But what are Pólya Vector Fields?

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Visualizing complex functions

Throughout our course, we've focused on one way of visualizing complex functions, namely as a **mapping of points in one complex plane to points in another**. We used two planes, the **pre-image** and the **image** to describe this mapping.

Visualizing complex functions

Our goal is to introduce a completely new way of visualizing complex functions, specifically focusing on how they can be used to **visualize complex integration**.

We hope to produce new insight into how to think about complex functions, and will end up finding that there are many surprising connections with physics!

The vector field interpretation

Let's visualize a complex function $f(z)$ in a single plane.

The input value z will be a point in our complex plane, but instead, *the value of $f(z)$ is pictured as a vector emanating from z.*

We now have a **vector field** of f!

The vector field interpretation

The vector field representation remedies a shortcoming in the pre-image and image complex mapping. We are able to get a better feel for the overall behavior of a complex function as we inspect its vector field.

Aside: flux and work

Suppose we have a complex function $f(z)$ with vector field $\mathbf{f}$, along with a curve C in the complex plane.

The work of $\mathbf{f}$ along C is the total magnitude of the field tangent to the curve, while the flux of $\mathbf{f}$ through C is the total magnitude of the field normal to the curve.

Complex integration and vector fields: a problem?

Consider the integral $\int_C f(z)dz$:

- $f(z) = |f(z)|e^{i\beta}$, where β is the angle $\mathbf{f}$ makes with the positive real axis at z.
- $dz = e^{i\alpha}ds$, where α is the angle $\mathbf{dz}$ makes with the positive real axis and ds is the differential length along C.

Thus this integral can be expressed as $\int_C |f(z)|e^{i(\alpha+\beta)}\, ds$.

Complex integration and vector fields: a problem?

Integrating this term involves the addition of angles, which is not easy to visualize. It is often easier to consider the difference between two angles $\theta = \alpha - \beta$, as this angle is used to calculate helpful quantities.

For example, the work done by $\mathbf{f}$ along a length dz of a curve at z is $\mathbf{f}(z) \cdot \mathbf{dz} = |f(z)||dz| \cos\theta$, where θ is the difference between the angles $\mathbf{f}(z)$ and $\mathbf{dz}$ make with the positive real axis. Similarly, the flux of $\mathbf{f}$ through this length at z is $\mathbf{f}(z) \times \mathbf{dz} = |f(z)||dz| \sin\theta$.

Pólya vector fields

Instead of drawing $\mathbf{f}(z)$ at z, what if we draw $\overline{\mathbf{f}(z)}$?

More formally, for a function $f(z) = u(x + iy) + iv(x + iy)$, we define the Pólya vector field to be:

$$\overline{\mathbf{f}(x, y)} = (u(x, y), -v(x, y)) \tag{1}$$

Re-expressing the integrand

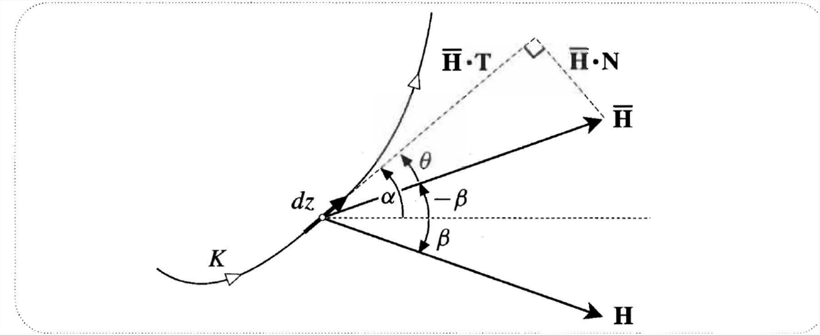
First, define $\overline{f} = |\overline{f}|e^{-i\beta}$.

We can then define:

$$
\begin{aligned}
f dz &= \left(|f|e^{i\beta}\right)\left(e^{i\alpha} ds\right) \\
&= |f|e^{i(\alpha+\beta)} ds \\
&= |\overline{f}|e^{i(\alpha-(-\beta))} ds \\
&= |\overline{f}|e^{i\theta} ds
\end{aligned}
$$

where θ is the angle between dz and $\overline{f}$.

Pólya vector fields



Defining our complex integral

Given that $f dz = |\overline{f}|e^{i\theta} ds$, then

$$\begin{aligned}\int_C f(z)dz &= \int_C |\overline{f}|e^{i\theta} ds \\ &= \int_C |\overline{f}| \left[\cos\theta + i\sin\theta\right] ds \\ &= \int_C \overline{\boldsymbol{f}} \cdot \boldsymbol{T} ds + i\int_C \overline{\boldsymbol{f}} \cdot \boldsymbol{N} ds\end{aligned}$$

Where $\boldsymbol{T}$ is a unit tangent vector in the direction of the path and $\boldsymbol{N}$ is unit normal vector in the direction of our path.

Defining our complex integral

More intuitively, we notice that the real and imaginary parts of each term in the integral are the **work and flux of the Pólya vector field for the corresponding element of the contour**.

