2021-Homework 4-5: Reference answer

Problem 1: When the mass matrix is orthonormalized into the identity matrix **I**, i.e., we can find a set of unit eigenvectors $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n$ which satisfy the orthonormal condition such that $\mathbf{U}^T\mathbf{M}\mathbf{U}=\mathbf{I}$, where $\mathbf{M}$ is the mass matrix, $\mathbf{U}=\left[\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_n\right]$. In such a scenario, please prove

$$\mathbf{U}^T\mathbf{K}\mathbf{U}=\mathbf{\Lambda}=\begin{bmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \cdots & \\ 0 & & & \lambda_n\end{bmatrix}$$

where $\mathbf{K}$ is the stiffness matrix, and $\lambda_i\left(i=1,\ldots,n\right)$ are the n eigenvalues. (5 points)

解：由多自由度系统动力学方程得出的特征值与特征向量满足

$$\left(\mathbf{K}-\lambda_i\mathbf{M}\right)\mathbf{u}_i=\mathbf{0}$$

如果我们找到一组特征向量使得特征向量矩阵$\mathbf{U}=\begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n\end{bmatrix}$满足

$$\mathbf{U}^T\mathbf{M}\mathbf{U}=\mathbf{I}_n$$

则有

$$\mathbf{u}_j^T\mathbf{M}\mathbf{u}_i=\begin{cases}1, i=j\\0, i\neq j\end{cases}$$

在第一个方程两端同时左乘$\mathbf{u}_j^T$，即

$$\mathbf{u}_j^T\mathbf{K}\mathbf{u}_i-\mathbf{u}_j^T\lambda_i\mathbf{M}\mathbf{u}_i=0$$

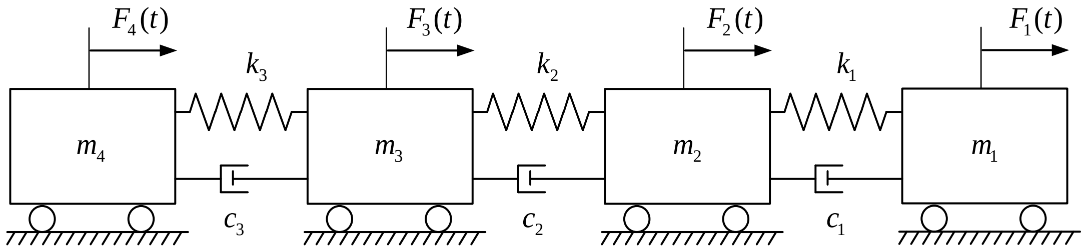
而常数λ_i可以单独提出

$$\mathbf{u}_j^T\mathbf{K}\mathbf{u}_i=\begin{cases}\lambda_i, i=j\\0, i\neq j\end{cases}$$

因此，

$$\mathbf{U}^T\mathbf{K}\mathbf{U}=\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\0 & 0 & \cdots & \lambda_n\end{bmatrix}$$

Problem 2: The system shown in the figure below consists of four lumped masses $m_i\left(i=1,2,3,4\right)$, connected by three springs and three dampers, that undergoing the horizontal displacements $x_i\left(t\right)\left(i=1,2,3,4\right)$. Each mass is also subjected to excitation forces $F_i\left(t\right)\left(i=1,2,3,4\right)$.



(1) Please derive the EOM by means of the Newton's second law **(5 points)**.

解：选取图中质量块的位移 $x_i\left(t\right), i=1,2,3,4$ 建立方程，以向右为正方向。分别对四个质量块作受力分析，可以得到，

$$\begin{aligned}
m_1\ddot{x}_1 &= F_1\left(t\right)-c_1\left(\dot{x}_1-\dot{x}_2\right)-k_1\left(x_1-x_2\right)\\
m_2\ddot{x}_2 &= F_2\left(t\right)-c_1\left(\dot{x}_2-\dot{x}_1\right)-c_2\left(\dot{x}_2-\dot{x}_3\right)-k_1\left(x_2-x_1\right)-k_2\left(x_2-x_3\right)\\
m_3\ddot{x}_3 &= F_3\left(t\right)-c_2\left(\dot{x}_3-\dot{x}_2\right)-c_3\left(\dot{x}_3-\dot{x}_4\right)-k_2\left(x_3-x_2\right)-k_3\left(x_3-x_4\right)\\
m_4\ddot{x}_4 &= F_4\left(t\right)-c_3\left(\dot{x}_4-\dot{x}_3\right)-k_3\left(x_4-x_3\right)
\end{aligned}$$

整理后有，

$$\begin{aligned}
&m_1\ddot{x}_1+c_1\left(\dot{x}_1-\dot{x}_2\right)+k_1\left(x_1-x_2\right)=F_1\left(t\right)\\
&m_2\ddot{x}_2+c_1\left(\dot{x}_2-\dot{x}_1\right)+c_2\left(\dot{x}_2-\dot{x}_3\right)+k_1\left(x_2-x_1\right)+k_2\left(x_2-x_3\right)=F_2\left(t\right)\\
&m_3\ddot{x}_3+c_2\left(\dot{x}_3-\dot{x}_2\right)+c_3\left(\dot{x}_3-\dot{x}_4\right)+k_2\left(x_3-x_2\right)+k_3\left(x_3-x_4\right)=F_3\left(t\right)\\
&m_4\ddot{x}_4+c_3\left(\dot{x}_4-\dot{x}_3\right)+k_3\left(x_4-x_3\right)=F_4\left(t\right)
\end{aligned}$$

写成矩阵形式有

$$\begin{bmatrix} m_1 & 0 & 0 & 0\\ 0 & m_2 & 0 & 0\\ 0 & 0 & m_3 & 0\\ 0 & 0 & 0 & m_4 \end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\\ \ddot{x}_4\end{Bmatrix}+\begin{bmatrix} c_1 & -c_1 & 0 & 0\\ -c_1 & c_1+c_2 & -c_2 & 0\\ 0 & -c_2 & c_2+c_3 & -c_3\\ 0 & 0 & -c_3 & c_3 \end{bmatrix}\begin{Bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{Bmatrix}+\begin{bmatrix} k_1 & -k_1 & 0 & 0\\ -k_1 & k_1+k_2 & -k_2 & 0\\ 0 & -k_2 & k_2+k_3 & -k_3\\ 0 & 0 & -k_3 & k_3 \end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{Bmatrix}=\begin{Bmatrix}F_1\left(t\right)\\ F_2\left(t\right)\\ F_3\left(t\right)\\ F_4\left(t\right)\end{Bmatrix}$$

(2) Derive the EOM for the system by means of the generalized Lagrange Equation **(13 points)**.

解：选取如图中质量快的位移 $x_i\left(t\right), i=1,2,3,4$ 为广义坐标，以向右为正方向，则系统的动能可以表示为

$$T=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2+\frac{1}{2}m_3\dot{x}_3^2+\frac{1}{2}m_4\dot{x}_4^2$$

系统的势能由弹簧提供，

$$V=\frac{1}{2}k_1\left(x_1-x_2\right)^2+\frac{1}{2}k_2\left(x_2-x_3\right)^2+\frac{1}{2}k_3\left(x_3-x_4\right)^2$$

Lagrangian 量为，

$$L=T-V=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2+\frac{1}{2}m_3\dot{x}_3^2+\frac{1}{2}m_4\dot{x}_4^2-\left[\frac{1}{2}k_1\left(x_1-x_2\right)^2+\frac{1}{2}k_2\left(x_2-x_3\right)^2+\frac{1}{2}k_3\left(x_3-x_4\right)^2\right]$$

系统中非保守力包括阻尼力和外激励

$$r_{F_1}=x_1, r_{F_2}=x_2, r_{F_3}=x_3, r_{F_4}=x_4,$$
$$F_{d1}=-c_1(\dot{x}_1-\dot{x}_2), r_{F_{d1}}=x_1-x_2,$$
$$F_{d2}=-c_2(\dot{x}_2-\dot{x}_3), r_{F_{d2}}=x_2-x_3,$$
$$F_{d3}=-c_3(\dot{x}_3-\dot{x}_4), r_{F_{d3}}=x_3-x_4.$$

因此与广义坐标 $x_i(t), i=1,2,3,4$ 相关的广义力可以表示为

$$Q_1=F_1(t)-c_1(\dot{x}_1-\dot{x}_2),$$
$$Q_2=F_2(t)-c_1(\dot{x}_1-\dot{x}_2)(-1)-c_2(\dot{x}_2-\dot{x}_3),$$
$$Q_3=F_3(t)-c_2(\dot{x}_2-\dot{x}_3)(-1)-c_3(\dot{x}_3-\dot{x}_4),$$
$$Q_4=F_4(t)-c_3(\dot{x}_3-\dot{x}_4)(-1).$$

