QQ plot, CLT

5 Estimation (continued), 6 Testing, 7 One Sample, ~~8 Two Sample, 9 Paired~~

Bootstrap for μ

not required

- Draw simple random sample of ~~size~~ n from the population. Find $\bar{x}$ and s.
- Resample $x_1^*, \ldots, x_n^*$ with replacement from data. Find $\bar{x}^*, s^*$ and $\hat{t} = \frac{\bar{x}^* - \bar{x}}{s^*/\sqrt{n}}$
- Repeat previous step B times to get many $\hat{t}$s.
- For interval, find $1 - \frac{\alpha}{2}$ and $\frac{\alpha}{2}$ upper critical values $\hat{t}_{(1-\alpha/2)}$ and $\hat{t}_{(\alpha/2)}$; $\left(\bar{x} - \hat{t}_{(\alpha/2)}\frac{s}{\sqrt{n}}, \bar{x} - \hat{t}_{(1-\alpha/2)}\frac{s}{\sqrt{n}}\right)$ contains μ for about $100(1-\alpha)\%$ of original samples
- To test $H_0 : \mu = \mu_0$, find $t_{\text{obs}} = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$ and use p-value $= \frac{m}{B}$, where
  $H_A : \mu > \mu_0 \implies m = \#\hat{t}$ with $\hat{t} > t_{\text{obs}}$
  $H_A : \mu < \mu_0 \implies m = \#\hat{t}$ with $\hat{t} < t_{\text{obs}}$
  $H_A : \mu \neq \mu_0 \implies m = \#\hat{t}$ with $|\hat{t}| > |t_{\text{obs}}|$

Inference patterns

Confidence interval for θ :

$\hat{\theta} \pm$ (table value for confidence) $\times\, \sigma_{\hat{\theta}}$

Test of $H_0 : \theta = \theta_0$: test statistic: $\dfrac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}}$

p-value $= P$(result at least as extreme as test statistic $|\, H_0$):

$H_A : \theta > \theta_0 \implies P$-value = right tail area
$H_A : \theta < \theta_0 \implies P$-value = left tail area
$H_A : \theta \neq \theta_0 \implies P$-value = both tails

$\alpha = P$(type I error) $= P$(reject $H_0 | H_0$ is true)
$\beta = P$(type II error) $= P$(do not reject $H_0 | H_0$ is false with $\theta = \theta_A$)
power $= 1 - \beta = P$(reject $H_0 | H_0$ is false with $\theta = \theta_A$)

Test-interval relationship :

Level-α test of $H_0 : \mu = \mu_0$ vs. $H_A : \mu \neq \mu_0$ retains $H_0 \iff \mu_0$ is inside a $1 - \alpha$ confidence interval for μ.

One mean, μ: normal or $n > 30$ and σ known: Z

$\bar{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$; ~~sample size~~ $\cancel{n = \left(\frac{z_{\alpha/2}\sigma}{m}\right)^2}$

$Z = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$

~~one-sided~~ $\cancel{H_A : \mu < \mu_0}$ ~~or~~ $\cancel{H_A : \mu > \mu_0 \implies \text{power}_{\mu_A} = P\left(Z < \frac{|\mu_0 - \mu_A|}{\sigma/\sqrt{n}} - z_\alpha\right)}$

~~two-sided~~ $\cancel{H_A : \mu \neq \mu_0 \implies \text{power}_{\mu_A} \approx P\left(Z < \frac{|\mu_0 - \mu_A|}{\sigma/\sqrt{n}} - z_{\alpha/2}\right)}$

~~for~~ $\cancel{H_A : \mu \neq \mu_0}$, ~~power~~ $\cancel{(1-\beta)_{\mu=\mu_A}}$ ~~requires~~ $\cancel{n \approx \left(\frac{\sigma(z_{\alpha/2} + z_\beta)}{\mu_0 - \mu_A}\right)^2}$

σ unknown: t

$\bar{X} \pm t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}$

$T = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t_{n-1}$

One median, M: sign test

To test $H_0 : M = M_0$, find $X_1 - M_0, \ldots, X_n - M_0$ and $B = \#$positive differences $\sim Bin(n, \frac{1}{2})$
p-value:
$H_A : M > M_0 \implies p$-value $= P(B \geq b)$
$H_A : M < M_0 \implies p$-value $= P(B \leq b)$
$H_A : M \neq M_0 \implies p$-value $= 2 \times \min(P(B \leq b), P(B \geq b))$

One proportion, π

$P = \hat{\pi} = \frac{X}{n}$, $E(P) = \pi$, $VAR(P) = \frac{\pi(1-\pi)}{n}$
need $n\pi > 5$, $n(1-\pi) > 5$ (use $P \approx \pi$ for interval, $\pi = \pi_0$ for test)

$P \pm z_{\alpha/2}\sqrt{\frac{P(1-P)}{n}}$

$Z = \dfrac{P - \pi_0}{\sqrt{\pi_0(1-\pi_0)/n}} \sim N(0,1)$

1. Suppose you are writing a contract between the producer of spliced ropes and the consumer, a parachute maker needing lines to attach a canopy to a harness.

