2021-Homework 1: Reference answer

PROBLEMS RAISED IN LECTURES

Problem 1: For a free SDOF system with the EOM $\ddot{x}+2\xi\omega_n\dot{x}+\omega_n^2x=0$, if the damping ratio $\xi<1$, the homogeneous solution (i.e., the transient response) can be expressed as the following complex form $x_h=e^{-\xi\omega_n t}\left(\alpha e^{i\omega_d t}+\beta e^{-i\omega_d t}\right)$. Based on this, please derive the homogeneous solution in real form. (5 points)

解：根据 Euler 公式可以将复数形式的解重新表示成，

$$\begin{aligned}x_h&=e^{-\xi\omega_n t}\left(\alpha e^{i\omega_d t}+\beta e^{-i\omega_d t}\right)\\&=e^{-\xi\omega_n t}\left[\alpha\left(\cos(\omega_d t)+i\sin(\omega_d t)\right)+\beta\left(\cos(-\omega_d t)+i\sin(-\omega_d t)\right)\right]\\&=e^{-\xi\omega_n t}\left[(\alpha+\beta)\cos(\omega_d t)+i(\alpha-\beta)\sin(\omega_d t)\right],\end{aligned}$$

其中$\alpha,\beta\in\mathbf{C}$。由于上述系统的解满足$x_h\in\mathbf{R}$，即$x_h=\bar{x}_h$，并且

$$\bar{x}_h=e^{-\xi\omega_n t}\left[\left(\bar{\alpha}+\bar{\beta}\right)\cos(\omega_d t)+\bar{i}\left(\bar{\alpha}-\bar{\beta}\right)\sin(\omega_d t)\right].$$

由正弦余弦函数的线性无关性以及时间的任意性，可以得到

$$\alpha+\beta=\bar{\alpha}+\bar{\beta},\alpha-\beta=\bar{\beta}-\bar{\alpha}.$$

将两式相加可以得到，

$$\alpha=\bar{\beta}.$$

因此我们可以选取实数 A 与 B （$A,B\in\mathbf{R}$）使得其满足，

$$A=\alpha+\beta,$$
$$B=i(\alpha-\beta).$$

可以得到实数形式的解为，

$$x_h=e^{-\xi\omega_n t}\left[A\cos(\omega_d t)+B\sin(\omega_d t)\right].$$

注：本体关键在于考虑复数形式解中两个常数的共轭性，然后利用 Euler 公式得到实数形式的解。

Problem 2: For a SDOF system subjected to harmonic excitations, the EOM is $m\ddot{x}+c\dot{x}+kx=F_0\sin\omega_f t$, the particular solution (i.e., the steady-state response) can be expressed as the following real form

$$x_p=\frac{F_0/k}{\sqrt{\left(1-\left(\frac{\omega_f}{\omega_n}\right)^2\right)^2+\left(2\xi\frac{\omega_f}{\omega_n}\right)^2}}\sin\left(\omega_f t-\phi_f\right),\text{ where }\phi_f=\arctan\left(\frac{2\xi\frac{\omega_f}{\omega_n}}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}\right).$$

Based on this, please derive the particular solution in complex form if the excitation force is $F_0e^{i\omega_f t}$. (5 points)

解：复数形式的激励力可以通过 Euler 公式表示成，

$$F_0e^{i\omega_f t}=F_0\left(\cos\omega_f t+i\sin\omega_f t\right)=F_0\left(\sin\left(\omega_f t+\frac{\pi}{2}\right)+i\sin\omega_f t\right).$$

题中给出了正弦谐波激励下稳态解的形式，由叠加原理可以得到激励为 $F_0e^{i\omega_f t}$ 时解的形式为，

$$\begin{aligned}x_p&=\frac{F_0/k}{\sqrt{\left(1-\left(\omega_f/\omega_n\right)^2\right)^2+\left(2\xi\omega_f/\omega_n\right)^2}}\left(\sin\left(\omega_f t+\frac{\pi}{2}-\phi_f\right)+i\sin\left(\omega_f t-\phi_f\right)\right)\\&=\frac{F_0/k}{\sqrt{\left(1-\left(\omega_f/\omega_n\right)^2\right)^2+\left(2\xi\omega_f/\omega_n\right)^2}}\left(\cos\left(\omega_f t-\phi_f\right)+i\sin\left(\omega_f t-\phi_f\right)\right)\end{aligned}$$

然后再次利用 Euler 公式可以得到

$$x_p=\frac{F_0/k}{\sqrt{\left(1-\left(\omega_f/\omega_n\right)^2\right)^2+\left(2\xi\omega_f/\omega_n\right)^2}}e^{i\left(\omega_f t-\phi_f\right)},\phi_f=\arctan\left(\frac{2\xi\omega_f/\omega_n}{1-\left(\omega_f/\omega_n\right)^2}\right).$$

注：本题将复数形式激励转化为三角函数的激励，利用已知实数形式解，通过叠加原理得到系统稳态解并转化为复数形式。注意不要从运动方程重新求解复数形式解。

Problem 3: If the damping ratio $\xi=1$, we can obtain one particular solution as $e^{-\omega_n t}$, please derive another particular solution so that the homogeneous solution can be constructed. (5 points)

解：临界阻尼条件下，系统的特征方程有两个相等的实根，其中一个特解为 $x_1(t)=e^{-\omega_n t}$ 时，需要保证另一个特解与其线性无关，即 $x_2(t)/x_1(t)\neq \text{const}$，可以将其设为

$$x_2(t)=u(t)x_1(t).$$

代入动力学方程 $\ddot{x}+2\omega_n\dot{x}+\omega_n^2x=0$ 中有，

$$\ddot{u}x_1+2\dot{u}\dot{x}_1+u\ddot{x}_1+2\omega_n(\dot{u}x_1+u\dot{x}_1)+\omega_n^2ux_1=0.$$

对上式进行必要的化简，并注意到 $\ddot{x}_1+2\omega_n\dot{x}_1+\omega_n^2x_1=0$，上式可以表示成，

$$\ddot{u}x_1+2\dot{u}(\dot{x}_1+\omega_nx_1)=0.$$

此外，由 $x_1(t)=e^{-\omega_n t}$ 可以得到 $\dot{x}_1=-\omega_ne^{-\omega_n t}=-\omega_nx_1$，上式可进一步简化为，

$$\ddot{u}x_1=0.$$

注意到 $x_1(t)=e^{-\omega_n t}$ 是恒大于 0 的。因此 $\ddot{u}x_1=0$ 的解可以表示为，

$$u(t)=A_1+A_2t,$$
$$x_2(t)=u(t)x_1(t)=(A_1+A_2t)e^{-\xi\omega_n t}.$$

在 $A_2\neq 0$ 时两个特解满足线性无关，因此系统在临界阻尼下解的形式为，

$$x(t)=B_1x_1(t)+B_2x_2(t)=(C_1+C_2t)e^{-\omega_n t}.$$

最简情况下可以取 $A_1=0, A_2=1$，即另一个特解为 $x_2(t)=te^{-\xi\omega_n t}$。

注：也可以通过求出方程通解的方式得到另一个特解：

$$\dot{x}+\omega_nx=c_1e^{-\xi\omega_n t},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(xe^{\xi\omega_n t}\right)=c_1,$$
$$xe^{\xi\omega_n t}=c_1t+c_2.$$

考虑到振动系统解在实数域内，有 $c_1,c_2\in\mathbf{R}$.

DERIVE EOM AND CALCULATE NATURAL FREQUENCY

Problem 4: Please solve Problem 2.19 in page 106 of the textbook. (5 points)

2.19 A disk of mass m and radius R rolls without slip while restrained by a dashpot with coefficient of viscous damping c in parallel with a spring of stiffness k, as shown in Fig. 2.22. Derive the differential equation for the displacement $x(t)$ of the disk mass center C and determine the viscous damping factor ζ and the frequency ω_n of undamped oscillation.

