Chapter 4: Random Variables and Distributions

Ott & Longnecker Sections: 4.6-4.10

Duzhe Wang

the Department of Statistics, UW-Madison

Part 3

What do we study?

We'll continue RV.

What do we study?

Key Concepts: normal distribution, standard normal distribution, z critical values, standardization, reverse standardization, Z table.

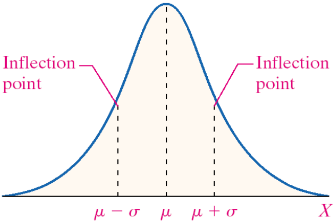
Outline

Continuous RV

We now discuss continous RVs. Recall that continuous RVs take values in specified ranges, and their probability distributions are called **probability density functions**, or pdfs, which are denoted $f(x)$. The area under the curve described by the pdf between any two possible realizations of the RV determines the probability that the RV will realize to a value in that range.

Outline

The bell curve



It's a special curve, called a **normal, Gaussian,** or **bell** curve. If the pdf of X is a bell curve, then we say X has the normal distribution. Lots of biological (and other) random variables are normal, for example, body weight, crop yield, protein content in soybean, density of blood components.

Outline

Normal distribution

If X is a normal RV, then the pdf is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- $\mu = E(X), \sigma^2 = VAR(X)$. We often denote a normal RV X with mean μ and variance σ^2 by $X \sim N(\mu, \sigma^2)$.
- Don't confuse this π, which is the fundamental math constant that is approximately 3.14, with the π that represents the chance of success in a Bernoulli trial!

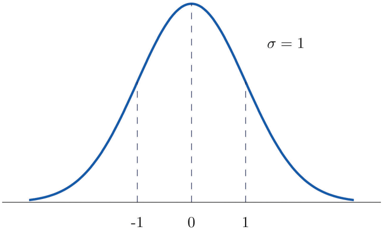
You can play with the bell curve at
https://dzwang.shinyapps.io/thebellcurve/

Properties of normal distribution

1. A normal RV can theoretically realize to any value between $-\infty$ and ∞.
2. The normal distribution is symmetric around the mean, μ.
3. The inflection points (points where the curve moves from concave downward to concave upward) are at $\mu \pm \sigma$.
4. The total area under the curve is 1.
5. The area under the curve between $\mu - \sigma$ and $\mu + \sigma$ is about 0.68; the area under the curve between $\mu - 2\sigma$ and $\mu + 2\sigma$ is about 0.95. Very little of the area is farther than three sds from the mean (about 0.003).

Outline

The standard normal distribution



We call the $N(0,1)$ distribution the **standard normal distribution** and we usually reserve Z to denote a standard normal RV.

The standard normal distribution

- $P(Z \leq 0) = ?$
- $P(Z = 0) = ?$
- $P(Z < 1) = ?$
- $P(0 \leq Z \leq 1) = ?$
- $P(-1 \leq Z \leq 1) = ?$
- $P(Z > 1.5) = ?$

The standard normal distribution

- $P(Z \leq 0) = ?$
- $P(Z = 0) = ?$
- $P(Z < 1) = ?$
- $P(0 \leq Z \leq 1) = ?$
- $P(-1 \leq Z \leq 1) = ?$
- $P(Z > 1.5) = ?$

$$P(Z \leq 0) = \frac{1}{2} \text{ (by symmetry)}$$

The standard normal distribution

- $P(Z \leq 0) =?$
- $P(Z = 0) =?$
- $P(Z < 1) =?$
- $P(0 \leq Z \leq 1) =?$
- $P(-1 \leq Z \leq 1) =?$
- $P(Z > 1.5) =?$

$$P(Z \leq 0) = \frac{1}{2} \text{ (by symmetry)}$$

$$P(Z = 0) = 0 \text{ (because Z is continuous)}$$

The standard normal distribution

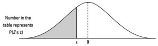
- $P(Z \leq 0) =?$
- $P(Z = 0) =?$
- $P(Z < 1) =?$
- $P(0 \leq Z \leq 1) =?$
- $P(-1 \leq Z \leq 1) =?$
- $P(Z > 1.5) =?$

$$P(Z \leq 0) = \tfrac{1}{2} \text{ (by symmetry)}$$

$$P(Z = 0) = 0 \text{ (because Z is continuous)}$$

But how do we calculate others?

Two tools



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
–3.6	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
–3.5	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
–3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
–3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
–3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
–3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
–3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
–2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
–2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
–2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
–2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
–2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
–2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
–2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
–2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
–2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
–2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
–1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
–1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
–1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
–1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
–1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
–1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
–1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
–1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
–1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
–1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
–0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
–0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
–0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
–0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
–0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
–0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
–0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
–0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
–0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
–0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Outline

