2　对于平方和立方非线性共存的系统

$$\begin{cases} \ddot{u}_1+\omega^2u_1+2\mu_1\dot{u}_1+a_{11}u_1^2+a_{12}u_1u_2+a_{13}u_2^2+ \\ \quad b_{11}u_1^3+b_{12}u_1^2u_2+b_{13}u_1u_2^2+b_{14}u_2^3=0 \\ \ddot{u}_2+\omega^2u_2+2\mu_2\dot{u}_2+a_{21}u_1^2+a_{22}u_1u_2+a_{23}u_2^2+ \\ \quad b_{21}u_1^3+b_{22}u_1^2u_2+b_{23}u_1u_2^2+b_{24}u_2^3=0 \end{cases}$$

求其自由振动的一次近似解。

解

阻尼为弱阻尼, 设

$$\mu_r=\varepsilon*\overline{\mu}_r \tag{1}$$

用多尺度法, 设

$$\begin{cases} u_1=\varepsilon u_{11}+\varepsilon^2u_{12} \\ u_2=\varepsilon u_{21}+\varepsilon^2u_{22} \end{cases} \tag{2}$$

代入原方程得

$$\begin{cases} D_0^2u_{11}+\omega^2u_{11}=0 \\ D_0^2u_{21}+\omega^2u_{21}=0 \end{cases} \tag{3}$$

$$\begin{cases} D_0^2u_{12}+\omega^2u_{12}=-2D_0D_1u_{11}-2\overline{\mu}_1D_0u_{11}-a_{11}u_{11}^2-a_{12}u_{11}u_{21}-a_{13}u_{21}^2 \\ D_0^2u_{22}+\omega^2u_{22}=-2D_0D_1u_{21}-2\overline{\mu}_2D_0u_{21}-a_{21}u_{11}^2-a_{22}u_{11}u_{21}-a_{23}u_{21}^2 \end{cases} \tag{4}$$

解得 (3)

$$\begin{cases} u_{11}=A_1e^{i\omega T_0}+\mathrm{cc} \\ u_{21}=A_2e^{i\omega T_0}+\mathrm{cc} \end{cases} \tag{5}$$

代入 (4) 得

$$\begin{cases} D_0^2u_{12}+\omega^2u_{12}=\left(-2i\omega D_1A_1-2\overline{\mu}_1i\omega A_1\right)e^{i\omega T_0}-(a_{11}A_1^2+a_{12}A_1A_2+a_{13}A_2^2)e^{i2\omega T_0} \\ \qquad -\left(a_{11}A_1\overline{A}_1+a_{12}A_1\overline{A}_2+a_{13}A_2\overline{A}_2\right)+\mathrm{cc} \\ D_0^2u_{22}+\omega^2u_{22}=\left(-2i\omega D_1A_2-2\overline{\mu}_2i\omega A_2\right)e^{i\omega T_0}-(a_{21}A_1^2+a_{22}A_1A_2+a_{23}A_2^2)e^{i2\omega T_0} \\ \qquad -\left(a_{21}A_1\overline{A}_1+a_{22}A_1\overline{A}_2+a_{23}A_2\overline{A}_2\right)+\mathrm{cc} \end{cases} \tag{6}$$

消除永年项

$$D_1A_r+\overline{\mu}_rA_r=0 \tag{7}$$

所以

$$A_r=a_re^{-\overline{\mu}_rT_1} \tag{8}$$

即

$$
\begin{aligned}
u_r &= \varepsilon A_r e^{i\omega T_0} + \mathrm{cc} \\
&= \varepsilon a_r e^{-\overline{\mu}_r \varepsilon t} \cos(\omega t + \varphi_r)
\end{aligned} \tag{9}
$$

3 考察由两个耦合 van der Pol 振子组成的自激振动系统

$$
\begin{cases}
\ddot{u}_1 + \omega_1^2 u_1 = \varepsilon(1-u_1^2)\dot{u}_1 + \varepsilon a_1 u_2 \\
\ddot{u}_2 + \omega_2^2 u_2 = \varepsilon(1-u_2^2)\dot{u}_2 + \varepsilon a_2 u_1
\end{cases}
$$

其中 $\omega_1 \approx \omega_2$，求系统自由振动的一次近似解。

解

用多尺度法, 设

$$
\begin{cases}
u_1 = \varepsilon u_{11} + \varepsilon^2 u_{12} \\
u_2 = \varepsilon u_{21} + \varepsilon^2 u_{22}
\end{cases} \tag{10}
$$