将其代入一般形式的 Lagrange 方程有

$$m_1\ddot{x}_1+k_1(x_1-x_2)=F_1(t)-c_1(\dot{x}_1-\dot{x}_2)$$
$$m_2\ddot{x}_2-k_1(x_1-x_2)+k_2(x_2-x_3)=F_2(t)-c_1(\dot{x}_2-\dot{x}_1)-c_2(\dot{x}_2-\dot{x}_3)$$
$$m_3\ddot{x}_3-k_2(x_2-x_3)+k_3(x_3-x_4)=F_3(t)-c_2(\dot{x}_3-\dot{x}_2)-c_3(\dot{x}_3-\dot{x}_4)$$
$$m_4\ddot{x}_4-k_3(x_3-x_4)=F_4(t)-c_3(\dot{x}_4-\dot{x}_3)$$

整理后有，

$$m_1\ddot{x}_1+c_1(\dot{x}_1-\dot{x}_2)+k_1(x_1-x_2)=F_1(t)$$
$$m_2\ddot{x}_2+c_1(\dot{x}_2-\dot{x}_1)+c_2(\dot{x}_2-\dot{x}_3)+k_1(x_2-x_1)+k_2(x_2-x_3)=F_2(t)$$
$$m_3\ddot{x}_3+c_2(\dot{x}_3-\dot{x}_2)+c_3(\dot{x}_3-\dot{x}_4)+k_2(x_3-x_2)+k_3(x_3-x_4)=F_3(t)$$
$$m_4\ddot{x}_4+c_3(\dot{x}_4-\dot{x}_3)+k_3(x_4-x_3)=F_4(t)$$

写成矩阵形式有

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{Bmatrix}+\begin{bmatrix} c_1 & -c_1 & 0 & 0 \\ -c_1 & c_1+c_2 & -c_2 & 0 \\ 0 & -c_2 & c_2+c_3 & -c_3 \\ 0 & 0 & -c_3 & c_3 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix}+\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}=\begin{Bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \\ F_4(t) \end{Bmatrix}$$

(3) Determine the equilibrium configuration **(2 points)**.

解：系统的平衡位置应当满足，

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

在弹簧刚度不为零($k_i \neq 0, i=1,2,3$)的条件下解得，

$$x_1=x_2=x_3=x_4$$

即四个质量块的位移相同，弹簧未拉伸且未压缩时均为平衡状态。

(4) Use the linear transformation $y_1=x_1-x_2, y_2=x_2-x_3, y_3=x_3-x_4$, and show how the system can be reduced to a 3-DOF system with elastic displacements $y_i(t)(i=1,2,3)$ **(10 points)**. Then, determine the equilibrium configuration **(5 points)** (Tips: $\det(\mathbf{K})\neq 0$), and write it into a matrix form **(5 points)**.

解：对于系统方程

$$
\begin{aligned}
&m_1\ddot{x}_1+c_1\left(\dot{x}_1-\dot{x}_2\right)+k_1\left(x_1-x_2\right)=F_1\left(t\right)\\
&m_2\ddot{x}_2+c_1\left(\dot{x}_2-\dot{x}_1\right)+c_2\left(\dot{x}_2-\dot{x}_3\right)+k_1\left(x_2-x_1\right)+k_2\left(x_2-x_3\right)=F_2\left(t\right)\\
&m_3\ddot{x}_3+c_2\left(\dot{x}_3-\dot{x}_2\right)+c_3\left(\dot{x}_3-\dot{x}_4\right)+k_2\left(x_3-x_2\right)+k_3\left(x_3-x_4\right)=F_3\left(t\right)\\
&m_4\ddot{x}_4+c_3\left(\dot{x}_4-\dot{x}_3\right)+k_3\left(x_4-x_3\right)=F_4\left(t\right)
\end{aligned}
$$

将二次项前面系数化为 1，并分别用前一式减去后一式，可以得到

$$
\ddot{x}_1-\ddot{x}_2+\frac{c_1}{m_1}\left(\dot{x}_1-\dot{x}_2\right)-\frac{c_1}{m_2}\left(\dot{x}_2-\dot{x}_1\right)-\frac{c_2}{m_2}\left(\dot{x}_2-\dot{x}_3\right)+\frac{k_1}{m_1}\left(x_1-x_2\right)-\frac{k_1}{m_2}\left(x_2-x_1\right)-\frac{k_2}{m_2}\left(x_2-x_3\right)=\frac{F_1\left(t\right)}{m_1}-\frac{F_2\left(t\right)}{m_2}
$$

$$
\left[\begin{aligned}
&\ddot{x}_2-\ddot{x}_3+\frac{c_1}{m_2}\left(\dot{x}_2-\dot{x}_1\right)+\frac{c_2}{m_2}\left(\dot{x}_2-\dot{x}_3\right)+\frac{k_1}{m_2}\left(x_2-x_1\right)+\frac{k_2}{m_2}\left(x_2-x_3\right)\\
&-\frac{c_2}{m_3}\left(\dot{x}_3-\dot{x}_2\right)-\frac{c_3}{m_3}\left(\dot{x}_3-\dot{x}_4\right)-\frac{k_2}{m_3}\left(x_3-x_2\right)-\frac{k_3}{m_3}\left(x_3-x_4\right)
\end{aligned}\right]=\frac{F_2\left(t\right)}{m_2}-\frac{F_3\left(t\right)}{m_3}
$$

$$
\ddot{x}_3-\ddot{x}_4+\frac{c_2}{m_3}\left(\dot{x}_3-\dot{x}_2\right)+\frac{c_3}{m_3}\left(\dot{x}_3-\dot{x}_4\right)+\frac{k_2}{m_3}\left(x_3-x_2\right)+\frac{k_3}{m_3}\left(x_3-x_4\right)-\frac{c_3}{m_4}\left(\dot{x}_4-\dot{x}_3\right)-\frac{k_3}{m_4}\left(x_4-x_3\right)=\frac{F_3\left(t\right)}{m_3}-\frac{F_4\left(t\right)}{m_4}
$$

利用线性变换 $y_1=x_1-x_2, y_2=x_2-x_3, y_3=x_3-x_4$，

$$
\begin{aligned}
&\ddot{y}_1+\frac{c_1}{m_1}\dot{y}_1-\frac{c_1}{m_2}\left(-\dot{y}_1\right)-\frac{c_2}{m_2}\dot{y}_2+\frac{k_1}{m_1}y_1-\frac{k_1}{m_2}\left(-y_1\right)-\frac{k_2}{m_2}y_2=\frac{F_1\left(t\right)}{m_1}-\frac{F_2\left(t\right)}{m_2}\\
&\ddot{y}_2+\frac{c_1}{m_2}\left(-\dot{y}_1\right)+\frac{c_2}{m_2}\dot{y}_2+\frac{k_1}{m_2}\left(-y_1\right)+\frac{k_2}{m_2}y_2-\frac{c_2}{m_3}\left(-\dot{y}_2\right)-\frac{c_3}{m_3}\dot{y}_3-\frac{k_2}{m_3}\left(-y_2\right)-\frac{k_3}{m_3}y_3=\frac{F_2\left(t\right)}{m_2}-\frac{F_3\left(t\right)}{m_3}\\
&\ddot{y}_3+\frac{c_2}{m_3}\left(-\dot{y}_2\right)+\frac{c_3}{m_3}\dot{y}_3+\frac{k_2}{m_3}\left(-y_2\right)+\frac{k_3}{m_3}y_3-\frac{c_3}{m_4}\left(-\dot{y}_3\right)-\frac{k_3}{m_4}\left(-y_3\right)=\frac{F_3\left(t\right)}{m_3}-\frac{F_4\left(t\right)}{m_4}
\end{aligned}
$$

