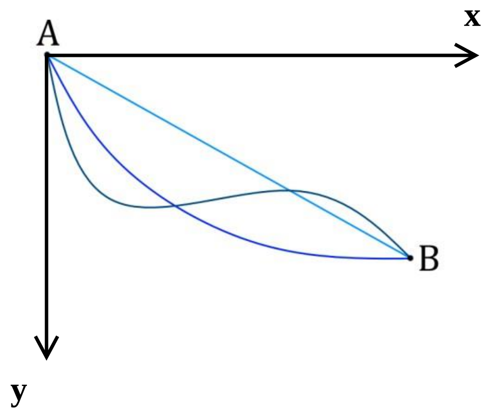
HOMEWORK 2

Problem 1: In mathematics and physics, a brachistochrone curve, or curve of fastest descent, is the one lying on the plane between a point A and a lower point B, where B is not directly below A, on which a bead slides frictionlessly under the influence of a uniform gravitational field to a given end point in the shortest time. The problem was posed by Johann Bernoulli in 1696. The brachistochrone problem was one of the earliest problems posed in the calculus of variations. Now, please read relative articles in Wikipedia or Zhihu, and derive the brachistochrone curve via the Least Action Principle. (15 points)



解：小球从 A 到 B 所需时间为：

$$T=\int_L \frac{1}{v}ds$$

其中：

$$ds=\sqrt{(dx)^2+(dy)^2}=dx\sqrt{1+y'^2}$$

由能量守恒得：$v=\sqrt{2gy}$

∴

$$T=\int_{x_A}^{x_B}\sqrt{\left(1+y'^2\right)/2gy}\,dx$$

根据最小作用量原理，要使泛函 T 取驻值，则令 $\delta T=0$。

设：

$$F\left(y,y',x\right)=\sqrt{\left(1+y'^2\right)/2gy}$$

则：

$$\int_{xA}^{xB}\left(\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y'}\delta y'\right)dx=0$$

$$\int_{x_A}^{x_B}\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y dx+\frac{\partial F}{\partial y'}\delta y\Big|_{x_A}^{x_B}=0$$

$$\int_{x_A}^{x_B}\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y dx=0$$

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$$

两端乘 y'，并做积分：

$$\int y' \frac{\partial F}{\partial y} dx = \int y' \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

$$\int y' \frac{\partial F}{\partial y} dx = y' \frac{\partial F}{\partial y'} - \int y'' \frac{\partial F}{\partial y'} dx$$

$$\int\left(y' \frac{\partial F}{\partial y} + y'' \frac{\partial F}{\partial y'}\right) dx = y' \frac{\partial F}{\partial y'} + C$$

$$\int\left(\frac{dF}{dx}\right) dx = y' \frac{\partial F}{\partial y'} + C$$

$$F - y' \frac{\partial F}{\partial y'} = C$$

将 F 代入:

$$\sqrt{\frac{1+y'^2}{2gy}} - \frac{y'^2}{\sqrt{2gy\left(1+y'^2\right)}} = C$$

$$\frac{1}{\sqrt{2gy\left(1+y'^2\right)}} = C$$

$$y = \frac{C_1}{1+y'^2}, \quad C_1 = \frac{1}{2gC^2}$$

令 $y' = \cot a$，则：

$$y = C_1 \sin^2 a$$

$$dx = \frac{dy}{y'}$$

$$= \frac{2C_1 \sin a \cos a da}{\cot a}$$

$$= C_1 \left(1 - \cos 2a\right) da$$

$$x = C_1 a - \frac{C_1 \sin 2a}{2} + C_2$$

由边界条件 $x_{(a=0)} = 0$ 得 $C_2 = 0$，则最速降线方程为：

$$\begin{cases} x = \dfrac{C_1}{2}\left(\theta - \sin\theta\right) \\ y = \dfrac{C_1}{2}\left(1 - \cos\theta\right) \end{cases}$$

这是一个半径为 $C_1/2$ 的圆滚动形成的摆线方程， θ 为滚动角。

参考：摆线的那些事儿——数学界的大型装逼事件 - TravorLZH 的文章 - 知乎

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Problem 2: Please solve Problem 6.1, 6.7, 6.9, and 6.11 in page 277-279 of the textbook. (15 points)

6.1. The system of Fig. 6.3 consists of a uniform rigid link of total mass m and length L and two linear springs of stiffnesses k_1 and k_2. When the springs are unstretched the link is horizontal. Use the principle of virtual work to calculate the angle θ corresponding to the equilibrium position.

