题 1

求解问题 7

考察下述 Coulomb 摩擦阻尼系统

$$\ddot{u}(t)+\mu N\operatorname{sgn}\dot{u}(t)+\omega_0^2u(t)=0$$

其中摩擦系数 μ 为小参数，用 KBM 法求解该系统自由振动的一次近似。

7　用多尺度法求题 5 中系统自由振动的一次近似解。

解

变形原方程为

$$\ddot{u}(t)+\omega_0^2u=-\varepsilon N\operatorname{sgn}\dot{u}(t)\quad \varepsilon=\mu \tag{1}$$

为求解一次近似解, 需要用两个时间尺度T_0, T_1

代入方程

$$\begin{cases}D_0^2u_0+\omega_0^2u_0=0\\ D_0^2u_1+\omega_0^2u_0=-2D_0D_1u_0+p(u_0,D_0u_0)\end{cases} \tag{2}$$

得

$$\begin{cases}D_0^2u_0+\omega_0^2u_0=0\\ D_0^2u_1+\omega_0^2u_0=-2D_0D_1u_0-N\operatorname{sgn}(D_0u_0)\end{cases} \tag{3}$$

解

$$D_0^2u_0+\omega_0^2u_0=0 \tag{4}$$

为

$$u_0=a(T_1)\cos(\omega_0T_0+\varphi(T_1)) \tag{5}$$

应用 Euler 公式写出复数形式

$$u_0=A(T_1)e^{j\omega_0T_0}+\mathrm{cc}\quad \mathrm{cc}\text{ 代表共轭项} \tag{6}$$

其中, $A=\frac{ae^{j\varphi}}{2}$

代入

$$D_0^2u_1+\omega_0^2u_0=-2D_0D_1u_0-N\operatorname{sgn}(D_0u_0) \tag{7}$$

计算

$$\begin{aligned}D_0D_1u_0&=D_1D_0u_0\\&=D_1D_0(A(T_1)e^{j\omega_0T_0}+\mathrm{cc})\\&=D_1D_0A(T_1)e^{j\omega_0T_0}+D_1D_0\ \mathrm{cc}\\&=j\omega_0e^{j\omega_0T_0}D_1A(T_1)+\mathrm{cc}\quad \mathrm{cc}\text{ 会随着变动, 保证实数}\end{aligned} \tag{8}$$

$$
\begin{aligned}
D_0 u_0 &== D_0(a(T_1)\cos(\omega_0 T_0 + \varphi(T_1))) \\
&= \omega_0 a \cos\psi
\end{aligned}
\tag{9}
$$

所以

$$
D_0^2 u_1 + \omega_0^2 u_0 = -2j\omega_0 e^{j\omega_0 T_0} D_1 A(T_1) - N\,\mathrm{sgn}(\omega_0 a \cos\psi) \tag{10}
$$

为消除永年项，要求不能含有$e^{\pm j\omega_0 T_0}$

则有

$$
\begin{cases}
D_1 a &= -\frac{1}{2\pi\omega_0}\int_0^{2\pi} p(a\cos\psi, -\omega_0 a\sin\psi)\sin\psi\,\mathrm{d}\psi \\
&= \frac{N}{2\pi\omega_0}\int_0^{2\pi}\mathrm{sgn}(-\omega_0 a\sin\psi)\sin\psi\,\mathrm{d}\psi \\
&= -\frac{2N}{\omega_0\pi}\,\mathrm{sgn}(a\omega_0) \\
D_1\varphi &= -\frac{1}{2\pi\omega_0 a}\int_0^{2\pi} p(a\cos\psi, -\omega_0 a\sin\psi)\cos\psi\,\mathrm{d}\psi \\
&= \frac{N}{2\pi\omega_0 a}\int_0^{2\pi}\mathrm{sgn}(-\omega_0 a\sin\psi)\cos\psi\,\mathrm{d}\psi \\
&= 0
\end{cases}
\tag{11}
$$

所以

$$
\begin{cases}
a &= -\frac{2N}{\omega_0\pi}\,\mathrm{sgn}(a\omega_0) T_1 + a_0 \\
\varphi &= \varphi_0
\end{cases}
\tag{12}
$$

