Locality-sensitive hashing

2IMW30 - Foundations of data mining
TU Eindhoven, Quartile 3, 2016

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Overview of this lecture

- Nearest-Neighbor rule
- Locality sensitive hashing
- Cosine distance
- Euclidean distance
- Jaccard Similarity
- Minhashing
- Banding
- Amplification

Random Partitions

Nearest-Neighbor-rule: Search among all labelled input elements for the one that minimizes a distance function (i.e., the *nearest neighbor*) and use this label as an estimator.

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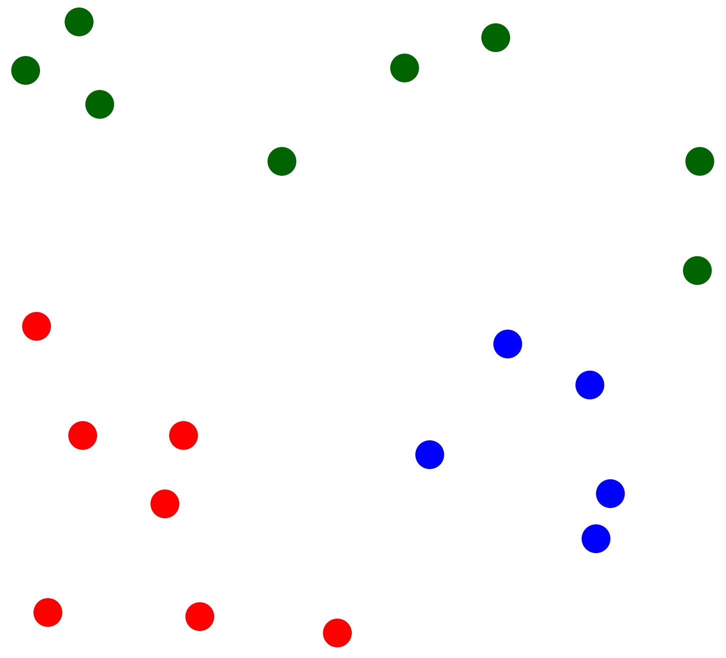

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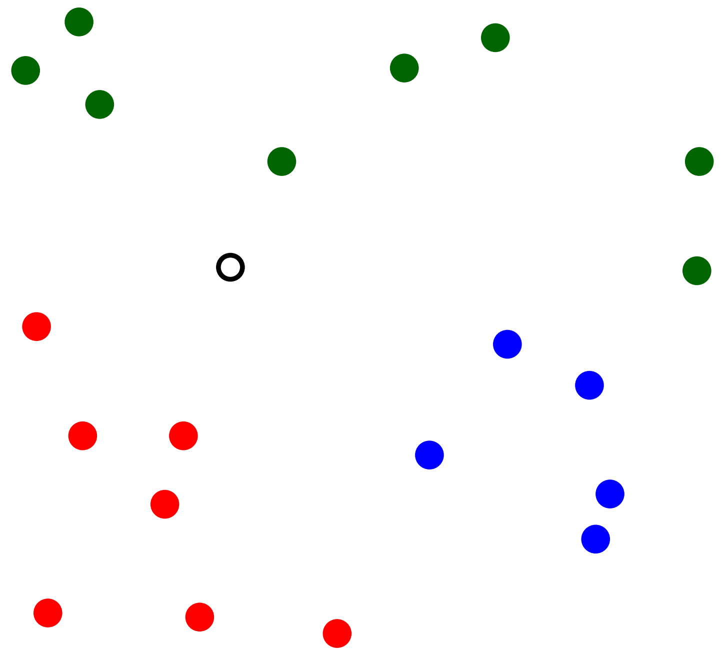

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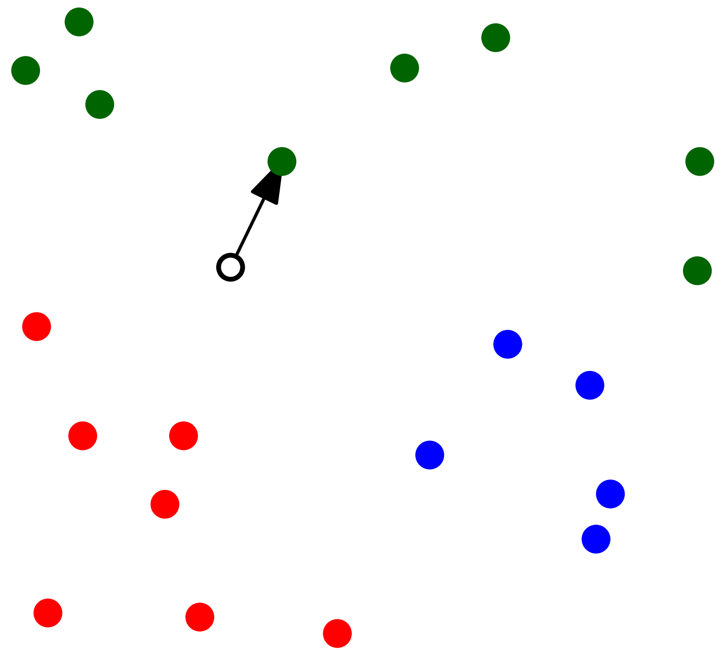

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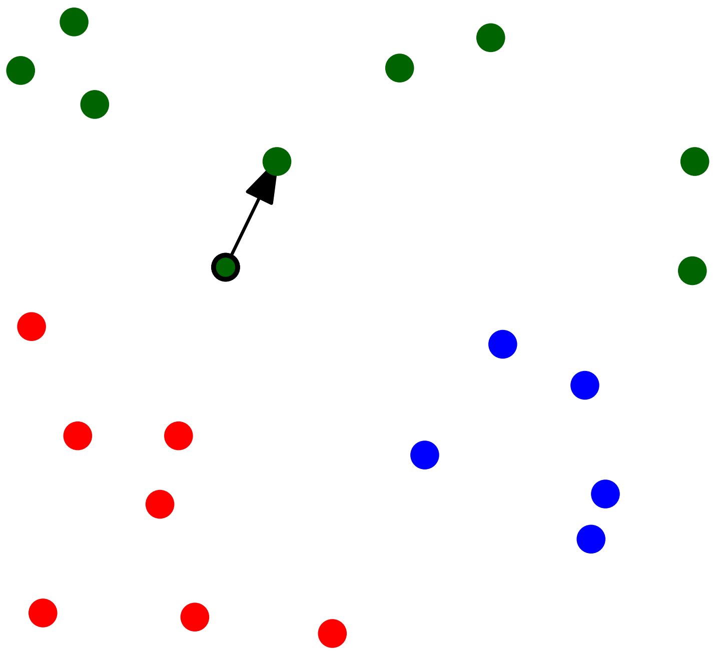

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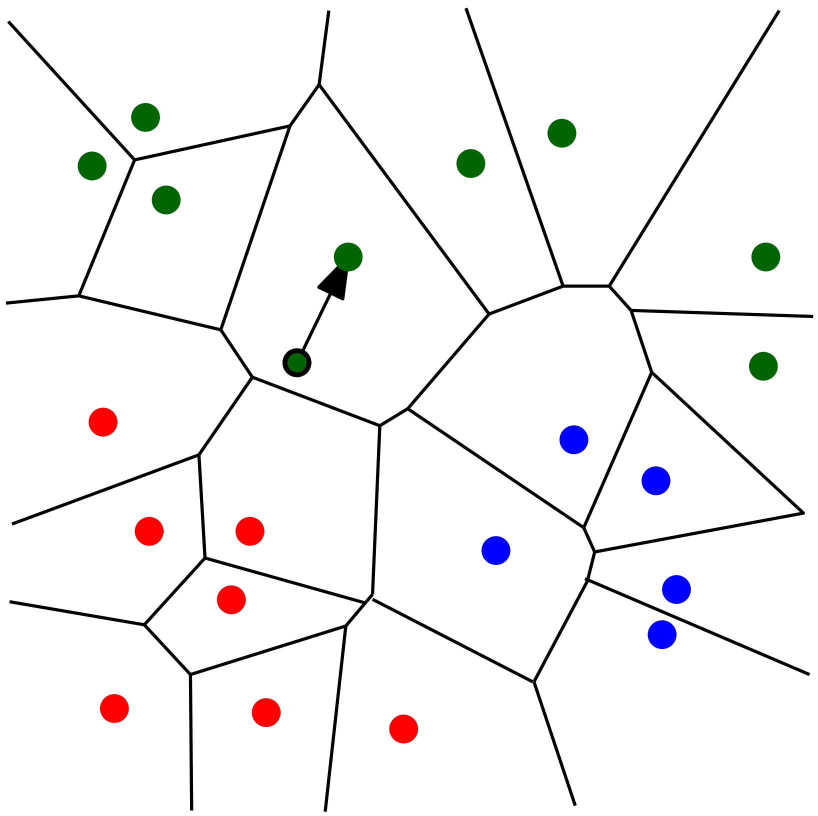

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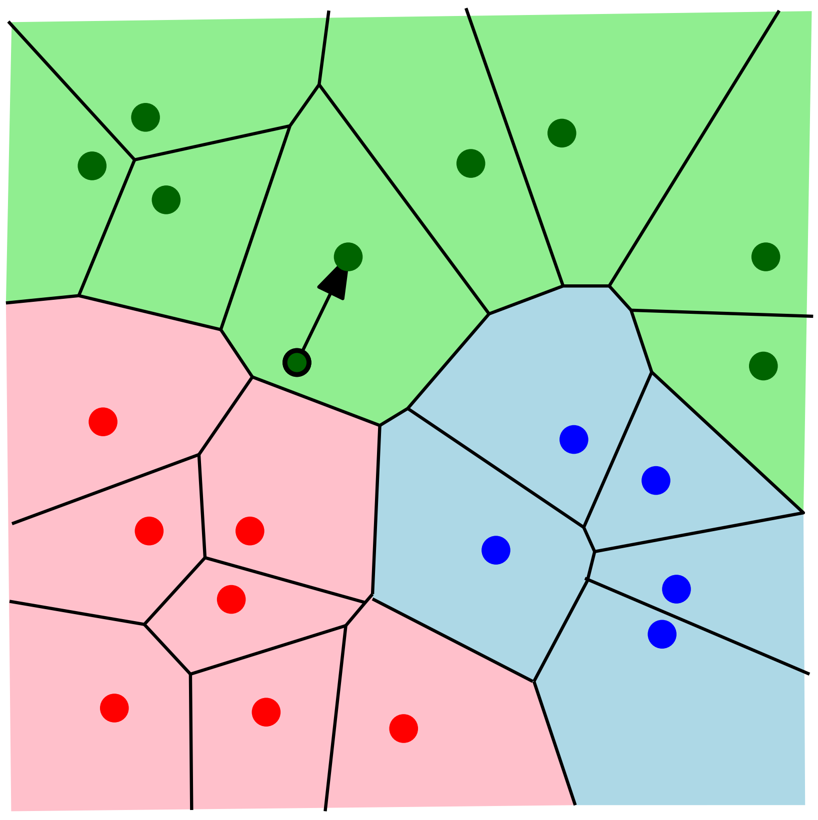

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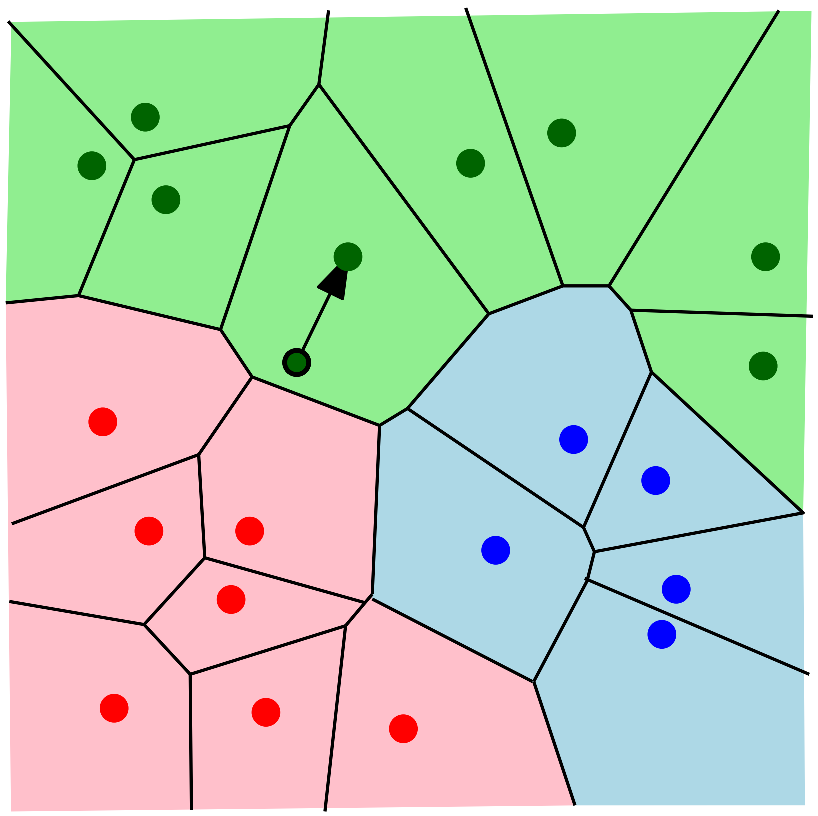

This induces a Voronoi partition with exponential growth in complexity

Can we use a random partition instead?