整理成矩阵形式有

$$
\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\ddot{y}_1\\\ddot{y}_2\\\ddot{y}_3\end{bmatrix}+\begin{bmatrix}\frac{c_1}{m_1}+\frac{c_1}{m_2}&-\frac{c_2}{m_2}&0\\-\frac{c_1}{m_2}&\frac{c_2}{m_2}+\frac{c_2}{m_3}&-\frac{c_3}{m_3}\\0&-\frac{c_2}{m_3}&\frac{c_3}{m_3}+\frac{c_3}{m_4}\end{bmatrix}\begin{bmatrix}\dot{y}_1\\\dot{y}_2\\\dot{y}_3\end{bmatrix}+\begin{bmatrix}\frac{k_1}{m_1}+\frac{k_1}{m_2}&-\frac{k_2}{m_2}&0\\-\frac{k_1}{m_2}&\frac{k_2}{m_2}+\frac{k_2}{m_3}&-\frac{k_3}{m_3}\\0&-\frac{k_2}{m_3}&\frac{k_3}{m_3}+\frac{k_3}{m_4}\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}\frac{F_1\left(t\right)}{m_1}-\frac{F_2\left(t\right)}{m_2}\\\frac{F_2\left(t\right)}{m_2}-\frac{F_3\left(t\right)}{m_3}\\\frac{F_3\left(t\right)}{m_3}-\frac{F_4\left(t\right)}{m_4}\end{bmatrix}
$$

系统的平衡位置应当满足，

$$
\begin{bmatrix}\frac{k_1}{m_1}+\frac{k_1}{m_2}&-\frac{k_2}{m_2}&0\\-\frac{k_1}{m_2}&\frac{k_2}{m_2}+\frac{k_2}{m_3}&-\frac{k_3}{m_3}\\0&-\frac{k_2}{m_3}&\frac{k_3}{m_3}+\frac{k_3}{m_4}\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}
$$

注意到，

$$
\begin{aligned}
\det\left(\mathbf{K}\right)&=\left(\frac{k_1}{m_1}+\frac{k_1}{m_2}\right)\left(\frac{k_2}{m_2}+\frac{k_2}{m_3}\right)\left(\frac{k_3}{m_3}+\frac{k_3}{m_4}\right)-\left(\frac{k_1}{m_1}+\frac{k_1}{m_2}\right)\frac{k_3}{m_3}\frac{k_2}{m_3}-\frac{k_1}{m_2}\frac{k_2}{m_2}\left(\frac{k_3}{m_3}+\frac{k_3}{m_4}\right)\\
&=k_1k_2k_3\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\left(\frac{1}{m_2m_3}+\frac{1}{m_2m_4}+\frac{1}{m_3m_4}\right)-k_1k_2k_3\frac{1}{m_2m_2}\left(\frac{1}{m_3}+\frac{1}{m_4}\right)\\
&=\frac{k_1k_2k_3}{m_1m_2m_3m_4}\left(m_1+m_2+m_3+m_4\right)
\end{aligned}
$$

在一般情况下（质量非负且不全为 0，刚度均非零）上式是不为零的，根据线性代数知识可以知道，平衡位置方程 $\mathbf{Ky}=\mathbf{0}$ 只有零解，即

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

在平衡位置附近线性化后得到的矩阵形式和原方程一致，

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{bmatrix} + \begin{bmatrix} \frac{c_1}{m_1}+\frac{c_1}{m_2} & -\frac{c_2}{m_2} & 0 \\ -\frac{c_1}{m_2} & \frac{c_2}{m_2}+\frac{c_2}{m_3} & -\frac{c_3}{m_3} \\ 0 & -\frac{c_2}{m_3} & \frac{c_3}{m_3}+\frac{c_3}{m_4} \end{bmatrix}\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} + \begin{bmatrix} \frac{k_1}{m_1}+\frac{k_1}{m_2} & -\frac{k_2}{m_2} & 0 \\ -\frac{k_1}{m_2} & \frac{k_2}{m_2}+\frac{k_2}{m_3} & -\frac{k_3}{m_3} \\ 0 & -\frac{k_2}{m_3} & \frac{k_3}{m_3}+\frac{k_3}{m_4} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{F_1(t)}{m_1}-\frac{F_2(t)}{m_2} \\ \frac{F_2(t)}{m_2}-\frac{F_3(t)}{m_3} \\ \frac{F_3(t)}{m_3}-\frac{F_4(t)}{m_4} \end{bmatrix}$$

(5) Assuming that all forces are equal to zero, i.e., $F_i(t)=0$. Please derive and solve the EVP (i.e., solve the eigenvalues and the corresponding eigenvectors) of the reduced 3-DOF system with $m_i = m(i=1,2,3,4)$, $k_i = k(i=1,2,3)$. Plot the modes of the reduced 3-DOF system and explain the nature of the mode shapes **(10 points)**.

解：在无阻尼，无激励且 $m_i = m, i=1,2,3,4$， $k_i = k, i=1,2,3$ 的条件下，原方程为

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{bmatrix} + \frac{k}{m}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

特征方程为

$$\det(K-\lambda M) = \frac{k}{m}\begin{vmatrix} 2-\frac{m}{k}\lambda & -1 & 0 \\ -1 & 2-\frac{m}{k}\lambda & -1 \\ 0 & -1 & 2-\frac{m}{k}\lambda \end{vmatrix} = 0$$

特征多项式为

$$\left(2-\frac{m}{k}\lambda\right)\left[\left(2-\frac{m}{k}\lambda\right)^2 - 2\right] = 0$$

求得特征值为

$$\lambda_1 = \left(2-\sqrt{2}\right)\frac{k}{m}, \lambda_2 = 2\frac{k}{m}, \lambda_3 = \left(2+\sqrt{2}\right)\frac{k}{m}$$

与之相对的固有频率和阵型分别为

$\lambda_1 = \left(2-\sqrt{2}\right)\frac{k}{m}$,

$$\omega_1 = \sqrt{2-\sqrt{2}\frac{k}{m}}, \frac{k}{m}\begin{bmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & -1 \\ 0 & -1 & \sqrt{2} \end{bmatrix} u_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, u_1 = \alpha_1[1,\sqrt{2},1]^T$$

$\lambda_2 = 2\frac{k}{m}$,

$$\omega_2=\sqrt{2}\frac{k}{m},\frac{k}{m}\begin{bmatrix}0 & -1 & 0\\ -1 & 0 & -1\\ 0 & -1 & 0\end{bmatrix}u_2=\begin{Bmatrix}0\\0\\0\end{Bmatrix},u_2=\alpha_2[1,0,-1]^T$$

$\lambda_3=\left(2+\sqrt{2}\right)\frac{k}{m}$,

$$\omega_3=\sqrt{2+\sqrt{2}}\frac{k}{m},\frac{k}{m}\begin{bmatrix}-\sqrt{2} & -1 & 0\\ -1 & -\sqrt{2} & -1\\ 0 & -1 & -\sqrt{2}\end{bmatrix}u_3=\begin{Bmatrix}0\\0\\0\end{Bmatrix},u_3=\alpha_3[1,-\sqrt{2},1]^T$$

振型绘制：略

第二种模态是反向振动模态，其中处于对称中心的点将会一直保持在静止状态；第一种和第三种模态是轴对称的，但是第一种模态下所有的点均处于平衡构型同一侧（同向振动），而第三种模态下相邻两个点分别处于平衡构型的两侧。

(6) Verify that the natural mode shapes in question (5) are orthogonal. Then orthonormalize the modes so as they satisfy $\mathbf{U}^T\mathbf{M}\mathbf{U}=\mathbf{I}$ and $\mathbf{U}^T\mathbf{K}\mathbf{U}=\mathbf{\Lambda}$ **(10 points).**

解：根据特征向量构造特征矩阵，

$$\mathbf{U}=[u_1,u_2,u_3]=\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \sqrt{2}\alpha_1 & 0 & -\sqrt{2}\alpha_3\\ \alpha_1 & -\alpha_2 & \alpha_3\end{bmatrix}$$

验证其是否正交，

$$\mathbf{U}^T\mathbf{M}\mathbf{U}=\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \sqrt{2}\alpha_1 & 0 & -\sqrt{2}\alpha_3\\ \alpha_1 & -\alpha_2 & \alpha_3\end{bmatrix}^T\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \sqrt{2}\alpha_1 & 0 & -\sqrt{2}\alpha_3\\ \alpha_1 & -\alpha_2 & \alpha_3\end{bmatrix}$$

$$=\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \sqrt{2}\alpha_1 & 0 & -\sqrt{2}\alpha_3\\ \alpha_1 & -\alpha_2 & \alpha_3\end{bmatrix}^T\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \sqrt{2}\alpha_1 & 0 & -\sqrt{2}\alpha_3\\ \alpha_1 & -\alpha_2 & \alpha_3\end{bmatrix}=\begin{bmatrix}4\alpha_1\alpha_1 & 0 & 0\\ 0 & 2\alpha_2\alpha_2 & 0\\ 0 & 0 & 4\alpha_3\alpha_3\end{bmatrix}$$

因此模态是正交的，且若取

$$4\alpha_1\alpha_1=2\alpha_2\alpha_2=4\alpha_3\alpha_3=1$$

即

$$u_1=\frac{1}{2}[1,\sqrt{2},1]^T,u_2=\frac{\sqrt{2}}{2}[1,0,-1]^T,u_3=\frac{1}{2}[1,-\sqrt{2},1]^T$$

时，可以使得

$$\mathbf{U}^T\mathbf{M}\mathbf{U}=\mathbf{I}\text{，}\quad \mathbf{U}^T\mathbf{K}\mathbf{U}=\Lambda$$

(7) Please read example 7.7 before answering this question. Based on questions (5) and (6), determine the response of the system to initial conditions for the three cases by means of modal analysis: (a) $\mathbf{y}(0)=[0,1,0]^T,\dot{\mathbf{y}}(0)=\mathbf{0}$, (b) $\mathbf{y}(0)=[-1,0,1]^T,\dot{\mathbf{y}}(0)=\mathbf{0}$, and (c) $\mathbf{y}(0)=\mathbf{0},\dot{\mathbf{y}}(0)=[0,0,1]^T$. Draw conclusions as to the modal participation in the response in each of the three cases **(15 points)**.

