HOMEWORK 4-5

Problem 1: (5 points)

解：由多自由度系统动力学方程得出的特征值与特征向量满足

$$\left(K-\lambda_i M\right)u_i=0$$

如果我们找到一组特征向量使得特征向量矩阵$U=\begin{bmatrix}u_1 & u_2 & \cdots & u_n\end{bmatrix}$满足

$$U^T M U=I_n$$

则有

$$u_j{}^T M u_i=\begin{cases}1, i=j\\ 0, i\neq j\end{cases}$$

在第一个方程两端同时左乘$u_j{}^T$，即

$$u_j{}^T K u_i-u_j{}^T\lambda_i M u_i=0$$

而常数λ_i可以单独提出

$$u_j{}^T K u_i=\begin{cases}\lambda_i, i=j\\ 0, i\neq j\end{cases}$$

因此我们证明了

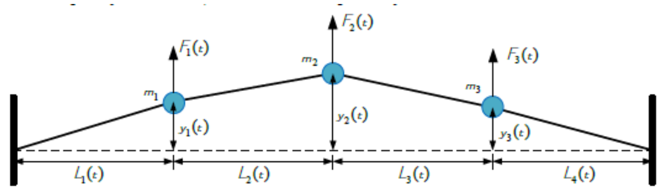
$$U^T K U=\begin{bmatrix}\lambda_1 & & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}$$

Problem 2:

Section 1：方程推导（1-4 小题）

(1) (10 points)

解：令四段细绳与虚线之间的夹角分别为$\theta_1,\theta_2,\theta_3,\theta_4$，且以逆时针方向为正方向



则有

$$\tan\theta_1=\frac{y_1}{L_1},\tan\theta_2=\frac{y_2-y_1}{L_2},\tan\theta_3=\frac{y_3-y_2}{L_3},\tan\theta_4=\frac{-y_3}{L_4} \qquad （1\text{-}1）$$

对于任意一个质点做受力分析，共受到重力竖直向下$m_i g$，左端细绳拉力T_i沿绳方向向左，右端细绳拉力T_{i+1}沿绳方向向右，以及外力$F_i(t)$。由牛顿第二定律有

$$\begin{aligned}&T_{i+1}\cos\theta_{i+1}-T_i\cos\theta_i=m_i\ddot{x}_i\\&F_i(t)-m_i g+T_{i+1}\sin\theta_{i+1}-T_i\sin\theta_i=m_i\ddot{y}_i\end{aligned} \qquad （1\text{-}2）$$

在小变形假设下，

$$\sin\theta\approx\theta,\tan\theta\approx\theta,\cos\theta\approx 1$$

若令细绳中的张力为恒定值

$$T_i=T$$

则方程（1-2）中第一个方程中$\ddot{x}_i=0$恒定成立。（1-2）中的第二项则有

$$F_i(t)-m_i g+T\sin\theta_{i+1}-T\sin\theta_i=m_i\ddot{y}_i$$

将方程（1-1）代入

$$\begin{aligned}&F_1(t)-m_1 g+T\frac{y_2-y_1}{L_2}-T\frac{y_1}{L_1}=m_1\ddot{y}_1\\&F_2(t)-m_2 g+T\frac{y_3-y_2}{L_3}-T\frac{y_2-y_1}{L_2}=m_2\ddot{y}_2\\&F_3(t)-m_3 g+T\frac{-y_3}{L_4}-T\frac{y_3-y_2}{L_3}=m_3\ddot{y}_3\end{aligned} \qquad （1\text{-}3）$$

假设系统的静平衡位置为y_{ei}，静平衡位置附近的微小振动为$\tilde{y}_i$，则

$$y_i=\tilde{y}_i+y_{ei}$$

代入方程（1-3）中得到平衡位置应满足的方程如下

$$\begin{aligned}&m_1 g+\left(\frac{T}{L_1}+\frac{T}{L_2}\right)y_{e1}-\frac{T}{L_2}y_{e2}=0\\&m_2 g-\frac{T}{L_1}y_{e1}+\left(\frac{T}{L_2}+\frac{T}{L_3}\right)y_{e2}-\frac{T}{L_3}y_{e3}=0\\&m_3 g-\frac{T}{L_3}y_{e2}+\left(\frac{T}{L_3}+\frac{T}{L_4}\right)y_{e3}=0\end{aligned} \qquad （1\text{-}4）$$

同时振动微分方程为

$$
\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}
\begin{Bmatrix} \ddot{\tilde{y}}_1 \\ \ddot{\tilde{y}}_2 \\ \ddot{\tilde{y}}_3 \end{Bmatrix}
+
\begin{bmatrix} \dfrac{T}{L_1}+\dfrac{T}{L_2} & -\dfrac{T}{L_2} & 0 \\ -\dfrac{T}{L_2} & \dfrac{T}{L_2}+\dfrac{T}{L_3} & -\dfrac{T}{L_3} \\ 0 & -\dfrac{T}{L_3} & \dfrac{T}{L_3}+\dfrac{T}{L_4} \end{bmatrix}
\begin{Bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \end{Bmatrix}
=
\begin{Bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{Bmatrix}
\tag{1-5}
$$

(2) (15 points)

解：以三个质点在竖直方向上的位移为广义坐标，以向上为正方向，忽略在水平方向上的位移。以图中虚线的高度作为零势能面，则动能和势能分别可以表示为

$$T=\frac{1}{2}m_1(\dot{y}_1)^2+\frac{1}{2}m_2(\dot{y}_2)^2+\frac{1}{2}m_3(\dot{y}_3)^2 \tag{2-1}$$

$$V=m_1gy_1+m_2gy_2+m_3gy_3 \tag{2-2}$$

将细绳中的张力作为广义力处理有

$$\underset{\sim}{r}_1=y_1\underset{\sim}{i},\ \underset{\sim}{r}_2=y_2\underset{\sim}{i},\ \underset{\sim}{r}_3=y_3\underset{\sim}{i}$$

$$\begin{aligned}Q_1&=\underset{\sim}{F}_1\cdot\underset{\sim}{i}+(-\underset{\sim}{T}_1)\cdot\underset{\sim}{i}+\underset{\sim}{T}_2\cdot\underset{\sim}{i}=F_1-T_1\sin\theta_1+T_2\sin\theta_2\\Q_2&=\underset{\sim}{F}_2\cdot\underset{\sim}{i}+(-\underset{\sim}{T}_2)\cdot\underset{\sim}{i}+\underset{\sim}{T}_3\cdot\underset{\sim}{i}=F_2-T_2\sin\theta_2+T_3\sin\theta_3\\Q_3&=\underset{\sim}{F}_3\cdot\underset{\sim}{i}+(-\underset{\sim}{T}_3)\cdot\underset{\sim}{i}+\underset{\sim}{T}_4\cdot\underset{\sim}{i}=F_3-T_3\sin\theta_3+T_4\sin\theta_4\end{aligned} \tag{2-3}$$