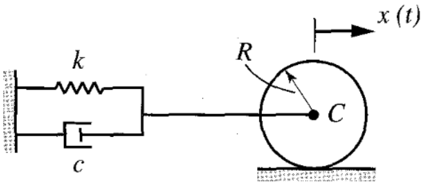


FIGURE 2.22

解：用 $x(t)$ 表示圆盘质心处的位移，则圆盘的动能可以表示为

$$T=\frac{1}{2}m\dot{x}^2+\frac{1}{2}J_C\dot{\omega}^2.$$

其中 J_C 表示圆盘质心处的转动惯量，ω 表示转动的角速度，分别有

$$J_C=\frac{1}{2}mR^2,\omega=\frac{\dot{x}}{R}.$$

即，

$$T=\frac{3}{4}m\dot{x}^2.$$

结构的势能可以表示为，

$$V=\frac{1}{2}kx^2.$$

阻尼力的广义力可以表示为

$$Q=c\dot{x}\frac{\partial r}{\partial x}.$$

其中 $r=-x$，将动能、势能以及广义力代入 Lagrange 方程，

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(T-V)}{\partial\dot{x}}\right)-\frac{\partial(T-V)}{\partial x}=Q.$$

得到动力学方程为

$$\frac{3}{2}m\ddot{x}+c\dot{x}+kx=0.$$

粘性阻尼系数为

$$\zeta=\frac{c}{2\sqrt{3km/2}}=\frac{c}{\sqrt{6km}}.$$

无阻尼固有频率为

$$\omega_n=\sqrt{\frac{k}{3m/2}}=\frac{1}{3}\sqrt{\frac{6k}{m}}.$$

注：动力学方程也可从动量矩定理得到，设圆盘与地面接触点为 O，

$$J_O\dot{\omega}=-R(kx+c\dot{x}).$$

其中 J_O 表示绕点 O 的转动惯量，由平行轴定理得到 $J_O=J_C+mR^2$；负号表示弹簧力与阻尼力合力对点 O 的矩为逆时针方向；ω 表示转动角速度 $\omega=\dot{x}/R$。

求出地面的摩擦力的方式同样合理。

CALCULATE THE VALUE OF CRITICAL DAMPING

Problem 5: Please solve Problem 2.20 in page 106 of the textbook. (5 points)

2.20 Calculate the frequency of damped oscillation of the system shown in Fig. 2.23 for the values $m=1750\ \mathrm{kg}$, $c=3500\ \mathrm{N\cdot s/m}$, $k=7\times10^5\ \mathrm{N/m}$, $a=1.25\ \mathrm{m}$ and $b=2.5\ \mathrm{m}$. Determine the value of the critical damping.

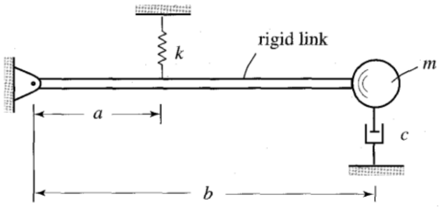


FIGURE 2.23

解：设端部质量 m 在纵向上的位移为 $y(t)$，则系统的动能和势能可以分别表示为

$$T=\frac{1}{2}m\dot{y}^2,$$

$$V=\frac{1}{2}k\left(\frac{a}{b}y\right)^2.$$

阻尼力的广义力可以表示为

$$Q=c\dot{y}\frac{\partial r}{\partial y}.$$

其中 $r=-y$，将动能、势能以及广义力代入 Lagrange 方程，

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(T-V)}{\partial\dot{y}}\right)-\frac{\partial(T-V)}{\partial y}=Q.$$

得到动力学方程为

$$m\ddot{y}+c\dot{y}+\frac{a^2}{b^2}ky=0.$$

无阻尼固有频率为

$$\omega_n=\sqrt{\frac{a^2k}{b^2m}}=\sqrt{\frac{1.25^2\times7\times10^5}{2.5^2\times1.75\times10^3}}=10\,\mathrm{rad/s}.$$

阻尼比为

$$\xi=\frac{c}{2\sqrt{kma^2/b^2}}=\frac{3.5\times10^3}{2\times\sqrt{7\times10^5\times1.75\times10^3\times1.25^2/2.5^2}}=0.1.$$

有阻尼频率为

$$\omega_d=\sqrt{1-\xi^2}\,\omega_n=\sqrt{1-0.1^2}\times10=9.95\,\mathrm{rad/s}.$$

临界阻尼时阻尼比满足 $\xi=1$，因此

$$\xi_c=\frac{c_c}{2\sqrt{kma^2/b^2}}=1.$$

即

$$c_c=2\sqrt{kma^2/b^2}=2\times\sqrt{7\times10^5\times1.75\times10^3\times1.25^2/2.5^2}=3.5\times10^4\ \mathrm{N\cdot s/m}.$$

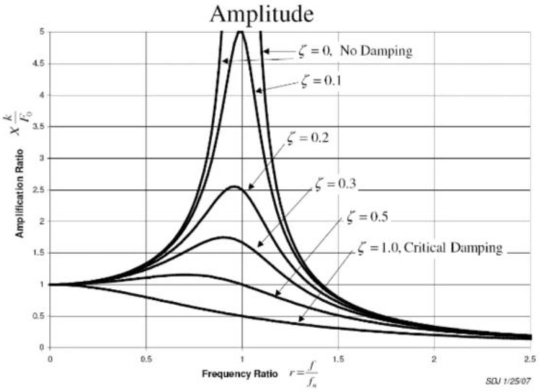
注：绕固定点的动量矩定理

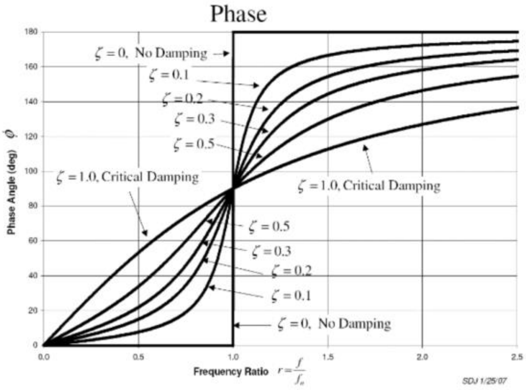
$$J\ddot{\theta}=-k\theta a\cdot a-c\dot{\theta}b\cdot b.$$

其中，忽略杆的质量时 $J=mb^2$；且在小振幅条件下可线性化为 $y=b\theta$。

GENERATE FREQUENCY-RESPONSE DIAGRAM

Problem 6: Using Mathematica/Matlab, please derive the steady-state response (i.e., the particular solution) of the following SDOF system $m\ddot{x}+c\dot{x}+kx=F_0\sin\left(2\pi f_f t\right)$. Based on the response, please regenerate the following figures. In addition to the printed results, the Mathematica/Matlab code should be uploaded to the eLearning system. (10 points)





解：单自由度系统的动力学方程可以表示为

$$\ddot{x}+2\xi\left(2\pi f_n\right)\dot{x}+\left(2\pi f_n\right)^2 x=\frac{F_0}{m}\sin\left(2\pi f_f t\right).$$

假设系统的稳态解为

$$x(t)=X\sin(2\pi f_f t-\phi).$$

将其代入单自由度动力学方程中有，

$$-\left(2\pi f_f\right)^2 X\sin(2\pi f_f t-\phi)+4\pi f_f\xi\left(2\pi f_n\right)X\cos(2\pi f_f t-\phi)+\left(2\pi f_n\right)^2 X\sin(2\pi f_f t-\phi)=\frac{F_0}{m}\sin\left(2\pi f_f t\right).$$

将上式进行整理可以得到

$$\left[\left(2\pi f_n\right)^2-\left(2\pi f_f\right)^2\right]\sin(2\pi f_f t-\phi)+\left(2\pi f_n\right)4\pi f_f\xi\cos(2\pi f_f t-\phi)=\frac{4\pi^2 f_n^2 F_0}{kX}\sin\left(2\pi f_f t\right).$$

将左边的三角函数展开后得到

$$\left[\left(f_n^2-f_f^2\right)\cos\phi+2\xi f_n f_f\sin\phi\right]\sin(2\pi f_f t)+\left[-\left(f_n^2-f_f^2\right)\sin\phi+2\xi f_n f_f\cos\phi\right]\cos(2\pi f_f t)=\frac{f_n^2 F_0}{kX}\sin\left(2\pi f_f t\right).$$

考虑到正弦函数与余弦函数的正交性，对应项前面系数应当相等，

$$4\pi^2\left(f_n^2-f_f^2\right)\cos\phi+8\pi^2\xi f_n f_f\sin\phi=\frac{4\pi^2 f_n^2 F_0}{kX},$$

$$-4\pi^2\left(f_n^2-f_f^2\right)\sin\phi+8\pi^2\xi f_n f_f\cos\phi=0.$$

得到所设稳态响应中幅值和相位分别为，

$$\left[\left(1-\left(f_f/f_n\right)^2\right)\right]^2+\left[2\xi f_f/f_n\right]^2=\left[\frac{F_0}{kX}\right]^2,$$

$$\tan\phi=\frac{2\xi f_f/f_n}{1-\left(f_f/f_n\right)^2}.$$

即

$$X=\frac{F_0}{k}\frac{1}{\sqrt{\left[\left(1-\left(f_f/f_n\right)^2\right)\right]^2+\left[2\xi f_f/f_n\right]^2}},$$