解：令

$$\boldsymbol{y}(t)=U\boldsymbol{\eta}(t)=\sum_{r=1}^{n}u_r\boldsymbol{\eta}_r(t)=\sum_{r=1}^{n}\boldsymbol{\eta}_r(t)u_r$$

对于整个方程有

$$U^T MU\ddot{\boldsymbol{\eta}}+U^T KU\boldsymbol{\eta}=0$$

又因为

$$\boldsymbol{\eta}(t)=U^{-1}\boldsymbol{y}(t)=U^T M\boldsymbol{y}(t)$$

则初始条件为

$$\boldsymbol{\eta}(0)=U^T M\boldsymbol{y}(0),\dot{\boldsymbol{\eta}}(0)=U^T M\dot{\boldsymbol{y}}(0)$$

因此方程的解为

$$\boldsymbol{y}(t)=\sum_{r=1}^{n}\boldsymbol{\eta}_r(t)u_r=\sum_{r=1}^{n}\left[u_r{}^T M\boldsymbol{y}(0)\cos\omega_r t+\frac{1}{\omega_r}u_r{}^T M\dot{\boldsymbol{y}}(0)\sin\omega_r t\right]u_r$$

（a）初始条件为：$y(0)=[0,1,0]^T,\dot{y}(0)=0$

$$\boldsymbol{y}(t)=\sum_{r=1}^{n}\boldsymbol{\eta}_r(t)u_r=\frac{1}{2}[1,\sqrt{2},1]\begin{pmatrix}0\\1\\0\end{pmatrix}\cos\omega_1 t\frac{1}{2}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}+\frac{\sqrt{2}}{2}[1,0,-1]\begin{pmatrix}0\\1\\0\end{pmatrix}\cos\omega_2 t\frac{\sqrt{2}}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}+\frac{1}{2}[1,-\sqrt{2},1]\begin{pmatrix}0\\1\\0\end{pmatrix}\cos\omega_3 t\frac{1}{2}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}$$

$$=\frac{\sqrt{2}}{4}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}\cos\omega_1 t-\frac{\sqrt{2}}{4}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}\cos\omega_3 t=\frac{\sqrt{2}}{4}\begin{pmatrix}\cos\omega_1 t-\cos\omega_3 t\\\sqrt{2}\left(\cos\omega_1 t+\cos\omega_3 t\right)\\\cos\omega_1 t-\cos\omega_3 t\end{pmatrix}$$

因为 $y(0)=[0,1,0]^T=\dfrac{\sqrt{2}}{2}\left(u_1-u_3\right)$，因此结果也只与第一、三这两个模态相关。

（b）初始条件为：$y(0)=[-1,0,1]^T,\dot{y}(0)=0$

$$\boldsymbol{y}(t)=\sum_{r=1}^{n}\boldsymbol{\eta}_r(t)u_r=\frac{1}{2}[1,\sqrt{2},1]\begin{pmatrix}-1\\0\\1\end{pmatrix}\cos\omega_1 t\frac{1}{2}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}+\frac{\sqrt{2}}{2}[1,0,-1]\begin{pmatrix}-1\\0\\1\end{pmatrix}\cos\omega_2 t\frac{\sqrt{2}}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}+\frac{1}{2}[1,-\sqrt{2},1]\begin{pmatrix}-1\\0\\1\end{pmatrix}\cos\omega_3 t\frac{1}{2}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}=\begin{pmatrix}-\cos\omega_2 t\\0\\\cos\omega_2 t\end{pmatrix}$$

因为 $y(0)=[-1,0,1]^T=-u_2$，因此结果也只与第二个模态相关。

（c）初始条件为：$y(0)=0,\dot{y}(0)=[0,0,1]^T$

$$\boldsymbol{y}(t)=\sum_{r=1}^{n}\boldsymbol{\eta}_r(t)u_r=\frac{1}{4\omega_1}[1,\sqrt{2},1]\begin{pmatrix}0\\0\\1\end{pmatrix}\sin\omega_1 t\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}+\frac{1}{2\omega_2}[1,0,-1]\begin{pmatrix}0\\0\\1\end{pmatrix}\sin\omega_2 t\begin{pmatrix}1\\0\\-1\end{pmatrix}+\frac{1}{4\omega_3}[1,-\sqrt{2},1]\begin{pmatrix}0\\0\\1\end{pmatrix}\sin\omega_3 t\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}$$

$$=\frac{1}{4\omega_1}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}\sin\omega_1 t-\frac{1}{2\omega_2}\begin{pmatrix}1\\0\\-1\end{pmatrix}\sin\omega_2 t+\frac{1}{4\omega_3}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}\sin\omega_3 t=\begin{pmatrix}\frac{1}{4\omega_1}\sin\omega_1 t-\frac{1}{2\omega_2}\sin\omega_2 t+\frac{1}{4\omega_3}\sin\omega_3 t\\\frac{\sqrt{2}}{4\omega_1}\sin\omega_1 t-\frac{\sqrt{2}}{4\omega_3}\sin\omega_3 t\\\frac{1}{4\omega_1}\sin\omega_1 t-\frac{1}{2\omega_2}\sin\omega_2 t+\frac{1}{4\omega_3}\sin\omega_3 t\end{pmatrix}$$

因为 $\dot{y}(0)=[0,0,1]^T=\dfrac{1}{2}\left(u_1+u_2-u_2\right)$，因此结果与三个模态都相关。

(8) Assuming that all forces are equal to zero, i.e., $F_i\left(t\right)=0$. Please derive and solve the EVP (i.e., solve the eigenvalues and the corresponding eigenvectors) for the case in which $m_1=m_4=m,m_2=m_3=2m$, $k_i=k\left(i=1,2,3\right)$. Plot the three modes of the reduced 3-DOF system. Compare the natural frequencies and mode shapes with those obtained in question (5) and explain the difference. **(10 points)**.

解：在无阻尼，无激励且 $m_1=m_4=m,m_2=m_3=2m$，$k_i=k,i=1,2,3$ 的条件下，原方程为

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\ddot{y}_1\\\ddot{y}_2\\\ddot{y}_3\end{bmatrix}+\frac{k}{m}\begin{bmatrix}3/2&-1/2&0\\-1/2&1&-1/2\\0&-1/2&3/2\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

特征方程为

$$\det\left(K-\lambda M\right)=\frac{k}{m}\begin{bmatrix}\frac{3}{2}-\frac{m}{k}\lambda & -\frac{1}{2} & 0\\ -\frac{1}{2} & 1-\frac{m}{k}\lambda & -\frac{1}{2}\\ 0 & -\frac{1}{2} & \frac{3}{2}-\frac{m}{k}\lambda\end{bmatrix}=0$$

特征多项式为

$$\left(\frac{3}{2}-\frac{m}{k}\lambda\right)\left[\left(1-\frac{m}{k}\lambda\right)\left(\frac{3}{2}-\frac{m}{k}\lambda\right)-\frac{1}{2}\right]=0$$

求得特征值为

$$\lambda_1=\frac{1}{2}\frac{k}{m},\lambda_2=\frac{3}{2}\frac{k}{m},\lambda_3=2\frac{k}{m}$$

与之相对的固有频率和阵型分别为

$\lambda_1=k/2m$

$$\omega_1=\frac{\sqrt{2}}{2}\sqrt{\frac{k}{m}},\frac{k}{m}\begin{bmatrix}1 & -1/2 & 0\\ -1/2 & 1/2 & -1/2\\ 0 & -1/2 & 1\end{bmatrix}u_1=\begin{Bmatrix}0\\0\\0\end{Bmatrix},u_1=\alpha_1[1,2,1]^T$$