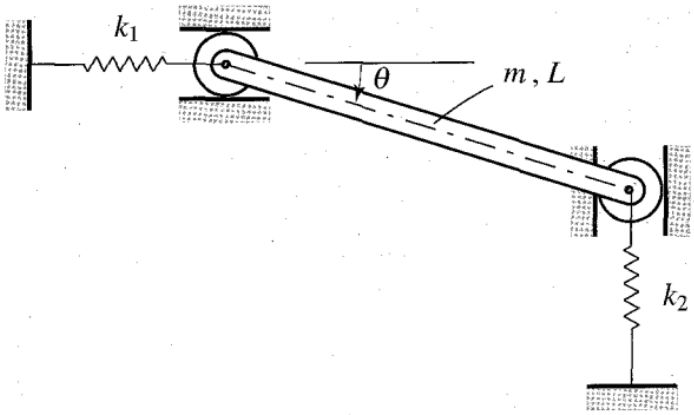


FIGURE 6.3
Rigid link supported by springs

解：

弹簧 k_1 水平位移 $x_1 = L(1-\cos\theta)$，弹簧 k_2 竖直位移 $y_2 = \sin\theta L$，杆质心的竖直位移为 $y_c = \frac{L}{2}\sin\theta$

由虚位移原理得：

$$\delta W = \sum_i^n F_i \delta r_i = 0$$

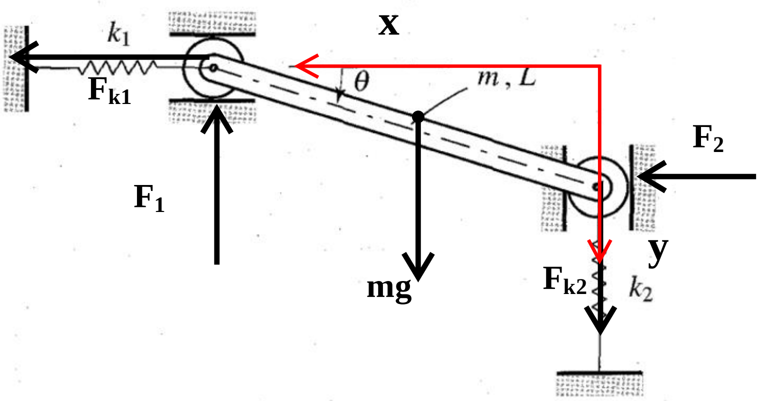
$$\left(mg\cos\theta\frac{L}{2} - k_1L^2(1-\cos\theta)\sin\theta - k_2L^2\sin\theta\cos\theta\right)\delta\theta = 0$$

$$\frac{1}{2}mg\cos\theta - k_1L\sin\theta - (k_2-k_1)L\sin\theta\cos\theta = 0$$

即平衡处的 θ 满足上述表达式，无法求出解析解。

6.7. Derive the equation of motion for the system of Problem 6.1 by means of d'Alembert's principle.

解：如图所示，分析杆的受力



其中杆质心的坐标：

$$x_c = \cos\theta\frac{L}{2},\ y_c = \sin\theta\frac{L}{2}$$

弹簧力：

$$F_{k1} = k_1L(1-\cos\theta),\ F_{k2} = k_1L\sin\theta$$

根据达朗贝尔原理：

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$$\begin{cases}\sum F_x=0,\ F_{k1}+F_2-\dfrac{mL}{2}\left(-\cos\theta\dot{\theta}^2-\sin\theta\ddot{\theta}\right)=0\\ \sum F_y=0, mg-F_{k2}-F_1-\dfrac{mL}{2}\left(-\sin\theta\dot{\theta}^2+\cos\theta\ddot{\theta}\right)=0\\ \sum M_c=0,-F_{k2}\cos\theta\dfrac{L}{2}+F_2\sin\theta-F_{k1}\sin\theta\dfrac{L}{2}+F_1\cos\theta\dfrac{L}{2}-\dfrac{mL^2\ddot{\theta}}{12}=0\end{cases}$$

联立解得：

$$-k_2L^2\sin\theta\cos\theta-k_1L^2\left(1-\cos\theta\right)\sin\theta+\frac{1}{2}mgL\cos\theta=\frac{mL\ddot{\theta}}{3}$$

在平衡位置 $\bar{\theta}$ 附近满足：

$$\frac{1}{2}mg\cos\bar{\theta}-k_1L\sin\bar{\theta}-\left(k_2-k_1\right)L\sin\bar{\theta}\cos\bar{\theta}=0$$

假设在平衡位置处小变形为 θ

$$\tilde{\theta}=\bar{\theta}+\theta$$
$$\cos\tilde{\theta}=\cos\bar{\theta}-\sin\bar{\theta}\theta$$
$$\sin\tilde{\theta}=\sin\bar{\theta}+\cos\bar{\theta}\theta$$

展开后可得

$$\frac{1}{3}mL\ddot{\theta}+k_1L(1-\cos\bar{\theta}+\sin\bar{\theta}\theta)(\sin\bar{\theta}+\cos\bar{\theta}\theta)+k_2L(\sin\bar{\theta}+\cos\bar{\theta}\theta)(\cos\bar{\theta}-\sin\bar{\theta}\theta)=\frac{1}{2}mg(\cos\bar{\theta}-\sin\bar{\theta}\theta)$$

忽略高阶小量，得到在平衡位置附近做小变形的线性运动方程：

$$\frac{1}{3}mL\ddot{\theta}+\left[k_1L(\cos\bar{\theta}-\cos2\bar{\theta})+k_2L\cos2\bar{\theta}+\frac{1}{2}mg\sin\bar{\theta}\right]\theta=0$$

