ISTA 421/521
Introduction to Machine Learning

Lecture 11:
The role of Priors

Clay Morrison
clayton@sista.arizona.edu
Gould-Simpson 819
Phone 621-6609

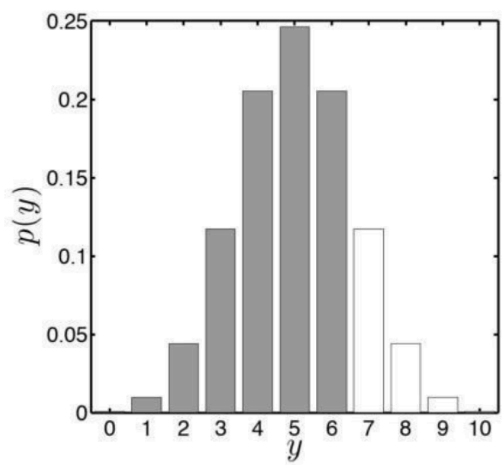
30 September 2014

The Coin Game

Place \$1 bet
Flip coin 10 times
6 or fewer heads, you win your \$1 + \$1
More than 6, you loose your \$1

Binomial Distribution

$$P(Y = y) = \binom{N}{y} r^y (1 - r)^{N-y}$$



Binomial & Beta are Conjugate !

$$P(Y=y) = \binom{N}{y} r^y (1-r)^{N-y} \qquad p(r) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} r^{\alpha-1}(1-r)^{\beta-1}$$

$$p(r|y_N) \propto P(y_N|r)p(r)$$

$$p(r|y_N) \propto \left[\binom{N}{y_N} r^{y_N}(1-r)^{N-y_N}\right] \times \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} r^{\alpha-1}(1-r)^{\beta-1}\right]$$

Computing the Posterior Directly

We can do this with the conjugate Beta prior and Binomial Likelihood

$$p(r|y_N) \propto \left[\binom{N}{y_N} r^{y_N}(1-r)^{N-y_N}\right] \times \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} r^{\alpha-1}(1-r)^{\beta-1}\right]$$

$$\begin{aligned} p(r|y_N) &\propto \left[\binom{N}{y_N} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \times \left[r^{y_N} r^{\alpha-1}(1-r)^{N-y_N}(1-r)^{\beta-1}\right] \\ &\propto r^{y_N+\alpha-1}(1-r)^{N-y_N+\beta-1} \\ &\propto r^{\delta-1}(1-r)^{\gamma-1} \end{aligned}$$

where $\delta = y_N + \alpha$ and $\gamma = N - y_N + \beta$.

Book misses this

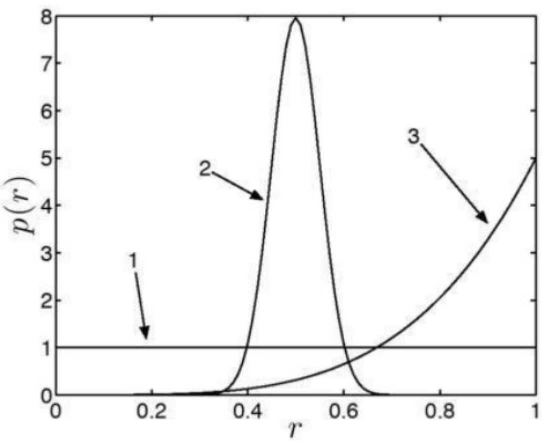
$$p(r|y_N) = \frac{\Gamma(\alpha+\beta+N)}{\Gamma(\alpha+y_N)\Gamma(\beta+N-y_N)} r^{\alpha+y_N-1}(1-r)^{\beta+N-y_N-1}$$

Effect of 3 Different Priors on Posterior

$$p(r|y_N) = \frac{P(y_N|r)p(r)}{P(y_N)}$$

(2) **The Prior**: $p(r)$

"Allows us to express any belief we have in the value of *r* ***before*** we see any data."



