Chapter 5: Estimation

(Ott & Longnecker Sections: 4.12, 4.14 and 5.2)

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https://dzwang91.github.io/stat371/

Part 4

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What do we study?

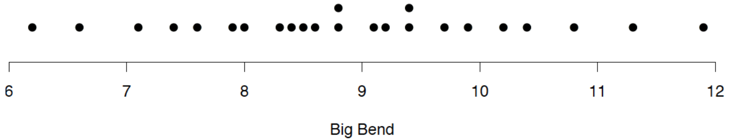
Key Concepts: t-distribution, confidence intervals

[1] Some of the slides in this lecture have been adapted/borrowed from materials developed by Cecile Ane.

Outline

1. Review of point estimation
2. The t-distribution
3. Confidence interval

Big bend lizards tail length



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```
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[16]  7.4  8.3  9.1  9.2  7.9  8.4 11.3  6.2  8.8
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[1] 8.895833
> sd(bigbend)
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> length(bigbend)
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- $\bar{X} = 8.896$ cm is one estimate for μ.
- How good is this estimate? How far is μ from 8.896 cm?

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- The standard error of $\bar{X}$ is $\frac{\sigma}{\sqrt{n}}$, but we don't know σ.
- estimated SE$=\frac{s}{\sqrt{n}}$ is the estimated standard error of the mean.
- s=1.43 and n=24, so estimated SE$=\frac{1.43}{\sqrt{24}} = 0.292$.
- The estimated SE gives us an idea of how far $\bar{X}$ is from μ typically.

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The t-distribution

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- If $X_1, ..., X_n$ have a normal distribution, then

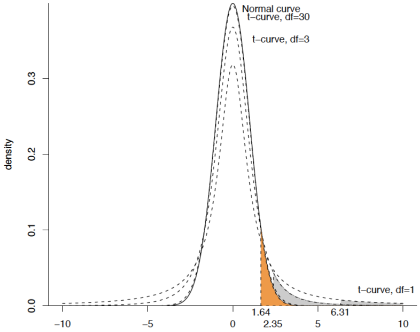
$$\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0, 1)$$

- When we replace $\sigma / \sqrt{n}$ by estimated SE$= s / \sqrt{n}$,

$$\frac{\bar{X} - \mu}{s / \sqrt{n}} \sim T_v$$

where $v = n - 1$ is called degrees of freedom and T_v is called t-distribution with degrees of freedom v.

The t-distribution



It looks very similar to a standard normal: it's symmetric and bell-shaped, but it is a little more spread out. The amount of additional spread decreases as the degrees of freedom (the sample size) increases.

T table

How do we find a value t s.t. $P(T_{10} \geq t) = 0.17$.

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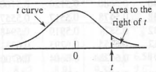
Table A.8 t Curve Tail Areas



t \ ν	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0.0	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500	.500
0.1	.468	.465	.463	.463	.462	.462	.462	.461	.461	.461	.461	.461	.461	.461	.461	.461	.461	.461
0.2	.437	.430	.427	.426	.425	.424	.424	.423	.423	.423	.423	.422	.422	.422	.422	.422	.422	.422
0.3	.407	.396	.392	.390	.388	.387	.386	.386	.386	.385	.385	.385	.384	.384	.384	.384	.384	.384
0.4	.379	.364	.358	.355	.353	.352	.351	.350	.349	.349	.348	.348	.348	.347	.347	.347	.347	.347
0.5	.352	.333	.326	.322	.319	.317	.316	.315	.315	.314	.313	.313	.313	.312	.312	.312	.312	.312
0.6	.328	.305	.295	.290	.287	.285	.284	.283	.282	.281	.280	.280	.279	.279	.279	.278	.278	.278
0.7	.306	.278	.267	.261	.258	.255	.253	.252	.251	.250	.249	.249	.248	.247	.247	.247	.247	.246
0.8	.285	.254	.241	.234	.230	.227	.225	.223	.222	.221	.220	.220	.219	.218	.218	.218	.217	.217
0.9	.267	.232	.217	.210	.205	.201	.199	.197	.196	.195	.194	.193	.192	.191	.191	.191	.190	.190
1.0	.250	.211	.196	.187	.182	.178	.175	.173	.172	.170	.169	.169	.168	.167	.167	.166	.166	.165
1.1	.235	.193	.176	.167	.162	.157	.154	.152	.150	.149	.147	.146	.146	.144	.144	.144	.143	.143
1.2	.221	.177	.158	.148	.142	.138	.135	.132	.130	.129	.128	.127	.126	.124	.124	.124	.123	.123
1.3	.209	.162	.142	.132	.125	.121	.117	.115	.113	.111	.110	.109	.108	.107	.107	.106	.105	.105
1.4	.197	.148	.128	.117	.110	.106	.102	.100	.098	.096	.095	.093	.092	.091	.091	.090	.090	.089
1.5	.187	.136	.115	.104	.097	.092	.089	.086	.084	.082	.081	.080	.079	.077	.077	.077	.076	.075
1.6	.178	.125	.104	.092	.085	.080	.077	.074	.072	.070	.069	.068	.067	.065	.065	.065	.064	.064
1.7	.169	.116	.094	.082	.075	.070	.065	.064	.062	.060	.059	.057	.056	.055	.055	.054	.054	.053
1.8	.161	.107	.085	.073	.066	.061	.057	.055	.053	.051	.050	.049	.048	.046	.046	.045	.045	.044
1.9	.154	.099	.077	.065	.058	.053	.050	.047	.045	.043	.042	.041	.040	.038	.038	.038	.037	.037
2.0	.148	.092	.070	.058	.051	.046	.043	.040	.038	.037	.035	.034	.033	.032	.032	.031	.031	.030