   - The producer promises that the mean breaking strength of a shipment of the lines is $\mu = 100$ pounds, with $\sigma = 16$.
   - An independent lab will find $\bar{X}$ from a SRS of $n = 10$ lines to test $H_0 : \mu = (\mu_0 = 100)$ vs. $H_1 : \mu < \mu_0$.
   - A draft contract specifies $\bar{x}_{\text{critical}} = 97$. If $\bar{X}$ is below 97, H_0 is rejected and neither payment nor shipment occurs. If $\bar{X}$ is above or equal to 97, H_0 is not rejected and both payment and shipment occur. (Hint: A picture may help.)

   (a) Suppose you work for the producer (the splicing shop). If H_0 is true and the test nevertheless rejects H_0, the shipment of lines will be discarded, and you will not be paid. This happens with which probability? Mark your choice with an "X":

   ____ $P(\text{type I error}) = \alpha$,
   ____ $P(\text{type II error}) = \beta$, or
   ____ power $= 1 - \beta$

   ANSWER:

   $P(\text{type I error}) = \alpha$

   (b) In which direction would you like to move $\bar{x}_{\text{critical}} = 97$ to reduce your risk of not being paid in this situation? Mark your choice with an "X":

   ____ I want to move $\bar{x}_{\text{critical}}$ so that $\bar{x}_{\text{critical}} > 97$, or
   ____ I want to move $\bar{x}_{\text{critical}}$ so that $\bar{x}_{\text{critical}} < 97$

   ANSWER:

   The second choice is correct (so we'll reject H_0 when it is true less often).

   (c) Suppose you work for the consumer (the parachute maker). You can't use the lines if $\mu = 95$ (unless you redesign your parachute to use more of the weaker lines). If H_0 is false because $\mu = (\mu_A = 95)$, what is the probability that the test will not reject H_0? (In this case, you'll use a defective shipment of lines, and then sell defective parachutes.) Mark your choice with an "X":

   ____ $P(\text{type I error}) = \alpha_{(\mu_A=95)}$
   ____ $P(\text{type II error}) = \beta_{(\mu_A=95)}$
   ____ power $= 1 - \beta_{(\mu_A=95)}$

   ANSWER:

   $P(\text{type II error}) = \beta_{(\mu_A=95)}$

   (d) Which way would you like to move $\bar{x}_{\text{critical}}$ to decrease your risk? Mark your choice with an "X":

   ____ I want to move $\bar{x}_{\text{critical}}$ so that $\bar{x}_{\text{critical}} > 97$, or
   ____ I want to move $\bar{x}_{\text{critical}}$ so that $\bar{x}_{\text{critical}} < 97$

ANSWER:

The first choice is correct (so we'll reject H_0 when it is false more often).

(e) Suppose the draft contract is abandoned. What sample size is required to have level .01 and power .9 when the true population mean strength is 95 pounds? (Note: Increasing the sample size may resolve the tension between producer and consumer.)

ANSWER:

$n \approx \left(\frac{\sigma(z_{\alpha/2}+z_\beta)}{\mu_0-\mu_A}\right)^2 = \left(\frac{16(z_{.005}+z_{.10})}{\mu_0-\mu_A}\right)^2 = \left(\frac{16(2.575+1.285)}{100-95}\right)^2 = 152.6$, round up to $n = 153$.

2. In a summer survey of Madison second grade children, 15 out of 48 had played outside in the past 24 hours.

   (a) Find a 90% confidence interval for the true proportion of students who played outside in the past 24 hours.

   ANSWER:
   The observed numbers of successes and failures, 15 and $33 = 48 - 15$, are both > 5. $P = \hat{\pi} = \frac{15}{48} = .3125$; margin $= z_{.05}\sqrt{\frac{.3125(1-.3125)}{48}} = 1.645(.0669) \approx .1101$; interval $=$ $.3125 \pm .1101$

   (b) Are the data strong evidence that the population proportion of children who played outside in the past 24 hours is less than 1/3? Do an appropriate test.