或者

由 d'Alembert's principle 原理有：

$$\sum_{i=1,2,3}\left(F_i-m_i\ddot{r}_i\right)\delta r_i=0$$

由于两个弹簧处不存在质量，因此惯性项只与杆质心处的位移及转动有关，

$$r_\theta=\frac{L}{2}\left(\cos\theta\boldsymbol{i}+\sin\theta\boldsymbol{j}\right)$$

$$\ddot{r}_\theta=-\frac{L}{2}\left(\cos\theta\dot{\theta}^2+\sin\theta\ddot{\theta}\right)\boldsymbol{i}+\frac{L}{2}\left(\cos\theta\ddot{\theta}-\sin\theta\dot{\theta}^2\right)\boldsymbol{j}$$

$$\left(-k_1L\left(1-\cos\theta\right)L\sin\theta-k_2L^2\sin\theta\cos\theta+mg\frac{L\cos\theta}{2}\right)\delta\theta-m\frac{L^2}{4}\left[\left(\cos\theta\dot{\theta}^2+\sin\theta\ddot{\theta}\right)\sin\theta+\left(\cos\theta\ddot{\theta}-\sin\theta\dot{\theta}^2\right)\cos\theta\right]\delta\theta-\frac{1}{12}mL^2\ddot{\theta}\delta\theta=0$$

化简后有

$$\frac{1}{3}mL\ddot{\theta}+k_1L\left(1-\cos\theta\right)\sin\theta+k_2L\sin\theta\cos\theta=\frac{1}{2}mg\cos\theta$$

令

$$\tilde{\theta}=\bar{\theta}+\theta$$

由于在平衡位置附近满足

$$k_1\left(1-\cos\bar{\theta}\right)L\sin\bar{\theta}+k_2L\sin\bar{\theta}\cos\bar{\theta}=mg\frac{\cos\bar{\theta}}{2}$$
$$\cos\tilde{\theta}=\cos\bar{\theta}-\sin\bar{\theta}\theta$$
$$\sin\tilde{\theta}=\sin\bar{\theta}+\cos\bar{\theta}\theta$$

展开后即有

$$\frac{1}{3}mL\ddot{\theta}+k_1L\left(1-\cos\bar{\theta}+\sin\bar{\theta}\theta\right)\left(\sin\bar{\theta}+\cos\bar{\theta}\theta\right)+k_2L\left(\sin\bar{\theta}+\cos\bar{\theta}\theta\right)\left(\cos\bar{\theta}-\sin\bar{\theta}\theta\right)=\frac{1}{2}mg\left(\cos\bar{\theta}-\sin\bar{\theta}\theta\right)$$

忽略高阶小量即有

$$\frac{1}{3}mL\ddot{\theta}+\left[k_1L\left(\cos\bar{\theta}-\cos2\bar{\theta}\right)+k_2L\cos2\bar{\theta}+\frac{1}{2}mg\sin\bar{\theta}\right]\theta=0$$

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6.9. Derive the equation of motion for the system of Problem 6.1 by means of Hamilton's principle.

解：由哈密顿原理得：

$$\int_{t_1}^{t_2}(\delta W+\delta T)dt=0$$

由 6.1 已知：

$$\delta W=\left(mg\cos\theta\frac{L}{2}-k_1L^2(1-\cos\theta)\sin\theta-k_2L^2\sin\theta\cos\theta\right)\delta\theta$$

系统的动能为：

$$T=\frac{1}{2}mv^2+\frac{1}{2}J\dot{\theta}^2=\frac{1}{6}mL^2\dot{\theta}^2$$

$$\delta T=\frac{1}{3}mL^2\dot{\theta}\delta\dot{\theta}$$

∴

$$\int_{t_1}^{t_2}\left(\frac{1}{3}mL^2\dot{\theta}\delta\dot{\theta}+\left[mg\cos\theta\frac{l}{2}-k_1L^2(1-\cos\theta)\sin\theta-k_2L^2\sin\theta\cos\theta\right]\delta\theta\right)dt=0$$

分部积分可得：

$$\frac{1}{3}mL^2\int_{t_1}^{t_2}\dot{\theta}\delta\dot{\theta}dt=\frac{1}{3}mL^2\dot{\theta}\delta\theta\Big|_{t_1}^{t_2}-\frac{1}{3}mL^2\int_{t_1}^{t_2}\ddot{\theta}\delta\theta dt$$

$$=-\frac{1}{3}mL^2\int_{t_1}^{t_2}\ddot{\theta}\delta\theta dt$$

原方程变为：

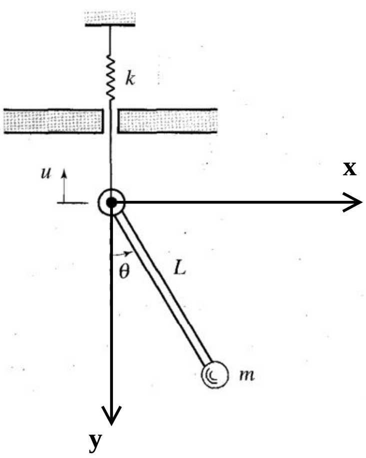
$$\int_{t_1}^{t_2}\left(\left[-\frac{1}{3}mL^2\ddot{\theta}+mg\cos\theta\frac{L}{2}-k_1L^2(1-\cos\theta)\sin\theta-k_2L^2\sin\theta\cos\theta\right]\delta\theta\right)dt=0$$

即可得运动方程：

$$-k_2L^2\sin\theta\cos\theta-k_1L^2(1-\cos\theta)\sin\theta+\frac{1}{2}mgL\cos\theta=\frac{mL\ddot{\theta}}{3}$$

线性化后的结果同 6.7。

6.11. The upper end of a pendulum is attached to a linear spring of stiffness k, where the spring is constrained so as to move in the vertical direction (Fig. 6.5). Derive the equations of motion for the vertical displacement u and the angular displacement θ by means of Hamilton's principle. Assume that u is measured from the equilibrium position and that θ is arbitrarily large.