$\lambda_2=3k/2m$

$$\omega_2=\frac{\sqrt{6}}{2}\sqrt{\frac{k}{m}},\frac{k}{m}\begin{bmatrix}0 & -1/2 & 0\\ -1/2 & -1/2 & -1/2\\ 0 & -1/2 & 0\end{bmatrix}u_2=\begin{Bmatrix}0\\0\\0\end{Bmatrix},u_2=\alpha_2[1,0,-1]^T$$

$\lambda_3=2k/m$

$$\omega_3=\sqrt{2}\sqrt{\frac{k}{m}},\frac{k}{m}\begin{bmatrix}-1/2 & -1/2 & 0\\ -1/2 & -1 & -1/2\\ 0 & -1/2 & -1/2\end{bmatrix}u_3=\begin{Bmatrix}0\\0\\0\end{Bmatrix},u_3=\alpha_3[1,-1,1]^T$$

振型绘制：略

比较固有频率和振型： $\omega_1\approx0.77\sqrt{\frac{k}{m}},\omega_2\approx1.41\sqrt{\frac{k}{m}},\omega_3\approx1.85\sqrt{\frac{k}{m}},u_1=\frac{1}{2}[1,\sqrt{2},1]^T,u_2=\frac{\sqrt{2}}{2}[1,0,-1]^T,u_3=\frac{1}{2}[1,-\sqrt{2},1]^T$

本题： $\omega_1\approx0.71\sqrt{\frac{k}{m}},\omega_2\approx1.22\sqrt{\frac{k}{m}},\omega_3=1.41\sqrt{\frac{k}{m}},u_1=\frac{1}{\sqrt{6}}[1,2,1]^T,u_2=\frac{\sqrt{2}}{2}[1,0,-1]^T,u_3=\frac{\sqrt{3}}{3}[1,-1,1]^T$

固有频率降低；同向振型中间单元振幅增大，反向振动模态不受影响，第三阶模态中间单元振幅降低。

解释区别：从力和惯性的角度解释

(9) Verify that the natural mode shapes in question (8) are orthogonal. Then orthonormalize the modes so as they satisfy $\mathbf{U}^T\mathbf{MU}=\mathbf{I}$ and $\mathbf{U}^T\mathbf{KU}=\mathbf{\Lambda}$ **(10 points).**

解：根据特征向量构造特征矩阵，

$$\mathbf{U}=[u_1,u_2,u_3]=\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ 2\alpha_1 & 0 & -\alpha_3\\ \alpha_1 & -\alpha_2 & \alpha_3\end{bmatrix}$$

验证其是否正交，

$$\mathbf{U}^T\mathbf{M}\mathbf{U}=\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ 2\alpha_1 & 0 & -\alpha_3\\ \alpha_1 & -\alpha_2 & \alpha_3\end{bmatrix}^T\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ 2\alpha_1 & 0 & -\alpha_3\\ \alpha_1 & -\alpha_2 & \alpha_3\end{bmatrix}$$

$$=\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ 2\alpha_1 & 0 & -\alpha_3\\ \alpha_1 & -\alpha_2 & \alpha_3\end{bmatrix}^T\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ 2\alpha_1 & 0 & -\alpha_3\\ \alpha_1 & -\alpha_2 & \alpha_3\end{bmatrix}=\begin{bmatrix}6\alpha_1\alpha_1 & 0 & 0\\ 0 & 2\alpha_2\alpha_2 & 0\\ 0 & 0 & 3\alpha_3\alpha_3\end{bmatrix}$$

因此模态是正交的，且若取

$$6\alpha_1\alpha_1=2\alpha_2\alpha_2=3\alpha_3\alpha_3=1$$

即

$$u_1=\frac{\sqrt{6}}{6}[1,2,1]^T,u_2=\frac{\sqrt{2}}{2}[1,0,-1]^T,u_3=\frac{\sqrt{3}}{3}[1,-1,1]^T$$

时，可以使得

$$\mathbf{U}^T\mathbf{M}\mathbf{U}=\mathbf{I}\text{，}\quad \mathbf{U}^T\mathbf{K}\mathbf{U}=\Lambda$$

(10)Based on questions (8) and (9), use the approach of modal analysis to determine the response of the system to harmonic excitation $F_1(t)=F_0\cos 0.7t, F_2(t)=F_3(t)=F_4(t)=0$. Solve this question again if the excitation frequency is 1.4 rad/s instead of 0.7 rad/s. Compare the results and draw conclusions **(20 points)**.

解：令

$$\boldsymbol{y}(t)=U\boldsymbol{\eta}(t)=\sum_{r=1}^{n}u_r\boldsymbol{\eta}_r(t)=\sum_{r=1}^{n}\boldsymbol{\eta}_r(t)u_r$$

对于整个方程有

$$U^TMU\ddot{\boldsymbol{\eta}}+U^TKU\boldsymbol{\eta}=U^TF$$

并且已经解耦。对于其中任意一个方程，

$$\ddot{\boldsymbol{\eta}}_r+\omega_r^2\boldsymbol{\eta}_r=u_r^TF\cos\omega t$$

方程的稳态解为

$$\boldsymbol{\eta}_r(t)=\frac{u_r^TF}{\omega_r^2-\omega^2}\cos\omega t$$

因此系统的稳态响应可以写为

$$\boldsymbol{y}(t)=\sum_{r=1}^{n}u_r\boldsymbol{\eta}_r(t)=\sum_{r=1}^{n}u_r\frac{u_r^TF}{\omega_r^2-\omega^2}\cos\omega t=\sum_{r=1}^{n}\frac{u_ru_r^T}{\omega_r^2-\omega^2}F\cos\omega t$$

对于本题 $F_1(t)=F_0\cos 0.7t,\ F_2(t)=F_3(t)=F_4(t)=0$，因此

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\ddot{y}_1\\ \ddot{y}_2\\ \ddot{y}_3\end{bmatrix}+\frac{k}{m}\begin{bmatrix}3/2 & -1/2 & 0\\ -1/2 & 1 & -1/2\\ 0 & -1/2 & 3/2\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}F_0/m\cos 0.7t\\ 0\\ 0\end{bmatrix}$$

由上一小题得到，

$$\omega_1\approx 0.71\sqrt{\frac{k}{m}},\omega_2\approx 1.22\sqrt{\frac{k}{m}},\omega_3=1.41\sqrt{\frac{k}{m}},u_1=\frac{1}{\sqrt{6}}[1,2,1]^T,u_2=\frac{\sqrt{2}}{2}[1,0,-1]^T,u_3=\frac{\sqrt{3}}{3}[1,-1,1]^T$$

（a） $F=\frac{1}{m}[F_0,0,0]^T$， $\omega=0.7$ 时，此处认为 $k=m$

$$\boldsymbol{y}(t)=\sum_{r=1}^{n}\frac{u_ru_r^T}{\omega_r^2-\omega^2}\frac{1}{m}\begin{pmatrix}F_0\\0\\0\end{pmatrix}\cos 0.7t=\left(\frac{1}{6m}\begin{pmatrix}1\\2\\1\end{pmatrix}\frac{1}{0.5-0.7^2}\begin{pmatrix}1\\2\\1\end{pmatrix}^T\begin{pmatrix}F_0\\0\\0\end{pmatrix}+\frac{1}{2m}\begin{pmatrix}1\\0\\-1\end{pmatrix}\frac{1}{1.5-0.7^2}\begin{pmatrix}1\\0\\-1\end{pmatrix}^T\begin{pmatrix}F_0\\0\\0\end{pmatrix}+\frac{1}{3m}\begin{pmatrix}1\\-1\\1\end{pmatrix}\frac{1}{2-0.7^2}\begin{pmatrix}1\\-1\\1\end{pmatrix}^T\begin{pmatrix}F_0\\0\\0\end{pmatrix}\right)\cos 0.7t$$

$$\approx\frac{F_0}{m}\cos 0.7t\left(16.67\begin{pmatrix}1\\2\\1\end{pmatrix}+0.50\begin{pmatrix}1\\0\\-1\end{pmatrix}+0.22\begin{pmatrix}1\\-1\\1\end{pmatrix}\right)=\begin{pmatrix}17.39\\33.12\\16.39\end{pmatrix}\frac{F_0}{m}\cos 0.7t$$

