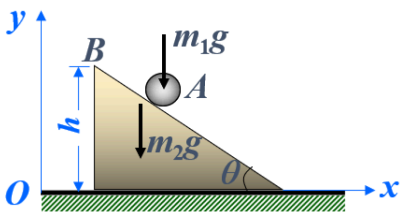
题 **1**

例: 质量为m_1的质点A，放在倾角为θ、质量为m_2的三角楔块的斜边上，楔块又可在水平面上滑动如图示。不计摩擦，试用第一类拉格朗日方程求质点和楔块的加速度以及他们所受的约束力。



设m_1的质心为(x_1,y_1), m_2的质心为(x_2,y_2)

考虑约束:

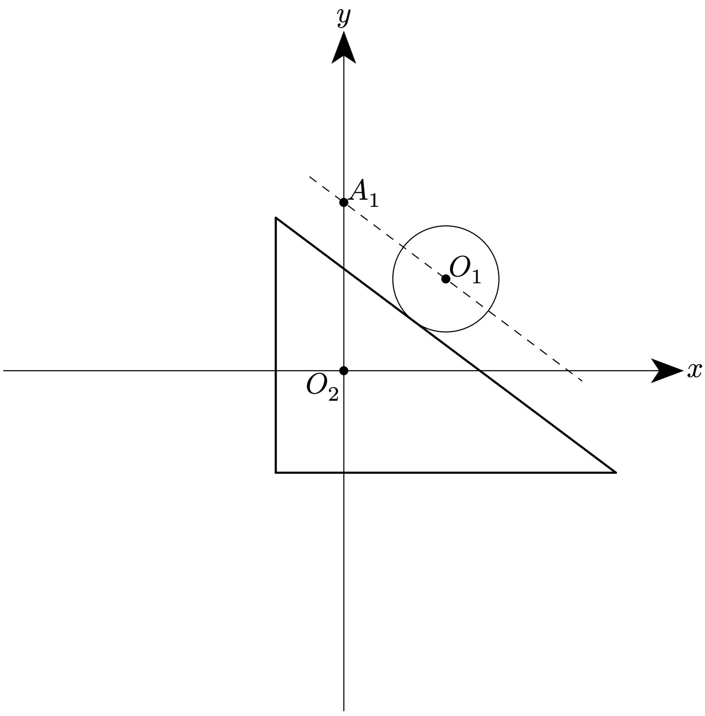
m_2的约束

$$y_2 = C_2$$

m_1的约束

需要变换参考系思考, m_1在m_2参考系的坐标为

$$x_1^{m_2} = x_1 - x_2$$
$$y_1^{m_2} = y_1 - y_2$$



设A坐标为$(0,C_A)$, 依据斜率恒定, 则有

$$-\tan\theta = k = \frac{y_1^{m_2} - C_A}{x_1^{m_2}}$$
$$\Leftrightarrow -x_1^{m_2}\tan\theta = y_1^{m_2} - C_A$$
$$\Leftrightarrow (x_1 - x_2)\tan\theta + (y_1 - y_2) - C_A = 0$$

综上, 约束方程为:

$$\begin{aligned} f_1 &= (x_1 - x_2)\tan\theta + (y_1 - y_2) - C_A &= 0 \\ f_2 &= y_2 - C_2 &= 0 \end{aligned}$$

力学方程

由第一类拉格朗日方程:

$$\begin{cases} F_{\mathrm{ix}} - m_i \ddot{x}_i + \sum_{j=0}^{r} \lambda_j \frac{\partial f_j}{\partial x_i} = 0 \\ F_{\mathrm{iy}} - m_i \ddot{y}_i + \sum_{j=0}^{r} \lambda_j \frac{\partial f_j}{\partial y_i} = 0 \\ (i = 1, 2, ..., n) \end{cases}$$

计算

$$\begin{cases} F_{1\mathrm{x}} = 0 \\ F_{1\mathrm{y}} = -m_1 g \end{cases} \qquad \begin{cases} F_{2\mathrm{x}} = 0 \\ F_{2\mathrm{y}} = -m_2 g \end{cases}$$

$$\begin{aligned} \boldsymbol{J} &= \begin{pmatrix} \frac{\partial f_1}{\partial x_1}, \frac{\partial f_1}{\partial y_1}, \frac{\partial f_1}{\partial x_2}, \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial x_1}, \frac{\partial f_2}{\partial y_1}, \frac{\partial f_2}{\partial x_2}, \frac{\partial f_2}{\partial y_2} \end{pmatrix} \\ &= \begin{pmatrix} \tan\theta & 1 & -\tan\theta & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

