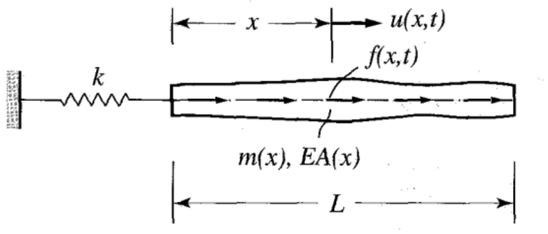
Homework 7-8 Reference answer

Problem 1 (60 points in total)

Problem 1.1:Please solve Problem 8.11 in page 460 of the textbook (10 points)

8.11. Derive the eigenvalue problem for the rod in axial vibration considered in Problem 8.3. Then, let $m(x)=m=\text{constant}$, $EA(x)=EA=\text{constant}$ and solve the eigenvalue problem for the two cases: 1) $k=0.5EA/L$ and 2) $k=2EA/L$. Plot the three lowest modes for each of the two cases and draw conclusions as to the effect of the spring stiffness k on the system.



解：

$$m\frac{\partial^2 u}{\partial t^2}-EA\frac{\partial^2 u}{\partial x^2}=0$$

分离变量法，设杆的振动解为

$$u(x,t)=U(x)T(t)$$

代入振动方程得到：

$$\frac{\partial^2 T}{\partial t^2}\frac{1}{T(t)}=\frac{EA}{m}\frac{\partial^2 U}{\partial x^2}\frac{1}{U(x)}=-\lambda,\ \ \lambda=\omega^2$$

$$\frac{\partial^2 T}{\partial t^2}+\lambda T(t)=0$$

$$\frac{\partial^2 U}{\partial x^2}+\lambda\frac{m}{EA}U(x)=0$$

解得：

$$U(x)=B\sin\beta x+C\cos\beta x,\quad where\ \ \beta=\sqrt{\frac{m\lambda}{EA}}$$

代入边界条件

$$EA\frac{\partial U^{'}(0)}{\partial x}=kU(0)$$

$$EA\frac{\partial U^{'}(L)}{\partial x}=0$$

即：

$$B\beta=\frac{k}{EA}C$$

$$B\beta\cos\beta L-C\beta\sin\beta L=0$$

特征方程为

$$\tan \beta L = \frac{k}{EA\beta} = \frac{B}{C}$$

（1） $k = 0.5EA / L$ 时，

$$2L\beta \tan \beta L = 1$$

可解得数值解，固有频率前三阶数值解分别为

$$\beta_1 L = 0.653, \beta_2 L = 3.292, \beta_1 L = 6.362$$

（2） $k = 2EA / L$ 时，

$$L\beta \tan \beta L = 2$$

前三节固有频率的解为：

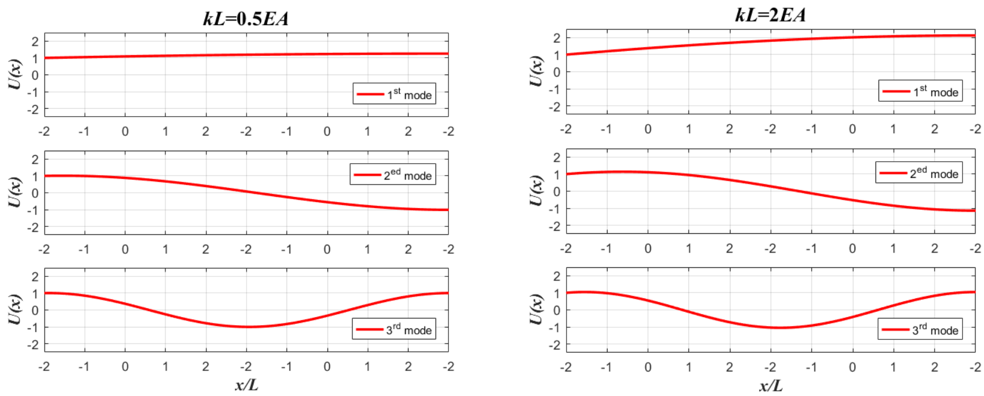
$$\beta_1 L = 1.077, \beta_2 L = 3.644, \beta_1 L = 6.578$$

前三阶主振型为

$$U_i(x) = \tan\left(\beta_i L\right)\sin\left(\beta_i x\right) + \cos\left(\beta_i x\right)$$

可发现刚度的提升会导致各阶主振型固有频率的提升。

主振型如下图所示



Problem 1.2: Please solve problem 8.17 in page 461 of the textbook (5 points).

8.17. Derive the orthogonality relations for the rod in axial vibration considered in Problem 8.11

解：将 $u(x,t) = U(x)\sin\left(\omega t + \varphi\right)$ 代入杆的振动方程：

$$m\frac{\partial^2 u}{\partial t^2} = EA\frac{\partial^2 u}{\partial x^2}$$

$$-\omega^2 mU\left(x\right) = EA\frac{\partial^2 U\left(x\right)}{\partial x^2}$$

可得：

$$-\omega_i^2 mU_i(x) = EA\frac{\partial^2 U_i(x)}{\partial x^2} \quad (1)$$

$$-\omega_j^2 mU_j(x) = EA\frac{\partial^2 U_j(x)}{\partial x^2} \quad (2)$$

对（1）左乘$U_j(x)$并在 0-L 上做积分：

$$-\omega_i^2\int_0^L mU_jU_i dx = \int_0^L EAU_j\frac{\partial^2 U_i}{\partial x^2}dx$$

分部积分，并由边界条件可化简：

$$-\omega_i^2\int_0^L mU_jU_i dx = EA\left(U_j\frac{\partial U_i}{\partial x}\bigg|_0^L - \int_0^L\frac{\partial U_j}{\partial x}\frac{\partial U_i}{\partial x}dx\right)$$

$$=\mathrm{EA}\left(-U_j(0)\frac{kU_i(0)}{EA} - \int_0^L\frac{\partial U_j}{\partial x}\frac{\partial U_i}{\partial x}dx\right)$$

同理由式（2）可得：

$$-\omega_j^2\int_0^L mU_jU_i dx = EA\left(-U_j(0)\frac{kU_i(0)}{EA} - \int_0^L\frac{\partial U_j}{\partial x}\frac{\partial U_i}{\partial x}dx\right)$$

与上式相减可得：

$$\left(\omega_i^2-\omega_j^2\right)\int_0^L U_jmU_i dx = 0$$

$$\int_0^L U_jmU_i dx = 0,\ (i\neq j)$$

即证明了主振型关于质量的正交性。

将此式回代得主振型关于刚度的正交性。

$$\int_0^L EAU_j\frac{\partial^2 U_i}{\partial x^2}dx = 0,\ (i\neq j)$$

Problem1.1 中已求解出主振型：

$$U_i(x) = \tan(\beta_i L)\sin(\beta_i x) + \cos(\beta_i x)$$

对主振型做归一化处理，使其满足：

$$\int_0^L c^2U_i^2 m dx = 1,\ i = 1,2,3\cdots$$

$$\int_0^L c^2 EAU_i\frac{\partial^2 U_i(x)}{\partial x^2}dx = -\omega_i^2,\ i = 1,2,3\cdots$$

$$where,\quad c = \left(\frac{2\beta_i}{m\left(\tan(L\beta_i) + L\beta_i\sec^2(L\beta_i)\right)}\right)^{0.5}$$

得到的结果称之为正则振型，如下：

$$U_i(x) = \sqrt{\frac{2\beta_i}{m\left(\tan(L\beta_i) + L\beta_i\sec^2(L\beta_i)\right)}}\left[\tan(\beta_i L)\sin(\beta_i x) + \cos(\beta_i x)\right]$$

Problem 1.3: Please solve Problem 8.31 in page 462 of the textbook (15 points).

8.31. Determine the response of the uniform rod of Problem 8.11 to the initial excitation $u(x,0)=0$, $\dot{u}(x,0)=v_0\delta(x)$, where $\delta(x)$ is a spatial Dirac delta function located at $x=0$.

解：振动方程

$$m\frac{\partial^2 u}{\partial t^2}-EA\frac{\partial^2 u}{\partial x^2}=0$$

根据类似多自由度系统的展开定理，将位移展开为正则振型的无穷级数，其中正则振型已在 1.2 中给出。

$$u(x,t)=\sum_{1}^{3}U_i(x)\eta_i(t)$$

$$U_i(x)=\sqrt{\frac{2\beta_i}{m\left(\tan\left(L\beta_i\right)+L\beta_i\sec^2\left(L\beta_i\right)\right)}}\left[\tan\left(\beta_i L\right)\sin\left(\beta_i x\right)+\cos\left(\beta_i x\right)\right]$$

初始条件：$u\left(x,0\right)=0,\dot{u}\left(x,0\right)=v_0\delta\left(x\right)$

根据正交性，左乘 mU_j 并积分，求出正则坐标下的初始条件。

$$\sum_{i=1}^{3}\int_0^L mU_jU_i\eta_i\left(0\right)\mathrm{dx}=\int_0^L mU_ju(x,0)dx$$

$$\eta_j\left(0\right)=0$$

$$\sum_{i=1}^{3}\int_0^L mU_jU_i\dot{\eta}_i\left(0\right)\mathrm{dx}=\int_0^L mU_j\dot{u}(x,0)dx$$

$$\dot{\eta}_j\left(0\right)=\int_0^L mU_jv_0\delta\left(x\right)dx$$

$$\dot{\eta}_j\left(0\right)=mU_j\left(0\right)v_0$$

$$\dot{\eta}_j\left(0\right)=mv_0\sqrt{\frac{2\beta_i}{m\left(\tan\left(L\beta_i\right)+L\beta_i\sec^2\left(L\beta_i\right)\right)}}$$

根据正交性，正则坐标下的动力学方程为：

$$\int_0^L\left(\sum_{i=1}^{3}mU_j\left(x\right)U_i(x)\frac{\partial^2\eta_i(t)}{\partial t^2}\right)dx=\int_0^L\left(\sum_{i=1}^{3}EAU_j\left(x\right)\frac{\partial^2 U_i\left(x\right)}{\partial x^2}\eta_i(t)\right)dx$$

$$\frac{\partial^2\eta_j(t)}{\partial t^2}+\omega_j^2\eta_j(t)=0,\ j=1,2,3$$

解得：

$$\eta_j(t)=\eta_j(0)\cos\left(\omega_j t\right)+\frac{\dot{\eta}_j(0)}{\omega_j}\sin\left(\omega_j t\right)$$

$$\eta_j(t)=\frac{mv_0\sqrt{\frac{2\beta_i}{m\left(\tan\left(L\beta_i\right)+L\beta_i\sec^2\left(L\beta_i\right)\right)}}}{\omega_j}\sin\left(\omega_j t\right),\quad where\ \beta=\sqrt{\frac{m\omega_i^2}{EA}}$$

则杆在初始条件下的响应可写成：

$$u(x,t)=\sum_1^3 U_i(x)\eta_i(t)$$
$$=\sum_{i=1}^3 \frac{2v_0\beta_i}{\omega_i\left(\tan\left(L\beta_i\right)+L\beta_i\sec^2\left(L\beta_i\right)\right)}\left[\tan\left(\beta_i L\right)\sin\left(\beta_i x\right)+\cos\left(\beta_i x\right)\right]\sin\left(\omega_i t\right)$$

Problem 1.4: Please solve Problem 8.36 in page 463 of the textbook (15 points).