解：如图，弹簧的伸长量为$\dfrac{mg}{k}-u$，小球质心位置为$(\sin\theta L,\cos\theta L-u)$

可得虚功表达式：

$$\delta W=-ku\delta u-mg\sin\theta L\delta\theta$$

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小球位置及速度表达式为：

$$r=\sin\theta L\vec{i}+\left(\cos\theta L-u\right)\vec{j},\qquad v=\sqrt{\dot\theta^2L^2+\dot u^2+2\sin\theta\dot\theta\dot uL}$$

动能： $T=\dfrac{m}{2}\left(\dot\theta^2L^2+\dot u^2+2\sin\theta\dot\theta\dot uL\right)$

∴

$$\delta T=\left(m\dot u+m\sin\theta\dot\theta\right)\delta\dot u+m\cos\theta\dot\theta\dot uL\delta\theta+\left(mL^2\dot\theta+m\sin\theta L\dot u\right)\delta\dot\theta$$

根据哈密顿原理：

$$\int_{t_1}^{t_2}\left(\delta W+\delta T\right)dt=0$$

$$\int_{t_1}^{t_2}\begin{bmatrix}-ku\delta u+\left(m\dot u+m\sin\theta L\dot\theta\right)\delta\dot u+\\ \left(m\cos\theta\dot\theta\dot uL-mg\sin\theta L\right)\delta\theta+\left(mL^2\dot\theta+m\sin\theta L\dot u\right)\delta\dot\theta\end{bmatrix}dt=0$$

利用分部积分可进一步化简：

$$\int_{t_1}^{t_2}\begin{bmatrix}\left(-ku-m\ddot u-mL\cos\theta\dot\theta^2-mL\sin\theta\ddot\theta\right)\delta u\\ \left(-mg\sin\theta L-mL^2\ddot\theta-mL\sin\theta\ddot u\right)\delta\theta\end{bmatrix}dt=0$$

由于虚位移和时间历程的任意性，可得运动方程：

$$ku+m\ddot u+mL\cos\theta\dot\theta^2+mL\sin\theta\ddot\theta=0$$

$$mg\sin\theta L+mL^2\ddot\theta+mL\sin\theta\ddot u=0$$

平衡位置附近满足： $\bar u=0,\bar\theta=0$，

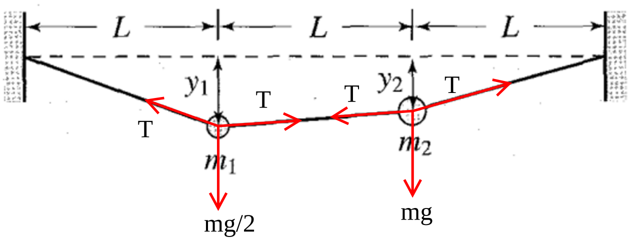
假设在平衡位置处小变形为 u,θ，令 $\tilde u=\bar u+u,\ \tilde\theta=\bar\theta+\theta$。

代入并忽略高阶小量，得到在平衡位置附近做小变形的线性运动方程：

$$\begin{bmatrix}m&0\\0&mL^2\end{bmatrix}\begin{bmatrix}\ddot u\\ \ddot\theta\end{bmatrix}+\begin{bmatrix}k&0\\0&mgL\end{bmatrix}\begin{bmatrix}u\\ \theta\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

Problem 3: Please solve Problem 6.2, 6.10, and 6.14 in page 278-279 of the textbook. (15 points)

6.2. Two masses, $m_1=0.5m$ and $m_2=m$, are suspended on a massless string of constant tension T (Fig. 6.4). Assume small displacements y_1 and y_2 and use the principle of virtual work to calculate the equilibrium position of the masses sagging under their own weight.



解：∵认为 y_1,y_2 为小变形

∴可认为细绳与虚线夹角的正弦值近似等于正切值，即：

$$\frac{y_1}{\sqrt{L^2+y_1^2}}\approx\frac{y_1}{L},\quad\frac{y_1-y_2}{\sqrt{L^2+\left(y_1-y_2\right)^2}}\approx\frac{y_1-y_2}{L},\quad\frac{y_2}{\sqrt{L^2+y_2^2}}\approx\frac{y_2}{L}$$

根据虚功原理：

$$\delta W=\sum_i^n F_i\delta r_i=0$$

$$\left(\frac{mg}{2}-T\frac{y_1}{L}-T\frac{y_1-y_2}{L}\right)\delta y_1+\left(mg-T\frac{y_2}{L}+T\frac{y_1-y_2}{L}\right)\delta y_2=0$$

