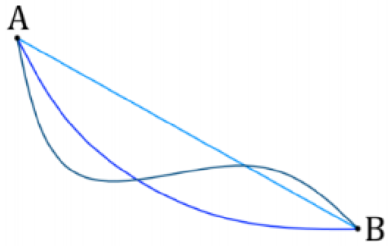
2021-Homework 2: Reference answer

LEAST ACTION PRINCIPLE

Problem 1: In mathematics and physics, a brachistochrone curve, or curve of fastest descent, is the one lying on the plane between a point A and a lower point B, where B is not directly below A, on which a bead slides frictionlessly under the influence of a uniform gravitational field to a given end point in the shortest time. The problem was posed by Johann Bernoulli in 1696. The brachistochrone problem was one of the earliest problems posed in the calculus of variations. Now, pleaser derive the brachistochrone curve via the Least Action Principle. (12 points)



解：对于任意从点 A 到点 B 的路径 $y=y(x)$，满足

$$x_A=0, y(x_A)=y_A=0, y(x_B)=y_B.$$

在 $y=y(x)$ 上取一微元 ds，则通过微元的时间可以表示为，

$$dt=ds/v.$$

其中 v 表示通过微元时的速率，由能量守恒定律可以得到

$$v(x)=\sqrt{2gy(x)}.$$

微元 ds 也可以确定，

$$ds=\sqrt{dx^2+dy^2}=\sqrt{1+y'(x)^2}dx.$$

对于路径 $y=y(x)$，从点 A 到点 B 所需时间为，

$$T=\int_{y=y(x)}\mathrm{d}t=\int_{x_A}^{x_B}\sqrt{\frac{1+y'(x)^2}{2gy(x)}}\,\mathrm{d}x.$$

根据最小作用量原理，系统在点 A 到点 B 之间的运动使得上述积分取极小值，即 $y=y(x)$ 使得泛函 $T(y)$ 取驻值。设 $F(y,y'(x),x)=\sqrt{1+y'(x)^2/2gy(x)}$，并令 $\delta T=0$，

$$\delta T=\int_{x_A}^{x_B}\left(\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y'}\delta y'\right)\mathrm{d}x=0.$$

对后一项进行分部积分，

$$\int_{x_A}^{x_B}\left(\frac{\partial F}{\partial y}\delta y-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\delta y\right)\mathrm{d}x+\left(\frac{\partial F}{\partial y'}\delta y\right)\Bigg|_{x_A}^{x_B}=0.$$

因为 $y(x_A)=y_A, y(x_B)=y_B$ 总是满足的，因此 $\delta y(x_A)=\delta y(x_B)=0$，上式可以写为

$$\int_{x_A}^{x_B}\left(\frac{\partial F}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right)\delta y\,\mathrm{d}x=0.$$

由于 δy 是任意的，要使上式满足则必须有（可以用反证法证明），

$$\frac{\partial F}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)=0.$$

将上式第二项移到等式右侧，并在两端同时乘以 y' 后进行积分，

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$$\int \frac{\partial F}{\partial y} y' \,\mathrm{d}\,x = \int y' \frac{\mathrm{d}}{\mathrm{d}\,x}\left(\frac{\partial F}{\partial y'}\right)\mathrm{d}\,x.$$

对等式右侧进行分部积分，

$$\int \frac{\partial F}{\partial y} y' \,\mathrm{d}\,x = y'\frac{\partial F}{\partial y'} + C - \int y'' \frac{\mathrm{d}}{\mathrm{d}\,x}\left(\frac{\partial F}{\partial y'}\right)\mathrm{d}\,x.$$

合并等式两侧的积分项有，

$$\int \left[\frac{\partial F}{\partial y} y' + y'' \frac{\mathrm{d}}{\mathrm{d}\,x}\left(\frac{\partial F}{\partial y'}\right)\right]\mathrm{d}\,x = y'\frac{\partial F}{\partial y'} + C.$$

因为式 $F\left(y, y'(x), x\right) = \sqrt{1 + y'(x)^2 \big/ 2gy(x)}$ 中不显含 x，因此

$$\frac{\mathrm{d}\,F}{\mathrm{d}\,x} = \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial x}.$$

即

$$\mathrm{d}\,F = \left(\frac{\partial F}{\partial y} y' + \frac{\partial F}{\partial y'} y''\right)\mathrm{d}\,x.$$

原等式可以表示成形式，

$$F - y'\frac{\partial F}{\partial y'} = C'.$$

将 $F\left(y, y'(x), x\right)$ 代入，

$$\sqrt{\frac{1+\left(y'\right)^2}{2gy}} - \frac{\left(y'\right)^2}{\sqrt{2gy\left(1+\left(y'\right)^2\right)}} = C'.$$

化简后有，

$$y\left(1+\left(y'\right)^2\right) = \frac{1}{2g\left(C'\right)^2}.$$

我们注意到在三角函数中有

$$\sin^2\alpha\left(1+\cot^2\alpha\right) = 1,$$

$$\cos^2\alpha\left(1+\tan^2\alpha\right) = 1.$$

令 $y = C_1 \sin^2\alpha, y' = \cot\alpha$，其中 $C_1 = 1\big/2g\left(C'\right)^2$ 。则

$$dx = \frac{dx}{dy}dy = \frac{1}{\cot\alpha}2C_1\sin\alpha\cos\alpha d\alpha = 2C_1\sin^2\alpha d\alpha.$$

两边同时积分有

$$x = C_1\alpha - \frac{C_1}{2}\sin 2\alpha + C_2,$$

$$y = \frac{C_1}{2} - \frac{C_1}{2}\cos 2\alpha.$$

代入边界条件 $x_A = 0, y\left(x_A\right) = y_A = 0$，从而得到

$$\left(-C_1\alpha_A - C_2\right)^2 + \left(-\frac{C_1}{2}\right)^2 = \left(\frac{C_1}{2}\right)^2.$$

又因为 C_1, C_2 均为常数，可以得到

$$\alpha_A = 0, C_2 = 0.$$

代入另一个边界条件可以通过数值求得 C_1 以及 α_B 的值。若初始时刻 $x_A \neq 0, y\left(x_A\right) = y_A \neq 0$，则可以通过坐标平移的方式得到其结果，

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$$x - x_A = C_1\alpha - \frac{C_1}{2}\sin 2\alpha,$$
$$y - y_A = \frac{C_1}{2} - \frac{C_1}{2}\cos 2\alpha.$$

注意到坐标轴取向会影响到α_B的正负，若取向右和向上为x，y方向正方向，则$x_B - x_A > 0, y_B - y_A < 0$，因此$C_1 < 0, \alpha_B < 0$。因此最速降线方程为

$$\begin{cases} x = C_1\alpha - \dfrac{C_1}{2}\sin 2\alpha + x_A \\ y = \dfrac{C_1}{2} - \dfrac{C_1}{2}\cos 2\alpha + y_A \end{cases}, \alpha \in [\alpha_B, 0].$$

其中C_1, α_B满足方程

$$x_B - x_A = C_1\alpha_B - \frac{C_1}{2}\sin 2\alpha_B,$$
$$y_B - y_A = \frac{C_1}{2} - \frac{C_1}{2}\cos 2\alpha_B.$$

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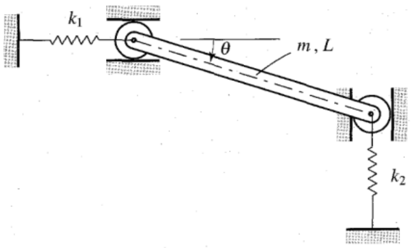
VIRTUAL WORK, D'ALEMBERT'S PRINCIPLE, HAMILTON'S PRINCIPLE, EULER-LAGRANGE EQUATION

Problem 2: Please solve Problem 6.1, 6.7, 6.9, and 6.13 in page 277-279 of the textbook. (16 points)
6.1 The system of Fig. 6.3 consists of a uniform rigid link of total mass m and length L and two linear springs of stiffnesses k_1 and k_2. When the springs are unstretched the link is horizontal. Use the principle of virtual work to calculate the angle θ corresponding to the equilibrium position.



6.7 Derive the equation of motion for the system of Problem 6.1 by means of d'Alembert's principle.
6.9 Derive the equation of motion for the system of Problem 6.1 by means of Hamilton's principle.
6.13 Derive the three Newton equations of motion and the single Lagrange equation for the system of Problem 6.1 and show how Newton's equations can be reduced to Lagrange's equation.

6.1 解：假设杆与水平方向之间的夹角为θ，顺时针方向为正，同时取$\boldsymbol{i}$和$\boldsymbol{j}$表示向右和向下的方向。以杆在水平方向时为系统的初始位置，则水平顶点和竖直顶点的位移可以表示为，

$$\boldsymbol{x} = L(1-\cos\theta)\boldsymbol{i},\ \boldsymbol{y} = L\sin\theta\boldsymbol{j}.$$

均匀杆的质心可表示为其几何中心，位移可以表示为

$$\boldsymbol{x}_C = L(1-\cos\theta)/2\boldsymbol{i},\ \boldsymbol{y}_C = L\sin\theta/2\boldsymbol{j}.$$

当系统静止于平衡位置时，其速度和加速度为 0。由虚功原理可得

$$\delta W = \boldsymbol{F}_{mg}\cdot\delta\boldsymbol{r}_c + \boldsymbol{F}_{k1}\cdot\delta\boldsymbol{x} + \boldsymbol{F}_{k2}\cdot\delta\boldsymbol{y} = 0.$$

其中，

$$\begin{aligned}
&\boldsymbol{F}_{mg} = mg\boldsymbol{j}, \boldsymbol{F}_{k1} = -k_1\boldsymbol{x}, \boldsymbol{F}_{k2} = -k_2\boldsymbol{y},\\
&\delta\boldsymbol{r}_c = \delta\boldsymbol{x}_c + \delta\boldsymbol{y}_c = \left(L\sin\theta/2\boldsymbol{i} + L\cos\theta/2\boldsymbol{j}\right)\delta\theta,\\
&\delta\boldsymbol{x} = L\sin\theta\delta\theta\boldsymbol{i}, \delta\boldsymbol{y} = L\cos\theta\delta\theta\boldsymbol{j}.
\end{aligned}$$

将其代入有，

$$\delta W = mg\,L\cos\theta/2\,\delta\theta - k_1L(1-\cos\theta)L\sin\theta\delta\theta - k_2L\sin\theta L\cos\theta\delta\theta = 0.$$

整理后可得

$$\left[mg\cos\theta/2 - k_1L(1-\cos\theta)\sin\theta - k_2\sin\theta L\cos\theta\right]\delta\theta = 0.$$

由虚位移的任意性，可以得到，

$$mg\cos\theta/2 - k_1L\sin\theta + (k_1-k_2)L\cos\theta\sin\theta = 0.$$

上式即为平衡位置应当满足的方程。

6.7 解：假设杆与水平方向之间的夹角为θ，顺时针方向为正，同时取$\boldsymbol{i}$和$\boldsymbol{j}$表示向右和向下的方向。以杆在水平方向时为系统的初始位置，杆的质心处的位移可以表示为，

$$\boldsymbol{r}_C = \frac{L}{2}\left((1-\cos\theta)\boldsymbol{i} + \sin\theta\boldsymbol{j}\right).$$