1) We don't know anything about the coins or the stall owner $\alpha = 1,\ \beta = 1$
2) We think the coin (and the stall owner) is fair $\alpha = 50,\ \beta = 50$
3) We think the coin (and the stall owner) is biased to give more heads $\alpha = 5,\ \beta = 1$

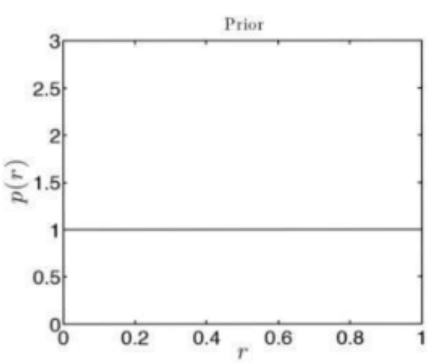
$$p(r) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} r^{\alpha-1}(1-r)^{\beta-1}$$

Scenario 1: Don't know anything

$$p(r) = \mathcal{B}(\alpha,\beta) \qquad \mathbf{E}_{p(r)}\{R\} = \frac{\alpha}{\alpha+\beta} \qquad \text{var}\{R\} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Scenario 1 prior: $\alpha = 1,\ \beta = 1$ $\mathbf{E}_{p(r)}\{R\} = \frac{\alpha}{\alpha+\beta} = \frac{1}{2}$ $\text{var}\{R\} = \frac{1}{12}$

General posterior: $\delta = \alpha + y_N$ $\gamma = \beta + N - y_N$ $p(r|y_N) = \mathcal{B}(\delta,\gamma)$



(a) $\alpha = 1, \beta = 1$

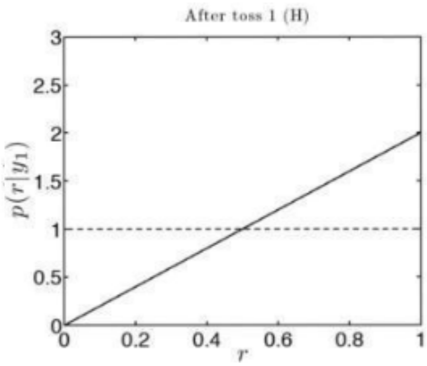
Observations:

Scenario 1: Don’t know anything

$$p(r) = \mathcal{B}(\alpha, \beta) \qquad \mathbf{E}_{p(r)}\{R\} = \frac{\alpha}{\alpha+\beta} \qquad \mathrm{var}\{R\} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Scenario 1 prior: $\alpha = 1, \beta = 1 \quad \mathbf{E}_{p(r)}\{R\} = \frac{\alpha}{\alpha+\beta} = \frac{1}{2} \quad \mathrm{var}\{R\} = \frac{1}{12}$

General posterior: $\delta = \alpha + y_N \quad \gamma = \beta + N - y_N \quad p(r|y_N) = \mathcal{B}(\delta, \gamma)$



(b) $\delta = 2, \gamma = 1$

Observations: H

$$\delta = 1 + 1 = 2$$
$$\gamma = 1 + 1 - 1 = 1$$

Posterior: $\mathbf{E}_{p(r|y_N)}\{R\} = \frac{2}{3}$

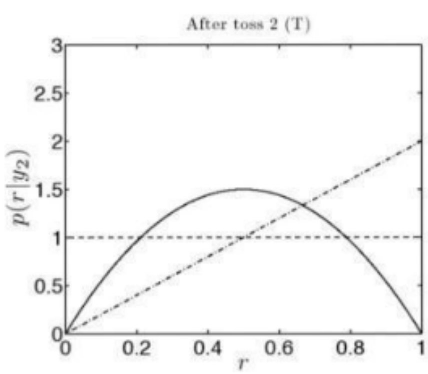
$\mathrm{var}\{R\} = \frac{1}{18}$

Scenario 1: Don’t know anything

$$p(r) = \mathcal{B}(\alpha, \beta) \qquad \mathbf{E}_{p(r)}\{R\} = \frac{\alpha}{\alpha+\beta} \qquad \mathrm{var}\{R\} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Scenario 1 prior: $\alpha = 1, \beta = 1 \quad \mathbf{E}_{p(r)}\{R\} = \frac{\alpha}{\alpha+\beta} = \frac{1}{2} \quad \mathrm{var}\{R\} = \frac{1}{12}$

General posterior: $\delta = \alpha + y_N \quad \gamma = \beta + N - y_N \quad p(r|y_N) = \mathcal{B}(\delta, \gamma)$



