Origami Axioms through GeoGebra

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Abstract

The origami folds corresponding to the axioms are available as GeoGebra tools. Some well known examples using the tools are presented.

1 Introduction

The origami folds corresponding to the seven *Huzita* axioms, [3], [6], [4], are given as GeoGebra tools. Set of rules to develop origami constructions are given in [1], [2], too.

GeoGebra is one of the leading mathematical software for learning and teaching, [7]. We suppose that the reader is familiarized with GeoGebra.

After presenting the tools, several examples are given.

We use *GeoGebra* 5 (*GeoGebra* classic). With given data the lines of a script must be copied in turn to the *Input* line and executed. Instead of using the mouse to pick command from menus we prefer to edit / copy commands in the *Input* line. That's amore programmable style of using *GeoGebra.* It would be useful for *GeoGebra* to offer the possibility to upload and process a command file.

Our developed tools are available at `https://github.com/e-scheiber/origami.git`.

2 The origami GeoGebra tools

A tool is contained in a GeoGebra `.ggb` file and the arguments are GeoGebra points. The first two arguments correspond to the bottom side of an origami square piece of paper. So, in $Hi[A, B, \ldots]$, $i \in \{1, \ldots, 7\}$, A and B denotes the left and the right side ends of the bottom line of a square.

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Axiom 1

Given two points P_1 and P_2, there is a unique fold passing through both of them.

If $E = P_1$ anf $F = P_2$ are two points then $H1(A, B, E, F)$ draws the fold through the line EF, Fig. 1.

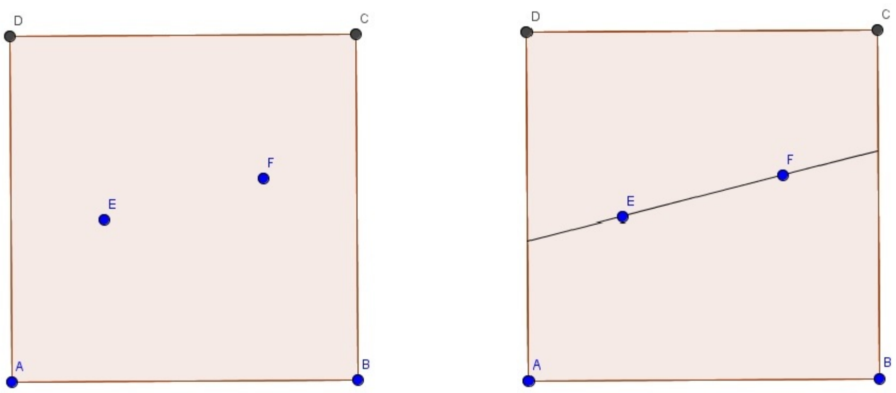

Fig. 1: Axiom 1.

The GeoGebra script is

```
p = polygon(A, B, 4)
G = intersect(line(E, F), p)
s = segment(G_1, G_2)
showlabel(p, false)
showlabel(s, false)
showlabel(G_1, false)
showlabel(G_2, false)
```

Axiom 2

Given two points P_1 and P_2, there is a unique fold placing P_1 onto P_2.

If $E = P_1$ anf $F = P_2$ are two points then $H2(A, B, E, F)$ draws the fold placing E onto F, Fig. 2.

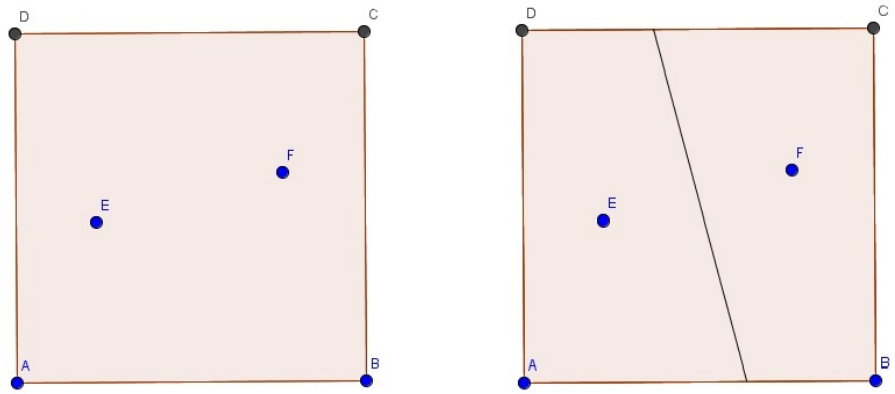

Fig. 2: Axiom 2.

