t_1} f''(a) dt_2 dt_1 + \int_a^x \int_a^{t_1} \int_a^{t_2} f'''(t_3) dt_3 dt_2 dt_1$$

- 1) We have shown this is true for first derivative $k = 1$.
- 2) Suppose it is true for k . Let's show it is true for $k + 1$.

$$f(x) = P_{k+1} + R_{k+1}$$

$$f(x) = P_k + f^{k+1}(a) \frac{(x-a)^{k+1}}{(k+1)!} + \int_a^x \int_a^{t_1} \dots \int_a^{t_{k+1}} f^{k+2}(t_{k+2}) dt_{k+2} \dots dt_1$$

Flip bounds on the remainder term and you realize it contains R_k and that the additional term cancels out the new f^{k+1} term.

Can obtain error bounds with computation of remainder. Because expansion around each point is necessarily finite as $k \rightarrow \infty$ remainder goes to zero.

More Fundamental Theorem of Calculus

Theorem

Suppose $f' : [a, b] \rightarrow \mathbb{R}$ is integrable on $[a, b]$ and suppose that its antiderivative is $f(x)$. Define $F(t) = \int_a^t f'(x)dx$ for $a \leq t \leq b$. Then, $F'(x_0)$ is $f'(x_0)$.

Proof.

intuition $F(t) = \int_a^t f'(x)dx = f(t) - f(a)$.

Now, we want to take the derivative of $F(t)$ and evaluate at x_0 and the derivative of $f(t) - f(a)$ and evaluate at x_0

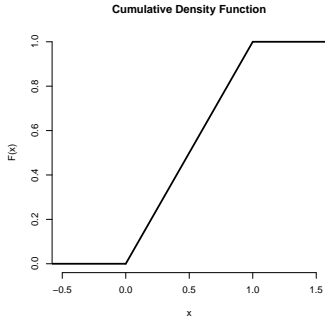
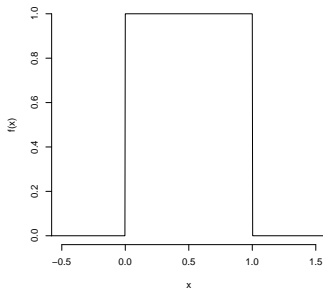
$$\begin{aligned}\frac{\partial}{\partial t} F(t) &= \frac{\partial}{\partial t} (f(t) - f(a)) \\ F'(t)|_{x_0} &= f'(t)|_{x_0} \\ F'(x_0) &= f'(x_0)\end{aligned}$$



Uniform Cumulative Density Function

Suppose that $f' \rightarrow \mathbb{R}$, $f'(x) = 1$ for $x \in [0, 1]$ and $f'(x) = 0$ otherwise. Define,

$$\begin{aligned} F(t) &= \int_0^t f'(x) dx \\ &= \int_0^t 1 dx = x \Big|_0^t \\ &= t \end{aligned}$$



Improper Integrals

Discount rates: valuing the future.

We'll do discrete time with infinite series, we can do them in continuous time with integrals

$$V = \int_0^{\infty} e^{-\delta t} dt$$

- How do we evaluate this integral?
- Improper integrals
- Continuous infinite series

Definition

Definition

Consider $f : [a, \infty) \rightarrow \mathbb{R}$. If the limit

$$\lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

exists then we will say $\int_a^\infty f(x) dx$ **converges** to L . Otherwise, we say it **diverges**.

Also apply definition for

- $\int_{-\infty}^a f(x) dx = \lim_{t \rightarrow -\infty} \int_t^a f(x) dx$
- $\int_{-\infty}^\infty f(x) dx = \lim_{t \rightarrow -\infty} \lim_{y \rightarrow \infty} \int_t^y f(x) dx$.

When do Integrals Converge?

Example 1

$$f(x) = 1/x.$$

$$\begin{aligned}\int_1^{\infty} \frac{1}{x} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx \\ &= \lim_{t \rightarrow \infty} (\log x) \Big|_1^t \\ &= \lim_{t \rightarrow \infty} (\log t) - \lim_{t \rightarrow \infty} (\log 1)\end{aligned}$$

Does not converge

When do Integrals Converge?

Example 2

$$f(x) = \frac{1}{x^2}$$

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2} dx \\ &= \lim_{t \rightarrow \infty} -\frac{1}{x} \Big|_1^t \\ &= \lim_{t \rightarrow \infty} -\frac{1}{t} + \frac{1}{1} \\ &= 0 + 1\end{aligned}$$

Substitution (slides borrowed from math.hmc.edu)

Sometimes, antidifferentiating is **hard**

$$\int (x^2 - 1)^4 2x dx$$

But we can use substitution to simplify. Suspend disbelief and set:

$$u = x^2 - 1$$

$$du = 2x dx$$

Rewriting the original,

$$\begin{aligned}\int (x^2 - 1)^4 (2x dx) &= u^4 du \\ &= \frac{u^5}{5} + c \\ &= \frac{(x^2 - 1)^5}{5} + c\end{aligned}$$

