# Random Projections

2IMW30 - Foundations of data mining
TU Eindhoven, Quartile 3, 2016-2017

Anne Driemel

# Why reduce the dimension?

Representation of input data often is often high dimensional (images, documents, etc.)

There are two main reasons to reduce the dimension:

- some algorithms have **running time** exponential in the dimension
- we want to **visualize** inherent structure in the data

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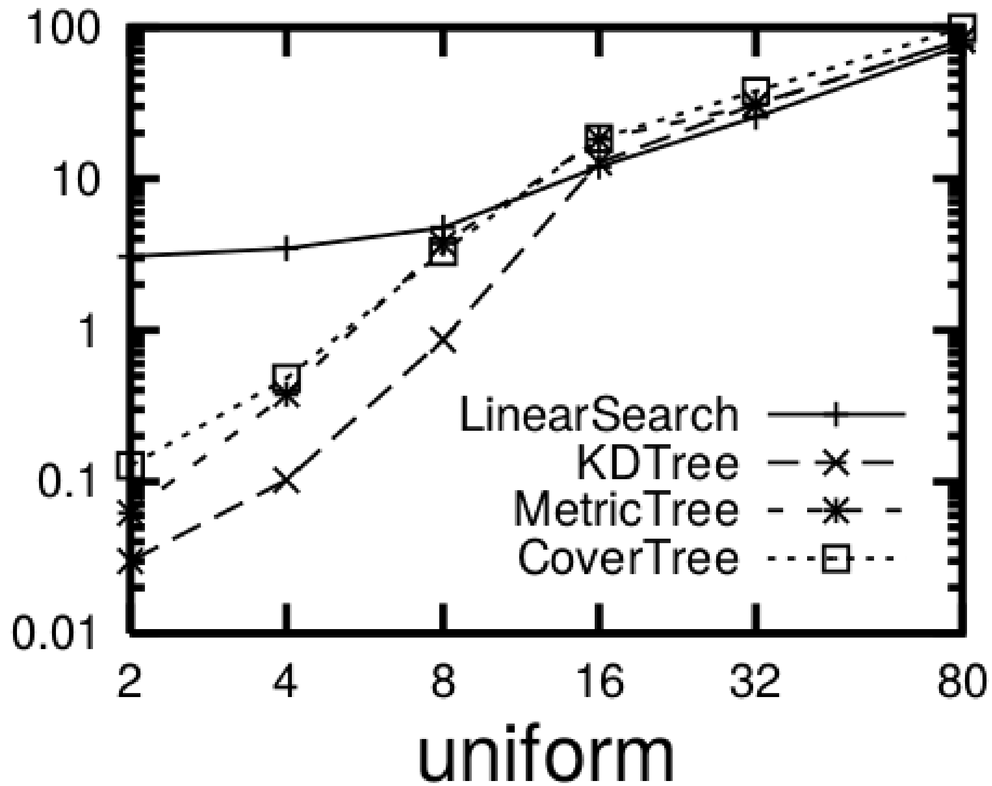
# Overview of this lecture

- Nearest-neighbor searching
- Embedding and Distortion
- Achlioptas' Random Projection
- Projection onto a subspace
- Random Rotation (Expectation)
- Analysis of a fixed distance (Expectation)
- Law of large numbers
- Concentration of measure
- Analysis of a fixed distance
- Analysis of the Distortion
- Alternative Projection Matrix

# Nearest neighbor searching

CPU-time to query the $k$-nearest neighbors vs. dimension of the data




**Source:** Ashraf M. Kibriya and Eibe Frank "An Empirical Comparison of Exact Nearest Neighbour Algorithms" PKDD 2007

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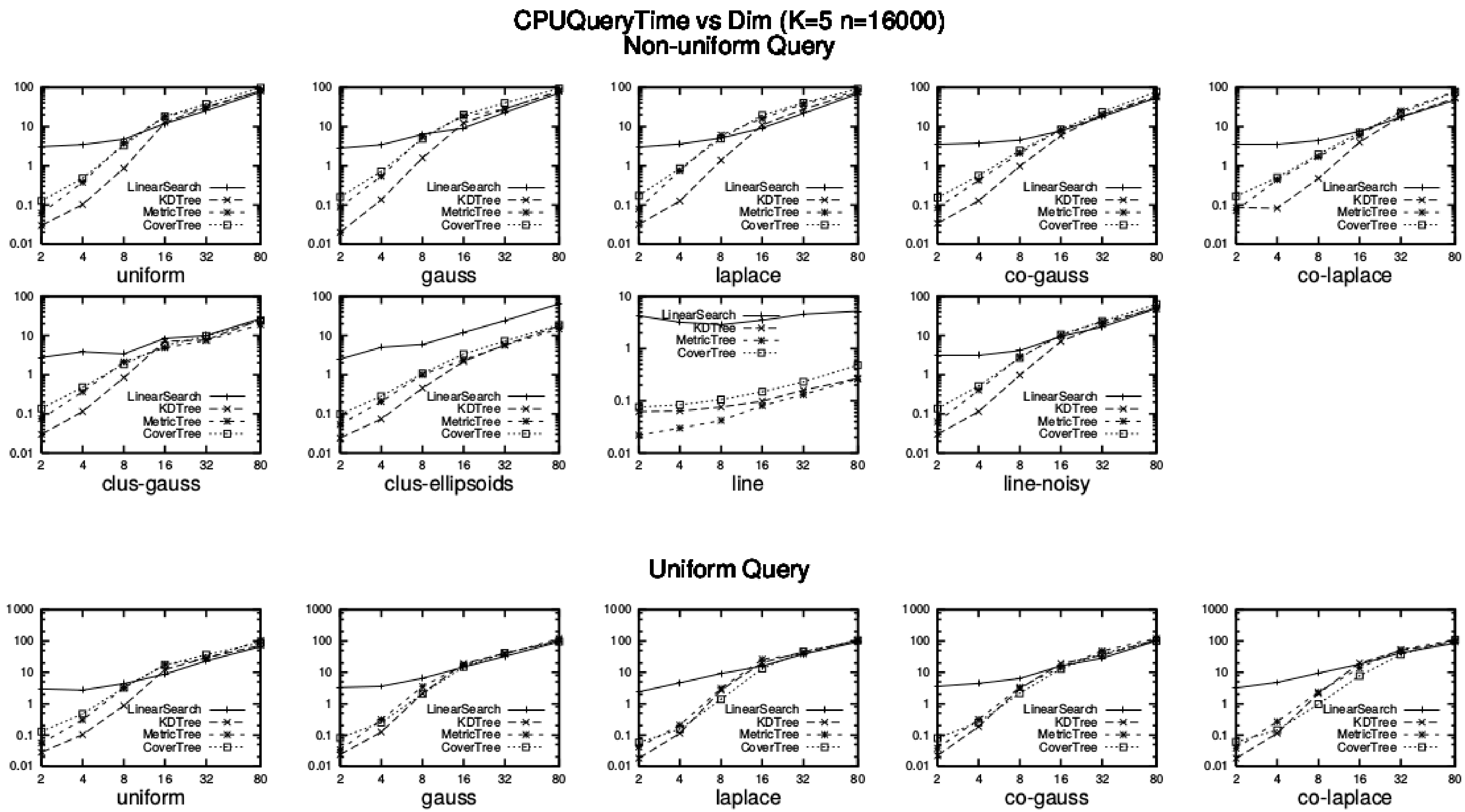


