Reconstruction of signals in separable union of cones by magnitude measurements

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Outline

1. Two important sampling problems
   - Phase retrieval
   - Sampling on a union of subspaces
2. Phase retrieval for the separable union of cones (UoC)
   - Separation
   - Recovery
3. Simulation
   - Phase-retrieval for a separable deterministic UoC
   - Phase-retrieval of a separable random UoC

Phase Retrieval

Magnitude measurement:

$$b_j := |\langle f, a_j\rangle|, j \in \Lambda \tag{1.1}$$

Objective: recover f via $\{b_j\}_{j\in\Lambda}$, up to a global phase.

Vector case:

- Fourier PR: $f \in \mathbb{C}^n, a_j = [e^{ijt_1}, e^{ijt_2}, \ldots, e^{ijt_n}]$. (X-ray crystallography....).
- Frame PR: $f \in \mathbb{C}^n$, $\{a_j\}_{j\in\Lambda}$ – – a frame of $\mathbb{C}^n$. (R. Balan, P.G. Casazza, and D. Edidin)

Function case:

- PR for bandlimited function f: (G. Thakur)
- PR for shift-invariant space: (Qiyu Sun)

Two issues of PR: amount of measurements and computational complexity

(I): sample complexity ratio: SCR=$\frac{m}{n}$, m phase retrievable measurement vectors, n = length(f).

- Real-valued: Real-valued matrix $A := [a_1, \ldots, a_m] \in \mathbb{R}^{n\times m}$ can do PR of $\mathbb{R}^n \Longrightarrow m \geq 2n-1 \Longrightarrow$ SCR $\geq 2 - 1/n$. [R. Balan, P.G. Casazza, and D. Edidin, Appl. Comp. Harm. Anal., 2006]
- Complex-valued: The minimum amount of vectors for PR of $\mathbb{C}^n$?

Two issues of PR: amount of measurements and computational complexity

(II): Computational complexity—-algorithm.

PR method	Wirtinger Flow	Alternating Minimization
Comp. Complexity	$O(n^2 \log \frac{1}{\epsilon})$	$O(n^2 \log^2 n(\log n + \log \frac{1}{\epsilon} \log\log \frac{1}{\epsilon}))$
PR method	BlockPR	PhaseLift
Comp. Complexity	$O(n \log^4 n)$	$O(n^3/\epsilon^2)$

Table: The computational complexity of different PR methods, where ϵ is the computing accuracy. [PhaseLift, Candés, Y.C. Eldar, T. Strohmer and V. Voroninski, SIAM Rev., 2015], [Wirtinger Flow, E. Candés, X. Li, M. Soltanolkotabi, IEEE. T. I.T, 2015], [Alternating Minimization, P. Netrapalli, P. Jain, and S. Sanghavi, IEEE, T.S.P.,2015], [BlockPR, M. A. Iwen, A. Viswanathan, and Y. Wang, SIAM Journal on Imaging Sciences, 2016].

Two signal models

- Traditional model: the target signal $f \in V := \text{span}\{\phi_k : k \in \Lambda\}$. (wavelet space, periodic signal space......)
- Union of subspaces (UoS model): the target signal $f \in \bigcup_{k=1}^{L} V_k$. (Advantage: By exploiting the structure of UoS, a signals in the bandlimited UoS can be recovered with the sampling rate lower than Nyquist. [Y. Lu and M. Do, 2008, IEE T. S. P], [Y. C. Eldar and M. Mishali, 2009, IEE T. I. T] and so on.)

Cones and their union

$$\mathcal{X}_k = [\mathbf{x}_{k,1}, \ldots, \mathbf{x}_{k,m_k}] \in \mathbb{R}^{n \times m_k},$$

$$\textbf{cone}(\mathcal{X}_k) = \{\mathcal{X}_k \Theta : \forall \Theta \geq 0 (\in \mathbb{R}^{m_k})\},$$

$$\text{UoC} : \bigcup_{k=1}^{L} \textbf{cone}(\mathcal{X}_k).$$

Applications: operations research ([R. Henrion, J. Outrata, OPTIMIZ., 2008]), representation theory ([V. Chari, Inventiones Mathematicae, 1986]), and compressed sensing ([Y. Traonmilin, R. Gribonval, Appl. Comput. Harmon. Anal., 2018]).