8.36. Determine the response of the rod of Problem 8.11 to the uniformly distributed harmonic force $f(x,t)=f_0\cos\Omega t$. Discuss the mode participation in the response.

解：振动方程

$$m\frac{\partial^2 u}{\partial t^2}-EA\frac{\partial^2 u}{\partial x^2}=f(x,t), 0<x<L$$

根据正则振型的正交性，正则坐标下的动力学方程为

$$\int_0^L\left(\sum_{i=1}^3 mU_j(x)U_i(x)\frac{\partial^2\eta_i(t)}{\partial t^2}\right)dx-\int_0^L\left(\sum_{i=1}^3 EAU_j(x)\frac{\partial^2 U_i(x)}{\partial x^2}\eta_i(t)\right)dx=\int_0^L\left[U_j(x)f(x,t)\right]dx$$
$$\ddot{\eta}_j(t)+\omega_j^2\eta_j(t)=N_j(t),\ j=1,2,3$$

其中：

$$N_j(t)=\int_0^L U_j(x)f(x,t)dx$$
$$=\int_0^L\left(\sqrt{\frac{2\beta_j}{m\left(\tan\left(L\beta_j\right)+L\beta_j\sec^2\left(L\beta_j\right)\right)}}\left[\tan\left(\beta_j L\right)\sin\left(\beta_j x\right)+\cos\left(\beta_j x\right)\right]f_0\cos\Omega t\right)dx$$
$$=C_j\cos\Omega t\ \ ,\ where\ \ C_j=\sqrt{\frac{2\beta_j}{m\left(\tan\left(L\beta_j\right)+L\beta_j\sec^2\left(L\beta_j\right)\right)}}\frac{\tan\left(\beta_j L\right)}{\beta_j}f_0$$

正则坐标下的稳态响应为：

$$\eta_j(t)=\frac{C_j}{\omega_j^2-\Omega^2}\cos\Omega t$$

回代即可得到杆在谐波激励下的响应：

$$u(x,t)=\sum_1^3 U_i(x)\eta_i(t)$$
$$=\sum_1^3\frac{2\left[\tan\left(\beta_i L\right)\sin\left(\beta_i x\right)+\cos\left(\beta_i x\right)\right]\tan\left(\beta_i L\right)}{m\left(\tan\left(L\beta_i\right)+L\beta_i\sec^2\left(L\beta_i\right)\right)}\frac{f_0\cos\Omega t}{\left(\omega_i^2-\Omega^2\right)}$$

仅讨论 1.1 中 $k=2EA/L$ 的情况，将 1.1 中解得的前三节固有频率代入

$$\beta_1 L=1.077,\beta_2 L=3.644,\beta_1 L=6.578$$

$$u(x,t)=0.559\left[1.858\sin(\beta_1 x)+\cos(\beta_1 x)\right]\frac{f_0\cos\Omega t}{m\left(\omega_1^2-\Omega^2\right)}+$$
$$0.208\left[0.549\sin(\beta_2 x)+\cos(\beta_2 x)\right]\frac{f_0\cos\Omega t}{m\left(\omega_2^2-\Omega^2\right)}+$$
$$0.081\left[0.304\sin(\beta_3 x)+\cos(\beta_3 x)\right]\frac{f_0\cos\Omega t}{m\left(\omega_3^2-\Omega^2\right)}$$

可发现各阶模态对响应的贡献程度与模态的阶数成反比。

Problem 1.5: Please solve Problem 8.42 in page 463 of the textbook (15 points).

8.42. Determine the response of the rod of Problem 8.11 to the impulsive force $F(t)=\hat{F}_0\delta(t)$ applied at $x=L$. Discuss the mode participation in the response.

解：正则坐标下的动力学方程为

$$\ddot{\eta}_j(t)+\omega_j^2\eta_j(t)=N_j(t),\ j=1,2,3$$

其中：

$$N_j(t)=\int_0^L U_j(x)f(x,t)dx$$
$$=\int_0^L\left(\sqrt{\frac{2\beta_j}{m\left(\tan(L\beta_j)+L\beta_j\sec^2(L\beta_j)\right)}}\left[\tan(\beta_j L)\sin(\beta_j x)+\cos(\beta_j x)\right]\hat{F}_0\delta(t)\delta(x-L)\right)dx$$
$$=\sqrt{\frac{2\beta_j}{m\left(\tan(L\beta_j)+L\beta_j\sec^2(L\beta_j)\right)}}\sec(\beta_j L)\hat{F}_0\delta(t)$$

根据脉冲激励的响应公式得到正则坐标下的稳态响应为：

$$\eta_j(t)=\sqrt{\frac{2\beta_j}{m\left(\tan(L\beta_j)+L\beta_j\sec^2(L\beta_j)\right)}}\frac{\sec(\beta_j L)\hat{F}_0}{\omega_j}\sin\omega_j t$$

回代即可得到杆在脉冲激励下的响应：

$$u(x,t)=\sum_1^3 U_i(x)\eta_i(t)$$
$$=\frac{2\sec(\beta_i L)}{\left(\tan(L\beta_i)+L\beta_i\sec^2(L\beta_i)\right)}\frac{\left[\tan(\beta_i L)\sin(\beta_i x)+\cos(\beta_i x)\right]\hat{F}_0}{m}\sqrt{\frac{m}{EA}}\sin\omega_i t$$

仅讨论 1.1 中 $k=2EA/L$ 的情况，将 1.1 中解得的前三节固有频率代入

$$\beta_1 L=1.077,\beta_2 L=3.644,\beta_1 L=6.578$$

$$u(x,t)=0.634\frac{\left[1.858\sin(\beta_1 x)+\cos(\beta_1 x)\right]\hat{F}_0}{m}\sqrt{\frac{m}{EA}}\sin\omega_1 t+$$
$$(-0.431)\frac{\left[0.549\sin(\beta_2 x)+\cos(\beta_2 x)\right]\hat{F}_0}{m}\sqrt{\frac{m}{EA}}\sin\omega_2 t+$$
$$0.279\frac{\left[0.304\sin(\beta_3 x)+\cos(\beta_3 x)\right]\hat{F}_0}{m}\sqrt{\frac{m}{EA}}\sin\omega_3 t$$

各阶模态对响应的贡献程度与模态的阶数成反比。

Problem 2: (30 points in total)

（此题 Bessel 函数的正交性等性质内容超纲，可自行查阅知乎等）

Problem 2.1: Based on Problem 4 in HOMEWORK 6 (i.e., Problem 8.7 in page 459 of the textbook), solve Problem 8.13 in page 460 of the textbook (10 points).

8.13. Derive and solve the eigenvalue problem for the hanging cable of Problem 8.7. Plot the three lowest modes. **Hints:** Devise a certain coordinate transformation capable of reducing the differential equation to a Bessel equation. Then, the boundary condition at the free end of the cable must be such as to permit elimination of the unacceptable solution.

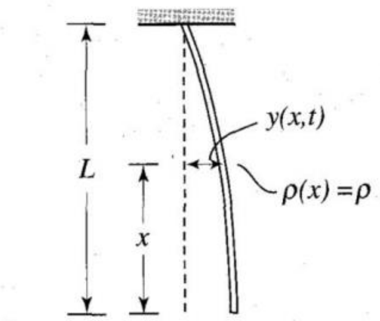


FIGURE 8.34

解：由

$$\frac{\partial^2 y}{\partial t^2}-g\frac{\partial y}{\partial x}-gx\frac{\partial^2 y}{\partial x^2}=0$$

设解并代入：

$$y(x,t)=Y(x)\sin\left(\omega t+\varphi\right)$$

$$Y\left(x\right)\omega^2+g\frac{\partial Y(x)}{\partial x}+gx\frac{\partial^2 Y(x)}{\partial x^2}=0$$

变量替换，令

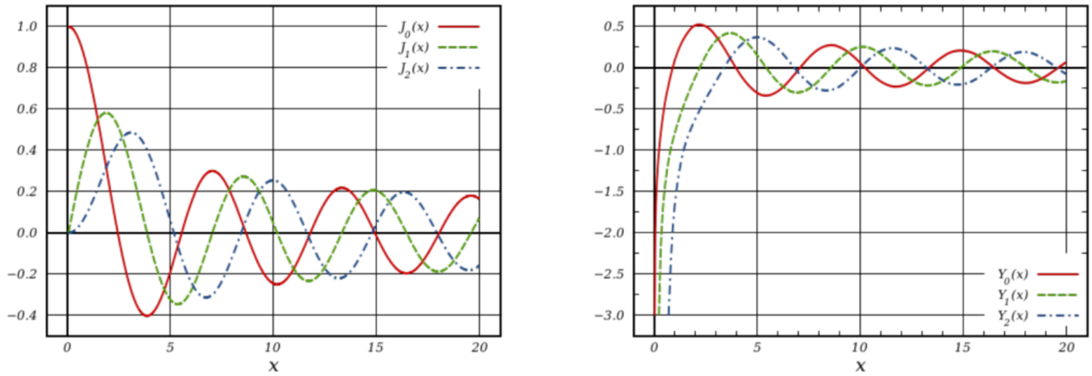
$$\xi=\frac{2\omega}{g}\sqrt{xg}$$

$$\xi^2\frac{\partial^2 Y}{\partial \xi^2}+\xi\frac{\partial Y}{\partial \xi}+\xi^2 Y=0$$