代入原方程得

$$
\begin{cases}
D_0^2 u_{11} + \omega_1^2 u_{11} = 0 \\
D_0^2 u_{21} + \omega_2^2 u_{21} = 0
\end{cases} \tag{11}
$$

$$
\begin{cases}
D_0^2 u_{12} + \omega_1^2 u_{12} = -2D_0 D_1 u_{11} + (1-u_{11}^2) D_0 u_{11} + a_1 u_{21} \\
D_0^2 u_{22} + \omega_2^2 u_{22} = -2D_0 D_1 u_{21} + (1-u_{21}^2) D_0 u_{21} + a_2 u_{11}
\end{cases} \tag{12}
$$

解得 (11)

$$
\begin{cases}
u_{11} = A_1 e^{i\omega_1 T_0} + \mathrm{cc} \\
u_{21} = A_2 e^{i\omega_2 T_0} + \mathrm{cc}
\end{cases} \tag{13}
$$

代入 (12) 得

$$
\begin{cases}
D_0^2 u_{12} + \omega^2 u_{12} = \left(-2i\omega D_1 A_1 + i\omega_1 A_1 - A_1^2 \overline{A}_1 i\omega_1\right) e^{i\omega_1 T_0} \\
\qquad\qquad -A_1^3 i\omega_1 e^{i3\omega_1 T_0} + a_1 A_2 e^{i\omega_2 T_0} + \mathrm{cc} \\
D_0^2 u_{22} + \omega^2 u_{22} = \left(-2i\omega D_1 A_2 + i\omega_2 A_2 - A_2^2 \overline{A}_2 i\omega_2\right) e^{i\omega_2 T_0} \\
\qquad\qquad -A_2^3 i\omega_2 e^{i3\omega_2 T_0} + a_2 A_1 e^{i\omega_1 T_0} + \mathrm{cc}
\end{cases} \tag{14}
$$

知$\omega_1 \approx \omega_2$, 可设

$$
\omega_1 = \omega_2 + \varepsilon\sigma \tag{15}
$$

消除永年项

$$
\begin{cases}
-\left(2D_1 A_1 - A_1 + A_1^2 \overline{A}_1\right) i\omega + a_1 A_2 e^{i\sigma T_1} = 0 \\
-\left(2D_1 A_2 - A_2 + A_2^2 \overline{A}_2\right) i\omega + a_2 A_1 e^{i\sigma T_1} = 0
\end{cases} \tag{16}
$$

分离虚实部

$$
\begin{cases}
D_1\alpha_1-\frac{\alpha_1}{2}+\frac{\alpha_1^3}{8}-\frac{a_1\alpha_2}{2\omega_1}\sin(\beta_2-\beta_1-\sigma T_1)=0\\
\alpha_1 D_1\beta_1+\frac{a_1\alpha_2}{2\omega_1}\cos(\beta_2-\beta_1-\sigma T_1) \qquad =0\\
D_1\alpha_2-\frac{\alpha_2}{2}+\frac{\alpha_2^3}{8}-\frac{a_2\alpha_1}{2\omega_2}\sin(\beta_1-\beta_2+\sigma T_1)=0\\
\alpha_2 D_1\beta_2+\frac{a_2\alpha_1}{2\omega_2}\cos(\beta_1-\beta_2+\sigma T_1) \qquad =0
\end{cases}
\tag{17}
$$

令$\varphi=\beta_2-\beta_1-\sigma T_1$

$$
\begin{cases}
D_1\alpha_1 \qquad =\frac{\alpha_1}{2}-\frac{\alpha_1^3}{8}+(a_1\alpha_2)\sin\varphi\\
D_1\alpha_2 \qquad =\frac{\alpha_2}{2}-\frac{\alpha_2^3}{8}+(a_2\alpha_1)\sin\varphi\\
\alpha_1\alpha_2 D_1\varphi=\left(\frac{a_1\alpha_2^2}{2\omega_1}-\frac{a_2\alpha_1^2}{2\omega_2}\right)\cos\varphi-\alpha_1\alpha_2\sigma
\end{cases}
\tag{18}
$$

求解三维自治方程得

$$
\begin{cases}
u_1=\alpha_1\cos(\omega_1 t+\beta_1)\\
u_2=\alpha_2\cos(\omega_2 t+\beta_2)
\end{cases}
\tag{19}
$$

