Homework 6 Reference answer

Problem 1: For a fixed-fixed undamped string, the tension of the string is uniform and constant *F*, the mass distribution of the string is also uniform, with linear density *m*. If there is a wave travels along the string, prove that the propagation speed of the wave along the string is $c=\sqrt{F/m}$. (10 points)

解：已知弦横向振动方程如下。其中 m 为单位长度的质量，T 为弦的张力

$$m\frac{\partial^2 y}{\partial^2 t}=T\frac{\partial^2 y}{\partial^2 x}$$
$$\frac{\partial^2 y}{\partial^2 t}=a^2\frac{\partial^2 y}{\partial^2 x},\quad a=\sqrt{\frac{T}{m}} \tag{1}$$

在弦上传播的波可以表示为 $f(x-ct)$ 的形式，其中 c 为波的传播速度

特征变换

$$\xi=x-ct \tag{2}$$

则：

$$\frac{\partial y}{\partial x}=\frac{\partial y}{\partial \xi}\frac{\partial \xi}{\partial x}=\frac{\partial y}{\partial \xi}$$
$$\frac{\partial^2 y}{\partial x^2}=\frac{\partial}{\partial \xi}\left(\frac{\partial y}{\partial \xi}\right)\frac{\partial \xi}{\partial x}=\frac{\partial^2 y}{\partial \xi^2} \tag{3}$$

同理可得：

$$\frac{\partial y}{\partial t}=\frac{\partial y}{\partial \xi}\frac{\partial \xi}{\partial t}=-c\left(\frac{\partial y}{\partial \xi}\right)$$
$$\frac{\partial^2 y}{\partial t^2}=-c\frac{\partial y}{\partial \xi}\left(\frac{\partial y}{\partial \xi}\right)\frac{\partial \xi}{\partial t}=c^2\left(\frac{\partial^2 y}{\partial \xi^2}\right) \tag{4}$$

将(3), (4)与(1)联立，可得

$$c^2\frac{\partial^2 y}{\partial^2 \xi}=a^2\frac{\partial^2 y}{\partial^2 \xi}, \tag{5}$$

即证明：$c=a=\sqrt{F/m}$

Ps：可通过行波法求出方程的通解，明确其物理意义：

特征变换：$\xi=x+at,\eta=x-at$

类似的，可得

$$\frac{\partial^2 y}{\partial x^2}=\frac{1}{a^2}\left(\frac{\partial^2 y}{\partial \xi^2}+\frac{\partial^2 y}{\partial \eta^2}+2\frac{\partial^2 y}{\partial \xi\partial \eta}\right),\quad \frac{\partial^2 y}{\partial t^2}=\frac{\partial^2 u}{\partial \xi^2}-2\frac{\partial^2 u}{\partial \xi\partial \eta}+\frac{\partial^2 u}{\partial \eta^2}$$

代入(1)，得到

$$\frac{\partial^2 u}{\partial\xi\partial\eta}=0$$

对η积分：

$$\frac{\partial u}{\partial\xi}=f\left(\xi\right)$$

对ξ积分，得到方程的通解：

$$\begin{aligned}\partial u&=\int f\left(\xi\right)d\xi+G\left(\eta\right)\\&=\mathrm{F}\left(\xi\right)+G\left(\eta\right)\\&=\mathrm{F}\left(x+at\right)+G\left(x-at\right)\end{aligned}$$

其物理意义为以速度 a 向 x 轴正负两个方向传播的行波函数之和。

Problem 2: For an undamped beam, four boundary conditions are possible based on the extended Hamilton's principle, namely $\delta w=0,\delta\left(\frac{\partial w}{\partial x}\right)=0,EI\left(\frac{\partial^2 w}{\partial x^2}\right)=0,EI\left(\frac{\partial^3 w}{\partial x^3}\right)=0$.

