Chapter 5: Estimation

Ott & Longnecker Sections: 4.12, 4.14 and 5.2

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Part 4

https://dzwang91.github.io/stat371/

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What do we study?

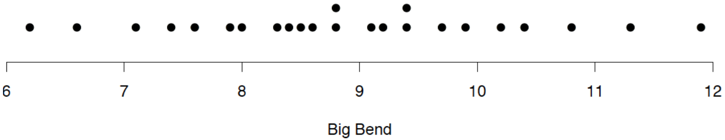
Key Concepts: t-distribution, confidence intervals

[1] Some of the slides in this lecture have been adapted/borrowed from materials developed by Cecile Ane.

Outline

1. Review of point estimation
2. The t-distribution
3. Confidence interval

Big bend lizards tail length



Big bend lizards tail length

We want to know μ, the mean tail length in the entire Big bend lizards.

```
> bigbend
 [1]  8.8  9.7 10.8  7.1  6.6  9.9 10.2  8.6 10.4 11.9  7.6  8.0  8.5
[16]  7.4  8.3  9.1  9.2  7.9  8.4 11.3  6.2  8.8
> mean(bigbend)
[1] 8.895833
> sd(bigbend)
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> length(bigbend)
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$\bar{X} = 8.896$ cm is our best estimate for μ.
How good is this estimate? How far is μ from 8.896 cm?

Estimated standard error of the mean

We know the standard error of $\bar{X}$ is $\frac{\sigma}{\sqrt{n}}$, but we don't know σ. Hopefully, the sample standard deviation s is close to σ. Therefore,

estimated SE$=\frac{s}{\sqrt{n}}$ is the estimated standard error of the mean. It gives us an idea of how far $\bar{X}$ is from μ typically.

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Here: s=1.43 and n=24, so estimated SE$=\frac{1.43}{\sqrt{24}} = 0.292$.

Outline

The t-distribution

If $X_1, ..., X_n$ have a normal distribution, $\bar{X}$ has one too, and

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

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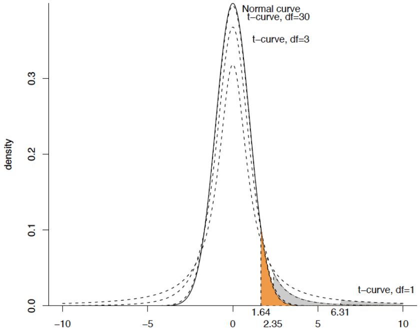
When we replace $\sigma/\sqrt{n}$ by estimated SE$=s/\sqrt{n}$,

$$\frac{\bar{X} - \mu}{s/\sqrt{n}} \sim t_v$$

where $v = n - 1$ is called **degrees of freedom** and t_v is called t-distribution with degrees of freedom v.

The t-distribution

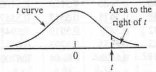


It looks very similar to a standard normal: it's symmetric and bell-shaped, but it is a little more spread out. The amount of additional spread decreases as the degrees of freedom (the sample size) increases.

T table

Question: find a value t s.t. $P(t_{10} \geq t) = 0.17$.

Table A.8 *t* Curve Tail Areas



t \ ν	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0.0	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500
0.1	.468	.465	.463	.463	.462	.462	.462	.461	.461	.461	.461	.461	.461	.461	.461	.461	.461	.461
0.2	.437	.430	.427	.426	.425	.424	.424	.423	.423	.423	.423	.422	.422	.422	.422	.422	.422	.422
0.3	.407	.396	.392	.390	.388	.387	.386	.386	.386	.385	.385	.385	.384	.384	.384	.384	.384	.384
0.4	.379	.364	.358	.355	.353	.352	.351	.350	.349	.349	.348	.348	.348	.347	.347	.347	.347	.347
0.5	.352	.333	.326	.322	.319	.317	.316	.315	.315	.314	.313	.313	.313	.312	.312	.312	.312	.312
0.6	.328	.305	.295	.290	.287	.285	.284	.283	.282	.281	.280	.280	.279	.279	.279	.278	.278	.278
0.7	.306	.278	.267	.261	.258	.255	.253	.252	.251	.250	.249	.249	.248	.247	.247	.247	.247	.246
0.8	.285	.254	.241	.234	.230	.227	.225	.223	.222	.221	.220	.220	.219	.218	.218	.218	.217	.217
0.9	.267	.232	.217	.210	.205	.201	.199	.197	.196	.195	.194	.193	.192	.191	.191	.191	.190	.190
