题 1

利用行波法（参看数理方程）求解波动方程的解

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0 & x\in\mathbb{R}, t>0 \\ u(x,0) & = \varphi(x) \\ u_t(x,0) & = \psi(x) \end{cases} \tag{1}$$

解

令

$$\begin{cases} \xi = x + at \\ \eta = x - at \end{cases} \tag{2}$$

- 物理含义: $u = f(\xi) + f(\eta)$ 波可以分解为两个方向
- $(1) \Leftrightarrow \left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right)u = 0$ 猜想(6)

则

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \\ &= \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \end{aligned} \tag{3}$$

进而有

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\right) \\ &= \left[\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right)\right] + \left[\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right)\right] \\ &= \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \\ &= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \end{aligned} \tag{4}$$

将$\frac{\partial u}{\partial \xi}$当作$u$代入(3)

同理可得

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right) \tag{5}$$

代入方程可得

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \tag{6}$$

可解得

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial\xi\partial\eta} = 0 \\
&\Rightarrow \int \frac{\partial^2 u}{\partial\xi\partial\eta}\,\mathrm{d}\xi = \int 0\,\mathrm{d}\xi \\
&\Rightarrow \frac{\partial u}{\partial\eta} = g(\eta) \\
&\Rightarrow \int \frac{\partial u}{\partial\eta}\,\mathrm{d}\eta = \int g(\eta)\,\mathrm{d}\eta \\
&\Rightarrow u = \int g(\eta)\,\mathrm{d}\eta + f(\xi) \\
&\Rightarrow u = F(\xi) + G(\eta)
\end{aligned} \tag{7}
$$

所以有

$$
u(x,t) = F(x+at) + G(x-at) \tag{8}
$$

代入初始条件则有

$$
\begin{cases} F(x) + G(x) & = \varphi(x) \\ a[F'(x) - G'(x)] & = \psi(x) \end{cases} \tag{9}
$$

对(9) 中 2 式两边积分则有

$$
F(x) - G(x) = \frac{1}{a}\int \psi(x)\,\mathrm{d}x + C \tag{10}
$$

联立 (10) 和 (9), 有

$$
\begin{cases} F(x) = \frac{1}{2}\varphi(x) + \frac{1}{2a}\int \psi(x)\,\mathrm{d}x + C \\ G(x) = \frac{1}{2}\varphi(x) - \frac{1}{2a}\int \psi(x)\,\mathrm{d}x + C \end{cases} \tag{11}
$$

代入到 (8) , 有

$$
\begin{aligned}
u(x,t) &= F(x+at) + G(x-at) \\
&= \frac{1}{2}\varphi(x+at) + \frac{1}{2a}\int_0^{x+at} \psi(y)\,\mathrm{d}y + \frac{1}{2}\varphi(x-at) - \frac{1}{2a}\int_0^{x-at} \psi(y)\,\mathrm{d}y + C \\
&= \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(y)\,\mathrm{d}y + C
\end{aligned} \tag{12}
$$

References

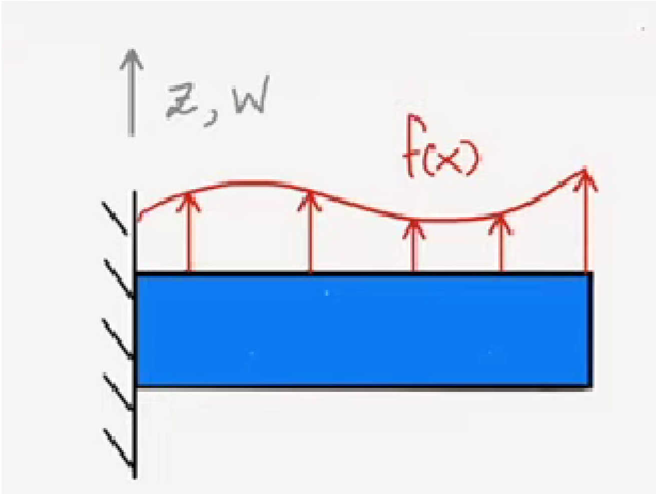
- 无界波动方程的定解——行波法
- 偏微分方程基础——特征线法/行波法/达朗贝尔法

