题 1

求解并画图比较 $\varepsilon=0.1$和ε=0.2数值、一次和二次近似结构，其中 $a_0=1,\omega_0=1$, 图取 $2.5\le t\le 5$

例 3.1.2 用直接摄动法求解如下 Duffing 系统自由振动的一次近似

$$\begin{cases}\ddot{u}+\omega_0^2u+\varepsilon\omega_0^2u^3=0\\ u(0)=a_0,\quad \dot{u}(0)=0\end{cases}$$

$$u(t,\varepsilon)=u_0(t)+\varepsilon u_1(t)+\varepsilon^2u_2(t)+\cdots \qquad u_0=a_0\cos\omega_0 t$$

$$\begin{cases}\ddot{u}_1+\omega_0^2u_1=-\dfrac{\omega_0^2a_0^3}{4}(3\cos\omega_0 t+\cos 3\omega_0 t)\\ u_1(0)=0,\quad \dot{u}_1(0)=0\end{cases}$$

共振

$$u_1(t)=\frac{a_0^3}{32}(-\cos\omega_0 t+\cos 3\omega_0 t)-\frac{3\omega_0a_0^3}{8}t\sin\omega_0 t$$

$$u(t)=a_0\cos\omega_0 t+\varepsilon[\frac{a_0^3}{32}(-\cos\omega_0 t+\cos 3\omega_0 t)-\frac{3\omega_0a_0^3}{8}t\sin\omega_0 t]$$

永年项、长期项、久期项(secular term)

<1>

解

二次近似

$$u(\varepsilon,t)=u_0(t)+\varepsilon u_1(t)+\varepsilon^2u_2(t) \tag{1}$$

代入方程

$$(\ddot{u}_0+\varepsilon\ddot{u}_1+\varepsilon^2\ddot{u}_2)+\omega_0^2(u_0+\varepsilon u_1+\varepsilon^2u_2)+\varepsilon\omega_0^2(u_0+\varepsilon u_1+\varepsilon^2u_2)^3=0 \tag{2}$$

可得

$$\begin{cases}\ddot{u}_0+\omega_0^2u_0=0\\ \ddot{u}_1+\omega_0^2u_1=-\omega_0^2u_0^3\\ \ddot{u}_2+\omega_0^2u_2=-3\omega_0^2u_0^2u_1\end{cases} \tag{3}$$

迭代

解

$$\ddot{u}_0+\omega_0^2u_0=0 \tag{4}$$

得

$$u_0=A\cos\omega_0 t+B\sin\omega_0 t \tag{5}$$

代入初值$u(0)=a_0,\dot{u}(0)=0$得到

$$u_0=a_0\cos\omega_0 t \tag{6}$$

代入

$$\ddot{u}_1+\omega_0^2u_1=-\omega_0^2u_0^3 \tag{7}$$

得

$$\ddot{u}_1+\omega_0^2u_1=-\omega_0^2(a_0\cos\omega_0t)^3 \tag{8}$$

由$\cos 3\theta=4\cos^3\theta-3\cos\theta$ 得

$$\ddot{u}_1+\omega_0^2u_1=-\frac{1}{4}\omega_0^2a_0^3(\cos 3\omega_0t+3\cos\omega_0t) \tag{9}$$

结合初值

$$\begin{cases}u_1=0\\ \dot{u}_1=0\end{cases} \tag{10}$$

解得

$$u_1(t)=\frac{a^3}{32}(\cos 3\omega_0t-\cos\omega_0t-12\omega_0t\sin\omega_0t) \tag{11}$$

所以一次近似系统解为

$$\begin{aligned}u(t)&=u_0(t)+\varepsilon u_1(t)\\&=a_0\cos\omega_0t+\frac{\varepsilon a^3}{32}(\cos 3\omega_0t-\cos\omega_0t-12\omega_0t\sin\omega_0t)\end{aligned} \tag{12}$$

代入到

$$\ddot{u}_2+\omega_0^2u_2=-3\omega_0^2u_0^2u_1 \tag{13}$$

得到

$$\ddot{u}_2+\omega_0^2u_2=-\frac{3\omega_0^2a^3}{32}(a_0\cos\omega_0t)^2(\cos 3\omega_0t-\cos\omega_0t-12\omega_0t\sin\omega_0t) \tag{14}$$

结合初值

$$\begin{cases}u_2=0\\ \dot{u}_2=0\end{cases} \tag{15}$$

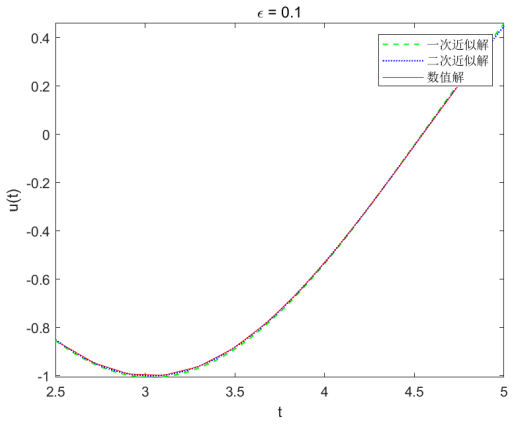
解得

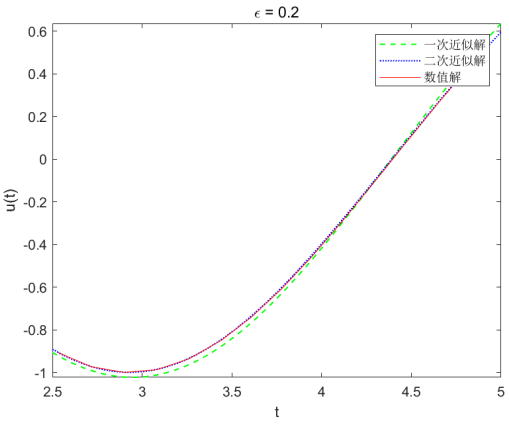
$$\begin{aligned}u_2(t)=\frac{a^5}{1024}&(\cos 5\omega_0t-24\cos 3\omega_0t+(23-72\omega_0^2t^2)\cos\omega_0t\\&-36\omega_0t\sin 3\omega_0t+96\omega_0t\sin\omega_0t)\end{aligned} \tag{16}$$

二次近似

$$\begin{aligned}u(t)&=u_0(t)+\varepsilon u_1(t)+\varepsilon^2u_2(t)\\&=a_0\cos\omega_0t+\frac{\varepsilon a^3}{32}(\cos 3\omega_0t-\cos\omega_0t-12\omega_0t\sin\omega_0t)\\&\quad+\frac{\varepsilon^2a^5}{1024}(\cos 5\omega_0t-24\cos 3\omega_0t+(23-72\omega_0^2t^2)\cos\omega_0t\\&\quad-36\omega_0t\sin 3\omega_0t+96\omega_0t\sin\omega_0t)\end{aligned} \tag{17}$$

比较


ϵ = 0.1
一次近似解
二次近似解
数值解
u(t)
t
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
2.5
3
3.5
4
4.5
5


ϵ = 0.2
一次近似解
二次近似解
数值解
u(t)
t
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
2.5
3
3.5
4
4.5
5