1) ***Solve the following system of equations*:** $\begin{cases} \mathbf{2x + 5y - 2z - 3t = 0} \\ \mathbf{x + 2y - 3z + t = 1} \\ \mathbf{x + 4y - 5z + 2t = 3} \end{cases}$

$$A|B = \left[\begin{array}{cccc|c} 2 & 5 & -2 & -3 & 0 \\ 1 & 2 & -3 & 1 & 1 \\ 1 & 4 & -5 & 2 & 3 \end{array}\right] = \begin{matrix} R_2 = 2R_2 - R_1 \\ R_3 = 2R_3 - R_1 \end{matrix} => \left[\begin{array}{cccc|c} 2 & 5 & -2 & -3 & 0 \\ 0 & -1 & -4 & 5 & 2 \\ 0 & 3 & -8 & 7 & 6 \end{array}\right] = R_3 = R_3 + 3R_2 =>$$

$$\left[\begin{array}{cccc|c} 2 & 5 & -2 & -3 & 0 \\ 0 & -1 & -4 & 5 & 2 \\ 0 & 0 & -20 & 22 & 12 \end{array}\right] = \begin{matrix} R_3 = R_3 \cdot -\frac{1}{20} \\ R_2 = R_2 + 4R_1 \end{matrix} => \left[\begin{array}{cccc|c} 2 & 5 & -2 & -3 & 0 \\ 0 & -1 & 0 & \frac{3}{5} & -\frac{2}{5} \\ 0 & 0 & 1 & -\frac{11}{10} & -\frac{3}{5} \end{array}\right] = R_1 = R_1 + 2R_3 =>$$

$$\left[\begin{array}{cccc|c} 2 & 5 & 0 & -\frac{26}{5} & -\frac{6}{5} \\ 0 & -1 & 0 & \frac{3}{5} & -\frac{2}{5} \\ 0 & 0 & 1 & -\frac{11}{10} & -\frac{3}{5} \end{array}\right] = R_1 = R_1 + 5R_2 => \left[\begin{array}{cccc|c} 2 & 0 & 0 & -\frac{11}{5} & -\frac{16}{5} \\ 0 & -1 & 0 & \frac{3}{5} & -\frac{2}{5} \\ 0 & 0 & 1 & -\frac{11}{10} & -\frac{3}{5} \end{array}\right] = \begin{matrix} R_2 = -R_2 \\ R_1 = \frac{1}{2}R_1 \end{matrix} =>$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{11}{10} & -\frac{8}{5} \\ 0 & 1 & 0 & -\frac{3}{5} & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{11}{10} & -\frac{3}{5} \end{array}\right]$$

Thus $\boldsymbol{x = -\frac{8}{5} + \frac{11}{10}t}$; $\boldsymbol{y = \frac{2}{5} + \frac{3}{5}t}$; $\boldsymbol{z = -\frac{3}{5} + \frac{11}{10}t}$; $t \in \mathbb{R}$

2) ***Solve the following system of equations*:** $\begin{cases} \mathbf{-2x + y + 6z = 3} \\ \mathbf{x + 4y + 3z = -2} \\ \mathbf{-x + 2y + 5z = 4} \end{cases}$

$$\mathrm{A} = \begin{bmatrix} -2 & 1 & 6 \\ 1 & 4 & 3 \\ -1 & 2 & 5 \end{bmatrix}; \quad \mathrm{A|B} = \begin{bmatrix} -2 & 1 & 6 & 3 \\ 1 & 4 & 3 & -2 \\ -1 & 2 & 5 & 4 \end{bmatrix}$$

$$\det \mathrm{A} = \begin{vmatrix} -2 & 1 & 6 \\ 1 & 4 & 3 \\ -1 & 2 & 5 \end{vmatrix} = -2 \cdot \begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 3 \\ -1 & 5 \end{vmatrix} + 6 \cdot \begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} = 2 \cdot 14 - 1 \cdot 8 + 6 \cdot 6 = 0$$

$$\begin{vmatrix} -2 & 1 \\ 1 & 4 \end{vmatrix} = (-2) \cdot 4 - 1 \cdot 1 = -9 \neq 0$$

Thus rank $\boldsymbol{A = 2}$

$$\begin{vmatrix} 1 & 6 & 3 \\ 4 & 3 & -2 \\ 2 & 5 & 4 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & -2 \\ 5 & 4 \end{vmatrix} - 6 \cdot \begin{vmatrix} 4 & -2 \\ 2 & 4 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix} = 1 \cdot 22 - 6 \cdot 20 + 3 \cdot 14 = -56 \neq 0$$

Thus rank $\boldsymbol{A|B = 3}$

Since the rank $\boldsymbol{A \neq}$ ***rank*** $\boldsymbol{A|B}$ ***the system does not have any solutions***

You can also use GAUSS method (like in ex.1)

3) $\boldsymbol{Solve\ the\ following\ system\ using\ the\ matrix\ elimination\ method}$: $\begin{cases} -x+3y-6z=3 \\ x-y+2z=-1 \\ 2x+y+z=3 \end{cases}$

$$\begin{bmatrix} -1 & 3 & -6 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