所以

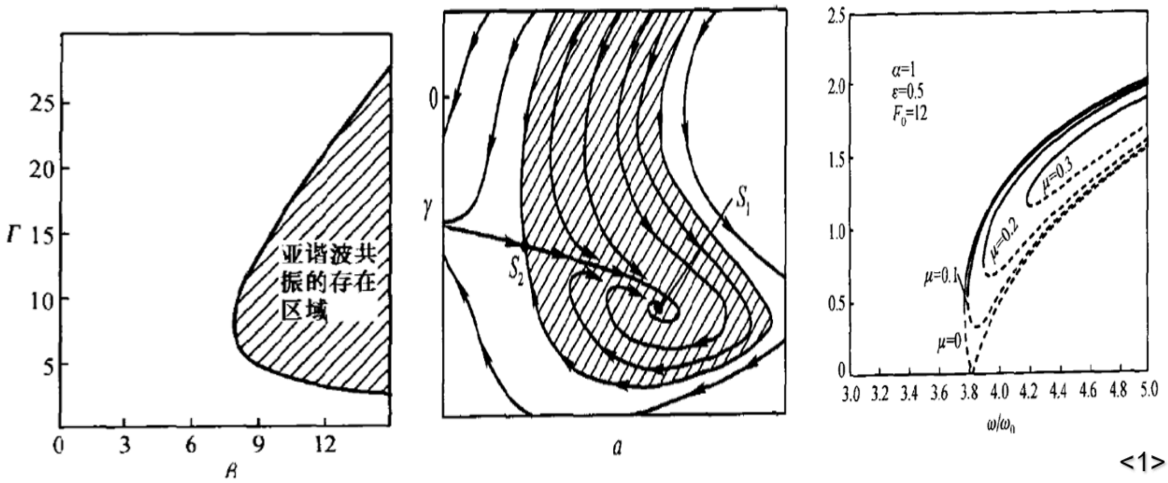
$$
\begin{aligned}
u &= \left(-\frac{2N}{\omega_0\pi}\,\mathrm{sgn}(a\omega_0) T_1 + a_0\right)\cos(\omega_0 T_0 + \varphi_0) \\
&= \left(-\frac{2N}{\omega_0\pi}\,\mathrm{sgn}(a\omega_0)\varepsilon t + a_0\right)\cos(\omega_0 t + \varphi_0)
\end{aligned}
\tag{13}
$$

题 2

另作业：对 $\ddot{x}+2\varepsilon\mu\dot{x}+\omega_0^2x+\varepsilon\alpha x^3=F_0\cos\omega t$

其中 $\varepsilon=0.5,\alpha=1,F_0=1.2$ 其他参数按照不同的图自定

还原亚谐波共振情形的下面三个图，需要提交程序，推导、程序和还原图贴在PPT上



<1>

解

采用多尺度的一次近似，需要用两个时间尺度T_0, T_1，有

$$\begin{cases} D_0^2x_0+\omega_0^2x_0=F_0\cos\omega T_0 \\ D_0^2x_1+\omega_0^2x_1=-2D_0D_1x_0-2\mu D_0x_0-\alpha x_0^3 \end{cases} \tag{14}$$

解

$$D_0^2x_0+\omega_0^2x_0=F_0\cos\omega T_0 \tag{15}$$

得

$$x_0=A(T_1)e^{j\omega_0T_0}+\Lambda e^{j\omega T_0}+\mathrm{cc}\quad \Lambda=\frac{F_0}{2(\omega_0^2-\omega^2)}\quad A=\frac{1}{2}ae^{j\varphi} \tag{16}$$

代入

$$\begin{aligned} D_0^2x_1+\omega_0^2x_1=&-\left[2i\omega_0(D_1A+\mu A)+6\alpha A\Lambda^2+3\alpha A^2\overline{A}\right]e^{j\omega_0T_0} \\ &-\alpha\Big(A^3e^{3j\omega_0T_0}+\Lambda^3e^{3j\omega T_0}+3A^2\Lambda e^{j(2\omega_0+\omega)T_0} \\ &+3\overline{A}^2\Lambda e^{j(\omega-2\omega_0)T_0}+3A\Lambda^2e^{3j(\omega_0+2\omega)T_0}+3A\Lambda^2e^{j(\omega_0-2\omega)T_0}\Big) \\ &-\Lambda\Big(-2j\mu\omega+3\alpha\Lambda^2+6\alpha A\overline{A}\Big)e^{j\omega T_0} \end{aligned} \tag{17}$$

亚谐波共振则有

$$\omega=3\omega_0+\varepsilon\sigma \tag{18}$$

代入

$$\begin{aligned} 3\overline{A}^2\Lambda e^{j(\omega-2\omega_0)T_1}&=3\overline{A}^2\Lambda e^{j(\omega_0+\varepsilon\sigma)T_1} \\ &=3\alpha\overline{A}^2\Lambda e^{j\sigma T_1}e^{j\omega_0T_1} \end{aligned} \tag{19}$$

$$2j\omega_0(D_1A+\mu A)+6\alpha A\Lambda^2+3\alpha A^2\overline{A}+3\alpha\overline{A}^2\Lambda e^{j\sigma T_1}=0 \tag{20}$$