容易观察, 第一类拉格朗日方程 可以写成矩阵形式:

$$\boldsymbol{F} - \boldsymbol{M}\boldsymbol{a} + \boldsymbol{J}^T \boldsymbol{\lambda} = 0$$

代入可得

$$\begin{aligned} m_1 \ddot{x}_1 &= \lambda_1 \tan\theta \\ m_1 \ddot{y}_1 &= -m_1 g + \lambda_1 \\ m_2 \ddot{x}_2 &= -\lambda_1 \tan\theta \\ m_2 \ddot{y}_2 &= -m_2 g - \lambda_1 + \lambda_2 \end{aligned}$$

解方程

考虑约束, 去掉常数, 考虑微分关系

$$\begin{aligned} (\ddot{x}_1 - \ddot{x}_2)\tan\theta + \left(\ddot{y}_1 - \ddot{y}_2\right) &= 0 \\ \ddot{y}_2 &= 0 \end{aligned}$$

联立, 解得

$$\begin{cases} \ddot{x}_1 = \frac{m_2 g \tan\theta}{\tan^2\theta m_1 + (1+\tan^2\theta) m_2} \\ \ddot{y}_1 = -\frac{(m_1+m_2) g \tan\theta}{\tan^2\theta m_1 + (1+\tan^2\theta) m_2} \\ \ddot{x}_2 = -\frac{m_1 g \tan\theta}{\tan^2\theta m_1 + (1+\tan^2\theta) m_2} \\ \ddot{y}_2 = 0 \\ \lambda_1 = \frac{m_1 m_2 g}{\tan^2\theta m_1 + (1+\tan^2\theta) m_2} \\ \lambda_2 = \frac{(m_1+m_2) m_2 g (1+\tan^2\theta)}{\tan^2\theta m_1 + (1+\tan^2\theta) m_2} \end{cases}$$

结论

加速度为

$$\boldsymbol{a}_1 = \ddot{x}_1 \mathtt{i} + \ddot{y}_1 \mathtt{j} = \frac{m_2 g \tan\theta}{\tan^2\theta m_1 + (1+\tan^2\theta) m_2} \mathtt{i} - \frac{(m_1+m_2) g \tan\theta}{\tan^2\theta m_1 + (1+\tan^2\theta) m_2} \mathtt{j}$$

$$\boldsymbol{a}_2 = \ddot{x}_2 \mathtt{i} + \ddot{y}_2 \mathtt{j} = -\frac{m_1 g \tan\theta}{\tan^2\theta m_1 + (1+\tan^2\theta) m_2} \mathtt{i}$$

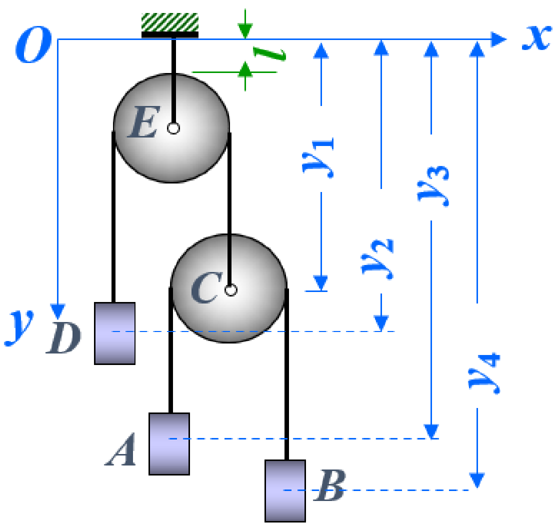
约束力为

$$\boldsymbol{F}_1 = \lambda_1 \left(\frac{\partial f_1}{\partial x_1} \mathtt{i} + \frac{\partial f_1}{\partial y_1} \mathtt{j} \right) = \frac{m_1 m_2 g}{\tan^2\theta m_1 + (1+\tan^2\theta) m_2} (\tan\theta \mathtt{i} + 1 \mathtt{j})$$

$$\boldsymbol{F}_2 = \lambda_2 \left(\frac{\partial f_2}{\partial x_2} \mathtt{i} + \frac{\partial f_2}{\partial y_2} \mathtt{j} \right) = \frac{m_1 m_2 g}{\tan^2\theta m_1 + (1+\tan^2\theta) m_2} \mathtt{j}$$

题 2

例: 机构如图所示,动滑轮C质量$m_1=m$，其上跨过长为l_1的悬线,两端分别悬挂着质量$m_3=2m$和$m_4=3m$的物块,动滑轮又用长l_2的悬线与另一质量$m_2=6m$的物块跨在定滑轮E上悬吊着。设约束是理想的,试用第一类拉格朗日方程求各物块的加速度。



约定

由题目可知, $A=m_3, B=m_4, C=m_1, D=m_2$.

