Introduction to the Model Theory of Higher-Order Logic

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Presentation Overview

1. A brief history of simple type theory
2. Alonzo: syntax and semantics
3. Standard semantics and categoricity
4. Henkin's Theorem and Skolem's Paradox

(1908) Russell's Ramified Theory of Types

Two distinct hierarchies [1]:

- Types
  - Stratifying and distinguishing individuals, predicates on individuals, and predicates on predicates on ...
  - Restrict variable scope and term formation
  - Ban set-theoretic paradoxes
- Orders
  - Stratifying properties based on argument order (predicativity)
  - Bans *impredicative* definitions which have a "vicious circle"
  - E.g. $y = \inf(X)$ iff

    $$\begin{aligned} &(1) \text{ For all } x \in X, y \leq x; \\ &(2) \text{ For all } z \in X, z \text{ is a lower bound, implies } z \leq y. \end{aligned}$$

  - Since we quantified over $y \in X$, $\inf(X)$ is impredicative

(1908) Russell's Axiom of Reducibility & STT

- (Axiom) Any propositional function can be expressed as a predicative one
- So why care about predicativity?!

$$\text{STT} = \text{Ramified Type Theory} + \text{Axiom of Reducibility}$$

(1940) Church's (Simple) Type Theory

- A simple theory of types with λ-conversion [2]
- Hierarchy of types via o, ι, $(\alpha\beta)$
- Great influence on computing as a practical function theory

(1940) Inference Rules of Church's Type Theory

I. *To replace any part* $\boldsymbol{M}_\alpha$ *of a formula by the result of substituting* $\boldsymbol{y}_\beta$ *for* $\boldsymbol{x}_\beta$ *throughout* $\boldsymbol{M}_\alpha$, *provided that* $\boldsymbol{x}_\beta$ *is not a free variable of* $\boldsymbol{M}_\alpha$ *and* $\boldsymbol{y}_\beta$ *does not occur in* $\boldsymbol{M}_\alpha$. (I.e., to infer from a given formula the formula obtained by this replacement.)

II. *To replace any part* $((\lambda \boldsymbol{x}_\beta \boldsymbol{M}_\alpha)\boldsymbol{N}_\beta)$ *of a formula by the result of substituting* $\boldsymbol{N}_\beta$ *for* $\boldsymbol{x}_\beta$ *throughout* $\boldsymbol{M}_\alpha$, *provided that the bound variables of* $\boldsymbol{M}_\alpha$ *are distinct both from* $\boldsymbol{x}_\beta$ *and from the free variables of* $\boldsymbol{N}_\beta$.

III. *Where* $\boldsymbol{A}_\alpha$ *is the result of substituting* $\boldsymbol{N}_\beta$ *for* $\boldsymbol{x}_\beta$ *throughout* $\boldsymbol{M}_\alpha$, *to replace any part* $\boldsymbol{A}_\alpha$ *of a formula by* $((\lambda \boldsymbol{x}_\beta \boldsymbol{M}_\alpha)\boldsymbol{N}_\beta)$, *provided that the bound variables of* $\boldsymbol{M}_\alpha$ *are distinct both from* $\boldsymbol{x}_\beta$ *and from the free variables of* $\boldsymbol{N}_\beta$.

IV. *From* $\boldsymbol{F}_{o\alpha}\boldsymbol{x}_\alpha$ *to infer* $\boldsymbol{F}_{o\alpha}\boldsymbol{A}_\alpha$, *provided that* $\boldsymbol{x}_\alpha$ *is not a free variable of* $\boldsymbol{F}_{o\alpha}$.

V. *From* $\boldsymbol{A}_o \supset \boldsymbol{B}_o$ *and* $\boldsymbol{A}_o$, *to infer* $\boldsymbol{B}_o$.

VI. *From* $\boldsymbol{F}_{o\alpha}\boldsymbol{x}_\alpha$ *to infer* $\Pi_{o(o\alpha)}\boldsymbol{F}_{o\alpha}$, *provided that* $\boldsymbol{x}_\alpha$ *is not a free variable of* $\boldsymbol{F}_{o\alpha}$.