可得

$$\begin{cases} D_1a &= -\left[\mu+\frac{3\alpha\Lambda}{4\omega_0}a\sin(\sigma T_1-3\varphi)\right]a \\ D_1\varphi &= \frac{3a}{\omega_0}\left[\Lambda^2+\frac{1}{8}a^2+\frac{\Lambda}{4}a\cos(\sigma T_1-3\varphi)\right] \end{cases} \tag{21}$$

令$\gamma=\sigma T_1-3\varphi$, 则有

$$\begin{cases} D_1a=-\left(\mu+\frac{3\alpha\Lambda}{4\omega_0}a\sin\gamma\right)a \\ D_1\gamma=\sigma-\frac{9a}{\omega_0}\left[\Lambda^2+\frac{1}{8}a^2+\frac{\Lambda}{4}a\cos\gamma\right] \end{cases} \tag{22}$$

令$D_1a=0, D_1\gamma=0$, 则有

$$\begin{cases} \mu &= -\frac{3\alpha\Lambda}{4\omega_0}a_s\sin\gamma_s \\ \sigma-\frac{9\alpha}{\omega_0}\left(\Lambda^2+\frac{a_s^2}{8}\right) &= \frac{9\alpha\Lambda}{4\omega_0}a_s\cos\gamma_s \end{cases} \tag{23}$$

消去γ_s, 则有

$$9\mu^2+\left(\sigma-\frac{9\alpha\Lambda^2}{\omega_0}-\frac{9\alpha}{8\omega_0}a_s^2\right)^2=\frac{81\alpha^2\Lambda^2}{16\omega_0^2}a_s^2 \tag{24}$$

得

$$a_s^4+2pa_s^2+q=0 \tag{25}$$

其中,

$$\begin{cases} p=\frac{8\omega_0\sigma}{9\alpha}-6\Lambda^2 \\ q=\left(\frac{8\omega_0}{9\alpha}\right)^2\left[9\mu^2+\left(\sigma-\frac{9\alpha\Lambda^2}{\omega_0}\right)^2\right] \end{cases} \tag{26}$$

解得

$$a_s^2=p\pm\sqrt{p^2-q} \tag{27}$$

若有实数解, 则要求

$$\begin{cases} p &> 0 \\ p^2 &> q \end{cases} \Leftrightarrow \begin{cases} \Lambda^2 &< \frac{4\omega_0\sigma}{27\alpha} \\ \frac{\alpha\Lambda^2}{\omega_0}\left(\sigma-\frac{63\alpha\Lambda^2}{8\omega_0}\right)-2\mu^2 &\geq 0 \end{cases} \tag{28}$$

引入参数

$$\begin{cases} \beta=\frac{\sigma}{\mu} \\ \Gamma=\frac{63\alpha\Lambda^2}{4\omega_0\mu} \end{cases} \tag{29}$$

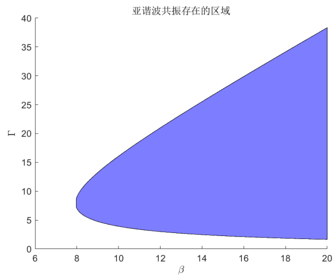
不等式变为

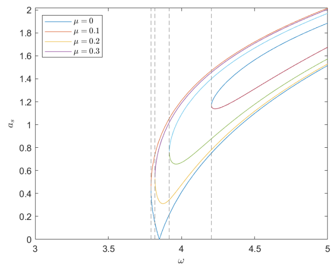
$$
\begin{cases} \Gamma & < \frac{7}{3}\beta \\ \Gamma^2 - 2\beta\mathrm{T} + 63 \le 0 \end{cases} \tag{30}
$$

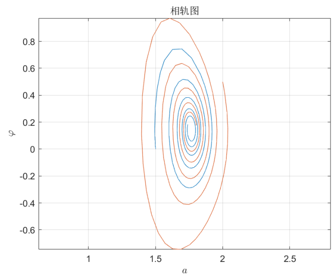
则给定σ振幅a_s有实数解的条件为

$$
\beta - (\beta^2 - 63)^{\frac{1}{2}} \le \Gamma \le \beta + (\beta^2 - 63)^{\frac{1}{2}} < \frac{7}{3}\beta \tag{31}
$$