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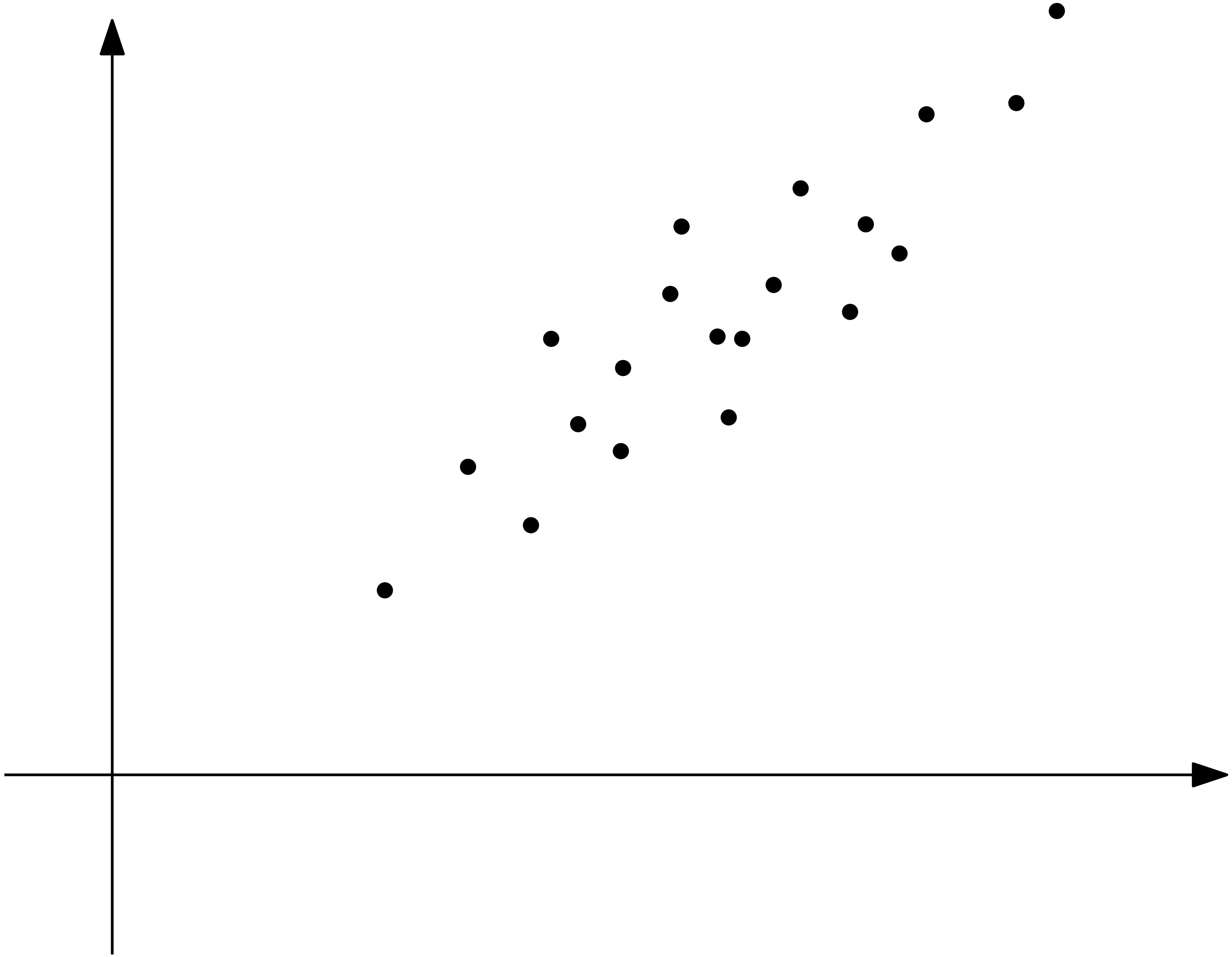
# Principal Component Analysis (PCA)

**Example:** $d = 2, k = 1$



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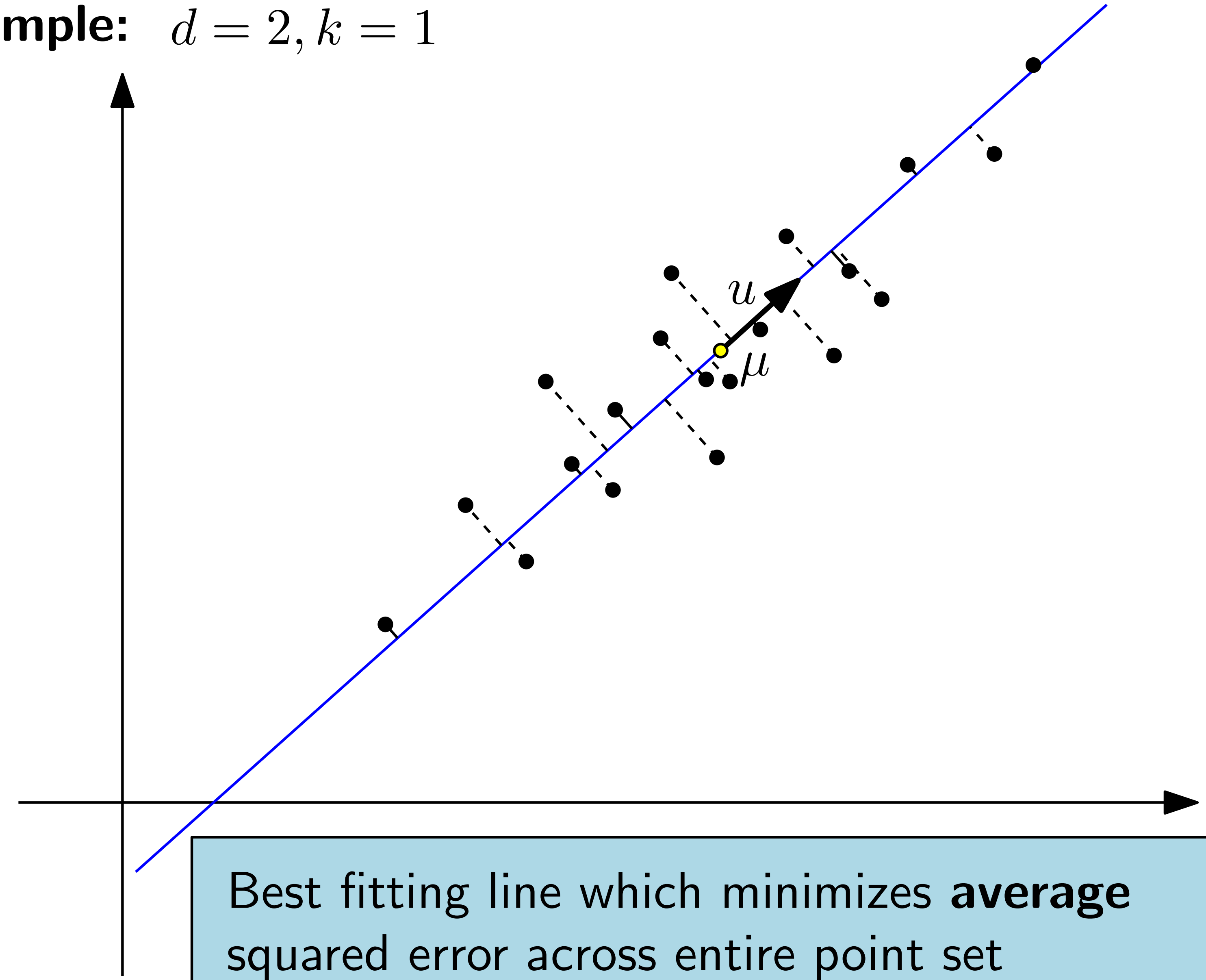


Best fitting line which minimizes **average** squared error across entire point set