Download T table from https://dzwang91.github.io/stat371/resource/.

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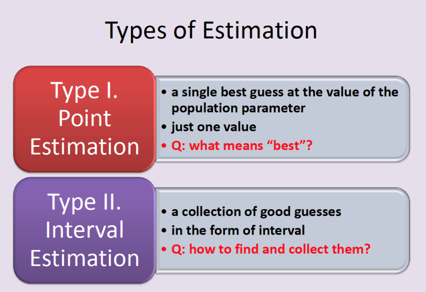
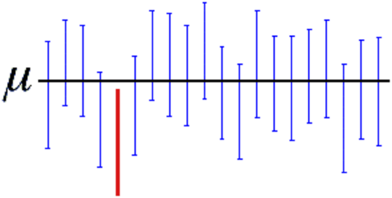
Confidence interval for population mean



- Point estimates are almost always wrong.
- Why not collect a lot of good guesses which form an interval, and let the interval cover the population mean with high probability?

Interpretation of a confidence interval



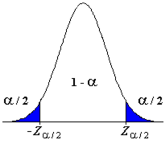
A 95% confidence interval indicates that 19 out of 20 samples (95%) from the same population will produce confidence intervals that contain the population parameter.

In confidence interval, the population mean μ is a fixed unknown constant, the interval is random.

Mechanics of a confidence interval: case 1

If we know the population standard deviation σ,

1. Choose a confidence level $1-\alpha$. Typically, if we require 95% confidence level, then $\alpha = 0.05$.
2. Use z table to find the $z_{\frac{\alpha}{2}}$ critical value such that $P(-z_{\frac{\alpha}{2}} \le Z \le z_{\frac{\alpha}{2}}) = 1-\alpha$.



3. Construct the interval: (L, U), where $L = \bar{X} - z_{\frac{\alpha}{2}} * \frac{\sigma}{\sqrt{n}}, U = \bar{X} + z_{\frac{\alpha}{2}} * \frac{\sigma}{\sqrt{n}}$. (Why do we construct this way?)
4. Conclude: $P(L \le \mu \le U) = 1-\alpha$. We are $(1-\alpha) \times 100\%$ confident that the population mean is between (L, U).

Mechanics of a confidence interval: case 2

If we don't know the population standard deviation σ,

1. Choose a confidence level $1-\alpha$. Typically, if we require 95% confidence level, then $\alpha = 0.05$.
2. Find the value t such that $P(-t \leq T_{n-1} \leq t) = 1-\alpha$. It also means $P(T_{n-1} \geq t) = \frac{\alpha}{2}$. Use t table with degrees of freedom n-1. We denote the value t as $t_{n-1,\alpha/2}$.
3. Construct the interval: (L, U), where $L = \bar{X} - t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}, U = \bar{X} + t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}$. (Why do we construct this way?)
4. Conclude: $P(L \leq \mu \leq U) = 1-\alpha$. We are $(1-\alpha) \times 100\%$ confident that the population mean is between (L, U).

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4. Conclude.

Confidence interval R simulation

See R codes from the course webpage.

What's the next?

In the next lecture, we'll discuss sample size and population proportions.