“How did the first person get it?”

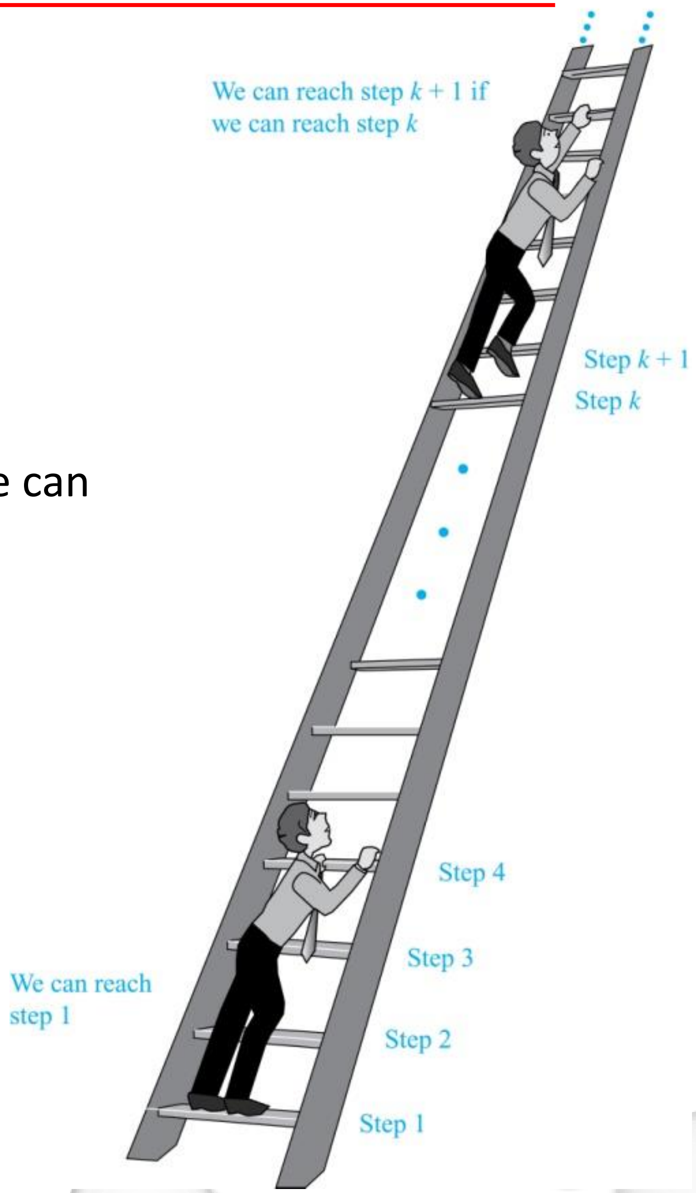
Intro to Mathematical Induction- 5.1

Overview

- Mathematical Induction
- Examples of Proof by Mathematical Induction
- Mistaken Proofs by Mathematical Induction
- Guidelines for Proofs by Mathematical Induction

Climbing an Infinite Ladder

Can we prove that you can climb an infinite ladder?



Suppose we have an infinite ladder:

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

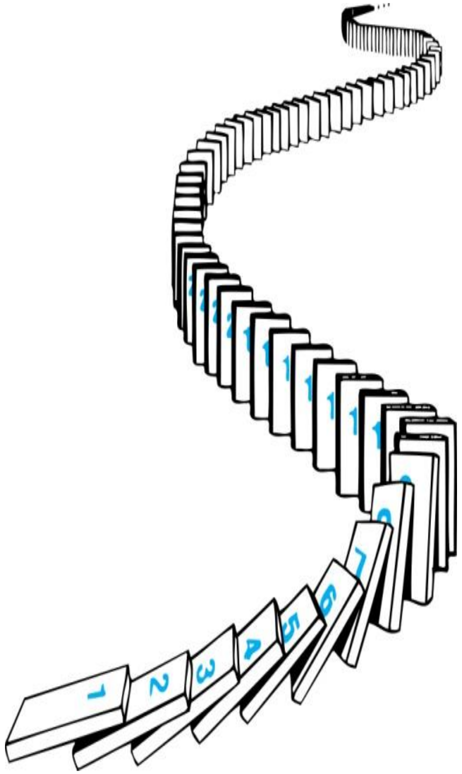
From (1), we can reach the first rung. Then by applying (2), we can reach the second rung. Applying (2) again, the third rung. And so on. We can apply (2) any number of times to reach any particular rung, no matter how high up.