# Random Projection

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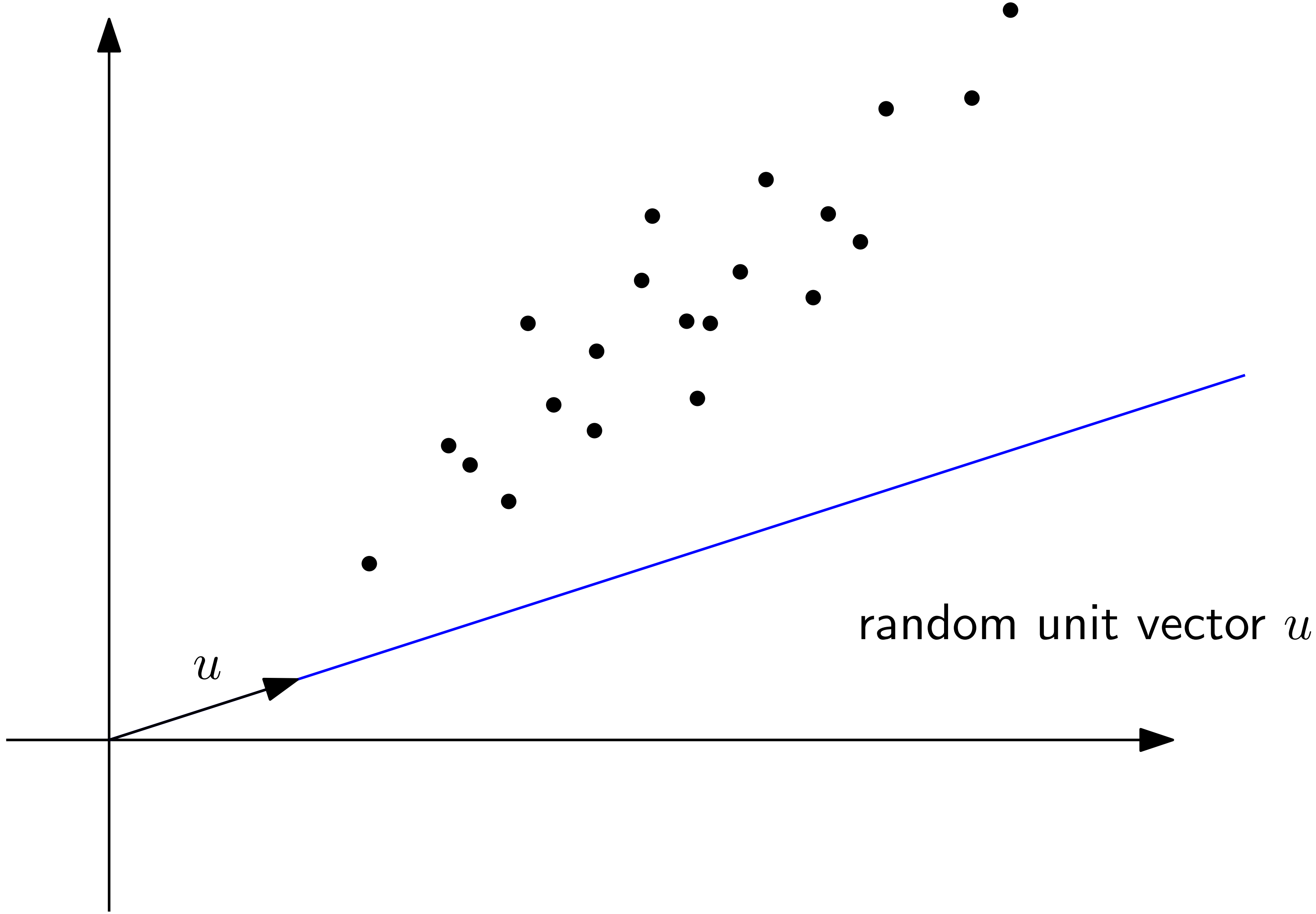

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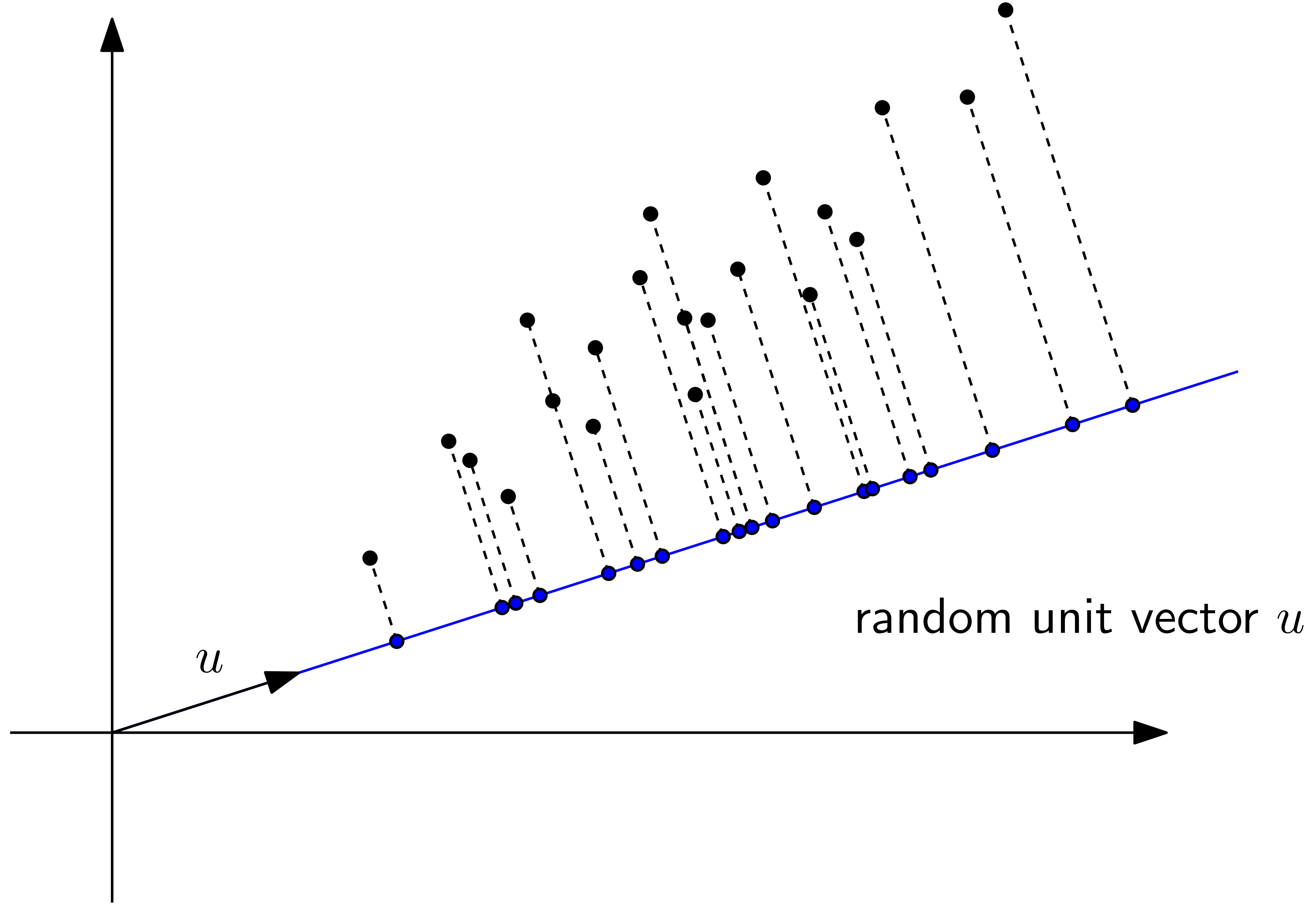

