Chapter 5: Estimation

Ott & Longnecker Sections: 5.8

Duzhe Wang

the Department of Statistics, UW-Madison

Part 6-Bootstrap Methods
https://dzwang91.github.io/stat371/

Summary and motivation

General form of CI: estimate $\pm$ multiplier $*$ estimated SE of the estimator

Population Distribution	$X \sim N(\mu, \sigma^2)$		$X \nsim N(\mu, \sigma^2)$	
subcase	σ is known	σ is unknown	n is large (like n>30)	n is small
CI	$\bar{X} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$	$\bar{X} \pm t_{n-1,\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$	$\bar{X} \pm z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$	Bootstrap ???

Example

Story

Secondhand smoke is of great health concern, especially for children. The level of exposure could be determined by measuring the urinary concentration of cotanine.

Primary Research Question

The Child Protective Services (CPS) in a city need to know the mean cotanine level for children in foster care.

$$\mu = ?$$

Sampling

15 children were selected randomly, and their urinary concentration of cotanine was measured.

$$n = 15 < 30$$

Observed Data

29, 30, 53, 75, 34, 21, 12, 58, 117, 119, 115, 134, 253, 289, 287

$$\bar{x} = 108.4$$

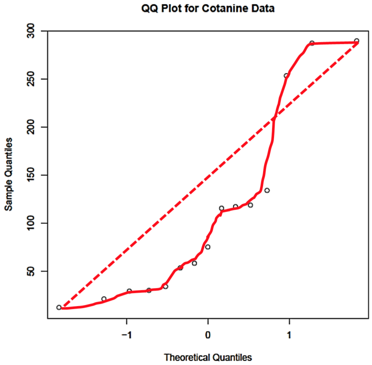
$$s = 95.6$$

Example

Question 1: do these data come from a normal population?

Example

Question 1: do these data come from a normal population? QQ plot



It looks pretty bad, and with the small sample size we may not be able to rely on the CLT as an accurate approximation to the distribution of the sample mean.

Example

Question 2: what's the challenge for us to make a CI of μ in this example?

Example

Question 2: what's the challenge for us to make a CI of μ in this example?

Without assuming normality of the population, the quantity

$$t = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}}$$

will not have a t-distribution.

Example

Question 3: how to get t's distribution?

Example

Question 3: how to get t's distribution?

By **simulations**, we call it **bootstrapping**.

The bootstrap

Given the original data set: $x_1, x_2, ..., x_n$.

1. Compute the sample mean $\bar{x}$ and sample standard deviation s of the **original** data.
2. Draw n data points from the original data set **with replacement**. Call these observations x_1^*, x_2^*, ..., x_n^*. (This is like treating your original sample as a population, and sampling iid from this new population!)
3. Compute the mean and sd of the **resampled** data. Call these things $\bar{x}^*$ and s^*.
4. Compute the statistic $\hat{t} = \frac{\bar{x}^* - \bar{x}}{\frac{s^*}{\sqrt{n}}}$. (If we treat our original sample like a population, $\bar{x}$ plays the role of the population mean!)
5. Repeat steps 2-4 a large number of times(say 1000 times), and compute $\hat{t}$ from each one. Put these values of $\hat{t}$ in order and throw them all into a density histogram. This is an approximation to the true sampling distribution of t!

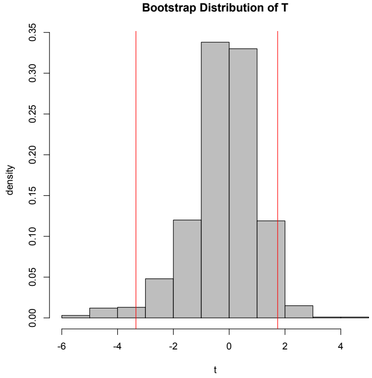
The bootstrap

After step 1-5, find the $\alpha/2$ and $1-\alpha/2$ critical values of the approximate sampling distribution you've generated with all these $\hat{t}$s. (For example if we use 1000 bootstrap samples and $\alpha = 0.05$, we could just take the 975th and 25th largest values of $\hat{t}$.) Call these critical values $\hat{t}_{(\alpha/2)}$ and $\hat{t}_{(1-\alpha/2)}$. An approximate $100(1-\alpha)\%$ CI for μ is now:

$$(\bar{x} - \hat{t}_{(\alpha/2)} \frac{s}{\sqrt{n}}, \bar{x} - \hat{t}_{(1-\alpha/2)} \frac{s}{\sqrt{n}}).$$

Example continued

For the secondhand smoke data, we find $\bar{x} = 108.4$ and $s = 95.6$. Bootstrapping 1000 times yields the following approximate distribution of t:



Example continued

You can see that this distribution is not very symmetric, and thus quite unlike a t or normal. Now let's compute a 95% confidence interval. Using R, we have $\hat{t}_{(1-0.05/2)} = -3.34$ and $\hat{t}_{0.025)} = 1.74$. Thus the 95% CI is:

$$(108.4 - 1.74\frac{95.6}{\sqrt{15}}, 108.4 - (-3.34)\frac{95.6}{\sqrt{15}}) = (65.53, 190.88).$$

See relevant R code from course website.

What's the next?

We'll start hypothesis testing in the next lecture.