Chapter 8: Comparing two independent populations

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Part 2: The Welch T-test

https://dzwang91.github.io/stat371/

Example

Concrete used for roadways or buildings is often reinforced with a material that is placed inside the setting concrete. A common example of this is called 'rebar' which is short for 'reinforcing bar' and is usually made out of steel. It is desirable that the reinforcing material is strong and corrosion resistant. Steel is strong, but tends to corrode over time, so experiments were conducted to test two corrosion resistant materials, one made of fiberglass and the other made of carbon.

8 beams with fiberglass reinforcement, and 11 beams with carbon reinforcement were poured, and each was then subjected to a load test, which measures the strength of the beam. Strength is measured in kN (kiloNewtons), which is a measure of the force required to break the beam.

Example continued

- The primary research question was, "Is there any difference in the strength of the two types of beams?" Thus we wish to test:

$$H_0 : \mu_{fiber} - \mu_{carbon} = 0, H_A : \mu_{fiber} - \mu_{carbon} \neq 0$$

- The data are as follows:
  - Fiberglass: 38.3, 29.6, 33.4, 33.6, 30.7, 32.7, 34.6, 32.3
  - Carbon: 48.8, 38.0, 42.2, 45.1, 32.8, 47.2, 50.6, 44.0, 43.9, 40.4, 45.8

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Fiber	8	33.15	2.63
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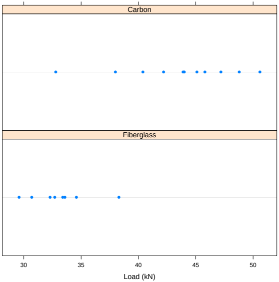
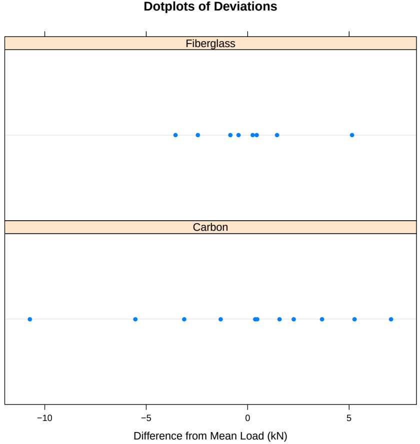
  $\frac{s_1}{s_2} = \frac{2.63}{5.06} = 0.52$, so while still strictly within the allowed limits, it is very close to the edge.
- By dotplot, the mean for carbon looks a bit higher, but the sd is also larger.



Example continued

- We can also look at differences from means.



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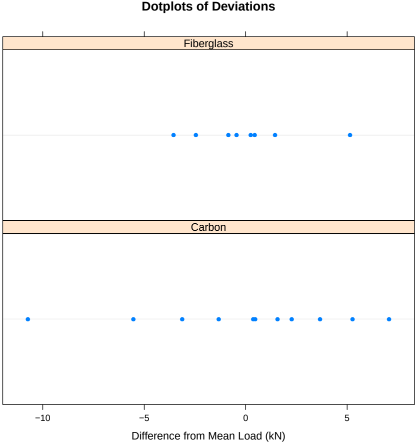
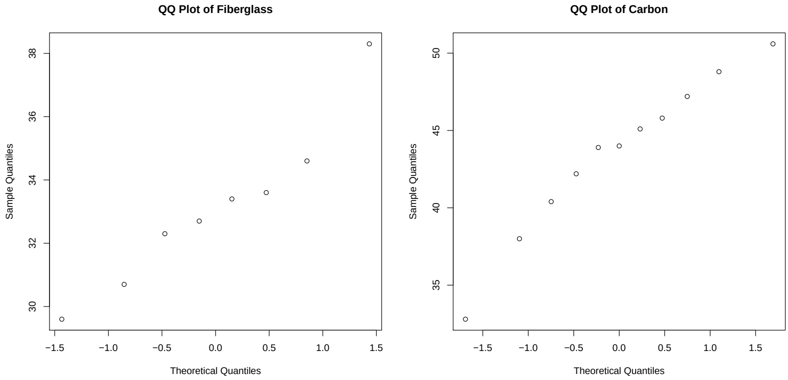


The constant variance assumption doesn't seem to be met, because there appears to be more spread in the carbon group. To be safe, we may not want to assume equal variances.

Example continued

- Separate QQ plots for each sample:



Each sample comes from a normal population.

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  - It is likely that the degrees of freedom will not be a whole number. It is common to round down to the nearest integer.

Example continued

- In the example:
  - $T_{obs} = \frac{33.15 - 43.53 - 0}{\sqrt{\frac{2.63^2}{8} + \frac{5.06^2}{11}}} = -5.81$
  - $\nu = \frac{\left(\frac{2.63^2}{8} + \frac{5.06^2}{11}\right)^2}{\frac{(2.63^2/8)^2}{8-1} + \frac{(5.06^2/11)^2}{11-1}} = 15.7$, round down to 15
- The p-value is $2 \times P(T_{15} > 5.81) < 0.001$, so there is evidence that the two kinds of materials are not equally strong. It seems the carbon is stronger and would be preferred.

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6. Draw a conclusion.

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  - If the variances are truly different, but they are assumed to be equal, the test can make wildly incorrect conclusions.
- Therefore, if there is any doubt about the equality of the variances, it's generally safer to allow them to differ.

What's the next?

We'll discuss how to compare two population proportions in the third part of Chapter 8.