(c) $\delta = 2, \gamma = 2$

Observations: H T

$$\delta = 1 + 1 = 2$$
$$\gamma = 1 + 2 - 1 = 2$$

Posterior: $\mathbf{E}_{p(r|y_N)}\{R\} = \frac{1}{2}$

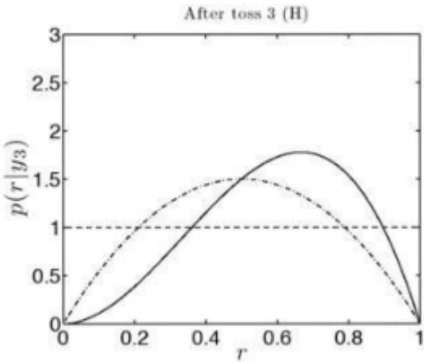
$\mathrm{var}\{R\} = \frac{1}{20}$

Scenario 1: Don't know anything

$$p(r) = \mathcal{B}(\alpha, \beta) \qquad \mathbf{E}_{p(r)}\{R\} = \frac{\alpha}{\alpha+\beta} \qquad \text{var}\{R\} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Scenario 1 prior: $\alpha = 1, \beta = 1$ $\mathbf{E}_{p(r)}\{R\} = \frac{\alpha}{\alpha+\beta} = \frac{1}{2}$ $\text{var}\{R\} = \frac{1}{12}$

General posterior: $\delta = \alpha + y_N$ $\gamma = \beta + N - y_N$ $p(r|y_N) = \mathcal{B}(\delta, \gamma)$



(d) $\delta = 3, \gamma = 2$

Observations: H T H

$$\delta = 1 + 2 = 3$$
$$\gamma = 1 + 3 - 2 = 2$$

Posterior: $\mathbf{E}_{p(r|y_N)}\{R\} = \frac{3}{5}$

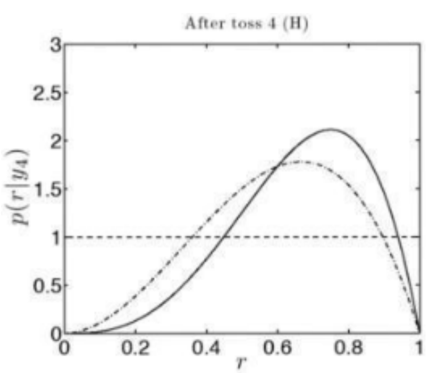
$\text{var}\{R\} = \frac{1}{25}$

Scenario 1: Don't know anything

$$p(r) = \mathcal{B}(\alpha, \beta) \qquad \mathbf{E}_{p(r)}\{R\} = \frac{\alpha}{\alpha+\beta} \qquad \text{var}\{R\} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Scenario 1 prior: $\alpha = 1, \beta = 1$ $\mathbf{E}_{p(r)}\{R\} = \frac{\alpha}{\alpha+\beta} = \frac{1}{2}$ $\text{var}\{R\} = \frac{1}{12}$

General posterior: $\delta = \alpha + y_N$ $\gamma = \beta + N - y_N$ $p(r|y_N) = \mathcal{B}(\delta, \gamma)$



(e) $\delta = 4, \gamma = 2$

Observations: H T H H

$$\delta = 1 + 3 = 4$$
$$\gamma = 1 + 4 - 3 = 2$$

Posterior: $\mathbf{E}_{p(r|y_N)}\{R\} = \frac{2}{3}$

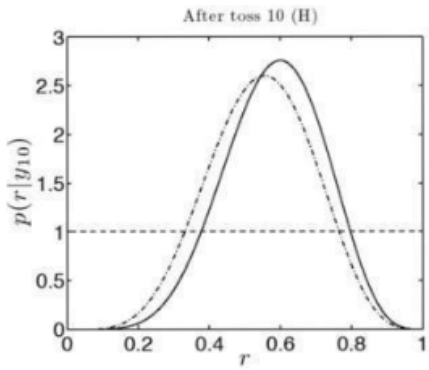
$\text{var}\{R\} = \frac{2}{63} = 0.0317$

Scenario 1: Don't know anything

$$p(r) = \mathcal{B}(\alpha, \beta) \qquad \mathbf{E}_{p(r)}\{R\} = \frac{\alpha}{\alpha+\beta} \qquad \text{var}\{R\} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Scenario 1 prior: $\alpha = 1, \beta = 1 \quad \mathbf{E}_{p(r)}\{R\} = \frac{\alpha}{\alpha+\beta} = \frac{1}{2} \quad \text{var}\{R\} = \frac{1}{12}$

