2021-Homework 6-8: Reference answer

Problem 1 (10 points): For a fixed-fixed undamped string, the tension of the string is uniform and constant *F*, the mass distribution of the string is also uniform, with linear density *m*. If there is a wave travels along the string, prove that the propagation speed of the wave along the string is $c=\sqrt{F/m}$. (10 points)

解：已知弦横向振动方程如下，

$$\frac{\partial^2 y}{\partial^2 t}=a^2\frac{\partial^2 y}{\partial^2 x},\quad a=\sqrt{\frac{F}{m}} \tag{1}$$

其中m为单位长度的质量，F为弦的张力。因此在弦上传播的波可以表示为$f(x-ct)$的形式，其中c为波的传播速度，进行特征变换

$$\xi=x-ct \tag{2}$$

则有，

$$\frac{\partial y}{\partial x}=\frac{\partial y}{\partial \xi}\frac{\partial \xi}{\partial x}=\frac{\partial y}{\partial \xi},\quad \frac{\partial^2 y}{\partial x^2}=\frac{\partial}{\partial \xi}\left(\frac{\partial y}{\partial \xi}\right)\frac{\partial \xi}{\partial x}=\frac{\partial^2 y}{\partial \xi^2} \tag{3}$$

同理可得：

$$\frac{\partial y}{\partial t}=\frac{\partial y}{\partial \xi}\frac{\partial \xi}{\partial t}=-c\left(\frac{\partial y}{\partial \xi}\right),\quad \frac{\partial^2 y}{\partial t^2}=-c\frac{\partial y}{\partial \xi}\left(\frac{\partial y}{\partial \xi}\right)\frac{\partial \xi}{\partial t}=c^2\frac{\partial^2 y}{\partial \xi^2} \tag{4}$$

将(3), (4)与(1)联立，可得

$$c^2\frac{\partial^2 y}{\partial^2 \xi}=a^2\frac{\partial^2 y}{\partial^2 \xi} \tag{5}$$

即证明

$$c=\pm a=\pm\sqrt{F/m}$$

其分别表示以速度c沿绳正负两个方向传播。

Problem 2 (10 points): For an undamped beam, four boundary conditions are possible based on the extended Hamilton's principle, namely, $\delta w=0$, $\delta\left(\frac{\partial w}{\partial x}\right)=0$, $EI\left(\frac{\partial^2 w}{\partial x^2}\right)=0$, $EI\left(\frac{\partial^3 w}{\partial x^3}\right)=0$. Please state the physical meaning of the four boundary conditions and explain why. (10 points)

解：边界条件的物理意义分别为：

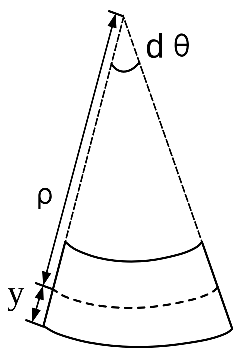
（1）$\delta w=0$，挠度为零，对应固定端或简支端。

（2）$\delta\left(\frac{\partial w}{\partial x}\right)=0$，斜率为零，对应固定端。

$$\arctan\theta=\frac{w+\frac{\partial w}{\partial x}dx-w}{dx}=\frac{\partial w}{\partial x}$$

因此$\delta\left(\frac{\partial w}{\partial x}\right)=0$意味着在边界处的斜率$\theta=0$，与固定边界条件相契合。

（3）$EI\left(\frac{\partial^2 w}{\partial x^2}\right)=0$，弯矩为零，对应自由端或简支端。



根据材料力学知识，弯曲梁截面上的正应变以及正应力为：

$$\varepsilon=\frac{(\rho+y)d\theta-\rho d\theta}{\rho d\theta}=\frac{y}{\rho},\qquad \sigma=E\varepsilon$$

在截面上积分，可求得该处的弯矩，

$$M=\int_A y\sigma dA=\frac{E}{\rho}\int_A y^2 dA=\frac{EI}{\rho}$$

由高等数学知识可得曲率为，

$$\frac{1}{\rho}\approx\frac{d^2 y}{dx^2}$$

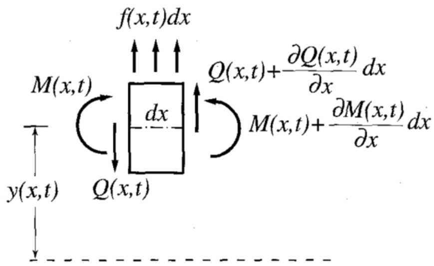
则弯矩公式可以表示为，

$$M=EI\frac{d^2 y}{dx^2}$$

边界处弯矩为零，对应于无穷大的曲率，也即在边界处不能弯曲，因此与自由端和简支端相符。

（4）$EI\left(\frac{\partial^3 w}{\partial x^3}\right)=0$，剪力为零，对应自由端。

如图，取微元分析，



假设单元惯性矩与角加速度乘积忽略不计，根据力矩平衡关系得到，

$$\left[M(x,t)+\frac{\partial M(x,t)}{\partial x}dx\right]-M(x,t)+\left[Q(x,t)+\frac{\partial Q(x,t)}{\partial x}dx\right]dx+f(x,t)dx\frac{dx}{2}=0,\ \ 0<x<L$$

略去高阶项，

$$\frac{\partial M(x,t)}{\partial x}+Q(x,t)=0, 0<x<L$$

将(3)中弯矩公式代入得到，

$$Q(x,t)=-EI\frac{\partial}{\partial x}\left(\frac{\partial^2 y(x,t)}{\partial x^2}\right)$$

因此 $EI\left(\frac{\partial^3 w}{\partial x^3}\right)=0$ 表示在边界处梁不能承受横向力，只有自由端符合条件。

Problem 3 (80 points in total)

Problem 3.1: Please solve Problem 8.5 in page 459 of the textbook (20 points).

Problem 3.2: Please solve Problem 8.11 in page 460 of the textbook (10 points).

Problem 3.3: Please solve problem 8.17 in page 461 of the textbook (5 points).

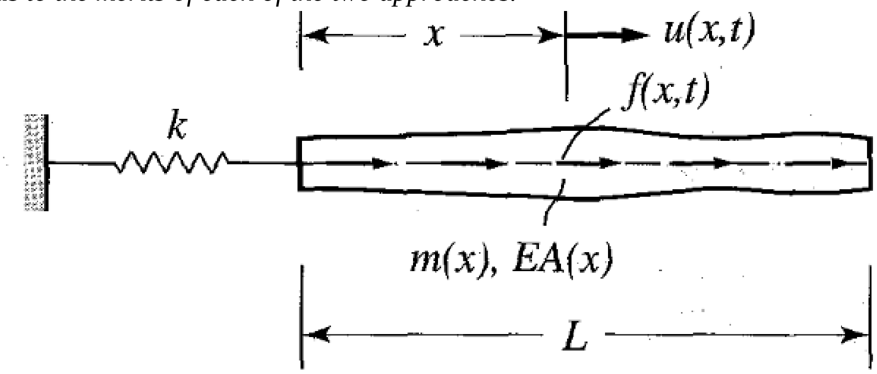
Problem 3.4: Please solve Problem 8.31 in page 462 of the textbook (15 points).

Problem 3.5: Please solve Problem 8.36 in page 463 of the textbook (15 points).

Problem 3.6: Please solve Problem 8.42 in page 463 of the textbook (15 points).

3.1 (8.3)

Solve Problem 8.3 by the extended Hamilton principle. Compare results with those obtained in Problem 8.3 and draw conclusions as to the merits of each of the two approaches.



解：设杆上任意一点的轴向位移为$u(x,t)$，则截面上任意一点处的速度可以表示为$\partial u(x,t)/\partial t$，而截面上的应变可以表示为，

$$\varepsilon_x = \frac{\partial u(x,t)}{\partial x}$$

杆的动能可以表示为，

$$T = \int_0^L \frac{1}{2} m(x)\dot{u}^2 dx$$

接下来考虑结构的应变，在小变形的条件下认为$\cos\theta \approx 1$，因此杆的势能为，

$$V = \int_0^L \frac{1}{2} EA(x)\left(\frac{\partial u}{\partial x}\right)^2 dx$$

对动能和势能求变分并在$0-t$内积分，可以得到

$$\begin{aligned}\int_0^t \delta T dt &= \int_0^t\int_0^L m(x)\left(\dot{u}\delta\dot{u}\right)dxdt = \int_0^L\int_0^t m(x)\left(\dot{u}\frac{\partial(\delta u)}{\partial t}\right)dtdx \\ &= \int_0^L m(x)\left(\dot{u}\delta u\right)dx\Big|_0^t - \int_0^L\int_0^t\left(\frac{\partial}{\partial t}\left(m(x)\dot{u}\right)\delta u\right)dtdx \\ &= -\int_0^L\int_0^t\left(\frac{\partial}{\partial t}\left(m(x)\dot{u}\right)\delta u\right)dtdx\end{aligned}$$

以及

$$\begin{aligned}\int_0^t \delta V dt &= \int_0^t\int_0^L \frac{1}{2} EA(x)\frac{\partial u}{\partial x}\delta\left(\frac{\partial u}{\partial x}\right)dxdt \\ &= \int_0^t EA(x)\frac{\partial u}{\partial x}\delta u\Big|_0^L dt - \int_0^t\int_0^L \frac{\partial}{\partial x}\left(EA(x)\frac{\partial u}{\partial x}\right)\delta u dxdt\end{aligned}$$

同时，杆受到轴向外力 $f(x,t)$，以及左端弹簧力 $ku\delta\left(x\right)$，利用广义 Hamilton 原理，

$$-\int_0^L\int_0^t\left(\frac{\partial}{\partial t}\left(m(x)\dot{u}\right)-\frac{\partial}{\partial x}\left(EA(x)\frac{\partial u}{\partial x}\right)-f(x,t)\right)\delta udtdx-\int_0^t\left(EA(x)\frac{\partial u}{\partial x}\delta u+ku\delta\left(x\right)\delta u\right)\Bigg|_0^L dt=0$$

因此杆的轴向振动方程为，

$$\frac{\partial}{\partial t}\left(m(x)\dot{u}\right)-\frac{\partial}{\partial x}\left(EA(x)\frac{\partial u}{\partial x}\right)-f(x,t)=0,\ \ 0<x<L$$

边界条件满足

$$EA(x)\frac{\partial u}{\partial x}-ku=0,\ \ x=0$$

$$EA(x)\frac{\partial u}{\partial x}=0,\ \ x=L$$

3.2 (8.11)

Derive the eigenvalue problem for the rod in axial vibration considered in Problem 8.3. Then, let $m\left(x\right)=m=\text{costant}, EA\left(x\right)=EA=\text{costant}$ *and solve the eigenvalue problem for the two cases: 1)* $k=0.5EA/L$ *and 2)* $k=2EA/L$*. Plot the three lowest modes for each of the two cases and draw conclusions as to the effect of the spring stiffness k on the system.*

解：在 $m\left(x\right)=m=\text{costant}, EA\left(x\right)=EA=\text{costant}$ 条件下，并不考虑外力时，系统方程为

$$m\frac{\partial^2 u}{\partial t^2}-EA\frac{\partial^2 u}{\partial x^2}=0$$

分离变量法，设杆的振动解为

$$u(x,t)=U(x)T(t)$$

代入振动方程得到，

$$\frac{\partial^2 T}{\partial t^2}\frac{1}{T\left(t\right)}=\frac{EA}{m}\frac{\partial^2 U}{\partial x^2}\frac{1}{U\left(x\right)}=-\lambda,\ \ \lambda=\omega^2$$

