

Introduction to the Model Theory of Higher-Order Logic

Dennis Y. Zvigelsky

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Presentation Overview

- 1 A brief history of simple type theory
- 2 Alonzo: syntax and semantics
- 3 Standard semantics and categoricity
- 4 Henkin's Theorem and Skolem's Paradox

(1908) Russell's Ramified Theory of Types

Two distinct hierarchies [1]:

- Types

- Stratifying and distinguishing individuals, predicates on individuals, and predicates on predicates on ...
- Restrict variable scope and term formation
- Ban set-theoretic paradoxes

- Orders

- Stratifying properties based on argument order (predicativity)
- Bans *impredicative* definitions which have a “vicious circle”
- E.g. $y = \inf(X)$ iff

(1) For all $x \in X, y \leq x$;

(2) For all $z \in X, z$ is a lower bound, implies $z \leq y$.

- Since we quantified over $y \in X$, $\inf(X)$ is impredicative

(1908) Russell's Axiom of Reducibility & STT

- (Axiom) Any propositional function can be expressed as a predicative one
- So why care about predicativity?!

STT = Ramified Type Theory + Axiom of Reducibility

(1940) Church's (Simple) Type Theory

- A simple theory of types with λ -conversion [2]
- Hierarchy of types via σ , ι , $(\alpha\beta)$
- Great influence on computing as a practical function theory

(1940) Inference Rules of Church's Type Theory

- I. To replace any part M_α of a formula by the result of substituting y_β for x_β throughout M_α , provided that x_β is not a free variable of M_α and y_β does not occur in M_α . (I.e., to infer from a given formula the formula obtained by this replacement.)
- II. To replace any part $((\lambda x_\beta M_\alpha) N_\beta)$ of a formula by the result of substituting N_β for x_β throughout M_α , provided that the bound variables of M_α are distinct both from x_β and from the free variables of N_β .
- III. Where A_α is the result of substituting N_β for x_β throughout M_α , to replace any part A_α of a formula by $((\lambda x_\beta M_\alpha) N_\beta)$, provided that the bound variables of M_α are distinct both from x_β and from the free variables of N_β .
- IV. From $F_{\alpha\alpha} x_\alpha$ to infer $F_{\alpha\alpha} A_\alpha$, provided that x_α is not a free variable of $F_{\alpha\alpha}$.
- V. From $A_\alpha \supset B_\alpha$ and A_α , to infer B_α .
- VI. From $F_{\alpha\alpha} x_\alpha$ to infer $\Pi_{\alpha(\alpha\alpha)} F_{\alpha\alpha}$, provided that x_α is not a free variable of $F_{\alpha\alpha}$.

Figure 1: Inference rules of CTT. (I–III) are λ -conversions. [2]

(1963) Henkin & Andrews' Practical Revision

- Henkin: logic [3]
- Andrews: proof system [4]
- Reformulated Church's type theory to a logic with:
 - *Definite description*
 - Equality
 - Function abstraction
 - Function application

The Duality of Simple Type Theory

- STT as a *weak set theory*: $\mathbf{A} + \mathbf{I}$ are equiconsistent with bounded Zermelo set theory and conversely [5].
- STT as a *strong predicate logic*: an ω -order logic.

(2023) Alonzo: Church's Type Theory With Undefinedness

The definitions below are taken from [6].

Let $L = (\mathcal{B}, \mathcal{C})$ be a language. A *frame* for L is a collection $\mathcal{D} = \{D_\alpha \mid \alpha \in \mathcal{T}(L)\}$ of *nonempty* domains (sets) of values such that:

- F1. *Domain of truth values:* $D_o = \mathbb{B} = \{\mathbf{F}, \mathbf{T}\}$.
 - F2. *Predicate domain:* $D_{\alpha \rightarrow o}$ is a set of *some* total functions from D_α to D_o for $\alpha \in \mathcal{T}(L)$.
 - F3. *Function domain:* $D_{\alpha \rightarrow \beta}$ is a set of *some* partial and total functions from D_α to D_β for $\alpha, \beta \in \mathcal{T}(L)$ with $\beta \neq o$.
 - F4. *Product domain:* $D_{\alpha \times \beta} = D_\alpha \times D_\beta$ for $\alpha, \beta \in \mathcal{T}(L)$.
- A frame is *full* if $D_{\alpha \rightarrow \beta}$ is full for all $\alpha, \beta \in \mathcal{T}(L)$.

Semantics of Alonzo

- An *interpretation* of L is a pair $M = (\mathcal{D}, I)$ where $I : \mathcal{C} \rightarrow \bigcup D_\alpha$.
- M is a *general model* of L if there is a partial binary valuation function V^M such that, for all $\varphi \in \text{assign}(M)$ and expressions \mathbf{C}_γ of L ,
 - 1 Either $V_\varphi^M(\mathbf{C}_\gamma) \in D_\gamma$ or $V_\varphi^M(\mathbf{C}_\gamma)$ is undefined
 - 2 Conditions V1 – V7 are satisfied