Please state the physical meaning of the four boundary conditions and explain why. (10 points)

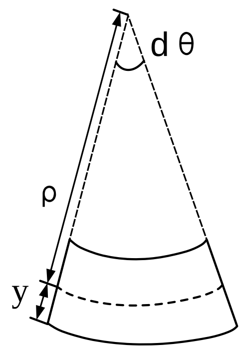
解：边界条件的物理意义分别为：

（1）$\delta w=0$，挠度为零，对应固定端或简支端

（2）$\delta\left(\frac{\partial w}{\partial x}\right)=0$，斜率为零，对应固定端或自由端。

$$\arctan\theta=\left(\omega+\frac{\partial\omega}{\partial x}dx-\omega\right)\Big/dx=\frac{\partial\omega}{\partial x}$$

（3）$EI\left(\frac{\partial^2 w}{\partial x^2}\right)=0$，弯矩为零，对应自由端或简支端



根据材料力学知识，弯曲梁截面上的正应变以及正应力为：

$$\varepsilon=\frac{(\rho+y)d\theta-\rho d\theta}{\rho d\theta}=\frac{y}{\rho},\quad \sigma=E\varepsilon$$

在截面上积分，可求得该处的弯矩：

$$M=\int_A y\sigma dA=\frac{E}{\rho}\int_A y^2 dA=\frac{EI}{\rho}$$

由高等数学知识可得曲率为：

$$\frac{1}{\rho}=\left|\frac{d^2y}{dx^2}\right|\Bigg/\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

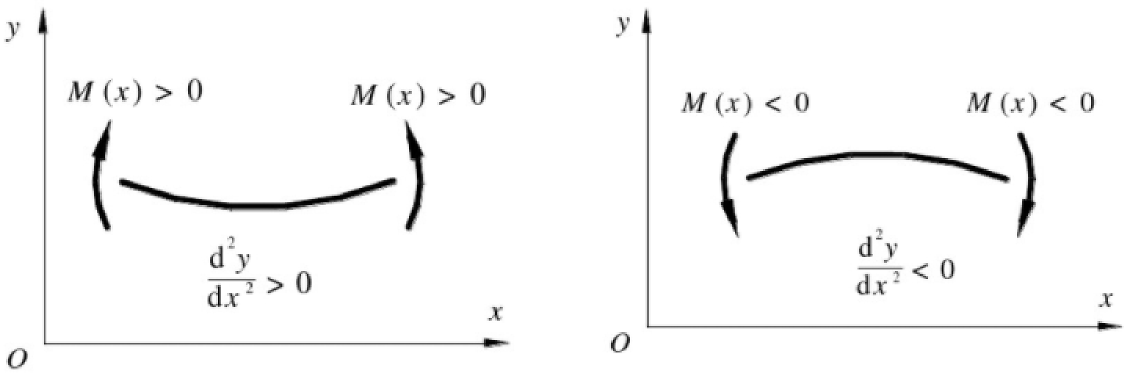
略去高阶项：

$$\frac{1}{\rho}=\left|\frac{d^2y}{dx^2}\right|$$

则弯矩公式为：

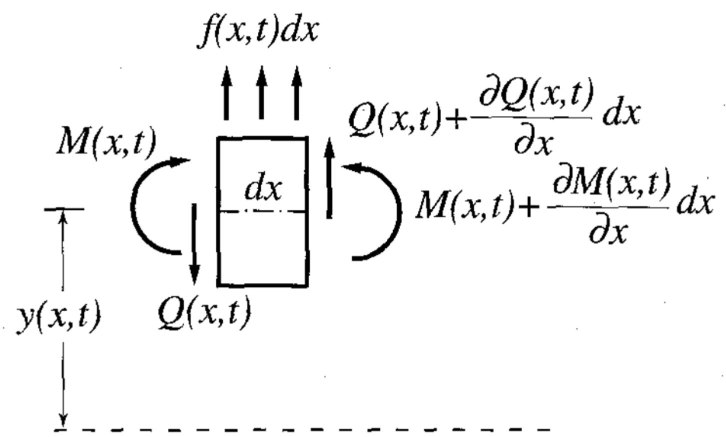
$$M=\mathrm{EI}\frac{d^2y}{dx^2}$$

弯矩正负规定如下图所示，上压（压缩）下拉（拉伸）为正。



（4）$EI\left(\frac{\partial^3 w}{\partial x^3}\right)=0$，剪力为零，对应自由端

如图，取微元分析，



假设单元惯性矩与角加速度乘积忽略不计，根据力矩平衡关系得到：

$$\left[M(x,t)+\frac{\partial M(x,t)}{\partial x}dx\right]-M(x,t)+\left[Q(x,t)+\frac{\partial Q(x,t)}{\partial x}dx\right]dx+f(x,t)dx\frac{dx}{2}=0,\ \ 0<x<L$$

