Assignment #3

1. Let $X \sim N(10, 25)$. Answer the following questions about X. Calculate these probabilities using Z table.

   (a) Find $P(X \leq 11)$. $P(X \leq 11) = P(Z \leq \frac{11-10}{\sqrt{25}}) = P(Z \leq 0.2) = 0.5793$.

   (b) Find $P(X \geq 8.4)$. $P(X \geq 8.4) = P(Z \geq \frac{8.4-10}{\sqrt{25}}) = P(Z \geq -0.32) = P(Z \leq 0.32) = 0.6255$.

   (c) Find x such that $P(X \leq x) = 0.95$. The z that makes $P(Z \leq z) = 0.95$ is 1.645, so reverse standardizing, this corresponds to $x = 1.645\sqrt{25} + 10 = 18.225$.

   (d) Find $P(7.5 \leq X \leq 15)$. $P(7.5 \leq X \leq 15) = P(\frac{7.5-10}{\sqrt{25}} \leq Z \leq \frac{15-10}{\sqrt{25}}) = P(-0.5 \leq Z \leq 1.0) = P(Z \leq 1.0) - P(Z \leq -0.5) = 0.8413 - (1 - 0.6915) = 0.5328$.

2. To celebrate their 30th birthdays, brothers Mario and Luigi wish to study the distribution of heights of their enemies, the Goombas. Their reasoning is that shorter Goombas are easier to jump on. (Goombas die when Mario and Luigi jump on them.)

   (a) If we assume that the population of Goomba heights is normally distributed with mean 12 inches and standard deviation 6 inches, what is the probability that a randomly chosen goomba has a height between 13 and 15 inches? Let X represent Goomba heights. Then $X \sim N(12, 36)$, so $P(13 \leq X \leq 15) = P(\frac{13-12}{\sqrt{36}} \leq Z \leq \frac{15-12}{\sqrt{36}}) = P(0.17 \leq Z \leq 0.50) = P(Z \leq 0.50) - P(Z \leq 0.17) = 0.6915 - 0.5675 = 0.124$.

   (b) Koopa Troopas, another enemy of Mario & Luigi, have normally distributed heights, with mean 15 inches and standard deviation 3 inches. What is the probability that a randomly selected Koopa Troopa is taller than the shortest 75% of Goombas? (**Hint: First compute the height such that 75% of Goombas are shorter than that height. Then compute the probability that a Koopa Troopa is taller than that height.**) We first must find the height such that 75% of Goombas are shorter than that height. This amounts to finding x such that $P(X \leq x) = 0.75$). Start by finding z such that $P(Z \leq z) = 0.75$. This occurs at $z = 0.675$. Reverse standardizing, we find that $x = 0.675 * \sqrt{36} + 12 = 16.05$. Thus a Koopa Troopa must be at least 16.05 inches tall to be taller than 75% of Goombas. Then, if we let Y represent Koopa Troopa heights, we have $Y \sim N(15, 9)$, so we want to find $P(Y \geq 16.05) = P(Z \geq \frac{16.05-15}{\sqrt{9}}) = P(Z \geq 0.35) = 1 - 0.6368 = 0.3632$.

   (c) What is the probability that a randomly chosen Goomba is taller than the shortest 75% of Koopa Troopas? We first must find the height such that 75% of Koopa Troopas are shorter than that height. This amounts to finding y such that $P(Y \leq y) = 0.75$). Start by finding z such that $P(Z \leq z) = 0.75$. This occurs at $z = 0.675$, as before. Reverse standardizing, we find that $y = 0.675 * \sqrt{9} + 15 = 17.025$. Thus a Goomba must be at least 17.025 inches tall to be taller than 75% of Koopa Troopas. Then, if we let X represent Goomba heights, we have $X \sim N(12, 36)$, so we want to find $P(X \geq 17.025) = P(Z \geq \frac{17.025-12}{\sqrt{36}}) = P(Z \geq 0.84) = 1 - 0.7995 = 0.2005$.

3. Let F be an RV that represents the operating temperature in Fahrenheit of one instance of a manufacturing process, and let $F \sim N(90, 25)$. Let C be an RV that represents the same process, but measured in Celsius. Fahrenheit can be converted to Celsius using $C = \frac{5}{9}(F - 32)$. (**I would suggest doing these with a hand calculator and normal table as practice for exam conditions, but you may check your answers in R if you wish.**)

   (a) Find the probability that one randomly selected instance of the process will have operating temperature greater than 93.8 Fahrenheit. Standardizing and using the normal table, $P(F \geq 93.8) = P(Z \geq \frac{(93.8-90)}{\sqrt{25}}) = P(Z \geq 0.76) = 1 - P(Z \leq 0.76) = 1 - 0.7764 = 0.2236$.

   (b) Find the distribution of C. The distribution will consist of the 'family' of distribution (for example, Bernoulli, Binomial, Normal, etc.) plus the relevant parameters associated with that family (π for a Bernoulli, n and π for a Binomial, μ and σ^2 for a Normal, etc.). Using our rules of E and VAR, $E(C) = (5/9)(E(F) - 32) = 32.22$, and $VAR(C) = (5/9)^2 VAR(F) = 7.72$. Since any linear function of normals is normal, the distribution of C is normal, thus $C \sim N(32.22, 7.72)$.

   (c) Find the probability that one randomly selected instance of the process will have operating temperature below 29 Celsius. Standardizing and using the normal table, $P(C \leq 29) = P(Z \leq \frac{(29-32.22)}{\sqrt{7.72}}) = P(Z \leq -1.16) = P(Z \geq 1.16) = 1 - P(Z \leq 1.16) = 1 - 0.8770 = 0.1230$.

4. Consider a large population which has population mean μ, and population variance σ^2. We take a sample of size $n = 3$ from this population, thinking of the samples as realizations of the RVs X_1, X_2, and X_3, where the X_i can be considered iid. We are interested in estimating μ.

   (a) Consider the estimator $\tilde{\mu} = X_1 + X_2 - X_3$. Is this estimator unbiased for μ? Explain your answer. The estimator $\tilde{\mu}$ is unbiased for μ if $E(\tilde{\mu}) = \mu$. By the rules of E, $E(\tilde{\mu}) = \mu + \mu - \mu = \mu$, so $\tilde{\mu}$ is indeed unbiased for μ.

   (b) Find the variance of $\tilde{\mu}$. By the rules of VAR, $VAR(\tilde{\mu}) = 3\sigma^2$.

   (c) When estimating μ, would you prefer the estimator $\tilde{\mu}$ or the estimator $\bar{X}$? Explain your answer. We would prefer $\bar{X}$, since both estimators are unbiased but the variance of $\bar{X}$ is smaller. $VAR(\bar{X}) = \frac{\sigma^2}{3} < 3\sigma^2 = VAR(\tilde{\mu})$.

   (d) Now consider the estimator $\mathring{\mu} = (X_1 + X_2 + X_3)/2$. Is this estimator unbiased for μ? Explain your answer. The estimator $\mathring{\mu}$ is unbiased for μ if $E(\mathring{\mu}) = \mu$. By the rules of E, $E(\mathring{\mu}) = (\mu + \mu + \mu)/2 = \frac{3}{2}\mu$, so $\mathring{\mu}$ is biased for μ. The bias is $\frac{1}{2}\mu$.

   (e) Compute the MSE of $\mathring{\mu}$. $MSE(\mathring{\mu}) = VAR(\mathring{\mu}) + Bias(\mathring{\mu})^2 = \frac{3}{4}\sigma^2 + \frac{1}{4}\mu^2$.