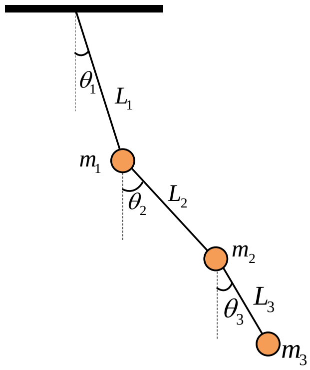
2021-Homework 3: Reference answer

Problem 1: Assuming small motions, (1) please derive the linear EOM of the following triple pendulum system based on the given generalized coordinates. (20 Points) (2) Let $L_1 = L_2 = L_3 = L$, $m_1 = m_2 = m, m_3 = 2m$, the acceleration of gravity g, calculate the natural frequencies and the mode shapes. (20 points)



解：（1）系统有三个自由度，选取如图所示的 $\theta_1,\theta_2,\theta_3$ 作为广义坐标，三个质点的位移可以表示成分量

$$\vec{r}_1 = L_1\sin\theta_1\vec{i} - L_1\cos\theta_1\vec{j}$$
$$\vec{r}_2 = (L_1\sin\theta_1 + L_2\sin\theta_2)\vec{i} - (L_1\cos\theta_1 + L_2\cos\theta_2)\vec{j}$$
$$\vec{r}_3 = (L_1\sin\theta_1 + L_2\sin\theta_2 + L_3\sin\theta_3)\vec{i} - (L_1\cos\theta_1 + L_2\cos\theta_2 + L_3\cos\theta_3)\vec{j}$$

则三个质点处的速度可以表示成

$$\dot{\vec{r}}_1 = L_1\cos\theta_1\dot{\theta}_1\vec{i} + L_1\sin\theta_1\dot{\theta}_1\vec{j}$$
$$\dot{\vec{r}}_2 = \left(L_1\cos\theta_1\dot{\theta}_1 + L_2\cos\theta_2\dot{\theta}_2\right)\vec{i} + \left(L_1\sin\theta_1\dot{\theta}_1 + L_2\sin\theta_2\dot{\theta}_2\right)\vec{j}$$
$$\dot{\vec{r}}_3 = \left(L_1\cos\theta_1\dot{\theta}_1 + L_2\cos\theta_2\dot{\theta}_2 + L_3\cos\theta_3\dot{\theta}_3\right)\vec{i} + \left(L_1\sin\theta_1\dot{\theta}_1 + L_2\sin\theta_2\dot{\theta}_2 + L_3\sin\theta_3\dot{\theta}_3\right)\vec{j}$$

因此系统的动能可以表示成

$$T = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 + \frac{1}{2}m_3\dot{\vec{r}}_3^2$$

所以

$$\begin{aligned}T =& \frac{1}{2}m_1L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right)\\ &+\frac{1}{2}m_3\left(L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + L_3^2\dot{\theta}_3^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + 2L_1L_3\dot{\theta}_1\dot{\theta}_3\cos(\theta_1-\theta_3) + 2L_2L_3\dot{\theta}_2\dot{\theta}_3\cos(\theta_2-\theta_3)\right)\end{aligned}$$

取上端固定点处为零势能点，则系统的势能为

$$V = -m_1gL_1\cos\theta_1 - m_2g(L_1\cos\theta_1 + L_2\cos\theta_2) - m_3g(L_1\cos\theta_1 + L_2\cos\theta_2 + L_3\cos\theta_3)$$

Lagrangian 量为，

$$L = T - V$$

系统不存在非保守力，因此可以通过 Euler-Lagrange 方程得到系统运动方程

$$\left[\begin{aligned}&(m_1+m_2+m_3)L_1^2\ddot{\theta}_1 + (m_2+m_3)L_1L_2\cos(\theta_1-\theta_2)\ddot{\theta}_2 + m_3L_1L_3\cos(\theta_1-\theta_3)\ddot{\theta}_3\\ &-(m_2+m_3)L_1L_2\sin(\theta_1-\theta_2)\left(\dot{\theta}_1\dot{\theta}_2-\dot{\theta}_2^2\right) - m_3L_1L_3\sin(\theta_1-\theta_3)\left(\dot{\theta}_1\dot{\theta}_3-\dot{\theta}_3^2\right) + (m_1+m_2+m_3)gL_1\sin\theta_1\end{aligned}\right] = 0$$

