题 1

1 分别用直接摄动法和 Lindstedt-Poincaré 摄动法求下述系统自由振动的二次近似解，并对结果进行一致有效展开检验

$$\begin{cases} \ddot{u}+u+\varepsilon u^2=0 \\ u(0)=a, \quad \dot{u}(0)=0 \end{cases}$$

直接摄动法

二次近似

$$u(\varepsilon,t)=u_0(t)+\varepsilon u_1(t)+\varepsilon^2 u_2(t) \tag{1}$$

代入方程

$$(\ddot{u}_0+\varepsilon\ddot{u}_1+\varepsilon^2\ddot{u}_2)+(u_0+\varepsilon u_1+\varepsilon^2 u_2)+\varepsilon(u_0+\varepsilon u_1+\varepsilon^2 u_2)^2=0 \tag{2}$$

可得

$$\begin{cases} \ddot{u}_0+u_0=0 \\ \ddot{u}_1+u_1=-u_0^2 \\ \ddot{u}_2+u_2=-2u_0u_1 \end{cases} \tag{3}$$

迭代

解

$$\begin{cases} \ddot{u}_0+u_0=0 \\ u_0(0) \quad =a \\ \dot{u}_0(0) \quad =0 \end{cases} \tag{4}$$

得

$$u_0=a\cos t \tag{5}$$

代入

$$\begin{cases} \ddot{u}_1+u_1=-u_0^2 \\ u_1(0) \quad =0 \\ \dot{u}_1(0) \quad =0 \end{cases} \tag{6}$$

解得

$$u_1(t)=\frac{a^2}{6}(\cos 2t+2\cos t-3) \tag{7}$$

代入到

$$\begin{cases} \ddot{u}_2+u_2=-2u_0u_1 \\ u_2(0) \quad =0 \\ \dot{u}_2(0) \quad =0 \end{cases} \tag{8}$$

解得

$$u_2(t) = \frac{a^3}{144}(3\cos 3t + 16\cos 2t + 29\cos t + 60t\sin t - 48) \tag{9}$$

二次近似

$$\begin{aligned} u(t) &= u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) \\ &= a\cos t + \frac{\varepsilon a^2}{6}(\cos 2t + 2\cos t - 3) \\ &+ \frac{\varepsilon^2 a^3}{144}(3\cos 3t + 16\cos 2t + 29\cos t + 60t\sin t - 48) \end{aligned} \tag{10}$$

LP 摄动法

做二次近似

$$\begin{aligned} u(\varepsilon, t) &= u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) \\ \omega(\varepsilon)^2 &= \omega_0^2 + \varepsilon b_1 + \varepsilon^2 b_2 \end{aligned} \tag{11}$$

则有

$$\omega_0^2 = \omega^2 - \varepsilon b_1 - \varepsilon^2 b_2 \tag{12}$$

代入方程合并同类项有:

$$\begin{cases} \ddot{u}_0 + \omega^2 u_0 = 0 \\ \ddot{u}_1 + \omega^2 u_1 = b_1 u_0 - u_0^2 \\ \ddot{u}_2 + \omega^2 u_2 = b_2 u_0 + b_1 u_1 - 2u_0 u_1 \end{cases} \tag{13}$$

解

$$\begin{cases} \ddot{u}_0 + \omega u_0 = 0 \\ u_0(0) \quad = a \\ \dot{u}_0(0) \quad = 0 \end{cases} \tag{14}$$

得

$$u_0 = a\cos\omega t \tag{15}$$

代入有

$$\begin{cases} \ddot{u}_1 + \omega u_1 = b_1 u_0 - u_0^2 \\ u_1(0) \quad = 0 \\ \dot{u}_1(0) \quad = 0 \end{cases} \tag{16}$$

其中$\cos\omega t$的系数为b_1, 为避免永年项, 取$b_1 = 0$

解得

$$u_1(t) = \frac{1}{6\omega^2}(2a^2\cos\omega t + a^2\cos 2\omega t - 3a^2) \tag{17}$$

代入

$$
\begin{cases}
\ddot{u}_2 + \omega^2 u_2 = b_2 u_0 + b_1 u_1 - 2u_0 u_1 \\
u_2(0) \qquad = 0 \\
\dot{u}_2(0) \qquad = 0
\end{cases}
\tag{18}
$$