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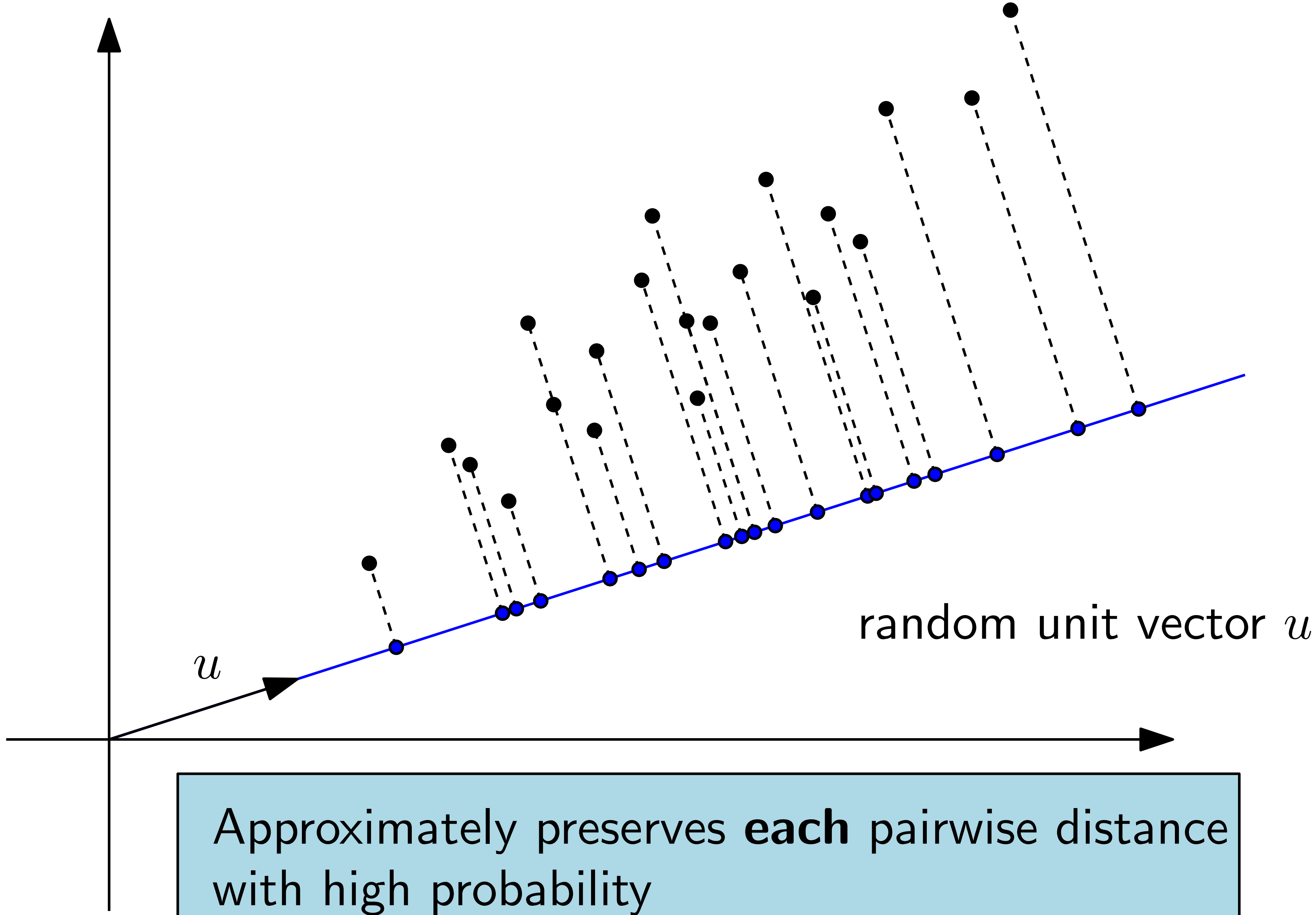


Approximately preserves **each** pairwise distance with high probability

# Embedding and Distortion

Given a point set $X \in \mathbb{R}^d$, we call a function $f : X \to \mathbb{R}^k$ an **embedding** of $X$. We define

$$\text{expansion}(f) = \max_{x,y \in X} \frac{\|f(x) - f(y)\|}{\|x - y\|}$$

$$\text{contraction}(f) = \max_{x,y \in X} \frac{\|x - y\|}{\|f(x) - f(y)\|}$$

The **distortion** of $f$ is defined as the product of the expansion and the contraction of $f$.

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Note that for all $x, y \in X$ we have

$$\frac{1}{\beta}\|x - y\| \leq \|f(x) - f(y)\| \leq \alpha\|x - y\|$$

where $\alpha$ denotes the expansion and $\beta$ denotes the contraction

# Achlioptas' Random Projection (Algorithm)

**Input:** set of points $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$, value of $k$
**Output:** set of points $Q = \{q_1, \ldots, q_n\} \subseteq \mathbb{R}^k$

---

**Algorithm:**

- Generate a random $k \times d$ matrix $\mathbf{R}$ by choosing

$$r_{i,j} = \begin{cases} +1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}$$

- For each $i = 1, \ldots, n$, compute $q_i = \frac{1}{\sqrt{k}} \mathbf{R} p_i$

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**Theorem:**

Let $k_0 = \frac{4+2\beta}{\varepsilon^2/2 - \varepsilon^3/3} \log n$, for given $\varepsilon, \beta > 0$. If $k \geq k_0$ then with probability at least $1 - \frac{1}{n^\beta}$, we have for all $p_i, p_j \in P$ that

$$(1-\varepsilon)\|p_i - p_j\|^2 \leq \|q_i - q_j\|^2 \leq (1+\varepsilon)\|p_i - p_j\|^2$$

# History: Embedding Lemma

Random projections were invented by Johnson and Lindenstrauss.

**Lemma (Johnson and Lindenstrauss, 1984)**:
Given $\varepsilon > 0$ and an integer $n$, let $k$ be a positive integer $k \geq k_0 = O\left(\frac{\log n}{\varepsilon^2}\right)$. For every set of points $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$ there exists $f : \mathbb{R}^d \to \mathbb{R}^k$ such that for all $p_i, p_j \in P$

$$(1-\varepsilon)\|p_i - p_j\|^2 \leq \|f(p_i) - f(p_j)\|^2 \leq (1+\varepsilon)\|p_i - p_j\|^2.$$

Note: The proof uses a random projection to show that $f$ exists. For historical reasons, the JL-lemma only talks about the existence of $f$.

# Linear Algebra: Rotation

**In general:**
A matrix is a rotation iff it is orthogonal

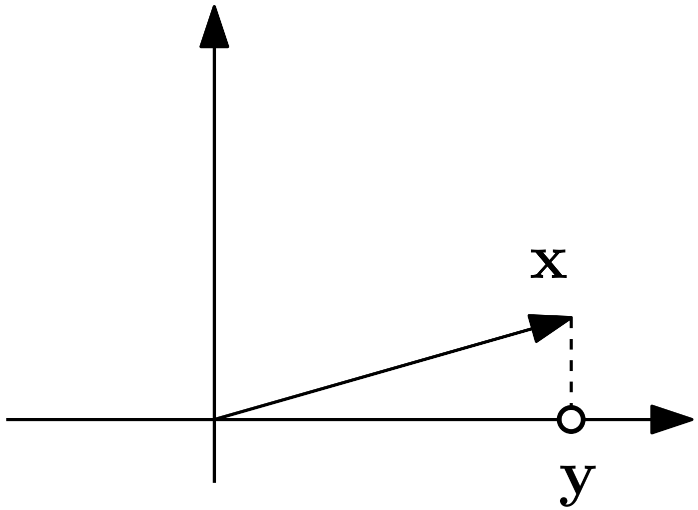
$$\mathbf{R} = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{pmatrix} = \begin{pmatrix} \mathbf{r_1} \\ \mathbf{r_2} \\ \mathbf{r_3} \end{pmatrix}$$

