题 **1**

补充作业：用Floquet理论证明图4. 6. 3中的阴影区域是零解的稳定区域

解

考虑具有阻尼的 Mathieu 方程

$$\ddot{u} + 2\zeta\dot{u} + (\delta + 2\varepsilon\cos 2t)u = 0, |\varepsilon| \ll 1, 0 \le \zeta \ll 1 \tag{1}$$

确定稳定性边界的条件

由 Liouville 公式可得

$$\begin{aligned}\det B &= e^{-\int_0^T p_1(t)\,\mathrm{d}t} \\ &= e^{-\int_0^T 2\zeta\,\mathrm{d}t} \\ &= e^{-2\pi\zeta}\end{aligned} \tag{2}$$

可得特征方程为

$$\lambda^2 - (\operatorname{tr} B)\lambda + e^{-2\pi\zeta} = 0 \tag{3}$$

解得

$$\lambda_{1,2} = \frac{\operatorname{tr} B}{2} \pm \sqrt{\left(\frac{\operatorname{tr} B}{2}\right)^2 - e^{-2\pi\zeta}} \tag{4}$$

由 Floquet 定理推论, 方程零件稳定的临界条件为三者之一:

$$\begin{gathered}|\lambda_1| = 1, |\lambda_2| < 1 \\ |\lambda_1| < 1, |\lambda_2| = 1 \\ |\lambda_1| = |\lambda_2| = 1\end{gathered} \tag{5}$$

注意到

$$\lambda_1\lambda_2 = e^{-2\pi\zeta} < 1 \tag{6}$$

所以条件 3 不成立

而根据前两个条件知, 根不可能为复数根, 因为这是实数方程, 复数根必然共轭, 模长相同

由此可得

- 情况 1:

$$\lambda_1 = \frac{\operatorname{tr} B}{2} + \sqrt{\left(\frac{\operatorname{tr} B}{2}\right)^2 - e^{-2\pi\zeta}} = 1, \lambda_2 = \frac{\operatorname{tr} B}{2} - \sqrt{\left(\frac{\operatorname{tr} B}{2}\right)^2 - e^{-2\pi\zeta}} < 1 \tag{7}$$

此时

$$\operatorname{tr} B = 1 + e^{-2\pi\zeta} \tag{8}$$

方程有一个以π为周期的正规解, 另一个正规解渐近稳定. 受扰零解趋于以π为周期的正规解

- 情况 2:

$$\lambda_1 = \frac{\operatorname{tr} B}{2} + \sqrt{\left(\frac{\operatorname{tr} B}{2}\right)^2 - e^{-2\pi\zeta}} > -1, \lambda_2 = \frac{\operatorname{tr} B}{2} - \sqrt{\left(\frac{\operatorname{tr} B}{2}\right)^2 - e^{-2\pi\zeta}} = -1 \tag{9}$$

此时

$$\operatorname{tr} B = -\left(1 + e^{-2\pi\zeta}\right) \tag{10}$$

方程有一个以2π为周期的正规解, 另一个正规解渐近稳定. 受扰零解趋于以2π为周期的正规解

稳定性边界的确定
设阻尼为

$$\zeta = \varepsilon\mu \tag{11}$$

由摄动法

$$\begin{cases} u(t) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \cdots \\ \delta(t) = \delta_0 + \varepsilon\delta_1 + \varepsilon^2\delta_2 + \cdots \end{cases} \tag{12}$$

代入方程得

$$\ddot{u}_0 + \delta_0 u_0 = 0 \tag{13}$$

$$\ddot{u}_1 + \delta_0 u_1 = -(\delta_1 + 2\cos 2t)u_0 - 2\mu\dot{u}_0 \tag{14}$$

$$\ddot{u}_2 + \delta_0 u_2 = -\delta_2 u_0 - (\delta_1 + 2\cos 2t)u_1 - 2\mu\dot{u}_1 \tag{15}$$

解得 (13) 为

$$u_0 = a\cos\sqrt{\delta_0}t + b\sin\sqrt{\delta_0}t \tag{16}$$

其中

$$\delta_0 = n^2, n = 0, 1, 2, \cdots \tag{17}$$

$\delta_0 = 0$
则(16) 为

$$u_0 = a = \text{const} \tag{18}$$

代入(14) 得

$$\ddot{u}_1 = -a(\delta_1 + 2\cos 2t) \tag{19}$$

消除永年项

$$\delta_1 = 0 \tag{20}$$

解得

$$u_1 = \frac{a}{2}\cos 2t \tag{21}$$

代入 (15)

$$\ddot{u}_2 = -a\left(\delta_2 + \frac{1}{2}\right) + 2\mu a \sin 2t - \frac{a}{2}\cos 4t \tag{22}$$

消除永年项

$$\delta_2 = -\frac{1}{2} \tag{23}$$

解得

$$u_2 = -\frac{\mu a}{2}\sin 2t + \frac{a}{32}\cos 4t \tag{24}$$

代入 (12), 可得$\delta_0 = 0$附近的稳定边界

$$\delta = -\frac{\varepsilon^2}{2} + O(\varepsilon^2) \tag{25}$$

$\delta_0 = 1$
则(16) 为

$$u_0 = a\cos t + b\sin t \tag{26}$$

代入(14) 得

$$\ddot{u}_1 + u_1 = -[(\delta_1 + 1)a + 2\mu b]\cos t + [2\mu a - (\delta_1 - 1)b]\sin t - a\cos 3t - b\sin 3t \tag{27}$$