因此系统的惯性力可以表示为

$$m\ddot{\boldsymbol{r}}_C = \frac{mL}{2}\left(\left(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2\right)\boldsymbol{i} + \left(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2\right)\boldsymbol{j}\right).$$

作用于系统的外力可以表示为

$$\begin{aligned}
&\boldsymbol{F}_{mg} = mg\boldsymbol{j}, \boldsymbol{F}_{k1} = -k_1L(1-\cos\theta)\boldsymbol{i}, \boldsymbol{F}_{k2} = -k_2L\sin\theta\boldsymbol{j},\\
&\boldsymbol{N}_1 = N_1\boldsymbol{j}, \boldsymbol{N}_2 = N_2\boldsymbol{i}.
\end{aligned}$$

其中 $\boldsymbol{N}_1, \boldsymbol{N}_2$ 分别表示上端以及右端约束对杆的支持力。由 d'Alembert 原理可以得到平衡方程如下，

$$\boldsymbol{F}_{mg} + \boldsymbol{F}_{k1} + \boldsymbol{F}_{k2} + \boldsymbol{N}_1 + \boldsymbol{N}_2 - m\ddot{\boldsymbol{r}}_C = \boldsymbol{0}.$$

即

$$mg\boldsymbol{j} - k_1 L(1-\cos\theta)\boldsymbol{i} - k_2 L\sin\theta\boldsymbol{j} + N_1\boldsymbol{j} + N_2\boldsymbol{i} - \frac{mL}{2}\left(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2\right)\boldsymbol{i} - \frac{mL}{2}\left(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2\right)\boldsymbol{j} = \boldsymbol{0}.$$

在两个正交方向上分别有，

$$-k_1 L(1-\cos\theta) + N_2 - \frac{mL}{2}\left(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2\right) = 0,$$

$$mg - k_2 L\sin\theta + N_1 - \frac{mL}{2}\left(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2\right) = 0.$$

此外杆转动的惯性可以表示为

$$\boldsymbol{M}_C = J_C\ddot{\theta}\boldsymbol{k}.$$

其中 J_C 表示杆绕质心的转动惯量 $J_C = mL^2/12$， $\boldsymbol{k}$ 表示垂直平面朝里的方向，满足 $\boldsymbol{k} = \boldsymbol{i}\times\boldsymbol{j}$ 。则在质心 C 处的力矩平衡方程为，

$$\boldsymbol{r}_1\times\boldsymbol{F}_{k1} + \boldsymbol{r}_2\times\boldsymbol{F}_{k2} + \boldsymbol{r}_1\times\boldsymbol{N}_1 + \boldsymbol{r}_2\times\boldsymbol{N}_2 - J_C\ddot{\theta}\boldsymbol{k} = \boldsymbol{0}.$$

矢量 $\boldsymbol{r}_1, \boldsymbol{r}_2$ 表示从质心 C 指向力的作用点，

$$\boldsymbol{r}_1 = -\frac{L\sin\theta}{2}\boldsymbol{j} - \frac{L\cos\theta}{2}\boldsymbol{i},$$

$$\boldsymbol{r}_2 = \frac{L\cos\theta}{2}\boldsymbol{i} + \frac{L\sin\theta}{2}\boldsymbol{j}.$$

代入后并合并同类项有

$$-k_1 LL\sin\theta(1-\cos\theta) - k_2 LL\cos\theta\sin\theta + mg\frac{L\cos\theta}{2} - \frac{mL^2}{4}\ddot{\theta} - \frac{mL^2}{12}\ddot{\theta} = 0.$$

整理后有

$$mL\ddot{\theta}/3 + k_1 L\sin\theta(1-\cos\theta) + k_2 L\cos\theta\sin\theta - mg\cos\theta/2 = 0.$$

假设系统的平衡位置为 $\bar{\theta}$ ，并令 $\theta = \bar{\theta} + \tilde{\theta}$ ，则有

$$mL\ddot{\tilde{\theta}}/3 + k_1 L\sin\left(\bar{\theta}+\tilde{\theta}\right)\left(1-\cos\left(\bar{\theta}+\tilde{\theta}\right)\right) + k_2 L\cos\left(\bar{\theta}+\tilde{\theta}\right)\sin\left(\bar{\theta}+\tilde{\theta}\right) - mg\cos\left(\bar{\theta}+\tilde{\theta}\right)/2 = 0.$$

并假设 $\tilde{\theta}$ 为小变形，有

$$\cos\theta = \cos\bar{\theta}\cos\tilde{\theta} - \sin\bar{\theta}\sin\tilde{\theta} \approx \cos\bar{\theta} - \sin\bar{\theta}\tilde{\theta},$$

$$\sin\theta = \sin\bar{\theta}\cos\tilde{\theta} + \cos\bar{\theta}\sin\tilde{\theta} \approx \sin\bar{\theta} + \cos\bar{\theta}\tilde{\theta}.$$

即有

$$mL\ddot{\tilde{\theta}}/3 + k_1 L\left(\sin\bar{\theta} + \cos\bar{\theta}\tilde{\theta}\right) + (k_2 - k_1)L\left(\sin\bar{\theta} + \cos\bar{\theta}\tilde{\theta}\right)\left(\cos\bar{\theta} - \sin\bar{\theta}\tilde{\theta}\right) - mg\left(\cos\bar{\theta} - \sin\bar{\theta}\tilde{\theta}\right)/2 = 0.$$

注意到系统在平衡位置处满足

$$mg\cos\bar{\theta}/2 - k_1 L\sin\bar{\theta} + (k_1 - k_2)L\cos\bar{\theta}\sin\bar{\theta} = 0.$$

联立后有

$$mL\ddot{\tilde{\theta}}/3 + k_1 L\cos\bar{\theta}\tilde{\theta} + (k_2 - k_1)L\left(\cos 2\bar{\theta}\tilde{\theta} - \sin\bar{\theta}\cos\bar{\theta}\tilde{\theta}^2\right) + mg\sin\bar{\theta}\tilde{\theta}/2 = 0.$$

忽略高阶小量后得到 EOM,

$$mL\ddot{\tilde{\theta}}/3 + \left(k_1 L\left(\cos\bar{\theta} - \cos 2\bar{\theta}\right) + k_2 L\cos 2\bar{\theta} + mg\sin\bar{\theta}/2\right)\tilde{\theta} = 0.$$

6.9 由 Hamilton 原理得：

$$\int_{t_1}^{t_2}(\delta W + \delta T)\,\mathrm{d}t = 0.$$

假设杆与水平方向之间的夹角为 θ，顺时针方向为正，同时取 $\boldsymbol{i}$ 和 $\boldsymbol{j}$ 表示向右和向下的方向。以杆在水平方向时为系统的初始位置，则水平顶点和竖直顶点的位移可以表示为，

$$\boldsymbol{x} = L(1-\cos\theta)\boldsymbol{i},\ \boldsymbol{y} = L\sin\theta\boldsymbol{j}.$$

均匀杆的质心可表示为其几何中心，位移可以表示为

$$\boldsymbol{x}_C = L(1-\cos\theta)/2\,\boldsymbol{i},\ \boldsymbol{y}_C = L\sin\theta/2\,\boldsymbol{j}.$$

所有主动力的虚功可以表示为

$$\delta W = \boldsymbol{F}_{mg}\cdot\delta\boldsymbol{r}_c + \boldsymbol{F}_{k1}\cdot\delta\boldsymbol{x} + \boldsymbol{F}_{k2}\cdot\delta\boldsymbol{y}.$$

其中，

$$\begin{aligned}&\boldsymbol{F}_{mg} = mg\boldsymbol{j}, \boldsymbol{F}_{k1} = -k_1\boldsymbol{x}, \boldsymbol{F}_{k2} = -k_2\boldsymbol{y},\\ &\delta\boldsymbol{r}_c = \delta\boldsymbol{x}_c + \delta\boldsymbol{y}_c = \left(L\sin\theta/2\,\boldsymbol{i} + L\cos\theta/2\,\boldsymbol{j}\right)\delta\theta,\\ &\delta\boldsymbol{x} = L\sin\theta\delta\theta\boldsymbol{i}, \delta\boldsymbol{y} = L\cos\theta\delta\theta\boldsymbol{j}.\end{aligned}$$

将其代入有，

$$\delta W = mg\,L\cos\theta/2\,\delta\theta - k_1L(1-\cos\theta)L\sin\theta\delta\theta - k_2L\sin\theta L\cos\theta\delta\theta.$$

此外，系统的动能为：

$$T = \frac{1}{2}m\dot{\boldsymbol{r}}_C\cdot\dot{\boldsymbol{r}}_C + \frac{1}{2}J_C\dot{\boldsymbol{\theta}}\cdot\dot{\boldsymbol{\theta}} = \frac{1}{6}mL^2\dot{\theta}^2.$$

则 $\delta T = \frac{1}{3}mL^2\dot{\theta}\delta\dot{\theta}$，将其代入 Hamilton 原理有，

$$\int_{t_1}^{t_2}\left(\frac{1}{3}mL^2\dot{\theta}\delta\dot{\theta} + \left[mg\cos\theta\frac{L}{2} - k_1L^2(1-\cos\theta)\sin\theta - k_2L^2\sin\theta\cos\theta\right]\delta\theta\right)\mathrm{d}t = 0.$$

对第一项进行分部积分可得：

$$\frac{1}{3}mL^2\int_{t_1}^{t_2}\dot{\theta}\delta\dot{\theta}\,\mathrm{d}t = \frac{1}{3}mL^2\dot{\theta}\delta\theta\Big|_{t_1}^{t_2} - \frac{1}{3}mL^2\int_{t_1}^{t_2}\ddot{\theta}\delta\theta\,\mathrm{d}t.$$

因为满足 $\delta\theta(t_1) = \delta\theta(t_2) = 0$，所以上式变为，

$$\int_{t_1}^{t_2}\left(-\frac{1}{3}mL^2\ddot{\theta} + mg\cos\theta\frac{L}{2} - k_1L^2(1-\cos\theta)\sin\theta - k_2L^2\sin\theta\cos\theta\right)\delta\theta\,\mathrm{d}t = 0.$$

由的任意性，可得运动方程为，

$$\frac{1}{3}mL\ddot{\theta} - \frac{mg\cos\theta}{2} + k_1L(1-\cos\theta)\sin\theta + k_2L\sin\theta\cos\theta = 0.$$

线性化过程及结果同 6.7。

6.13 解：（1）假设杆与水平方向之间的夹角为 θ，顺时针方向为正；以弹簧处于原长（杆处于水平位置）时左端点为原点，取向右和向上的方向为 x，y 轴的正方向建立坐标系。则杆的质心处的位移可以表示为，

$$x_C = \frac{L}{2}(1-\cos\theta), y_C = -\frac{L}{2}\sin\theta.$$

由 Newton 第二定律可以得到动力学方程为，

$$\begin{aligned}&-k_1L(1-\cos\theta) + N_2 = m\ddot{x}_C,\\ &k_2L\sin\theta + N_1 - mg = m\ddot{y}_C.\end{aligned}$$