$$\phi=\arctan\frac{2\xi f_f/f_n}{1-\left(f_f/f_n\right)^2}.$$

图形绘制：

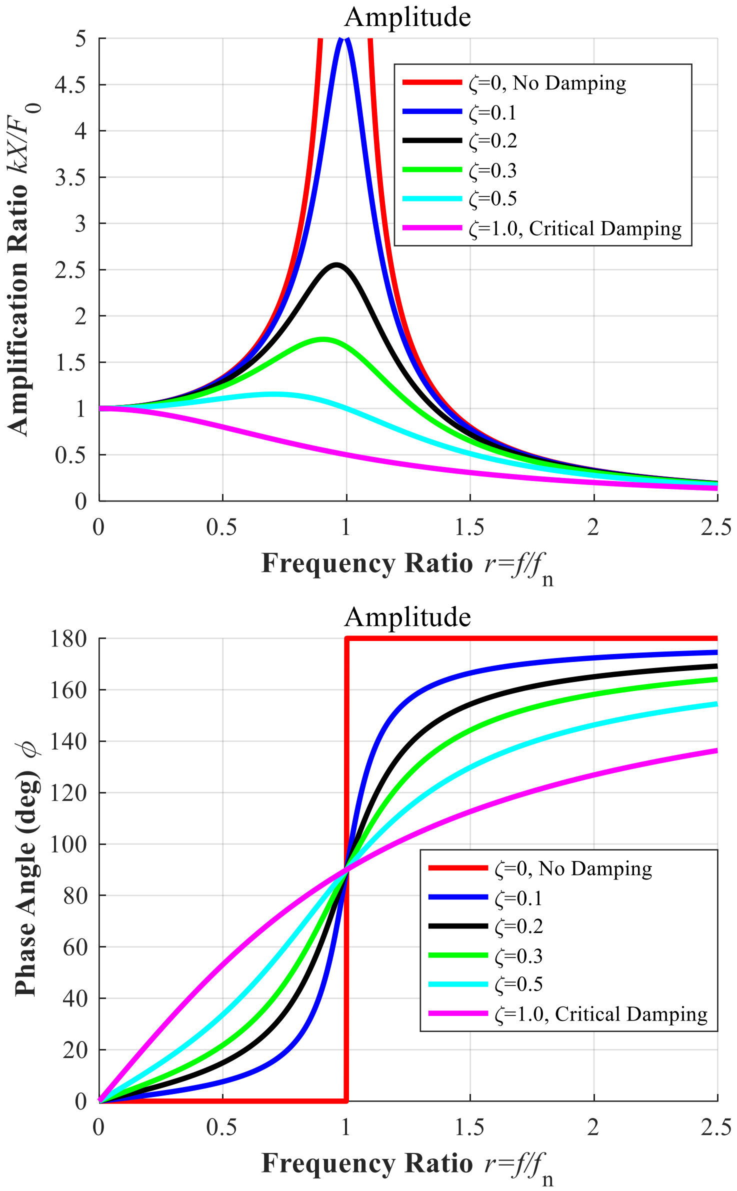


Fig 1. The amplitude and phase of the steady-state responses.

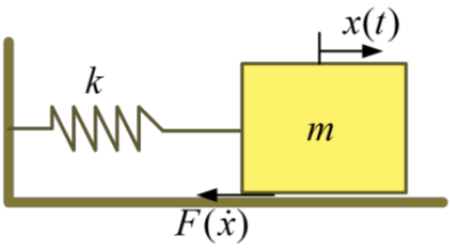
注：绘图时需要注意相位角的范围在$[0,\pi]$之间。

RESPONSE OF A NON-SMOOTH SDOF SYSTEM

Problem 7: Consider the following SDOF spring-mass system suffering to Coulomb dry friction. Its equation of motion can be written as $m\ddot{x}+F_d(\dot{x})+kx=0$, where

$$F_d(\dot{x})=\begin{cases}\mu_k mg\,\mathrm{sgn}(\dot{x}),\dot{x}\neq 0,\\ -kx,\dot{x}=0\,\&\,|kx|<\mu_k mg.\end{cases}$$

$\mu_k>0$ is the kinetic friction coefficient, sgn() denotes the *signum function* and represents a function having the value +1 if the argument $\dot{x}$ is positive and the value −1 if the argument is negative. Mathematically, the function can be expressed as $\mathrm{sgn}(\dot{x})=\dot{x}/|\dot{x}|$ if $\dot{x}\neq 0$. Please read section 2.4 of the textbook and use Mathematica/Matlab to derive its transient response (i.e., the homogeneous solution) under initial conditions $x(0)=x_0,\dot{x}(0)=0$ (x_0 is sufficiently large that the restoring force in the spring exceeds the static friction force). Please regenerate Figure 2.15 in the textbook. In addition to the printed results, the Mathematica/Matlab code should be uploaded to the eLearning system. (15 points)



解：系统的动力学方程可以根据的符号分成两种情况，当$\dot{x}>0$时，

$$m\ddot{x}+kx=-\mu_k mg,$$

当$\dot{x}<0$时，

$$m\ddot{x}+kx=\mu_k mg.$$

采用分段的方式求解库伦干摩擦下的瞬态解：假设第i段的瞬态解为$x_i(t)$，第i段需要的总时间为Δt_i，则初始条件可以表示成如下的形式，

$$x_i(t_i^0)=\begin{cases}x_0,i=1\\ x_{i-1}(t_{i-1}^0+\Delta t_i),i\geq 2\end{cases}.$$

(1) 当$\dot{x}>0$时，方程的瞬态解可以写成

$$x_i(t)=\left(x_i(t_i^0)+\frac{\mu_k mg}{k}\right)\cos\left(\omega_n\left(t-t_i^0\right)\right)-\frac{\mu_k mg}{k}.$$

初始时刻$t=t_i^0$速度为 0，

$$\dot{x}_i(t_i^0)=-\omega_n\left(x_i(t_i^0)+\frac{\mu_k mg}{k}\right)\sin(0)=0.$$

经历Δt_i的时间后$t=t_i^0+\Delta t_i$，系统速度降为 0，

$$\dot{x}_i(t_i^0+\Delta t_i)=-\omega_n\left(x_i(t_i^0)+\frac{\mu_k mg}{k}\right)\sin(\omega_n\Delta t_i)=0.$$

考虑到中间过程满足$\dot{x}>0$，可以得到如下结果，

$$\Delta t_i=\frac{\pi}{\omega_n}.$$

得到$t=t_i^0+\Delta t_i$时的位移为，

$$x_i(t_i^0+\Delta t_i)=\left(x_i(t_i^0)+\frac{\mu_k mg}{k}\right)\cos(\omega_n\Delta t_i)-\frac{\mu_k mg}{k}=-x_i(t_i^0)-\frac{2\mu_k mg}{k}.$$

(2) 当$\dot{x}<0$时，方程的瞬态解可以写成

$$x_i(t)=\left(x_i(t_i^0)-\frac{\mu_k mg}{k}\right)\cos\left(\omega_n\left(t-t_i^0\right)\right)+\frac{\mu_k mg}{k}.$$

初始时刻$t=t_i^0$速度为 0，

$$\dot{x}_i(t_i^0)=-\omega_n\left(x_i(t_i^0)-\frac{\mu_k mg}{k}\right)\sin(0)=0.$$

经历Δt_i的时间后$t=t_i^0+\Delta t_i$，系统速度降为 0，

$$\dot{x}_i(t_i^0+\Delta t_i)=-\omega_n\left(x_i(t_i^0)-\frac{\mu_k mg}{k}\right)\sin\left(\omega_n\Delta t_i\right)=0.$$

考虑到中间过程满足$\dot{x}<0$，可以得到如下结果，

$$\Delta t_i=\frac{\pi}{\omega_n}.$$

得到$t=t_i^0+\Delta t_i$时的位移为，

$$x_i(t_i^0+\Delta t_i)=\left(x_i(t_i^0)-\frac{\mu_k mg}{k}\right)\cos\left(\omega_n\Delta t_i\right)+\frac{\mu_k mg}{k}=-x_i(t_i^0)+\frac{2\mu_k mg}{k}.$$

在上述分析中我们发现每一段的时间长度均为$\Delta t_i=\pi/\omega_n$，因此

$$t_i^0=(i-1)\frac{\pi}{\omega_n}.$$

且每一阶段初始时刻的位移满足，

$$x_i\left(t_i^0\right)=(-1)^{i+1}\left(x_0-\operatorname{sgn}\left(x_0\right)2(i-1)\mu_k mg/k\right),\ i=1,2,3,\ldots$$

因此上述两种情况下的瞬态解可以写为

$$x_i(t)=x_i(t_i^0)\cos\left(\omega_n\left(t-t_i^0\right)\right)+\operatorname{sgn}(\dot{x})\mu_k mg/k\left(\cos\left(\omega_n\left(t-t_i^0\right)\right)-1\right).$$