Z table

Standard Normal Probabilities

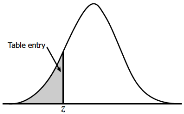


Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148

Z table

Standard Normal Probabilities

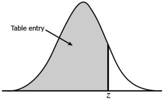


Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441

$$P(Z < 1) = ?$$

Z table

Standard Normal Probabilities

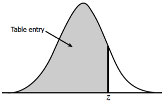


Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441

$$P(Z < 1) = ?$$

0.8413

Z table

$$P(0 \leq Z \leq 1) = ?$$

Z table

$$P(0 \leq Z \leq 1) = ?$$

$$P(0 \leq Z \leq 1) = P(Z \leq 1) - P(Z \leq 0) = 0.8413 - 0.5 = 0.3413$$

Z table

$$P(0 \leq Z \leq 1) = ?$$

$$P(0 \leq Z \leq 1) = P(Z \leq 1) - P(Z \leq 0) = 0.8413 - 0.5 = 0.3413$$

$$P(-1 \leq Z \leq 1) = ?$$

Z table

$$P(0 \leq Z \leq 1) = ?$$

$$P(0 \leq Z \leq 1) = P(Z \leq 1) - P(Z \leq 0) = 0.8413 - 0.5 = 0.3413$$

$$P(-1 \leq Z \leq 1) = ?$$

$$P(-1 \leq Z \leq 1) = P(Z \leq 1) - P(Z \leq -1) = 0.8413 - 0.1587 = 0.6826$$

Z table

$$P(0 \leq Z \leq 1) =?$$

$$P(0 \leq Z \leq 1) = P(Z \leq 1) - P(Z \leq 0) = 0.8413 - 0.5 = 0.3413$$

$$P(-1 \leq Z \leq 1) =?$$

$$P(-1 \leq Z \leq 1) = P(Z \leq 1) - P(Z \leq -1) = 0.8413 - 0.1587 = 0.6826$$

$$P(Z > 1.5) =?$$

Z table

$$P(0 \leq Z \leq 1) = ?$$

$$P(0 \leq Z \leq 1) = P(Z \leq 1) - P(Z \leq 0) = 0.8413 - 0.5 = 0.3413$$

$$P(-1 \leq Z \leq 1) = ?$$

$$P(-1 \leq Z \leq 1) = P(Z \leq 1) - P(Z \leq -1) = 0.8413 - 0.1587 = 0.6826$$

$$P(Z > 1.5) = ?$$

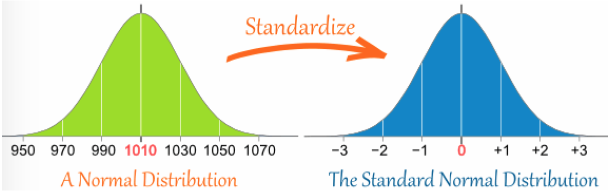
$$P(Z > 1.5) = 1 - P(Z \leq 1.5) = 1 - 0.9332 = 0.0668$$

But what if...

Example. $Y \sim N(3, 0.25^2)$, how do we calculate $P(Y \leq 2.8)$?

But what if...

Example. $Y \sim N(3, 0.25^2)$, how do we calculate $P(Y \leq 2.8)$?



Outline

Standardization

$$\text{If } X \sim N(\mu, \sigma^2), \text{ then } Z = \frac{X-\mu}{\sigma} \sim N(0,1).$$

Note: this is a very **important** technique, we'll use this fact a lot of times in this course.

Standardization

Example. $Y \sim N(3, 0.25^2)$, how do we calculate $P(Y \leq 2.8)$?

Standardization

Example. $Y \sim N(3, 0.25^2)$, how do we calculate $P(Y \leq 2.8)$?

$$P(Y \leq 2.8) = P(\frac{Y-3}{0.25} \leq \frac{2.8-3}{0.25}),$$

then we have

$$P(Y \leq 2.8) = P(Z \leq -0.8).$$

We can use the Z table to compute $P(Z \leq -0.8)$, and we get 0.21.

But what if...

Example. $Y \sim N(3, 0.25^2)$, how do we **find y** such that

$$P(Y \geq y) = 0.25$$

Outline

Reverse standardization

$$\text{If } Z \sim N(0,1), \text{ then } X = Z\sigma + \mu \sim N(\mu, \sigma^2).$$

Note: this is also a very **important** technique.

z critical value

Let $Z \sim N(0,1)$, α is given, then the value z such that $P(Z \geq z) = \alpha$ is called z_α. We call z_α a *z critical value*. It can be thought of as the $1-\alpha$ quantile of the standard normal distribution (recall the definition of quantile from the section on descriptive statistics!).

Example

Example. $Y \sim N(3, 0.25^2)$, how do we **find y** such that

$$P(Y \geq y) = 0.25$$

Example

Example. $Y \sim N(3, 0.25^2)$, how do we **find y** such that

$$P(Y \geq y) = 0.25$$

step 1:

$$\text{Find } z \text{ such that } P(Z \geq z) = 0.25,$$

Use Z table, we find the 0.25 critical value of the standard normal distribution to be $z_{0.25} = 0.67$.

Example

Example. $Y \sim N(3, 0.25^2)$, how do we **find y** such that

$$P(Y \geq y) = 0.25$$

step 1:

$$\text{Find } z \text{ such that } P(Z \geq z) = 0.25,$$

Use Z table, we find the 0.25 critical value of the standard normal distribution to be $z_{0.25} = 0.67$.

step 2:

Use reverse standardization to find the appropriate value of y

Reverse standardizing gives $0.25 * 0.67 + 3 = 3.17$.

A reminder

You need to know how to use Z table to calculate probability and find the z critical value, particularly for exams. It's very **important**, so feel free to do many different examples until you feel confident and comfortable.

Doing calculations with R

```
> pnorm(1)
[1] 0.8413447
> pnorm(2)-pnorm(-2)
[1] 0.9544997
> pnorm(3)-pnorm(-3)
[1] 0.9973002
> 1- pnorm(137, mean=112, sd=10)
[1] 0.006209665
> qnorm(.95, mean=112, sd=10)
[1] 128.4485
> pbinom(1, size=6, prob=1/6)
[1] 0.7367755
> dbinom(0:6, size=6, prob=1/6)
[1] 0.335 0.402 0.201 0.054 0.008 0.001 0.000
```

`norm`	normal distribution
`binom`	binomial
`p`	probability: $\mathbb{P}\{Y \leq \ldots\}$
`q`	quantile
`d`	density, or probability mass function: $\mathbb{P}\{Y = \ldots\}$

What's the next?

In the next lecture, we'll discuss the distributions of functions of RVs and concepts of estimation.