Normal Distribution

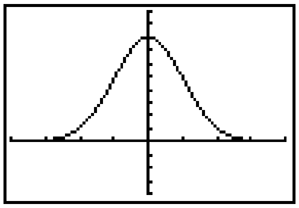
Normal distribution is a continuous, symmetric, bell-shaped distribution of a variable.

μ = mean σ = standard deviation

Summary of Properties

1) The normal distribution curve is bell-shaped
2) The mean, median, and mode are equal and located at the center of the distribution
3) The curve is symmetrical about the mean
4) The curve never touches the x-axis
5) The total area under the curve is equal to 1.00
6) The area under the curve that lies within one standard deviation of the mean is 0.68 or 68%; within two standard deviations is 0.95 or 95%; within three standard deviations is 0.997 or 99.7%

Standard Normal Distribution is a normal distribution with a mean of 0 (μ = 0) and a standard deviation of 1 (σ = 1).



- Any normally distributed variable can be transformed into a standard normal distribution by using the formula for the standard score or z-value

  $$z = \frac{x-\mu}{\sigma}$$

  where the z-value is the number of standard deviations that a particular x value is away from the mean.

Example: Find the z-value for a normally distributed variable with a mean of 4, standard deviation of 2, and an x value of 8.

$$z = \frac{8-4}{2} = \frac{4}{2} = 2$$

Note: We have found that z = 2, which tells us that the x-value of 8 is 2 standard deviations away from the mean of 4 when the standard deviation is 2.

Application: Using the z-value, we can use the standard normal distribution table to find the area under the curve. The area is used to solve problems such as finding percentages or probabilities.

Finding the Area Under the Standard Normal Distribution Curve

1) Between 0 and any z-value:

a) Look up the z-value in the table to get the area

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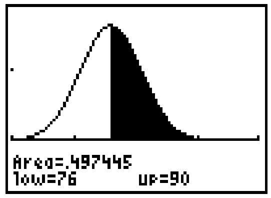
Example: The average score on Dr. Vladamere's math test was a 76 with a standard deviation of 5. To get an A, a student must have a score of 90 or higher; a B is 80-90; a C is 70-80; and a D is 60-70. What is the probability a randomly selected student scored between a 76 and 90? Assume the variable is normally distributed.

First draw the normal curve. The average score of 76 tells us that we have a mean of 76.

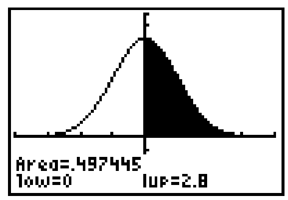
$\mu = 76$, $\sigma = 5$, $x = 90$



Next find the z-value:

$$z = \frac{x-\mu}{\sigma} = \frac{90-76}{5} = 2.8$$

Then draw the standard normal curve with the found z-value:



Lastly, look up the z-value on the table. (A sample table is on the last page of this handout.):
With a z-value of 2.8, we find the area to be .4974.

Therefore, the probability a randomly selected student scored between 76 and 90 is <u>.4974</u> or <u>49.74%</u>.

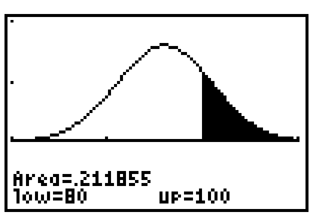
2) In any tail:

a) Look up the z-value to get the area
b) Subtract the area from 0.500

Example: Using the same information from the previous example, what is the probability a randomly selected student will have received at least a B?

First draw the normal curve:

$\mu = 76$, $\sigma = 5$, $x = 80$



Note: To receive at least a B, a student must score an 80 or better.

Next find the z-value:

$$z = \frac{x-\mu}{\sigma} = \frac{80-76}{5} = 0.8$$

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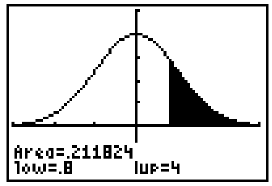
Then draw the standard normal curve:



Next, look up the z-value on the table. (A sample table is on the last page of this handout.):

A z-value of 0.8 gives an area of 0.2881

Lastly, subtract that area from 0.500.

