Chapter 5: Estimation

Ott & Longnecker Sections: 5.3, 10.2

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Part 5

https://dzwang91.github.io/stat371/

Outline

Determining sample size

"Is it possible to have a 100% CI?"

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This kind of confidence interval is uninformative.

Determine sample size

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For example, we want to get a 95% confidence interval with width 5(U-L=5), then what's the required sample size?

Determine sample size

$$H = z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

$$n = \left(z_{\frac{\alpha}{2}} \frac{\sigma}{H} \right)^2$$

Case 1: σ is known

$$H = t_{n-1,\frac{\alpha}{2}} \frac{s}{\sqrt{n}} \approx z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$$

$$n \approx \left(z_{\frac{\alpha}{2}} \frac{s}{H} \right)^2$$

Case 2: σ is unknown

Example

Example: We want a 95% CI for μ. We desire the half-width to be no larger than 0.1mm. Then what's the sample size?

Case 1: if σ, the true population standard deviation is known, since $z_{\alpha/2} = 1.96$, we would just need to solve the equation

$$0.1 = 1.96 * \frac{\sigma}{\sqrt{n}},$$

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Case 2: if σ is unknown, in this case, we solve the equation

$$0.1 = t_{(n-1,\alpha/2)} \frac{s}{\sqrt{n}} \approx z_{\alpha/2} \frac{s}{\sqrt{n}}$$

So if we are given $s = 0.3385$ mm.

$$0.1 = 1.96(0.3385/\sqrt{n}),$$

which gives:

$$n = \frac{(1.96^2)(0.3385^2)}{0.1^2} = 44.01, \text{ which we } \textbf{round up} \text{ to } 45.$$

Outline

1. Determining sample size

2. Estimation and inference for population proportions

What have we learnt so far

We've talked about the estimation of **population means**

1. Point estimate: sample mean
2. interval estimate: σ is given and σ is unknown.

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We'll now discuss estimation of **population proportions**.

Example

An accounting firm has a large list of clients (the population), and each client has a file with information about that client. The firm has noticed errors in some of these files, and has decided that it would be worthwhile to know the proportion of files that contain an error. Call the population proportion of files in error π. It was decided to take a simple random sample of size $n = 50$, and use the results of the sample to **estimate** π. Each selected file was thoroughly reviewed, and classified as either containing an error (call this 1), or not (call this 0). The results are as follows:

Files with an error: 10; Files without any errors, 40.

Example

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The procedure by which the files were selected is a **binomial process**. Let the random variable Y_i be the *indicator* that the ith file sampled had errors: that is, Y_i is 1 if the file contains an error and 0 otherwise. The pmf of Y_i for all i is:

Y_i	$p(Y_i)$
0	$1-\pi$
1	π

Then the random variable $B = Y_1 + Y_2 + ... + Y_n = \sum_{i=1}^{n} Y_i \sim Bin(n, \pi)$. Observe that B just counts the number of files with errors. (In the example, we happened to realize $b = 10$ errors out of $n = 50$ files sampled.)

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Recall $E(Y_i) = \pi$ and $VAR(Y_i) = \pi(1-\pi)$. Hence:

$$E(P) = \pi,\ VAR(P) = \frac{\pi(1-\pi)}{n}, SE(P) = \sqrt{\frac{\pi(1-\pi)}{n}}.$$

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- The estimator P is **unbiased** for π.
- Note that if $\pi = 0$ or 1 the standard error is 0. Does this make sense?
- We can get the estimated standard error of P by plugging in our estimator of π:

$$\text{Estimated standard error of P: } \widehat{SE(P)} = \sqrt{\frac{P(1-P)}{n}}.$$

Example

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Question 2: what's the exact distribution of P?
The exact distribution of P is related to a binomial, but it turns out that making an exact CI based on this fact is very mathematically challenging and difficult.
Question 3: do we have any other tools to overcome this challenge?

YES, CLT!!!

So long as the sample size is large enough, all the conditions of the CLT are met, because the Y_i are iid, and P is just a sample mean of a bunch of zeros and ones. Thus, for large samples, P **is approximately distributed as a normal**:

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Question 4: what is the $100(1-\alpha)\%$ **CI for** π**?**

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Question 4: what is the $100(1-\alpha)\%$ CI for π?
Recall the general form of CI: estimate $\pm$ multiplier $\times$ estimated SE of the estimator. This means that an approximate $100(1-\alpha)\%$ CI for π would be of the form:

$$P \pm z_{\alpha/2}\sqrt{\frac{P(1-P)}{n}}.$$

Example

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Generally, a rule of thumb is that if $n\pi > 5$ and $n(1 - \pi) > 5$, the approximation will be good. In this expression π can be approximated by p as estimated by the sample. The rule then becomes, you should have observed at least 5 successes and at least 5 failures.

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Generally, a rule of thumb is that if $n\pi > 5$ and $n(1-\pi) > 5$, the approximation will be good. In this expression π can be approximated by p as estimated by the sample. The rule then becomes, you should have observed at least 5 successes and at least 5 failures.

Returning to the audit data, our estimate would be $p = 10/50 = 0.2$, with estimated standard error $\sqrt{(0.2 * 0.8)/50} = 0.057$. The CLT should be a good approximation since we have 10 successes and 40 failures, more than 5 each. Thus an approximate 95% CI for π would be $0.2 \pm 1.96 * 0.057$, or $(0.088, 0.312)$.

Example

Question 6: Can the CI for a proportion go below 0 or above 1 using the CLT method?

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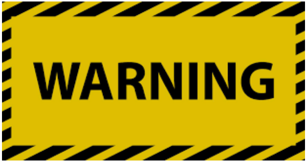
Yes, it will happen because the interval is **approximate**. Practically, you would probably use a lower or upper bound of 0 or 1, rather then extending the interval into a range that is physically impossible.

What's the next?



We'll talk about the bootstrap method in next lecture. This is VERY challenging and I hope you feel good.