This means its row vectors are..
(1) pairwise orthogonal: $\mathbf{r_i} \cdot \mathbf{r_j} = 0$
(2) unit vectors: $\|\mathbf{r_i}\| = 1$

Furthermore, it holds that $\mathbf{R}^{-1} = \mathbf{R}^{\mathbf{T}}$ and that the length of any vector is preserved under $\mathbf{R}$

# Linear Algebra: Axis-orthogonal Projection

Project a vector $\mathbf{x}$ into first dimension:




$\mathbf{P} \cdot \mathbf{x} = \mathbf{y}$

Transformation matrix:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

**In general:**

An axis-orthogonal projection is an identity matrix with some 'collapsed' dimensions

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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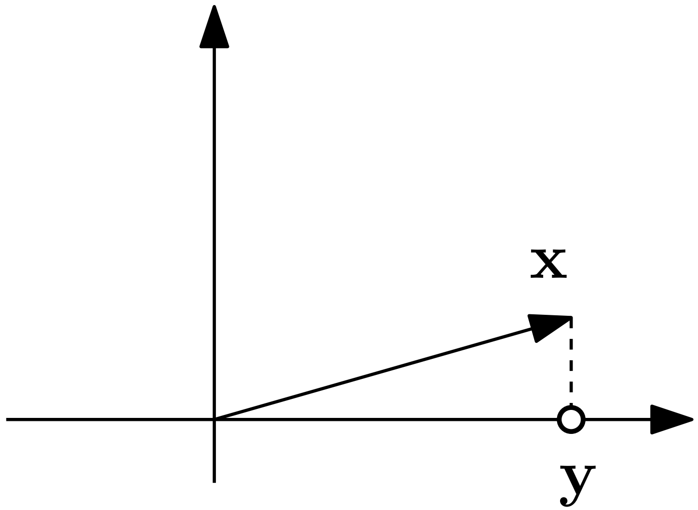


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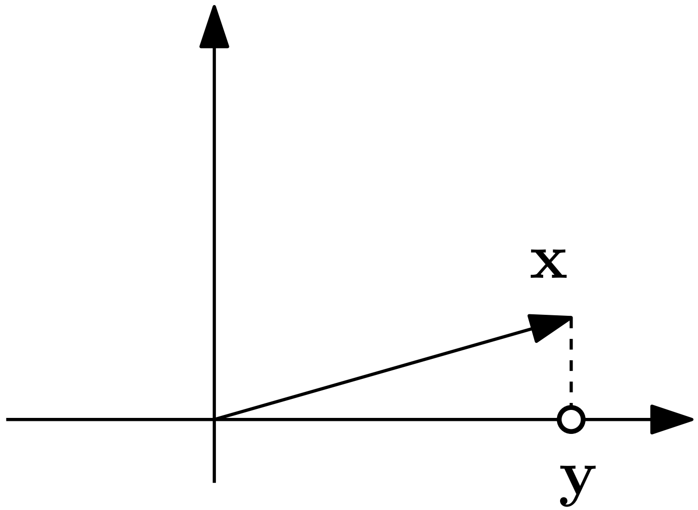


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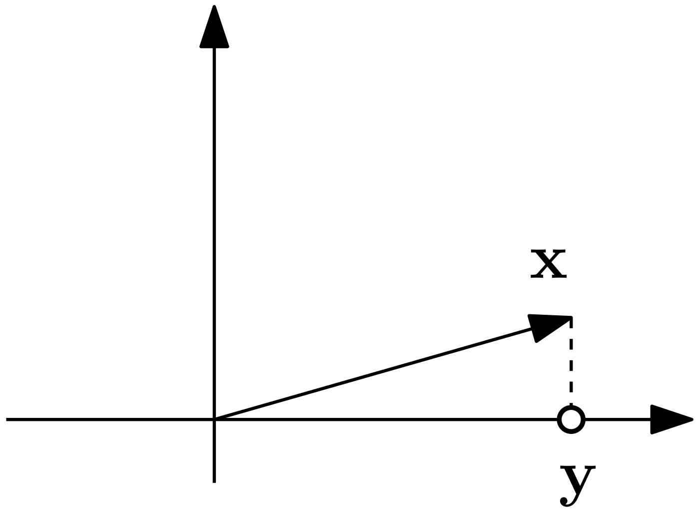


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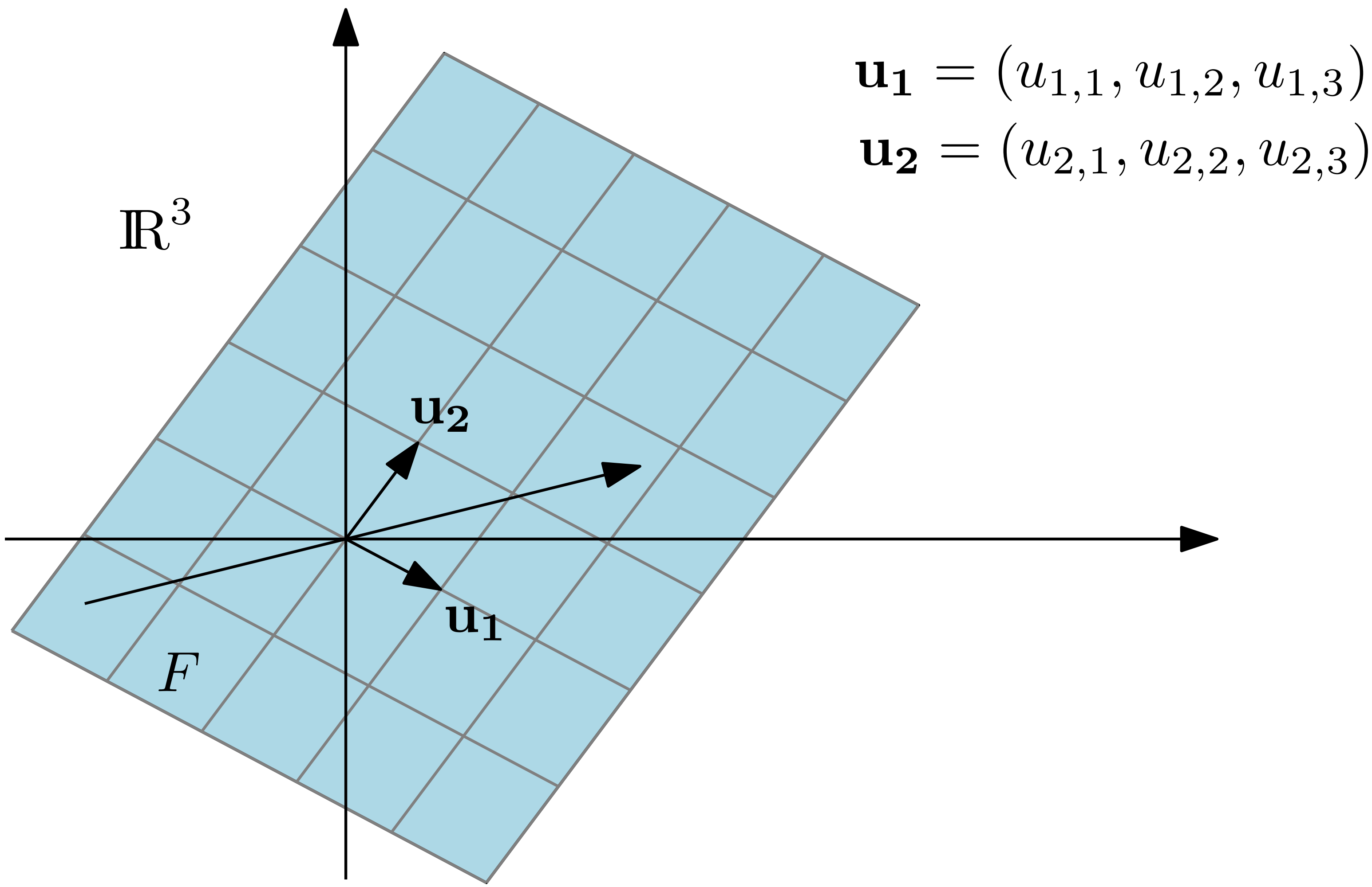
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

# Linear Algebra: Projection onto subspace

Let $F$ be a $k$-dimensional linear subspace of $\mathbb{R}^d$ spanned by orthonormal vectors $\mathbf{u_1}, \ldots, \mathbf{u_k}$ and let $\mathbf{R}$ be the projection onto $F$



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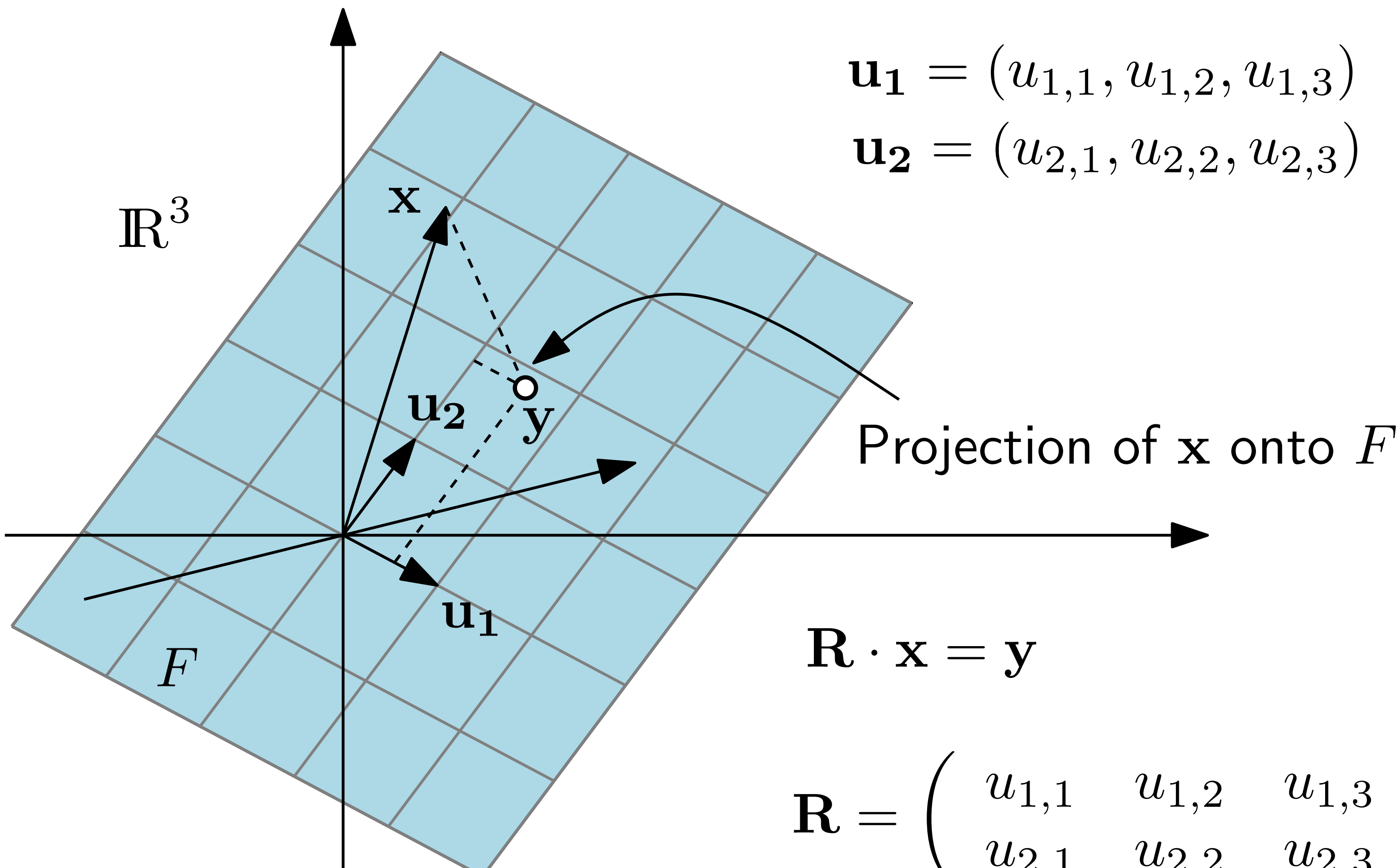


$$\mathbf{u_1} = (u_{1,1}, u_{1,2}, u_{1,3})$$
$$\mathbf{u_2} = (u_{2,1}, u_{2,2}, u_{2,3})$$

$$\mathbf{R} \cdot \mathbf{x} = \mathbf{y}$$

$$\mathbf{R} = \begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \end{pmatrix}$$

# Linear Algebra: Projection onto subspace

A projection onto a subspace can be viewed as rotation followed by an axis-orthogonal projection

To see this, let's rewrite $\mathbf{R} = \mathbf{P} \cdot \mathbf{M}$ with

- $\mathbf{M}$ : rotation to align each $\mathbf{u_i}$ with standard basis vector $\mathbf{v_i}$
- $\mathbf{P}$ : orthogonal projection onto first $k$ coordinates

To find the rotation matrix $\mathbf{M}$, note that for $i = 1, \ldots, k$

$$\mathbf{M} \cdot \mathbf{u_i} = \mathbf{v_i} \quad \Leftrightarrow \quad \mathbf{M^{-1}} \cdot \mathbf{v_i} = \mathbf{u_i} \quad \Leftrightarrow \quad \mathbf{M^T} \cdot \mathbf{v_i} = \mathbf{u_i}$$

- since $\mathbf{v_i}$ is the i'th standard basis vector, $\mathbf{M^T v_i}$ is the $i$th column vector of $\mathbf{M^T}$
- thus, $\mathbf{u_i}$ is the $i$th row vector of $\mathbf{M}$ for $i = 1, \ldots, k$

# Analysis of Achlioptas' Random Rotation

We can think of Achlioptas transformation as a rotation $\mathbf{M}$ followed by a projection $\mathbf{P}$ onto the first $k$ dimensions.

$$f(p) = \frac{1}{\sqrt{k}}\mathbf{R}p = \frac{\sqrt{d}}{\sqrt{k}}\frac{1}{\sqrt{d}}\mathbf{R}p = \frac{\sqrt{d}}{\sqrt{k}}\mathbf{PM}p$$

**Example:** $k = 2, d = 4$

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{M} = \frac{1}{\sqrt{d}} \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} & r_{1,4} \\ r_{2,1} & r_{2,2} & r_{2,3} & r_{2,4} \\ r_{3,1} & r_{3,2} & r_{3,3} & r_{3,4} \\ r_{4,1} & r_{4,2} & r_{4,3} & r_{4,4} \end{pmatrix}$$

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But is $\mathbf{M}$ a rotation?