$$\left[\begin{aligned}&(m_2+m_3)L_1L_2\cos(\theta_1-\theta_2)\ddot{\theta}_1 + (m_2+m_3)L_2^2\ddot{\theta}_2 + m_3L_2L_3\cos(\theta_2-\theta_3)\ddot{\theta}_3\\ &-(m_2+m_3)L_1L_2\sin(\theta_1-\theta_2)\left(\dot{\theta}_1^2-\dot{\theta}_1\dot{\theta}_2\right) - m_3L_2L_3\sin(\theta_2-\theta_3)\left(\dot{\theta}_2\dot{\theta}_3-\dot{\theta}_3^2\right) + (m_2+m_3)gL_2\sin\theta_2\end{aligned}\right] = 0$$

$$\left[\begin{aligned}&m_3L_1L_3\cos(\theta_1-\theta_3)\ddot{\theta}_1 + m_3L_2L_3\cos(\theta_2-\theta_3)\ddot{\theta}_2 + m_3L_3^2\ddot{\theta}_3\\ &-m_3L_1L_3\sin(\theta_1-\theta_3)\left(\dot{\theta}_1^2-\dot{\theta}_1\dot{\theta}_3\right) - m_3L_2L_3\sin(\theta_2-\theta_3)\left(\dot{\theta}_2^2-\dot{\theta}_2\dot{\theta}_3\right) + m_3gL_3\sin\theta_3\end{aligned}\right] = 0$$

而系统的平衡位置应当满足

$$\left(m_1+m_2+m_3\right)gL_1\sin\bar{\theta}_1=0$$
$$\left(m_2+m_3\right)gL_2\sin\bar{\theta}_2=0$$
$$m_3gL_3\sin\bar{\theta}_3=0$$

易得：$\bar{\theta}_1=\bar{\theta}_2=\bar{\theta}_3=0$。在平衡位置附近做 Taylor 展开，并忽略高阶小量可以得到系统的线性化运动方程为

$$\begin{bmatrix}\left(m_1+m_2+m_3\right)L_1^2 & \left(m_2+m_3\right)L_1L_2 & m_3L_1L_3\\ \left(m_2+m_3\right)L_1L_2 & \left(m_2+m_3\right)L_2^2 & m_3L_2L_3\\ m_3L_1L_3 & m_3L_2L_3 & m_3L_3^2\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\\ \ddot{\theta}_3\end{Bmatrix}+\begin{bmatrix}\left(m_1+m_2+m_3\right)gL_1 & 0 & 0\\ 0 & \left(m_2+m_3\right)gL_2 & 0\\ 0 & 0 & m_3gL_3\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

若取 $L_1=L_2=L_3=L,m_1=m_2=m,m_3=2m$，则有

$$\begin{bmatrix}4&3&2\\3&3&2\\2&2&2\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\\ \ddot{\theta}_3\end{Bmatrix}+\frac{g}{L}\begin{bmatrix}4&0&0\\0&3&0\\0&0&2\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

其特征值满足

$$\det\left(K-\lambda M\right)=\begin{vmatrix}4\frac{g}{L}-4\lambda & -3\lambda & -2\lambda\\ -3\lambda & 3\frac{g}{L}-3\lambda & -2\lambda\\ -2\lambda & -2\lambda & 2\frac{g}{L}-2\lambda\end{vmatrix}=0$$

特征多项式为

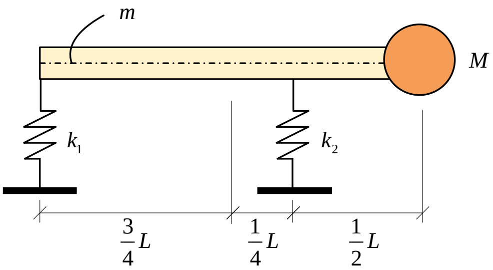
$$12\left(\frac{g}{L}\right)^3-36\lambda\left(\frac{g}{L}\right)^2+13\frac{g}{L}\lambda^2-\lambda^3=0$$