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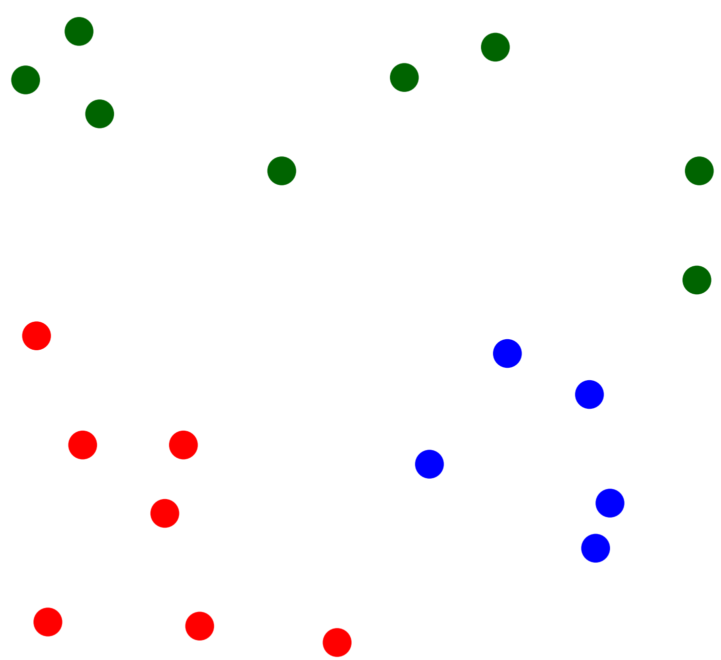

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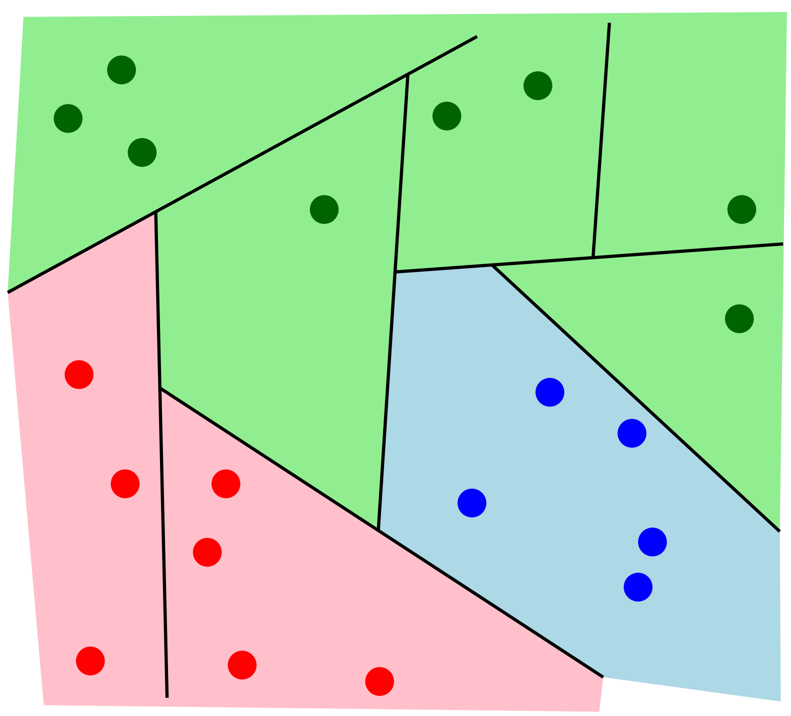

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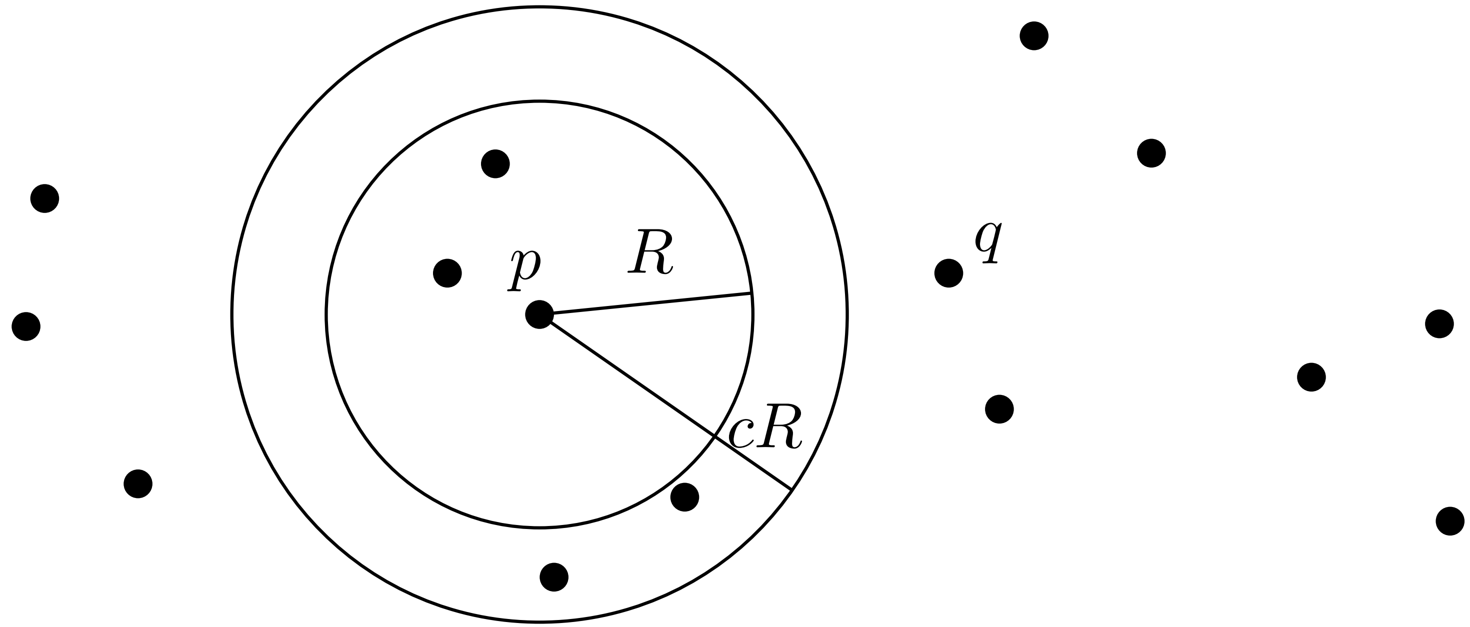
Can we use a random partition instead?

Locality-sensitive hashing (LSH)

Definition:
A family of hash functions H is called (R, cR, P_1, P_2)-locality-sensitive if for $p, q \in \mathbb{R}^d$:

$$(a) \quad \text{if } \mathrm{d}(p,q) \leq R \quad \text{then} \quad \Pr\left[h(p) = h(q)\right] \geq P_1$$

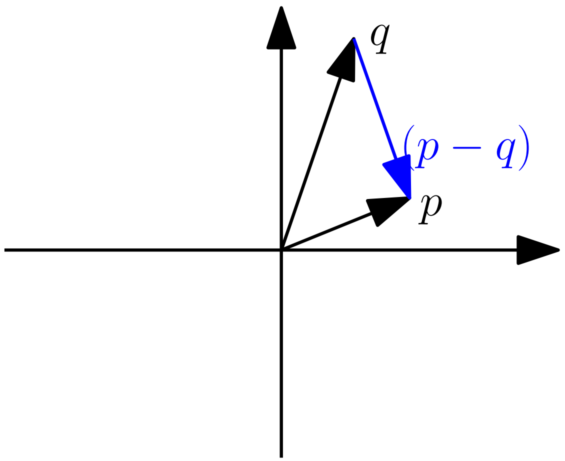
$$(b) \quad \text{if } \mathrm{d}(p,q) \geq cR \quad \text{then} \quad \Pr\left[h(p) = h(q)\right] \leq P_2$$



Commonly used distance functions

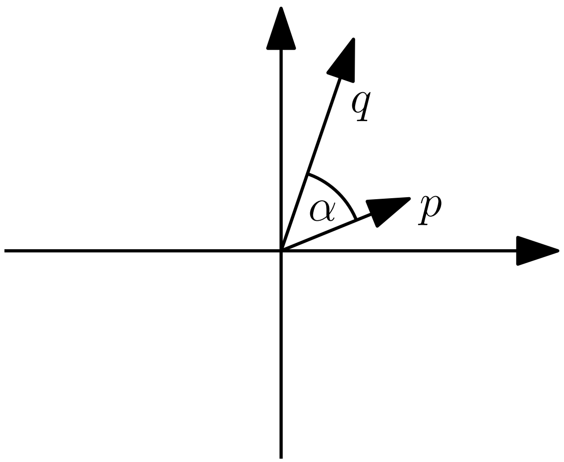
Euclidean distance:
for $p = (x_1, \ldots, x_d)$ and $q = (y_1, \ldots, y_d)$