其中 N_1, N_2 分别表示上端以及右端约束对杆的支持力。同时由动量矩定理有，

$$-k_1L(1-\cos\theta)L\sin\theta/2 - N_2\,L\sin\theta/2 - k_2L\sin\theta L\cos\theta/2 + N_1\,L\cos\theta/2 = J_C\ddot{\theta}.$$

（2）系统的动能可以表示为

$$T = \frac{1}{2}mv_C^2 + \frac{1}{2}J_C\dot{\theta}^2 = \frac{1}{6}mL^2\dot{\theta}^2.$$

以水平位置为杆的重力势能零点，则系统势能可以表示为，

$$V = \frac{1}{2}k_1(L - L\cos\theta)^2 + \frac{1}{2}k_2(L\sin\theta)^2 - mg\frac{L}{2}\sin\theta.$$

Lagrangian 可以表示成，

$$L = T - V = \frac{1}{6}mL^2\dot{\theta}^2 - \frac{1}{2}k_1\left(L - L\cos\theta\right)^2 - \frac{1}{2}k_2\left(L\sin\theta\right)^2 + mg\frac{L}{2}\sin\theta.$$

系统没有受到非保守力的作用，因此可以通过 Euler-Lagrange 方程推导其运动方程，

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0.$$

代入后有，

$$\frac{1}{3}mL\ddot{\theta} + k_1L\left(1-\cos\theta\right)\sin\theta + k_2L\sin\theta\cos\theta - \frac{mg}{2}\cos\theta = 0.$$

（3）为了将牛顿方程简化，首先从方程中求出支持力的大小，

$$N_2 = m\ddot{x}_C + k_1L\left(1-\cos\theta\right),$$

$$N_1 = m\ddot{y}_C + mg - k_2L\sin\theta.$$

然后将其带入力矩方程中，整理后得到，

$$-k_1L^2\left(1-\cos\theta\right)\sin\theta - k_2L^2\sin\theta\cos\theta + \frac{1}{2}mgL\cos\theta + \frac{mL}{2}\left(\ddot{y}_C\cos\theta - \ddot{x}_C\sin\theta\right) = \frac{mL^2}{12}\ddot{\theta}.$$

然后将 $x_C = \frac{L}{2}\left(1-\cos\theta\right), y_C = -\frac{L}{2}\sin\theta$ 代入

$$-k_1L^2\left(1-\cos\theta\right)\sin\theta - k_2L^2\sin\theta\cos\theta + \frac{1}{2}mgL\cos\theta + \frac{mL}{2}\left(-\frac{L\ddot{\theta}}{2}\right) = \frac{mL^2}{12}\ddot{\theta}.$$

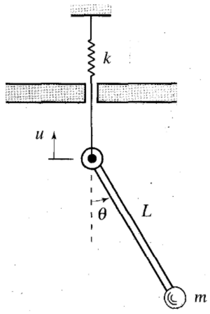
简化后得到，

$$\frac{mL^2}{3}\ddot{\theta} + k_1L\left(1-\cos\theta\right)\sin\theta + k_2L\sin\theta\cos\theta - \frac{1}{2}mg\cos\theta = 0.$$

线性化过程及结果同 6.7。

Problem 3: Please solve Problem 6.11 and 6.15 in page 278-279 of the textbook. (12 points)

6.11 The upper end of a pendulum is attached to a linear spring of stiffness k, where the spring is constrained so as to move in the vertical direction (Fig. 6.5). Derive the equations of motion for the vertical displacement u and the angular displacement θ by means of Hamilton's principle. Assume that u is measured from the equilibrium position and that θ is arbitrarily large.



6.15 Derive Newton's and Lagrange's equation of motion for the system of Problem 6.11, discuss differences and show how Newton's equations can be reduced to Lagrange's equations.

6.11 解：假设杆与竖直方向之间的夹角为θ，逆时针方向为正，同时取$\boldsymbol{i}$和$\boldsymbol{j}$表示**向右**和**向上**的方向。以平衡位置为初始位置，则弹簧力和重力可以分别表示为，

$$\boldsymbol{F}_k = -k\left(u-u_0\right)\boldsymbol{j}, \boldsymbol{F}_{mg} = -mg\boldsymbol{j}.$$

其中$u=u_0$时表示弹簧的原长。力的作用点的位移为，

$$\boldsymbol{r}_k = u\boldsymbol{j}, \boldsymbol{r}_{mg} = L\sin\theta\boldsymbol{i} + \left(u - L\cos\theta\right)\boldsymbol{j}.$$

则虚功表示为，

$$\delta W = \boldsymbol{F}_k\cdot\delta\boldsymbol{r}_k + \boldsymbol{F}_{mg}\cdot\delta\boldsymbol{r}_{mg} = \left(-mg-k\left(u-u_0\right)\right)\delta u - mgL\sin\theta\delta\theta.$$

小球的速度为

$$\dot{\boldsymbol{r}}_{mg} = L\cos\theta\dot{\theta}\boldsymbol{i} + \left(\dot{u} + L\sin\theta\dot{\theta}\right)\boldsymbol{j}.$$

其动能为

$$T = \frac{1}{2}m\dot{\boldsymbol{r}}_{mg}\cdot\dot{\boldsymbol{r}}_{mg} = \frac{1}{2}m\left(L\cos\theta\dot{\theta}\right)^2 + \frac{1}{2}m\left(\dot{u}+L\sin\theta\dot{\theta}\right)^2 = \frac{1}{2}m\left[L^2\dot{\theta}^2 + \dot{u}^2 + 2\dot{u}L\sin\theta\dot{\theta}\right].$$

因此

$$\delta T = \left(m\dot{u} + mL\sin\theta\dot{\theta}\right)\delta\dot{u} + mL\cos\theta\dot{u}\dot{\theta}\delta\theta + \left(mL^2\dot{\theta} + mL\sin\theta\dot{u}\right)\delta\dot{\theta}.$$

将其代入广义 Hamilton 原理有，

$$\int_{t_1}^{t_2}\left(\delta W + \delta T\right)\mathrm{d}t = \int_{t_1}^{t_2}\begin{pmatrix}\left(-mg-k\left(u-u_0\right)\right)\delta u - mgL\sin\theta\delta\theta \\ +\left(m\dot{u}+mL\sin\theta\dot{\theta}\right)\delta\dot{u} + mL\cos\theta\dot{u}\dot{\theta}\delta\theta + \left(mL^2\dot{\theta}+mL\sin\theta\dot{u}\right)\delta\dot{\theta}\end{pmatrix}\mathrm{d}t$$

$$= \int_{t_1}^{t_2}\left(\left(-mg-k\left(u-u_0\right)\right)\delta u + \left(m\dot{u}+mL\sin\theta\dot{\theta}\right)\delta\dot{u} + \left(mL\cos\theta\dot{u}\dot{\theta} - mgL\sin\theta\right)\delta\theta + \left(mL^2\dot{\theta}+mL\sin\theta\dot{u}\right)\delta\dot{\theta}\right)\mathrm{d}t$$

$$= \int_{t_1}^{t_2}\begin{pmatrix}\left(-mg-k\left(u-u_0\right)\right)\delta u - \left(m\ddot{u}+mL\sin\theta\ddot{\theta}+mL\cos\theta\dot{\theta}^2\right)\delta u \\ +\left(mL\cos\theta\dot{u}\dot{\theta}-mgL\sin\theta\right)\delta\theta - \left(mL^2\ddot{\theta}+mL\sin\theta\ddot{u}+mL\cos\theta\dot{u}\dot{\theta}\right)\delta\theta\end{pmatrix}\mathrm{d}t + \left[\begin{matrix}\left(m\dot{u}+mL\sin\theta\dot{\theta}\right)\delta u \\ +\left(mL^2\dot{\theta}+mL\sin\theta\dot{u}\right)\delta\theta\end{matrix}\right]_{t_0}^{t_1}$$

$$= \int_{t_1}^{t_2}\begin{pmatrix}\left(-mg-k\left(u-u_0\right)-m\ddot{u}-mL\sin\theta\ddot{\theta}-mL\cos\theta\dot{\theta}^2\right)\delta u \\ +\left(mL\cos\theta\dot{u}\dot{\theta}-mgL\sin\theta-mL^2\ddot{\theta}-mL\sin\theta\ddot{u}-mL\cos\theta\dot{u}\dot{\theta}\right)\delta\theta\end{pmatrix}\mathrm{d}t$$

由于δu与$\delta\theta$是任意的，因此

$$-mg-k\left(u-u_0\right)-m\ddot{u}-mL\sin\theta\ddot{\theta}-mL\cos\theta\dot{\theta}^2=0,$$

$$-mgL\sin\theta-mL^2\ddot{\theta}-mL\sin\theta\ddot{u}=0.$$

由于$u=0$时系统处于平衡位置，因此

$$\delta W=\left(-mg-k\left(0-u_0\right)\right)\delta u-mgL\sin\theta\delta\theta=0,$$

同样由δu与$\delta\theta$的任意性

$$-mg+ku_0=0, mgL\sin\theta=0.$$

即$u_0=mg/k,\theta=0$，上述方程可整理为，

$$m\ddot{u}+mL\sin\theta\ddot{\theta}+mL\cos\theta\dot{\theta}^2+ku=0,$$

$$m\sin\theta\ddot{u}+mL\ddot{\theta}+mg\sin\theta=0.$$

接下来考虑在平衡位置$\bar{u}=0,\bar{\theta}=0$附近的小振幅振动，代入上述方程并忽略高阶小量有

$$m\ddot{u}+ku=0,$$

$$L\ddot{\theta}+g\theta=0.$$

方程所示系统转化为两个独立的系统：弹簧质量系统、摆球系统。

6.15 解：（1）以$u=0$时杆的上端点为坐标原点，**向右**和**向上**为 x，y 正方向建立坐标系。则杆端小球的坐标可以写为，

$$x=L\sin\theta, y=u-L\cos\theta.$$

弹簧在x方向上受到限制，设在x方向上的约束力为N，方向**向右**。则弹簧传递到杆上的力可以表示为，

$$F_{rod}\cos\theta=F_k=-k\left(u-u_0\right),$$

$$F_{rod}\sin\theta=-N.$$

因此杆端小球的运动方程为

$$-N=-m\ddot{x},$$

$$-k\left(u-u_0\right)-mg=m\ddot{y}.$$

因此整个系统方程为

$$-N=-m\ddot{x},-k\left(u-u_0\right)-mg=m\ddot{y},$$

$$F_{rod}\cos\theta=F_k=-k\left(u-u_0\right),F_{rod}\sin\theta=-N,x=L\sin\theta, y=u-L\cos\theta.$$

（2）杆端小球在x与y方向上的速度分量分别为

$$v_x=L\cos\theta\dot{\theta}, v_y=\dot{u}+L\sin\theta\dot{\theta}.$$

其动能为

$$T=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2=\frac{1}{2}m\left(L^2\dot{\theta}^2+\dot{u}^2+2L\sin\theta\dot{u}\dot{\theta}\right).$$