上述为零阶贝塞尔方程，其通解为$Y=C_1J_0\left(\xi\right)+C_2Y_0\left(\xi\right)$

$J_0\left(\xi\right),Y_0\left(\xi\right)$线性无关，分别为第一类和第二类贝塞尔函数。如下图所示。

因为$Y_0\left(\xi\right)$在$\xi=0$处有一条渐近线，对应于链的固定端，不满足物理意义，因此有意义的解只有第一类贝塞尔函数。



第一类/第二类贝塞尔函数

第一类贝塞尔函数表达式为

$$J_n(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{n+2m}$$

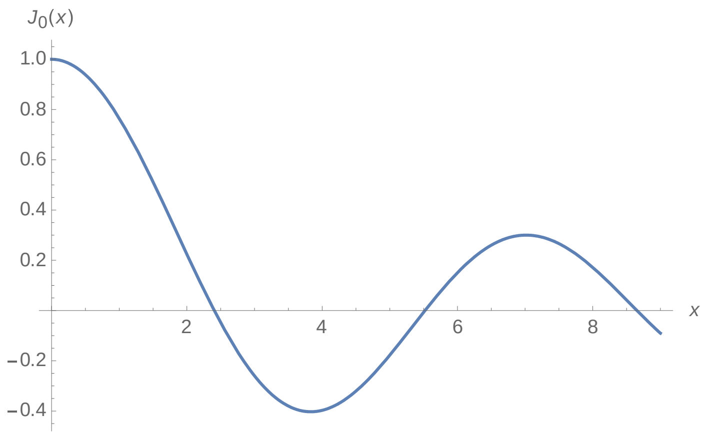
$$where\quad J_0(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!m!}\left(\frac{x}{2}\right)^{2m}$$

因此有意义的解为：

$$Y=CJ_0(\xi)=CJ_0\left(\frac{2\omega}{g}\sqrt{xg}\right)=C\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{\omega}{g}\sqrt{xg}\right)^{2m}}{m!m!}$$

代入边界条件：$Y\big|_{x=L}=0$

$$J_0\left(2\omega\sqrt{\frac{L}{g}}\right)=0$$



零阶第一类贝塞尔函数

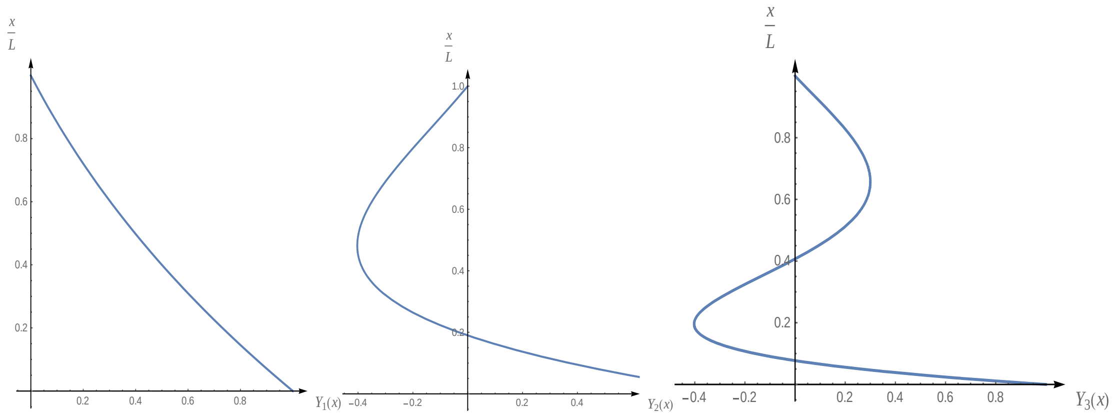
前三阶解为：

$$2\omega_i\sqrt{\frac{L}{g}}=2.40483,2\omega_2\sqrt{\frac{L}{g}}=5.52008,2\omega_3\sqrt{\frac{L}{g}}=8.65373$$

因此主阵型为：

$$Y_i\left(x\right)=J_0\left(\xi_i\right)=J_0\left(\frac{2\omega_i}{g}\sqrt{xg}\right)=\sum_{m=0}^{\infty}\frac{(-1)^m\left(\omega_i\sqrt{\frac{x}{g}}\right)^{2m}}{m!m!}$$

前三阶主阵型如下图所示：



前三阶主阵型（从左到右）

Problem 2.2: Please solve problem 8.19 in page 461 of the textbook (5 points).

8.19. Verify that the modes of the hanging cable obtained in Problem 8.13 are indeed orthogonal.

解：固有频率满足

$$\sum_{m=0}^{\infty}\frac{(-1)^m\left(\omega_i\sqrt{\frac{L}{g}}\right)^{2m}}{m!m!}=0$$

主阵型：

$$Y_i\left(x\right)=J_0\left(\xi\right)=J_0\left(\frac{2\omega_i}{g}\sqrt{xg}\right)=\sum_{m=0}^{\infty}\frac{(-1)^m\left(\omega_i\sqrt{\frac{x}{g}}\right)^{2m}}{m!m!}$$

$$where\ \ 2\omega_i\sqrt{\frac{L}{g}}=2.40483,2\omega_2\sqrt{\frac{L}{g}}=5.52008,2\omega_3\sqrt{\frac{L}{g}}=8.65373$$

正交性验证：

当$i\neq j$时

$$\int_0^L \rho Y_i(x) Y_j(x) dx = \int_0^L \rho J_0(\xi_i) J_0(\xi_j) dx$$

$$= \rho \int_0^L \sum_{m=0}^{\infty} \frac{(-1)^m \left(\omega_i \sqrt{\frac{L}{g}} * \sqrt{\frac{x}{L}}\right)^{2m}}{m!m!} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\omega_j \sqrt{\frac{L}{g}} * \sqrt{\frac{x}{L}}\right)^{2m}}{m!m!} dx$$

令$x_1 = \sqrt{x}, L_1 = \sqrt{L}$，即$\frac{x_1}{L_1} = \sqrt{\frac{x}{L}}$，得到：

$$= 2\rho \int_0^{L_1} x_1 \sum_{m=0}^{\infty} \frac{(-1)^m \left(\omega_i \sqrt{\frac{L}{g}} * \frac{x_1}{L_1}\right)^{2m}}{m!m!} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\omega_j \sqrt{\frac{L}{g}} * \frac{x_1}{L_1}\right)^{2m}}{m!m!} dx_1$$

$$= 2\rho \int_0^{L_1} x_1 J_0\left(2\omega_i \sqrt{\frac{L}{g}} * \frac{x_1}{L_1}\right) J_0\left(2\omega_j \sqrt{\frac{L}{g}} * \frac{x_1}{L_1}\right) dx_1$$

根据零阶 Bessel 函数的正交性（证明略），上式为零。

将其回代到振动方程中，得以下关系式：

$$\rho g \frac{\partial Y_i(x)}{\partial x} + \rho g x \frac{\partial^2 Y_i(x)}{\partial x^2} = -\omega_i^2 \rho Y_i(x)$$

$$\int_0^L \left(\rho g \frac{\partial Y_i(x)}{\partial x} + \rho g x \frac{\partial^2 Y_i(x)}{\partial x^2}\right) Y_j(x) dx = -\omega_i^2 \int_0^L \rho Y_i(x) Y_j(x) dx$$

$$\int_0^L \left(\rho g \frac{\partial Y_i(x)}{\partial x} + \rho g x \frac{\partial^2 Y_i(x)}{\partial x^2}\right) Y_j(x) dx = 0, \quad where\ i \neq j$$

当$i = j$时，同理：

$$\int_0^L \rho Y^2{}_i(x) dx = \int_0^L \rho J_0{}^2(\xi_i) dx$$

$$= \rho \int_0^L \left[\sum_{m=0}^{\infty} \frac{(-1)^m \left(\omega_i \sqrt{\frac{L}{g}} * \sqrt{\frac{x}{L}}\right)^{2m}}{m!m!}\right]^2 dx$$

$$set\ \ x_1 = \sqrt{x}, L_1 = \sqrt{L}, \frac{x_1}{L_1} = \sqrt{\frac{x}{L}}, \quad then$$

$$= 2\rho \int_0^{L_1} x_1 \left[\sum_{m=0}^{\infty} \frac{(-1)^m \left(\omega_i \sqrt{\frac{L}{g}} * \frac{x_1}{L_1}\right)^{2m}}{m!m!}\right]^2 dx_1 = 2\rho \int_0^{L_1} x_1 J_0{}^2\left(2\omega_i \sqrt{\frac{L}{g}} * \frac{x_1}{L_1}\right) dx_1$$

根据 Bessel 函数的性质，上式等于：

$$\int_0^L \rho Y^2{}_i\left(x\right)dx = 2\rho\int_0^{L_1} x_1 J_0{}^2\left(2\omega_i\sqrt{\frac{L}{g}} * \frac{x_1}{L_1}\right)dx_1$$

$$= 2\rho\frac{L_1{}^2}{2}J_1{}^2\left(2\omega_i\sqrt{\frac{L}{g}}\right)$$

$$= \rho L\left[\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\left(m+1\right)!}\left(\omega_i\sqrt{\frac{L}{g}}\right)^{2m+1}\right]^2$$

对主振型做归一化处理，使其满足：

$$c_i{}^2\int_0^L \rho Y_i\left(x\right)^2 dx = 1$$

解得：

$$c_1{}^2 = \frac{1}{0.2695\rho L},\ c_2{}^2 = \frac{1}{0.1158\rho L},\ c_3{}^2 = \frac{1}{0.0737\rho L}$$

得到正则振型：

$$Y_i\left(x\right) = c_i\sum_{m=0}^{\infty}\frac{(-1)^m\left(\omega_i\sqrt{\frac{x}{g}}\right)^{2m}}{m!m!}$$

$$where\ \ 2\omega_i\sqrt{\frac{L}{g}} = 2.40483, 2\omega_2\sqrt{\frac{L}{g}} = 5.52008, 2\omega_3\sqrt{\frac{L}{g}} = 8.65373$$

而且，正则振型满足：

$$\rho g\frac{\partial Y_i(x)}{\partial x} + \rho gx\frac{\partial^2 Y_i(x)}{\partial x^2} = -\omega_i^2\rho Y_i\left(x\right)$$

$$\int_0^L\left(\rho g\frac{\partial Y_i(x)}{\partial x} + \rho gx\frac{\partial^2 Y_i(x)}{\partial x^2}\right)Y_i\left(x\right)dx = -\omega_i^2\int_0^L \rho Y_i\left(x\right)^2 dx$$

$$\int_0^L\left(\rho g\frac{\partial Y_i(x)}{\partial x} + \rho gx\frac{\partial^2 Y_i(x)}{\partial x^2}\right)Y_i\left(x\right)dx = -\omega_i^2$$

Problem 2.3: Please solve Problem 8.32 in page 462 of the textbook (15 points).

8.32. The hanging cable of Problem 8.13 is displaced initially according to $y(x, 0) = y_0(1 - x/L)$. Determine the response of the cable subsequent to being released from rest in the displaced position.

