

题 1

利用行波法（参看数理方程）求解波动方程的解

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 & x \in \mathbb{R}, t > 0 \\ u(x, 0) = \varphi(x) \\ u_t(x, 0) = \psi(x) \end{cases} \quad (1)$$

解
令

$$\begin{cases} \xi = x + at \\ \eta = x - at \end{cases} \quad (2)$$

- 物理含义: $u = f(\xi) + f(\eta)$
波可以分解为两个方向
- (1) $\Leftrightarrow \left(\frac{\partial}{\partial t} + a \frac{\partial}{\partial x}\right) \left(\frac{\partial}{\partial t} - a \frac{\partial}{\partial x}\right) u = 0$
猜想(6)

则

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} \\ &= \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \end{aligned} \quad (3)$$

进而有

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial \xi} \right) + \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial \eta} \right) \\ &= \left[\frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial \xi} \right) \right] + \left[\frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \eta} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial \eta} \right) \right] \\ &= \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \\ &= \frac{\partial^2 u}{\partial \xi^2} + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \end{aligned} \quad (4)$$

将 $\frac{\partial u}{\partial \xi}$ 当作 u 代入(3)

同理可得

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial \xi^2} - 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \right) \quad (5)$$

代入方程可得

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \quad (6)$$

可解得

$$\begin{aligned}
& \frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \\
& \Rightarrow \int \frac{\partial^2 u}{\partial \xi \partial \eta} d\xi = \int 0 d\xi \\
& \Rightarrow \frac{\partial u}{\partial \eta} = g(\eta) \\
& \Rightarrow \int \frac{\partial u}{\partial \eta} d\eta = \int g(\eta) d\eta \\
& \Rightarrow u = \int g(\eta) d\eta + f(\xi) \\
& \Rightarrow u = F(\xi) + G(\eta)
\end{aligned} \tag{7}$$

所以有

$$u(x, t) = F(x + at) + G(x - at) \tag{8}$$

代入初始条件则有

$$\begin{cases} F(x) + G(x) = \varphi(x) \\ a[F'(x) - G'(x)] = \psi(x) \end{cases} \tag{9}$$

对(9)中2式两边积分则有

$$F(x) - G(x) = \frac{1}{a} \int \psi(x) dx + C \tag{10}$$

联立(10)和(9), 有

$$\begin{cases} F(x) = \frac{1}{2}\varphi(x) + \frac{1}{2a} \int \psi(x) dx + C \\ G(x) = \frac{1}{2}\varphi(x) - \frac{1}{2a} \int \psi(x) dx + C \end{cases} \tag{11}$$

代入到(8), 有

$$\begin{aligned}
u(x, t) &= F(x + at) + G(x - at) \\
&= \frac{1}{2}\varphi(x + at) + \frac{1}{2a} \int_0^{x+at} \psi(y) dy + \frac{1}{2}\varphi(x - at) - \frac{1}{2a} \int_0^{x-at} \psi(y) dy + C \\
&= \frac{1}{2}[\varphi(x + at) + \varphi(x - at)] + \frac{1}{2a} \int_{x-at}^{x+at} \psi(y) dy + C
\end{aligned} \tag{12}$$

References

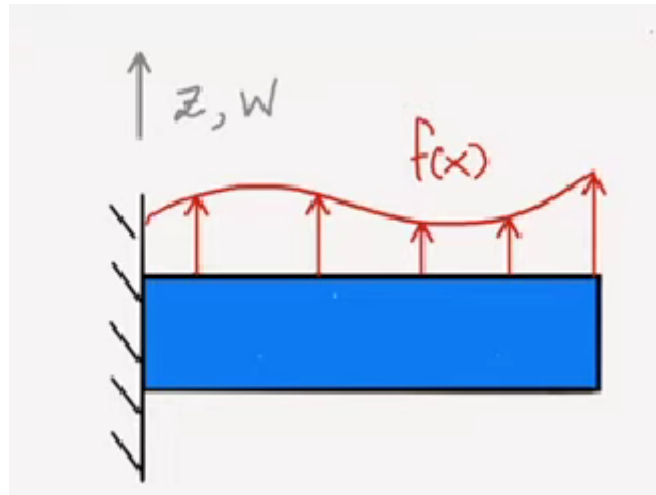
- [无界波动方程的定解——行波法](#)
- [偏微分方程基础——特征线法/行波法/达朗贝尔法](#)