利用 MATLAB 求解得到特征值、特征向量和固有频率分别为，

$$\lambda_1=0.3854\frac{g}{L},u_1=\alpha_1[0.18,0.21,0.25]^T,\omega_1=0.6208\sqrt{\frac{g}{L}},$$
$$\lambda_2=3.3673\frac{g}{L},u_2=\alpha_2[-0.58,-0.12,1.00]^T,\omega_2=1.8350\sqrt{\frac{g}{L}},$$
$$\lambda_3=9.2473\frac{g}{L},u_3=\alpha_3[0.80,-1.39,0.67]^T,\omega_3=3.0409\sqrt{\frac{g}{L}}$$

Problem 2: (Problem 6 in HW 2) A rigid bar of mass per unit length m carries a point mass M at its right end. The bar is supported by two springs, as shown below. Assuming small motions, please derive the linear EOM for the translation and rotation of the mass center around the system's equilibrium configuration.

Based on the derived linear EOM, let $k_1 = k, k_2 = 3k, M = 2mL$， the acceleration of gravity g, calculate the natural frequencies and the mode shapes. Plot the mode shapes. (15 points)



解：系统的运动方程为

$$\frac{9L^2}{32}(2M+mL)\ddot{\theta}+\frac{3ML\cos\theta}{4}\ddot{y}+\frac{(9k_1+k_2)L^2}{16}\sin\theta\cos\theta+\frac{(k_2-3k_1)L}{4}\cos\theta y+\frac{3MgL}{4}\cos\theta=0$$

$$\left(M+\frac{3mL}{2}\right)\ddot{y}+\frac{3ML\cos\theta}{4}\ddot{\theta}-\frac{3ML\sin\theta}{4}\dot{\theta}^2+\frac{(k_2-3k_1)L}{4}\sin\theta+(k_1+k_2)y+Mg+\frac{3mgL}{2}=0$$

假设平衡位置为(0,0)，在其附近线性化得动力学方程：

$$\begin{bmatrix} \frac{9L^2}{32}(2M+mL) & \frac{3ML}{4} \\ \frac{3ML}{4} & M+\frac{3mL}{2} \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{y} \end{bmatrix}+\begin{bmatrix} \frac{(9k_1+k_2)L^2}{16} & \frac{(k_2-3k_1)L}{4} \\ \frac{(k_2-3k_1)L}{4} & (k_1+k_2) \end{bmatrix}\begin{bmatrix} \theta \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

令 $k_1 = k, k_2 = 3k, M = 2mL$， 则有，

$$mL\begin{bmatrix} \frac{45L^2}{32} & \frac{3L}{2} \\ \frac{3L}{2} & \frac{7}{2} \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{y} \end{bmatrix}+k\begin{bmatrix} \frac{3L^2}{4} & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \theta \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

其特征值满足

$$\det(K-\lambda M)=\begin{vmatrix} \frac{k}{mL}\frac{3L^2}{4}-\frac{45L^2}{32}\lambda & -\frac{3L}{2}\lambda \\ -\frac{3L}{2}\lambda & 4\frac{k}{mL}-\frac{7}{2}\lambda \end{vmatrix}=0$$

特征多项式为

$$\frac{k^2}{m^2}-\frac{22kL\lambda}{8m}+\frac{57L^2}{64}\lambda^2=0$$

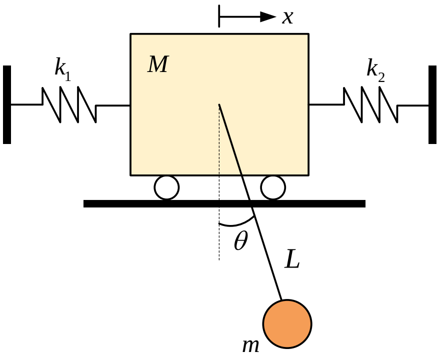
求解得到特征值、特征向量和固有频率分别为，

$$\lambda_1=\frac{8k}{19mL}, u_1=\alpha_1[4,L]^T, \omega_1=\sqrt{\frac{8k}{19mL}}\approx 0.65\sqrt{\frac{k}{mL}},$$

$$\lambda_2=\frac{8k}{3mL}, u_2=\alpha_2[-\frac{4}{3},L]^T, \omega_2=\sqrt{\frac{8k}{3mL}}\approx 1.63\sqrt{\frac{k}{mL}},$$

注：本题不具体分析平衡位置问题，求解过程正确即可。

Problem 3: (Problem 7 in HW 2) Assuming small motions, please derive the linear EOM for the system shown below. Based on the derived EOM, let $k_1 = 2k, k_2 = k, M = 3m$, the acceleration of gravity g, calculate the natural frequencies and the natural modes. Plot the mode shapes (15 points).