以时小球位置为其势能零点，则系统势能表示为

$$V=\frac{1}{2}k\left(u-u_0\right)^2+mg\left(u-L\cos\theta+L\right).$$

Lagrangian 为

$$L_s=T-V=\frac{1}{2}m\left(L^2\dot{\theta}^2+\dot{u}^2+2L\sin\theta\dot{u}\dot{\theta}\right)-\frac{1}{2}k\left(u-u_0\right)^2-mg\left(u-L\cos\theta+L\right).$$

将其代入 Lagrange 方程

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_s}{\partial\dot{u}}\right)-\frac{\partial L}{\partial u}=m\ddot{u}+mL\sin\theta\ddot{\theta}+mL\cos\theta\dot{\theta}^2-\left(-k\left(u-u_0\right)-mg\right)=0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_s}{\partial\dot{\theta}}\right)-\frac{\partial L_s}{\partial\theta}=mL^2\ddot{\theta}+mL\sin\theta\ddot{u}+mL\cos\theta\dot{\theta}\dot{u}-\left(mL\cos\theta\dot{\theta}\dot{u}-mgL\sin\theta\right)=0.$$

代入$u_0=mg/k$，并整理后得到，

$$m\ddot{u}+mL\sin\theta\ddot{\theta}+mL\cos\theta\dot{\theta}^2+ku=0,$$

$$mL\ddot{\theta}+m\sin\theta\ddot{u}+mg\sin\theta=0.$$

（3）Newton 定律得出的方程包含系统各部分之间的内力以及边界处的约束力；而在 Lagrange 方程中微分方程的个数取决于系统的自由度数目，并且系统的约束体现在方程中自由度的减少。

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（4）为了将 Newton 定律得到的方程化为 Lagrange 方程，将 x 与 y 的表达式代入并消去 F_{rod} 与 N，

$$-k\left(u-u_0\right)\sin\theta = -m\left(L\cos\theta\ddot{\theta} - L\sin\theta\dot{\theta}^2\right)\cos\theta,$$

$$-k\left(u-u_0\right) - mg = m\left(\ddot{u} + L\sin\theta\ddot{\theta} + L\cos\theta\dot{\theta}^2\right).$$

然后整理后有

$$-mL\cos\theta\cos\theta\ddot{\theta} + mL\cos\theta\sin\theta\dot{\theta}^2 + ku\sin\theta - mg\sin\theta = 0,$$

$$m\ddot{u} + mL\sin\theta\ddot{\theta} + mL\cos\theta\dot{\theta}^2 + ku = 0.$$

后式与 Lagrange 方程第一式相同，然后将后式乘以 $\sin\theta$ 并与前式相减可以得到，

$$m\ddot{u} + mL\sin\theta\ddot{\theta} + mL\cos\theta\dot{\theta}^2 + ku = 0,$$

$$mL\ddot{\theta} + m\ddot{u}\sin\theta + mg\sin\theta = 0.$$

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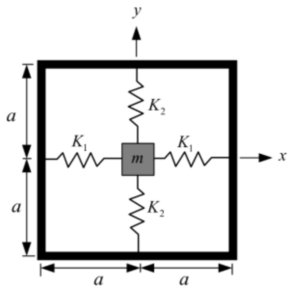
Problem 4: Consider a 2-dimensional harmonic oscillator shown below. A mass m is free to move in the xy plane. It is coupled to the walls by two unstretched massless springs of spring constant K_1 oriented along x and by two unstretched massless springs of spring constant K_2 oriented along y. Please note that the gravity effect can be ignored in this problem.

(1) Please derive the EOM for the mass m based on the Euler-Lagrange Equation. (8 points)

(2) In the small-oscillation approximation, please derive the linearized EOM and prove that the motion along the x and y directions are uncoupled, and each is a harmonic oscillation with its own frequency. (7 points)



解：（1）质量块可以在平面内运动，为两自由度系统。取横向方向与竖直方向的位移 x 和 y 为广义坐标，并设如图所示的弹簧无拉伸状态为 $x=0, y=0$，则质量快的速度为

$$v_x = \dot{x}, v_y = \dot{y}.$$

系统的动能可以表示为

$$T = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right).$$

弹簧的原长均为 a，势能可以表示为弹簧伸长量的函数，

$$V = \frac{1}{2}K_1\left(\sqrt{(x+a)^2+y^2}-a\right)^2 + \frac{1}{2}K_1\left(\sqrt{(x-a)^2+y^2}-a\right)^2 + \frac{1}{2}K_2\left(\sqrt{x^2+(y+a)^2}-a\right)^2 + \frac{1}{2}K_2\left(\sqrt{x^2+(y-a)^2}-a\right)^2.$$

则 Lagrangian 可以表示为

$$\begin{aligned} L = T - V = {} & \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) - \frac{1}{2}K_1\left(\sqrt{(x+a)^2+y^2}-a\right)^2 - \frac{1}{2}K_1\left(\sqrt{(x-a)^2+y^2}-a\right)^2 \\ & -\frac{1}{2}K_2\left(\sqrt{x^2+(y+a)^2}-a\right)^2 - \frac{1}{2}K_2\left(\sqrt{x^2+(y-a)^2}-a\right)^2. \end{aligned}$$

将其代入 Euler-Lagrange 方程有

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} + \left(\begin{aligned} & K_1\left(\sqrt{(x+a)^2+y^2}-a\right)\frac{x+a}{\sqrt{(x+a)^2+y^2}} + K_1\left(\sqrt{(x-a)^2+y^2}-a\right)\frac{x-a}{\sqrt{(x-a)^2+y^2}} \\ & + K_2\left(\sqrt{x^2+(y+a)^2}-a\right)\frac{x}{\sqrt{x^2+(y+a)^2}} + K_2\left(\sqrt{x^2+(y-a)^2}-a\right)\frac{x}{\sqrt{x^2+(y-a)^2}} \end{aligned}\right) = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = m\ddot{y} + \left(\begin{aligned} & K_1\left(\sqrt{(x+a)^2+y^2}-a\right)\frac{y}{\sqrt{(x+a)^2+y^2}} + K_1\left(\sqrt{(x-a)^2+y^2}-a\right)\frac{y}{\sqrt{(x-a)^2+y^2}} \\ & + K_2\left(\sqrt{x^2+(y+a)^2}-a\right)\frac{y+a}{\sqrt{x^2+(y+a)^2}} + K_2\left(\sqrt{x^2+(y-a)^2}-a\right)\frac{y-a}{\sqrt{x^2+(y-a)^2}} \end{aligned}\right) = 0.$$

上式可以化为

$$m\ddot{x} + K_1\left(2x - a\frac{x+a}{\sqrt{(x+a)^2+y^2}} - a\frac{x-a}{\sqrt{(x-a)^2+y^2}}\right) + K_2\left(2x - a\frac{x}{\sqrt{x^2+(y+a)^2}} - a\frac{x}{\sqrt{x^2+(y-a)^2}}\right) = 0,$$

$$m\ddot{y} + K_1\left(2y - a\frac{y}{\sqrt{(x+a)^2+y^2}} - a\frac{y}{\sqrt{(x-a)^2+y^2}}\right) + K_2\left(2y - a\frac{y+a}{\sqrt{x^2+(y+a)^2}} - a\frac{y-a}{\sqrt{x^2+(y-a)^2}}\right) = 0.$$

（2）假设质量块以小振幅振动，则

$$\frac{x+a}{\sqrt{(x+a)^2+y^2}} \approx \frac{x+a}{|x+a|}, \frac{x-a}{\sqrt{(x-a)^2+y^2}} \approx \frac{x-a}{|x-a|}, \frac{x}{\sqrt{x^2+(y+a)^2}} \approx \frac{x}{|y+a|}, \frac{x}{\sqrt{x^2+(y-a)^2}} \approx \frac{x}{|y-a|},$$

$$a\frac{y}{\sqrt{(x+a)^2+y^2}} \approx \frac{y}{|x+a|}, \frac{y}{\sqrt{(x-a)^2+y^2}} \approx \frac{y}{|x-a|}, \frac{y+a}{\sqrt{x^2+(y+a)^2}} \approx \frac{y+a}{|y+a|}, \frac{y-a}{\sqrt{x^2+(y-a)^2}} \approx \frac{y-a}{|y-a|}.$$

注意到 $x, y \to 0$ 且 $a > 0\left(a > |x|, |y|\right)$ 时，

$$|x+a| = x+a, |x-a| = a-x, |y+a| = y+a, |y+a| = a-y.$$

将上述两式代入运动方程中有，

$$m\ddot{x} + K_1\left(2x - a\frac{x+a}{|x+a|} - a\frac{x-a}{|x-a|}\right) + K_2\left(2x - a\frac{x}{|y+a|} - a\frac{x}{|y-a|}\right) = 0,$$

$$m\ddot{y} + K_1\left(2y - a\frac{y}{|x+a|} - a\frac{y}{|x-a|}\right) + K_2\left(2y - a\frac{y+a}{|y+a|} - a\frac{y-a}{|y-a|}\right) = 0.$$

即

$$m\ddot{x} + K_1\left(2x - a - a(-1)\right) + K_2\left(2x - a\left(\frac{x}{y+a} + \frac{x}{a-y}\right)\right) = 0,$$

$$m\ddot{y} + K_1\left(2y - a\left(\frac{y}{x+a} + \frac{y}{a-x}\right)\right) + K_2\left(2y - a - a(-1)\right) = 0.$$

进一步地，忽略其中的高阶小量，

$$\frac{x}{y+a} + \frac{x}{a-y} = x\frac{2a}{a^2-y^2} \approx \frac{2x}{a},$$

$$\frac{y}{x+a} + \frac{y}{a-x} = y\frac{2a}{a^2-x^2} \approx \frac{2y}{a}.$$

将其代入

$$m\ddot{x} + 2xK_1 + K_2\left(2x - a\frac{2x}{a}\right) = 0,$$

$$m\ddot{y} + K_1\left(2y - a\frac{2y}{a}\right) + 2yK_2 = 0.$$

得到最终的线性化方程，

$$m\ddot{x} + 2K_1x = 0,$$

$$m\ddot{y} + 2K_2y = 0.$$

在两个方向上没有耦合，且其固有频率分别为，

$$\omega_x = \sqrt{\frac{2K_1}{m}}, \omega_y = \sqrt{\frac{2K_2}{m}}.$$

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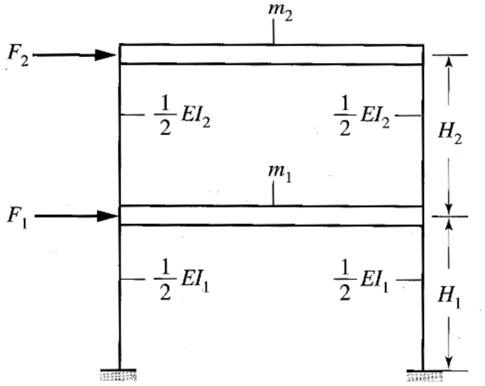
Problem 5: Please solve Problem 5.8 on page 258 of the textbook through the Euler-Lagrange equation. (15 points)
5.8. Figure 5.27 depicts a two-story building. Assume that the horizontal members are rigid and that the columns are massless beams clamped at both ends and derive the differential equations for the horizontal translation of the masses.