0.500 – 0.2881 = 0.2119

Therefore, the probability a randomly selected student received at least a B is 0.2119 or 21.19%.

3) Between two z-values on opposite sides of the mean:

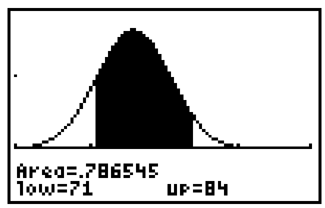
a) Look up both z-values

b) Add the areas

Example: Again using the same information, what is the probability a randomly selected student will score between a 71 and 84?

First draw the normal curve:

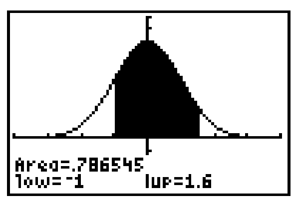
$\mu = 76, \sigma = 5, \ x_1 = 71, \ x_2 = 84$



Next find the z-values:

$z_1 = \frac{71-76}{5} = -1$ $z_2 = \frac{84-76}{5} = 1.6$

Then draw the standard normal curve:



Next look up the z-values on the table (A sample table is on the last page of this handout.):

A z-value of -1 gives us and area of 0.3413 and a z-value of 1.6 gives an area of 0.4452

Note: Even though z_1 is negative, we look up 1 on the table.

Lastly, add the areas together:

0.4452 + 0.3413 = 0.7865

Therefore, the probability a randomly selected student scored between 71 and 84 is 0.7865 or 78.65%.

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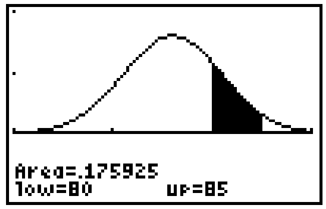
4) Between two z-values on the same side of the mean:
   a) Look up both z values
   b) Subtract the smaller area from the larger

Example: Still using the same information, what is the probability a randomly selected student will score between an 80 and 85?

First draw the normal curve:

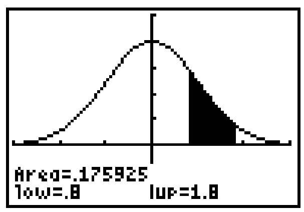


$\mu = 76, \sigma = 5, x_1 = 80, x_2 = 85$

Next find the z-values:

$$z_1 = \frac{80-76}{5} = 0.8 \qquad z_2 = \frac{85-76}{5} = 1.8$$

Then draw the standard normal curve:



Next look up the z-values on the table (A sample table is on the last page of this handout.):
   A z-value of 0.8 gives an area of 0.2881 and a value of 1.8 gives 0.4641

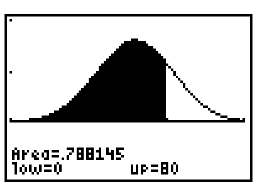
Lastly, subtract the smaller area from the larger:
   0.4641 – 0.2881 = .1760

Therefore, the probability a randomly selected student will score between an 80 and 85 is .1760 or 17.60%.

5) The left of any z-value, where z is greater than the mean:
   a) Look up the z-value to get the area
   b) Add 0.500 to the area.

Example: Still using the same information, what is the probability a randomly selected student will not receive an A or B?

First draw the normal curve:



$\mu = 76, \sigma = 5, x = 80$

Note: To get an A or B, a student must score an 80 or above. We want the probability of not getting an A or B so look at everything below an 80.

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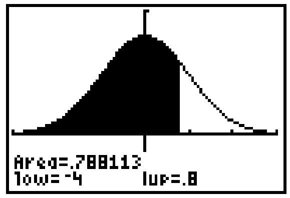
Next find the z-value:

$$z = \frac{80 - 76}{5} = 0.8$$

Then draw the standard normal curve:



Next look up the z-value on the table (A sample table is on the last page of this handout.):

A z-value of 0.8 has an area of 0.2881.