Recovery of signals on UoC $\bigcup_{k=1}^{L}$ cone$(\mathcal{X}_k)$ by magnitude measurements

Objective: Using few measurements and fast recovery?
Two-step PR-scheme: PR = separation + recovery.

Some problems:
P1: For the target $f \in$ UoC, how can we use the **magnitude measurements** to identify **cone**$(\mathcal{X}_{k_f})$ such that $f \in$ **cone**$(\mathcal{X}_{k_f})$?
P2: If **P1** holds, then how the separation can be performed?
P3: How to exploit the structure of the cone to reduce measurements and establish fast recovery?

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Definition: separable union of cones

We say that a UoC $\bigcup_{k=1}^{L}$ **cone**$(\mathcal{X}_k)$ is separable with respect to a separating strategy, if there exists a so-called identifier $G := [g_1, \ldots, g_\gamma] \in \mathbb{R}^{n\times\gamma}$ such that for any nonzero target vector $f \in \bigcup_{k=1}^{L}$ **cone**$(\mathcal{X}_k)$, by applying the strategy to the measurement $|Gf| := [|\langle g_1, f\rangle|, \ldots, |\langle g_\gamma, f\rangle|]$, the unique cone **cone**$(\mathcal{X}_{k_f})$ where f lies in can be identified.

Lemma1(Necessity and sufficiency for a separable union of two cones)

Suppose that **cone**$(\mathcal{X}_1)$ and **cone**$(\mathcal{X}_2)$ are the two cones generated by the columns vectors of the matrices $\mathcal{X}_1 \in \mathbb{R}^{n\times m_1}$ and $\mathcal{X}_2 \in \mathbb{R}^{n\times m_2}$, respectively. Then the UoC $\bigcup_{k=1}^{2}$ **cone**$(\mathcal{X}_k)$ is separable with respect to a vector identifier if and only if either

$$\text{invim}(\mathcal{R}(\mathcal{X}_1^T) \cap \mathbb{R}^{+,m_1}) \cap \mathcal{N}(X_2^T) \neq \emptyset \tag{2.1}$$

or

$$\text{invim}(\mathcal{R}(\mathcal{X}_2^T) \cap \mathbb{R}^{+,m_2}) \cap \mathcal{N}(X_1^T) \neq \emptyset, \tag{2.2}$$

where $\mathbb{R}^{+,m} := \{(z_1, \dots, z_m) : (z_1, \dots, z_m) \succ 0\}, X_1^T$ is the transpose.

Theorem 1 (General separable UoC $\bigcup_{k=1}^{L}$ cone$(\mathcal{X}_k)$)

A UoC $\bigcup_{k=1}^{L}$ **cone**$(\mathcal{X}_k)$, where $\mathcal{X}_k = [\mathbf{x}_{k,1}, \ldots, \mathbf{x}_{k,m_k}] \subseteq \mathbb{R}^{n \times m_k}$, is separable if and only if for every k we have either