The GeoGebra script is

1	$p = polygon(A, B, 4)$
2	$G = intersect(perpendicularbisector(E, F), p)$
3	$s = segment(G_1, G_2)$
4	$showlabel(p, false)$
5	$showlabel(s, false)$
6	$showlabel(G_1, false)$
7	$showlabel(G_2, false)$

Axiom 3

Given two lines L_1 and L_2 there is a fold placing L_1 onto L_2.

If $line(E, F) = L_1$ and $line(G, H) = L_2$ are the two lines defined by the points in the parentheses then $H3(A, B, E, F, G, H)$ draws the fold placing L_1 onto L_2, Fig. 3.

The programming problem is to find out if the lines L_1, L_2 are concurrent or parallel. If the lines are concurrent then let I be their intersection point and the folding line is their angle bisector. If the lines are parallel then the GeoGebra command *anglebisector* gives the parallel line in the middle of their distance.

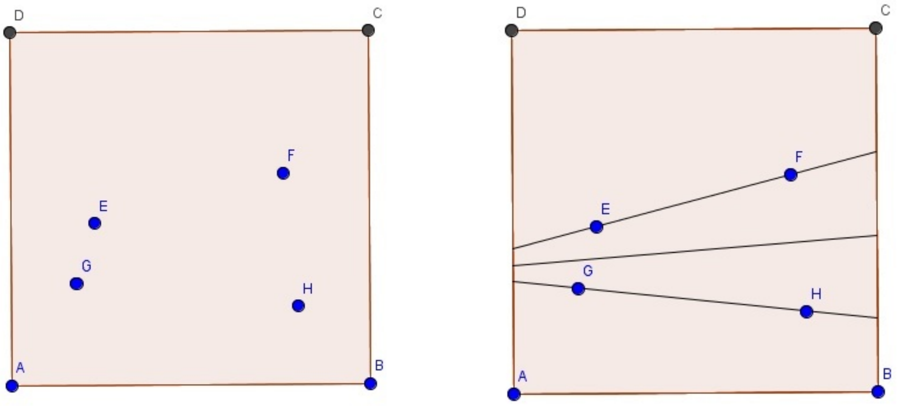

Fig. 3: Axiom 3.

The GeoGebra script is

```
I = intersect(line(E, F), line(G, H))
lst = removeundefined({I})
a = if(length(lst) == 0, 1, 2)
h = if(a == 1, anglebisector(line(E, F), line(G, H)),
anglebisector(G, I, if(angle(G, I, F) < 90°, F, E)))
p = polygon(A, B, 4)
R = intersect(h, p)
r = segment(R_1, R_2)
K = intersect(line(E, F), p)
k = segment(K_1, K_2)
P = intersect(line(G, H), p)
t = segment(P_1, P_2)
setvisibleinview(h, 1, false)
setvisibleinview(I, 1, false)
setvisibleinview(lst, 1, false)
setvisibleinview(R_1, 1, false)
setvisibleinview(R_2, 1, false)
setvisibleinview(K_1, 1, false)
setvisibleinview(K_2, 1, false)
setvisibleinview(T_1, 1, false)
setvisibleinview(T_2, 1, false)
showlabel(p, false)
showlabel(r, false)
showlabel(k, false)
showlabel(t, false)
```

Axiom 4

Given a point P and a line L, there is a unique fold perpendicular to L passing through P.

Let be the point $E = P$ and the $line(F, G) = L$, the tool $H4(A, B, E, F, G)$ draws the fold line passing through E and perpendicular to $line(F, G)$, Fig. 4.