代入 Lagrange 方程中有

$$\begin{aligned}m_1\ddot{y}_1+m_1g&=F_1-T_1\sin\theta_1+T_2\sin\theta_2\\m_2\ddot{y}_2+m_2g&=F_2-T_2\sin\theta_2+T_3\sin\theta_3\\m_3\ddot{y}_3+m_3g&=F_3-T_3\sin\theta_3+T_4\sin\theta_4\end{aligned}$$

忽略绳中张力的差别，即认为$T_i=T$，

$$\begin{aligned}m_1\ddot{y}_1+m_1g+T\sin\theta_1-T\sin\theta_2&=F_1\\m_2\ddot{y}_2+m_2g+T\sin\theta_2-T\sin\theta_3&=F_2\\m_3\ddot{y}_3+m_3g+T\sin\theta_3-T\sin\theta_4&=F_3\end{aligned} \tag{2-4}$$

其中

$$\tan\theta_1=\frac{y_1}{L_1},\tan\theta_2=\frac{y_2-y_1}{L_2},\tan\theta_3=\frac{y_3-y_2}{L_3},\tan\theta_4=\frac{-y_3}{L_4} \tag{1-1}$$

(3) (5 points)

解：系统的平衡位置应当满足

$$\begin{aligned}m_1g+T\sin\theta_1-T\sin\theta_2&=0\\m_2g+T\sin\theta_2-T\sin\theta_3&=0\\m_3g+T\sin\theta_3-T\sin\theta_4&=0\end{aligned}$$

在任意大位移的情况下无法解析解出。

(4) (10 points)

解：在小振幅的假设下，绳中张力可以认为是相等的，且

$$\sin\theta \approx \theta, \tan\theta \approx \theta, \cos\theta \approx 1$$

并且平衡位置满足

$$\begin{aligned} m_1 g + T\sin\bar{\theta}_1 - T\sin\bar{\theta}_2 &= 0 \\ m_2 g + T\sin\bar{\theta}_2 - T\sin\bar{\theta}_3 &= 0 \\ m_3 g + T\sin\bar{\theta}_3 - T\sin\bar{\theta}_4 &= 0 \end{aligned}$$

可以得到小振幅下的振动方程

$$\begin{aligned} m_1\ddot{y}_1 + m_1 g + T\cos\bar{\theta}_1\theta_1 - T\cos\bar{\theta}_2\theta_2 &= F_1 \\ m_2\ddot{y}_2 + m_2 g + T\cos\bar{\theta}_2\theta_2 - T\cos\bar{\theta}_3\theta_3 &= F_2 \\ m_3\ddot{y}_3 + m_3 g + T\cos\bar{\theta}_3\theta_3 - T\cos\bar{\theta}_4\theta_4 &= F_3 \end{aligned}$$

若系统偏离水平位置的位移很小，我们可以认为（**温馨提示：作业可以这样假设，其余时刻还需慎重，如果可以的话尽量数值求解得出较为精确的结果**）

$$\sin\bar{\theta} \approx \bar{\theta}, \tan\bar{\theta} \approx \bar{\theta}, \cos\bar{\theta} \approx 1$$

则得到了平衡位置

$$\begin{bmatrix} \frac{T}{L_1}+\frac{T}{L_2} & -\frac{T}{L_2} & 0 \\ -\frac{T}{L_2} & \frac{T}{L_2}+\frac{T}{L_3} & -\frac{T}{L_3} \\ 0 & -\frac{T}{L_3} & \frac{T}{L_3}+\frac{T}{L_4} \end{bmatrix} \begin{Bmatrix} y_{e1} \\ y_{e2} \\ y_{e3} \end{Bmatrix} = \begin{Bmatrix} -m_1 g \\ -m_2 g \\ -m_3 g \end{Bmatrix}$$

同时振动微分方程为

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{\tilde{y}}_1 \\ \ddot{\tilde{y}}_2 \\ \ddot{\tilde{y}}_3 \end{Bmatrix} + \begin{bmatrix} \frac{T}{L_1}+\frac{T}{L_2} & -\frac{T}{L_2} & 0 \\ -\frac{T}{L_2} & \frac{T}{L_2}+\frac{T}{L_3} & -\frac{T}{L_3} \\ 0 & -\frac{T}{L_3} & \frac{T}{L_3}+\frac{T}{L_4} \end{bmatrix} \begin{Bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{Bmatrix}$$

Section 2.1 无阻尼，初条件响应。

(5) (10 points)

解：在 $m_i = m, L_i = L$ 的条件下，平衡位置满足

$$\begin{Bmatrix} y_{e1} \\ y_{e2} \\ y_{e3} \end{Bmatrix} = \frac{mgL}{T} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}^{-1} \begin{Bmatrix} -1 \\ -1 \\ -1 \end{Bmatrix} = \begin{Bmatrix} -1.5 \\ -2 \\ -1.5 \end{Bmatrix} \frac{mgL}{T}$$

同时振动微分方程为

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\tilde{y}}_1 \\ \ddot{\tilde{y}}_2 \\ \ddot{\tilde{y}}_3 \end{Bmatrix} + \frac{T}{mL} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

特征方程为

$$\begin{vmatrix} 2-\lambda\frac{mL}{T} & -1 & 0 \\ -1 & 2-\lambda\frac{mL}{T} & -1 \\ 0 & -1 & 2-\lambda\frac{mL}{T} \end{vmatrix} = \left(2-\lambda\frac{mL}{T}\right)\left(2+\sqrt{2}-\lambda\frac{mL}{T}\right)\left(2-\sqrt{2}-\lambda\frac{mL}{T}\right) = 0$$

