

CS456: Algorithm Design and Analysis

Elikem Asudo Tsatsu Gale-Zoyiku

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Assignment 2

Question 1

1. Constant Factor in Execution Time: The data shows that the count, representing the algorithm's execution time, consistently falls within a narrow range (between 11 and 13 times the input size) across various input sizes.

- For instance, with an input size of 2000, the count is 24,303, and with an input size of 3000, the count is approximately 39,992.
- Calculating the ratio of count to input size for each data point reveals a nearly constant factor, around 12, regardless of the input size.
- Consider the data point with an input size of 1000: Count is approximately 11,966. Calculating the ratio of count to input size: $\frac{11966}{1000} \approx 11.966$. Similarly, for the data point with an input size of 2000: Count is approximately 24,303, and the ratio is $\frac{24303}{2000} \approx 12.152$. Despite doubling the input size from 1000 to 2000, the ratio remains close to 12, indicating a nearly constant factor.

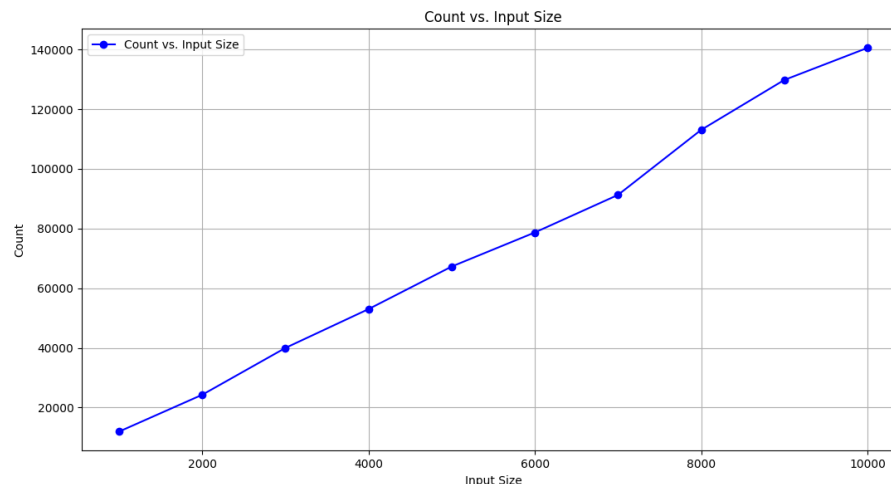
2. Linear Relationship:

- Doubling the input size approximately doubles the count, demonstrating a linear relationship between the input size and the count. For example, when the input size increases from 1000 to 2000, the count roughly doubles from 11,966 to 24,303.
- Take the example of doubling the input size from 2000 to 4000. With an input size of 2000, the count is around 24,303, and with an input size of 4000, the count is approximately 53,010. The ratio of counts is $\frac{53010}{24303} \approx 2$. Doubling the input size approximately doubles the count, demonstrating a linear relationship. This consistent doubling behavior across different input sizes indicates that the execution time scales linearly with the size of the input data.

These two observations suggest that the algorithm's execution time has a linear relationship with the input size, with a nearly constant factor. This implies that the algorithm has a time complexity of $O(n)$, where n is the input size.

3. Graphical Representation:

To visually illustrate these observations, the data points have been plotted on a graph with the input size on the x-axis and the count on the y-axis. The graph clearly depicts a linear relationship, where the data points form a straight line with a positive slope. This linear behavior on the graph further reinforces the hypothesis of linear time complexity ($O(n)$) for the algorithm, as it demonstrates that the execution time increases linearly with the input size. The graph can be seen below:



Question 2

a

This algorithm computes the minimum value in an array of real numbers. It recursively searches for the minimum value by comparing elements of the array. The base case is when the array contains only one element, in which case the algorithm returns that element as the minimum value. Otherwise, it recursively finds the minimum value in the subarray $A[0 \dots n-2]$ and compares it with the last element of the array ($A[n-1]$). If the minimum value from the subarray is less than or equal to the last element, it returns the minimum value from the subarray. Otherwise, it returns the last element of the array as the minimum value.

b

Let the basic operation count of the algorithm as $T(n)$, where n is the size of the input array A .

The basic operations in this algorithm are comparisons and recursive calls.