题 2

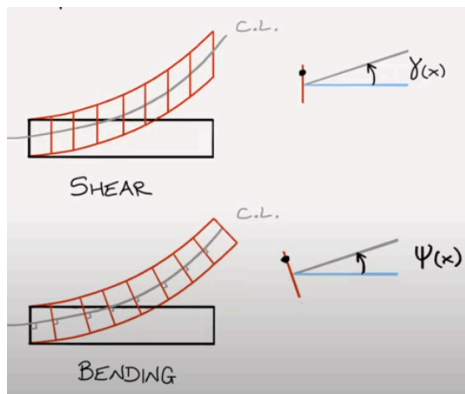
利用微元法建立均质铁摩辛科 (Timoshienko) 梁无外激的线性自由振动线性方程

解

考虑一般形式



slope of center line



形变为弯矩和剪力的叠加

$$\frac{\partial w}{\partial x} = \gamma + \varphi \quad (13)$$

displacements

$$\begin{aligned} u_x(x, y, z, t) &= -z\varphi(x, t) \\ u_y(x, y, z, t) &= 0 \\ u_z(x, y, z, t) &= w(x, t) \end{aligned} \quad (14)$$

where u_x, u_y, u_z are the components of the displacement vector in the three coordinate directions

strain

由 [Infinitesimal strain theory](#) 知

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (15)$$

$$u_{i,j} = \frac{\partial u_i}{\partial j}$$

可得

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{2}(u_{x,x} + u_{x,x}) \\ &= u_{x,x} = \frac{\partial u_x}{\partial x} \\ &= -z \frac{\partial \varphi}{\partial x} \end{aligned} \quad (16)$$

$$\begin{aligned}
 \varepsilon_{xz} &= \frac{1}{2}(u_{x,z} + u_{z,x}) = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\
 &= \frac{1}{2}\left(-\varphi + \frac{\partial w}{\partial x}\right) \quad \text{依据 (13)} \\
 &= \frac{1}{2}\gamma
 \end{aligned} \tag{17}$$

同理

$$\begin{aligned}
 \varepsilon_{yy} &= 0 \\
 \varepsilon_{zz} &= 0 \\
 \varepsilon_{yz} &= 0 \\
 \varepsilon_{xy} &= 0
 \end{aligned} \tag{18}$$

由 [Hooke's law](#):

$$\sigma_{xx} = E\varepsilon_{xx} \quad \text{泊松比 } \nu = 0 \tag{19}$$

xz 为角度变形, 设剪切系数为 k , 则

$$\begin{aligned}
 \sigma_{xz} &= G\gamma(xz) \\
 &= kG\gamma(x)
 \end{aligned} \tag{20}$$

- ε 被称为 tensor strain
- γ 被称为 engineering strain
- 二者的关系由(17) 联系
- Hooke's law 矩阵为 tensor strain 形式

Hamilton

由 [Hamilton's principle](#) 知

$$\int_{t_1}^{t_2} (\delta T - \delta U + \delta W_E) dt = 0 \tag{21}$$

strain energy

$$U = \int_V W dV \tag{22}$$

由 [Strain energy density function](#)

$$W = \frac{1}{2} \sum \sum \sigma_{ij} \varepsilon_{ij} \tag{23}$$

代入 (16) (17) (18) (19) (20) :

$$\begin{aligned}
 W &= \frac{1}{2} \sum \sum \sigma_{ij} \varepsilon_{ij} \\
 &= \frac{1}{2} \sigma_{xx} \varepsilon_{xx} + \sigma_{xz} \varepsilon_{xz} \\
 &= \frac{1}{2} E \varepsilon_{xx}^2 + \frac{1}{2} k G \gamma^2 \\
 &= \frac{1}{2} E z^2 \left(\frac{\partial \varphi}{\partial x} \right)^2 + \frac{1}{2} k G \gamma^2
 \end{aligned} \tag{24}$$

所以 (22) 变为

$$\begin{aligned}
U &= \int_V \left(\frac{1}{2} E z^2 \left(\frac{\partial \varphi}{\partial x} \right)^2 + \frac{1}{2} k G \gamma^2 \right) dV \\
&= \frac{1}{2} \int_0^l \int_A \left(E z^2 \left(\frac{\partial \varphi}{\partial x} \right)^2 + k G \gamma^2 \right) dA dx \\
&= \frac{1}{2} \left[\left(\int_0^l E \left(\int_A z^2 dA \right) \frac{\partial \varphi}{\partial x} dx \right) + \int_0^l k G A \gamma^2 dx \right] \quad \boxed{I = \int_A z^2 dA} \quad (25) \\
&= \frac{1}{2} \int_0^l EI \left(\frac{\partial \varphi}{\partial x} \right)^2 dx + \frac{1}{2} \int_0^l k G A \gamma^2 dx
\end{aligned}$$

代入 (13), 可得

$$U = \frac{1}{2} \int_0^l EI \left(\frac{\partial \varphi}{\partial x} \right)^2 dx + \frac{1}{2} \int_0^l k G A \left(\frac{\partial w}{\partial x} - \varphi \right)^2 dx \quad (26)$$

故

$$\begin{aligned}
\delta U &= \int_0^l EI \frac{\partial \varphi}{\partial x} \delta \left(\frac{\partial \varphi}{\partial x} \right) dx + \int_0^l k G A \left(\frac{\partial w}{\partial x} - \varphi \right) \left(\delta \left(\frac{\partial w}{\partial x} \right) - \delta \varphi \right) dx \\
&= \int_0^l EI \frac{\partial \varphi}{\partial x} \delta \left(\frac{\partial \varphi}{\partial x} \right) dx \\
&\quad + \int_0^l k G A \left(\frac{\partial w}{\partial x} - \varphi \right) \delta \left(\frac{\partial w}{\partial x} \right) dx \\
&\quad - \int_0^l k G A \left(\frac{\partial w}{\partial x} - \varphi \right) \delta \varphi dx \quad (27) \\
&= EI \frac{\partial \varphi}{\partial x} \delta \varphi \Big|_0^l - \int_0^l \frac{\partial}{\partial x} \left(EI \frac{\partial \varphi}{\partial x} \right) \delta \varphi dx \\
&\quad + k G A \left(\frac{\partial w}{\partial x} - \varphi \right) \delta w \Big|_0^l - \int_0^l \frac{\partial}{\partial x} \left[k G A \left(\frac{\partial w}{\partial x} - \varphi \right) \right] \delta w dx \\
&\quad - \int_0^l k G A \left(\frac{\partial w}{\partial x} - \varphi \right) \delta \varphi dx
\end{aligned}$$

kinetic energy

translational kinetic energy + rotational kinetic energy

$$T = \frac{1}{2} \int_0^l \rho A \left(\frac{\partial w}{\partial t} \right)^2 dx + \frac{1}{2} \int_0^l \rho I \left(\frac{\partial \varphi}{\partial t} \right)^2 dx \quad (28)$$

故

$$\delta T = \int_0^l m \frac{\partial w}{\partial t} \delta \left(\frac{\partial w}{\partial t} \right) dx + \int_0^l J \frac{\partial \varphi}{\partial t} \delta \left(\frac{\partial \varphi}{\partial t} \right) dx \quad \boxed{m = \rho A, J = \rho I} \quad (29)$$

则

$$\begin{aligned}
\int_{t_1}^{t_2} \delta T &= \int_{t_1}^{t_2} \left[\int_0^l m \frac{\partial w}{\partial t} \delta \left(\frac{\partial w}{\partial t} \right) dx + \int_0^l J \frac{\partial \varphi}{\partial t} \delta \left(\frac{\partial \varphi}{\partial t} \right) dx \right] dt \\
&= \int_0^l \int_{t_1}^{t_2} m \frac{\partial w}{\partial t} \delta \left(\frac{\partial w}{\partial t} \right) dt dx + \int_0^l \int_{t_1}^{t_2} J \frac{\partial \varphi}{\partial t} \delta \left(\frac{\partial \varphi}{\partial t} \right) dt dx \\
&= \int_0^l \left(m \frac{\partial w}{\partial t} \delta w \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(m \frac{\partial w}{\partial t} \right) \delta w dt \right) dx \\
&\quad + \int_0^l \left(J \frac{\partial \varphi}{\partial t} \delta \varphi \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(J \frac{\partial \varphi}{\partial t} \right) \delta \varphi dt \right) dx \\
&= - \int_0^l \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(m \frac{\partial w}{\partial t} \right) \delta w dt dx - \int_0^l \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(J \frac{\partial \varphi}{\partial t} \right) \delta \varphi dt dx \\
&= - \int_{t_1}^{t_2} \int_0^l \frac{\partial}{\partial t} \left(m \frac{\partial w}{\partial t} \right) \delta w dx dt - \int_{t_1}^{t_2} \int_0^l \frac{\partial}{\partial t} \left(J \frac{\partial \varphi}{\partial t} \right) \delta \varphi dx dt
\end{aligned} \tag{30}$$

external work

$$W_E = \int_0^l q w dx \tag{31}$$

故

$$\delta W_E = \int_0^l q \delta w dx \tag{32}$$

代入 **Hamilton**

由 (27) (30) (32) 可得

$$\begin{aligned}
\int_{t_1}^{t_2} (\delta T - \delta U + \delta W_E) dt &= - \int_{t_1}^{t_2} \int_0^l \frac{\partial}{\partial t} \left(m \frac{\partial w}{\partial t} \right) \delta w dx dt \\
&\quad - \int_{t_1}^{t_2} \int_0^l \frac{\partial}{\partial t} \left(J \frac{\partial \varphi}{\partial t} \right) \delta \varphi dx dt \\
&\quad - \int_{t_1}^{t_2} \left(EI \frac{\partial \varphi}{\partial x} \delta \varphi \Big|_0^l - \int_0^l \frac{\partial}{\partial x} \left(EI \frac{\partial \varphi}{\partial x} \right) \delta \varphi dx \right. \\
&\quad \left. kGA \left(\frac{\partial w}{\partial x} - \varphi \right) \delta w \Big|_0^l - \int_0^l \frac{\partial}{\partial x} \left[kGA \left(\frac{\partial w}{\partial x} - \varphi \right) \right] \delta w dx \right. \\
&\quad \left. - \int_0^l kGA \left(\frac{\partial w}{\partial x} - \varphi \right) \delta \varphi dx \right) dt \\
&\quad + \int_{t_1}^{t_2} \int_0^l q \delta w dx dt \\
&= \int_{t_1}^{t_2} \int_0^l \left[\left(-\frac{\partial}{\partial t} \left(m \frac{\partial w}{\partial t} \right) + \frac{\partial}{\partial x} \left[kGA \left(\frac{\partial w}{\partial x} - \varphi \right) \right] + q \right) \delta w \right. \\
&\quad \left. \left(-\frac{\partial}{\partial t} \left(J \frac{\partial \varphi}{\partial t} \right) + \frac{\partial}{\partial x} \left(EI \frac{\partial \varphi}{\partial x} \right) + kGA \left(\frac{\partial w}{\partial x} - \varphi \right) \right) \delta \varphi \right] dx dt \\
&\quad - EI \frac{\partial \varphi}{\partial x} \delta \varphi \Big|_0^l - kGA \left(\frac{\partial w}{\partial x} - \varphi \right) \delta w \Big|_0^l \\
&= 0
\end{aligned} \tag{33}$$

由任意性可得

$$\begin{cases} -\frac{\partial}{\partial t} \left(m \frac{\partial w}{\partial t} \right) + \frac{\partial}{\partial x} \left[kGA \left(\frac{\partial w}{\partial x} - \varphi \right) \right] + q = 0 \\ -\frac{\partial}{\partial t} \left(J \frac{\partial \varphi}{\partial t} \right) + \frac{\partial}{\partial x} \left(EI \frac{\partial \varphi}{\partial x} \right) + kGA \left(\frac{\partial w}{\partial x} - \varphi \right) = 0 \end{cases} \tag{34}$$

边界条件为

$$\begin{cases} EI \frac{\partial \varphi}{\partial x} \delta \varphi \Big|_0^l = 0 \\ kGA \left(\frac{\partial w}{\partial x} - \varphi \right) \delta w \Big|_0^l = 0 \end{cases} \tag{35}$$

根据均质条件

由(34) 可得

$$-m \frac{\partial^2 w}{\partial t^2} + kGA \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \varphi}{\partial x} \right) + q = 0 \tag{36}$$

$$-J \frac{\partial^2 \varphi}{\partial t^2} + EI \frac{\partial^2 \varphi}{\partial x^2} + kGA \left(\frac{\partial w}{\partial x} - \varphi \right) = 0 \tag{37}$$

