Chapter 5: Estimation

Ott & Longnecker Sections: 4.12, 4.14 and 5.2

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Part 2

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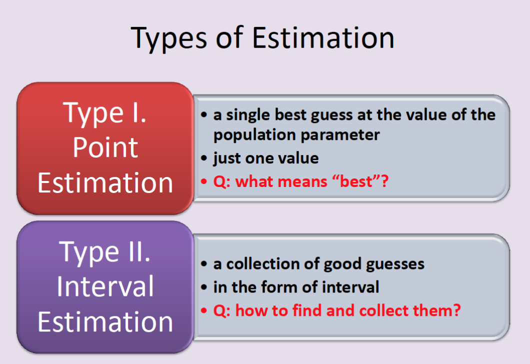
What do we study?

Key Concepts: estimator, statistic, estimate, bias, unbiased estimator, standard error, mean squared error

Types of estimation



We'll focus on point estimation in this lecture.

Outline

1. Sample mean
2. Estimator and estimate
3. Bias
4. Standard error
5. MSE
6. Sample variance and sample standard deviation
7. Estimated standard error

Example

A car manufacturer uses an automatic device to apply paint to engine blocks. Since engine blocks get very hot, the paint must be heat-resistant, and it is important that the amount applied is of a certain minimum thickness. A warehouse contains thousands of blocks painted by the automatic device. The manufacturer wants to know the average amount of paint applied by the device, so 16 blocks are selected at random, and the paint thickness is measured in mm. The results are below: 1.29, 1.12, 0.88, 1.65, 1.48, 1.59, 1.04, 0.83, 1.76, 1.31, 0.88, 1.71, 1.83, 1.09, 1.62, 1.49

Example

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How might we go about using this **sample** to make **inferences** about the **population mean** paint thickness of the entire population of blocks, which we call μ?

Sample mean

We think of each of the numbers listed above as the realization of an RV. In particular, they are the realizations of 16 iid RVs, that we might call X_1, X_2, ..., X_{16}. For now, we don't care what the exact distribution of these RVs is, but let's call the expectation $E(X_i) = \mu$, and variance $VAR(X_i) = \sigma^2$. Note that since these RVs are iid, the expectations and variances are the same for every one.

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Our intuition serves us well here, the sample mean of these observations, which we define below, will be a good estimate of the population mean μ:

$$\text{Sample mean: } \hat{\mu} = \bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

Outline

Estimator and estimate

The formula that describes how data from a sample would be used to compute a guess about a population parameter is called an **estimatOR**, or a **statistic**. The sample mean formula given above is an example of an estimator. The numerical value computed using the estimator once the data is collected is called an **estimATE**. The sample mean of the 16 data points given above, which we compute shortly, is an example of an estimate. **An estimator is a RV, and an estimate is a realization of that RV**.

Estimator and estimate

It is **very important** to understand that the sample mean, being an estimator, is itself an RV - it is an RV constructed as a function of other RVs. In order to get one realization of the RV that represents the sample mean, we would randomly sample n elements from the population in an iid fashion, and calculate their mean.

A question

Why is sample mean a good estimator to the population mean μ?

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Before answering this question, let's first define some useful concepts.

Outline

Bias

The **bias** in an estimator $\hat{\theta}$ is defined as:

$$bias(\hat{\theta}) = E(\hat{\theta}) - \theta.$$

If the bias is equal to zero, the estimator $\hat{\theta}$ is called **unbiased** for θ. All other things being equal, smaller bias is better.

Outline

Standard error

The variance of an estimator $\hat{\theta}$ is defined as $VAR(\hat{\theta})$. All other things being equal, smaller variance is better. The square root of the variance is usually called the standard deviation or SD. However, when we are talking about estimating a parameter, we instead use the term **standard error** or **SE**, to remind us that this is the amount of error in estimation. Thus the square root of the variance of an estimator will be denoted $SE(\hat{\theta})$. Standard error provides a measure of accuracy.

Outline

Mean squared error

The **mean squared error**, or **MSE**, of an estimator $\hat{\theta}$ can be calculated as:

$$MSE(\hat{\theta}) = VAR(\hat{\theta}) + \left(bias(\hat{\theta})\right)^2.$$

All other things being equal, smaller MSE is better. Note how MSE incorporates information about both bias and variance.

