Discrete Structures and Theory (Spring 2023)
Revision for Final Exam

Exercise 1:

Use logical equivalences to prove that: $(p \rightarrow q) \vee (p \rightarrow r)$ and $p \rightarrow (q \vee r)$ are logically equivalent.

Exercise 2:

Prove the first absorption law for propositional logic $p \vee (p \wedge q) \equiv p$ using logical equivalences. (Do not use the first absorption law).

Exercise 3:

Let $I(x)$ be the statement "x has an Internet connection" and $C(x, y)$ be the statement "x and y have chatted over the Internet," where the domain for the variables x and y consists of all students in your class. Use quantifiers to express each of these statements.

a) Jerry does not have an Internet connection.
b) Rachel has not chatted over the Internet with Chelsea.
c) No one in the class has chatted with Bob.
d) Sanjay has chatted with everyone.
e) Someone in your class does not have an Internet connection.
f) Not everyone in your class has an Internet connection.
g) Everyone in your class with an Internet connection has chatted over the Internet with at least one other student in your class.
h) Someone in your class has an Internet connection but has not chatted with anyone else in your class.
i) There is a student in your class who has chatted with everyone in your class over the Internet.

Exercise 4:

Let $Q(x, y)$ be the statement "$x + y = x - y$." If the domain for both variables consists of all integers, what are the truth values?

a) $Q(1, 1)$ b) $Q(2, 0)$ c) $\forall y Q(1, y)$ d) $\exists x Q(x, 2)$ e) $\exists x \exists y Q(x, y)$
f) $\forall x \exists y Q(x, y)$ g) $\exists y \forall x Q(x, y)$ h) $\forall y \exists x Q(x, y)$ i) $\forall x \forall y Q(x, y)$

Exercise 5:

For the following argument, explain which rules of inference are used for each step.

"All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. Therefore, there is a wonderful movie about coal miners."

Exercise 6:
n is an integer. Prove that n is even if and only if n^2 is even.

Exercise 7:
A and B are sets. Prove that $A - B$ is a subset of A.

Exercise 8:
A and B are sets. Use set identities to show that $(A \cap B) \cup (A \cap B) = A$.

Exercise 9:

a) Use a truth table to express the values of the Boolean function $F(x, y, z) = x + z$.
b) Using the result from (a), write $F(x, y, z)$ in conjunctive normal form.
c) Using the result from (a), write $F(x, y, z)$ in disjunctive normal form.
d) Using Boolean identities, write $F(x, y, z)$ in disjunctive normal form.

Exercise 10:
Prove by induction that $\sum_{j=1}^{n} \frac{1}{2^j} = \frac{(2^n - 1)}{2^n}$ where n is a positive integer.

Exercise 11:
Use mathematical induction to prove that $n^3 - n$ is divisible by 3, for every positive integer n.

Exercise 12:
What is the minimum number of students, each of whom comes from one of the 54 African countries, who must be enrolled at Ashesi university to guarantee that there are at least 5 who come from the same country?

Exercise 13:
Thirteen people on a softball team show up for a game.

a) How many ways are there to choose 10 players to take the field?
b) How many ways are there to assign the 10 positions by selecting players from the 13 people who show up?
c) Of the 13 people who show up, three are women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?

TABLE 5 **Boolean Identities.**

Identity	*Name*
$\overline{\overline{x}} = x$	Law of the double complement
$x + x = x$ $x \cdot x = x$	Idempotent laws
$x + 0 = x$ $x \cdot 1 = x$	Identity laws
$x + 1 = 1$ $x \cdot 0 = 0$	Domination laws
$x + y = y + x$ $xy = yx$	Commutative laws
$x + (y + z) = (x + y) + z$ $x(yz) = (xy)z$	Associative laws
$x + yz = (x + y)(x + z)$ $x(y + z) = xy + xz$	Distributive laws
$\overline{(xy)} = \overline{x} + \overline{y}$ $\overline{(x + y)} = \overline{x}\,\overline{y}$	De Morgan's laws
$x + xy = x$ $x(x + y) = x$	Absorption laws
$x + \overline{x} = 1$	Unit property
$x\overline{x} = 0$	Zero property

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws