Chapter 7: One sample tests
(Ott & Longnecker Sections: 5.4-5.7)

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Part 1
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What do we study

Key concepts: One-tailed test, two-tailed test, sample mean test, Z test, T test, p-value

Review of hypothesis testing

1. State the null hypothesis H_0 and the alternative hypothesis H_A. The goal is to test if H_A is "likely" true.
2. Choose a significance level α. Typically 0.05, 0.01.
3. Choose the test statistic $T_n(X_1, ..., X_n)$ and establish the rejection region. (How do we choose the test statistic and rejection region?)
4. Data $X_1, ..., X_n$ are gathered, compute the realization of the test statistic. If the test statistic is in the rejection region, we reject H_0 and accept H_A(because of sufficient evidence in the sample in favor of H_A), otherwise we do not reject H_0(because of insufficient evidence to support H_A).

One-tailed test

- If
  - $H_0 : \theta = \theta_0$
  - $H_A : \theta > \theta_0$

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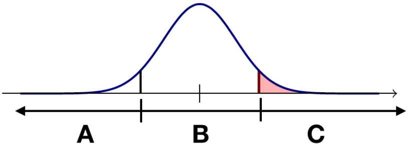
- If this is the distribution of the test statistic given the H_0 is true, then which part of A, B, C is the rejection region?



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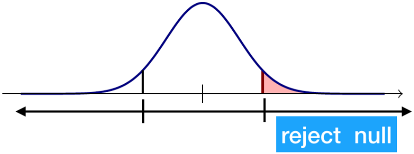
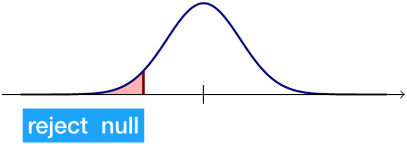


One-tailed test continued

- If
  - $H_0 : \theta = \theta_0$
  - $H_A : \theta < \theta_0$

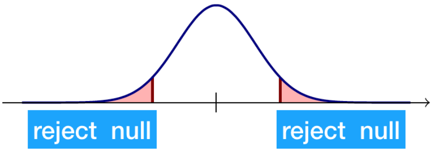
  then this is also a one-tailed test.
- rejection region is in the left tail of the test statistic.



Two-tailed test

- If
  - $H_0 : \theta = \theta_0$
  - $H_A : \theta \neq \theta_0$

  then this is a two-tailed test.
- rejection region is in both of the left and right tails of the test statistic.



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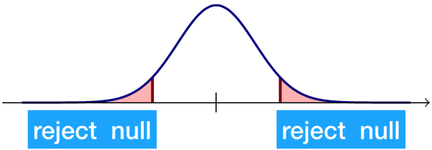

- The rejection region corresponds to the alternative hypothesis.

An example of two-tailed test

- Consider a production line of resistors that are supposed to be 100 Ohms. Assume $\sigma = 8$, so the hypotheses are
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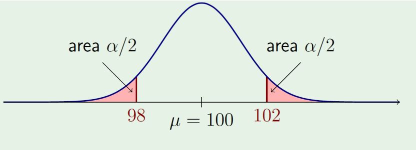
98 102

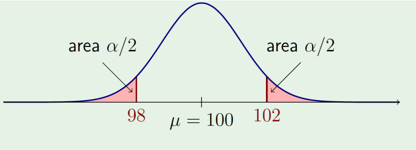
- In this case the test statistic is the sample mean. What is the sampling distribution of $\bar{X}$?

An example of two-tailed test continued



- The sampling distribution of $\bar{X}$ is a normal distribution with mean μ and standard deviation $\sigma/\sqrt{n} = 0.8$ due to the central limit theorem.

An example of two-tailed test continued



- The sampling distribution of $\bar{X}$ is a normal distribution with mean μ and standard deviation $\sigma/\sqrt{n} = 0.8$ due to the central limit theorem.
- Then the probability of the Type I error is

$$
\begin{aligned}
\alpha &= \Pr(\bar{X} < 98 \text{ when } \mu = 100) + \Pr(\bar{X} > 102 \text{ when } \mu = 100) \\
&= \Pr\left(Z < \frac{98-100}{8/\sqrt{100}}) + \Pr(Z > \frac{102-100}{8/\sqrt{100}}\right) \\
&= \Pr(Z < -2.5) + \Pr(Z > 2.5) \\
&= 2 \times \Pr(Z < -2.5) = 2 \times 0.0062 = 0.0124.
\end{aligned}
$$

Two-tailed sample mean test

- In general we are often interested in testing
  - $H_0 : \mu = \mu_0$
  - $H_A : \mu \neq \mu_0$

  based on the sample mean $\bar{X}$ from samples $X_1, ..., X_n$ with known population variance σ^2.

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- Under $H_0 : \mu = \mu_0$, the probability of Type I error is computed using the sampling distribution of $\bar{X}$, which is normal distributed with mean μ_0 and standard deviation $\sigma/\sqrt{n}$.

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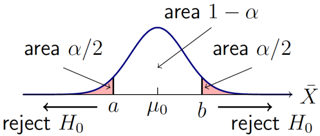
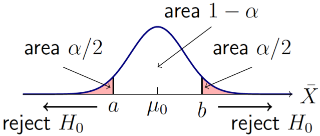
- Under $H_0 : \mu = \mu_0$, the probability of Type I error is computed using the sampling distribution of $\bar{X}$, which is normal distributed with mean μ_0 and standard deviation $\sigma/\sqrt{n}$.
- How do we decide the rejection region at the level of significance α?

"Use the distribution of the test statistic to determine a rejection region that limits the type I error at significance level α"

Two-tailed sample mean test continued

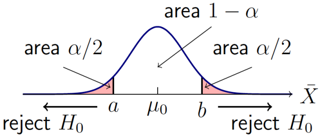


Two-tailed sample mean test continued



$$P(-z_{\alpha/2} < \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} < z_{\alpha/2}) = 1-\alpha$$

Two-tailed sample mean test continued



$$P(-z_{\alpha/2} < \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} < z_{\alpha/2}) = 1-\alpha$$

- Therefore, to design a test at the level of significance α we choose the critical values a and b as

$$a = \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$b = \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

- we collect the sample, compute the sample mean $\bar{X}$ and reject H_0 if $\bar{X} < a$ or $\bar{X} > b$.

An equivalent two-tailed Z test

$$H_0 : \mu = \mu_0, \text{ vs. } H_A : \mu \neq \mu_0$$

- Equivalently, we can choose the test statistic $Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$
- To design a test at the level of significance α, choose the rejection region $Z > z_{\alpha/2}$ and $Z < -z_{\alpha/2}$.

An example

A batch of 100 resistors have an average of 102 Ohms. Assuming a population standard deviation of 8 Ohms, test whether the population mean is 100 Ohms at a significance level $\alpha = 0.05$.

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- Step 1: $H_0 : \mu = 100$, $H_A : \mu \neq 100$.
- Step 2: Choose $\alpha = 0.05$

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- Step 1: $H_0 : \mu = 100$, $H_A : \mu \neq 100$.
- Step 2: Choose $\alpha = 0.05$
- Step 3: In this case, reject H_0 if $\bar{X} < a$ or $\bar{X} > b$ with

$$a = \mu_0 - z_{0.025} \frac{\sigma}{\sqrt{100}} = 100 - 1.96 \frac{8}{10} = 98.432$$

$$b = \mu_0 + z_{0.025} \frac{\sigma}{\sqrt{100}} = 100 + 1.96 \frac{8}{10} = 101.568$$

An example

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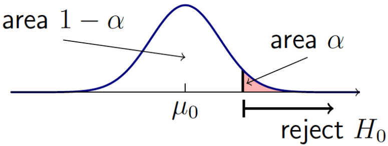
$$b = \mu_0 + z_{0.025} \frac{\sigma}{\sqrt{100}} = 100 + 1.96 \frac{8}{10} = 101.568$$

- Step 4: $\bar{X} = 102 > b$, therefore reject H_0.

One-tailed sample mean test, right tail

We are interested in testing $H_0 : \mu = \mu_0$ vs. $H_A : \mu > \mu_0$ at the significance level α assuming the variance is known.



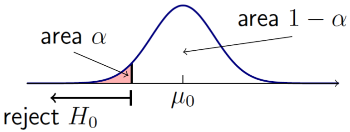
- Under $H_0 : \mu = \mu_0$, the probability of a Type I error is

$$P(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > z_\alpha) = \alpha$$

- Thus the rejection region is $\bar{X} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$. Note we use z_α instead of $z_{\alpha/2}$.

One-tailed sample mean test, left tail

We are interested in testing $H_0 : \mu = \mu_0$ vs. $H_A : \mu < \mu_0$ at the significance level α assuming the variance is known.



- Under $H_0 : \mu = \mu_0$, the probability of a Type I error is

$$P(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} < -z_\alpha) = \alpha$$

- Thus the rejection region is $\bar{X} < \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}}$. Note we use z_α instead of $z_{\alpha/2}$.

An example

A quality control engineer finds that a sample of 100 light bulbs had an average life-time of 470 hours. Assuming a population standard deviation of $\sigma = 25$ hours, test whether the population mean is 480 hours vs. the alternative hypothesis $\mu < 480$ at a significance level of $\alpha = 0.05$.

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- Step 1: $H_0 : \mu = 480$, $H_A : \mu < 480$.
- Step 2: Choose $\alpha = 0.05$
- Step 3: In this case, reject H_0 if

$$\bar{X} < \mu_0 - z_{0.05} \frac{\sigma}{\sqrt{n}} = 480 - 1.645 \frac{25}{10} = 475.9$$

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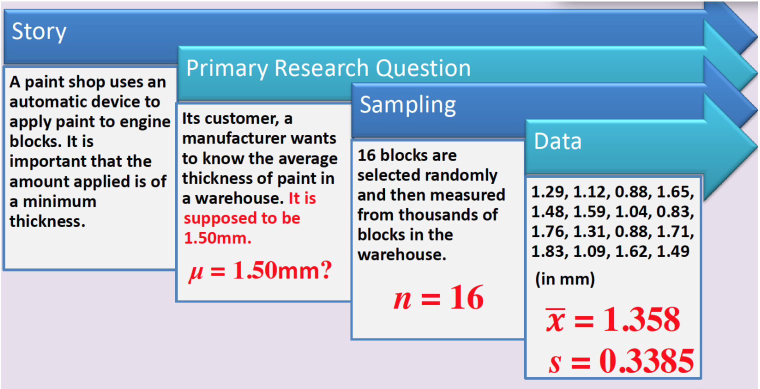
- Step 4: $\bar{X} = 470 < 475.9$, therefore reject H_0.

Let's make life harder...

What if the population standard deviation is unknown?

An example



Story
A paint shop uses an automatic device to apply paint to engine blocks. It is important that the amount applied is of a minimum thickness.
Primary Research Question
Its customer, a manufacturer wants to know the average thickness of paint in a warehouse. It is supposed to be 1.50mm.
$\mu = 1.50\text{mm}?$
Sampling
16 blocks are selected randomly and then measured from thousands of blocks in the warehouse.
$n = 16$
Data
1.29, 1.12, 0.88, 1.65, 1.48, 1.59, 1.04, 0.83, 1.76, 1.31, 0.88, 1.71, 1.83, 1.09, 1.62, 1.49
(in mm)
$\bar{x} = 1.358$
$s = 0.3385$

Statistical assumptions

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- Normality?

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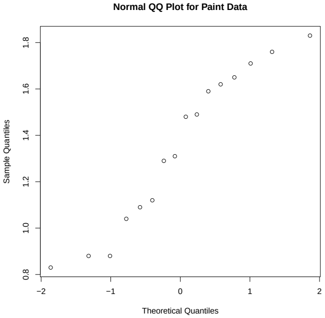

Statistical assumptions

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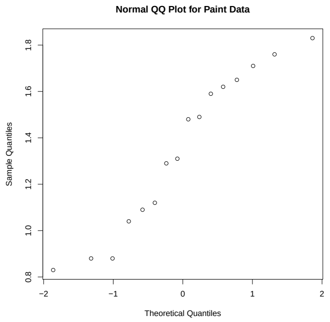

- Is population standard deviation σ known?

Statistical assumptions

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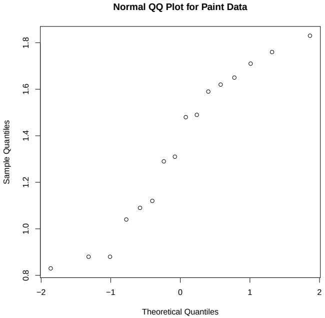

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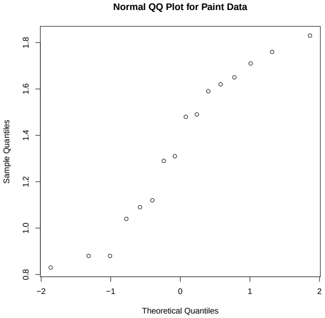

- Is population standard deviation σ known?
  NO
- Is the sample size n large?

Statistical assumptions

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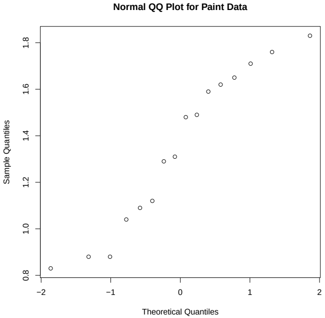

- Is population standard deviation σ known?
  NO
- Is the sample size n large?
  NO

Two-tailed T test

We are interested in testing $H_0 : \mu = \mu_0$ vs. $H_A : \mu \neq \mu_0$ at the significance level α based a sample $X_1, ..., X_n$ from a normal distribution, but with unknown variance σ^2.