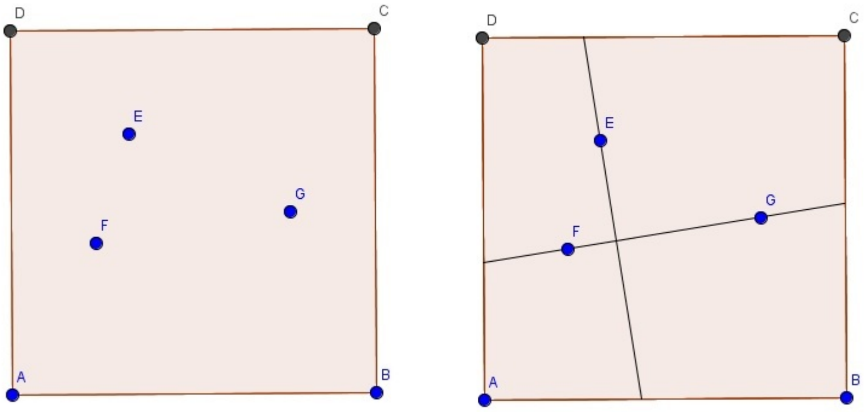

Fig. 4: Axiom 4.

The GeoGebra script is

1	$g = perpendicularline(E, line(F, G))$
2	$p = polygon(A, B, 4)$
3	$K = intersect(p, line(F, G))$
4	$k = segment(K_1, K_2)$
5	$Q = intersect(g, p)$
6	$q = segment(Q_1, Q_2)$
7	$setvisibleinview(g, 1, false)$
8	$setvisibleinview(Q_1, 1, false)$
9	$setvisibleinview(Q_2, 1, false)$
10	$setvisibleinview(K_1, 1, false)$
11	$setvisibleinview(K_2, 1, false)$
12	$showlabel(p, false)$
13	$showlabel(q, false)$
14	$showlabel(k, false)$

Axiom 5

Given two points P_1 and P_2 and a line L then whenever possible there is a fold placing P_1 onto L and passing through P_2.

If there are given the points $E = P_1,\ F = P_2$ and the $line(G, H) = L$ then the tool $H5(A, B, E, F, G, H)$ draws the fold passing through F and placing E onto the line $line(G, H)$, Fig. 5.

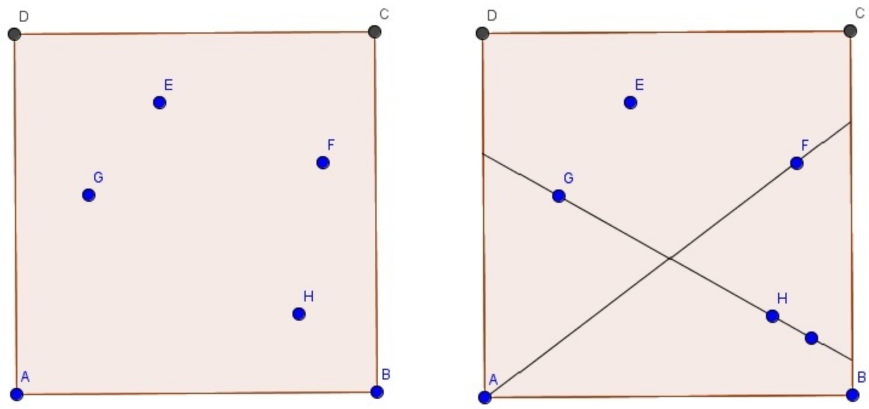

Fig. 5: Axiom 5.

The point without label on the line $line(G, H)$ is the reflection of the point E about the fold line.

The construction involves the intersection of a line with a circle. Let $\mathcal{C}(F, |EF|)$ be the circle with center F through E and I an intersection point of the circle with the segment of $line(G, H)$ contained in the origami square. The fold line is the angle bisector of $\widehat{EFI}$. If the intersection is the empty set then the fold line doesn't exists.

The GeoGebra script is

```
p = polygon(A, B, 4)
j = line(G, H)
J = intersect(j, p)
s = segment(J_1, J_2)
I = intersect(s, circle(F, segment(F, E)))
I = (x(I_1), y(I_1))
k = anglebisector(E, F, I)
K = intersect(k, p))
t = segment(K_1, K_2)
setvisibleinview(j, 1, false)
setvisibleinview(k, 1, false)
setvisibleinview(K_1, 1, false)
setvisibleinview(K_2, 1, false)
setvisibleinview(J_1, 1, false)
setvisibleinview(J_2, 1, false)
setvisibleinview(J, 1, false)
showlabel(p, false)
showlabel(t, false)
showlabel(s, false)
showlabel(I_1, false)
```

Axiom 6

Given two points P_1 and P_2 and two lines L_1 and L_2, there is a fold placing P_1 onto L_1 and P_2 onto L_2.