令$\cos\omega t$的系数为 0

$$
\frac{5a^3 + 6a\omega^2 b_2}{6\omega^2} = 0 \tag{19}
$$

解得

$$
b_2 = -\frac{5}{6}\frac{a^2}{\omega^2} \tag{20}
$$

解得方程为

$$
u_2(t) = \frac{1}{144\omega^2}(3a^3\cos 3\omega t + 16a^3\cos 2\omega t + 29a^3\cos\omega t - 48a^3) \tag{21}
$$

可得

$$
\begin{aligned}
\omega &= \sqrt{\omega_0^2 + \varepsilon b_1 + \varepsilon^2 b_2} \\
&= \sqrt{1 - \varepsilon^2 \frac{5a^2}{6\omega^2}}
\end{aligned}
\tag{22}
$$

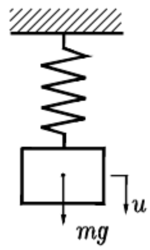
题 2

2 考察图示重力场中的单自由度无阻尼系统，其非线性弹簧恢复力q与变形δ间关系为

$$
q(\delta) = k\delta + \varepsilon k\delta^3, \qquad k>0, \qquad 0<\varepsilon<<1
$$

(1) 以系统静平衡位置为位移原点，建立系统的振动微分方程；

(2) 用 Lindstedt-Poincaré 摄动法分析重力对系统自由振动频率的影响。



题 2 图

1 建立方程

由牛顿定律

$$
\begin{cases}
mg - q(u_g) \qquad = 0 \\
mg - q(u_g + u) = m\ddot{u}
\end{cases}
\tag{23}
$$

消掉mg得

$$
m\ddot{u} + ku + \varepsilon k(u^3 + 3u_g u^2 + 3u u_g^2) = 0 \tag{24}
$$

标准化

$$
\ddot{u} + \omega_0^2 u + \varepsilon\omega_0^2(u^3 + 3u_g u^2 + 3u u_g^2) = 0 \quad \omega_0 = \sqrt{\frac{k}{m}} \tag{25}
$$

LP 摄动法

做二次近似

$$\begin{aligned}u(\varepsilon,t)&=u_0(t)+\varepsilon u_1(t)+\varepsilon^2u_2(t)\\ \omega(\varepsilon)^2&=\omega_0^2+\varepsilon b_1+\varepsilon^2 b_2\end{aligned} \tag{26}$$

则有

$$\omega_0^2=\omega^2-\varepsilon b_1-\varepsilon^2 b_2 \tag{27}$$

代入方程合并同类项有:

$$\begin{cases}\ddot{u}_0+\omega^2u_0=0\\ \ddot{u}_1+\omega^2u_1=-3u_g^2\omega^2u_0+b_1u_0-3u_g\omega^2u_0^2-\omega^2u_0^3\\ \ddot{u}_2+\omega^2u_2=3u_g^2b_1u_0+b_2u_0+3u_gb_1u_0^2+b_1u_0^3-3u_g^2\omega^2u_1+b_1u_1-6u_g\omega^2u_0u_1-3\omega^2u_0^2u_1\end{cases} \tag{28}$$

解

$$\begin{cases}\ddot{u}_0+\omega u_0&=0\\ u_0(0)&=a\\ \dot{u}_0(0)&=0\end{cases} \tag{29}$$

得

$$u_0=a\cos\omega t \tag{30}$$

代入有

$$\begin{cases}\ddot{u}_1+\omega^2u_1&=-3u_g^2\omega^2u_0+b_1u_0-3u_g\omega^2u_0^2-\omega^2u_0^3\\ u_1(0)&=0\\ \dot{u}_1(0)&=0\end{cases} \tag{31}$$

令$\cos\omega t$的系数为 0

$$\frac{1}{4}\left(3a^3\omega^2+12au_g^2\omega^2-4ab_1\right)=0 \tag{32}$$

得

$$b_1=\frac{3a^3+12au_g^2}{4a}\omega^2 \tag{33}$$

解得

$$u_1(t)=\frac{1}{32}\left(-48a^2u_g-a^3\cos\omega t+32a^2u_g\cos\omega t+16a^2u_g\cos 2\omega t+a^3\cos 3\omega t\right) \tag{34}$$

代入

$$\begin{cases}\ddot{u}_2+\omega^2u_2&=3u_g^2b_1u_0+b_2u_0+3u_gb_1u_0^2+b_1u_0^3-3u_g^2\omega^2u_1+b_1u_1-6u_g\omega^2u_0u_1-3\omega^2u_0^2u_1\\ u_2(0)&=0\\ \dot{u}_2(0)&=0\end{cases} \tag{35}$$

