

Inducing π -partial characters with a given vertex

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Abstract

Let G be a solvable group. Let p be a prime and let Q be a p -subgroup of a subgroup V . Suppose $\varphi \in \text{IBr}(G)$. If either $|G|$ is odd or $p = 2$, we prove that the number of Brauer characters of H inducing φ with vertex Q is at most $|\text{N}_G(Q) : \text{N}_V(Q)|$.

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1 Introduction

Throughout this note, G is a finite group, and $\text{Irr}(G)$ is the set of irreducible characters of G . Suppose $\chi \in \text{Irr}(G)$ and H is a subgroup of G . It is easy to obtain an upper bound on the number of characters in $\text{Irr}(H)$ that induce χ . Let $\varphi_1, \dots, \varphi_n \in \text{Irr}(H)$ be the characters so that $\varphi_i^G = \chi$. Evaluating at 1, we obtain $\varphi_i(1) = \chi(1)/|G : H|$ for each i . By Frobenius reciprocity (Lemma 5.2 of [?]), each φ_i is a constituent of χ_H with multiplicity 1. Since there are n such characters occurring as constituents of χ_H , it follows that $n(\chi(1)/|G : H|) \leq \chi(1)$. We deduce that $n \leq |G : H|$, and we have an upper bound. If H is normal in G , this bound is obtained, and it is not particularly difficult to find nonnormal subgroups where this bound is obtained.

We now turn our attention to Brauer characters. Fix a prime p . We will write $\text{IBr}(G)$ for the irreducible p -Brauer characters of G . If $\varphi \in \text{IBr}(G)$, then it is easy to adapt the above proof to show that φ is induced by at most $|G : H|$ Brauer characters of H . However, associated with φ are certain p -subgroups of G called the vertex subgroups. When G is a p -solvable group, a p -subgroup Q of G is defined to be a vertex for φ if there is a subgroup U of G so that φ is induced by a Brauer character of U with p' -degree and Q is a Sylow subgroup of U . It is known that all the vertex subgroups of φ are conjugate in G . If φ is induced from $\tau \in \text{IBr}(H)$, it is easy to see that a vertex for τ is a vertex for φ . Thus, H contains some vertex Q for φ . Now, different Brauer characters of H that induce φ may have vertex subgroups that are not conjugate in H but are necessarily conjugate in G . Hence, one can ask the following question: Suppose $\varphi \in \text{IBr}(G)$ has vertex Q , and $Q \leq H$, how many

characters in $IBr(H)$ with vertex Q induce φ ? When either $|G|$ is odd or G is solvable and $p = 2$, we can obtain an upper bound for this question.

Theorem 1. *Let G be a solvable group and p a prime. Assume either $|G|$ is odd or $p = 2$. Let Q be a p -subgroup of H . If $\varphi \in IBr(G)$, then the number of Brauer characters of H with vertex Q that induce φ is at most $|N_G(Q) : N_H(Q)|$.*

At this time, we are not able to determine whether or not this theorem is true if we loosen the hypothesis that either $|G|$ is odd or $p = 2$. In other words, is the conclusion still true if G is a solvable group of even order and p is an odd prime.

- (1) * Alex mit-gebracht gestern einen Freund hat.
Alex along-brought yesterday a friend has
- (2) Was ist mit dem Buch passiert? ‘What happened to the book?’
 - a. Meiner FREUNDIN hab ich ’s geschenkt.
my.DAT girlfriend have I it given
‘I gave it to my girlfriend as a present.’
 - b. [Meiner Freundin geschenkt] hab ich’s.
- (3) Da ist nichts mehr zu machen. ‘Nothing can be done anymore.’
 - a. Es sieht so aus, also ob [der Löffel jetzt endgültig ab-gegeben ist].
it looks so out as if the spoon now definitively on-passed is
‘It looks like it is definitely over now.’
 - b. Der Löffel ist jetzt endgültig ab-gegeben.
‘It is definitely over now.’