Lastly, add the area to 0.500:

0.500 + 0.2881 = 0.7881

Therefore, the probability a randomly selected student did not receive an A or B is <u>0.7881</u> or <u>78.81%</u>.

6) To the right of any z-value, where z is less than the mean:

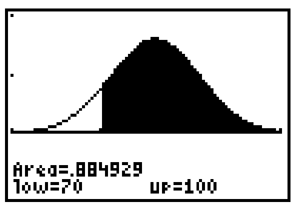
a) Look up the z-value to get the area

b) Add 0.500 to the area

Example: Again using the same information, what is the probability a randomly selected student will score at least a 70?

First draw the normal curve:

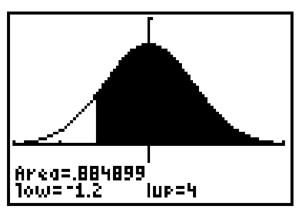
$\mu = 76$, $\sigma = 5$, $x = 70$



Next, find the z-value:

$$z = \frac{70 - 76}{5} = -1.2$$

Then, draw the standard normal curve:



Next, look up the z-value (A sample table is on the last page of this handout.):

A z-value of -1.2 has an area of 0.3849

Lastly, add the area to 0.500:

0.500 + 0.3849 = .8849

Therefore, the probability a randomly selected student will score at least a 70 is <u>0.8849</u> or <u>88.49%</u>.

Reverse Normal Distribution Solutions

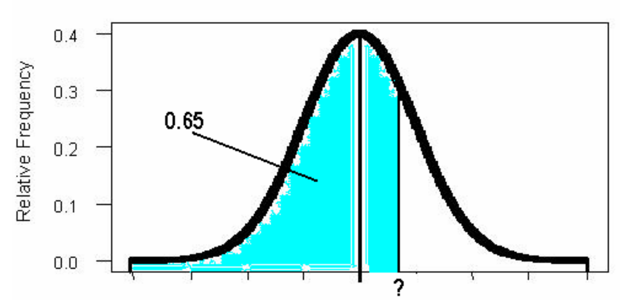
For problems 1 through 3 let z be distributed as a standard normal variable:

1) What is the z-score that marks the 65th percentile of the standard normal distribution?

The left tail area is 65% or 0.65. Look this value up in the z-table to get your z-score.

Z = 0.38 or 0.39

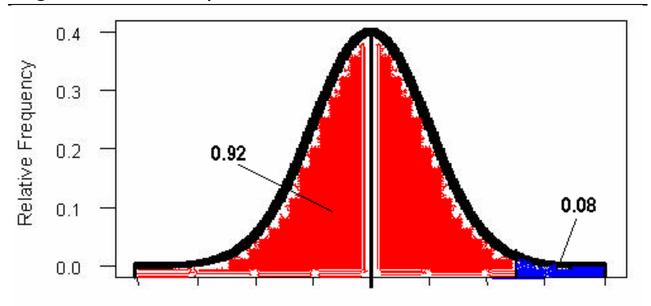
Figure 1: Accompanies Problem 1



2) What is the z-score that marks the top 8% of the standard normal distribution?

In order to answer this question you need to find the left tail area. Since we know that the right tail area is 8% and that there is 100% under the normal density curve, the left tail area is 100-8= 92%. Look up 0.92 in the body of Table A and find z to be 1.40 or 1.41

Figure 2: Accompanies Problem 2



3) Find a value 'a' such that Pr{z $\geq$ a} = 0.8238?

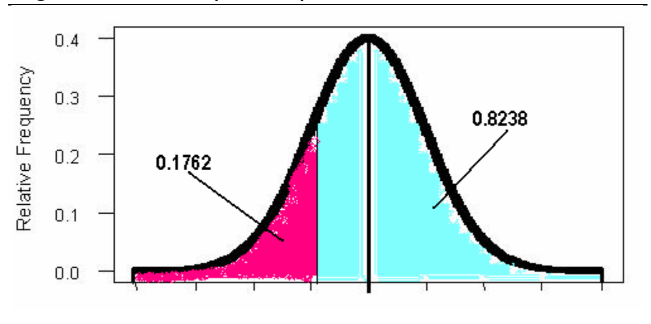
Start in the right hand tail and move into the distribution until you've accounted for 82.38% of the area. You'll need to go to the left of the mean (why?).