题 2

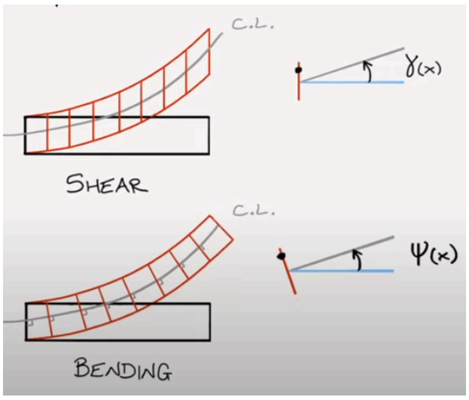
利用微元法建立均质铁摩辛科（Timoshienko）梁无外激的线性自由振动线性方程

解

考虑一般形式



slope of center line



形变为弯矩和剪力的叠加

$$\frac{\partial w}{\partial x} = \gamma + \varphi \tag{13}$$

displacements

$$\begin{aligned} u_x(x, y, z, t) &= -z\varphi(x, t) \\ u_y(x, y, z, t) &= 0 \\ u_z(x, y, z, t) &= w(x, t) \end{aligned} \tag{14}$$

where u_x, u_y, u_z are the components of the displacement vector in the three coordinate directions

strain

由 [Infinitesimal strain theory] 知

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad u_{i,j} = \frac{\partial u_i}{\partial j} \tag{15}$$

可得

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{2}(u_{x,x} + u_{x,x}) \\ &= u_{x,x} = \frac{\partial u_x}{\partial x} \\ &= -z\frac{\partial \varphi}{\partial x} \end{aligned} \tag{16}$$

$$
\begin{aligned}
\varepsilon_{xz} &= \frac{1}{2}(u_{x,z} + u_{z,x}) = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial z}\right) \\
&= \frac{1}{2}\left(-\varphi + \frac{\partial w}{\partial x}\right) \\
&= \frac{1}{2}\gamma
\end{aligned}
\tag{17}
$$

依据 (13)

同理

$$
\begin{aligned}
\varepsilon_{yy} &= 0 \\
\varepsilon_{zz} &= 0 \\
\varepsilon_{yz} &= 0 \\
\varepsilon_{xy} &= 0
\end{aligned}
\tag{18}
$$

由 Hooke's law:

$$
\sigma_{xx} = E\varepsilon_{xx} \tag{19}
$$

泊松比$\nu = 0$

xz为角度变形, 设剪切系数为k, 则

$$
\begin{aligned}
\sigma_{xz} &= G\gamma(xz) \\
&= kG\gamma(x)
\end{aligned}
\tag{20}
$$

- ε被称为 tensor strain
- γ被称为 engineering strain

二者的关系由(17) 联系

Hooke's law 矩阵为 tensor strain 形式

Hamilton

由 Hamilton's principle 知

$$
\int_{t_1}^{t_2} (\delta T - \delta U + \delta W_E)\,\mathrm{d}t = 0 \tag{21}
$$

strain energy

$$
U = \int_V W\,\mathrm{d}V \tag{22}
$$

由 Strain energy density function

$$
W = \frac{1}{2}\sum\sum \sigma_{ij}\varepsilon_{ij} \tag{23}
$$

代入 (16) (17) (18) (19) (20) :

$$
\begin{aligned}
W &= \frac{1}{2}\sum\sum \sigma_{ij}\varepsilon_{ij} \\
&= \frac{1}{2}\sigma_{xx}\varepsilon_{xx} + \sigma_{xz}\varepsilon_{xz} \\
&= \frac{1}{2}E\varepsilon_{xx}^2 + \frac{1}{2}kG\gamma^2 \\
&= \frac{1}{2}Ez^2\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{1}{2}kG\gamma^2
\end{aligned}
\tag{24}
$$

