

Linear Algebra Basics

A **vector space** over \mathbb{R} is a set of vectors V closed under the operations vector addition and scalar multiplication.

vector addition:

$$\forall \mathbf{x}, \mathbf{y} \in V: \quad \mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \end{pmatrix} \in V$$

scalar multiplication:

$$\forall \lambda \in \mathbb{R}, \mathbf{x} \in V: \quad \lambda \mathbf{x} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \end{pmatrix} \in V$$

Linear Algebra Basics

A **basis** of the vector space is a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d\}$, such that every $\mathbf{x} \in V$ can be expressed as a linear combination of the basis vectors:

$$\mathbf{x} = x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \dots + x_d \mathbf{v}_d$$

Standard basis: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \dots \mathbf{v}_d = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

Linear Algebra Basics

The **inner product** of two vectors:

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^d x_i y_i$$

The **outer product** of two vectors:

$$\mathbf{x} \otimes \mathbf{y} := \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_d \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_d \\ \vdots & \ddots & & \\ x_d y_1 & x_d y_2 & \cdots & x_d y_d \end{pmatrix}$$

Linear Algebra Basics

Matrix Multiplication $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$

$$\begin{array}{c}
 \mathbf{A} \\
 k \quad \ell
 \end{array}
 \begin{pmatrix}
 a_{1,1} & a_{1,2} & a_{1,3} \\
 a_{2,1} & a_{2,2} & a_{2,3} \\
 a_{3,1} & a_{3,2} & a_{3,3} \\
 a_{4,1} & a_{4,2} & a_{4,3}
 \end{pmatrix}
 \begin{array}{c}
 \mathbf{B} \\
 \ell \quad m
 \end{array}
 \begin{pmatrix}
 b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\
 b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \\
 b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4}
 \end{pmatrix}
 \begin{array}{c}
 \mathbf{C} \\
 k \quad m
 \end{array}
 \begin{pmatrix}
 c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} \\
 c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} \\
 c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \\
 c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4}
 \end{pmatrix}$$

The diagram illustrates the calculation of the entry $c_{3,2}$ in matrix \mathbf{C} . A horizontal arrow points from the third row of \mathbf{A} (highlighted in light blue) to the entry $c_{3,2}$ in \mathbf{C} (also highlighted in light blue). A vertical arrow points from the second column of \mathbf{B} (highlighted in light blue) down to the same entry $c_{3,2}$ in \mathbf{C} .

Each entry of \mathbf{C} is defined as an inner product

$$c_{ij} = \sum_{1 \leq p \leq \ell} a_{i,p} \cdot b_{p,j} = \langle \mathbf{a}_i, \mathbf{b}_j \rangle$$

\mathbf{a}_i is the i 'th row vector of \mathbf{A} and \mathbf{b}_j is the j 'th column vector of \mathbf{B}

Linear Algebra Basics

Matrix Multiplication $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$

$$\begin{array}{c} \mathbf{A} \\ k \end{array} \begin{array}{c} \ell \\ \left(\begin{array}{ccc} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{array} \right) \end{array} \begin{array}{c} \mathbf{B} \\ m \\ \left(\begin{array}{cccc} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} \end{array} \right) \end{array} \begin{array}{c} \mathbf{C} \\ \left(\begin{array}{cccc} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} \end{array} \right) \end{array}$$

We can also write \mathbf{C} as a sum of outer products

$$\mathbf{C} = \sum_{1 \leq i \leq \ell} \mathbf{a}_i \otimes \mathbf{b}_i$$

\mathbf{a}_i is the i 'th column vector of \mathbf{A} and \mathbf{b}_i is the i 'th row vector of \mathbf{B}

Linear Algebra Basics

Transpose of a matrix switches row and column indices

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} \Rightarrow \begin{pmatrix} a_{1,1} & a_{2,1} & a_{3,1} & a_{4,1} \\ a_{1,2} & a_{2,2} & a_{3,2} & a_{4,2} \\ a_{1,3} & a_{2,3} & a_{3,3} & a_{4,3} \end{pmatrix}$$