Now find the left tail area by subtracting 0.8238 from 1.

Left tail = 1 - 0.8238 = 0.1762

Now look 0.1762 up in the z-table to get the z-score.
Z = -0.93

Figure 3: Accompanies problem 3

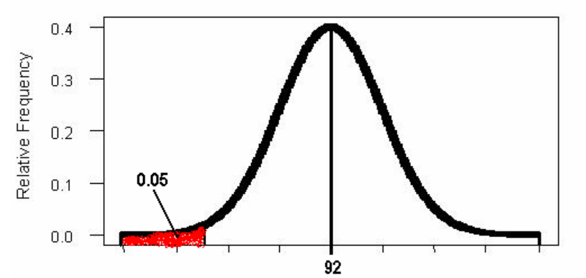


Reverse Normal Distribution Solutions

Use the information in the following setting to answer questions 4 and 5:

An important characteristic of woven fabric is the tensile strength of the threads used to produce the fabric. Let's say that a polypropylene manufacturing process produces rolls of fabric with an average tensile strength of 92 pounds per square inch (psi) with a standard deviation of 4 psi. Assume also that the distribution of this variable is normal.

Figure 4: Accompanies problem 4



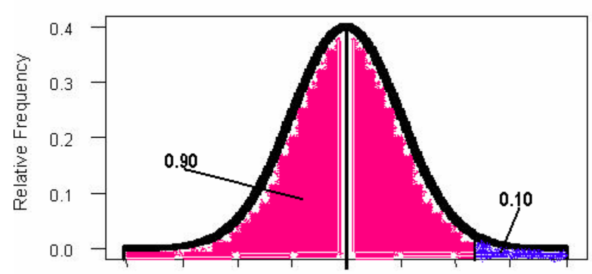
4) Rolls of fabric in the weakest 5% of the population are discarded. What is the minimum strength a roll of fabric can have so that it is not discarded?

We will eventually be using the relation: $x = \mu + z\sigma$ so we need to know z. Everything else on the right hand side is given.

Go to Table A and find that z = -1.65

So now it's plug-N-chug:
$x = 92 - 1.65(4) = 85.4$ psi

Figure 5: Accompanies problem 5



5) Rolls whose strength tests in the top 10% are sold at a premium. What is the minimum strength a roll can have in order to command a premium price?

An upper tail area of 10% corresponds to a lower tail area of 90%.

Go to the z-table and look up 0.90 to find the corresponding z-score.

Z=1.28

Now use the formula from problem 4 to find the strength that is required in order for a fabric to be in the top 10%

x = 92 + 1.28(4) = 97.12 psi

Additional Normal Distribution Practice Problems

First, use either a Standard Normal Distribution (Z Table) or your calculator to solve the following problems. Then verify your answers using the Excel Normal Distribution Functions.

1. For a Standard Normal Distribution (i.e., Mean $\mu = 0$ and Variance $\sigma^2 = 1$ Standard Deviation $\sigma = 1$)

Find:	Find:	Find z, such that
$P(Z < 1.43)$	$P(-1.43 < Z < +1.37)$	$P(Z < z) = 0.98$
$P(Z > 1.43)$	$P(-1.37 < Z < +1.43)$	$P(Z > z) = 0.98$
$P(Z > -1.43)$	$P(-1.43 < Z < -1.37)$	$P(Z < z) = 0.02$
$P(Z < -1.43)$	$P(+1.37 < Z < +1.43)$	$P(Z > z) = 0.02$

2. Normal Distribution Mean = 120, Standard Deviation = 32
Find:
$P(X < 80)$
$P(X > 160)$
$P(80 < X < 160)$
x such that $P(X < x) = 50\%$
x such that $P(X > x) = 50\%$
x such that $P(X < x) = 25\%$
x such that $P(X > x) = 25\%$
x such that $P(X > x) = 98\%$
x such that $P(X < x) = 98\%$

3. Assume the detection of a digital signal imbedded in background noise is normally distributed with mean = 2.70 volts and standard deviation = 0.45 volts. If the signal level exceeds 3.60 volts, the system reports that a digital 1 has been transmitted. What is the probability of reporting a digital 1 if no digital signal was sent. Note: This is the probability of a false detection (false positive).