注意到$\operatorname{sgn}(\dot{x})$可以由初始位移$x_0$符号以及阶段数$i$进行表示，

$$\operatorname{sgn}(\dot{x})=(-1)^i\operatorname{sgn}\left(x_0\right).$$

因此，

$$\begin{aligned}
x_i(t)&=(-1)^{i+1}\left(x_0-\operatorname{sgn}\left(x_0\right)2(i-1)\mu_k mg/k\right)\cos\left(\omega_n t-(i-1)\pi\right)+(-1)^i\operatorname{sgn}\left(x_0\right)\mu_k mg/k\left(\cos\left(\omega_n t-(i-1)\pi\right)-1\right)\\
&=(-1)^{i+1}\left(x_0-\operatorname{sgn}\left(x_0\right)2(i-1)\mu_k mg/k\right)(-1)^{i-1}\cos\left(\omega_n t\right)+(-1)^i\operatorname{sgn}\left(x_0\right)\mu_k mg/k\left((-1)^{i-1}\cos\left(\omega_n t\right)-1\right)\\
&=\left(x_0-\operatorname{sgn}\left(x_0\right)2(i-1)\mu_k mg/k-\operatorname{sgn}\left(x_0\right)\mu_k mg/k\right)\cos\left(\omega_n t\right)-(-1)^i\operatorname{sgn}\left(x_0\right)\mu_k mg/k\\
&=\left(x_0-\operatorname{sgn}\left(x_0\right)(2i-1)\mu_k mg/k\right)\cos\left(\omega_n t\right)-(-1)^i\operatorname{sgn}\left(x_0\right)\mu_k mg/k.
\end{aligned}$$

接下来考虑系统最终停下来时的条件，

$$\dot{x}=0\ \&\ |kx|<\mu_k mg.$$

由上面的分析我们知道当$t_n=n\pi/\omega_n$时满足$\dot{x}=0$，此时的位移为

$$\begin{aligned}
x_i(t_n)&=\left(x_0-\operatorname{sgn}\left(x_0\right)(2n-1)\mu_k mg/k\right)\cos(n\pi)-(-1)^n\operatorname{sgn}\left(x_0\right)\mu_k mg/k\\
&=x_0\cos(n\pi)-\operatorname{sgn}\left(x_0\right)\mu_k mg/k\left[(2n-1)\cos(n\pi)+(-1)^n\right].
\end{aligned}$$

注意到$\cos(n\pi)=(-1)^n$以及$x_0=\operatorname{sgn}\left(x_0\right)\left|x_0\right|$，则上式可表示为

$$x_i(t_n)=(-1)^n\operatorname{sgn}\left(x_0\right)\left(\left|x_0\right|-2n\mu_k mg/k\right).$$

因此上述停止条件可写为，

$$\left|(-1)^n\operatorname{sgn}\left(x_0\right)\left(\left|x_0\right|-2n_{end}\mu_k mg/k\right)\right|<\mu_k mg/k.$$

即

$$\left|\left|x_0\right|-2n_{end}\mu_k mg/k\right|<\mu_k mg/k.$$

得到

$$\frac{1}{2}\left(\frac{k|x_0|}{\mu_k mg}-1\right)<n_{end}<\frac{1}{2}\left(\frac{k|x_0|}{\mu_k mg}+1\right).$$

取 $m=400\,\mathrm{kg},\ k=1.4\times10^5\ \mathrm{N/m},\ \mu_k=0.1,\ x_0=0.03\,\mathrm{m}$，绘图如下，

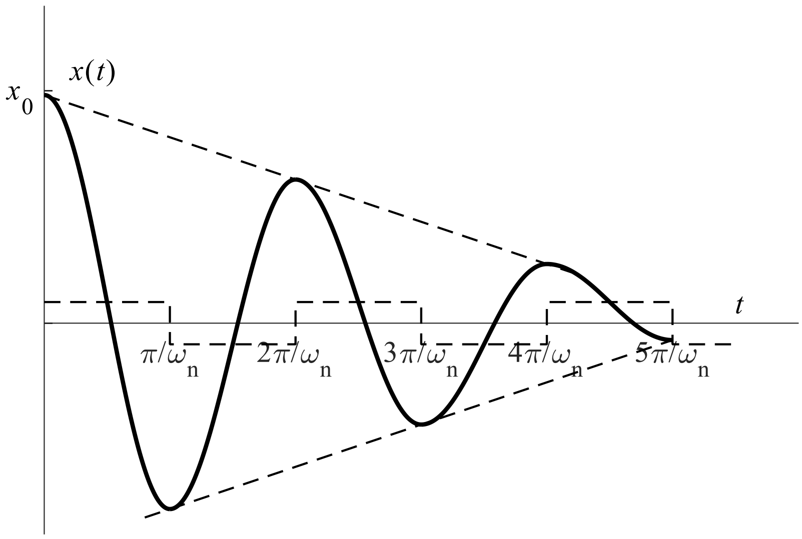


Fig 2. Response of system subjected to Coulomb dry frication.

注：本题需要注意停止条件。

RESPONSE UNDER INITIAL CONDITION

Problem 8: Please solve Problem 2.22 in page 107 of the textbook. (10 points)

2.22 A projectile of mass $m = 10\ \text{kg}$ traveling with the velocity $v = 50\ \text{m/s}$ strikes and becomes embedded in a massless board supported by a spring of stiffness $k = 6.4\times10^4\ \text{N/m}$ in parallel with a dashpot with the coefficient of viscous damping $c = 400\ \text{N}\cdot\text{s/m}$ (Fig. 2.24). Determine the time required for the board to reach the maximum displacement and the value of the maximum displacement.

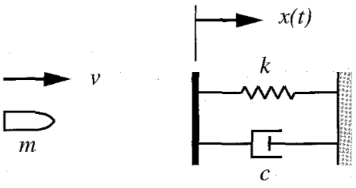


FIGURE 2.24

解：在相撞时由动量定理可以得到，

$$mv + 0 = (m+0) * v_0.$$

相撞后系统的动力学方程为

$$m\ddot{x} + c\dot{x} + kx = 0.$$

初始条件为

$$x(0) = 0, \dot{x}(0) = v_0.$$

系统的阻尼比为，

$$\xi = \frac{c}{2\sqrt{km}} = 0.25.$$

因为$\xi < 1$，系统的瞬态响应为

$$x(t) = e^{-\xi\omega_n t}\left(x(0)\cos(\omega_d t) + \frac{\dot{x}(0) + \xi\omega_n x(0)}{\omega_d}\sin(\omega_d t)\right).$$

代入初始条件有，

$$x(t) = \frac{v}{\omega_d} e^{-\xi\omega_n t}\sin(\omega_d t).$$

板在达到最大位移时应当满足，

$$\dot{x}(t) = \left(-\xi\omega_n \sin(\omega_d t) + \omega_d \cos(\omega_d t)\right)\frac{v}{\omega_d} e^{-\xi\omega_n t} = 0.$$

即

$$-\zeta\sin(\omega_d t) + \sqrt{1-\xi^2}\cos(\omega_d t) = 0.$$

得到

$$\cos(\omega_d t) = \xi.$$

由于欠阻尼系统的峰值会逐渐衰减，因此在第一次满足上式时位移取得最大值，

$$t = \frac{1}{\omega_d}\cos^{-1}(0.25) \approx 0.0170\ \text{s}.$$

最大位移为

$$x_{\max} = \frac{5}{8}e^{-20t} \approx 0.4447\ \text{m}.$$

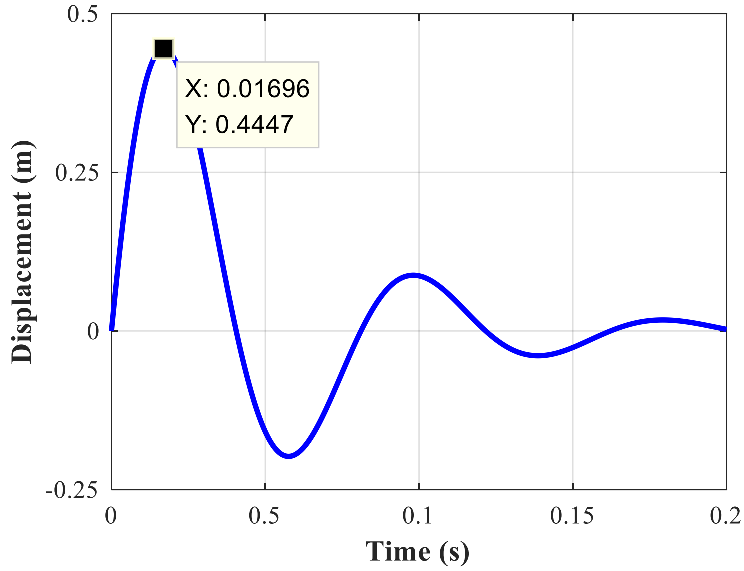


Fig 3. The displacement response obtained by ODE45.

注：初始条件由动量定理得到；最大位移一定在驻值处，但是并不是充分必要条件。

RESPONSE UNDER HARMONIC DISPLACEMENT EXCITATION

Problem 9: Please solve Problem 3.14 in page 156 of the textbook. (10 points)

3.14 The system shown in Fig. 3.33 simulates a vehicle traveling on a rough road. Let the vehicle velocity be uniform, $v = \text{const}$, and calculate the response $z(t)$, as well as the force transmitted to the vehicle.