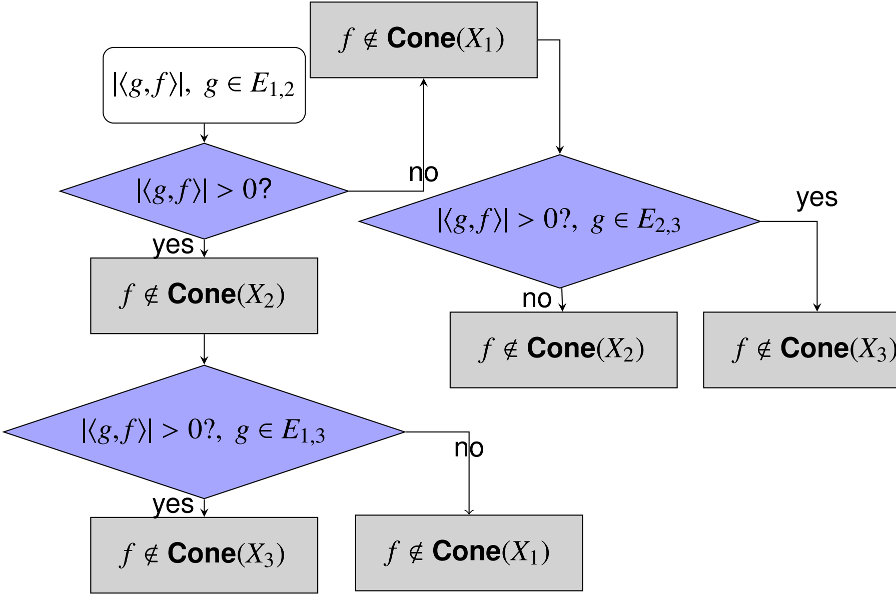
$$\text{invim}(\mathcal{R}(\mathcal{X}_l^T) \cap \mathbb{R}^{+,m_l}) \cap \mathcal{N}(\mathcal{X}_k^T) \neq \emptyset, \tag{2.3}$$

or

$$\text{invim}(\mathcal{R}(\mathcal{X}_k^T) \cap \mathbb{R}^{+,m_k}) \cap \mathcal{N}(\mathcal{X}_l^T) \neq \emptyset, \tag{2.4}$$

where $l = 1, \ldots, k-1$.

$E_{k,l} := \text{invim}(\mathcal{R}(\mathcal{X}_k^T) \cap \mathbb{R}^{+,m_k}) \cap \mathcal{N}(\mathcal{X}_l^T)$.


$f \notin$ **Cone**(X_1)
$|\langle g, f\rangle|, \ g \in E_{1,2}$
$|\langle g, f\rangle| > 0?$
no
yes
$f \notin$ **Cone**(X_2)
$|\langle g, f\rangle| > 0?, \ g \in E_{2,3}$
yes
no
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$|\langle g, f\rangle| > 0?, \ g \in E_{1,3}$
no
yes
$f \notin$ **Cone**(X_3)
$f \notin$ **Cone**(X_1)

Remark: separation of UoC $\bigcup_{k=1}^{L}$ cone$(\mathcal{X}_k)$

- $(L-1)$ exclusions.
- $(L-1)$ measurements.
- $O((L-1)n)$ operations.
- *There are at least $(L-1)$ cones satisfying the following overlap property*

$$\mathcal{R}(\mathcal{X}_i^T) \cap \mathbb{R}^{+,m_i} \neq \emptyset \tag{2.5}$$

Theorem 2: (Phase-retrieval on a cone satisfying the overlap property (2.5))

Let **cone**$(\mathcal{X})$ be a cone with $\mathcal{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_m] \in \mathbb{R}^{n\times m}$. Suppose that $\mathcal{R}(\mathcal{X}^T) \cap \mathbb{R}^{+,m} \neq \emptyset$. Then there exist γ-vectors $\{\mathsf{F}_k\}_{k=1}^{\gamma}$ such that $\{|\langle f, \mathsf{F}_1\rangle|, \ldots, |\langle f, \mathsf{F}_\gamma\rangle|\}$ determines f (up to a unimodular scalar) for any $f \in$ **cone**$(\mathcal{X})$, where $\mathsf{F}_1 \in \mathrm{invim}(\mathcal{R}(\mathcal{X}^T) \cap \mathbb{R}^{+,m})$ and $\gamma = \min\{\mathrm{rank}[\mathcal{X}, \mathsf{F}_1], n\}$. Moreover, $\{\mathsf{F}_k\}_{k=1}^{\gamma}$ can be designed in such a way that the recovery of any vector in **cone**$(\mathcal{X})$ requires only $O(\gamma \log \gamma)$-number of operations, i.e., the cost is FFT-time.