This result was motivated by our work with J. P. Cossey. If we could prove the conclusion of Theorem ?? when p is odd, then we would be able to prove J. P. Cossey’s conjecture that the number of lifts of a Brauer character is bounded by the index of a vertex subgroup in the vertex subgroup when p is odd. Our argument can be found in the preprint [?].

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2 Results

We will in the more general setting of irreducible π -partial characters of a π -separable group G . We here briefly mention that if π is a set of primes and G is a π -separable group, one can define (see [?] for more details) a set of class functions $I_\pi(G)$ from the set G° (which consists of the elements of G whose order is divisible by only the primes in π) to \mathbb{C} that plays the role of $IBr(G)$, and in fact $I_\pi(G) = IBr(G)$ if $\pi = \{p'\}$, the complement of the prime p .

We start by considering vertices in Clifford correspondence (see Proposition 3.2 of [?]). Let G be a π -separable group. Let N be a normal subgroup of G . Fix $\varphi \in I_\pi(G)$. If $\alpha \in I_\pi(N)$ is a constituent of φ_N , then we write G_α for the stabilizer of α in G , and we write φ_α for the Clifford correspondent of φ with respect to α . In particular, the vertices of the Clifford correspondent form an orbit under the action of the normalizer of a particular vertex.

Lemma 2. *Let G be a π -separable group. Let N be a normal subgroup of G . Suppose that $\alpha \in I_\pi(N)$. Let $\varphi \in I_\pi(G)$ and $\hat{\varphi} \in I_\pi(G_\alpha)$ so that $\hat{\varphi}^G = \varphi$. Suppose that Q is a vertex for $\hat{\varphi}$. Then Q is a vertex for $\hat{\varphi}^g$ if and only if there exists $n \in N_G(Q)$ so that $G_\alpha g = G_\alpha n$.*

Proof. We first suppose that there exists $n \in N_G(Q)$ so that $G_\alpha g = G_\alpha n$. Thus, $g = tn$ for some $t \in G_\alpha$. We see that $\hat{\varphi}^g = \hat{\varphi}^{tn} = \hat{\varphi}^n$. We see that $Q = Q^n$ is a vertex for $\hat{\varphi}^n = \hat{\varphi}^g$.

Conversely, suppose that Q is a vertex for $\hat{\varphi}^g$. Then $Q^{g^{-1}}$ is a vertex for $\hat{\varphi}$. Since Q is also a vertex for $\hat{\varphi}$, we have $Q^{g^{-1}} = Q^t$ for some $t \in G_\alpha$. It follows that $Q = Q^{tg}$, and so, $tg \in N_G(Q)$. This implies that $tg = n$ for some $n \in N_G(Q)$. This implies that $n \in G_\alpha g$, and we conclude that $G_\alpha n = G_\alpha g$. \square

We continue to work in the context of the Clifford correspondence. In this case, we can get an exact count of the number of partial characters in N whose Clifford correspondent has vertex Q .

Corollary 3. *Let G be a π -separable group. Let N be a normal subgroup of G , let $\varphi \in I_\pi(G)$ have vertex Q , and suppose that β is an irreducible constituent of φ_N so that φ_β has vertex Q . Then $|\{\alpha \in I_\pi(N) \mid \varphi_\alpha \text{ has vertex } Q\}| = |N_G(Q) : N_{G_\beta}(Q)|$.*

Proof. By Lemma ??, we see that φ_α has Q as a vertex if and only if $\alpha = \beta^g$ where $g \in G$ and $g \in G_\beta n$ for some $n \in N_G(Q)$. Finally, we observe that $G_\beta n_1 = G_\beta n_2$ if and only if $N_{G_\beta}(Q)n_1 = N_{G_\beta}(Q)n_2$ for $n_1, n_2 \in N_G(Q)$. We have $|\{\alpha \in I_\pi(N) \mid \varphi_\alpha \text{ has vertex } Q\}| = |\{G_\beta n \mid n \in N_G(Q)\}| = |N_G(Q) : N_{G_\beta}(Q)|$. \square