将其表示为，

$$\frac{\partial^2 T}{\partial t^2}+\lambda T\left(t\right)=0$$

$$\frac{\partial^2 U}{\partial x^2}+\lambda\frac{m}{EA}U\left(x\right)=0$$

设解的形式为，

$$U(x)=B\sin\beta x+C\cos\beta x,\quad \beta=\sqrt{\frac{m\lambda}{EA}}$$

代入边界条件

$$EA\frac{\partial U(x)}{\partial x}\bigg|_{x=0}=kU(0)$$

$$EA\frac{\partial U(x)}{\partial x}\bigg|_{x=L}=0$$

即，

$$EAB\beta=kC$$

$$B\beta\cos\beta L-C\beta\sin\beta L=0$$

特征方程为

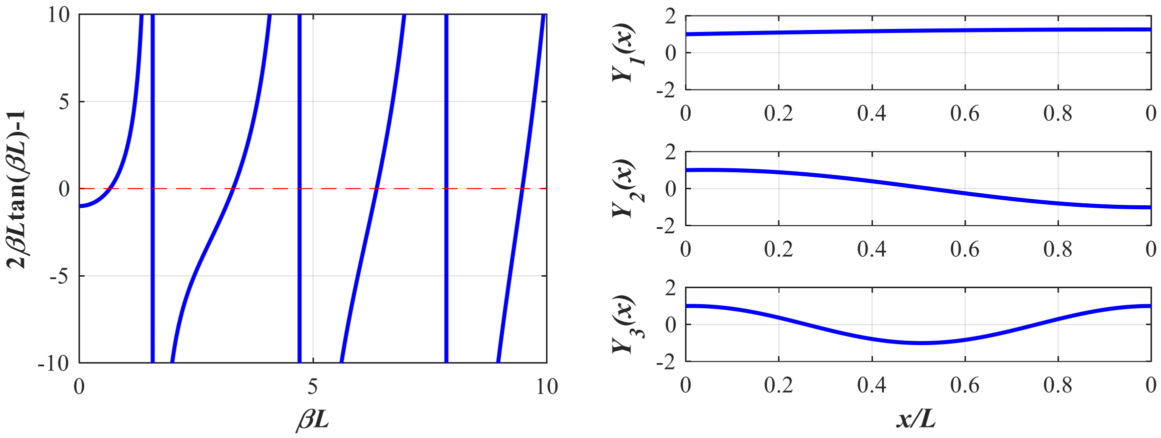
$$\tan\beta L=\frac{k}{EA\beta}=\frac{B}{C}$$

（1） $k=0.5EA/L$ 时，

$$2L\beta\tan\beta L=1$$

可解得数值解，固有频率前三阶数值解分别为，

$$\beta_1 L=0.653,\beta_2 L=3.292,\beta_1 L=6.362$$

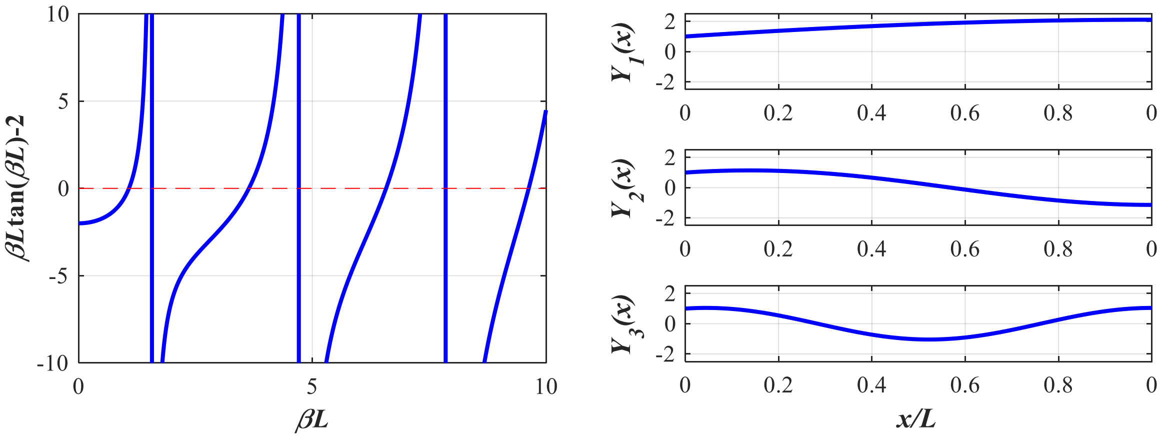


（2） $k=2EA/L$ 时，

$$L\beta\tan\beta L=2$$

前三阶固有频率的解为，

$$\beta_1 L=1.077,\beta_2 L=3.644,\beta_1 L=6.578$$



主振型为

$$U_i(x)=\tan\left(\beta_i L\right)\sin\left(\beta_i x\right)+\cos\left(\beta_i x\right)$$

可发现刚度的提升会导致各阶主振型固有频率的提升。

3.3 (8.17)

Derive the orthogonality relations for the rod in axial vibration considered in Problem 8.11.

解：将 $u(x,t)=U(x)\sin\left(\omega t+\varphi\right)$ 代入杆的振动方程，

$$m\frac{\partial^2 u}{\partial t^2}=EA\frac{\partial^2 u}{\partial x^2}$$

$$-\omega^2 mU(x)=EA\frac{\partial^2 U(x)}{\partial x^2}$$

对于第 i 阶以及第 j 阶主振型有

$$-\omega_i^2 mU_i(x)=EA\frac{\partial^2 U_i(x)}{\partial x^2} \qquad (1)$$

$$-\omega_j^2 mU_j(x)=EA\frac{\partial^2 U_j(x)}{\partial x^2} \qquad (2)$$

对（1）左乘 $U_j(x)$ 并在 $0-L$ 上做积分，

$$-\omega_i^2\int_0^L mU_jU_i dx=\int_0^L EAU_j\frac{\partial^2 U_i}{\partial x^2}dx$$

分部积分，并由边界条件可化简，

$$-\omega_i^2\int_0^L mU_jU_i dx=EA\left(U_j\frac{\partial U_i}{\partial x}\bigg|_0^L-\int_0^L\frac{\partial U_j}{\partial x}\frac{\partial U_i}{\partial x}dx\right)=EA\left(-U_j(0)\frac{kU_i(0)}{EA}-\int_0^L\frac{\partial U_j}{\partial x}\frac{\partial U_i}{\partial x}dx\right)$$

同理由式（2）可得，

$$-\omega_j^2\int_0^L mU_jU_i dx=EA\left(-U_j(0)\frac{kU_i(0)}{EA}-\int_0^L\frac{\partial U_j}{\partial x}\frac{\partial U_i}{\partial x}dx\right)$$

与上式相减可得，

$$\left(\omega_i^2-\omega_j^2\right)\int_0^L U_j mU_i dx=0$$

由于固有频率 $\omega_i^2-\omega_j^2=0$，因此有

$$\int_0^L U_j mU_i dx=0,\ (i\neq j)$$

即主振型关于质量的正交性。将此式回代得主振型关于刚度的正交性，

$$\int_0^L EAU_j\frac{\partial^2 U_i}{\partial x^2}dx=0,\ (i\neq j)$$

上题中已求解出主振型，

$$U_i(x)=\tan(\beta_i L)\sin(\beta_i x)+\cos(\beta_i x)$$

对主振型做归一化处理，使其满足，

$$\int_0^L c^2 U_i{}^2 m dx=1,\ i=1,2,3\cdots$$

$$\int_0^L c^2 EAU_i\frac{\partial^2 U_i(x)}{\partial x^2}dx=-\omega_i^2,\ i=1,2,3\cdots$$

解得

$$c=\left(\frac{2\beta_i}{m\left(\tan(L\beta_i)+L\beta_i\sec^2(L\beta_i)\right)}\right)^{0.5}$$

得到的结果称之为正则振型，如下，

$$U_i(x)=\sqrt{\frac{2\beta_i}{m\left(\tan(\beta_i L)+\beta_i L\sec^2(\beta_i L)\right)}}\left[\tan(\beta_i L)\sin(\beta_i x)+\cos(\beta_i x)\right]$$

3.4 (8.31)

Determine the response of the uniform rod of Problem 8.11 to the initial excitation $u(x,0)=0,\dot{u}(x,0)=v_0\delta(x)$, *where* $\delta(x)$ *is a spatial Dirac delta function located at* $x=0$.

解：振动方程

$$m\frac{\partial^2 u}{\partial t^2}-EA\frac{\partial^2 u}{\partial x^2}=0$$

根据类似多自由度系统的展开定理，将位移展开为正则振型的无穷级数，其中正则振型已在 3.3 中给出，

$$u(x,t)=\sum_{i=1}^{n}U_i(x)\eta_i(t)$$

$$U_i(x)=\sqrt{\frac{2\beta_i}{m\left(\tan\left(\beta_i L\right)+\beta_i L\sec^2\left(\beta_i L\right)\right)}}\left[\tan\left(\beta_i L\right)\sin\left(\beta_i x\right)+\cos\left(\beta_i x\right)\right]$$

初始条件为

$$u\left(x,0\right)=0,\dot{u}\left(x,0\right)=v_0\delta\left(x\right)$$

根据正交性，左乘 mU_j 并积分，求出正则坐标下的初始条件，

$$\sum_{i=1}^{n}\int_0^L mU_jU_i\eta_i\left(0\right)dx=\int_0^L mU_ju(x,0)dx$$

$$\sum_{i=1}^{n}\int_0^L mU_jU_i\dot{\eta}_i\left(0\right)dx=\int_0^L mU_j\dot{u}(x,0)dx$$

因此有

$$\eta_j\left(0\right)=0$$

$$\dot{\eta}_j\left(0\right)=mv_0\sqrt{\frac{2\beta_i}{m\left(\tan\left(\beta_i L\right)+\beta_i L\sec^2\left(\beta_i L\right)\right)}}$$

根据正交性，正则坐标下的动力学方程为，

$$\int_0^L\left(\sum_{i=1}^{n}mU_j\left(x\right)U_i(x)\frac{\partial^2\eta_i(t)}{\partial t^2}\right)dx=\int_0^L\left(\sum_{i=1}^{3}EAU_j\left(x\right)\frac{\partial^2 U_i\left(x\right)}{\partial x^2}\eta_i(t)\right)dx$$

$$\frac{\partial^2\eta_j(t)}{\partial t^2}+\omega_j{}^2\eta_j(t)=0,\ j=1,2,3,\ldots$$

解得，

$$\eta_j(t)=\eta_j(0)\cos\left(\omega_j t\right)+\frac{\dot{\eta}_j(0)}{\omega_j}\sin\left(\omega_j t\right)$$

$$\eta_j(t)=\frac{mv_0\sqrt{\frac{2\beta_i}{m\left(\tan\left(L\beta_i\right)+L\beta_i\sec^2\left(L\beta_i\right)\right)}}}{\omega_j}\sin\left(\omega_j t\right),\quad where\ \beta=\sqrt{\frac{m\omega_i^2}{EA}}$$

仅考虑前三阶，则杆在初始条件下的响应可写成，

$$u(x,t)=\sum_{i=1}^{3}U_i(x)\eta_i(t)=\sum_{i=1}^{3}\frac{2v_0\beta_i}{\omega_i\left(\tan\left(\beta_i L\right)+\beta_i L\sec^2\left(\beta_i L\right)\right)}\left[\tan\left(\beta_i L\right)\sin\left(\beta_i x\right)+\cos\left(\beta_i x\right)\right]\sin\left(\omega_i t\right)$$

3.5 (8.36)

Determine the response of the rod of Problem 8.1 1 to the uniformly distributed harmonic force $f\left(x,t\right)=f_0\cos\Omega t$ *.*

Discuss the mode participation in the response.

解：振动方程

$$m\frac{\partial^2 u}{\partial t^2}-EA\frac{\partial^2 u}{\partial x^2}=f(x,t),0<x<L$$

根据正则振型的正交性，正则坐标下的动力学方程为

$$\int_0^L\left(\sum_{i=1}^{n} mU_j(x)U_i(x)\frac{\partial^2\eta_i(t)}{\partial t^2}\right)dx-\int_0^L\left(\sum_{i=1}^{n} EAU_j(x)\frac{\partial^2 U_i(x)}{\partial x^2}\eta_i(t)\right)dx=\int_0^L\left[U_j(x)f(x,t)\right]dx$$

可以得到

$$\ddot{\eta}_j(t)+\omega_j^{\,2}\eta_j(t)=N_j(t),\ j=1,2,3,\ldots$$

其中：

$$\begin{aligned}N_j(t)&=\int_0^L U_j(x)f(x,t)dx\\&=\int_0^L\left(\sqrt{\frac{2\beta_j}{m\left(\tan(\beta_jL)+\beta_jL\sec^2(\beta_jL)\right)}}\left[\tan(\beta_jL)\sin(\beta_jx)+\cos(\beta_jx)\right]f_0\cos\Omega t\right)dx=C_j\cos\Omega t\end{aligned}$$

其中

$$C_j=\sqrt{\frac{2\beta_j}{m\left(\tan(L\beta_j)+L\beta_j\sec^2(L\beta_j)\right)}}\frac{\tan(\beta_jL)}{\beta_j}f_0$$

正则坐标下的稳态响应为，

$$\eta_j(t)=\frac{C_j}{\omega_j^2-\Omega^2}\cos\Omega t$$

回代即可得到杆在谐波激励下的响应，

$$\begin{aligned}u(x,t)&=\sum_{i=1}^{n}U_i(x)\eta_i(t)\\&=\sum_{i=1}^{n}\frac{2\left[\tan(\beta_iL)\sin(\beta_ix)+\cos(\beta_ix)\right]\tan(\beta_iL)}{m\left(\tan(\beta_iL)+\beta_iL\sec^2(\beta_iL)\right)}\frac{f_0\cos\Omega t}{\left(\omega_i^{\,2}-\Omega^2\right)}\end{aligned}$$