消除永年项

$$\begin{cases} (\delta_1 + 1)a + 2\mu b = 0 \\ 2\mu a - (\delta_1 - 1)b = 0 \end{cases} \tag{28}$$

即

$$\delta_1 = \pm\sqrt{1 - 4\mu^2} \quad |\mu| < \frac{1}{2} \tag{29}$$

解得

$$u_1 = \frac{1}{8}(a\cos 3t + b\sin 3t) \tag{30}$$

代入 (15)

$$\begin{aligned} \ddot{u}_2 + u_2 = -\left(\delta_2 + \frac{1}{8}\right)(a\cos t + b\sin t) - \left(\frac{\delta_1 a}{8} + \frac{3\mu b}{4}\right)\cos 3t \\ + \left(\frac{3\mu a}{4} - \frac{\delta_1 b}{8}\right)\sin 3t - \frac{a}{8}\cos 5t - \frac{b}{8}\sin 5t \end{aligned} \tag{31}$$

消除永年项

$$\delta_2 = -\frac{1}{8} \tag{32}$$

解得

$$u_2 = \frac{1}{64}[(\delta_1 a + 6\mu a)\cos 3t + (\delta_1 b - 6\mu a)\sin 3t] + \frac{1}{192}(a\cos 5t + b\sin 5t) \tag{33}$$

代入 (12), 可得$\delta_0 = 1$附近的稳定边界

$$\delta = 1 \pm \sqrt{\varepsilon^2 - 4\zeta^2} - \frac{1}{8}\varepsilon^2 + O(\varepsilon^3) \tag{34}$$

$\delta_0 = 4$
则(16) 为

$$u_0 = a\cos 2t + b\sin 2t \tag{35}$$

代入(14) 得

$$\ddot{u}_1 + u_1 = -a - (\delta_1 a + 4\mu b)\cos 2t + (4\mu a - \delta_1 b)\sin 2t - a\cos 4t - b\sin 4t \tag{36}$$

消除永年项

$$\begin{cases} \delta_1 a + 4\mu b = 0 \\ 4\mu a - \delta_1 b = 0 \end{cases} \tag{37}$$

即

$$\begin{cases} \delta_1 = 0 \\ \mu \ = 0 \end{cases} \tag{38}$$

说明$O(\varepsilon)$量级的阻尼过大, 需要用更小的阻尼

$$\zeta = \varepsilon^2 \hat{\mu} \tag{39}$$

此时

$$\ddot{u}_0 + \delta_0 u_0 = 0 \tag{40}$$

$$\ddot{u}_1 + \delta_0 u_1 = -(\delta_1 + 2\cos 2t)u_0 \tag{41}$$

$$\ddot{u}_2 + \delta_0 u_2 = -\delta_2 u_0 - (\delta_1 + 2\cos 2t)u_1 - 2\hat{\mu}\dot{u}_0 \tag{42}$$

重新将解代入 (41) 得

$$\ddot{u}_1 + u_1 = -a - \delta_1(a\cos 2t + b\sin 2t) - a\cos 4t - b\sin 4t \tag{43}$$

消除永年项

$$\delta_1 = 0 \tag{44}$$

解得

$$u_1 = -\frac{a}{4} + \frac{a}{12}\cos 4t + \frac{b}{12}\sin 4t \tag{45}$$

代入 (42)

$$
\begin{aligned}
\ddot{u}_2 + u_2 = -\left[\left(\delta_2 - \frac{5}{12}\right)a + 4\hat{\mu}a\right]\cos 2t + \left[4\hat{\mu}a - \left(\delta_2 + \frac{1}{12}\right)b\right]\sin 2t \\
-\frac{a}{12}\cos 6t - \frac{b}{12}\sin 6t
\end{aligned} \tag{46}
$$

消除永年项

$$
\begin{cases}
(\delta_2 - \frac{5}{12})a + 4\hat{\mu}a = 0 \\
4\hat{\mu}a - (\delta_2 + \frac{1}{12})b = 0
\end{cases} \tag{47}
$$

非零解条件为

$$
\delta_2^2 - \frac{1}{3}\delta_2 - \left(\frac{5}{144} - 16\hat{\mu}^2\right) = 0 \tag{48}
$$

可得

$$
\delta_2 = \frac{1}{6} \pm \sqrt{\frac{1}{16} - 16\hat{\mu}^2} \tag{49}
$$

此时, 解得

$$
u_2 = \frac{1}{384}(a\cos 6t + b\sin 6t) \tag{50}
$$

代入 (12), 可得$\delta_0 = 4$附近的稳定边界

$$
\delta = 4 + \frac{\varepsilon^2}{6} \pm \sqrt{\frac{\varepsilon^4}{16} - 16\zeta^2} + O(\varepsilon^3) \tag{51}
$$

作图