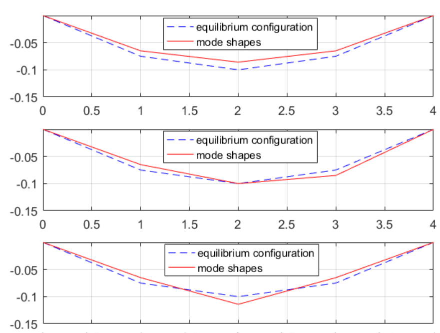
特征值分别为

$$\lambda_1 = \frac{\left(2-\sqrt{2}\right)T}{mL}, \lambda_2 = \frac{2T}{mL}, \lambda_3 = \frac{\left(2+\sqrt{2}\right)T}{mL}$$

$$\omega_1 = 0.7654\sqrt{\frac{T}{mL}}, \omega_2 = 1.4142\sqrt{\frac{T}{mL}}, \omega_3 = 1.8478\sqrt{\frac{T}{mL}}$$

特征向量分别为

$$u_1 = \begin{pmatrix} 1 & \sqrt{2} & 1 \end{pmatrix}^T, u_2 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T, u_3 = \begin{pmatrix} 1 & -\sqrt{2} & 1 \end{pmatrix}^T$$



第二种模态是反向振动模态，其中处于对称中心的点将会一直保持在静止状态；第一种和第三种模态是轴对称的，但是第一种模态下所有的点均处于平衡构型同一侧（同向振动），而第三种模态下相邻两个点分别处于平衡构型的两侧。

注：在涉及符号运算时，可以先将惯性阵前面的符号消去；这样在运算过程中只会影响正则化时的矩阵，但是运算过程会方便很多。

(6) (10 points)

解：令

$$U=\begin{bmatrix}1 & 1 & 1\\ \sqrt{2} & 0 & -\sqrt{2}\\ 1 & -1 & 1\end{bmatrix}$$

$$U^TMU=\begin{bmatrix}1 & \sqrt{2} & 1\\ 1 & 0 & -1\\ 1 & -\sqrt{2} & 1\end{bmatrix}I\begin{bmatrix}1 & 1 & 1\\ \sqrt{2} & 0 & -\sqrt{2}\\ 1 & -1 & 1\end{bmatrix}=\begin{bmatrix}4 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 4\end{bmatrix}$$

因此模态是正交的。如果取

$$u_1=\frac{1}{2}\begin{pmatrix}1 & \sqrt{2} & 1\end{pmatrix}^T,u_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & -1\end{pmatrix}^T,u_3=\frac{1}{2}\begin{pmatrix}1 & -\sqrt{2} & 1\end{pmatrix}^T$$

则能满足

$$U^TMU=I,U^TKU=\Lambda$$

(7) (15 points)

解：令

$$y(t)=U\eta(t)=\sum_{r=1}^{n}u_r\eta_r(t)=\sum_{r=1}^{n}\eta_r(t)u_r$$

$$\eta(t)=U^{-1}y(t)=U^TMy(t)$$

对于整个方程有

$$U^TMU\ddot{\eta}+U^TKU\eta=0$$

并且已经解耦，初始条件为

$$\eta(0)=U^TMy(0),\dot{\eta}(0)=U^TM\dot{y}(0)$$

方程的解为

$$y(t)=\sum_{r=1}^{n}\eta_r(t)u_r=\sum_{r=1}^{n}\left[u_r{}^TMy(0)\cos\omega_rt+\frac{1}{\omega_r}u_r{}^TM\dot{y}(0)\sin\omega_rt\right]u_r$$

（a）初始条件为： $y(0)=[0,1,0]^T,\dot{y}(0)=0$

$$y(t)=\sum_{r=1}^{n}\eta_r(t)u_r=\left[\frac{1}{4}\begin{pmatrix}1 & \sqrt{2} & 1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\cos\omega_1t\right]\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}+\left[\frac{1}{2}\begin{pmatrix}1 & 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\cos\omega_2t\right]\begin{pmatrix}1\\0\\-1\end{pmatrix}+\left[\frac{1}{4}\begin{pmatrix}1 & -\sqrt{2} & 1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\cos\omega_3t\right]\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}$$
$$=\frac{\sqrt{2}}{4}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}\cos\omega_1t-\frac{\sqrt{2}}{4}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}\cos\omega_3t=\frac{\sqrt{2}}{4}\begin{pmatrix}\cos\omega_1t-\cos\omega_3t\\\sqrt{2}\left(\cos\omega_1t+\cos\omega_3t\right)\\\cos\omega_1t-\cos\omega_3t\end{pmatrix}$$

因为 $y(0)=[0,1,0]^T=\frac{\sqrt{2}}{2}\left(u_1-u_3\right)$，因此结果也只与第一、三这两个模态相关。

（b）初始条件为： $y(0)=[-1,0,1]^T,\dot{y}(0)=0$

$$y(t)=\sum_{r=1}^{n}\eta_r(t)u_r=\left[\frac{1}{4}\begin{pmatrix}1 & \sqrt{2} & 1\end{pmatrix}\begin{pmatrix}-1\\0\\1\end{pmatrix}\cos\omega_1t\right]\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}+\left[\frac{1}{2}\begin{pmatrix}1 & 0 & -1\end{pmatrix}\begin{pmatrix}-1\\0\\1\end{pmatrix}\cos\omega_2t\right]\begin{pmatrix}1\\0\\-1\end{pmatrix}+\left[\frac{1}{4}\begin{pmatrix}1 & -\sqrt{2} & 1\end{pmatrix}\begin{pmatrix}-1\\0\\1\end{pmatrix}\cos\omega_3t\right]\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}=\begin{pmatrix}-\cos\omega_2t\\0\\\cos\omega_2t\end{pmatrix}$$

因为 $y(0)=[-1,0,1]^T=-u_2$，因此结果也只与第二个模态相关。

（c）初始条件为： $y(0)=0,\dot{y}(0)=[0,0,1]^T$

$$y(t)=\sum_{r=1}^{n}\eta_r(t)u_r=\left[\frac{1}{4\omega_1}\begin{pmatrix}1 & \sqrt{2} & 1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\sin\omega_1t\right]\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}+\left[\frac{1}{2\omega_2}\begin{pmatrix}1 & 0 & -1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\sin\omega_2t\right]\begin{pmatrix}1\\0\\-1\end{pmatrix}+\left[\frac{1}{4\omega_3}\begin{pmatrix}1 & -\sqrt{2} & 1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\sin\omega_3t\right]\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}$$
$$=\frac{1}{4\omega_1}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}\sin\omega_1t-\frac{1}{2\omega_2}\begin{pmatrix}1\\0\\-1\end{pmatrix}\sin\omega_2t+\frac{1}{4\omega_3}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}\sin\omega_3t$$