- Comparisons: There are $(n-1)$ comparisons made in the worst case scenario when checking each element of the array to find the minimum value.
- Recursive Calls: The algorithm makes one recursive call for each element of the array except for the first one. Therefore, there are $(n-1)$ recursive calls in total.

Hence, the recurrence relation for the algorithm is:

$$T(n) = T(n-1) + (n-1)$$

Solving:

$$\begin{aligned} T(n) &= T(n-1) + (n-1) \\ &= [T(n-2) + (n-2)] + (n-1) \\ &= T(n-2) + (n-2) + (n-1) \\ &= T(n-3) + (n-3) + (n-2) + (n-1) \\ &\quad \vdots \\ &= T(n-(n-1)) + (n-(n-1)) + \dots + (n-3) + (n-2) + (n-1) \\ &= T(1) + 1 + 2 + 3 + \dots + (n-1) \end{aligned}$$

The sum of the first $n-1$ natural numbers can be expressed as $\frac{n(n-1)}{2}$, so:

$$T(n) = T(1) + \frac{n(n-1)}{2}$$

The base case is when $n = 1$ and the algorithm simply returns the single element of the array, $T(1) = c$ where c is a constant time operation so $T(1)$ is dropped from the equation.

$$T(n) = \frac{n(n-1)}{2}$$

Therefore, the closed form format for the recurrence relation of the algorithm's basic operation count is $T(n) = \frac{n(n-1)}{2}$.

Prooving by induction:

1. Base Case ($n = 1$): When $n = 1$, $T(1) = 0$ (as there are no comparisons or recursive calls needed for an array of size 1). Therefore, the base case holds true.
2. Inductive Step: Assume that the closed form solution $T(k) = \frac{k(k-1)}{2}$ holds for some arbitrary integer $k \geq 1$. and prove that $T(k+1) = \frac{(k+1)k}{2}$.

Using the recurrence relation:

$$T(k+1) = T(k) + k$$

Substituting the assumed closed form solution for $T(k)$:

$$T(k+1) = \frac{k(k-1)}{2} + k$$

Simplifying:

$$T(k+1) = \frac{k(k-1)}{2} + \frac{2k}{2}$$

$$T(k+1) = \frac{k^2 - k + 2k}{2}$$

$$T(k+1) = \frac{k^2 + k}{2}$$

$$T(k+1) = \frac{(k+1)k}{2}$$

Hence, the closed form solution holds for $k+1$.

Therefore, the recurrence relation has the closed form solution $T(n) = \frac{n(n-1)}{2}$.

Question 3

The Min2 algorithm recursively divides an array into two halves to find the minimum value.

1. Number of Recursions:

- Each recursive call divides the array into two halves.
- Recursion continues until the base case is reached, where the array size becomes 1.
- Therefore, the number of recursive calls made by the Min2 algorithm for an input array of size n can be expressed as $\log_2 n$.

2. Number of Comparisons:

- At each level of recursion, a constant number of comparisons is made to determine the minimum value within each half of the array.
- At each level of recursion, the work increases by 2, but the problem size decreases by 2, so the work done at each level remains constant.

3. Work Done at Each Level:

- At each level of recursion, the array is divided into two halves.
- The algorithm makes a total of $O(\log n)$ levels of recursion, with each level performing a constant number of comparisons.

Based on these observations, the recurrence relation for the Min2 algorithm's basic operation count:

$$T(n) = 2T\left(\frac{n}{2}\right) + O(1)$$

Solving the recurrence relation

$$T(n) = 2T\left(\frac{n}{2}\right) + O(1)$$

$$T(n) = \log n * n$$

$$T(n) = n \log n$$

b

The Min2 algorithm has a time complexity of $O(n \log n)$, where n is the size of the input array. The Min1 algorithm has a time complexity of $O(n^2)$. The Min2 algorithm has a better time complexity than the Min1 algorithm, especially for large input sizes, so the Min2 algorithm is faster than the Min1 algorithm.

A more efficient algorithm to find the minimum value in an array is using a MinHeap to sort the array and select the top element. The MinHeap method involves constructing a MinHeap from the given unsorted array and then extracting the minimum element from the heap. A MinHeap is a binary tree data structure where the parent node's value is less than or equal to its children's values. By maintaining this property, the minimum element is always at the root of the heap. The procedure is:

1. Construct MinHeap: First, the unsorted array is used to construct a MinHeap. This operation rearranges the elements of the array such that the heap property is satisfied.
2. Extract Minimum: Once the MinHeap is constructed, the minimum element can be extracted from the root of the heap in constant time.

