题 1

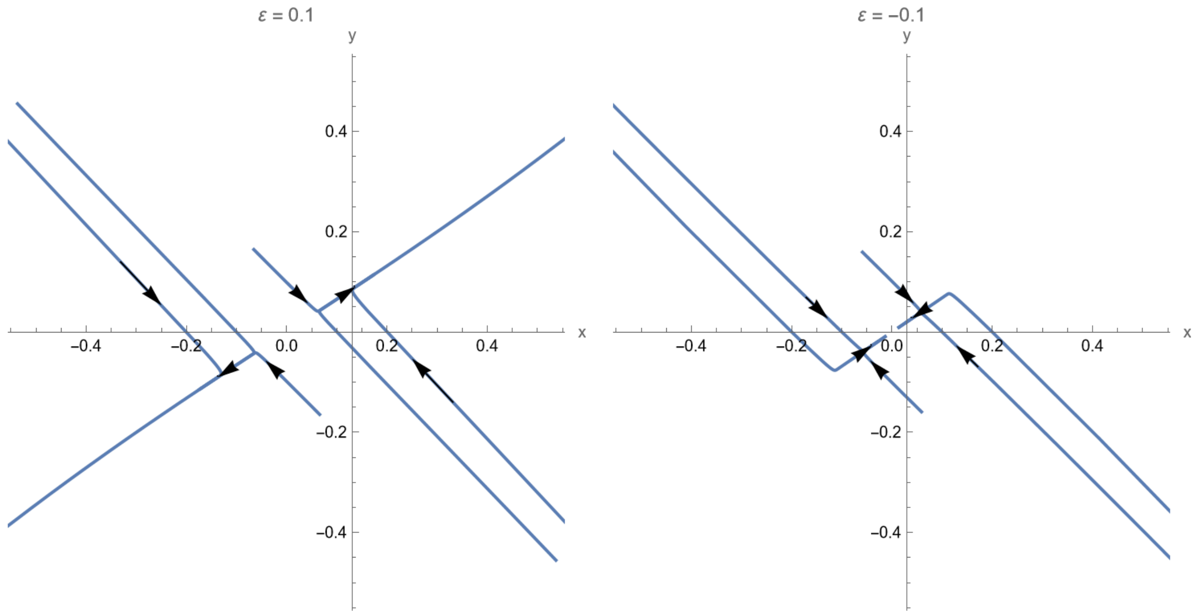
4. Consider the following parametrized families of vector fields with parameter $\varepsilon \in \mathbb{R}^1$. For $\varepsilon = 0$, the origin is a fixed point of each vector field. Study the dynamics near the origin for ε small. Draw phase portraits. Compute the one-parameter family of center manifolds and describe the dynamics on the center manifolds. How do the dynamics depend on ε? Note that, for $\varepsilon = 0$, e.g., a) and a$'$) reduce to a) in the previous exercise. Discuss the role played by a parameter by comparing these cases. In, for example, a) and a$'$), the parameter ε multiplies a linear and nonlinear term, respectively. Discuss the differences in these two cases in the most general setting possible.

f) $$\begin{aligned} \dot{x} &= -2x + 3y + \varepsilon x + y^3, \\ \dot{y} &= 2x - 3y + x^3, \end{aligned} \qquad (x, y) \in \mathbb{R}^2.$$

i) $$\begin{aligned} \dot{x} &= -2x + y + z + \varepsilon x - y^2 z, \\ \dot{y} &= x - 2y + z + \varepsilon x + x z^2, \\ \dot{z} &= x + y - 2z + \varepsilon x + x^2 y, \end{aligned} \qquad (x, y, z) \in \mathbb{R}^3.$$

4.f

Draw phase portraits



Compute the one-parameter family of center manifolds and describe the dynamics on the center manifolds

Jacobian Matrix at fixed point:

$$J = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$$

the corresponding eigenvalues and eigenvectors are:

eigenvalues	eigenvectors
$\lambda_1 = 0$	$u_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$
$\lambda_2 = -5$	$u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

adopt the following transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

then the system becomes:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \frac{1}{5} \begin{pmatrix} \varepsilon(3u+v) + (2u-v)^3 + (3u+v)^3 \\ 2\varepsilon(3u+v) + 2(2u-v)^3 - 3(3u+v)^3 \end{pmatrix}$$

the center manifold is locally represented by

$$W^{c(0)} = \left\{ (u, v, \varepsilon) \in \mathbb{R}^3 \mid v = h(u, \varepsilon), h(0,0) = 0, Dh(0,0) = 0 \right\}$$

Differentiating $v = h(u)$ with respect to time implies that the $(\dot{u}, \dot{v})$ coordinates of any point on $W^c(0)$ must satisfy

$$\dot{v} = Dh(u)\dot{u}$$

assuming that

$$h(u, \varepsilon) = a_1 u^2 + a_2 \varepsilon u + a_3 \varepsilon^2 + \cdots$$

substituting the equation above, it can be obtained that

$$-5h + \frac{1}{5}\left[2\varepsilon(3u+h) + 2(2u-h)^3 - 3(3u+h)^3\right] = \frac{1}{5}(2a_1 u + a_2 \varepsilon + \cdots)\left[\varepsilon(3u+h) + (2u-h)^3 + (3u+h)^3\right]$$

Equating coefficients of like powers to zero gives

$$\begin{aligned} u^2 &: 5a_1 = 0 \qquad \Rightarrow a_1 = 0 \\ \varepsilon u &: 5a_2 - \frac{6}{5} = 0 \Rightarrow a_2 = \frac{6}{25} \end{aligned}$$

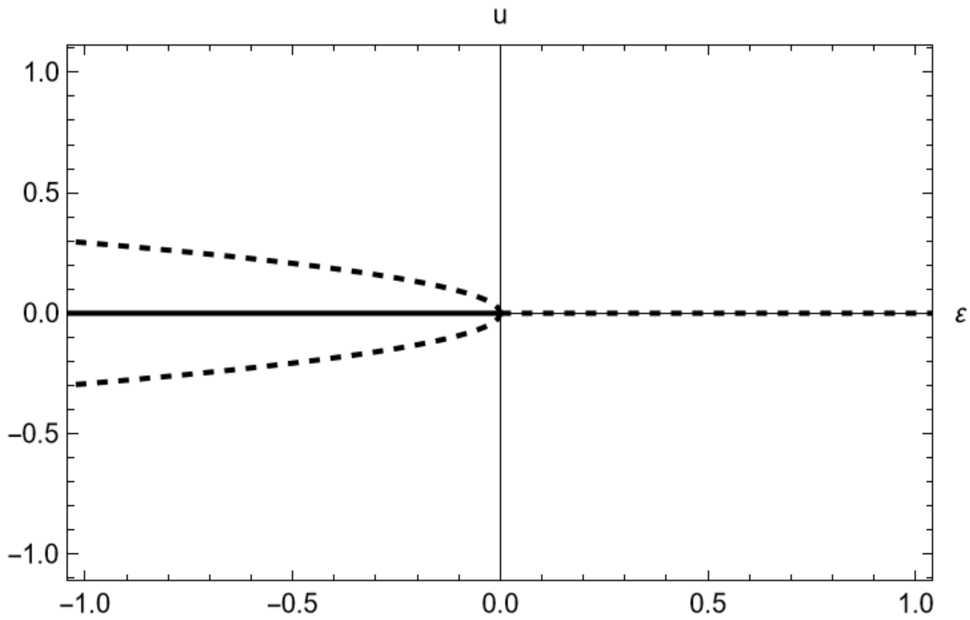
Thus, the center manifold is given by the graph of

$$h(u, \varepsilon) = \frac{6}{25}\varepsilon u + \cdots$$

the map restricted to the center manifold is given by

$$\dot{u} = \frac{1}{5}u(3\varepsilon + 35u^2 + \cdots)$$

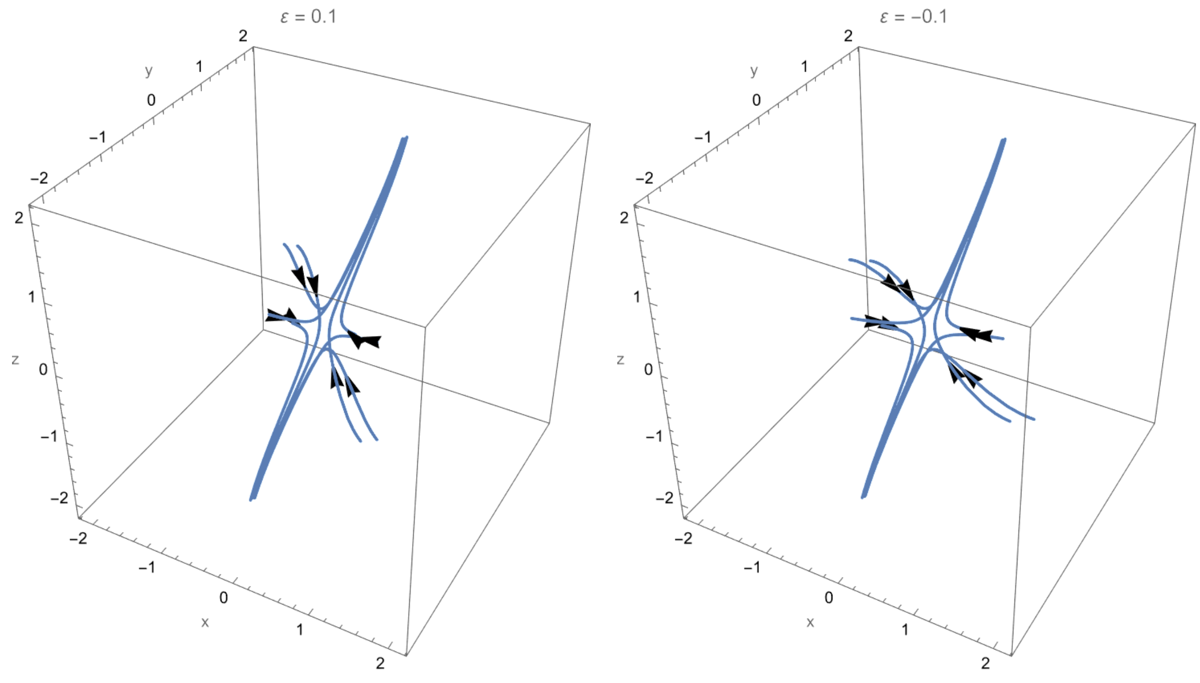
How do the dynamics depend on ε



- if $\varepsilon \geq 0$, there is only one unstable fixed point $(0, 0)$
- if $\varepsilon < 0$, $(0, 0)$ turns into a stable fixed point and two more unstable fixed points appear

4.i

Draw phase portraits



Compute the one-parameter family of center manifolds and describe the dynamics on the center manifolds

Jacobian Matrix at fixed point:

$$J = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

the corresponding eigenvalues and eigenvectors are:

$$\begin{array}{cc} \text{eigenvalues} & \text{eigenvectors} \\ \lambda_1 = 0 & u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ \lambda_2 = -3 & u_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\ \lambda_2 = -3 & u_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \end{array}$$

adopt the following transformation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

then the system becomes:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 3\varepsilon(u-v-w) - (u+w)^2(u+v) + (u-v-w)(u+v)^2 + (u-v-w)^2(u+w) \\ (u+w)^2(u+v) - (u-v-w)(u+v)^2 + 2(u-v-w)^2(u+w) \\ (u+w)^2(u+v) + 2(u-v-w)(u+v)^2 - (u-v-w)^2(u+w) \end{pmatrix}$$

the center manifold is locally represented by

$$W^{c(0)} = \left\{(u, v, w, \varepsilon) \in \mathbb{R}^3 \mid v = h_1(u, \varepsilon), w = h_2(u, \varepsilon), h_{i(0,0)} = 0, Dh_{i(0,0)} = 0, i = 1, 2\right\}$$