This example motivates proof by mathematical induction.

Mathematical Induction- One more example

Consider an infinite sequence of dominoes, labeled 1,2,3, …, where each domino is standing.

Let $P(n)$ be the proposition that the nth domino is knocked over.



We know that the first domino is knocked down, i.e., $P(1)$ is true .

We also know that if whenever the kth domino is knocked over, it knocks over the $(k + 1)$st domino, i.e, $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.

Hence, all dominos are knocked over.

$P(n)$ is true for all positive integers n.

Principle of Mathematical Induction

Principle of Mathematical Induction: To prove that $P(n)$ is true for all positive integers n, we complete these steps:

- ***Basis Step***: Show that $P(1)$ is true.
- ***Inductive Step:*** Show that $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.

To complete the inductive step, assuming the *inductive hypothesis* that $P(k)$ holds for an arbitrary integer k, show that must $P(k + 1)$ be true.

Climbing an Infinite Ladder Example:

- **Basis Step:** By (1), we can reach rung 1.
- **Inductive Step:** Assume the inductive hypothesis that we can reach rung k. Then by (2), we can reach rung $k + 1$.

Hence, $P(k) \rightarrow P(k + 1)$ is true for all positive integers k. We can reach every rung on the ladder.

Important Points About Using Mathematical Induction

- Mathematical induction can be expressed as the rule of inference $(P(1) \land \forall k\,(P(k) \rightarrow P(k+1))) \rightarrow \forall n\, P(n)$, where the domain is the set of positive integers.

- In a proof by mathematical induction, we don't assume that $P(k)$ is true for all positive integers! Rather we show that **if we assume that** $\boldsymbol{P(k)}$ **is true**, then $P(k+1)$ must also be true.
- Proofs by mathematical induction do not always start at the integer 1. In such a case, the basis step begins at a starting point b where b is an integer.

Summation Formula by Mathematical Induction

Example: Show that if n is a positive integer, then $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.

Solution:

- **BASIS STEP**: $P(1)$ is true since $1 = 1(1 + 1)/2$.
- **INDUCTIVE STEP**: Assume $P(k)$ true.

Then the inductive hypothesis is $1 + 2 + \cdots + k = \dfrac{k(k+1)}{2}$.

Under this assumption,

$$1 + 2 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

More Examples

Prove that the following is true for all positive integers n by using the Principle of Mathematical Induction: $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

<u>BASIS STEP</u>: $P(1)$ is true since $1^2 = 1$.

<u>INDUCTIVE STEP</u>: $P(k) \rightarrow P(k + 1)$ for every positive integer n.

Assume the inductive hypothesis holds and then show that $P(n + 1)$ holds.

Inductive Hypothesis: $1 + 3 + 5 + \cdots + (2k - 1) = k^2$

$$
\begin{aligned}
1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) &= [1 + 3 + 5 + \cdots + (2k - 1)] + (2k + 1) \\
&= k^2 + (2k + 1) \quad (\textit{inductive hypothesis}) \\
&= k^2 + 2k + 1 \\
&= (k + 1)^2
\end{aligned}
$$

Therefore *P(n)*, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ **holds for all positive integers *n*.**

Proving Inequalities

Example: Use mathematical induction to prove that $2^n < n!$, for every integer $n \geq 4$.

Solution: Let $P(n)$ be the proposition that $2^n < n!$.

- BASIS STEP: $P(4)$ is true since $2^4 = 16 < 4! = 24$.
- INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $2^k < k!$ for an arbitrary integer $k \geq 4$.
- To show that $P(k + 1)$ holds:

$$
\begin{aligned}
2^{k+1} &= 2 \cdot 2^k \\
&< 2 \cdot k! \qquad \textit{(by the inductive hypothesis)} \\
&< (k + 1)k! \\
&= (k + 1)!
\end{aligned}
$$

Therefore, $2^n < n!$ holds, for every integer $n \geq 4$.

Note that here the basis step is $P(4)$, since $P(1)$, $P(2)$, and $P(3)$ are all false.

Guidelines: Mathematical Induction Proofs

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words **"Basis Step."** Then show that $P(b)$ is true, taking care that the correct value of b is used. **This completes the first part of the proof**.
3. Write out the words "**Inductive Step**."
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, **write out what $P(k + 1)$ says**.

6. *Prove the statement $P(k + 1)$ making use the assumption $P(k)$.* Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.

7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."

8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.