# Analysis of Achlioptas' Random Rotation

$\mathbf{M}$ is a rotation if and only if the product of $\mathbf{M}$ and its transpose is the identity (i.e., $\mathbf{M}$ is orthogonal)

$$\mathbf{M}\mathbf{M}^{\mathbf{T}} = \mathbf{I}$$

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(1) each pair of row vectors is orthogonal

(2) each row vector has unit length

Let $\mathbf{r_i} = \frac{1}{\sqrt{d}}(r_{i,1}, \ldots, r_{i,d})$ be the $i$th row vector of $\mathbf{M}$

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# Analysis of Achlioptas' Random Rotation

(continued)

Recall that each $r_{i,j}$ is a discrete random variable:

$$r_{i,j} = \begin{cases} +1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}$$

Let's analyze expected values of the matrix entries of $\mathbf{MM^T}$

We will need the expected value of $r_{i,j}$:

$$\forall i, j : \mathrm{E}\left[r_{i,j}\right] = -1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 0$$

# Analysis of Achlioptas' Random Rotation

(continued)

(1) the expected angle between each pair of row vectors is orthogonal: $\forall i \neq j : \mathrm{E}\left[\langle \mathbf{r_i}, \mathbf{r_j} \rangle\right] = 0$

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**Proof:**

By linearity of expectation

$$\mathrm{E}\left[\langle \mathbf{r_i}, \mathbf{r_j} \rangle\right] = \mathrm{E}\left[\sum_{t=1}^{d} \frac{r_{i,t}}{\sqrt{d}} \cdot \frac{r_{j,t}}{\sqrt{d}}\right] = \frac{1}{d} \sum_{t=1}^{d} \mathrm{E}\left[r_{i,t} \cdot r_{j,t}\right]$$

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for $i \neq j$ it holds that $r_{i,t}$ and $r_{j,t}$ are independent random variables, therefore

$$\mathrm{E}\left[\langle \mathbf{r_i}, \mathbf{r_j} \rangle\right] = \frac{1}{d} \sum_{t=1}^{d} \mathrm{E}\left[r_{i,t}\right] \cdot \mathrm{E}\left[r_{j,t}\right] = 0$$

□

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(continued)

(2) the expected squared length of each row vector is $1$:

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Note that $r_{i,t}^2$ is also a random variable and its expected value is:

$$\mathrm{E}\left[r_{i,t}^2\right] = (-1)^2 \cdot \tfrac{1}{2} + (1)^2 \cdot \tfrac{1}{2} = 1$$

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**Proof:**

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$$\mathrm{E}\left[\langle \mathbf{r_i}, \mathbf{r_i} \rangle\right] = \mathrm{E}\left[\sum_{t=1}^{d} \frac{r_{i,t}}{\sqrt{d}} \cdot \frac{r_{i,t}}{\sqrt{d}}\right] = \frac{1}{d} \sum_{t=1}^{d} \mathrm{E}\left[r_{i,t}^2\right]$$

Note that $r_{i,t}^2$ is also a random variable and its expected value is:

$$\mathrm{E}\left[r_{i,t}^2\right] = (-1)^2 \cdot \tfrac{1}{2} + (1)^2 \cdot \tfrac{1}{2} = 1$$

Therefore, $\mathrm{E}\left[\langle \mathbf{r_i}, \mathbf{r_i} \rangle\right] = 1$ $\square$

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- Next we want to analyze the effect of the random projection on a distance between two points
- Therefore we analyze for fixed $p_i, p_j \in P$ the expectation of its squared length in the projection

$$\mathrm{E}\left[\|f(p_i) - f(p_j)\|^2\right]$$

# Analysis of a fixed distance $d_{ij}$ (Expectation)

**Claim:** For fixed $p_i, p_j \in P : \mathrm{E}\left[\|f(p_i) - f(p_j)\|^2\right] = \|p_i - p_j\|^2$

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By the definition of $f$

$$\|f(\alpha)\|^2 = \left\|\frac{\sqrt{d}}{\sqrt{k}}\mathbf{PM}\alpha\right\|^2 = \frac{d}{k}\|\mathbf{PM}\alpha\|^2 = \frac{d}{k}\sum_{i=1}^{k}(\mathbf{r_i}\alpha)^2$$

where $\mathbf{r_i} = \frac{1}{\sqrt{d}}(r_{i,1}, \ldots, r_{i,d})$ is the $i$th row vector of $\mathbf{M}$, as defined earlier.

# Analysis of a fixed distance $d_{ij}$ (Expectation)

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by linearity of expectation

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$$\mathrm{E}\left[(\mathbf{r_i}\alpha)^2\right] = \mathrm{E}\left[\left(\sum_{j=1}^{d}\frac{1}{\sqrt{d}}r_{i,j}a_j\right)^2\right] = \mathrm{E}\left[\frac{1}{d}\left(\sum_{j=1}^{d}r_{i,j}a_j\right)^2\right]$$

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We recursively expand the inner quadratic expression

$$\begin{aligned}\left(\sum_{j=1}^{d}r_{i,j}a_j\right)^2 &= (r_{i,1}a_1)^2 + 2r_{i,1}a_1\left(\sum_{j=2}^{d}r_{i,j}a_j\right) + \left(\sum_{j=2}^{d}r_{i,j}a_j\right)^2 \\ &= \ldots \\ &= \sum_{j=1}^{d}(r_{i,j}a_j)^2 + \sum_{j=1}^{d-1}\sum_{l=j+1}^{d}2r_{i,j}a_jr_{i,l}a_l\end{aligned}$$

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Plugging back into the equation..

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Now, plugging back the definition of $\alpha$

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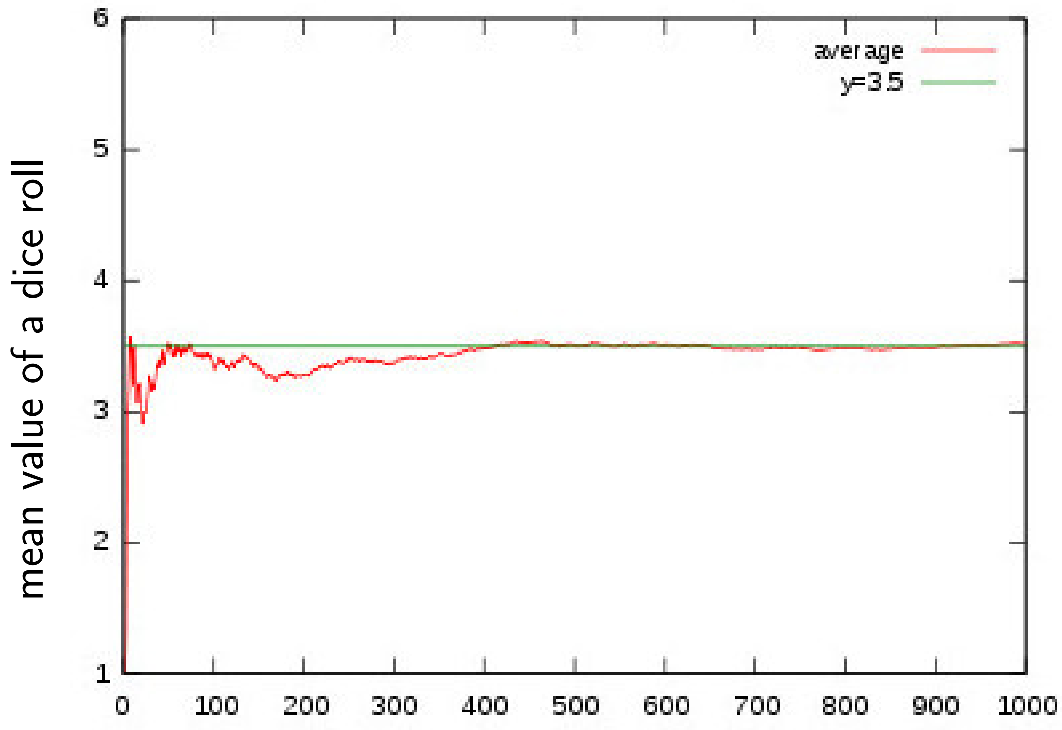
$\square$

