WREATH PRODUCT OF GROUPS WITH INFINITE CONJUGACY CLASSES

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ABSTRACT. We characterise the group property of being with infinite conjugacy classes for wreath products of groups.

1. INTRODUCTION

A group is said to be with *infinite conjugacy classes* (or *icc*) if it is non trivial, and if all its conjugacy classes except $\{1\}$ are infinite. This property is motivated by the theory of Von Neumann algebra, since for any group Γ, a necessary and sufficient condition for its Von Neumann algebra to be a type $II - 1$ factor is that Γ be icc (cf. [ROIV]). The property of being icc has been characterized in several classes of groups : 3-manifolds and $PD(3)$ groups in [HP], and groups acting on Bass-Serre trees in [Co]. We will focus here on groups defined as a wreath product.

In the following, D, Q are groups, Ω is a Q-set and the group G is the wreath product $G = D \wr_\Omega Q$. That is, let us denote by $D^{(\Omega)}$ the group of maps from Ω to D having a finite support and by $\lambda : Q \longrightarrow Aut(D^{(\Omega)})$ the homomorphism defined by $\forall x \in Q$, $\forall \phi \in D^{(\Omega)}$, $\lambda(q)(\phi)(x) = \phi(q^{-1}x)$; the group G is defined to be the split extension $G = D^{(\Omega)} \rtimes Q$ associated with λ, in the sense that $\forall \phi \in D^{(\Omega)}$, $\forall q \in Q$, $q\phi q^{-1} = \lambda(q)(\phi)$.

Theorem 1. *Let $G = D \wr_\Omega Q$, with $D \neq \{1\}$; a necessary and sufficient condition for G to be icc is that on the one hand condition (i) is satisfied :*

   (i) 1 is the only element of $FC(Q)$ which fixes Ω pointwise.

and on the other at least one of the following conditions is satisfied :

   (ii) D is icc,
   (iii) all Q-orbits in Ω are infinite.

In particular, if the action of Q on Ω is free, then G is icc if and only if either D is icc or Q is infinite.

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2. Proof of the theorem

Let us first give some notations : if G is a group and x, y are element of G, then x^y is the element of G defined by $x^y = y^{-1}xy$. If H is a subgroup of G, then $x^H = \{x^y \mid y \in H\}$; in particular x^G denote the conjugacy class of x in G. The set of elements of G having a finite conjugacy class is a normal subgroup of G that we denote $FC(G)$.

(1) **German**

    a. Peter **hat** einen   Hund gekauft.
       PN   has a.ACC.M dog   bought

     'Peter bought a dog.'

    b. **Hat** Peter einen   Hund gekauft?
       has PN   a.ACC.M dog   bought

     'Did Peter buy a dog?'

(2) **German**

   Bist   du verrückt?
   be.PRS.IND.2SG 2SG crazy

   'Are you crazy?'[1]

Proof. In the following ε will denote the neutral element of $D^{(\Omega)}$, *i.e.* the element of $D^{(\Omega)}$ defined by $\forall x \in \Omega$, $\varepsilon(x) = 1$. Given $y \in \Omega$ and $d \in D$, the element ζ_d^y of $D^{(\Omega)}$ is defined by $\zeta_d^y(x) = d$ if $x = y$ and $\zeta_d^y(x) = 1$ otherwise.

We first suppose that G is icc and prove the necessary part of the assumption. Necessarily condition (i) is satisfied ; otherwise there would exist $q_0 \neq 1$ in $FC(Q)$ fixing Ω pointwise, and $\{(\varepsilon, q_0^q) \in G \mid q \in Q\}$ would be a finite subset of G invariant under conjugacy, contradicting that G is icc. We now prove that if condition (iii) does not hold then condition (ii) must hold. Suppose that Ω has a finite Q-orbit $\mathcal{O}$. If D would contain a finite conjugacy class $\xi \neq \{1\}$, then the set Φ of maps from Ω to ξ having their support in $\mathcal{O}$ would be finite and non empty, and the subset $\{(\phi, 1) \in G \mid \phi \in \Phi\}$ of G would be finite and invariant under conjugacy, which is impossible. Hence if G is icc, either condition (ii) or condition (iii) is satisfied, which proves the necessary part of the assumption.

We now prove the sufficient part of the assumption. So we suppose in the following that condition (i) is satisfied.

Suppose that condition (iii) is satisfied, *i.e.* all Q-orbits in Ω are infinite. Let $g = (\phi, q) \in G$; suppose first that $\phi \neq \varepsilon$. Its support is non empty and has an infinite Q-orbit so that $\phi \in D^{(\Omega)}$ has an infinity of translated under the action of $\lambda(Q)$, and it follows that g^Q and hence also g^G is infinite. Suppose now that $g = (\varepsilon, q)$ is non trivial in G. If $q \notin FC(Q)$ then g^Q and hence also g^G is infinite. If $q \in FC(Q)$ let $y \in \Omega$ be an element that q does not fix (existence follows from condition (i)), and $d \neq 1$ be an element of D. Consider the element $g' = (\zeta_d^y, 1)^{-1} g (\zeta_d^y, 1) = (\phi, q)$ of G ; g' is a conjugate of g and $\phi = \zeta_{d^{-1}}^y \zeta_d^{qy} \neq \varepsilon$. Hence the above argument applies to show that g^G is infinite. It follows that G is icc.

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[1]Colloquially, German also has the adjective *meschugge* 'crazy'.

Suppose that condition (ii) is satisfied, *i.e.* D is icc. Obviously each element $(\phi, 1)$ with $\phi \neq \varepsilon$ has an infinite conjugacy class. Let $g = (\phi, q)$ with $q \neq 1$. If $q \notin FC(Q)$ then g^Q and g^G are infinite. If $q \in FC(Q)$, condition (i) implies that q does not fix some element $y \in \Omega$. Consider for any $d \in D$ the conjugate of g that we denote by $g_d = (\zeta_d^y, 1)^{-1}(\phi, q)\ (\zeta_d^y, 1)$. If y does not lie in the support $Supp(\phi)$ of ϕ, then $g_d = (\phi \zeta_{d^{-1}}^y \zeta_d^{qy}, q)$. If $y \in Supp(\phi)$, say $\phi = \phi_0 \zeta_c^y$ and $y \notin Supp(\phi_0)$, then $g_d = (\phi_0 \zeta_{d^{-1}c}^y \zeta_d^{qy}, q)$. In any case, since $qy \neq y$ all g_d for $d \in D$ are distinct. Since D is icc D is infinite, and so g has an infinite conjugacy class. Hence G is icc. $\qquad \square$

References

[Co]   Y. DE CORNULIER, *Infinite conjugacy classes in groups acting on trees*, Pr publication 2005.

[HP]   P. DE LA HARPE et J.-P. PR AUX, *Groupes fondamentaux des vari t s de dimension 3 et alg bres d'op rateurs*, Pr publication arXiv:math.GR/0509449 v1 (2005).

[ROIV] F.J. MURRAY et J. VON NEUMANN, *On rings of operators*, IV, Annals of Math. **44** (1943), 716–808.