所以 (22) 变为

$$
\begin{aligned}
U &= \int_V \left(\frac{1}{2} E z^2 \left(\frac{\partial \varphi}{\partial x} \right)^2 + \frac{1}{2} k G \gamma^2 \right) \mathrm{d}V \\
&= \frac{1}{2} \int_0^l \int_A \left(E z^2 \left(\frac{\partial \varphi}{\partial x} \right)^2 + k G \gamma^2 \right) \mathrm{d}A \mathrm{d}x \\
&= \frac{1}{2} \left[\left(\int_0^l E \left(\int_A z^2 \, \mathrm{d}A \right) \frac{\partial \varphi}{\partial x} \, \mathrm{d}x \right) + \int_0^l k G A \gamma^2 \, \mathrm{d}x \right] \\
&= \frac{1}{2} \int_0^l E I \left(\frac{\partial \varphi}{\partial x} \right)^2 \mathrm{d}x + \frac{1}{2} \int_0^l k G A \gamma^2 \, \mathrm{d}x
\end{aligned}
\tag{25}
$$

$I = \int_A z^2 \, \mathrm{d}A$

代入 (13) , 可得

$$
U = \frac{1}{2} \int_0^l E I \left(\frac{\partial \varphi}{\partial x} \right)^2 \mathrm{d}x + \frac{1}{2} \int_0^l k G A \left(\frac{\partial w}{\partial x} - \varphi \right)^2 \mathrm{d}x \tag{26}
$$

故

$$
\begin{aligned}
\delta U &= \int_0^l E I \frac{\partial \varphi}{\partial x} \delta \left(\frac{\partial \varphi}{\partial x} \right) \mathrm{d}x + \int_0^l k G A \left(\frac{\partial w}{\partial x} - \varphi \right) \left(\delta \left(\frac{\partial w}{\partial x} \right) - \delta \varphi \right) \mathrm{d}x \\
&= \int_0^l E I \frac{\partial \varphi}{\partial x} \delta \left(\frac{\partial \varphi}{\partial x} \right) \mathrm{d}x \\
&\quad + \int_0^l k G A \left(\frac{\partial w}{\partial x} - \varphi \right) \delta \left(\frac{\partial w}{\partial x} \right) \mathrm{d}x \\
&\quad - \int_0^l k G A \left(\frac{\partial w}{\partial x} - \varphi \right) \delta \varphi \, \mathrm{d}x \\
&= E I \frac{\partial \varphi}{\partial x} \delta \varphi \, |_0^l - \int_0^l \frac{\partial}{\partial x} \left(E I \frac{\partial \varphi}{\partial x} \right) \delta \varphi \, \mathrm{d}x \\
&\quad k G A \left(\frac{\partial w}{\partial x} - \varphi \right) \delta w \, |_0^l - \int_0^l \frac{\partial}{\partial x} \left[k G A \left(\frac{\partial w}{\partial x} - \varphi \right) \right] \delta w \, \mathrm{d}w \\
&\quad - \int_0^l k G A \left(\frac{\partial w}{\partial x} - \varphi \right) \delta \varphi \, \mathrm{d}x
\end{aligned}
\tag{27}
$$

kinetic energy

translatinal kinetic energy + rotational kinetic energy

$$
T = \frac{1}{2} \int_0^l \rho A \left(\frac{\partial w}{\partial t} \right)^2 \mathrm{d}x + \frac{1}{2} \int_0^l \rho I \left(\frac{\partial \varphi}{\partial t} \right)^2 \mathrm{d}x \tag{28}
$$

