STATIC POTENTIALS ON ASYMPTOTICALLY FLAT MANIFOLDS

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Abstract. We consider the question whether a static potential on an asymptotically flat 3-manifold can have nonempty zero set which extends to the infinity. We prove that this does not occur if the metric is asymptotically Schwarzschild with nonzero mass. If the asymptotic assumption is relaxed to the usual assumption under which the total mass is defined, we prove that the static potential is unique up to scaling unless the manifold is flat. We also provide some discussion concerning the rigidity of complete asymptotically flat 3-manifolds without boundary that admit a static potential.

1. introduction

In [**?**], Corvino studied localized scalar curvature deformation of a Riemannian metric and introduced the following definition:

Definition 1. *A Riemannian metric g is called* static *on a manifold M if the linearized scalar curvature map at g has a nontrivial cokernel, i.e. if there exists a nontrivial function f on M such that*

$$-(\Delta f)g + \nabla^2 f - f Ric = 0. \tag{1.1}$$

Here ∇^2, Δ and Ric denote the Hessian, the Laplacian and the Ricci curvature of g respectively.

We call a nontrivial solution f to (**??**) a *static potential* if it exists. In [**?**, Theorem 1], Corvino proved that if (M, g) does not have a static potential, one can deform the scalar curvature of g through variations having compact support in M.

It is known that a static metric (as defined above) must have constant scalar curvature (cf. [**?**, Proposition 2.3]). When this constant is zero

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(which is always the case for an asymptotically flat, static metric), (**??**) becomes

$$\nabla^2 f = f\mathrm{Ric} \quad \text{and} \quad \Delta f = 0. \tag{1.2}$$

It is this equation that explains the implications of Corvino's result in mathematical relativity, where a vacuum static spacetime is a 4-dimensional Lorentz manifold that is isometric to $(\mathbb{R}^1 \times M, -N^2 dt^2 + g)$, where (M, g) is a 3-dimensional Riemannian manifold, $N > 0$ is a function on M, and the pair (g, N) satisfies

$$\nabla^2 N = N\mathrm{Ric} \quad \text{and} \quad \Delta N = 0. \tag{1.3}$$

By (**??**) and (**??**), one knows if f is a static potential on a manifold (M, g) of zero scalar curvature, then $(\mathbb{R}^1 \times \tilde{M}, -f^2 dt^2 + g)$ is a vacuum static spacetime, where $\tilde{M} = M \setminus f^{-1}(0)$.

There exists a vast amount of literature concerning 3-dimensional asymptotically flat manifolds which admit a *positive* solution N to (**??**) in the asymptotic region (see e.g. [**?**, **?**, **?**, **?**, **?**, **?**, **?**]). Since the positivity of N is always assumed in these works, it is natural to ask:

Question 1. *Suppose (E, g) is a 3-dimensional asymptotically flat end on which there exists a static potential f. Under what conditions, is f free of zeros near infinity?*

We recall the definition of an asymptotically flat 3-manifold.

Definition 2. *A Riemannian 3-manifold (M, g) (perhaps with boundary) is said to be asymptotically flat if there exists a compact set K such that $M \setminus K$ consists of a finite number of components $E_1, \ldots, E_k$, called the ends of (M, g), such that each end E_i is diffeomorphic to $\mathbb{R}^3$ minus a ball and, under this diffeomorphism, the metric g on E_i satisfies*

$$g_{ij} = \delta_{ij} + b_{ij} \text{ with } b_{ij} = O_2(|x|^{-\tau}) \tag{1.4}$$

for some constant $\tau > \frac{1}{2}$. Here $x = (x_1, x_2, x_3)$ denotes the standard coordinate on $\mathbb{R}^3$ and a function ϕ satisfies $\phi = O_l(|x|^{-\tau})$ provided $|\partial^i \phi| \leq C|x|^{-\tau-i}$ for $0 \leq i \leq l$ and some constant C.

We first describe a necessary condition for f to be positive near infinity. On an end E of an asymptotically flat (M, g), suppose f is a static potential and $f > 0$ near infinity, it is known (cf. [**?**, **?**]) that there exists a coordinate chart $\{x_1, x_2, x_3\}$ on E near infinity in which the metric g satisfies

$$g_{ij} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} + p_{ij}, \tag{1.5}$$

where $|p_{ij}| = O_2(|x|^{-2})$ and m is a constant that equals the ADM mass ([**?**]) of (M, g) at the end E. Metrics satisfying the fall-off condition given in (**??**) is often called *asymptotically Schwarzchild* (AS).

Our main result in answering Question **??** is that the AS condition is also a sufficient condition for the zero set $f^{-1}(0)$ to be bounded, provided the mass is nonzero.

Theorem 1.1. *Let (M, g) be an asymptotically flat 3-manifold with or without boundary. If g is asymptotically Schwarzschild on an end E which has nonzero mass, then any static potential f on E must be bounded and is either positive or negative outside a compact set.*

The main tool in our proof of Theorem **??** is Proposition **??**, which describes the asymptotic behavior of the zero set of f assuming it is unbounded. We also make use of an observation in Lemma **??** (iii) that the Ricci curvature of g, when restricted to the zero set of f, is a multiple of the induced metric.

In relation to the question of its positivity, we also ask "how many" static potentials may exist. We prove

Theorem 1.2. *Let (M, g) be a connected, asymptotically flat 3-manifold with or without boundary. Let $\mathcal{F}$ be the space of all solutions to (**??**). Let $\dim(\mathcal{F})$ be the dimension of $\mathcal{F}$. Then $\dim(\mathcal{F}) \leq 1$ unless (M, g) is flat.*

In the proof of Theorem **??**, beside Proposition **??**, we also use a local result that describes the dimension of $\mathcal{F}$ on any open set. We prove the following result using some techniques by Tod [**?**].

Theorem 1.3. *Let (M, g) be a connected, 3-dimensional Riemannian manifold of zero scalar curvature. Let $\mathcal{F}$ be the space of static potentials on (M, g). Then*

(i) $\dim(\mathcal{F}) \leq 2$ *unless (M, g) is flat.*
(ii) *If there exist two linearly independent functions $f_1, f_2 \in \mathcal{F}$ such that $f_1^{-1}(0) \cap f_2^{-1}(0) \neq \emptyset$, then (M, g) is flat.*

Our method in proving Theorem **??** and Theorem **??** also allow us to obtain some rigidity results for complete, asymptotically flat manifolds without boundary which admit a static potential. For instance, a direct corollary of Theorem **??**, Theorem **??** in Section 4 and the Riemannian positive mass theorem [**?**, **?**] is that

Corollary 1.1. *Let (M, g) be a complete, connected, asymptotically flat 3-manifold without boundary. Suppose (M, g) is asymptotically Schwarzschild at each end. If there is a static potential on (M, g), then*

(M,g) is isometric to either the Euclidean space $(\mathbb{R}^3, g_0)$ or a spatial Schwarzschild manifold $(\mathbb{R}^3 \setminus \{0\}, (1+\frac{m}{2|x|})^4 g_0)$ with $m>0$.

After the initial draft of this paper was completed, we were informed by Piotr Chruściel and Greg Galloway that there in fact exists a spacetime approach toward Question **??**. Namely, using results in [**?**] on Cauchy development, results in [**?**, **?**, **?**] on vacuum KID development (also see [**?**]), results in [**?**] concerning existence of boost-type domains, and in particular the result of Beig-Chruściel in [**?**, Theorem 1.1] which excludes boost-type Killing vector fields under appropriate conditions, Question 1 can also be approached in the spacetime setting.

We deem this spacetime method a very natural, important and physically motivated way to understand the structure of the zero set of static potentials. Comparatively, our approach toward Question **??** is a purely initial data based method and our method is more elementary.

The organization of the paper is as follows. In Section 2, we discuss local properties of static metrics and prove Theorem **??**. In Section 3, we analyze static potentials on an asymptotically flat end and prove Theorems **??** and **??**. In Section 4, we provide some discussion of rigidity questions for complete asymptotically flat 3-manifolds which admits a static potential.

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2. Local properties of static metrics

In this section, we assume that (M,g) is a 3-dimensional, connected, smooth Riemannian manifold whose scalar curvature R is zero. By (**??**), a nontrivial function f is a static potential on (M,g) if

$$\nabla^2 f = f \mathrm{Ric}. \tag{2.1}$$

In [**?**], Tod studied the question when a spatial metric could give rise to a static spacetime in more than one way. In our work, we often need to apply Proposition 2 (ii), Corollary 3 (i) and equation (15) in [**?**]. For convenience, we list these results of Tod in the next Proposition. We also sketch the proof.

Proposition 2.1 (Tod [**?**])**.** *Let $\{e_1, e_2, e_3\}$ be an orthonormal frame that diagonalizes the Ricci curvature at a given point p.*

(i) *Suppose f is a static potential. Then*

$$f(R_{33;1} - R_{31;3}) = (R_{22} - R_{33})f_{;1}$$
$$f(R_{11;2} - R_{12;1}) = (R_{33} - R_{11})f_{;2}$$
$$f(R_{22;3} - R_{23;2}) = (R_{11} - R_{22})f_{;3}.$$

(ii) *Suppose $\{R_{11}, R_{22}, R_{33}\}$ are distinct and suppose N, V are two positive static potentials. Then $V = cN$ for some constant c.*

(iii) *Suppose $R_{11} = R_{22} \neq R_{33}$ and suppose N is a positive static potential. If f is another static potential, then $Z = N^{-1}f$ satisfies $Z_{;1} = Z_{;2} = 0$.*

Proof. (i) Let $\{a, b, c, \dots\}$ denote indices that run through $\{1, 2, 3\}$. Differentiating the static equation, one has

$$f_{;abc} = f_{;c}R_{ab} + fR_{ab;c}.$$

Let ${R^d}_{acb}$ be the curvature tensor. (In our notation, ${R^d}_{acb}$ is given by

$$\nabla_{\partial_c}\nabla_{\partial_b}\partial_a - \nabla_{\partial_b}\nabla_{\partial_c}\partial_a = {R^d}_{acb}\partial_d$$

in a local coordinate chart.) Then

$$\begin{aligned} {R^d}_{abc}f_{;d} &= f_{;abc} - f_{;acb} \\ &= f_{;c}R_{ab} - f_{;b}R_{ac} + f(R_{ab;c} - R_{ac;b}). \end{aligned} \tag{2.2}$$

In 3-dimension, the curvature tensor and the Ricci curvature are related by

$${R^d}_{abc} = \delta^d_b R_{ac} - \delta^d_c R_{ab} + g_{ac}R^d_b - g_{ab}R^d_c + \frac{1}{2}R(\delta^d_c g_{ab} - \delta^d_b g_{ac}). \tag{2.3}$$

It follows from (**??**), (**??**) and the fact $R = 0$ that

$$2(f_{;b}R_{ac} - f_{;c}R_{ab}) + g_{ac}f_{;d}R^d_b - g_{ab}f_{;d}R^d_c = f(R_{ab;c} - R_{ac;b}). \tag{2.4}$$

Take $a = b \neq c$ and use the fact $\{e_1, e_2, e_3\}$ diagonalizes Ric, one has

$$f_{;c}(-2R_{aa} - R_{cc}) = f(R_{aa;c} - R_{ac;a}). \tag{2.5}$$

Now (i) follows from (**??**) and the fact $R = 0$.

(ii) The assumption on Ric implies that Ric has distinct eigenvalues in an open set U. Hence, $\nabla \log N = \nabla \log V$ on U by (i), which shows $V = cN$ for some constant c on U. Since V, N are both harmonic functions, $V = cN$ on M by unique continuation.

(iii) Apply (i) to N and $f = ZN$, one has

$$(R_{22} - R_{33})NZ_{;1} = (R_{33} - R_{11})NZ_{;2} = (R_{11} - R_{22})NZ_{;3} = 0.$$

The claim then follows from the fact $N \neq 0$ and $R_{11} = R_{22} \neq R_{33}$. $\square$

The zero set of a static potential, if nonempty, was known to be a totally geodesic hypersurface (cf. [**?**, Proposition 2.6]). In the next lemma, we give more geometric properties of this zero set.

Lemma 2.1. *Suppose f is a static potential with nonempty zero set. Let $\Sigma = f^{-1}(0)$.*

- (i) *Σ is a totally geodesic hypersurface and $|\nabla f|$ is a positive constant on each connected component of Σ.*
- (ii) *At any $p \in \Sigma$, ∇f is an eigenvector of Ric.*
- (iii) *At any $p \in \Sigma$, let $\{e_1, e_2, e_3\}$ be an orthonormal frame that diagonalizes Ric such that e_3 is normal to Σ. Then $R_{11} = R_{22}$.*
- (iv) *Let K be the Gaussian curvature of Σ at p. Using the same notations in (iii), one has $K = 2R_{11} = 2R_{22} = -R_{33}$. In particular, K is zero if and only if (M, g) is flat at p.*

Proof. (i) Let $p \in \Sigma$. If $\nabla f(p) = 0$, then along any geodesic $\gamma(t)$ emanating from p, $f(\gamma(t))$ satisfies $f'' = \mathrm{Ric}(\gamma', \gamma') f$ and $f(0) = f'(0) = 0$. This implies f is zero near p. By unique continuation, $f = 0$ on M, thus a contradiction. Hence, $\nabla f(p) \neq 0$, which implies that Σ is an embedded surface. On Σ, the static equation shows $\nabla^2 f(X, Y) = 0$ and $\nabla^2 f(X, \nabla f) = 0$ for any tangent vectors X, Y tangential to Σ, which readily implies that Σ is totally geodesic and $\nabla_X |\nabla f|^2 = 0$.

(ii) Since Σ is totally geodesic, it follows from the Codazzi equation that $\mathrm{Ric}(\nu, X) = 0$ for all X tangent to Σ, where ν is the unit normal of Σ. Therefore, $\nabla f = \frac{\partial f}{\partial \nu} \nu$ is an eigenvector of Ric.

(iii) Apply Proposition **??** (i), one has

$$(R_{11} - R_{22}) f_{;3} = f(R_{22;3} - R_{23;2}) = 0.$$

Since $|f_{;3}| = |\nabla f| > 0$, one concludes $R_{11} = R_{22}$.

(iv) It follows from the Gauss equation, the fact $R = 0$ and (iii) that $K = -R_{33} = 2R_{11} = 2R_{22}$. As a result, $K = 0 \Leftrightarrow \mathrm{Ric} = 0$ at p. □

In what follows, we let $\mathcal{F} = \{f \mid \nabla^2 f = f\mathrm{Ric}\}$.

Lemma 2.2. *If the Ricci curvature of g has distinct eigenvalues at a point, then $\dim(\mathcal{F}) \leq 1$. Here $\dim(\mathcal{F})$ denotes the dimension of $\mathcal{F}$.*

Proof. The assumption on Ric implies there is an open set U such that Ric has distinct eigenvalues everywhere in U. By Lemma **??** (iii), a static potential f is either positive or negative in U. The claim now follows from Proposition **??** (ii). □

Given two static potentials, if one of them is positive, one can look at their quotient.

Lemma 2.3. *Suppose f and N are two static potentials. Suppose N is positive. Let $Z = f/N$. Then either Z is a constant or ∇Z never vanishes. In the latter case, one has*

(i) *each level set of Z is a totally geodesic hypersurfaces.*
(ii) *$N^2|\nabla Z|^2$ equals a constant on each connected component of the level set of Z.*
(iii) *(M, g) is locally isometric to $((-\epsilon, \epsilon) \times \Sigma, N^2 dt^2 + g_0))$ where Σ is a 2-dimensional surface, Z is a constant on each $\Sigma_t = \{t\} \times \Sigma$ and g_0 is a fixed metric on Σ.*

Proof. Let $\{x_i\}$ be local coordinates on M. Since N and $f = NZ$ both are solutions to (**??**), we have

$$\begin{aligned} NZR_{ij} &= (NZ)_{;ij} \\ &= NZR_{ij} + NZ_{;ij} + N_{;i}Z_{;j} + N_{;j}Z_{;i}. \end{aligned}$$

Therefore, $NZ_{;ij} = -N_{;i}Z_{;j} - N_{;j}Z_{;i}$ or equivalently

$$N\nabla^2 Z(v, w) = -\langle \nabla N, v\rangle\langle \nabla Z, w\rangle - \langle \nabla N, w\rangle\langle \nabla Z, v\rangle \tag{2.6}$$

for any tangent vectors v, w.

Suppose $\nabla Z = 0$ at some point p. Similar to the proof of Lemma **??** (i), we consider an arbitrary geodesic $\gamma(t)$ emanating from p. Taking $v = w = \gamma'$ in (**??**), we have $NZ(\gamma(t))'' = -2N(\gamma(t))'Z(\gamma(t))'$. As $N > 0$ and $Z(\gamma(t))'|_{t=0} = 0$, we have $Z(\gamma(t))' = 0$, $\forall\ t$. Hence Z is a constant near p. By unique continuation [**?**], Z is a constant on M.

Next, suppose $\nabla Z \neq 0$ everywhere. In this case, every level set $Z^{-1}(t)$, if nonempty, is an embedded hypersurface. Let v and w be tangent vectors tangent to $Z^{-1}(t)$, (**??**) implies $N\nabla^2 Z(v, w) = 0$. As $N > 0$ and $\nabla^2 Z(v, w) = \langle \nabla_v(\nabla Z), w\rangle = |\nabla Z|\mathbb{II}(v, w)$, where $\mathbb{II}(\cdot, \cdot)$ is the second fundamental form of $Z^{-1}(t)$ with respect to $\nu = \nabla Z/|\nabla Z|$, we have $\mathbb{II} = 0$. Hence $Z^{-1}(t)$ is totally geodesic, which proves (i).

To prove (ii), let $v = \nabla Z$ and w be tangent to $Z^{-1}(t)$ in (**??**), we have $Nw\left(|\nabla Z|^2\right) = -2w(N)|\nabla Z|^2$, which implies $w\left(N^2|\nabla Z|^2\right) = 0$. Hence $N^2|\nabla Z|^2$ equals a constant on each connected component of $Z^{-1}(t)$.

For (iii), let $X = \nabla Z/|\nabla Z|^2$ which is a nowhere vanishing vector field. Given any point $p \in M$, let Σ be a connected hypersurface passing p on which Z is a constant. By considering the integral curves of X starting from Σ and shrinking Σ if necessary, one knows there exists an open neighborhood U of p, diffeomorphic to $(-\epsilon, \epsilon) \times \Sigma$ for some $\epsilon > 0$, on which the metric g takes the form

$$g = \frac{1}{|\nabla Z|^2} dt^2 + g_t$$

where $\partial_t = X$, Z is a constant on each $\Sigma_t = \{t\} \times \Sigma$ and g_t is the induced metric on Σ_t. Consider a background metric

$$\bar{g} = dt^2 + g_t$$

on $U = (-\epsilon, \epsilon) \times \Sigma$. Let $\mathbb{II}, \overline{\mathbb{II}}$ be the second fundamental form of Σ_t in (U, g), $(U, \bar{g})$ respectively with respect to ∂_t. Then $\mathbb{II} = |\nabla Z| \overline{\mathbb{II}}$. Since $\mathbb{II} = 0$ by (i), we have $\overline{\mathbb{II}} = 0$. Hence $\frac{d}{dt} g_t = 0$ by the fact $\overline{\mathbb{II}} = \frac{1}{2} \frac{d}{dt} g_t$. This shows, for each t, $g_t = g_0$ which is a fixed metric on Σ. By (ii), $N|\nabla Z|$ is a constant on Σ_t. Let $\phi(t) = N|\nabla Z|$. Then

$$g = \frac{N^2}{\phi(t)^2} dt^2 + g_0.$$

Replacing t by $\int \frac{1}{\phi(t)} dt$, we have $g = N^2 dt^2 + g_0$. This proves (iii). $\square$

Proposition 2.2. *If (M, g) is not flat at a point, then* $\dim(\mathcal{F}) \leq 2$.

Proof. Suppose $\dim(\mathcal{F}) > 2$. Let f_1, f_2, f_3 be three linearly independent static potentials. Let U be an open set such that g is not flat at every point in U. By Lemma **??**, $U \setminus \cup_{i=1}^3 f_i^{-1}(0)$ is nonempty. Hence one can find a connected open set $V \subset U$ such that each f_i is nowhere vanishing on V. Let $\{\lambda_1, \lambda_2, \lambda_3\}$ denote the eigenvalues of Ric in V. $\{\lambda_1, \lambda_2, \lambda_3\}$ can not be distinct by Proposition **??** (ii). The fact g is not flat and $R = 0$ shows $\{\lambda_1, \lambda_2, \lambda_3\}$ can not be identical. Therefore, one may assume $\lambda_1 = \lambda_2 \neq \lambda_3$ in V. Let $Z_1 = f_1/f_3$, $Z_2 = f_2/f_3$. By Proposition **??** (iii), both ∇Z_1 and ∇Z_2 are parallel to the eigenvector of Ric with eigenvalue λ_3. Therefore, at a point $q \in V$, $\nabla Z_1 + \alpha \nabla Z_2 = 0$ for some constant α. By Lemma **??**, $\nabla Z_1 + \alpha \nabla Z_2 \equiv 0$ in V. So $Z_1 + \alpha Z_2$ is a constant in V. Hence, $f_1 + \alpha f_2 = \beta f_3$ for some constant β, which is a contradiction. $\square$

When the zero set of a given static potential is not empty, we can consider the behavior of another static potential along such a set.

Lemma 2.4. *Suppose f and $\tilde{f}$ are two static potentials. Suppose $\tilde{f}$ has nonempty zero set. Let $\Sigma = \tilde{f}^{-1}(0)$. Then*

$$\nabla^2_\Sigma f = \frac{1}{2} K f \gamma \tag{2.7}$$

along Σ. Here ∇^2_Σ is the Hessian on Σ, γ is the induced metric on Σ, and K is the Gaussian curvature of (Σ, γ). Consequently, $K f^3$ equals a constant along each connected component of Σ.

Proof. By Lemma **??** (iii), $\mathrm{Ric}(X, Y) = \lambda \gamma(X, Y)$, $\forall\ X, Y$ tangent to Σ, where $2\lambda + \mathrm{Ric}(\nu, \nu) = 0$ and ν is a unit normal to Σ. Therefore, $\nabla^2 f(X, Y) = f \lambda \gamma(X, Y)$ along Σ. On the other hand, $\nabla^2 f(X, Y) =$

$\nabla^2_\Sigma f(X,Y)$ since Σ is totally geodesic. Hence $\nabla^2_\Sigma f = f\lambda\gamma = \frac{1}{2}fK\gamma$, where we have used $K = 2\lambda$ by Lemma **??** (iv).

Let $\{x_\alpha\}$ be local coordinates on Σ. Taking divergence and trace of (**??**), we have

$$(\Delta_\Sigma f)_{;\alpha} + Kf_{;\alpha} = \frac{1}{2}(Kf)_{;\alpha} \quad \text{and} \quad \Delta_\Sigma f = Kf \tag{2.8}$$

where Δ_Σ is the Laplacian on (Σ, γ). It follows from (**??**) that

$$K_{;\alpha} f + 3Kf_{;\alpha} = 0,$$

which implies $(Kf^3)_{;\alpha} = 0$. Hence, Kf^3 is a constant on each connected component of Σ. $\square$

To prove the main result in this section, we need an additional lemma in connection with Lemma **??** (iii).

Lemma 2.5. *Suppose (Σ_0, g_0) is a flat surface. If $\dim(\mathcal{F}) \geq 2$ on*

$$(M,g) = ((-\epsilon,\epsilon)\times\Sigma, N^2dt^2 + g_0)$$

where N is a positive function on M and g has zero scalar curvature, then (M,g) is flat.

Proof. Take any $(t,q) \in (-\epsilon,\epsilon)\times\Sigma$, the surface $\Sigma_t = \{t\}\times\Sigma$ has zero Gaussian curvature and is totally geodesic in (M,g). Let $\{e_1,e_2,e_3\}$ be an orthonormal frame at (t,q) which diagonalizes the Ricci curvature and satisfies $e_3 \perp \Sigma_t$. Then $R_{33} = 0$ by the Gaussian equation. Hence, $R_{11} + R_{22} = 0$. If $R_{11} \neq R_{22}$, then Ric has distinct eigenvalues at (t,q) and Lemma **??** implies $\dim(\mathcal{F}) \leq 1$, contradicting to the assumption $\dim(\mathcal{F}) \geq 2$. Therefore $R_{11} = R_{22} = 0$ by Lemma **??** (iii). We conclude that g has zero curvature at (t,q). $\square$

Proposition 2.3. *Suppose $\dim(\mathcal{F}) \geq 2$. Let f_1 and f_2 be two linearly independent static potentials. Let P_1, P_2 be a connected component of $f_1^{-1}(0)$, $f_2^{-1}(0)$ respectively. If $P_1 \cap P_2 \neq \emptyset$, then*

(i) *(M,g) is flat along $P_1 \cup P_2$.*
(ii) *(M,g) is flat in an open set which contains $P_1 \setminus f_2^{-1}(0)$ and $P_2 \setminus f_1^{-1}(0)$.*

Proof. First we note that $f_1^{-1}(0) \cap f_2^{-1}(0)$ is an embedded curve (hence a geodesic since both P_1 and P_2 are totally geodesic). This is because f_1 and f_2 are linearly independent, which implies ∇f_1 and ∇f_2 are linearly independent at any point in $f_1^{-1}(0) \cap f_2^{-1}(0)$.

Now let K_1, K_2 be the Gaussian curvature of P_1, P_2 respectively. By Lemma **??**, $K_1 f_2^3 = C$ for some constant C on P_1 and $K_2 f_1^3 = D$ for some constant D on P_2. Since $f_1 = f_2 = 0$ on $P_1 \cap P_2$, we have

$C = D = 0$. As $P_1 \cap f_2^{-1}(0)$, $P_2 \cap f_1^{-1}(0)$ consists of embedded curves, we conclude $K_1 = 0$ on P_1 and $K_2 = 0$ on P_2. Consequently g is flat along $P_1 \cup P_2$ by Lemma **??** (iv). This proves (i).

To prove (ii), let p be an arbitrary point in $P_1 \setminus f_2^{-1}(0)$, then f_2 does not vanish in an open set U containing p. Consider $Z = f_1/f_2$ on U. We have $Z = 0$ on $P_1 \cap U$. By Lemma **??** (iii), there exists an open neighborhood W of p, diffeomorphic to $(-\epsilon, \epsilon) \times \Sigma$, where Σ is a small piece of P_1 containing p, and Z is a constant on each $\{t\} \times \Sigma$, such that on W the metric g takes the form of

$$g = f_2^2 dt^2 + g_0$$

where g_0 is the induced metric on Σ. By (i), (Σ, g_0) has zero Gaussian curvature. Since $\dim(\mathcal{F}) \geq 2$ on (W, g), Lemma **??** implies that g is flat in W. Similarly, we know g is flat in an open neighborhood of any point in $P_2 \setminus f_1^{-1}(0)$. Therefore, (ii) is proved. $\square$

To end this section, we apply the analyticity of a static metric to improve Proposition **??**. It is known that, if (M, g) admits a static potential f, then g is analytic in harmonic coordinates around any point p with $f(p) \neq 0$ (cf. [**?**, Proposition 2.8]).

Theorem 2.1. *Suppose* $\dim(\mathcal{F}) \geq 2$. *Let* f_1 *and* f_2 *be two linearly independent static potentials. If* $f_1^{-1}(0) \cap f_2^{-1}(0)$ *is nonempty, then* (M, g) *is flat.*

Proof. Let $S = f_1^{-1}(0) \cap f_2^{-1}(0)$. Given any $p \in M \setminus S$, either $f_1(p) \neq 0$ or $f_2(p) \neq 0$, hence there exists an open set containing p in which g is analytic. As f_1 and f_2 are linearly independent, S is an embedded curve. In particular $M \setminus S$ is path-connected. Therefore, by Proposition **??** (ii), we conclude that g is flat in $M \setminus S$, hence flat in M. $\square$

Remark 2.1. We note that a much stronger analytic property of static metrics was shown by Chruściel in [**?**, Section 4]. Theorem **??** also follows from Proposition **??** and the result of Chruściel in [**?**].

3. STATIC POTENTIALS ON AN ASYMPTOTICALLY FLAT END

In this section, unless otherwise stated, we assume that M is diffeomorphic to $\mathbb{R}^3 \setminus B(\rho)$, where $B(\rho)$ is an open Euclidean ball centered at the origin with radius $\rho > 0$, and g is a smooth metric on M such that with respect to the standard coordinates $\{x_i\}$ on $\mathbb{R}^3$, g satisfies

$$g_{ij} = \delta_{ij} + b_{ij} \quad \text{with} \quad b_{ij} = O_2(|x|^{-\tau}) \tag{3.1}$$

for some constant $\tau \in (\frac{1}{2}, 1]$. We also assume that g has zero scalar curvature.

On such an (M, g), a static potential f is necessarily smooth up to ∂M by (**??**) and the assumption that g is smooth up to ∂M (cf. [**?**, Proposition 2.5]). The following lemma shows that at infinity f has at most linear growth.

Lemma 3.1. *Suppose f is a static potential on (M, g). Then f has at most linear growth, i.e. there exists $C > 0$ such that $|f(x)| \leq C|x|$.*

Proof. Let Rm denote the Riemann curvature tensor of g. By the AF condition (**??**), we have

$$r^{2+\tau}|\mathrm{Rm}| = O(1) \tag{3.2}$$

where $r = |x|$. Therefore, given any $\epsilon > 0$, there is $r_0 > \rho$ such that

$$|\mathrm{Rm}|(x) \leq \frac{1}{2}\epsilon|x|^{-2} \leq \epsilon(d(x) + r_0)^{-2}$$

if $|x| > r_0$. Here $d(x) = \mathrm{dist}(x, S_{r_0})$, where $S_{r_0} = \partial B(r_0)$, the Euclidean sphere with radius r_0. Given any x outside S_{r_0}, let $\gamma(t)$, $t \in [r_0, T]$, be a minimal geodesic parametrized by arc length connecting x and S_{r_0} with $\gamma(r_0) \in S_{r_0}$ and $\gamma(T) = x$. Then $f(t) = f(\gamma(t))$ satisfies

$$f''(t) = h(t)f(t),$$

where $h(t) = \mathrm{Ric}(\gamma'(t), \gamma'(t))$ satisfies

$$|h(t)| \leq \epsilon t^{-2}.$$

Let $\alpha = \frac{1}{2}(1+\sqrt{1+4\epsilon})$ and $a = \sup_{S_{r_0}}(|f|+|\nabla f|)$. Define $w(t) = At^{\alpha}$, where $A > 0$ is chosen so that $Ar_0^{\alpha} > a$ and $A\alpha r_0^{\alpha-1} > a$, then $w(t)$ satisfies

$$w''(t) = \epsilon t^{-2}w, \ |f(r_0)| < w(r_0) \text{ and } |f'(r_0)| < w'(r_0).$$

Suppose $|f(t)| > w(t)$ for some $t \in [r_0, T]$. Let

$$t_1 = \inf\{t \in [r_0, T] \mid |f(t)| > w(t)\}.$$

Then $t_1 > r_0$ and $|f(t_1)| = w(t_1)$. On $[r_0, t_1]$, we have

$$|f''(t)| = |h(t)f(t)| \leq \epsilon t^{-2}w = w''(t).$$

Therefore, $\forall\ t \in [r_0, t_1]$,

$$-w'(t) + w'(r_0) \leq f'(t) - f'(r_0) \leq w'(t) - w'(r_0)$$

which implies $-w'(t) < f'(t) < w'(t)$ because $|f'(r_0)| < w'(r_0)$. Integrating again, we have

$$-w(t) + w(r_0) < f(t) - f(r_0) < w(t) - w(r_0),$$

which shows $-w(t) < f(t) < w(t)$ because $|f(t_0)| < w(t_0)$. Therefore, $|f(t_1)| < w(t_1)$, which is a contradiction. Hence we have

$$|f(t)| \le At^{\alpha}, \ \forall\ t. \tag{3.3}$$

Now choose ϵ such that $\alpha < 1 + \frac{\tau}{2}$. It follows from (**??**) and (**??**) that

$$|f''(t)| = |h(t)f(t)| \le A|h(t)|t^{1+\frac{\tau}{2}}$$

where $|h(t)| \le C_1 t^{-2-\tau}$ for some C_1 independent on x and t. This shows $|f'(t)| \le C_2$ for some constant C_2 independent on x. Hence

$$|f(x)| \le a + C_2(|x| - r_0),$$

which proves that f has at most linear growth. □

Using Lemma **??**, we now present the following structure result for static potentials near infinity (cf. [**?**, Proposition 2.1] and Remark **??**).

Proposition 3.1. *Suppose f is a static potential on (M, g). Then*

(i) *there exists a tuple (a_1, a_2, a_3) such that*

$$f = a_1x_1 + a_2x_2 + a_3x_3 + h$$

where h satisfies $\partial h = O_1(|x|^{-\tau})$ and

$$|h| = \begin{cases} O(|x|^{1-\tau}) & \text{when } \tau < 1, \\ O(\ln|x|) & \text{when } \tau = 1. \end{cases}$$

(ii) *$(a_1, a_2, a_3) = (0, 0, 0)$ if and only if f is bounded. In this case, either $f > 0$ near infinity or $f < 0$ near infinity; moreover, upon rescaling,*

$$f = 1 - \frac{m}{|x|} + o(|x|^{-1})$$

for some constant m.

Proof. By (**??**) and Lemma **??**, $|\nabla^2 f| = |f\mathrm{Ric}| = O(r^{-1-\tau})$ where $r = |x|$. Let $\phi = |\nabla f|^2$, then

$$|\nabla\phi|^2 \le 4|\nabla^2 f|^2\phi \le C_1 r^{-2-2\tau}\phi \tag{3.4}$$

for some constant C_1. By considering ϕ restricted to a minimal geodesic emanating from the boundary, as in the proof of Lemma **??**, it is not hard to see that (**??**) implies ϕ is bounded. Hence

$$|\partial_{x_i}\partial_{x_j} f| = |f_{;ij} + \Gamma_{ij}^k \partial_{x_k} f| = O(r^{-1-\tau}), \tag{3.5}$$

where " ; " denotes covariant derivative and Γ_{ij}^k are the Christoffel symbols. It follows from (**??**) that, for each i, $\lim_{x\to\infty}\partial_{x_i} f$ exists and is finite. Let $a_i = \lim_{x\to\infty}\partial_{x_i} f$ and define $\lambda = \sum_{i=1}^3 a_i x_i$, then

$$|\partial_{x_i}\partial_{x_j}(f - \lambda)| = |\partial_{x_i}\partial_{x_j} f| = O(r^{-1-\tau})$$

and $\lim_{x\to\infty} \partial_{x_i}(f-\lambda) = 0$. This implies

$$|\partial_{x_i}(f-\lambda)| = O(r^{-\tau}),$$

which then shows

$$f - \lambda = \begin{cases} O(r^{1-\tau}) & \text{when } \tau < 1, \\ O(\ln r) & \text{when } \tau = 1. \end{cases} \tag{3.6}$$

Let $h = f - \lambda$. This proves (i).

To prove (ii), first suppose $a_1 = a_2 = a_3 = 0$. Let τ' be any fixed constant with $\tau > \tau' > \frac{1}{2}$. Then $|f| = |h| = O(r^{1-\tau'})$, hence $|\nabla^2 f| = |f\mathrm{Ric}| = O(r^{-1-2\tau'})$. This combined with $|\partial_{x_i} f| = O(r^{-\tau})$ implies $|\partial_{x_i}\partial_{x_j} f| = O(r^{-1-2\tau'})$, which in turns shows $|\partial_{x_i} f| = O(r^{-2\tau'})$. Since $2\tau' > 1$, we conclude that f has a finite limit as $x \to \infty$. In particular, f is bounded.

Next, suppose f is bounded. Then a_1, a_2, a_3 must be zero since h grows slower than a linear function. Moreover, $\lim_{x\to\infty} \phi = 0$ since $|\partial_{x_i} f| = O(r^{-\tau})$. Let $\Sigma = f^{-1}(0)$. By Lemma **??**(i), Σ is an embedded totally geodesic surface and ϕ is a positive constant on any connected component of Σ. We want to prove that Σ is bounded.

Let P be any connected component of Σ, then P must be bounded (hence compact), for otherwise contradicting to the fact $\lim_{x\to\infty} \phi = 0$ and ϕ is a positive constant on P. Next, note that there is $R_0 > 0$ such that $\partial B(R)$, $\forall\ R \geq R_0$, has positive mean curvature in (M, g). Therefore, for each fixed P, $P \cap \{|x| > R_0\} = \emptyset$ by the maximum principle and the fact that P is a compact embedded minimal surface. Since R_0 is independent of P, this implies $\Sigma \cap \{|x| > R_0\} = \emptyset$, therefore either $f > 0$ or $f < 0$ on $\{|x| > R_0\}$.

To complete the proof, let $a = \lim_{x\to\infty} f$ (which was shown to exists). Since $\Delta f = 0$, we have $f = a + A|x|^{-1} + o(|x|^{-1})$ for some constant A (cf. [**?**]). We want to show $a \neq 0$. Suppose $a = 0$. By what we have proved, we may assume $f > 0$ near infinity. Let $R > 0$ be a constant such that $f > 0$ on $S_R = \partial B(R)$. Let ψ be a harmonic function outside S_R such that $\psi = \inf_{S_R} f > 0$ on S_R and $\lim_{x\to\infty} \psi = 0$. Then $f \geq \psi$ by the maximum principle. Since ψ behaves like the Green's function which has a decay order of $\frac{1}{|x|}$, we have $A > 0$. On the other hand, the assumption $a = 0$ implies $f = O(|x|^{-1})$, hence $|\nabla^2 f| = O(r^{-3-\tau})$. Since $|\partial_{x_i} f| = O(r^{-2\tau'})$, we have $|\partial_{x_i}\partial_{x_j} f| = O(r^{-3-\tau}) + O(r^{-1-\tau-2\tau'})$ which implies $|\partial_{x_i} f| = O(r^{-3\tau'})$. Iterating this argument and using the fact τ' can be chosen arbitrarily close to τ, we conclude $|\partial_{x_i}\partial_{x_j} f| = O(r^{-3-\tau})$ and $|\partial_{x_i} f| = O(r^{-2-\tau})$. This together with $a = 0$ shows $|f| = O(r^{-1-\tau})$, contradicting the fact $A > 0$. Therefore, $a \neq 0$. Multiplying f by a nonzero constant, we conclude $f = 1 - m|x|^{-1} + o(|x|^{-1})$ for some constant m. This complete the proof of (ii). $\square$

Remark 3.1. Proposition **??** was also stated in a more general setting by Beig and Chruściel in [**?**, Proposition 2.1] for KID (Killing initial data). The proof of [**?**, Proposition 2.1] was briefly outlined in Appendix C in [**?**]. For the convenience of the reader, we have presented a detailed proof of Proposition **??**.

The next proposition describes the zero set of a static potential f near infinity in the case that f is unbounded.

Proposition 3.2. *Suppose f is an unbounded static potential on (M, g). There exists a new set of coordinates $\{y_i\}$ on $\mathbb{R}^3 \setminus B(\rho)$ obtained by a rotation of $\{x_i\}$ such that, outside a compact set, $f^{-1}(0)$ is given by the graph of a smooth function $q = q(y_2, y_3)$ over*

$$\Omega_C = \{(y_2, y_3) \mid y_2^2 + y_3^2 > C^2\}$$

for some constant $C > 0$, where q satisfies

$$\partial q = O_1(|\bar{y}|^{-\tau}) \quad \text{and} \quad |q| = \begin{cases} O(|\bar{y}|^{1-\tau}) & \text{when } \tau < 1 \\ O(\ln |\bar{y}|) & \text{when } \tau = 1. \end{cases} \tag{3.7}$$

Here $\bar{y} = (y_2, y_3)$. As a result, if $\gamma_R \subset f^{-1}(0)$ is the curve given by

$$\gamma_R = \{(q(y_2, y_3), y_2, y_3) \mid y_2^2 + y_3^2 = R^2\}$$

and κ is the geodesic curvature of γ_R in $f^{-1}(0)$, then

$$\lim_{R \to \infty} \int_{\gamma_R} \kappa = 2\pi. \tag{3.8}$$

Proof. Let (a_1, a_2, a_3) and h be given by Proposition **??** such that $f = \sum_{i=1}^3 a_i x_i + h$. As f is unbounded, $(a_1, a_2, a_3) \neq (0, 0, 0)$. We can rescale f so that $\sum_{i=1}^3 a_i^2 = 1$. Hence, there exists new coordinates $\{y_i\}$ obtained by a rotation of $\{x_i\}$ such that

$$f = y_1 + h(y_1, y_2, y_3) \tag{3.9}$$

where h satisfies

$$\partial h = O_1(|y|^{-\tau}) \quad \text{and} \quad |h| = \begin{cases} O(|y|^{1-\tau}) & \text{when } \tau < 1 \\ O(\ln |y|) & \text{when } \tau = 1. \end{cases} \tag{3.10}$$

It follows from (**??**) and (**??**) that

$$\frac{\partial f}{\partial y_1} = 1 + \frac{\partial h}{\partial y_1} = 1 + O(|y|^{-\tau}).$$

Therefore there exists a constant $C > 0$ such that

$$\frac{\partial f}{\partial y_1} > \frac{1}{2}, \quad \forall\ (y_2, y_3) \in \Omega_C = \{(y_2, y_3) \mid |\bar{y}| > C\}.$$

For any fixed $(y_2, y_3) \in \Omega_C$, (**??**) and (**??**) imply

$$\lim_{y_1 \to -\infty} f = -\infty, \quad \lim_{y_1 \to \infty} f = \infty.$$

Hence the set $f^{-1}(0) \cap \{(y_1, y_2, y_3) \mid (y_2, y_3) \in \Omega_C\} \neq \emptyset$ and is given by the graph of some function $q = q(y_2, y_3)$ defined on Ω_C. Since $\nabla f \neq 0$ on $f^{-1}(0)$, q is a smooth function by the implicit function theorem. Given the constant C, (**??**) and (**??**) imply there exists another constant $C_1 > 0$ such that

$$|f| \geq \frac{1}{2}|y_1| > 0 \ \text{ whenever } \ |\bar{y}| \leq C \text{ and } |y_1| > C_1.$$

Therefore,

$$\begin{aligned} & f^{-1}(0) \cap \{(y_1, y_2, y_3) \mid (y_2, y_3) \in \Omega_C\} \\ = \ & f^{-1}(0) \setminus \{(y_1, y_2, y_3) \mid |y_1| \leq C_1, \ |\bar{y}| \leq C\}. \end{aligned}$$

This proves that, outside a compact set, $f^{-1}(0)$ is given by the graph of q over Ω_C.

Next we estimate q and its derivatives. The equation

$$q + h(q, y_2, y_3) = 0 \tag{3.11}$$

and (**??**) imply that, if $|\bar{y}|$ is large,

$$|q| = |h(q, y_2, y_3)| \leq \left\{ \begin{array}{ll} C_2 \left(|q| + |\bar{y}|\right)^{1-\tau}, & \tau < 1 \\ C_2 \ln\left(|q| + |\bar{y}|\right), & \tau = 1 \end{array} \right.$$

for some constant $C_2 > 0$. This in turn implies, as $|\bar{y}| \to \infty$,

$$|q| = O(|\bar{y}|^{1-\tau}) \text{ if } \tau < 1 \ \text{ and } \ |q| = O(\ln |\bar{y}|) \ \text{ if } \tau = 1.$$

Let $\alpha, \beta \in \{2, 3\}$. Taking derivative of (**??**), we have

$$\frac{\partial q}{\partial y_\alpha} = -\frac{\frac{\partial h}{\partial y_\alpha}}{1 + \frac{\partial h}{\partial y_1}} = O(|\bar{y}|^{-\tau}). \tag{3.12}$$

Similarly, by taking derivative of (**??**), we have $\dfrac{\partial^2 q}{\partial y_\beta y_\alpha} = O(|\bar{y}|^{-1-\tau})$.

To verify (**??**), we consider the pulled back metric $\sigma = F^*(g)$ on Ω_C where $F : \Omega_C \to \mathbb{R}^3$ is given by $F(y_2, y_3) = (q(y_2, y_3), y_2, y_3)$. It follows from (**??**) and (**??**) that

$$\sigma_{\alpha\beta} = \delta_{\alpha\beta} + h_{\alpha\beta} \tag{3.13}$$

where $\sigma_{\alpha\beta} = \sigma(\partial_{y_\alpha}, \partial_{y_\beta})$ and $h_{\alpha\beta}$ satisfies

$$|h_{\alpha\beta}| + |\bar{y}||\partial h_{\alpha\beta}| = O(|\bar{y}|^{-\tau}). \tag{3.14}$$

Direct calculation using (**??**) and (**??**) then shows

$$\kappa = R^{-1} + O(R^{-1-\tau}) \tag{3.15}$$

while the length of C_R is $2\pi R + O(R^{1-\tau})$. From this, we conclude that (**??**) holds. □

Remark 3.2. In [**?**], Beig and Schoen solved static n-body problem in the case that there exists a closed, noncompact, totally geodesic surface disjoint from the bodies. One may compare Proposition **??** with Proposition 2.1 in [**?**].

Now we are ready to prove the main results of this section.

Theorem 3.1. *Let (M, g) be a connected, asymptotically flat 3-manifold with or without boundary. If $\dim(\mathcal{F}) \geq 2$, then (M, g) is flat.*

Proof. It suffices to prove this result on an end of (M, g). So we assume M is diffeomorphic to $\mathbb{R}^3$ minus an open ball. Suppose f and $\tilde{f}$ are two linearly independent static potentials. We have the following three cases:

Case 1. Suppose both f and $\tilde{f}$ are bounded. By Proposition **??** (ii), after rescaling, we have

$$f = 1 - \frac{m}{|x|} + o(|x|^{-1}), \quad \tilde{f} = 1 - \frac{\tilde{m}}{|x|} + o(|x|^{-1})$$

for some constants $m, \tilde{m}$. Therefore, $f - \tilde{f}$ is a bounded static potential satisfying $f - \tilde{f} = -\frac{m-\tilde{m}}{|x|} + o(|x|^{-1})$. This contradicts Proposition **??** (ii). Hence, this case does not occur.

Case 2. Suppose f is bounded and $\tilde{f}$ is unbounded. By Proposition **??**, upon a rotation of coordinates and scaling, we may assume that $\tilde{f} = x_1 + h$, where h satisfies the properties in Proposition **??**(i), and $f = 1 - \frac{m}{|x|} + o(|x|^{-1})$ for some constant m. Let $r_0 > \rho$ be a fixed constant such that $f > \frac{1}{2}$ on $\{|x| \geq r_0\}$, and $S_r = \partial B(r)$ has positive mean curvature $\forall\ r \geq r_0$. Let $\lambda_0 > 0$ be another constant such that if $\lambda > \lambda_0$, $\tilde{f}_\lambda := \tilde{f} - \lambda f$ will be negative on S_{r_0}. For each $\lambda > \lambda_0$, let $\Sigma_\lambda = \{x \mid \tilde{f}_\lambda(x) = 0,\ |x| \geq r_0\}$. Then $\Sigma_\lambda \neq \emptyset$ by Proposition **??**. As $\tilde{f}_\lambda < 0$ on S_{r_0}, Σ_λ does not intersect S_{r_0}. Hence Σ_λ is a surface without boundary. Let P be any connected component of Σ_λ. Since (M, g) is foliated by positive mean curvature surfaces $\{S_r\}$ outside S_{r_0} and P is an embedded minimal surface without boundary, P cannot be compact by the maximum principle. By Proposition **??**, we have $P = \Sigma_\lambda$. Let

K be the Gaussian curvature of Σ_λ. By Lemma **??**, $Kf^3 = C$ for some constant C along Σ_λ. Note that $\lim_{x\to\infty} K = 0$ because g is asymptotically flat and Σ_λ is totally geodesic. This implies $C = 0$ since f is bounded. Hence $Kf^3 = 0$ on Σ_λ. As $f > 0$ outside S_{r_0}, we conclude $K = 0$. Hence, (M, g) is flat along Σ_λ by Lemma **??**(iv).

Thus we have proved that (M, g) is flat at every point in the set

$$U = \bigcup_{\lambda>\lambda_0} \{x \mid \tilde{f}(x) - \lambda f(x) = 0,\ |x| > r_0\}.$$

By the growth condition on h, we know that there exists a constant $a > 0$ such that for all $x_1 > a$ and all $(x_2, x_3) \in \mathbb{R}^2$ with $x_2^2 + x_3^2 < 1$,

$$\tilde{f}(x_1, x_2, x_3) > \lambda_0 f(x_1, x_2, x_3) > 0.$$

Clearly this implies that these points $(x_1, x_2, x_3) \in U$ and U contains a nonempty interior. Let $\hat{M} = M \setminus (f^{-1}(0) \cap \tilde{f}^{-1}(0))$. $\hat{M}$ is either M itself or M minus an embedded curve, hence $\hat{M}$ is path-connected. Since g is analytic on $\hat{M}$ which intersects U, we conclude that g is flat on $\hat{M}$, hence flat everywhere in M.

Case 3. Suppose both f and $\tilde{f}$ are unbounded. By the proof of Proposition **??**, upon a rotation of coordinates and scaling, we may assume $f = x_1 + h$, $\tilde{f} = a_1x_1 + a_2x_2 + a_3x_3 + \tilde{h}$, where $h = O(|x|^\theta)$, $\tilde{h} = O(|x|^\theta)$ for some constant $0 < \theta < 1$, and a_i, $i = 1, 2, 3$, are some constants. Moreover, we can assume that $f^{-1}(0)$, outside a compact set, is given by the graph of $q = q(x_2, x_3)$ where $q = O(|x_2|^\theta + |x_3|^\theta)$.

Replacing $\tilde{f}$ by $\tilde{f} - a_1 f$, we may assume $a_1 = 0$. In this case, if $a_2 = a_3 = 0$, then Proposition **??** (ii) implies that $\tilde{f}$ is bounded and we are back to Case 2. Therefore we may assume $(a_2, a_3) \neq (0, 0)$. Without loss of generality, we can assume $a_2 = 1$ upon rescaling $\tilde{f}$ so that $\tilde{f} = x_2 + a_3x_3 + \tilde{h}$. Given any large positive number a, consider the point $x_+ = (q(a, 0), a, 0)$ which lies in $f^{-1}(0)$. We have

$$\begin{aligned} \tilde{f}(x_+) &= a + \tilde{h}(q(a,0), a, 0) \\ &= a + O(|a|^{\theta^2} + |a|^\theta). \end{aligned} \tag{3.16}$$

Hence $\tilde{f}(x_+) > 0$ if a is sufficiently large. Similarly, we have $\tilde{f}(x_-) < 0$, where $x_- = (q(-a, 0), -a, 0)$, for large a. Since x_+ and x_- can be joint by a curve that is contained in the graph of q, hence in $f^{-1}(0)$, we conclude

$$f^{-1}(0) \cap \tilde{f}^{-1}(0) \neq \emptyset.$$

Therefore (M, g) is flat by Theorem **??**. □

Theorem 3.2. *Let g be a smooth metric on $M = \mathbb{R}^3 \setminus B(\rho)$, where $B(\rho)$ is an open ball, such that*

$$g_{ij}(x) = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} + p_{ij} \tag{3.17}$$

where $p_{ij}(x) = O_2(|x|^{-2})$ and $m \neq 0$ is a constant. If f is a static potential of (M, g), then f does not vanish outside a compact set.

Proof. By Proposition **??** (ii), it suffices to prove that f is bounded. Suppose f is unbounded, by Proposition **??** there exists a new set of coordinates $\{y_i\}$, obtained by a rotation of $\{x_i\}$, such that the zero set of f which we denote by Σ, outside a compact set, is given by the graph of a smooth function $q = q(y_2, y_3)$ defined on

$$\Omega_C = \{(y_2, y_3) \mid \ y_2^2 + y_3^2 > C^2\}$$

for some constant $C > 0$. Here q satisfies (**??**) with $\tau = 1$.

Since $\{y_i\}$ differs from $\{x_i\}$ only by a rotation, the asymptotically Schwarzschild condition (**??**) is preserved in the $\{y_i\}$ coordinates, i.e.

$$g_{ij}(y) = \left(1 + \frac{m}{2|y|}\right)^4 \delta_{ij} + p_{ij} \tag{3.18}$$

where $p_{ij}(y) = O_2(|y|^{-2})$. The Ricci curvature of g now can be estimated explicitly in terms of y. By [**?**, Lemma 1.2], (**??**) implies

$$\mathrm{Ric}(\partial_{y_i}, \partial_{y_j}) = \frac{m}{|y|^3}\phi(y)^{-2}\left(\delta_{ij} - 3\frac{y_i y_j}{|y|^2}\right) + O(|y|^{-4}), \tag{3.19}$$

where $\phi(y) = 1 + \frac{m}{2|y|}$.

Given any $\bar{y} = (y_2, y_3) \in \Omega_C$, let $y = (q(\bar{y}), y_2, y_3)$ and $T_y\Sigma$ be the tangent space to Σ at y. As a subspace in $T_y\mathbb{R}^3$, $T_y\Sigma$ is spanned by

$$v = (\partial_{y_2} q)\partial_{y_1} + \partial_{y_2}, \ w = (\partial_{y_3} q)\partial_{y_1} + \partial_{y_3}.$$

Let $|v|_g$, $|w|_g$ be the length of v, w with respect to g respectively. Define $\tilde{v} = |v|_g^{-1} v$, $\tilde{w} = |w|_g^{-1} w$, we want to compare

$$\mathrm{Ric}(\tilde{v}, \tilde{v}) \ \text{ and } \ \mathrm{Ric}(\tilde{w}, \tilde{w})$$

when $|\bar{y}|$ is large. By (**??**) and (**??**), we have

$$\begin{aligned} \mathrm{Ric}(v, v) &= \frac{m}{|y|^3}\phi(y)^{-2}\left[1 + (\partial_{y_2} q)^2 - \frac{3}{|y|^2}\left[(\partial_{y_2} q)q + y_2\right]^2\right] + O(|\bar{y}|^{-4}) \\ &= \frac{m}{|y|^3}\phi(y)^{-2}\left(1 - \frac{3y_2^2}{|y|^2}\right) + O(|\bar{y}|^{-4}). \end{aligned} \tag{3.20}$$

Similarly,

$$\mathrm{Ric}(w,w) = \frac{m}{|y|^3}\phi(y)^{-2}\left(1 - \frac{3y_3^2}{|y|^2}\right) + O(|\bar{y}|^{-4}). \tag{3.21}$$

On the other hand, (**??**) and (**??**) imply

$$|v|_g^2 = \phi(y)^4 + O(|\bar{y}|^{-2}),\ |w|_g^2 = \phi(y)^4 + O(|\bar{y}|^{-2}). \tag{3.22}$$

Therefore, it follows from (**??**) – (**??**) that

$$\mathrm{Ric}(\tilde{v},\tilde{v}) - \mathrm{Ric}(\tilde{w},\tilde{w}) = \frac{3m}{\phi(y)^6}\frac{(y_3^2 - y_2^2)}{|y|^5} + O(|\bar{y}|^{-4}). \tag{3.23}$$

Together with (**??**), this shows that there exists (y_2, y_3) such that $\mathrm{Ric}(\tilde{v},\tilde{v}) \neq \mathrm{Ric}(\tilde{w},\tilde{w})$ when $|\bar{y}|$ is large. For instance, let $y_2 = 0$ and $y_3 \to +\infty$, then

$$|y_3|^2(\mathrm{Ric}(\tilde{v},\tilde{v}) - \mathrm{Ric}(\tilde{w},\tilde{w})) \longrightarrow 3m \neq 0. \tag{3.24}$$

This is a contradiction to Lemma **??** (iii). We conclude that f must be bounded. $\square$

4. Rigidity of static asymptotically flat manifolds

In this section, we consider a complete, asymptotically flat 3-manifold without boundary, with finitely many ends, on which there exists a static potential f. Two basic examples are

Example 1. The Euclidean space $(\mathbb{R}^3, g_0)$. Here $f = a_0 + \sum_{i=1}^3 a_i x_i$ and $\{a_i\}$ are constants.

Example 2. A spatial Schwarzschild manifold with mass $m > 0$, i.e. $(\mathbb{R}^3 \setminus \{0\}, (1 + \frac{m}{2|x|})^4 g_0)$. In this case, $f = \frac{1-\frac{m}{2|x|}}{1+\frac{m}{2|x|}}$.

A natural question is whether these are the only examples of such manifolds? We start by showing that f must have nonempty zero set unless the manifold is $(\mathbb{R}^3, g_0)$.

Lemma 4.1. *Let (M,g) be a complete, connected, asymptotically flat 3-manifold without boundary. If (M,g) has a static potential f, then $f^{-1}(0)$ is nonempty unless (M,g) is isometric to $(\mathbb{R}^3, g_0)$.*

Proof. By Bochner's formula and the static equation (**??**),

$$\begin{aligned} \frac{1}{2}\Delta|\nabla f|^2 &= |\nabla^2 f|^2 + f^{-1}\nabla^2 f(\nabla f, \nabla f) \\ &= |\nabla^2 f|^2 + \frac{1}{2} f^{-1}\nabla f(|\nabla f|^2) \end{aligned} \tag{4.1}$$

wherever $f \neq 0$. Suppose $f^{-1}(0)$ is empty, then Proposition **??** implies f is bounded. By Proposition **??** (ii), $\lim_{x\to\infty} |\nabla f| = 0$ at each end of (M, g). Hence there is $p \in M$ such that $|\nabla f|^2(p) = \sup_M |\nabla f|^2$. By (**??**) and the strong maximum principle, $|\nabla f|^2$ must be a constant and hence is identically zero. Therefore, f is a nonzero constant and $\mathrm{Ric} = 0$ everywhere. This shows (M, g) is flat and hence isometric to $(\mathbb{R}^3, g_0)$ by volume comparison as (M, g) is asymptotically flat. □

In [**?**], Bunting and Masood-ul-Alam proved that if (M, g) is an asymptotically flat 3-manifold with boundary, with one end, on which there is a static potential f which goes to 1 at ∞ and is 0 on ∂M, then (M, g) is isometric to a spatial Schwarzschild manifold with positive mass outside its horizon. By examining the proof in [**?**], we observe that the result in [**?**] holds on manifolds with any number of ends.

Proposition 4.1. *Let (M, g) be a complete, connected, asymptotically flat 3-manifold with nonempty boundary, with possibly more than one end. Suppose f is a static potential such that $f > 0$ in the interior and $f = 0$ on ∂M. Then (M, g) is isometric to a spatial Schwarzschild manifold with positive mass outside its horizon.*

Proof. Since $f > 0$ away from the boundary, f must be bounded by Proposition **??**. Upon scaling, we may assume $\sup_M f = 1$. Suppose (M, g) has k ends $E_1, \ldots, E_k$, $k \geq 1$. For each $1 \leq i \leq k$, Proposition **??** (ii) implies $\lim_{x\to\infty, x\in E_i} f(x) = a_i$ for some constant $0 < a_i \leq 1$. By the maximal principal, $a_i = 1$ for some i. Without losing generality, we may assume $a_1 = 1$.

We proceed as in [**?**]. Define $\gamma^+ = (1 + f)^4 g$ and $\gamma^- = (1 - f)^4 g$. Then the following are true:

- γ^+ and γ^- have zero scalar curvature (cf. Lemma 1 in [**?**]).
- If $a_j = 1$, then E_j is an asymptotically flat end in (M, γ^+) and the mass of (M, γ^+) at E_j is zero; on the other hand, E_j gets compactified in (M, γ^-) in the sense that if p_j is the point of infinity at E_j, then there is a $W^{2,q}$ extension of γ^- to $E_j \cup \{p_j\}$ (cf. Lemma 2 and 3 in [**?**])
- If $a_j < 1$, then clearly E_j is an asymptotically flat end in both (M, γ^+) and (M, γ^-).

Glue (M, γ^+) and (M, γ^-) along ∂M to obtain a manifold $(\tilde{M}, \tilde{g})$, then $\tilde{g}$ is $C^{1,1}$ across ∂M in $\tilde{M}$ (cf. Lemma 4 in [**?**]). Apply the Riemannian positive mass theorem as stated in [**?**, Theorem 1] and use the fact that the mass of E_1 in $(\tilde{M}, \tilde{g})$ is zero, we conclude that $(\tilde{M}, \tilde{g})$ is isometric to $(\mathbb{R}^3, g_0)$. In particular, this shows that (M, g) only has one end. The rest now follows from the main theorem in [**?**]. □

Proposition **??** can be used to answer the rigidity question in the case that f is bounded.

Theorem 4.1. *Let (M,g) be a complete, connected, asymptotically flat 3-manifold without boundary, with finitely many ends. If there exists a bounded static potential on (M,g), then (M,g) is isometric to either $(\mathbb{R}^3, g_0)$ or a spatial Schwarzschild manifold $(\mathbb{R}^3 \setminus \{0\}, (1+\frac{m}{2|x|})^4 g_0)$ with $m > 0$.*

Proof. Let f be a bounded static potential. If (M,g) has only one end, then f must be a constant by Proposition **??** (ii) and the fact $\Delta f = 0$. Hence, (M,g) is flat and is isometric to $(\mathbb{R}^3, g_0)$.

Next suppose (M,g) has more than one end, in particular (M,g) is not isometric to $(\mathbb{R}^3, g_0)$. By Lemma **??**, $f^{-1}(0) \neq \emptyset$. By Lemma **??** (i) and Proposition **??** (ii), $f^{-1}(0)$ is a closed totally geodesic hypersurface (possibly disconnected); moreover f changes sign near $f^{-1}(0)$. Let N_1 be a component of $\{f > 0\}$, then N_1 is unbounded as $f = 0$ on ∂N. Since f is either positive or negative near the infinity of each end of (M,g), N_1 must be asymptotically flat, with possibly more than one end, with nonempty boundary Σ on which $f = 0$. By Proposition **??** and [?], (N_1, g) is isometric to $\left(\{x \in \mathbb{R}^3 \mid |x| > \frac{m_1}{2}\}, \left(1 + \frac{m_1}{2|x|}\right)^4 \delta_{ij}\right)$ with some constant $m_1 > 0$.

Similarly, let N_2 be the component of $\{f < 0\}$ whose boundary contains Σ. By the same argument, we know that (N_2, g) is isometric to $\left(\{y \in \mathbb{R}^3 \mid 0 < |y| < \frac{m_2}{2}\}, \left(1 + \frac{m_2}{2|y|}\right)^4 \delta_{ij}\right)$ for some $m_2 > 0$. Since M is connected, we conclude that $M = N_1 \cup N_2 \cup \Sigma$.

Now we have $\Sigma = \{|x| = 2m_1\} = \{|y| = 2m_2\}$. As the area of Σ is given by $16\pi m_1^2$ and $16\pi m_2^2$ respectively, we have $m_1 = m_2$. This proves that (M,g) is isometric to a spatial Schwarzschild manifold with positive mass. □

Next, we consider the rigidity question without the boundedness assumption of f. We recall that, by Proposition **??** (ii) and Proposition **??**, the zero set of a static potential on an asymptotically flat manifold has only finitely many components.

Proposition 4.2. *Let (M,g) be a complete, connected, asymptotically flat 3-manifold without boundary, with finitely many ends E_1, …, E_k. Suppose there exists a static potential f on (M,g). Then*

(i) $\displaystyle\int_M f|Ric|^2 = 0.$

(ii) $\displaystyle\int_M |f|\,|Ric|^2 = 4\pi\left[\sum_\alpha c_\alpha(\chi(\Sigma_\alpha)-k_\alpha)+\sum_\beta \tilde c_\beta \chi(\tilde\Sigma_\beta)\right]$. *Here* $\{\Sigma_\alpha \mid 0\le\alpha\le m\}$ *and* $\{\tilde\Sigma_\beta \mid 0\le\beta\le n\}$ *are the sets of unbounded components and bounded components of* $f^{-1}(0)$ *respectively.* $c_\alpha>0$ *and* $\tilde c_\beta>0$ *are the constants which equal* $|\nabla f|$ *on* Σ_α *and* $\tilde\Sigma_\beta$ *respectively. For each* α, $k_\alpha\ge 1$ *is the number of ends* E_i *with* $E_i\cap\Sigma_\alpha\ne\emptyset$. $\chi(\Sigma_\alpha)$ *and* $\chi(\tilde\Sigma_\beta)$ *denote the Euler characteristic of* Σ_α *and* $\tilde\Sigma_\beta$.

Proof. At each end E_i, $1\le i\le k$, let $\{y_1,y_2,y_3\}$ be a set of coordinates in which g satisfies (**??**). If f is unbounded in E_i, we require that $\{y_1,y_2,y_3\}$ be given by Proposition **??**. For any large $r>0$, let S_r^i be the coordinate sphere $\{|y|=r\}$ in E_i. Let U_r be the region bounded by S_r^1, …, S_r^k in M.

By Lemma **??** and (**??**), $|f|=O(r)$ and $|\mathrm{Ric}|=O(r^{-2-\tau})$ in each E_i. Hence, the integrals in (i) and (ii) exist and are finite. The static equation (**??**) implies

$$f|\mathrm{Ric}|^2=\langle\nabla^2 f,\mathrm{Ric}\rangle. \tag{4.2}$$

Integrating (**??**) over U_r and doing integration by parts, we have

$$\int_{U_r} f|\mathrm{Ric}|^2=\sum_{i=1}^k\int_{S_r^i}\mathrm{Ric}(\nabla f,\nu) \tag{4.3}$$

where ν is the unit outward normal to S_r^i and we also have used the fact g has zero scalar curvature. Since $|\nabla f|$ is bounded by Proposition **??**, $|\mathrm{Ric}|=O(r^{-2-\tau})$, and the area of S_r^i is of order r^2, we conclude that (i) holds by letting $r\to\infty$ in (**??**).

To prove (ii), we first choose r sufficient large so that $\tilde\Sigma_\beta\subset U_r$, $\forall\ \beta$. If f is unbounded, we assume it is unbounded in the ends E_1, …, E_l, $1\le l\le k$, and bounded in the other ends. We then choose r large enough so that outside each S_r^i in E_i, $1\le i\le l$, $f^{-1}(0)$ is the graph of some function $q=q(\bar y)$ given by Proposition **??**; moreover, by (**??**) we can assume the graph of $q(\bar y)$ always intersects S_r^i transversally. Hence, the set $U_r^+=U_r\cap\{f>0\}$ has Lipschitz boundary. Integrating (**??**) over U_r^+ gives

$$\begin{aligned}\int_{U_r^+} f|\mathrm{Ric}|^2=&\int_{U_r\cap\left(\cup_{\alpha=1}^m\Sigma_\alpha\right)}\mathrm{Ric}(\nabla f,\nu)+\int_{\cup_{\beta=0}^n\tilde\Sigma_\beta}\mathrm{Ric}(\nabla f,\nu)\\&+\int_{\partial U_r\cap\{f>0\}}\mathrm{Ric}(\nabla f,\nu).\end{aligned} \tag{4.4}$$

Here ν denotes the outward unit normal to ∂U_r^+. As in (i),

$$\lim_{r\to\infty}\int_{\partial U_r\cap\{f>0\}} \mathrm{Ric}(\nabla f,\nu)=0. \tag{4.5}$$

On each $\tilde{\Sigma}_\beta$ or Σ_α, by the fact $\nu=-\frac{\nabla f}{|\nabla f|}$, we have

$$\mathrm{Ric}(\nabla f,\nu)=-|\nabla f|\mathrm{Ric}(\nu,\nu)=|\nabla f|K,$$

where K is the Gaussian curvature of $\tilde{\Sigma}_\beta$ or Σ_α by Lemma **??** (iv). Hence,

$$\int_{\cup_{\beta=0}^n\tilde{\Sigma}_\beta}\mathrm{Ric}(\nabla f,\nu)=2\pi\sum_{\beta=0}^n\tilde{c}_\beta\chi(\tilde{\Sigma}_\beta), \tag{4.6}$$

by the Gauss-Bonnet theorem, and

$$\int_{U_r\cap\left(\cup_{\alpha=1}^m\Sigma_\alpha\right)}\mathrm{Ric}(\nabla f,\nu)=\sum_{\alpha=0}^m c_\alpha\int_{U_r\cap\Sigma_\alpha}K. \tag{4.7}$$

Note that Σ_α is totally geodesic, hence (**??**) implies that $|K|$ decays on Σ_α in the order of $O(|y|^{-2-\tau})$ in each end E_i with $\Sigma_\alpha\cap E_i\neq\emptyset$. But (**??**) implies that, on $\Sigma_\alpha\cap E_i$, $|y|$ is equivalent to the intrinsic distance function to a fixed point in Σ_α. Therefore,

$$\int_{\Sigma_\alpha}|K|<\infty. \tag{4.8}$$

Let $C_{_R}^i$ be the curve in $\Sigma_\alpha\cap E_i$ which is the graph of q over the circle $\{|\bar{y}|=R\}$ (see the definition of $C_{_R}$ in Proposition **??**). Let κ denote the geodesic curvature of $C_{_R}^i$ in Σ_α. By the Gauss-Bonnet theorem and Proposition **??**, we have

$$\begin{aligned}\int_{\Sigma_\alpha}K&=\lim_{R\to\infty}\left(2\pi\chi(\Sigma_\alpha)-\sum_{i\in\Lambda_\alpha}\int_{C_{_R}^i}\kappa\right)\\&=2\pi\chi(\Sigma_\alpha)-2\pi k_\alpha,\end{aligned} \tag{4.9}$$

where Λ_α is the set of indices i such that $\Sigma_\alpha\cap E_i\neq\emptyset$. It follows from (**??**) – (**??**) that

$$\lim_{r\to\infty}\int_{U_r\cap\left(\cup_{\alpha=1}^m\Sigma_\alpha\right)}\mathrm{Ric}(\nabla f,\nu)=2\pi\sum_{\alpha=0}^m c_\alpha(\chi(\Sigma_\alpha)-k_\alpha). \tag{4.10}$$

By (**??**) – (**??**) and (**??**), we conclude that

$$\int_{\{f>0\}}f|\mathrm{Ric}|^2=2\pi\sum_{\alpha=0}^m c_\alpha(\chi(\Sigma_\alpha)-k_\alpha)+2\pi\sum_{\beta=0}^n\tilde{c}_\beta\chi(\tilde{\Sigma}_\beta). \tag{4.11}$$

(ii) now follows from (**??**) and (i). $\square$

Remark 4.1. From (**??**) and (**??**), one can show $\lim_{r\to\infty}\frac{A(r)}{r^2}=\pi$, where $A(r)$ is the area of $D(r)\cap E_i$, $i\in\Lambda_\alpha$, for a geodesic ball $D(r)$ with radius r in Σ_α. Therefore, the fact $\int_{\Sigma_\alpha} K = 2\pi(\chi(\Sigma_\alpha)-k_\alpha)$ also follows from results in [**?**, **?**].

Proposition **??** implies that (M,g) must be $(\mathbb{R}^3,g_0)$ if M has simple topology.

Theorem 4.2. *Let (M,g) and f be given as in Proposition **??**. If M is orientable and every 2-sphere in M is the boundary of a bounded domain, then (M,g) is isometric to $(\mathbb{R}^3,g_0)$. In particular, if M is homeomorphic to $\mathbb{R}^3$, then (M,g) is isometric to $(\mathbb{R}^3,g_0)$.*

Proof. Suppose $\tilde\Sigma_\beta$ is a compact component of $f^{-1}(0)$. Since M is orientable and $\tilde\Sigma_\beta$ is two-sided (with a nonzero normal ∇f), $\tilde\Sigma_\beta$ is orientable. If $\chi(\tilde\Sigma_\beta)>0$, then $\tilde\Sigma_\beta$ is a 2-sphere. Hence $\tilde\Sigma_\beta=\partial\Omega$ for some bounded domain Ω in M by the assumption. This implies $f\equiv 0$ in Ω by the maximum principal and therefore $f\equiv 0$ in M by unique continuation [**?**]. Thus, (M,g) is flat and is isometric to $(\mathbb{R}^3,g_0)$. However, $(\mathbb{R}^3,g_0)$ does not contain any closed minimal surface. Hence, we must have $\chi(\tilde\Sigma_\beta)\le 0$ for all compact components $\tilde\Sigma_\beta$ of $f^{-1}(0)$ if such a component exists. On the other hand, if Σ_α is a noncompact component of $f^{-1}(0)$, then $\chi(\Sigma_\alpha)\le 1$. By Proposition **??** (ii), we have

$$\int_M |f|\,|\mathrm{Ric}|^2\le 0.$$

This implies $\mathrm{Ric}\equiv 0$ and therefore (M,g) is isometric to $(\mathbb{R}^3,g_0)$. $\square$

In what follows, we replace the topological assumption in Theorem **??** by an assumption that f has no critical points. For this purpose, we analyze the behavior of integral curves of the gradient of a static potential. We formulate the results in a setting similar to that in Proposition **??**.

Proposition 4.3. *Let (M,g) be a complete, connected, asymptotically flat 3-manifold with nonempty boundary, with finitely many ends E_1, …, E_k. Suppose there exists a static potential f with $f=0$ on ∂M. Given any point $p\in Int(M)$, the interior of M, let $\gamma_p(t)$ be the integral curve of ∇f with $\gamma_p(0)=p$. Let (α,β) be the maximal interval of existence of γ_p inside $Int(M)$.*

(a) *If $\beta<\infty$, then $\lim_{t\to\beta}\gamma_p(t)=x$ for some $x\in\partial M$; if $\alpha>-\infty$, then $\lim_{t\to\alpha}\gamma_p(t)=y$ for some $y\in\partial M$. Consequently, either $\alpha=-\infty$ or $\beta=\infty$.*

(b) *If* $\beta = \infty$*, then* $\lim_{t\to\infty} f(\gamma_p(t)) = b > -\infty$*. Moreover,*

 (i) *if* $b = \infty$*, then, as* $t \to \infty$*,* $\gamma_p(t)$ *tends to infinity in an end* E_i *on which* f *is unbounded;*

 (ii) *if* $b < \infty$*, then* $b \neq 0$ *and* $\lim_{t\to\infty} |\nabla f|(\gamma_p(t)) = 0$*;*

 (iii) *if* $b < 0$*, then* $\bigcap_{t>0} \overline{\{\gamma_p(s)|\ s > t\}} \neq \emptyset$ *and consists of critical points of* f*.*

(c) *If* $\alpha = -\infty$*, then* $\lim_{t\to-\infty} f(\gamma_p(t)) = a < \infty$*. Moreover,*

 (i) *if* $a = -\infty$*, then, as* $t \to -\infty$*,* $\gamma_p(t)$ *tends to infinity in an end* E_i *on which* f *is unbounded;*

 (ii) *if* $a > -\infty$*, then* $a \neq 0$ *and* $\lim_{t\to-\infty} |\nabla f|(\gamma_p(t)) = 0$*;*

 (iii) *if* $a > 0$*, then* $\bigcap_{t<0} \overline{\{\gamma(s)|\ s < t\}} \neq \emptyset$ *and consists of critical points of* f*.*

Proof. If p is a critical point of f, then $\gamma_p(t) = p$, $\forall\ t \in (-\infty, \infty)$. Also $f(p) \neq 0$ by Lemma **??** (i). The proposition is obviously true in this case. In the following, we assume $\nabla f(p) \neq 0$. Then $\nabla f(\gamma_p(t)) \neq 0$ for all t and

$$\frac{d}{dt} f(\gamma_p(t)) = |\nabla f|^2(\gamma_p(t)) > 0. \tag{4.12}$$

By Proposition **??**, $\lim_{x\to\infty} |\nabla f|$ exists and is finite at each end E_i. Therefore,

$$|\nabla f|(x) < B,\ \forall\ x \in M \tag{4.13}$$

for some constant $B > 0$. Suppose $\beta < \infty$, then for $t_2 > t_1 > 0$,

$$d(\gamma_p(t_1), \gamma_p(t_2)) \leq \int_{t_1}^{t_2} |\gamma_p'(s)| ds \leq\ (t_2 - t_1)B,$$

where $d(\cdot, \cdot)$ denotes the distance on (M, g). Hence $\lim_{t\to\beta} \gamma_p(t) = x$ for some $x \in M$. Since (α, β) is the maximal interval of existence of $\gamma_p(t)$ in $\mathrm{Int}(M)$, we conclude $x \in \partial M$. Similarly, if $\alpha > -\infty$, then $\lim_{t\to\beta} \gamma_p(t) = y$, for some $y \in \partial M$. If $\alpha > -\infty$ and $\beta < \infty$, then $f(x) = 0 = f(y)$, which contradicts (**??**). This proves (a).

To prove (b), we note that (**??**) implies $f(\gamma_p(t))$ is increasing, hence $\lim_{t\to\infty} f(\gamma_p(t)) = b$ exists and $b > -\infty$. If $b = \infty$, then there exists $t_n \to \infty$ such that $\gamma_p(t_n) \to \infty$ in some end E_i on which f is unbounded. Let $\{t_n'\}$ be any other sequence with $t_n' \to \infty$. We claim that $\gamma_p(t_n')$ must tend to infinity in E_i as well. Otherwise, passing to subsequence, we may assume that $\gamma_p(t_n')$ tends to infinity in another end E_j with $j \neq i$. But this implies that, for large n, there exists t_n'' between

t_n and t'_n such that $\gamma_p(t''_n)$ lies in a fixed compact set K of M (for instance the set K used in Definition **??**). This contradicts the fact $\lim_{n\to\infty} f(\gamma_p(t''_n)) \to b = \infty$. Therefore, $\gamma_p(t)$ tends to infinity in E_i as $t \to \infty$, which proves (i) in (b).

Next, suppose $b < \infty$. Let $\{t_n\}$ be any sequence such that $t_n \to \infty$. Given any fixed number $0 < \delta < \frac{1}{B}$, we have

$$\int_{t_n-\delta}^{t_n+\delta} |\nabla f|^2(\gamma_p(t))dt = f(\gamma_p(t_n+\delta)) - f(\gamma_p(t_n-\delta)) \to 0,\ n\to\infty.$$

Hence there exists $t'_n \in [t_n-\delta, t_n+\delta]$ such that $|\nabla f|(\gamma_p(t'_n)) \to 0$. Define $B_{\gamma_p(t_n)}(1) = \{q \in M \mid d(q,\gamma_p(t_n)) < 1\}$. For large n, (**??**) implies $|f| < 2|b| + 2B$ on $B_{\gamma(t_n))}(1)$. This together with the fact $\nabla^2 f = f\mathrm{Ric}$ and (M,g) is asymptotically flat implies

$$|\nabla^2 f| \le C_1 \tag{4.14}$$

on $B_{\gamma(t_n))}(1)$ for some constant C_1 independent on n and δ. Now let $\phi = |\nabla f|^2$, then $\nabla\phi$ is dual to the 1-form $2\nabla^2 f(\nabla f,\cdot)$. By (**??**) and (**??**), we conclude

$$|\nabla\phi| \le C_2$$

on $B_{\gamma(t_n))}(1)$ by a constant C_2 independent on n and δ. Note that $d(\gamma(t_n),\gamma(t'_n)) \le \delta B < 1$, we therefore have

$$\phi(\gamma_p(t_n)) \le \phi(\gamma_p(t'_n)) + 2\delta B C_2.$$

Since $\phi(\gamma_p(t'_n)) \to 0$ and δ can be arbitrarily chosen, we conclude that $\phi(\gamma_p(t_n)) \to 0$ as $n \to \infty$.

We also want to show $b \ne 0$. Let $\{t_n\}$ be given as above. Suppose $\{\gamma_p(t_n)\}$ is unbounded, then passing to a subsequence we may assume $\gamma_p(t_n) \to \infty$ in some end E_j. If f is unbounded in E_j, we would have $|\nabla f|(\gamma_p(t_n)) \ge C_3$ for some $C_3 > 0$ independent of n by Proposition **??** (i), contradicting to the fact $|\nabla f|(\gamma_p(t_n)) \to 0$. Hence, f is bounded in E_j. By Proposition **??** (ii), we have $b = \lim_{x\to\infty,\ x\in E_j} f \ne 0$. Next, suppose $\{\gamma_p(t_n)\}$ is bounded. Passing to a subsequence, we may assume $\gamma_p(t_n) = q \in M$. Then q is a critical point of f since $|\nabla f|(\gamma_p(t_n)) \to 0$. Therefore, $b = f(q) \ne 0$ by Lemma **??** (i). This completes the proof of (ii) in (b).

To prove (iii) of (b), it is sufficient to prove that if $b < 0$ and if $\{t_n\}$ is a sequence tending to ∞, then $\{\gamma_p(t_n)\}$ must be bounded, hence containing a subsequence converging to a critical point in M. Suppose $\{\gamma(t_n)\}$ is unbounded, then passing to a subsequence we may assume $\gamma(t_n) \to \infty$ in an end E_j where f is bounded by the proof in (ii) above.

On E_j, Proposition **??** (ii) implies

$$f = b - \frac{A}{|x|} + o(|x|^{-1}), \ |x| \to \infty \tag{4.15}$$

where A is a constant such that

$$\frac{A}{b} = m$$

which is the mass of (M, g) at the end of E_j (cf. [**?**, **?**]). By the positive mass theorem [**?**, **?**], we have $m > 0$ (which can be seen by reflecting (M, g) through ∂M since ∂M is totally geodesic). Therefore, $A < 0$ because $b < 0$. As a result, $f(\gamma_p(t_n)) > b$ for large n by (**??**). But this leads to a contradiction to the fact that $b = \lim_{n\to\infty} f(\gamma_p(t_n))$ and $f(\gamma_p(t))$ is strictly increasing in t. Therefore, $\{\gamma_p(t_n)\}$ must be bounded. This proves (iii) of (b).

Claim (c) follows from (b) by replacing f by $-f$. □

Using Proposition **??**, we obtain an analogue of Proposition **??** with the assumption $f > 0$ replaced by that f has no critical points.

Corollary 4.1. *Let (M, g) be a complete, connected, asymptotically flat 3-manifold with nonempty boundary ∂M, with finitely many ends. Suppose there exists a static potential f without critical points such that $f = 0$ on ∂M. Then (M, g) is isometric to a spatial Schwarzschild manifold with positive mass outside its horizon.*

Proof. By definition, ∂M is compact. Let Σ be a component of ∂M. Since $\nabla f \neq 0$ at Σ by Lemma **??** (i), we may assume that ∇f is inward pointing at Σ. Consider the map $F : \Sigma \times (0, \infty) \to \mathrm{Int}(M)$ given by $F(x, t) = \gamma_x(t)$ which is the integral curve of ∇f such that $\gamma_x(0) = x \in \Sigma$. By Proposition **??** (a), γ_x is defined on $[0, \infty)$. The fact $f = 0$ and $\nabla f \neq 0$ at Σ implies that F is one-to-one. Hence, by the invariance of domain, the image N of F is open in $\mathrm{Int}(M)$. We want to prove that N is also closed in $\mathrm{Int}(M)$.

Let $y \in \mathrm{Int}(M)$ be a point that lies in the closure of N in $\mathrm{Int}(M)$. Then there exist $x_i \in \Sigma$ and $t_i > 0$ such that $\tilde{x}_i = \gamma_{x_i}(t_i)$ converge to y. Passing to a subsequence, we may assume that $x_i \to x \in \Sigma$ and $t_i \to a$ with $0 \le a \le \infty$. We claim that $a < \infty$. If this is true, we will have $y = \lim_{i\to\infty} \gamma_{x_i}(t_i) = \gamma_x(a) \in N$. Suppose $a = \infty$. Consider the integral curve $\gamma_{\tilde{x}_i}(t) = \gamma_{x_i}(t + t_i)$, which is defined on $(-t_i, 0]$. Let $\gamma_y(t)$ be the integral curve of ∇f with $\gamma_y(0) = y$. Since $t_i \to \infty$, $\{\gamma_{\tilde{x}_i}(t)\}$ converge uniformly to $\gamma_y(t)$ on $[-n, 0]$ for any $n > 0$. In particular, $\gamma_y(t)$ is defined on $(-\infty, 0]$. On the other hand, $f(\gamma_{x_i}(t))$ is strictly increasing in t for all i. Hence, $f(\gamma_{\tilde{x}_i}(t)) > 0$ on $(-t_i, 0]$, which implies

$f(\gamma_y(t)) \geq 0$ on $(-\infty, 0]$. By Proposition **??** (c), there exists a critical point of f in M, contradicting the assumption that f has no critical points.

Therefore, N is closed in $\mathrm{Int}(M)$ and hence $N = \mathrm{Int}(M)$. Since $f > 0$ along each $\gamma_x(t)$ on $(0, \infty)$, we conclude that $f > 0$ in $N = \mathrm{Int}(M)$. Hence, (M, g) is isometric to a spatial Schwarzschild manifold with positive mass outside its horizon by [**?**] or Proposition **??**. $\square$

Corollary **??** implies the following rigidity theorem.

Theorem 4.3. *Let (M, g) be a complete, connected, asymptotically flat 3-manifold without boundary, with finitely many ends. If there exists a static potential f on (M, g) which has no critical points, then (M, g) is isometric to either $(\mathbb{R}^3, g_0)$ or a spatial Schwarzschild manifold $(\mathbb{R}^3 \setminus \{0\}, (1 + \frac{m}{2|x|})^4 g_0)$ with $m > 0$.*

Proof. If $f^{-1}(0)$ has no compact component, then (M, g) is isometric to $(\mathbb{R}^3, g_0)$ by Proposition **??** (ii) (cf. the proof of Theorem **??**). Next, suppose $f^{-1}(0)$ has a compact component Σ. Cutting M along Σ, and let $(\tilde{M}, \tilde{g})$ be the metric completion of $(M \setminus \Sigma, g)$. Then either $\tilde{M}$ has two components whose boundary is isometric to Σ, or $\tilde{M}$ is connected with two boundary components that are isometric to Σ. Applying Corollary **??** to each component of $(\tilde{M}, \tilde{g})$ shows that $(\tilde{M}, \tilde{g})$ can not be connected, and hence has two components each of which is isometric to a spatial Schwarzschild manifold with positive mass outside its horizon. Since their boundaries are isometric, we conclude that (M, g) itself is isometric to a complete spatial Schwarzschild manifold with positive mass. $\square$

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