

# Chapter 7: One sample tests

(Ott & Longnecker Sections: 5.4-5.7, 5.9)

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Part 2

<https://dzwang91.github.io/stat371/>



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**Key concepts:** Power calculation, sample size calculation, sign test

## Definition

The **power** of a test is the probability of rejecting  $H_0$  given that a specific alternative hypothesis is true. That is

$$\text{Power} = 1 - \beta$$

where  $\beta$  is the Type II error.

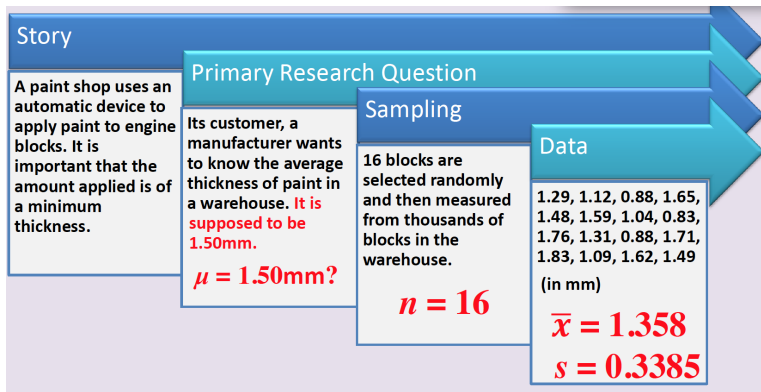
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- How do we calculate the power of a test given a specific alternative hypothesis?



- Step 1:

$$H_0: \mu = 1.5, H_A: \mu \neq 1.5$$

- Step 2: Choose  $\alpha = 0.05$

- Step 3: Use the T test statistic:  $T = \frac{\bar{X} - \mu_0}{\frac{s}{\sqrt{n}}}$ . In this example,

$$t_{obs} = \frac{1.348 - 1.50}{\frac{0.3385}{\sqrt{16}}} = -1.796. \text{ The rejection region is}$$

$T < -t_{n-1, \alpha/2}, T > t_{n-1, \alpha/2}$ . In this example,  $t_{15, 0.025} = 2.13$ , so the rejection region is  $T < -2.13$  or  $T > 2.13$ .

- Step 4: Make a conclusion: since  $t_{obs} = -1.796$  does not fall in the rejection region, so we do not reject the null.

- Suppose  $H_A : \mu = 1.4$ ,

$$\begin{aligned}\text{Power} &= P(\text{Reject } H_0 | H_0 \text{ false}) = P(|T| > 2.13 | \mu = 1.4) \\ &= P\left(\left|\frac{\bar{X} - 1.5}{S/\sqrt{n}}\right| > 2.13 \middle| \mu = 1.4\right) = ?\end{aligned}$$

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- We rewrite  $\frac{\bar{X}-1.5}{S/\sqrt{n}} = \frac{\bar{X}-1.4}{S/\sqrt{n}} + \frac{1.4-1.5}{S/\sqrt{n}}$

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- Since  $S = 0.3385$ ,  $n = 16$ , so

$$\begin{aligned} \text{Power} &= P\left(\left|\frac{\bar{X} - 1.4}{S/\sqrt{n}} - 1.18\right| > 2.13 \mid \mu = 1.4\right) \\ &= P\left(\frac{\bar{X} - 1.4}{S/\sqrt{n}} > 3.31 \mid \mu = 1.4\right) + P\left(\frac{\bar{X} - 1.4}{S/\sqrt{n}} < -0.95 \mid \mu = 1.4\right) \\ &= P(T_{15} < -0.95) + P(T_{15} > 3.31) = 0.179 + 0.002 = 0.181 \end{aligned}$$

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- This power is low. This means that if the true population mean really was  $\mu = 1.4$ , we would be very unlikely to reject the null based on a sample of size 16. **How can we increase the power?**

- We can control the probability of Type I and Type II error simultaneously provided we can collect the requisite number of samples. **What is the required sample size to achieve some power?**
- It can be shown that if  $\sigma$  is known, the sample size  $n$  required to achieve power  $1 - \beta$  for a test of  $H_0 : \mu = \mu_0$  vs.  $H_A : \mu \neq \mu_0$  at a given alternative  $\mu = \mu_A$  at level  $\alpha$  is approximately:

$$n = \left( \frac{\sigma(z_{\alpha/2} + z_{\beta})}{\mu_0 - \mu_A} \right)^2$$

where  $z_{\alpha/2}$  and  $z_{\beta}$  are the  $\alpha/2$  and  $\beta$  right-tailed critical values of the standard normal distribution.