Figure 1: Inference rules of CTT. (I–III) are λ-conversions. [2]

(1963) Henkin & Andrews' Practical Revision

- Henkin: logic [3]
- Andrews: proof system [4]
- Reformulated Church's type theory to a logic with:
  - *Definite description*
  - Equality
  - Function abstraction
  - Function application

The Duality of Simple Type Theory

- STT as a *weak set theory*: $\mathbf{A} + \mathbf{I}$ are equiconsistent with bounded Zermelo set theory and conversely [5].
- STT as a *strong predicate logic*: an ω-order logic.

(2023) Alonzo: Church's Type Theory With Undefinedness

The definitions below are taken from [6].
Let $L = (\mathcal{B}, \mathcal{C})$ be a language. A *frame* for L is a collection $\mathcal{D} = \{D_\alpha \mid \alpha \in \mathcal{T}(L)\}$ of *nonempty* domains (sets) of values such that:

F1. *Domain of truth values*: $D_o = \mathbb{B} = \{\mathrm{F}, \mathrm{T}\}$.

F2. *Predicate domain*: $D_{\alpha \to o}$ is a set of *some* total functions from D_α to D_o for $\alpha \in \mathcal{T}(L)$.

F3. *Function domain*: $D_{\alpha \to \beta}$ is a set of *some* partial and total functions from D_α to D_β for $\alpha, \beta \in \mathcal{T}(L)$ with $\beta \neq o$.

F4. *Product domain*: $D_{\alpha \times \beta} = D_\alpha \times D_\beta$ for $\alpha, \beta \in \mathcal{T}(L)$.

A frame is *full* if $D_{\alpha \to \beta}$ is full for all $\alpha, \beta \in \mathcal{T}(L)$.

Semantics of Alonzo

- An *interpretation* of L is a pair $M = (\mathcal{D}, I)$ where $I : \mathcal{C} \to \bigcup D_\alpha$.
- M is a *general model* of L if there is a partial binary valuation function V^M such that, for all $\varphi \in \text{assign}(M)$ and expressions $\mathbf{C}_\gamma$ of L,
  1. Either $V^M_\varphi(\mathbf{C}_\gamma) \in D_\gamma$ or $V^M_\varphi(\mathbf{C}_\gamma)$ is undefined
  2. Conditions V1 – V7 are satisfied

Semantics of Alonzo

V1. $V_\varphi^M((\mathbf{x} : \alpha)) = \varphi((\mathbf{x} : \alpha))$.

V2. $V_\varphi^M(\mathbf{c}_\alpha) = I(\mathbf{c}_\alpha)$.

V3. $V_\varphi^M(\mathbf{A}_\alpha = \mathbf{B}_\alpha) = \mathrm{T}$ if $V_\varphi^M(\mathbf{A}_\alpha)$ is defined, $V_\varphi^M(\mathbf{B}_\alpha)$ is defined, and $V_\varphi^M(\mathbf{A}_\alpha) = V_\varphi^M(\mathbf{B}_\alpha)$. Otherwise, $V_\varphi^M(\mathbf{A}_\alpha = \mathbf{B}_\alpha) = \mathrm{F}$.

V4. $V_\varphi^M(\mathbf{F}_{\alpha\to\beta}\,\mathbf{A}_\alpha) = V_\varphi^M(\mathbf{F}_{\alpha\to\beta})(V_\varphi^M(\mathbf{A}_\alpha))$ — i.e., the application of the function $V_\varphi^M(\mathbf{F}_{\alpha\to\beta})$ to the argument $V_\varphi^M(\mathbf{A}_\alpha)$ — if $V_\varphi^M(\mathbf{F}_{\alpha\to\beta})$ is defined, $V_\varphi^M(\mathbf{A}_\alpha)$ is defined, and $V_\varphi^M(\mathbf{F}_{\alpha\to\beta})$ is defined at $V_\varphi^M(\mathbf{A}_\alpha)$. Otherwise, $V_\varphi^M(\mathbf{F}_{\alpha\to\beta}\,\mathbf{A}_\alpha) = \mathrm{F}$ if $\beta = o$ and $V_\varphi^M(\mathbf{F}_{\alpha\to\beta}\,\mathbf{A}_\alpha)$ is undefined if $\beta \neq o$.