因为 $\dot{y}(0)=[0,0,1]^T=\frac{1}{2}\left(u_1+u_2-u_2\right)$，因此结果与三个模态都相关。

(8) (10 points)

解：解：在 $m_1 = m_2 = m, m_3 = 2m, L_i = L$ 的条件下，平衡位置满足

$$\begin{Bmatrix} y_{e1} \\ y_{e2} \\ y_{e3} \end{Bmatrix} = \frac{mgL}{T}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}^{-1}\begin{Bmatrix} -1 \\ -1 \\ -2 \end{Bmatrix} = \frac{mgL}{4T}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{Bmatrix} -1 \\ -1 \\ -2 \end{Bmatrix} = \begin{Bmatrix} -1.75 \\ -2.5 \\ -2.25 \end{Bmatrix}\frac{mgL}{T}$$

同时振动微分方程为

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{\tilde{y}}_1 \\ \ddot{\tilde{y}}_2 \\ \ddot{\tilde{y}}_3 \end{Bmatrix} + \frac{T}{mL}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{Bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

特征方程为

$$\begin{vmatrix} 2-\lambda\dfrac{mL}{T} & -1 & 0 \\ -1 & 2-\lambda\dfrac{mL}{T} & -1 \\ 0 & -1 & 2-2\lambda\dfrac{mL}{T} \end{vmatrix} = 0$$

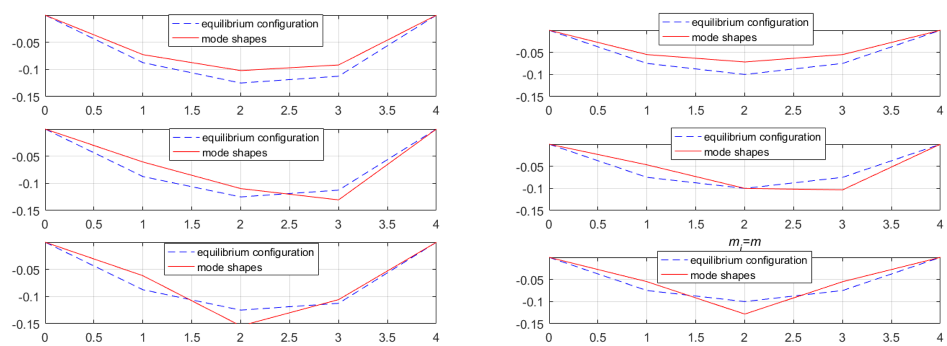
特征值分别为

$$\lambda_1 = \frac{0.4486T}{mL}, \lambda_2 = \frac{1.4268T}{mL}, \lambda_3 = \frac{3.1246T}{mL}$$

$$\omega_1 = 0.6698\sqrt{\frac{T}{mL}}, \omega_2 = 1.1945\sqrt{\frac{T}{mL}}, \omega_3 = 1.7676\sqrt{\frac{T}{mL}}$$

特征向量分别为

$$u_1 = \begin{pmatrix} 0.3685 & 0.5717 & 0.5184 \end{pmatrix}^T, u_2 = \begin{pmatrix} 0.6696 & 0.3838 & -0.4496 \end{pmatrix}^T, u_3 = \begin{pmatrix} 0.6449 & -0.7252 & 0.1707 \end{pmatrix}^T$$



左为第八题，右为第五题。具体的对比：略。

(9) (10 points)

解：令

$$U=\begin{bmatrix}0.3685 & 0.6696 & 0.6449\\ 0.5717 & 0.3838 & -0.7252\\ 0.5184 & -0.4496 & 0.1707\end{bmatrix}$$

$$U^TMU=\begin{bmatrix}0.3685 & 0.5717 & 0.5184\\ 0.6696 & 0.3838 & -0.4496\\ 0.6449 & -0.7252 & 0.1707\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}0.3685 & 0.6696 & 0.6449\\ 0.5717 & 0.3838 & -0.7252\\ 0.5184 & -0.4496 & 0.1707\end{bmatrix}=I$$

$$\begin{aligned}U^TKU&=\begin{bmatrix}0.3685 & 0.5717 & 0.5184\\ 0.6696 & 0.3838 & -0.4496\\ 0.6449 & -0.7252 & 0.1707\end{bmatrix}\frac{T}{mL}\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}\begin{bmatrix}0.3685 & 0.6696 & 0.6449\\ 0.5717 & 0.3838 & -0.7252\\ 0.5184 & -0.4496 & 0.1707\end{bmatrix}^T\\ &=\frac{T}{mL}\begin{bmatrix}0.4486 & 0 & 0\\ 0 & 1.4268 & 0\\ 0 & 0 & 3.1246\end{bmatrix}=\Lambda\end{aligned}$$

已经能满足

$$U^TMU=I,U^TKU=\Lambda$$

Section 2.2 无阻尼，谐波激励和任意外激励。

(10) (20 points)

解：令

$$\boldsymbol{y}(t)=U\boldsymbol{\eta}(t)=\sum_{r=1}^{n}u_r\boldsymbol{\eta}_r(t)=\sum_{r=1}^{n}\boldsymbol{\eta}_r(t)u_r$$

$$\boldsymbol{\eta}(t)=U^{-1}\boldsymbol{y}(t)=U^TM\boldsymbol{y}(t)$$

对于整个方程有

$$U^TMU\ddot{\boldsymbol{\eta}}+U^TKU\boldsymbol{\eta}=U^TQ$$

并且已经解耦，

$$\ddot{\boldsymbol{\eta}}_r+\omega_r^2\boldsymbol{\eta}_r=u_r^TF\cos\omega t$$

方程的解为

$$\boldsymbol{\eta}_r(t)=\frac{u_r^TF}{\omega_r^2-\omega^2}\cos\omega t$$

因此系统的稳态响应可以写为

$$\boldsymbol{y}(t)=\sum_{r=1}^{n}u_r\boldsymbol{\eta}_r(t)=\sum_{r=1}^{n}u_r\frac{u_r^TF}{\omega_r^2-\omega^2}\cos\omega t=\sum_{r=1}^{n}\frac{u_ru_r^T}{\omega_r^2-\omega^2}F\cos\omega t$$

对于本题先将常数 m 从惯性阵中消掉，

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}\begin{Bmatrix}\ddot{\tilde{y}}_1\\\ddot{\tilde{y}}_2\\\ddot{\tilde{y}}_3\end{Bmatrix}+\frac{T}{mL}\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}\begin{Bmatrix}\tilde{y}_1\\\tilde{y}_2\\\tilde{y}_3\end{Bmatrix}=\frac{F_0}{m}\cos\omega t$$