Let be given the points $E = P_1,\ F = P_2$ and the lines $line(G, H) = L_1$ and $line(I, J) = L_2$. The tool $H6(A, B, E, F, G, H, I, J)$ draws the fold placing E on $line(G, H)$ and F on $line(I, J)$, Fig. 6.

The points without labels on the $line(G, H)$ and $line(I, J)$ are the reflections about the fold line of the point E and respectively F.

The construction related to this axiom leads to compute the roots of a third order polynomial equation.

The reflection of the point $P(a, b)$ about the line $y = mx + n$ has the coordinates $(\frac{a+2bm-am^2-2mn}{1+m^2}, \frac{-b+2am+bm^2+2n}{1+m^2})$.

If $P_i(x_i, y_i)$ are two points and $y = m_i x + n_i$ are the equations of the lines $L_i,\ i = 1, 2$, then the conditions that the fold $y = mx + n$ place P_i on the line L_i is

$$-n_i-\frac{m_i(-2mn+x_i-m^2x_i+2my_i)}{1+m^2}+\frac{2n+2mx_i-y_i+m^2y_i}{1+m^2}=0,\ i=1,2. \tag{1}$$

We shall eliminate n between this two equations. The computations performed with CAS *Mathematica* are given on the next page. In the *Mathematica* codes we denoted $m \to M, m \to N$ and $m_i \to m\_, n_i \to n\_$, at least in the first setting. This setting corresponds to (1), after the denominator was eliminated.

```
In[1]:= Eq[x_, y_, m_, n_] := 2 N + 2 M x - y + M ^2 y - m (-2 M N + x - M^2 x + 2 M y) - n (1 + M^2)

In[2]:= E1 = Eq[x1, y1, m1, n1]
```

Out[2]= $2\,N - \left(1 + M^2\right) n1 + 2\,M\,x1 - y1 + M^2\,y1 - m1\,\left(-2\,M\,N + x1 - M^2\,x1 + 2\,M\,y1\right)$

```
In[3]:= E2 = Eq[x2, y2, m2, n2]
```

Out[3]= $2\,N - \left(1 + M^2\right) n2 + 2\,M\,x2 - y2 + M^2\,y2 - m2\,\left(-2\,M\,N + x2 - M^2\,x2 + 2\,M\,y2\right)$

```
In[4]:= s = Eliminate[{E1 == 0, E2 == 0}, N]
```

Out[4]=
$$n2 + M^2\,n2 + M\,m1\,n2 + M^3\,m1\,n2 + 2\,M\,x1 - m1\,x1 + M^2\,m1\,x1 + 2\,M^2\,m2\,x1 - M\,m1\,m2\,x1 + M^3\,m1\,m2\,x1 - 2\,M\,x2 - 2\,M^2\,m1\,x2 + m2\,x2 - M^2\,m2\,x2 + M\,m1\,m2\,x2 - M^3\,m1\,m2\,x2 - y1 + M^2\,y1 - 2\,M\,m1\,y1 - M\,m2\,y1 + M^3\,m2\,y1 - 2\,M^2\,m1\,m2\,y1 + y2 - M^2\,y2 + M\,m1\,y2 - M^3\,m1\,y2 + 2\,M\,m2\,y2 + 2\,M^2\,m1\,m2\,y2 == \left(1 + M^2 + M\,m2 + M^3\,m2\right) n1$$

```
In[5]:= lhs = First[s];

In[6]:= rhs = Last[s];

In[7]:= Collect[rhs - lhs, M]
```

Out[7]=
$$n1 - n2 + m1\,x1 - m2\,x2 + y1 - y2 + M^3\,(m2\,n1 - m1\,n2 - m1\,m2\,x1 + m1\,m2\,x2 - m2\,y1 + m1\,y2) + M\,(m2\,n1 - m1\,n2 - 2\,x1 + m1\,m2\,x1 + 2\,x2 - m1\,m2\,x2 + 2\,m1\,y1 + m2\,y1 - m1\,y2 - 2\,m2\,y2) + M^2\,(n1 - n2 - m1\,x1 - 2\,m2\,x1 + 2\,m1\,x2 + m2\,x2 - y1 + 2\,m1\,m2\,y1 + y2 - 2\,m1\,m2\,y2)$$