令$\cos\omega t$的系数为 0

$$
\begin{aligned}
&-\frac{3}{64}a^5\omega^2+\frac{9}{4}a^4u_g\omega^2-\frac{243}{32}a^3u_g^2\omega^2+3a^2u_g^3\omega^2+\frac{23}{128}a^2\left(-3a^3\omega^2-12au_g^2\omega^2\right)\\
&+\frac{1}{4}au_g\left(-3a^3\omega^2-12au_g^2\omega^2\right)+\frac{3}{4}u_g^2\left(-3a^3\omega^2-12au_g^2\omega^2\right)-ab_2=0
\end{aligned}
\tag{36}
$$

解得

$$
b_2=-\frac{75a^5\omega^2-192a^4u_g\omega^2+1536a^3u_g^2\omega^2+1152au_g^4\omega^2}{128a}
\tag{37}
$$

解得方程为

$$
\begin{aligned}
u_2(t)=\frac{1}{1024}&\left(2016a^4u_g-3072a^3u_g^2+4608a^2u_g^3+23a^5\cos\omega t-1120a^4u_g\cos\omega t\right.\\
&+1952a^3u_g^2\cos\omega t-3072a^2u_g^3\cos\omega t-1024a^4u_g\cos 2\omega t\\
&+1024a^3u_g^2\cos 2\omega t-1536a^2u_g^3\cos 2\omega t-24a^5\cos 3\omega t+96a^4u_g\cos 3\omega t\\
&\left.+96a^3u_g^2\cos 3\omega t+32a^4u_g\cos 4\omega t+a^5\cos 5\omega t\right)
\end{aligned}
\tag{38}
$$

可得

$$
\begin{aligned}
\omega&=\sqrt{\omega_0^2+\varepsilon b_1+\varepsilon^2 b_2}\\
&=\sqrt{\omega_0^2+\varepsilon\frac{3a^3+12au_g^2}{4a}\omega^2+\varepsilon^2\frac{75a^5\omega^2-192a^4u_g\omega^2+1536a^3u_g^2\omega^2+1152au_g^4\omega^2}{128a}}
\end{aligned}
\tag{39}
$$

可知, 重力大, u_g 大, ω 也会增大

题 3

3 用平均法求下述保守系统周期振动的一阶近似解

(1) $\ddot{u}+\sin u=0$

(2) $\ddot{u}+u+\varepsilon(u^2+u^3)=0,\quad 0<\varepsilon<<1$

1

近似

$$
\sin u\approx u-\frac{1}{3!}u^3
\tag{40}
$$

代入则有

$$
\ddot{u}+u=\frac{1}{6}u^3
\tag{41}
$$

令

$$
\begin{cases}
u=a(t)\cos(\omega_0 t+\varphi(t))\\
\dot{u}=-\omega_0 a(t)\sin(\omega_0 t+\varphi(t))
\end{cases}
\tag{42}
$$

求导得

$$\begin{cases} \dot{u} = \dot{a}\cos(\omega_0 t + \varphi) - a\sin(\omega_0 t + \varphi)(\omega_0 + \dot{\varphi}) \\ \ddot{u} = -\omega_0[\dot{a}\sin(\omega_0 t + \varphi) + a\cos(\omega_0 t + \varphi)(\omega_0 + \dot{\varphi})] \end{cases} \tag{43}$$

则有

$$\begin{cases} \dot{a}\cos(\omega_0 t + \varphi) - a\sin(\omega_0 t + \varphi)\dot{\varphi} = 0 \\ \dot{a}\sin(\omega_0 t + \varphi) + a\cos(\omega_0 t + \varphi)\dot{\varphi} = -\frac{1}{\omega_0}\frac{1}{6}a^3\cos^3(\omega_0 t + \varphi) \end{cases} \tag{44}$$

解得

$$\begin{cases} \dot{a} \quad\;\; = -\frac{1}{\omega_0}\frac{1}{6}a^3\cos^3(\omega_0 t + \varphi)\sin(\omega_0 t + \varphi) \\ \varphi(a) = -\frac{1}{\omega_0}\frac{1}{6}a^3\cos^3(\omega_0 t + \varphi)\cos(\omega_0 t + \varphi) \end{cases} \tag{45}$$