由上一小题得到：

$$u_1=\begin{pmatrix}0.3685&0.5717&0.5184\end{pmatrix}^T,u_2=\begin{pmatrix}0.6696&0.3838&-0.4496\end{pmatrix}^T,u_3=\begin{pmatrix}0.6449&-0.7252&0.1707\end{pmatrix}^T$$

$$\omega_1=0.6698\sqrt{\frac{T}{mL}},\omega_2=1.1945\sqrt{\frac{T}{mL}},\omega_3=1.7676\sqrt{\frac{T}{mL}}$$

（a） $F=\frac{1}{m}[0\quad F_0\quad 0]^T$ 时， $\omega=0.65$ 此处认为 $T=mL$ 对比才有意义。

$$\boldsymbol{y}(t)=\sum_{r=1}^{n}\frac{u_ru_r^T}{\omega_r^2-\omega^2}\frac{F_0}{m}\cos\omega t=\begin{pmatrix}8.1496\\12.8560\\11.1307\end{pmatrix}\frac{F_0}{m}\cos 0.65t$$

与第一阶振型比较接近（缩放后为（8.0668,12.5147,11.3484））；

（b） $F=\frac{1}{m}[0\quad F_0\quad 0]^T$ 时， $\omega=1.2$

$$\boldsymbol{y}(t)=\sum_{r=1}^{n}\frac{u_ru_r^T}{\omega_r^2-\omega^2}\frac{F_0}{m}\cos\omega t=\begin{pmatrix}-19.9855\\-11.1919\\12.7180\end{pmatrix}\frac{F_0}{m}\cos 1.2t$$

(11) (25 points)

解：单自由度系统对任意激励的响应为

$$x(t)=e^{-\xi\omega_n t}\left(x_0\cos\omega_d t+\frac{\dot{x}_0+\xi\omega_n x_0}{\omega_d}\sin\omega_d t\right)+\int_0^t P(\tau)h(t-\tau)d\tau$$

其中第一项为初值响应；第二项中

$$h(t)=\frac{1}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t$$

是脉冲响应函数。并且脉冲响应函数 $h(t)$ 和阶跃响应函数 $g(t)$ 满足

$$h(t)=\frac{dg(t)}{dt}$$

$$g(t)=\frac{1}{k}\left[1-e^{-\zeta\omega_n t}\left(\cos\omega_d t+\frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right)\right]$$

因此无初值响应为

$$\begin{aligned}x(t)&=\int_0^t P(\tau)h(t-\tau)d\tau\\&=-P(\tau)g(t-\tau)\big|_0^t+\int_0^t P(\tau)\frac{dg(t-\tau)}{d\tau}d\tau+\int_0^t \dot{P}(\tau)g(t-\tau)d\tau+\int_0^t P(\tau)h(t-\tau)d\tau\\&=-P(\tau)g(t-\tau)\big|_0^t+\int_0^t \dot{P}(\tau)g(t-\tau)d\tau\end{aligned}$$

对于任意的连续激励均可以使用，若激励不连续则分段使用。对斜坡函数 $r(t)$ 的响应为

$$\begin{aligned}x(t)&=\int_0^t \tau\frac{1}{m\omega_d}e^{-\xi\omega_n(t-\tau)}\sin\omega_d(t-\tau)d\tau\\&=\frac{1}{k}\left(t-\frac{2\zeta}{\omega_n}+\frac{e^{-\zeta\omega_n t}}{\omega_n}\left(2\zeta\cos\omega_d t+\frac{(\zeta\omega_n)^2-\omega_d^2}{\omega_d\omega_n}\sin\omega_d t\right)\right)\end{aligned}$$

对于本题的无阻尼系统有

$$x(t)=\frac{1}{k}\left(t-\frac{\sin\omega_n t}{\omega_n}\right)$$

令

$$\boldsymbol{y}(t)=U\boldsymbol{\eta}(t)=\sum_{r=1}^{n}u_r\boldsymbol{\eta}_r(t)=\sum_{r=1}^{n}\boldsymbol{\eta}_r(t)u_r$$

对于整个方程有

$$U^TMU\ddot{\boldsymbol{\eta}}+U^TKU\boldsymbol{\eta}=U^TQ$$

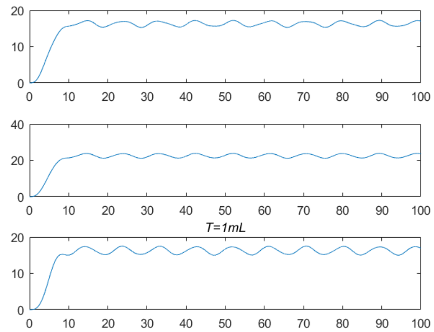
其中外激励为

$$Q=\frac{F_0}{10m}[r(t)-r(t-10)]\begin{pmatrix}1\\1.25\\1\end{pmatrix}=\frac{F_0}{10m}\begin{pmatrix}1\\1.25\\1\end{pmatrix}r(t)-\frac{F_0}{10m}\begin{pmatrix}1\\1.25\\1\end{pmatrix}r(t-10)$$

并且已经解耦，

$$\ddot{\boldsymbol{\eta}}_r+\omega_r^2\boldsymbol{\eta}_r=u_r^TQ$$

$$\boldsymbol{y}(t)=\sum_{r=1}^{n}u_r\boldsymbol{\eta}_r(t)=\sum_{r=1}^{n}u_r\frac{F_0}{10m}u_r^T\begin{pmatrix}1\\1.25\\1\end{pmatrix}\frac{1}{\omega_r^2}\left[\left(t-\frac{\sin\omega_r t}{\omega_r}\right)-\left(t-10-\frac{\sin\omega_r(t-10)}{\omega_r}\right)\varepsilon(t-10)\right]$$



其中 $\varepsilon(t-10)$ 是阶跃函数。而 $u_r^T\begin{pmatrix}1\\1.25\\1\end{pmatrix}\frac{1}{\omega_r^2}$ 表征了各阶模态的参与情况，大小分别为 3.5697,0.4904,-0.0291。

Section 3 有阻尼情形。

(12) (15 points)