We now look at the conditions of a minimal counterexample. For this we need to review and develop more notation. We make use of the canonical set of π -lifts, $B_\pi(G)$, that was defined in [?] by Isaacs. In other words, $B_\pi(G) \subseteq \text{Irr}(G)$ and the map $\chi \mapsto \chi^\circ$ is a bijection from $B_\pi(G)$ to $I_\pi(G)$. Closely related to this set is the subnormal nucleus which also was defined in [?]. To define the subnormal nucleus, we need the π -special characters. Let G be a π -separable group. A character $\chi \in \text{Irr}(G)$ is π -special if $\chi(1)$ is a π -number and for every subnormal group M of G , the irreducible constituents of χ_M have determinants that have π -order. Many of the basic results of π -special characters can be found in Section 40 of [?] and Chapter VI of [?]. One result that is proved is that if α is π -special and β is π' -special, then $\alpha\beta$ is necessarily irreducible. We say that χ is **factored** if $\chi = \alpha\beta$ where α is π -special and β is π' -special. We also note that if $\chi \in B_\pi(G)$ and N is normal in G , then the irreducible constituents of χ_N lie in $B_\pi(N)$.

If $\chi \in \text{Irr}(G)$, Isaacs constructs the subnormal vertex as follows. Let M be maximal so that M is subnormal in G and the irreducible constituents of χ_M are factored. Let μ be an irreducible constituent of χ_M and let T be the stabilizer of (M, μ) in G . Isaacs proved in [?] that there is a Clifford theorem for T . In other words, there is a unique character $\tau \in \text{Irr}(T \mid \mu)$ so that $\tau^G = \chi$. He also proved that (M, μ) is unique up to conjugacy, and so, (T, τ) is unique up to conjugacy. If $T = G$, then χ is π -factored and we take (G, χ) to be the subnormal nucleus of χ . If $T < G$, then inductively, the subnormal nucleus for τ is the subnormal nucleus for χ . We write (W, γ) for the subnormal nucleus of χ , and Isaacs showed that $\gamma^G = \chi$, γ is factored, and (W, γ) is unique up to conjugacy. A character $\chi \in \text{Irr}(G)$ is in $B_\pi(G)$ if and only if the character of its nucleus is π -special.

If Q is a π' -subgroup of G , then we use $I_\pi(G \mid Q)$ to denote the π -partial characters in $I_\pi(G)$ that have vertex Q . If $\varphi \in I_\pi(G)$ and $V \leq G$, then we write $I_\varphi(V \mid Q) = \{\eta \in I_\pi(V \mid Q) \mid \eta^G = \varphi\}$. We now find details about properties of a minimal counterexample. We will see that a counterexample cannot occur when either $|G|$ is odd or 2 is not in π . Our goal is find enough information so that we can either find a contradiction or build an example when $|G|$ is even and $2 \in \pi$.

Theorem 4. *Let G be a solvable group. Assume $\varphi \in I_\pi(G)$ has vertex Q , let V be a subgroup of G , and let N be the core of V in G . If G and V are chosen so that $|G| + |G : V|$ is minimal subject to the condition that $|I_\varphi(V \mid Q)| > |N_G(Q) : N_V(Q)|$, then the following are true:*

1. V is a nonnormal maximal subgroup of G ,
2. $|G : V|$ is a power of 2,
3. $2 \in \pi$,
4. $Q \leq V$,
5. $\varphi_N = a\alpha$ for some $\alpha \in IBr(N)$,
6. $\alpha(1)$ is a π -number,
7. if K is normal in G so that K/N is a chief factor for G , then α is fully ramified with respect to K/N .

Proof. If either $V = G$ or $I_\varphi(V \mid Q)$ is empty, then $|I_\varphi(V \mid Q)| \leq |N_G(Q) : N_V(Q)|$ contradicting the hypotheses. Thus, $V < G$ and $I_\varphi(V \mid Q)$ is not empty, and so, $Q \leq V$ and there exist characters in $I_\pi(V)$ that induce φ and have vertex Q .