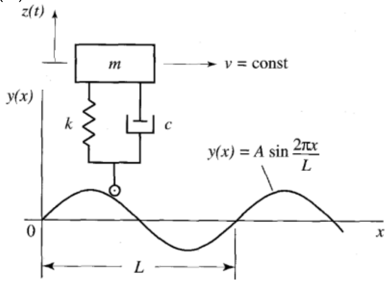


FIGURE 3.33

解：车辆纵向的位移 $z(t)$ 满足，

$$m\ddot{z} = -k(z-y) - c(\dot{z}-\dot{y}).$$

代入 $y = A\sin 2\pi x/L$ 以及 $x = vt$，有

$$m\ddot{z} + c\dot{z} + kz = kA\sin 2\pi vt/L + ckA 2\pi v/L \cos 2\pi vt/L.$$

设系统的解为 $z(t) = Z\sin(2\pi vt/L - \phi)$，则有

$$\left[kZ - mZ(2\pi v/L)^2\right]\sin(2\pi vt/L - \phi) + cZ\,2\pi v/L\cos(2\pi vt/L - \phi) = kA\sin 2\pi vt/L + cA 2\pi v/L\cos 2\pi vt/L.$$

整理后得到

$$\sqrt{\left(kZ - mZ(2\pi v/L)^2\right)^2 + (cZ\,2\pi v/L)^2}\sin(2\pi vt/L - \phi + \varphi_1) = \sqrt{(kA)^2 + (cA 2\pi v/L)^2}\sin(2\pi vt/L + \varphi_2).$$

$$\tan\varphi_1 = \frac{cZ\,2\pi v/L}{kZ - mZ(2\pi v/L)^2}, \tan\varphi_2 = \frac{cA 2\pi v/L}{kA}.$$

对应项应当相等，即

$$Z\sqrt{\left(k - m(2\pi v/L)^2\right)^2 + (c2\pi v/L)^2} = A\sqrt{k^2 + (c2\pi v/L)^2},$$

$$-\phi + \varphi_1 = \varphi_2.$$

因此车辆运动稳态解的幅值和相位分别为

$$Z = \frac{A\sqrt{k^2 + (c2\pi v/L)^2}}{\sqrt{\left(k - m(2\pi v/L)^2\right)^2 + (c2\pi v/L)^2}},$$

$$\tan\phi = \tan(\varphi_1 - \varphi_2) = \frac{\tan\varphi_1 - \tan\varphi_2}{1 + \tan\varphi_1\tan\varphi_2} = \frac{\dfrac{c2\pi v/L}{k - m(2\pi v/L)^2} - \dfrac{c2\pi v/L}{k}}{1 + \dfrac{c2\pi v/L}{k - m(2\pi v/L)^2}\dfrac{c2\pi v/L}{k}}.$$

利用 $\omega_n = \sqrt{k/m}$ 以及 $\omega = 2\pi v/L$ 对上式进行化简，

$$Z = \frac{A\sqrt{1 + (2\xi\,\omega/\omega_n)^2}}{\sqrt{\left(1 - (\omega/\omega_n)^2\right)^2 + (2\xi\,\omega/\omega_n)^2}},$$

$$\tan\phi = \frac{2\xi(\omega/\omega_n)^3}{1 - (\omega/\omega_n)^2 + (2\xi\,\omega/\omega_n)^2}.$$

由 Newton 第二定律可知，传递到车辆上的力可以表示为

$$
\begin{aligned}
& f = m\ddot{z} = -\omega^2 Z \sin(\omega t - \phi), \\
& \omega = 2\pi v / L, \\
& Z = \frac{A\sqrt{1 + (2\xi\,\omega/\omega_n)^2}}{\sqrt{\left(1 - (\omega/\omega_n)^2\right)^2 + (2\xi\,\omega/\omega_n)^2}}, \\
& \phi = \arctan\frac{2\xi(\omega/\omega_n)^3}{1 - (\omega/\omega_n)^2 + (2\xi\,\omega/\omega_n)^2}.
\end{aligned}
$$

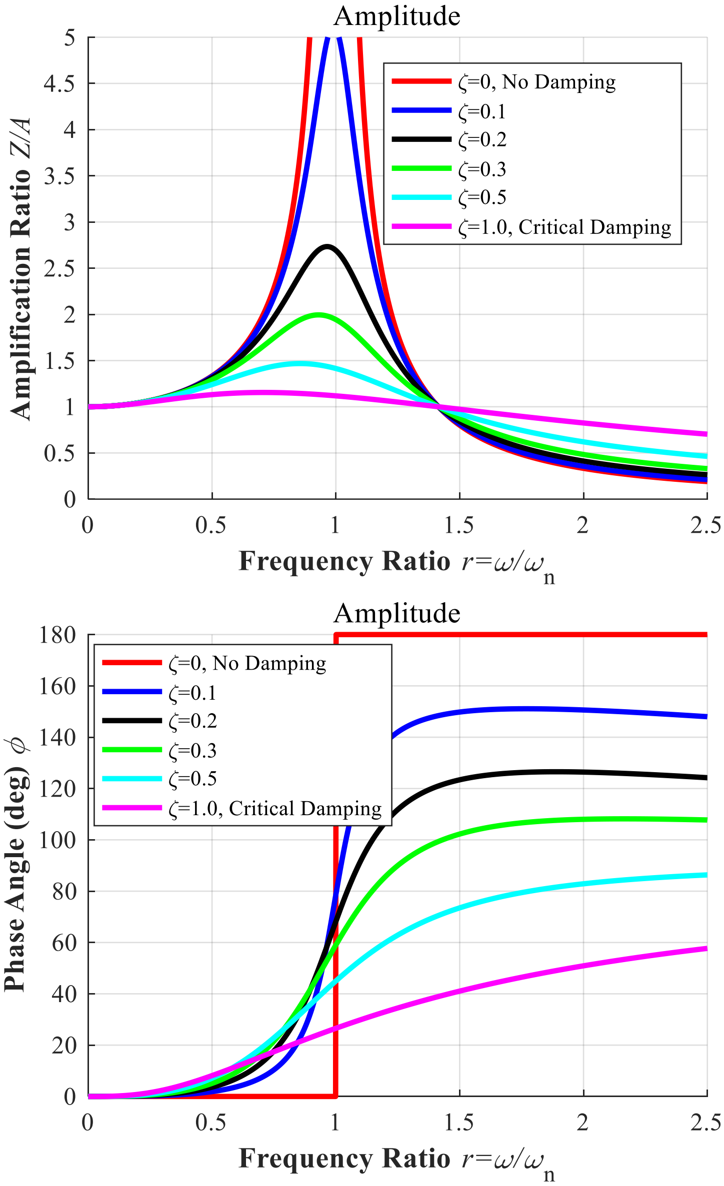


Fig 4. The amplitude and phase of the steady-state responses.

注：本题也可以分别求解右侧两项的稳态解，通过叠加原理得到系统的稳态解。

RESPONSE UNDER RANDOM PERIODIC EXCITATION VIA FOURIER ANALYSIS

Problem 10: Please solve Problem 3.19 in page 156 of the textbook. (15 points)

3.19 Solve the differential equation

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=kf(t)$$

by means of a Fourier analysis, where $f(t)$ is the periodic function shown in Fig. 3.36.

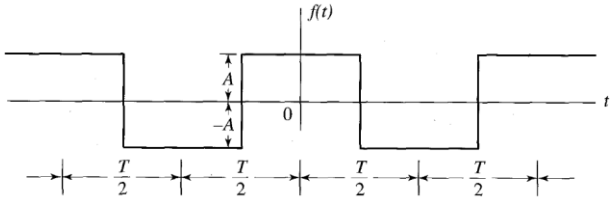


FIGURE 3.36

解：将激励 $f(t)$ 进行 Fourier 展开

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{2\pi nt}{T}\right)+b_n\sin\left(\frac{2\pi nt}{T}\right)\right),$$

其中

$$a_0=\frac{2}{T}\int_0^T f(t)\mathrm{d}t=0,$$

$$a_n=\frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi nt}{T}\right)\mathrm{d}t=\frac{4A}{n\pi}\sin\frac{n\pi}{2},$$

$$b_n=\frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi nt}{T}\right)\mathrm{d}t=0.$$

因此有，

$$f(t)=\sum_{n=1}^{\infty}\left(\frac{4A}{n\pi}\sin\frac{n\pi}{2}\cos\left(\frac{2\pi nt}{T}\right)\right).$$

根据叠加原理可以得到方程 $\ddot{x}+2\xi\omega_n\dot{x}+\omega_n^2x=\omega_n^2 f(t)$ 的稳态解为

$$x(t)=\frac{4A}{\pi}\sum_{n=1,3,5\cdots}^{\infty}\frac{\sin\frac{n\pi}{2}}{n\sqrt{\left(1-n^2\lambda^2\right)^2+(2\xi n\lambda)}}\cos\left(n\omega t-\phi_n\right),$$

其中，

$$\omega=\frac{2\pi}{T},\ \omega_n=\sqrt{\frac{k}{m}},\ \lambda=\frac{\omega}{\omega_n},\ \xi=\frac{c}{2m\omega_n},\ \phi_n=\arctan\frac{2\xi n\lambda}{1-n^2\lambda^2}.$$

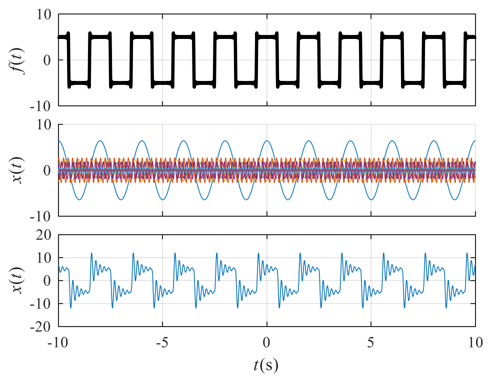


Fig 5. The amplitude and phase of the steady-state responses.