约束

绳长不变

考虑第一段绳子

$$\begin{aligned} (y_2-l)+(y_1-l) &= l_2 \\ \Leftrightarrow f_1 = y_2+y_1-2l-l_2 \quad &= 0 \end{aligned}$$

考虑第二段绳子

$$\begin{aligned} (y_3-y_1)+(y_4-y_1) &= l_1 \\ \Leftrightarrow f_2 = y_3+y_4-2y_1-l_1 \ &= 0 \end{aligned}$$

综上

$$\begin{cases} f_1 = y_2+y_1-2l-l_2 \quad = 0 \\ f_2 = y_3+y_4-2y_1-l_1 = 0 \end{cases}$$

方程

由第一类拉格朗日方程:

$$\begin{cases} F_{\text{ix}} - m_i\ddot{x}_i + \sum_{j=0}^{r} \lambda_j \frac{\partial f_j}{\partial x_i} = 0 \\ F_{\text{iy}} - m_i\ddot{y}_i + \sum_{j=0}^{r} \lambda_j \frac{\partial f_j}{\partial y_i} = 0 \\ (i=1,2,...,n) \end{cases}$$

驱动力

$$\begin{cases} F_{1x} = 0 \\ F_{1y} = mg \end{cases} \qquad \begin{cases} F_{2x} = 0 \\ F_{2y} = 6mg \end{cases} \qquad \begin{cases} F_{3x} = 0 \\ F_{3y} = 2mg \end{cases} \qquad \begin{cases} F_{4x} = 0 \\ F_{4y} = 3mg \end{cases}$$

约束力

$$\begin{aligned} \boldsymbol{J} &= \begin{pmatrix} \frac{\partial f_1}{\partial x_1}, \frac{\partial f_1}{\partial y_1}, \frac{\partial f_1}{\partial x_2}, \frac{\partial f_1}{\partial y_2}, \frac{\partial f_1}{\partial x_3}, \frac{\partial f_1}{\partial y_3}, \frac{\partial f_1}{\partial x_4}, \frac{\partial f_1}{\partial y_4} \\ \frac{\partial f_2}{\partial x_1}, \frac{\partial f_2}{\partial y_1}, \frac{\partial f_2}{\partial x_2}, \frac{\partial f_2}{\partial y_2}, \frac{\partial f_2}{\partial x_3}, \frac{\partial f_2}{\partial y_3}, \frac{\partial f_2}{\partial x_4}, \frac{\partial f_2}{\partial y_4} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{aligned}$$

代入

$$
\begin{cases}
m\ddot{x}_1 = 0 \\
m\ddot{y}_1 = mg + \lambda_1 - 2\lambda_2 \\
6m\ddot{x}_2 = 0 \\
6m\ddot{y}_2 = 6mg + \lambda_1 \\
2m\ddot{x}_3 = 0 \\
2m\ddot{y}_3 = 2mg + \lambda_2 \\
3m\ddot{x}_4 = 0 \\
3m\ddot{y}_4 = 3mg + \lambda_2
\end{cases}
$$

约束等价变换

考虑二阶导

$$
\begin{cases}
\ddot{y}_2 + \ddot{y}_1 \qquad\quad = 0 \\
\ddot{y}_3 + \ddot{y}_4 - 2\ddot{y}_1 = 0
\end{cases}
$$

解方程

$$
\begin{cases}
\ddot{x}_1 = 0 \\
\ddot{y}_1 = -\frac{g}{59} \\
\ddot{x}_2 = 0 \\
\ddot{y}_2 = \frac{g}{59} \\
\ddot{x}_3 = 0 \\
\ddot{y}_3 = -\frac{13g}{59} \\
\ddot{x}_4 = 0 \\
\ddot{y}_4 = \frac{11g}{59} \\
\lambda_1 = -\frac{348gm}{59} \\
\lambda_2 = -\frac{144gm}{59}
\end{cases}
$$