6　考察处于内共振条件下的立方非线性参激振动系统

$$
\begin{cases}
\ddot{u}_1+\omega_1^2u_1+2\mu_1\dot{u}_1+a_{11}u_1^3+a_{12}u_1^2u_2+a_{13}u_1u_2^2+a_{14}u_2^3\\
\qquad +2(b_{11}u_1+b_{12}u_2)\cos\omega t=0\\
\ddot{u}_2+\omega_2^2u_2+2\mu_2\dot{u}_2+a_{21}u_1^3+a_{22}u_1^2u_2+a_{23}u_1u_2^2+a_{24}u_2^3\\
\qquad +2(b_{21}u_1+b_{22}u_2)\cos\omega t=0
\end{cases}
$$

其中$\omega_2\approx3\omega_1$。分别求下述三种条件下系统参激振动的一次近似解：

(1)　$\omega\approx2\omega_1$；

(2)　$\omega\approx2\omega_2$；

(3)　$\omega\approx\omega_1+\omega_2$。

解

由微小假设

$$
\begin{aligned}
\mu_r&=\overline{\mu}_r\\
a_r&=\overline{a}_r\\
b_r&=\overline{b}_r
\end{aligned}
\tag{20}
$$

用多尺度法, 设

$$
\begin{cases}
u_1=\varepsilon u_{11}+\varepsilon^2u_{12}\\
u_2=\varepsilon u_{21}+\varepsilon^2u_{22}
\end{cases}
\tag{21}
$$

代入原方程得

$$
\begin{cases}
D_0^2 u_{11} + \omega_1^2 u_{11} = 0 \\
D_0^2 u_{21} + \omega_2^2 u_{21} = 0
\end{cases} \tag{22}
$$

$$
\begin{cases}
D_0^2 u_{12} + \omega_1^2 u_{12} = -2D_0 D_1 u_{11} - 2\overline{\mu}_1 D_0 u_{11} - \overline{a}_{11} u_{11}^3 - \overline{a}_{12} u_{11}^2 u_{21} - \overline{a}_{14} u_{21}^3 \\
\qquad\qquad -\left(\overline{b}_{11} u_{11} + \overline{b}_{12} u_{21}\right) e^{i\omega T_0} + \mathrm{cc} \\
D_0^2 u_{22} + \omega_2^2 u_{22} = -2D_0 D_1 u_{21} - 2\overline{\mu}_2 D_0 u_{21} - \overline{a}_{21} u_{11}^3 - \overline{a}_{22} u_{11}^2 u_{21} - \overline{a}_{24} u_{21}^3 \\
\qquad\qquad -\left(\overline{b}_{21} u_{11} + \overline{b}_{22} u_{21}\right) e^{i\omega T_0} + \mathrm{cc}
\end{cases} \tag{23}
$$

解得 (22)

$$
\begin{cases}
u_{11} = A_1 e^{i\omega_1 T_0} + \mathrm{cc} \\
u_{21} = A_2 e^{i\omega_2 T_0} + \mathrm{cc}
\end{cases} \tag{24}
$$

代入 (23) 得

$$
\begin{cases}
D_0^2 u_{12} + \omega^2 u_{12} = \left(-2i\omega D_1 A_1 - 2\overline{\mu}_1 i\omega_1 A_1 - 3\overline{a}_{11} A_1^2 \overline{A}_1 - 2\overline{a}_{13} A_1 A_2 \overline{A}_2\right) e^{i\omega_1 T_0} \\
\qquad -\left(2\overline{a}_{12} A_1 \overline{A}_1 A_2 + 3\overline{a}_{14} A_2^2 \overline{A}_2\right) e^{i\omega_2 T_0} - \overline{a}_{11} A_1^3 i\omega_1 e^{i3\omega_1 T_0} \\
\qquad -\overline{a}_{12} A_1^2 A_2 e^{i(2\omega_1+\omega_2)T_0} - \overline{a}_{12} \overline{A}_1^2 A_2 e^{i(\omega_2-2\omega_1)T_0} - \overline{a}_{13} A_1 A_2^2 e^{i(\omega_1+2\omega_2)T_0} \\
\qquad -\overline{a}_{13} A_1 \overline{A}_2^2 e^{i(\omega_1-2\omega_2)T_0} - \overline{a}_{14} A_2^3 e^{i3\omega_2 T_0} \\
\qquad -\sum_{r=1}^3 \overline{b}_{1r} A_r \left(e^{i(\omega_r+\omega)T_0} + e^{i(\omega_r-\omega)T_0}\right) + \mathrm{cc} \\
D_0^2 u_{22} + \omega^2 u_{22} = \left(-2i\omega D_1 A_2 - 2\overline{\mu}_2 i\omega_2 A_2 - 3\overline{a}_{21} A_2^2 \overline{A}_2 - 2\overline{a}_{23} A_1 A_2 \overline{A}_1\right) e^{i\omega_2 T_0} \\
\qquad -\left(2\overline{a}_{22} A_2 \overline{A}_2 A_1 + 3\overline{a}_{24} A_1^2 \overline{A}_1\right) e^{i\omega_1 T_0} - \overline{a}_{21} A_2^3 i\omega_2 e^{i3\omega_2 T_0} \\
\qquad -\overline{a}_{22} A_2^2 A_1 e^{i(2\omega_2+\omega_1)T_0} - \overline{a}_{22} \overline{A}_2^2 A_1 e^{i(\omega_1-2\omega_2)T_0} - \overline{a}_{23} A_2 A_1^2 e^{i(\omega_2+2\omega_1)T_0} \\
\qquad -\overline{a}_{23} A_2 \overline{A}_1^2 e^{i(\omega_2-2\omega_1)T_0} - \overline{a}_{24} A_1^3 e^{i3\omega_1 T_0} \\
\qquad -\sum_{r=1}^3 \overline{b}_{2r} A_r \left(e^{i(\omega_r+\omega)T_0} + e^{i(\omega_r-\omega)T_0}\right) + \mathrm{cc}
\end{cases} \tag{25}
$$