General posterior: $\delta = \alpha + y_N \quad \gamma = \beta + N - y_N \quad p(r|y_N) = \mathcal{B}(\delta, \gamma)$



(f) $\delta = 7, \gamma = 5$

Observations: H T H H H H T T T H

$$\delta = 1 + 6 = 7$$
$$\gamma = 1 + 10 - 6 = 5$$

Posterior: $\mathbf{E}_{p(r|y_N)}\{R\} = \frac{7}{12} = 0.5833$

$\text{var}\{R\} = 0.0187$

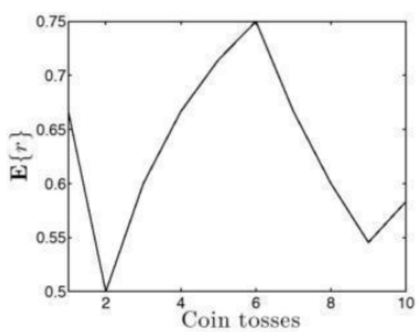
Scenario 1: Don't know anything

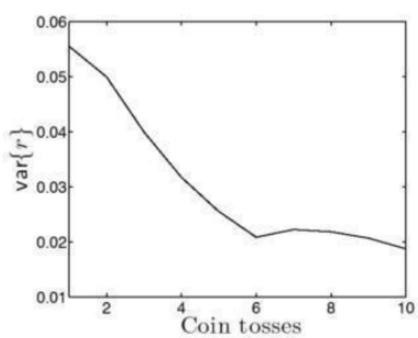
$$p(r) = \mathcal{B}(\alpha, \beta) \qquad \mathbf{E}_{p(r)}\{R\} = \frac{\alpha}{\alpha+\beta} \qquad \text{var}\{R\} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Scenario 1 prior: $\alpha = 1, \beta = 1 \quad \mathbf{E}_{p(r)}\{R\} = \frac{\alpha}{\alpha+\beta} = \frac{1}{2} \quad \text{var}\{R\} = \frac{1}{12}$

General posterior: $\delta = \alpha + y_N \quad \gamma = \beta + N - y_N \quad p(r|y_N) = \mathcal{B}(\delta, \gamma)$

Observations: H T H H H H T T T H





Uses of our Posterior Density of *r*

The posterior density encapsulates ***all*** of the information we have about *r*

$$p(r|y_N) = \mathcal{B}(\delta, \gamma)$$

We can use a **point estimate** of *r* by extracting a single value $\hat{r}$ from the posterior density.

We can then compare the expected probability of winning with the probability of winning computed from the single value of *r*.

What should be our single value $\hat{r}$ chosen from the posterior distribution of *r* ?

6 H out of 10 flips
$\delta = 7$, $\gamma = 5$

$$\hat{r} = \mathbf{E}_{p(r|y_N)}\{R\} = \frac{\delta}{\delta+\gamma} = \frac{7}{12}$$

$$\begin{aligned} P(Y_{\text{new}} \leq 6|\hat{r}) &= 1 - \sum_{y_{\text{new}}=7}^{10} P(Y_{\text{new}} = y_{\text{new}}|\hat{r}) \\ &= 1 - 0.3414 \\ &= 0.6586 \end{aligned}$$

SISTA 13

Uses of our Posterior Density of *r*

Let's use ***all*** of the posterior information**!**

$$\begin{aligned} \mathbf{E}_{p(r|y_N)}\{P(Y_{\text{new}} \leq 6|r)\} &= \mathbf{E}_{p(r|y_N)}\{1 - P(Y_{\text{new}} \geq 7|r)\} \\ &= 1 - \mathbf{E}_{p(r|y_N)}\{P(Y_{\text{new}} \geq 7|r)\} \\ &= 1 - \mathbf{E}_{p(r|y_N)}\left\{\sum_{y_{\text{new}}=7}^{y_{\text{new}}=10} P(Y_{\text{new}} = y_{\text{new}}|r)\right\} \\ &= 1 - \sum_{y_{\text{new}}=7}^{y_{\text{new}}=10} \mathbf{E}_{p(r|y_N)}\{P(Y_{\text{new}} = y_{\text{new}}|r)\}. \end{aligned}$$

$$\begin{aligned} \mathbf{E}_{p(r|y_N)}\{P(Y_{\text{new}} = y_{\text{new}}|r)\} &= \int_{r=0}^{r=1} P(Y_{\text{new}} = y_{\text{new}}|r)p(r|y_N)\, dr \\ &= \int_{r=0}^{r=1} \left[\binom{N_{\text{new}}}{y_{\text{new}}} r^{y_{\text{new}}}(1-r)^{N_{\text{new}}-y_{\text{new}}}\right]\left[\frac{\Gamma(\delta+\gamma)}{\Gamma(\delta)\Gamma(\gamma)} r^{\delta-1}(1-r)^{\gamma-1}\right] dr \\ &= \binom{N_{\text{new}}}{y_{\text{new}}} \frac{\Gamma(\delta+\gamma)}{\Gamma(\delta)\Gamma(\gamma)} \int_{r=0}^{r=1} r^{y_{\text{new}}+\delta-1}(1-r)^{N_{\text{new}}-y_{\text{new}}+\gamma-1}\, dr. \end{aligned}$$

$$\int_{r=0}^{r=1} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} r^{\alpha-1}(1-r)^{\beta-1} dr = 1 \qquad \int_{r=0}^{r=1} r^{\alpha-1}(1-r)^{\beta-1} dr = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\mathbf{E}_{p(r|y_N)}\{P(Y_{\text{new}} = y_{\text{new}}|r)\} = \binom{N_{\text{new}}}{y_{\text{new}}} \frac{\Gamma(\delta+\gamma)}{\Gamma(\delta)\Gamma(\gamma)} \frac{\Gamma(\delta+y_{\text{new}})\Gamma(\gamma+N_{\text{new}}-y_{\text{new}})}{\Gamma(\delta+\gamma+N_{\text{new}})}$$

14

Uses of our Posterior Density of *r*

Let's use ***all*** of the posterior information**!**

After 10 tosses, we have 6 heads and 4 tails; so N=10, δ=7, γ=5. Plug in!

$$\begin{aligned}\mathbf{E}_{p(r|y_N)}\left\{P(Y_{\text{new}} \le 6|r)\right\} &= 1 - \sum_{y_{\text{new}}=7}^{y_{\text{new}}=10} \mathbf{E}_{p(r|y_N)}\left\{P(Y_{\text{new}} = y_{\text{new}}|r)\right\} \\ &= 1 - 0.3945 \\ &= 0.6055.\end{aligned}$$

Comparing this with the point estimate (0.6586), we see that both predict we will win more often than not.

This agrees with the evidence: the one person we have fully observed got 6 heads, 4 tails

The point estimate gives a higher probability; ignoring the posterior uncertainty makes it more likely that we will win.

$$\mathbf{E}_{p(r|y_N)}\left\{P(Y_{\text{new}} = y_{\text{new}}|r)\right\} = \binom{N_{\text{new}}}{y_{\text{new}}} \frac{\Gamma(\delta+\gamma)}{\Gamma(\delta)\Gamma(\gamma)} \frac{\Gamma(\delta+y_{\text{new}})\Gamma(\gamma+N_{\text{new}}-y_{\text{new}})}{\Gamma(\delta+\gamma+N_{\text{new}})}$$

Uses of our Posterior Density of *r*

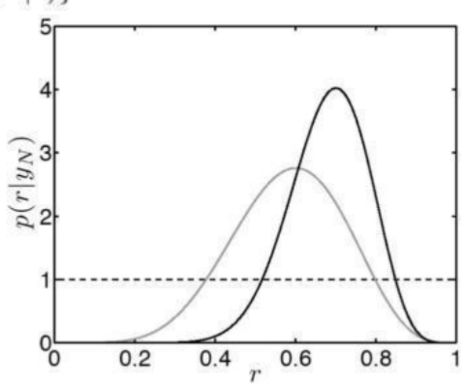
Observations: H T H H H H T T T H H H T T H H H H H H

After 10 MORE tosses, we have a total of 14 heads and 6 tails: N=20, δ=15, γ=7. Plug in!