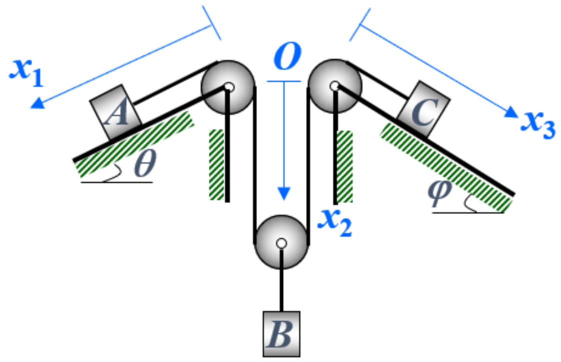
结论

加速度为

$$
\begin{cases}
\boldsymbol{a}_1 = \ddot{x}_1\mathtt{i} + \ddot{y}_1\mathtt{j} = -\frac{g}{59}\mathtt{j} \\
\boldsymbol{a}_2 = \ddot{x}_2\mathtt{i} + \ddot{y}_2\mathtt{j} = \frac{g}{59}\mathtt{j} \\
\boldsymbol{a}_3 = \ddot{x}_3\mathtt{i} + \ddot{y}_3\mathtt{j} = -\frac{13g}{59}\mathtt{j} \\
\boldsymbol{a}_4 = \ddot{x}_4\mathtt{i} + \ddot{y}_4\mathtt{j} = \frac{11g}{59}\mathtt{j}
\end{cases}
$$

题 3

例: **在图示的系统中，已知：物块A、B、C的质量均为m，用一不可伸长的轻绳通过滑轮系住。各滑轮质量不计，A、C物块分别放在倾角为θ、φ的光滑斜面上。物块B吊在动滑轮上。利用罗斯方程试求各物块的加速度。**



约定

如图采用极坐标系

约束

由绳长不变

$$
\begin{aligned}
& x_1+2x_2+x_3 &= l \\
\Leftrightarrow\ & f=x_1+2x_2+x_3-l=0
\end{aligned}
$$

方程

由第一类拉格朗日方程:

$$
\begin{cases}
F_{\mathrm{ix}}-m_i\ddot{x}_i+\sum_{j=0}^{r}\lambda_j\frac{\partial f_j}{\partial x_i}=0 \\
F_{\mathrm{iy}}-m_i\ddot{y}_i+\sum_{j=0}^{r}\lambda_j\frac{\partial f_j}{\partial y_i}=0 \\
(i=1,2,...,n)
\end{cases}
$$

驱动力

此时采用的是广义坐标, 通过能量来求驱动力

$$\delta W=mg\sin\theta\delta x_1+mg\delta x_2+mg\sin\varphi\delta x_3$$

因此, 驱动力为

$$
\begin{cases}
Q_1=mg\sin\theta \\
Q_2=mg \\
Q_3=mg\sin\varphi
\end{cases}
$$

约束力

$$
\begin{aligned}
\boldsymbol{J} &= \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}
\end{aligned}
$$

代入

$$
\begin{cases}
m\ddot{x}_1 = mg\sin\theta + \lambda \\
m\ddot{x}_2 = mg + 2\lambda \\
m\ddot{x}_3 = mg\sin\varphi + \lambda
\end{cases}
$$

约束等价变换

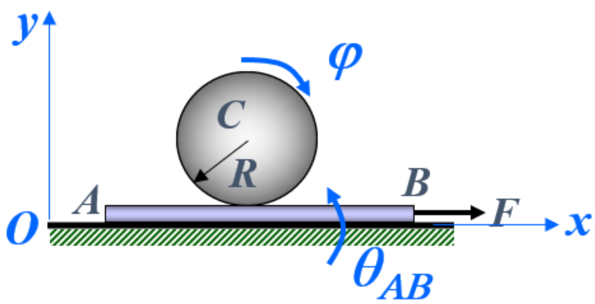
$$
\ddot{x}_1 + 2\ddot{x}_2 + \ddot{x}_3 = 0
$$

解方程

$$
\begin{cases}
\ddot{x}_1 = \frac{1}{6}(-2 - \sin\varphi + 5\sin\theta)g \\
\ddot{x}_2 = \frac{1}{3}(1 - \sin\varphi - \sin\theta)g \\
\ddot{x}_3 = \frac{1}{6}(-2 + 5\sin\varphi - \sin\theta)g \\
\lambda = \frac{1}{6}(-2 - \sin\varphi + 5\sin\theta)mg
\end{cases}
$$