解：将响应展开为正则振型的无穷级数。取前三阶分析

$$y(x,t) = \sum_1^3 Y_i(x)\eta_i(t)$$

初始条件：

$$y\left(x,0\right)=y_0\left(1-\frac{x}{L}\right),\dot{y}\left(x,0\right)=0$$

根据 2.2 所推导的正则振型的正交性，得到正则坐标下的振动方程：

$$\rho\frac{\partial^2 y}{\partial t^2}-\rho g\frac{\partial y}{\partial x}-\rho gx\frac{\partial^2 y}{\partial x^2}=0$$

$$\int_0^L\rho\sum_1^3 Y_i\left(x\right)\ddot{\eta}_i(t)Y_j\left(x\right)dx-\int_0^L\left(\rho g\sum_1^3\frac{\partial Y_i\left(x\right)}{\partial x}\eta_i(t)+\rho gx\sum_1^3\frac{\partial^2 Y_i\left(x\right)}{\partial x^2}\eta_i(t)\right)Y_j\left(x\right)dx=0$$

$$\ddot{\eta}_j(t)+\omega_j{}^2\eta_j(t)=0$$

正则坐标下的初始条件为：

$$\sum_{i=1}^3\int_0^L\rho Y_jY_i\eta_i\left(0\right)\mathrm{dx}=\int_0^L\rho Y_j y(x,0)dx$$

$$\eta_j\left(0\right)=\rho c_j y_0\int_0^L\sum_{m=0}^\infty\frac{(-1)^m\left(\omega_j\sqrt{\frac{x}{g}}\right)^{2m}}{m!m!}\left(1-\frac{x}{L}\right)dx$$

$$\sum_{i=1}^3\int_0^L\rho Y_jY_i\dot{\eta}_i\left(0\right)\mathrm{dx}=\int_0^L\rho Y_j\dot{y}(x,0)dx$$

$$\dot{\eta}_j\left(0\right)=0$$

解得正则坐标下的响应：

$$\eta_i\left(t\right)=\eta_i\left(0\right)\cos\omega_i t$$

则绳索在初始条件下的响应可写为：

$$y(x,t)=\sum_1^3 Y_i(x)\eta_i(t)$$

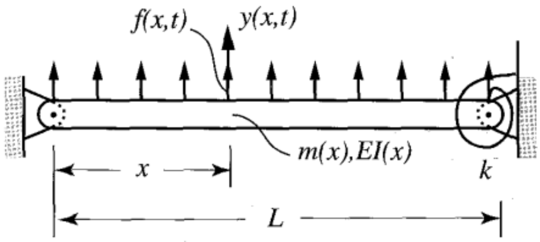
$$=\sum_1^3\left(c_i{}^2\rho y_0\left(\int_0^L\sum_{m=0}^\infty\frac{(-1)^m\left(\omega_j\sqrt{\frac{x}{g}}\right)^{2m}}{m!m!}\left(1-\frac{x}{L}\right)dx\cos\omega_i t\right)\sum_{m=0}^\infty\frac{(-1)^m\left(\omega_i\sqrt{\frac{x}{g}}\right)^{2m}}{m!m!}\right)$$

其中第二项 Bessel 函数在 0-L 上的积分可通过 Bessel 函数的性质来计算，内容超纲，在此不继续进行运算。

Problem 3 (45 points in total)

Problem 3.1: Based on Problem 5 in HOMEWORK 6 (i.e., Problem 8.8 in page 460 of the textbook), solve Problem 8.16 in page 461 of the textbook (10 points).

8.16. Derive the eigenvalue problem for the beam of Problem 8.8. Then, let $m(x) = m =$ constant, $EI(x) = EI =$ constant, $k = 0.5EI/L$, solve the eigenvalue problem and plot the three lowest modes.



解：梁振动方程为

$$m\frac{\partial^2 y(x,t)}{\partial t^2}+EI\frac{\partial^4 y(x,t)}{\partial x^4}=0, 0<x<L$$

设解并代入：$y(x,t)=Y(x)b\sin(\omega t+\varphi)$

$$\frac{\partial^4 Y}{\partial x^4}=\lambda^4 Y,\ \ where\ \ \lambda^4=\frac{m\omega^2}{EI}$$

通解为：$Y(x)=A\cos\lambda x+B\sin\lambda x+C\cosh\lambda x+D\sinh\lambda x$

将边界条件代入：

$$\delta y=0, EI\frac{\partial^2 y}{\partial x^2}=0\ \ at\ \ x=0$$

$$\delta y=0, EI\frac{\partial^2 y}{\partial x^2}=-k\theta\ \ at\ \ x=L$$

在 x=0 处：（结构不存在刚体模态）

$$\begin{cases} A+C=0 \\ \lambda^2(-A+C)=0 \end{cases} \Rightarrow A=C=0$$

在 x=L 处：

$$B\sin\lambda L+D\sinh\lambda L=0$$
$$2L\lambda(-B\sin\lambda L+D\sinh\lambda L)+(B\cos\lambda L+D\cosh\lambda L)=0$$

要使上式成立且有非零解，则：

$$Det\begin{bmatrix} \sin\lambda L & \sinh\lambda L \\ \cos\lambda L-2L\lambda\sin\lambda L & \cosh\lambda L+2L\lambda\sinh\lambda L \end{bmatrix}=0$$

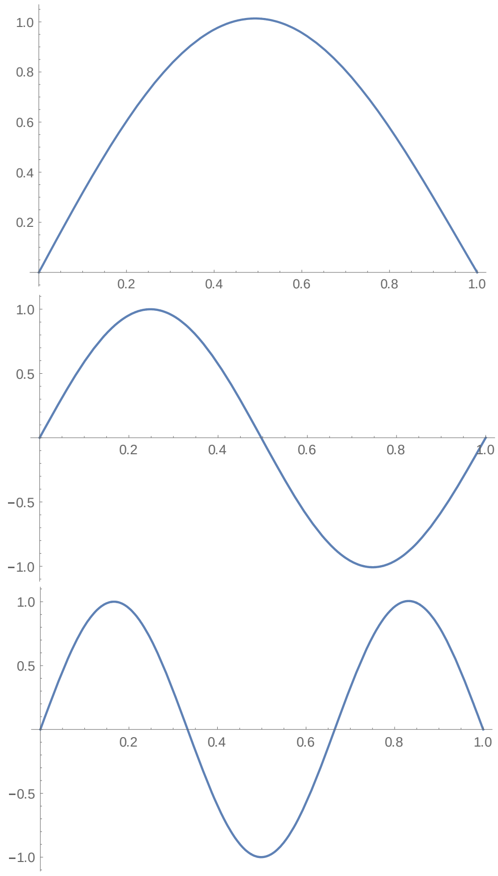
解得前三阶固有频率：

$$\lambda_1 L = 3.21, \lambda_2 L = 6.32, \lambda_3 L = 9.45, \ \ where\ \ \omega_i = \sqrt{\frac{EI\lambda_i^4}{m}}$$

主振型为：

$$Y_i(x) = \sin\lambda_i x - \frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i x$$

前三阶主振型分别如下图所示：



Problem 3.2: Please solve problem 8.22 in page 461 of the textbook (5 points).

8.22. Derive the orthogonality relations for the beam of Problem 8.16

解：由 3. 1 可得：

$$EI\frac{\partial^4 Y}{\partial x^4} = m\omega^2 Y$$

即：

$$EI\frac{\partial^4 Y_i}{\partial x^4} = m\omega_i^2 Y_i \ (1)$$

$$EI\frac{\partial^4 Y_j}{\partial x^4} = m\omega_j^2 Y_j \ (2)$$

对（1）左乘 Y_j 并在 0-L 上积分，并分部积分可得：

$$\int_0^L EIY_j \frac{\partial^4 Y_i}{\partial x^4}dx = \omega_i^2 \int_0^L Y_j m Y_i dx$$

$$EIY_j \frac{\partial^3 Y_i}{\partial x^3}\Big|_0^L - \int_0^L EI \frac{\partial Y_j}{\partial x}\frac{\partial^3 Y_i}{\partial x^3}dx = \omega_i^2 \int_0^L Y_j m Y_i dx$$

$$EIY_j \frac{\partial^3 Y_i}{\partial x^3}\Big|_0^L - EI\frac{\partial Y_j}{\partial x}\frac{\partial^2 Y_i}{\partial x^2}\Big|_0^L + \int_0^L EI \frac{\partial^2 Y_j}{\partial x^2}\frac{\partial^2 Y_i}{\partial x^2}dx = \lambda_i \int_0^L Y_j m Y_i dx$$

根据边界条件可得：

$$\frac{\partial Y_j(L)}{\partial x}K\frac{\partial Y_i(L)}{\partial x} + \int_0^L EI \frac{\partial^2 Y_j}{\partial x^2}\frac{\partial^2 Y_i}{\partial x^2}dx = \lambda_i \int_0^L Y_j m Y_i dx$$

同理有（2）得：

$$\frac{\partial Y_j(L)}{\partial x}K\frac{\partial Y_i(L)}{\partial x} + \int_0^L EI \frac{\partial^2 Y_j}{\partial x^2}\frac{\partial^2 Y_i}{\partial x^2}dx = \lambda_j \int_0^L Y_j m Y_i dx$$

两式相减得到：

$$\left(\lambda_j - \lambda_i\right)\int_0^L Y_j m Y_i dx = 0$$

$$\int_0^L Y_j m Y_i dx = 0 \quad \left(i \neq j\right)$$

上式验证了主振型关于质量的正交性，将其回代可得到主振型关于刚度的正交性。

$$\int_0^L EIY_j \frac{\partial^4 Y_i}{\partial x^4}dx \quad \left(i \neq j\right)$$

对主振型做归一化处理，使其满足：

$$\int_0^L c^2 Y_i^2 m dx = 1,\ i = 1,2,3\cdots$$

$$\int_0^L c^2 EIY_i \frac{\partial^4 Y_i(x)}{\partial x^4}dx = \lambda_i,\ i = 1,2,3\cdots$$

$$where,\quad c_1 = \frac{1}{\sqrt{0.510mL}},\ c_1 = \frac{1}{\sqrt{0.503mL}},\ c_1 = \frac{1}{\sqrt{0.501mL}},$$

得到正则振型：

$$Y_i(x) = c_i\left(\sin\lambda_i x - \frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i x\right)$$

Problem 3.3: Please solve Problem 8.34 in page 463 of the textbook (15 points).

8.34. Determine the response of the uniform beam of Problem 8.16 to the initial excitation $y(x,0) = y_0[13(x/L) - 27(x/L)^3 + 14(x/L)^4]$, $\dot{y}(x,0) = 0$. Discuss the mode participation in the response.