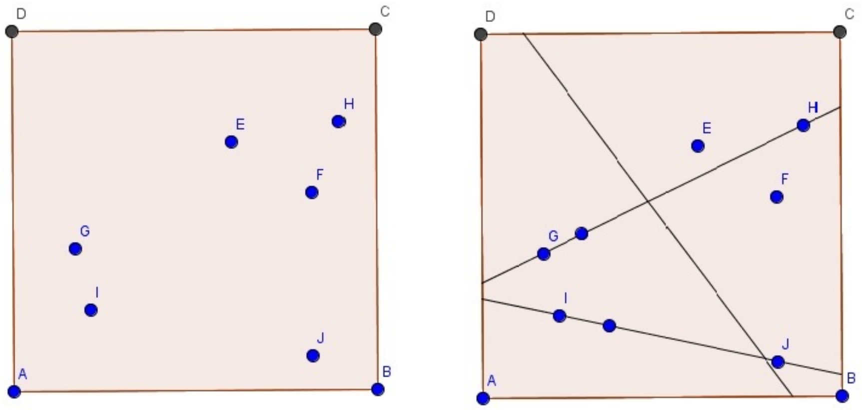

Fig. 6: Axiom 6.

Using the last result it results that m is a root of the polynomial

$$P(m) = p_0 + p_1 m + p_2 m^2 + p_3 m^3 \tag{2}$$

where

$$\begin{aligned}
p_0 &= n_1 - n_2 + m_1 x_1 - m_2 x_2 + y_1 - y_2 \\
p_1 &= m_2 n_1 - m_1 n_2 - 2x_1 + m_1 m_2 x_1 + 2x_2 - m_1 m_2 x_2 + 2m_1 y_1 + m_2 y_1 - \\
&\quad - m_1 y_2 - 2m_2 y_2 \\
p_2 &= n_1 - n_2 - m_1 x_1 - 2m_2 x_1 + 2m_1 x_2 + m_2 x_2 - y_1 + 2m_1 m_2 y_1 + \\
&\quad + y_2 - 2m_1 m_2 y_2 \\
p_3 &= m_2 n_1 - m_1 n_2 - m_1 m_2 x_1 + m_1 m_2 x_2 - m_2 y_1 + m_1 y_2
\end{aligned}$$

Within GeoGebra, the real roots m of (2) are computed as

$$m = x(Root(p)).$$

From (1), for $i = 1$, it results that

$$n = \frac{(1+m^2)n_1 + m_1 x_1 + y_1 - m^2(m_1 x_1 + y_1) - 2m(x_1 - m_1 y_1)}{2 + 2mm_1}.$$

This fold is the reason that in origami it is possible to solve some problems which cannot be solved with the ruler and the compass.

It results the following GeoGebra script

```
k = line(G, H)
j = line(I, J)
x1 = x(E)
x2 = x(F)
y1 = y(E)
y2 = y(F)
m1 = slope(k)
n1 = y(intersect(k, yAxis))
m2 = slope(j)
n2 = y(intersect(j, yAxis))
p0 = n1 − n2 + m1 ∗ x1 − m2 ∗ x2 + y1 − y2
p3 = m2 ∗ n1 − m1 ∗ n2 − m1 ∗ m2 ∗ x1 + m1 ∗ m2 ∗ x2 − m2 ∗ y1 + m1 ∗ y2
m2 ∗ n1 − m1 ∗ n2 − 2 ∗ x1 + m1 ∗ m2 ∗ x1 + 2 ∗ x2 − m1 ∗ m2 ∗ x2 + 2 ∗ m1 ∗ y1 + m2 ∗ y1 − m1 ∗ y2 − 2 ∗ m2 ∗ y2
p2 = n1 − n2 − m1 ∗ x1 − 2 ∗ m2 ∗ x1 + 2 ∗ m1 ∗ x2 + m2 ∗ x2 − y1 + 2 ∗ m1 ∗ m2 ∗ y1 + y2 − 2 ∗ m1 ∗ m2 ∗ y2
P(m) = p0 + p1 ∗ m + p2 ∗ m^2 + p3 ∗ m^3
m = x(root(P))
n = (n1 + m^2 ∗ n1 − 2 ∗ m ∗ x1 + m1 ∗ x1 − m^2 ∗ m1 ∗ x1 + y1 − m^2 ∗ y1 + 2 ∗ m ∗ m1 ∗ y1)/(2 ∗ (1 + m ∗ m1))
g : y = m ∗ x + n
p = polygon(A, B, 4)
K = intersect(k, p)
sk = segment(K_1, K_2)
S = intersect(j, p)
sj = segment(S_1, S_2)
Q = intersect(g, p)
sq = segment(Q_1, Q_2)
E1 = reflect(E, g)
F1 = reflect(F, g)
setvisibleinview(k, 1, false)
setvisibleinview(j, 1, false)
setvisibleinview(g, 1, false)
setvisibleinview(P, 1, false)
setvisibleinview(K_1, 1, false)
setvisibleinview(K_2, 1, false)
setvisibleinview(S_1, 1, false)
setvisibleinview(S_2, 1, false)
setvisibleinview(Q_1, 1, false)
setvisibleinview(Q_2, 1, false)
showlabel(m1, false)
showlabel(m2, false)
showlabel(sk, false)
showlabel(sq, false)
showlabel(sj, false)
showlabel(E1, false)
showlabel(F1, false)
```