略去高阶项：

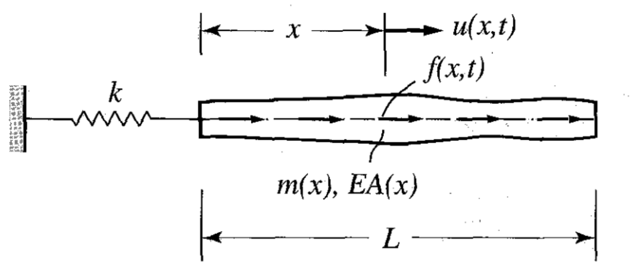
$$\frac{\partial M(x,t)}{\partial x}+Q(x,t)=0,0<x<L$$

将(3)中弯矩公式代入得到：

$$Q(x,t)=-\frac{\partial EI}{\partial x}\left[\frac{\partial^2 y(x,t)}{\partial x^2}\right]$$

Problem 3: Please solve Problem 8.5 in page 459 of the textbook (20 points)

Use the Newtonian approach to derive the boundary-value problem for a rod in axial vibration attached to a spring of stiffness k at $x=0$ and free at $x=L$ by regarding the system as distributed from the onset (Fig. 8.32). The rod is subjected to the force per unit length $f(x,t)$, its mass per unit length is $m(x)$ and its axial stiffness is $EA(x)$, where E is the modulus of elasticity and $A(x)$ the cross-sectional area.



Extended Hamilton's Princlple→EOM, then compared to Newtonian method

解：设杆上任意一点的轴向位移为$u(x,t)$，则截面上任意一点处的速度可以表示为$\partial u(x,t)/\partial t$，而截面上的应变可以表示为

$$\varepsilon_x = \frac{\partial u(x,t)}{\partial x}$$

杆的动能可以表示为

$$T = \int_0^L \frac{1}{2} m(x) \dot{u}^2 dx$$

接下来考虑结构的应变，在小变形的条件下认为$\cos\theta \approx 1$，

$$V = \int_0^L \frac{1}{2} EA(x) \left(\frac{\partial u}{\partial x} \right)^2 dx$$

则有

$$\begin{aligned}
\int_0^t \delta T dt &= \int_0^t \int_0^L m(x) \left(\dot{u} \delta \dot{u} \right) dx dt = \int_0^L \int_0^t m(x) \left(\dot{u} \frac{\partial \left(\delta u \right)}{\partial t} \right) dt dx \\
&= \int_0^L m(x) \left(\dot{u} \delta u \right) dx \Big|_0^t - \int_0^L \int_0^t \left(\frac{\partial}{\partial t} \left(m(x) \dot{u} \right) \delta u \right) dt dx \\
&= -\int_0^L \int_0^t \left(\frac{\partial}{\partial t} \left(m(x) \dot{u} \right) \delta u \right) dt dx \\
\int_0^t \delta V dt &= \int_0^t \int_0^L \frac{1}{2} EA(x) \frac{\partial u}{\partial x} \delta \left(\frac{\partial u}{\partial x} \right) dx dt \\
&= \int_0^t EA(x) \frac{\partial u}{\partial x} \delta u \Big|_0^L dt - \int_0^t \int_0^L \frac{\partial}{\partial x} \left(EA(x) \frac{\partial u}{\partial x} \right) \delta u dx dt
\end{aligned}$$

同时，杆受到轴向外力 $f(x,t)$ ，以及左端弹簧力 $ku\delta(x)$ ，利用 Extended Hamilton principle，

$$-\int_0^L \int_0^t \left(\frac{\partial}{\partial t} \left(m(x) \dot{u} \right) - \frac{\partial}{\partial x} \left(EA(x) \frac{\partial u}{\partial x} \right) - f(x,t) \right) \delta u dt dx - \int_0^t \left(EA(x) \frac{\partial u}{\partial x} \delta u + ku \delta(x) \delta u \right) \Big|_0^L dt = 0$$

左右两边的边界条件满足

$$EA(x) \frac{\partial u}{\partial x} - ku = 0 \quad at \ x = 0$$

$$EA(x) \frac{\partial u}{\partial x} = 0 \quad at \quad x = L$$

动力学方程为

$$\frac{\partial}{\partial t} \left(m(x) \dot{u} \right) - \frac{\partial}{\partial x} \left(EA(x) \frac{\partial u}{\partial x} \right) - f(x,t) = 0, 0 < x < L$$

Problem 4: Please solve Problem 8.7 in page 459 of the textbook (20 points).

