1) Solve by Crammer's rule: $\begin{cases} 6x + y - 2z = -2 \\ x + y + 3z = 1 \\ -5x + 2y - 3z = 0 \end{cases}$

$$|A| = \begin{vmatrix} 6 & 1 & -2 \\ 1 & 1 & 3 \\ -5 & 2 & -3 \end{vmatrix} = +6 \cdot \begin{vmatrix} 1 & 3 \\ 2 & -3 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 3 \\ -5 & -3 \end{vmatrix} + (-2) \cdot \begin{vmatrix} 1 & 1 \\ -5 & 2 \end{vmatrix} = \dots = -80$$

$$D_1 = \begin{vmatrix} -2 & 1 & -2 \\ 1 & 1 & 3 \\ 0 & 2 & -3 \end{vmatrix} = +(-2) \cdot \begin{vmatrix} 1 & 3 \\ 2 & -3 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 3 \\ 0 & -3 \end{vmatrix} + (-2) \cdot \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = \dots = 17$$

$$D_2 = \begin{vmatrix} 6 & -2 & -2 \\ 1 & 1 & 3 \\ -5 & 0 & -3 \end{vmatrix} = +6 \cdot \begin{vmatrix} 1 & 3 \\ 0 & -3 \end{vmatrix} - (-2) \cdot \begin{vmatrix} 1 & 3 \\ -5 & -3 \end{vmatrix} + (-2) \cdot \begin{vmatrix} 1 & 1 \\ -5 & 0 \end{vmatrix} = \dots = -4$$

$$D_3 = \begin{vmatrix} 6 & 1 & -2 \\ 1 & 1 & 1 \\ -5 & 2 & 0 \end{vmatrix} = +6 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ -5 & 0 \end{vmatrix} + (-2) \cdot \begin{vmatrix} 1 & 1 \\ -5 & 2 \end{vmatrix} = \dots = -31$$

Thus $x = -\frac{17}{80}$; $y = \frac{4}{80}$; $z = \frac{31}{80}$

2) Solve using the unit column method: $\begin{cases} x - 3y + z = 3 \\ 3x - 6y + 5z = 2 \\ -x + 2y - 2z = 1 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 3 \\ 3 & -6 & 5 & 2 \\ -1 & 2 & -2 & 1 \end{array}\right] \begin{array}{c} R_2 = R_2 - 3R_1 \\ R_3 = R_3 + R1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 1 & 3 \\ 0 & 3 & 2 & -7 \\ 0 & -1 & -1 & 4 \end{array}\right] \quad R_3 = -R_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 1 & 3 \\ 0 & 3 & 2 & -7 \\ 0 & 1 & 1 & -4 \end{array}\right]$$

$$\begin{array}{c} R_1 = R_1 - R_3 \\ R_2 = R_2 - 2R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & -4 & 0 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & -4 \end{array}\right] \quad \begin{array}{c} R_3 = R_3 - R_2 \\ R_1 = R_1 + 4R_2 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

Thus by the unit column method $x = 11$; $y = 1$; $z = -5$

3) ***Find the rank of*** $A = \begin{bmatrix} 1 & 5 & 2 & -2 \\ 2 & 3 & -2 & 1 \\ -3 & -1 & 6 & 0 \end{bmatrix}$ $\quad A_{3\times 4} \Rightarrow R(A) \leq 3$

$$\begin{vmatrix} 1 & 5 & 2 \\ 2 & 3 & -2 \\ -3 & -1 & 6 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & -2 \\ -1 & 6 \end{vmatrix} - 5 \cdot \begin{vmatrix} 2 & -2 \\ -3 & 6 \end{vmatrix} + 2 \cdot \begin{vmatrix} 2 & 3 \\ -3 & -1 \end{vmatrix} = 1 \cdot 16 - 5 \cdot 6 + 2 \cdot 7 = 0$$

$$\begin{vmatrix} 1 & 5 & -2 \\ 2 & 3 & 1 \\ -3 & -1 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} - 5 \cdot \begin{vmatrix} 2 & 1 \\ -3 & 0 \end{vmatrix} + (-2) \cdot \begin{vmatrix} 2 & 3 \\ -3 & -1 \end{vmatrix} = 1 \cdot \quad 1 - 5 \cdot \quad 3 - 2 \cdot 7 \neq 0$$

Therefore the rank of A is 3

4) ***Find the values of the parameter p, for which the given system is nonsingular*:**

$$\begin{cases} -3x - 5y + pz = -3 \\ -x - 4y + 2z = p \\ px + 3\ \ y + z = p \end{cases}$$

A system is non − singular when the coefficient matrix has a non − zero determinant .

$$|A| = \begin{vmatrix} -3 & -5 & p \\ -1 & -4 & 2 \\ p & 3 & 1 \end{vmatrix} = +(-3)\cdot\begin{vmatrix} -4 & 2 \\ 3 & 1 \end{vmatrix} - (-5)\cdot\begin{vmatrix} -1 & 2 \\ p & 1 \end{vmatrix} + p\cdot\begin{vmatrix} -1 & -4 \\ p & 3 \end{vmatrix} = \ldots \quad = 4p^2 - 13p + 25 = 0$$

$\Delta < 0$

$|A| = 0 \Leftrightarrow p \in \emptyset$

The system is nonsingular for any values of p ∈ $\mathbb{R}$