In simple terms, let v = the voltage level, find $P(v > 3.60)$
Note: $Z = (v - \mu) / \sigma$; so $Z = (3.60 - 2.70) / 0.45 = 0.90 / 0.45 = 2.00$
$P(v > 3.60) = P(Z > 2.00) = 1 - P(Z < 2.00) = 1 - 0.9772 = 0.0228$
So the probability of saying a digital signal 1 was sent when no digital signal was sent equals $0.0228 \approx 3\%$

4. Assume the life of a semiconductor laser at constant power is normally distributed with mean of 7000 hours and a standard deviation of 600 hours.

What is the probability that a laser fails before 6000 hours?
$P(X < 6000) = P(Z < [X - \mu] / \sigma) = P(Z < [6000 - 7000] / 600) = P(Z < -1.67) = 0.0475 \approx 5\%$

What is the laser operating life (in hours) for which 95% of all laser are expected to exceed?
$P(X > x) = 0.95$ which is the same as $P(X < x) = 1 - 0.95 = 0.05$
Find z such that $P(Z < z) = 0.05$ Answer $z = -1.645$
and from $Z = (X - \mu) / \sigma$ we have $X = \mu + Z\sigma = 7000 + (-1.645)600 = 7000 - 987 = 6013$ hours

What are the symmetrical lower and upper bounds on the 99% of laser operating life (in hours)?
Note: Since we are asked to find the ***symmetric*** lower and upper bounds $P(-z < Z < +z) = 0.99$;
z can be found by $P(Z < -z) = (1 - 0.99) / 2 = 0.005$; hence $z = -2.575$
And $X_{Lower} = \mu - |Z|\sigma = 7000 - (2.575)(600) = 7000 - 1545 = 5455$
And $X_{Upper} = \mu + |Z|\sigma = 7000 + (2.575)(600 = 7000 + 1545 = 8545$
QED the 99% symmetrical bounds on the laser operating life is 5455 to 8545 hours.
Would you expect the 95% bounds to be wider or narrower than the 99% bounds?
Is this counter-intuitive?

Additional Normal Distribution Practice Problems (Answers)

1. For a Standard Normal Distribution (i.e., Mean $\mu = 0$ and Variance $\sigma^2 = 1$ Standard Deviation $\sigma = 1$)

$P(Z < 1.43) = x$	$P(-1.43 < Z < +1.37) = x$	$P(Z < z) = 0.98$ $z = +2.05$
$P(Z > 1.43) = x$	$P(-1.37 < Z < +1.43) = x$	$P(Z > z) = 0.98$ $z = -2.05$
$P(Z > -1.43 = x)$	$P(-1.43 < Z < -1.37) = x$	$P(Z < -z) = 0.98$ $z = -2.05$
$P(Z < -1.43) = x$	$P(+1.37 < Z < +1.43 = x)$	$P(Z > -z) = 0.98$ $z = +2.05$

2. Normal Distribution Mean = 120, Standard Deviation = 32
Find:
$P(X < 80) = 0.1056$
$P(X > 160) = 0.1056$
$P(80 < X < 120) = 0.8944 - 0.1056 = 0.7888$
x such that $P(X < x) = 50\%$ $x = 120.0$
x such that $P(X > x) = 50\%$ $x = 120.0$
x such that $P(X < x) = 25\%$ $x = 98.4$
x such that $P(X > x) = 25\%$ $x = 141.6$
x such that $P(X > x) = 98\%$ $x = 54.3$
x such that $P(X < x) = 98\%$ $x = 185.7$