Chapter 11: Simple linear regression
(Ott & Longnecker Sections: 11.1-11.5)

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Part 2

https://dzwang91.github.io/stat371/

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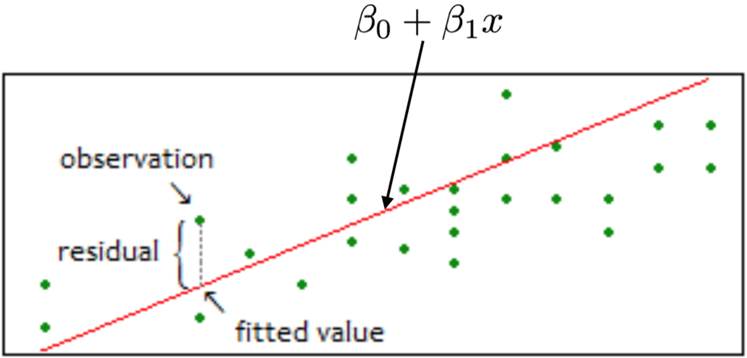
Review of last lecture

- Denote the height of son i by y_i, the height of father i by x_i, and the random error by ϵ_i, so that the model becomes:
$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i.$$
- Our goal is to estimate the values of β_0 and β_1 from data.

Estimation of β_0 and β_1



- We want the vertical distance from the line to the points to be small.

Estimation of β_0 and β_1

- Suppose $\hat{\beta}_0$ and $\hat{\beta}_1$ are our estimates, then the estimated (fitted) value y for any given x is:

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i.$$

- The difference between this and the actual y_i (observed) is $y_i - \hat{y}_i$, we call this the residual.
- For given choices of $\hat{\beta}_0$ and $\hat{\beta}_1$, we can measure how well the line fits by measuring the residual sum of squares (RSS) produced by these choices:

$$RSS(\hat{\beta}_0, \hat{\beta}_1) = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i\right)^2.$$

Estimation of β_0 and β_1

- By some calculations,

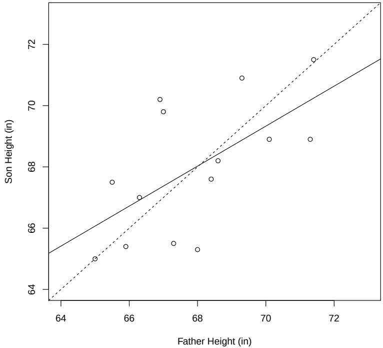
$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}.$$

- These choices of $\hat{\beta}_0$ and $\hat{\beta}_1$ produce what is called the least squares line.
- The residual sum of squares for the least squares line has a special name: the sum of squared errors or SSE. SSE is the smallest possible residual sum of squares in the universe of all possible lines.

Example

- For the father and son data, these values work out to: $\hat{\beta}_1 = 0.65$ and $\hat{\beta}_0 = 23.64$.
- To assess the quality of the fit, we can add the line (solid) to the scatterplot. We also add a line with slope 1 and intercept zero (dashed) for comparison purposes:



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  4. The random error around the true line is normal.
- Assumptions 2-4 are equivalent to $\epsilon_i \sim iid\ N(0, \sigma^2)$.

Hypothesis testing in regression model continued

- t-test statistic:

$$t = \frac{\hat{\beta}_1}{\widehat{SE(\hat{\beta}_1)}}.$$

where

$$\widehat{SE(\hat{\beta}_1)} = \frac{\hat{\sigma}}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}}$$

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$$\hat{\sigma}^2 = \frac{SSE}{n-2}.$$

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- If H_0 is true, then t has a T-distribution on $n-2$ degrees of freedom.
- For the father and son data, $\hat{\sigma} = 1.78$, so $\widehat{SE(\hat{\beta}_1)} = 0.24$, and $t = 2.70$. Comparing this to a t_{12}, the p-value is 0.0193. So we would reject at the 5% level, and conclude that father's height is related to son's height.

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$$SE(\hat{y}|x^*) = \sigma\sqrt{\frac{1}{n} + \frac{(x^* - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}.$$

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- Example: Suppose we want to predict the average son's height when the father is $x^* = 70$ inches tall. Our estimate and standard error would be:

$$\hat{y}|(x^* = 70) = 23.64 + 0.65 * 70 = 69.14$$

$$\text{estimated } SE(\hat{y}|(x^* = 70)) = 1.78\sqrt{1/14 + \tfrac{(70-67.93)^2}{54.25}} = 0.69.$$

R^2

- If we define $SSTot = \sum_{i=1}^{n}(y_i - \bar{y})^2$, we can create a quantity called R^2:

$$R^2 = \frac{SSTot - SSE}{SSTot}.$$

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- For the father and son data, $R^2 = 0.38$. So we can say that about 38% of the variability in sons' heights can be explained by fathers' heights.