Differentiating $v = h(u)$ with respect to time implies that the $(\dot{u}, \dot{v})$ coordinates of any point on $W^c(0)$ must satisfy

$$\dot{v} = Dh(u)\dot{u}$$

assuming that

$$h_1(u, \varepsilon) = a_1u^2 + a_2\varepsilon u + a_3\varepsilon^2 + \cdots$$
$$h_2(u, \varepsilon) = b_1u^2 + b_2\varepsilon u + b_3\varepsilon^2 + \cdots$$

substituting the equation above, it can be obtained that

$$-3h_1 + \frac{1}{3}\left[(u+h_2)^2(u+h_1) - (u-h_1-h_2)(u+h_1)^2 + 2(u-h_1-h_2)^2(u+h_2)\right]$$
$$= \frac{1}{3}(2a_1u + a_2\varepsilon + \cdots)\left[3\varepsilon(u-h_1-h_2) - (u+h_2)^2(u+h_1) + (u-h_1-h_2)(u+h_1)^2 + (u-h_1-h_2)^2(u+h_2)\right]$$

$$-3h_2 + \frac{1}{3}\left[(u+h_2)^2(u+h_1) + (u-h_1-h_2)(u+h_1)^2 + 2(u-h_1-h_2)(u+h_2)^2\right]$$
$$= \frac{1}{3}(2b_1u + b_2\varepsilon + \cdots)\left[3\varepsilon(u-h_1-h_2) - (u+h_2)^2(u+h_1) + (u-h_1-h_2)(u+h_1)^2 + (u-h_1-h_2)^2(u+h_2)\right]$$

Equating coefficients of like powers to zero gives

$$u^2 : \begin{cases} -3a_1 = 0 \\ -3b_1 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ b_1 = 0 \end{cases}$$

$$\varepsilon u : \begin{cases} -3a_2 = 0 \\ -3b_2 = 0 \end{cases} \Rightarrow \begin{cases} a_2 = 0 \\ b_2 = 0 \end{cases}$$

$$\varepsilon^2 : \begin{cases} -3a_3 = 0 \\ -3b_3 = 0 \end{cases} \Rightarrow \begin{cases} a_3 = 0 \\ b_3 = 0 \end{cases}$$

$$u^3 : \begin{cases} -3a_4 + \frac{2}{3} = 0 \\ -3b_4 + \frac{2}{3} = 0 \end{cases} \Rightarrow \begin{cases} a_4 = \frac{2}{9} \\ b_4 = \frac{2}{9} \end{cases}$$

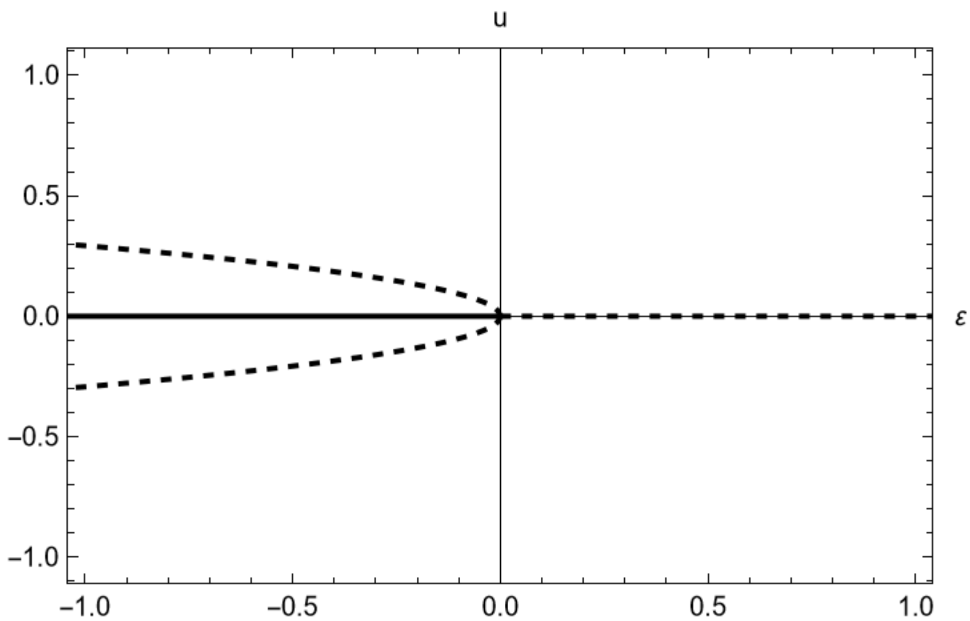
Thus, the center manifold is given by the graph of