与第一阶振型比较接近；

（b） $F=\frac{1}{m}[F_0,0,0]^T$， $\omega=1.4$ 时

$$y(t)=\sum_{r=1}^{n}\frac{u_r u_r^T}{\omega_r^2-\omega^2}\frac{1}{m}\begin{pmatrix}F_0\\0\\0\end{pmatrix}\cos 1.4t$$
$$=\left(\frac{1}{6m}\begin{pmatrix}1\\2\\1\end{pmatrix}\frac{1}{0.5-1.4^2}\begin{pmatrix}1\\2\\1\end{pmatrix}^T\begin{pmatrix}F_0\\0\\0\end{pmatrix}+\frac{1}{2m}\begin{pmatrix}1\\0\\-1\end{pmatrix}\frac{1}{1.5-1.4^2}\begin{pmatrix}1\\0\\-1\end{pmatrix}^T\begin{pmatrix}F_0\\0\\0\end{pmatrix}+\frac{1}{3m}\begin{pmatrix}1\\-1\\1\end{pmatrix}\frac{1}{2-1.4^2}\begin{pmatrix}1\\-1\\1\end{pmatrix}^T\begin{pmatrix}F_0\\0\\0\end{pmatrix}\right)\cos 1.4t$$
$$\approx\left(-0.11\begin{pmatrix}1\\2\\1\end{pmatrix}-1.09\begin{pmatrix}1\\0\\-1\end{pmatrix}+8.33\begin{pmatrix}1\\-1\\1\end{pmatrix}\right)\frac{F_0}{m}\cos 1.4t\approx\begin{pmatrix}7.13\\-8.56\\9.31\end{pmatrix}\frac{F_0}{m}\cos 1.4t$$

与第三阶阵型接近。

(11)Based on questions (8) and (9), determine the response of the system to the excitation $F_1(t)=F_3(t)=(F_0/10)[r(t)-r(t-10)], F_2(t)=1.4(F_0/10)[r(t)-r(t-10)]$, where F_0 is a constant and $r(t)$ is the unit ramp function. Discuss the mode participation in the response **(25 points)**.

解：单自由度系统对斜坡函数 $r(t)$ 的响应为

$$h(t)=\int_0^t\tau\frac{1}{m\omega_d}e^{-\xi\omega_n(t-\tau)}\sin\omega_d(t-\tau)d\tau$$
$$=\frac{1}{k}\left(t-\frac{2\xi}{\omega_n}+\frac{e^{-\xi\omega_n t}}{\omega_n}\left(2\xi\cos\omega_d t+\frac{(\xi\omega_n)^2-\omega_d^2}{\omega_d\omega_n}\sin\omega_d t\right)\right)$$

对于本题的无阻尼系统有

$$h(t)=\frac{1}{k}\left(t-\frac{\sin\omega_n t}{\omega_n}\right)$$

系统方程为，

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\ddot{y}_1\\\ddot{y}_2\\\ddot{y}_3\end{bmatrix}+\frac{k}{m}\begin{bmatrix}3/2&-1/2&0\\-1/2&1&-1/2\\0&-1/2&3/2\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\frac{1}{m}\begin{bmatrix}F_0/10-1.4(F_0/10)\\1.4(F_0/10)-(F_0/10)\\(F_0/10)\end{bmatrix}[r(t)-r(t-10)]$$

令

$$\boldsymbol{y}(t)=U\boldsymbol{\eta}(t)=\sum_{r=1}^{n}u_r\boldsymbol{\eta}_r(t)=\sum_{r=1}^{n}\boldsymbol{\eta}_r(t)u_r$$

对于整个方程有

$$U^TMU\ddot{\boldsymbol{\eta}}+U^TKU\boldsymbol{\eta}=U^TQ$$

其中外激励为

$$Q=\frac{F_0}{10m}[r(t)-r(t-10)]\begin{pmatrix}-0.4\\0.4\\1\end{pmatrix}=\frac{F_0}{10m}\begin{pmatrix}-0.4\\0.4\\1\end{pmatrix}r(t)-\frac{F_0}{10m}\begin{pmatrix}-0.4\\0.4\\1\end{pmatrix}r(t-10)$$

并且已经解耦，

$$y(t)=\sum_{r=1}^{n}u_r\boldsymbol{\eta}_r(t)=\sum_{r=1}^{n}u_r\frac{F_0}{10m}u_r^T\begin{pmatrix}-0.4\\0.4\\1\end{pmatrix}\frac{1}{\omega_r^2}\left(\left(t-\frac{\sin\omega_r t}{\omega_r}\right)\mu(t)-\left(t-10-\frac{\sin\omega_r(t-10)}{\omega_r}\right)\mu(t-10)\right)$$
$$=\frac{F_0}{10m}\left(\frac{1}{6}\frac{1}{0.5}\begin{pmatrix}1\\2\\1\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix}^T\begin{pmatrix}-0.4\\0.4\\1\end{pmatrix}+\frac{1}{2}\frac{1}{1.5}\begin{pmatrix}1\\0\\-1\end{pmatrix}\begin{pmatrix}1\\0\\-1\end{pmatrix}^T\begin{pmatrix}-0.4\\0.4\\1\end{pmatrix}+\frac{1}{3}\frac{1}{2}\begin{pmatrix}1\\-1\\1\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}^T\begin{pmatrix}-0.4\\0.4\\1\end{pmatrix}\right)\left(\left(t-\frac{\sin\omega_r t}{\omega_r}\right)\mu(t)-\left(t-10-\frac{\sin\omega_r(t-10)}{\omega_r}\right)\mu(t-10)\right)$$
$$=\frac{F_0}{10m}\left(\frac{1.4}{3}\begin{pmatrix}1\\2\\1\end{pmatrix}+\frac{-1.4}{3}\begin{pmatrix}1\\0\\-1\end{pmatrix}+\frac{0.2}{6}\begin{pmatrix}1\\-1\\1\end{pmatrix}\right)\left(\left(t-\frac{\sin\omega_r t}{\omega_r}\right)\mu(t)-\left(t-10-\frac{\sin\omega_r(t-10)}{\omega_r}\right)\mu(t-10)\right)$$
$$\approx\frac{F_0}{10m}\begin{pmatrix}0.03\\0.90\\0.97\end{pmatrix}\left(\left(t-\frac{\sin\omega_r t}{\omega_r}\right)\mu(t)-\left(t-10-\frac{\sin\omega_r(t-10)}{\omega_r}\right)\mu(t-10)\right)$$

而 $u_r^T\begin{pmatrix}-0.4\\0.4\\1\end{pmatrix}\frac{1}{\omega_r^2}$ 表征了各阶模态的参与情况，大小分别为 0.4667,-0.4667,0.0333。

(12)The system is immersed in a fluid generating resistance forces proportional to the velocities of the masses, where the proportionality constants are $c_i = 0.1m\left(i=1,2,3\right)$. Based on questions (5) and (6), use the approach of modal analysis to determine the response to initial conditions for the three cases by means of modal analysis: (a) $\mathbf{y}\left(0\right)=\left[0,1,0\right]^T,\dot{\mathbf{y}}\left(0\right)=\mathbf{0}$, (b) $\mathbf{y}\left(0\right)=\left[-1,0,1\right]^T,\dot{\mathbf{y}}\left(0\right)=\mathbf{0}$, and (c) $\mathbf{y}\left(0\right)=\mathbf{0},\dot{\mathbf{y}}\left(0\right)=\left[0,0,1\right]^T$. Compare the results with those obtained in question (7) **(15 points)**.