Sketch of the proof of Theorem 2

Let **cone**$(\mathcal{X})$ with $\mathcal{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_m] \in \mathbb{R}^{n\times m}$ and $\gamma = \min\{\text{rank}[\mathcal{X}, \mathsf{F}_1], n\}$. Define an isometry $\mathfrak{P} : \text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathsf{F}_1\} \longrightarrow \mathbb{R}^\gamma$. Based on $\mathfrak{P}$, design phase retrievable vectors $\{\mathsf{F}_k\}_{k=1}^{\gamma} \subseteq \text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathsf{F}_1\}$. Then $f \in$ **cone**$(\mathcal{X})$ can be reconstructed by

$$\mathfrak{P}f = \mathsf{FFT}\Big(\text{diag}^{-1}(\mathsf{FFT}(\mathfrak{P}\mathsf{F}_1^T))\mathsf{IFFT}\Big(\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\delta_2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\delta_\gamma & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} |\langle f, \mathsf{F}_1\rangle| \\ |\langle f, \mathsf{F}_2\rangle| \\ \vdots \\ |\langle f, \mathsf{F}_\gamma\rangle| \end{bmatrix}\Big)\Big), \tag{2.6}$$

where δ_k are some constants dependent on $\mathcal{X}$ and the measurement vectors $\{\mathsf{F}_k\}_{k=1}^{\gamma}$.

Proposition: the amount of phase retrievable measurement vectors for separable UoC.

Let the separable UoC $\bigcup_{k=1}^{L} \mathbf{cone}(\mathcal{X}_k)$ be as previously. Consequently, there are at least $(L-1)$ cones satisfying the overlap property (2.5).

(i) If there exists $\mathbf{cone}(\mathring{\mathcal{X}})$ (only one indeed) not satisfying (2.5), then the two-step PR-scheme of the target vector on this cone can be performed by $L + 2\text{rank}(\mathring{\mathcal{X}}) - 2$ measurements.

(ii) If all the L cones satisfy the overlap property (2.5). Then using the two-step PR-scheme, any target signal in the UoC can be determined by at most $L + \gamma + 1$ magnitude measurements, where $\gamma = \max_k\{\text{rank}(\mathcal{X}_k)\}$. Moreover, our scheme costs $O(Ln)+O(\gamma \log \gamma)$-number operations.

Computational complexity

When L is a constant and $\gamma \log \gamma \lesssim n$,

$$O(Ln) + O(\gamma \log \gamma) = O(n).$$

When L is a constant and $\gamma \approx n$,

$$O(Ln) + O(\gamma \log \gamma) = O(n \log n).$$

Therefore, if all the L cones satisfy the overlap property (2.5), then our comp. complex. is not larger than $O(n \log n)$.

PR method	Wirtinger Flow	Alternating Minimization
Comp. Complexity	$O(n^2 \log \frac{1}{\epsilon})$	$O(n^2 \log^2 n(\log n + \log \frac{1}{\epsilon} \log \log \frac{1}{\epsilon}))$
PR method	BlockPR	PhaseLift
Comp. Complexity	$O(n \log^4 n)$	$O(n^3/\epsilon^2)$

Numerical Simulation I: two-step PR-scheme for a separable deterministic UoC

Let $\widetilde{\mathcal{X}}_1 = [\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{1,n}] \in \mathbb{R}^{n\times n}$, where

$$\mathbf{x}_{1,1}^T = [1, \frac{-1}{3\times 2^3 \times 1}, \frac{1}{3\times 3^3\times 2}, \ldots, \frac{(-1)^{n-1}}{3\times n^3 \times (n-1)}],$$

and $\mathbf{x}_{1,k}^T = \mathbf{x}_{1,1}^T \circ \mathrm{e}_k$ where $\mathrm{e}_k = [1, \ldots, 1, -1, 1, \ldots, 1]$ with -1 being the k-th element, and $\circ$ being the element-wise product of two vectors. Define

$$[\mathbf{x}_{1,n+1}, \ldots, \mathbf{x}_{1,2n-1}] := \widetilde{\mathcal{X}}_1 \begin{bmatrix} b & b & \cdots & b \\ -a & 0 & \cdots & 0 \\ 0 & -a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -a \end{bmatrix}_{n\times(n-1)},$$

where $a = 0.115$, $b = 0.8850$.