\mathbf{A} \mathbf{A}^T

It holds that $(\mathbf{A} \cdot \mathbf{B})^T = \mathbf{B}^T \cdot \mathbf{A}^T$

Linear Algebra Basics


The **length** of a vector:

$$\|\mathbf{x}\| := \sqrt{\sum_{i=1}^d x_i^2}$$

Vector \mathbf{x} has **unit length** if and only if $\langle \mathbf{x}, \mathbf{x} \rangle = 1$

The **law of cosine**:

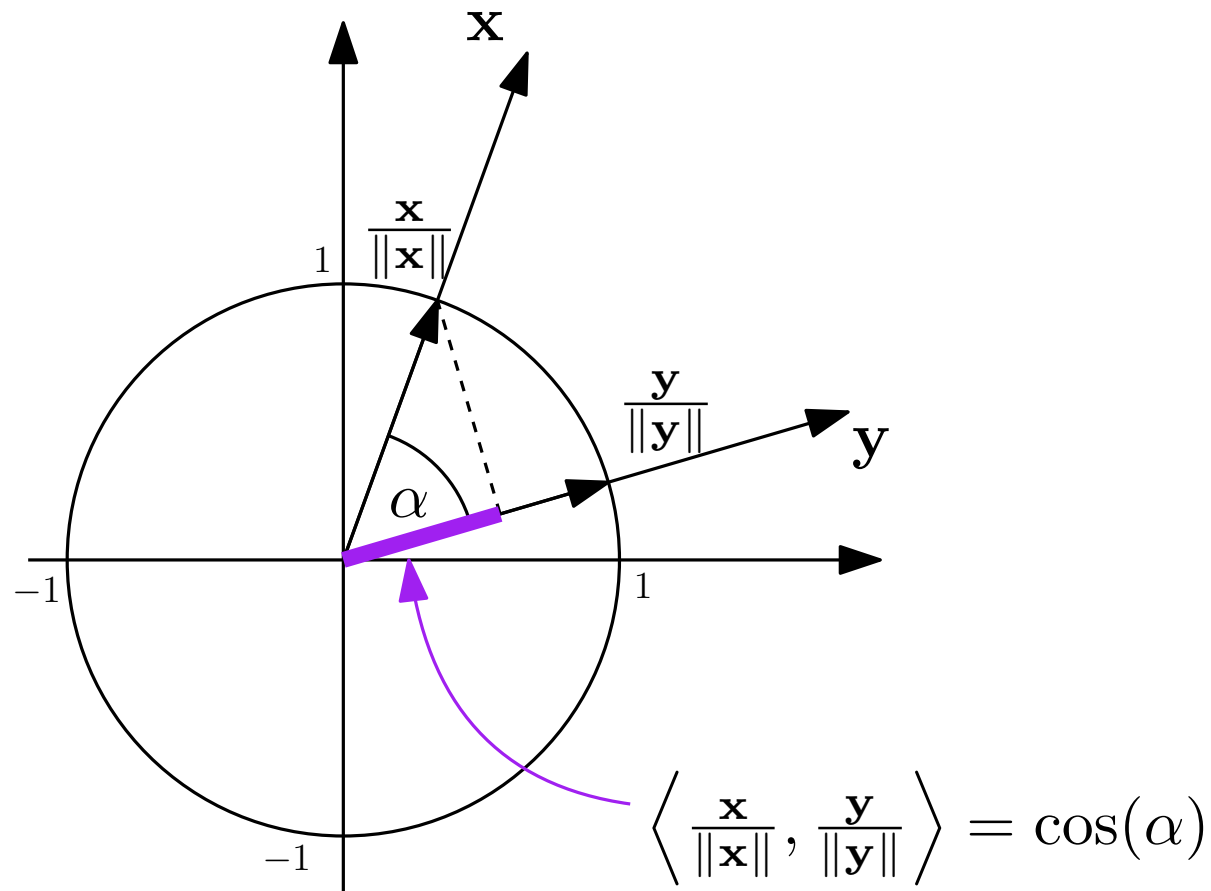
$$\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cos(\alpha)$$

