Math Camp - Homework 1 Solutions

1. Using the sets...

$$\begin{aligned} A &= \{2,3,7,9,13\} \\ B &= \{x : 4 \leq x \leq 8 \text{ and } x \in \mathbb{Z}\} \\ C &= \{x : 2 < x < 25 \text{ and } x \text{ is prime}\} \\ D &= \{1,4,9,16,25,\ldots\} \end{aligned}$$

identify the following:

(a) $A \cup B$
$E = \{2,3,4,5,6,7,8,9,13\}$, combine all integers between 4 and 8 inclusive with the numbers in set A

(b) $(A \cup B) \cap C$
$F = \{3,5,7,13\}$, Since C is only prime numbers greater than 2 and less than 25, we take all the prime numbers that are also included in E, but remember to drop out 2 since it is not included in C

(c) $C \cap D$
$G = \emptyset$, there are no prime numbers in D, so nothing is shared between C and D

2. Simplify the following:

(a) $k^{x-y} * k^{-x-y}$
Using our rules for working with exponents, $k^{(x-y)+(-x-y)} = k^{-2y} = \frac{1}{k^{2y}}$

(b) $\left(\frac{z^{4v+6}}{z^{v+9}}\right)$
We can rewrite this to simplify things and then work again with exponents,
$z^{4v+6} * z^{-(v+9)} = z^{3v-3}$

(c) $(a^{b^0} + a^{0^b} - a^{-1} * a^2)^b$
Important to remember for this, anything raised to the "0" power is equal to 1. This will help us simply things inside the parenthesis.
$(a^1 + 1 - a^{-1+2})^b = 1^b = 1$

3. Express each of the following as a single logarithm:

(a) $\log(x) + \log(y) - \log(z)$
Applying the log rules, we combine logs that are added through multiplication and then combine logs that are subtracted with division.
$\log(xy) - \log(z)$
$\log(\frac{xy}{z})$

(b) $2\log(x) + 1$
Here its important to remember that anything multiplying a log can be moved inside the parenthesis as an exponent and that $1 = \log(e)$
$\log(x^2) + \log(e) = \log(ex^2)$

(c) $\log(x) - 2$
$\log(x) - 2\log(e) = \log(x) - \log(e^2) = \log(\frac{x}{e^2})$

NB: There are multiple ways to prove both 4 and 5.

4. Prove that $n! > n^2$ for integers $n \geq 4$. (Hint: try using induction.)

Let's call our proposition P_n (e.g., P_{10} is that $10! > 10^2$). Recall that induction is a strategy to prove a mathematical proposition for all integers greater than some starting value. In this case, we want to prove that $P_4, P_5, \ldots, P_n, \ldots$ are all true. Induction requires two steps.

First, we prove a *base case* for some particular integer. Here, our base case is P_4, since we want to prove the proposition for $n \geq 4$. It is easy to see that $4! = 4 \times 3 \times 2 \times 1 = 24$ is greater than $4^2 = 16$. So P_4 serves as our basis for induction.

Next, we need to prove the *inductive step*. That is, we *assume* that P_k is true, and prove that if P_k is true, then P_{k+1} is also true. For this problem, we assume that $k! > k^2$. Then note:

$$\begin{aligned} (k+1)! &= (k+1)k! && (1) \\ &> (k+1)k^2 \quad \text{by the inductive hypothesis} && (2) \end{aligned}$$

If we can show that $k^2 > k + 1$, then our proof will be complete. This inequality holds for $k \geq 2$. To see this, divide both sides of the inequality by k to get $k > 1 + \frac{1}{k}$ – which is true for integers $k \geq 2$. Thus, we have

$$(k+1)! > (k+1)k^2 > (k+1)(k+1) \tag{3}$$

So we have proven that a base case P_4 is true, and we have proven that whenever P_k is true, P_{k+1} is also true. Therefore, by the principle of induction, we have now proven that $n! > n^2$ for all $n \geq 4$.

5. A number is *rational* if it can be written as the quotient of two integers — e.g., if $x = \frac{p}{q}$, with $p, q \in \mathbb{Z}$, then x is rational. (The set of rational numbers is often denoted $\mathbb{Q}$.) A number is *irrational* if it is not rational. Prove that $\sqrt{2}$ is irrational. (Hint: try writing a proof by contradiction.)

Recall that in a proof by contradiction, we show that if we assume the truth of a proposition that is actually false, we generate a contradiction. This shows that our original assumption must be wrong.

Suppose, by way of contradiction, that $\sqrt{2}$ is rational. Then we can write $\sqrt{2} = \frac{p}{q}$, where p and q are integers and have no common factors (i.e., we can't simplify the fraction further — otherwise we can just cancel out common factors until this is true). Then we have:

$$\sqrt{2} = \frac{p}{q} \tag{4}$$

$$2 = \frac{p^2}{q^2} \tag{5}$$

$$2q^2 = p^2 \tag{6}$$

Note that by definition, the square of a number is a number multiplied by itself. It must therefore have an even number of prime factors. So p^2 and q^2 both have an even number of prime factors. But this points to a contradiction, because the left hand side is a prime number (2) multiplied by an even number of prime factors (q^2), implying that p^2 has an odd number of prime factors. Thus, our original hypothesis is false and $\sqrt{2} \notin \mathbb{Q}$.