用平均值近似, 且由题意$\omega_0 = 1$

$$\begin{cases} \dot{a} \quad\;\; = -\frac{1}{2\pi\omega_0}\int_0^{2\pi}\frac{1}{6}a^3\cos^3(\omega_0 t + \varphi)\sin(\omega_0 t + \varphi)\,\mathrm{d}t = 0 \\ \varphi(a) = -\frac{1}{2\pi\omega_0 a}\int_0^{2\pi}\frac{1}{6}a^3\cos^3(\omega_0 t + \varphi)\cos(\omega_0 t + \varphi)\,\mathrm{d}t = \frac{1}{16}a^2 \end{cases} \tag{46}$$

所以

$$\begin{cases} a \;= a_0 \\ \varphi = \frac{1}{16}a^2 t \end{cases} \tag{47}$$

所以

$$u = a_0\cos\left(t + \frac{1}{16}a^2 t\right) \tag{48}$$

2

令

$$\begin{cases} u = a(t)\cos(\omega_0 t + \varphi(t)) \\ \dot{u} = -\omega_0 a(t)\sin(\omega_0 t + \varphi(t)) \end{cases} \tag{49}$$

则有

$$\begin{cases} \dot{a} \;= -\frac{\varepsilon}{2\pi\omega}\int_0^{\pi} p(u,\dot{u})\sin\psi\,\mathrm{d}\psi \\ \dot{\varphi} = -\frac{\varepsilon}{2\pi\omega a}\int_0^{\pi} p(u,\dot{u})\cos\psi\,\mathrm{d}\psi \end{cases} \tag{50}$$

代入

$$\begin{cases} \dot{a} \;= \frac{\varepsilon}{2\pi}\int_0^{\pi}(a^2\cos^3\psi + a^3\cos^3\psi)\sin\psi\,\mathrm{d}\psi = 0 \\ \dot{\varphi} = \frac{\varepsilon}{2\pi a}\int_0^{\pi}(a^2\cos^3\psi + a^3\cos^3\psi)\cos\psi\,\mathrm{d}\psi = \frac{3}{8}\varepsilon a^2 \end{cases} \tag{51}$$

所以

$$\begin{cases} a = a_0 \\ \varphi = \frac{3}{8}\varepsilon a^2 t \end{cases} \tag{52}$$

所以

$$u = a_0 \cos\left(t + \frac{3}{8}\varepsilon a^2 t\right) \tag{53}$$

题 4

4　试论证可否用平均法求解下述立方非线性系统的周期振动

$$\ddot{u} + u^3 = 0$$

变形为

$$\ddot{u} + \omega_0^2 u = \varepsilon(\omega_0^2 u - u^3) \quad \varepsilon = 1 \tag{54}$$

则有

$$\begin{cases} \dot{a} = -\frac{\varepsilon}{2\pi\omega}\int_0^\pi p(u,\dot{u})\sin\psi\,\mathrm{d}\psi \\ \dot{\varphi} = -\frac{\varepsilon}{2\pi\omega a}\int_0^\pi p(u,\dot{u})\cos\psi\,\mathrm{d}\psi \end{cases} \tag{55}$$

代入则有

$$\begin{cases} \dot{a} = -\frac{1}{2\pi\omega}\int_0^\pi (\omega_0^2 a\cos\psi - a_0^3\cos\psi^3)\sin\psi\,\mathrm{d}\psi = 0 \\ \dot{\varphi} = -\frac{1}{2\pi\omega a}\int_0^\pi (\omega_0^2 a\cos\psi - a^3\cos\psi^3)\cos\psi\,\mathrm{d}\psi = -\frac{1}{2}\omega_0 + \frac{3}{8}\frac{a^2}{\omega_0} \end{cases} \tag{56}$$

得

$$\begin{cases} a = a_0 \\ \varphi = -\frac{1}{2}\omega_0 t + \frac{3}{8}\frac{a_0^2}{\omega_0} t \end{cases} \tag{57}$$

$$u = a_0 \cos\left(t + \frac{3}{8}\frac{a_0^2}{\omega_0}t - \frac{1}{2}\omega_0 t\right) \tag{58}$$