The MinHeap method has a time complexity of $O(n)$ for constructing the heap and $O(1)$ for extracting the minimum element. Overall, the MinHeap method has a time complexity of $O(1+n) = O(n)$ making it better than the Min2 algorithm..

Question 4

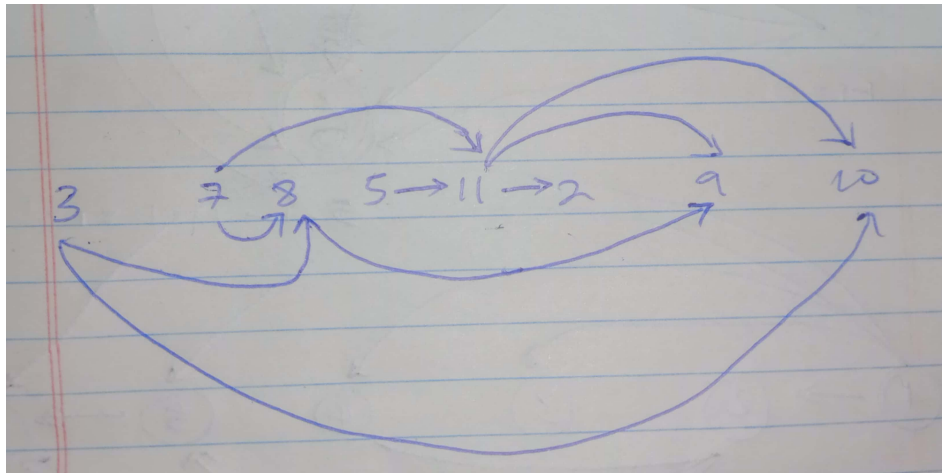
a

```
unvisited = new List;  
visited = new Stack;
```

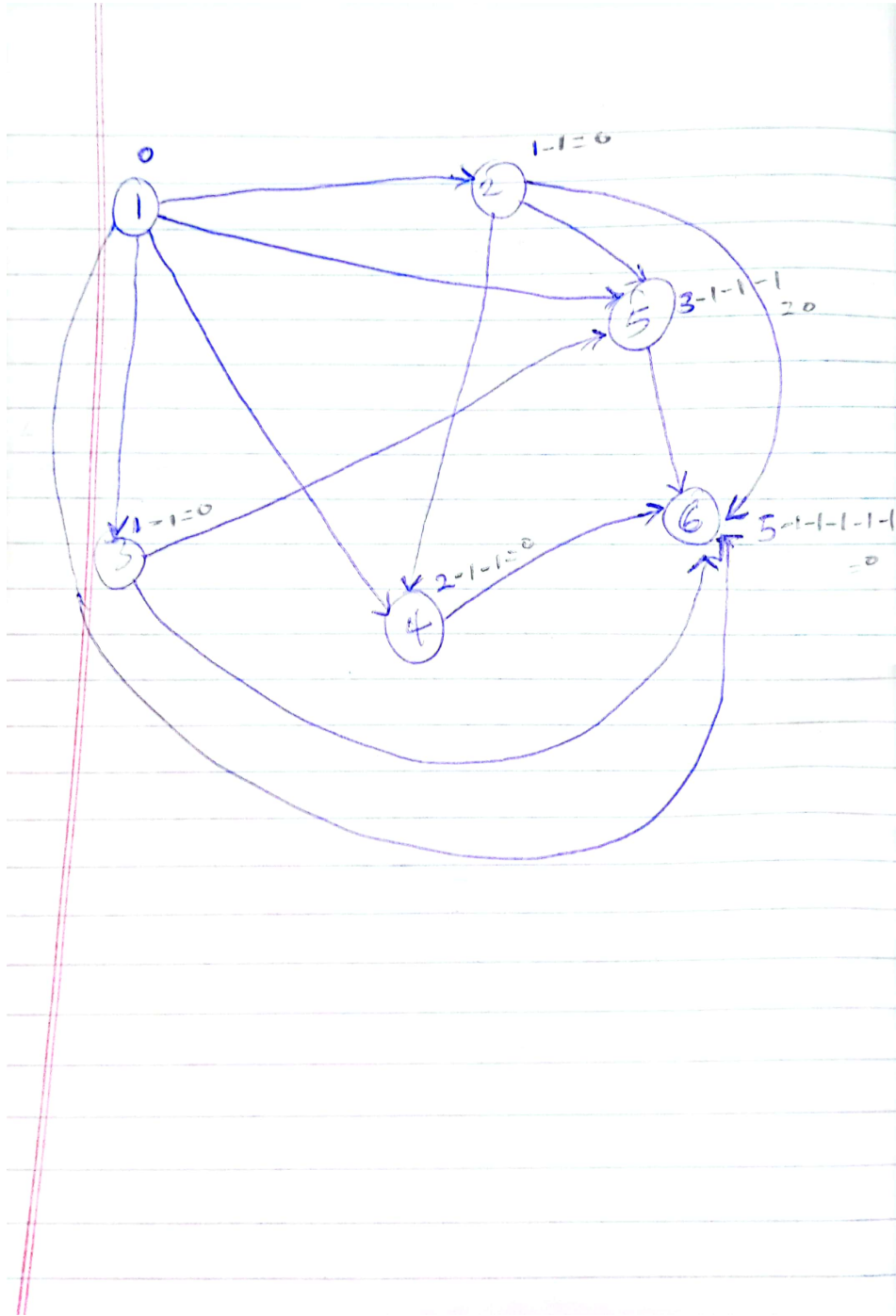
```
unvisited = [5, 11, 2, 9, 8, 7, 3, 10]
```