仅讨论 $k=2EA/L$ 的情况，将前三节固有频率代入，

$$\begin{aligned}u(x,t)=0.559&\left[1.858\sin(\beta_1x)+\cos(\beta_1x)\right]\frac{f_0\cos\Omega t}{m\left(\omega_1^{\,2}-\Omega^2\right)}\\+0.208&\left[0.549\sin(\beta_2x)+\cos(\beta_2x)\right]\frac{f_0\cos\Omega t}{m\left(\omega_2^{\,2}-\Omega^2\right)}\\+0.081&\left[0.304\sin(\beta_3x)+\cos(\beta_3x)\right]\frac{f_0\cos\Omega t}{m\left(\omega_3^{\,2}-\Omega^2\right)}\end{aligned}$$

可发现各阶模态对响应的贡献程度与模态的阶数成反比。

3.6 (8.42)

Determine the response of the rod of Problem 8.11 to the impulsive force $F(t)=\hat{F}_0\delta(t)$ *applied at* $x=L$. *Discuss the mode participation in the response.*

解：正则坐标下的动力学方程为

$$\ddot{\eta}_j(t)+\omega_j^{\,2}\eta_j(t)=N_j(t),\ j=1,2,3$$

其中，

$$\begin{aligned}N_j(t)&=\int_0^L U_j(x)f(x,t)dx\\&=\int_0^L\left(\sqrt{\frac{2\beta_j}{m\left(\tan(\beta_jL)+\beta_jL\sec^2(\beta_jL)\right)}}\left[\tan(\beta_jL)\sin(\beta_jx)+\cos(\beta_jx)\right]\hat{F}_0\delta(t)\delta(x-L)\right)dx\\&=\sqrt{\frac{2\beta_j}{m\left(\tan(\beta_jL)+\beta_jL\sec^2(\beta_jL)\right)}}\sec(\beta_jL)\hat{F}_0\delta(t)\end{aligned}$$

根据脉冲激励的响应公式得到正则坐标下的稳态响应为：

$$\eta_j(t)=\sqrt{\frac{2\beta_j}{m\left(\tan\left(L\beta_j\right)+L\beta_j\sec^2\left(L\beta_j\right)\right)}}\frac{\sec\left(\beta_j L\right)\hat{F}_0}{\omega_j}\sin\omega_j t$$

回代即可得到杆在脉冲激励下的响应，

$$u(x,t)=\sum_{i=1}^{n}U_i(x)\eta_i(t)=\frac{2\sec\left(\beta_i L\right)}{\left(\tan\left(\beta_i L\right)+\beta_i L\sec^2\left(\beta_i L\right)\right)}\frac{\left[\tan\left(\beta_i L\right)\sin\left(\beta_i x\right)+\cos\left(\beta_i x\right)\right]\hat{F}_0}{m}\sqrt{\frac{m}{EA}}\sin\omega_i t$$

仅讨论 $k=2EA/L$ 的情况，将解得的前三节固有频率代入

$$\begin{aligned}u(x,t)=0.634&\frac{\left[1.858\sin\left(\beta_1 x\right)+\cos\left(\beta_1 x\right)\right]\hat{F}_0}{m}\sqrt{\frac{m}{EA}}\sin\omega_1 t\\ -0.431&\frac{\left[0.549\sin\left(\beta_2 x\right)+\cos\left(\beta_2 x\right)\right]\hat{F}_0}{m}\sqrt{\frac{m}{EA}}\sin\omega_2 t\\ +0.279&\frac{\left[0.304\sin\left(\beta_3 x\right)+\cos\left(\beta_3 x\right)\right]\hat{F}_0}{m}\sqrt{\frac{m}{EA}}\sin\omega_3 t\end{aligned}$$

各阶模态对响应的贡献程度与模态的阶数成反比。

Problem 4 (50 points in total)

Problem 4.1: Please solve Problem 8.7 in page 459 of the textbook (20 points).

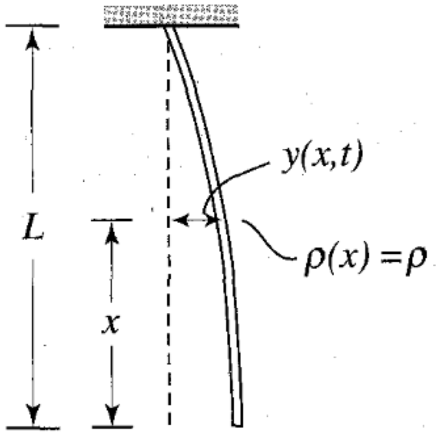
Problem 4.2: Please solve Problem 8.13 in page 460 of the textbook (10 points).

Problem 4.3: Please solve problem 8.19 in page 461 of the textbook (5 points).

Problem 4.4: Please solve Problem 8.32 in page 462 of the textbook (15 points).

4.1 (8.7)

A cable of uniform mass per unit length, $\rho(x)=\rho=\text{costant}$, *hangs freely from the ceiling, as shown in Fig. 8.34. Assume that the cable possesses no flexural stiffness and derive the boundary-value problem for the transverse vibration. Hint: The boundary condition at* $x=0$, *ordinarily associated with a free end, is satisfied trivially in the case at hand, without involving the displacement. Hence, it must be replaced by a different boundary condition, based on physical considerations and the nature of the solution (see also Problem 8.13).*



解：绳子的动能为，

$$T=\frac{1}{2}\int_0^L \rho\left(\frac{\partial y}{\partial t}\right)^2 dx$$

其中绳子的变形量为，

$$ds-dx=\sqrt{dx^2+\left(\frac{\partial y}{\partial x}dx\right)^2}-dx=dx\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}-dx$$

$$=dx\left(1+\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2+\cdots\right)-dx=\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

设绳子的张力为 $F(x)$，在重力的作用下，弦上的张力处处不相等。势能为，

$$V=\int_0^L F(x)(ds-dx)dx=\frac{1}{2}\int_0^L F(x)\left(\frac{\partial y}{\partial x}\right)^2 dx$$

根据广义 Hamilton 原理得到

$$\int_0^L\int_0^t\left(\rho\frac{\partial y}{\partial t}\delta\left(\frac{\partial y}{\partial t}\right)-F(x)\frac{\partial y}{\partial x}\delta\left(\frac{\partial y}{\partial x}\right)\right)dtdx=0$$

分部积分得到

$$\int_0^L\int_0^t\left(\rho\frac{\partial^2 y}{\partial t^2}-\frac{\partial y}{\partial x}\left(F(x)\frac{\partial y}{\partial x}\right)\right)\delta ydtdx+\int_0^t\left(F(x)\frac{\partial y}{\partial x}\delta y\Big|_0^L\right)dt=0$$

因此绳的方程为

$$\rho\frac{\partial^2 y}{\partial t^2}-\frac{\partial y}{\partial x}\left(F(x)\frac{\partial y}{\partial x}\right)=0$$

边界条件满足，

$$F(x)\frac{\partial y}{\partial x}=0,\ \ x=0$$

$$y=0,\ \ x=L$$

对竖直放置做横向微小摆动的弦来说，可近似认为弦上某处的张力等于其下端弦的重力，张力可表达为，

$$F(x)=\rho g x$$

代入可得振动方程，

$$\frac{\partial^2 y}{\partial t^2}-g\frac{\partial y}{\partial x}-gx\frac{\partial^2 y}{\partial x^2}=0$$

此外，由于自由端张力为零，边界条件满足平凡解，并未涉及到位移，将 $x=0$ 代入振动方程，可得自由端的边界条件满足，

$$\left.\left(\frac{\partial^2 y}{\partial t^2}-g\frac{\partial y}{\partial x}\right)\right|_{x=0}=0$$

4.2 (8.13)

Derive and solve the eigenvalue problem for the hanging cable of Problem 8.7. Plot the three lowest modes. Hints: Devise a certain coordinate transformation capable of reducing the differential equation to a Bessel equation. Then, the boundary condition at the free end of the cable must be such as to permit elimination of the unacceptable solution.

解：由

$$\frac{\partial^2 y}{\partial t^2}-g\frac{\partial y}{\partial x}-gx\frac{\partial^2 y}{\partial x^2}=0$$

设解，

$$y(x,t)=Y(x)\sin(\omega t+\phi)$$

可以得到

$$\omega^2 Y(x)+g\frac{\partial Y(x)}{\partial x}+gx\frac{\partial^2 Y(x)}{\partial x^2}=0$$

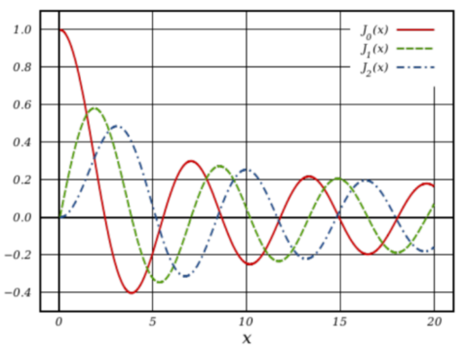
变量替换，令 $\xi=2\omega/g\sqrt{xg}$ ，可以得到

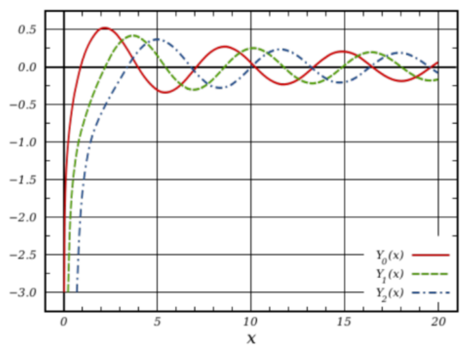
$$\xi^2\frac{\partial^2 Y}{\partial \xi^2}+\xi\frac{\partial Y}{\partial \xi}+\xi^2 Y=0$$

上述为零阶贝塞尔方程，其通解为

$$Y=C_1 J_0(\xi)+C_2 Y_0(\xi)$$

其中 $J_0(\xi),Y_0(\xi)$ 线性无关，分别为第一类和第二类贝塞尔函数。如下图所示，





因为$Y_0\left(\xi\right)$在$\xi=0$处有一条渐近线，对应于链的固定端，不满足物理意义，因此有意义的解只有第一类贝塞尔函数，函数表达式为

$$J_n\left(x\right)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma\left(n+m+1\right)}\left(\frac{x}{2}\right)^{n+2m}$$

其中

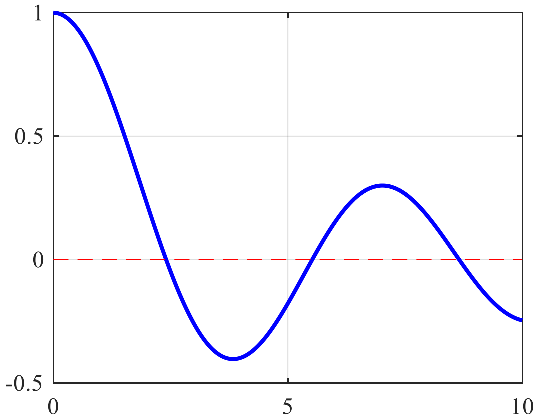
$$J_0\left(x\right)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!m!}\left(\frac{x}{2}\right)^{2m}$$

因此有意义的解为，

$$Y=CJ_0\left(\xi\right)=CJ_0\left(\frac{2\omega}{g}\sqrt{xg}\right)=C\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{\omega}{g}\sqrt{xg}\right)^{2m}}{m!m!}$$

代入边界条件$Y|_{x=L}=0$可以得到

$$J_0\left(2\omega\sqrt{L/g}\right)=0$$



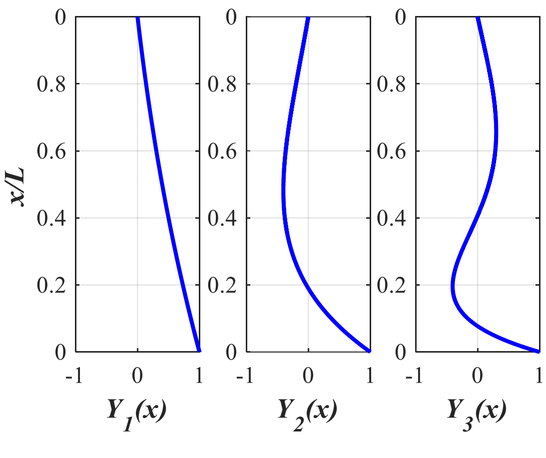
前三阶解为，

$$2\omega_1\sqrt{L/g}=2.405,2\omega_2\sqrt{L/g}=5.520,2\omega_3\sqrt{L/g}=8.654$$

因此主阵型为，

$$Y_i\left(x\right)=J_0\left(\xi_i\right)=J_0\left(2\omega_i/g\sqrt{xg}\right)=\sum_{m=0}^{\infty}\frac{(-1)^m\left(\omega_i\sqrt{x/g}\right)^{2m}}{m!m!}=\sum_{m=0}^{\infty}\frac{(-1)^m\left(2\omega_i\sqrt{L/g}\cdot0.5\sqrt{x/L}\right)^{2m}}{m!m!}$$

前三阶主阵型如下图所示，



4.3 (8.19)

Verify that the modes of the hanging cable obtained in Problem 8.13 are indeed orthogonal.