$\boldsymbol{R_2=R_2+R_1}$:

$$E_{21}=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};\quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} -1 & 3 & -6 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix}=\begin{bmatrix} -1 & 3 & -6 \\ 0 & 2 & -4 \\ 2 & 1 & 1 \end{bmatrix}$$

$$\begin{cases} -x+3y-6z=3 \\ 2y-4z=2 \\ 2x+y+z=3 \end{cases}$$

$\boldsymbol{R_3=R_3+2R_1}$:

$$E_{31}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix};\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} -1 & 3 & -6 \\ 0 & 2 & -4 \\ 2 & 1 & 1 \end{bmatrix}=\begin{bmatrix} -1 & 3 & -6 \\ 0 & 2 & -4 \\ 0 & 7 & -11 \end{bmatrix}$$

$$\begin{cases} -x+3y-6z=3 \\ 2y-4z=2 \\ 7y-11z=9 \end{cases}$$

$\boldsymbol{R_3=R_3-\frac{7}{2}R_2}$:

$$E_{32}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{7}{2} & 1 \end{bmatrix};\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{7}{2} & 1 \end{bmatrix}\cdot\begin{bmatrix} -1 & 3 & -6 \\ 0 & 2 & -4 \\ 0 & 7 & -11 \end{bmatrix}=\begin{bmatrix} -1 & 3 & -6 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{cases} -x+3y-6z=3 \\ 2y-4z=2 \\ 3z=2 \end{cases}$$

$\boldsymbol{Thus}$:

$$\begin{bmatrix} -1 & 3 & -6 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}=E_{32}\cdot E_{31}\cdot E_{21}\cdot\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

$\boldsymbol{The\ solution\ is}$: $\boldsymbol{x=0}$; $\boldsymbol{y=\frac{7}{3}}$; $\boldsymbol{z=\frac{2}{3}}$

4) $\boldsymbol{Find\ the\ relationship\ between\ the\ value\ of\ parameter\ p\ and\ the\ number\ of\ solutions}$

$\boldsymbol{using\ the\ Kronecker-Capelli\ theorem}$: $\begin{cases} -px+5y+3z=3 \\ 2x-4y-z=p \\ x+3py+pz=p \end{cases}$

$$A=\begin{bmatrix} -p & 5 & 3 \\ 2 & -4 & -1 \\ 1 & 3p & p \end{bmatrix};\quad A|B=\begin{bmatrix} -p & 5 & 3 & 3 \\ 2 & -4 & -1 & p \\ 1 & 3p & p & p \end{bmatrix}$$

$$|A| = \begin{vmatrix} -p & 5 & 3 \\ 2 & -4 & -1 \\ 1 & 3p & p \end{vmatrix} = -p \cdot \begin{vmatrix} -4 & -1 \\ 3p & p \end{vmatrix} - 5 \cdot \begin{vmatrix} 2 & -1 \\ 1 & p \end{vmatrix} + 3 \cdot \begin{vmatrix} 2 & -4 \\ 1 & 3p \end{vmatrix} = -p(-p) - 5(2p+1) +$$

$$3(6p+4) = \ p^2 + 8p + 7 => \boldsymbol{p = \ -1}; \ \ \boldsymbol{p = \ -7}$$

$$\begin{vmatrix} 5 & 3 & 3 \\ -4 & -1 & p \\ 3p & p & p \end{vmatrix} = 5 \cdot \begin{vmatrix} -1 & p \\ p & p \end{vmatrix} - 3 \cdot \begin{vmatrix} -4 & p \\ 3p & p \end{vmatrix} + 3 \cdot \begin{vmatrix} -4 & -1 \\ 3p & p \end{vmatrix} = 5(-p-p^2) - 3(-4p-3p^2) + 3(-p) =$$

$$4p^2 + 4p => \ \ \boldsymbol{p = -1} \ ; \ \boldsymbol{p = 0}$$

$$\begin{vmatrix} -p & 5 & 3 \\ 2 & -4 & p \\ 1 & 3p & p \end{vmatrix} = -p \cdot \begin{vmatrix} -4 & p \\ 3p & p \end{vmatrix} - 5 \cdot \begin{vmatrix} 2 & p \\ 1 & p \end{vmatrix} + 3 \cdot \begin{vmatrix} 2 & -4 \\ 1 & 3p \end{vmatrix} = -p(-4p-3p^2) - 5p + 3(6p+4) =$$

$$4p^2 + 3p^3 + 13p + 12 = 0 => \boldsymbol{p = -1}$$

$$\begin{vmatrix} -p & 3 & 3 \\ 2 & -1 & p \\ 1 & p & p \end{vmatrix} = \ \ -p \cdot \begin{vmatrix} -1 & p \\ p & p \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & p \\ 1 & p \end{vmatrix} + 3 \cdot \begin{vmatrix} 2 & -1 \\ 1 & p \end{vmatrix} = -p(-p-p^2) - 3p + 3(2p+1) =$$

$$p^2 + p^3 + 3p + 3 => \boldsymbol{p = -1}$$

$at\ p = -1\ the\ rank\ of\ A = 2\ ,rank\ \ of\ A|B = 2$

$at\ p = -7\ the\ rank\ of\ A = 2\ ,rank\ of\ A|B = 3$

$Otherwise\ the\ rank\ of\ A = 3, rank\ of\ A|B = 3$

$\boldsymbol{Thus\ the\ system\ of\ equations}$:

$$\begin{cases} \boldsymbol{Has\ no\ solutions\ when\ p = -7} \\ \boldsymbol{Has\ infinetly\ many\ solution\ when\ p = -1} \\ \boldsymbol{Has\ a\ unique\ solution\ when\ p \neq \ -7, p \neq \ -1} \end{cases}$$