Substitution Rule (slides borrowed from math.hmc.edu)

Just chain rule in reverse. We know that the antiderivative of

$$\int f(g(x))g'(x)dx = F(g(x))$$

So, with substitution rule, we look for ways to set up chain rule

Substitution Rule (slides borrowed from math.hmc.edu)

$$\int -e^{-x} dx$$

$$u = -x$$

$$du = -dx$$

$$\int e^u du = e^u + c$$

$$= e^{-x} + c$$

Substitution Rule (slides borrowed from math.hmc.edu)

We can also multiply by 1 (creatively) to set up substitution rule

$$\begin{aligned}\int e^{-2x} dx &= -\frac{1}{2} \int -2e^{-2x} dx \\ u &= -2x \\ du &= -2dx \\ -\frac{1}{2} \int e^u du &= -\frac{1}{2} e^u + c \\ &= -\frac{1}{2} e^{-2x} + c\end{aligned}$$

Example: Exponential Distribution

Suppose $f : [0, \infty) \rightarrow \mathbb{R}$, with $f(x) = e^{-x}$. Evaluate

$$\begin{aligned}\int_0^{\infty} e^{-x} dx &= \lim_{t \rightarrow \infty} \int_0^t e^{-x} dx \\ &= \lim_{t \rightarrow \infty} -e^{-x} \Big|_0^t \\ &= \lim_{t \rightarrow \infty} -e^{-t} + 1 \\ &= 0 + 1\end{aligned}$$

We will call $f(x) = e^{-x}$ the **exponential** distribution

Integration by Parts

Consider:

$$\int x \cos(x) dx$$

That is hard to integrate.

Instead we'll use **Integration by parts**

Integration by Parts

Define:

$$g(x) = u(x)v(x)$$

Let's differentiate $g(x)$

$$g'(x) = u'(x)v(x) + v'(x)u(x)$$

So, antidifferentiating $g'(x)$ yields

$$\begin{aligned}\int g'(x)dx &= \int u'(x)v(x)dx + \int u(x)v'(x)dx \\ u(x)v(x) + c - \int u'(x)v(x)dx &= \int u(x)v'(x)dx \\ uv - \int vdu &= \int u dv\end{aligned}$$

Integration by Parts

$$\begin{aligned} \int x \cos(x) dx \\ u &= x \\ du &= 1 \\ dv &= \cos(x) \\ v &= \sin(x) \end{aligned}$$

$$\begin{aligned} \int x \cos(x) dx &= x \sin(x) - \int \sin(x) 1 dx \\ &= x \sin(x) + \cos(x) \end{aligned}$$

Integration by Parts

Challenge:

$$\int \exp(x) \cos(x) dx$$

$$\int \log(x) dx$$

$$\int \arctan(x) dx$$

Wolfram Alpha (briefly)

Monte Carlo and Integration (via Jackman)

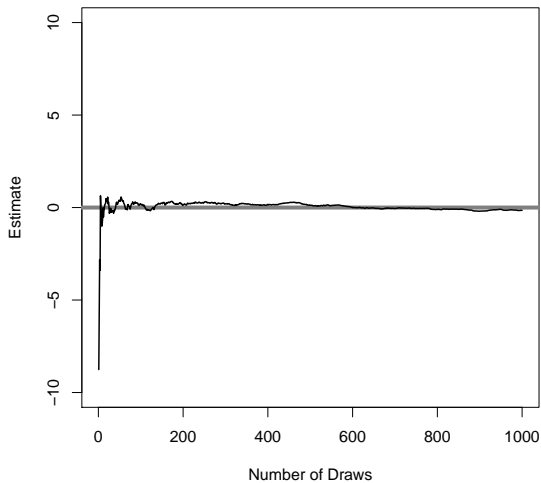
Suppose that we want to compute some integral $\int_{-\infty}^{\infty} xf(x)dx$, but $f(x)$ is **complicated**.

$$f(x) = \frac{\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{\sqrt{2\pi}}$$

Suppose we can generate random draws from $f(x)$.
 $d_1, d_2, d_3, \dots, d_T$, then we can approximate this with:

$$\text{Expected Value} = \frac{1}{T} \sum_{i=1}^T d_i$$

as $T \rightarrow \infty$, Expected value $\rightarrow \int_{-\infty}^{\infty} xf(x)dx$



R code for quantiles! MonteCarlo.R

Infinite Series

- Interactions are often **repeated**
 - **Countries**: Fight now or fight later
 - **Congress**: Caro, LBJ, and the Southern Strategy
 - **FDA**: Do I approve this drug?
 - **Bargain**: Do I make a deal now, or wait?
- General idea :
 - Actions have **continuation value**:
 - Value in the present time
 - Stream of benefits in the future
- **Infinite Series** to model

Formal definition \rightsquigarrow Heuristics \rightsquigarrow example problem (from JF)

Infinite Series

Definition

An infinite series is a pair $(\{a_n\}_{n=1}^{\infty}, \{S_n\}_{n=1}^{\infty})$ where $\{a_n\}_{n=1}^{\infty}$ is a sequence and $S_n = \sum_{k=1}^n a_k$.