解：固有频率满足

$$\sum_{m=0}^{\infty}\frac{(-1)^m\left(\omega_i\sqrt{L/g}\right)^{2m}}{m!m!}=0$$

主阵型，

$$Y_i\left(x\right)=J_0\left(\xi\right)=J_0\left(\frac{2\omega_i}{g}\sqrt{xg}\right)=\sum_{m=0}^{\infty}\frac{(-1)^m\left(\omega_i\sqrt{x/g}\right)^{2m}}{m!m!}$$

正交性验证：当$i\neq j$时

$$\int_0^L\rho Y_i\left(x\right)Y_j\left(x\right)dx=\int_0^L\rho J_0\left(\xi_i\right)J_0\left(\xi_j\right)dx=\rho\int_0^L\sum_{m=0}^{\infty}\frac{(-1)^m\left(\omega_i\sqrt{L/g}*\sqrt{x/L}\right)^{2m}}{m!m!}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\omega_j\sqrt{L/g}*\sqrt{x/L}\right)^{2m}}{m!m!}dx$$

令$x_1=\sqrt{x},L_1=\sqrt{L}$，即$\frac{x_1}{L_1}=\sqrt{\frac{x}{L}}$，得到：

$$\begin{aligned}\int_0^L\rho Y_i\left(x\right)Y_j\left(x\right)dx&=2\rho\int_0^{L_1}x_1\sum_{m=0}^{\infty}\frac{(-1)^m\left(\omega_i\sqrt{L/g}*x_1/L_1\right)^{2m}}{m!m!}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\omega_j\sqrt{L/g}*x_1/L_1\right)^{2m}}{m!m!}dx_1\\&=2\rho\int_0^{L_1}x_1J_0\left(2\omega_i\sqrt{L/g}*x_1/L_1\right)J_0\left(2\omega_j\sqrt{L/g}*x_1/L_1\right)dx_1\end{aligned}$$

根据零阶 Bessel 函数的正交性（证明略），上式为零。将其回代到振动方程中，得以下关系式，

$$\int_0^L\left(\rho g\frac{\partial Y_i(x)}{\partial x}+\rho gx\frac{\partial^2Y_i(x)}{\partial x^2}\right)Y_j\left(x\right)dx=-\omega_i^2\int_0^L\rho Y_i\left(x\right)Y_j\left(x\right)dx=0,\quad i\neq j$$

当$i=j$时，同理

$$\int_0^L\rho Y^2{}_i\left(x\right)dx=\int_0^L\rho J_0{}^2\left(\xi_i\right)dx=\rho\int_0^L\left[\sum_{m=0}^{\infty}\frac{(-1)^m\left(\omega_i\sqrt{L/g}*\sqrt{x/L}\right)^{2m}}{m!m!}\right]^2dx$$

$$=2\rho\int_0^{L_1}x_1\left[\sum_{m=0}^{\infty}\frac{(-1)^m\left(\omega_i\sqrt{L/g}*x_1/L_1\right)^{2m}}{m!m!}\right]^2dx_1=2\rho\int_0^{L_1}x_1J_0{}^2\left(2\omega_i\sqrt{L/g}*x_1/L_1\right)dx_1$$

根据 Bessel 函数的性质，上式满足

$$\begin{aligned}\int_0^L\rho Y^2{}_i\left(x\right)dx&=2\rho\int_0^{L_1}x_1J_0{}^2\left(2\omega_i\sqrt{L/g}*x_1/L_1\right)dx_1=2\rho\frac{L_1{}^2}{2}J_1{}^2\left(2\omega_i\sqrt{L/g}\right)\\&=\rho L\left[\sum_{m=0}^{\infty}\frac{(-1)^m}{m!(m+1)!}\left(\omega_i\sqrt{L/g}\right)^{2m+1}\right]^2\end{aligned}$$

对主振型做归一化处理，使其满足，

$$c_i^2\int_0^L\rho Y_i\left(x\right)^2dx=1$$

解得，

$$c_1^2=\frac{1}{0.2695\rho L},\ c_2^2=\frac{1}{0.1158\rho L},\ c_3^2=\frac{1}{0.0737\rho L}$$

得到正则振型，

$$Y_i\left(x\right)=c_i\sum_{m=0}^{\infty}\frac{(-1)^m\left(\omega_i\sqrt{x/g}\right)^{2m}}{m!m!}$$

而且，正则振型满足：

$$\int_0^L\left(\rho g\frac{\partial Y_i(x)}{\partial x}+\rho gx\frac{\partial^2Y_i(x)}{\partial x^2}\right)Y_i\left(x\right)dx=-\omega_i^2$$

4.4 (8.32)

The hanging cable of Problem 8.13 is displaced initially according to $y\left(x,0\right)=y_0\left(1-x/L\right)$. *Determine the response of the cable subsequent to being released from rest in the displaced position.*

解：将响应展开为正则振型的无穷级数，

$$y(x,t)=\sum_{i=1}^{n}Y_i(x)\eta_i(t)$$

初始条件，

$$y\left(x,0\right)=y_0\left(1-\frac{x}{L}\right),\ \dot{y}\left(x,0\right)=0$$

根据所推导的正则振型的正交性，得到正则坐标下的振动方程，

$$\int_0^L\left(\sum_{i=1}^{n}\rho Y_i\left(x\right)\ddot{\eta}_i\left(t\right)Y_j\left(x\right)\right)dx-\int_0^L\left(\rho g\sum_{i=1}^{n}\frac{\partial Y_i\left(x\right)}{\partial x}\eta_i(t)+\rho gx\sum_{i=1}^{n}\frac{\partial^2Y_i\left(x\right)}{\partial x^2}\eta_i(t)\right)Y_j\left(x\right)dx=0$$

即

$$\ddot{\eta}_j(t)+\omega_j^{\,2}\eta_j(t)=0$$

正则坐标下的初始条件为，

$$\sum_{i=1}^{3}\int_0^L\rho Y_jY_i\eta_i\left(0\right)\mathrm{dx}=\int_0^L\rho Y_jy(x,0)dx$$

$$\sum_{i=1}^{3}\int_0^L\rho Y_jY_i\dot{\eta}_i\left(0\right)\mathrm{dx}=\int_0^L\rho Y_j\dot{y}(x,0)dx$$

即

$$\eta_j\left(0\right)=\rho c_jy_0\int_0^L\sum_{m=0}^{\infty}\frac{(-1)^m\left(\omega_j\sqrt{x/g}\right)^{2m}}{m!m!}\left(1-\frac{x}{L}\right)dx,\quad \dot{\eta}_j\left(0\right)=0$$

解得正则坐标下的响应，

$$\eta_i\left(t\right)=\eta_i\left(0\right)\cos\omega_it$$

则绳索在初始条件下的响应可写为：

$$y(x,t)=\sum_{i=1}^{n}Y_i(x)\eta_i(t)=\sum_{i=1}^{n}\left(c_i^2\rho y_0\left(\int_0^L\sum_{m=0}^{\infty}\frac{(-1)^m\left(\omega_j\sqrt{x/g}\right)^{2m}}{m!m!}\left(1-\frac{x}{L}\right)dx\cos\omega_it\right)\sum_{m=0}^{\infty}\frac{(-1)^m\left(\omega_i\sqrt{x/g}\right)^{2m}}{m!m!}\right)$$

Problem 5 (85 points in total)

Problem 5.1: Please solve Problem 8.8 in page 460 of the textbook (20 points).

Problem 5.2: Please solve Problem 8.9 in page 460 of the textbook (20 points).

Problem 5.3: Please solve Problem 8.16 in page 461 of the textbook (10 points).

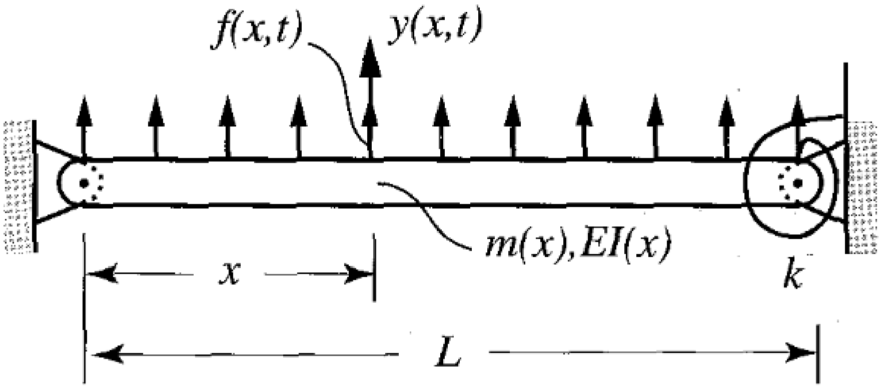
Problem 5.4: Please solve problem 8.22 in page 461 of the textbook (5 points).

Problem 5.5: Please solve Problem 8.34 in page 463 of the textbook (15 points).

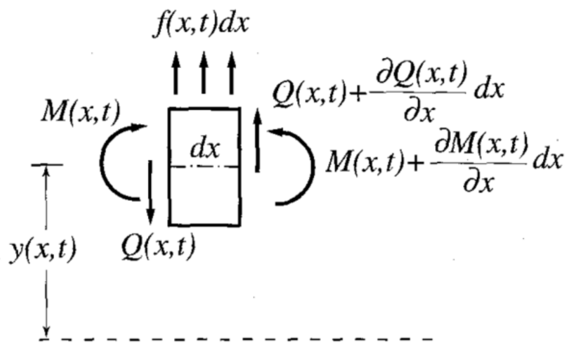
Problem 5.6: Please solve Problem 8.40 in page 463 of the textbook (15 points).

5.1 (8.8)

Use the Newtonian approach to derive the boundary-value problem for the bending vibration of a beam pinned at $x=0$ *and pinned but with the slope to the deflection curve restrained by a spring at* $x=L$*, as shown in Fig. 8.35.*



解：取微元进行分析，



竖直方向上的力平衡方程为，

$$\left[Q(x,t)+\frac{\partial Q(x,t)}{\partial x}dx\right]-Q(x,t)+f(x,t)dx=m(x)dx\frac{\partial^2 y(x,t)}{\partial t^2},0<x<L$$

忽略微元体惯性矩与角加速度乘积，力矩平衡方程如下，

$$\left[M(x,t)+\frac{\partial M(x,t)}{\partial x}dx\right]-M(x,t)+\left[Q(x,t)+\frac{\partial Q(x,t)}{\partial x}dx\right]dx+f(x,t)dx\frac{dx}{2}=0,\ \ 0<x<L$$

略去高阶项，

$$\frac{\partial M(x,t)}{\partial x}+Q(x,t)=0,0<x<L$$

将其代入力平衡方程，得到，

$$-\frac{\partial^2 M(x,t)}{\partial x^2}+f(x,t)=m(x)\frac{\partial^2 y(x,t)}{\partial t^2},0<x<L$$

根据弯矩与挠度的关系，动力学方程变为：

$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]+f(x,t)=m(x)\frac{\partial^2 y(x,t)}{\partial t^2},0<x<L$$

边界条件： $x=0$ 处简支，挠度和弯矩为零； $x=L$ 处简支，附着有刚度为 k 的扭簧，挠度为零，弯矩大小等于扭簧所提供的弯矩，符号为负，

$$\delta y=0,EI\frac{\partial^2 y}{\partial x^2}=0,\ \ x=0$$

$$\delta y=0,EI\frac{\partial^2 y}{\partial x^2}=-k\theta,\ \ x=L$$

5.2 (8.9)

Solve Problem 8.8 by the extended Hamilton principle.