$$\mathrm{d}(p,q) := \|p-q\| = \sqrt{\sum_{i=1}^{d}(x_i - y_i)^2}$$

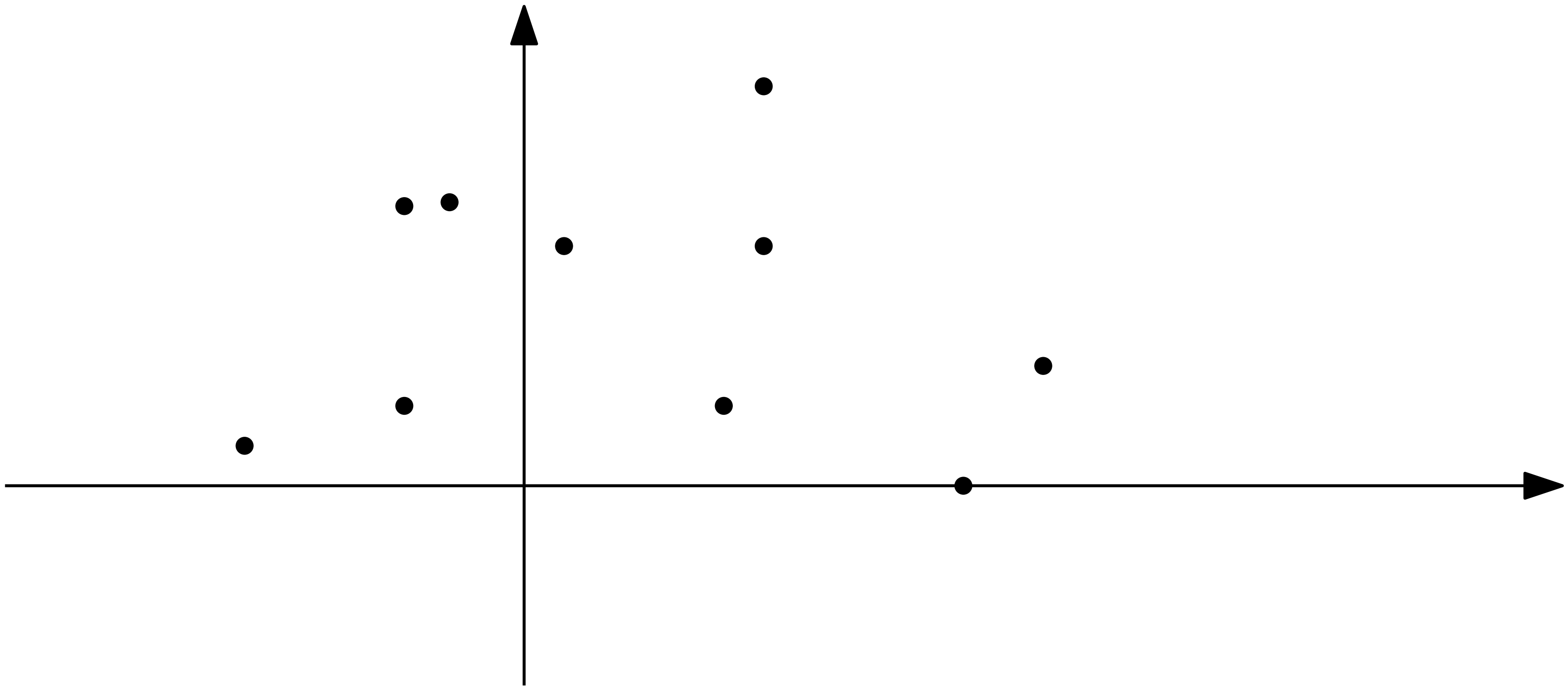


Arccos distance:
for $p = (x_1, \ldots, x_d)$ and $q = (y_1, \ldots, y_d)$

$$\mathrm{d}(p,q) := \arccos\left(\frac{p \cdot q}{\|p\| \; \|q\|}\right)$$



Locality sensitive hashing: Arccos distance



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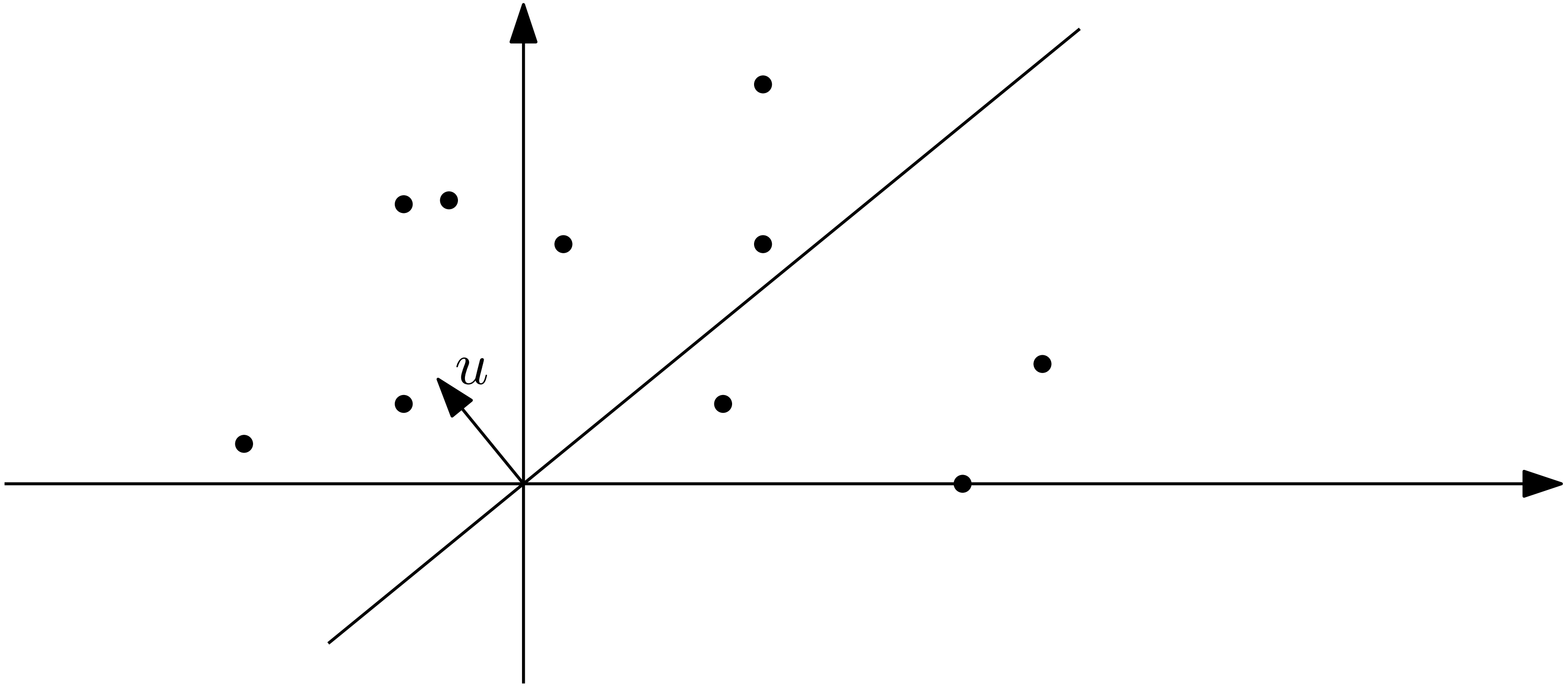
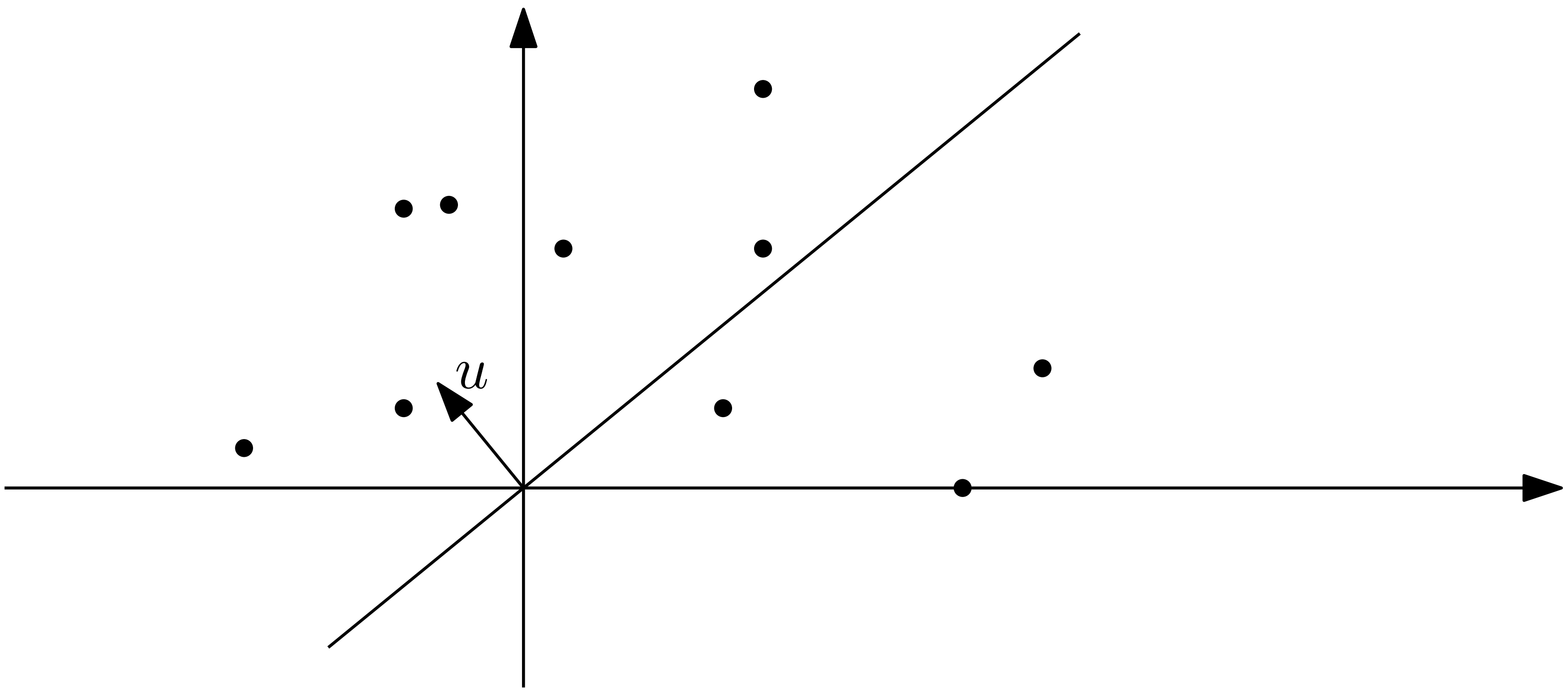


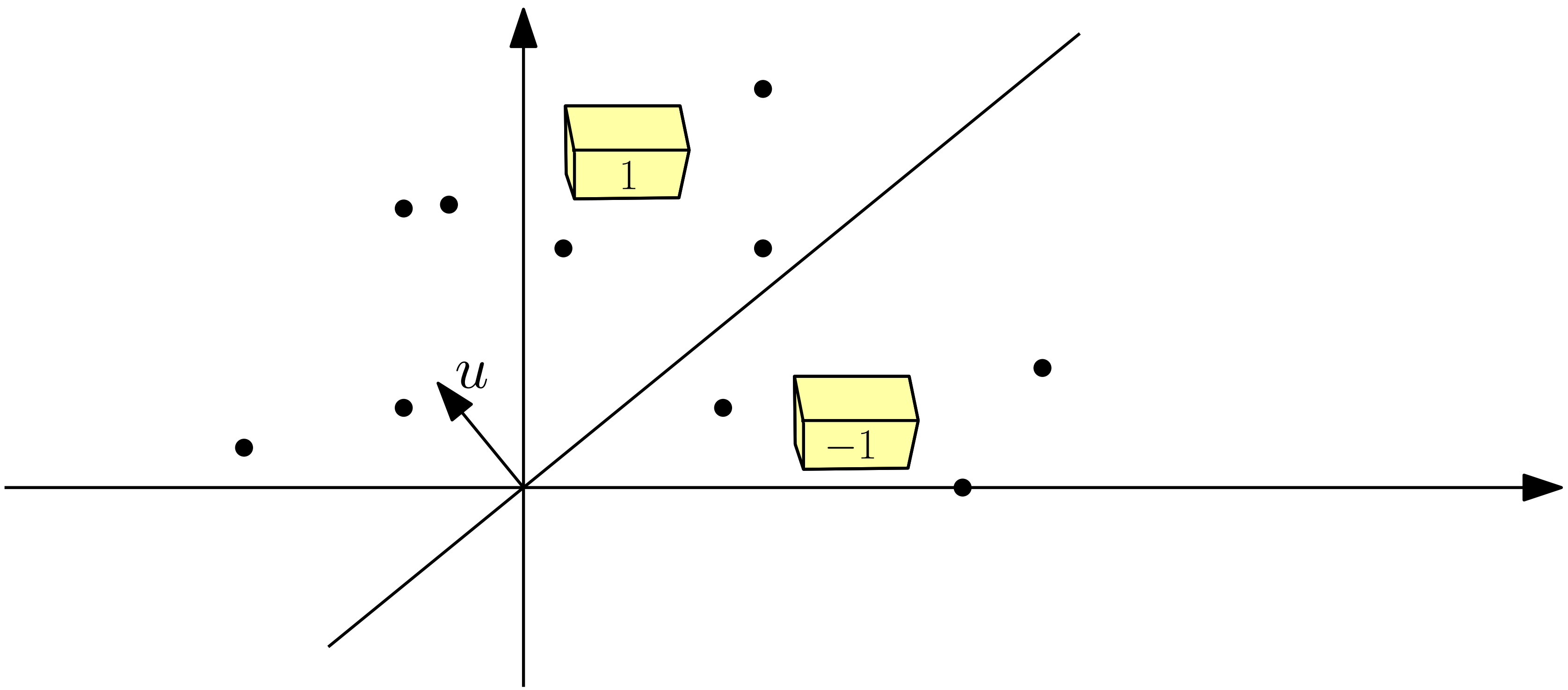
- randomly sample a hyperplane by choosing a normal vector u

Locality sensitive hashing: Arccos distance



- randomly sample a hyperplane by choosing a normal vector u
- for each p_i compute the sign of $p_i \cdot u$ to find the side of the hyperplane thay p_i lies on

Locality sensitive hashing: Arccos distance



- randomly sample a hyperplane by choosing a normal vector u
- for each p_i compute the sign of $p_i \cdot u$ to find the side of the hyperplane thay p_i lies on
- $h(p_i) = \text{sign}(p_i \cdot u)$

Locality sensitive hashing: Arccos distance

Claim:

For any p_i, p_j, it holds that

$$\Pr\left[h(p_i) = h(p_j)\right] = \frac{2\pi - \alpha}{2\pi}$$

where $\alpha = \arccos\left(\frac{p_i \cdot p_j}{\|p_i\| \ \|p_j\|}\right)$.

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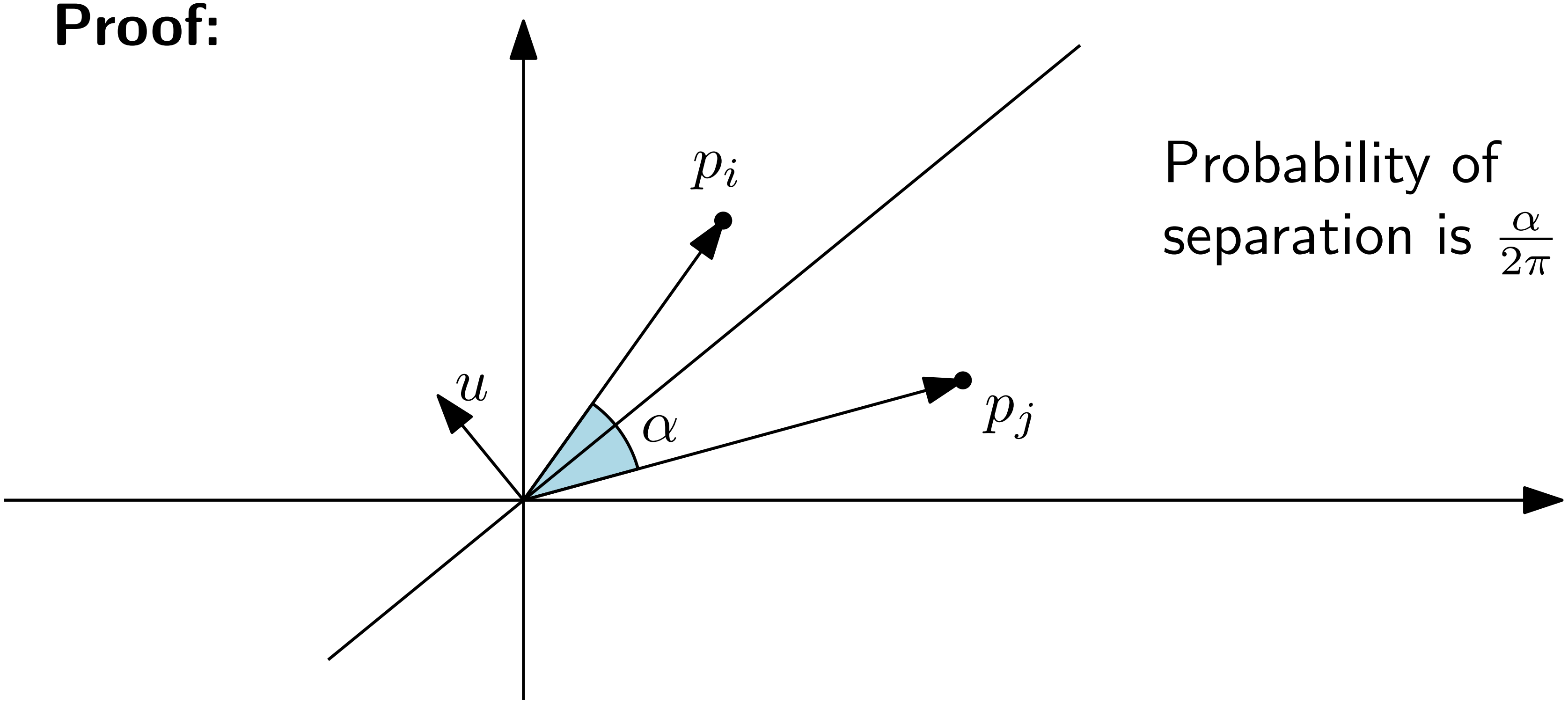
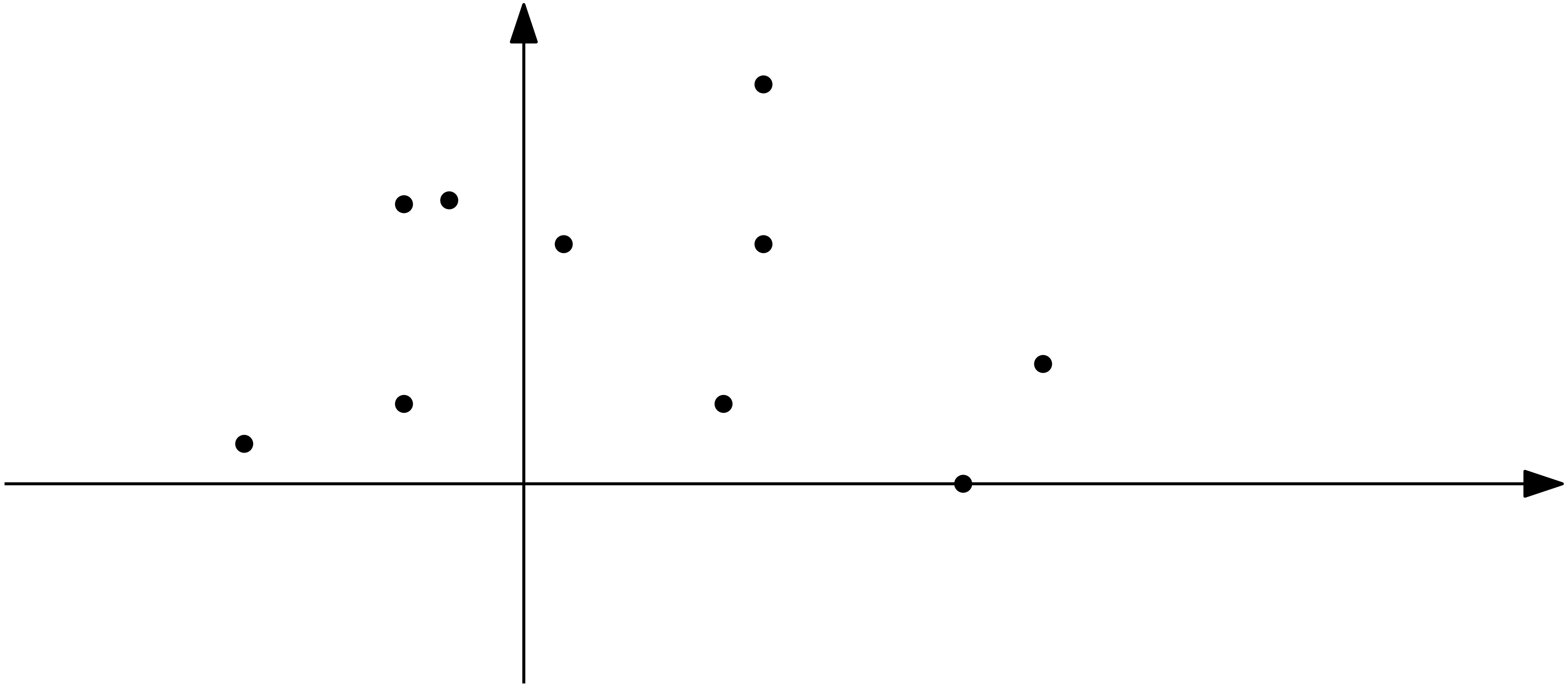
Proof:



Probability of separation is $\frac{\alpha}{2\pi}$

Locality-sensitive hashing: Euclidean distance



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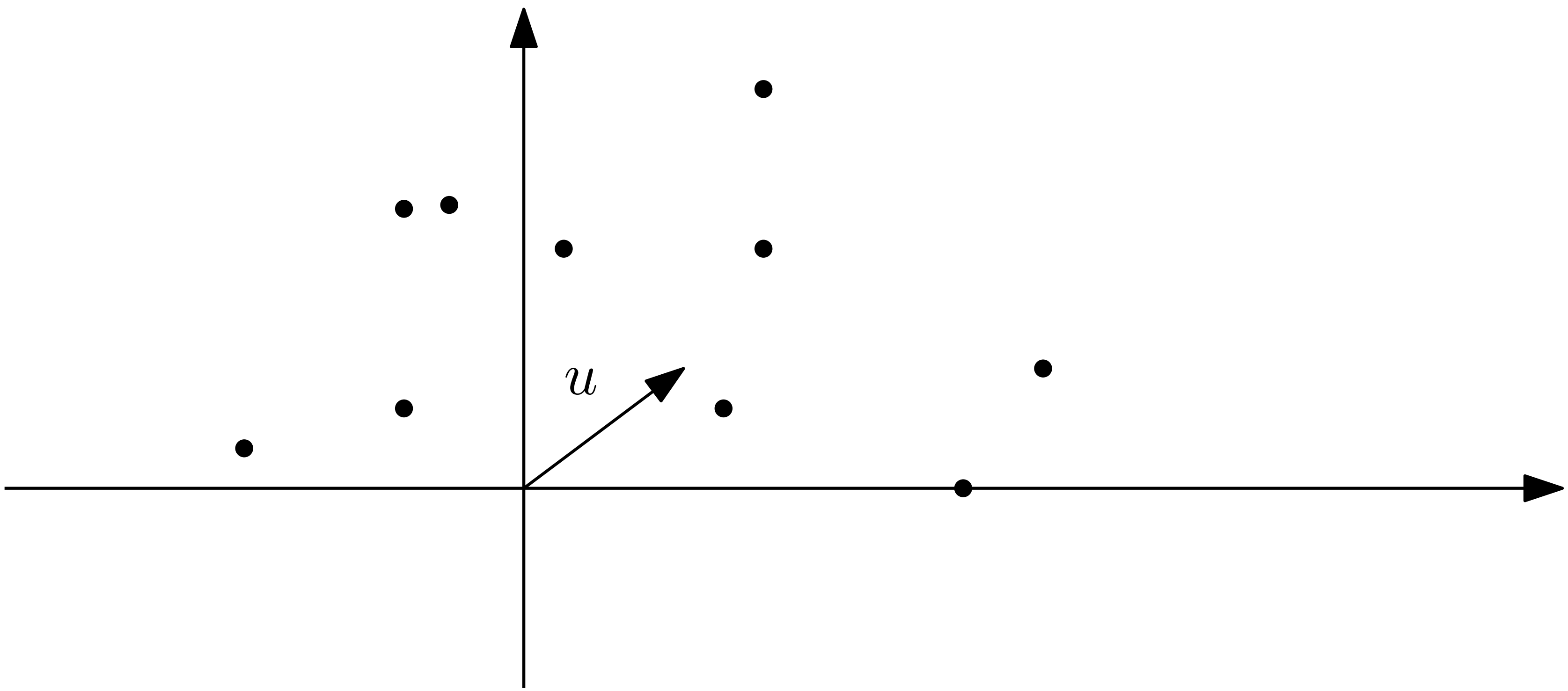

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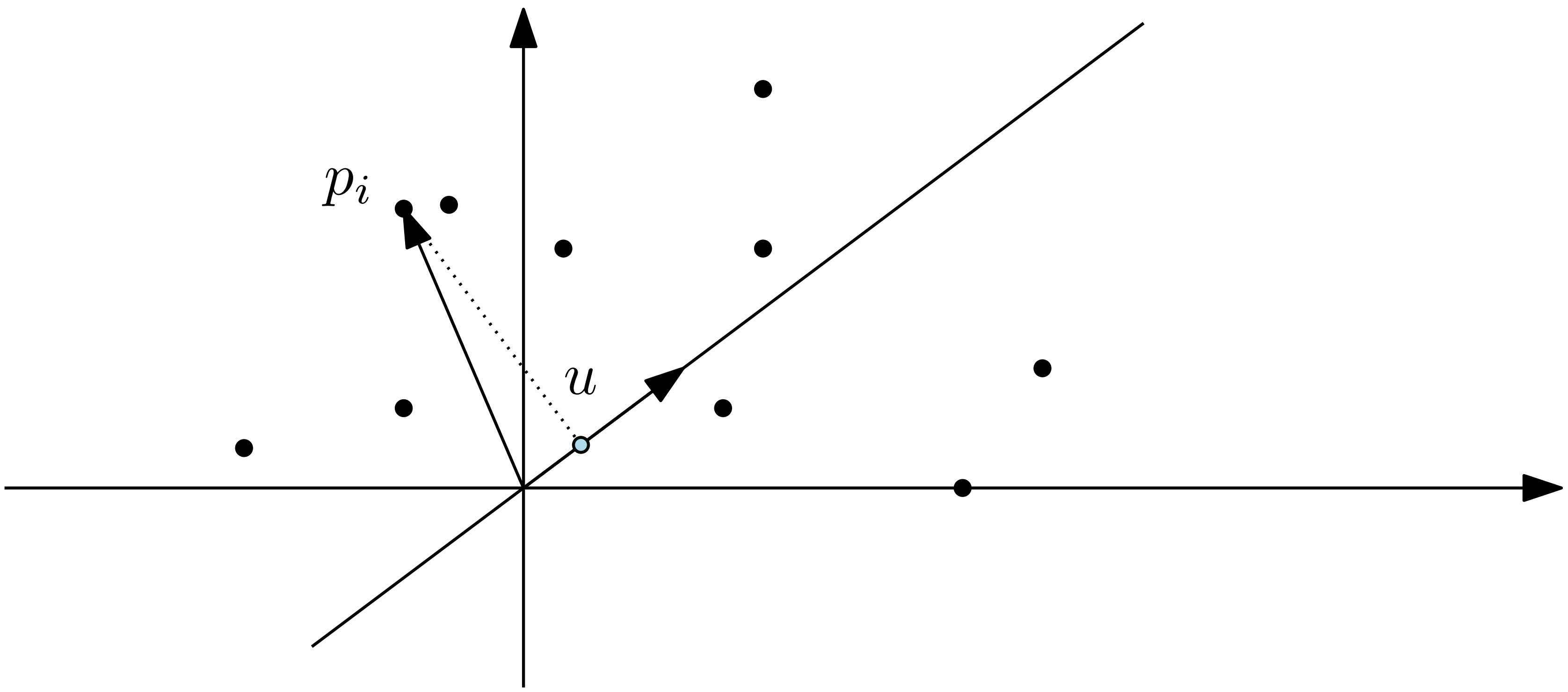

- randomly sample a unit vector u
- project onto u by computing the dot product $p_i \cdot u$