- Under $H_0 : \mu = \mu_0$, the sampling distribution of $\frac{\bar{X}-\mu_0}{s/\sqrt{n}}$ is a t-distribution with degrees of freedom $n-1$.
- Choose test statistic $T = \frac{\bar{X}-\mu_0}{s/\sqrt{n}}$.
- Reject H_0 if $T < -t_{\alpha/2,n-1}$ or $T > t_{\alpha/2,n-1}$.
- For a one-sided test, $t_{\alpha/2,n-1}$ is replaced by $t_{\alpha,n-1}$ as usual.

An example revisited

- Step 1:

$$H_0\text{: } \mu = 1.5,\ H_A\text{: } \mu \neq 1.5$$

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$$H_0\colon\ \mu = 1.5,\ H_A\colon\ \mu \neq 1.5$$

- Step 2: Choose $\alpha = 0.05$

An example revisited

- Step 1:

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- Step 2: Choose $\alpha = 0.05$
- Step 3: Use the T test statistic: $T = \frac{\bar{X} - \mu_0}{\frac{S}{\sqrt{n}}}$. In this example, $t_{obs} = \frac{1.348 - 1.50}{\frac{0.3385}{\sqrt{16}}} = -1.796$. The rejection region is $T < -t_{n-1,\alpha/2}$, $T > t_{n-1,\alpha/2}$. In this example, $t_{15,0.025} = 2.13$, so the rejection region is $T < -2.13$ or $T > 2.13$.

An example revisited

- Step 1:
$$H_0\colon \mu = 1.5,\ H_A\colon \mu \neq 1.5$$
- Step 2: Choose $\alpha = 0.05$
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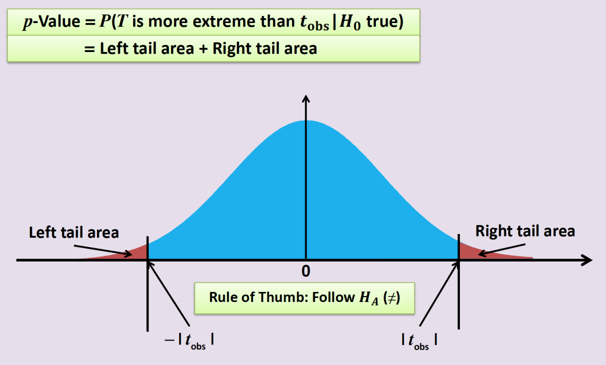
- Step 4: Make a conclusion: since $t_{obs} = -1.796$ does not fall in the rejection region, so we do not reject the null.

Hypothesis testing using the p-value

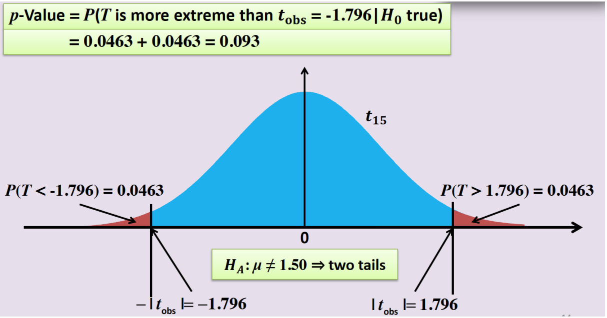
- In the approach we have taken so far, the significance level is pre-selected up front, either by choosing a given value of α or setting the rejection region explicitly.
- Suppose a hypothesis test is performed at a significance level of 0.05, but someone else wants to test with a stricter significance level of 0.01, this requires recomputing the rejection region.
- Using p-value is better than using rejection region.

Hypothesis testing using the p-value

- " If the p-value is smaller than the given significance level α, we would reject the null, otherwise we would not reject the null."
- For a two-sided test,

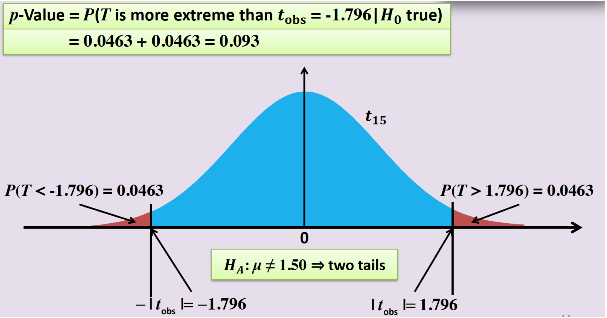


An example revisited



- If we choose $\alpha = 0.05$, then since $0.094 > 0.05$, we would not reject the null. However, if we had chosen $\alpha = 0.1$, we would have rejected.

An example revisited



- If we choose $\alpha = 0.05$, then since $0.094 > 0.05$, we would not reject the null. However, if we had chosen $\alpha = 0.1$, we would have rejected.
- p-value is a measure of evidence in the sense that it is the minimum probability of a Type I error with which H_0 can still be rejected.

What's the next?

We'll discuss how to calculate the sample size and power in hypothesis testing in the next lecture.