解：将梁的挠度展开为正则振型的无穷级数。

$$y(x,t) = \sum_1^3 Y_i(x)\eta_i(t)$$

$$Y_i(x) = c_i\left(\sin\lambda_i x - \frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i x\right)$$

初始条件： $y\left(x,0\right) = 0, \dot{y}\left(x,0\right) = 0$

根据正交性求出正则坐标下的初始条件。

$$\sum_{i=1}^{3}\int_{0}^{L} mU_jU_i\eta_i\left(0\right)\mathrm{dx}=\int_{0}^{L} mU_jy(x,0)dx$$

$$\eta_j\left(0\right)=\int_{0}^{L} mU_jy_0\left[13(x/L)-27(x/L)^3+14(x/L)^4\right]dx$$

$$where\ \ \eta_1\left(0\right)=2.023mLc_1y_0,\eta_2\left(0\right)=-0.0012mLc_2y_0,\eta_3\left(0\right)=0.009mLc_3y_0,$$

$$\sum_{i=1}^{3}\int_{0}^{L} mU_jU_i\dot{\eta}_i\left(0\right)\mathrm{dx}=\int_{0}^{L} mU_j\dot{y}(x,0)dx$$

$$\dot{\eta}_j\left(0\right)=0$$

根据正交性，正则坐标下的动力学方程为：

$$\int_0^L\left(\sum_{i=1}^{3} mY_j\left(x\right)Y_i(x)\ddot{\eta}_i(t)\right)dx+\int_0^L\left(\sum_{i=1}^{3} EIY_j\left(x\right)\frac{\partial^4 Y_i\left(x\right)}{\partial x^4}\eta_i(t)\right)dx=0$$

$$\ddot{\eta}_j(t)+\omega_j^{\ 2}\eta_j(t)=0,\ j=1,2,3$$

解得：

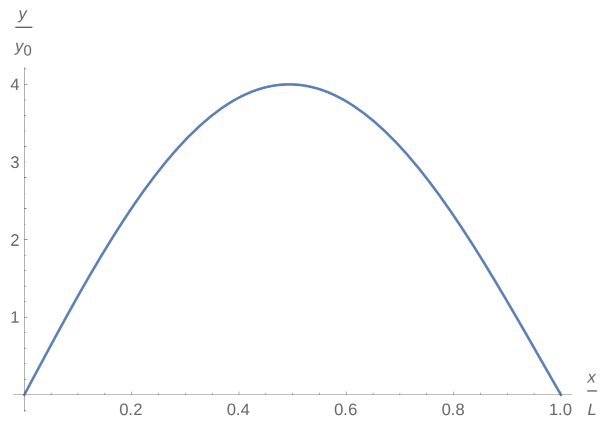
$$\eta_j(t)=\eta_j(0)\cos\left(\omega_jt\right)$$

则杆在初始条件下的响应可写成：

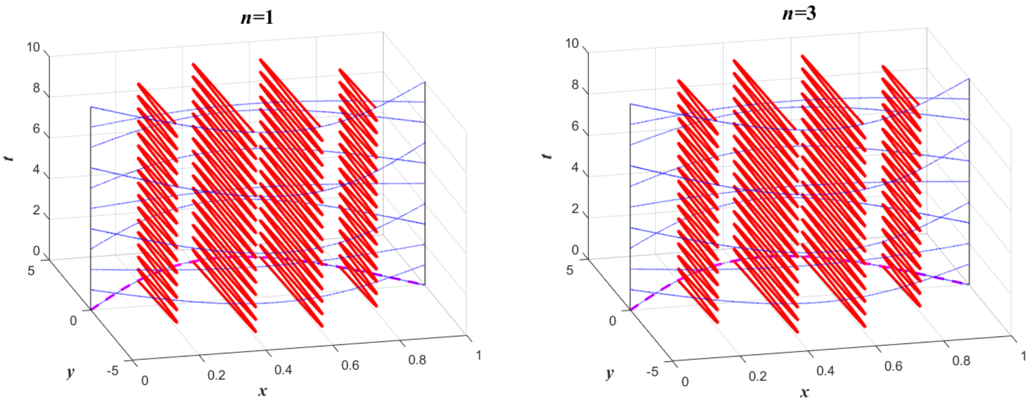
$$\begin{aligned}
y(x,t)&=\sum_{1}^{3}Y_i(x)\eta_i(t)\\
&=\sum_{1}^{3}c_i\left(\sin\lambda_ix-\frac{\sin\lambda_iL}{\sinh\lambda_iL}\sinh\lambda_ix\right)\eta_i(0)\cos\left(\omega_it\right)\\
&\approx 3.967y_0\left(\sin\lambda_1x-\frac{\sin\lambda_1L}{\sinh\lambda_1L}\sinh\lambda_1x\right)\cos\left(\omega_1t\right)\\
&\quad -0.00239y_0\left(\sin\lambda_2x-\frac{\sin\lambda_2L}{\sinh\lambda_2L}\sinh\lambda_2x\right)\cos\left(\omega_2t\right)\\
&\quad +0.018y_0\left(\sin\lambda_3x-\frac{\sin\lambda_3L}{\sinh\lambda_3L}\sinh\lambda_3x\right)\cos\left(\omega_3t\right)
\end{aligned}$$

可发现前三阶模态中第一阶模态对响应的贡献最大，第二阶模态对响应贡献极小。

这是因为:初始位移是关于 0.4935L$\left(\text{接近 }\mathbf{L/2}\right)$对称的（如下图），而 1,3 阶模态关于 L/2 对称，2 阶模态关于 L/2 反对称（见 3.1）。因此所给定初始条件很难激发反对称的二阶模态。且 3 阶模态相对 1 阶模态较难激发。



初始位移



初始条件的振动形态与第一阶模态较为接近，增加振型叠加的项数对于结果的影响很小。

Problem 3.4: Please solve Problem 8.40 in page 463 of the textbook (15 points).

8.40. Determine the response of the beam of Problem 8.16 to a concentrated force expressed as distributed in the form $f(x,t) = F_0\delta(x-3L/4)[r(t)-r(t-T)]$, where $\delta(x-3L/4)$ is a spatial Dirac delta function located at $x=3L/4$ and $r(t)$ is the unit ramp function. Discuss the mode participation in the response.

解：振动方程为：

$$m\frac{\partial^2 y(x,t)}{\partial t^2}+EI\frac{\partial^4 y(x,t)}{\partial x^4}=f\left(x,t\right),0<x<L$$

根据振型的正交性，正则坐标下的动力学方程为：

$$m\int_0^L Y_iY_i dx\ddot{\eta}_i + EI\int_0^L Y_i'''' Y_i dx\eta_i = \int_0^L f(x,t)Y_i(x)dx$$

$$\ddot{\eta}_i+\omega_i^2\eta_i=\int_0^L F_0\delta(x-3L/4)[r(t)-r(t-T)]Y_i(x)dx$$

$$\ddot{\eta}_i+\omega_i^2\eta_i=F_0[r(t)-r(t-T)]c_i\left(\sin\lambda_i\frac{3}{4}L-\frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i\frac{3}{4}L\right)$$

$$where,\quad c_1=\frac{1}{\sqrt{0.510mL}},c_1=\frac{1}{\sqrt{0.503mL}},c_1=\frac{1}{\sqrt{0.501mL}},$$

由斜坡响应叠加可得正则坐标下的响应为：

$$\eta_i(t)=\frac{F_0 c_i}{\omega_i^3 T}\left(\sin\lambda_i\frac{3}{4}L-\frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i\frac{3}{4}L\right)\left[(\omega_i t-\sin\omega_i t)u(t)-\left(\omega_i(t-T)-\sin\omega_i(t-T)\right)u(t-T)\right]$$

梁的稳态响应为：

$$\lambda_1 L=3.21,\lambda_2 L=6.32,\lambda_3 L=9.45,\ \ where\ \ \omega_i=\sqrt{\frac{EI\lambda_i^4}{m}}$$

$$c_1=\frac{1}{\sqrt{0.510mL}},c_1=\frac{1}{\sqrt{0.503mL}},c_1=\frac{1}{\sqrt{0.501mL}},$$

$$Y_i(x)=c_i\left(\sin\lambda_i x-\frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i x\right)$$

$$\begin{aligned}
y(x,t)&=\sum_1^3 Y_i(x)\eta_i(t)\\
&=\sum_1^3\begin{matrix}\dfrac{c_i^{\,2}}{\lambda_i^{\,4}}\left(\sin\lambda_i\dfrac{3}{4}L-\dfrac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i\dfrac{3}{4}L\right)\\ \dfrac{mF_0}{EIT}\left(\sin\lambda_i x-\dfrac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i x\right)\left[\left(t-\dfrac{\sin\omega_i t}{\omega_i}\right)u(t)-\left(t-T-\dfrac{\sin\omega_i(t-T)}{\omega_i}\right)u(t-T)\right]\end{matrix}\\
&\approx 0.0129\frac{L^3F_0}{EIT}\left(\sin\lambda_1 x-\frac{\sin\lambda_1 L}{\sinh\lambda_1 L}\sinh\lambda_1 x\right)\left[\left(t-\frac{\sin\omega_1 t}{\omega_1}\right)u(t)-\left(t-T-\frac{\sin\omega_1(t-T)}{\omega_1}\right)u(t-T)\right]\\
&\quad-0.0013\frac{L^3F_0}{EIT}\left(\sin\lambda_2 x-\frac{\sin\lambda_2 L}{\sinh\lambda_2 L}\sinh\lambda_2 x\right)\left[\left(t-\frac{\sin\omega_2 t}{\omega_2}\right)u(t)-\left(t-T-\frac{\sin\omega_2(t-T)}{\omega_2}\right)u(t-T)\right]\\
&\quad+0.0002\frac{L^3F_0}{EIT}\left(\sin\lambda_3 x-\frac{\sin\lambda_3 L}{\sinh\lambda_3 L}\sinh\lambda_3 x\right)\left[\left(t-\frac{\sin\omega_3 t}{\omega_3}\right)u(t)-\left(t-T-\frac{\sin\omega_3(t-T)}{\omega_3}\right)u(t-T)\right]
\end{aligned}$$

可发现各阶模态对响应的贡献随着阶数的增加而降低。

Problem 4 (45 points in total)

Problem 4.1: Please solve Problem 8.14 in page 460 of the textbook. Note: you can direct use the EOM of the beam, however, you need to choose the correct boundary conditions (10 points).

8.14. Derive the eigenvalue problem for the bending vibration of a beam pinned at $x = 0$ and free at $x = L$ (Fig. 8.36). Then, let $m(x) = m =$ constant, $EI(x) = EI =$ constant, solve the eigenvalue problem and plot the three lowest modes. **Hint:** The system is only semidefinite, as it admits a rigid-body rotation.