注：绘图时需要注意相位角的范围在 $[0,\pi]$ 之间。

RESPONSE UNDER RANDOM NON-PERIODIC EXCITATION VIA SUPERPOSITION PRINCIPLE

Problem 11: Please solve Problem 4.6 in page 204 of the textbook. (10 points)

4.6 Use the superposition principle to derive the response of a viscously damped single-degree-of-freedom system to the triangular pulse shown in Fig. 4.32.

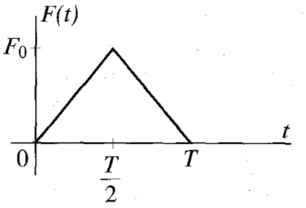


FIGURE 4.32

解：三角形脉冲可以表示成斜坡函数的求和形式

$$F(t)=\frac{F_0}{T}\left(2r(t)-4r\left(t-\frac{T}{2}\right)+2r\left(t-T\right)\right).$$

由叠加原理得到三角形脉冲激励下的响应为

$$x(t)=\frac{F_0}{T}\left(2v(t)-4v\left(t-\frac{T}{2}\right)+2v\left(t-T\right)\right),$$

其中$v\left(t\right)$表示斜坡函数的响应，由卷积定理可以将其表示成阶跃函数响应$h\left(t\right)$的积分

$$\begin{aligned}v(t)&=\int_0^t h\left(\tau\right)\mathrm{d}\tau\\&=\int_0^t\frac{1}{k}\left[1-e^{-\xi\omega_n\tau}\left(\cos\omega_d\tau+\frac{\xi\omega_n}{\omega_d}\sin\omega_d\tau\right)\right]\mathrm{d}\tau\\&=\frac{1}{k}\left(t-\frac{2\xi}{\omega_n}+\frac{e^{-\xi\omega_n t}}{\omega_n}\left(2\xi\cos\omega_d t+\frac{\left(\xi\omega_n\right)^2-\omega_d^2}{\omega_d\omega_n}\sin\omega_d t\right)\right)u\left(t\right).\end{aligned}$$

其中$u\left(t\right)$为阶跃函数。据此求得$v\left(t-T/2\right)$及$v\left(t-T\right)$分别为，

$$v\left(t-T/2\right)=\frac{1}{k}\left(t-T/2-\frac{2\xi\omega_n}{\omega_d^2+\left(\xi\omega_n\right)^2}+\frac{e^{-\xi\omega_n(t-T/2)}}{\omega_d^2+\left(\xi\omega_n\right)^2}\left(2\xi\omega_n\cos\omega_d\left(t-T/2\right)+\frac{\left(\xi\omega_n\right)^2-\omega_d^2}{\omega_d}\sin\omega_d\left(t-T/2\right)\right)\right)u\left(t-T/2\right),$$

$$v\left(t-T\right)=\frac{1}{k}\left(t-T-\frac{2\xi}{\omega_n}+\frac{e^{-\xi\omega_n(t-T)}}{\omega_n^2}\left(2\xi\omega_n\cos\omega_d\left(t-T\right)+\frac{\left(\xi\omega_n\right)^2-\omega_d^2}{\omega_d}\sin\omega_d\left(t-T\right)\right)\right)u\left(t-T\right).$$

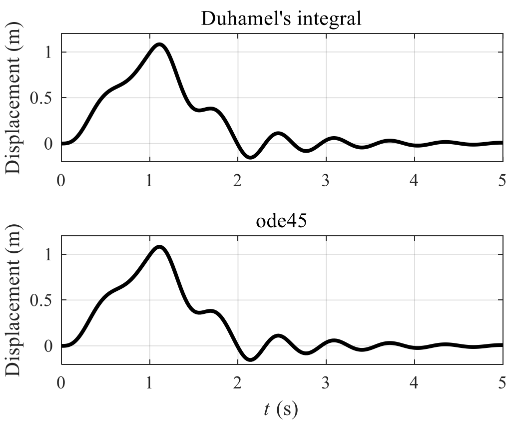


Fig 6. The transient response obtained by the superposition principle and ode45.

RESPONSE UNDER RANDOM NON-PERIODIC EXCITATION VIA CONVOLUTION INTEGRAL

Problem 12: Please solve Problem 4.10 in page 205 of the textbook. (10 points)

4.10 Derive and plot the response of the system of Problem 4.9 to the force $F(t)=F_0\left(1-e^{-\alpha t}\right)u(t)$ by means of the convolution integral. Then, use Eq. (4.22) to plot the response to a step function of magnitude $F_0=200\ \mathrm{N}$, compare results and draw conclusions.

4.9 Derive the response of a viscously damped single-degree-of-freedom system to the force $F(t)=F_0e^{-\alpha t}u(t)$ by means of the convolution integral. Plot the response for the system parameters $m=12\ \mathrm{kg}$, $c=24\ \mathrm{N\cdot s/m}$, $k=4800\ \mathrm{N/m}$ and the force parameters $F_0=200\ \mathrm{N}$, $\alpha=1$.

解：单自由度系统在脉冲函数下的响应为

$$g(t)=\frac{1}{m\omega_d}e^{-\xi\omega_n t}\sin(\omega_d t).$$

通过卷积积分得到系统的响应

$$\begin{aligned}
x(t)&=\int_{-\infty}^{t}F(\tau)g(t-\tau)\mathrm{d}\tau\\
&=\int_0^t F_0\left(1-e^{-\alpha\tau}\right)\frac{1}{m\omega_d}e^{-\xi\omega_n(t-\tau)}\sin\left(\omega_d(t-\tau)\right)\mathrm{d}\tau\\
&=\frac{F_0}{m\omega_d}\int_0^t\left(e^{-\xi\omega_n(t-\tau)}-e^{-\alpha\tau}e^{-\xi\omega_n(t-\tau)}\right)\frac{e^{i\omega_d(t-\tau)}-e^{-i\omega_d(t-\tau)}}{2i}\mathrm{d}\tau\\
&=\frac{F_0}{2im\omega_d}\int_0^t\left(e^{-\xi\omega_n(t-\tau)}-e^{-\alpha\tau}e^{-\xi\omega_n(t-\tau)}\right)\left(e^{i\omega_d(t-\tau)}-e^{-i\omega_d(t-\tau)}\right)\mathrm{d}\tau\\
&=\frac{F_0}{2im\omega_d}\int_0^t\left[e^{i\omega_d(t-\tau)-\xi\omega_n(t-\tau)}-e^{i\omega_d(t-\tau)-\alpha\tau-\xi\omega_n(t-\tau)}-e^{-\xi\omega_n(t-\tau)-i\omega_d(t-\tau)}+e^{-\alpha\tau-\xi\omega_n(t-\tau)-i\omega_d(t-\tau)}\right]\mathrm{d}\tau\\
&=\frac{F_0}{2im\omega_d}\left[\begin{aligned}&\frac{1}{-i\omega_d+\xi\omega_n}e^{i\omega_d(t-\tau)-\xi\omega_n(t-\tau)}-\frac{1}{-i\omega_d+\xi\omega_n-\alpha}e^{i\omega_d(t-\tau)-\alpha\tau-\xi\omega_n(t-\tau)}\\&-\frac{1}{\xi\omega_n+i\omega_d}e^{-\xi\omega_n(t-\tau)-i\omega_d(t-\tau)}+\frac{1}{-\alpha+\xi\omega_n+i\omega_d}e^{-\alpha\tau-\xi\omega_n(t-\tau)-i\omega_d(t-\tau)}\end{aligned}\right]_0^t\\
&=\frac{F_0}{2im\omega_d}\left[\begin{aligned}&\frac{1}{-i\omega_d+\xi\omega_n}-\frac{1}{\xi\omega_n+i\omega_d}\\&+\left(-\frac{1}{-i\omega_d+\xi\omega_n-\alpha}+\frac{1}{-\alpha+\xi\omega_n+i\omega_d}\right)e^{-\alpha t}\end{aligned}\right]-\frac{F_0}{2im\omega_d}\left[\begin{aligned}&\left(\frac{1}{-i\omega_d+\xi\omega_n}-\frac{1}{-i\omega_d+\xi\omega_n-\alpha}\right)e^{i\omega_d t-\xi\omega_n t}\\&+\left(-\frac{1}{\xi\omega_n+i\omega_d}+\frac{1}{-\alpha+\xi\omega_n+i\omega_d}\right)e^{-\xi\omega_n t-i\omega_d t}\end{aligned}\right]\\
&=\frac{F_0}{m}\left[\frac{1}{\omega_n^2}+\frac{-1}{(-\alpha+\xi\omega_n)^2+(\omega_d)^2}e^{-\alpha t}\right]-\frac{F_0}{2im\omega_d}\left[\begin{aligned}&\frac{-\alpha}{(-i\omega_d+\xi\omega_n)(-i\omega_d+\xi\omega_n-\alpha)}\left(\cos(\omega_d t)+i\sin(\omega_d t)\right)\\&+\frac{\alpha}{(-\alpha+\xi\omega_n+i\omega_d)(\xi\omega_n+i\omega_d)}\left(\cos(\omega_d t)-i\sin(\omega_d t)\right)\end{aligned}\right]e^{-\xi\omega_n t}\\
&=\frac{F_0}{m}\left[\frac{1}{\omega_n^2}+\frac{-1}{(-\alpha+\xi\omega_n)^2+(\omega_d)^2}e^{-\alpha t}\right]-\frac{F_0}{2im\omega_d}\left[\begin{aligned}&\frac{-4i\alpha\xi\omega_n\omega_d+2i\alpha^2\omega_d}{\omega_n^2\left((-\alpha+\xi\omega_n)^2+(\omega_d)^2\right)}\cos(\omega_d t)\\&-\frac{2\alpha\left((\xi\omega_n)^2-(\omega_d)^2\right)-2\xi\alpha^2\omega_n}{\omega_n^2\left((-\alpha+\xi\omega_n)^2+(\omega_d)^2\right)}i\sin(\omega_d t)\end{aligned}\right]e^{-\xi\omega_n t}\\
&=\frac{F_0}{m}\left[\frac{1}{\omega_n^2}+\frac{-1}{(-\alpha+\xi\omega_n)^2+(\omega_d)^2}e^{-\alpha t}\right]+\frac{F_0}{m\omega_d}\frac{1}{\omega_n^2\left((-\alpha+\xi\omega_n)^2+(\omega_d)^2\right)}\left[\begin{aligned}&\left(2\alpha\xi\omega_n\omega_d-\alpha^2\omega_d\right)\cos(\omega_d t)\\&+\left(\alpha\left((\xi\omega_n)^2-(\omega_d)^2\right)-\xi\alpha^2\omega_n\right)\sin(\omega_d t)\end{aligned}\right]e^{-\xi\omega_n t}.
\end{aligned}$$