解：在推导动力学方程之前，我们首先给出两端固支梁的等效刚度。根据基本梁理论，一端固定，另一端的横向位移 $w(x)$ 满足微分方程，

$$EI(x)\frac{\mathrm{d}w^2(x)}{\mathrm{d}x^2}=M(x),0<x<L.$$

假设梁端受到横向方向的力为 F ，伴随着弯矩 M_0 ，则轴向上一点处的弯矩为，

$$M(x)=M_0+F(L-x),0<x<L.$$

然后考虑边界条件，

$$w(0)=0,\frac{\mathrm{d}w}{\mathrm{d}x}=0,x=0,$$

$$\frac{\mathrm{d}w}{\mathrm{d}x}=0,x=L.$$

将横向位移微分方程一次积分后有，

$$\frac{\mathrm{d}w(x)}{\mathrm{d}x}=\int_0^x\frac{M(\xi)}{EI(\xi)}\mathrm{d}\xi,0<x<L.$$

假设截面惯性矩和杨氏模量为定值，则有，

$$\frac{\mathrm{d}w(x)}{\mathrm{d}x}=\frac{1}{EI}\int_0^x M_0+F(L-\xi)\mathrm{d}\xi=\frac{1}{EI}\left((M_0+FL)x-\frac{F}{2}x^2+C\right),0<x<L.$$

代入边界条件后有 $C=0,M_0=-FL/2$ ，

$$\frac{\mathrm{d}w(x)}{\mathrm{d}x}=\frac{F}{EI}\left(\frac{L}{2}x-\frac{1}{2}x^2\right),0<x<L.$$

然后对上式进行第二次积分

$$w(x)=\int_0^x\frac{F}{EI}\left(\frac{L}{2}x-\frac{1}{2}x^2\right)\mathrm{d}x=\frac{F}{EI}\left(\frac{1}{4}Lx^2-\frac{1}{6}x^3+C\right),0<x<L.$$

代入固定端边界条件后有 $C=0$

$$w(x)=\frac{F}{EI}\left(\frac{1}{4}Lx^2-\frac{1}{6}x^3\right),0<x<L.$$

因此两端固支的梁弯曲时横向的等效刚度为，

$$k_{eq}=\frac{F}{w(L)}=F\left(\frac{FL^3}{12EI}\right)^{-1}=\frac{12EI}{L^3}.$$

接下来考虑本题，竖直方向的四条梁中，左右是完全一致的，因此只用考虑上下梁的区别，其等效刚度分别为，

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$$k_{eq}^{\text{top}}=\frac{12\left(EI_2/2\right)}{\left(H_2\right)^3}=\frac{6EI_2}{H_2^3},k_{eq}^{\text{bottom}}=\frac{12\left(EI_1/2\right)}{\left(H_1\right)^3}=\frac{6EI_1}{H_1^3}.$$

系统的两根横梁运动独立，是一个两自由度系统。取两根横梁的横向方向位移 x_1,x_2 为广义坐标，则系统的动能可以表示成，

$$T=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2.$$

系统的势能为，

$$V=2\times\frac{1}{2}k_{eq}^{\text{top}}\left(x_2-x_1\right)^2+2\times\frac{1}{2}k_{eq}^{\text{bottom}}x_1^2=\frac{6EI_2}{H_2^3}\left(x_2-x_1\right)^2+\frac{6EI_1}{H_1^3}x_1^2.$$

则 Lagrangian 为，

$$L=T-V=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2-\frac{6EI_2}{H_2^3}\left(x_2-x_1\right)^2-\frac{6EI_1}{H_1^3}x_1^2.$$

非保守外力的广义力可以表示为，

$$Q_1=F_1\frac{\partial x_1}{\partial x_1}+F_2\frac{\partial x_2}{\partial x_1}=F_1,Q_2=F_1\frac{\partial x_1}{\partial x_2}+F_2\frac{\partial x_2}{\partial x_2}=F_2.$$

将上述结果代入 Lagrange 方程有，

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}=m_1\ddot{x}_1+\frac{12EI_2}{H_2^3}\left(x_1-x_2\right)+\frac{12EI_1}{H_1^3}x_1=Q_1,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\frac{\partial L}{\partial x_2}=m_2\ddot{x}_2+\frac{12EI_2}{H_2^3}\left(x_2-x_1\right)=Q_2.$$

整理后有，

$$m_1\ddot{x}_1+\left(\frac{12EI_2}{H_2^3}+\frac{12EI_1}{H_1^3}\right)x_1-\frac{12EI_2}{H_2^3}x_2=F_1,$$
$$m_2\ddot{x}_2-\frac{12EI_2}{H_2^3}x_1+\frac{12EI_2}{H_2^3}x_2=F_2.$$

写成矩阵形式

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} \frac{12EI_2}{H_2^3}+\frac{12EI_1}{H_1^3} & -\frac{12EI_2}{H_2^3} \\ -\frac{12EI_2}{H_2^3} & \frac{12EI_2}{H_2^3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}.$$

选取广义坐标不同时的结果为

$$\begin{bmatrix} m_1+m_2 & m_2 \\ m_2 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} \frac{12EI_2}{H_2^3} & 0 \\ 0 & \frac{12EI_2}{H_2^3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} F_1+F_2 \\ F_2 \end{bmatrix}.$$

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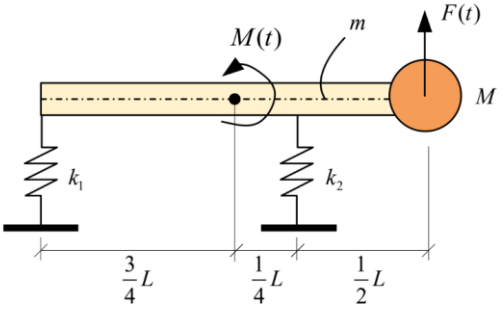
Problem 6: A rigid bar of mass per unit length m carries a point mass M at its right end. The bar is supported by two springs, as shown below. Assuming small motions, please derive the linear EOM for the translation and rotation of the mass center around the system's equilibrium configuration. (15 points)



解：假设如图所示的弹簧为原长，系统是一个两自由度系统。选取杆质心处竖直方向的位移 y （向上为正方向）以及质心处的转角 θ （逆时针方向为正方向）为广义坐标， $y=0,\theta=0$ 对应于如图所示的位置。则弹簧的形变以及小球的位移可以分别表示为，

$$y_1 = y - \frac{3L}{4}\sin\theta, y_2 = y + \frac{L}{4}\sin\theta,$$
$$x_M = -\left(\frac{L}{4}+\frac{L}{2}\right)\cos\theta, y_M = y + \left(\frac{L}{4}+\frac{L}{2}\right)\sin\theta.$$

则系统的动能可以表示为，

$$T = \frac{1}{2}M\left(\dot{x}_M^2 + \dot{y}_M^2\right) + \frac{1}{2}\left(m\frac{3L}{2}\right)\dot{y}^2 + \frac{1}{2}J_C\dot{\theta}^2.$$

其中 J_C 为杆绕质心的转动惯量，其形式为，

$$J_C = \frac{1}{12}\frac{3mL}{2}\left(\frac{3L}{2}\right)^2 = \frac{9mL^3}{32}.$$

假设如图所示位置为杆和小球的重力势能零势能面，则系统势能可以表示为，

$$V = \frac{1}{2}k_1\left(y - \frac{3L\sin\theta}{4}\right)^2 + \frac{1}{2}k_2\left(y + \frac{L\sin\theta}{4}\right)^2 + Mg\left(y + \frac{3L\sin\theta}{4}\right) + \frac{3}{2}mgLy.$$

则 Lagrangian 为，

$$L = T - V = \frac{9L^2}{64}(mL+2M)\dot{\theta}^2 + \frac{3}{4}ML\cos\theta\dot{y}\dot{\theta} + \frac{1}{4}(2M+3mL)\dot{y}^2 +$$
$$-\frac{1}{2}k_1\left(y - \frac{3L\sin\theta}{4}\right)^2 - \frac{1}{2}k_2\left(y + \frac{L\sin\theta}{4}\right)^2 - Mg\left(y + \frac{3L\sin\theta}{4}\right) - \frac{3}{2}mgLy.$$

系统受到非保守力的广义力为，

$$Q_y = F(t)\frac{\partial y_M}{\partial y} = F(t), Q_\theta = F(t)\frac{\partial y_M}{\partial \theta} + M(t)\frac{\partial\theta}{\partial\theta} = \frac{3L\cos\theta}{4}F(t) + M(t).$$

其运动方程由 Euler-Lagrange 方程给出如下，

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = \left[\begin{array}{l} \frac{3}{4}ML\cos\theta\ddot{\theta} - \frac{3}{4}ML\sin\theta\dot{\theta}^2 + \frac{1}{2}(2M+3mL)\ddot{y} \\ +k_1\left(y - \frac{3L\sin\theta}{4}\right) + k_2\left(y + \frac{L\sin\theta}{4}\right) + Mg + \frac{3}{2}mgL \end{array}\right] = F(t),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \left[\begin{array}{l} \frac{9L^2}{32}(mL+2M)\ddot{\theta} + \frac{3}{4}ML\cos\theta\ddot{y} - \frac{3}{4}ML\sin\theta\dot{y}\dot{\theta} - \left(-\frac{3L\sin\theta}{4}\dot{y}\dot{\theta}\right) \\ +k_1\frac{3L\cos\theta}{4}\left(\frac{3L\sin\theta}{4} - y\right) + k_2\frac{L\cos\theta}{4}\left(y + \frac{L\sin\theta}{4}\right) + Mg\frac{3L\cos\theta}{4} \end{array}\right] = \frac{3L\cos\theta}{4}F(t) + M(t).$$

对上式进行化简有，

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$$\left(M+\frac{3mL}{2}\right)\ddot{y}+\frac{3}{4}ML\cos\theta\ddot{\theta}-\frac{3}{4}ML\sin\theta\dot{\theta}^2+\left(k_1+k_2\right)y+\frac{\left(k_2-3k_1\right)L}{4}\sin\theta+Mg+\frac{3}{2}mgL=F\left(t\right),$$

$$\frac{3}{4}ML\cos\theta\ddot{y}+\frac{9L^2}{32}\left(mL+2M\right)\ddot{\theta}+\frac{\left(k_2-3k_1\right)L}{4}\cos\theta y+\frac{\left(9k_1+k_2\right)L^2}{16}\sin\theta\cos\theta+\frac{3MgL\cos\theta}{4}=\frac{3L\cos\theta}{4}F\left(t\right)+M\left(t\right).$$