We begin by showing that V is a maximal subgroup. Suppose that $V < M < G$ for some subgroup M . Let $I_\varphi(M \mid Q) = \{\eta_1, \dots, \eta_m\}$. Using minimality, we have $m = |I_\varphi(M \mid Q)| \leq |N_G(Q) : N_M(Q)|$. Suppose that $\zeta \in I_\varphi(V \mid Q)$, then $\zeta^M \in I_\pi(M)$ and ζ^M has Q as a vertex. Since $(\zeta^M)^G = \zeta^G = \varphi$, we see that $\zeta^M \in I_\varphi(M \mid Q)$. It follows that $\zeta^M = \eta_i$ for some i . We conclude that $|I_\varphi(V \mid Q)| \leq \sum_{i=1}^m |I_{\eta_i}(V \mid Q)|$. Since this contradicts our hypothesis, we obtain $|I_{\eta_i}(V \mid Q)| \leq |N_M(Q) : N_V(Q)|$. We deduce that

$$|I_\varphi(V \mid Q)| \leq m|N_M(Q) : N_V(Q)| \leq |N_G(Q) : N_M(Q)||N_M(Q) : N_V(Q)| = |N_G(Q) : N_V(Q)|.$$

Since this violates the hypotheses, V is maximal in G .

If V is normal in G , then either φ is induced from V or φ restricts irreducibly to V . If φ is induced from V , then we can apply Corollary ?? to see that $|I_\varphi(V \mid Q)| \leq |N_G(Q) : N_V(Q)|$ in violation of the hypotheses. If φ restricts irreducibly, then it cannot be induced from V , and we have seen that this is also a contradiction. We conclude that V is not normal in G .

Suppose $\alpha \in I_\pi(N)$ is a constituent of φ_N . We use $\varphi_\alpha \in I_\pi(G_\alpha \mid \alpha)$ to denote the Clifford correspondent for φ with respect to α (see Proposition 3.2 of [?] again). Write $\{\alpha \in I_\pi(N) \mid \varphi_\alpha \text{ has vertex } Q\} = \{\alpha_1, \dots, \alpha_k\}$, and let $\varphi_i = \varphi_{\alpha_i}$ and $G_i = G_{\alpha_i}$. By Lemma ??, we know that $k = |N_G(Q) : N_{G_i}(Q)|$.

Suppose $\eta \in \mathcal{I}_\varphi(V \mid Q)$. Denote $\{\beta \in \mathcal{I}_\pi(N) \mid \eta_\beta \text{ has vertex } Q\} = \{\beta_1, \dots, \beta_l\}$, and let $\eta_j = \eta_{\beta_j}$ and $V_j = V_{\beta_j}$. By Lemma ??, $l = |\mathcal{N}_V(Q) : \mathcal{N}_{V_i}(Q)|$.

We see that $(\eta_j)^G = ((\eta_j)^V)^G = \eta^G = \varphi$. This implies that $(\eta_j)^{G_{\beta_j}}$ is irreducible and has vertex Q . It follows that $\beta_j = \alpha_{i_j}$ for some i_j . We obtain $G_{\beta_j} = G_{i_j}$ and $(\beta_j)^{G_{i_j}} = \alpha_{i_j}$. Observe that $V_j = G_{i_j} \cap V$, and we denote this subgroup by $V_{i_j}^*$.

Now, we assume that $k > 1$, and we start to count. We see that $\eta \in \mathcal{I}_\varphi(G \mid Q)$ is induced by $|\mathcal{N}_V(Q) : \mathcal{N}_{V_{i_j}^*}(Q)|$ partial characters in $\bigcup \mathcal{I}_{\varphi_i}(V_{i_j}^* \mid Q)$. Because $G_i < G$, we may use minimality of $|G| + |G : V|$ to deduce $|\mathcal{I}_{\varphi_i}(V_{i_j}^* \mid Q)| \leq |\mathcal{N}_{G_i}(Q) : \mathcal{N}_{V_{i_j}^*}(Q)|$. We compute

$$|\mathcal{I}_\varphi(V \mid Q)| = \sum_{i=1}^k \frac{1}{|\mathcal{N}_V(Q) : \mathcal{N}_{V_i^*}(Q)|} |\mathcal{I}_{\varphi_i}(V_{i_j}^* \mid Q)| \leq \sum_{i=1}^k \frac{1}{|\mathcal{N}_V(Q) : \mathcal{N}_{V_i^*}(Q)|} |\mathcal{N}_{G_i}(Q) : \mathcal{N}_{V_i^*}(Q)|.$$

We determine that

$$\frac{1}{|\mathcal{N}_V(Q) : \mathcal{N}_{V_i^*}(Q)|} |\mathcal{N}_{G_i}(Q) : \mathcal{N}_{V_i^*}(Q)| = \frac{|\mathcal{N}_{G_i}(Q)|}{|\mathcal{N}_V(Q)|},$$

for each i . Notice that $|\mathcal{N}_{G_i}(Q)| = |\mathcal{N}_{G_1}(Q)|$ for all i and $k = |\mathcal{N}_G(Q) : \mathcal{N}_{G_1}(Q)|$. This yields

$$|\mathcal{I}_\varphi(V \mid Q)| \leq \sum_{i=1}^k k \frac{|\mathcal{N}_{G_1}(Q)|}{|\mathcal{N}_V(Q)|} = \frac{|\mathcal{N}_G(Q) : \mathcal{N}_{G_1}(Q)| |\mathcal{N}_{G_1}(Q)|}{|\mathcal{N}_V(Q)|} = |\mathcal{N}_G(Q) : \mathcal{N}_V(Q)|.$$

This contradicts the hypothesis. We deduce that $k = 1$, and α is invariant in G .

Set $\alpha = \alpha_1$, and let α^* be the character in $\mathcal{B}_\pi(N)$ satisfying $(\alpha^*)^o = \alpha$. Write $(W, \hat{\alpha})$ for the nucleus of α^* , and take T to be the stabilizer of $(W, \hat{\alpha})$ in G . By Lemma 2.3 of [?], there is a unique character $\hat{\varphi} \in \mathcal{I}(T \mid \hat{\alpha})$ so that $\hat{\varphi}^G = \varphi$ and Q is a vertex for $\hat{\varphi}$. Similarly, if $\eta \in \mathcal{I}_\varphi(V \mid Q)$, then there is a unique character $\hat{\eta} \in \mathcal{I}(T \cap V \mid \hat{\alpha})$ so that $\hat{\eta}^V = \eta$ and Q is a vertex for $\hat{\eta}$. Observe that $\hat{\eta}^T \in \mathcal{I}(T \mid \hat{\alpha})$ and induces φ , so $\hat{\eta}^T = \hat{\varphi}$. It follows that $|\mathcal{I}_\varphi(V \mid Q)| = |\mathcal{I}_{\hat{\varphi}}(T \cap V \mid Q)|$. If $T < G$, then we can use the minimality of $|G| + |G : V|$ to see that $|\mathcal{I}_{\hat{\varphi}}(T \cap V \mid Q)| \leq |\mathcal{N}_T(Q) : \mathcal{N}_{V \cap T}(Q)|$. By the diamond lemma, we have $|\mathcal{N}_T(Q) : \mathcal{N}_{V \cap T}(Q)| = |\mathcal{N}_T(Q) : V \cap \mathcal{N}_T(Q)| \leq |\mathcal{N}_G(Q) : \mathcal{N}_V(Q)|$. This contradicts the hypotheses, and so $T = G$.

We now have that $(W, \hat{\alpha})$ is G -invariant. By the construction of the subnormal, this implies that $W = N$. Since $\alpha^* \in \mathcal{B}_\pi(N)$, the nucleus for α^* has a character that is π -special. Thus, $\hat{\alpha}$ is π -special, and since $W = N$, we see that $\hat{\alpha} = \alpha^*$. In particular, $\hat{\alpha}$ is π -special. We deduce that $\alpha(1)$ is a π -number.

Take K normal in G so that K/N is a chief factor for G . This is the point where we use the fact that G is solvable to see that $G = VK$ and $V \cap K = N$ where K/N is an elementary abelian p -group for some prime p . (This is the only place we use the hypothesis that G is solvable in place of G being π -separable.) Let L/K be a chief factor for G . We know that $(|L : K|, |K : N|) = 1$ and $\mathbf{C}_{L \cap V/N}(K/N)$. (See Lemma 5.1 of [?] for a proof of this.) By Problem 6.12 of [?], either α^* extends to K or α^* is fully-ramified with respect to K/N .

Suppose first that α^* extends to K . Notice that multiplication by $\text{Irr}(K/L)$ is a transitive action on the irreducible constituents of $(\alpha^*)^K$. Also, $(V \cap K)/L$ acts compatibly on the irreducible constituents of $(\alpha^*)^K$ and on $\text{Irr}(K/L)$ where the action on $\text{Irr}(K/L)$ is coprime.

We can use Glauberman's lemma (Lemma 13.8 of [?]) to see that α^* has a $V \cap L$ -invariant extension. The corollary to Glauberman's lemma (Corollary 13.9 of [?]) can be applied to see that α^* has a unique $V \cap L$ -invariant extension δ . Since V permutes the $V \cap L$ -extensions of α^* , it follows that δ is V -invariant. We now use Corollary 4.2 of [?] to see that restriction is a bijection from $\text{Irr}(G \mid \beta)$ to $\text{Irr}(V \mid \alpha^*)$.

Let $\eta \in I_\varphi(V \mid Q)$ so that $\eta^G = \varphi$. We can find $\eta^* \in B_\pi(V)$ so that $(\eta^*)^o = \eta$. Since $(\eta^{*G})^o = (\eta^{*o})^G = \eta^G = \varphi \in \text{IBr}(G)$, we see that η^G is irreducible. On the other hand, $(\eta^{*o})_N = (\eta_N)^o = b\alpha$ for some integer b . Since the irreducible constituents of η^*_N lie in $B_\pi(N)$, we deduce that $\eta^* \in \text{Irr}(V \mid \alpha^*)$. But we saw that this implies that η^* extends to G . Since $V < G$, it is not possible for η^* to both extend to G and induce irreducibly. Therefore, we have a contradiction. We see that α^* (and hence, α) is fully ramified with respect to K/N . Notice that if p is not in π , then Corollary 6.28 of [?] applies and α^* extends to K . Therefore, $p \in \pi$.

We suppose that p is odd, and we work for a contradiction. Since α^* is fully-ramified with respect to K/N and $|K : N|$ has odd order, main theorem of [?] implies that no character in $\text{Irr}(V \mid \alpha)$ induces irreducibly to G . (A stronger theorem is proved in [?].) As in the previous paragraph, this implies that φ is not induced from V which contradicts the assumption that $I_\varphi(V \mid Q)$ is not empty. (This strongly uses the fact that p is odd. When $p = 2$, it is tempting to try use the correspondence in [?], but that correspondence does not preclude inducing characters in $\text{Irr}(G \mid \alpha)$ from V . In fact, $\text{GL}_2(3)$ is an example where this occurs.) We conclude that $p = 2$. Since $|G : V| = |K : N|$, we see that $|G : V|$ is a power of 2. This proves the theorem. \square

As a corollary, we obtain Theorem ?? stated for π -partial characters.

Corollary 5. *Let G be a solvable group. Assume either $|G|$ is odd or $2 \notin \pi$. Let Q be a π' -subgroup of G and suppose that $Q \leq V$. If $\varphi \in I_\pi(G)$, then $|I_\varphi(V \mid Q)| \leq |N_G(Q) : N_V(Q)|$.*

Proof. We suppose the result is not true. Let G be a counterexample with $|G| + |G : V|$ as in Theorem ?. By that result, we have that $|G : V|$ is a nontrivial power of 2 which is a contradiction if $|G|$ is odd. We also have $2 \in \pi$ which is a contradiction to $2 \notin \pi$. This proves the corollary. \square

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