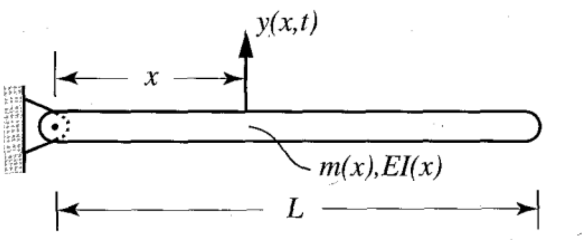


FIGURE 8.36
Beam in bending pinned at $x = 0$ and free at $x = L$

解：梁的振动方程为

$$\frac{\partial}{\partial t}\left(m(x)\dot{y}\right)+\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right)=0, 0<x<L$$

设梁的振动通解为 $y(x,t)=Y(x)b\sin(\omega t+\varphi)$

代入动力学方程：

$$Y'''' = \frac{m\omega^2}{EI}Y$$

用简谐函数和双曲函数表示通解

$$Y(x)=A\cos\lambda x+B\sin\lambda x+C\cosh\lambda x+D\sinh\lambda x,\ \ where\ \lambda^4=\frac{m\omega^2}{EI}$$

左端简支，右端自由的边界条件可以表示为

$$x=0, Y=0, EIY''=0$$

$$x=L, EIY''=0, EIY'''=0$$

在 $x=0$ 处有

$$\begin{cases} A+C=0 \\ \lambda^2(-A+C)=0 \end{cases}$$

可以得到 $A=-C, \lambda^2 C=0$。在 $x=L$ 处有

$$\lambda^2(-A\cos\lambda L-B\sin\lambda L+C\cosh\lambda L+D\sinh\lambda L)=0$$

$$\lambda^3(A\sin\lambda L-B\cos\lambda L+C\sinh\lambda L+D\cosh\lambda L)=0$$

将 $A=-C, \lambda^2 C=0$ 代入有

$$\lambda^2(-B\sin\lambda L + D\sinh\lambda L) = 0$$
$$\lambda^3(-B\cos\lambda L + D\cosh\lambda L) = 0$$

当$\lambda = 0$时存在刚体模态，可以取$Y(x) = A' + B'x$，代入边界条件得到$Y(x) = B'x$。

当$\lambda \neq 0$时，有

$$A = C = 0$$
$$-B\sin\lambda L + D\sinh\lambda L = 0$$
$$-B\cos\lambda L + D\cosh\lambda L = 0$$
$$\sin\lambda L\cosh\lambda L - \cos\lambda L\sinh\lambda L = 0$$

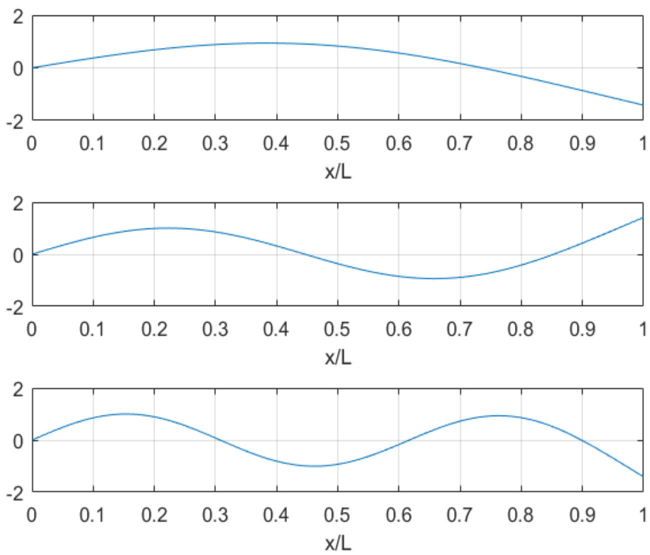
解得前三阶固有频率：

$$\lambda_1 L = 3.9266, \lambda_2 L = 7.0686, \lambda_3 L = 10.2102, \ \ where\ \omega_i = \lambda_i^2\sqrt{\frac{EI}{m}}$$

主振型为：

$$Y_i(x) = \sin\lambda_i x + \frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i x$$

前三阶振型如下图：



Problem 4.2: Please solve problem 8.20 in page 461 of the textbook (5 points).

8.20. Derive the orthogonality relations for the pinned-free beam of Problem 8.14. Then, verify that the modes obtained in Problem 8.14 are indeed orthogonal and explain the meaning of the fact that the rigid-body mode is orthogonal to the remaining modes.

解：**1.正交关系的推导**类似 3.2，由 3.2 可得

$$EIY_j\frac{\partial^3 Y_i}{\partial x^3}\bigg|_0^L - EI\frac{\partial Y_j}{\partial x}\frac{\partial^2 Y_i}{\partial x^2}\bigg|_0^L + \int_0^L EI\frac{\partial^2 Y_j}{\partial x^2}\frac{\partial^2 Y_i}{\partial x^2}dx = \lambda_i\int_0^L Y_j m Y_i dx$$

根据边界条件可得：

$$\int_0^L EI\frac{\partial^2 Y_j}{\partial x^2}\frac{\partial^2 Y_i}{\partial x^2}dx = \lambda_i\int_0^L Y_j m Y_i dx$$

同理有：

$$\int_0^L EI\frac{\partial^2 Y_j}{\partial x^2}\frac{\partial^2 Y_i}{\partial x^2}dx = \lambda_j\int_0^L Y_j m Y_i dx$$

因此

$$\left(\lambda_j - \lambda_i\right)\int_0^L Y_j m Y_i dx = 0$$

$$\int_0^L Y_j m Y_i dx = 0 \quad \left(i \neq j\right)$$

回代得到：

$$\int_0^L EIY_j\frac{\partial^4 Y_i}{\partial x^4}dx \quad \left(i \neq j\right)$$

2.验证各阶模态互相正交

主阵型为

$$Y_i(x) = \sin\lambda_i x + \frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i x$$

当$i \neq j$时

$$\int_0^L mY_i(x)Y_j(x)dx = m\int_0^L\left(\sin\lambda_i L\frac{x}{L} + \frac{\sin\lambda_i L}{\sinh\lambda_i L}\sinh\lambda_i L\frac{x}{L}\right)\left(\sin\lambda_j L\frac{x}{L} + \frac{\sin\lambda_j L}{\sinh\lambda_j L}\sinh\lambda_j L\frac{x}{L}\right)dx$$

$$= m\left[\int_0^L\sin\lambda_i x\sin\lambda_j xdx + \frac{\sin\lambda_i L}{\sinh\lambda_i L}\frac{\sin\lambda_j L}{\sinh\lambda_j L}\int_0^L\sinh\lambda_i x\sinh\lambda_j xdx + \frac{\sin\lambda_i L}{\sinh\lambda_i L}\int_0^L\sinh\lambda_i x\sin\lambda_j xdx + \frac{\sin\lambda_j L}{\sinh\lambda_j L}\int_0^L\sin\lambda_i x\sinh\lambda_j xdx\right]$$

第一项为三角函数乘积的积分

$$\int_0^L\sin\lambda_i x\sin\lambda_j xdx = \left.\frac{\sin\left(\lambda_i - \lambda_j\right)x}{2\left(\lambda_i - \lambda_j\right)} + \frac{\sin\left(\lambda_i + \lambda_j\right)x}{2\left(\lambda_i + \lambda_j\right)}\right|_0^L$$

$$= \left(\frac{\sin\left(\lambda_i - \lambda_j\right)L}{2\left(\lambda_i - \lambda_j\right)L} + \frac{\sin\left(\lambda_i + \lambda_j\right)L}{2\left(\lambda_i + \lambda_j\right)L}\right)L$$

$$= \frac{\cos\lambda_i L\cos\lambda_j L}{\left(\lambda_i^2 - \lambda_j^2\right)L^2}\left(\lambda_j L\tan\lambda_i - \lambda_i L\tan\lambda_j\right)L$$

第二项为双曲函数乘积的积分

$$\frac{\sin\lambda_i L}{\sinh\lambda_i L}\frac{\sin\lambda_j L}{\sinh\lambda_j L}\int_0^L\sinh\lambda_i x\sinh\lambda_j xdx = \frac{\sin\lambda_i L}{\sinh\lambda_i L}\frac{\sin\lambda_j L}{\sinh\lambda_j L}\frac{\lambda_i\cosh\lambda_i L\sinh\lambda_j L - \lambda_j\sinh\lambda_i L\cosh\lambda_j L}{\left(\lambda_i^2 - \lambda_j^2\right)}$$

$$= \frac{\cos\lambda_i L\cos\lambda_j L}{\left(\lambda_i^2 - \lambda_j^2\right)}\left(-\lambda_j\tan\lambda_i + \lambda_i\tan\lambda_j\right)$$

后面两项为三角函数与双曲函数的乘积的积分

$$\frac{\sin\lambda_i L}{\sinh\lambda_i L}\int_0^L \sinh\lambda_i x\sin\lambda_j x dx=\frac{\sin\lambda_i L}{\sinh\lambda_i L}\frac{\lambda_i\cosh\lambda_i L\sin\lambda_j L-\lambda_j\sinh\lambda_i L\cos\lambda_j L}{\left(\lambda_i^2+\lambda_j^2\right)}$$

$$=\frac{\cos\lambda_i L\cos\lambda_j L\left(\lambda_i\tan\lambda_j L-\lambda_j\tan\lambda_i L\right)}{\left(\lambda_i^2+\lambda_j^2\right)}$$

$$\frac{\sin\lambda_j L}{\sinh\lambda_j L}\int_0^L \sin\lambda_i x\sinh\lambda_j x dx=\frac{\sin\lambda_j L}{\sinh\lambda_j L}\frac{-\lambda_i\cos\lambda_i L\sinh\lambda_j L+\lambda_j\sin\lambda_i L\cosh\lambda_j L}{\left(\lambda_i^2+\lambda_j^2\right)}$$

$$=\frac{\cos\lambda_i L\cos\lambda_j L\left(-\lambda_i\tan\lambda_j L+\lambda_j\tan\lambda_i L\right)}{\left(\lambda_i^2+\lambda_j^2\right)}$$