   ANSWER:

   - Hypotheses: $H_0 : \pi = 1/3$ vs. $H_A : \pi < 1/3$
   - Assumptions: We have a SRS. $n = 48$ and $\pi = 1/3 \implies$ the expected numbers of successes and failures are $16 = 48(1/3)$ and $32 = 48(1 - 1/3)$, which are both > 5. I'll use the test for one proportion.
   - Test statistic: $P = \hat{\pi} = \frac{15}{48} = .3125 \implies z = \frac{.3125-1/3}{\sqrt{\frac{(1/3)(1-1/3)}{48}}} \approx -0.31$.
   - P-value: $P(Z < -0.31) = .3783$
   - Conclusion: Do not reject H_0. The data are not strong evidence that the population proportion of children played outside in the past 24 hours is less than 1/3.

3. A simple random sample of 8 pills was taken and the time (in minutes) to dissolve in water was measured for each pill. Here are the data:

   24 72 294 18 54 66 120 48

   A few summary statistics include $n = 8$, $\bar{x} = 87$, and $s = 89.4$. Plots suggest that the population is not normal, so a bootstrap is used. 10000 resamples with replacement are taken, with a $\hat{t}$ calculated for each resample. Here are some quantiles of the resampling distribution of $\hat{t}$:

probability	.001	.005	.01	.05	.10	.50	.90	.95	.99	.995	.999
quantile	-10.2	-8.4	-7.6	-5.7	-3.7	-0.1	1.0	1.3	1.9	2.1	2.7

   (a) Find a 90% confidence interval for the population mean dissolve time for the pills.

   ANSWER:
   For a 90% interval, we have $1 - \alpha = .90 \implies \alpha = .10 \implies \alpha/2 = .05$ and we need the .05 and .95 quantiles, $\hat{t}_{.05} = -5.7$ and $\hat{t}_{.95} = 1.3$. Here is the interval:
   $\left(\bar{x} - \hat{t}_{(\alpha/2)}\frac{s}{\sqrt{n}}, \bar{x} - \hat{t}_{(1-\alpha/2)}\frac{s}{\sqrt{n}}\right) = \left(87 - (1.3)\frac{89.4}{\sqrt{8}}, 87 - (-5.7)\frac{89.4}{\sqrt{8}}\right) = (45.9, 267.2)$

   (b) Estimate the p-value for a bootstrap test of $H_0 : \mu = 100$ vs. $H_A : \mu < 100$, where μ is the population mean dissolve time, at level $\alpha = .05$. Draw a conclusion. (Hint: You cannot use R to estimate the p-value as we did in class. You can, nevertheless, estimate the p-value and draw a conclusion.)

   ANSWER:

$$
\begin{aligned}
p\text{-value} &= P(T < t_{obs}) \\
&= P(T < \frac{87 - 100}{89.4/\sqrt{8}}) \\
&= P(T < -.41) \\
&= \text{between } .10 \text{ and } .50, \\
&\quad \text{because the .10 quantile is } -3.7 \text{ and the .50 quantile is } -0.1 \\
&> (\alpha = .05)
\end{aligned}
$$

   Do not reject H_0. The data are not strong evidence that the population mean dissolve time is less than 100 seconds.

4. A large hospital took a simple random sample of 10 babies delivered at the hospital whose mothers intended to breastfeed. Here are the number of days until weaning for each baby: 210, 217, 240, 270, 273, 289, 324, 330, 339, 530. Is the population median number of days to weaning less than 365 (one year)? Run an appropriate test.

ANSWER:

Test $H_0 : M = 365$ vs. $H_A : M < 365$ via a sign test.

Regarding assumptions, note that we have a SRS of babies delivered at the hospital whose mothers intended to breastfeed.

Let B = # babies in sample of 10 who are breastfed with days to weaning, minus 365 days, positive. This is the same as the number of days to weaning > 365. Under H_0, $B \sim Bin(n = 10, \pi = .5)$.

Our test statistic is $b = 1$.

The p-value is $P(B \leq 1) = P(B = 0) + P(B = 1) = \binom{10}{0}.5^0(1-.5)^{10-0} + \binom{10}{1}.5^1(1-.5)^{10-1} = .5^{10} + 10(.5^{10}) \approx 0.01$.

Since $0.01 < 0.05$, we reject H_0. The data are strong evidence the median #days to weaning is less than 365