故

$$
\delta T = \int_0^l m \frac{\partial w}{\partial t} \delta \left(\frac{\partial w}{\partial t} \right) \mathrm{d}x + \int_0^l J \frac{\partial \varphi}{\partial t} \delta \left(\frac{\partial \varphi}{\partial t} \right) \mathrm{d}x \tag{29}
$$

$m = \rho A, J = \rho I$

则

$$
\begin{aligned}
\int_{t_1}^{t_2} \delta T &= \int_{t_1}^{t_2} \left[\int_0^l m \frac{\partial w}{\partial t} \delta \left(\frac{\partial w}{\partial t} \right) \mathrm{d}x + \int_0^l J \frac{\partial \varphi}{\partial t} \delta \left(\frac{\partial \varphi}{\partial t} \right) \mathrm{d}x \right] \mathrm{d}t \\
&= \int_0^l \int_{t_1}^{t_2} m \frac{\partial w}{\partial t} \delta \left(\frac{\partial w}{\partial t} \right) \mathrm{d}t \, \mathrm{d}x + \int_0^l \int_{t_1}^{t_2} J \frac{\partial \varphi}{\partial t} \delta \left(\frac{\partial \varphi}{\partial t} \right) \mathrm{d}t \, \mathrm{d}x \\
&= \int_0^l \left(m \frac{\partial w}{\partial t} \delta w \mid_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(m \frac{\partial w}{\partial t} \right) \delta w \, \mathrm{d}t \right) \mathrm{d}x \\
&\quad + \int_0^l \left(J \frac{\partial \varphi}{\partial t} \delta \varphi \mid_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(J \frac{\partial \varphi}{\partial t} \right) \delta \varphi \, \mathrm{d}t \right) \mathrm{d}x \\
&= - \int_0^l \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(m \frac{\partial w}{\partial t} \right) \delta w \, \mathrm{d}t \, \mathrm{d}x - \int_0^l \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(J \frac{\partial \varphi}{\partial t} \right) \delta \varphi \, \mathrm{d}t \, \mathrm{d}x \\
&= - \int_{t_1}^{t_2} \int_0^l \frac{\partial}{\partial t} \left(m \frac{\partial w}{\partial t} \right) \delta w \, \mathrm{d}x \, \mathrm{d}t - \int_{t_1}^{t_2} \int_0^l \frac{\partial}{\partial t} \left(J \frac{\partial \varphi}{\partial t} \right) \delta \varphi \, \mathrm{d}x \, \mathrm{d}t
\end{aligned} \tag{30}
$$