 α is the smaller angle spanned by the two vectors

Vectors \mathbf{x} and \mathbf{y} are **orthogonal** if and only if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$

Linear Algebra Basics

The law of cosine: $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cos(\alpha)$



Linear Algebra Basics

The **span** of a set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ is the set of all possible linear combinations of these vectors

$$\left\{ \sum_{i=1}^k \lambda_i \mathbf{v}_i \ : \ \lambda_i \in \mathbb{R} \right\}$$

The **rank** of a matrix is the dimension of the space spanned by its column vectors (or row vectors)

Linear Algebra Basics

A **linear map** is a mapping between two vector spaces $f : V \rightarrow W$ that satisfies the following two conditions:

$$(1) f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$$

$$(2) \forall \gamma \in \mathbb{R} : f(\gamma \mathbf{x}) = \gamma f(\mathbf{x})$$

Any matrix \mathbf{A} defines a **linear map**:

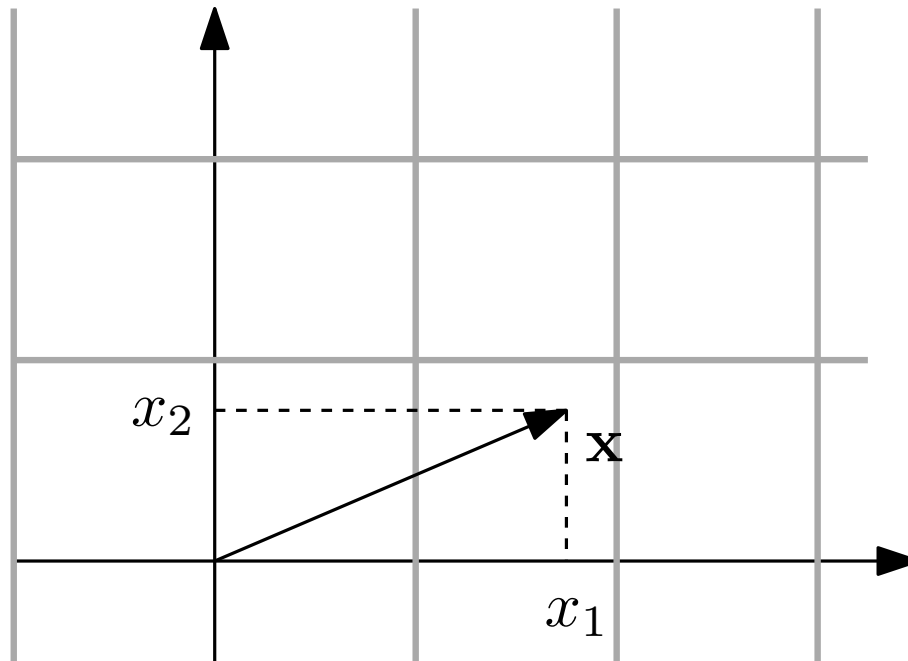
$$f : V \rightarrow W$$

$$f(\mathbf{x}) = \mathbf{A} \cdot \mathbf{x}$$

Linear Algebra Basics

Using linearity, we can expand $\mathbf{A} \cdot \mathbf{x}$ using the standard basis:

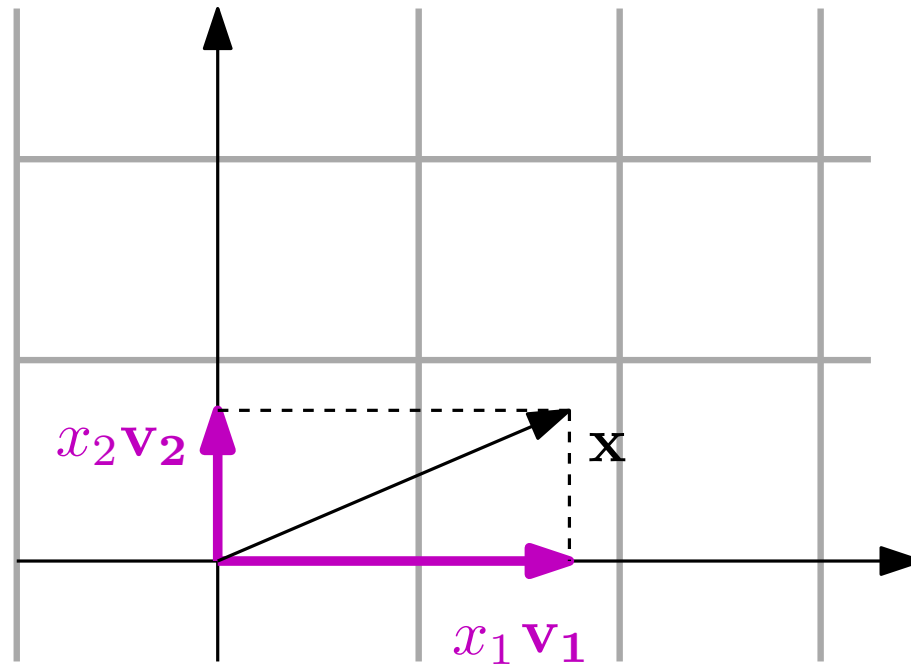
$$\mathbf{A}\mathbf{x} = \mathbf{A}(x_1\mathbf{v}_1 + x_2\mathbf{v}_2) = x_1\mathbf{A}\mathbf{v}_1 + x_2\mathbf{A}\mathbf{v}_2$$
$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \left(x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = x_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix} + x_2 \begin{pmatrix} a_{1,2} \\ a_{2,2} \end{pmatrix}$$



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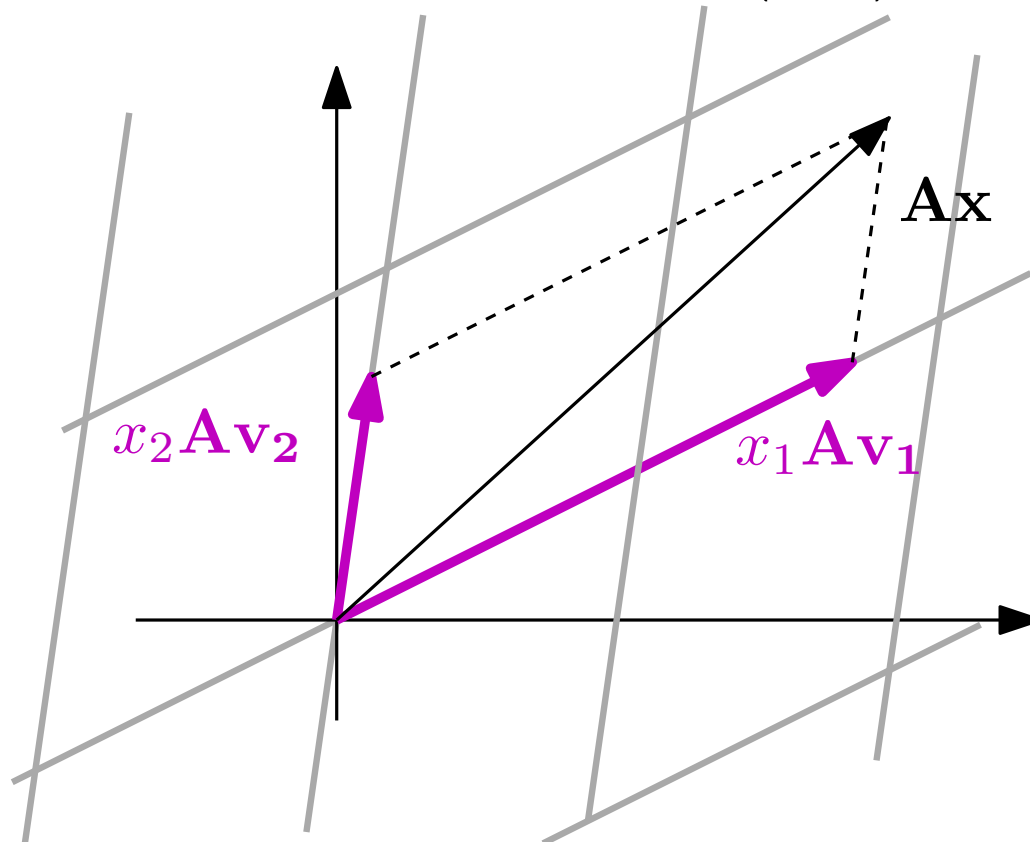
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Linear Algebra Basics