Our integral can now be expressed as:

$$\int_C f(z)dz = \text{Work}\left[\overline{\boldsymbol{f}}, C\right] + i \cdot \text{Flux}\left[\overline{\boldsymbol{f}}, C\right] \tag{2}$$

Visualizing a complex integral

Given this definition, we can examine a certain contour along a Pólya vector field and quickly get a feel for the value of the integral by **looking at how much the field flows *along* and *across* the contour**!!

Pólya vector fields of analytic functions

We've now defined a visual intuition for complex integration.
But let's pose another question:

Given a Pólya vector field, how can we tell whether or not f is analytic?

Aside: curl and divergence

Suppose we have a vector field $\mathbf{f}$.

The curl of $\mathbf{f}$ ($\nabla \times \mathbf{f}$) is a measure of the tendency of $\mathbf{f}$ to rotate.

The divergence of $\mathbf{f}$ ($\nabla \cdot \mathbf{f}$) is a measure of how much of $\mathbf{f}$ flows in or out (whether the field is a sink or a source).

Theorem

Let $f(z)$ be a complex mapping. Then the Pólya vector field of $f(z)$ is divergence-less (sourceless) and curl-less (irrotational) if and only if $f(z)$ is an analytic function.

Proof of analytic Pólya theorem

Proof: Express the complex function $f(z) = u(x, y) + iv(x, y)$, as a vector in the complex plane:

$$\mathbf{f}(z) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$$

The Pólya vector field of $f(z)$ then has vector form

$$\overline{\mathbf{f}}(z) = \begin{pmatrix} u(x, y) \\ -v(x, y) \end{pmatrix}$$

Finally, the gradient operator can be expressed as a vector with partial derivative operations as entries:

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

Proof of analytic Pólya theorem (pt. 2)

The divergence of $\overline{\mathbf{f}}(z)$ is then

$$\begin{aligned}\nabla \cdot \overline{\mathbf{f}}(z) &= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} u(x,y) \\ -v(x,y) \end{pmatrix} \\ &= \frac{\partial u}{\partial x} + \frac{\partial(-v)}{\partial y} \\ &= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\end{aligned}$$

while its curl is

$$\begin{aligned}\nabla \times \overline{\mathbf{f}}(z) &= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \times \begin{pmatrix} u(x,y) \\ -v(x,y) \end{pmatrix} \\ &= \frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y} \\ &= -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\end{aligned}$$

Proof of analytic Pólya theorem (pt. 3)

If $\overline{\mathbf{f}}(z)$ has zero divergence and zero curl, then the previous conditions become

$$\begin{aligned} \nabla \cdot \overline{\mathbf{f}}(z) = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= 0 \Rightarrow \\ \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \nabla \times \overline{\mathbf{f}}(z) = -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 0 \Rightarrow \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

These are exactly the Cauchy-Riemann equations, so they are satisfied when $f(z)$ is holomorphic. Thus, because all holomorphic functions are analytic, $\overline{\mathbf{f}}(z)$ is divergence-less and curl-less if and only if $f(z)$ is analytic.

Cauchy's Theorem: visualized

If $f(z)$ is analytic everywhere inside a simple loop K bounding a region R, its Pólya vector field in R will have (as a flow) **zero flux density** and (as a force field) **zero work density**.

Intuitively, this means that there is no net flux of "fluid" out of R, and that a puck fired round K returns with its kinetic energy unchanged.

The integral of f around K must also vanish.

z is holomoprhic, while $\overline{z}$ is not

Using the logic above, the functions $f_1(z) = z = x + iy$ and $f_2(z) = \overline{z} = x - iy$ have Pólya vector fields

$$\overline{\mathbf{f_1}}(z) = \begin{pmatrix} x \\ -y \end{pmatrix}, \quad \overline{\mathbf{f_2}}(z) = \begin{pmatrix} x \\ y \end{pmatrix}$$

The divergence of both fields is therefore

$$\begin{aligned} \nabla \times \overline{\mathbf{f_1}}(z) &= \frac{\partial(-y)}{\partial x} - \frac{\partial x}{\partial y} \\ &= 0 - 0 = 0 \\ \nabla \times \overline{\mathbf{f_2}}(z) &= \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \\ &= 0 + 0 = 0 \end{aligned}$$

z is holomoprhic, while $\overline{z}$ is not

However, the curl of both fields is

$$\begin{aligned} \nabla \cdot \overline{\mathbf{f_1}}(z) &= \frac{\partial x}{\partial x} + \frac{\partial(-y)}{\partial y} \\ &= 1 - 1 = 0 \\ \nabla \cdot \overline{\mathbf{f_2}}(z) &= \frac{\partial x}{\partial x} + \frac{\partial(y)}{\partial y} \\ &= 1 + 1 = 2 \end{aligned}$$

Therefore the Pólya vector field of $f_1(z) = z$ is divergence-less and curl-less, which makes $f(z) = z$ an analytic, and therefore harmonic, function. On the other hand, the Pólya vector field of $f_2(z) = \overline{z}$ has a non-zero curl, so $f(z) = \overline{z}$ is not analytic, and is therefore not harmonic.