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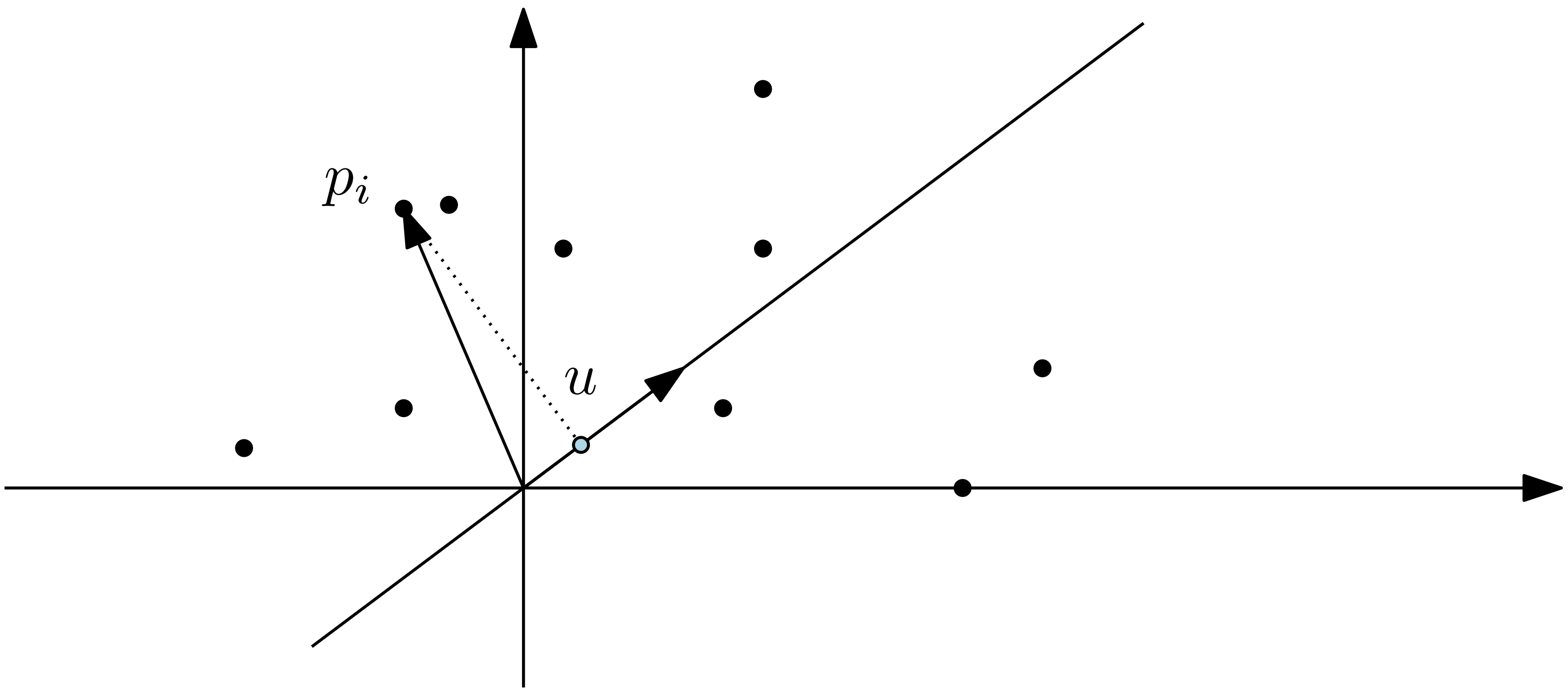

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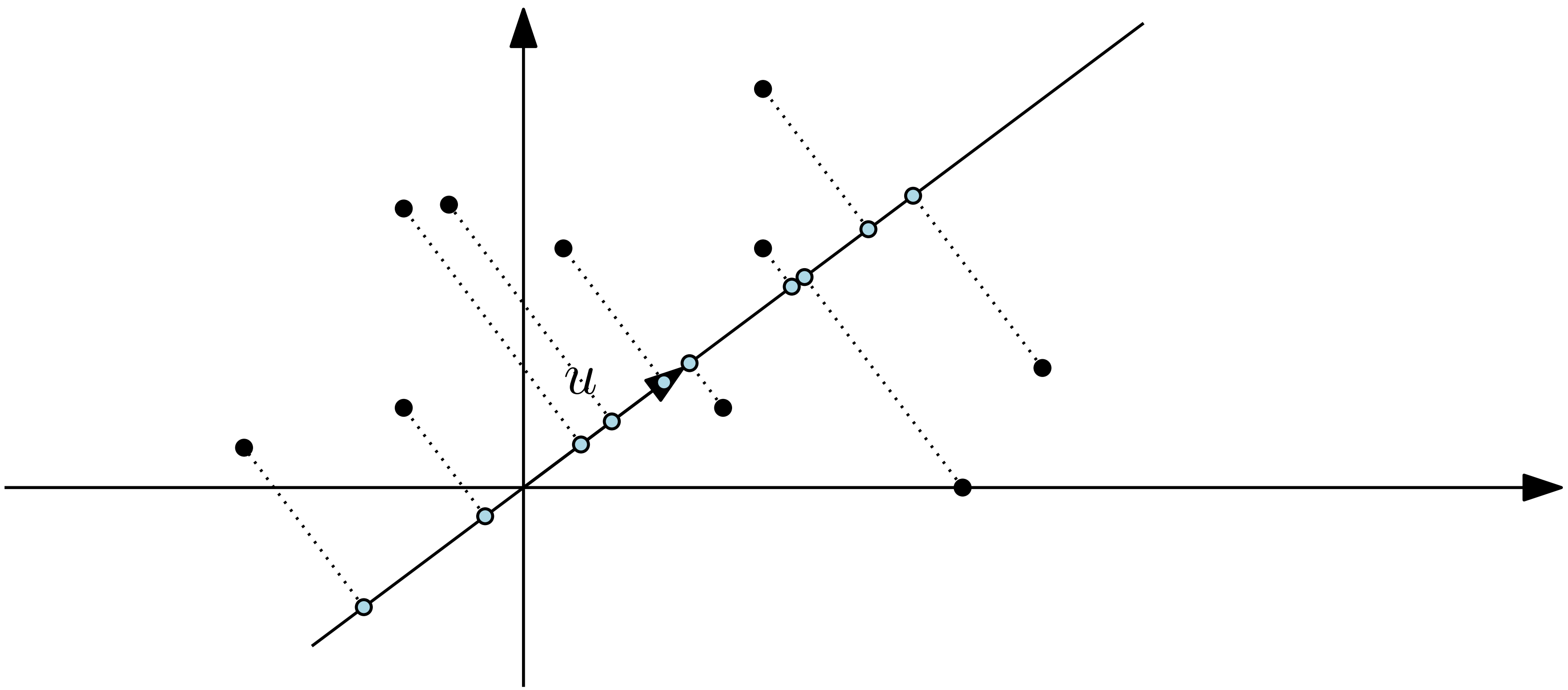

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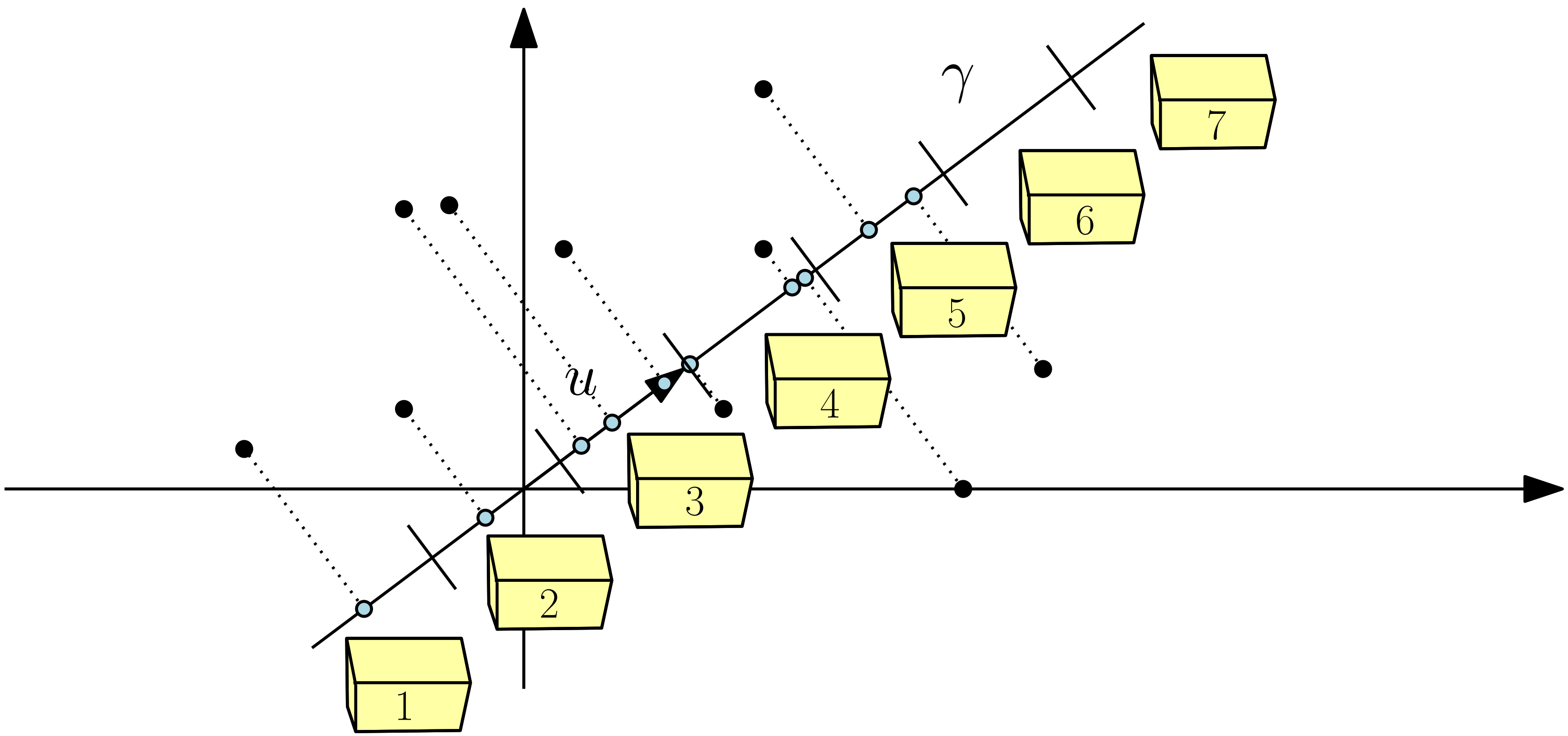

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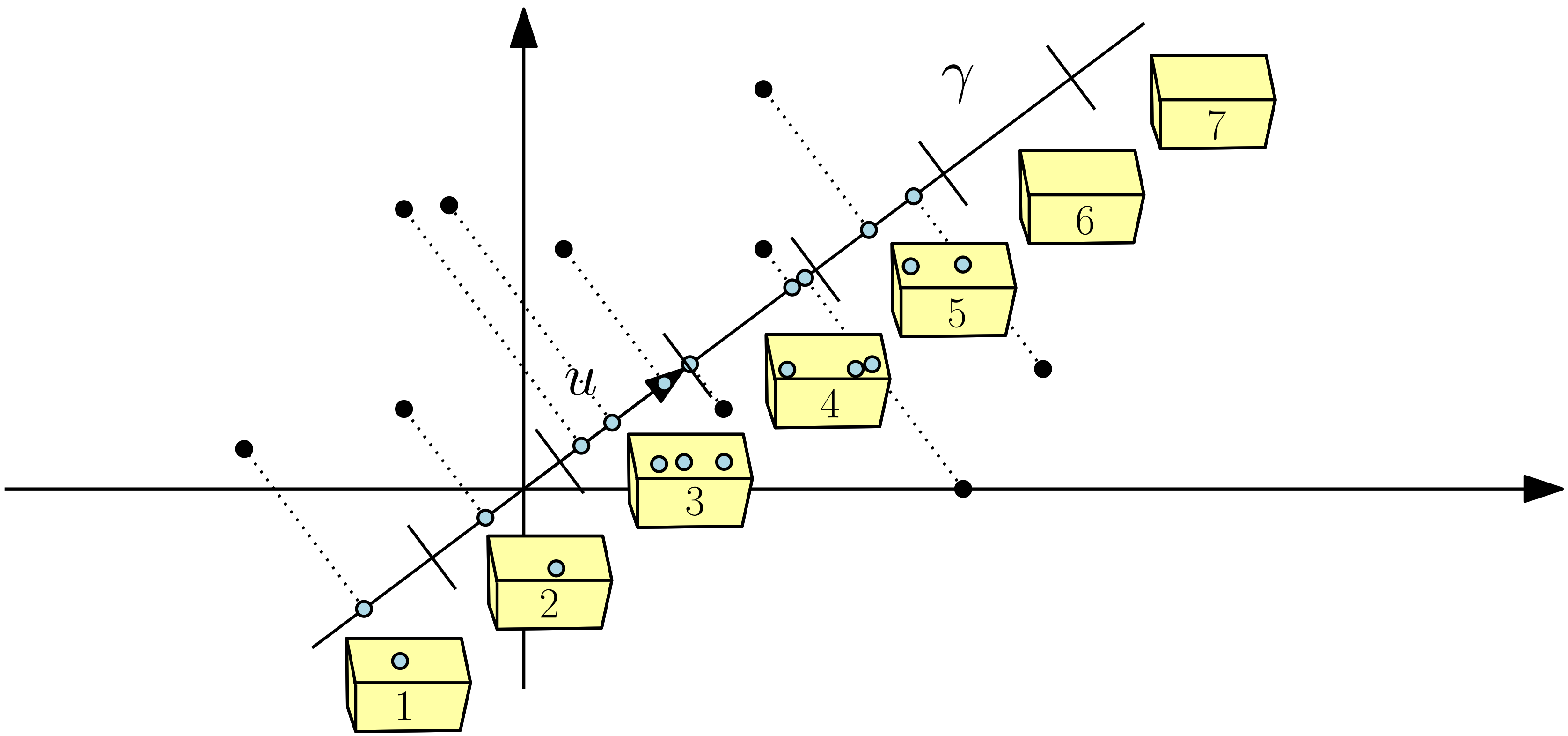