$$\mathbf{E}_{p(r|y_N)}\left\{R\right\} = 0.6818, \text{var}\{R\} = 0.0094$$

$$\mathbf{E}_{p(r|y_N)}\left\{P(Y_{\text{new}} \le 6|r)\right\} = 0.4045$$

($P(Y_{\text{new}} \le 6|\widehat{r}) = 0.3994$ The not-fully-Bayesian estimate)



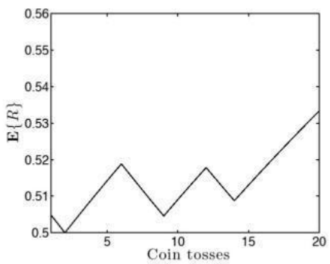
$$\mathbf{E}_{p(r|y_N)}\left\{P(Y_{\text{new}} = y_{\text{new}}|r)\right\} = \binom{N_{\text{new}}}{y_{\text{new}}} \frac{\Gamma(\delta+\gamma)}{\Gamma(\delta)\Gamma(\gamma)} \frac{\Gamma(\delta+y_{\text{new}})\Gamma(\gamma+N_{\text{new}}-y_{\text{new}})}{\Gamma(\delta+\gamma+N_{\text{new}})}$$

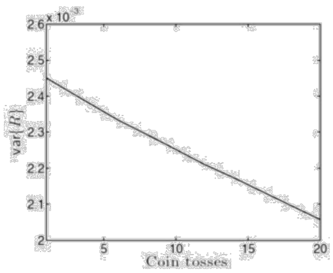
Scenario 2: Fair Coin

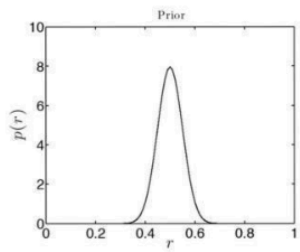
$\alpha = \beta = 50$

Observations:

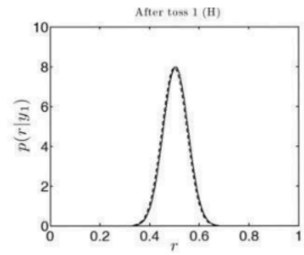
H T H H H H T T T H
H H T T H H H H H H



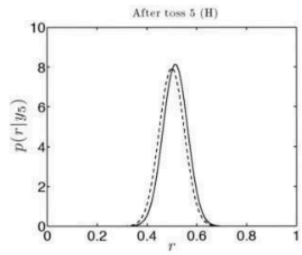




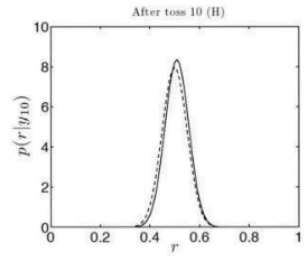
(a) $\alpha = 50, \beta = 50$



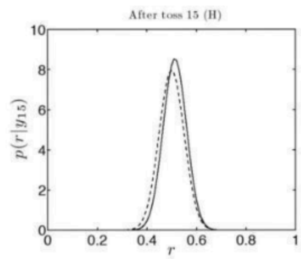
(b) $\delta = 51, \gamma = 50$



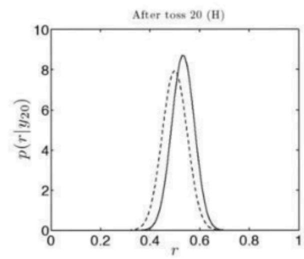
(c) $\delta = 54, \gamma = 51$



(d) $\delta = 56, \gamma = 54$



(e) $\delta = 59, \gamma = 56$



(f) $\delta = 64, \gamma = 56$

Scenario 2: Fair Coin

$\alpha = \beta = 50$

Observations:

H T H H H H T T T H
H H T T H H H H H H

$$\mathbf{E}_{p(r|y_N)} \{P(Y_{\text{new}} \leq 6|r)\}$$

$$\delta = \alpha + y_N = 50 + 14 = 64$$
$$\gamma = \beta + N - y_N = 50 + 20 - 14 = 56$$

$$\widehat{r} = 64/(64 + 56) = 0.5333$$
$$P(Y_{\text{new}} \leq 6|\widehat{r}) = 0.7680$$

Compare differences between point estimate and proper expectation

Scenario 1: $|\mathbf{E}_{p(r|y_N)} \{P(Y_{\text{new}} \leq 6|r)\} - P(Y_{\text{new}} \leq 6|\widehat{r})| = 0.0531$