解：由前面的内容可以推导出有阻尼的初值问题响应为

令

$$\boldsymbol{y}(t)=U\boldsymbol{\eta}(t)=\sum_{r=1}^{n}u_r\boldsymbol{\eta}_r(t)=\sum_{r=1}^{n}\boldsymbol{\eta}_r(t)u_r$$

$$\boldsymbol{\eta}(t)=U^{-1}\boldsymbol{y}(t)=U^{T}M\boldsymbol{y}(t)$$

对于整个方程有

$$U^{T}MU\ddot{\boldsymbol{\eta}}+U^{T}CU+U^{T}KU\boldsymbol{\eta}=0$$

$$\ddot{\boldsymbol{\eta}}_r+0.1\dot{\boldsymbol{\eta}}_r+\omega_r^2\boldsymbol{\eta}_r=0$$

阻尼比和阻尼固有频率为

$$\xi_r=\frac{c_r}{2m_r\omega_r}=\frac{1}{20\omega_r}$$

$$\omega_{rd}=\sqrt{1-\xi_r^2}\,\omega_r$$

并且已经解耦，初始条件为

$$\boldsymbol{\eta}(0)=U^{T}M\boldsymbol{y}(0),\dot{\boldsymbol{\eta}}(0)=U^{T}M\dot{\boldsymbol{y}}(0)$$

方程的初值响应为

$$\begin{aligned}\boldsymbol{y}(t)=\sum_{r=1}^{n}\boldsymbol{\eta}_r(t)u_r&=\sum_{r=1}^{n}e^{-\xi\omega_r t}\left(u_r{}^{T}M\boldsymbol{y}(0)\cos\omega_{rd}t+\frac{u_r{}^{T}M\dot{\boldsymbol{y}}(0)+\xi_r\omega_r u_r{}^{T}M\boldsymbol{y}(0)}{\omega_{rd}}\sin\omega_{rd}t\right)u_r\\&=\sum_{r=1}^{n}e^{-\xi\omega_r t}\left[u_r{}^{T}M\boldsymbol{y}(0)\left(\cos\omega_{rd}t+\frac{\xi_r\omega_r}{\omega_{rd}}\sin\omega_{rd}t\right)+u_r{}^{T}M\dot{\boldsymbol{y}}(0)\frac{1}{\omega_{rd}}\sin\omega_{rd}t\right]u_r\end{aligned}$$

（a）初始条件为： $y(0)=[0,1,0]^T,\dot{y}(0)=0$

$$\begin{aligned}\boldsymbol{y}(t)&=\sum_{r=1}^{n}e^{-\xi\omega_r t}\left[u_r{}^{T}M\boldsymbol{y}(0)\left(\cos\omega_{rd}t+\frac{\xi_r\omega_r}{\omega_{rd}}\sin\omega_{rd}t\right)+u_r{}^{T}M\dot{\boldsymbol{y}}(0)\frac{1}{\omega_{rd}}\sin\omega_{rd}t\right]u_r\\&=e^{-\xi_1\omega_1 t}\left[\frac{1}{4}\begin{pmatrix}1&\sqrt{2}&1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\left(\cos\omega_{1d}t+\frac{\xi_1\omega_1}{\omega_{1d}}\sin\omega_{1d}t\right)\right]\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}+e^{-\xi_2\omega_2 t}\left[\frac{1}{2}\begin{pmatrix}1&0&-1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\left(\cos\omega_{2d}t+\frac{\xi_2\omega_2}{\omega_{2d}}\sin\omega_{2d}t\right)\right]\begin{pmatrix}1\\0\\-1\end{pmatrix}+e^{-\xi_3\omega_3 t}\left[\frac{1}{4}\begin{pmatrix}1&-\sqrt{2}&1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\left(\cos\omega_{3d}t+\frac{\xi_3\omega_3}{\omega_{3d}}\sin\omega_{3d}t\right)\right]\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}\\&=\frac{\sqrt{2}}{4}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}e^{-\xi_1\omega_1 t}\left(\cos\omega_{1d}t+\frac{\xi_1\omega_1}{\omega_{1d}}\sin\omega_{1d}t\right)-\frac{\sqrt{2}}{4}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}e^{-\xi_3\omega_3 t}\left(\cos\omega_{3d}t+\frac{\xi_3\omega_3}{\omega_{3d}}\sin\omega_{3d}t\right)\\&=\frac{\sqrt{2}}{4}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}e^{-0.05t}\left(\cos\omega_{1d}t+\frac{0.05}{\omega_{1d}}\sin\omega_{1d}t\right)-\frac{\sqrt{2}}{4}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}e^{-0.05t}\left(\cos\omega_{3d}t+\frac{0.05}{\omega_{3d}}\sin\omega_{3d}t\right)\end{aligned}$$

（b）初始条件为： $y(0)=[-1,0,1]^T,\dot{y}(0)=0$

$$
\begin{aligned}
\boldsymbol{y}(t)&=\sum_{r=1}^{n}e^{-\xi\omega_r t}\left[u_r^{\ T}M\boldsymbol{y}(0)\left(\cos\omega_{rd}t+\frac{\xi_r\omega_r}{\omega_{rd}}\sin\omega_{rd}t\right)+u_r^{\ T}M\dot{\boldsymbol{y}}(0)\frac{1}{\omega_{rd}}\sin\omega_{rd}t\right]u_r\\
&=e^{-\xi_1\omega_1 t}\left[\frac{1}{4}\begin{pmatrix}1&\sqrt{2}&1\end{pmatrix}\begin{pmatrix}-1\\0\\1\end{pmatrix}\left(\cos\omega_{1d}t+\frac{\xi_1\omega_1}{\omega_{1d}}\sin\omega_{1d}t\right)\right]\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}+e^{-\xi_2\omega_2 t}\left[\frac{1}{2}\begin{pmatrix}1&0&-1\end{pmatrix}\begin{pmatrix}-1\\0\\1\end{pmatrix}\left(\cos\omega_{2d}t+\frac{\xi_2\omega_2}{\omega_{2d}}\sin\omega_{2d}t\right)\right]\begin{pmatrix}1\\0\\-1\end{pmatrix}+e^{-\xi_3\omega_3 t}\left[\frac{1}{4}\begin{pmatrix}1&-\sqrt{2}&1\end{pmatrix}\begin{pmatrix}-1\\0\\1\end{pmatrix}\left(\cos\omega_{3d}t+\frac{\xi_3\omega_3}{\omega_{3d}}\sin\omega_{3d}t\right)\right]\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}\\
&=\begin{pmatrix}-1\\0\\1\end{pmatrix}e^{-\xi_2\omega_2 t}\left(\cos\omega_{2d}t+\frac{\xi_2\omega_2}{\omega_{2d}}\sin\omega_{2d}t\right)=\begin{pmatrix}-1\\0\\1\end{pmatrix}e^{-0.05t}\left(\cos\omega_{2d}t+\frac{0.05}{\omega_{2d}}\sin\omega_{2d}t\right)
\end{aligned}
$$