解：与无阻尼初值问题相比，方程变为

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\ddot{y}_1\\\ddot{y}_2\\\ddot{y}_3\end{bmatrix}+0.1\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}\begin{bmatrix}\dot{y}_1\\\dot{y}_2\\\dot{y}_3\end{bmatrix}+\frac{k}{m}\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

令

$$\boldsymbol{y}(t)=U\boldsymbol{\eta}(t)=\sum_{r=1}^{n}u_r\boldsymbol{\eta}_r(t)=\sum_{r=1}^{n}\boldsymbol{\eta}_r(t)u_r$$

$$\boldsymbol{\eta}(t)=U^{-1}\boldsymbol{y}(t)=U^TM\boldsymbol{y}(t)$$

对于整个方程有

$$U^TMU\ddot{\boldsymbol{\eta}}+U^TCU+U^TKU\boldsymbol{\eta}=0$$

$$\ddot{\boldsymbol{\eta}}_r+0.1\frac{m\omega_r^2}{k}\dot{\boldsymbol{\eta}}_r+\omega_r^2\boldsymbol{\eta}_r=0$$

且

$$\omega_1\approx0.77\sqrt{\frac{k}{m}},\omega_2\approx1.41\sqrt{\frac{k}{m}},\omega_3\approx1.85\sqrt{\frac{k}{m}},u_1=\frac{1}{2}[1,\sqrt{2},1]^T,u_2=\frac{\sqrt{2}}{2}[1,0,-1]^T,u_3=\frac{1}{2}[1,-\sqrt{2},1]^T$$

阻尼比和阻尼固有频率为

$$\xi_r=\frac{c_r}{2m_r\omega_r}=\frac{1}{2\omega_r}0.1\frac{m\omega_r^2}{k}=\frac{m\omega_r}{20k}$$

$$\omega_{rd}=\sqrt{1-\xi_r^2}\omega_r$$

初始条件为

$$\boldsymbol{\eta}(0)=U^TM\boldsymbol{y}(0),\dot{\boldsymbol{\eta}}(0)=U^TM\dot{\boldsymbol{y}}(0)$$

假设其为欠阻尼，方程的初值响应为

$$\boldsymbol{y}(t)=\sum_{r=1}^{n}\boldsymbol{\eta}_r(t)u_r=\sum_{r=1}^{n}e^{-\xi\omega_rt}\left(u_r^TM\boldsymbol{y}(0)\cos\omega_{rd}t+\frac{u_r^TM\dot{\boldsymbol{y}}(0)+\xi_r\omega_ru_r^TM\boldsymbol{y}(0)}{\omega_{rd}}\sin\omega_{rd}t\right)u_r$$

$$=\sum_{r=1}^{n}e^{-\xi\omega_rt}\left(u_r^TM\boldsymbol{y}(0)\left(\cos\omega_{rd}t+\frac{\xi_r\omega_r}{\omega_{rd}}\sin\omega_{rd}t\right)+u_r^TM\dot{\boldsymbol{y}}(0)\frac{1}{\omega_{rd}}\sin\omega_{rd}t\right)u_r$$

（a）初始条件为： $y(0)=[0,1,0]^T,\dot{y}(0)=0$ 时，

$$\boldsymbol{y}(t)=\sum_{r=1}^{n}e^{-\xi\omega_r t}\left(u_r^T \boldsymbol{M}\boldsymbol{y}(0)\left(\cos\omega_{rd}t+\frac{\xi_r\omega_r}{\omega_{rd}}\sin\omega_{rd}t\right)+u_r^T \boldsymbol{M}\dot{\boldsymbol{y}}(0)\frac{1}{\omega_{rd}}\sin\omega_{rd}t\right)u_r$$

$$=\frac{e^{-\xi_1\omega_1 t}}{4}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}^T\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}\left(\cos\omega_{1d}t+\frac{\xi_1\omega_1}{\omega_{1d}}\sin\omega_{1d}t\right)\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}+\frac{e^{-\xi_2\omega_2 t}}{2}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}^T\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}\left(\cos\omega_{2d}t+\frac{\xi_2\omega_2}{\omega_{2d}}\sin\omega_{2d}t\right)\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}+\frac{e^{-\xi_3\omega_3 t}}{4}\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}^T\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}\left(\cos\omega_{3d}t+\frac{\xi_3\omega_3}{\omega_{3d}}\sin\omega_{3d}t\right)\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}$$

$$=\frac{\sqrt{2}}{4}e^{-\xi_1\omega_1 t}\left(\cos\omega_{1d}t+\frac{\xi_1\omega_1}{\omega_{1d}}\sin\omega_{1d}t\right)\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}-\frac{\sqrt{2}}{4}e^{-\xi_3\omega_3 t}\left(\cos\omega_{3d}t+\frac{\xi_3\omega_3}{\omega_{3d}}\sin\omega_{3d}t\right)\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}$$

（b）初始条件为： $y(0)=[-1,0,1]^T,\ \dot{y}(0)=0$

$$\boldsymbol{y}(t)=\sum_{r=1}^{n}e^{-\xi\omega_r t}\left(u_r^T \boldsymbol{M}\boldsymbol{y}(0)\left(\cos\omega_{rd}t+\frac{\xi_r\omega_r}{\omega_{rd}}\sin\omega_{rd}t\right)+u_r^T \boldsymbol{M}\dot{\boldsymbol{y}}(0)\frac{1}{\omega_{rd}}\sin\omega_{rd}t\right)u_r$$

$$=\frac{e^{-\xi_1\omega_1 t}}{4}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}^T\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}\left(\cos\omega_{1d}t+\frac{\xi_1\omega_1}{\omega_{1d}}\sin\omega_{1d}t\right)\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}+\frac{e^{-\xi_2\omega_2 t}}{2}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}^T\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}\left(\cos\omega_{2d}t+\frac{\xi_2\omega_2}{\omega_{2d}}\sin\omega_{2d}t\right)\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}+\frac{e^{-\xi_3\omega_3 t}}{4}\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}^T\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}\left(\cos\omega_{3d}t+\frac{\xi_3\omega_3}{\omega_{3d}}\sin\omega_{3d}t\right)\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}$$

$$=-e^{-\xi_2\omega_2 t}\left(\cos\omega_{2d}t+\frac{\xi_2\omega_2}{\omega_{2d}}\sin\omega_{2d}t\right)\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}$$

（c）初始条件为： $y(0)=0,\ \dot{y}(0)=[0,0,1]^T$

$$\boldsymbol{y}(t)=\sum_{r=1}^{n}e^{-\xi\omega_r t}\left(u_r^T \boldsymbol{M}\boldsymbol{y}(0)\left(\cos\omega_{rd}t+\frac{\xi_r\omega_r}{\omega_{rd}}\sin\omega_{rd}t\right)+u_r^T \boldsymbol{M}\dot{\boldsymbol{y}}(0)\frac{1}{\omega_{rd}}\sin\omega_{rd}t\right)u_r$$

$$=\frac{e^{-\xi_1\omega_1 t}}{4}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}^T\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\left(\frac{1}{\omega_{1d}}\sin\omega_{1d}t\right)\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}+\frac{e^{-\xi_2\omega_2 t}}{2}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}^T\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\left(\frac{1}{\omega_{2d}}\sin\omega_{2d}t\right)\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}+\frac{e^{-\xi_3\omega_3 t}}{4}\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}^T\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\left(\frac{1}{\omega_{3d}}\sin\omega_{3d}t\right)\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}$$

$$=\frac{1}{4\omega_{1d}}e^{-\xi_1\omega_1 t}\sin\omega_{1d}t\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}-\frac{1}{2\omega_{2d}}e^{-\xi_2\omega_2 t}\sin\omega_{2d}t\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}-\frac{1}{4\omega_{3d}}e^{-\xi_3\omega_3 t}\sin\omega_{3d}t\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}$$

与（7）对比可以发现，改变阻尼对于初值响应来说仅仅是改变了解的组成部分的具体表达式，对于解的构成形式（模态的组合方式）没有任何影响（只取决于初值与模态之间的线性表出的关系式）。

(13) The system is immersed in a fluid generating resistance forces proportional to the velocities of the masses, where the proportionality constants are $c_i=0.1m(i=1,2,3)$. Based on questions (8) and (9), use the approach of modal analysis to determine the response of the system to harmonic excitation $F_1(t)=F_0\cos 0.7t$, $F_2(t)=F_3(t)=F_4(t)=0$. Compare the results with those obtained in question (10) **(15 points).**

解：原方程为

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\ddot{y}_1\\ \ddot{y}_2\\ \ddot{y}_3\end{bmatrix}+0.1\begin{bmatrix}3/2&-1/2&0\\-1/2&1&-1/2\\0&-1/2&3/2\end{bmatrix}\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{bmatrix}+\frac{k}{m}\begin{bmatrix}3/2&-1/2&0\\-1/2&1&-1/2\\0&-1/2&3/2\end{bmatrix}=\begin{bmatrix}F_0/m\cos 0.7t\\0\\0\end{bmatrix}$$

令

$$\boldsymbol{y}(t)=\boldsymbol{U}\boldsymbol{\eta}(t)=\sum_{r=1}^{n}u_r\boldsymbol{\eta}_r(t)=\sum_{r=1}^{n}\boldsymbol{\eta}_r(t)u_r$$

对于整个方程有

$$U^TMU\ddot{\boldsymbol{\eta}}+U^TCU\dot{\boldsymbol{\eta}}+U^TKU\boldsymbol{\eta}=U^TQ$$

并且已经解耦，

$$\ddot{\eta}_r + 0.1\frac{m\omega_r^2}{k}\dot{\eta}_r + \omega_r^2\eta_r = u_r^T F\cos\omega t$$