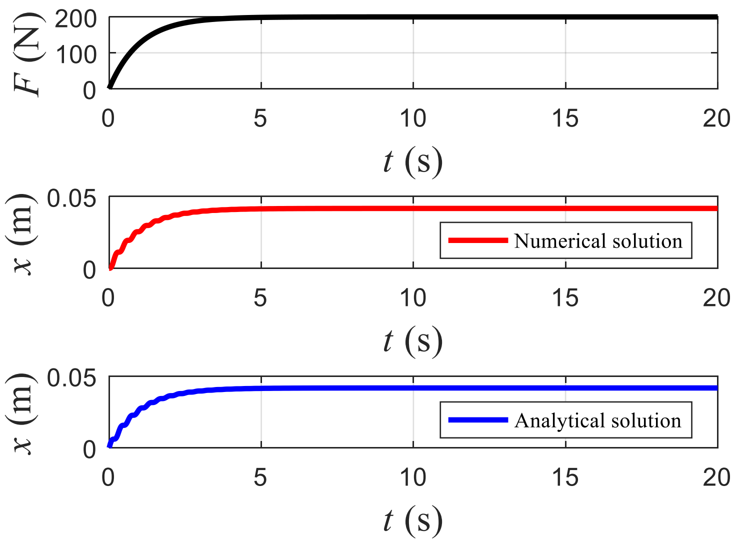


Fig 7: Response of the system by means of the convolution integral.

阶跃函数的响应为

$$h(t)=\frac{F_0}{k}\left[1-e^{-\xi\omega_n t}\left(\cos\omega_d t+\frac{\xi\omega_n}{\omega_d}\sin\omega_d t\right)\right]u(t).$$

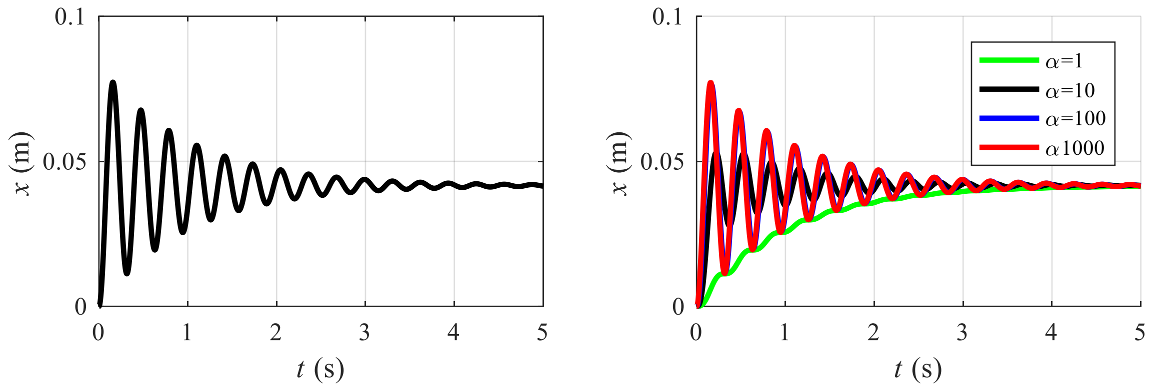


Fig 8: Response of the system to a step function.

比较：随着 α 的增大，激励 $F(t)=F_0\left(1-e^{-\alpha t}\right)u(t)$ 的响应趋近于阶跃函数的响应。

结论：可以通过取一个较大的 α 值，将阶跃函数 $F(t)=F_0u(t)$ 变成连续的函数 $F(t)=F_0\left(1-e^{-\alpha t}\right)u(t)$。

ADDITIONAL PROBLEMS (INVERSE PROBLEM, LAPLACE TRANSFORMATION)

Additional Problem 1: Please solve Problem 2.25 in page 108 of the textbook. (10 points)

2.25. Measurements of the response peak amplitudes of a vibrating single-degree-of-freedom system are as follows: $x_1 = 24.86$, $x_2 = 22.98$, $x_3 = 21.49$, $x_4 = 20.55$, $x_5 = 19.14$, $x_6 = 17.80$, $x_7 = 17.03$ and $x_8 = 15.97$. Use the least squares method described in Sec. 2.3 to determine the "best" viscous damping factor ζ.

解：在理想情况下，单自由度系统瞬态响应衰减时峰值的对数是呈线性的，因此可以构造如下

$$\ln \hat{x}_j = a y_j + b.$$

与实际数据差距的平方和为

$$\varepsilon = \sum_{j=1}^{8}\left(\ln x_j - \ln \hat{x}_j\right)^2 = \sum_{j=1}^{8}\left(\ln x_j - a y_j - b\right)^2.$$

要使上式最小，应当满足

$$\frac{\partial \varepsilon}{\partial a} = -2\sum_{j=1}^{8}\left(\ln x_j - a y_j - b\right) y_j = 0,$$

$$\frac{\partial \varepsilon}{\partial b} = -2\sum_{j=1}^{8}\left(\ln x_j - a y_j - b\right) = 0.$$

可以得到关于未知常数 a 与 b 的关系式，

$$\left(\sum_{j=1}^{8} y_j^2\right)a + \left(\sum_{j=1}^{8} y_j\right)b = \left(\sum_{j=1}^{8} y_j \ln x_j\right),$$

$$\left(\sum_{j=1}^{8} y_j\right)a + \left(\sum_{j=1}^{8} 1\right)b = \left(\sum_{j=1}^{8} \ln x_j\right).$$

代入上述数值有，

$$140a + 28b = 80.95,$$

$$28a + 8b = 23.88.$$

可得 a 与 b 的值为

$$\ln \hat{x}_j = -0.062 y_j + 3.20.$$

拟合结果如下

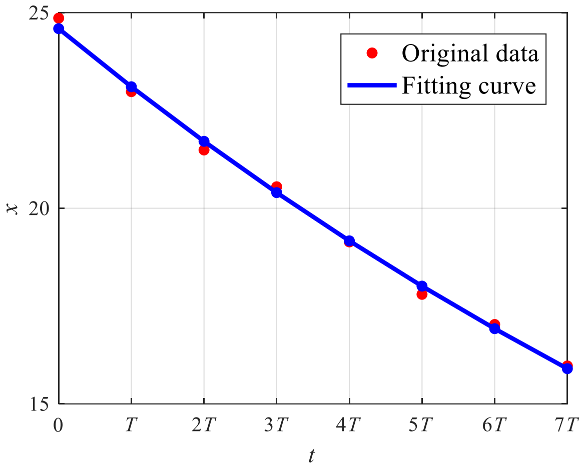


Fig 9. Original data and fitted curve.