（c）初始条件为： $y(0)=0,\dot{y}(0)=[0,0,1]^T$

$$
\begin{aligned}
\boldsymbol{y}(t)&=\sum_{r=1}^{n}e^{-\xi\omega_r t}\left[u_r^{\ T}M\boldsymbol{y}(0)\left(\cos\omega_{rd}t+\frac{\xi_r\omega_r}{\omega_{rd}}\sin\omega_{rd}t\right)+u_r^{\ T}M\dot{\boldsymbol{y}}(0)\frac{1}{\omega_{rd}}\sin\omega_{rd}t\right]u_r\\
&=e^{-\xi_1\omega_1 t}\left[\frac{1}{4}\begin{pmatrix}1&\sqrt{2}&1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\left(\frac{1}{\omega_{1d}}\sin\omega_{1d}t\right)\right]\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}+e^{-\xi_2\omega_2 t}\left[\frac{1}{2}\begin{pmatrix}1&0&-1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\left(\frac{1}{\omega_{2d}}\sin\omega_{2d}t\right)\right]\begin{pmatrix}1\\0\\-1\end{pmatrix}+e^{-\xi_3\omega_3 t}\left[\frac{1}{4}\begin{pmatrix}1&-\sqrt{2}&1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\left(\frac{1}{\omega_{3d}}\sin\omega_{3d}t\right)\right]\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}\\
&=\frac{1}{4\omega_{1d}}e^{-0.05t}\sin\omega_{1d}t\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}-\frac{1}{2\omega_{2d}}e^{-0.05t}\sin\omega_{2d}t\begin{pmatrix}1\\0\\-1\end{pmatrix}-\frac{1}{4\omega_{3d}}e^{-\xi_3\omega_3 t}\sin\omega_{3d}t\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}
\end{aligned}
$$

与（7）对比可以发现，改变阻尼对于初值响应来说仅仅是改变了解的组成部分的具体表达式，对于解的构成形式（模态的组合方式）没有任何影响（只取决于初值与模态之间的线性表出的关系式）。

(13) (15 points)

解：令

$$\boldsymbol{y}(t)=U\boldsymbol{\eta}(t)=\sum_{r=1}^{n}u_r\boldsymbol{\eta}_r(t)=\sum_{r=1}^{n}\boldsymbol{\eta}_r(t)u_r$$

$$\boldsymbol{\eta}(t)=U^{-1}\boldsymbol{y}(t)=U^TM\boldsymbol{y}(t)$$

对于整个方程有

$$U^TMU\ddot{\boldsymbol{\eta}}+U^TCU\dot{\boldsymbol{\eta}}+U^TKU\boldsymbol{\eta}=U^TQ$$

并且已经解耦，

$$\ddot{\boldsymbol{\eta}}_r+0.1\dot{\boldsymbol{\eta}}_r+\omega_r^2\boldsymbol{\eta}_r=u_r^TF\cos\omega t$$

阻尼比和阻尼固有频率为

$$\xi_r=\frac{c_r}{2m_r\omega_r}=\frac{1}{20\omega_r}$$

$$\omega_{rd}=\sqrt{1-\xi_r^2}\omega_r$$

方程的解为

$$\boldsymbol{\eta}_r(t)=\frac{u_r^TF}{\omega_r^2\sqrt{\left(1-\lambda_r^2\right)^2+\left(2\lambda_r\xi_r\right)^2}}\cos\left(\omega t-\varphi_r\right)$$

$$\varphi_r=\arctan\left(\frac{2\lambda_r\xi_r}{1-\lambda_r^2}\right)$$

因此系统的稳态响应可以写为

$$\begin{aligned}\boldsymbol{y}(t)&=\sum_{r=1}^{n}u_r\boldsymbol{\eta}_r(t)=\sum_{r=1}^{n}\frac{u_ru_r^T}{\omega_r^2\sqrt{\left(1-\lambda_r^2\right)^2+\left(2\lambda_r\xi_r\right)^2}}F\cos\left(\omega t-\varphi_r\right)\\&=\sum_{r=1}^{n}\frac{u_ru_r^T}{\sqrt{\left(\omega_r^2-\omega^2\right)^2+\left(\omega_r/10\right)^2}}\begin{pmatrix}0\\1\\0\end{pmatrix}\frac{F_0}{m}\cos\left(\omega t-\varphi_r\right)\\\varphi_r&=\arctan\left(\frac{2\lambda_r\xi_r}{1-\lambda_r^2}\right)=\arctan\left(\frac{\omega_r/10}{\omega_r^2-\omega^2}\right)\end{aligned}$$

代入数据之后

$$u_1=\left(0.3685\quad 0.5717\quad 0.5184\right)^T,u_2=\left(0.6696\quad 0.3838\quad -0.4496\right)^T,u_3=\left(0.6449\quad -0.7252\quad 0.1707\right)^T$$

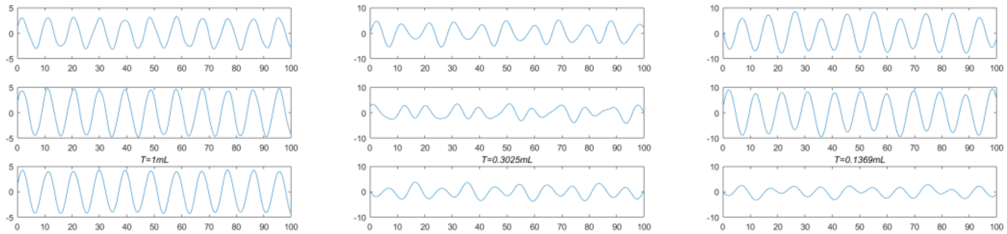
$$\omega_1=0.6698\sqrt{\frac{T}{mL}},\omega_2=1.1945\sqrt{\frac{T}{mL}},\omega_3=1.7676\sqrt{\frac{T}{mL}}$$