Scenario 2: $|\mathbf{E}_{p(r|y_N)} \{P(Y_{\text{new}} \leq 6|r)\} - P(Y_{\text{new}} \leq 6|\widehat{r})| = 0.0101$

Theoretical note: So, what is the relationship of the point estimate to the full integral over r? Imagine the variance decreasing to such an extent that there was a single value of *r* that had probability 1 of occurring with $p(r|y_N)$ zero everywhere else.

$$\mathbf{E}_{p(r|y_N)} \{P(Y_{\text{new}} \leq 6|r)\} = \int_{r=0}^{r=1} P(Y_{\text{new}} \leq 6|r) p(r|y_N) \, dr$$
$$= P(Y_{\text{new}} \leq 6|\widehat{r})$$

Scenario 3: Biased Coin

$\alpha = 5, \ \beta = 1$

Observations:

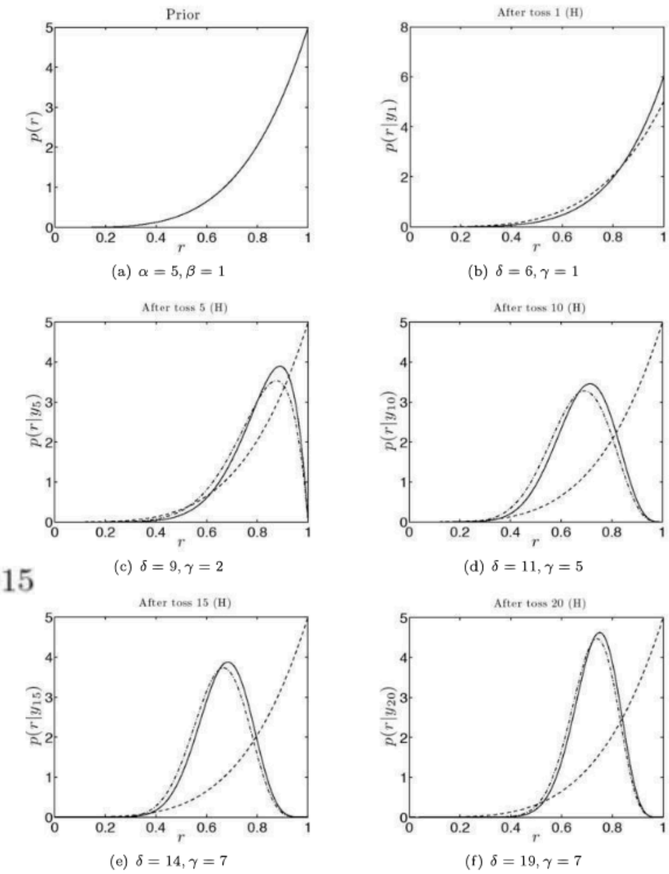
`H T H H H H T T T H`
`H H T T H H H H H H`

Initially:

$$\mathbf{E}_{p(r)}\{R\} = \frac{\alpha}{\alpha+\beta} = 5/6$$

$$\delta = \alpha + y_N = 5 + 14 = 19$$
$$\gamma = 1 + N - y_N = 1 + 20 - 14 = 7$$
$$\mathbf{E}_{p(r|y_N)}\{P(Y_{\text{new}} \leq 6|r)\} = 0.2915$$

$$\widehat{r} = 19/(19+7) = 0.7308$$
$$P(Y_{\text{new}} \leq 6|\widehat{r}) = 0.2707$$



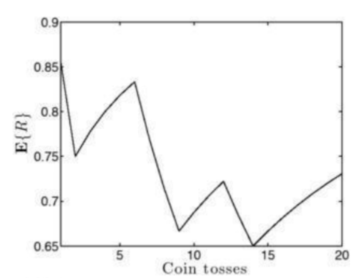
(a) $\alpha = 5, \beta = 1$ (b) $\delta = 6, \gamma = 1$ (c) $\delta = 9, \gamma = 2$ (d) $\delta = 11, \gamma = 5$ (e) $\delta = 14, \gamma = 7$ (f) $\delta = 19, \gamma = 7$