作图







代码

```
% 定义 beta 的范围
beta = linspace(0, 20, 400);  % 包括负值和正值的区间

% 计算对应的 Gamma 值
Gamma_lower = beta - sqrt(beta.^2 - 63);
Gamma_upper = beta + sqrt(beta.^2 - 63);

% 为了防止复数结果, 我们只在 beta^2 - 63 >= 0 的情况下计算 Gamma
valid_indices = beta.^2 >= 63;
beta_valid = beta(valid_indices);
Gamma_lower_valid = Gamma_lower(valid_indices);
Gamma_upper_valid = Gamma_upper(valid_indices);

% 绘制 Gamma 的下界和上界
figure;
hold on;
fill([beta_valid, fliplr(beta_valid)], [Gamma_lower_valid,
fliplr(Gamma_upper_valid)], 'b', 'FaceAlpha', 0.5);
xlabel('$\beta$', 'Interpreter', 'latex');
ylabel('$\Gamma$', 'Interpreter', 'latex');
title('亚谐波共振存在的区域');
hold off;
saveas(gcf, '../figure/figure-1.png');
%%
% 载入符号计算工具箱
syms mu sigma alpha Lambda omega_0 a_s omega

% 定义方程
eq = 9*mu^2 + (sigma - (9*alpha*Lambda^2)/omega_0 - (9*alpha)/(8*omega_0)*a_s^2)^2 ==
(81*alpha^2*Lambda^2)/(16*omega_0^2)*a_s^2

% 给定的参数值
omega_0_val = 1;
epsilon_val = 0.5;
```

```matlab
F_0_val = 12;
alpha_val = 1;
sigma_val = (omega - 3) / epsilon_val;
Lambda_val = F_0_val / (2 * (omega_0_val^2 - omega^2));

eq_substituted = subs(eq, {omega_0, alpha, sigma, Lambda}, ...
                      {omega_0_val, alpha_val, sigma_val, Lambda_val})

% 解方程，求 mu 和 sigma
solution = solve(eq_substituted, a_s)

sol = [solution(1); solution(2)]

sol_mu_0 = subs(sol, mu, 0.0);
sol_mu_1 = subs(sol, mu, 0.1);
sol_mu_2 = subs(sol, mu, 0.2);
sol_mu_3 = subs(sol, mu, 0.3);

omega_range = [3 5];

fplot(sol_mu_0, omega_range)
hold on;
fplot(sol_mu_1, omega_range)
fplot(sol_mu_2, omega_range)
fplot(sol_mu_3, omega_range)

% 设置图例
legend('$\mu = 0$', '$\mu = 0.1$', '$\mu = 0.2$', '$\mu = 0.3$', 'Interpreter',
'latex', 'Location', 'northwest');

% 设置坐标轴标签
xlabel('$\omega$', 'Interpreter', 'latex');
ylabel('$a_s$', 'Interpreter', 'latex');
hold off;
saveas(gcf, '../figure/figure-2.png');

%%
% 参数定义
mu = 0.1;
alpha = 1;
omega_0 = 1;
epsilon = 0.5;
omega = 3.5;
F_0 = 12;
sigma = omega / epsilon; % 由给出的 omega 和 epsilon 计算 sigma
Lambda = F_0 / (2 * (omega_0^2 - omega^2)); % 计算 Lambda

% 定义微分方程系统
odefun = @(t, y) [- (mu + 3 * alpha * Lambda / (4 * omega_0) * y(1) * sin(y(2))) *
y(1);
                   sigma - (9 * y(1) / omega_0) * (Lambda^2 + (1/8) * y(1)^2 +
(Lambda/4) * y(1) * cos(y(2)))];

% 设置图形窗口
figure;
```

```matlab
% 求解微分方程并绘图，对于两个不同的初始条件
for initial_conditions = [1.5, 0; 2, 0.5;]'
    [t, y] = ode45(odefun, [0 10], initial_conditions);
    % 将 y(:,2) 的值调整到 [-pi, pi] 范围内
    y(:,2) = mod(y(:,2) + pi, 2*pi) - pi;
    plot(y(:,1), y(:,2));
    hold on;
end

% 设置坐标轴标签和标题
xlabel('$a$', 'Interpreter', 'latex');
ylabel('$\varphi$', 'Interpreter', 'latex');
title('相轨图');

% 显示网格并保持坐标轴比例相同
grid on;
axis equal;

% 关闭持续绘图模式
hold off;

% 保存图像
saveas(gcf, '../figure/phase_portrait.png');
```