# Law of Large Numbers

Let $X_1, \ldots, X_n$ be $n$ samples of a random variable $X$.
The law of large numbers states that

$$\Pr\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mathrm{E}[X]\right| \geq \varepsilon\right] \leq \frac{\mathrm{Var}(X)}{n \cdot \varepsilon^2}$$




$n =$ number of trials

# Concentration of mass in high dimensions

The unit hypercube: $[0,1]^d = \{(x_1, \ldots, x_d) \mid x_i \in [0,1]\}$

Random vector $\mathbf{x} = (x_1, \ldots, x_d) \in [0,1]^d$
(choose $x_i$ independently and uniformly random in $[0,1]$)

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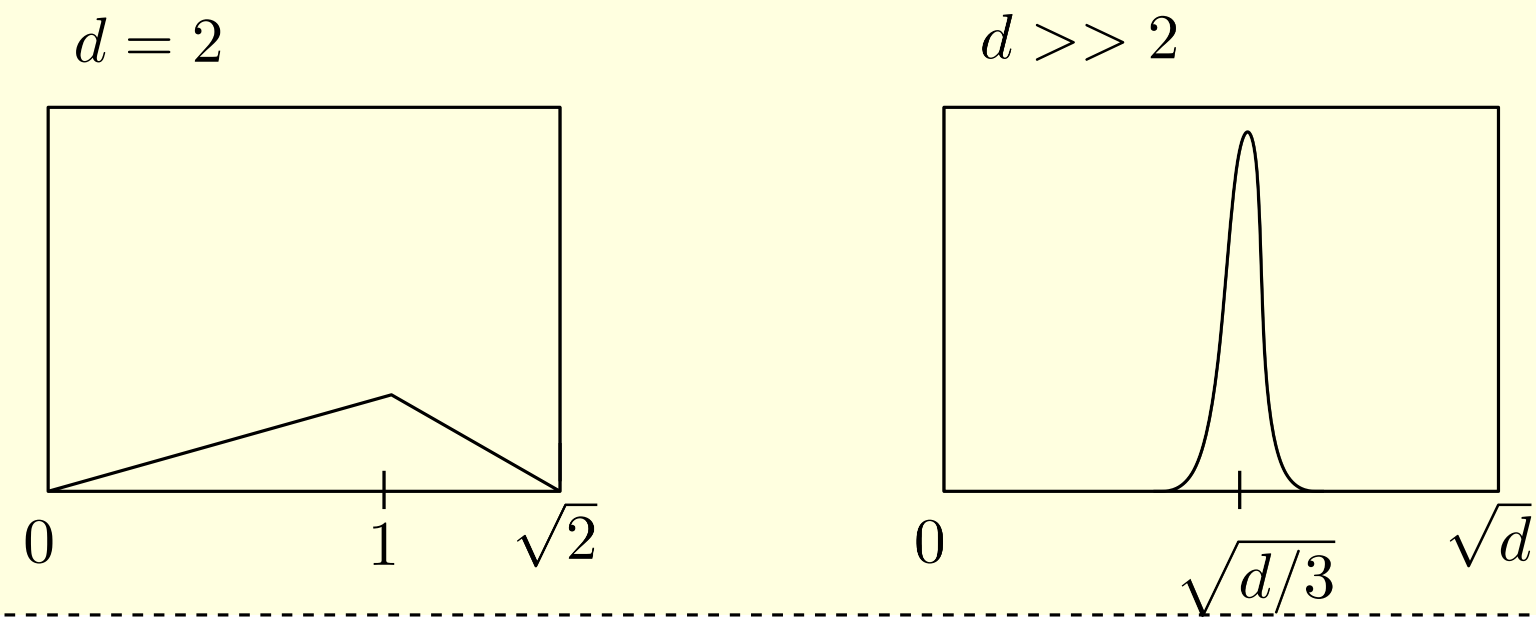
Distribution of the length of $\mathbf{x}$ in low vs. high dimensions



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Failure probability for two fixed points $p_i, p_j$:

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define $\alpha = \frac{p_i - p_j}{\|p_i - p_j\|}$ (note that $\alpha$ is a fixed unit vector)

$$= \Pr\left[\|f(\alpha)\|^2 \notin [1-\varepsilon, 1+\varepsilon]\right]$$

# Analysis of a fixed distance $d_{ij}$

(continued)

$$\Pr\left[\left\|f\left(\alpha\right)\right\|^{2}\notin\left[1-\varepsilon,1+\varepsilon\right]\right]$$

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squared length of a **random unit vector**, projected onto the first $k$ coordinates

# Analysis of a fixed distance $d_{ij}$

Using concentration of measure, Achlioptas shows the following lemma (we omit the full proof):

**Lemma:**

Let $r_{i,j}$ be chosen uniformly random from $\{-1, 1\}$, then for any $\varepsilon > 0$ and any unit vector $\alpha \in \mathbb{R}^d$,

$$\Pr\left[\|f(\alpha)\|^2 \notin [1-\varepsilon, 1+\varepsilon]\right] < 2 \cdot e^{(-\frac{k}{2}(\varepsilon^2/2 - \varepsilon^3/3))}$$

Therefore, choosing

$$k \geq \frac{4+2\beta}{\varepsilon^2/2 + \varepsilon^3/3} \log n$$

is sufficient to ensure

$$\Pr\left[\frac{\|f(p_i) - f(p_j)\|^2}{\|p_i - p_j\|^2} \notin [1-\varepsilon, 1+\varepsilon]\right] = \Pr\left[\|f(\alpha)\|^2 \notin [1-\varepsilon, 1+\varepsilon]\right] < \frac{2}{n^{2+\beta}}$$

# Analysis of the Distortion

**Theorem:**

Let $k_0 = \frac{4+2\beta}{\varepsilon^2/2 - \varepsilon^3/3} \log n$, for given $\varepsilon, \beta > 0$. If $k \geq k_0$ then with probability at least $1 - \frac{1}{n^\beta}$, we have for all $p_i, p_j \in P$ that

$$(1-\varepsilon)\|p_i - p_j\|^2 \leq \|f(p_i) - f(p_j)\|^2 \leq (1+\varepsilon)\|p_i - p_j\|^2$$

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**Proof :**

- there are $\binom{n}{2}$ pairs of points in $P$
- bound the failure probability for two fixed points:

$$\Pr\left[\frac{\|f(p_i) - f(p_j)\|^2}{\|p_i - p_j\|^2} \notin [1-\varepsilon, 1+\varepsilon]\right] < \frac{2}{n^{2+\beta}}$$

- apply the union bound: $\binom{n}{2} \cdot \frac{2}{n^{2+\beta}} < \frac{n^2}{2} \cdot \frac{2}{n^{2+\beta}} = \frac{1}{n^\beta}$

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$$r_{i,j} = \sqrt{3} \begin{cases} +1 & \text{with probability } \frac{1}{6} \\ 0 & \text{with probability } \frac{2}{3} \\ -1 & \text{with probability } \frac{1}{6} \end{cases}$$

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What would be the advantage of this?

This generates a sparser matrix

# Summary

- Nearest-neighbor searching
- Embedding and Distortion
- Achlioptas' Random Projection
- Projection onto a subspace
- Random Rotation (Expectation)
- Analysis of a fixed distance (Expectation)
- Law of large numbers
- Concentration of measure
- Analysis of a fixed distance
- Analysis of the Distortion
- Alternative Projection Matrix

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