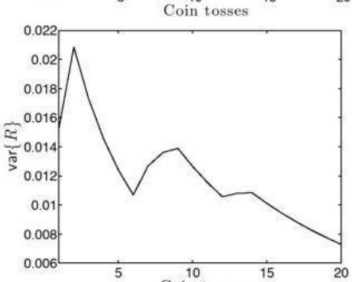
Scenario 3: Biased Coin

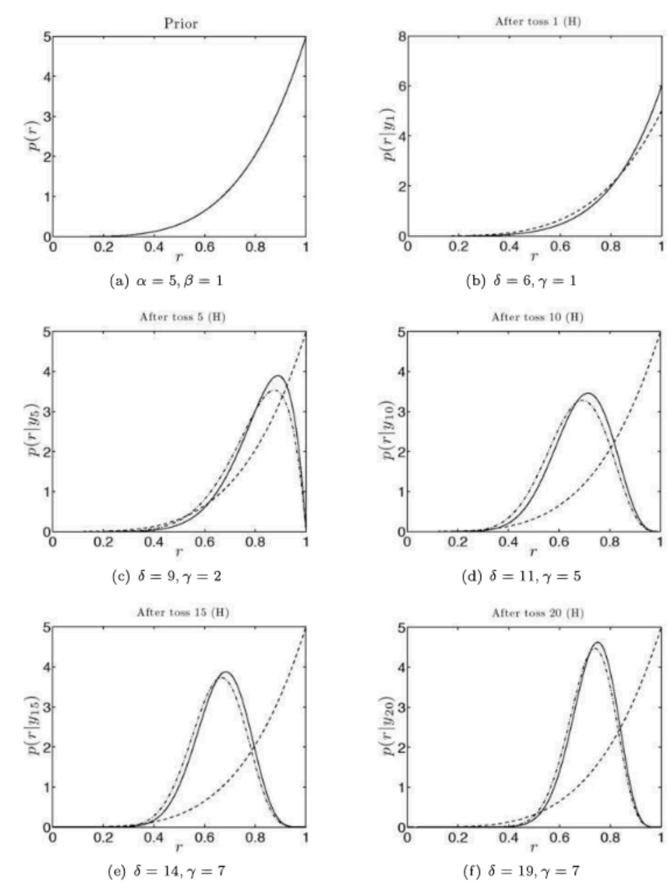
$\alpha = 5, \ \beta = 1$

Observations:

`H T H H H H T T T H`
`H H T T H H H H H H`







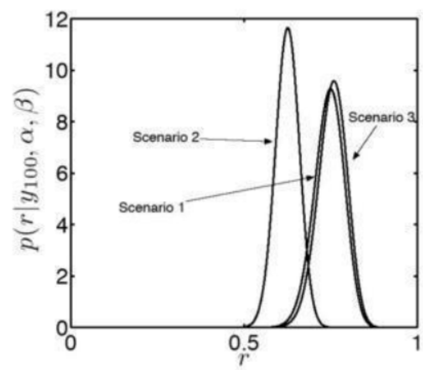
(a) $\alpha = 5, \beta = 1$ (b) $\delta = 6, \gamma = 1$ (c) $\delta = 9, \gamma = 2$ (d) $\delta = 11, \gamma = 5$ (e) $\delta = 14, \gamma = 7$ (f) $\delta = 19, \gamma = 7$

Summary

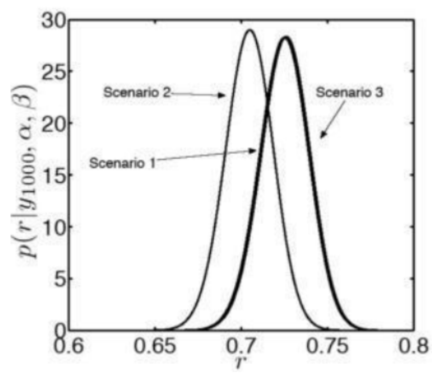
1. No prior knowledge: $\mathbf{E}_{p(r|y_N)}\left\{P(Y_{\text{new}} \leq 6|r)\right\} = 0.4045$

2. Fair coin: $\mathbf{E}_{p(r|y_N)}\left\{P(Y_{\text{new}} \leq 6|r)\right\} = 0.7579$

3. Biased coin: $\mathbf{E}_{p(r|y_N)}\left\{P(Y_{\text{new}} \leq 6|r)\right\} = 0.2915.$



After 100 tosses



After 1000 tosses