$$h_1(u, \varepsilon) = \frac{2}{9}u^3 + \cdots$$
$$h_2(u, \varepsilon) = \frac{2}{9}u^3 + \cdots$$

the map restricted to the center manifold is given by

$$\dot{u} = u\left(3\varepsilon + \frac{1}{3}u^2 + \cdots\right)$$

How do the dynamics depend on ε



- if $\varepsilon \geq 0$, there is only one unstable fixed point $(0, 0)$
- if $\varepsilon < 0$, $(0, 0)$ turns into a stable fixed point and two more unstable fixed points appear

1. This exercise comes from Marsden and McCracken [1976]. Consider the following vector fields

   a) $\begin{aligned} \dot{r} &= -r(r-\mu)^2, \\ \dot{\theta} &= 1, \end{aligned} \qquad (r,\theta) \in \mathbb{R}^+ \times S^1.$

   b) $\begin{aligned} \dot{r} &= r(\mu - r^2)(2\mu - r^2)^2, \\ \dot{\theta} &= 1. \end{aligned}$

   c) $\begin{aligned} \dot{r} &= r(r+\mu)(r-\mu), \\ \dot{\theta} &= 1. \end{aligned}$

   d) $\begin{aligned} \dot{r} &= \mu r(r^2 - \mu), \\ \dot{\theta} &= 1. \end{aligned}$

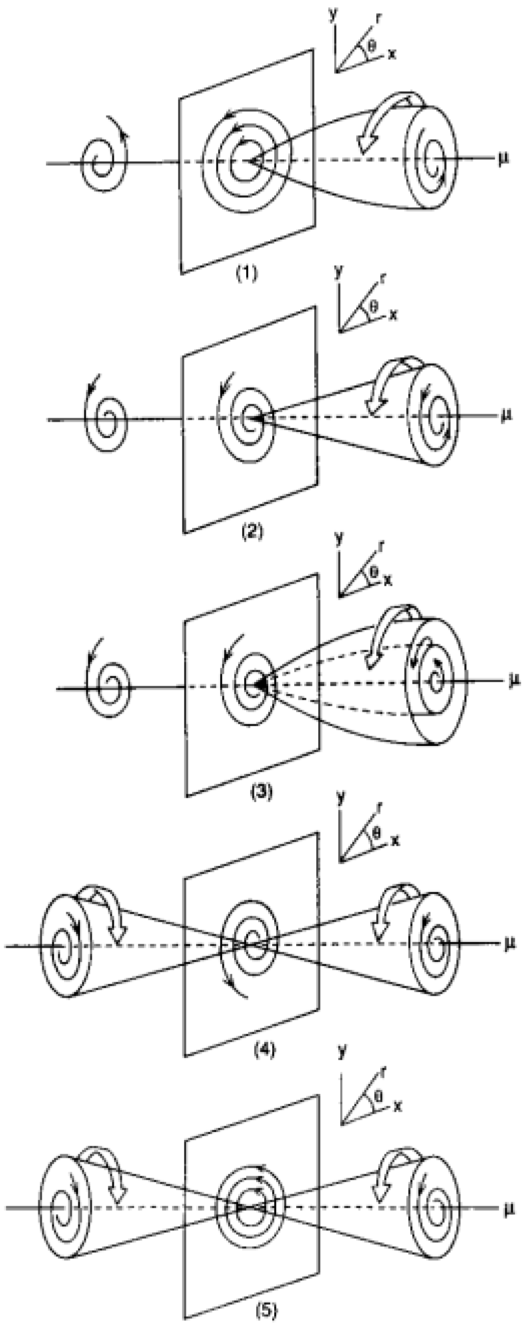


FIGURE 20.2.6.