external work

$$
W_E = \int_0^l q w \, \mathrm{d}x \tag{31}
$$

故

$$
\delta W_E = \int_0^l q \delta w \, \mathrm{d}x \tag{32}
$$

代入 **Hamilton**

由 (27) (30) (32) 可得

$$
\begin{aligned}
\int_{t_1}^{t_2}(\delta T-\delta U+\delta W_E)\,\mathrm{d}t = & -\int_{t_1}^{t_2}\int_0^l \frac{\partial}{\partial t}\left(m\frac{\partial w}{\partial t}\right)\delta w\,\mathrm{d}x\,\mathrm{d}t \\
& -\int_{t_1}^{t_2}\int_0^l \frac{\partial}{\partial t}\left(J\frac{\partial \varphi}{\partial t}\right)\delta \varphi\,\mathrm{d}x\,\mathrm{d}t \\
& -\int_{t_1}^{t_2}\left(EI\frac{\partial \varphi}{\partial x}\delta\varphi\mid_0^l-\int_0^l\frac{\partial}{\partial x}\left(EI\frac{\partial \varphi}{\partial x}\right)\delta\varphi\,\mathrm{d}x\right. \\
& kGA\left(\frac{\partial w}{\partial x}-\varphi\right)\delta w\mid_0^l-\int_0^l\frac{\partial}{\partial x}\left[kGA\left(\frac{\partial w}{\partial x}-\varphi\right)\right]\delta w\,\mathrm{d}w \\
& \left.-\int_0^l kGA\left(\frac{\partial w}{\partial x}-\varphi\right)\delta\varphi\,\mathrm{d}x\right)\mathrm{d}t \\
& +\int_{t_1}^{t_2}\int_0^l q\delta w\,\mathrm{d}x\,\mathrm{d}t \\
= & \int_{t_1}^{t_2}\int_0^l\left[\left(-\frac{\partial}{\partial t}\left(m\frac{\partial w}{\partial t}\right)+\frac{\partial}{\partial x}\left[kGA\left(\frac{\partial w}{\partial x}-\varphi\right)\right]+q\right)\delta w\right. \\
& \left.\left(-\frac{\partial}{\partial t}\left(J\frac{\partial \varphi}{\partial t}\right)+\frac{\partial}{\partial x}\left(EI\frac{\partial \varphi}{\partial x}\right)+kGA\left(\frac{\partial w}{\partial x}-\varphi\right)\right)\delta\varphi\right]\mathrm{d}x\,\mathrm{d}t \\
& -EI\frac{\partial \varphi}{\partial x}\delta\varphi\mid_0^l-kGA\left(\frac{\partial w}{\partial x}-\varphi\right)\delta w\mid_0^l \\
= & \, 0
\end{aligned}
\tag{33}
$$

由任意性可得

$$
\begin{cases}
-\frac{\partial}{\partial t}\left(m\frac{\partial w}{\partial t}\right)+\frac{\partial}{\partial x}\left[kGA\left(\frac{\partial w}{\partial x}-\varphi\right)\right]+q & =0 \\
-\frac{\partial}{\partial t}\left(J\frac{\partial \varphi}{\partial t}\right)+\frac{\partial}{\partial x}\left(EI\frac{\partial \varphi}{\partial x}\right)+kGA\left(\frac{\partial w}{\partial x}-\varphi\right) & =0
\end{cases}
\tag{34}
$$

边界条件为

$$
\begin{cases}
EI\frac{\partial \varphi}{\partial x}\delta\varphi\mid_0^l & =0 \\
kGA\left(\frac{\partial w}{\partial x}-\varphi\right)\delta w\mid_0^l & =0
\end{cases}
\tag{35}
$$

根据均质条件

由(34) 可得

$$
-m\frac{\partial^2 w}{\partial t^2}+kGA\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial \varphi}{\partial x}\right)+q=0 \tag{36}
$$

$$
-J\frac{\partial^2 \varphi}{\partial t^2}+EI\frac{\partial^2 \varphi}{\partial x^2}+kGA\left(\frac{\partial w}{\partial x}-\varphi\right)=0 \tag{37}
$$

现将(36) 代入(37) 消掉 φ

由 (36) 可得

$$
\begin{cases}
\frac{\partial\varphi}{\partial x} &= \frac{q}{kGA} - \frac{m}{kGA}\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} \\
\frac{\partial^3\varphi}{\partial x^3} &= \frac{1}{kGA}\frac{\partial^2 q}{\partial x^2} - \frac{m}{kGA}\frac{\partial^4 w}{\partial x^2\partial t^2} + \frac{\partial^3 w}{\partial x^3} \\
\frac{\partial^3\varphi}{\partial x\partial t^2} &= \frac{1}{kGA}\frac{\partial^2 q}{\partial t^2} - \frac{m}{kGA}\frac{\partial^4 w}{\partial t^4} + \frac{\partial^4 w}{\partial x^2\partial t^2}
\end{cases}
\tag{38}
$$

由 (37) 可得

$$
J\frac{\partial^3 w}{\partial x\partial t^2} = EI\frac{\partial^3 w}{\partial x^3} + kGA\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\varphi}{\partial x}\right) = 0 \tag{39}
$$