8.7. A cable of uniform mass per unit length, $\rho(x) = \rho =$ constant, hangs freely from the ceiling, as shown in Fig. 8.34. Assume that the cable possesses no flexural stiffness and derive the boundary-value problem for the transverse vibration. **Hint:** The boundary condition at $x = 0$, ordinarily associated with a free end, is satisfied trivially in the case at hand, without involving the displacement. Hence, it must be replaced by a different boundary condition, based on physical considerations and the nature of the solution (see also Problem 8.13).

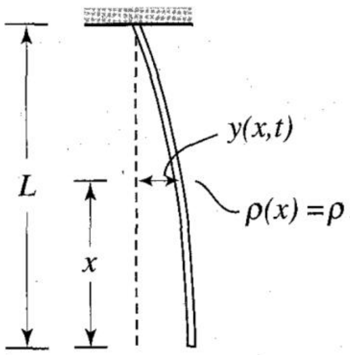


FIGURE 8.34
Cable in transverse vibration hanging freely from the ceiling

解：绳子的动能为：

$$T = \frac{1}{2}\int_0^L \rho\left(\frac{\partial y}{\partial t}\right)^2 dx$$

其中绳子的变形量为：

$$ds - dx = \sqrt{dx^2 + \left(\frac{\partial y}{\partial x}dx\right)^2} - dx = dx\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} - dx$$

$$= dx\left(1+\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 + \cdots\right) - dx = \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

设绳子的张力为 F(x)，在重力的作用下，弦上的张力处处不相等。势能为：

$$V = \int_0^L F(x)(ds - dx)dx$$

$$= \frac{1}{2}\int_0^L F(x)\left(\frac{\partial y}{\partial x}\right)^2 dx$$

根据 extended Hamilton’s principle，得到

$$\int_0^L\int_0^t\left(\rho\frac{\partial y}{\partial t}\delta\left(\frac{\partial y}{\partial t}\right)-F(x)\frac{\partial y}{\partial x}\delta\left(\frac{\partial y}{\partial x}\right)\right)dtdx=0$$

$$\int_0^L\left(\rho\frac{\partial y}{\partial t}\delta y\bigg|_0^t-\int_0^t\rho\frac{\partial^2 y}{\partial t^2}\delta ydt\right)dx-\int_0^t\left(F(x)\frac{\partial y}{\partial x}\delta y\bigg|_0^L-\int_0^L\frac{\partial y}{\partial x}\left(F(x)\frac{\partial y}{\partial x}\right)\delta ydx\right)dt=0$$

$$\int_0^L\int_0^t\left(\rho\frac{\partial^2 y}{\partial t^2}-\frac{\partial y}{\partial x}\left(F(x)\frac{\partial y}{\partial x}\right)\right)\delta ydtdx+\int_0^t\left(F(x)\frac{\partial y}{\partial x}\delta y\bigg|_0^L\right)dt=0$$

边界条件满足：

$$F(x)\frac{\partial y}{\partial x}=0 \quad at \quad x=0$$
$$\partial y=0 \quad at \quad x=L$$

动力学方程为：

$$\rho\frac{\partial^2 y}{\partial t^2}-\frac{\partial y}{\partial x}\left(F(x)\frac{\partial y}{\partial x}\right)=0$$

对竖直放置的做横向微小摆动的弦来说，可近似认为弦上某处的张力等于其下端弦的重力，张力的表达式为：

$$F(x)=\rho gx$$

代入可得振动方程：

$$\frac{\partial^2 y}{\partial t^2}-g\frac{\partial y}{\partial x}-gx\frac{\partial^2 y}{\partial x^2}=0$$

此外，由于自由端张力为零，边界条件满足平凡解，并未涉及到位移。

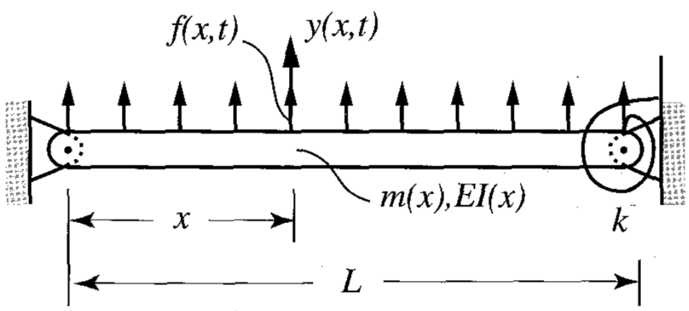
$$F(0)=0,\quad F(0)\frac{\partial y(0)}{\partial x}=0$$