阻尼比为

$$\xi_r = \frac{c_r}{2m_r\omega_r} = \frac{1}{2\omega_r}0.1\frac{m\omega_r^2}{k} = \frac{m\omega_r}{20k}$$

方程的解为

$$\eta_r(t) = \frac{u_r^T F}{\omega_r^2\sqrt{\left(1-\lambda_r^2\right)^2+\left(2\lambda_r\xi_r\right)^2}}\cos\left(\omega t-\varphi_r\right)$$

$$\varphi_r = \arctan\left(\frac{2\lambda_r\xi_r}{1-\lambda_r^2}\right)$$

因此系统的稳态响应可以写为

$$y(t) = \sum_{r=1}^{n} u_r\eta_r(t) = \sum_{r=1}^{n}\frac{u_r}{\omega_r^2\sqrt{\left(1-\lambda_r^2\right)^2+\left(2\lambda_r\xi_r\right)^2}}u_r^T\begin{pmatrix}F/m\\0\\0\end{pmatrix}\cos\left(0.7t-\varphi_r\right)$$

$$=\frac{F}{m}\cos\left(0.7t-\varphi_r\right)\left(\begin{array}{l}\dfrac{\frac{1}{6}[1,2,1]^T}{0.5\sqrt{\left(1-\frac{0.7^2}{0.5}\right)^2+\left(2\cdot 0.5\frac{0.71}{20}\right)^2}}[1,2,1]\begin{pmatrix}1\\0\\0\end{pmatrix}+\dfrac{\frac{1}{2}[1,0,-1]^T}{1.5\sqrt{\left(1-\frac{0.7^2}{1.5}\right)^2+\left(2\cdot 1.5\frac{1.22}{20}\right)^2}}[1,0,-1]\begin{pmatrix}1\\0\\0\end{pmatrix}\\ +\dfrac{\frac{1}{3}[1,-1,1]^T}{2\sqrt{\left(1-\frac{0.7^2}{2}\right)^2+\left(2\cdot 2\frac{1.41}{20}\right)^2}}[1,-1,1]\begin{pmatrix}1\\0\\0\end{pmatrix}\end{array}\right)$$

$$=\frac{F}{m}\cos\left(0.7t-\varphi_r\right)\left(8.21\begin{pmatrix}1\\2\\1\end{pmatrix}+0.48\begin{pmatrix}1\\0\\-1\end{pmatrix}+0.21\begin{pmatrix}1\\-1\\1\end{pmatrix}\right)=\begin{pmatrix}8.90\\16.21\\7.94\end{pmatrix}\frac{F}{m}\cos\left(0.7t-\varphi_r\right)$$

$$\varphi_r = \arctan\left(\frac{2\omega_r^2\frac{m\omega_r}{20k}}{1-\lambda_r^2}\right) = \arctan\left(\frac{m\omega_r^3}{10k\left(1-\lambda_r^2\right)}\right)$$

模态幅值均降低，模态参与情况改变很小

(14) The system is immersed in a fluid generating resistance forces proportional to the velocities of the masses, where the proportionality constants are $c_i = 0.1m\left(i=1,2,3\right)$. Based on questions (5) and (6), use the approach of modal analysis to determine the response to the forces $F_1(t)=0, F_2(t)=F_0\left[\mu(t)-\mu(t-5)\right], F_3(t)=F_4(t)=0$, where F_0 is a constant and $\mu(t)$ is the unit step function **(25 points)**.

Unit ramp function: $r(t)=\begin{cases}t, t\ge 0\\ 0, t<0\end{cases}$; Unit step function: $\mu(t)=\begin{cases}1, t\ge 0\\ 0, t<0\end{cases}$

解：单位阶跃函数响应为

$$g(t)=\frac{1}{k}\left(1-e^{-\xi\omega_n t}\left(\cos\omega_d t+\frac{\xi\omega_n}{\omega_d}\sin\omega_d t\right)\right)$$

原方程为

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\ddot{y}_1\\\ddot{y}_2\\\ddot{y}_3\end{bmatrix}+0.1\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}\begin{bmatrix}\dot{y}_1\\\dot{y}_2\\\dot{y}_3\end{bmatrix}+\frac{k}{m}\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}-1\\1\\0\end{bmatrix}\frac{F_0}{m}\left[\mu(t)-\mu(t-5)\right]$$

令

$$\boldsymbol{y}(t)=U\boldsymbol{\eta}(t)=\sum_{r=1}^{n}u_r\boldsymbol{\eta}_r(t)=\sum_{r=1}^{n}\boldsymbol{\eta}_r(t)u_r$$

对于整个方程有

$$U^TMU\ddot{\boldsymbol{\eta}}+U^TCU\dot{\boldsymbol{\eta}}+U^TKU\boldsymbol{\eta}=U^TQ$$

并且已经解耦，

$$\ddot{\boldsymbol{\eta}}_r+0.1\frac{m\omega_r^2}{k}\dot{\boldsymbol{\eta}}_r+\omega_r^2\boldsymbol{\eta}_r=u_r^TF\cos\omega t$$

阻尼比和阻尼固有频率为

$$\xi_r=\frac{c_r}{2m_r\omega_r}=\frac{1}{2\omega_r}0.1\frac{m\omega_r^2}{k}=\frac{m\omega_r}{20k}$$

$$\omega_{rd}=\sqrt{1-\xi_r^2}\,\omega_r$$

并且已经解耦，

$$\ddot{\boldsymbol{\eta}}_r+0.1\frac{m\omega_r^2}{k}\dot{\boldsymbol{\eta}}_r+\omega_r^2\boldsymbol{\eta}_r=u_r^TQ$$

系统的解为

$$\boldsymbol{y}(t)=\sum_{r=1}^{n}u_r\boldsymbol{\eta}_r(t)=\sum_{r=1}^{n}u_r\frac{F_0}{m}u_r^T\begin{pmatrix}-1\\1\\0\end{pmatrix}\frac{1}{\omega_r^2}\left(\left(1-e^{-\xi_r\omega_r t}\left(\cos\omega_{rd}t+\frac{\xi_r\omega_r}{\omega_{rd}}\sin\omega_{rd}t\right)\right)-\left(1-e^{-\xi_r\omega_r(t-5)}\left(\cos\omega_{rd}(t-5)+\frac{\xi_r\omega_r}{\omega_{rd}}\sin\omega_{rd}(t-5)\right)\right)\right)$$

$$=\frac{F_0}{m}\left(\left(1-e^{-\xi_r\omega_r t}\left(\cos\omega_{rd}t+\frac{\xi_r\omega_r}{\omega_{rd}}\sin\omega_{rd}t\right)\right)-\left(1-e^{-\xi_r\omega_r(t-5)}\left(\cos\omega_{rd}(t-5)+\frac{\xi_r\omega_r}{\omega_{rd}}\sin\omega_{rd}(t-5)\right)\right)\right)$$

$$\cdot\left(\frac{1}{4}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}^T\begin{pmatrix}-1\\1\\0\end{pmatrix}\frac{1}{2-\sqrt{2}}+\frac{1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}\begin{pmatrix}1\\0\\-1\end{pmatrix}^T\begin{pmatrix}-1\\1\\0\end{pmatrix}\frac{1}{2}+\frac{1}{4}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}^T\begin{pmatrix}-1\\1\\0\end{pmatrix}\frac{1}{2+\sqrt{2}}\right)$$

$$=\frac{F_0}{m}\left(\left(1-e^{-\xi_r\omega_r t}\left(\cos\omega_{rd}t+\frac{\xi_r\omega_r}{\omega_{rd}}\sin\omega_{rd}t\right)\right)-\left(1-e^{-\xi_r\omega_r(t-5)}\left(\cos\omega_{rd}(t-5)+\frac{\xi_r\omega_r}{\omega_{rd}}\sin\omega_{rd}(t-5)\right)\right)\right)\cdot\left(\frac{\sqrt{2}}{8}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}-\frac{1}{4}\begin{pmatrix}1\\0\\-1\end{pmatrix}-\frac{\sqrt{2}}{8}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}\right)$$

$$=\frac{1}{4}\begin{pmatrix}-1\\2\\1\end{pmatrix}\frac{F_0}{m}\left(\left(1-e^{-\xi_r\omega_r t}\left(\cos\omega_{rd}t+\frac{\xi_r\omega_r}{\omega_{rd}}\sin\omega_{rd}t\right)\right)-\left(1-e^{-\xi_r\omega_r(t-5)}\left(\cos\omega_{rd}(t-5)+\frac{\xi_r\omega_r}{\omega_{rd}}\sin\omega_{rd}(t-5)\right)\right)\right)$$