解：运动方程，

$$mL^2\ddot{\theta}_1 + mL\cos\theta_1\ddot{x} + mgL\sin\theta_1 = 0$$
$$M\ddot{x} + m\ddot{x} + mL\cos\theta_1\ddot{\theta}_1 - mL\sin\theta_1\dot{\theta}_1^2 + k_1x + k_2x = 0$$

系统的平衡位置为 $x=0,\theta=0$，在平衡位置附近的线性运动方程为，

$$\begin{bmatrix} mL^2 & mL \\ mL & M+m \end{bmatrix}\begin{Bmatrix} \ddot{\theta} \\ \ddot{x} \end{Bmatrix} + \begin{bmatrix} mLg & 0 \\ 0 & k_1+k_2 \end{bmatrix}\begin{Bmatrix} \theta \\ x \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

代入 $k_1 = 2k, k_2 = k, M = 3m$ 有，

$$\begin{bmatrix} L^2 & L \\ L & 4 \end{bmatrix}\begin{Bmatrix} \ddot{\theta} \\ \ddot{x} \end{Bmatrix} + \begin{bmatrix} Lg & 0 \\ 0 & 3k/m \end{bmatrix}\begin{Bmatrix} \theta \\ x \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

其特征值满足

$$\det(K - \lambda M) = \begin{vmatrix} gL - L^2\lambda & -L\lambda \\ -L\lambda & 3k/m - 4\lambda \end{vmatrix} = 0$$

特征多项式为

$$\frac{3kgL}{m} - \left(\frac{3k}{m}L^2 + 4gL\right)\lambda + 3L^2\lambda^2 = 0$$

求解得到特征值、特征向量和固有频率分别为，

$$\lambda_1 = \frac{(3kL + 4mg) - \sqrt{(9k^2L^2 - 12kmgL + 16m^2g^2)}}{6mL}, \omega_1 = \sqrt{\lambda_1},$$
$$\lambda_2 = \frac{(3kL + 4mg) + \sqrt{(9k^2L^2 - 12kmgL + 16m^2g^2)}}{6mL}, \omega_2 = \sqrt{\lambda_2},$$

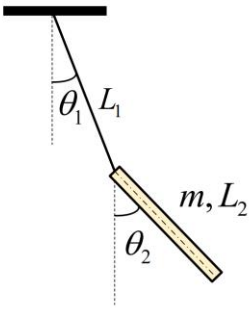
注意到 $\lambda_1\lambda_2 = 1$，特征向量可表示为

$$u_1 = \alpha_1[\frac{3k}{mL\lambda_1} - \frac{4}{L}, 1]^T, u_2 = \alpha_2[\frac{3k}{mL\lambda_2} - \frac{4}{L}, 1]^T$$

取 $mg = kL$，

$$\lambda_1 = 0.57\frac{k}{m}, u_1 = \alpha_1[-0.80, 0.35]^T, \omega_1 = 0.75\sqrt{\frac{k}{m}},$$
$$\lambda_2 = 1.77\frac{k}{m}, u_2 = \alpha_2[0.60, 0.46]^T, \omega_2 = 1.33\sqrt{\frac{k}{m}},$$

Problem 4: A uniform thin rod is suspended by a string. (1) Assuming small motion, derive the linear EOM of the system around the equilibrium configuration **(15 points)** (2) Let $L_1 = L_2 = L$, the acceleration of gravity g, calculate the natural frequencies and the natural modes. Plot the mode shapes. (15 points)