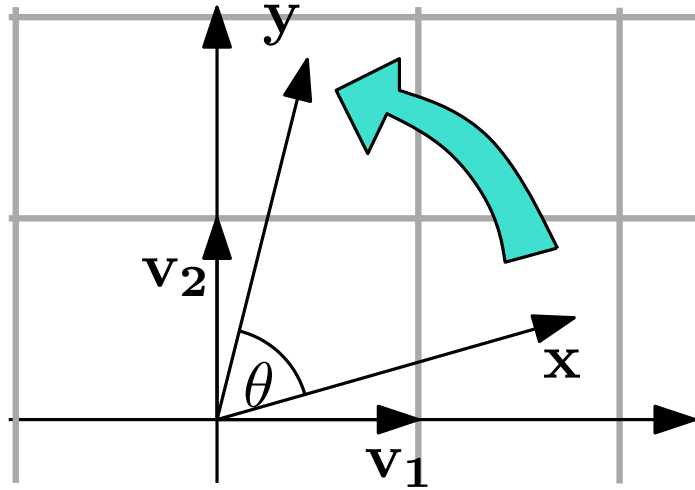
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Linear Algebra: Rotation

Rotate a vector \mathbf{x} by angle θ :



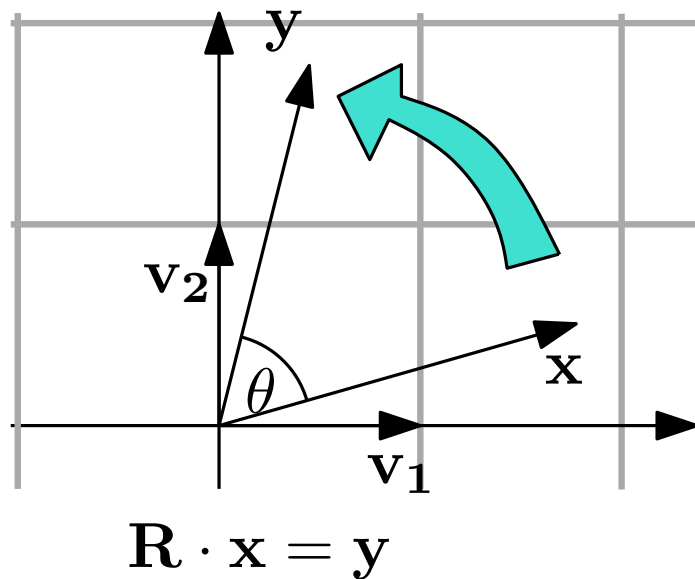
$$\mathbf{R} \cdot \mathbf{x} = \mathbf{y}$$

Transformation matrix:

$$\mathbf{R} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

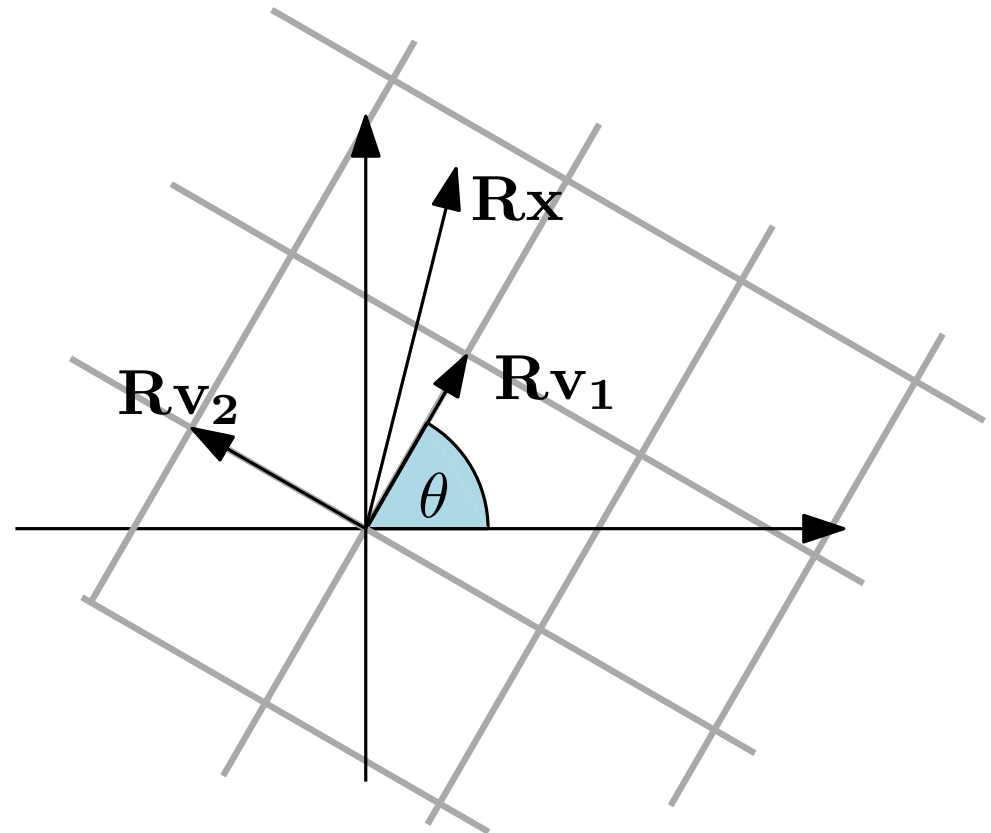
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$$\mathbf{R}\mathbf{v}_1 = (\cos \theta, \sin \theta)$$

$$\mathbf{R}\mathbf{v}_2 = (-\sin \theta, \cos \theta)$$

Linear Algebra: Rotation

In general:

A matrix is a rotation iff it is orthogonal

$$\mathbf{R} = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{pmatrix} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{pmatrix}$$

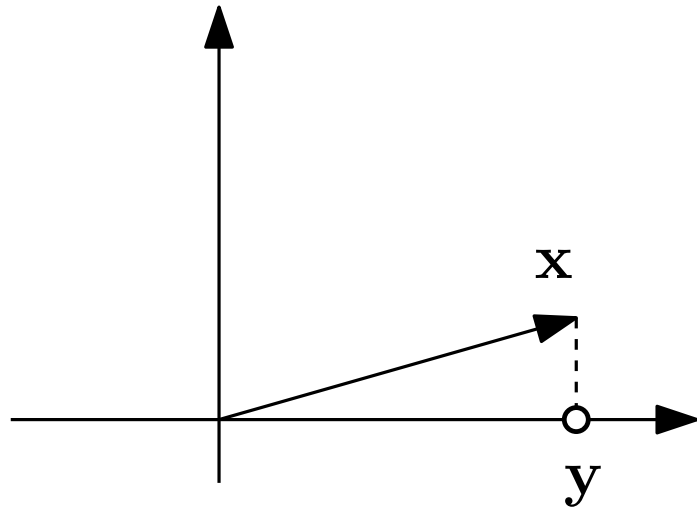
This means its row vectors are..

- (1) pairwise orthogonal: $\mathbf{r}_i \cdot \mathbf{r}_j = 0$
- (2) unit vectors: $\|\mathbf{r}_i\| = 1$

Furthermore, it holds that $\mathbf{R}^{-1} = \mathbf{R}^T$
and that the length of any vector is
preserved under \mathbf{R}