Semantics of Alonzo

V5. $V_{\varphi}^{M}(\lambda\, \mathbf{x} : \alpha \,.\, \mathbf{B}_{\beta})$ is the (partial or total) function $f \in D_{\alpha\rightarrow\beta}$ such that, for each $d \in D_{\alpha}$, $f(d) = V_{\varphi[(\mathbf{x}:\alpha)\mapsto d]}^{M}(\mathbf{B}_{\beta})$ if $V_{\varphi[(\mathbf{x}:\alpha)\mapsto d]}^{M}(\mathbf{B}_{\beta})$ is defined and $f(d)$ is undefined if $V_{\varphi[(\mathbf{x}:\alpha)\mapsto d]}^{M}(\mathbf{B}_{\beta})$ is undefined.

V6. $V_{\varphi}^{M}(\mathrm{I}\, \mathbf{x} : \alpha \,.\, \mathbf{A}_{o})$ is the $d \in D_{\alpha}$ such that $V_{\varphi[(\mathbf{x}:\alpha)\mapsto d]}^{M}(\mathbf{A}_{o}) = \mathrm{T}$ if there is exactly one such d. Otherwise, $V_{\varphi}^{M}(\mathrm{I}\, \mathbf{x} : \alpha \,.\, \mathbf{A}_{o})$ is undefined.

V7. $V_{\varphi}^{M}((\mathbf{A}_{\alpha}, \mathbf{B}_{\beta})) = (V_{\varphi}^{M}(\mathbf{A}_{\alpha}), V_{\varphi}^{M}(\mathbf{B}_{\beta}))$ if $V_{\varphi}^{M}(\mathbf{A}_{\alpha})$ and $V_{\varphi}^{M}(\mathbf{B}_{\beta})$ are defined. Otherwise, $V_{\varphi}^{M}((\mathbf{A}_{\alpha}, \mathbf{B}_{\beta}))$ is undefined.

Some Definitions

Let M be a general model of $L = (\mathcal{B}, \mathcal{C})$.

- The *size* of a model, $|M|$, is the cardinality of $\bigcup_{\mathbf{a} \in \mathcal{B}} D^M_{\mathbf{a}}$
- The *power* of a model, $\|M\|$, is the least cardinal κ such that $|D^M_\alpha| \leq \kappa$ for all $\alpha \in \mathcal{T}$
- An interpretation $N = (\mathcal{D}, I)$ of L is a *standard model* of L if $\mathcal{D}$ is full.

Notational definitions:

- T_o stands for $(\lambda\, x : o \,.\, x) = (\lambda\, x : o \,.\, x)$
- F_o stands for $(\lambda\, x : o \,.\, T_o) = (\lambda\, x : o \,.\, x)$
- $\neg_{o \to o}$ stands for $(\lambda\, x : o \,.\, x) = F_o$
- $(\forall\, \mathbf{x} : \alpha \,.\, \mathbf{A}_o)$ stands for $(\lambda\, x : \alpha \,.\, T_o) = (\lambda\, \mathbf{x} : \alpha \,.\, \mathbf{A}_o)$
- $(\exists\, \mathbf{x} : \alpha \,.\, \mathbf{A})$ stands for $\neg(\forall\, \mathbf{x} : \alpha \,.\, \neg \mathbf{A}_o)$

References I

[1] B. Russell, "Mathematical logic as based on the theory of types," *American journal of mathematics*, vol. 30, no. 3, pp. 222–262, 1908.

[2] A. Church, "A formulation of the simple theory of types," *The journal of symbolic logic*, vol. 5, no. 2, pp. 56–68, 1940.

[3] L. Henkin, "A theory of propositional types," *Fundamenta Mathematicae*, vol. 52, pp. 323–334, 1963.

[4] P. Andrews, "A reduction of the axioms for the theory of propositional types," *Fundamenta Mathematicae*, vol. 52, pp. 345–350, 1963.

[5] W. M. Farmer, "The seven virtues of simple type theory," *Journal of Applied Logic*, vol. 6, no. 3, pp. 267–286, 2008.

References II

[6] W. M. Farmer, *Simple type theory: a practical logic for expressing and reasoning about mathematical ideas*. Springer Nature, 2023.