由于虚位移的任意性，可得平衡构型：

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$$\begin{cases} \dfrac{mg}{2} - T\dfrac{\bar{y}_1}{L} - T\dfrac{\bar{y}_1 - \bar{y}_2}{L} = 0 \\ mg - T\dfrac{\bar{y}_2}{L} + T\dfrac{\bar{y}_1 - \bar{y}_2}{L} = 0 \end{cases}$$

$$\bar{y}_1 = \frac{2mgL}{3T} \quad, \bar{y}_2 = \frac{5mgL}{6T}$$

6.10. Derive the equations of motion for the system of Problem 6.2 by means of Hamilton's principle.

解：由 6.2 已得：

$$\delta W = \left(\frac{mg}{2} - T\frac{y_1}{L} - T\frac{y_1 - y_2}{L}\right)\delta y_1 + \left(mg - T\frac{y_2}{L} + T\frac{y_1 - y_2}{L}\right)\delta y_2$$

且系统的动能（此 T 非绳子的张力）

$$T_{sys} = \frac{m}{4}\dot{y}_1^2 + \frac{m}{2}\dot{y}_2^2$$

$$\delta T_{sys} = \frac{m}{2}\dot{y}_1\delta\dot{y}_1 + m\dot{y}_2\delta\dot{y}_2$$

哈密顿原理：

$$\int_{t_1}^{t_2}\left(\delta W + \delta T_{sys}\right)dt = 0$$

$$\int_{t_1}^{t_2}\left(\begin{array}{l}\left(\dfrac{mg}{2} - T\dfrac{y_1}{L} - T\dfrac{y_1 - y_2}{L} - \dfrac{m}{2}\ddot{y}_1\right)\delta y_1 + \\ \left(mg - T\dfrac{y_2}{L} + T\dfrac{y_1 - y_2}{L} - m\ddot{y}_2\right)\delta y_2\end{array}\right)dt = 0$$

可得系统的运动方程为：

$$\frac{mg}{2} - T\frac{y_1}{L} - T\frac{y_1 - y_2}{L} = \frac{m}{2}\ddot{y}_1$$

$$mg - T\frac{y_2}{L} + T\frac{y_1 - y_2}{L} = m\ddot{y}_2$$

即：

$$\begin{bmatrix} \dfrac{m}{2} & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} \dfrac{2T}{L} & -\dfrac{T}{L} \\ -\dfrac{T}{L} & \dfrac{2T}{L} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} mg/2 \\ mg \end{bmatrix}$$

∵平衡位置处构型：

$$\begin{cases} \dfrac{mg}{2} - T\dfrac{\bar{y}_1}{L} - T\dfrac{\bar{y}_1 - \bar{y}_2}{L} = 0 \\ mg - T\dfrac{\bar{y}_2}{L} + T\dfrac{\bar{y}_1 - \bar{y}_2}{L} = 0 \end{cases}$$

假设在平衡位置处做微小振动 $\tilde{y}_1 = \bar{y}_1 + y_1, \tilde{y}_2 = \bar{y}_2 + y_2$。 y_1, y_2 为平衡位置附近的微小振动。

得到在平衡位置附近做小变形的线性运动方程：

$$\begin{bmatrix} \dfrac{m}{2} & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} \dfrac{2T}{L} & -\dfrac{T}{L} \\ -\dfrac{T}{L} & \dfrac{2T}{L} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

6.14. Derive the equations of motion for the system of Problem 6.2 by means of Lagrange's equations

解：选取广义坐标 y_1, y_2， 以向下为正。

广义力:

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$$Q_1 = -T\left(\frac{y_1}{L} + \frac{y_1 - y_2}{L}\right)$$

$$Q_2 = -T\left(\frac{y_2}{L} - \frac{y_1 - y_2}{L}\right)$$

系统动能：$T_{sys} = \frac{1}{4} m\dot{y}_1{}^2 + \frac{1}{2} m\dot{y}_2{}^2$

以虚线处为零势能面，则系统势能：$V = -\frac{1}{2} mgy_1 - mgy_2$

代入拉格朗日方程

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right) - \frac{\partial L}{\partial y_i} = Q_i$$

得到系统的运动方程：

$$\frac{m}{2}\ddot{y}_1 - \frac{mg}{2} = Q_1$$

$$m\ddot{y}_2 - mg = Q_2$$

即：

$$\begin{bmatrix} \frac{m}{2} & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} \frac{2T}{L} & -\frac{T}{L} \\ -\frac{T}{L} & \frac{2T}{L} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} mg/2 \\ mg \end{bmatrix}$$

在平衡位置附近做微小振动的运动方程同 6.10

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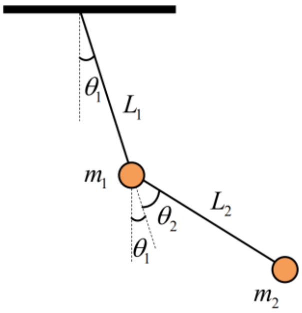
Problem 4: Assuming small motions, please derive the linear EOM of the following double pendulum system based on the given generalized coordinates. Please check with the example in the class where different generalized coordinates is used. (10 points)



解：如图，广义坐标选θ_1, θ_2。

两个质点的速度可表示为：

$$\boldsymbol{v_1} = L_1\dot{\theta}_1\left(\cos\theta_1 \boldsymbol{i} + \sin\theta_1 \boldsymbol{j}\right)$$

$$\boldsymbol{v_2} = \boldsymbol{v_1} + L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\left(\cos\left(\theta_1+\theta_2\right)\boldsymbol{i} + \sin\left(\theta_1+\theta_2\right)\boldsymbol{j}\right)$$

