Exercise 1. Check if the following set W is a linear subspace of V if:

a) $W = \{[0, y, z] \in R^3 : yz = 0\},\ V = R^3$. **NO**

$yz = 0 \Rightarrow y = 0 \lor z = 0$

$[0, 1, 0], [0, 0, 1] \in W$

$[0,1,0] + [0,0,1] = [0,1,1] \notin W$

b) $W = \{[x, y, z] \in R^3 : x + 3y = y - 2z = 0\},\ V = R^3$. **YES**

$$\begin{cases} x = -3y \\ y = 2z \end{cases}$$

$\underline{v} = [-3y, y, \frac{1}{2}y] \in W$

1. $W \neq \phi$

   $[-3, 1, \frac{1}{2}] \in W$

2. $\underline{v}_1, \underline{v}_2 \in W,\ \alpha, \beta \in \mathbb{R},\ \alpha\underline{v}_1 + \beta\underline{v}_2 \in W$

   $\underline{v}_1 = [-3y_1, y_1, \frac{1}{2}y_1],\ \ \underline{v}_2 = [-3y_2, y_2, \frac{1}{2}y_2]$

   $\alpha\underline{v}_1 + \beta\underline{v}_2 = [-3\alpha y_1 - 3\beta y_2, \alpha y_1 + \beta y_2, \frac{1}{2}\alpha y_1 + \frac{1}{2}\beta y_2]$

Let $k = \alpha y_1 + \beta y_2$

   $\alpha\underline{v}_1 + \beta\underline{v}_2 = [-3k, k, \frac{1}{2}k] \in W$

   W is a linear subset of V.

Exercise 2. The vectors {u, v, w} are linearly independent. Determine, using the definition, whether the vectors {v, u - v + w, u - 2v + 2w} are linearly independent.

There are scalars c_1, c_2 and c_3, such that:

$c_1\underline{v} + c_2(\underline{u} - \underline{v} + \underline{w}) + c_3(\underline{u} - 2\underline{v} + 2\underline{w}) = 0$

$\underline{u}(c_2 + c_3) + \underline{v}(c_1 - c_2 - 2c_3) + \underline{w}(c_2 + 2c_3) = 0$

Since vectors {$\underline{u}$, $\underline{v}$, $\underline{w}$} are linearly independent

$$\begin{cases} c_2 + 2c_3 = 0 \\ c_2 + c_3 = 0 \\ c_1 - 2c_3 - c_2 = 0 \end{cases} \Longrightarrow$$

$\Rightarrow$ $c_1 = 0$ $\quad$ $c_2 = 0$ $\quad$ $c_3 = 0$

Thus vectors $\underline{v}$, $\underline{u - v + w}$, $\underline{u} - 2\underline{v} + 2\underline{w}$ are also linearly independent.

Exercise 3. Determine, using the definition, whether the vectors: [1, 0,-1], [0, 1, 3], [-2, 3, 2] are linearly independent in R^3.

Vectors $\underline{a_1}$ = [1, 0,-1], $\underline{a_2}$= [0, 1, 3], $\underline{a_3}$ = [-2 , 3, 2] are linearly independent if and only if, for every system of real numbers c_1, c_2, c_3 if $c_1\underline{a_1} + c_2\underline{a_2} + c_3\underline{a_3} = 0$, then $c_1 = c_2 = c_3 = 0$.

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -1 & 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} c1 \\ c2 \\ c3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -1 & 3 & 2 \end{bmatrix} \quad + R1$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 3 & 0 \end{bmatrix} \quad - 3R2$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & -9 \end{bmatrix} \cdot \begin{bmatrix} c1 \\ c2 \\ c3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-9c_3 = 0 \Rightarrow c_3 = 0$$

$$c_2 + 3 \cdot 0 = 0 \Rightarrow c_2 = 0$$

$$c_1 + 0 - 2 \cdot 0 = 0 \Rightarrow c_1 = 0$$

The vectors [1, 0,-2], [0, 1, 3], [- 1, 3, 2] are linearly independent in R^3.

Exercise 4. Present the vector [1, 2, -5] as a linear combination of vectors: [1, 0,-2], [0, 1, 3], [- 1, 3, 2].

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ -2 & 3 & 2 & -5 \end{array}\right] \begin{array}{l} \\ \\ +\ 2R1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 3 & 0 & -3 \end{array}\right] \begin{array}{l} \\ \\ -\ 3R2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -9 & -9 \end{array}\right] \begin{array}{l} \\ \\ \div\ (-\ 9) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} +\ R3 \\ -\ 3R3 \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

[1, 2, -5] $=\ 2\ \cdot$ [1, 0,-2] $-\ 1\ \cdot$ [0, 1, 3] $+$ [- 1, 3, 2]