Semantics of Alonzo

- V1. $V_{\varphi}^M((\mathbf{x} : \alpha)) = \varphi((\mathbf{x} : \alpha))$.
- V2. $V_{\varphi}^M(\mathbf{c}_{\alpha}) = I(\mathbf{c}_{\alpha})$.
- V3. $V_{\varphi}^M(\mathbf{A}_{\alpha} = \mathbf{B}_{\alpha}) = \mathbf{T}$ if $V_{\varphi}^M(\mathbf{A}_{\alpha})$ is defined, $V_{\varphi}^M(\mathbf{B}_{\alpha})$ is defined, and $V_{\varphi}^M(\mathbf{A}_{\alpha}) = V_{\varphi}^M(\mathbf{B}_{\alpha})$. Otherwise, $V_{\varphi}^M(\mathbf{A}_{\alpha} = \mathbf{B}_{\alpha}) = \mathbf{F}$.
- V4. $V_{\varphi}^M(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_{\alpha}) = V_{\varphi}^M(\mathbf{F}_{\alpha \rightarrow \beta})(V_{\varphi}^M(\mathbf{A}_{\alpha}))$
— i.e., the application of the function $V_{\varphi}^M(\mathbf{F}_{\alpha \rightarrow \beta})$ to the argument $V_{\varphi}^M(\mathbf{A}_{\alpha})$ — if $V_{\varphi}^M(\mathbf{F}_{\alpha \rightarrow \beta})$ is defined, $V_{\varphi}^M(\mathbf{A}_{\alpha})$ is defined, and $V_{\varphi}^M(\mathbf{F}_{\alpha \rightarrow \beta})$ is defined at $V_{\varphi}^M(\mathbf{A}_{\alpha})$. Otherwise, $V_{\varphi}^M(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_{\alpha}) = \mathbf{F}$ if $\beta = o$ and $V_{\varphi}^M(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_{\alpha})$ is undefined if $\beta \neq o$.

Semantics of Alonzo

- V5. $V_{\varphi}^M(\lambda \mathbf{x} : \alpha . \mathbf{B}_{\beta})$ is the (partial or total) function $f \in D_{\alpha \rightarrow \beta}$ such that, for each $d \in D_{\alpha}$, $f(d) = V_{\varphi[(\mathbf{x}:\alpha) \mapsto d]}^M(\mathbf{B}_{\beta})$ if $V_{\varphi[(\mathbf{x}:\alpha) \mapsto d]}^M(\mathbf{B}_{\beta})$ is defined and $f(d)$ is undefined if $V_{\varphi[(\mathbf{x}:\alpha) \mapsto d]}^M(\mathbf{B}_{\beta})$ is undefined.
- V6. $V_{\varphi}^M(\mathbf{I} \mathbf{x} : \alpha . \mathbf{A}_o)$ is the $d \in D_{\alpha}$ such that $V_{\varphi[(\mathbf{x}:\alpha) \mapsto d]}^M(\mathbf{A}_o) = \mathbf{T}$ if there is exactly one such d . Otherwise, $V_{\varphi}^M(\mathbf{I} \mathbf{x} : \alpha . \mathbf{A}_o)$ is undefined.
- V7. $V_{\varphi}^M((\mathbf{A}_{\alpha}, \mathbf{B}_{\beta})) = (V_{\varphi}^M(\mathbf{A}_{\alpha}), V_{\varphi}^M(\mathbf{B}_{\beta}))$ if $V_{\varphi}^M(\mathbf{A}_{\alpha})$ and $V_{\varphi}^M(\mathbf{B}_{\beta})$ are defined. Otherwise, $V_{\varphi}^M((\mathbf{A}_{\alpha}, \mathbf{B}_{\beta}))$ is undefined.

Some Definitions

Let M be a general model of $L = (\mathcal{B}, \mathcal{C})$.

- The *size* of a model, $|M|$, is the cardinality of $\bigcup_{a \in \mathcal{B}} D_a^M$
- The *power* of a model, $\|M\|$, is the least cardinal κ such that $|D_\alpha^M| \leq \kappa$ for all $\alpha \in \mathcal{T}$
- An interpretation $N = (\mathcal{D}, I)$ of L is a *standard model* of L if \mathcal{D} is full.

Notational definitions:

- T_o stands for $(\lambda x : o . x) = (\lambda x : o . x)$
- F_o stands for $(\lambda x : o . T_o) = (\lambda x : o . x)$
- $\neg_{o \rightarrow o}$ stands for $(\lambda x : o . x) = F_o$
- $(\forall \mathbf{x} : \alpha . \mathbf{A}_o)$ stands for $(\lambda x : \alpha . T_o) = (\lambda \mathbf{x} : \alpha . \mathbf{A}_o)$
- $(\exists \mathbf{x} : \alpha . \mathbf{A}_o)$ stands for $\neg(\forall \mathbf{x} : \alpha . \neg \mathbf{A}_o)$

References I

- [1] B. Russell, “Mathematical logic as based on the theory of types,” *American journal of mathematics*, vol. 30, no. 3, pp. 222–262, 1908.
- [2] A. Church, “A formulation of the simple theory of types,” *The journal of symbolic logic*, vol. 5, no. 2, pp. 56–68, 1940.
- [3] L. Henkin, “A theory of propositional types,” *Fundamenta Mathematicae*, vol. 52, pp. 323–334, 1963.
- [4] P. Andrews, “A reduction of the axioms for the theory of propositional types,” *Fundamenta Mathematicae*, vol. 52, pp. 345–350, 1963.
- [5] W. M. Farmer, “The seven virtues of simple type theory,” *Journal of Applied Logic*, vol. 6, no. 3, pp. 267–286, 2008.

References II

- [6] W. M. Farmer, *Simple type theory: a practical logic for expressing and reasoning about mathematical ideas*. Springer Nature, 2023.