现将(36) 代入(37) 消掉 φ

由 (36) 可得

$$\begin{cases} \frac{\partial \varphi}{\partial x} = \frac{q}{kGA} - \frac{m}{kGA} \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^3 \varphi}{\partial x^3} = \frac{1}{kGA} \frac{\partial^2 q}{\partial x^2} - \frac{m}{kGA} \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^3 w}{\partial x^3} \\ \frac{\partial^3 \varphi}{\partial x \partial t^2} = \frac{1}{kGA} \frac{\partial^2 q}{\partial t^2} - \frac{m}{kGA} \frac{\partial^4 w}{\partial t^4} + \frac{\partial^4 w}{\partial x^2 \partial t^2} \end{cases} \quad (38)$$

由 (37) 可得

$$J \frac{\partial^3 w}{\partial x \partial t^2} = EI \frac{\partial^3 w}{\partial x^3} + kGA \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \varphi}{\partial x} \right) = 0 \quad (39)$$

将(38) 代入(39) 可得

$$EI \frac{\partial^4 w}{\partial x^4} + m \frac{\partial^2 w}{\partial t^2} - \left(J + \frac{mEI}{kGA} \right) \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{mJ}{kGA} \frac{\partial^4 w}{\partial t^4} = q + \frac{J}{kGA} \frac{\partial^2 q}{\partial t^2} - \frac{EI}{kGA} \frac{\partial^2 q}{\partial x^2} \quad (40)$$

由题意, 外部激励 $q = 0$, 所以

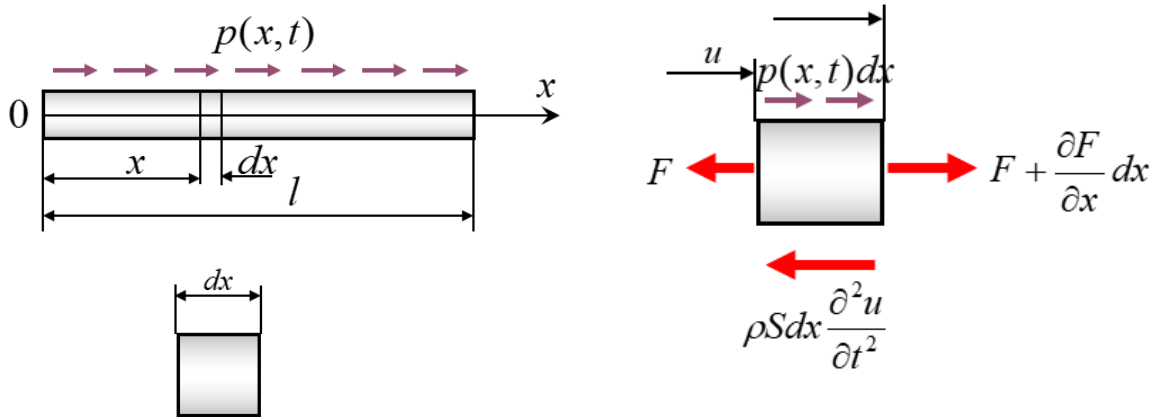
$$EI \frac{\partial^4 w}{\partial x^4} + m \frac{\partial^2 w}{\partial t^2} - \left(J + \frac{mEI}{kGA} \right) \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{mJ}{kGA} \frac{\partial^4 w}{\partial t^4} = 0 \quad (41)$$

Reference

- [Structural Dynamics](#): 很不错的合集 关于 Timoshenko Beam 的里面:
 - [Timoshenko Beam Theory Part 2 of 3: Hamilton's Principle](#)
 - [Timoshenko Beam Theory Part 3 of 3: Equations of Motion](#)
- [Timoshenko–Ehrenfest beam theory](#)

题 3

建立均质杆纵向振动的几何非线性振动方程（建模方法自由选择）



根据泰勒公式, 保留应变一定的高阶项, 则有

$$\varepsilon = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 \quad (42)$$

则轴向力为

$$\begin{aligned}
F &= \sigma S \\
&= E\varepsilon S \\
&= ES \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 \right]
\end{aligned} \tag{43}$$

由达朗贝尔原理可知

$$\rho S \, dx \frac{\partial^2 u}{\partial t^2} = \left(F + \frac{\partial F}{\partial x} dx \right) - F + p(x, t) \tag{44}$$

代入可得

$$\rho S \frac{\partial^2 u}{\partial t^2} = ES \frac{\partial^2 u}{\partial x^2} \left(1 + \frac{\partial u}{\partial x} \right) + p(x, t) \tag{45}$$

存在 $ES \frac{\partial^2 u}{\partial x^2} \frac{\partial u}{\partial x}$ 项, 为非线性项, 故为非线性方程