则系统的动能为：

$$T = \frac{1}{2}m_1 L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(\left(L_1^2+L_2^2\right)\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + 2L_2^2\dot{\theta}_1\dot{\theta}_2 + 2L_1L_2\dot{\theta}_1\left(\dot{\theta}_1+\dot{\theta}_2\right)\cos\left(\theta_2\right)\right)$$

取上端固定处为零势能面，则系统的势能为：

$$V = -m_1 g L_1\cos\theta_1 - m_2 g\left(L_1\cos\theta_1 + L_2\cos\left(\theta_1+\theta_2\right)\right)$$

系统为保守系统，代入 Lagrange 方程:

$$\begin{cases} \left(m_1L_1^2 + m_2\left(L_1^2+L_2^2\right)\right)\ddot{\theta}_1 + m_2L_2^2\ddot{\theta}_2 + m_2L_1L_2\cos\theta_2\left(2\ddot{\theta}_1+\ddot{\theta}_2\right) - m_2L_1L_2\sin\theta_2\left(\dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\right) + \left(m_1+m_2\right)gL_1\sin\theta_1 + m_2gL_2\sin\left(\theta_1+\theta_2\right) = 0 \\ m_2L_2^2\ddot{\theta}_2 + m_2L_2^2\ddot{\theta}_1 + m_2L_1L_2\cos\theta_2\ddot{\theta}_1 + m_2L_1L_2\sin\theta_2\dot{\theta}_1^2 + m_2gL_2\sin\left(\theta_1+\theta_2\right) = 0 \end{cases}$$

系统的平衡位置满足$\sin\left(\bar{\theta}_1+\bar{\theta}_2\right)=\sin\left(\bar{\theta}_1\right)=0$，其中稳定平衡位置为

$$\bar{\theta}_1 = \bar{\theta}_2 = 0$$

在平衡位置附近做小振动，略去高阶项：

$$\begin{bmatrix} m_1L_1 + m_2L_1 + m_2L_2 & m_2L_2 \\ m_2L_2 + m_2L_1 & m_2L_2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} \left(m_1+m_2\right)g & 0 \\ m_2 g & m_2 g \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Ps: 若选取广义坐标为$\theta_3 = \theta_1 + \theta_2$,则:

$$\begin{bmatrix} m_1L_1 + m_2L_1 + m_2L_2 & m_2L_2 \\ m_2L_2 + m_2L_1 & m_2L_2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_3 - \ddot{\theta}_1 \end{Bmatrix} + \begin{bmatrix} \left(m_1+m_2\right)g & 0 \\ m_2 g & m_2 g \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_3 - \theta_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

化简得：

$$\begin{bmatrix} L_1 & \dfrac{m_2L_2}{m_1+m_2} \\ L_1 & L_2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

与课上所讲结果一致。

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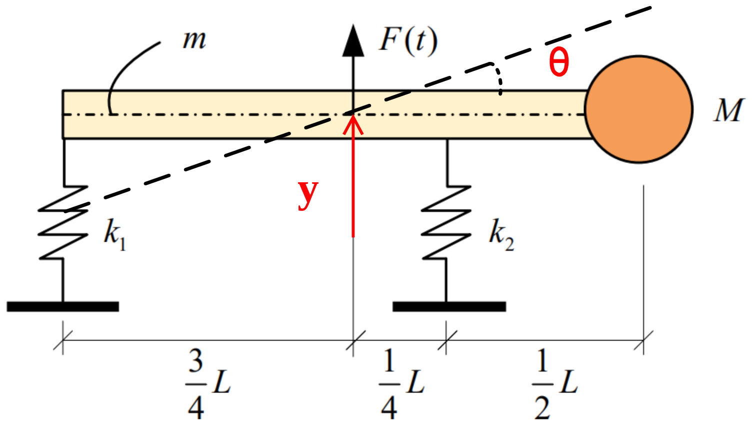
Problem 5: A rigid bar of mass per unit length m carries a point mass M at its right end. The bar is supported by two springs, as shown below. Assuming small motions, please derive the linear EOM for the translation and rotation of the mass center around the system's equilibrium configuration. (15 points)



解：如图所示，选取广义坐标为杆质心处向上的位移 y（从弹簧处于原长时开始测量）以及质心处的转角 θ（逆时针旋转为正）。

假设初始位置为零势能面，可得系统势能：

$$V=\frac{1}{2}k_1\left(\frac{3L\sin\theta}{4}-y\right)^2+\frac{1}{2}k_2\left(\frac{L\sin\theta}{4}+y\right)^2+Mg\left(\frac{3L\sin\theta}{4}+y\right)+\frac{3}{2}mgLy$$

杆的转动惯量为：

$$J_c=\frac{1}{12}\frac{3mL}{2}\left(\frac{3L}{2}\right)^2=\frac{9}{32}mL^3$$

质点 M 的速度为：

$$v_m=-\frac{3L}{4}\sin\theta\dot{\theta}\boldsymbol{i}+\left(\dot{y}+\cos\theta\dot{\theta}\frac{3L}{4}\right)\boldsymbol{j}$$