- randomly sample a unit vector u
- project onto u by computing the dot product $p_i \cdot u$
- create bins of size γ in $\mathbb{R}^1$ with random shift in $[0, \gamma)$
- $h(p) =$ index of the bin that p_i is projected into

Locality-sensitive hashing: Euclidean distance

Claim:

This hashing scheme is locality-sensitive with

(a) if $\|p_i - p_j\| \leq \frac{\gamma}{2}$ then $\Pr[h(p_i) = h(p_j)] \geq \frac{1}{2}$, and

(b) if $\|p_i - p_j\| \geq 2\gamma$ then $\Pr[h(p_i) = h(p_j)] \leq \frac{1}{3}$

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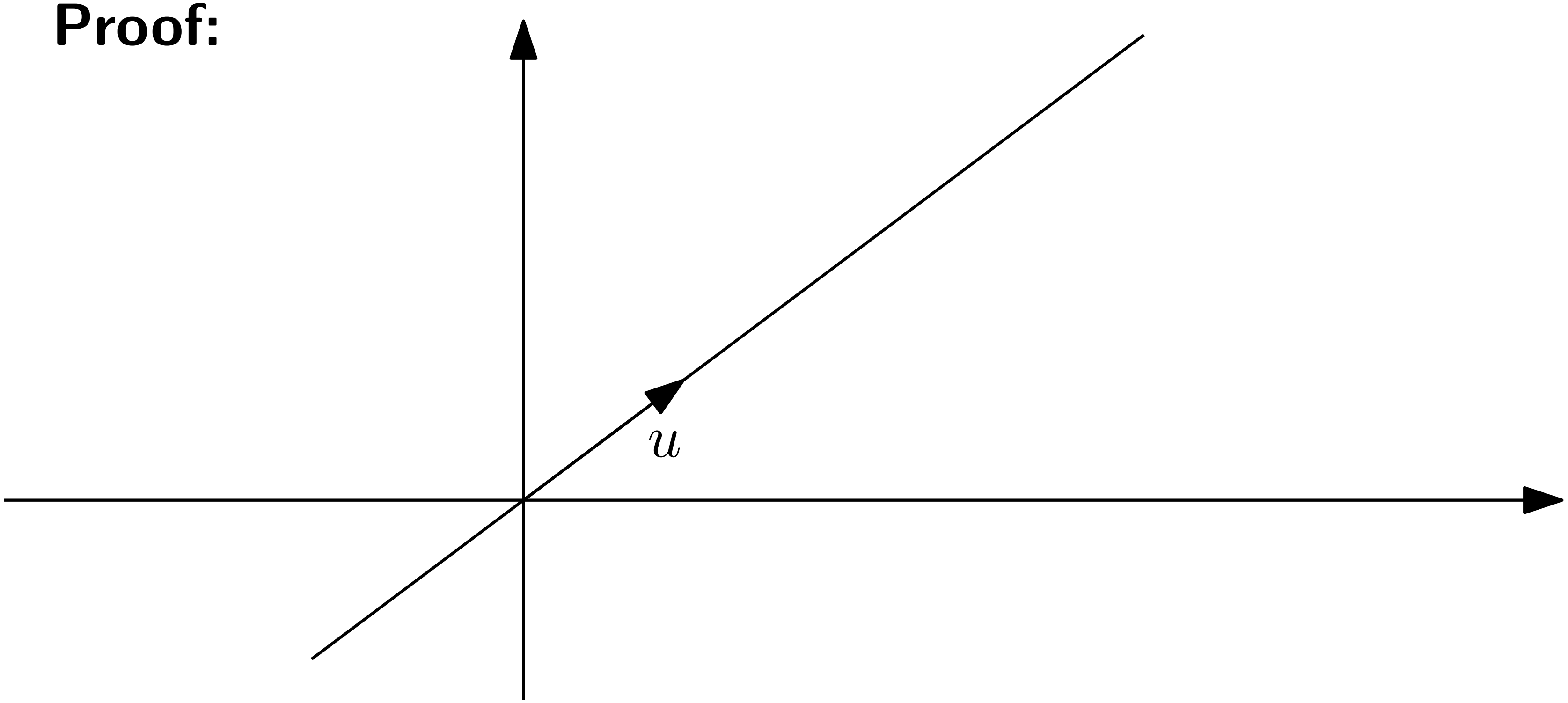

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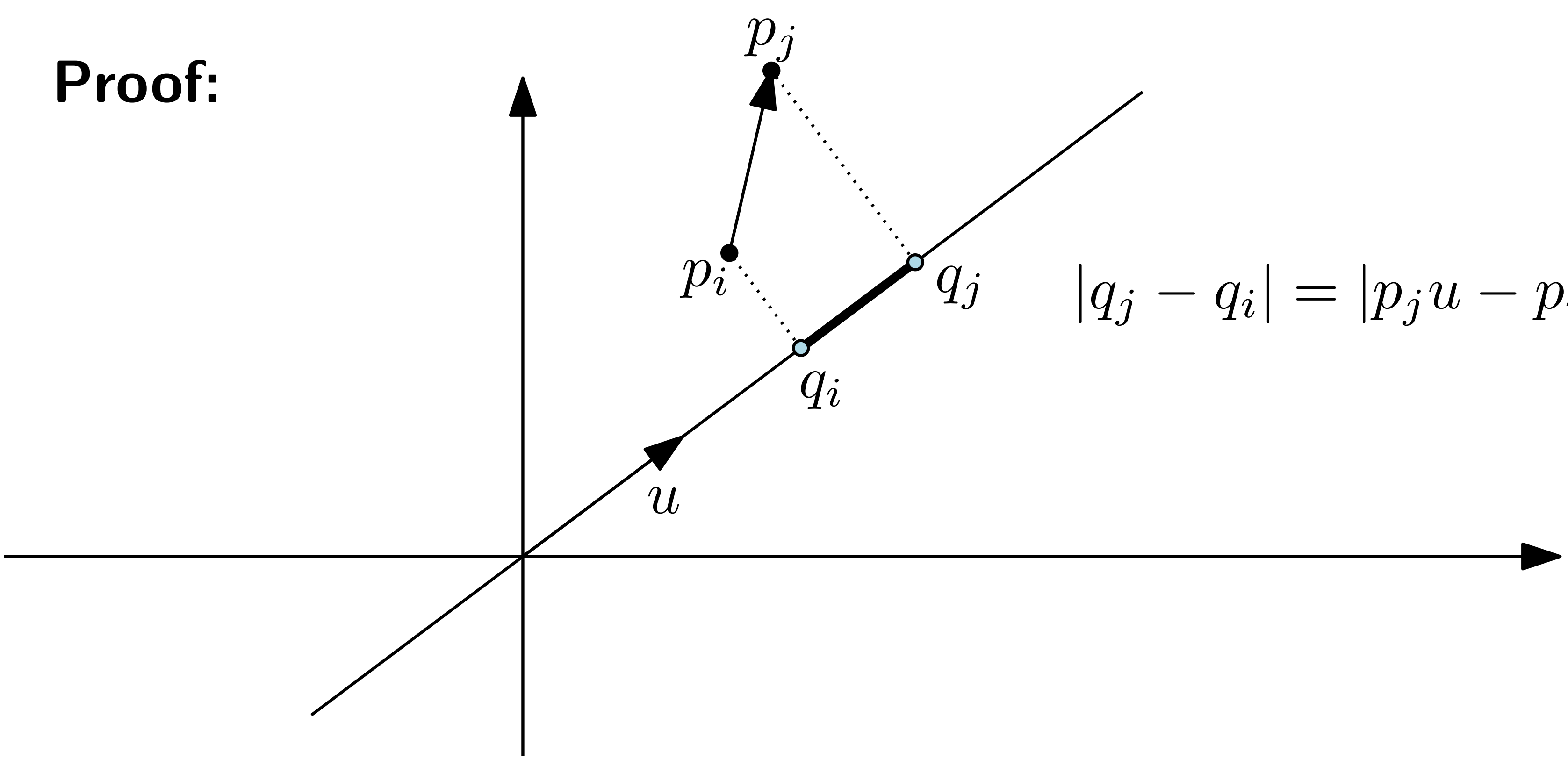

$$|q_j - q_i| = |p_j u - p_i u|$$

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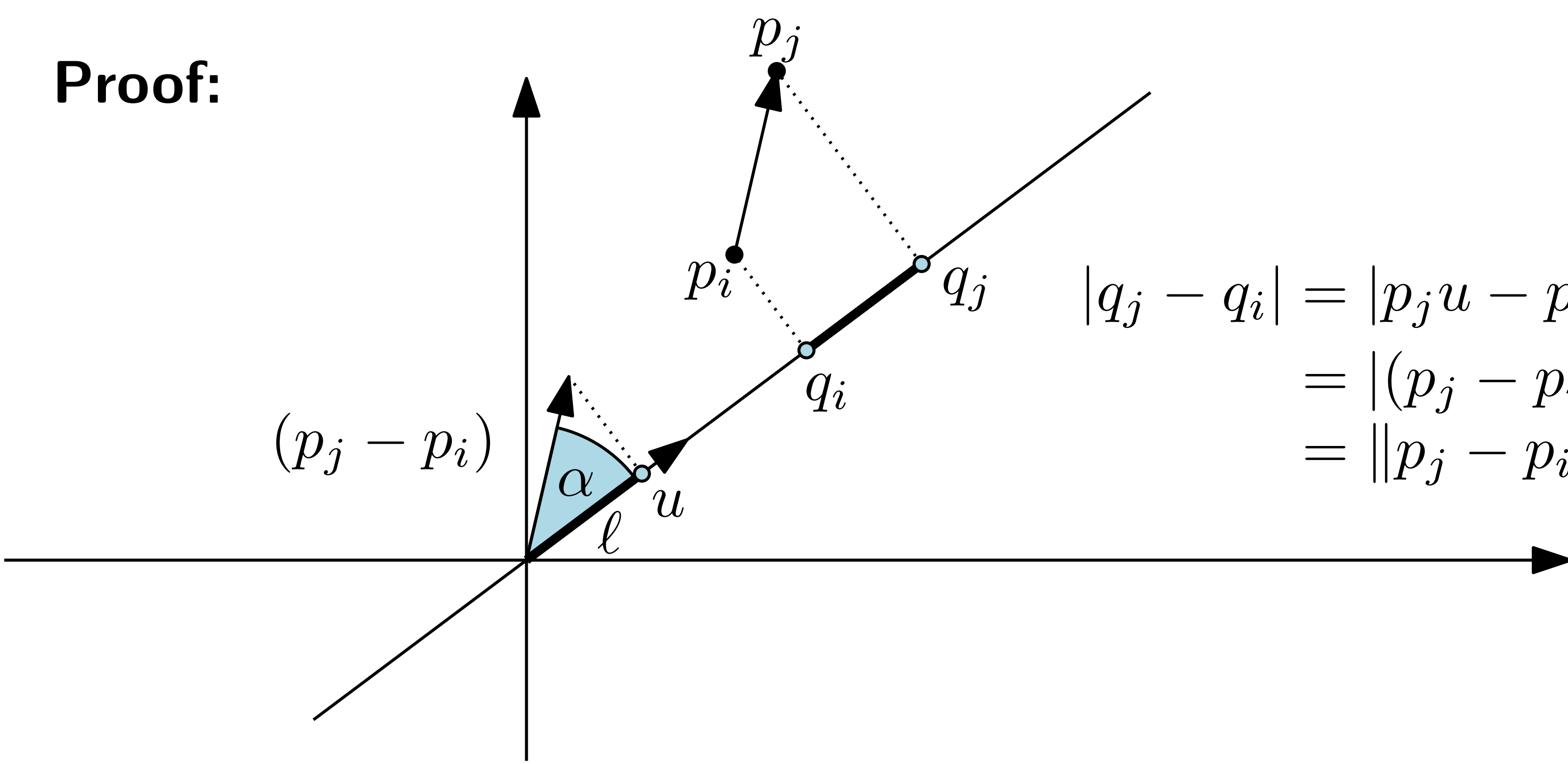

$$\begin{aligned} |q_j - q_i| &= |p_j u - p_i u| \\ &= |(p_j - p_i)u| \\ &= \|p_j - p_i\| |\cos(\alpha)| \end{aligned}$$