假设在 $T=mL$ 的条件下

$$\omega_r=\left[0.6698\quad 1.1945\quad 1.7676\right]$$

$$\xi_r=\left[0.0747\quad 0.0419\quad 0.0283\right]$$

$$\omega_{rd} = \begin{bmatrix} 0.6679 & 1.1934 & 1.7669 \end{bmatrix}$$



三图分别展示了激励频率接近三阶阵型时的稳态响应。

与（10）的比较略。

(14) (25 points)

解：同（11）第二项脉冲响应函数

$$h(t)=\frac{1}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t$$

和阶跃响应函数 $g(t)$ 满足

$$h(t)=\frac{dg(t)}{dt}$$

阶跃函数响应为

$$g(t)=\frac{1}{k}\left[1-e^{-\zeta\omega_n t}\left(\cos\omega_d t+\frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right)\right]$$

令

$$\boldsymbol{y}(t)=U\boldsymbol{\eta}(t)=\sum_{r=1}^{n}u_r\boldsymbol{\eta}_r(t)=\sum_{r=1}^{n}\boldsymbol{\eta}_r(t)u_r$$

对于整个方程有

$$U^TMU\ddot{\boldsymbol{\eta}}+U^TCU\dot{\boldsymbol{\eta}}+U^TKU\boldsymbol{\eta}=U^TQ$$

其中外激励为

$$Q=\frac{F_0}{m}[\mu(t)-\mu(t-5)]\begin{pmatrix}0\\1\\0\end{pmatrix}=\frac{F_0}{m}\begin{pmatrix}0\\1\\0\end{pmatrix}\mu(t)-\frac{F_0}{m}\begin{pmatrix}0\\1\\0\end{pmatrix}\mu(t=5)$$

并且已经解耦，

$$\ddot{\boldsymbol{\eta}}_r+0.1\dot{\boldsymbol{\eta}}_r+\omega_r^2\boldsymbol{\eta}_r=u_r^TQ$$

$$\boldsymbol{y}(t)=\sum_{r=1}^{n}u_r\boldsymbol{\eta}_r(t)=\sum_{r=1}^{n}u_r\frac{F_0}{m}u_r^T\begin{pmatrix}0\\1\\0\end{pmatrix}\frac{1}{\omega_r^2}\left[\left(1-e^{-\zeta_r\omega_r t}\left(\cos\omega_{rd}t+\frac{\zeta_r\omega_r}{\omega_{rd}}\sin\omega_{rd}t\right)\right)-\left(1-e^{-\zeta_r\omega_r(t-5)}\left(\cos\omega_{rd}(t-5)+\frac{\zeta_r\omega_r}{\omega_{rd}}\sin\omega_{rd}(t-5)\right)\right)\right]$$

$$=\frac{F_0}{m}\sum_{r=1}^{n}\frac{u_ru_r^T}{\omega_r^2}\begin{pmatrix}0\\1\\0\end{pmatrix}e^{-\zeta_r\omega_r t}\left(e^{5\zeta_r\omega_r}\left(\cos\omega_{rd}(t-5)+\frac{\zeta_r\omega_r}{\omega_{rd}}\sin\omega_{rd}(t-5)\right)-\left(\cos\omega_{rd}t+\frac{\zeta_r\omega_r}{\omega_{rd}}\sin\omega_{rd}t\right)\right)$$

在问题（6）中得到

$u_1=\frac{1}{2}\begin{pmatrix}1 & \sqrt{2} & 1\end{pmatrix}^T, u_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & -1\end{pmatrix}^T, u_3=\frac{1}{2}\begin{pmatrix}1 & -\sqrt{2} & 1\end{pmatrix}^T$

$\lambda_1=\frac{\left(2-\sqrt{2}\right)T}{mL}, \lambda_2=\frac{2T}{mL}, \lambda_3=\frac{\left(2+\sqrt{2}\right)T}{mL}$

$$\boldsymbol{y}(t)=\frac{F_0}{m}\sum_{r=1}^{n}\frac{u_r u_r^T}{\omega_r^2}\begin{pmatrix}0\\1\\0\end{pmatrix}e^{-\zeta_r\omega_r t}\left(e^{5\zeta_r\omega_r}\left(\cos\omega_{rd}(t-5)+\frac{\zeta_r\omega_r}{\omega_{rd}}\sin\omega_{rd}(t-5)-1\right)\mu(t-5)-\left(\cos\omega_{rd}t+\frac{\zeta_r\omega_r}{\omega_{rd}}\sin\omega_{rd}t-1\right)\right)$$

$$=\frac{F_0}{m}\frac{mL}{4T}\begin{pmatrix}\sqrt{2}+1\\\sqrt{2}+2\\\sqrt{2}+1\end{pmatrix}e^{-0.05t}\left(e^{0.25}\left(\cos\omega_{1d}(t-5)+\frac{0.05}{\omega_{1d}}\sin\omega_{1d}(t-5)-1\right)\mu(t-5)-\left(\cos\omega_{1d}t+\frac{0.05}{\omega_{1d}}\sin\omega_{1d}t-1\right)\right)$$

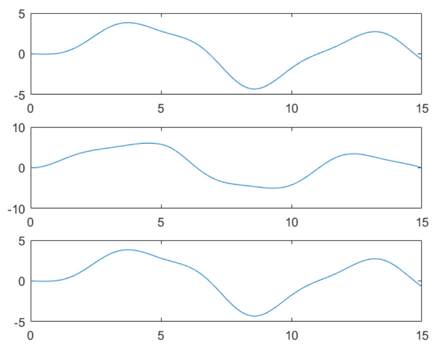
$$+\frac{F_0}{m}\frac{mL}{4T}\begin{pmatrix}-\sqrt{2}+1\\-\sqrt{2}+2\\-\sqrt{2}+1\end{pmatrix}e^{-0.05t}\left(e^{0.25}\left(\cos\omega_{3d}(t-5)+\frac{0.05}{\omega_{3d}}\sin\omega_{3d}(t-5)-1\right)\mu(t-5)-\left(\cos\omega_{3d}t+\frac{0.05}{\omega_{3d}}\sin\omega_{3d}t-1\right)\right)$$

假设在$T=mL$的条件下

$\omega_r=\begin{bmatrix}0.7654 & 1.4142 & 1.8478\end{bmatrix}$

$\xi_r=\begin{bmatrix}0.0653 & 0.0354 & 0.0271\end{bmatrix}$

$\omega_{rd}=\begin{bmatrix}0.7637 & 1.4133 & 1.8471\end{bmatrix}$



其余的情况下