Axiom 7

Given a point P and two concurrent lines L_1 and L_2, there is a fold placing P onto L_1 and perpendicular to L_2.

If $E = P$ is a point and $line(F, G) = L_1, line(H, I) = L_2$ are two lines then the tool $H7(A, B, E, F, G, H, I)$ draws the fold placing E on $line(F, G)$ and perpendicular to $line(H, I)$, Fig. 7.

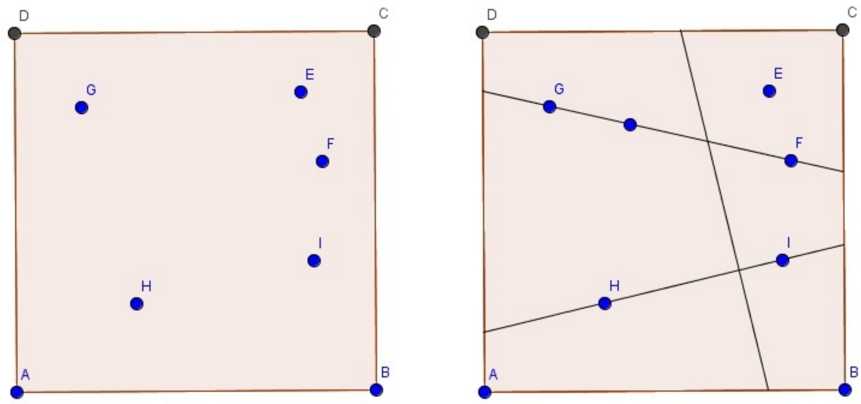

Fig. 7: Axiom 7.

Again, the point without label on the line $line(F, G)$ is the reflection of the point E about the fold line.

Let J be the intersection between the parallel to $line(H, I)$ through E with $line(F, G)$. The fold line will be the perpendicular bisector of the the segment EJ.

The GeoGebra script is

```
k = line(F, G)
j = line(H, I)
i = line(E, j)
q = perpendicularbisector(E, intersect(k, i))
p = polygon(A, B, 4)
K = intersect(k, p)
sk = segment(K_1, K_2)
J = intersect(j, p)
sj = segment(J_1, J_2)
Q = intersect(q, p)
sq = segment(Q_1, Q_2)
E1 = reflect(E, q)
setvisibleinview(k, 1, false)
setvisibleinview(j, 1, false)
setvisibleinview(i, 1, false)
setvisibleinview(q, 1, false)
setvisibleinview(K_1, 1, false)
setvisibleinview(K_2, 1, false)
setvisibleinview(J_1, 1, false)
setvisibleinview(J_2, 1, false)
setvisibleinview(Q_1, 1, false)
setvisibleinview(Q_2, 1, false)
showlabel(sk, false)
showlabel(sj, false)
showlabel(sq, false)
showlabel(E1, false)
```

3 Examples of the use of the tools

We advise the reader to introduce the calling statement of a tool through the input line.