由于 $\varepsilon = 1$, 解与自由振动频率相差较大, 所以不用平均法

题 5

5　考察下述 Coulomb 摩擦阻尼系统

$$\ddot{u}(t) + \mu N \operatorname{sgn}\dot{u}(t) + \omega_0^2 u(t) = 0$$

其中摩擦系数 μ 为小参数，用 KBM 法求解该系统自由振动的一次近似。

变形为

$$\ddot{u} + \omega_0^2 u = -\varepsilon N \operatorname{sgn}\dot{u} \quad \varepsilon = \mu \tag{59}$$

则有

$$\begin{cases} \dot{a} = -\frac{\varepsilon}{2\pi\omega} \int_0^{\pi} p(u, \dot{u}) \sin\psi \, \mathrm{d}\psi \\ \dot{\varphi} = -\frac{\varepsilon}{2\pi\omega a} \int_0^{\pi} p(u, \dot{u}) \cos\psi \, \mathrm{d}\psi \end{cases} \tag{60}$$

代入则有

$$\begin{cases} \dot{a} = \frac{\varepsilon}{2\pi\omega} \int_0^{\pi} (N \operatorname{sgn}(-\omega_0 a \sin\psi)) \sin\psi \, \mathrm{d}\psi = -\frac{2\varepsilon N}{\pi\omega_0} \operatorname{sgn} a \\ \dot{\varphi} = \frac{\varepsilon}{2\pi\omega a} \int_0^{\pi} (N \operatorname{sgn}(-\omega_0 a \sin\psi)) \cos\psi \, \mathrm{d}\psi = 0 \end{cases} \tag{61}$$

得

$$\begin{cases} a = a_0 - \frac{2\varepsilon N}{\pi\omega_0} \operatorname{sgn}(a) t \\ \varphi = 0 \end{cases} \tag{62}$$

所以

$$u = \left(a_0 - \frac{2\mu N}{\pi\omega_0} \operatorname{sgn}(a) t \right) \cos\omega_0 t \tag{63}$$

题 6

设 $\ddot{x} + \omega_0^2 x = \varepsilon f(x, \dot{x})$

用矩阵变换推导平均方程

解

矩阵形式

$$\begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} + \begin{pmatrix} \omega_0^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 \\ \varepsilon f(x, \dot{x}) \end{pmatrix} \tag{64}$$

令

$$\begin{cases} x = a(t) \cos\psi \\ \dot{x} = -\omega_0 a(t) \sin\psi \\ \psi = \omega_0 t + \varphi(t) \end{cases} \tag{65}$$

可得

$$\begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} \dot{a}\cos\psi - a\cos\psi(\omega_0 + \dot{\varphi}) \\ -\omega_0(\dot{a}\sin\psi + a\cos\psi(\omega_0 + \dot{\varphi})) \end{pmatrix} \tag{66}$$

代入则有

$$\begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{pmatrix}\begin{pmatrix}\dot a\\ a\dot\varphi\end{pmatrix}=\begin{pmatrix}0\\ -\frac{\varepsilon}{\omega_0}f(x,\dot x)\end{pmatrix} \tag{67}$$

知$\begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{pmatrix}$为旋转矩阵, 即正交矩阵, 有

$$A^{-1}=A^T \tag{68}$$

所以

$$\begin{aligned}\begin{pmatrix}\dot a\\ a\dot\varphi\end{pmatrix}&=\begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{pmatrix}^{-1}\begin{pmatrix}0\\ -\frac{\varepsilon}{\omega_0}f(x,\dot x)\end{pmatrix}\\ &=\begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{pmatrix}^{T}\begin{pmatrix}0\\ -\frac{\varepsilon}{\omega_0}f(x,\dot x)\end{pmatrix}\\ &=\begin{pmatrix}\cos\psi & \sin\psi\\ -\sin\psi & \cos\psi\end{pmatrix}\begin{pmatrix}0\\ -\frac{\varepsilon}{\omega_0}f(x,\dot x)\end{pmatrix}\\ &=\begin{pmatrix}-\frac{\varepsilon}{\omega_0}f(x,\dot x)\sin\psi\\ -\frac{\varepsilon}{\omega_0}f(x,\dot x)\cos\psi\end{pmatrix}\end{aligned} \tag{69}$$

所以

$$\begin{pmatrix}\dot a\\ \dot\varphi\end{pmatrix}=\begin{pmatrix}-\frac{\varepsilon}{\omega_0}f(x,\dot x)\sin\psi\\ -\frac{\varepsilon}{\omega_0 a}f(x,\dot x)\cos\psi\end{pmatrix} \tag{70}$$