1.0	.250	.211	.196	.187	.182	.178	.175	.173	.172	.170	.169	.169	.168	.167	.167	.166	.166	.165
1.1	.235	.193	.176	.167	.162	.157	.154	.152	.150	.149	.147	.146	.146	.144	.144	.144	.143	.143
1.2	.221	.177	.158	.148	.142	.138	.135	.132	.130	.129	.128	.127	.126	.124	.124	.124	.123	.123
1.3	.209	.162	.142	.132	.125	.121	.117	.115	.113	.111	.110	.109	.108	.107	.107	.106	.105	.105
1.4	.197	.148	.128	.117	.110	.106	.102	.100	.098	.096	.095	.093	.092	.091	.091	.090	.090	.089
1.5	.187	.136	.115	.104	.097	.092	.089	.086	.084	.082	.081	.080	.079	.077	.077	.077	.076	.075
1.6	.178	.125	.104	.092	.085	.080	.077	.074	.072	.070	.069	.068	.067	.065	.065	.065	.064	.064
1.7	.169	.116	.094	.082	.075	.070	.065	.064	.062	.060	.059	.057	.056	.055	.055	.054	.054	.053
1.8	.161	.107	.085	.073	.066	.061	.057	.055	.053	.051	.050	.049	.048	.046	.046	.045	.045	.044
1.9	.154	.099	.077	.065	.058	.053	.050	.047	.045	.043	.042	.041	.040	.038	.038	.038	.037	.037
2.0	.148	.092	.070	.058	.051	.046	.043	.040	.038	.037	.035	.034	.033	.032	.032	.031	.031	.030

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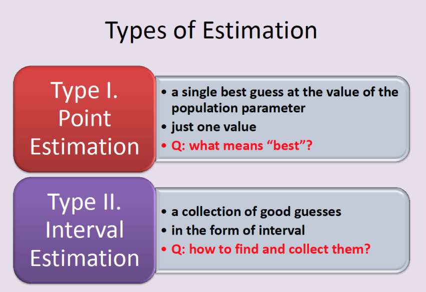
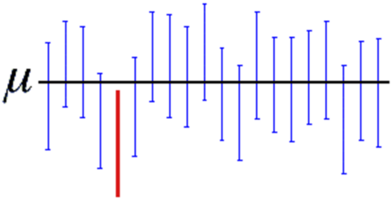
Confidence interval for population mean



Point estimates are almost always wrong, so why not collect a lot of good guesses which form an interval, and let the interval cover the population mean with high probability?

Interpretation of a confidence interval



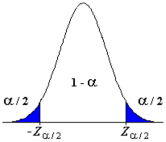
A 95% confidence interval indicates that 19 out of 20 samples (95%) from the same population will produce confidence intervals that contain the population parameter.

Note: in confidence interval, the population mean μ is a fixed unknown constant, the interval is **random**.

Mechanics of a confidence interval: case 1

If we know the population standard deviation σ,

1. Choose a confidence level $1-\alpha$. Typically, if we require 95% confidence level, then $\alpha = 0.05$.
2. Use z table to find the z critical value such that $P(-z_{\frac{\alpha}{2}} \leq Z \leq z_{\frac{\alpha}{2}}) = 1-\alpha$.



3. Construct the interval: (L, U), where $L = \bar{X} - z_{\frac{\alpha}{2}} * \frac{\sigma}{2}, U = \bar{X} + z_{\frac{\alpha}{2}} * \frac{\sigma}{2}$. (**Why do we construct this way?**)
4. Conclude: $P(L \leq \mu \leq U) = 1-\alpha$. We are $(1-\alpha) \times 100\%$ confident that the population mean is between (L, U).

Mechanics of a confidence interval: case 2

If we don't know the population standard deviation σ,

1. Choose a confidence level $1-\alpha$. Typically, if we require 95% confidence level, then $\alpha = 0.05$.
2. Find the value t such that $P(-t \leq T \leq t) = 1-\alpha$. It also means $P(T \geq t) = \frac{\alpha}{2}$. Use t table with degrees of freedom n-1. We denote the value t as $t_{n-1,\alpha/2}$.
3. Construct the interval: (L, U), where $L = \bar{X} - t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}, U = \bar{X} + t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}$. (**Why do we construct this way?**)
4. Conclude: $P(L \leq \mu \leq U) = 1-\alpha$. We are $(1-\alpha) \times 100\%$ confident that the population mean is between (L, U).

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4. Conclude.

Confidence interval R simulation

See R codes from the course webpage.

What's the next?

In the next lecture, we'll discuss sample size and population proportions.