Chapter 5: Estimation

Ott & Longnecker Sections: 4.11-4.12

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Part 1

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What do we study?

Key Concepts: Independence and dependence of RVs, simple random sample, independent and identically distributed (iid) RVs

Outline

1. Independence and dependence of RVs
2. Properties of expectation and variance
3. Simple random sample
4. iid

Independence and dependence of RVs

Two RVs are said to be **independent** if the realization of one of them does not change the probability distribution of the other, and vice versa. If two RVs are not independent, then they are **dependent**.

Example

Recall the ant farm with 20 ants, of which 5 are poisonous. You select two ants at random. Let X_1 be 1 if the first ant is poisonous, and 0 otherwise. Let X_2 be 1 if the second ant is poisonous, and 0 otherwise. What distribution does X_1 have?

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What distribution does X_2 have?

- If the two ants are selected **with replacement**, then X_1 and X_2 are independent since knowledge of whether $X_1 = 1$ (poisonous) or $X_1 = 0$ (non-poisonous) won't change the distribution of X_2 – it's an identical draw from the same population, so X_2 is still Bernoulli$(1/4)$.

Example

- If the two ants are selected **without replacement**, then X_1 and X_2 are dependent. If we know $X_1 = 1$ (poisonous), then now $X_2 \sim Ber(4/19)$. If $X_1 = 0$ (not poisonous), then now $X_2 \sim Ber(5/19)$. Knowing the outcome of the first ant changed the probability distribution of the second!

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Properties of expectation and variance

We now continue with some properties of expectation and variance. Let X and Y be any RVs, and let c be a constant.

1. $E(c) = c$.
2. $E(c * X) = c * E(X)$.
3. $E(X + c) = E(X) + c$.
4. $E(X + Y) = E(X) + E(Y)$.
5. $VAR(c) = 0$.
6. $VAR(c * X) = c^2 VAR(X)$.
7. $VAR(X + c) = VAR(X)$.
8. If X and Y are independent, $VAR(X + Y) = VAR(X) + VAR(Y)$.

Example

We can use these properties to get more intuition about some results from the previous chapter. We stated that:

$$\text{If } X \sim N(\mu, \sigma^2), \text{ then } Z = \frac{X-\mu}{\sigma} \sim N(0,1).$$

Recall that $E(X) = \mu$ and $VAR(X) = \sigma^2$. It's not trivial to show that Z is normal, but it is relatively easy to show that $E(Z) = 0$ and $VAR(Z) = 1$.

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$$E\left(\frac{X-\mu}{\sigma}\right) = \frac{E(X)-\mu}{\sigma} = \frac{0}{\sigma} = 0, \text{ by properties (1), (2), and (3).}$$
$$VAR\left(\frac{X-\mu}{\sigma}\right) = \frac{VAR(X)-0}{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1, \text{ by properties (5), (6), and (7).}$$

Example

When discussing the binomial, we stated that:

If $B \sim Bin(n, \pi)$, then $E(B) = n\pi$, and $VAR(B) = n\pi(1 - \pi)$.

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Recall that a binomial is a sum of n iid Bernoulli RVs, call them X_1, X_2, ..., X_n, then $B = \sum_{i=1}^{n} X_i$. Since each of these Bernoulli RVs has expectation π and variance $\pi(1-\pi)$, then we have

$$E(B) = E(\sum_{i=1}^{n} X_i) = n\pi, \text{ by repeated use of property (4).}$$

$$VAR(B) = VAR(\sum_{i=1}^{n} X_i) = n\pi(1-\pi), \text{ by repeated use of property (8).}$$

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Population and sample

- **Population**: collection of all items which is of interest for some question or experiment. For example, we are interested in the weight of UW-Madison students, then the population is collection of all weights of UW-Madison students.
- **Random sample**: a randomly selected subset of the population. For the above example, we can randomly select 100 students in this classroom and have their weights, these weights is a random sample.
- **Sampling**: process of randomly selecting sample from population is called sampling.

More examples

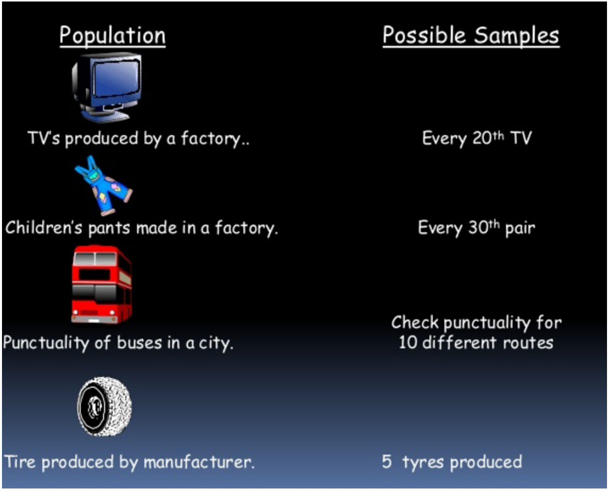

Figure: image from https://www.slideshare.net/dennyese/theo-37920004

Why sampling

Why do we take a sample?