解：设梁上中性轴的竖直位移为 $y(x,t)$ ，中性轴的转动角度为 $\theta(x,t)$ ，则梁上任意一点处的横向位移和纵向位移可以分别表示为，

$$u(x,t)=-z\sin\theta(x,t),\ \ w(x,t)=y(x,t)+z\left(1-\cos\theta(x,t)\right)$$

其中 z 为梁上任意一点到中性轴的有向距离。中性轴的位移与转动之间有关系，

$$\tan\theta(x,t)=\frac{\partial y(x,t)}{\partial x}$$

梁的动能可以表示为

$$T=\int_0^L\frac{1}{2}m(x)\left(\dot{u}^2+\dot{w}^2\right)dx=\int_0^L\frac{1}{2}m(x)\int_{-h/2}^{h/2}\frac{1}{h}\left(\dot{u}^2+\dot{w}^2\right)dzdx$$

$$=\int_0^L\frac{1}{2}m(x)\int_{-h/2}^{h/2}\frac{1}{h}\left(\dot{y}^2+z^2\dot{\theta}^2+2\dot{y}\dot{\theta}\sin\theta z\right)dzdx$$

其中最后一项在截面上积分为 0，在小变形的条件下第二项可忽略，因此动能为

$$T=\int_0^L\frac{1}{2}m(x)\dot{y}^2dx$$

接下来考虑结构的应变

$$\varepsilon_x=\frac{\partial u}{\partial x}=-z\cos\theta\frac{\partial\theta}{\partial x},\ \ \varepsilon_z=\frac{\partial w}{\partial z}=1-\cos\theta,\ \ \gamma_{xz}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}=\frac{\partial y}{\partial x}+z\sin\theta\frac{\partial\theta}{\partial x}-\sin\theta$$

忽略剪应力，梁的势能可以表达为

$$V=\int_0^L\int_S\frac{1}{2}E\left(\varepsilon_x^2+\varepsilon_z^2\right)dSdx=\int_0^L\int_S\frac{1}{2}E\left(\left(-z\cos\theta\frac{\partial\theta}{\partial x}\right)^2+\left(1-\cos\theta\right)^2\right)dSdx$$

$$=\int_0^L\left[\frac{1}{2}EI\left(\cos\theta\frac{\partial\theta}{\partial x}\right)^2+\frac{1}{2}EA\left(1-\cos\theta\right)^2\right]dx$$

在小变形的条件下认为 $\cos\theta\approx 1$ ，第二项为 0，因此

$$V=\int_0^L\frac{1}{2}EI\left(\frac{\partial\theta}{\partial x}\right)^2dx$$

对动能势能求变分并积分后有，

$$\int_0^t\delta Tdt=\int_0^t\int_0^L m(x)\left(\dot{y}\delta\dot{y}\right)dxdt=\int_0^L\int_0^t m(x)\left(\dot{y}\frac{\partial\left(\delta y\right)}{\partial t}\right)dtdx$$

$$=\int_0^L m(x)\left(\dot{y}\delta y\right)dx\Big|_0^t-\int_0^L\int_0^t\left(\frac{\partial}{\partial t}\left(m(x)\dot{y}\right)\delta y\right)dtdx=-\int_0^L\int_0^t\left(\frac{\partial}{\partial t}\left(m(x)\dot{y}\right)\delta y\right)dtdx$$

以及

$$\int_0^t \delta V dt = \int_0^t\int_0^L \frac{1}{2} EI\frac{\partial\theta}{\partial x}\delta\left(\frac{\partial\theta}{\partial x}\right)dxdt = \int_0^t EI\frac{\partial\theta}{\partial x}\delta\theta\bigg|_0^L dt - \int_0^t\int_0^L \frac{\partial}{\partial x}\left(EI\frac{\partial\theta}{\partial x}\right)\delta\theta dxdt$$

$$= \int_0^t\left(EI\frac{\partial\theta}{\partial x}\delta\theta - \frac{\partial}{\partial x}\left(EI\frac{\partial\theta}{\partial x}\right)\delta y\right)\bigg|_0^L dt + \int_0^t\int_0^L \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial\theta}{\partial x}\right)\delta y dtdx$$

同时梁还受到外力 $f(x,t)$，以及力矩 $k\theta\delta(x-L)$，利用广义 Hamilton 原理有，

$$-\int_0^L\int_0^t\left(\frac{\partial}{\partial t}(m(x)\dot{y}) + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial\theta}{\partial x}\right) - f(x,t)\right)\delta y dtdx - \int_0^t\left(EI\frac{\partial\theta}{\partial x}\delta\theta - \frac{\partial}{\partial x}\left(EI\frac{\partial\theta}{\partial x}\right)\delta y + k\theta\delta(x-L)\delta\theta\right)\bigg|_0^L dt = 0$$

动力学方程为

$$\frac{\partial}{\partial t}(m(x)\dot{y}) + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial\theta}{\partial x}\right) = f(x,t)$$

在小变形条件下，中性轴的转动角度 $\theta(x,t)$ 可表示为

$$\theta(x,t) = \frac{\partial y(x,t)}{\partial x}$$

左右两边的边界条件满足

$$\delta y = 0, EI\frac{\partial^2 y}{\partial x^2} = 0,\ \ x = 0$$

$$\delta y = 0, EI\frac{\partial^2 y}{\partial x^2} = -k\theta,\ \ x = L$$

5.3 (8.16)

Derive the eigenvalue problem for the beam of Problem 8.8. Then, let $m(x) = m = \text{costant}, EI(x) = EI = \text{costant}$ *,* $k = 0.5EI/L$ *, solve the eigenvalue problem and plot the three lowest modes.*

解：在 $m(x) = m = \text{costant}, EI(x) = EI = \text{costant}$ 条件下，不考虑外力时梁振动方程为

$$m\frac{\partial^2 y(x,t)}{\partial t^2} + EI\frac{\partial^4 y(x,t)}{\partial x^4} = 0, 0 < x < L$$

设解 $y(x,t) = Y(x)b\sin(\omega t + \varphi)$ 并代入，有

$$\frac{\partial^4 Y}{\partial x^4} = \lambda^4 Y,\quad \lambda^4 = \frac{m\omega^2}{EI}$$

通解为

$$Y(x) = A\cos\lambda x + B\sin\lambda x + C\cosh\lambda x + D\sinh\lambda x$$

将边界条件代入，在 $x = 0$ 处

$$\begin{cases} A + C = 0 \\ \lambda^2(-A + C) = 0 \end{cases} \Rightarrow A = C = 0$$

在 $x = L$ 处，

$$B\sin\lambda L + D\sinh\lambda L = 0$$

$$2L\lambda(-B\sin\lambda L + D\sinh\lambda L) + (B\cos\lambda L + D\cosh\lambda L) = 0$$

要使上式成立且有非平凡解，则，

$$\det\left(\begin{bmatrix} \sin\lambda L & \sinh\lambda L \\ \cos\lambda L - 2L\lambda\sin\lambda L & \cosh\lambda L + 2L\lambda\sinh\lambda L \end{bmatrix}\right) = 0$$

其特征方程为

$$\sin\lambda L\cosh\lambda L - \sinh\lambda L\cos\lambda L + 4\lambda L\sinh\lambda L\sin\lambda L = 0$$

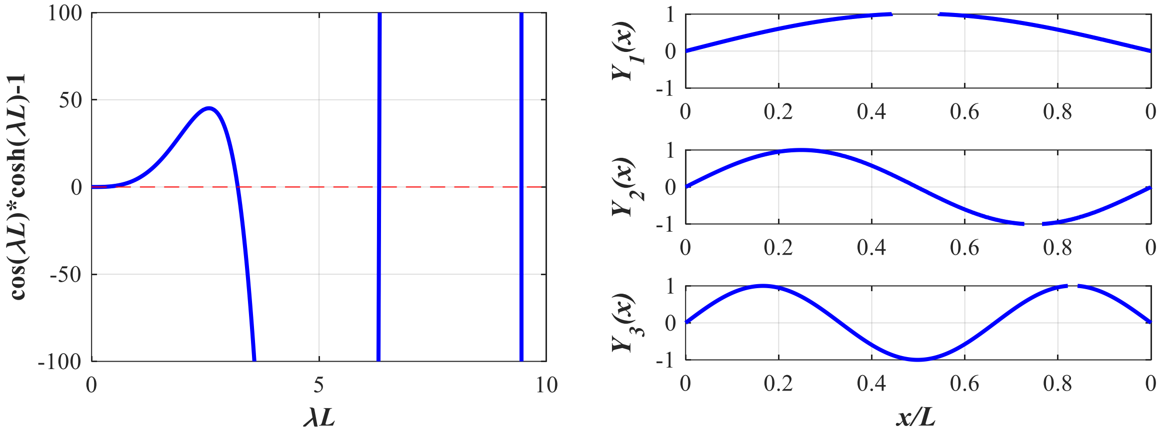
解得前三阶固有频率，

$$\lambda_1 L = 3.21, \lambda_2 L = 6.32, \lambda_3 L = 9.45$$

主振型为：

$$Y_i(x) = \sin\lambda_i x - \frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i x$$

前三阶主振型分别如下图所示：



5.4 (8.22)

Derive the orthogonality relations for the beam of Problem 8.16.

解：由上一题可得，

$$EI\frac{\partial^4 Y}{\partial x^4} = m\omega^2 Y$$

即，

$$EI\frac{\partial^4 Y_i}{\partial x^4} = m\omega_i^{\ 2} Y_i \qquad (1)$$

$$EI\frac{\partial^4 Y_j}{\partial x^4} = m\omega_j^{\ 2} Y_j \qquad (2)$$

对（1）左乘Y_j并在$0-L$上积分，

$$\int_0^L EIY_j\frac{\partial^4 Y_i}{\partial x^4}dx = \omega_i^{\ 2}\int_0^L Y_j mY_i dx$$

分部积分并代入边界条件可得，

$$\frac{\partial Y_j(L)}{\partial x}K\frac{\partial Y_i(L)}{\partial x} + \int_0^L EI\frac{\partial^2 Y_j}{\partial x^2}\frac{\partial^2 Y_i}{\partial x^2}dx = \lambda_i\int_0^L Y_j mY_i dx$$

同理由（2）得，

$$\frac{\partial Y_j(L)}{\partial x}K\frac{\partial Y_i(L)}{\partial x} + \int_0^L EI\frac{\partial^2 Y_j}{\partial x^2}\frac{\partial^2 Y_i}{\partial x^2}dx = \lambda_j\int_0^L Y_j mY_i dx$$

两式相减得到，

$$\left(\lambda_j - \lambda_i\right)\int_0^L Y_j mY_i dx = 0$$

也即

$$\int_0^L Y_j mY_i dx = 0 \ \ (i \neq j)$$

上式验证了主振型关于质量的正交性，将其回代可得到主振型关于刚度的正交性，

$$\int_0^L EIY_j\frac{\partial^4 Y_i}{\partial x^4}dx \ \ (i \neq j)$$

对主振型做归一化处理，使其满足，

$$\int_0^L c^2 Y_i^2 m dx = 1,\ i = 1,2,3\cdots$$

$$\int_0^L c^2 EIY_i \frac{\partial^4 Y_i(x)}{\partial x^4} dx = \lambda_i,\ i = 1,2,3\cdots$$

得到正则振型，

$$Y_i(x) = c_i\left(\sin\lambda_i x - \frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i x\right)$$

其中前三阶满足

$$c_1 = \frac{1}{\sqrt{0.510mL}},\ c_1 = \frac{1}{\sqrt{0.503mL}},\ c_1 = \frac{1}{\sqrt{0.501mL}}$$

5.5 (8.34)

Determine the response of the uniform beam of Problem 8.16 to the initial excitation $y(x,0) = y_0\left[13(x/L) - 27(x/L)^3 + 14(x/L)^4\right], \dot{y}(x,0) = 0$. *Discuss the mode participation in the response.*

解：将梁的挠度展开为正则振型的无穷级数。

$$y(x,t) = \sum_{i=1}^{n} Y_i(x)\eta_i(t)$$

初始条件

$$y(x,0) = 0,\ \dot{y}(x,0) = 0$$

根据正交性求出正则坐标下的初始条件，

$$\sum_{i=1}^{n}\int_0^L mU_jU_i\eta_i(0)dx = \int_0^L mU_j y(x,0)dx$$

$$\sum_{i=1}^{n}\int_0^L mU_jU_i\dot{\eta}_i(0)dx = \int_0^L mU_j \dot{y}(x,0)dx$$

由此求得

$$\eta_j(0) = \int_0^L mU_j y_0\left[13(x/L) - 27(x/L)^3 + 14(x/L)^4\right]dx$$

$$\dot{\eta}_j(0) = 0$$

根据正交性，正则坐标下的动力学方程为，

$$\ddot{\eta}_j(t) + \omega_j^2\eta_j(t) = 0,\ j = 1,2,3$$

解得，

$$\eta_j(t) = \eta_j(0)\cos(\omega_j t)$$

则杆在初始条件下的响应可写成，

$$y(x,t) = \sum_{i=1}^{n} Y_i(x)\eta_i(t) = \sum_{i=1}^{n} c_i\left(\sin\lambda_i x - \frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i x\right)\eta_i(0)\cos(\omega_i t)$$