Locality-sensitive hashing: Euclidean distance

(Continued)

Proof of (a) "near points have high collision probability"

The probability of separation is

$$\Pr\left[h(p_i) \neq h(q_j)\right] = \frac{|q_j - q_i|}{\gamma}$$

Locality-sensitive hashing: Euclidean distance

(Continued)

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Locality-sensitive hashing: Euclidean distance

(Continued)

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Locality-sensitive hashing: Euclidean distance

(Continued)

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&= \frac{\|p_j - p_i\| |\cos\alpha|}{\gamma} \\
&\leq \frac{\|p_j - p_i\|}{\gamma} \qquad \text{(since } |\cos\alpha| \leq 1\text{)} \\
&\leq \frac{\gamma/2}{\gamma} = \frac{1}{2}
\end{aligned}$$

Locality-sensitive hashing: Euclidean distance

(Continued)

Proof of (b) "far points have low collision probability"

If $h(p_j) = h(p_i)$ then

$$\gamma \geq |q_j - q_i|$$

Locality-sensitive hashing: Euclidean distance

(Continued)

Proof of (b) "far points have low collision probability"

If $h(p_j) = h(p_i)$ then

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Locality-sensitive hashing: Euclidean distance

(Continued)

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Locality-sensitive hashing: Euclidean distance

(Continued)

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Locality-sensitive hashing: Euclidean distance

(Continued)

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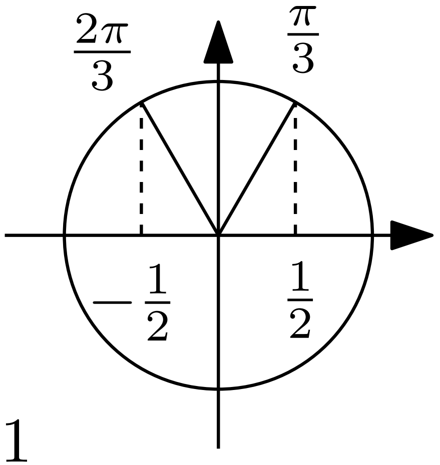

This implies

$$|\cos\alpha| \leq \tfrac{1}{2}$$

Since α is uniformly random in $(0, \pi)$, we have

$$\Pr\left[|\cos\alpha| \leq \frac{1}{2}\right] = \Pr\left[\alpha \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]\right] = \frac{1}{3}$$

Locality-sensitive hashing

A family of hash functions H is called (R, cR, P_1, P_2)-locality-sensitive if for $p, q \in \mathbb{R}^d$:

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We saw

- $(\frac{\gamma}{2}, 2\gamma, \frac{1}{2}, \frac{1}{3})$-locality-sensitive hashing scheme for the Euclidean distance
- $(\alpha_1, \alpha_2, \frac{2\pi - \alpha_1}{2\pi}, \frac{2\pi - \alpha_2}{2\pi})$-locality-sensitive hashing scheme for the Arccos distance

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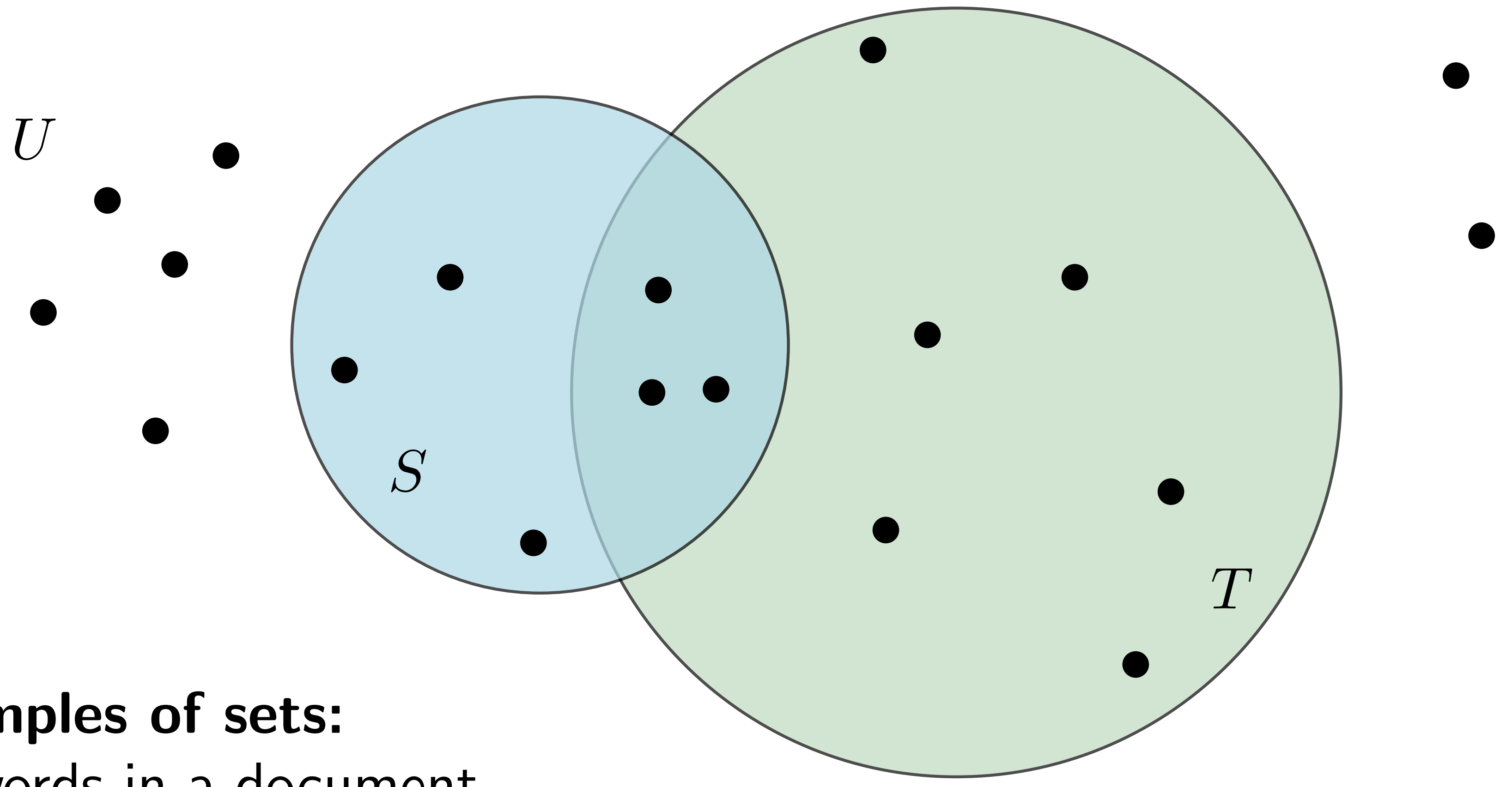
What about other distance measures?

Jaccard Similarity

Similarity function to compare sets.



Examples of sets:

- words in a document
- products in a shopping basket
- movies liked by a person

Definition:

$$\mathrm{sim}_{\mathcal{J}}(S,T) := \frac{|S \cap T|}{|S \cup T|}$$

Jaccard Similarity

We represent the sets $S, T \subseteq U$ using indicator vectors.

Example

- Given set of documents $D_1, \ldots, D_n$.
- Let S_i be the set of words contained in D_i
- Indicator vector for S_i is a $(0,1)$-vector over the dictionary U

$$v_i = (\ldots, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, \ldots)$$

'cat'

'catastrophy'

'caterpillar' $\in S_i$

'category' $\notin S_i$

Jaccard Similarity – Minhashing

Minhashing for estimating the Jaccard similarity

Characteristic matrix with indicator vectors as columns

	S_1	S_2	S_3	S_4
a	1	0	0	1
b	0	0	1	0
c	0	1	0	1
d	1	0	1	1
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Minhashing for estimating the Jaccard similarity

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randomly permute rows

	S_1	S_2	S_3	S_4
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a	1	0	0	1
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	S_1	S_2	S_3	S_4
b	0	0	$\boxed{1}$	0
a	$\boxed{1}$	0	0	$\boxed{1}$
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Minhash $h(S_i)$ is the index of first row from the top which has a 1

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Minhashing for estimating the Jaccard similarity

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Minhash $h(S_i)$ is the index of first row from the top which has a 1

Claim:

$$\Pr\left[h(S_i) = h(S_j)\right] = \mathrm{sim}_{\mathcal{J}}(S_i, S_j)$$

Jaccard Similarity – Minhashing

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Jaccard Similarity – Minhashing

Claim: $\Pr\left[h(S_i) = h(S_j)\right] = \mathrm{sim}_{\mathcal{J}}(S_i, S_j)$

Is it true for S_1 and S_2?

	S_1	S_2	S_3	S_4
b	0	0	1	0
a	1	0	0	1
c	0	1	0	1
e	0	0	1	0
d	1	0	1	1

Jaccard Similarity – Minhashing

Claim: $\Pr[h(S_i) = h(S_j)] = \text{sim}_{\mathcal{J}}(S_i, S_j)$

Is it true for S_1 and S_2?

$\text{sim}_{\mathcal{J}}(S_1, S_2) = 0$

$\Pr[h(S_1) = h(S_2)] = 0$

	S_1	S_2	S_3	S_4
b	0	0	1	0
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$\Pr\left[h(S_3) = h(S_4)\right] = \frac{1}{5}$

	S_1	S_2	S_3	S_4
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c	0	1	0	1
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Proof:

$x := |S_i \cap S_j|$ (i.e., number of $(1,1)$ rows)

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$x := |S_i \cap S_j|$ (i.e., number of $(1,1)$ rows)

$y := |S_i \cup S_j| - |S_i \cap S_j|$ (i.e., number of $(0,1)$ and $(1,0)$ rows)

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	S_1	S_2	S_3	S_4
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Proof:

$x := |S_i \cap S_j|$ (i.e., number of $(1,1)$ rows)
$y := |S_i \cup S_j| - |S_i \cap S_j|$ (i.e., number of $(0,1)$ and $(1,0)$ rows)

$$\mathrm{sim}_{\mathcal{J}}(S_i, S_j) = \frac{x}{x+y} = Pr\left[h(S_i) = h(S_j)\right]$$

$\square$

Jaccard Similarity – Signature Matrix

Now repeat and create hash functions $h_1, h_2, \ldots, h_m$

	S_1	S_2	S_3	S_4	h_1	h_2	$\ldots$
0	1	0	0	1	1	1	$\vdots$
1	0	0	1	0	2	4	
2	0	1	0	1	3	2	
3	1	0	1	1	4	0	
4	0	0	1	0	0	3	

For each set we obtain a **minhash signature**:

	S_1	S_2	S_3	S_4
h_1 :				
h_2 :				
$\vdots$				

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	S_1	S_2	S_3	S_4	h_1	h_2	$\ldots$
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0	1	0	0	1	1	1	$\vdots$
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	S_1	S_2	S_3	S_4
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0	1	0	0	1	1	1	$\vdots$
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	S_1	S_2	S_3	S_4
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	S_1	S_2	S_3	S_4	h_1	h_2	$\ldots$
0	1	0	0	1	1	1	$\vdots$
1	0	0	1	0	2	4	
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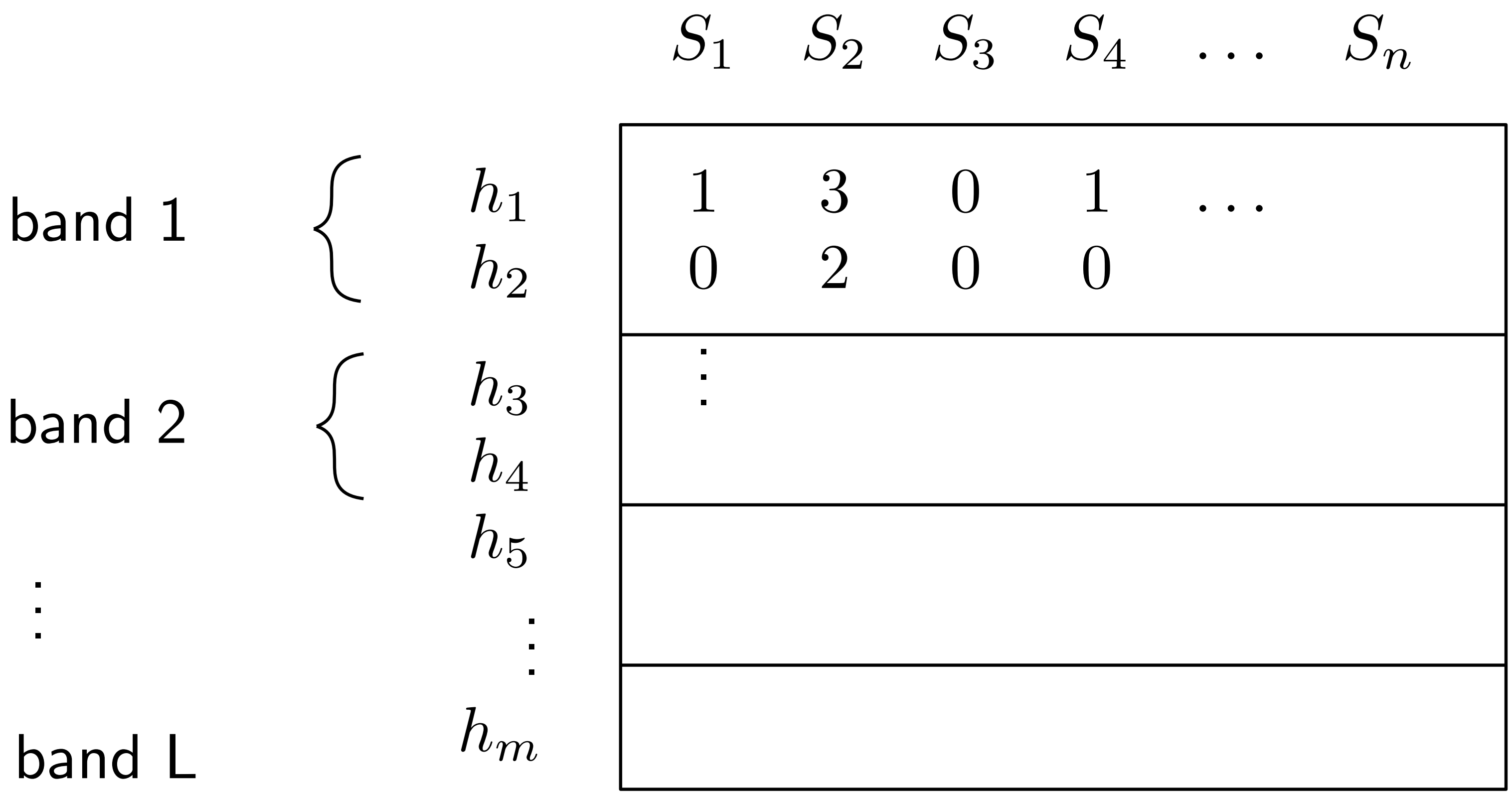
For each set we obtain a **minhash signature**:

	S_1	S_2	S_3	S_4
h_1 :	1	3	0	1
h_2 :	0	2	0	0
$\vdots$				

minhash signature of S_2 is $(3, 2)$

Jaccard Similarity – Banding technique



Divide the rows of the signature matrix into bands of size k

Jaccard Similarity – Banding technique

		S_1	S_2	S_3	S_4	$\dots$	S_n
band 1	h_1	1	3	0	1	$\dots$	
	h_2	0	2	0	0		
band 2	h_3	$\vdots$					
	h_4						
	h_5						
$\vdots$	$\vdots$						
band L	h_m						

Divide the rows of the signature matrix into bands of size k

If S_i and S_j have equal minhash signature within some band, we consider them as **candidates**

Jaccard Similarity – Banding technique

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What is the probability of two sets becoming candidates?

Jaccard Similarity – Banding technique

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What is the probability of two sets becoming candidates?

Let $s = \mathrm{sim}_{\mathcal{J}}(S_i, S_j)$

Event	**Probability**
They agree in all rows of a particular band:	
They do not agree in a particular band:	
They do not agree in any of the bands:	
They become candidates:	

Jaccard Similarity – Banding technique

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What is the probability of two sets becoming candidates?

Let $s = \mathrm{sim}_{\mathcal{J}}(S_i, S_j)$

Event	**Probability**
They agree in all rows of a particular band:	s^k
They do not agree in a particular band:	
They do not agree in any of the bands:	
They become candidates:	

Jaccard Similarity – Banding technique

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What is the probability of two sets becoming candidates?

Let $s = \mathrm{sim}_{\mathcal{J}}(S_i, S_j)$

Event	**Probability**
They agree in all rows of a particular band:	s^k
They do not agree in a particular band:	$1 - s^k$
They do not agree in any of the bands:	
They become candidates:	

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Let $s = \mathrm{sim}_{\mathcal{J}}(S_i, S_j)$

Event	**Probability**
They agree in all rows of a particular band:	s^k
They do not agree in a particular band:	$1 - s^k$
They do not agree in any of the bands:	$(1 - s^k)^L$
They become candidates:	

Jaccard Similarity – Banding technique

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What is the probability of two sets becoming candidates?

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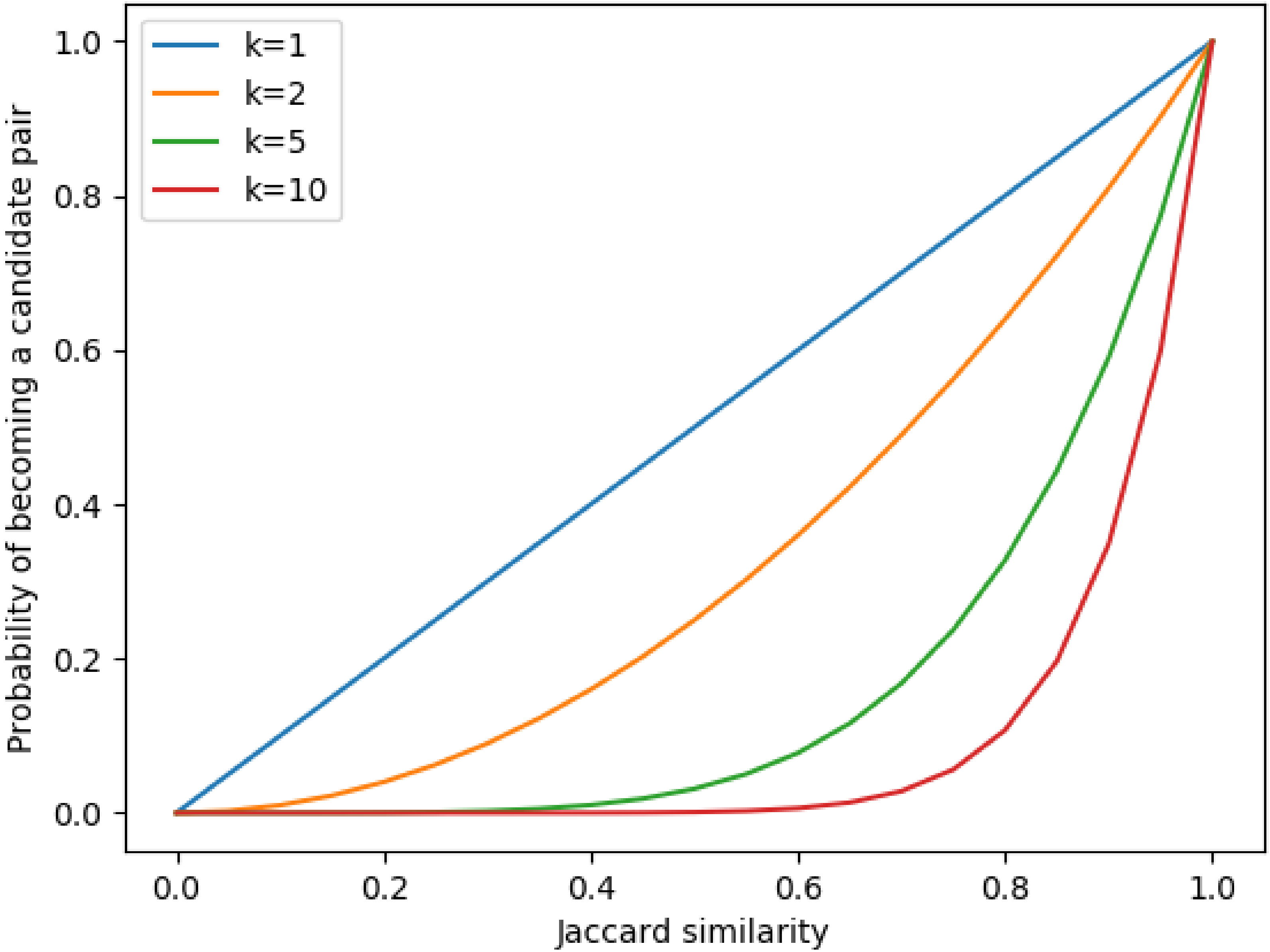
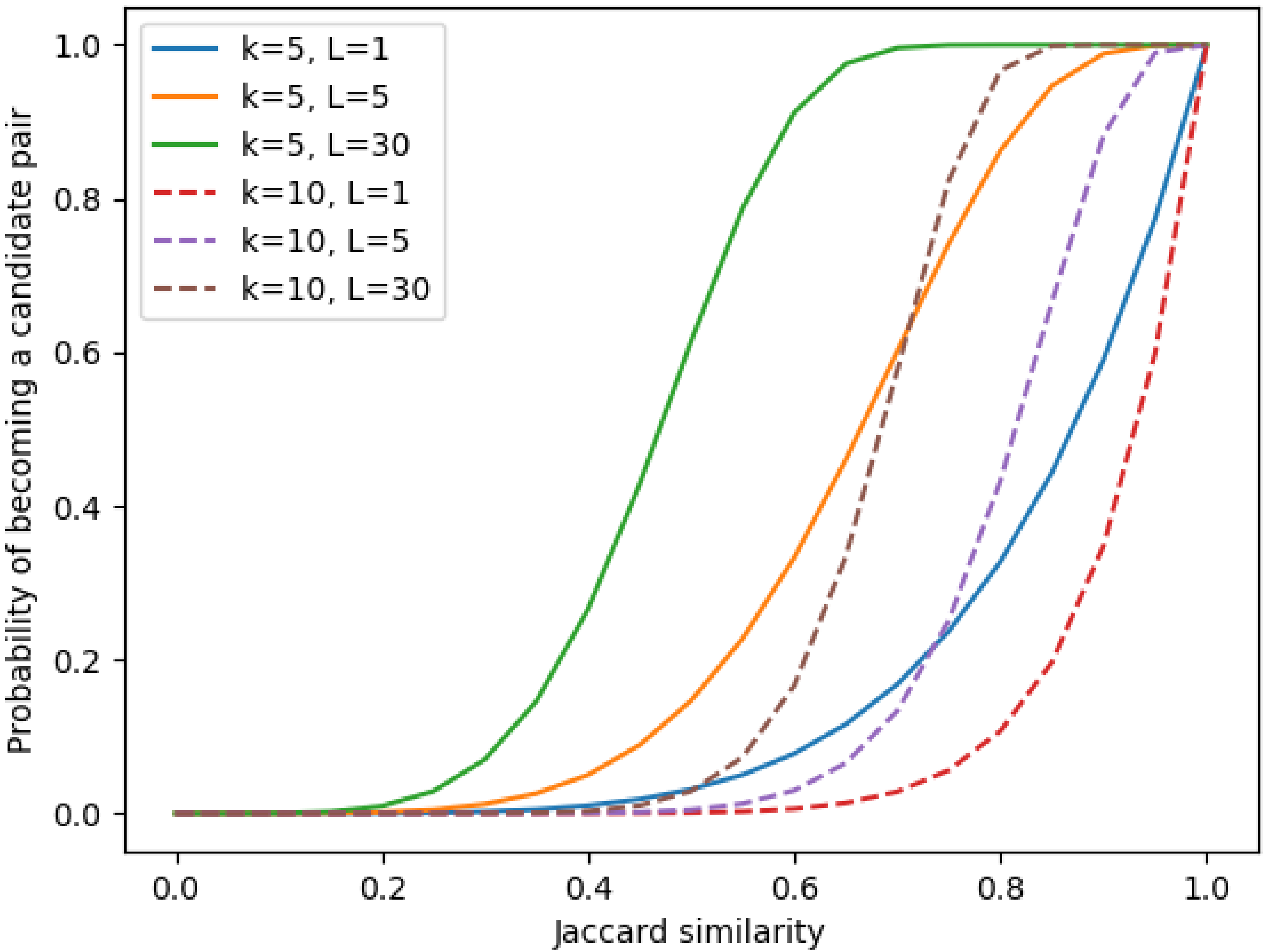
Event	**Probability**
They agree in all rows of a particular band:	s^k
They do not agree in a particular band:	$1 - s^k$
They do not agree in any of the bands:	$(1 - s^k)^L$
They become candidates:	$1 - (1 - s^k)^L$

Jaccard Similarity – Banding technique



Jaccard Similarity – Banding technique



Amplification of an LSH

In general, this process is called **amplification** (we "amplify" the success probabilities)

Let H be a (d_1, d_2, p_1, p_2)-sensitive family of hash functions

AND-construction:

$g(x) = g(y)$ if and only if $h_i(x) = h_i(y)$ for all $1 \leq i \leq r$

yields a (d_1, d_2, p_1^r, p_2^r)-sensitive family

OR-construction:

$g(x) = g(y)$ if and only if $h_i(x) = h_i(y)$ for some $1 \leq i \leq L$

yields a $(d_1, d_2, 1 - (1 - p_1)^L, 1 - (1 - p_2)^L)$-sensitive family

Summary

- Nearest-Neighbor rule
- Locality sensitive hashing
- Cosine distance
- Euclidean distance
- Jaccard Similarity
- Minhashing
- Banding
- Amplification

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