- Now suppose we wanted to determine the sample size to have power 0.8 to reject when the true mean was  $\mu_A = 1.4$ .
- Thus  $\beta$  is  $1-0.8 = 0.2$ . Use  $S = 0.3385$  as an estimator of  $\sigma$ . Using the standard normal table we have  $z_{0.05/2} = 1.96$  and  $z_{0.2} = 0.84$ . By formula we would need:

$$n = \left( \frac{0.3385(1.96+0.84)}{1.5-1.4} \right)^2 = 89.8, \text{ round up to } 90.$$



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- **Example:** The sanitation department in a large city is considering separating recyclable material out of the trash to save on landfill space and make money selling the recyclables. Based on data from other cities, it is determined that if at least half of the households in the city have 4.6 lbs or more of reclaimable recyclable material, then the separation will be profitable for the city. A random sample of 11 households yields the following data on pounds of recyclable material found in the trash:

14.2, 5.3, 2.9, 4.2, 1.2, 4.3, 1.1, 2.6, 6.7, 7.8, 25.9

Note that the median of the sample is 4.3 lbs. We are interested in whether the median of the whole city is 4.6lbs.



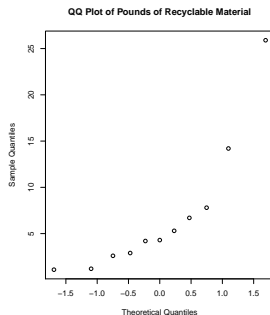
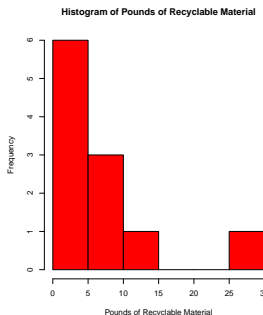
- Let  $m$  be the population test, we want to test:  $H_0 : m = 4.6$  vs.  $H_A : m > 4.6$ .



# Example continued



- Let  $m$  be the population test, we want to test:  $H_0 : m = 4.6$  vs.  $H_A : m > 4.6$ .
- A QQ plot and histogram:



The histogram looks neither normal nor symmetric, and the QQ plot also does not support normality.

- Since we only have  $n = 11$ , the CLT is risky.

- If the null hypothesis is correct, then by the definition of **median**, the sample should have about half of the observations greater than 4.6 and half less than 4.6.
- Thus a natural choice of test statistic is the number of observations in the sample that are greater than 4.6, which we might call  $B$ .
- If the null hypothesis is correct,  $B \sim \text{Bin}(11, 0.5)$ .
- Our observed test statistic is  $B_{\text{obs}} = 5$ . Hence, the p-value is  $P(B \geq 5) = 0.726$ . This was found by adding Binomial(11, 0.5) probabilities for  $B = 5, B = 6, \dots, B = 11$  when  $n = 11$  and  $p = 0.5$ .
- Set  $\alpha = 0.05$ , then there is sufficient evidence to reject the null. There is not strong evidence that separation of the recyclables would be profitable in this city.

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- **Hypothesis.** We wish to test  $H_0 : m = m_0$  vs.  $H_A$ . Possible alternative hypotheses:  $H_A : m > m_0$  (one-tailed),  $H_A : m < m_0$  (one-tailed), or  $H_A : m \neq m_0$  (two-tailed).

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- **Test statistic.**  $B$  = number of data values greater than  $m_0$ . (Ignore values tied with  $m_0$ .) Note that if  $H_0$  is true,  $B \sim \text{Binomial}(n^*, 0.5)$ , where  $n^*$  is the number of data points not equal to  $m_0$ .

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- **p-value.** Let  $b$  be the observation of  $B$ . If:
  - $H_A : m > m_0$ : p-value is  $P(B \geq b) = P(B = b) + P(B = b + 1) + \dots + P(B = n^*)$ .
  - $H_A : m < m_0$ : p-value is  $P(B \leq b) = P(B = b) + P(B = b - 1) + \dots + P(B = 1) + P(B = 0)$ .
  - $H_A : m \neq m_0$ : p-value is  $2 \min\{P(B \geq b), P(B \leq b)\}$ .

# What's the next?



We'll discuss how to test population proportions in the next lecture.