以前三阶为例，有

$$\begin{aligned}
y(x,t) \approx\ & 3.967y_0\left(\sin\lambda_1 x - \frac{\sin\lambda_1 L}{\sinh\lambda_1 L}\sinh\lambda_1 x\right)\cos(\omega_1 t) \\
& -0.00239y_0\left(\sin\lambda_2 x - \frac{\sin\lambda_2 L}{\sinh\lambda_2 L}\sinh\lambda_2 x\right)\cos(\omega_2 t) \\
& +0.018y_0\left(\sin\lambda_3 x - \frac{\sin\lambda_3 L}{\sinh\lambda_3 L}\sinh\lambda_3 x\right)\cos(\omega_3 t)
\end{aligned}$$

可发现前三阶模态中第一阶模态对响应的贡献最大，第二阶模态对响应贡献极小。这是因为：初始位移是关于0.4935L对称的，而 1,3 阶模态关于0.5L对称，2 阶模态关于0.5L反对称。因此所给定初始条件很难激发反对称的二阶模态，且 3 阶模态相对 1 阶模态较难激发。

5.6 (8.40)

Determine the response of the beam of Problem 8.16 to a concentrated force expressed as distributed in the form $f(x,t)=F_0\delta(x-3L/4)\left[r(t)-r(t-T)\right]$*, where* $\delta(x-3L/4)$ *is a spatial Dirac delta function located at* $x=3L/4$ *and* $r(t)$ *is the unit ramp function. Discuss the mode participation in the response.*

解：振动方程为，

$$m\frac{\partial^2 y(x,t)}{\partial t^2}+EI\frac{\partial^4 y(x,t)}{\partial x^4}=f(x,t),\quad 0<x<L$$

根据振型的正交性，正则坐标下的动力学方程为：

$$\ddot{\eta}_i+\omega_i^2\eta_i=\int_0^L F_0\delta(x-3L/4)[r(t)-r(t-T)]Y_i(x)dx$$

代入之后有，

$$\ddot{\eta}_i+\omega_i^2\eta_i=F_0[r(t)-r(t-T)]c_i\left(\sin\lambda_i\frac{3}{4}L-\frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i\frac{3}{4}L\right)$$

由斜坡响应叠加可得正则坐标下的响应为：

$$\eta_i(t)=\frac{F_0c_i}{\omega_i^3T}\left(\sin\lambda_i\frac{3}{4}L-\frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i\frac{3}{4}L\right)\left[(\omega_i t-\sin\omega_i t)u(t)-\left(\omega_i(t-T)-\sin\omega_i(t-T)\right)u(t-T)\right]$$

梁的稳态响应为，

$$\begin{aligned}y(x,t)&=\sum_{i=1}^{n}Y_i(x)\eta_i(t)\\&=\sum_{1}^{3}\frac{c_i^2}{\lambda_i^4}\left(\sin\lambda_i\frac{3}{4}L-\frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i\frac{3}{4}L\right)\frac{mF_0}{EIT}\left(\sin\lambda_i x-\frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i x\right)\left[\left(t-\frac{\sin\omega_i t}{\omega_i}\right)u(t)-\left(t-T-\frac{\sin\omega_i(t-T)}{\omega_i}\right)u(t-T)\right]\end{aligned}$$

以前三阶为例有

$$\begin{aligned}y(x,t)\approx\ &0.0129\frac{L^3F_0}{EIT}\left(\sin\lambda_1 x-\frac{\sin\lambda_1 L}{\sinh\lambda_1 L}\sinh\lambda_1 x\right)\left[\left(t-\frac{\sin\omega_1 t}{\omega_1}\right)u(t)-\left(t-T-\frac{\sin\omega_1(t-T)}{\omega_1}\right)u(t-T)\right]\\&-0.0013\frac{L^3F_0}{EIT}\left(\sin\lambda_2 x-\frac{\sin\lambda_2 L}{\sinh\lambda_2 L}\sinh\lambda_2 x\right)\left[\left(t-\frac{\sin\omega_2 t}{\omega_2}\right)u(t)-\left(t-T-\frac{\sin\omega_2(t-T)}{\omega_2}\right)u(t-T)\right]\\&+0.0002\frac{L^3F_0}{EIT}\left(\sin\lambda_3 x-\frac{\sin\lambda_3 L}{\sinh\lambda_3 L}\sinh\lambda_3 x\right)\left[\left(t-\frac{\sin\omega_3 t}{\omega_3}\right)u(t)-\left(t-T-\frac{\sin\omega_3(t-T)}{\omega_3}\right)u(t-T)\right]\end{aligned}$$

可发现各阶模态对响应的贡献随着阶数的增加而降低。

Problem 6 (45 points in total)

Problem 6.1: Please solve Problem 8.14 in page 460 of the textbook. Note: you can direct use the EOM of the beam, however, you need to choose the correct boundary conditions (10 points).

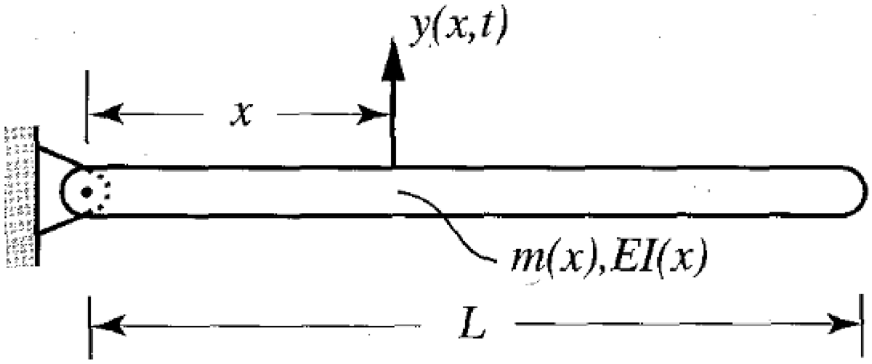
Problem 6.2: Please solve problem 8.20 in page 461 of the textbook (5 points).

Problem 6.3: Please solve Problem 8.33 in page 462 of the textbook (15 points).

Problem 6.4: Please solve Problem 8.37 in page 463 of the textbook (15 points).

6.1 (8.14)

Derive the eigenvalue problem for the bending vibration of a beam pinned at $x=0$ and free at $x=L$ (Fig. 8.36). Then, let $m(x)=m=\text{costant}$, $EI(x)=EI=\text{costant}$, solve the eigenvalue problem and plot the three lowest modes. Hint: The system is only semidefinite, as it admits a rigid-body rotation.



解：梁的振动方程为

$$\frac{\partial}{\partial t}\left(m(x)\dot{y}\right)+\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right)=0,0<x<L$$

设梁的振动通解为

$$y(x,t)=Y(x)b\sin(\omega t+\varphi)$$

代入动力学方程，

$$Y^{(4)}=\frac{m\omega^2}{EI}Y$$

用简谐函数和双曲函数表示通解

$$Y(x)=A\cos\lambda x+B\sin\lambda x+C\cosh\lambda x+D\sinh\lambda x,\ \ \lambda^4=\frac{m\omega^2}{EI}$$

左端简支，右端自由的边界条件可以表示为

$$x=0,\quad Y=0,EIY^{(2)}=0$$
$$x=L,\quad EIY^{(2)}=0,EIY^{(3)}=0$$

在 $x=0$ 处有

$$\begin{cases} A+C=0 \\ \lambda^2(-A+C)=0 \end{cases}$$

可以得到 $A=-C,\lambda^2C=0$，在 $x=L$ 处有

$$\lambda^2(-A\cos\lambda L-B\sin\lambda L+C\cosh\lambda L+D\sinh\lambda L)=0$$
$$\lambda^3(A\sin\lambda L-B\cos\lambda L+C\sinh\lambda L+D\cosh\lambda L)=0$$

将 $A=-C,\lambda^2C=0$ 代入有

$$\lambda^2(-B\sin\lambda L + D\sinh\lambda L) = 0$$
$$\lambda^3(-B\cos\lambda L + D\cosh\lambda L) = 0$$

当$\lambda = 0$时存在刚体模态，考虑边界条件得到$Y(x) = A'x$。

当$\lambda \neq 0$时，有

$$A = C = 0$$
$$-B\sin\lambda L + D\sinh\lambda L = 0$$
$$-B\cos\lambda L + D\cosh\lambda L = 0$$

因此特征方程为

$$\sin\lambda L\cosh\lambda L - \cos\lambda L\sinh\lambda L = 0$$

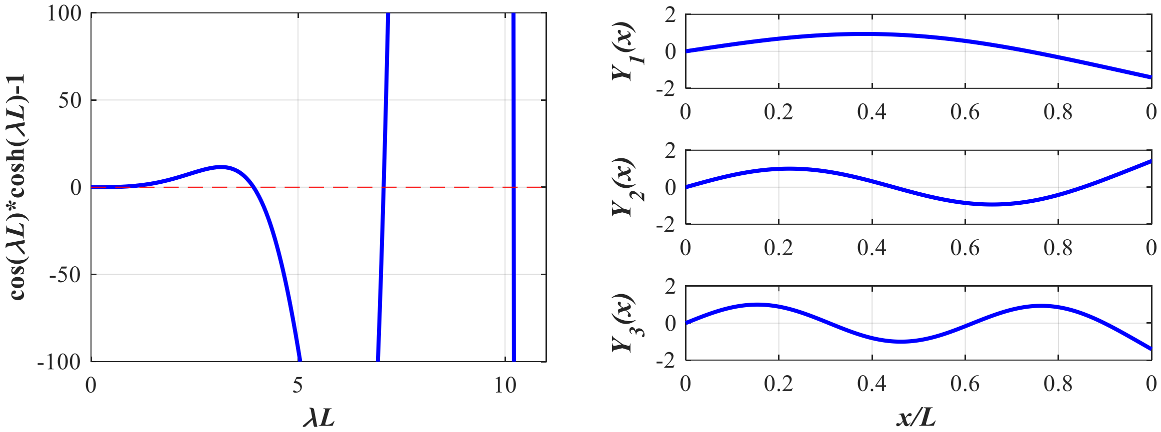
解得前三阶固有频率，

$$\lambda_1 L = 3.927, \lambda_2 L = 7.069, \lambda_3 L = 10.210$$

主振型为，

$$Y_i(x) = \sin\lambda_i x + \frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i x$$

前三阶振型如下图，



6.2 (8.20)

Derive the orthogonality relations for the pinned-free beam of Problem 8.14. Then, verify that the modes obtained in Problem 8.14 are indeed orthogonal and explain the meaning of the fact that the rigid-body mode is orthogonal to the remaining modes.

解：正交关系的推导：对方程进行积分后有，

$$EIY_j\frac{\partial^3 Y_i}{\partial x^3}\bigg|_0^L - EI\frac{\partial Y_j}{\partial x}\frac{\partial^2 Y_i}{\partial x^2}\bigg|_0^L + \int_0^L EI\frac{\partial^2 Y_j}{\partial x^2}\frac{\partial^2 Y_i}{\partial x^2}dx = \lambda_i\int_0^L Y_j m Y_i dx$$

根据边界条件可得，

$$\int_0^L EI\frac{\partial^2 Y_j}{\partial x^2}\frac{\partial^2 Y_i}{\partial x^2}dx = \lambda_i\int_0^L Y_j m Y_i dx$$

同理有，

$$\int_0^L EI\frac{\partial^2 Y_j}{\partial x^2}\frac{\partial^2 Y_i}{\partial x^2}dx = \lambda_j\int_0^L Y_j m Y_i dx$$

因此

$$\left(\lambda_j - \lambda_i\right)\int_0^L Y_j m Y_i dx = 0$$

$$\int_0^L Y_j m Y_i dx = 0 \quad \left(i \neq j\right)$$

回代得到，

$$\int_0^L EIY_j \frac{\partial^4 Y_i}{\partial x^4} dx \ \left(i \neq j\right)$$

验证各阶模态互相正交：主阵型为

$$Y_i(x) = \sin\lambda_i x + \frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i x$$

当 $i \neq j$ 时

$$\int_0^L mY_i(x)Y_j(x)dx = m\int_0^L \left(\sin\lambda_i L\frac{x}{L} + \frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i L\frac{x}{L}\right)\left(\sin\lambda_j L\frac{x}{L} + \frac{\sin\lambda_j L}{\sinh\lambda_j L}\sinh\lambda_j L\frac{x}{L}\right)dx$$

$$= m\left[\int_0^L \sin\lambda_i x\sin\lambda_j xdx + \frac{\sin\lambda_i L}{\sinh\lambda_i L}\frac{\sin\lambda_j L}{\sinh\lambda_j L}\int_0^L \sinh\lambda_i x\sinh\lambda_j xdx + \frac{\sin\lambda_i L}{\sinh\lambda_i L}\int_0^L \sinh\lambda_i x\sin\lambda_j xdx + \frac{\sin\lambda_j L}{\sinh\lambda_j L}\int_0^L \sin\lambda_i x\sinh\lambda_j xdx\right]$$