Linear Algebra: Axis-orthogonal Projection

Project a vector \mathbf{x} into first dimension:



$$\mathbf{P} \cdot \mathbf{x} = y$$

Transformation matrix:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

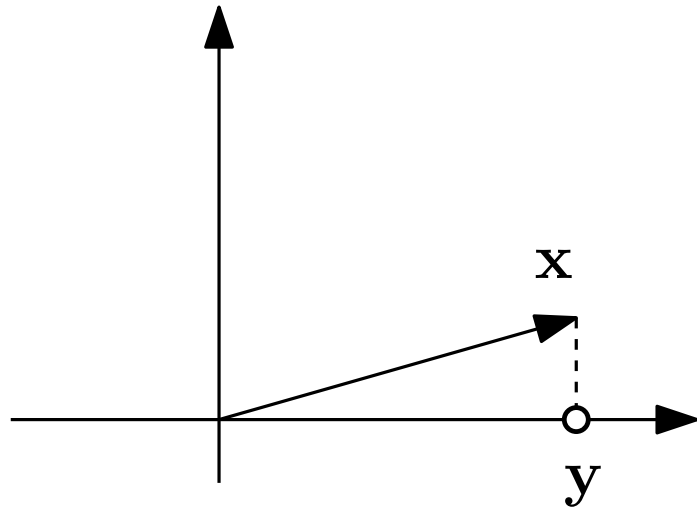
In general:

An axis-orthogonal projection is an identity matrix with some 'collapsed' dimensions