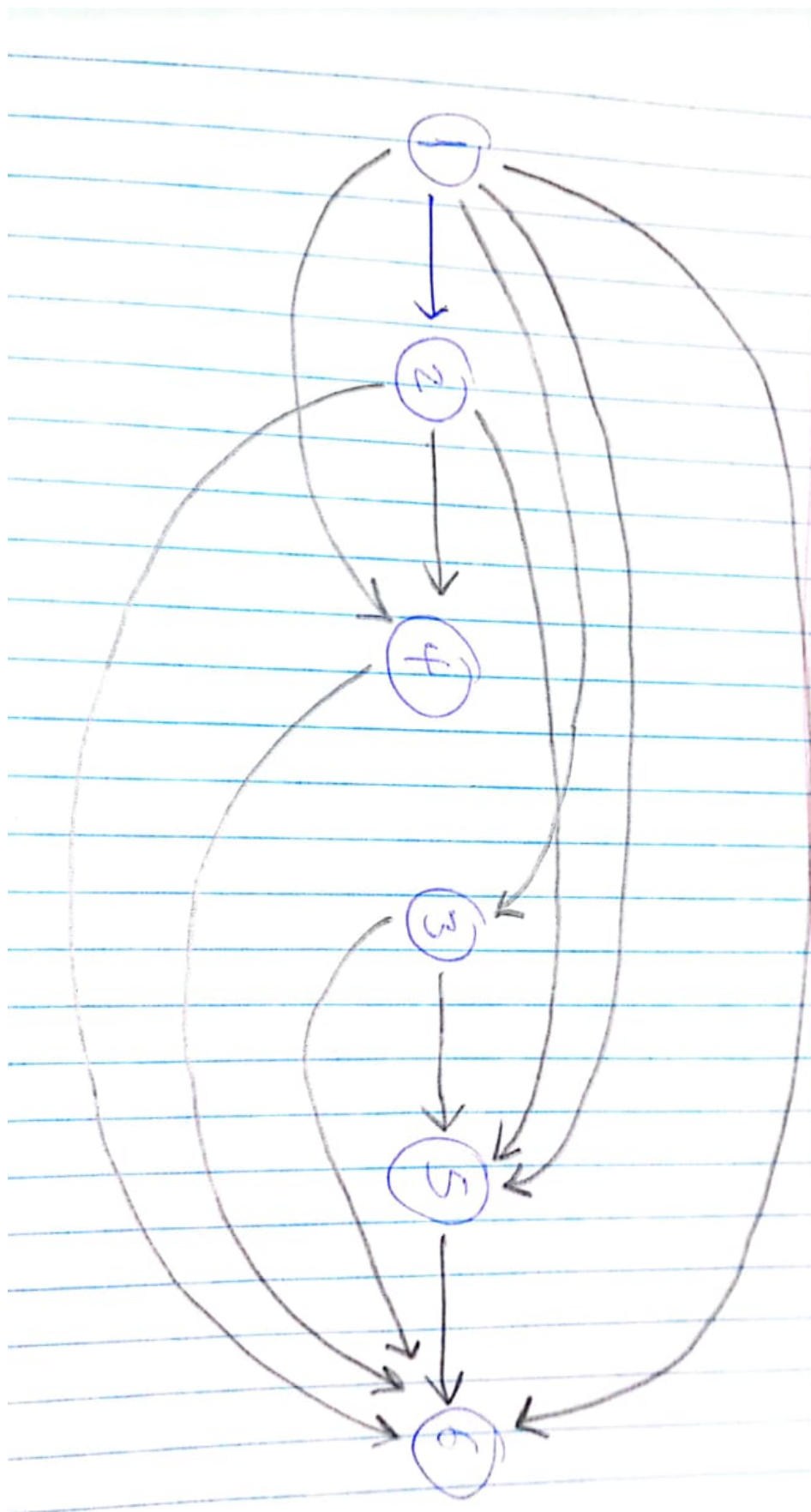
```
visited = [10, 9, 2, 11, 5, 8, 7, 3]
```

```
reversed_stack = [3, 7, 8, 5, 11, 2, 9, 10]
```