第一项为三角函数乘积的积分

$$\int_0^L \sin\lambda_i x\sin\lambda_j xdx = \left(\frac{\sin\left(\lambda_i - \lambda_j\right)x}{2\left(\lambda_i - \lambda_j\right)} + \frac{\sin\left(\lambda_i + \lambda_j\right)x}{2\left(\lambda_i + \lambda_j\right)}\right)\Bigg|_0^L = \left(\frac{\sin\left(\lambda_i - \lambda_j\right)L}{2\left(\lambda_i - \lambda_j\right)L} + \frac{\sin\left(\lambda_i + \lambda_j\right)L}{2\left(\lambda_i + \lambda_j\right)L}\right)L$$

$$= \frac{\cos\lambda_i L\cos\lambda_j L}{\left(\lambda_i^2 - \lambda_j^2\right)L^2}\left(\lambda_j L\tan\lambda_i - \lambda_i L\tan\lambda_j\right)L$$

第二项为双曲函数乘积的积分

$$\frac{\sin\lambda_i L}{\sinh\lambda_i L}\frac{\sin\lambda_j L}{\sinh\lambda_j L}\int_0^L \sinh\lambda_i x\sinh\lambda_j xdx = \frac{\sin\lambda_i L}{\sinh\lambda_i L}\frac{\sin\lambda_j L}{\sinh\lambda_j L}\frac{\lambda_i\cosh\lambda_i L\sinh\lambda_j L - \lambda_j\sinh\lambda_i L\cosh\lambda_j L}{\left(\lambda_i^2 - \lambda_j^2\right)}$$

$$= \frac{\cos\lambda_i L\cos\lambda_j L}{\left(\lambda_i^2 - \lambda_j^2\right)}\left(-\lambda_j\tan\lambda_i + \lambda_i\tan\lambda_j\right)$$

后面两项为三角函数与双曲函数的乘积的积分

$$\frac{\sin\lambda_i L}{\sinh\lambda_i L}\int_0^L \sinh\lambda_i x\sin\lambda_j xdx = \frac{\sin\lambda_i L}{\sinh\lambda_i L}\frac{\lambda_i\cosh\lambda_i L\sin\lambda_j L - \lambda_j\sinh\lambda_i L\cos\lambda_j L}{\left(\lambda_i^2 + \lambda_j^2\right)}$$

$$= \frac{\cos\lambda_i L\cos\lambda_j L\left(\lambda_i\tan\lambda_j L - \lambda_j\tan\lambda_i L\right)}{\left(\lambda_i^2 + \lambda_j^2\right)}$$

$$\frac{\sin\lambda_j L}{\sinh\lambda_j L}\int_0^L \sin\lambda_i x\sinh\lambda_j xdx = \frac{\sin\lambda_j L}{\sinh\lambda_j L}\frac{-\lambda_i\cos\lambda_i L\sinh\lambda_j L + \lambda_j\sin\lambda_i L\cosh\lambda_j L}{\left(\lambda_i^2 + \lambda_j^2\right)}$$

$$= \frac{\cos\lambda_i L\cos\lambda_j L\left(-\lambda_i\tan\lambda_j L + \lambda_j\tan\lambda_i L\right)}{\left(\lambda_i^2 + \lambda_j^2\right)}$$

前两项相加为 0，后两项也互为相反数，验证了非刚体模态的正交性。
对主阵型做归一化处理，令，

$$\int_0^L mc_r^2 Y_r^2(x)dx = 1, r = 1,2,\cdots$$

代入振型，

$$mc_r^2\int_0^L \left(\sin\lambda_r L\frac{x}{L} + \frac{\sin\lambda_r L}{\sinh\lambda_r L}\sinh\lambda_r L\frac{x}{L}\right)^2 dx = mc_r^2\left[\left(\frac{x}{2} - \frac{1}{4\lambda_r}\sin 2\lambda_r x\right) + \left(\frac{\sin\lambda_r L}{\sinh\lambda_r L}\right)^2\left(-\frac{x}{2} + \frac{1}{4\lambda_r}\sinh 2\lambda_r x\right) + \frac{\sin\lambda_r L}{2\lambda_r\sinh\lambda_r L}\left(2\sin\lambda_r x\cosh\lambda_r x - 2\cos\lambda_r x\sinh\lambda_r x\right)\right]\Bigg|_0^L$$

$$= mc_r^2 L\left[\left(\frac{1}{2} - \frac{1}{4\lambda_r L}\sin 2\lambda_r L\right) + \left(\frac{\sin\lambda_r L}{\sinh\lambda_r L}\right)^2\left(-\frac{1}{2} + \frac{1}{4\lambda_r L}\sinh 2\lambda_r L\right) + \frac{\sin\lambda_r L}{2\lambda_r L\sinh\lambda_r L}\left(2\sin\lambda_r L\cosh\lambda_r L - 2\cos\lambda_r L\sinh\lambda_r L\right)\right]$$

满足特征方程

$$\sin\lambda_r L\cosh\lambda_r L - \cos\lambda_r L\sinh\lambda_r L = 0$$

有

$$mc_r^2L\left[\left(\frac{1}{2}-\frac{1}{4\lambda_r L}\sin 2\lambda_r L\right)+\left(\frac{\sin\lambda_r L}{\sinh\lambda_r L}\right)^2\left(-\frac{1}{2}+\frac{1}{4\lambda_r L}\sinh 2\lambda_r L\right)\right]=1$$

计算可得前三阶固有频率下

$$\left(\frac{1}{2}-\frac{1}{4\lambda_r L}\sin 2\lambda_r L\right)+\left(\frac{\sin\lambda_r L}{\sinh\lambda_r L}\right)^2\left(-\frac{1}{2}+\frac{1}{4\lambda_r L}\sinh 2\lambda_r L\right)$$

的值分别为

$$0.50091, 0.50030, 0.49997$$

因此正则化后的振型可以表示为

$$Y_r(x)=c_r\left(\sin\lambda_r x+\frac{\sin\lambda_r L}{\sinh\lambda_r L}\sinh\lambda_r x\right), 0<x<L, r=1,2,\cdots$$

其中

$$c_1=\sqrt{\frac{1}{0.50091mL}}, c_2=\sqrt{\frac{1}{0.50030mL}}, c_3=\sqrt{\frac{1}{0.49997mL}}$$

刚体模态：刚体模态可以取为$Y(x)=A'x$，则有

$$\int_0^L mA'xY_j(x)dx=0$$

$$mA'A_j\int_0^L x\left(\sin\lambda_j x+\frac{\sin\lambda_j L}{\sinh\lambda_j L}\sinh\lambda_j x\right)dx=mA'A_j\frac{1}{\lambda_j^2}\left(\sin\lambda_j L-\lambda_j L\cos\lambda_j L+\frac{\sin\lambda_j L}{\sinh\lambda_j L}\left(-\sinh\lambda_j L+\lambda_j L\cosh\lambda_j L\right)\right)$$

$$=mA'A_j\frac{1}{\lambda_j^2}\left(\sin\lambda_j L-\lambda_j L\cos\lambda_j L+\cos\lambda_j L\left(-\tan\lambda_j L+\lambda_j L\right)\right)=0$$

正则化的刚体模态，

$$\int_0^L m\left(A'\right)^2x^2dx=\frac{1}{3}mL^3\left(A'\right)^2=1$$

$$Y_0(x)=\sqrt{\frac{3}{mL^3}}x$$

意义：

(a) 刚体模态可以看作是$\omega=0$，因此满足

$$Y^{(4)}=\frac{m\omega^2}{EI}Y=0$$

因此与其余模态必然正交。

(b) 利用振型叠加法$y(x,t)=\sum_{r=1}^{\infty}Y_r(x)\eta_r(t)$时，刚体模态与其余模态正交时才能将各个模态解耦独立出来。

6.3 (8.33)

Determine the response of the uniform pinned-free beam of Problem 8.14 subsequent to being released from rest in the deformed configuration $y(x,0)=y_0\left(x/L\right)^2$. *Discuss the mode participation in the response.*

解：响应可以写为如下形式

$$y(x,t)=\sum_{r=1}^{\infty}Y_r(x)\eta_r(t)$$

其中

$$Y_r(x)=c_r\left(\sin\lambda_r x+\frac{\sin\lambda_r L}{\sinh\lambda_r L}\sinh\lambda_r x\right), 0<x<L, r=1,2,\cdots$$

$$Y_0(x)=\sqrt{\frac{3}{mL^3}}x$$

由初始条件

$$y(x,0)=y_0(x/L)^2, \dot{y}(x,0)=0$$

得正则坐标下的初始条件，

$$\eta_r(0)=\int_0^L mY_r(x)y(x,0)dx=my_0c_r\int_0^L(x/L)^2\left(\sin\lambda_r x+\frac{\sin\lambda_r L}{\sinh\lambda_r L}\sinh\lambda_r x\right)dx$$

$$=\frac{my_0}{L^2}c_r\left[\int_0^L x^2\sin\lambda_r xdx+\frac{\sin\lambda_r L}{\sinh\lambda_r L}\int_0^L x^2\sinh\lambda_r xdx\right]$$

$$=\frac{my_0}{\lambda_r^3L^2}c_r\left[2\cos\lambda_r L-(\lambda_r L)^2\cos\lambda_r L+2\lambda_r L\sin\lambda_r L-2\right]+\frac{my_0}{\lambda_r^3L^2}c_r\frac{\sin\lambda_r L}{\sinh\lambda_r L}\left[2\cosh\lambda_r L+(\lambda_r L)^2\cosh\lambda_r L-2\lambda_r L\sinh\lambda_r L-2\right]$$

$$=\frac{2my_0}{\lambda_r^3L^2}c_r\left[2\cos\lambda_r L-1-\frac{\cos\lambda_r L}{\cosh\lambda_r L}\right]\quad,r\neq 0$$

$$\eta_0(0)=\int_0^L mY_0(x)y(x,0)dx=my_0\int_0^L(x/L)^2\sqrt{\frac{3}{mL^3}}xdx=\frac{mL^2y_0}{4}\sqrt{\frac{3}{mL^3}}$$

$$\dot\eta_r(0)=\int_0^L mY_0(x)\dot y_0(x)dx=0$$

正则坐标下梁的振动方程

$$\ddot\eta_i+\omega_i^2\eta_i=0$$

正则坐标下的响应，

$$\eta_r(t)=\eta_r(0)\cos\omega_r t,\ \ r=1,2,3\cdots$$

$$\eta_0(t)=\frac{mL^2y_0}{4}\sqrt{\frac{3}{mL^3}}$$

梁振动的响应可以表示为

$$y(x,t)=\sum_{r=1}^{\infty}Y_r(x)\eta_r(t)$$

$$=\frac{mL^2y_0}{4}\sqrt{\frac{3}{mL^3}}\sqrt{\frac{3}{mL^3}}x+\sum_{r=1}^{\infty}\left\{c_r\left(\sin\lambda_r x+\frac{\sin\lambda_r L}{\sinh\lambda_r L}\sinh\lambda_r x\right)\left[\eta_r(0)\cos(\omega_r t)\right]\right\}$$

$$=\frac{3xy_0}{4}+2y_0\sum_{r=1}^{\infty}\left\{c_r^2\frac{m}{\lambda_r^3L^2}\left(2\cos\lambda_r L-1-\frac{\cos\lambda_r L}{\cosh\lambda_r L}\right)\left(\sin\lambda_r x+\frac{\sin\lambda_r L}{\sinh\lambda_r L}\sinh\lambda_r x\right)\cos(\omega_r t)\right\}$$

以前三阶为例

$$y(x,t)=\frac{3xy_0}{4}-0.1574(\sin\lambda_1 x-0.0279\sinh\lambda_1 x)\cos(\omega_1 t)$$
$$+0.0047(\sin\lambda_2 x+0.0012\sinh\lambda_2 x)\cos(\omega_2 t)$$
$$-0.0091(\sin\lambda_3 x-0.0001\sinh\lambda_3 x)\cos(\omega_3 t)$$

主要由第一阶模态以及刚体模态决定了梁的初值响应。

6.4 (8.37)

Determine the response of the beam of Problem 8.14 to the concentrated harmonic force $F(t)=F_0\cos\Omega t$ *applied at* $x=L$*. Discuss the mode participation in the response. Note that the concentrated force can be represented as the distributed force* $f(x,t)=F_0\delta(x-L)\cos\Omega t$ *, where* $\delta(x-L)$ *is a spatial Dirac delta function (see Eq. (8.263)).*