e) $\dot{r} = -\mu^2 r(r+\mu)^2(r-\mu)^2,$
$\dot{\theta} = 1.$

Match each of these vector fields to the appropriate phase portrait in Figure 20.2.6 and explain which hypotheses (if any) of the Poincaré–Andronov–Hopf bifurcation theorem are violated.

1.a

It's easy to integrate the equation

$$\dot{r} = -r(r-\mu)^2 \Leftrightarrow -\frac{1}{r(r-\mu)^2}\,\mathrm{d}r = \mathrm{d}t$$

Obviously, fixed point $r = 0$ is stable by the solution of the equation.

And $r = \mu$ is a periodic orbit.

So, (2) is the phase portrait of (a).

the equation can be written as

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} -\mu^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r \\ \theta \end{pmatrix} + \begin{pmatrix} -r^3 + 2\mu r^2 \\ 1 \end{pmatrix}$$

the eigenvalues of the Jacobian matrix at fixed point $(0, 0)$ are

$$\lambda_1 = -\mu^2, \lambda_2 = 0$$

- $f(0, \mu) = 0$ and $\mathrm{Re}\ (\lambda(0)) = 0$ are satisfied
- $\frac{\mathrm{d}\ \mathrm{Re}\ (\lambda)}{\mathrm{d}\mu}\ |_{\mu=0} = 0$ voilates the hypothese that $\frac{\mathrm{d}\ \mathrm{Re}\ (\lambda)}{\mathrm{d}\mu}\ |_{\mu=0} \neq 0$

1.c

It's easy to integrate the equation

$$\dot{r} = r(r+\mu)(r-\mu) \Leftrightarrow \frac{1}{r(r+\mu)(r-\mu)}\,\mathrm{d}r = \mathrm{d}t$$

Obviously, fixed point $r = 0$ is unstable by the solution of the equation.

And $r = \pm\mu$ is a periodic orbit.

So, (4) is the phase portrait of (c).

the equation can be written as

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \mu^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r \\ \theta \end{pmatrix} + \begin{pmatrix} r^3 \\ 1 \end{pmatrix}$$

the eigenvalues of the Jacobian matrix at fixed point $(0, 0)$ are

$$\lambda_1 = \mu^2, \lambda_2 = 0$$

- $f(0, \mu) = 0$ and $\mathrm{Re}\ (\lambda(0)) = 0$ are satisfied
- $\frac{\mathrm{d}\ \mathrm{Re}\ (\lambda)}{\mathrm{d}\mu}\ |_{\mu=0} = 0$ voilates the hypothese that $\frac{\mathrm{d}\ \mathrm{Re}\ (\lambda)}{\mathrm{d}\mu}\ |_{\mu=0} \neq 0$

3. For the Poincaré–Andronov–Hopf bifurcation, compute the expression for the coefficient a given in (20.2.14).

given the example of the system

$$\begin{cases} \dot{x} = \varepsilon x - y + x^2 \\ \dot{y} = x + \varepsilon y + x^2 \end{cases}$$

then we know

$$f^1 = x^2, f^2 = x^2, \omega = 1$$

so

$$\begin{aligned}
a &= \frac{1}{16}\left(f^1_{xxx} + f^1_{xyy} + f^2_{xxy} + f^2_{yyy}\right) + \frac{1}{16\omega}\left(f^1_{xy}\left(f^1_{xx} + f^1_{yy}\right) - f^2_{xy}\left(f^2_{xx} + f^2_{yy}\right) - f^1_{xx}f^2_{xx} - f^1_{yy}f^2_{yy}\right) \\
&= \frac{1}{16}(2 \times 2) \\
&= \frac{1}{4}
\end{aligned}$$