当忽略外力 $F\left(t\right)$ 的影响 $\dot{y}=0,\dot{\theta}=0$ 时可以得到系统的平衡位置，

$$\frac{\left(9k_1+k_2\right)L^2}{16}\sin\bar{\theta}+\frac{\left(k_2-3k_1\right)L}{4}\bar{y}+\frac{3MgL}{4}=0,$$

$$\frac{\left(k_2-3k_1\right)L}{4}\sin\bar{\theta}+\left(k_1+k_2\right)\bar{y}+Mg+\frac{3mgL}{2}=0.$$

假设系统在平衡位置附近做微小振动 $\tilde{y}=y-\bar{y},\tilde{\theta}=\theta-\bar{\theta}$，代入运动方程中有，

$$\begin{bmatrix}\left(M+\frac{3mL}{2}\right)\ddot{\tilde{y}}+\frac{3}{4}ML\cos\left(\tilde{\theta}+\bar{\theta}\right)\ddot{\tilde{\theta}}-\frac{3}{4}ML\sin\left(\tilde{\theta}+\bar{\theta}\right)\dot{\tilde{\theta}}^2\\+\left(k_1+k_2\right)\left(\tilde{y}+\bar{y}\right)+\frac{\left(k_2-3k_1\right)L}{4}\sin\left(\tilde{\theta}+\bar{\theta}\right)+Mg+\frac{3}{2}mgL\end{bmatrix}=F\left(t\right),$$

$$\begin{bmatrix}\frac{3}{4}ML\cos\left(\tilde{\theta}+\bar{\theta}\right)\ddot{\tilde{y}}+\frac{9L^2}{32}\left(mL+2M\right)\ddot{\tilde{\theta}}+\frac{\left(k_2-3k_1\right)L}{4}\cos\left(\tilde{\theta}+\bar{\theta}\right)\left(\tilde{y}+\bar{y}\right)\\+\frac{\left(9k_1+k_2\right)L^2}{16}\sin\left(\tilde{\theta}+\bar{\theta}\right)\cos\left(\tilde{\theta}+\bar{\theta}\right)+\frac{3MgL\cos\left(\tilde{\theta}+\bar{\theta}\right)}{4}\end{bmatrix}=\frac{3L\cos\left(\tilde{\theta}+\bar{\theta}\right)}{4}F\left(t\right)+M\left(t\right).$$

在小振幅假设下有

$$\cos\left(\tilde{\theta}+\bar{\theta}\right)=\cos\bar{\theta}\cos\tilde{\theta}-\sin\bar{\theta}\sin\tilde{\theta}\approx\cos\bar{\theta}-\sin\bar{\theta}\tilde{\theta},$$

$$\sin\left(\tilde{\theta}+\bar{\theta}\right)=\sin\bar{\theta}\cos\tilde{\theta}+\cos\bar{\theta}\sin\tilde{\theta}\approx\sin\bar{\theta}+\cos\bar{\theta}\tilde{\theta}.$$

代入平衡位置的方程并进行化简，

$$\left(M+\frac{3mL}{2}\right)\ddot{\tilde{y}}+\frac{3}{4}ML\cos\bar{\theta}\ddot{\tilde{\theta}}+\left(k_1+k_2\right)\tilde{y}+\frac{\left(k_2-3k_1\right)L}{4}\cos\bar{\theta}\tilde{\theta}=F\left(t\right),$$

$$\begin{bmatrix}\frac{3}{4}M\cos\bar{\theta}\ddot{\tilde{y}}+\frac{9L}{32}\left(mL+2M\right)\ddot{\tilde{\theta}}+\frac{\left(k_2-3k_1\right)}{4}\cos\bar{\theta}\tilde{y}\\+\left(\frac{\left(9k_1+k_2\right)L}{16}\cos 2\bar{\theta}-\frac{\left(k_2-3k_1\right)}{4}\sin\bar{\theta}\bar{y}-\frac{3Mg\sin\bar{\theta}}{4}\right)\tilde{\theta}\end{bmatrix}=\frac{3\cos\bar{\theta}}{4}F\left(t\right)+M\left(t\right).$$

代入 $\frac{\left(9k_1+k_2\right)L^2}{16}\sin\bar{\theta}+\frac{\left(k_2-3k_1\right)L}{4}\bar{y}+\frac{3MgL}{4}=0$ 可得矩阵形式方程为，

$$\begin{bmatrix}M+\frac{3mL}{2} & \frac{3}{4}ML\cos\bar{\theta}\\ \frac{3}{4}M\cos\bar{\theta} & \frac{9L}{32}\left(mL+2M\right)\end{bmatrix}\begin{bmatrix}\ddot{\tilde{y}}\\ \ddot{\tilde{\theta}}\end{bmatrix}+\begin{bmatrix}k_1+k_2 & \frac{\left(k_2-3k_1\right)L}{4}\cos\bar{\theta}\\ \frac{\left(k_2-3k_1\right)}{4}\cos\bar{\theta} & \frac{\left(9k_1+k_2\right)L}{16}\cos^2\bar{\theta}\end{bmatrix}\begin{bmatrix}\tilde{y}\\ \tilde{\theta}\end{bmatrix}=\begin{bmatrix}F\left(t\right)\\ \frac{3\cos\bar{\theta}}{4}F\left(t\right)+M\left(t\right)\end{bmatrix}.$$

注意，若选取的广义坐标为 y',θ' 满足，

$$\left(M+\frac{3mL}{2}\right)x'=\frac{3ML}{4},$$

$$y'=y+x'\sin\theta',\theta'=\theta.$$

则 y',θ' 表示整个系统质心处的竖向位移及旋转角度，上述方程化为，

$$\left[\begin{aligned}&\left(M+\frac{3mL}{2}\right)\left(\ddot{y}'-x'\cos\theta'\ddot{\theta}'+x'\sin\theta'\dot{\theta}'^2\right)+\frac{3}{4}ML\cos\theta'\ddot{\theta}'-\frac{3}{4}ML\sin\theta'\dot{\theta}'^2\\&+\left(k_1+k_2\right)\left(y'-x'\sin\theta'\right)+\frac{\left(k_2-3k_1\right)L}{4}\sin\theta'+Mg+\frac{3}{2}mgL\end{aligned}\right]=F(t),$$

$$\left[\begin{aligned}&\frac{3}{4}ML\cos\theta'\left(\ddot{y}'-x'\cos\theta'\ddot{\theta}'+x'\sin\theta'\dot{\theta}'^2\right)+\frac{9L^2}{32}\left(mL+2M\right)\ddot{\theta}'\\&+\frac{\left(k_2-3k_1\right)L}{4}\cos\theta'\left(y'-x'\sin\theta'\right)+\frac{\left(9k_1+k_2\right)L^2}{16}\sin\theta'\cos\theta'+\frac{3MgL\cos\theta'}{4}\end{aligned}\right]=\frac{3L\cos\theta'}{4}F(t)+M(t).$$

对上式进行线性化可得，

$$\left(M+\frac{3mL}{2}\right)\ddot{\tilde{y}}+\left(k_1+k_2\right)\tilde{y}+\left[\frac{\left(k_2-3k_1\right)L}{4}-\frac{3ML\left(k_1+k_2\right)}{4M+6mL}\right]\cos\bar{\theta}\tilde{\theta}=F(t),$$

$$\left[\begin{aligned}&\frac{3}{4}M\cos\bar{\theta}\,\ddot{\tilde{y}}+\left(\frac{9L\left(mL+2M\right)}{32}-\frac{9M^2L\cos^2\bar{\theta}}{16M+24mL}\right)\ddot{\tilde{\theta}}\\&+\frac{\left(k_2-3k_1\right)}{4}\cos\bar{\theta}\,\tilde{y}+\left(\frac{\left(9k_1+k_2\right)L}{16}-\frac{3ML\left(k_2-3k_1\right)}{16M+24mL}\right)\cos^2\bar{\theta}\tilde{\theta}\end{aligned}\right]=\frac{3\cos\bar{\theta}}{4}F(t)+M(t).$$

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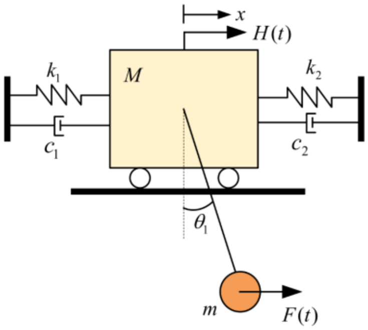
Problem 7: Assuming small motions, please derive the linear EOM for the system shown below. (15 points)



解：假设如图所示的弹簧为原长，系统是一个两自由度系统。选取小车质心处横向方向的位移 x（向右为正方向）以及小球的转角 θ_1（逆时针方向为正方向）为广义坐标， $x=0$ 对应于如图所示的位置，而 $\theta_1=0$ 对应于小球竖直向下的状态。小球的位移可以表示为，

$$x_m = x + L\sin\theta_1, y_m = -L\cos\theta.$$

系统的动能为，

$$T = \frac{1}{2}m\left(\dot{x}_m^2+\dot{y}_m^2\right)+\frac{1}{2}M\dot{x}^2 = \frac{1}{2}M\dot{x}^2+\frac{1}{2}m\left(\dot{x}^2+L^2\dot{\theta}_1^2\right)+mL\cos\theta_1\dot{\theta}_1\dot{x}.$$

假设零势能点位于小车的质心处，则系统势能为，

$$V = \frac{1}{2}k_1x^2+\frac{1}{2}k_2x^2-mgL\cos\theta_1.$$

则 Lagrangian 可以表示为，

$$L = T-V = \frac{1}{2}M\dot{x}^2+\frac{1}{2}m\left(\dot{x}^2+L^2\dot{\theta}_1^2\right)+mL\cos\theta_1\dot{\theta}_1\dot{x}-\frac{1}{2}k_1x^2-\frac{1}{2}k_2x^2+mgL\cos\theta_1.$$

非保守力的广义力由阻尼器和两个外力提供，

$$Q_\theta = F(t)\frac{\partial x_m}{\partial\theta_1}+H(t)\frac{\partial x}{\partial\theta_1}-c_1\dot{x}\frac{\partial x}{\partial\theta_1}-c_2\dot{x}\frac{\partial x}{\partial\theta_1} = F(t)L\cos\theta_1,$$

$$Q_x = F(t)\frac{\partial x_m}{\partial x}+H(t)\frac{\partial x}{\partial x}-c_1\dot{x}\frac{\partial x}{\partial x}-c_2\dot{x}\frac{\partial x}{\partial x} = F(t)+H(t)-(c_1+c_2)\dot{x}.$$

将其代入第二类 Lagrange 方程有，

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right)-\frac{\partial L}{\partial\theta_1} = mL^2\ddot{\theta}_1+mL\cos\theta_1\ddot{x}-mL\sin\theta_1\dot{x}\dot{\theta}_1-\left(-mL\sin\theta_1\dot{\theta}_1\dot{x}-mgL\sin\theta_1\right) = F(t)L\cos\theta_1,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{x}}\right)-\frac{\partial L}{\partial x} = M\ddot{x}+m\ddot{x}+mL\cos\theta_1\ddot{\theta}_1-mL\sin\theta_1\dot{\theta}_1^2-\left(-k_1x-k_2x\right) = F(t)+H(t)-(c_1+c_2)\dot{x}.$$