Definition

The infinite series $(\{a_n\}_{n=1}^{\infty}, \{S_n\}_{n=1}^{\infty})$ **converges** if the **sequence** $\{S_n\}_{n=1}^{\infty}$ converges to S . We'll write this as,

$$\sum_{n=1}^{\infty} a_n = S$$

Infinite Series

- Example 1

- $a_n = \{0, 1, 0, 1, 0, 1, \dots\}$
- $S_n = \sum_{i=1}^n a_i = \frac{n}{2}$ as $n \rightarrow \infty$ clearly diverges

- Example 2

- $a_n = \left\{ \frac{1}{n(n+1)} \right\}_{n=1}^{\infty}$
- We know that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$
- So, for $n = m$,

$$\begin{aligned} S_m &= \sum_{i=1}^m a_i \\ &= \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \dots + \frac{1}{(m)(m+1)} \\ S_m &= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \dots + \left(\frac{1}{m} - \frac{1}{m+1} \right) \\ &= 1 - \frac{1}{m+1} \end{aligned}$$

So S_m converges on 1. (the **sequence** S_m converges, just like we prove other sequence convergence)

How Do We Assess Convergence?

Theorem

If $\{S_n\}_{n=1}^{\infty}$ converges then $\{a_n\}_{n=1}^{\infty}$ is converges to zero

- Necessary!
- But not sufficient

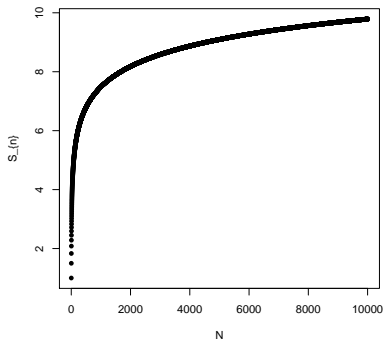
Infinite Series Convergence

Example 1:

$$- a_n = \frac{1}{n}. S_n,$$

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Does this converge?



Infinite Series Convergence

Suppose $n = 2^k$

$$\begin{aligned} S_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{2^k} \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^{k-1}+1} + \dots + \frac{1}{2^k}\right) \\ &> 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \dots + 2^{k-1}\left(\frac{1}{2^k}\right) = 1 + \frac{k}{2} \end{aligned}$$

We know that $1 + \frac{k}{2}$ does not converge.

And we know that $S_n > 1 + \frac{k}{2} \rightsquigarrow$ **does not converge** (!!)

Theorem

$\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges **if and only if** $p > 1$.

Geometric Series and Discount Rates

Definition

A geometric series is an infinite series such that $a_n = cr^n$ and that
$$S_n = \sum_{k=0}^n cr^k = c + cr + cr^2 + cr^3 + \dots cr^n$$

Geometric Series and Discount Rates

Theorem

If $|r| < 1$, then the geometric series converges to $\frac{c}{1-r}$. If $|r| > 1$, the geometric series diverges

Proof.

$$\begin{aligned}(1-r)S_n &= (1-r) \sum_{k=0}^n cr^k \\(1-r) \sum_{k=0}^n cr^k &= c + cr + cr^2 + \dots + cr^n - \\&\quad (cr + cr^2 + cr^3 + \dots + cr^n + cr^{n+1}) \\&= c - cr^{n+1} \\S_n &= c \left(\frac{1 - r^{n+1}}{1 - r} \right)\end{aligned}$$

$$\begin{aligned} S_n &= c \left(\frac{1 - r^{n+1}}{1 - r} \right) \\ &= c \left(\frac{1}{1 - r} \right) - c \left(\frac{r^{n+1}}{1 - r} \right) \end{aligned}$$

$c \left(\frac{r^{n+1}}{1-r} \right)$ converges **if and only if** $|r| < 1$.

Discount Rates and IR (Fearon, Part 2)

Suppose states are choosing between **attacking** another country to obtain a short time gain, or **cooperating** for peace

	C	D
C	20,20	10,25
D	25,10	15,15

Grim-trigger: cooperate, until defect. Then defect forever

Suppose states **discount** future $\delta \in [0, 1]$.

$$\begin{aligned}V(C) &= 20 + \delta 20 + \delta^2 20 + \delta^3 20 + \dots \\&= \frac{20}{1 - \delta} \\V(D) &= 25 + \delta 15 + \delta^2 15 + \delta^3 15 \\&= 25 + \delta \frac{15}{1 - \delta}\end{aligned}$$

When Will States Cooperate? (Fearon, Part 2)

$$V(C) > V(D)$$

$$\frac{20}{1-\delta} > 25 + \delta \frac{15}{1-\delta}$$

$$\frac{1}{1-\delta}(20 - \delta 15) > 25$$

$$(20 - \delta 15) > 25(1 - \delta)$$

$$10\delta > 5$$

$$\delta > \frac{1}{2}$$

Linear Algebra Tuesday!