Numerical Simulation I: two-step PR-scheme for a separable deterministic UoC

Extend $\widetilde{\mathcal{X}}_1$ to $\mathcal{X}_1 := [\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{1,2n-1}] \in \mathbb{R}^{n\times(2n-1)}$. Define

$$\mathcal{X}_2 := [\mathbf{x}_{2,1}, \ldots, \mathbf{x}_{2,n}] = \begin{bmatrix} 2 & 2 & \cdots & 2 \\ -1 & -1 & \cdots & -1 \\ (-1)^{3+1} & (-1)^{3+2} & \cdots & (-1)^{3+n} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1} & (-1)^{n+2} & \cdots & (-1)^{n+n} \end{bmatrix}_{n\times n},$$

$$\mathcal{X}_3 := [\mathbf{x}_{3,1}, \ldots, \mathbf{x}_{3,n}] = \begin{bmatrix} -1.5 & -1.5 & \cdots & -1.5 \\ 1 & 1 & \cdots & 1 \\ (-\frac{1}{4})^{3+1} & (-\frac{1}{5})^{3+2} & \cdots & (-\frac{1}{n+3})^{3+n} \\ \vdots & \vdots & \ddots & \vdots \\ (-\frac{1}{n+1})^{n+1} & (-\frac{1}{n+2})^{n+2} & \cdots & (-\frac{1}{2n})^{n+n} \end{bmatrix}_{n\times n}.$$

Numerical Simulation I: two-step PR-scheme of a separable deterministic UoC

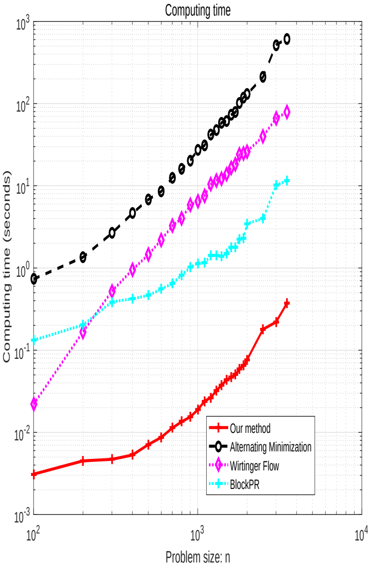
Pick $g_1 := [1, 2, 0, \ldots, 0]^T$ and $g_2 := [1, 1.5, 0, \ldots, 0]^T$. By direct computation,

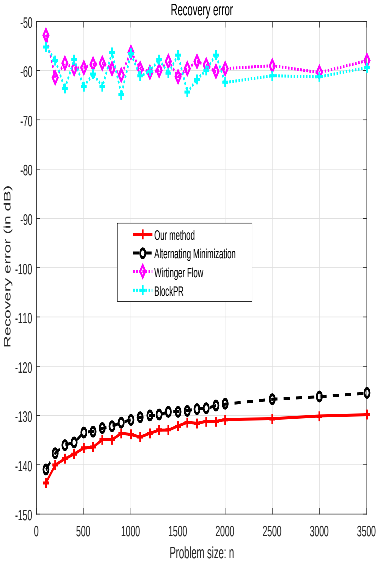
$$\mathcal{X}_1^T g_1 \succ 0, \mathcal{X}_2^T g_1 = 0, \mathcal{X}_1^T g_2 \succ 0, \mathcal{X}_2^T g_2 \succ 0, \mathcal{X}_3^T g_2 = 0. \tag{3.1}$$

That is, the UoC $\bigcup_{k=1}^{3} \textbf{cone}(\mathcal{X}_k)$ is separable. As an example, pick $f = 0.8815\mathbf{x}_{1,1} - 0.115\mathbf{x}_{1,2}$ to check the efficiency. The error is defined as follows and reported in dB,

$$\text{error} := 10 \log_{10}[\min\{\|f - f_r\|_2 / \|f\|_2, \|f + f_r\|_2 / \|f\|_2\}], \tag{3.2}$$

where f_r is the recovery result.