前两项相加为 0，后两项也互为相反数，验证了非刚体模态的正交性。

对主阵型做归一化处理，令：

$$\int_0^L m{c_r}^2 Y_r^2(x)dx=1, r=1,2,\cdots$$

$$m{c_r}^2\int_0^L\left(\sin\lambda_r L\frac{x}{L}+\frac{\sin\lambda_r L}{\sinh\lambda_r L}\sinh\lambda_r L\frac{x}{L}\right)^2 dx$$

$$=m{c_r}^2\left[\left(\frac{x}{2}-\frac{1}{4\lambda_r}\sin 2\lambda_r x\right)+\left(\frac{\sin\lambda_r L}{\sinh\lambda_r L}\right)^2\left(-\frac{x}{2}+\frac{1}{4\lambda_r}\sinh 2\lambda_r x\right)+\frac{\sin\lambda_r L}{2\lambda_r\sinh\lambda_r L}\left(2\sin\lambda_r x\cosh\lambda_r x-2\cos\lambda_r x\sinh\lambda_r x\right)\right]\Bigg|_0^L$$

$$=m{c_r}^2 L\left[\left(\frac{1}{2}-\frac{1}{4\lambda_r L}\sin 2\lambda_r L\right)+\left(\frac{\sin\lambda_r L}{\sinh\lambda_r L}\right)^2\left(-\frac{1}{2}+\frac{1}{4\lambda_r L}\sinh 2\lambda_r L\right)+\frac{\sin\lambda_r L}{2\lambda_r L\sinh\lambda_r L}\left(2\sin\lambda_r L\cosh\lambda_r L-2\cos\lambda_r L\sinh\lambda_r L\right)\right]$$

满足特征方程

$$\sin\lambda_r L\cosh\lambda_r L-\cos\lambda_r L\sinh\lambda_r L=0$$

有

$$m{c_r}^2 L\left[\left(\frac{1}{2}-\frac{1}{4\lambda_r L}\sin 2\lambda_r L\right)+\left(\frac{\sin\lambda_r L}{\sinh\lambda_r L}\right)^2\left(-\frac{1}{2}+\frac{1}{4\lambda_r L}\sinh 2\lambda_r L\right)\right]=1$$

计算可得前三阶固有频率下

$$\left(\frac{1}{2}-\frac{1}{4\lambda_r L}\sin 2\lambda_r L\right)+\left(\frac{\sin\lambda_r L}{\sinh\lambda_r L}\right)^2\left(-\frac{1}{2}+\frac{1}{4\lambda_r L}\sinh 2\lambda_r L\right)$$

的值分别为

$$0.50091, 0.50030, 0.49997$$

因此正则化后的振型可以表示为

$$Y_r(x)=c_r\left(\sin\lambda_r x+\frac{\sin\lambda_r L}{\sinh\lambda_r L}\sinh\lambda_r x\right), 0<x<L, r=1,2,\cdots$$

$$where\ c_1=\sqrt{\frac{1}{0.50091mL}}, c_2=\sqrt{\frac{1}{0.50030mL}}, c_3=\sqrt{\frac{1}{0.49997mL}}$$

3、刚体模态

刚体模态可以取为 $Y(x)=B^{'}x$，则有

$$\int_0^L mB' x Y_j(x)dx = 0$$

$$\begin{aligned}
& mB' A_j \int_0^L x\left(\sin\lambda_j x + \frac{\sin\lambda_j L}{\sinh\lambda_j L}\sinh\lambda_j x\right)dx \\
& = mB' A_j \frac{1}{\lambda_j^2}\left(\sin\lambda_j L - \lambda_j L\cos\lambda_j L + \frac{\sin\lambda_j L}{\sinh\lambda_j L}\left(-\sinh\lambda_j L + \lambda_j L\cosh\lambda_j L\right)\right) \\
& = mB' A_j \frac{1}{\lambda_j^2}\left(\sin\lambda_j L - \lambda_j L\cos\lambda_j L + \cos\lambda_j L\left(-\tan\lambda_j L + \lambda_j L\right)\right) = 0
\end{aligned}$$

正则化的刚体模态：

$$\int_0^L mB^2 x^2 dx = \frac{1}{3}mL^3 B^2 = 1$$

$$Y_0(x) = \sqrt{\frac{3}{mL^3}}x$$

意义：

(a) 刚体模态可以看作是 $\omega = 0$，因此满足

$$Y'''' = \frac{m\omega^2}{EI}Y = 0$$

因此与其余模态必然正交。

b) 利用振型叠加法 $y(x,t) = \sum_{r=1}^{\infty} Y_r(x)\eta_r(t)$ 时，刚体模态与其余模态正交时才能将各个模态解耦独立出来。

Problem 4.3: Please solve Problem 8.33 in page 462 of the textbook (15 points).

8.33. Determine the response of the uniform pinned-free beam of Problem 8.14 subsequent to being released from rest in the deformed configuration $y(x,0) = y_0(x/L)^2$. Discuss the mode participation in the response.

解：响应可以写为如下形式

$$y(x,t) = \sum_{r=1}^{\infty} Y_r(x)\eta_r(t)$$

其中

$$Y_r(x) = c_r\left(\sin\lambda_r x + \frac{\sin\lambda_r L}{\sinh\lambda_r L}\sinh\lambda_r x\right), 0 < x < L, r = 1, 2, \cdots$$

$$Y_0(x) = \sqrt{\frac{3}{mL^3}}x$$

由初始条件

$$y(x,0) = y_0(x/L)^2, \dot{y}(x,0) = 0$$

得正则坐标下的初始条件：

$$
\begin{aligned}
\eta_r(0) &= \int_0^L mY_r(x)y(x,0)dx \\
&= my_0c_r\int_0^L (x/L)^2\left(\sin\lambda_r x+\frac{\sin\lambda_r L}{\sinh\lambda_r L}\sinh\lambda_r x\right)dx \\
&= \frac{my_0}{L^2}c_r\left[\int_0^L x^2\sin\lambda_r xdx+\frac{\sin\lambda_r L}{\sinh\lambda_r L}\int_0^L x^2\sinh\lambda_r xdx\right] \\
&= \frac{my_0}{\lambda_r^3L^2}c_r\left[2\cos\lambda_r L-\left(\lambda_r L\right)^2\cos\lambda_r L+2\lambda_r L\sin\lambda_r L-2\right] \\
&\quad+\frac{my_0}{\lambda_r^3L^2}c_r\frac{\sin\lambda_r L}{\sinh\lambda_r L}\left[2\cosh\lambda_r L+\left(\lambda_r L\right)^2\cosh\lambda_r L-2\lambda_r L\sinh\lambda_r L-2\right] \\
&= \frac{2my_0}{\lambda_r^3L^2}c_r\left[2\cos\lambda_r L-1-\frac{\cos\lambda_r L}{\cosh\lambda_r L}\right]\quad ,r\neq 0
\end{aligned}
$$

$$
\eta_0(0)=\int_0^L mY_0(x)y(x,0)dx=my_0\int_0^L (x/L)^2\sqrt{\frac{3}{mL^3}}xdx=\frac{mL^2y_0}{4}\sqrt{\frac{3}{mL^3}}
$$

$$
\dot{\eta}_r(0)=\int_0^L mY_0(x)\dot{y}_0(x)dx=0
$$

正则坐标下梁的振动方程： $\ddot{\eta}_i+\omega_i^2\eta_i=0$

正则坐标下的响应：

$$
\eta_r(t)=\eta_r(0)\cos\omega_r t,\ \ r=1,2,3\cdots
$$

$$
\eta_0(t)=\frac{mL^2y_0}{4}\sqrt{\frac{3}{mL^3}}
$$

梁振动的响应可以表示为

$$
\begin{aligned}
y(x,t) &= \sum_{r=1}^{\infty}Y_r(x)\eta_r(t) \\
&= \frac{mL^2y_0}{4}\sqrt{\frac{3}{mL^3}}\sqrt{\frac{3}{mL^3}}x+\sum_{r=1}^{\infty}\left\{c_r\left(\sin\lambda_r x+\frac{\sin\lambda_r L}{\sinh\lambda_r L}\sinh\lambda_r x\right)\left[\eta_r(0)\cos\left(\omega_r t\right)\right]\right\} \\
&= \frac{3xy_0}{4}+2y_0\sum_{r=1}^{\infty}\left\{c_r{}^2\frac{m}{\lambda_r^3L^2}\left(2\cos\lambda_r L-1-\frac{\cos\lambda_r L}{\cosh\lambda_r L}\right)\left(\sin\lambda_r x+\frac{\sin\lambda_r L}{\sinh\lambda_r L}\sinh\lambda_r x\right)\cos\left(\omega_r t\right)\right\} \\
&= \frac{3xy_0}{4}+\left(-0.1574\right)\left(\sin\lambda_1 x-0.0279\sinh\lambda_1 x\right)\cos\left(\omega_1 t\right) \\
&\quad+0.0047\left(\sin\lambda_2 x+0.0012\sinh\lambda_2 x\right)\cos\left(\omega_2 t\right) \\
&\quad-0.0091\left(\sin\lambda_3 x-0.0001\sinh\lambda_3 x\right)\cos\left(\omega_3 t\right)
\end{aligned}
$$

$$
where\quad \lambda_1 L=3.9266,\lambda_2 L=7.0686,\lambda_3 L=10.2102,\quad \omega_i=\lambda_i^2\sqrt{\frac{EI}{m}}
$$

主要由第一阶模态以及刚体模态决定了梁的初值响应。

Problem 4.4: Please solve Problem 8.37 in page 463 of the textbook (15 points).

8.37. Determine the response of the beam of Problem 8.14 to the concentrated harmonic force $F(t)=F_0\cos\Omega t$ applied at $x=L$. Discuss the mode participation in the response. Note that the concentrated force can be represented as the distributed force $f(x,t)=F_0\delta(x-L)\cos\Omega t$, where $\delta(x-L)$ is a spatial Dirac delta function (see Eq. (8.263)).