b





Question 5

a

- Base Case ($n = 1$): For a magic square of order 1, it consists of a single cell with the value a_{11} . The sum of any row, column, or diagonal is simply a_{11} , which equals the only number present in the magic square. This sum is $1 \times (1^2 + 1)/2 = 1$, satisfying the formula.
- Inductive Step: Assume that the sum is equal to $n(n^2 + 1)/2$ for a magic square of order $n = k$, where $k \geq 1$.
- Consider a magic square of order $n = k + 1$. The sum must equal $(k + 1)((k + 1)^2 + 1)/2$

Consider the sum of the elements in the magic square of order $n = k + 1$. Each row, column, and diagonal in this magic square can be split into two parts: the elements from the magic square of order $n = k$ and the additional elements in the last row and last column.

By the induction hypothesis, the sum of the elements from the magic square of order $n = k$ is $k(k^2 + 1)/2$.

Consider the sum of the additional elements in the last row and last column. Since the numbers in these positions increase by $k + 1$ from the corresponding positions in the magic square of order $n = k$, the sum of the elements in the last row and last column is $(k + 1)(k + 1) = (k + 1)^2$.

Therefore, the total sum of the elements in the magic square of order $n = k + 1$ is $k(k^2 + 1)/2 + (k + 1)^2$.

Simplifying:

$$\begin{aligned} \frac{k(k^2 + 1)}{2} + (k + 1)^2 &= \frac{k^3 + k}{2} + (k^2 + 2k + 1) = \frac{k^3 + 3k^2 + 3k + 1}{2} \\ &= \frac{(k + 1)((k + 1)^2 + 1)}{2} \end{aligned}$$

Thus, the sum for a magic square of order $n = k + 1$ is also equal to $(k + 1)((k + 1)^2 + 1)/2$.

By the principle of mathematical induction, the statement holds for all positive integers n .

Therefore, if a magic square of order n exists, the sum must be equal to $n(n^2 + 1)/2$.

b

```
function GenerateMagicSquares(n):
    // Initialize an empty n x n matrix
    matrix = empty n x n matrix

    // Initialize a list to store all magic squares
    magicSquares = empty list

    // Call the generateSquares function to start generating magic squares
    generateSquares(matrix, 0, 0, magicSquares)

    // Return the list of all magic squares
    return magicSquares

function generateSquares(matrix, row, col, magicSquares):
    // If the entire matrix is filled up
    if row == matrix.length and col == 0:
        // If the matrix is a magic square, add it to the list
        if isMagicSquare(matrix):
            magicSquares.add(copy of matrix)
        return
```

```

// Calculate the next row and column
nextRow, nextCol = (row, col + 1) if col + 1 < matrix.length else (row + 1, 0)

// Try placing each number from 1 to n^2 in the current cell
for num in 1 to n^2:
    // If the number is not already in the matrix
    if num not in matrix:
        // Place the number in the current cell
        matrix[row][col] = num

        // Recursively generate squares for the rest of the cells
        generateSquares(matrix, nextRow, nextCol, magicSquares)

        // Remove the number from the current cell (backtrack)
        matrix[row][col] = 0

function isMagicSquare(matrix):
    magicConstant = sum of the first row of matrix

    for i in 0 to matrix.length - 1:
        if sum of row i != magicConstant or sum of column i != magicConstant:
            return false

    if sum of main diagonal != magicConstant or sum of secondary diagonal != magicConstant:
        return false

    // If all checks passed, the matrix is a magic square
    return true