解：（1）选取广义坐标为θ_1, θ_2，则杆的质心位移为

$$\vec{r} = \left(L_1 \sin\theta_1 + \frac{L_2}{2}\sin\theta_2\right)\vec{i} - \left(L_1\cos\theta_1 + \frac{L_2}{2}\cos\theta_2\right)\vec{j}$$

杆质心的速度为

$$\dot{\vec{r}} = \left(L_1\cos\theta_1\dot{\theta}_1 + \frac{L_2}{2}\cos\theta_2\dot{\theta}_2\right)\vec{i} + \left(L_1\sin\theta_1\dot{\theta}_1 + \frac{L_2}{2}\sin\theta_2\dot{\theta}_2\right)\vec{j}$$

系统的动能为

$$T = \frac{1}{24}mL_2^2\dot{\theta}_2^2 + \frac{1}{2}m\left(L_1^2\dot{\theta}_1^2 + \frac{L_2^2\dot{\theta}_2^2}{4} + \cos(\theta_1 - \theta_2)L_1L_2\dot{\theta}_1\dot{\theta}_2\right) = \frac{1}{6}mL_2^2\dot{\theta}_2^2 + \frac{1}{2}mL_1^2\dot{\theta}_1^2 + \frac{1}{2}m\cos(\theta_1-\theta_2)L_1L_2\dot{\theta}_1\dot{\theta}_2$$

设顶端为零势能面，系统势能为

$$V = -mg\left(\cos\theta_1 L_1 + \cos\theta_2\frac{L_2}{2}\right)$$

Lagrangian 量为，

$$L = T - V = \frac{1}{6}mL_2^2\dot{\theta}_2^2 + \frac{1}{2}mL_1^2\dot{\theta}_1^2 + \frac{1}{2}m\cos(\theta_1-\theta_2)L_1L_2\dot{\theta}_1\dot{\theta}_2 + mg\left(\cos\theta_1 L_1 + \cos\theta_2\frac{L_2}{2}\right)$$

代入 Euler-Lagrange 方程：

$$mL_1^2\ddot{\theta}_1 + \frac{1}{2}mL_1L_2\cos(\theta_1-\theta_2)\ddot{\theta}_2 + \frac{1}{2}mL_1L_2\sin(\theta_1-\theta_2)\dot{\theta}_2^2 + \sin\theta_1 L_1 mg = 0$$

$$\frac{1}{2}mL_1L_2\cos(\theta_1-\theta_2)\ddot{\theta}_1 + \frac{1}{3}mL_2^2\ddot{\theta}_2 - \frac{1}{2}mL_1L_2\sin(\theta_1-\theta_2)\dot{\theta}_1^2 + \sin\theta_2\frac{L_2}{2}mg = 0$$

平衡位置$\theta_1 = \theta_2 = 0$，在其附近线性化得，

$$\begin{bmatrix} L_1^2 & \frac{1}{2}L_1L_2 \\ \frac{1}{2}L_1L_2 & \frac{1}{3}L_2^2 \end{bmatrix}\begin{bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix} gL_1 & 0 \\ 0 & \frac{1}{2}gL_2\end{bmatrix}\begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

令$L_1 = L_2 = L$，得到

$$L\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix} g & 0 \\ 0 & \frac{1}{2}g\end{bmatrix}\begin{bmatrix}\theta_1\\\theta_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

其特征值满足

$$\det\left(K-\lambda M\right)=\begin{vmatrix} g-L\lambda & -\frac{1}{2}L\lambda \\ -\frac{1}{2}L\lambda & \frac{g}{2}-\frac{1}{3}L\lambda \end{vmatrix}=0$$

特征多项式为

$$6g^2-10gL\lambda+L^2\lambda^2=0$$

求解得到特征值、特征向量和固有频率分别为，

$$\lambda_1=\left(5-\sqrt{19}\right)\frac{g}{L}\approx 0.6411\frac{g}{L},\omega_1=0.8007\sqrt{\frac{g}{L}},u_1\approx\alpha_1[-0.63,-0.70]^T,$$

$$\lambda_2=\left(5+\sqrt{19}\right)\frac{g}{L}\approx 9.3589\frac{g}{L},\omega_2=3.0592\sqrt{\frac{g}{L}},u_2\approx\alpha_2[-1.90,3.39]^T,$$