Sample mean

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$$E(\bar{X}) = E(\frac{X_1+X_2+...+X_n}{n}) = \frac{\mu+\mu+...+\mu}{n} = \mu.$$
$$VAR(\bar{X}) = VAR(\frac{X_1+X_2+...+X_n}{n}) = \frac{\sigma^2+\sigma^2+...+\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$
$$SE(\bar{X}) = \sqrt{VAR(\bar{X})} = \frac{\sigma}{\sqrt{n}}.$$

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What do you find?

- We can see that $\bar{X}$ is unbiased for μ, since $E(\bar{X}) = \mu$. Among the unbiased estimators of μ, the sample mean is proven to be the **best**.
- As n increases, the SE decreases. This is intuitive, since as we take a larger sample, we should do a better job of estimating.

Example

Examples ($n = 3$)

iid RVs in the Sample	• X_1, X_2, X_3 with $E(X_i) = \mu$, $SD(X_i) = \sigma$
Goal	• estimate μ
Estimator 1 (unbiased): $\hat{\mu}_1 = \bar{X} = \frac{1}{3}(X_1 + X_2 + X_3)$	• $bias(\bar{X}) = \mu - \mu = 0$; $VAR(\bar{X}) = \frac{\sigma^2}{3}$ • $MSE(\bar{X}) = 0^2 + \frac{\sigma^2}{3} = \frac{\sigma^2}{3}$
Estimator 2 (unbiased): $\hat{\mu}_2 = \frac{1}{4}X_1 + \frac{1}{2}X_2 + \frac{1}{4}X_3$	• $bias(\hat{\mu}_2) = E(\frac{1}{4}X_1 + \frac{1}{2}X_2 + \frac{1}{4}X_3) - \mu = \mu - \mu = 0$ • $VAR(\hat{\mu}_2) = VAR(\frac{1}{4}X_1 + \frac{1}{2}X_2 + \frac{1}{4}X_3) = (\frac{1}{16} + \frac{1}{4} + \frac{1}{16})\ \sigma^2 = \frac{3\sigma^2}{8}$ • $MSE(\bar{X}) = 0^2 + \frac{3\sigma^2}{8} = \frac{3\sigma^2}{8} > MSE(\bar{X})$
Estimator 3 (biased): $\hat{\mu}_3 = X_1 + \frac{1}{2}X_2 - X_3$	• $bias(\hat{\mu}_3) = E(X_1 + \frac{1}{2}X_2 - X_3) - \mu = \frac{1}{2}\mu - \mu = -\frac{1}{2}\mu$ • $VAR(\hat{\mu}_3) = VAR(X_1 + \frac{1}{2}X_2 - X_3) = (1 + \frac{1}{4} + 1)\ \sigma^2 = \frac{9\sigma^2}{4}$ • $MSE(\bar{X}) = (-\frac{1}{2}\mu)^2 + \frac{9\sigma^2}{4} = \frac{\mu^2}{4} + \frac{9\sigma^2}{4}$

Outline

Estimate of population variance and standard deviation

- Sample variance of X: $\hat{\sigma}^2 = S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$
  Note that this, like the sample mean, is an estimator, and thus an RV. In this case S^2 is an estimator of σ^2. The reason we use $n-1$ in the denominator is that this makes the estimator S^2 unbiased for σ^2.
- Sample standard deviation of X: $\hat{\sigma} = S = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}}$
  In this case, S is an estimator of σ, but it's **biased**.

Outline

Estimate standard error of $\bar{X}$

$$SE(\bar{X}) = \frac{\sigma}{n}$$

Unfortunately, in most cases we don't know the value of σ^2, and therefore, we need to estimate the standard error of $\bar{X}$.

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We plug in the estimate of σ into the formula for the standard error, we find:

$$\text{Estimated standard error of } \bar{X}\text{: } \widehat{SE(\bar{X})} = \frac{\hat{\sigma}}{\sqrt{n}} = \frac{S}{\sqrt{n}}.$$

For the paint data, we find $S = 0.3385$ mm, so $\widehat{SE(\bar{X})} = \frac{0.3385}{\sqrt{16}} = 0.085$ mm.

An overview

Note the difference between the terms standard deviation and standard error. The **standard deviation** is a property of the distribution of the X_i, whereas the **standard error** of $\bar{X}$ is a property of the estimator $\bar{X}$. Also, note that the **estimated standard error** is an estimator of the standard error.

Trust me, I know how confusing this is!!

What's the next?

In the next lecture, we'll discuss normality, the central limit theorem and confidence intervals.