整理后得到，

$$mL^2\ddot{\theta}_1+mL\cos\theta_1\ddot{x}+mgL\sin\theta_1 = F(t)L\cos\theta_1,$$

$$(M+m)\ddot{x}+mL\cos\theta_1\ddot{\theta}_1-mL\sin\theta_1\dot{\theta}_1^2+(c_1+c_2)\dot{x}+(k_1+k_2)x = F(t)+H(t).$$

在系统的平衡位置 $\bar{x}=0,\bar{\theta}_1=0$ 进行展开 $\sin\theta_1\approx\theta_1,\cos\theta_1\approx1$，并略去高阶项可得线性运动方程，

$$mL\ddot{\theta}_1+m\ddot{x}+mg\theta_1 = F(t),$$

$$(M+m)\ddot{x}+mL\ddot{\theta}_1+(c_1+c_2)\dot{x}+(k_1+k_2)x = F(t)+H(t).$$

写成矩阵形式，

$$\begin{bmatrix} mL & m \\ mL & M+m \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{x} \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & c_1+c_2 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{x} \end{bmatrix}+\begin{bmatrix} mg & 0 \\ 0 & k_1+k_2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ x \end{bmatrix}=\begin{bmatrix} F(t) \\ F(t)+H(t) \end{bmatrix}.$$

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LAGRANGE EQUATION OF THE FIRST KIND

Additional Problem: Please derive the Lagrange equation of the first kind for non-holonomic system. (20 points)

解：假设存在受约束的n个质点组成的质点系，完全独立时的自由度为$N=3n$，约束方程中可以写成广义坐标函数的显示形式的方程数为n_f，剩下的中完整约束方程数目为n_h，非完整约束方程数量为n_{nh}。由牛顿定律，第i个质点应当满足，

$$\sum_j \boldsymbol{F}_{ij} = m_i \ddot{\boldsymbol{r}}_i.$$

其中$\boldsymbol{F}_{ij}$表示作用在第i个质点上的第j个力矢量，$\boldsymbol{r}_i$表示第i个质点的位移矢量。因此整个系统的方程可以表示成，

$$\begin{aligned}
\sum_j \boldsymbol{F}_{ij} &= m_i \ddot{\boldsymbol{r}}_i, i=1,\ldots,n.\\
r_s &= f_s\left(r_1,r_2,\ldots,r_{N-n_f}\right), s=N-n_f+1,\ldots,N.\\
g_s &= g_s\left(r_1,r_2,\ldots,r_N,t\right)=0, s=1,2,\ldots,n_h.\\
h_s &= h_s\left(r_1,r_2,\ldots,r_N,\dot{r}_1,\dot{r}_2,\ldots,\dot{r}_N,t\right)=0, s=1,2,\ldots,n_h.
\end{aligned}$$

首先我们来看可以写出显示形式的约束，这部分约束可以将其重新写成完整约束的形式，

$$f_s = f_s\left(r_1,r_2,\ldots,r_{N-n_f}\right)-r_s=0, s=N-n_f+1,\ldots,N.$$

也可以在后边再进行处理，因此我们可以先假设$n_f=0$。

我们注意到上式中的力矢量可以表示成，

$$\sum_j \boldsymbol{F}_{ij} = \sum_j \alpha_j \boldsymbol{F}_{ij}^{\mathrm{a}} + \sum_{j=1}^{n_h} \beta_j \boldsymbol{F}_{ij}^{\mathrm{r}} + \sum_{j=1}^{n} \boldsymbol{F}_{ij}^{\mathrm{m}}.$$

三项$\boldsymbol{F}_{ij}^{\mathrm{a}},\boldsymbol{F}_{ij}^{\mathrm{r}},\boldsymbol{F}_{ij}^{\mathrm{m}}$分别表示外界施加的主动力、约束力以及其它质点的作用力，而系数$\alpha_j,\beta_j\in\{0,1\}$分别表示主动力和约束力是否直接作用于第$i$个质点。首先我们假设约束面是光滑的，则约束力可以表示成，

$$\boldsymbol{F}_{ij}^{\mathrm{r}} = F_{ij}^{\mathrm{r}}\left(\frac{\partial g_j}{\partial \tilde{q}_1}\boldsymbol{e}_1 + \frac{\partial g_j}{\partial \tilde{q}_2}\boldsymbol{e}_2 + \frac{\partial g_j}{\partial \tilde{q}_3}\boldsymbol{e}_3\right).$$

其中$\left(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\right)^T$为曲线坐标系中的一组正交基。在系统运动方程左右两侧同时乘以质点的虚位移$\delta\boldsymbol{r}_i$，则有

$$\left(\sum_j \alpha_j \boldsymbol{F}_{ij}^{\mathrm{a}} + \sum_{j=1}^{n_h} \beta_j \boldsymbol{F}_{ij}^{\mathrm{r}} + \sum_{j=1}^{n} \boldsymbol{F}_{ij}^{\mathrm{m}}\right)\cdot\delta\boldsymbol{r}_i = m_i\ddot{\boldsymbol{r}}_i\cdot\delta\boldsymbol{r}_i.$$

其中$\delta\boldsymbol{r}_i$满足所有约束方程，而$\boldsymbol{F}_{ij}^{\mathrm{r}}$与当前位置约束方程切平面的法线共线，即$\boldsymbol{F}_{ij}^{\mathrm{r}}\cdot\delta\boldsymbol{r}_i=0$。所以上式变成，

$$\left(\sum_j \alpha_j \boldsymbol{F}_{ij}^{\mathrm{a}} + \sum_{j=1}^{n} \boldsymbol{F}_{ij}^{\mathrm{m}}\right)\cdot\delta\boldsymbol{r}_i = m_i\ddot{\boldsymbol{r}}_i\cdot\delta\boldsymbol{r}_i.$$

接下来考虑其它质点的作用力，由相互作用力原理可以知道，

$$\boldsymbol{F}_{ij}^{\mathrm{m}} = -\boldsymbol{F}_{ji}^{\mathrm{m}}.$$

对运动方程左右求和可以得到，

$$\sum_{i=1}^{n}\sum_j \alpha_j \boldsymbol{F}_{ij}^{\mathrm{a}}\cdot\delta\boldsymbol{r}_i + \sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{F}_{ij}^{\mathrm{m}}\cdot\delta\boldsymbol{r}_i - \sum_{i=1}^{n} m_i\ddot{\boldsymbol{r}}_i\cdot\delta\boldsymbol{r}_i = 0.$$

第一项为所有外力在虚位移上做的虚功（势力场中的场力以及对系统施加的其他外力），第二项可以写为

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{F}_{ij}^{\mathrm{m}}\cdot\delta\boldsymbol{r}_i = \sum_{i=1}^{n}\sum_{j=1}^{i}\boldsymbol{F}_{ij}^{\mathrm{m}}\cdot\left(\delta\boldsymbol{r}_i-\delta\boldsymbol{r}_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{i}\boldsymbol{F}_{ij}^{\mathrm{m}}\cdot\delta\left(\boldsymbol{r}_i-\boldsymbol{r}_j\right).$$

例如最常见的弹簧力、阻尼器的阻尼力等。第三项可以写为，

$$-\sum_{i=1}^{n} m_i\ddot{\boldsymbol{r}}_i\cdot\delta\boldsymbol{r}_i = -\sum_{i=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i\dot{\boldsymbol{r}}_i\cdot\delta\boldsymbol{r}_i\right)-\left(m_i\dot{\boldsymbol{r}}_i\cdot\delta\dot{\boldsymbol{r}}_i\right)\right] = -\sum_{i=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i\dot{\boldsymbol{r}}_i\cdot\delta\boldsymbol{r}_i\right)-\delta\left(\frac{1}{2}m_i\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i\right)\right].$$

我们注意到上式中出现了微分形式。对求和以后的运动方程左右从 t_0 到 t_1 进行积分，

$$\int_{t_0}^{t_1}\left\{\sum_{i=1}^{n}\sum_{j}\alpha_j \boldsymbol{F}_{ij}^{\mathrm{a}}\cdot\delta\boldsymbol{r}_i+\sum_{i=1}^{n}\sum_{j=1}^{i}\boldsymbol{F}_{ij}^{\mathrm{m}}\cdot\delta\left(\boldsymbol{r}_i-\boldsymbol{r}_j\right)-\sum_{i=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i\dot{\boldsymbol{r}}_i\cdot\delta\boldsymbol{r}_i\right)-\delta\left(\frac{1}{2}m_i\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i\right)\right]\right\}\mathrm{d}t=0.$$

则有，

$$\int_{t_0}^{t_1}\left\{\sum_{i=1}^{n}\sum_{j}\alpha_j \boldsymbol{F}_{ij}^{\mathrm{a}}\cdot\delta\boldsymbol{r}_i+\sum_{i=1}^{n}\sum_{j=1}^{i}\boldsymbol{F}_{ij}^{\mathrm{m}}\cdot\delta\left(\boldsymbol{r}_i-\boldsymbol{r}_j\right)+\sum_{i=1}^{n}\delta\left(\frac{1}{2}m_i\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i\right)\right\}\mathrm{d}t-\sum_{i=1}^{n}\left(m_i\dot{\boldsymbol{r}}_i\cdot\delta\boldsymbol{r}_i\right)\Big|_{t_0}^{t_1}=0.$$

选取广义坐标为 $\boldsymbol{q}=\left(q_1,q_2,\ldots,q_{3n}\right)$，最后一项表示为

$$-\sum_{i=1}^{n}\left(m_i\dot{\boldsymbol{r}}_i\cdot\delta\boldsymbol{r}_i\right)\Big|_{t_0}^{t_1}=-\sum_{i=1}^{n}\left(m_i\sum_{j=1}^{3n}\frac{\partial\dot{\boldsymbol{r}}_i}{\partial q_j}\cdot\sum_{j=1}^{3n}\left(\frac{\partial\boldsymbol{r}_i}{\partial q_j}\delta q_j\right)\right)\Bigg|_{t_0}^{t_1}.$$

在 t_0 与 t_1 时刻满足，

$$\delta q_j\left(t_0\right)=\delta q_j\left(t_1\right)=0.$$

所以上式的结果为 0。因此系统的运动方程为，

$$\int_{t_0}^{t_1}\left[\sum_{i=1}^{n}\sum_{j}\alpha_j \boldsymbol{F}_{ij}^{\mathrm{a}}\cdot\delta\boldsymbol{r}_i+\sum_{i=1}^{n}\sum_{j=1}^{i}\boldsymbol{F}_{ij}^{\mathrm{m}}\cdot\delta\left(\boldsymbol{r}_i-\boldsymbol{r}_j\right)+\sum_{i=1}^{n}\delta\left(\frac{1}{2}m_i\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i\right)\right]\mathrm{d}t=0.$$