题 4

例: 在图示的系统中，已知：薄平板AB的质量均为m_{AB}，放置在光滑的水平面上，匀质圆盘C的质量为m_C、半径为R，相对板作纯滚动，在板上作用了常力F，试用第一类拉格朗日方程求系统的所有未知量。



约定

设小球的质心坐标为(x_c, y_c), 薄平板的质心坐标为(x_{AB}, y_{AB})

约束

由于水平放置, 所以

$$
\theta_{AB} = 0
$$

直接以水平考虑

薄板

$$
y_{AB} = C_1
$$

小球

变换坐标系思考, 考虑小球相对薄板的运动

$$\begin{cases} x_C^{AB} = x_c - x_{AB} \\ y_C^{AB} = y_c - y_{AB} \end{cases}$$

接触的约束为

$$\begin{aligned} y_C^{AB} = C_2 \\ \Leftrightarrow y_c - y_{AB} = C_2 \end{aligned}$$

纯滚动的约束为

$$\begin{aligned} x_C^{AB} = R\varphi \\ \Leftrightarrow x_c - x_{AB} = R\varphi \end{aligned}$$

综上

$$\begin{cases} f_1 = y_{AB} - C_1 & = 0 \\ f_2 = y_c - y_{AB} - C_2 & = 0 \\ f_3 = x_c - x_{AB} - R\varphi & = 0 \end{cases}$$

方程

由第一类拉格朗日方程:

$$\begin{cases} F_{\text{ix}} - m_i \ddot{x}_i + \sum_{j=0}^{r} \lambda_j \frac{\partial f_j}{\partial x_i} = 0 \\ F_{\text{iy}} - m_i \ddot{y}_i + \sum_{j=0}^{r} \lambda_j \frac{\partial f_j}{\partial y_i} = 0 \\ (i = 1, 2, ..., n) \end{cases}$$

驱动力

$$\begin{cases} F_{2x} = 0 \\ F_{2y} = m_C g \\ M = 0 \end{cases} \qquad \begin{cases} F_{1x} = 0 \\ F_{1y} = m_{AB} g \end{cases}$$

容易忽略转动

约束力

$$\begin{aligned} \boldsymbol{J} &= \begin{pmatrix} \frac{\partial f_1}{\partial x_C}, \frac{\partial f_1}{\partial y_C}, \frac{\partial f_1}{\partial \varphi}, \frac{\partial f_1}{\partial x_{AB}}, \frac{\partial f_1}{\partial y_{AB}} \\ \frac{\partial f_2}{\partial x_C}, \frac{\partial f_2}{\partial y_C}, \frac{\partial f_2}{\partial \varphi}, \frac{\partial f_2}{\partial x_{AB}}, \frac{\partial f_2}{\partial y_{AB}} \\ \frac{\partial f_3}{\partial x_C}, \frac{\partial f_3}{\partial y_C}, \frac{\partial f_3}{\partial \varphi}, \frac{\partial f_3}{\partial x_{AB}}, \frac{\partial f_3}{\partial y_{AB}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & -R & -1 & 0 \end{pmatrix} \end{aligned}$$

代入

$$\begin{cases} m_C\ddot{x}_C &= \lambda_3 \\ m_C\ddot{y}_C &= m_C g + \lambda_2 \\ \frac{1}{2}m_C R^2\ddot{\varphi} &= -R\lambda_3 \\ m_{AB}\ddot{x}_{AB} &= F - \lambda_3 \\ m_{AB}\ddot{y}_{AB} &= m_{AB}g + \lambda_1 - \lambda_2 \end{cases}$$

约束等价变换

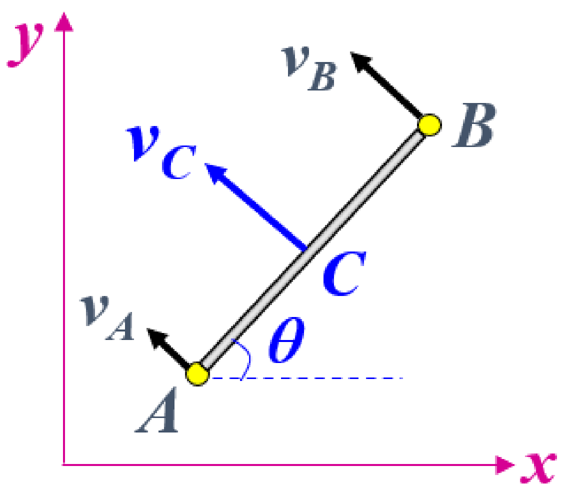
$$\begin{cases} \ddot{y}_{AB} &= 0 \\ \ddot{y}_C - \ddot{y}_{AB} &= 0 \\ \ddot{x}_C - \ddot{x}_{AB} - R\ddot{\varphi} &= 0 \end{cases}$$

解方程

$$\begin{cases} \ddot{x}_C &= \frac{F}{3m_{AB}+m_C} \\ \ddot{y}_C &= 0 \\ \ddot{\varphi} &= -\frac{2F}{(3m_{AB}+m_C)R} \\ \ddot{x}_{AB} &= \frac{3F}{3m_{AB}+m_C} \\ \ddot{y}_{AB} &= 0 \\ \lambda_1 &= -(m_{AB}+m_C)g \\ \lambda_2 &= -m_C g \\ \lambda_3 &= \frac{Fm_C}{3m_{AB}+m_C} \end{cases}$$

题 5

两质点A、B有相同的质量m，由长为l的无重刚杆相连。在点A和点B处都装有小刀刃支承，使得两点的绝对速度矢方向必须始终与刚杆垂直如图示。设系统保持在光滑水平面上运动，且刚杆以匀角速度ω转动，试用罗斯方程确定系统的运动。



约定

设质心为(x_c, y_c), 杆转动的角度为θ

约束

易知, 对A, B的约束等价于对C的约束: V_c垂直于杆, 即

$$\begin{aligned}\left(\dot{x}_c, \dot{y}_c\right) \cdot (\cos\theta, \sin\theta) &= 0 \\ \Leftrightarrow \dot{x}_c \cos\theta + \dot{y}_c \sin\theta &= 0 \\ \Leftrightarrow \cos\theta\delta x + \sin\theta\delta y &= 0\end{aligned}$$

综上

$$\delta f = \cos\theta\delta x + \sin\theta\delta y = 0$$

方程

由罗斯方程:

$$Q_j - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial q_i} + \sum_{i=1}^{r+s} \lambda_i B_{ij} = 0 \quad j = 1, 2, ..., n$$

能量

动能

$$\begin{aligned}T &= \frac{1}{2} m_c v_c^2 + \frac{1}{2} J_c \omega^2 \\ &= \frac{1}{2}(2m)\left(\dot{x}_c^2 + \dot{y}_c^2\right) + \frac{1}{2}\left(m\left(\frac{l}{2}\right)^2 + m\left(\frac{l}{2}\right)^2\right)\dot{\theta}^2 \\ &= m\left(\dot{x}_c^2 + \dot{y}_c^2\right) + \frac{1}{4} m l^2 \dot{\theta}^2\end{aligned}$$

势能

$$V = 0$$

所以

$$L = m\left(\dot{x}_c^2 + \dot{y}_c^2\right) + \frac{1}{4} m l^2 \dot{\theta}^2$$

广义力

$$Q = 0$$

约束

$$\begin{aligned}\boldsymbol{J} &= \begin{pmatrix} \frac{\partial f}{\partial x_c} & \frac{\partial f}{\partial y_c} & \frac{\partial f}{\partial \theta} \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & \sin\theta & 0 \end{pmatrix}\end{aligned}$$

代入

$$\begin{cases} 2m\ddot{x}_c &= \lambda\cos\theta \\ 2m\ddot{y}_c &= \lambda\sin\theta \\ \frac{1}{2} m l^2 \ddot{\theta} = 0 \end{cases}$$

解方程
知$\theta=\omega t$

所以

$$
\begin{cases}
2m\dot{x}_c = \frac{\lambda}{\omega}\sin\omega t \\
2m\dot{y}_c = -\frac{\lambda}{\omega}\cos\omega t
\end{cases}
$$

代入约束中, 恒成立

考虑初值, 则有

$$
\begin{aligned}
2m\dot{y}_c(0) &= -\frac{\lambda}{\omega} \\
\Leftrightarrow \lambda &= -2m\dot{y}_c(0)\omega
\end{aligned}
$$

则有

$$
\begin{cases}
x_c = -\frac{\lambda}{2m\omega^2}\cos\omega t + x_0 = \frac{\dot{y}_c(0)}{\omega}\cos\omega t + x_0 \\
y_c = -\frac{\lambda}{2m\omega^2}\sin\omega t + y_0 = \frac{\dot{y}_c(0)}{\omega}\sin\omega t + y_0
\end{cases}
$$