Ex. 1. Let $\underline{v}_1 = [1,2,3]$, $\underline{v}_2 = [-1,1,0]$, $\underline{v}_3 = [1,1,1]$. The coordinates in the unit basis are the same.

$$\begin{bmatrix} 1 & 3 & -1 \\ 3 & 1 & 2 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ 7 \end{bmatrix} \Rightarrow f(\underline{v}_1) = [4,11,7] = a_1\,\underline{v}_1 + b_1\,\underline{v}_2 + c_1\,\underline{v}_3$$

$$\begin{bmatrix} 1 & 3 & -1 \\ 3 & 1 & 2 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix} \Rightarrow f(\underline{v}_2) = [2,-2,-4] = a_2\,\underline{v}_1 + b_2\,\underline{v}_2 + c_2\,\underline{v}_3$$

$$\begin{bmatrix} 1 & 3 & -1 \\ 3 & 1 & 2 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix} \Rightarrow f(\underline{v}_3) = [3,6,5] = a_3\,\underline{v}_1 + b_3\,\underline{v}_2 + c_3\,\underline{v}_3$$

Thus the transformation matrix relative to basis $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$. Now it is enough to find these constants.

Ex. 2. The coordinates of vector $2\underline{u}_1 - 6\underline{u}_2 + 3\underline{u}_3$ in this basis are [*2,-6,3*].

$$\begin{bmatrix} 0 & 2 & -4 \\ 2 & -1 & 2 \\ -3 & 1 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix} = \begin{bmatrix} -24 \\ 16 \\ 3 \end{bmatrix}$$

Thus $f(2\underline{u}_1 - 6\underline{u}_2 + 3\underline{u}_3) = -24\underline{u}_1 + 16\underline{u}_2 + 3\underline{u}_3$.

Ex. 3. For $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow y = -2x \Leftrightarrow \operatorname{Ker} f = \begin{bmatrix} x \\ -2x \end{bmatrix}, x \in R$

Ex. 4. $\det\begin{bmatrix} 3-\lambda & -1 & 0 \\ 6 & -2-\lambda & 0 \\ 2 & -1 & 1-\lambda \end{bmatrix} = -\lambda(\lambda-1)^2 = 0 \Rightarrow$ eigenvalues $\lambda_1 = 0, \lambda_2 = \lambda_3 = 1$.

Eigenvectors

for $\lambda_1 = 0$: $\begin{bmatrix} 3 & -1 & 0 \\ 6 & -2 & 0 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} y = 3x \\ z = x \end{cases} \Leftrightarrow \underset{\smile}{\underline{x}_1} = \begin{bmatrix} x \\ 3x \\ x \end{bmatrix} = x\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ e.g. $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

for $\lambda_2 = \lambda_3 = 1$: $\begin{bmatrix} 2 & -1 & 0 \\ 6 & -3 & 0 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow y = 2x \Leftrightarrow \underline{x}_2 = \begin{bmatrix} x \\ 2x \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ e.g. $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$