将 $x=0$ 代入振动方程，可得自由端的边界条件满足：

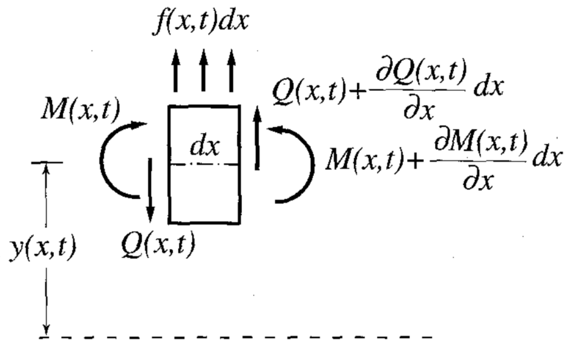
$$\frac{\partial^2 y(0)}{\partial t^2}-g\frac{\partial y(0)}{\partial x}=0$$

Problem 5: Please solve Problem 8.8 in page 460 of the textbook (20 points).

8.8. Use the Newtonian approach to derive the boundary-value problem for the bending vibration of a beam pinned at $x=0$ and pinned but with the slope to the deflection curve restrained by a spring at $x=L$, as shown in Fig. 8.35.



解：



如图所示，取梁的微元体进行受力分析，竖直方向上的力平衡方程为：

$$\left[Q(x,t)+\frac{\partial Q(x,t)}{\partial x}dx\right]-Q(x,t)+f(x,t)dx=m(x)dx\frac{\partial^2 y(x,t)}{\partial t^2},0<x<L$$

忽略微元体惯性矩与角加速度乘积，力矩平衡方程如下：

$$\left[M(x,t)+\frac{\partial M(x,t)}{\partial x}dx\right]-M(x,t)+\left[Q(x,t)+\frac{\partial Q(x,t)}{\partial x}dx\right]dx+f(x,t)dx\frac{dx}{2}=0,\ \ 0<x<L$$

略去高阶项：

$$\frac{\partial M(x,t)}{\partial x}+Q(x,t)=0,0<x<L$$

将其代入力平衡方程，得到：

$$-\frac{\partial^2 M(x,t)}{\partial x^2}+f(x,t)=m(x)\frac{\partial^2 y(x,t)}{\partial t^2},0<x<L$$

根据弯矩与挠度的关系，动力学方程变为：

$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]+f(x,t)=m(x)\frac{\partial^2 y(x,t)}{\partial t^2},0<x<L$$

边界条件：

x=0 处简支，挠度和弯矩为零；x=0 处简支，附着有刚度为 k 的扭簧，挠度为零，弯矩大小等于扭簧所提供的弯矩，符号为负。

$$\delta y=0,EI\frac{\partial^2 y}{\partial x^2}=0\quad at\quad x=0$$

$$\delta y=0,EI\frac{\partial^2 y}{\partial x^2}=-k\theta\quad at\quad x=L$$

Problem 6: Please solve Problem 8.9 in page 460 of the textbook (20 points).

Solve Problem 8.8 by the extended Hamilton principle.

解：设梁上中性轴的竖直位移为 $y(x,t)$，中性轴的转动角度为 $\theta(x,t)$，则梁上任意一点处的位移可以表示为

$$u(x,t)=-z\sin\theta(x,t)$$

$$w(x,t)=y(x,t)+z\left(1-\cos\theta(x,t)\right)$$

其中 z 为梁上任意一点到中性轴的有向距离。中性轴的位移与转动之间有关系

$$\tan\theta(x,t)=\frac{\partial y(x,t)}{\partial x}$$

梁的动能可以表示为

$$T=\int_0^L\frac{1}{2}m(x)\left(\dot{u}^2+\dot{w}^2\right)dx=\int_0^L\frac{1}{2}m(x)\int_{-h/2}^{h/2}\frac{1}{h}\left(\dot{u}^2+\dot{w}^2\right)dzdx$$
$$=\int_0^L\frac{1}{2}m(x)\int_{-h/2}^{h/2}\frac{1}{h}\left(\dot{y}^2+z^2\dot{\theta}^2+2\dot{y}\dot{\theta}\sin\theta z\right)dzdx$$

其中最后一项在截面上积分为 0，在小变形的条件下忽略第二项，因此动能为

$$T=\int_0^L\frac{1}{2}m(x)\dot{y}^2dx$$

接下来考虑结构的应变

$$\varepsilon_x=\frac{\partial u}{\partial x}=-z\cos\theta\frac{\partial\theta}{\partial x}$$

$$\varepsilon_z=\frac{\partial w}{\partial z}=1-\cos\theta$$

$$\gamma_{xz}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}=\frac{\partial y}{\partial x}+z\sin\theta\frac{\partial\theta}{\partial x}-\sin\theta$$

忽略剪应力，梁的势能可以表达为

$$V=\int_0^L\int_S\frac{1}{2}E\left(\varepsilon_x^2+\varepsilon_z^2\right)dSdx=\int_0^L\int_S\frac{1}{2}E\left(\left(-z\cos\theta\frac{\partial\theta}{\partial x}\right)^2+\left(1-\cos\theta\right)^2\right)dSdx$$
$$=\int_0^L\left[\frac{1}{2}EI\left(\cos\theta\frac{\partial\theta}{\partial x}\right)^2+\frac{1}{2}EA\left(1-\cos\theta\right)^2\right]dx$$

在小变形的条件下认为 $\cos\theta\approx1$，

$$V=\int_0^L\frac{1}{2}EI\left(\frac{\partial\theta}{\partial x}\right)^2dx$$

则有

$$\int_0^t\delta Tdt=\int_0^t\int_0^L m(x)\left(\dot{y}\delta\dot{y}\right)dxdt=\int_0^L\int_0^t m(x)\left(\dot{y}\frac{\partial\left(\delta y\right)}{\partial t}\right)dtdx$$
$$=\int_0^L m(x)\left(\dot{y}\delta y\right)dx\bigg|_0^t-\int_0^L\int_0^t\left(\frac{\partial}{\partial t}\left(m(x)\dot{y}\right)\delta y\right)dtdx$$
$$=-\int_0^L\int_0^t\left(\frac{\partial}{\partial t}\left(m(x)\dot{y}\right)\delta y\right)dtdx$$

$$\int_0^t \delta V dt = \int_0^t\int_0^L \frac{1}{2} EI \frac{\partial\theta}{\partial x}\delta\left(\frac{\partial\theta}{\partial x}\right) dxdt$$
$$= \int_0^t EI\frac{\partial\theta}{\partial x}\delta\theta\Big|_0^L dt - \int_0^t\int_0^L \frac{\partial}{\partial x}\left(EI\frac{\partial\theta}{\partial x}\right)\delta\theta dxdt$$
$$= \int_0^t \left(EI\frac{\partial\theta}{\partial x}\delta\theta - \frac{\partial}{\partial x}\left(EI\frac{\partial\theta}{\partial x}\right)\delta y\right)\Bigg|_0^L dt + \int_0^t\int_0^L \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial\theta}{\partial x}\right)\delta y dtdx$$

同时梁还受到外力 $f(x,t)$，以及力矩 $k\theta\delta(x-L)$，利用 extended Hamilton principle，

$$-\int_0^L\int_0^t\left(\frac{\partial}{\partial t}\left(m(x)\dot{y}\right) + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial\theta}{\partial x}\right) - f(x,t)\right)\delta y dtdx - \int_0^t\left(EI\frac{\partial\theta}{\partial x}\delta\theta - \frac{\partial}{\partial x}\left(EI\frac{\partial\theta}{\partial x}\right)\delta y + k\theta\delta(x-L)\delta\theta\right)\Bigg|_0^L dt = 0$$

左右两边的边界条件满足

$$\delta y = 0, EI\frac{\partial^2 y}{\partial x^2} = 0 \quad at \quad x = 0$$
$$\delta y = 0, EI\frac{\partial^2 y}{\partial x^2} = -k\theta \quad at \quad x = L$$

动力学方程为

$$\frac{\partial}{\partial t}\left(m(x)\dot{y}\right) + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial\theta}{\partial x}\right) = f(x,t)$$

在小变形条件下，中性轴的转动角度 $\theta(x,t)$ 可表示为

$$\theta(x,t) = \frac{\partial y(x,t)}{\partial x}$$