解：已知主阵型：

$$Y_r(x)=c_r\left(\sin\lambda_r x+\frac{\sin\lambda_r L}{\sinh\lambda_r L}\sinh\lambda_r x\right),0<x<L,r=1,2,\cdots,$$

$$where\ c_1=\sqrt{\frac{1}{0.50091mL}},c_2=\sqrt{\frac{1}{0.50030mL}},c_3=\sqrt{\frac{1}{0.49997mL}}$$

$$Y_0(x)=\sqrt{\frac{3}{mL^3}}x$$

正则坐标下梁振动方程：

$$m\int_0^L Y_r Y_r dx\ddot{\eta}_r+EI\int_0^L Y_r'''' Y_r dx\eta_r=\int_0^L F(x,t)Y_r(x)dx$$

$$\ddot{\eta}_r+\omega_r^2\eta_r=\int_0^L F_0\cos\Omega t\delta(x-L)Y_r(x)dx$$

$$\ddot{\eta}_r+\omega_r^2\eta_r=F_0\cos\Omega tY_r(L)$$

对于刚体振型有：

$$\ddot{\eta}_0=F_0\cos\Omega t\sqrt{\frac{3}{mL}}$$

正则坐标下的稳态响应为：

$$\eta_r(t)=\frac{1}{\omega_r^2-\Omega^2}F_0c_r\left(2\sin\lambda_r L\right)\cos\Omega t,r=1,2,3\cdots$$

$$\eta_0(t)=\frac{F_0}{-\Omega^2}\sqrt{\frac{3}{mL}}\cos\Omega t$$

梁的稳态响应为：

$$y(x,t)=\sum_{r=1}^{\infty}Y_r(x)\eta_r(t)$$

$$=\sqrt{\frac{3}{mL^3}}x\frac{F_0}{-\Omega^2}\sqrt{\frac{3}{mL}}\cos\Omega t+\sum_{r=1}^{\infty}c_r{}^2F_0\left(2\sin\lambda_r L\right)\frac{\left(\sin\lambda_r x+\frac{\sin\lambda_r L}{\sinh\lambda_r L}\sinh\lambda_r x\right)\cos\Omega t}{\omega_r^2-\Omega^2}$$

$$\approx\frac{3F_0x}{-mL^2\Omega^2}\cos\Omega t$$

$$-\frac{2.8222F_0}{ml\left(\omega_1^2-\Omega^2\right)}\left(\sin\lambda_1 x-0.0279\sinh\lambda_1 x\right)\cos\Omega t$$

$$+\frac{2.8233F_0}{ml\left(\omega_2^2-\Omega^2\right)}\left(\sin\lambda_2 x+0.0012\sinh\lambda_2 x\right)\cos\Omega t$$

$$-\frac{2.8234F_0}{ml\left(\omega_3^2-\Omega^2\right)}\left(\sin\lambda_3 x-0.0001\sinh\lambda_3 x\right)\cos\Omega t$$

可发现，当外激励Ω频率接近于零时受刚体模态影响，当Ω接近第i阶固有频率时，第i阶模态对响应的贡献就越大。

Problem 5 (20 points in total)

Problem 5.1: Please solve Problem 8.15 in page 461 of the textbook. Note: you can direct use the EOM of the beam, however, you need to choose the correct boundary conditions (15 points).

8.15. Derive the eigenvalue problem for the bending vibration of a beam free at both ends (Fig. 8.37). Then, let $m(x) = m =$ constant, $EI(x) = EI =$ constant, solve the eigenvalue problem and plot the four lowest modes. **Hint:** The system is only semidefinite and it admits two rigid-body motions, one representing transverse translation and the other rotation.

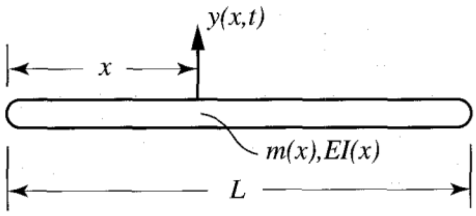


FIGURE 8.37
Free-free beam in bending

解：梁的振动方程为

$$\frac{\partial}{\partial t}\left(m(x)\dot{y}\right)+\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right)=0, 0<x<L$$

边界条件应当满足

$$EI\frac{\partial\theta}{\partial x}\delta\theta-\frac{\partial}{\partial x}\left(EI\frac{\partial\theta}{\partial x}\right)\delta y=0$$

设梁的振动通解为

$$y(x,t)=Y(x)b\sin(\omega t+\varphi)$$

将其代入动力学方程中，

$$Y'''' = \frac{m\omega^2}{EI}Y$$

用简谐函数和双曲函数表示通解

$$Y(x)=A\cos\lambda x+B\sin\lambda x+C\cosh\lambda x+D\sinh\lambda x,\ \ where\ \lambda^4=\frac{m\omega^2}{EI}$$

左端自由，右端自由的边界条件可以表示为

$$x=0, EIY''=0, EIY'''=0$$
$$x=L, EIY''=0, EIY'''=0$$

在 $x=0$ 处有

$$\lambda^2(-A+C)=0$$
$$\lambda^3(-B+D)=0$$

在 $x=L$ 处有

$$-A\cos\lambda L - B\sin\lambda L + A\cosh\lambda L + B\sinh\lambda L = 0$$
$$A\sin\lambda L - B\cos\lambda L + A\sinh\lambda L + B\cosh\lambda L = 0$$

当$\lambda = 0$时存在刚体模态，可以取

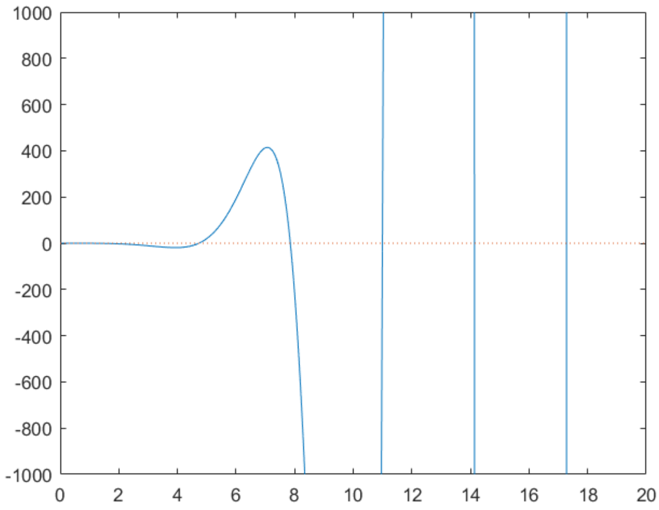
$$Y(x) = A' + B'x$$

当$\lambda \neq 0$时，可以得到$A = C, B = D$。

得到特征方程为

$$\cos\lambda L * \cosh\lambda L - 1 = 0$$

曲线如图所示



四个零点分别为

$$\lambda_1 L = 4.73, \lambda_2 L = 7.85, \lambda_3 L = 11.00, \lambda_4 L = 14.14$$

因此梁的前三阶固有频率分别为：

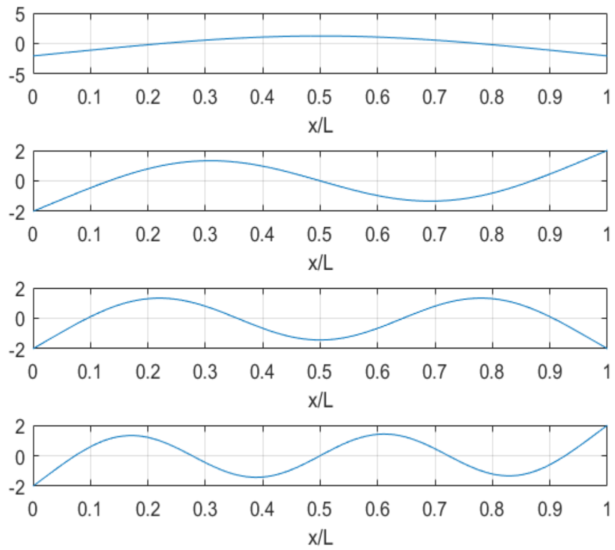
$$\omega = (\lambda L)^2 \sqrt{\frac{EI}{mL^4}}$$

$$\omega_1 = 22.37\sqrt{\frac{EI}{mL^4}} \quad \omega_2 = 61.62\sqrt{\frac{EI}{mL^4}} \quad \omega_3 = 121.00\sqrt{\frac{EI}{mL^4}} \quad \omega_4 = 199.94\sqrt{\frac{EI}{mL^4}}$$

各阶主阵型为

$$Y_i(x) = \frac{-\cosh\lambda_i L + \cos\lambda_i L}{\sinh\lambda_i L + \sin\lambda_i L}\left(\cos\lambda_i x + \cosh\lambda_i x\right) + \left(\sin\lambda_i x + \sinh\lambda_i x\right), i = 1, 2, 3\cdots$$

前四阶主阵型如下图：



Prblem 5.2: Please solve problem 8.21 in page 461 of the textbook (5 points)

8.21. Derive the orthogonality relations for the free-free beam of Problem 8.15. Make sure that the modes obtained in Problem 8.15 are indeed orthogonal and explain the meaning of the fact that each of the two rigid-body modes is orthogonal to the remaining modes, including the other rigid-body mode.

解：梁的**正交关系的推导**类似 3.2，由 3.2 可得

$$EIY_j \frac{\partial^3 Y_i}{\partial x^3}\bigg|_0^L - EI\frac{\partial Y_j}{\partial x}\frac{\partial^2 Y_i}{\partial x^2}\bigg|_0^L + \int_0^L EI\frac{\partial^2 Y_j}{\partial x^2}\frac{\partial^2 Y_i}{\partial x^2}dx = \lambda_i \int_0^L Y_j m Y_i dx$$

根据边界条件可得：

$$\int_0^L EI\frac{\partial^2 Y_j}{\partial x^2}\frac{\partial^2 Y_i}{\partial x^2}dx = \lambda_i \int_0^L Y_j m Y_i dx$$

同理有：

$$\int_0^L EI\frac{\partial^2 Y_j}{\partial x^2}\frac{\partial^2 Y_i}{\partial x^2}dx = \lambda_j \int_0^L Y_j m Y_i dx$$

因此

$$\left(\lambda_j - \lambda_i\right)\int_0^L Y_j m Y_i dx = 0$$

$$\int_0^L Y_j m Y_i dx = 0 \quad \left(i \neq j\right)$$

回代得到：

$$\int_0^L EIY_j \frac{\partial^4 Y_i}{\partial x^4}dx \quad \left(i \neq j\right)$$

正交性验证：满足特征方程： $\cos\lambda_i L * \cosh\lambda L - 1 = 0$

1.非刚体模态之间（过程略）：

$$Y_i(x) = \frac{-\cosh\lambda_i L + \cos\lambda_i L}{\sinh\lambda_i L + \sin\lambda_i L}\left(\cos\lambda_i x + \cosh\lambda_i x\right) + \left(\sin\lambda_i x + \sinh\lambda_i x\right)$$

满足

$$\int_0^L mY_iY_jdx = 0 \quad i \neq j \neq 0$$

2.刚体模态与非刚体模态之间：

$$\int_0^L mBxY_idx = \frac{2-2\cos(L\lambda_i)\cosh(L\lambda_i)}{\lambda_i^2\left(\sin(L\lambda_i)+\sinh(L\lambda_i)\right)}$$

根据特征方程，上式为零。

且：

$$\int_0^L mA'Y_idx = 0$$

3.两个刚体模态之间

两个刚体模态对应的固有频率均为零，为了验证其正交性，需自行设置合适的主振型函数。

设两个刚体模态为：

$$Y_0 = A+\frac{BL}{2}, \hat{Y}_0 = B\left(x-\frac{L}{2}\right)$$

满足：

$$\int_0^L m\left(A+\frac{BL}{2}\right)B\left(x-\frac{L}{2}\right)dx = 0,$$

$$\int_0^L EI\hat{Y}_0\frac{\partial^4 Y_0}{\partial x^4}dx = \omega_0{}^2\int_0^L \hat{Y}_0 mY_0 dx = 0$$

两个刚体模态分别表示梁的平移和梁的转动，相互独立。

意义同 4.2。