Example 3.1 *Construct a fold (line) parallel to a given line through any given point using origami.*

In the square $ABCD$, let be the point E and the line $line(F, G)$. The purpose is to construct a parallel line through E to the $line(F, G)$.

No.	Commands and actions
1	Construct the points A, B.
2	$p = polygon(A, B, 4)$
3	Construct the points $E, F, G \in p$
4	$H4(A, B, E, F, G)$
5	Let J be the intersection between the fold line and FG
6	$H4(A, B, E, E, J)$

The final result is given in Fig. 8.

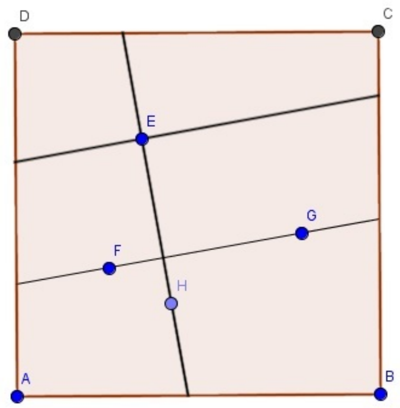

Fig. 8: Parallel line to $line(F, G)$ through the point E.

Example 3.2 *Construct an equilateral triangle with the base AB, the bottom side of the origami piece of paper, [5].*

The sequence of the commands are:

No.	Commands and actions
1	Construct the points A, B.
2	$p = polygon(A, B, 4)$
3	$G = midpoint(A, B)$
4	$H = midpoint(C, D)$
5	$H5(A, B, D, A, G, H)$
6	Let K be the label of the reflection point placed on $line(G, H)$
7	$H1(A, B, A, K)$
8	$H1(A, B, B, K)$

The final result is given in Fig. 9.

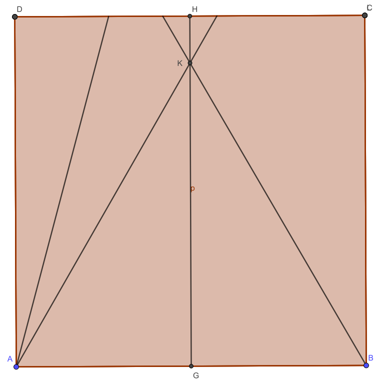

Fig. 9: Construction of the equilateral triangle.

Example 3.3 *The trisection of an acute angle [3], [6].*

The construction is:

No.	Commands and actions
1	Construct the points A, B.
2	$p = polygon(A, B, 4)$
3	Select a point E on the segment AD and a point F on CD.
4	$H1(A, B, A, F)$ The angle to be trisect will be $\widehat{BAF}$.
5	$H4(A, B, E, A, D)$
6	$H2(A, B, A, E)$
7	Let the intersection of the fold line with the $line(A, D)$ and the $line(B, C)$ be the points H and respectively I
8	$H6(A, B, A, E, H, I, A, F)$
9	Let M be the intersection point between the fold line with $line(H, I)$
10	$H1(A, B, A, M)$

Then $\widehat{MAF} = \frac{1}{3}\widehat{BAF}$, see [8] for the proof.
The final result is given in Fig. 10.

References

[1] Alperin R.C., 2000, *A Mathematical Theory of Origami Constructions and Numbers.* New York Journal of Mathematics, **6**, 119-133.

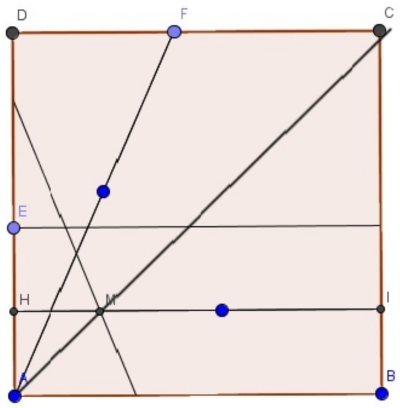

Fig. 10: The trisection of an acute angle.

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[4] Hatori K., Origami versus Straight-Edge-and-Compass. `http://origami.ousaan.com/index.html`, K's Origami → Fractional Library → Origami Construction.

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[7] * * *, `www.geogebra.org`.

[8] * * *, 2015, Trisecting the angle using origami – proof. `https://plus.maths.org/content/os/issue53/features/newton/angle`.