Exercise 4:

Let $B = \{\underline{u}, \underline{v}, \underline{w}\} = \{[u_1, u_2, u_3], [v_1, v_2, v_3], [w_1, w_2, w_3]\}$ a basis of $\mathbb{R}^3$.

$$[2, -1, 4] = -2[u_1, u_2, u_3] + 3[v_1, v_2, v_3] + 0[w_1, w_2, w_3]$$

$$\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$$

Therefore the following restrictions are found:

$$\begin{cases} 2 = -2u_1 + 3v_1 \\ -1 = -2u_2 + 3v_2 \\ 4 = -2u_3 + 3v_3 \end{cases}$$

For example,

$$\underline{u} = [2,2,1] \rightarrow \underline{v} = [2,1,2]$$

Finally, for $\underline{w}$, the fact that $\{\underline{u}, \underline{v}, \underline{w}\}$ must be linearly independent is applied

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \neq 0 \rightarrow \begin{vmatrix} 2 & 2 & w_1 \\ 2 & 1 & w_2 \\ 1 & 2 & w_3 \end{vmatrix} = 2w_3 + 2w_2 + 4w_1 - w_1 - 4w_2 - 4w_3 = 3w_1 - 2w_2 - 2w_3 \neq 0$$

For example,

$$\underline{w} = [0,0,1]$$

Therefore,

$$\boxed{\boldsymbol{B} = \{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}}\} = \{[\mathbf{2}, \mathbf{2}, \mathbf{1}], [\mathbf{2}, \mathbf{1}, \mathbf{2}], [\mathbf{0}, \mathbf{0}, \mathbf{1}]\}}$$