```

c

There are two popular methods for generating magic squares: the Siamese method for odd-order magic squares and the Square of Squares method for even-order magic squares. The Siamese method is a simple algorithm that starts by placing the number 1 at the middle of the top row and then fills in the numbers sequentially using specific rules. The Square of Squares method decomposes the square into four sub-squares and fills them with numbers sequentially to form a complete magic square.

```

function GenerateMagicSquare(n):
    if n is odd:
        return SiameseMethod(n)
    else:
        return SquareOfSquaresMethod(n)

function SiameseMethod(n):
    // Initialize an empty n x n matrix
    matrix = empty n x n matrix

    // Initialize the position for the first number
    row = 0
    col = n // 2

    // Fill in the numbers in the matrix using the Siamese method
    for num in 1 to n*n:
        matrix[row][col] = num
        // Calculate the next position based on the Siamese method rules
        nextRow = (row - 1 + n) % n

```

```

        nextCol = (col + 1) % n
        // If the next position is already filled, move down instead
        if matrix[nextRow][nextCol] != 0:
            nextRow = (row + 1) % n
            nextCol = col
        // Update the current position
        row = nextRow
        col = nextCol

    return matrix

function SquareOfSquaresMethod(n):
    // Initialize an empty n x n matrix
    matrix = empty n x n matrix

    // Calculate the size of each sub-square
    subSize = n / 4

    // Generate sub-squares for each quadrant
    subMatrix1 = GenerateSubSquare(subSize)
    subMatrix2 = GenerateSubSquare(subSize)
    subMatrix3 = GenerateSubSquare(subSize)
    subMatrix4 = GenerateSubSquare(subSize)

    // Fill the main matrix with sub-squares
    for i in 0 to subSize-1:
        for j in 0 to subSize-1:
            matrix[i][j] = subMatrix1[i][j]
            matrix[i][j + subSize*2] = subMatrix2[i][j]
            matrix[i + subSize*2][j] = subMatrix3[i][j]
            matrix[i + subSize*2][j + subSize*2] = subMatrix4[i][j]

    return matrix

function GenerateSubSquare(subSize):
    // Initialize an empty subSize x subSize matrix
    subMatrix = empty subSize x subSize matrix
    // Fill the sub-square with numbers sequentially
    num = 1
    for i in 0 to subSize-1:
        for j in 0 to subSize-1:
            subMatrix[i][j] = num
            num = num + 1
    return subMatrix

```

Explanation

- The GenerateMagicSquare function takes the order n of the magic square as input. If n is odd, it calls the SiameseMethod function to generate an odd-order magic square using the Siamese method. If n is even, it calls the SquareOfSquaresMethod function to generate an even-order magic square using the Square of Squares method.
- The SiameseMethod function generates an odd-order magic square using the Siamese method. It initializes an empty n x n matrix and starts by placing the number 1 at the middle of the top row. It then iteratively fills in the numbers in the matrix according to the Siamese method rules.
- The SquareOfSquaresMethod function generates an even-order magic square using the Square

of Squares method. It decomposes the square into four sub-squares and fills them with numbers using the `GenerateSubSquare` function. It then arranges these sub-squares in the main matrix to form a complete magic square.

- The `GenerateSubSquare` function generates a sub-square of a specified size and fills it with consecutive numbers starting from 1. This function is used to generate each of the four sub-squares in the `SquareOfSquaresMethod`.

d

- The exhaustive search algorithm was only able to generate magic squares of order 4 under 1 minute.
- The non-exhaustive search algorithm (Siamese method for odd numbers, Square of Square method for even orders) went up to order 876 under 1 minute.