系统的动能：

$$\begin{aligned}T&=\frac{1}{2}J_c\dot{\theta}^2+\frac{1}{2}\frac{3mL}{2}\dot{y}^2+\frac{1}{2}Mv^2\\&=\frac{9L^2}{64}(2M+mL)\dot{\theta}^2+\frac{1}{4}(2M+3mL)\dot{y}^2+\frac{3ML\cos\theta}{4}\dot{\theta}\dot{y}\end{aligned}$$

广义力：

$$Q_1=0, Q_2=F(t)$$

代入 Lagrange 方程得：

$$\frac{9L^2}{32}(2M+mL)\ddot{\theta}+\frac{3ML\cos\theta}{4}\ddot{y}+\frac{(9k_1+k_2)L^2}{16}\sin\theta\cos\theta+\frac{(k_2-3k_1)L}{4}\cos\theta y+\frac{3MgL}{4}\cos\theta=0$$

$$\left(M+\frac{3mL}{2}\right)\ddot{y}+\frac{3ML\cos\theta}{4}\ddot{\theta}-\frac{3ML\sin\theta}{4}\dot{\theta}^2+\frac{(k_2-3k_1)L}{4}\sin\theta+(k_1+k_2)y+Mg+\frac{3mgL}{2}=F(t)$$

系统的平衡位置满足：

$$\frac{(9k_1+k_2)L^2}{16}\sin\bar{\theta}+\frac{(k_2-3k_1)L}{4}\bar{y}+\frac{3MgL}{4}=0$$

$$\frac{(k_2-3k_1)L}{4}\sin\bar{\theta}+(k_1+k_2)\bar{y}+Mg+\frac{3mgL}{2}=0$$

设在平衡位置附近小振动： $\tilde{y}=\bar{y}+u, \tilde{\theta}=\bar{\theta}+\theta$

代入并化简：

$$\frac{9L^2}{32}(2M+mL)\ddot{\theta}+\frac{3ML}{4}\ddot{y}\left(\cos\bar{\theta}-\sin\bar{\theta}\theta\right)+\frac{(9k_1+k_2)L^2}{16}\left(\cos2\theta\cdot\theta-\sin\theta\cos\theta\theta^2\right)+\frac{(k_2-3k_1)L}{4}\left(\cos\bar{\theta}y-\sin\bar{\theta}\bar{y}\theta-\sin\bar{\theta}\theta y\right)-\frac{3MgL}{4}\sin\bar{\theta}\theta=0$$

$$\left(M+\frac{3mL}{2}\right)\ddot{y}+\frac{3ML}{4}\ddot{\theta}\left(\cos\bar{\theta}-\sin\bar{\theta}\theta\right)-\frac{3ML}{4}\dot{\theta}^2\left(\sin\bar{\theta}+\cos\bar{\theta}\theta\right)+\frac{(k_2-3k_1)L}{4}\cos\bar{\theta}\theta+(k_1+k_2)y=F(t)$$

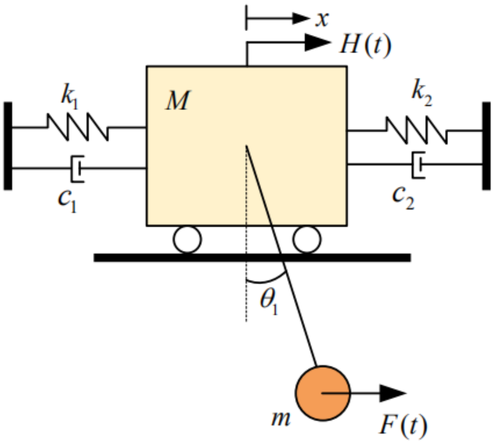
做泰勒展开并忽略高阶项，可得线性运动方程：

$$\begin{bmatrix} \frac{9L}{32}(2M+mL) & \frac{3M}{4}\cos\bar{\theta} \\ \frac{3ML}{4}\cos\bar{\theta} & M+\frac{3mL}{2} \end{bmatrix}\begin{Bmatrix} \ddot{\theta} \\ \ddot{y} \end{Bmatrix}+\begin{bmatrix} \frac{(9k_1+k_2)L}{16}\cos 2\bar{\theta}-\frac{k_2-3k_1}{4}\sin\bar{\theta}\,\bar{y}-\frac{3Mg}{4}\sin\bar{\theta} & \frac{k_2-3k_1}{4}\cos\bar{\theta} \\ \frac{(k_2-3k_1)L}{4}\cos\bar{\theta} & k_1+k_2 \end{bmatrix}\begin{Bmatrix} \theta \\ y \end{Bmatrix}=\begin{Bmatrix} 0 \\ F(t) \end{Bmatrix}$$

可进一步化简：

$$\begin{bmatrix} \frac{9L}{32}(2M+mL) & \frac{3M}{4}\cos\bar{\theta} \\ \frac{3ML}{4}\cos\bar{\theta} & M+\frac{3mL}{2} \end{bmatrix}\begin{Bmatrix} \ddot{\theta} \\ \ddot{y} \end{Bmatrix}+\begin{bmatrix} \frac{(9k_1+k_2)L}{16}\cos^2\bar{\theta} & \frac{k_2-3k_1}{4}\cos\bar{\theta} \\ \frac{(k_2-3k_1)L}{4}\cos\bar{\theta} & k_1+k_2 \end{bmatrix}\begin{Bmatrix} \theta \\ y \end{Bmatrix}=\begin{Bmatrix} 0 \\ F(t) \end{Bmatrix}$$