由题意可知$\omega_2 \approx 3\omega_1$, 可设

$$
\omega_2 = 3\omega_1 + \varepsilon\sigma \tag{26}
$$

(1)

当$\omega \approx 2\omega_1$, 可设

$$
\omega = 2\omega_1 + \varepsilon\sigma_1 \tag{27}
$$

消除永年项

$$
\begin{cases}
-2i\omega_1 D_1 A_1 - 2\overline{\mu}_1 i\omega_1 A_1 - 3\overline{a}_{11} A_1^2 \overline{A}_1 - 2\overline{a}_{13} A_1 A_2 \overline{A}_2 - \overline{a}_{12} \overline{A}_1^2 A_2 e^{i\sigma T_1} - \overline{b}_{11} \overline{A}_1 e^{i\sigma T_1} \\
\qquad\qquad -\overline{b}_{12} A_2 e^{i(\sigma-\sigma_1)T_1} = 0 \\
-2i\omega_1 D_1 A_2 - 2\overline{\mu}_2 i\omega_2 A_2 - 3\overline{a}_{24} A_2^2 \overline{A}_2 - 2\overline{a}_{22} A_1 \overline{A}_1 A_2 - \overline{a}_{21} A_1^3 e^{i\sigma T_1} - \overline{b}_{21} A_1 e^{i\sigma_1 T_1} = 0
\end{cases} \tag{28}
$$

可以解得A_1, A_2, 有近似解

$$\begin{cases} u_1 = A_1 e^{i\omega_1 T_0} + \mathrm{cc} \\ u_2 = A_2 e^{i\omega_2 T_0} + \mathrm{cc} \end{cases} \tag{29}$$

(2)

当$\omega \approx 2\omega_2$, 可设

$$\omega = 2\omega_2 + \varepsilon\sigma_2 = 6\omega_1 + 2\varepsilon\sigma + \varepsilon\sigma_2 \tag{30}$$

消除永年项

$$\begin{cases} -2i\omega_1 D_1 A_1 - 2\overline{\mu}_1 i\omega_1 A_1 - 3\overline{a}_{11} A_1^2 \overline{A}_1 - 2\overline{a}_{13} A_1 A_2 \overline{A}_2 - \overline{a}_{12} \overline{A}_1^2 A_2 e^{i\sigma T_1} = 0 \\ -2i\omega_1 D_1 A_2 - 2\overline{\mu}_2 i\omega_2 A_2 - 3\overline{a}_{24} A_2^2 \overline{A}_2 - 2\overline{a}_{22} A_1 \overline{A}_1 A_2 - \overline{a}_{21} A_1^3 e^{-i\sigma T_1} \\ \qquad - \overline{b}_{22} \overline{A}_2 e^{i\sigma_2 T_1} = 0 \end{cases} \tag{31}$$

可以解得A_1, A_2, 有近似解

$$\begin{cases} u_1 = A_1 e^{i\omega_1 T_0} + \mathrm{cc} \\ u_2 = A_2 e^{i\omega_2 T_0} + \mathrm{cc} \end{cases} \tag{32}$$

(3)