将(38) 代入(39) 可得

$$
EI\frac{\partial^4 w}{\partial x^4} + m\frac{\partial^2 w}{\partial t^2} - \left(J + \frac{mEI}{kGA}\right)\frac{\partial^4 w}{\partial x^2\partial t^2} + \frac{mJ}{kGA}\frac{\partial^4 w}{\partial t^4} = q + \frac{J}{kGA}\frac{\partial^2 q}{\partial t^2} - \frac{EI}{kGA}\frac{\partial^2 q}{\partial x^2} \tag{40}
$$

由题意, 外部激励$q = 0$, 所以

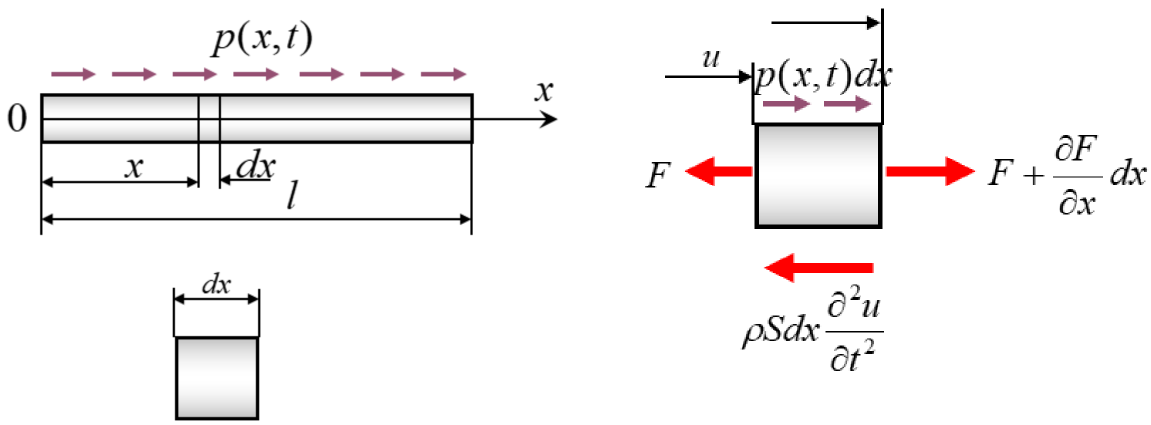
$$
EI\frac{\partial^4 w}{\partial x^4} + m\frac{\partial^2 w}{\partial t^2} - \left(J + \frac{mEI}{kGA}\right)\frac{\partial^4 w}{\partial x^2\partial t^2} + \frac{mJ}{kGA}\frac{\partial^4 w}{\partial t^4} = 0 \tag{41}
$$

Reference

- Structural Dynamics: 很不错的合集 关于 Timoshenko Beam 的里面:
  - Timoshenko Beam Theory Part 2 of 3: Hamilton's Principle
  - Timoshenko Beam Theory Part 3 of 3: Equations of Motion
- Timoshenko–Ehrenfest beam theory

题 3

建立均质杆纵向振动的几何非线性振动方程（建模方法自由选择）



根据泰勒公式, 保留应变一定的高阶项, 则有

$$
\varepsilon = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 \tag{42}
$$

则轴向力为

$$
\begin{aligned}
F &= \sigma S \\
&= E\varepsilon S \\
&= ES\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right]
\end{aligned} \tag{43}
$$

由达朗贝尔原理可知

$$
\rho S \,\mathrm{d}x \frac{\partial^2 u}{\partial t^2} = \left(F + \frac{\partial F}{\partial x}\,\mathrm{d}x\right) - F + p(x,t)\,\mathrm{d}x \tag{44}
$$

代入可得

$$
\rho S \frac{\partial^2 u}{\partial t^2} = ES\frac{\partial^2 u}{\partial x^2}\left(1 + \frac{\partial u}{\partial x}\right) + p(x,t) \tag{45}
$$

存在$ES\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial x}$项, 为非线性项, 故为非线性方程