可以得到

$$\delta = -a = 0.062,$$

$$\xi = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{0.062}{\sqrt{(2\pi)^2 + 0.062^2}} = 0.0099.$$

Additional Problem 2: Please read Section 4.6 of the textbook and solve Problem 4.18 in page 206 of the textbook. (10 points)

4.18. Solve Problem 4.16 for the sawtooth pulse shown in Fig. 4.20.

4.16. Derive the response of a viscously damped single-degree-of-freedom system to the force $F(t)=F_0e^{-\alpha t}u(t)$ by means of the Laplace transformation method.

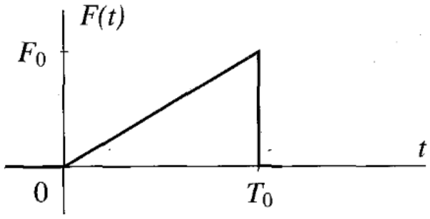


FIGURE 4.20

解：系统的运动方程为，

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=F(t).$$

外激励有如下的形式，

$$F(t)=\begin{cases}\dfrac{F_0}{T_0}t, 0<t<T_0\\ 0, t\geq T_0\end{cases}.$$

系统的响应可以写成如下逆 Laplace 变换，

$$x(t)=\mathbf{L}^{-1}\left(G(s)F(s)\right).$$

其中$G(s)$表示传递函数，

$$G(s)=\frac{1}{ms^2+cs+k}=\frac{1}{2mi\omega_d}\left(\frac{1}{s-(\xi\omega_n+i\omega_d)}-\frac{1}{s-(\xi\omega_n-i\omega_d)}\right).$$

其中 $\omega_n=\sqrt{k/m},\omega_d=\sqrt{1-\xi^2}\omega_n,\xi=c/2m\omega_n$。$F(s)$表示激励的 Laplace 变换，

$$F(s)=\int_0^\infty e^{-st}F(t)\mathrm{d}t=\frac{F_0}{T_0}\int_0^{T_0}e^{-st}t\,\mathrm{d}t=\frac{F_0}{T_0}\left(\frac{te^{-st}}{-s}\bigg|_0^{T_0}-\int_0^{T_0}\frac{e^{-st}}{-s}\mathrm{d}t\right)$$

$$=\frac{F_0}{T_0}\left(\frac{te^{-st}}{-s}-\frac{e^{-st}}{(-s)^2}\right)\bigg|_0^{T_0}=\frac{F_0}{T_0}\left(\frac{T_0e^{-sT_0}}{-s}-\frac{e^{-sT_0}}{s^2}+\frac{1}{s^2}\right).$$

因此

$$X(s)=G(s)F(s)=\frac{F_0}{2mi\omega_dT_0}\left(\frac{1}{s-(-\xi\omega_n+i\omega_d)}-\frac{1}{s-(-\xi\omega_n-i\omega_d)}\right)\left(\frac{T_0e^{-sT_0}}{-s}-\frac{e^{-sT_0}}{s^2}+\frac{1}{s^2}\right).$$

取 $s_1=-\xi\omega_n+i\omega_d,s_2=-\xi\omega_n-i\omega_d$，则

$$X(s)=\frac{F_0}{2mi\omega_dT_0}\left(\frac{1}{s-s_1}-\frac{1}{s-s_2}\right)\left(\frac{T_0e^{-sT_0}}{-s}-\frac{e^{-sT_0}}{s^2}+\frac{1}{s^2}\right).$$

分别计算

$$\mathbf{L}^{-1}\left(\frac{1}{s(s-s_0)}\right)=\frac{1}{s_0}\mathbf{L}^{-1}\left(\frac{1}{(s-s_0)}-\frac{1}{s}\right)=\frac{1}{s_0}\left(e^{s_0t}-1\right),$$

$$\mathbf{L}^{-1}\left(\frac{1}{s^2(s-s_0)}\right)=\frac{1}{s_0}\mathbf{L}^{-1}\left(\frac{1}{s(s-s_0)}-\frac{1}{s^2}\right)=\frac{1}{s_0^2}\mathbf{L}^{-1}\left(\frac{1}{(s-s_0)}-\frac{1}{s}\right)+\frac{1}{s_0}\mathbf{L}^{-1}\left(-\frac{1}{s^2}\right)=\frac{1}{s_0^2}\left(e^{s_0t}-1\right)-\frac{t}{s_0}.$$

分别用 s_1,s_2 代替上式中的 s_0，可以得到最终解的形式为

$$
\begin{aligned}
x(t) &= \mathbf{L}^{-1}\left(X(s)\right) = \mathbf{L}^{-1}\left[\frac{F_0}{2mi\omega_d T_0}\left(\frac{1}{s-s_1}-\frac{1}{s-s_2}\right)\left(\frac{T_0 e^{-sT_0}}{-s}-\frac{e^{-sT_0}}{s^2}+\frac{1}{s^2}\right)\right] \\
&= \frac{F_0}{2mi\omega_d T_0}\mathbf{L}^{-1}\left(\frac{1}{s-s_1}\frac{T_0 e^{-sT_0}}{-s}+\frac{1}{s-s_1}\frac{e^{-sT_0}}{-s^2}+\frac{1}{s-s_1}\frac{1}{s^2}-\frac{1}{s-s_2}\frac{T_0 e^{-sT_0}}{-s}-\frac{1}{s-s_2}\frac{e^{-sT_0}}{-s^2}-\frac{1}{s-s_2}\frac{1}{s^2}\right) \\
&= \frac{F_0}{2mi\omega_d T_0}\left[\begin{array}{l} -\frac{T_0}{s_1}\left(e^{s_1(t-T_0)}-1\right)u(t-T_0)-\frac{1}{s_1^2}\left(e^{s_1(t-T_0)}-s_1 t-1\right)u(t-T_0)+\frac{1}{s_1^2}\left(e^{s_1 t}-s_1 t-1\right)u(t) \\ +\frac{T_0}{s_2}\left(e^{s_2(t-T_0)}-1\right)u(t-T_0)+\frac{1}{s_2^2}\left(e^{s_2(t-T_0)}-s_2 t-1\right)u(t-T_0)-\frac{1}{s_2^2}\left(e^{s_2 t}-s_2 t-1\right)u(t) \end{array}\right] \\
&= \frac{F_0}{2mi\omega_d T_0}\left[\begin{array}{l} T_0\left(-\frac{e^{s_1(t-T_0)}-1}{s_1}+\frac{e^{s_2(t-T_0)}-1}{s_2}\right)u(t-T_0)+\left(\frac{e^{s_1 t}-s_1 t-1}{s_1^2}-\frac{e^{s_2 t}-s_2 t-1}{s_2^2}\right)u(t) \\ \left(-\frac{e^{s_1(t-T_0)}-s_1 t-1}{s_1^2}+\frac{e^{s_2(t-T_0)}-s_2 t-1}{s_2^2}\right)u(t-T_0) \end{array}\right].
\end{aligned}
$$

取值为$T_0=10\ \mathrm{s}$，$F_0=100\ \mathrm{N}$，$m=1\,\mathrm{kg}$，$c=1\ \mathrm{N\cdot s/m}$，$k=100\ \mathrm{N/m}$时得到数值解与解析解一致。

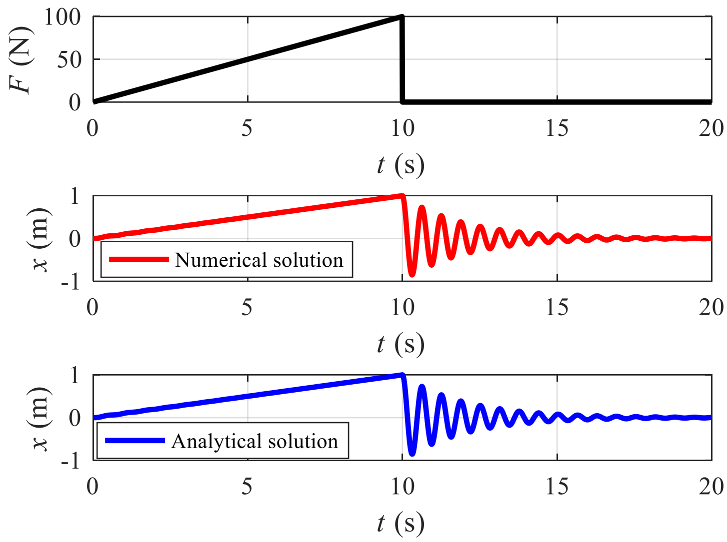


Fig 10. Response of the system by ode45 and by means of the Laplace transformation method.