将上式的变分项进行类似的处理，为了简化形式，记 $T=\dfrac{1}{2}m_i\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i$，

$$\int_{t_0}^{t_1}\left[\sum_{i=1}^{n}\sum_{j}\alpha_j \boldsymbol{F}_{ij}^{\mathrm{a}}\cdot\sum_{r=1}^{3n}\frac{\partial\boldsymbol{r}_i}{\partial q_r}\delta q_r+\sum_{i=1}^{n}\sum_{j=1}^{i}\boldsymbol{F}_{ij}^{\mathrm{m}}\cdot\sum_{r=1}^{3n}\frac{\partial\left(\boldsymbol{r}_i-\boldsymbol{r}_j\right)}{\partial q_r}\delta q_r+\sum_{r=1}^{3n}\left(\frac{\partial T}{\partial\dot{q}_r}\delta\dot{q}_r+\frac{\partial T}{\partial q_r}\delta q_r\right)\right]\mathrm{d}t=0.$$

最后一项为

$$\int_{t_0}^{t_1}\sum_{r=1}^{3n}\left(\frac{\partial T}{\partial\dot{q}_r}\delta\dot{q}_r+\frac{\partial T}{\partial q_r}\delta q_r\right)\mathrm{d}t=\int_{t_0}^{t_1}\sum_{r=1}^{3n}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_r}\right)\delta q_r+\frac{\partial T}{\partial q_r}\delta q_r\right)\mathrm{d}t+\frac{\partial T}{\partial\dot{q}_r}\delta q_r\Big|_{t_0}^{t_1}=0.$$

由 $\delta q_j\left(t_0\right)=\delta q_j\left(t_1\right)=0$ 可知上式中最后一项为 0，

$$\int_{t_0}^{t_1}\sum_{r=1}^{3n}\left(\sum_{i=1}^{n}\sum_{j}\left(\alpha_j\boldsymbol{F}_{ij}^{\mathrm{a}}\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_r}+\boldsymbol{F}_{ij}^{\mathrm{m}}\cdot\frac{\partial\left(\boldsymbol{r}_i-\boldsymbol{r}_j\right)}{\partial q_r}\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_r}\right)+\frac{\partial T}{\partial q_r}\right)\delta q_r\mathrm{d}t=0.$$

对式中第一项进行简化，假设系统共受到 n_e 个外力 $\boldsymbol{F}_j^e, j=1,\ldots,n_e$。首先假设所有外力（包括场力）均作用在质点上，且外力 $\boldsymbol{F}_j^e$ 的作用点的位移为 $\boldsymbol{r}_j^e$，

$$\int_{t_0}^{t_1}\sum_{r=1}^{3n}\left(\sum_{j=1}^{n_e}\boldsymbol{F}_j^e\cdot\frac{\partial\boldsymbol{r}_j^e}{\partial q_r}+\sum_{i=1}^{n}\sum_{j}\boldsymbol{F}_{ij}^{\mathrm{m}}\cdot\frac{\partial\left(\boldsymbol{r}_i-\boldsymbol{r}_j\right)}{\partial q_r}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_r}\right)+\frac{\partial T}{\partial q_r}\right)\delta q_r\mathrm{d}t=0.$$

对于第二项，同样可以假设系统共受到 n_i 对相互作用力 $\boldsymbol{F}_j^i, j=1,\ldots,n_i$，且两个质点的位移差为 $\boldsymbol{r}_j^i$，

$$\int_{t_0}^{t_1}\sum_{r=1}^{3n}\left(\sum_{j=1}^{n_e}\boldsymbol{F}_j^e\cdot\frac{\partial\boldsymbol{r}_j^e}{\partial q_r}+\sum_{j=1}^{n_e}\boldsymbol{F}_j^i\cdot\frac{\partial\boldsymbol{r}_j^i}{\partial q_r}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_r}\right)+\frac{\partial T}{\partial q_r}\right)\delta q_r\mathrm{d}t=0.$$

需要注意，由于受到约束，所有的广义坐标 $\boldsymbol{q}=\left(q_1,q_2,\ldots,q_{3n}\right)$ 并不是完全独立的，对完整约束方程进行变分处理，

$$\delta g_s=\sum_{r=1}^{3n}\frac{\partial g_s}{\partial q_r}\delta q_r, s=1,2,\ldots,n_h.$$

对所有的 n_h 个完整约束方程进行上式所示变分操作，然后进行加权求和并代入运动方程，

$$\int_{t_0}^{t_1}\sum_{r=1}^{3n}\left(\sum_{j=1}^{n_e}\boldsymbol{F}_j^e\cdot\frac{\partial\boldsymbol{r}_j^e}{\partial q_r}+\sum_{j=1}^{n_e}\boldsymbol{F}_j^i\cdot\frac{\partial\boldsymbol{r}_j^i}{\partial q_r}+\sum_{s=1}^{n_h}\lambda_s\frac{\partial g_s}{\partial q_r}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_r}\right)+\frac{\partial T}{\partial q_r}\right)\delta q_r\mathrm{d}t=0.$$

虽然 $\boldsymbol{q}=\left(q_1,q_2,\ldots,q_{3n}\right)$ 并不是完全独立的，但是通过适当选择 $\lambda_s, s=1,\ldots,n_h$ 使得 δq_r 前面的系数为 0，即

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$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_r}\right)-\frac{\partial T}{\partial q_r}=\sum_{j=1}^{n_e}\boldsymbol{F}_j^e\cdot\frac{\partial \boldsymbol{r}_j^e}{\partial q_r}+\sum_{j=1}^{n_e}\boldsymbol{F}_j^i\cdot\frac{\partial \boldsymbol{r}_j^i}{\partial q_r}+\sum_{s=1}^{n_h}\lambda_s\frac{\partial g_s}{\partial q_r}.$$

而对于非完整约束，

$$h_s=h_s\left(r_1,r_2,\ldots,r_N,\dot{r}_1,\dot{r}_2,\ldots,\dot{r}_N,t\right)=0,s=1,2,\ldots,n_h.$$

进行同样的处理，

$$\delta h_s=\sum_{r=1}^{3n}\left(\frac{\partial h_s}{\partial \dot{q}_r}\delta\dot{q}_r+\frac{\partial h_s}{\partial q_r}\delta q_r\right),s=1,2,\ldots,n_{nh}.$$

加权求和，并代入运动方程，然后进行一定的化简，

$$\int_{t_0}^{t_1}\sum_{r=1}^{3n}\left(\sum_{j=1}^{n_e}\boldsymbol{F}_j^e\cdot\frac{\partial \boldsymbol{r}_j^e}{\partial q_r}+\sum_{j=1}^{n_e}\boldsymbol{F}_j^i\cdot\frac{\partial \boldsymbol{r}_j^i}{\partial q_r}+\sum_{s=1}^{n_h}\lambda_s\frac{\partial g_s}{\partial q_r}+\sum_{s=1}^{n_{nh}}\left(\lambda_s^*\frac{\partial h_s}{\partial q_r}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda_s^*\frac{\partial h_s}{\partial \dot{q}_r}\right)\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_r}\right)+\frac{\partial T}{\partial q_r}\right)\delta q_r\mathrm{d}t=0.$$

在引入 $\lambda_s,s=1,\ldots,n_h$ 和 $\lambda_s^*,s=1,\ldots,n_{nh}$ 并选择合适的值时可以使得 δq_r 前面的系数为 0，

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_r}\right)-\frac{\partial T}{\partial q_r}=\sum_{j=1}^{n_e}\boldsymbol{F}_j^e\cdot\frac{\partial \boldsymbol{r}_j^e}{\partial q_r}+\sum_{j=1}^{n_e}\boldsymbol{F}_j^i\cdot\frac{\partial \boldsymbol{r}_j^i}{\partial q_r}+\sum_{s=1}^{n_h}\lambda_s\frac{\partial g_s}{\partial q_r}+\sum_{s=1}^{n_{nh}}\left(\lambda_s^*\frac{\partial h_s}{\partial q_r}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda_s^*\frac{\partial h_s}{\partial \dot{q}_r}\right)\right).$$

推导过程中的一些问题：

（1）在**非光滑情形**下可以将其表示成

$$\boldsymbol{F}_{ij}^{\mathrm{r}}=F_{ij}^{\mathrm{r}}\left(\frac{\partial g_j}{\partial \tilde{q}_1}\boldsymbol{e}_1+\frac{\partial g_j}{\partial \tilde{q}_2}\boldsymbol{e}_2+\frac{\partial g_j}{\partial \tilde{q}_3}\boldsymbol{e}_3\right)+\boldsymbol{F}_{ij}^{\mathrm{r,plane}}.$$

其中 $\boldsymbol{F}_{ij}^{\mathrm{r,plane}}$ 表示约束力在约束平面内的分量。最终这一项可以表示成，

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_r}\right)-\frac{\partial T}{\partial q_r}=\sum_{j=1}^{n_e}\boldsymbol{F}_j^e\cdot\frac{\partial \boldsymbol{r}_j^e}{\partial q_r}+\sum_{j=1}^{n_e}\boldsymbol{F}_j^i\cdot\frac{\partial \boldsymbol{r}_j^i}{\partial q_r}+\left[\sum_{j=1}^{n_p}\boldsymbol{F}_j^{\mathrm{r,plane}}\cdot\frac{\partial \boldsymbol{r}_j^{\mathrm{plane}}}{\partial q_r}\right]+\sum_{s=1}^{n_h}\lambda_s\frac{\partial g_s}{\partial q_r}+\sum_{s=1}^{n_{nh}}\left(\lambda_s^*\frac{\partial h_s}{\partial q_r}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda_s^*\frac{\partial h_s}{\partial \dot{q}_r}\right)\right).$$

会增加括号中的一项。

（2）**增加未知量之后的效果**

对于方程，

$$\sum_{r=1}^{3n}f_r\delta q_r=0.$$

其特征函数为

$$\begin{bmatrix}f_1 & \cdots & f_{3n}\end{bmatrix}\begin{bmatrix}\delta q_1 & \cdots & \delta q_{3n}\end{bmatrix}^T=0.$$

将两者理解为向量，即两个向量垂直，也就是说 $\begin{bmatrix}f_1 & \cdots & f_{3n}\end{bmatrix}$ 垂直于此时刻此点所有的约束切线平面。

在增加未知函数之后变为，

$$\left(\begin{bmatrix}f_1 & \cdots & f_{3n}\end{bmatrix}+\begin{bmatrix}\lambda_1 & \cdots & \lambda_s\end{bmatrix}\begin{bmatrix}\dfrac{\partial g_1}{\partial q_1} & \cdots & \dfrac{\partial g_1}{\partial q_{3n}}\\ \vdots & \ddots & \vdots\\ \dfrac{\partial g_s}{\partial q_1} & \cdots & \dfrac{\partial g_s}{\partial q_{3n}}\end{bmatrix}\right)\begin{bmatrix}\delta q_1\\ \vdots\\ \delta q_{3n}\end{bmatrix}=0.$$

注意式中矩阵表示了 s 个切线平面的法向方向且线性无关，因此可以表示所有垂直于约束方程切线平面的向量。即存在一组 $\lambda_j,j=1,\ldots,s$ 使得 $f_i+\sum_{j=1}^{s}\lambda_j\dfrac{\partial g_j}{\partial q_i}=0,i=1,2,\ldots,3n$。