Computing time
Computing time (seconds)
Problem size: n
Our method
Alternating Minimization
Wirtinger Flow
BlockPR


Recovery error
Recovery error (in dB)
Problem size: n
Our method
Alternating Minimization
Wirtinger Flow
BlockPR

Method	measurements	time (seconds)	error
WF	7000	6.4158	-59.5304
AM	7000	27.1701	-130.3796
BlockPR	7000	1.147	-61.1413
Our method	1002	0.0188	-133.8534

Table: When the length of the signal is $n = 1000$.

Numerical Simulation II: two-step PR-scheme of a separable random UoC

Define

$$\begin{aligned}
\mathcal{X}_1 &= [\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{1,J}] \\
&= \begin{bmatrix} \epsilon_{1,1} & \epsilon_{1,2} & \cdots & \epsilon_{1,J} \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon_{n-1,1} & \epsilon_{n-1,2} & \cdots & \epsilon_{n-1,J} \\ \eta_1 - \sum_{k=1}^{n-1} \alpha_k \epsilon_{k,1} & \eta_2 - \sum_{k=1}^{n-1} \alpha_k \epsilon_{k,2} & \ldots & \eta_n - \sum_{k=1}^{n-1} \alpha_k \epsilon_{k,J} \end{bmatrix}_{n \times J},
\end{aligned} \tag{3.3}$$

where the i.i.d standard Gaussian distribution $\epsilon_{k,l} \sim \mathbf{N}(0, 1)$, and the i.i.d uniformly distribution $\alpha_k, \eta_k \sim \mathbf{U}(a, b)$. For the random vector $g = (\alpha_1, \ldots, \alpha_{n-1}, 1)^T$, we have

$$\mathcal{X}_1^T g = (\eta_1, \ldots, \eta_n)^T \succ 0 \tag{3.4}$$

with probability 1.

Numerical Simulation II: two-step PR-scheme of a separable random UoC

Define

$$\mathbf{x}_{2,k} = (0, 0, \ldots, 1, 0, \ldots, -\alpha_2)^T \tag{3.5}$$

with 1 being the kth element, and

$$\mathcal{X}_2 := [\mathbf{x}_{2,1}, \ldots, \mathbf{x}_{2,n-1}]. \tag{3.6}$$

It is easy to check that the random UoC $\mathbf{cone}(\mathcal{X}_1) \cup \mathbf{cone}(\mathcal{X}_2)$ is separable with respect to the random identifier g. Moreover, $\mathbf{x}_{2,k}(1, \ldots, 1, 0)^T = 1$ for any k. That is, with the probability 1, both $\mathbf{cone}(\mathcal{X}_1)$ and $\mathbf{cone}(\mathcal{X}_2)$ satisfy the overlap property (2.5).

If the error is smaller than -50, then we say that the recovery is successful.

Method	J	a	b	m	success rate	time (seconds)
WF	1000	0	1	4000	0.8600	3.1079 ± 0.0617
AM	–	–	–	3500	1	7.5310 ± 0.8921
BlockPR	–	–	–	3500	0.8800	0.9332 ± 0.0576
Our method	–	–	–	501	1	$0.006 \pm (3.78e - 7)$

Table: the problem size $n = 500$; $f = \mathcal{X}_1(\mathbf{U}(0,1))^J$; 100 trials; m measurements. – the same value as above.

Thank You!