解：已知主阵型：

$$Y_r(x)=c_r\left(\sin\lambda_r x+\frac{\sin\lambda_r L}{\sinh\lambda_r L}\sinh\lambda_r x\right),0<x<L,r=1,2,\cdots,$$

$$Y_0(x)=\sqrt{\frac{3}{mL^3}}x$$

正则坐标下梁振动方程，

$$\ddot\eta_r+\omega_r^2\eta_r=\int_0^L F_0\cos\Omega t\delta(x-L)Y_r(x)dx=F_0\cos\Omega tY_r(L)$$

对于刚体振型有，

$$\ddot{\eta}_0 = F_0 \cos\Omega t\sqrt{\frac{3}{mL}}$$

正则坐标下的稳态响应为，

$$\eta_r(t) = \frac{1}{\omega_r^2 - \Omega^2} F_0 c_r \left(2\sin\lambda_r L\right)\cos\Omega t,\ r = 1,2,3\cdots$$

$$\eta_0(t) = \frac{F_0}{-\Omega^2}\sqrt{\frac{3}{mL}}\cos\Omega t$$

梁的稳态响应为，

$$y(x,t) = \sum_{r=1}^{\infty} Y_r(x)\eta_r(t) = \sqrt{\frac{3}{mL^3}}x\frac{F_0}{-\Omega^2}\sqrt{\frac{3}{mL}}\cos\Omega t + \sum_{r=1}^{\infty} c_r{}^2 F_0\left(2\sin\lambda_r L\right)\frac{\left(\sin\lambda_r x + \dfrac{\sin\lambda_r L}{\sinh\lambda_r L}\sinh\lambda_r x\right)\cos\Omega t}{\omega_r^2 - \Omega^2}$$

以前三阶为例有，

$$\begin{aligned} y(x,t) \approx{}& \frac{3F_0 x}{-mL^2\Omega^2}\cos\Omega t - \frac{2.8222F_0}{ml\left(\omega_1^2 - \Omega^2\right)}\left(\sin\lambda_1 x - 0.0279\sinh\lambda_1 x\right)\cos\Omega t \\ &+ \frac{2.8233F_0}{ml\left(\omega_2^2 - \Omega^2\right)}\left(\sin\lambda_2 x + 0.0012\sinh\lambda_2 x\right)\cos\Omega t \\ &- \frac{2.8234F_0}{ml\left(\omega_3^2 - \Omega^2\right)}\left(\sin\lambda_3 x - 0.0001\sinh\lambda_3 x\right)\cos\Omega t \end{aligned}$$

可发现，当外激励Ω频率接近于零时受刚体模态影响，当Ω接近第i阶固有频率时，第i阶模态对响应的贡献就越大。

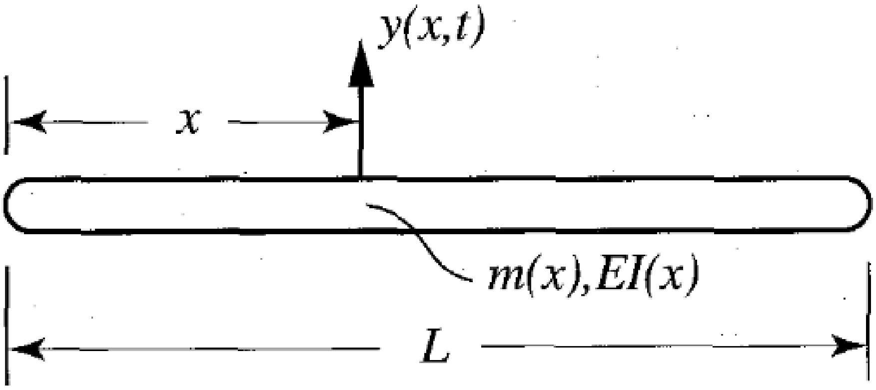
Problem 7 (20 points in total)

Problem 7.1: Please solve Problem 8.15 in page 461 of the textbook. Note: you can direct use the EOM of the beam, however, you need to choose the correct boundary conditions (15 points).

Problem 7.2: Please solve problem 8.21 in page 461 of the textbook (5 points).

7.1 (8.15)

Derive the eigenvalue problem for the bending vibration of a beam free at both ends (Fig. 8.37). Then, let $m(x)=m=\text{costant}, EI(x)=EI=\text{costant}$ *, solve the eigenvalue problem and plot* ***the four lowest modes****. Hint: The system is only semidefinite and it admits two rigid-body motions, one representing transverse translation and the other rotation.*



解：梁的振动方程为

$$m(x)\frac{\partial^2 y}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y}{\partial x^2}\right)=0, 0<x<L$$

边界条件应当满足

$$EI(x)\frac{\partial\theta}{\partial x}\delta\theta-\frac{\partial}{\partial x}\left(EI(x)\frac{\partial\theta}{\partial x}\right)\delta y=0$$

考虑 $m(x)=m=\text{costant}, EI(x)=EI=\text{costant}$ ，并设梁的振动通解为

$$y(x,t)=Y(x)b\sin(\omega t+\varphi)$$

将其代入动力学方程中，

$$Y^{(4)}=\frac{m\omega^2}{EI}Y$$

用简谐函数和双曲函数表示通解

$$Y(x)=A\cos\lambda x+B\sin\lambda x+C\cosh\lambda x+D\sinh\lambda x,\quad \lambda^4=\frac{m\omega^2}{EI}$$

左端自由，右端自由的边界条件可以表示为

$$x=0：EIY^{(2)}=0, EIY^{(3)}=0$$
$$x=L：EIY^{(2)}=0, EIY^{(3)}=0$$

在 $x=0$ 处有

$$\lambda^2(-A+C)=0$$
$$\lambda^3(-B+D)=0$$

在 $x=L$ 处有

$$-A\cos\lambda L-B\sin\lambda L+C\cosh\lambda L+D\sinh\lambda L=0$$
$$A\sin\lambda L-B\cos\lambda L+C\sinh\lambda L+D\cosh\lambda L=0$$

当$\lambda \neq 0$时，可以得到

$$A = C, B = D$$

将其代入边界$x = L$

$$A(\cosh \lambda L - \cos \lambda L) + B(\sinh \lambda L - \sin \lambda L) = 0$$

$$B(\cosh \lambda L - \cos \lambda L) + A(\sinh \lambda L + \sin \lambda L) = 0$$

由行列式系数为 0，得到特征方程为

$$\cos \lambda L * \cosh \lambda L - 1 = 0$$

四个零点分别为

$$\lambda_1 L = 4.73, \lambda_2 L = 7.85, \lambda_3 L = 11.00, \lambda_4 L = 14.14$$

由于

$$\omega = (\lambda L)^2 \sqrt{\frac{EI}{mL^4}}$$

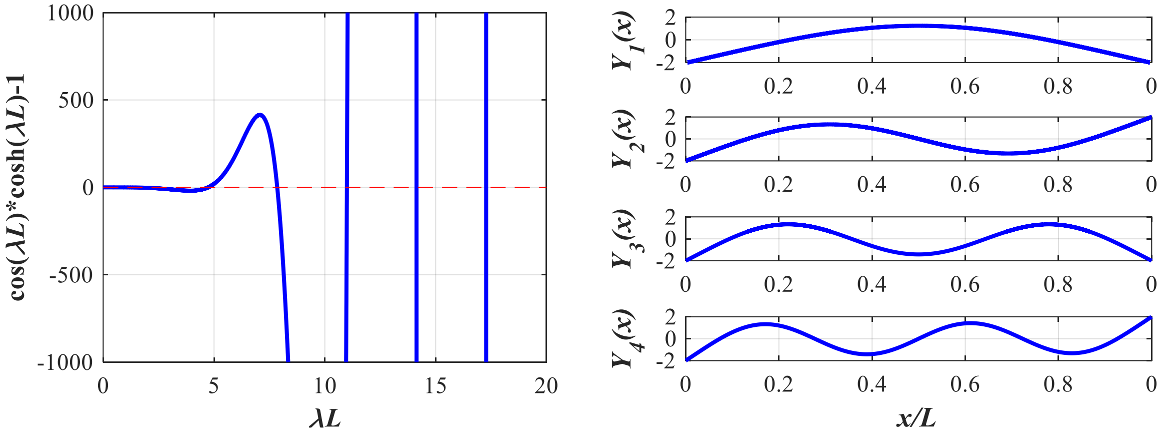
因此梁的前三阶固有频率分别为：

$$\omega_1 = 22.37\sqrt{\frac{EI}{mL^4}} \quad \omega_2 = 61.62\sqrt{\frac{EI}{mL^4}} \quad \omega_3 = 121.00\sqrt{\frac{EI}{mL^4}} \quad \omega_4 = 199.94\sqrt{\frac{EI}{mL^4}}$$

各阶主阵型为

$$Y_i(x) = \frac{-\cosh \lambda_i L + \cos \lambda_i L}{\sinh \lambda_i L + \sin \lambda_i L}(\cos \lambda_i x + \cosh \lambda_i x) + (\sin \lambda_i x + \sinh \lambda_i x), i = 1, 2, 3 \cdots$$

前四阶主阵型如下图：



当$\lambda = 0$时存在刚体模态，同样可以得到

$$-A + C = 0$$

$$-B + D = 0$$

代回通解之后可以得到一个刚体模态为

$$Y(x) = 2A_0$$

根据边界条件可以取另一个刚体模态为

$$Y(x) = A' + B'x$$

7.2 (8.21)

Derive the orthogonality relations for the free-free beam of Problem 8.15. Make sure that the modes obtained in Problem 8.15 are indeed orthogonal and explain the meaning of the fact that each of the two rigid-body modes is orthogonal to the remaining modes, including the other rigid-body mode.

解：梁的正交关系的推导

$$EIY_j\frac{\partial^3 Y_i}{\partial x^3}\bigg|_0^L - EI\frac{\partial Y_j}{\partial x}\frac{\partial^2 Y_i}{\partial x^2}\bigg|_0^L + \int_0^L EI\frac{\partial^2 Y_j}{\partial x^2}\frac{\partial^2 Y_i}{\partial x^2}dx = \lambda_i\int_0^L Y_j m Y_i dx$$

根据边界条件可得，

$$\int_0^L EI\frac{\partial^2 Y_j}{\partial x^2}\frac{\partial^2 Y_i}{\partial x^2}dx = \lambda_i\int_0^L Y_j m Y_i dx$$

同理有，

$$\int_0^L EI\frac{\partial^2 Y_j}{\partial x^2}\frac{\partial^2 Y_i}{\partial x^2}dx = \lambda_j\int_0^L Y_j m Y_i dx$$

因此

$$\left(\lambda_j - \lambda_i\right)\int_0^L Y_j m Y_i dx = 0$$

$$\int_0^L Y_j m Y_i dx = 0 \quad \left(i \neq j\right)$$

回代得到，

$$\int_0^L EIY_j\frac{\partial^4 Y_i}{\partial x^4}dx \quad \left(i \neq j\right)$$

正交性验证：

1.非刚体模态之间（过程略）：

$$Y_i(x) = \frac{-\cosh\lambda_i L + \cos\lambda_i L}{\sinh\lambda_i L + \sin\lambda_i L}\left(\cos\lambda_i x + \cosh\lambda_i x\right) + \left(\sin\lambda_i x + \sinh\lambda_i x\right)$$

满足

$$\int_0^L mY_iY_j dx = 0 \quad i \neq j$$

2.刚体模态与非刚体模态之间，

$$\int_0^L m\left(A' + B'x\right)Y_i\left(x\right)dx = \frac{-\cosh\lambda_i L + \cos\lambda_i L}{\sinh\lambda_i L + \sin\lambda_i L}m\int_0^L\left(A' + B'x\right)\left[\left(\cos\lambda_i x + \cosh\lambda_i x\right) + \left(\sin\lambda_i x + \sinh\lambda_i x\right)\right]dx$$

根据特征方程 $\cos\lambda L * \cosh\lambda L - 1 = 0$，上式为零。

且，

$$\int_0^L mA_0 Y_i\left(x\right)dx = \frac{-\cosh\lambda_i L + \cos\lambda_i L}{\sinh\lambda_i L + \sin\lambda_i L}\frac{mA_0}{\lambda_i}\left[\left(\sin\lambda_i L + \sinh\lambda_i L\right) + \left(-\cos\lambda_i L + \cosh\lambda_i L\right)\right] = 0$$

3.两个刚体模态之间

两个刚体模态对应的固有频率均为零，为了验证其正交性，需自行设置合适的主振型函数。

设两个刚体模态为：

$$Y_0 = A_0, \hat{Y}_0 = A' + B'x$$

满足，

$$\int_0^L mA_0\left(A' + B'x\right)dx = mA_0\left(A'L + \frac{B'}{2}L^2\right) = 0,$$

因此可以取

$$Y_0 = \frac{1}{\sqrt{mL}}, \hat{Y}_0 = \sqrt{\frac{3}{mL^3}}\left(L - 2x\right)$$

两个刚体模态分别表示梁的平移和梁的转动，相互独立。