当$\omega \approx \omega_1 + \omega_2$, 可设

$$\omega = \omega_1 + \omega_2 + \varepsilon\sigma_3 \tag{33}$$

消除永年项

$$\begin{cases} -2i\omega_1 D_1 A_1 - 2\overline{\mu}_1 i\omega_1 A_1 - 3\overline{a}_{11} A_1^2 \overline{A}_1 - 2\overline{a}_{13} A_1 A_2 \overline{A}_2 - \overline{a}_{12} \overline{A}_1^2 A_2 e^{i\sigma T_1} - \overline{b}_{22} \overline{A}_1 e^{i\sigma_3 T_1} = 0 \\ -2i\omega_1 D_1 A_2 - 2\overline{\mu}_2 i\omega_2 A_2 - 3\overline{a}_{24} A_2^2 \overline{A}_2 - 2\overline{a}_{22} A_1 \overline{A}_1 A_2 - \overline{a}_{21} A_1^3 e^{i\sigma T_1} - \overline{b}_{21} A_1 e^{i\sigma_3 T_1} = 0 \end{cases} \tag{34}$$

可以解得A_1, A_2, 有近似解

$$\begin{cases} u_1 = A_1 e^{i\omega_1 T_0} + \mathrm{cc} \\ u_2 = A_2 e^{i\omega_2 T_0} + \mathrm{cc} \end{cases} \tag{35}$$

已知 $\begin{cases} D_0^2 u_{11} - cD_0 u_{21} + k_{11} u_{11} + k_{12} u_{21} = 0 \\ D_0^2 u_{21} + cD_0 u_{11} + k_{21} u_{11} + k_{22} u_{21} = 0 \end{cases}$ **和** $\begin{cases} u_{11} = A_1(T_1) \mathrm{e}^{\mathrm{j}\omega_1 T_0} + A_2(T_1) \mathrm{e}^{\mathrm{j}\omega_2 T_0} + \mathrm{cc} \\ u_{21} = \Gamma_1 A_1(T_1) \mathrm{e}^{\mathrm{j}\omega_1 T_0} + \Gamma_2 A_2(T_1) \mathrm{e}^{\mathrm{j}\omega_2 T_0} + \mathrm{cc} \end{cases}$

证明 $j\omega_1, j\omega_2$ **是特征方程**

$$\det \begin{bmatrix} k_{11} + \lambda^2 & k_{12} - c\lambda \\ k_{21} + c\lambda & k_{22} + \lambda^2 \end{bmatrix} = \lambda^4 + (k_{11} + k_{22} + c^2)\lambda^2 + k_{11}k_{22} - k_{12}k_{21} = 0$$

的根，并且 $\Gamma_r \overset{\text{def}}{=} \dfrac{k_{12} - \mathrm{j}c\omega_r}{\omega_r^2 - k_{22}}, \qquad r = 1,2$

证明

令$A_2=0$, 则有

$$\begin{cases} u_{11}=A_1 e^{j\omega_1 T_0}+\mathrm{cc} \\ u_{21}=\Gamma_1 A_1 e^{j\omega_1 T_0}+\mathrm{cc} \end{cases} \tag{36}$$

代入方程组有

$$\begin{cases} (-\omega_1^2 A_1 - cj\omega_1 \Gamma_1 A_1 + k_{11}A_1 + k_{12}\Gamma_1 A_1)e^{j\omega_1 T_0}=0 \\ (-\omega_1^2 \Gamma_1 A_1 + cj\omega_1 A_1 + k_{21}A_1 + k_{22}\Gamma_1 A_1)e^{i\omega_1 T_0}=0 \end{cases} \tag{37}$$

令$\lambda=j\omega_1$, 则有

$$\begin{bmatrix} k_{11}+\lambda^2 & k_{12}-c\lambda \\ k_{21}+c\lambda & k_{22}+\lambda^2 \end{bmatrix} \begin{bmatrix} A_1 \\ \Gamma_1 A_1 \end{bmatrix}=0 \tag{38}$$

有非零解, 则

$$\begin{vmatrix} k_{11}+\lambda^2 & k_{12}-c\lambda \\ k_{21}+c\lambda & k_{22}+\lambda^2 \end{vmatrix}=0 \tag{39}$$

同时由

$$(k_{21}+c\lambda)A_1+(k_{22}+\lambda)\Gamma_1 A_1=0 \tag{40}$$

可得

$$\Gamma_1=-\frac{k_{21}+cj\omega_1}{\omega_1^2-k_{22}} \tag{41}$$

同理令$A_1=0$有

$$\begin{vmatrix} k_{11}+\lambda^2 & k_{12}-c\lambda \\ k_{21}+c\lambda & k_{22}+\lambda^2 \end{vmatrix}=0 \tag{42}$$

$$\Gamma_2=-\frac{k_{21}+cj\omega_2}{\omega_2^2-k_{22}} \tag{43}$$