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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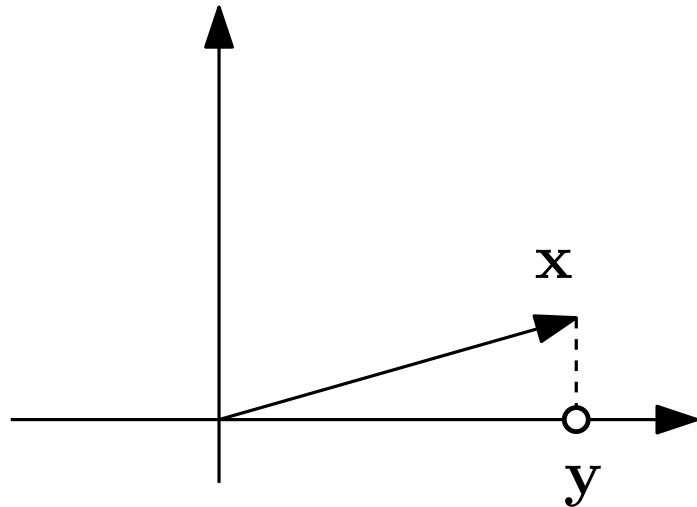
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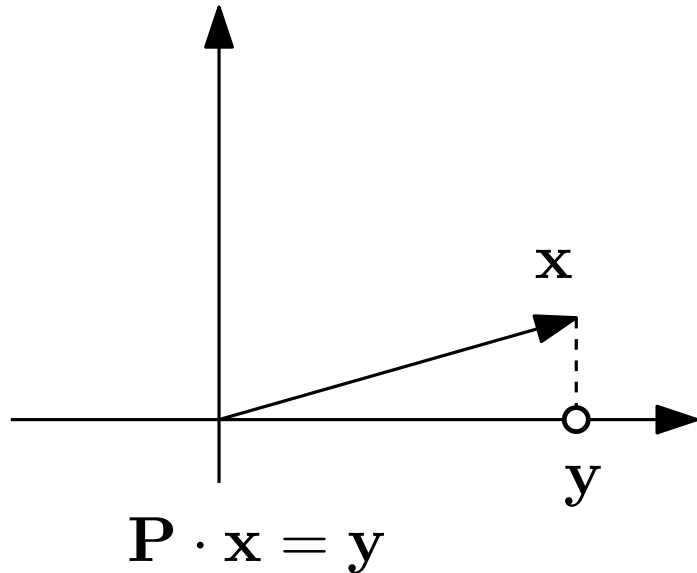
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$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \text{---} 0 & \text{---} 0 & \text{---} 0 & \text{---} 0 \\ \text{---} 0 & \text{---} 0 & \text{---} 0 & \text{---} 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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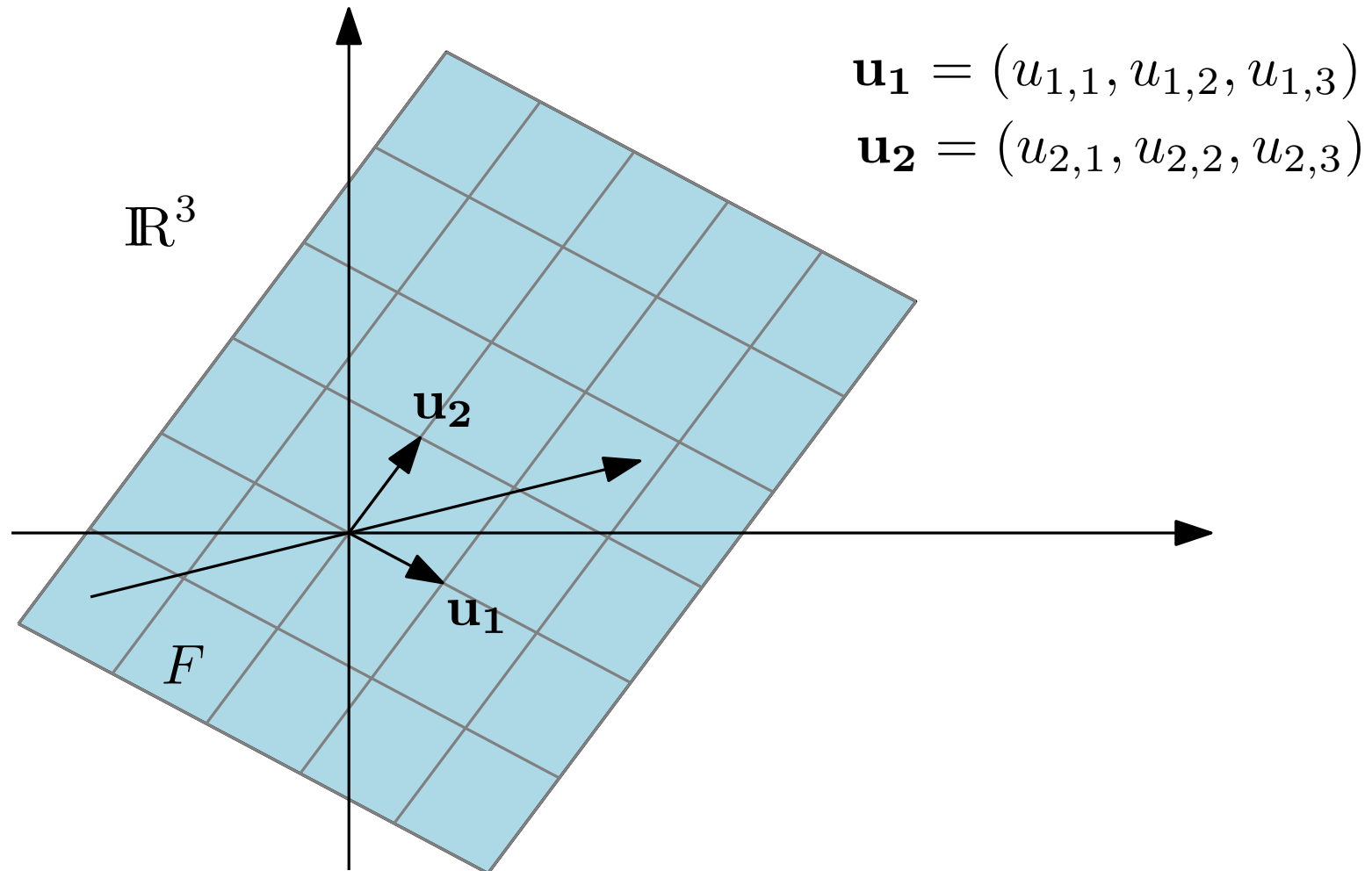
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An axis-orthogonal projection is an identity matrix with some 'collapsed' dimensions

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Linear Algebra: Projection onto subspace

Let F be a k -dimensional linear subspace of \mathbb{R}^d spanned by orthonormal vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ and let \mathbf{R} be the projection onto F



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