Problem 6: Assuming small motions, please derive the linear EOM for the system shown below. (15 points)



解：如图，选取广义坐标为 x （从弹簧原长时开始测量）和 θ_1

小球 m 的速度为：

$$v=\left(L\dot{\theta}_1\cos\theta+\dot{x}\right)i+\left(L\dot{\theta}_1\sin\theta\right)j$$

系统的动能为：

$$T=\frac{1}{2}M\dot{x}^2+\frac{1}{2}m\left(L^2\dot{\theta}_1^2+\dot{x}^2\right)+mL\dot{x}\dot{\theta}_1\cos\theta_1$$

令 M 的质心处于零势能面上，系统势能为：

$$V=\frac{1}{2}k_1x^2+\frac{1}{2}k_2x^2-mgL\cos\theta_1$$

质量块 M 和小球 M 的位移分别为：

$$r_M=x\boldsymbol{i},\quad r_m=\left(x+L\sin\theta_1\right)\boldsymbol{i}+\left(L-L\cos\theta_1\right)\boldsymbol{j}$$

广义力：

$$Q_\theta=F\left(t\right)L\cos\theta_1$$
$$Q_x=H\left(t\right)+F\left(t\right)-\left(c_1+c_2\right)\dot{x}$$

代入 Euler-Lagrange 方程得到：

$$mL^2\ddot{\theta}_1+mL\cos\theta_1\ddot{x}+mgL\sin\theta_1=F(t)L\cos\theta_1$$
$$M\ddot{x}+m\ddot{x}+mL\cos\theta_1\ddot{\theta}_1-mL\sin\theta_1\dot{\theta}_1^2+k_1x+k_2x=F(t)+H(t)-\left(c_1+c_2\right)\dot{x}$$

平衡位置为： $\bar{x}=0,\bar{\theta}_1=0$ ，在平衡位置附近做泰勒展开并略去高阶项可得线性运动方程：

$$\begin{bmatrix} mL & m \\ mL & M+m \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{x} \end{Bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & c_1+c_2 \end{bmatrix}\begin{Bmatrix} \dot{\theta}_1 \\ \dot{x} \end{Bmatrix}+\begin{bmatrix} mg & 0 \\ 0 & k_1+k_2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ x \end{Bmatrix}=\begin{Bmatrix} F(t) \\ F(t)+H(t) \end{Bmatrix}$$

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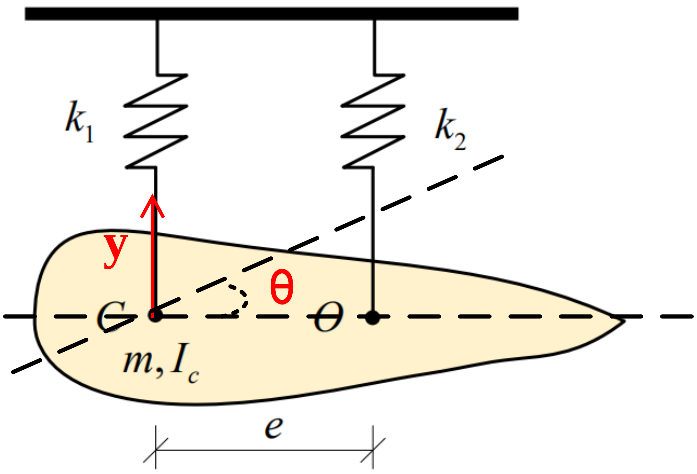
Problem 7: The system shown below represents an airfoil section being tested in a wind tunnel. Let the airfoil have total mass *m* and moment of inertia I_c about the mass center C. Assuming small motions and assume that the springs can only deform along the vertical direction, please derive the EOM. (15 points)



解：如图所示，选取广义坐标为 y （从弹簧处于平衡位置开始测量）和 θ 。
系统的动能为：

$$T = \frac{1}{2} I_C \dot{\theta}^2 + \frac{1}{2} m \dot{y}^2$$

假设初始时刻质心位于零势能面上，系统的势能为：

$$V = \frac{1}{2} k_1 y^2 + \frac{1}{2} k_2 \left(y + e \sin\theta \right)^2 + mgy$$

代入 Lagrange 方程得到：

$$I_C \ddot{\theta} + k_2 e^2 \sin\theta \cos\theta + k_2 e y \cos\theta = 0$$
$$m\ddot{y} + k_2 e \sin\theta + k_1 y + k_2 y + mg = 0$$

平衡位置为：

$$e \sin\bar{\theta} + \bar{y} = 0$$
$$k_2 e \sin\bar{\theta} + k_1 \bar{y} + k_2 \bar{y} + mg = 0$$

平衡位置附近泰勒展开并忽略高阶项，得到线性运动方程，化简得到：

$$\begin{bmatrix} I_C & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} k_2 e^2 \cos^2\bar{\theta} & k_2 e \cos\bar{\theta} \\ k_2 e \cos\bar{\theta} & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} \theta \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$