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Too expensive and too time consuming to survey an entire population.

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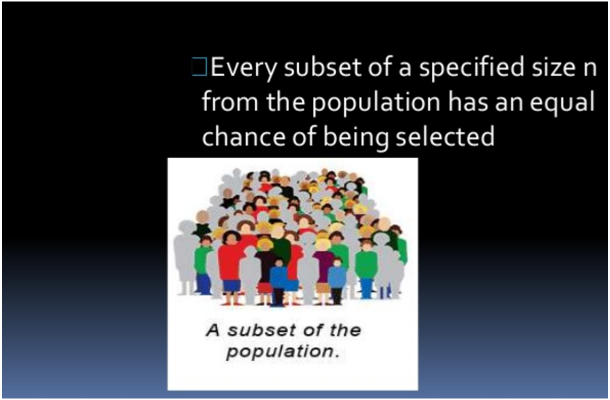
That is equivalently to say, we hope **every possible sample is equally likely to be drawn**.

- A random sample of size n from a population is called a **simple random sample** if every possible sample of size n is equally likely to be drawn.
- The process of selecting simple random sample is called **simple random sampling**.

Simple random sample



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How can we perform simple random sampling

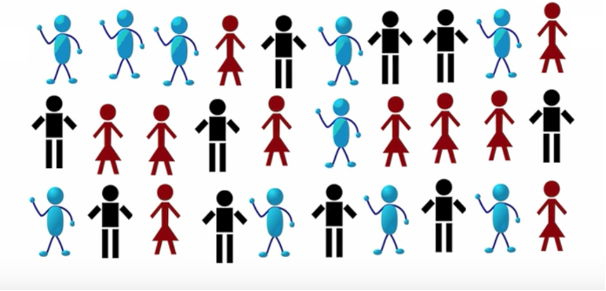
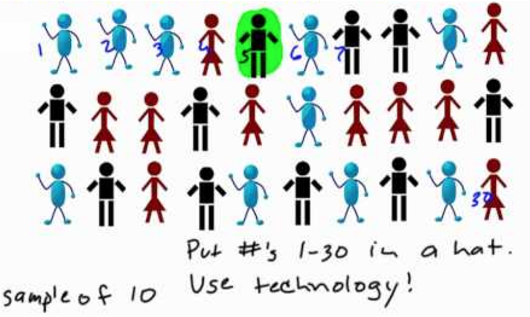


Figure: how do we have a simple random sample of size 10?

How can we perform simple random sampling



Method 1: number all people from 1 to 30 and put these numbers in a hat, then pick up 10 numbers from the hat. **Method 2**: use sample function in R: **sample(1:30, 10)**

Outline

A random sample of n RVs X_1, X_2, ..., X_n are said to be **independent and identically distributed**, or **iid**, if:

- the RVs are all independent of one another, that is, the realization of any one of them does not change the probability distribution of any other one;
- they all have exactly the same probability distribution.

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Note: An iid sample of size n can be generated by randomly drawing n samples from a population **with replacement.**

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Example: the results of repeated flips of a coin, or rolls of a die, are i.i.d. The outcome of a single flip (roll) doesn't affect the probabilities of the outcomes of any other, and it's the same coin (die), so the distribution in each trial is the same.

An overview

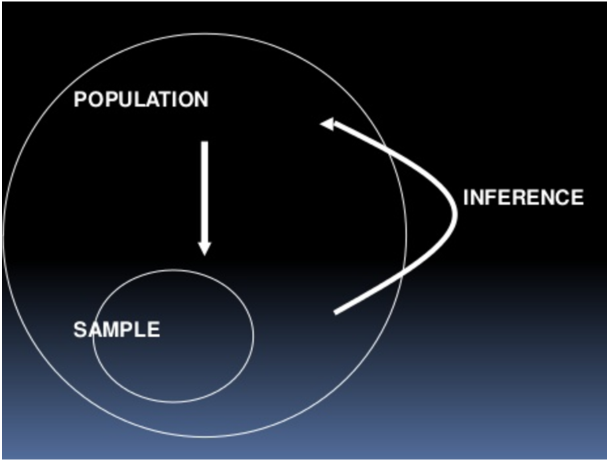

Figure: we use the sample to make inference about the population.

What's the next?

In the next lecture, we'll discuss concepts of estimation.